UNIVERSIDADE FEDERAL DO PARANÁ

ANTONIO GUIMARÃES LEITE

ASYMPTOTIC BEHAVIOR IN THREE LOCALLY DISSIPATIVE ELASTIC MODELS

CURITIBA
2025

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Tese apresentada ao Curso de Pós-Graduação em Matemática, Setor de Exatas, da Universidade Federal do Paraná, como requisito parcial à obtenção do título de Doutor em Matemãtica.

Orientador: Prof. Dr. Higidio Portillo Oquendo
Coorientadora: Profa. Dra. María Rosario Astudillo Rojas

CURITIBA
2025

DADOS INTERNACIONAIS DE CATALOGAÇÃO NA PUBLICAÇÃO (CIP)
UNIVERSIDADE FEDERAL DO PARANÁ
SISTEMA DE BIBLIOTECAS – BIBLIOTECA DE CIÊNCIA E TECNOLOGIA

Leite, Antônio Guimarães
Asymptotic behavior in three locally dissipative elastic models / Antônio Guimarães Leite. – Curitiba, 2025.
1 recurso on-line : PDF.

Tese (Doutorado) - Universidade Federal do Paraná, Setor de Exatas, Programa de Pós-Graduação em Matemática.

Orientador: Higidio Portillo Oquendo
Coorientador: María Rosario Astudillo Rojas

1. Operadores polinomiais. 2. Funções exponenciais. 3. Elasticidade. 4. Termoelasticidade. I. Universidade Federal do Paraná. II. Programa de Pós-Graduação em Matemática. III. Oquendo, Higidio Portillo. IV. Rojas, María Rosario Astudillo. V. Título.

Bibliotecário: Douglas Lenon da Silva CRB-9/1892

ATA Nº55-D

ATA DE SESSÃO PÚBLICA DE DEFESA DE DOUTORADO PARA A OBTENÇÃO DO GRAU DE DOUTOR EM MATEMÁTICA

No dia doze de agosto de dois mil e vinte e cinco às 13:30 horas, na sala virtual, Plataforma Teams, e sala PA 300, Bloco PA, Centro Politécnico, UFPR, foram instaladas as atividades pertinentes ao rito de defesa de tese do doutorando **ANTONIO GUIMARÃES LEITE**, intitulada: **Asymptotic behavior in three locally dissipative elastic models**, sob orientação do Prof. Dr. HIGIDIO PORTILLO OQUENDO. A Banca Examinadora, designada pelo Colegiado do Programa de Pós-Graduação MATEMÁTICA da Universidade Federal do Paraná, foi constituída pelos seguintes Membros: HIGIDIO PORTILLO OQUENDO (UNIVERSIDADE FEDERAL DO PARANÁ), CARLOS ALBERTO RAPOSO DA CUNHA (UNIVERSIDADE FEDERAL DO PARÁ), MARCELO MOREIRA CAVALCANTI (UNIVERSIDADE ESTADUAL DE MARINGÁ), OCTAVIO PAULO VERA VILLAGRÁN (UNIVERSIDAD DE TARAPACá), MARCIO ANTONIO JORGE DA SILVA (UNIVERSIDADE ESTADUAL DE LONDRINA). A presidência iniciou os ritos definidos pelo Colegiado do Programa e, após exarados os pareceres dos membros do comitê examinador e da respectiva contra argumentação, ocorreu a leitura do parecer final da banca examinadora, que decidiu pela APROVAÇÃO. Este resultado deverá ser homologado pelo Colegiado do programa, mediante o atendimento de todas as indicações e correções solicitadas pela banca dentro dos prazos regimentais definidos pelo programa. A outorga de título de doutor está condicionada ao atendimento de todos os requisitos e prazos determinados no regimento do Programa de Pós-Graduação. Nada mais havendo a tratar a presidência deu por encerrada a sessão, da qual eu, HIGIDIO PORTILLO OQUENDO, lavrei a presente ata, que vai assinada por mim e pelos demais membros da Comissão Examinadora.

CURITIBA, 12 de Agosto de 2025.

Assinatura Eletrônica
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HIGIDIO PORTILLO OQUENDO
Presidente da Banca Examinadora

Assinatura Eletrônica
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CARLOS ALBERTO RAPOSO DA CUNHA
Avaliador Externo (UNIVERSIDADE FEDERAL DO PARÁ)

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Avaliador Externo (UNIVERSIDADE ESTADUAL DE MARINGÁ)

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MINISTÉRIO DA EDUCAÇÃO
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TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação MATEMÁTICA da Universidade Federal do Paraná foram convocados para realizar a arguição da tese de Doutorado de **ANTONIO GUIMARÃES LEITE**, intitulada: **Asymptotic behavior in three locally dissipative elastic models**, sob orientação do Prof. Dr. HIGIDIO PORTILLO OQUENDO, que após terem inquirido o aluno e realizada a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.
A outorga do título de doutor está sujeita à homologação pelo colegiado, ao atendimento de todas as indicações e correções solicitadas pela banca e ao pleno atendimento das demandas regimentais do Programa de Pós-Graduação.

CURITIBA, 12 de Agosto de 2025.

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13/08/2025 11:40:17.0
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ACKNOWLEDGEMENTS

The journey toward the completion of this thesis was marked by challenges, learning experiences, and, above all, by people who left a profound impact on my academic and personal development.

First and foremost, I thank God for the strength, health, and resilience throughout every moment of this journey.

To my advisor, Higidio, and my co-advisor, Maria, for their steady guidance, patience, constant encouragement, and valuable scientific contributions. Your dedication and commitment were fundamental to the development of this work and to my growth as a researcher.

To the professors and colleagues from PPGM-UFPR and PROMAT-UFS, for the stimulating environment, academic discussions, and support throughout the course.

To my family, for all the love, unconditional support, and understanding during this process. To my parents, Pedro and Lourdes, for teaching me the value of education and honesty. To my siblings — Daguia, João, and Socorro — for all their emotional support. To the love of my life, Anne, for being by my side at every stage, even during the hardest moments. To her parents, Angerleide and Naldo, for welcoming me with such warmth and encouragement throughout this journey.

To the Coordination for the Improvement of Higher Education Personnel (CAPES), for the financial support, and to the university, for being my second home during this period.

Finally, I thank all those who, in one way or another, contributed to making this work possible. To each one of you, my most sincere thank you.

RESUMO

Este trabalho investiga três problemas de transmissão distintos em estruturas unidimensionais compostas por regiões com propriedades mecânicas heterogêneas. O modelo considerado envolve, em todos os casos, três subdomínios: um material elástico sem dissipação, um material viscoelástico governado pela lei constitutiva do tipo Kelvin-Voigt e um terceiro componente com dissipação variável, assumindo ora o caráter friccional, ora o efeito de memória, ora ainda um comportamento termoelástico. A análise desenvolvida tem como objetivo compreender a influência do tipo e da localização desses mecanismos dissipativos na evolução temporal da energia do sistema. Os resultados obtidos mostram que a taxa de decaimento da energia não depende apenas da presença da dissipação, mas também da posição em que esta é introduzida, o que evidencia aspectos sutis na formulação e no tratamento de problemas de transmissão. Dessa forma, este estudo contribui para o avanço na compreensão da estabilidade assintótica em estruturas compostas, oferecendo subsídios teóricos relevantes para aplicações em engenharia e ciências aplicadas.

Palavras-chave: problema de transmissão, mecanismos dissipativos, decaimento exponencial, decaimento polinomial, kelvin-voigt, friccional, elástico, efeito de memória, termoelástico.

ABSTRACT

This work investigates three distinct transmission problems in one-dimensional structures composed of regions with heterogeneous mechanical properties. In all cases, the model involves three subdomains: an elastic material without dissipation, a viscoelastic material governed by a Kelvin–Voigt constitutive law, and a third dissipative component, which varies between a frictional mechanism, a memory effect, or a thermoelastic behavior. The analysis aims to understand the influence of both the type and the location of these dissipative mechanisms on the temporal evolution of the system's energy. The results show that the energy decay rate depends not only on the presence of dissipation but also on its spatial placement, thereby highlighting subtle aspects in the formulation and treatment of transmission problems. In this sense, the study contributes to advancing the understanding of asymptotic stability in composite structures, providing relevant theoretical insights for applications in engineering and applied sciences.

Keywords: transmission problem, dissipative mechanisms, exponential decay, polynomial decay, Kelvin-Voigt, frictional, elastic, memory effect, thermoelastic.

Contents

Introduction

Wave propagation occurs when a vibrating source disturbs a medium, transmitting oscillations from particle to particle. To mitigate these vibrations, several dissipative mechanisms can be incorporated into mathematical models, among which the following stand out: Kelvin-Voigt damping, representing the material's viscoelastic behavior; frictional damping, associated with energy dissipation through velocity-dependent forces; memory effect damping, which accounts for the influence of past deformations on the system's current response; and thermoelastic damping, related to energy loss due to heat exchange with the environment.

In recent years, there has been growing interest within the scientific community in problems involving these types of damping, both local and global, with a strong focus on the asymptotic behavior of solutions. Numerous studies have established exponential and polynomial energy decay rates, with particular emphasis on the optimality of the latter. An extensive body of literature highlights the relevance and timeliness of this topic. See, for example, the extensive list of contributions addressing these topics: [3], [4], [11], [12], [14],[17], [18], [19], [20], [24], [27],[29], [40]), and references therein.

Taking into account the results mentioned above, it becomes relevant to investigate the behavior of solutions in elastic systems where both Kelvin-Voigt damping and frictional or memory-based damping act simultaneously. These dissipative mechanisms may operate jointly within the same region of the domain or separately in different parts of the medium.

In this thesis, we study three transmission problems involving localized Kelvin-Voigt viscoelasticity. We consider a string composed of three distinct components. In the first problem, one part exhibits viscoelastic behavior, another is purely elastic (i.e., without any damping mechanism), and the third features frictional damping. In the second problem, we replace the frictional component with one governed by memory effects. In both cases, the string is divided into four or five subdomains, and we demonstrate that the spatial positioning of these components plays a critical role in the stabilization analysis.

In the third problem, we consider a bar composed of three different components: one viscoelastic, one purely elastic, and one thermoelastic. The main result in this case reveals that the position of the thermoelastic component also plays a decisive role in the asymptotic behavior of the system, further highlighting the relevance of how dissipative mechanisms are spatially distributed.

Our main analytical tool to address these problems is the Semigroup Theory. To establish the well-posedness of the systems under consideration, we apply the classical Hille–Yosida and Lumer–Phillips theorems, which ensure the generation of contraction semigroups in suitable Hilbert spaces. Furthermore, in the analysis of exponential stability and polynomial decay of energy, we rely on the results of Prüss and the Borichev–Tomilov theorem, which provide essential spectral and frequency-domain criteria to characterize the asymptotic behavior of solutions.

This thesis is organized as follows:

In Chapter 1, we briefly present the notation and preliminary results required throughout the work, including key properties of Sobolev spaces, some functional inequalities, relevant spectral properties, and foundational concepts from semigroup theory.

In Chapter 2, we investigate a wave equation system with Kelvin-Voigt damping combined with frictional damping. In Section 2.2, we establish the well-posedness of the system using a semigroup approach. Section 2.3 analyzes the case of exponential decay of solutions, which occurs when all elastic components are connected to the frictional damping region. In Section 2.4, we study the case in which an elastic component is connected solely to the region with Kelvin-Voigt damping, leading to slower decay. Section 2.5 addresses the optimality of the decay rates obtained.

In Chapter 3, we study a wave equation system involving memory damping in combination with Kelvin-Voigt damping. In Section 3.2, we prove the well-posedness of the model via semigroup theory. Section 3.3 covers the case of exponential stabilization, occurring when all elastic components are connected to the memory damping region. Section 3.4 focuses on the situation where one elastic component interacts only with the Kelvin-Voigt damping region. Section 3.5 discusses the optimality of the resulting decay rates.

In Chapter 4, we examine a system involving a bar with thermoelastic damping. In Section 4.2, we establish the well-posedness of the model, while Section 4.3 proves that the associated semigroup is exponentially stable, provided the viscoelastic component is not located at the center of the bar. Finally, Section 4.4 demonstrates that, in the absence of exponential stability, the system exhibits polynomial energy decay at the rate of t^{-2}.

Chapter 1

Preliminaries

In this chapter, we present key definitions and results essential for understanding the development of this work. For detailed proofs and further discussion on the results introduced here, we refer the reader to [8], [15], [16], and [33].

1.1 Notations

Let $\mathbb{X}$ and $\mathbb{Y}$ be two normed vector spaces. We denote by $\mathcal{B}(\mathbb{X}, \mathbb{Y})$ the space of continuous (=bounded) linear operators from $\mathbb{X}$ into $\mathbb{Y}$. As usual, one writes $\mathcal{B}(\mathbb{X})$ instead of $\mathcal{B}(\mathbb{X}, \mathbb{X})$.

If G is a linear subspace of a (possibly infinite dimensional) vector space $\mathbb{Y}$ then the codimension of G in $\mathbb{Y}$ is the dimension (possibly infinite) of the quotient space $\mathbb{Y}/G$. This agrees with the previous definition

$$\text{codim}(G) = \text{dim}(\mathbb{Y}/G).$$

Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary denoted by $\partial\Omega$. We define a multi-index $\alpha = \{\alpha_1, \alpha_1, \cdots, \alpha_n\} \in \mathbb{N}^n$, with $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and for a function $u : \Omega \to \mathbb{R}$ the α-derivative

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

The **gradient** operator is defined by

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right),$$

and the **Laplacian** operator by

$$\Delta u = \nabla \cdot \nabla u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

Lef $f, g :]0, \infty[\to \mathbb{R}$. In this work, the following notations will appear frequently:

1. $|f| \lesssim |g|$, if there exist $C > 0$ sucht that $|f| \le C|g|$;
2. $|f| \gtrsim |g|$, if $|g| \lesssim |f|$;
3. $f \approx g$, if $|f| \lesssim |g| \lesssim |f|$.

1.2 Sobolev spaces and inequalities

In this section, we will review Lebesgue spaces and introduce the definition of Sobolev spaces. Let μ be an integrable, non-increasing function defined in Ω, and let $\mathbb{X}$ be a normed space with the norm denoted as $\|\cdot\|$.

Definition 1.2.1. *We define for $1 \le p < \infty$ the space*

$$L^p_\mu(\Omega, \mathbb{X}) = \left\{ u : \Omega \to \mathbb{X} \mid \mu \textit{ is measuable and } \int_\Omega \mu(x)\|u(x)\|^p_{\mathbb{X}} dx < \infty \right\}.$$

The usual norm in space is

$$\|u\|_{L^p_\mu} = \left(\int_\Omega \mu(x)\|u(x)\|^p_{\mathbb{X}} dx \right)^{\frac{1}{p}}.$$

If $\mathbb{X}$ is a Hilbert space and $p = 2$ then $L^p_\mu(\Omega, \mathbb{X})$ is a Hilbert space with following inner product

$$(u, v)_{L^p_\mu} = \int_\Omega \mu(x)(u(x), v(x))_{\mathbb{X}} dx.$$

When $\mu = 1$ we just write $L^p(\Omega, \mathbb{X})$ and when $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{C}$ we write $L^p_\mu(\Omega)$.

Now we will define the Sobolev spaces:

Definition 1.2.2. *Let $m \in \mathbb{N}$ and $1 \le p < \infty$, the Sobolev spaces are defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \textit{ for each multi-index } |\alpha| \le m\}.$$

It is possible to prove that $W^{m,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

When $p = 2$, we denote the space $W^{m,p}(\Omega)$ by $H^m(\Omega)$(or eventually just H^m) where this is a Hilbert space. Note that $H^0(\Omega) = L^2(\Omega)$. Moreover, $C^k(\Omega)(1 \le k \le \infty)$ will denote the space of k times continuously differentiable functions on Ω and $C^k_0(\Omega) = \{u \in C^k(\Omega) \mid \text{supp}(u) \subset \Omega\}$, where supp represents the support of a function, that is, the closure of the subset of Ω where the function is not zero.

Definition 1.2.3. *The space $W^{m,p}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ in $W^{m,p}(\Omega)$.*

In other words, $W^{m,p}_0(\Omega)$ consists in all functions $u \in W^{m,p}(\Omega)$ such that "$D^\alpha u = 0$ on $\partial\Omega$", for $|\alpha| \le m-1$. However, this idea is more general because it involves trace theory. For a thorough analysis, a relevant reference is [8].

The upcoming theorems concern inequalities that we will frequently utilize throughout this thesis.

Theorem 1.2.1. ***(Hölder inequality)*** *Let $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, where $1 < p < \infty$, $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We have $uv \in L^1(\Omega)$ and also*

$$\int_\Omega |uv| dx \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Proof. See [8]. □

Theorem 1.2.2. ***(Young inequality)*** *Let a and b nonnegative real numbers with $1 < p < \infty$ and $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that*

$$ab \leq \epsilon a^p + C(\epsilon) b^q.$$

Proof. See [8]. □

Theorem 1.2.3. ***(Poincaré inequality)*** *If Ω is a bounded domain in $\mathbb{R}^n$ and $u \in H_0^1(\Omega)$, then there exists a positive constant $C > 0$, depending only on Ω, such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega).$$

Proof. See [8]. □

1.3 Functional analysis

In this section, we present some classical definitions in functional analysis.

Definition 1.3.1. ***(Sesquilinear form)*** *Let $\mathcal{H}$ be a complex vector space. A map $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a sesquilenar form if for all $x, y, z, w \in \mathcal{H}$ and for all $\alpha, \varrho \in \mathbb{C}$,*

i) $a(x + y, z + w) = a(x, z) + a(x, w) + a(y, z) + a(y, w)$

ii) $a(\alpha x, \varrho y) = \alpha \overline{\varrho} a(x, y)$.

Definition 1.3.2. *A map $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is called **bounded** if*

$$|a(x, y)| \lesssim \|x\|_{\mathcal{H}} \|y\|_{\mathcal{H}}, \quad \forall x, y \in \mathcal{H}.$$

Definition 1.3.3. *A map $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is called **coercive** if*

$$Re|a(x, x)| \gtrsim \|x\|^2_{\mathcal{H}}, \quad \forall x \in \mathcal{H}.$$

Theorem 1.3.1. ***(Lax-Milgram)*** *Let $\mathcal{H}$ be a complex Hilbert and $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ a sesquilinear form, bounded and coercive on $\mathcal{H}$. If $f \in H'$, where $\mathcal{H}'$ denotes the dual space of $\mathcal{H}$, then there exists a unique $x \in \mathcal{H}$ such that*

$$a(x, y) = (f, y), \quad \forall y \in \mathcal{H}.$$

Proof. See [8, Page 140]. □

Theorem 1.3.2. *If $\mathcal{M}$ is a closed subspace of the Hilbert space $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, that is, each $u \in \mathcal{H}$ admits a unique representation in the form*

$$u = p + q, \quad \text{with } p \in \mathcal{M} \text{ and } q \in \mathcal{M}^{\perp},$$

where $\mathcal{M}^{\perp} = \{q \in \mathcal{H} : (p, q) = 0, \ \forall p \in \mathcal{M}\}$.

Proof. See [9, Page 111]. □

Lemma 1.3.1. *Let $T : \mathbb{X} \to \mathbb{Y}$ be a bijective mapping with a closed graph in $\mathbb{X} \times \mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are normed spaces. Then T^{-1} also has a closed graph in $\mathbb{Y} \times \mathbb{X}$.*

Proof. Let (x, y) be an element of the closure of the graph of T^{-1} denoted as $\overline{G(T^{-1})}$. Then, there exists a sequence (x_n, y_n) in the graph of T^{-1}, which is a subset of $\mathbb{Y} \times \mathbb{X}$, such that (x_n, y_n) converges to (x, y) in $\mathbb{Y} \times \mathbb{X}$. It should be noted that $y_n = T^{-1}(x_n)$, which implies that $T(y_n) = x_n$. Therefore, we have $(y_n, T(y_n)) \in G(T)$, and $(y_n, T(y_n))$ converges to (y, x). So, we have $(y, x) \in \overline{G(T)}$. Since $G(T)$ is closed, we can conclude that $(y, x) \in G(T)$. This implies $x = T(y)$, and consequently $y = T^{-1}(x)$. Therefore, $(x, y) \in G(T^{-1})$. □

Let $g \in L^1([0, \infty)) \cap C^1([0, \infty))$ be a positive function such that $g(0) > 0$ and $g'(s) \leq -c_0 g(s)$, for some $c_0 > 0$, $\forall s \geq 0$. Consider the weighted space $\mathcal{G}_g \doteq L^2_g((0, \infty); H^1_0(0, L))$ with the inner product

$$(\eta^1, \eta^2)_{\mathcal{G}_g} := \int_0^L \int_0^\infty g(s)\partial_x \eta^1 \overline{\partial_x \eta^2} ds dx, \quad \forall\, \eta^1, \eta^2 \in \mathcal{G}_g.$$

Lemma 1.3.2. *If $\eta, \eta_s \in \mathcal{G}_g$ and $\eta(\cdot, 0) = 0$ in $(0, L) \times (0, \infty)$, then*

$$-\mathit{Re} \int_0^L \int_0^\infty g(s)\partial_s \partial_x \eta(\cdot, s)\overline{\partial_x \eta(\cdot, s)} ds dx = \frac{1}{2}\int_0^\infty g'(s)\|\partial_x \eta(\cdot, s)\|^2_{L^2} ds$$

Proof. Note that

$$\begin{aligned} -\mathrm{Re} \int_0^L \int_0^\infty g(s)\partial_s \partial_x \eta(\cdot, s)\overline{\partial_x \eta(\cdot, s)} ds dx &= -\frac{1}{2}\int_0^L \int_0^\infty g(s)\frac{d}{ds}|\partial_x \eta(\cdot, s)|^2 ds dx \\ &= -\lim_{y \to 0^+} \int_y^{1/y} g(s)\frac{d}{ds}\|\partial_x \eta(\cdot, s)\|^2_{L^2} ds. \end{aligned}$$

Using integration by parts, we get

$$\begin{aligned} &-\int_0^L \int_0^\infty g(s)\partial_s \partial_x \eta(\cdot, s)\overline{\partial_x \eta(\cdot, s)} ds dx \\ &\qquad = -\lim_{y \to 0^+} \left[g(1/y)\|\partial_x \eta(\cdot, 1/y)\|^2_{L^2} - g(y)\|\partial_x \eta(\cdot, y)\|^2_{L^2}\right] \\ &\qquad\quad + \lim_{y \to 0^+} \int_y^{1/y} g'(s)\|\partial_x \eta(\cdot, s)\|^2_{L^2} ds. \end{aligned} \tag{1.1}$$

Since $\eta \in \mathcal{G}_g$, then $g\|\partial_x \eta(\cdot, s)\|^2_{L^2} \in L^1(0, \infty)$ and thus

$$\lim_{y \to 0^+} g(1/y)\|\partial_x \eta(\cdot, 1/y)\|^2_{L^2} = \lim_{\tau \to +\infty} g(\tau)\|\partial_x \eta(\cdot, \tau)\|^2_{L^2} = 0. \tag{1.2}$$

Furthermore, as $\eta(\cdot, 0) = 0$ in $(0, L)$ and g is non-increasing and positive, it follows from the Hölder's inequality that

$$\begin{aligned} g(y)\|\partial_x \eta(\cdot, y)\|^2_{L^2} &= g(y)\left\|\int_0^y \partial_s \partial_x \eta(\cdot, s) ds\right\|^2_{L^2} \\ &\lesssim g(y)\left(\int_0^y \|\partial_s \partial_x \eta(\cdot, s)\|_{L^2} ds\right)^2 \\ &\lesssim \left(\int_0^y g^{\frac{1}{2}}(s)\|\partial_s \partial_x \eta(\cdot, s)\|_{L^2} ds\right)^2 \\ &\lesssim y\int_0^y g(s)\|\partial_s \partial_x \eta(\cdot, s)\|^2_{L^2} ds, \quad \forall\, y \in (0, \infty). \end{aligned}$$

Given that $\partial_s\eta \in \mathcal{G}_g$, we get

$$0 \lesssim g(y)\|\partial_x\eta(\cdot,y)\|^2_{L^2} \lesssim \lim_{y\to 0^+} y\int_0^y g(s)\|\partial_x\partial_s\eta(\cdot,s)\|^2_{L^2}ds = \lim_{y\to 0^+} y\|\partial_s\eta(\cdot,s)\|^2_{\mathcal{G}_g} = 0,$$

this is,

$$\lim_{y\to 0^+} g(y)\|\partial_x\eta(\cdot,y)\|^2_{L^2} = 0. \tag{1.3}$$

Inserting (1.2) and (1.3) in (1.1), we obtain the desired result. □

Definition 1.3.4. ***(Fredholm operators)*** *Given two Hilbert spaces $\mathbb{X}$ and $\mathbb{Y}$, one says $\mathcal{A} \in \mathcal{L}(\mathbb{X},\mathbb{Y})$ is a Fredholm operator if it satisfies:*

1. *N($\mathcal{A}$) is finite-dimensional;*
2. *R($\mathcal{A}$) is closed and has finite codimension.*

The index of $\mathcal{A}$ is defined by

$$\mathit{ind}\mathcal{A} = \mathit{dim}N(\mathcal{A}) - \mathit{codim}R(\mathcal{A})$$

Lemma 1.3.3. *If $\mathcal{A}$ is a Fredholm operator and K is a compact operator, then $\mathcal{A}+K$ is a Fredholm operator and*

$$\mathit{ind}(\mathcal{A}+K) = \mathit{ind}(\mathcal{A})$$

Proof. See [8, Page 169]. □

The following theorem is used to show the well-posed of partial differential equations.

Lemma 1.3.4. *Let $\lambda \in \mathbb{R}$ such $\lambda \neq 0$ and $\varrho > 0$. Also, consider $0 \le \alpha < \beta < \infty$ and*

$$\begin{cases} \lambda^2 u + \rho\partial_{xx}u = 0, \quad \textit{in } (\alpha,\beta) \\ \\ u(\alpha) = \partial_x u(\alpha) = 0 \textit{ or } u(\beta) = \partial_x u(\beta) = 0. \end{cases}$$

Then, $u \equiv 0$ in (α,β).

Proof. We will do it for $u(\alpha) = \partial_x u(\alpha) = 0$, the case $u(\beta) = \partial_x u(\beta) = 0$ follows in a similar way. Solving the equation

$$\lambda^2 u + \rho\partial_{xx}u = 0, \quad \text{in } (\alpha,\beta)$$

we obtain

$$u(x) = c_1\cos\left(\sqrt{\frac{\lambda^2}{\rho}}x\right) + c_2\sin\left(\sqrt{\frac{\lambda^2}{\rho}}x\right), \quad \text{for } x \in (\alpha,\beta).$$

Then, using $u(\alpha) = \partial_x u(\alpha) = 0$ and squaring it, we get

$$0 = c_1^2\cos^2\left(\sqrt{\frac{\lambda^2}{\rho}}\alpha\right) + c_2^2\sin^2\left(\sqrt{\frac{\lambda^2}{\rho}}\alpha\right) + 2c_1c_2\cos\left(\sqrt{\frac{\lambda^2}{\rho}}\alpha\right)\sin\left(\sqrt{\frac{\lambda^2}{\rho}}\alpha\right),$$

$$0 = c_1^2\sin^2\left(\sqrt{\frac{\lambda^2}{\rho}}\alpha\right) + c_2^2\cos^2\left(\sqrt{\frac{\lambda^2}{\rho}}\alpha\right) - 2c_1c_2\cos\left(\sqrt{\frac{\lambda^2}{\rho}}\alpha\right)\sin\left(\sqrt{\frac{\lambda^2}{\rho}}\alpha\right).$$

Adding the two equations above, we get $c_1 = c_2 = 0$. Therefore, $u \equiv 0$. □

1.4 Some definitions about Semigroups Theory and Spectral Properties

In this section, we present some definitions concerning the semigroup theory of operators. These properties are crucial for establishing the well-posedness and understanding the asymptotic behavior of all problems addressed in this thesis.

Definition 1.4.1. *Let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|$. A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators in $\mathbb{X}$ is called a strong continuous semigroup (or C_0 -semigroup) if:*

(i) $T(0) = I$, *where I is the identity operator in the set of all bounded linear operator in $\mathbb{X}$;*

(ii) $T(t+s) = T(t)T(s), \forall t, s \in \mathbb{R}^+$;

(iii) For each $x \in \mathbb{X}$, $\lim\limits_{t\to 0^+} \|T(t)x - x\| = 0$.

Theorem 1.4.1. *If $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup then there exists $M \geq 1$ and $\omega \geq 0$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \forall t \geq 0.$$

Proof. See [33]. □

Definition 1.4.2. *A semigroup $T(t)$ is called a* **semigroup of contractions** *if for all $t > 0$ we have*

$$\|T(t)\| \leq 1.$$

Theorem 1.4.2. *If $x \in \mathbb{X}$ and $\{T(t)\}$ is a C_0-semigroup, then the function $t \longmapsto T(t)x$ is continuous on $[0, +\infty)$.*

Proof. See [33]. □

From Theorem 1.4.2 it is possible to define:

Definition 1.4.3. *Let $\mathbb{X}$ be a Banach space and $\{T(t)\}_{t\geq 0}$ a C_0-semigroup. The linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X} \to \mathbb{X}$ defined by*

$$D(\mathcal{A}) = \left\{ x \in \mathbb{X} \text{ such that } \exists \lim_{t\to 0^+} \frac{T(t)x - x}{t} \right\},$$

and

$$\mathcal{A}x := \lim_{t\to 0^+} \frac{T(t)x - x}{t}, \quad \forall x \in D(\mathcal{A}),$$

is called the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$.

The next theorem gives us the answer to how it is possible to solve the abstract problem

Definition 1.4.4. *Let $\mathcal{A}$ be a linear operator in a Hilbert space $\mathcal{H}$. The resolvent set of an operator $\mathcal{A}$ is*

$$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \lambda I - \mathcal{A} \text{ is injective}; R(\lambda I - \mathcal{A}) = \mathcal{H}; (\lambda I - \mathcal{A})^{-1} \text{ is bounded}\}$$

Moreover, the set $\sigma(\mathcal{A}) = \mathbb{C} \setminus \rho(\mathcal{A})$ is called the spectrum of $\mathcal{A}$.

Lemma 1.4.1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a continuos linear operator with continuos inverse. If $S \in \mathcal{L}(\mathcal{H})$ and*

$$\|S\|_{\mathcal{L}(\mathcal{H})} < \frac{1}{\|T^{-1}\|_{\mathcal{L}(\mathcal{H})}},$$

then $T + S$ is a continuos linear operator with continuous inverse.

Proof. See [8]. □

Lemma 1.4.2. *Let an unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$. If $0 \in \rho(\mathcal{A})$, then there is $\lambda > 0$ such that $\lambda \in \rho(\mathcal{A})$.*

Proof. Let $|\lambda| < 1/\|\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})}$ and note that

$$\lambda I - \mathcal{A} = \mathcal{A}(\lambda \mathcal{A}^{-1} - I).$$

If $T \doteq -I$ and $S \doteq \lambda \mathcal{A}^{-1}$, we have T, T^{-1} are continuos and $S \in \mathcal{L}(\mathcal{H})$, because $0 \in \rho(\mathcal{A})$ with

$$\|S\|_{\mathcal{L}(\mathcal{H})} = \|\lambda \mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} = |\lambda| \|\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} < \frac{1}{\|T^{-1}\|_{\mathcal{L}(\mathcal{H})}}$$

From Lemma 1.4.1, we can conclude that $\lambda \mathcal{A}^{-1} - I$ is a continuous linear operator with a continuous inverse. Furthermore, we can assert that $\lambda I - \mathcal{A}$ is also a continuous linear operator with a continuous inverse, as it is the composition of two continuous and invertible operators. Therefore, we get $\lambda \in \rho(\mathcal{A})$. □

Definition 1.4.5. *Let $\mathcal{H}$ be a Hilbert space. The operator $\mathcal{A}$ é called dissipative operator when for all $x \in D(A)$,*

$$Re(\mathcal{A}x, x) \leq 0.$$

Lemma 1.4.3. *Let an unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$. If $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, then $D(\mathcal{A})$ is dense in $\mathcal{A}$, that is, $\overline{D(\mathcal{A})} = \mathcal{H}$.*

Proof. According to Theorem 1.3.2, we can write $\mathcal{H} = \overline{D(\mathcal{A})} \oplus \overline{D(\mathcal{A})}^{\perp}$. We will show $\overline{D(\mathcal{A})}^{\perp} = \{0\}$. In fact, suppose $U \in \overline{D(\mathcal{A})}^{\perp}$, This implies that $(U, V)_{\mathcal{H}} = 0$, for all $V \in \overline{D(A)}$. In particular, we have

$$(U, V)_{\mathcal{H}} = 0, \quad \forall \, V \in D(\mathcal{A}).$$

Since $0 \in \rho(\mathcal{A})$, by Lemma 1.4.2, there exists $\lambda > 0$ such that $\mathrm{Im}(\lambda I - \mathcal{A}) = \mathcal{H}$. Therefore, there exists $V_0 \in D(\mathcal{A})$ such that $U = \lambda V_0 - \mathcal{A}V_0$. Hence, we have

$$0 = (U, V_0)_{\mathcal{H}} = (\lambda V_0, V_0)_{\mathcal{H}} - (\mathcal{A}V_0, V_0)_{\mathcal{H}}.$$

Taking the real part and using the fact that $\mathcal{A}$ is dissipative, we get

$$\lambda (V_0, V_0)_{\mathcal{H}} = \mathrm{Re}(\mathcal{A}V_0, V_0)_{\mathcal{H}} \leq 0.$$

Thus, we obtain $V_0 = 0$ and consequently, $U = 0$. Therefore, we get $\overline{D(\mathcal{A})}^{\perp} = \{0\}$. □

1.5 Some notions and stability theorems

We recall in this short section some notions and stability results used in this work.

Theorem 1.5.1. ***(Lumer-Phillips's Theorem)*** *Let $\mathcal{A}$ be a linear operator in a Hilbert space $\mathcal{H}$ with dense domain $D(\mathcal{A})$ in $\mathcal{H}$. If $\mathcal{A}$ is dissipative and there exists $\lambda_0 > 0$ such that Im$(\lambda_0 I - \mathcal{A}) = \mathcal{H}$, then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof. See [33, Theorem 4.3, page 14]. □

As a collorary of the above theorem, the following result will be frequently used

Theorem 1.5.2. ***(Variant of Lummer-Phillips's Theorem)*** *Let A be a linear operator with domain $D(A)$ dense in a Hilbert space $\mathcal{H}$. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof. See [33, Theorem 1.2.4, page 3]. □

We will present some results about asymptotic behavior.

Definition 1.5.1. *Assume that que $\mathcal{A}$ is the generator of C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. The C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is said to be*

(1) *Strongly stable if*
$$\lim_{t\to+\infty} \|e^{tA}U_0\|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.$$

(2) *Exponentially (or uniformly) stable if there exists two positive constants M and ε such that*
$$\|e^{tA}U_0\|_{\mathcal{H}} \leq Me^{-\varepsilon t}\|U_0\|_{\mathcal{H}}, \quad \forall t > 0, \forall U_0 \in \mathcal{H}.$$

(3) *Polynomially stable if there exists two positive constans C and α such that*
$$\|e^{tA}U_0\|_{\mathcal{H}} \leq Ct^{-\alpha}\|U_0\|_{D(\mathcal{A})}, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}).$$

In that case, one says that the semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ decays at a rate $t^{-\alpha}$. The C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is said to be polynomially stable with optmial decay rate $t^{-\alpha}$(with $\alpha > 0$) if it is polynomially stable with decay rate $t^{-\alpha}$ and, for any $\varepsilon > 0$ small enough, the semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ does not decay at a rate $t^{-(\alpha-\varepsilon)}$.

Concerning the characterisation of exponential stability of C_0 semigroup of contraction $(e^{tA})_{t\geq 0}$ we rely on the following result due to Huang and Pruss.

Theorem 1.5.3. ***(Huang and Pruss' Theorem)*** *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ generates a C_0 semigroup of contractions $(e^{tA})_{t\geq 0}$ on $\mathcal{H}$. Assume that $i\mathbb{R} \subset \rho(\mathcal{A})$. Then, the C_0-semigroup $(e^{tA})_{t\geq 0}$ is exponentially stable if and only if*

$$\limsup_{\lambda\in\mathbb{R},|\lambda|\to+\infty} \|(i\lambda I - A)^{-1}\|_{\mathcal{H}} < \infty.$$

Concerning the charaterization of polinomial stability of a C_0-semigroup of contraction $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ we rely on the following result due Borichev and Tomilov([7]).

Theorem 1.5.4. ***(Borichev and Tomilov's Theorem)*** *Let $\mathcal{A}$ be the generator of a bounded C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If $i\mathbb{R} \subset \rho(\mathcal{A})$, then for a fixed $\ell > 0$, we have*

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \leq \frac{C}{t^{\frac{1}{\ell}}}\|U_0\|_{D(\mathcal{A})}, \quad \forall t > 0, U_0 \in D(\mathcal{A}), \textit{ some constant } C > 0,$$

if and only if,

$$\limsup_{\lambda\in\mathbb{R},|\lambda|\to\infty} \frac{1}{|\lambda|^{\ell}}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. See [7]. □

Chapter 2

Wave Equation with Kelvin-Voigt and Frictional Damping: Analysis of Asymptotic Stability

2.1 Introduction to the problem

In studies of vibrating systems modeled by wave equations, beams or plates, it is known that Kelvin-Voigt damping mechanisms, when distributed globally, stabilize the solutions of these systems exponentially. Furthermore, this damping mechanism is so strong that it tends to regularize the solutions. The situation may be completely different if this type of damping acts only on a part of the body as was shown by K. Liu and Z. Liu in [24] (see also [11]). These authors proved that if Kelvin-Voigt damping acts locally in a wave equation with discontinuous coefficient then the solutions of the equation are not exponentially stable.

Later, Alves et al [3], studied the stabilizing force that Kelvin-Voigt damping exerts on a transmission problem. This time, two dissipative mechanisms act on different parts of the body. In one part, Kelvin-Voigt damping and in the other, frictional damping. Even with the collaboration of frictional damping, the authors showed that Kelvin-Voigt damping can predominate in the decay of the solutions, not allowing the exponential decay of the solutions. However, the authors showed that the solutions decay polynomially with the optimal decay rate t^{-2}.

Problems with localized Kelvin-Voigt damping have aroused the interest of several researchers in the last two decades and several results have been obtained. The problem

$$u_{tt}(x,t) - u_{xx}(x,t) - \big(b(x)u_{xt}(x,t)\big)_x = 0,$$

was studied by Liu and Zhang [27] in the interval $(-1,1)$ (see also [43]). They showed that if the coefficient $b(x)$ is zero in $(-1,0]$, positive in $(0,1)$ and has a behavior like x around zero then the solution of this problem is exponentially stable. Also, if the behavior of $b(x)$ around zero is x^α, $\alpha > 1$, the solution is polynomially stable with a decay rate depending on α. A result with sharp stability $t^{-\frac{2-\alpha}{1-\alpha}}$ were obtained by Han et al in [19] (see also [18, 27]).

When the coefficient $b(x)$ is discontinuous, Liu et al. [24] had shown the solution does not decay exponentially. A few years later, this same problem was studied by Rivera et al. [2] where they showed that the solutions of the system decay polynomially with the optimal rate t^{-2} (see also [17, 20, 29, 40]).

Taking into account the results mentioned above, it is interesting to study the behavior of solutions in elastic systems where both Kelving-Voigt and frictional damping act simultaneously on the body. These dissipative mechanisms can act jointly on a part of the body or on separate parts. In this work we try to answer which of the dissipative mechanisms prevails: the frictional damping that stabilizes the system exponentially, or the Kelving-Voigt damping, which, being discontinuous, stabilizes the system more slowly.

Therefore, in this article we consider the following problem: to study the asymptotic behavior of the solutions of the equation

$$u_{tt}(x,t) - \left(\rho u_x(x,t) + b(x)u_{xt}(x,t)\right)_x + a(x)u_t = 0, \quad (x,t) \in (0,L)\times(0,\infty). \tag{2.1}$$

satisfying the Dirichlet boundary conditions

$$u(0,t) = u(L,t) = 0, \ \ t \geq 0, \tag{2.2}$$

and initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \ \ x \in (0,L). \tag{2.3}$$

Here, L and ρ are positive real numbers.

The coefficients $a(x)$ and $b(x)$ are characteristic functions whose supports are subintervals of $[0,L]$. These supports can overlap, be disjoint, or even contain one another.

Given the variety of possible configurations for the supports of $a(x)$ and $b(x)$, we will focus on three specific cases. In the first case, the supports do not overlap, and all purely elastic components are in contact with the component containing frictional damping. In this scenario, we define:

$$b(x) = b_0\chi_{[0,L_1]}(x), \quad a(x) = a_0\chi_{[L_2,L_3]}(x), \quad a_0, b_0 > 0,$$

where $0 < L_1 < L_2 < L_3 < L$. This model is referred to as the KEFE model.

In the second case, the supports of $a(x)$ and $b(x)$ remain disjoint, but there is a purely elastic component that only interacts with the Kelvin-Voigt component, without any contact with the frictional component. In this case, we have:

$$b(x) = b_0\chi_{[L_1,L_2]}(x), \quad a(x) = a_0\chi_{[L_3,L_4]}(x), \quad a_0, b_0 > 0,$$

where $0 < L_1 < L_2 < L_3 < L_4 < L$. This model is referred to as the EKEFE model.

Finally, in the third case, the supports of $a(x)$ and $b(x)$ overlap, but there is still a purely elastic component that interacts exclusively with the Kelvin-Voigt component. In this configuration, we define:

$$b(x) = b_0\chi_{[L_1,L_3]}(x), \quad a(x) = a_0\chi_{[L_2,L_4]}(x), \quad a_0, b_0 > 0.$$

This model is referred to as the EKIFE model.

Geometric description of the functions $a(x)$ and $b(x)$ in each model is described in Figure 1 below.

The main results we obtain in this work are the following:

- If all the purely elastic components are in contact with the frictional damping component, whether or not they contact the component with Kelvin-Voigt damping, then the solutions of the system (2.1)-(2.3) decay exponentially

- If there is a purely elastic component that contacts only the component with Kelvin-Voigt damping then the solutions of the system (2.1)-(2.3) do not decay exponentially. However, it is proven that the solutions decay polynomially with the rate t^{-2}.
- In case of non-exponential decay of the solutions, it is proven that the polynomial decay rate t^{-2} is optimal.

The remaining part of this chapter is organized as follows: In Section 2.2, we study the well-posedness of the system (2.1)-(2.3) using a semigroup approach. In Section 2.3, we study the case when the solutions of the system decay exponentially, that is, when all the purely elastic components are in contact with the frictional damping component. In Section 2.4, we study the case when there exists a purely elastic in contact only with the Kelvin-Voigt damping component. Finally, Section 2.5 deals with the optimality of the decay rates obtained in the previous section.

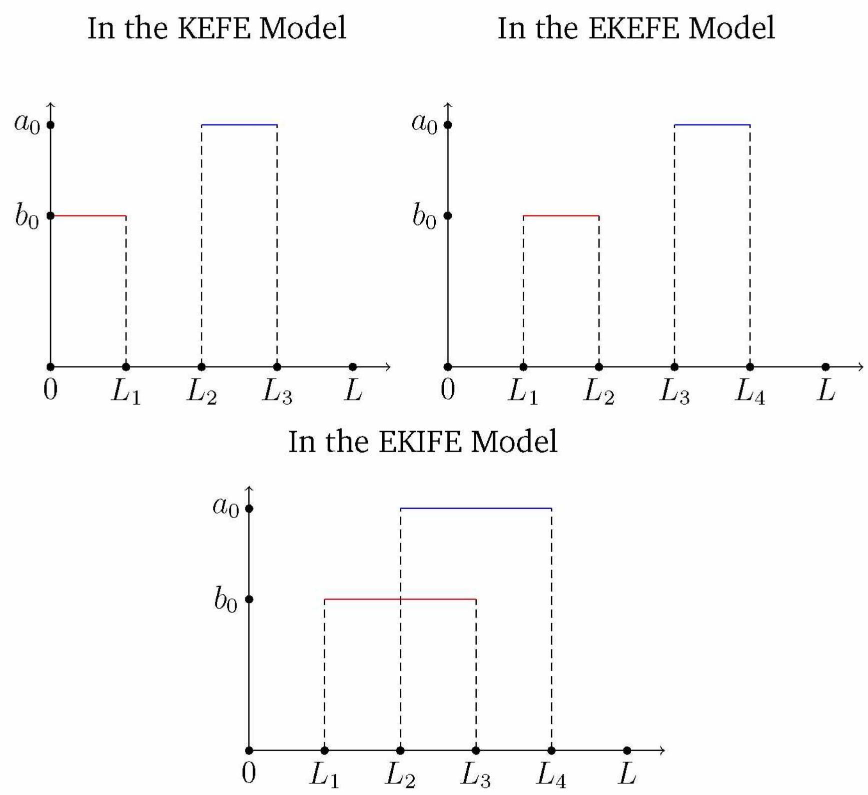

Figure 2.1: Geometric description of the functions b(x) and a(x)

2.2 Existence of solutions

In this section, we will establish the well-posedness of problem (2.1)-(2.3) by using a semigroup approach.

$$\mathcal{H} := H_0^1(0,L) \times L^2(0,L)$$

The Hilbert space $\mathcal{H}$ is equipped with the inner product defined by

$$(U_1, U_2)_{\mathcal{H}} = \int_0^L \rho u_x^1 \overline{u_x} dx + \int_0^L v^1 \overline{v} dx$$

for all $U_1 = (u^1, v^1)$ and $U_2 = (u^2, v^2)$ in $\mathcal{H}$. We use $\|U\|_{\mathcal{H}}$ to denote the corresponding norm. We define the unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$D(\mathcal{A}) = \left\{ U = (u, v)^\top \in \mathcal{H} \mid v \in H_0^1(0, L), \quad \text{and} \quad (\rho u + b(\cdot)v) \in H^2(0, L) \right\},$$

and

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ (\rho u_x + b(\cdot)v_x)_x - a(\cdot)v \end{pmatrix},$$

for all $U = (u, v)^\top \in D(\mathcal{A})$.
If $U = (u, u_t)^\top$ is the state of System (2.1)-(2.3), then this system is transformed into the first order evolution on the Hilbert space $\mathcal{H}$ given by

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \tag{2.4}$$

where $U_0 = (u_0, u_1)^\top \in \mathcal{H}$. We have the following result on the well-posedness of system (2.4).

Proposition 2.2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $\mathcal{A}$ generates a C_0 semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$.*

Proof. First, note that the $\mathcal{A}$ is a dissipative operator in the energy space $\mathcal{H}$. In fact, let $U = (u, v)^\top \in D(\mathcal{A})$. Using the inner product in $\mathcal{H}$, integration by parts, and the boundary conditions (2.2), we have

$$(\mathcal{A}U, U)_{\mathcal{H}} = \left(\int_0^L v_x \overline{u_x} dx - \int_0^L u_x \overline{v_x} dx \right) - \int_0^L a(\cdot)|v|^2 dx - \int_0^L b(\cdot)|v_x|^2 dx$$

Using $-(z, w)_{L^2} + \overline{(z, w)_{L^2}} = -2\mathrm{Im}(z, w)_{L^2}$, for $z, w \in L^2$, and taking the real part, we get

$$\mathrm{Re}(\mathcal{A}U, U)_{\mathcal{H}} = -\int_0^L a(\cdot)|v|^2 dx - \int_0^L b(\cdot)|v_x|^2 dx \leq 0 \tag{2.5}$$

therefore, $\mathcal{A}$ is dissipative.

Now, we will prove $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, this is, $\mathcal{A}$ is bijective and $\mathcal{A}^{-1}$ is bounded. In fact, let $F = (f, g)^\top \in \mathcal{H}$. We will show that there is unique $U = (u, v)^\top \in D(\mathcal{A})$ such that

$$-\mathcal{A}U = F. \tag{2.6}$$

The solver equation (2.6) in terms of its components is equivalent to the following system of differential equations

$$-v = f, \tag{2.7}$$

$$-[\rho u_x + b(\cdot)v_x]_x + a(\cdot)v = g, \tag{2.8}$$

with the boundary conditions

$$u(0) = u(L) = 0, \quad \text{in} \quad (0, L). \tag{2.9}$$

First, note that from (2.7) we have $v = -f \in H_0^1(0, L)$. Now we need to show that u, $(\rho u_x + b(\cdot)v_x) \in H_0^1(0, L)$. In fact, from (2.8), we have

$$-(\rho u_x + b(\cdot)f_x)_x = g + a(\cdot)f. \tag{2.10}$$

Multiplying (2.10) by $\overline{\phi} \in H_0^1(0,L)$, integrating over $(0,L)$ and using integration by parts, we get

$$\int_0^L \rho u_x \overline{\phi_x} dx = \int_0^L (g + a(\cdot) f)\overline{\phi} dx - \int_0^L b(\cdot) f_x \overline{\phi_x} dx. \tag{2.11}$$

Define $\Upsilon : H_0^1(0,L) \times H_0^1(0,L) \to \mathbb{C}$ and $J : H_0^1(0,L) \to \mathbb{C}$, such that

$$\Upsilon(u,\phi) = \int_0^L \rho u_x \overline{\phi_x} dx, \quad \forall \phi \in H_0^1(0,L), \tag{2.12}$$

and

$$J(\phi) = \int_0^L (g + a(\cdot) f)\overline{\phi} dx - \int_0^L b(\cdot) f_x \overline{\phi_x} dx, \quad \forall \phi \in H_0^1(0,L). \tag{2.13}$$

From (2.11), we get

$$\Upsilon(u,\phi) = J(\phi), \quad \forall \phi \in H_0^1(0,L), \tag{2.14}$$

Note that Υ is a sesquilinear form on $H_0^1(0,L) \times H_0^1(0,L)$. Moreover, by Hölder's inequality and Poincaré's inequality, Υ is a continuous and coercive form on $H_0^1(0,L) \times H_0^1(0,L)$, because

$$\Upsilon(U_1,U_2) := \left| \langle U_1, U_2 \rangle_{H_0^1(0,L)} \right| \lesssim \|U_1\|_{H_0^1(0,L)} \|U_2\|_{H_0^1(0,L)}, \quad \forall U_1, U_2 \in H_0^1(0,L),$$

and

$$\Upsilon(U,U) := \langle U, U \rangle_{H_0^1(0,L)} = \|U\|^2_{H_0^1(0,L)}, \quad \forall U \in H_0^1(0,L).$$

On the other hand, J is a antilinear functional on $H_0^1(0,L)$ and using Hölder's inequality, we get J is a continuos functional on $H_0^1(0,L)$. Therefore, by Lax-Milgram theorem we have that (2.14) admits a unique solution $u \in H_0^1(0,L)$. By taking test function $\phi \in C_0^\infty(0,L)$, we deduce that

$$-(\rho u_x + b(\cdot) v_x)_x = -(\rho u_x + b(\cdot) f_x)_x = g + a(\cdot) f \in L^2(0,L).$$

Therefore, $U \in D(\mathcal{A})$ is a unique solution of (2.6). This tells us that $\mathcal{A}$ is bijective and there is $\mathcal{A}^{-1}$. So, to conclude that $0 \in \rho(\mathcal{A})$, we just need to show that $\mathcal{A}^{-1}$ is bounded. For this, as there is only one $U \in D(\mathcal{A})$ such that $-\mathcal{A}U = F$, we need to show $\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}}$. In fact, since $v = -f$, then by Poincaré's inequality, we have

$$\|v\|^2_{L^2(0,L)} \lesssim \|f_x\|^2_{L^2(0,L)} \lesssim \|F\|^2_{\mathcal{H}}. \tag{2.15}$$

Furthermore, from (2.14) and by Poincaré's inequality, we get

$$\begin{aligned} \int_0^L \rho |u_x|^2 dx &= \int_0^L (g + a(\cdot) f)\overline{u} dx - \int_0^L b(\cdot) f_x \overline{u_x} dx \\ &\lesssim (\|g\|_{L^2(0,L)} + \|f\|_{L^2(0,L)} + \|f_x\|_{L^2(0,L)}) \|u_x\|_{L^2(0,L)} \\ &\lesssim \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} \end{aligned} \tag{2.16}$$

Thus, from (2.15)-(2.16), we obtain

$$\|U\|^2_{\mathcal{H}} = \int_0^L \rho |u_x|^2 dx + \int_0^L |v|^2 dx \lesssim \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}},$$

using Young's inequality, we get $\|U\|^2_{\mathcal{H}} \lesssim \|F\|^2_{\mathcal{H}}$, how we wanted to show.

Then, since $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, then $D(\mathcal{A})$ is dense in $\mathcal{H}$. Therefore, the operator $\mathcal{A}$ satisfies the conditions of Lumer-Phillips's Theorem (see Pazy [25]) and the result of the proposition follows. □

The well-posedness of the problem (2.4) and therefore of problem (2.1)-(2.3) is a consequence of semigroup theory and we state this result in the following theorem.

Theorem 2.2.1. *For $U_0 = (u_0, u_1) \in \mathcal{H}$, the problem (2.4) admits a unique mild solution U satisfying*

$$U = e^{t\mathcal{A}}U_0 \in C^0([0,\infty[;\mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the solutions belong to the following space:

$$U \in C^0([0,\infty[;D(\mathcal{A})) \cap C^1([0,\infty[;\mathcal{H}).$$

2.3 Asymptotic behavior: Exponential Stability

We will study the asymptotic behavior of the semigroup $e^{t\mathcal{A}}$ associated to the system (2.1)-(2.3). The results will be obtained using the spectral characterizations for exponential stability of semigroups (see [21] or [32]).

We will demonstrate the exponential stability of system (2.1)-(2.3) in the case that every elastic part of the string either connects only with the frictional part or connects with both types of dampings. Since the proof of the decay rate is similar in all these cases, we focus on the proof considering the KEFE model, which is given by (2.1)-(2.3) considering

$$b(x) = b_0\chi_{[0,L_1]}(x) \text{ and } a(x) = a_0\chi_{[L_2,L_3]}(x), \tag{2.17}$$

where $a_0, b_0 > 0$.

The main result of this section is Theorem 2.3.1 and to prove this theorem we will need to introduce some technical lemmas.
Let $\lambda \in \mathbb{R}$ and $F = (f, g)^\top \in \mathcal{H}$. In what follows, the stationary problem

$$(i\lambda I - \mathcal{A})U = F, \tag{2.18}$$

will be considered several times. Note that $U = (u, v)$ is a solution of this problem if the following equations are satisfied:

$$i\lambda u - v = f, \tag{2.19}$$
$$i\lambda v - [\rho u_x + b(\cdot)v_x]_x + a(\cdot)v = g, \tag{2.20}$$

with the following boundary conditions

$$u(0) = u(L) = 0. \tag{2.21}$$

Note that

$$((i\lambda U - \mathcal{A}U), U)_{\mathcal{H}} = i\lambda\|U\|^2_{\mathcal{H}} - (\mathcal{A}U, U)_{\mathcal{H}},$$

so, we have

$$-\mathrm{Re}(\mathcal{A}U, U)_{\mathcal{H}} \lesssim \|(i\lambda U - AU)\|_{\mathcal{H}}\|U\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Therefore, from (2.5), we get

$$\int_0^L a(x)|v|^2dx + \int_0^L b(x)|v_x|^2dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{2.22}$$

From (2.19) and (2.22), we get

$$\begin{aligned}\int_0^L \rho b(x)|u_x|^2dx &\lesssim |\lambda|^{-2}\left(\int_0^L \rho b(x)|v_x|^2dx + \int_0^L \rho b(x)|f_x|^2dx\right)\\ &\lesssim |\lambda|^{-2}(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}).\end{aligned} \tag{2.23}$$

Furthermore, by Poincaré's inequality and (2.22), we have

$$\int_0^L b(x)|v|^2dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{2.24}$$

Then, from (2.23) and (2.24), for $|\lambda|$ large, we have

$$\int_0^{L_1} b_0(|v|^2 + \rho|u_x|^2)dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + |\lambda|^{-2}\|F\|^2_{\mathcal{H}}. \tag{2.25}$$

The following lemma will be used in the next lemmas.

Lemma 2.3.1. *Let us $[\alpha_0, \beta_0] \subset [0, L] \setminus$ supp $b(x)$. For $|\lambda|$ large and $\epsilon > 0$ small, we have*

$$\begin{aligned}&|u_x(\alpha_0)|^2 + |v(\alpha_0)|^2 + |u_x(\beta_0)|^2 + |v(\beta_0)|^2\\ &\qquad \lesssim \int_{\alpha_0}^{\beta_0} (|v|^2 + \rho|u_x|^2)dx + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \epsilon\|U\|^2_{\mathcal{H}}.\end{aligned} \tag{2.26}$$

Furthermore, check the following inequalities:

$$\int_{\alpha_0}^{\beta_0} (|v|^2 + |u_x|^2)dx \lesssim |u_x(\alpha_0)|^2 + |v(\alpha_0)|^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \epsilon\|U\|^2_{\mathcal{H}}, \tag{2.27}$$

$$\int_{\alpha_0}^{\beta_0} (|v|^2 + |u_x|^2)dx \lesssim |u_x(\beta_0)|^2 + |v(\beta_0)|^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \epsilon\|U\|^2_{\mathcal{H}}, \tag{2.28}$$

for $\epsilon > 0$ small.

Proof. Let $[\alpha_0, \beta_0] \subset [0, L] \setminus \operatorname{supp} b(x)$. First, let's prove the inequality (2.26). In fact, multiplying (2.20) by $\left(x - \frac{\alpha_0+\beta_0}{2}\right)\overline{u_x}$ and using (2.19), we have

$$\begin{aligned}-\left(x - \frac{\alpha_0+\beta_0}{2}\right)\overline{v_x}v - \rho\left(x - \frac{\alpha_0+\beta_0}{2}\right)\overline{u_x}u_{xx} + a(x)\left(x - \frac{\alpha_0+\beta_0}{2}\right)\overline{u_x}v&\\ = \left(x - \frac{\alpha_0+\beta_0}{2}\right)\overline{u_x}g - \left(x - \frac{\alpha_0+\beta_0}{2}\right)\overline{f_x}v,& \quad \text{in } [\alpha_0, \beta_0].\end{aligned}$$

Using integration by parts in $[\alpha_0, \beta_0]$ and taking real part, we obtain

$$\begin{aligned}&|u_x(\alpha_0)|^2 + |v(\alpha_0)|^2 + |u_x(\beta_0)|^2 + |v(\beta_0)|^2\\ &\qquad \lesssim \int_{\alpha_0}^{\beta_0} (|v|^2 + \rho|u_x|^2)dx + \text{Re}\left\{\int_{\alpha_0}^{\beta_0}\left(x - \frac{\alpha_0+\beta_0}{2}\right)\overline{u_x}g dx\right\}\\ &\qquad\quad - \text{Re}\left\{\int_{\alpha_0}^{\beta_0} a(x)\left(x - \frac{\alpha_0+\beta_0}{2}\right)\overline{u_x}v dx\right\}\\ &\qquad\quad - \text{Re}\left\{\int_{\alpha_0}^{\beta_0}\left(x - \frac{\alpha_0+\beta_0}{2}\right)\overline{f_x}v dx\right\}.\end{aligned} \tag{2.29}$$

Note that $\left|x-\frac{\alpha_0+\beta_0}{2}\right| \leq \frac{\beta_0-\alpha_0}{2}$ for all $x \in [\alpha_0, \beta_0]$ and using Holder's inequality, Young's inequality and (2.22), we get

$$\begin{aligned}
-\text{Re}\left\{\int_{\alpha_0}^{\beta_0} a(x)\left(x-\frac{\alpha_0+\beta_0}{2}\right)\overline{u_x}v dx\right\} &\lesssim \frac{\beta_0-\alpha_0}{2}\int_{L_1}^{L_2} a_0|\overline{u_x}v|dx \\
&\lesssim \epsilon\int_{L_1}^{L_2}|u_x|^2dx + C(\epsilon)\int_{L_1}^{L_2}|v|^2dx \\
&\lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \qquad (2.30)
\end{aligned}$$

for $\epsilon > 0$ small.
Therefore, using (2.30) and Holder's inequality to estimate the other terms on the right-hand side of inequality (2.29), we obtain

$$\begin{aligned}
|u_x(\alpha_0)|^2 + |v(\alpha_0)|^2 + &|u_x(\beta_0)|^2 + |v(\beta_0)|^2 \\
&\lesssim \int_{\alpha_0}^{\beta_0}(|v|^2+\rho|u_x|^2)dx + \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},
\end{aligned}$$

for $\epsilon > 0$ small. Now, let's prove (2.27). In fact, multiplying (2.20) by $(x-\beta_0)\overline{u_x}$, using integration by parts in $[\alpha_0, \beta_0]$, taking real part and using (2.19), we get

$$\begin{aligned}
\int_{\alpha_0}^{\beta_0}(|v|^2+\rho|u_x|^2)dx \lesssim |u_x(\beta_0)|^2 + |v(\beta_0)|^2 + \text{Re}\left\{\int_{\alpha_0}^{\beta_0}(x-\beta_0)\overline{u_x}g dx\right\} \\
-\text{Re}\left\{\int_{\alpha_0}^{\beta_0} a(x)(x-\beta_0)\overline{u_x}v dx\right\} - \text{Re}\left\{\int_{\alpha_0}^{\beta_0}(x-\beta_0)\overline{f_x}v dx\right\}.
\end{aligned}$$

Note that $|x-\beta_0| \leq \beta_0$ for all $x \in [\alpha_0, \beta_0]$, using Holder's inequality and (2.30), we get

$$\int_{\alpha_0}^{\beta_0}(|v|^2+\rho|u_x|^2)dx \lesssim |u_x(\beta_0)|^2 + |v(\beta_0)|^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \epsilon\|U\|_{\mathcal{H}}^2,$$

for $\epsilon > 0$ small.
The proof of (2.28) follows in a similar manner, multiplying (2.20) by $(x-\alpha_0)\overline{u_x}$. $\square$

Lemma 2.3.2. *For $|\lambda|$ large and $\epsilon > 0$ small, we have*

$$\int_{L_2}^{L_3}\rho|u_x|^2dx \lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2.$$

Proof. Multiplying (2.20) by $\overline{u}$, we have

$$\overline{-i\lambda u v} - \rho u_{xx}\overline{u} + \frac{ia_0}{\lambda}v\overline{i\lambda u} = g\overline{u}, \quad \text{in} \quad [L_2, L_3].$$

Using integration by parts in $[L_2, L_3]$, taking real part and using (2.19), we get

$$\begin{aligned}
\int_{L_2}^{L_3}\rho|u_x|^2dx = &\text{Re}\{\rho u_x(L_3)\overline{u}(L_3)\} - \text{Re}\{\rho u_x(L_2)\overline{u}(L_2)\} + \int_{L_2}^{L_3}|v|^2dx + \text{Re}\left\{\int_{L_2}^{L_3}v\overline{f}dx\right\} \\
&+ \text{Re}\left\{\int_{L_2}^{L_3}g\overline{u}dx\right\} - \text{Re}\left\{\frac{ia_0}{\lambda}\int_{L_2}^{L_3}v\overline{f}dx\right\}.
\end{aligned}$$

Using (2.22) and by Holder's inequality, we get

$$\int_{L_2}^{L_3} \rho|u_x|^2 dx \lesssim |\rho u_x(L_3)\overline{u}(L_3)| + |\rho u_x(L_2)\overline{u}(L_2)| + \frac{1}{|\lambda|}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{2.31}$$

for $|\lambda|$ large. Finally, using Lemma 2.3.1 and that $H^1(L_2, L_3) \subset L^\infty(L_2, L_3)$ with continuous injection, we get

$$\begin{aligned}|\rho u_x(L_3)\overline{u}(L_3)| + |\rho u_x(L_2)\overline{u}(L_2)| &= \frac{\rho}{|\lambda|}\left\{|u_x(L_3)||\overline{i\lambda u}(L_3)| + |u_x(L_2)||\overline{i\lambda u}(L_2)|\right\} \\ &= \frac{\rho}{|\lambda|}\{|u_x(L_3)||v(L_3) + f(L_3)| + |u_x(L_2)||v(L_2) + f(L_2)|\} \\ &\lesssim \frac{1}{|\lambda|}\int_{L_2}^{L_3}(|v|^2 + \rho|u_x|^2)dx + \frac{C(\epsilon)}{|\lambda|}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \\ &\quad + \frac{\epsilon}{|\lambda|}\|U\|_{\mathcal{H}}^2 + \frac{1}{|\lambda|}\|F\|_{\mathcal{H}}^2, \end{aligned} \tag{2.32}$$

for $|\lambda|$ large and $\epsilon > 0$ small. Therefore, from (2.31)-(2.32) and using (2.22), we obtain

$$\int_{L_2}^{L_3} \rho|u_x|^2 dx \lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2,$$

provided that λ is large enough and $\epsilon > 0$ small. □

From Lemma 2.3.2 and (2.22), we have

$$\int_{L_2}^{L_3} \left(|v|^2 + \rho|u_x|^2\right) dx \lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2, \tag{2.33}$$

for $|\lambda|$ large and $\epsilon > 0$ small.

Lemma 2.3.3. *For $|\lambda|$ large and $\epsilon > 0$ small, we have*

$$\int_{L_1}^{L_2} (|v|^2 + \rho|u_x|^2)dx \lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2.$$

Proof. Using Lemma 2.3.1 twice and (2.33), we obtain

$$\begin{aligned}\int_{L_1}^{L_2} (|v|^2 + \rho|u_x|^2)dx &\lesssim |u_x(L_2)|^2 + |v(L_2)|^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \epsilon\|U\|_{\mathcal{H}}^2 \\ &\lesssim \int_{L_2}^{L_3}(|v|^2 + \rho|u_x|^2)dx + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \epsilon\|U\|_{\mathcal{H}}^2 \\ &\lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2, \end{aligned} \tag{2.34}$$

for $|\lambda|$ large and $\epsilon > 0$ small. □

Lemma 2.3.4. *For $|\lambda|$ large and $\epsilon > 0$ small, we have*

$$\int_{L_3}^{L} (|v|^2 + \rho|u_x|^2)dx \lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2.$$

Proof. Again, using Lemma 2.3.1 twice and (2.33), we obtain

$$\begin{aligned}\int_{L_3}^{L}(|v|^2+\rho|u_x|^2)dx &\lesssim |u_x(L_3)|^2+|v(L_3)|^2+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\epsilon\|U\|_{\mathcal{H}}^2\\ &\lesssim \int_{L_2}^{L_3}(|v|^2+\rho|u_x|^2)dx+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\epsilon\|U\|_{\mathcal{H}}^2\\ &\lesssim \epsilon\|U\|_{\mathcal{H}}^2+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^2, \end{aligned} \tag{2.35}$$

for $|\lambda|$ large and $\epsilon > 0$ small. □

The main result of this section is given by the following theorem.

Theorem 2.3.1. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, considering the conditions of the KEFE model. Then, the system* (2.4) *is exponentially stable, that is, there exists a positive constant ϵ such that*

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \lesssim e^{-\epsilon t}\|U_0\|_{\mathcal{H}}, \quad \forall\, U_0 \in \mathcal{H}, \quad t>0,$$

Proof. The proof is based on Theorem 1.5.3, for which we need to prove that $i\mathbb{R} \subset \rho(\mathcal{A})$. First, let us prove that $i\mathbb{R} \subset \rho(\mathcal{A})$. In fact, since $\mathcal{A}$ is a closed operator and $D(\mathcal{A})$ has compact embedding over the phase space $\mathcal{H}$, then the spectrum $\sigma(\mathcal{A})$ contains only eigenvalues. Thus, it suffices to prove that $\sigma(\mathcal{A})$ does not contain any imaginary eigenvalues. To do this, we will use the argument by contradiction. Let us suppose that there exists an imaginary eigenvalue $i\lambda$ with $\lambda \in \mathbb{R}$ such that $i\lambda U - \mathcal{A}U = 0$, with $0 \neq U \in D(\mathcal{A})$. From (2.19)-(2.22) with $F = 0$, we get

$$\int_0^L a(x)|v|^2dx+\int_0^L b(x)|v_x|^2dx=0,$$

which implies in

$$v=0 \text{ in } (L_2,L_3) \quad \text{and} \quad v_x=0 \text{ in } (0,L_1). \tag{2.36}$$

From (2.19) and (2.36), we have

$$u=0 \text{ in } (L_2,L_3) \quad \text{and} \quad u_x=0 \text{ in } (0,L_1). \tag{2.37}$$

Since $U \in D(\mathcal{A})$, then $(\rho u_x + b(\cdot)v_x)_x \in L^2(0,L)$, and using (2.36) we have

$$u \in H^2(0,L), \quad \text{and consequently} \quad u \in C^1([0,L]). \tag{2.38}$$

From (2.37) and (2.38), we get

$$u(L_1)=u_x(L_1)=u(L_2)=u_x(L_2)=u(L_3)=u_x(L_3)=0. \tag{2.39}$$

Now, using (2.19)-(2.20), we have

$$\lambda^2 u+\rho\partial_{xx}u=0, \quad \text{in } (0,L).$$

Then, from (2.39), we obtain $u = 0$ in $(0,L)$, and consequently, from (2.19), we have $v = 0$ in $(0,L)$. Therefore, $U = 0$. But this is a contradiction, and therefore there are no imaginary eigenvalues. Thus, $i\mathbb{R} \subset \rho(\mathcal{A})$.

Now, let $F \in \mathcal{H}$, consider $U = (u, v)$ solution of $(i\lambda I - \mathcal{A})U = F$, i.e, the system (2.19)-(2.20) is satisfied. To show the exponential decay, according Theorem 1.5.3 is sufficient to show $\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}}$, for $|\lambda|$ large. In fact, from (2.25),(2.33) and from Lemmas 2.3.3-2.3.4, we get

$$\|U\|^2_{\mathcal{H}} = \int_0^L \left(|v|^2 + \rho|u_x|^2\right) dx \lesssim \epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}},$$

for $|\lambda|$ large and $\epsilon > 0$ small. Thus, using Young's inequality, taking $|\lambda|$ large and a suitably small $\epsilon > 0$, we obtain

$$\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}},$$

as we desired to prove. □

2.4 Asymptotic behavior: Polynomial stability

To show the polynomial decay of the solution for the system (2.19)-(2.20) we use a result due to Borichev and Tomilov ([7]).

Here we consider the cases where at least one of the elastic parts of the string (with no dissipation) connects with only the Kelvin-Voigt damping. We focus in investigating the stability of the EKIFE model because the EKEFE model and other cases are similar. We recall that the EKIFE model is given by considering

$$b(x) = b_0\chi_{[L_1,L_3]}(x) \text{ and } a(x) = a_0\chi_{[L_2,L_4]}(x), \quad a_0, b_0 > 0. \tag{2.40}$$

The main result of this section is Theorem 2.4.1 and to prove this theorem we will need to introduce some technical lemmas.
From (2.22) and (2.40), we have

$$\int_{L_2}^{L_4} a_0|v|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{2.41}$$

and

$$\int_{L_1}^{L_3} b_0|v_x|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{2.42}$$

Now, from (2.19) and (2.42), we get

$$\begin{aligned}\int_{L_1}^{L_3} b_0|u_x|^2 dx &\lesssim |\lambda|^{-2}\left(\int_{L_1}^{L_3} b_0|v_x|^2 dx + \int_{L_1}^{L_3} b_0|f_x|^2 dx\right)\\ &\lesssim |\lambda|^{-2}(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}). \end{aligned}\tag{2.43}$$

Lemma 2.4.1. *For $|\lambda|$ large, we have*

$$\int_{L_1}^{L_3} |\rho u_x + b_0 v_x|^2 \, dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.$$

Proof. From (2.42) and (2.43), we get

$$\int_{L_1}^{L_3} |\rho u_x + b_0 v_x|^2 \, dx \lesssim \int_{L_1}^{L_3} \rho\, |u_x|^2 \, dx + \int_{L_1}^{L_3} b_0\, |v_x|^2 \, dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}},$$

for $|\lambda|$ large. □

Lemma 2.4.2. *For $|\lambda|$ large, we have*

$$|\lambda|\int_{L_1}^{L_3}|v|^2dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{3/2}\|F\|_{\mathcal{H}}^{1/2}+\|F\|_{\mathcal{H}}^2.$$

Proof. Consider $H^{-1}(L_1,L_3)$ the dual space of $H^1(L_1,L_3)$. From (2.20),(2.41),(2.43), we get

$$\begin{aligned}
&|\lambda|\|v\|_{H^{-1}(L_1,L_3)}\\
&\quad\lesssim \|u_x\|_{L^2(L_1,L_3)}+\|b(x)^{\frac{1}{2}}v_x\|_{L^2(L_1,L_3)}+\|a_0^{\frac{1}{2}}v\|_{L^2(L_2,L_3)}+\|g\|_{L^2(L_1,L_3)}\\
&\quad\lesssim \|F\|_{\mathcal{H}}^{\frac{1}{2}}\|U\|_{\mathcal{H}}^{\frac{1}{2}}+\|F\|_{\mathcal{H}}.
\end{aligned} \tag{2.44}$$

Using Interpolation and inequality (2.44), we obtain

$$\begin{aligned}
\|v\|_{L^2(L_1,L_3)}^2 &\lesssim \|v\|_{H^{-1}(L_1,L_3)}\|v\|_{H^1(L_1,L_3)}\\
&\lesssim \frac{1}{|\lambda|}\left(\|F\|_{\mathcal{H}}^{\frac{1}{2}}\|U\|_{\mathcal{H}}^{\frac{1}{2}}+\|F\|_{\mathcal{H}}\right)\left(\|\partial_x v\|_{L^2(L_1,L_3)}+\|v\|_{L^2(L_1,L_3)}\right)\\
&\lesssim \frac{1}{|\lambda|}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{\frac{3}{2}}\|U\|_{\mathcal{H}}^{\frac{1}{2}}\right)+\frac{C(\epsilon)}{|\lambda|^2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^2\right)+\epsilon\|v\|_{L^2(L_1,L_3)}^2,
\end{aligned}$$

for $|\lambda|$ large and $\epsilon>0$ small. Considering an appropriate small $\epsilon>0$, we obtain

$$|\lambda|\|v\|_{L^2(L_1,L_3)}^2 \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{\frac{3}{2}}\|U\|_{\mathcal{H}}^{\frac{1}{2}}+\|F\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large. □

Lemma 2.4.3. *We have*

$$\int_0^{L_1}(\rho|u_x|^2+|v|^2)dx \lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right)+\|F\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large.

Proof. Multiplying (2.20) by $(L_2-x)\overline{(\rho u_x+b_0v_x)}$, integrating over (L_1,L_2) and taking the real part, we arrive at

$$\begin{aligned}
&\text{Re}\left\{\int_{L_1}^{L_2} i\lambda(L_2-x)v\overline{(\rho u_x+b_0v_x)}dx\right\}-\text{Re}\left\{\frac{1}{2}\int_{L_1}^{L_2}(L_2-x)\frac{d}{dx}|\rho u_x+b_0v_x|^2dx\right\}\\
&=\text{Re}\left\{\int_{L_1}^{L_2}(L_2-x)g\overline{(\rho u_x+b_0v_x)}dx\right\}.
\end{aligned} \tag{2.45}$$

By (2.19), we have

$$\begin{aligned}
&\text{Re}\left\{\rho\int_{L_1}^{L_2}(L_2-x)v\overline{(-i\lambda u_x)}dx\right\}\\
&\quad=-\text{Re}\left\{\rho\int_{L_1}^{L_2}(L_2-x)v\overline{v_x}dx\right\}-\text{Re}\left\{\beta\int_{L_1}^{L_2}(L_2-x)v\overline{f_x}dx\right\}\\
&\quad=\frac{\rho(L_2-L_1)}{2}|v(L_1)|^2-\frac{\rho}{2}\int_{L_1}^{L_2}|v|^2dx-\text{Re}\left\{\rho\int_{L_1}^{L_2}\delta_2(L_2-x)v\overline{f_x}dx\right\},
\end{aligned} \tag{2.46}$$

and

$$\text{Re}\left\{\frac{1}{2}\int_{L_1}^{L_2}(L_2-x)\ \frac{d}{dx}|\rho u_x+b_0v_x|^2dx\right\}$$
$$=\frac{1}{2}(L_2-L_1)|\rho u_x(L_1)+b_0v_x(L_1)|^2-\frac{1}{2}\int_{L_1}^{L_2}|\rho u_x+b_0v_x|^2dx. \tag{2.47}$$

On the other hand, by Lemma 2.4.2 and using (2.20), we get

$$\begin{aligned}
\text{Re}\Big\{b_0\int_{L_1}^{L_2}i\lambda(L_2-x)v\overline{v_x}dx\Big\}
&\lesssim b_0|\lambda|^{1/2}\int_{L_1}^{L_2}b_0|v_x|(|\lambda|^{1/2}|v|)dx\\
&\lesssim |\lambda|^{1/2}\left(\int_{L_1}^{L_2}b_0|v_x|^2dx\right)^{1/2}\left(|\lambda|\int_{L_1}^{L_2}|v|^2dx\right)^{1/2}\\
&\lesssim |\lambda|^{1/2}\|F\|_{\mathcal{H}}^{1/2}\|U\|_{\mathcal{H}}^{1/2}\left(\|F\|_{\mathcal{H}}^{1/2}\|U\|_{\mathcal{H}}^{1/2}+\|F\|_{\mathcal{H}}^{\frac{3}{4}}\|U\|_{\mathcal{H}}^{\frac{1}{4}}+\|F\|_{\mathcal{H}}\right)\\
&\lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right).
\end{aligned} \tag{2.48}$$

We define the functional

$$\varphi_u=\frac{(L_2-L_1)}{2}\left[\rho|v(L_1)|^2+|\rho u_x(L_1)+b_0v_x(L_1)|^2\right].$$

So, from (2.45)-(2.48) and using Lemma 2.4.1 and Lemma 2.4.2, we get

$$\begin{aligned}
\varphi_u=&\frac{\rho}{2}\int_{L_1}^{L_2}|v|^2dx+\frac{1}{2}\int_{L_1}^{L_2}|\rho u_x+b_0v_x|^2dx+\text{Re}\left\{\rho\int_{L_1}^{L_2}(L_2-x)v\overline{f_x}dx\right\}\\
&+\text{Re}\left\{\int_{L_1}^{L_2}(L_2-x)g\overline{(\rho u_x+b_0v_x)}dx\right\}-\text{Re}\left\{b_0\int_{L_1}^{L_2}i\lambda(L_2-x)v\overline{v_x}dx\right\}\\
\lesssim& |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right)+\|F\|_{\mathcal{H}}^2,
\end{aligned} \tag{2.49}$$

for $|\lambda|$ large.
On the other hand, multiplying (2.20) by $x\overline{u_x}$, integrating over $(0,L_1)$ and taking the real part, we get

$$\text{Re}\left\{i\lambda\int_0^{L_1}xv\overline{u_x}dx\right\}-\text{Re}\left\{\int_0^{L_1}\rho x\overline{u_x}u_{xx}dx\right\}=\text{Re}\left\{\int_0^{L_1}xg\overline{u_x}dx\right\}.$$

Using Equation(2.19), we get

$$\begin{aligned}
\text{Re}\left\{i\lambda\int_0^{L_1}xv\overline{u_x}dx\right\}&=-\text{Re}\left\{\int_0^{L_1}xv\overline{v_x}dx\right\}-\text{Re}\left\{\int_0^{L_1}xv\overline{f_x}dx\right\}\\
&=\frac{L_1}{2}|v(L_1)|^2-\int_0^{L_1}|v|^2dx-\text{Re}\left\{\int_0^{L_1}xv\overline{f_x}dx\right\}.
\end{aligned} \tag{2.50}$$

Also, we have

$$-\text{Re}\left\{\int_0^{L_1}\rho x\overline{u_x}u_{xx}dx\right\}=\frac{L_1}{2}\rho|u_x(L_1)|^2-\int_0^{L_1}\rho|u_x|^2dx. \tag{2.51}$$

Therefore, from (2.49)-(2.51), we have

$$\begin{aligned}\int_0^{L_1}(\rho|u_x|^2+|v|^2)dx &= \frac{L_1}{2}\left[\rho|u_x(L_1)|^2+|v(L_1)|^2\right]-\text{Re}\left\{\int_0^{L_1}xv\overline{f_x}dx\right\}-\text{Re}\left\{\int_0^{L_1}xg\overline{u_x}dx\right\}\\ &\lesssim \varphi_u+\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\\ &\lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right)+\|F\|_{\mathcal{H}}^2,\end{aligned}$$

for $|\lambda|$ large. □

So, from (2.43) and from Lemma (2.4.2), we have

$$\int_{L_1}^{L_3}\left(|v|^2+\rho|u_x|^2\right)dx\lesssim|\lambda|^{-1}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{\frac{3}{2}}\|U\|_{\mathcal{H}}^{\frac{1}{2}}+\|F\|_{\mathcal{H}}^2\right), \tag{2.52}$$

for $|\lambda|$ large.

Lemma 2.4.4. *Let $h\in C^1([0,L])$ be a function with $h(0)=h(L)=0$ and $\epsilon>0$ small. We have*

$$\begin{aligned}&\int_0^L h'(x)(|v|^2+|\rho u_x+b(\cdot)v_x|^2)dx\\ &\lesssim \mathit{Re}\left\{\int_0^L h(x)g\overline{(\rho u_x+b(\cdot)v_x)}dx\right\}-\mathit{Re}\left\{i\lambda\int_0^L b(x)h(x)v\overline{v_x}dx\right\}\\ &\quad-\mathit{Re}\left\{\int_0^L\rho h(x)a(x)v\overline{u_x}dx\right\}+\epsilon\|U\|_{\mathcal{H}}^2+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.\end{aligned}$$

Proof. Multiplying (2.20) by $h(x)\overline{(\rho u_x+b(\cdot)v_x)}$ and integrating over $(0,L)$, we get

$$\begin{aligned}&i\lambda\int_0^L h(x)v\overline{(\rho u_x+b(\cdot)v_x)}dx-\int_0^L h(x)\overline{(\rho u_x+b(\cdot)v_x)}(\rho u_x+b(\cdot)v_x))_xdx\\ &=\int_0^L h(x)g\overline{(\rho u_x+b(\cdot)v_x)}dx-\int_0^L a(\cdot)h(x)v\overline{(\rho u_x+b(\cdot)v_x)}dx.\end{aligned}\tag{2.53}$$

Using (2.19), we notice that the first term on the left side of the above equation can be rewritten as follows:

$$\begin{aligned}i\lambda\int_0^L h(x)v\overline{(\rho u_x+b(\cdot)v_x)}dx&=\int_0^L\rho h(x)v\overline{(-i\lambda u_x)}dx+i\lambda\int_0^L b(\cdot)h(x)v\overline{v_x}dx\\ &=-\int_0^L\rho h(x)v\overline{v_x}dx-\int_0^L\rho h(x)v\overline{f_x}dx+i\lambda\int_0^L b(\cdot)h(x)v\overline{v_x}dx\end{aligned}\tag{2.54}$$

Taking the real part in (2.53) and using $h(0)=h(L)$ and (2.54), we obtain

$$\begin{aligned}\int_0^L h'(x)(|v|^2+|\rho u_x+b(\cdot)v_x|^2)dx\lesssim&\ \text{Re}\left\{\int_0^L h(x)g\overline{(\rho u_x+b(\cdot)v_x)}dx\right\}+\text{Re}\left\{\int_0^L\rho h(x)v\overline{f_x}dx\right\}\\ &-\text{Re}\left\{\int_0^L\rho h(x)a(\cdot)v\overline{u_x}dx\right\}-\text{Re}\left\{i\lambda\int_0^L b(\cdot)h(x)v\overline{v_x}dx\right\}\\ &-\text{Re}\left\{\int_0^L a(\cdot)b(\cdot)h(x)v\overline{v_x}dx\right\}.\end{aligned}\tag{2.55}$$

Let's estimate some terms on the left side of the above equation. First, by Holder's inequality, we have

$$-\mathrm{Re}\left\{\int_0^L \rho h(x) v\overline{f_x} dx\right\} \lesssim \|v\|_{L^2(0,L)}\|f_x\|_{L^2(0,L)} \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{2.56}$$

Furthermore, since $\mathrm{supp}(b(x)) \cap \mathrm{supp}(a(x)) = (L_2, L_3)$, then by Young's inequality and (2.22), we get

$$\begin{aligned}
\mathrm{Re}\left\{\int_0^L a(\cdot)b(\cdot)h(x)v\overline{v_x}dx\right\} &\lesssim \max|h(x)| \int_{L_2}^{L_3} a_0 b_0 |v\overline{v_x}| dx \\
&\lesssim \epsilon \int_{L_2}^{L_3} a_0 |v|^2 dx + C(\epsilon)\int_{L_2}^{L_3} b_0 |v_x|^2 dx \\
&\lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},
\end{aligned} \tag{2.57}$$

for $\epsilon > 0$ small.
Therefore, using (2.56)-(2.57) in (2.55), we get

$$\begin{aligned}
\int_0^L h'(x)\left(|v|^2 + |\rho u_x + b(\cdot)v_x|^2\right)dx \lesssim \mathrm{Re}\left\{\int_0^L h(x) g\overline{(\rho u_x + b(\cdot)v_x)}dx\right\} - \mathrm{Re}\left\{\int_0^L \rho h(x) a(\cdot) v\overline{u_x}dx\right\} \\
- \mathrm{Re}\left\{i\lambda \int_0^L b(\cdot)h(x)v\overline{v_x}dx\right\} + \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},
\end{aligned}$$

for $\epsilon > 0$ small. $\square$

Lemma 2.4.5. *For $|\lambda|$ large, we have*

$$\begin{aligned}
&\int_{L_3}^L \left(|v|^2 + \rho|u_x|^2\right)dx \\
&\qquad \lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right) + \|F\|_{\mathcal{H}}^2.
\end{aligned}$$

Proof. Let us introduce $\tilde{L}_3 \in (L_2, L_3)$ and $q_1 \in C^1(0, L)$ such that $0 \le q_1(x) \le 1$, for all $x \in [0, L]$ and $q_1(x) = 0$, if $x \in [0, \tilde{L}_3]$ and $q_1(x) = 1$ if $x \in [L_3, L]$. Using the result of Lemma 2.4.4 with $h(x) = (x - L)q_1(x)$, we obtain

$$\begin{aligned}
&\int_{L_3}^L \left(|v|^2 + \rho|u_x|^2\right)dx \\
&\lesssim -\int_{\tilde{L}_3}^{L_3} \left(q_1 + (x-L)q_1'\right)\left(|v|^2 + |\rho u_x + b_0 v_x|^2\right)dx \\
&\quad + \mathrm{Re}\left\{\int_0^L (x-L)q_1(x) g\overline{(\rho u_x + b(\cdot)v_x)}dx\right\} \\
&\quad - \mathrm{Re}\left\{i\lambda\int_0^L b(x)(x-L)q_1(x)v\overline{v_x}dx\right\} - \mathrm{Re}\left\{\int_0^L \rho(x-L)q_1(x)a(x)v\overline{u_x}dx\right\} \\
&\quad + \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.
\end{aligned} \tag{2.58}$$

Let us estimate the terms on the right side of the above equation.

Note that, using $q_1 \in C^1([0,L])$ and from the Lemma (2.4.1) and from the Lemma (2.4.2), we have

$$\begin{aligned}
&-\int_{\tilde{L}_3}^{L_3} (q_1 + (x-L)q_1') \left(|v|^2 + |\rho u_x + b_0 v_x|^2\right) dx \\
&\qquad \lesssim (1 + (L-\tilde{L}_3)\text{max}|q_1'|) \int_{\tilde{L}_3}^{L_3} \left(|v|^2 + |\rho u_x + b_0 v_x|^2\right) dx \\
&\qquad \lesssim |\lambda|^{-1} \left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2} + \|F\|_{\mathcal{H}}^2\right) \\
&\qquad\quad + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2, \qquad (2.59)
\end{aligned}$$

for $|\lambda|$ large.
Using the Holder's inequality and the Lemma (2.4.1), we have

$$\begin{aligned}
&\text{Re}\left\{\int_0^L h(x) g\overline{(\rho u_x + b(\cdot)v_x)} dx\right\} \\
&\lesssim \int_{\tilde{L}_3}^{L_3} |(x-L)q_1 g\overline{(\rho u_x + b_0 v_x)}| dx + \int_{L_3}^{L} |(x-L)g\rho\overline{u}_x| dx \\
&\lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2. \qquad (2.60)
\end{aligned}$$

By Lemma(2.4.2) and from (2.42), we get

$$\begin{aligned}
\text{Re}\left\{i\lambda \int_0^L b(\cdot)h(x)v\overline{v_x}dx\right\} &\lesssim b_0\text{max}|h|\lambda|^{1/2} \int_{\tilde{L}_3}^{L_3} |v_x|(|\lambda|^{1/2}|v|)dx \\
&\lesssim |\lambda|^{1/2}\left(\int_{\tilde{L}_3}^{L_3} |v_x|^2 dx\right)^{1/2} \left(|\lambda|\int_{\tilde{L}_3}^{L_3} |v|^2 dx\right)^{1/2} \\
&\lesssim |\lambda|^{1/2}\|F\|_{\mathcal{H}}^{1/2}\|U\|_{\mathcal{H}}^{1/2}\left(\|F\|_{\mathcal{H}}^{1/2}\|U\|_{\mathcal{H}}^{1/2} + \|F\|_{\mathcal{H}}^{3/4}\|U\|_{\mathcal{H}}^{1/4} + \|F\|_{\mathcal{H}}\right) \\
&\lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right), \qquad (2.61)
\end{aligned}$$

for $|\lambda|$ large.
Finally, from (2.41), we have

$$\begin{aligned}
&\text{Re}\left\{\int_0^L \rho h(x) a(\cdot) v\overline{u_x} dx\right\} \\
&\lesssim C(\epsilon)\int_{\tilde{L}_3}^{L_4} |v|^2 dx + \epsilon \int_{\tilde{L}_3}^{L_4} \rho|u_x|^2 dx \\
&\lesssim \epsilon\|U\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \qquad (2.62)
\end{aligned}$$

Therefore, from (2.58)-(2.62), we get

$$\begin{aligned}
&\int_{L_3}^{L} \left(|v|^2 + \rho|u_x|^2\right) dx \\
&\lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right) + \|F\|_{\mathcal{H}}^2,
\end{aligned}$$

for $|\lambda|$ large. □

The main result of this section is given by the following theorem.

Theorem 2.4.1. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. Then, the EKIFE model is not exponentially stable. Moreover, the semigroup $e^{t\mathcal{A}}$ of system* (2.4) *decays polynomially with the rate t^{-2}, that is*

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \lesssim t^{-2}\|U_0\|_{D(\mathcal{A})}, \quad \forall\, U_0 \in D(\mathcal{A}), \quad t \geq 1,$$

Proof. The proof is based on Theorem 1.5.4. Using the same arguments as in the proof of Theorem 2.3.1 we can show that $i\mathbb{R} \subset \rho(\mathcal{A})$.

Now, let $F \in \mathcal{H}$, consider $U = (u, v)$ solution of $(i\lambda I - \mathcal{A})U = F$, i.e, the system (2.19)-(2.20) is satisfied. To show the polynomial decay with the rate t^{-2}, according the Theorem 1.5.4, is sufficient to show

$$\|U\|_{\mathcal{H}} \lesssim |\lambda|^{1/2}\|F\|_{\mathcal{H}},$$

for $|\lambda|$ large.
Therefore, from Lemmas 2.4.3, 2.4.5 and Equation (2.52), we obtain

$$\begin{aligned}\|U\|_{\mathcal{H}}^2 &= \int_0^{L_1} \left(|v|^2 + \rho|u_x|^2\right) dx + \int_{L_1}^{L_3} \left(|v|^2 + \rho|u_x|^2\right) dx + \int_{L_3}^{L} \left(|v|^2 + \rho|u_x|^2\right) dx \\ &\lesssim |\lambda|^{1/2} \left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right) + \|F\|_{\mathcal{H}}^2,\end{aligned}$$

for $|\lambda|$ large. Thus, using Young's inequality, we obtain

$$\|U\|_{\mathcal{H}}^2 \lesssim \left(|\lambda| + |\lambda|^{4/5} + |\lambda|^{2/3} + 1\right)\|F\|_{\mathcal{H}}^2 \lesssim |\lambda|\|F\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large. Therefore, we get

$$\|U\|_{\mathcal{H}} \lesssim |\lambda|^{1/2}\|F\|_{\mathcal{H}},$$

for $|\lambda|$ large. □

2.5 Optimality of the decay rate

In this section we show that the decay rate obtained in Theorem 2.4.1 is the best. We recall that Theorem 2.4.1 was proved for the EKIFE model, but the proof is similar for the cases where at least one of the elastic parts of the string is connected only with the Kelvin-Voigt damping leading to the same decay rate. In the proof of optimality we will address the EKIFE model, with the proof being similar for other models. The EKIFE model, already introduced, considering in problem (2.1)-(2.3) that

$$b(x) = b_0\chi_{[L_1,L_3](x)} \text{ and } a(x) = a_0\chi_{[L_2,L_4](x)}, \quad a_0, b_0 > 0.$$

Thus, both the Kelvin-Voigt and frictional type dampings are effective in the interval (L_2, L_3).

Theorem 2.5.1. *The polynomial decay rate obtained in Theorem 2.4.1 for the EKIFE model is optimal in the sense that the semigroup does not decay with the rate t^{-s} for $s > 2$.*

To prove the above theorem, we need to state and prove some important results.
Let $\lambda \in \mathbb{R}$ and $F = (f, g)^\top \in \mathcal{H}$. In what follows, the stationary problem

$$(i\lambda I - \mathcal{A})U = F, \tag{2.63}$$

will be considered several times. Note that $U = (u, v)$ is a solution of this problem if the following equations are satisfied:

$$i\lambda u - v = f, \tag{2.64}$$
$$i\lambda v - [\rho u_x + b(\cdot)v_x]_x + a(\cdot)v = g, \tag{2.65}$$

with the following boundary conditions

$$u(0) = u(L) = 0 \quad \text{in} \quad (0, \infty), \tag{2.66}$$

and

$$b(x) = b_0\chi_{[L_1,L_3](x)} \text{ and } a(x) = a_0\chi_{[L_2,L_4](x)}.$$

For $i = 1, \dots, 5$, we consider

$$u_i = u\chi_{[L_{i-1},L_i]}, \quad L_0 = 0, \quad L_5 = L.$$

We have the following lemma

Lemma 2.5.1. *Consider a particular solution, defined by $v = i\lambda u$, $f = 0$ and $g \doteq (\rho h)\chi_{(0,L_1)}$, where h will be chosen later Then, the system* (2.63) *can be written as*

$$\begin{array}{ll} \partial_x^2 u_1 + \ell_1^2 u_1 = -h(x), & x \in (0, L_1); \\ \partial_x^2 u_i + \ell_i^2 u_i = 0, & x \in (L_{i-1}, L_i),\ i = 2, \dots, 5. \end{array}$$

with boundary conditions

$$u(0) = u(L) = 0, \quad u_i(L_i) = u_{i+1}(L_i), \quad \sigma_i \partial_x u_i(L_i) = \sigma_{i+1}\partial_x u_{i+1}(L_i), \quad i = 1 \dots 4.$$

where

$$\ell_1^2 = \ell_5^2 = \frac{\lambda^2}{\rho}, \quad , \quad \ell_2^2 = \frac{\lambda^2}{\rho + i\lambda b_0} \quad \ell_3^2 = \frac{\lambda^2 - i\lambda a_0}{\rho + ib_0\lambda}, \quad \ell_4^2 = \frac{\lambda^2 - i\lambda a_0}{\rho};$$
$$\sigma_1 = \sigma_4 = \sigma_3 = \sigma_5 = \rho, \quad \sigma_2 = \rho + b_0 i\lambda.$$

Proof. Let a particular solution, defined by $f = 0$ and $g \doteq (\rho h)\chi_{(0,L_1)}$, where h will be chosen later.
From (2.64) - (2.65), we have $v = i\lambda u$ and

$$\begin{cases} \partial_x^2 u_1 + \frac{\lambda^2}{\rho} u_1 = -h, & \text{in} \quad (0, L_1); \\ \partial_x^2 u_2 + \frac{\lambda^2}{\rho + ib_0\lambda} u_2 = 0, & \text{in} \quad (L_1, L_2); \\ \partial_x^2 u_3 + \frac{\lambda^2 - i\lambda a_0}{\rho + ib_0\lambda} u_3 = 0, & \text{in} \quad (L_2, L_3); \\ \partial_x^2 u_4 + \frac{\lambda^2 - a_0 i\lambda}{\rho} u_4 = 0, & \text{in} \quad (L_3, L_4); \\ \partial_x^2 u_5 + \frac{\lambda^2}{\rho} u_5 = 0, & \text{in} \quad (L_4, L). \end{cases}$$

with transmission condition

$$\begin{aligned} &u_1(L_1) = u_2(L_1), \qquad \rho\partial_x u_1(L_1) = \rho\partial_x u_2(L_1) + b_0\partial_x v_2(L_1), \\ &u_2(L_2) = u_3(L_2), \qquad \rho\partial_x u_2(L_2) + b_0\partial_x v_2(L_2) = \rho\partial_x u_3(L_2), \\ &u_3(L_3) = u_4(L_3), \qquad \rho\partial_x u_3(L_3) = \rho\partial_x u_4(L_3), \\ &u_4(L_4) = u_5(L_4), \qquad \rho\partial_x u_4(L_4) = \rho\partial_x u_5(L_4), \end{aligned}$$

and boudary condition $u_1(0) = 0$, $u_5(L) = 0$. □

The following theorem will help us find an estimate for the lemma system (2.5.1).

Lemma 2.5.2. *Let $h \in L^2(0, L_1)$. For $i = 1, \ldots, 5$, let us consider the following system*

$$\begin{array}{ll} \partial_x^2 u_1 + \ell_1^2 u_1 = -h(x), & x \in (L_0, L_1), \\ \partial_x^2 u_i + \ell_i^2 u_i = 0, & x \in (L_{i-1}, L_i), \ i = 2, \ldots, 5 \end{array}$$

with boundary conditions

$$u(0) = u(L) = 0, \quad u_i(L_i) = u_{i+1}(L_i), \quad \sigma_i\partial_x u_i(L_i) = \sigma_{i+1}\partial_x u_{i+1}(L_i), \quad i = 1 \ldots 4.$$

where

$$\ell_1^2 = \ell_5^2 = \frac{\lambda^2}{\rho}, \quad \ell_2^2 = \frac{\lambda^2}{\rho + i\lambda b_0}, \quad \ell_3^2 = \frac{\lambda^2 - i\lambda a_0}{\rho + ib_0\lambda}, \quad \ell_4^2 = \frac{\lambda^2 - i\lambda a_0}{\rho}; \tag{2.67}$$

$$\sigma_1 = \sigma_3 = \sigma_4 = \sigma_5 = \rho, \quad \sigma_2 = \rho + b_0 i\lambda.$$

The solution of the system is given by

$$u_1(x) = u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} + \frac{1}{\ell_1}\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} H_2(L_1) - \frac{1}{\ell_1} H_2(x). \tag{2.68}$$

where we have the estimate

$$|u(L_1)| \approx \frac{|\sigma_1 H_1(L_1)|}{|W_1 \sin(\ell_1 L_1)|}, \tag{2.69}$$

with $W_1 = \sigma_1\ell_1\cot(\ell_1 L_1) - \sigma_2\ell_2\cot(\ell_2(L_1 - L_2))$ and

$$H_1(x) = \int_0^x \sin(\ell_1 s)h(s)\, ds, \quad H_2(x) = \int_0^x \sin(\ell_1(x - s))h(s)\, ds.$$

Proof. Solving the stationary differential equations with boundary conditions $u(0) = u(L) = 0$, we have

$$\begin{aligned}
u_1(x) &= u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} + \frac{1}{\ell_1}\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)}H_2(L_1) - \frac{1}{\ell_1}H_2(x),\\
u_2(x) &= u(L_1)\frac{\sin(\ell_2(x-L_2))}{\sin(\ell_2(L_1-L_2))} + u(L_2)\frac{\sin(\ell_2(x-L_1))}{\sin(\ell_2(L_2-L_1))},\\
u_3(x) &= u(L_2)\frac{\sin(\ell_3(x-L_3))}{\sin(\ell_3(L_2-L_3))} + u(L_3)\frac{\sin(\ell_3(x-L_2))}{\sin(\ell_3(L_3-L_2))}\\
u_4(x) &= u(L_3)\frac{\sin(\ell_4(x-L_4))}{\sin(\ell_4(L_3-L_4))} + u(L_4)\frac{\sin(\ell_4(x-L_3))}{\sin(\ell_3(L_3-L_4))}\\
u_5(x) &= u(L_4)\frac{\sin(\ell_5(x-L_5))}{\sin(\ell_5(L_4-L_5))}.
\end{aligned}$$

Using the transmission conditions $\sigma_1\partial_x u_1(L_1) = \sigma_2\partial_x u_2(L_1)$ we have

$$\begin{aligned}
\sigma_1\ell_1 u(L_1)\cot(\ell_1 L_1) + \sigma_1\left\{\cot(\ell_1 L_1)H_2(L_1) - \frac{1}{\ell_1}H_2'(L_1)\right\}&\\
= \sigma_2\ell_2 u(L_1)\cot(\ell_2(L_1-L_2)) + \frac{\sigma_2\ell_2 u(L_2)}{\sin(\ell_2(L_2-L_1))}&
\end{aligned}$$

Since

$$\cot(\ell_1 x)H_2(x) - \frac{1}{\ell_1}H_2'(x) = -\frac{1}{\sin(\ell_1 x)}H_1(x),$$

we obtain

$$\sigma_1\ell_1 u(L_1)\cot(\ell_1 L_1) - \frac{\sigma_1 H_1(L_1)}{\sin(\ell_1 L_1)} = \sigma_2\ell_2 u(L_1)\cot(\ell_2(L_1-L_2)) + \frac{\sigma_2\ell_2 u(L_2)}{\sin(\ell_2(L_2-L_1))},$$

this is

$$W_1 u(L_1) + U_2 u(L_2) = \frac{\sigma_1 H_1(L_1)}{\sin(\ell_1 L_1)}, \tag{2.70}$$

where

$$W_1 = \sigma_1\ell_1\cot(\ell_1 L_1) - \sigma_2\ell_2\cot(\ell_2(L_1-L_2)), \quad U_2 = \frac{\sigma_2\ell_2}{\sin(\ell_2(L_2-L_1))}.$$

Using the transmission conditions $\sigma_2\partial_x u_2(L_2) = \sigma_3\partial_x u_3(L_2)$, we get

$$\begin{aligned}
\frac{\sigma_2\ell_2 u(L_1)}{\sin(\ell_2(L_1-L_2))} + \sigma_2\ell_2 u(L_2)\cot(\ell_2(L_2-L_1))&\\
= \sigma_3\ell_3 u(L_2)\cot(\ell_3(L_2-L_3)) + \frac{\sigma_3\ell_3 u(L_3)}{\sin(\ell_3(L_3-L_2))}&
\end{aligned}$$

from where follows that

$$U_2 u(L_1) + W_2 u(L_2) + U_3 u(L_3) = 0, \tag{2.71}$$

where

$$W_2 = \sigma_2\ell_2\cot(\ell_2(L_2-L_1)) - \sigma_3\ell_3\cot(\ell_3(L_2-L_3)), \quad U_3 = \frac{\sigma_3\ell_3}{\sin(\ell_3(L_3-L_2))}.$$

Using the transmission conditions $\sigma_3\partial_x u_3(L_3) = \sigma_4\partial_x u_4(L_3)$ we get

$$\frac{\sigma_3\ell_3 u(L_2)}{\sin(\ell_3(L_2-L_3))} + \sigma_3\ell_3 u(L_3)\cot(\ell_3(L_3-L_2)) = \sigma_4\ell_4 u(L_3)\cot(\ell_4(L_3-L_4)) + \frac{\sigma_4\ell_4 u(L_4)}{\sin(\ell_4(L_4-L_3))}$$

from where follows that

$$U_3 u(L_2) + W_3 u(L_3) + U_4 u(L_4) = 0, \tag{2.72}$$

where

$$W_3 = \sigma_3\ell_3\cot(\ell_3(L_3-L_2)) - \sigma_4\ell_4\cot(\ell_4(L_3-L_4)), \quad U_4 = \frac{\sigma_4\ell_4}{\sin(\ell_4(L_4-L_3))}.$$

Using the transmission conditions $\sigma_4\partial_x u_4(L_4) = \sigma_5\partial_x u_5(L_4)$ we get

$$\frac{\sigma_4\ell_4 u(L_3)}{\sin(\ell_4(L_3-L_4))} + \sigma_4\ell_4 u(L_4)\cot(\ell_4(L_4-L_3)) = \sigma_5\ell_5 u(L_4)\cot(\ell_5(L_4-L_5))$$

from where follows that

$$U_4 u(L_3) + W_4 u(L_4) = 0, \tag{2.73}$$

where

$$W_4 = \sigma_4\ell_4\cot(\ell_4(L_4-L_3)) - \sigma_5\ell_5\cot(\ell_5(L_4-L_5)).$$

Solving $u(L_3)$ (from (2.72)-(2.73)) in terms of $u(L_2)$, we get

$$\begin{bmatrix} W_3 & U_4 \\ U_4 & W_4 \end{bmatrix} \begin{bmatrix} u(L_3) \\ u(L_4) \end{bmatrix} = \begin{bmatrix} -U_3 u(L_2) \\ 0 \end{bmatrix}$$

this is

$$u(L_3) = \frac{-U_3W_4}{W_3W_4 - U_4^2} u(L_2)$$

Solving $u(L_1)$ (from (2.70)-(2.71)), we get

$$\begin{bmatrix} W_1 & U_2 \\ U_2 & W_2 - \frac{U_3^2 W_4}{W_3W_4 - U_4^2} \end{bmatrix} \begin{bmatrix} u(L_1) \\ u(L_2) \end{bmatrix} = \begin{bmatrix} H_0 \\ 0 \end{bmatrix}, \quad H_0 := \frac{\sigma_1 H_1(L_1)}{\sin(\ell_1 L_1)}$$

this is

$$u(L_1) = \frac{H_0 W_0}{W_1W_0 - U_2^2}, \quad \text{with} \quad W_0 := W_2 - \frac{U_3^2 W_4}{W_3W_4 - U_4^2}.$$

We consider $a_0 > 0$, such that

$$\frac{a_i}{\pi}\left(2\pi n + \frac{a_0}{\sqrt{n}}\right) \notin \mathbb{Z}, \quad i = 1, 2, \tag{2.74}$$

where

$$a_1 = \frac{L_3 - L_2}{L_1}, \quad a_2 = \frac{L_5 - L_4}{L_1}.$$

We take $\lambda = \lambda_n$, where

$$\lambda_n := \frac{\sqrt{\rho}}{L_1}\left(2\pi n + \frac{a_0}{\sqrt{n}}\right) \quad \Rightarrow \quad \lambda_n \approx n.$$

From (2.67), note that

$$\sigma_i \ell_i = \frac{\lambda_n^2}{\ell_i}, \quad i = 1, \ldots, 5.$$

• Our objective will now be to estimate the terms that involve: ℓ_1 and ℓ_5. Note that

$$\ell_1 \approx \ell_5 \approx \lambda_n.$$

Furthermore, $\ell_1 L_1 = 2\pi n + \frac{a_0}{\sqrt{n}}$, we have

$$\sin(\ell_1 L_1) = \sin\left(\frac{a_0}{\sqrt{n}}\right) \approx \lambda_n^{-1/2} \quad \Rightarrow \quad \sigma_1 \ell_1 \cot(\ell_1 L_1) \approx \lambda_n^{3/2}.$$

• Our objective will now be to estimate the terms that involve: ℓ_2 e ℓ_4 .
Note that, if

$$\ell = x + iy \quad \text{and} \quad \ell^2 = A + iB = x^2 - y^2 + i2xy,$$

then

$$x^2 = \frac{A + \sqrt{A^2 + B^2}}{2}, \quad y^2 = \frac{-A + \sqrt{A^2 + B^2}}{2} \overset{A \geq 0}{=} \frac{B^2}{2(A + \sqrt{A^2 + B^2})}.$$

Since

$$\ell_2^2 = \frac{\lambda_n^2}{\rho + b_0 i \lambda_n} = \frac{\lambda_n^2(\rho - b_0 i \lambda_n)}{\rho^2 + (b_0 \lambda_n)^2} = A_2 + iB_2,$$

we have $A_2 \approx 1$ and $B_2 \approx \lambda_n$. Then

$$\text{Re}\ell_2 \approx \lambda_n^{1/2}, \quad \text{Im}\ell_2 \approx \lambda_n^{1/2}. \tag{2.75}$$

Since

$$\ell_4^2 = \frac{\lambda_n^2 - i\lambda_n a_0}{\rho} = A_4 + iB_4,$$

we have that $A_4 \approx \lambda_n^2$ and $B_4 \approx \lambda_n$. Then

$$\text{Re}\ell_4 \approx \lambda_n, \quad \text{Im}\ell_4 \approx 1.$$

On the other hand, since $\sigma_i \ell_i = \frac{\lambda_n^2}{\ell_i}$, $i = 2, 4$, from the previous estimates we have

$$\sigma_2 \ell_2 \approx \lambda_n^{3/2}(1 + i), \quad \sigma_4 \ell_4 \approx \lambda_n + i.$$

In that follows, we use the identities

$$|\sin(\mu+i\eta)|^2=\sin^2(\mu)+\sinh^2(\eta),\quad \text{and}\quad \cot(\mu+i\eta)=\frac{\cos(\mu)\sin(\mu)-i\cosh(\eta)\sinh(\eta)}{\cosh^2(\eta)-\cos^2(\mu)}. \tag{2.76}$$

Since $\ell_2\approx\lambda_n^{1/2}(1+i)$ we can write $\ell_2(L_2-L_1)=\lambda_n^{1/2}(\mu_{2,n}+i\eta_{2,n})$ with $\mu_{2,n}\approx 1\approx\eta_{2,n}$. Then, we have

$$|\sin(\ell_2(L_2-L_1))|\gtrsim|\sinh(\lambda_n^{1/2}\eta_{2,n})|\quad\Rightarrow\quad |U_2|\lesssim\frac{1}{|\sinh(\lambda_n^{1/2}\eta_{2,n})|}\to 0.$$

Moreover, we also have

$$\cot(\ell_2(L_2-L_1))\approx i\quad\Longrightarrow\quad \sigma_2\ell_2\cot(\ell_2(L_2-L_1))\approx\lambda_n^{3/2}(1+i). \tag{2.77}$$

Since $\ell_4\approx\lambda_n+i$ we can write $\ell_4(L_4-L_3)=\lambda_n\mu_{4,n}+i\eta_{4,n}$ with $\mu_{4,n}\approx 1\approx\eta_{4,n}$. Then

$$|\sin(\ell_4(L_4-L_3))|\approx 1\quad\Rightarrow\quad |U_4|\approx\lambda_n.$$

Writing $\cot(\ell_4(L_4-L_3))=p_n+iq_n$, we have

$$|p_n|\lesssim 1,\quad q_n\approx 1.$$

Since $\sigma_4\ell_4\approx\lambda_n+i$, we can write $\sigma_4\ell_4=r_n\lambda_n+is_n$ where $r_n\approx 1\approx s_n$, then

$$\begin{aligned}\sigma_4\ell_4\cot(\ell_4(L_4-L_3))&=(r_n\lambda_n+is_n)(p_n+iq_n)\\&=\lambda_nr_np_n-s_nq_n+i(\lambda_nr_nq_n+s_np_n).\end{aligned}$$

From where follows

$$|\mathrm{Re}\,\{\sigma_4\ell_4\cot(\ell_4(L_4-L_3))\}\,|\lesssim\lambda_n,\quad |\mathrm{Im}\,\{\sigma_4\ell_4\cot(\ell_4(L_4-L_3))\}\,|\approx\lambda_n$$

Therefore

$$|\sigma_4\ell_4\cot(\ell_4(L_4-L_3))|\approx\lambda_n.$$

• Our objective will now be to estimate the terms that involve: ℓ_3.
Note that

$$\ell_3^2=\frac{\lambda_n^2-i\lambda_na_0}{\rho+i\lambda_nb_0}=\frac{\lambda_n^2(\rho-a_0b_0)-i(\lambda_n^3b_0+\lambda_na_0\rho)}{\rho^2+\lambda_n^2b_0^2},$$

then $\mathrm{Re}(\ell_3^2)\approx 1$ and $\mathrm{Im}(\ell_3^2)\approx\lambda_n$. Then

$$\mathrm{Re}(\ell_3)\approx\lambda_n^{1/2},\quad \mathrm{Im}(\ell_3)\approx\lambda_n^{1/2}.$$

On the other hand, since $\sigma_3\ell_3=\frac{\lambda_n^2}{\ell_3}$, from the previous estimates we have

$$\cot(\ell_3(L_3-L_2))\approx i\quad\Longrightarrow\quad \sigma_3\ell_3\cot(\ell_3(L_3-L_3))\approx\lambda_n^{3/2}(1+i).$$

In that follows, we use the identities (2.76).

Since $\ell_3 \approx \lambda_n^{1/2}(1+i)$ we can write $\ell_3(L_3-L_2) = \lambda_n^{1/2}(\mu_{3,n} + i\eta_{3,n})$ with $\mu_{3,n} \approx 1 \approx \eta_{3,n}$. Then we have

$$|\sin(\ell_3(L_3-L_2))| \gtrsim |\sinh(\lambda_n^{1/2}\eta_{3,n})| \quad \Rightarrow \quad |U_3| \lesssim \frac{\lambda_n^{3/2}}{|\sinh(\lambda_n^{1/2}\eta_{3,n})|} \to 0,$$

• Our objective will now be to estimate the terms that involve: ℓ_1 and ℓ_5. Note that

$$\ell_1 \approx \ell_5 \approx \lambda_n.$$

Furthermore, $\ell_1 L_1 = 2\pi n + \frac{a_0}{\sqrt{n}}$, we have

$$\sin(\ell_1 L_1) = \sin\left(\frac{a_0}{\sqrt{n}}\right) \approx \lambda_n^{-1/2} \quad \Rightarrow \quad \sigma_1\ell_1\cot(\ell_1 L_1) \approx \lambda_n^{3/2}. \tag{2.78}$$

Now, since $\ell_5(L_5-L_4) = a_2\left(2\pi n + \frac{a_0}{\sqrt{n}}\right)$, we have

$$\sin\left(2a_2\pi n + \frac{a_2a_0}{\sqrt{n}}\right) = \sin(2a_2\pi n)\cos\left(\frac{a_2a_0}{\sqrt{n}}\right) + \cos(2a_2\pi n)\sin\left(\frac{a_2a_0}{\sqrt{n}}\right),$$

From the condition (2.74), note that the sequence

$$\Lambda_{2,n} := \sin(\ell_5(L_5-L_4))$$

is a non-zero bounded sequence (note that maybe it could converge to zero).
Note that, if $\Lambda_{2,n} \to 0$, then $\cos(2a_2\pi n) \nrightarrow 0$ and we get

$$|\sigma_5\ell_5\cot(\ell_5(L_5-L_4))| \approx \frac{\lambda_n}{|\Lambda_{2,n}|}.$$

On the other hand, if $\Lambda_{2,n} \nrightarrow 0$, then

$$|\sigma_5\ell_5\cot(\ell_5(L_5-L_4))| \approx \frac{\lambda_n|\cos(2a_2\pi n)|}{|\Lambda_{2,n}|}.$$

• Our objective will now be to estimate the terms: W_1 and W_4.
From the previous estimates, we immediately obtain that $|W_1| \approx \lambda_n^{3/2}$.
If $\Lambda_{2,n} \to 0$ and since $|\Lambda_{2,n}| = |\sin(\ell_5(L_5-L_4))| \lesssim 1$, we obtain

$$|W_4| = |\sigma_4\ell_4\cot(\ell_4(L_4-L_3)) - \sigma_5\ell_5\cot(\ell_5(L_4-L_5))| \approx \frac{\lambda_n}{|\Lambda_{2,n}|}.$$

On the other hand, if $\Lambda_{2,n} \nrightarrow 0$, we have

$$|W_4| = |\sigma_4\ell_4\cot(\ell_4(L_4-L_3)) - \sigma_5\ell_5\cot(\ell_5(L_4-L_5))| \lesssim \lambda_n\left(1 + \frac{|\cos(2a_2\pi n)|}{|\Lambda_{2,n}|}\right) \lesssim \lambda_n.$$

and

$$|W_4| = |\sigma_4\ell_4\cot(\ell_4(L_4-L_3)) - \sigma_5\ell_5\cot(\ell_5(L_4-L_5))| \gtrsim \lambda_n\left(1 - \frac{|\cos(2a_2\pi n)|}{|\Lambda_{2,n}|}\right) \gtrsim \lambda_n.$$

It is important to note that this last inequality cannot occur if $|W_4| \approx 0$, but in this case, the proof to estimate $|u(L_1)|$ follows in an analogous way.

Then, if $\Lambda_{2,n} \nrightarrow 0$, we have $|W_4| \approx \lambda_n$. So, we get

$$|W_4| \approx \frac{\lambda_n}{|\Lambda_{2,n}|}, \text{ if } \quad \Lambda_{2,n} \to 0 \quad \text{and} \quad |W_4| \approx \lambda_n, \text{ if } \quad \Lambda_{2,n} \nrightarrow 0.$$

• Term estimates: W_2 and W_3.
Note that

$$|W_3| \approx \lambda_n^{3/2} \quad \text{and} \quad |W_2| \lesssim \lambda_n^{3/2}.$$

• Now let's estimate the term $u(L_1)$.
First, suppose that $\Lambda_{2,n} \to 0$.
Let us first estimate the term

$$W_0 = W_2 - \frac{U_3^2 W_4}{W_3 W_4 - U_4^2} = \frac{W_4(W_2 W_3 - U_3^2) - W_2 U_4^2}{W_3 W_4 - U_4^2}. \tag{2.79}$$

Since

$$|W_3||W_4| \approx \frac{\lambda_n^{5/2}}{|\Lambda_{2,n}|} \quad \text{and} \quad |U_4|^2 \approx \lambda_n^2,$$

we have

$$|W_3 W_4 - U_4^2| \approx \frac{\lambda_n^{5/2}}{|\Lambda_{2,n}|}.$$

On the other hand, since $|U_3| \to 0$, note that

$$\begin{aligned}\text{Im}(W_2 W_3 - U_3^2) &\approx \text{Im}(W_2 W_3)\\ &= \text{Im}\{AB + (A+B)C + B^2 - U_3^2)\}\\ &= \text{Im}\{AB + (A+B)C - U_3^2)\}\end{aligned}$$

where

$$A = \sigma_2 \ell_2 \cot(\ell_2(L_2 - L_1)), \quad B = \sigma_3 \ell_3 \cot(\ell_3(L_3 - L_2)), \quad C = \sigma_4 \ell_4 \cot(\ell_4(L_4 - L_3))$$

Since

$$|\text{Im}(AB)| = |\text{Re}(A)\text{Im}(B) + \text{Re}(B)\text{Im}(A)| \approx \lambda_n^3,$$

and

$$|\text{Im}(A+B)C| \leq |A||C| + |B||C| \lesssim \lambda_n^{5/2},$$

from where follows that

$$|W_2 W_3 - U_3^2| \gtrsim |\text{Im}(W_2 W_3 - U_3^2)| \gtrsim |\text{Im}(AB)| \approx \lambda_n^3.$$

Provided that

$$|W_4||W_2 W_3 - U_3^2| \gtrsim \frac{\lambda_n^4}{|\Lambda_{2,n}|} \quad \text{and} \quad |W_2||U_4|^2 \approx \lambda_n^{7/2}$$

follows

$$|W_4(W_2 W_3 - U_3^2) - W_2 U_4^2| \gtrsim \frac{\lambda_n^4}{|\Lambda_{2,n}|}$$

Using all previous estimates we have

$$|W_0| = \frac{|W_4(W_2W_3 - U_3^2) - W_2U_4^2|}{|W_3W_4 - U_4^2|} \gtrsim \lambda_n^{3/2}.$$

Therefore,

$$\frac{|U_2^2|}{|W_0|} \to 0.$$

Now, suppose that $\Lambda_{2,n} \nrightarrow 0$.
Again, note that

$$W_0 = W_2 - \frac{U_3^2W_4}{W_3W_4 - U_4^2} = \frac{W_4(W_2W_3 - U_3^2) - W_2U_4^2}{W_3W_4 - U_4^2}.$$

Since

$$|W_3||W_4| \approx \lambda_n^{5/2} \quad \text{and} \quad |U_4|^2 \approx \lambda_n^2,$$

we have

$$|W_3W_4 - U_4^2| \approx \lambda_n^{5/2}.$$

On the other hand, since $|U_3| \to 0$, note that

$$\begin{aligned} \text{Im}(W_2W_3 - U_3^2) &\approx \text{Im}(W_2W_3) \\ &= \text{Im}\{AB + (A+B)C + B^2 - U_3^2)\} \\ &= \text{Im}\{AB + (A+B)C - U_3^2)\} \end{aligned}$$

where

$$A = \sigma_2\ell_2\cot(\ell_2(L_2 - L_1)), \quad B = \sigma_3\ell_3\cot(\ell_3(L_3 - L_2)), \quad C = \sigma_4\ell_4\cot(\ell_4(L_4 - L_3))$$

Since

$$|\text{Im}(AB)| = |\text{Re}(A)\text{Im}(B) + \text{Re}(B)\text{Im}(A)| \approx \lambda_n^3$$

and

$$|\text{Im}(A+B)C| \leq |A||C| + |B||C| \lesssim \lambda_n^{5/2},$$

from where follows that

$$|W_2W_3 - U_3^2| \gtrsim |\text{Im}(W_2W_3 - U_3^2)| \approx |\text{Im}(AB)| \approx \lambda_n^3.$$

Provided that

$$|W_4||W_2W_3 - U_3^2| \gtrsim \lambda_n^4 \quad \text{and} \quad |W_2||U_4|^2 \approx \lambda_n^{7/2}$$

follows

$$|W_4(W_2W_3 - U_3^2) - W_2U_4^2| \ \gtrsim \ \lambda_n^4$$

Using all previous estimates we have

$$|W_0| = \frac{|W_4(W_2W_3 - U_3^2) - W_2U_4^2|}{|W_3W_4 - U_4^2|} \gtrsim \lambda_n^{3/2}.$$

Therefore,

$$\frac{|U_2^2|}{|W_0|} \to 0.$$

$\square$

Lemma 2.5.3. *Take $h(s) = \sin(\ell s)$ in the previous Lemma 2.5.1. If $\ell \in \mathbb{R}$ is such that $\sin(\ell L_1) = 0$ and $\cos(\ell L_1) = 1$, then the solution in (2.68) satisfies*

$$u_1(x) = u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} - \frac{1}{\ell_1^2 - \ell^2}\sin(\ell x).$$

Proof. Take $h(s) = \sin(\ell s)$, we have

$$H_1(x) = \int_0^x \sin(\ell_1 s)\sin(\ell s)\, ds, \quad H_2(x) = \int_0^x \sin(\ell_1(x-s))\sin(\ell s)\, ds.$$

Since

$$\sin(\ell s)\sin(\ell_1(x-s)) = \sin(\ell_1 x)\cos(\ell_1 s)\sin(\ell s) - \cos(\ell_1 x)\sin(\ell_1 s)\sin(\ell s)],$$

$$\sin(\ell_1 x)\int_0^x \cos(\ell_1 s)\sin(\ell s) ds = \sin(\ell_1 x)\left[\frac{\ell_1 \sin(\ell_1 x)\sin(\ell x) + \ell\cos(\ell_1 x)\cos(\ell x) - \ell}{\ell_1^2 - \ell^2}\right],$$

$$\cos(\ell_1 x)\int_0^x \sin(\ell_1 s)\sin(\ell s) ds = \cos(\ell_1 x)\left[\frac{\ell \sin(\ell_1 x)\cos(\ell x) - \ell_1\cos(\ell_1 x)\sin(\ell x)}{\ell_1^2 - \ell^2}\right].$$

Using the hypothesis, we get

$$H_1(L_1) = \frac{\ell\sin(\ell_1 L_1)\cos(\ell L_1) - \ell_1\cos(\ell_1 L_1)\sin(\ell L_1)}{\ell_1^2 - \ell^2} = \frac{\ell \sin(\ell_1 L_1)}{\ell_1^2 - \ell^2},$$

and

$$\begin{aligned} H_2(x) =& \frac{\ell_1 \sin(\ell x)[\sin^2(\ell_1 x) + \cos^2(\ell_1 x)]}{\ell_1^2 - \ell^2} \\ &- \frac{\ell\sin(\ell_1 x)[\cos(\ell_1 x)\cos(\ell x) - \cos(\ell_1 x)\cos(\ell x)]}{\ell_1^2 - \ell^2} - \frac{\ell\sin(\ell_1 x)}{\ell_1^2 - \ell^2} \\ =& \frac{\ell_1 \sin(\ell x) - \ell \sin(\ell_1 x)}{\ell_1^2 - \ell^2}. \end{aligned}$$

Note that

$$\begin{aligned} &\frac{1}{\ell_1}\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)}H_2(L_1) - \frac{1}{\ell_1}H_2(x) \\ &\qquad = \frac{1}{\ell_1}\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)}\left[\frac{-\ell\sin(\ell_1 L_1)}{\ell_1^2 - \ell^2}\right] - \frac{1}{\ell_1}\left[\frac{\ell_1\sin(\ell x) - \ell\sin(\ell_1 x)}{\ell_1^2 - \ell^2}\right] \\ &\qquad = -\frac{1}{\ell_1^2 - \ell^2}\sin(\ell x). \end{aligned}$$

Thereby, we have

$$\begin{aligned} u_1(x) &= u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} + \frac{1}{\ell_1}\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)}H_2(L_1) - \frac{1}{\ell_1}H_2(x) \\ &= u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} - \frac{1}{\ell_1^2 - \ell^2}\sin(\ell x). \end{aligned}$$

$\square$

Lemma 2.5.4. *Assuming the same conditions as the Lemma 2.5.3 and considering $\ell = \ell_n = \frac{2n\pi}{L_1}$, we have*

$$\mathit{Im}(u(L_1)) \approx \mathit{Im}\left(\frac{H_0}{W_1}\right).$$

Proof. We know

$$u(L_1) = \frac{H_0}{W_1 - \frac{U_2^2}{W_0}}, \quad \text{with} \quad \left|\frac{U_2^2}{W_0}\right| \to 0.$$

We remember $H_0 := \frac{\sigma_1 H_1(L_1)}{\sin(\ell_1 L_1)}$, where $H_1(L_1) = \frac{\ell \sin(\ell_1 L_1)}{\ell_1^2 - \ell^2}$. Then, from $\ell_1 \approx \ell \approx \lambda_n \approx n$, we have

$$H_0 = \frac{\ell \sigma_1}{\ell_1^2 - \ell^2} \approx \lambda_n^{1/2}.$$

Since $W_1 = \sigma_1 \ell_1 \cot(\ell_1 L_1) - \sigma_2 \ell_2 \cot(\ell_2(L_1 - L_2))$, then

$$|\mathrm{Im}(W_1)| = |\mathrm{Im}(\sigma_2 \ell_2 \cot(\ell_2(L_1 - L_2)))| \approx \lambda_n^{3/2}.$$

Let $\tilde{h}_n \doteq H_0$, $W_1 = \mu_n + i\eta_n$, with $\eta_n \to \infty$ as $n \to \infty$ and $\frac{U_2^2}{W_0} = \jmath_{1,n} + i\jmath_{2,n}$ where $\jmath_{1,n}, \jmath_{2,n} \to 0$ as $n \to \infty$. Note that $\tilde{h}_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. From

$$\begin{aligned} u(L_1) =& \frac{\tilde{h}_n}{(\mu_n + i\eta_n) - (\jmath_{1,n} + i\jmath_{2,n})} \\ =& \frac{\tilde{h}_n}{(\mu_n - \jmath_{1,n}) + i(\eta_n - \jmath_{2,n})} \\ =& \frac{\tilde{h}_n((\mu_n - \jmath_{1,n}) - i(\eta_n - \jmath_{2,n}))}{(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2}, \end{aligned}$$

we get

$$\mathrm{Im}(u_1(L_1)) = \frac{-\tilde{h}_n(\eta_n - \jmath_{2,n})}{(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2}.$$

Note that

$$\frac{-\tilde{h}_n(\eta_n - \jmath_{2,n})}{-h_n\eta_n} = 1 - \frac{\jmath_{2,n}}{\eta_n} \approx 1 \quad \Rightarrow \quad -\tilde{h}_n(\eta_n - \jmath_{2,n}) \approx -\tilde{h}_n\eta_n.$$

and

$$\begin{aligned} \frac{(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2}{\mu_n^2 + \eta_n^2} &= \frac{\mu_n^2 - 2\mu_n\jmath_{1,n} + \jmath_{1,n}^2 + \eta_n^2 - 2\eta_n\jmath_{2,n} + \jmath_{2,n}^2}{\mu_n^2 + \eta_n^2} \\ &= 1 + \frac{-2\mu_n\jmath_{1,n} + \jmath_{1,n}^2 - 2\eta_n\jmath_{2,n} + \jmath_{2,n}^2}{\mu_n^2 + \eta_n^2} \approx 1, \end{aligned}$$

this is,

$$(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2 \approx \mu_n^2 + \eta_n^2.$$

Therefore, we get

$$\mathrm{Im}(u_1(L_1)) = \frac{-\tilde{h}_n(\eta_n - \jmath_{2,n})}{(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2} \approx \frac{-\tilde{h}_n\eta_n}{\mu_n^2 + \eta_n^2} = \mathrm{Im}\left(\frac{H_0}{W_1}\right).$$

$\square$

Lemma 2.5.5. *Let $\ell_1 L_1 = 2n\pi + \frac{a_0}{\sqrt{n}}$. We have*

$$\mathit{Im}\left(\frac{1}{W_1}\right) \approx \lambda_n^{-3/2}.$$

Proof. Let $\ell_1 L_1 = 2n\pi + \frac{a_0}{\sqrt{n}}$. We know that $W_1 = \sigma_1 \ell_1 \cot(\ell_1 L_1) + \sigma_2 \ell_2 \cot(\ell_2(L_2 - L_1))$ and we saw in (2.78) and (2.77) that $\sigma_1 \ell_1 \cot(\ell_1 L_1) \approx \lambda_n^{3/2}$ and $\sigma_2 \ell_2 \cot(\ell_2(L_2 - L_1)) \approx \lambda_n^{3/2}(1+i)$, then

$$\mathrm{Im}(W_1) \approx \mathrm{Im}(\sigma_2 \ell_2 \cot(\ell_2(L_2 - L_1))) \approx \lambda_n^{3/2} \quad \text{and} \quad \mathrm{Re}(W_1) \approx \lambda_n^{3/2}.$$

Therefore, we get

$$\mathrm{Im}\left(\frac{1}{W_1}\right) = \frac{-\mathrm{Im}(W_1)}{\mathrm{Re}(W_1)^2 + \mathrm{Im}(W_1)^2} \approx \lambda_n^{-3/2}.$$

□

Lemma 2.5.6. *There exists $(\lambda_n)_n \in \mathbb{R}_+^*$ with $\lambda_n \to \infty$, $(U_n)_n \subset D(\mathcal{A})$ and $(F_n)_n \in \mathcal{H}$ such that $(i\lambda_n I - \mathcal{A})U_n = F_n$ is bounded in $\mathcal{H}$ with*

$$\lambda_n \|U_n\|_{\mathcal{H}}^2 \gtrsim 1, \quad \text{for } n \text{ large}.$$

Proof. For $n \in \mathbb{N}$, we consider $\lambda = \lambda_n = \frac{\sqrt{\varrho}}{L_1}\left(2n\pi + \frac{1}{\sqrt{n}}\right)$, $F_n = (0, (\varrho h_n)\mathbb{1}_{(0,L_1)}) \in \mathcal{H}$, $h_n(s) = \sin(\ell_n s)$, $s \in (0, L_1)$ with $\ell_n = \frac{2n\pi}{L_1}$ and $U_n = (u_n, v_n) \in D(\mathcal{A})$. Therefore $v_n = i\lambda_n u_n$. In what follows, we will avoid putting the subindex n in some variables that depend on n. Thus, from the lemma 2.5.1, the coordinate u of solution $U = (u, v)$ satisfies the hypotheses of Lemma 2.5.2. Moreover, note that λ satisfies the conditions of Lemma 2.5.2. Hence, by Lemmas 2.5.3 and 2.5.4, u_1 satisfies the following estimate

$$\begin{aligned}&|\lambda_n u_1(x)| \\ &\gtrsim |\mathrm{Im}(\lambda_n u_1(x))| = \left|\mathrm{Im}\left(\lambda_n u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)}\right)\right| = \left|\mathrm{Im}\left(\frac{1}{W_1}\right)\frac{\lambda_n H_0 \sin(\ell_1 x)}{\sin(\ell_1 L_1)}\right|, \end{aligned} \tag{2.80}$$

On the other hand

$$\|F_n\|_{\mathcal{H}}^2 = \varrho^2 \|h_n\|_{L^2(0,L_1)}^2 = \varrho^2 \left(\frac{L_1}{2} - \frac{\sin(2\ell_n L_1)}{4\ell_n}\right) \lesssim \varrho^2 \left(\frac{L_1}{2} + \frac{1}{4\ell_n}\right) \approx \frac{\varrho^2 L_1}{2},$$

which implies that F_n is bounded.
Now, note that $\ell_1 \approx \ell \approx \lambda_n \approx n$, and as $\ell_1 L_1 = 2n\pi + \frac{a_0}{\sqrt{n}}$, then

$$\sin(\ell_1 L_1) = \sin\left(\frac{a_0}{\sqrt{n}}\right) \approx \frac{a_0}{\sqrt{n}} \approx \lambda_n^{-1/2},$$

Using in Equation (2.80) the previous estimate, the estimate obtained in Lemma 2.5.5 and the fact that we already deduced in Lemma 2.5.4 that $H_0 \approx \lambda_n^{1/2}$, we get

$$|\lambda_n u_1(x)| \gtrsim \left|\mathrm{Im}\left(\frac{1}{W_1}\right)\frac{\lambda_n H_0 \sin(\ell_1 x)}{\sin(\ell_1 L_1)}\right| \approx \lambda_n^{1/2} |\sin(\ell_1 x)|. \tag{2.81}$$

Since $\|\sin(\ell_1 \cdot)\|_{L^2} \gtrsim \sqrt{L_1}/2$, from Equation (2.81), we conclude

$$\|U_n\|_{\mathcal{H}} \gtrsim \|\lambda_n u_1(x)\|_{L^2} \gtrsim \lambda_n^{1/2},$$

□

Finally, let us prove the main result of the section.

Proof of Theorem 2.5.1. Suppose that the rate t^{-2} is not optimal. Then, we can improve the decay rate, say to $t^{\frac{-2}{1-\epsilon}}$. By Borichev-Tomilov theorem, subsequently

$$\|U_n\|_{\mathcal{H}} \lesssim |\lambda_n|^{\frac{1-\epsilon}{2}} \|F_n\|_{\mathcal{H}}.$$

Thus, for n large, we obtain

$$\|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \lesssim |\lambda_n|^{\frac{1-\epsilon}{2}}. \tag{2.82}$$

On the other hand, by Lemma 2.5.6, there exists $(\lambda_n)_n \in \mathbb{R}_+^*$ with $\lambda_n \to \infty$, $(U_n)_n \subset D(\mathcal{A})$ and $(F_n)_n \in \mathcal{H}$ such that $(i\lambda_n I - \mathcal{A})U_n = F_n$ is bounded in $\mathcal{H}$ and

$$\frac{1}{\lambda_n}\|U_n\|_{\mathcal{H}}^2 \gtrsim 1 \Rightarrow \frac{1}{\lambda_n^{1-\epsilon}}\|U_n\|_{\mathcal{H}}^2 \gtrsim \lambda_n^{\epsilon} \Rightarrow \frac{1}{\lambda_n^{1-\epsilon}}\|(i\lambda_n I - \mathcal{A})^1 F_n\|_{\mathcal{H}}^2 \gtrsim \lambda_n^{\epsilon},$$

we get

$$\frac{1}{\lambda_n^{\frac{1-\epsilon}{2}}}\|(i\lambda_n I - \mathcal{A})^1 F_n\|_{\mathcal{H}} \gtrsim \lambda_n^{\frac{\epsilon}{2}} \to +\infty \ \text{ as } \ n \to \infty,$$

what is a contradiction with (2.82). Note that this inequality also implies the lack of exponential stability. $\square$

Chapter 3

Asymptotic stability for wave equation with Kelvin-Voigt damping and Memory Effect

3.1 Introduction to the problem

In studies of vibrating systems modeled by wave equations, beams or plates, it is known that Kelvin-Voigt damping mechanisms, when distributed globally, stabilize the solutions of these systems exponentially. Furthermore, this damping mechanism is so strong that it tends to regularize the solutions. The situation may be completely different if this type of damping acts only on a part of the body as was shown by K. Liu and Z. Liu in [24] (see also [11]). These authors proved that if Kelvin-Voigt damping acts locally in a wave equation with discontinuous coefficient then the solutions of the equation are not exponentially stable.

Later, Alves et al [3], studied the stabilizing force that Kelvin-Voigt damping exerts on a transmission problem. This time, two dissipative mechanisms act on different parts of the body. In one part, Kelvin-Voigt damping and in the other, frictional damping. Even with the collaboration of frictional damping, the authors showed that Kelvin-Voigt damping can predominate in the decay of the solutions, not allowing the exponential decay of the solutions. However, the authors showed that the solutions decay polynomially with the optimal decay rate $t^{-1/2}$. In the literature we have not found a study on the behavior of solutions where Kelvin-Voigt damping acts collaboratively with memory effects and this was what motivated this work.

Problems with localized Kelvin-Voigt damping have aroused the interest of several researchers in the last two decades and several results have been obtained. The problem

$$u_{tt}(x,t) - u_{xx}(x,t) - \big(b(x)u_{xt}(x,t)\big)_x = 0,$$

was studied by Liu and Zhang [27] in the interval $(-1,1)$ (see also [43]). They showed that if the coefficient $b(x)$ is zero in $(-1,0]$, positive in $(0,1)$ and has a behavior like x around zero then the solution of this problem is exponentially stable. Also, if the behavior of $b(x)$ around zero is x^α, $\alpha > 1$, the solution is polynomially stable with a decay rate depending on α. A result with sharp stability $t^{-\frac{2-\alpha}{1-\alpha}}$ were obtained by Han et al in [19] (see also [18, 27]).

When the coefficient $b(x)$ is discontinuous, Liu et al. [24] had shown the solution does not decay exponentially. A few years later, this same problem was studied by

Rivera et al. [2] where they showed that the solutions of the system decay polynomially with the optimal rate t^{-2} (see also [17, 20, 29, 40]).

Elastic equations with memory effects have been widely studied in recent decades. It is known that memory damping can be strong enough for the solutions of the system to tend exponentially to zero even if the damping is locally distributed. This problem was studied by Liu et al [26] who considered the equation

$$u_{tt}(x,t)-\Big(u_x(x,t)-a(x)\int_0^\infty g_s(s)(u_x(t)-u_x(x,t-s))ds\Big)_x=0.$$

In this case, if the kernel of the memory is exponentially decreasing, the authors showed that it does not matter if the coefficient $a(x)$ is discontinuous that the system remains exponentially stable. A similar result had already been obtained in [38] when considering a transmission problem partially damped by memory effects. Other problems with memory effects, locally or globally distributed, can be found in [4, 14, 30, 31, 36, 42].

Taking into account the results mentioned above, it is interesting to study the behavior of solutions in elastic systems where both Kelving-Voigt and memory damping act simultaneously on the body. These dissipative mechanisms can act jointly on a part of the body or on separate parts. In this article we try to answer which of the dissipative mechanisms prevails: the memory damping that stabilizes the system exponentially, or the Kelving-Voigt damping, which, being discontinuous, stabilizes the system more slowly.

Therefore, in this article we consider the following problem: to study the asymptotic behavior of the solutions of the equation

$$u_{tt}(x,t)-\varrho u_{xx}(x,t)+\int_0^\infty g(s)\big(a(x)u_x(x,t-s)\big)_x ds-\big(b(x)u_{xt}(x,t)\big)_x=0, \quad (3.1)$$

$x\in(0,L)$, $t>0$, satisfying the boundary conditions

$$u(0,t)=u(L,t)=0, \quad t>0 \quad (3.2)$$

and initial data

$$u(x,0)=u_0(x);\ u_t(x,0)=u_1(x),\ u(x,-s)=\phi_0(s), \quad (x,s)\in(0,L)\times(0,\infty). \quad (3.3)$$

Here, ϱ is positive and the kernel of the memory, g, is a positive function with exponential decreasing behavior. The coefficients $a(x)$ e $b(x)$ are characteristic functions whose supports are subintervals of $[0,L]$. These supports can intersect, be disjoint or even contain each other.

Since the possibilities of location of the supports of the coefficients $a(x)$ and $b(x)$ can be diverse, we will focus on 3 cases: The first case that we will study is when the supports do not intersect and all the purely elastic components are in contact with the component that contains the memory damping, that is

$$b(x)=b_0\chi_{[0,L_1]}(x) \text{ and } a(x)=a_0\chi_{[L_2,L_3]}(x), \quad a_0,b_0>0,$$

where $0<L_1<L_2<L_3<L$. We refer to this model as the KEME model. In the second case, we still keep the supports of the coefficients $a(x)$, $b(x)$ disjoint, but there is a purely elastic component in contact only with the Kelvin-Voigt component (without contact with the memory component), that is

$$b(x)=b_0\chi_{[L_1,L_2]}(x) \text{ and } a(x)=a_0\chi_{[L_3,L_4]}(x), \quad a_0,b_0>0,$$

where $0 < L_1 < L_2 < L_3 < L_4 < L$. This model is referred to as the EKEME model. In the last case we consider that the supports of the coefficients $a(x)$, $b(x)$ intersect, but there is still a purely elastic component in contact only with the Kelvin-Voigt component, that is

$$b(x) = b_0\chi_{[L_1,L_3]}(x) \text{ and } a(x) = a_0\chi_{[L_2,L_4]}(x), \quad a_0, b_0 > 0.$$

We refer to this model as EKIME model. The KEME, EKEME and EKIME models are shown in Figure 3.1 below.

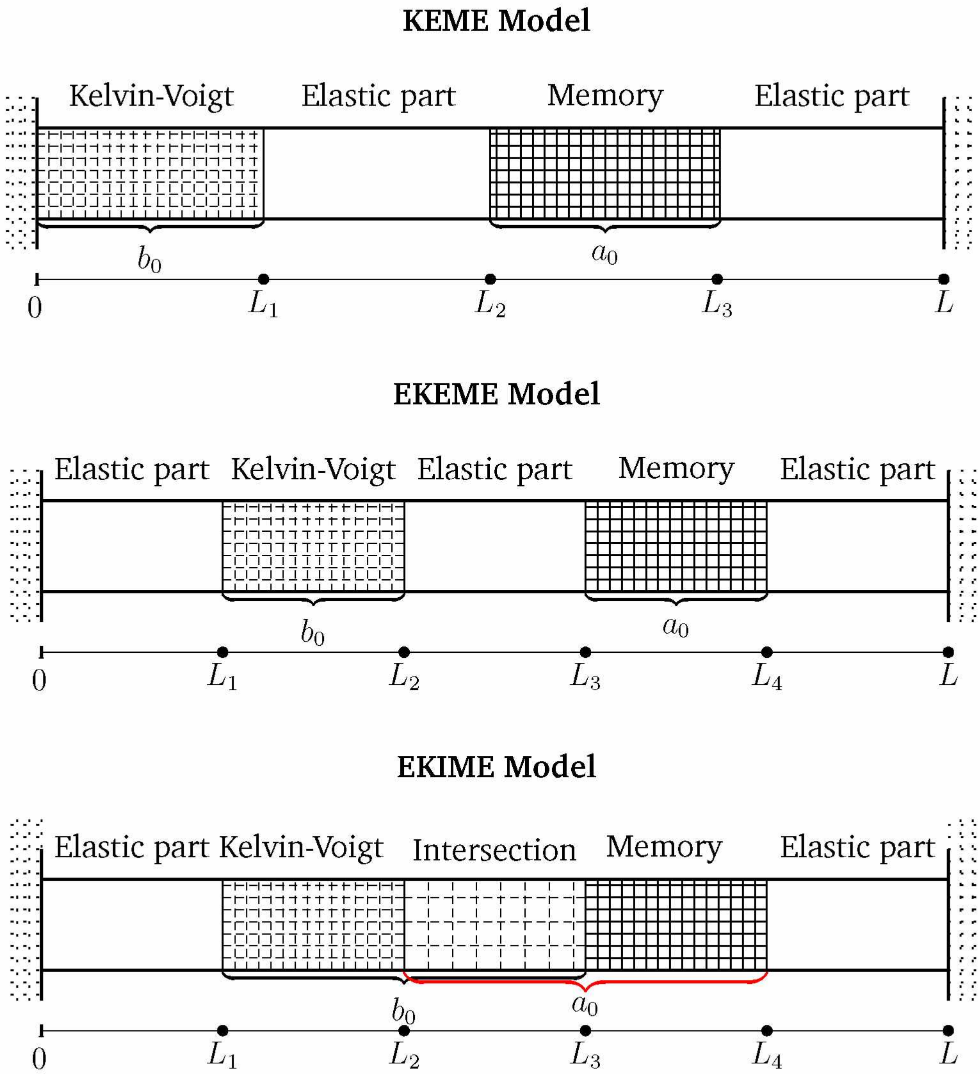

Figure 3.1: Different partial viscoelastic materials

We will use the following assumptions for the memory kernel

$$\begin{cases} g \in L^1([0,\infty)) \cap C^1([0,\infty)) \text{ is a positive function such that} \\ g(0) := g_0 > 0, \int_0^\infty g(s)ds := \tilde{g}, \tilde{a}(x) := \varrho - a(x)\tilde{g} > 0, \text{and} \\ g'(s) \leq -c_0 g(s), \text{ for some } c_0 > 0, \forall s \geq 0. \end{cases} \tag{3.4}$$

The main results we obtain in this problem are the following:

- If all the purely elastic components are in contact with the memory damping component, whether or not they contact the component with Kelvin-Voigt damping, then the solutions of the system (3.1)-(3.3) decay exponentially

- If there is a purely elastic component that contacts only the component with Kelvin-Voigt damping then the solutions of the system (3.1)-(3.3) do not decay exponentially. However, it is proven that the solutions decay polynomially with the rate t^{-2}.

- In case of non-exponential decay of the solutions, it is proven that the polynomial decay rate t^{-2} is optimal.

The remaining part of this chapter is organized as follows: In Section 3.2, we study the well-posedness of the system (3.1)-(3.3) using a semigroup approach. In Section 3.3, we study the case when the solutions of the system decay exponentially, that is, when all the purely elastic components are in contact with the memory damping component. In Section 3.4, we study the case when there exists a purely elastic in contact only with the Kelvin-Voigt damping component. Finally, Section 3.5 deals with the optimality of the decay rates obtained in the previous section.

3.2 Existence of solutions

In this section, we will establish the well-posedness of problem (3.1)-(3.3) by using a semigroup approach. To this aim, as in Dafermos [14], we introduce the following auxiliary change of variable

$$\eta(x,s,t) := u(x,t) - u(x,t-s), \quad (x,s,t) \in (0,L) \times (0,\infty) \times (0,\infty). \tag{3.5}$$

Then, system (3.1) becomes

$$u_{tt}(x,t) - \left[\tilde{a}(x)u_x(x,t) + \int_0^\infty a(x)g(s)\eta_x(x,s)ds + b(x)u_{xt}(x,t)\right]_x = 0, \tag{3.6}$$

$$\eta_t(x,s,t) + \eta_s(x,s,t) - u_t(x,t) = 0, \tag{3.7}$$

for $(x,s,t) \in (0,L) \times (0,\infty) \times (0,\infty)$, satisfying the boundary conditions

$$\begin{cases} u(0,t) = u(L,t) = 0, & t > 0, \\ \eta(\cdot,0,t) = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \eta(0,s,t) = \eta(L,s,t) = 0, & (s,t) \in (0,\infty) \times (0,\infty), \end{cases} \tag{3.8}$$

and the following initial conditions

$$u(\cdot,-s) = \phi_0(s), \; u(\cdot,0) = u_0(\cdot), \; u_t(\cdot,0) = u_1(\cdot), \tag{3.9}$$

$$\eta(\cdot,s,0) = \eta_0(\cdot,s) := u_0(\cdot) - \phi_0(s), \tag{3.10}$$

with $(x,s) \in (0,L) \times (0,\infty)$.

The problem (3.6)-(3.10) is dissipative in the sense that its energy is a non-increasing function with respect to the time variable t. Let us define the energy space $\mathcal{H}$ by

$$\mathcal{H} := H_0^1(0,L) \times L^2(0,L) \times \mathcal{G}_g,$$

where $\mathcal{G}_g$ is the weighted space $L_g^2((0,\infty); H_0^1(0,L))$, with the inner product

$$(\eta^1,\eta^2)_{\mathcal{G}_g} := \int_0^L \int_0^\infty a(\cdot)g(s)\eta_x^1\overline{\eta_x^2}dsdx, \; \forall\, \eta^1,\eta^2 \in \mathcal{G}_g.$$

The Hilbert space $\mathcal{H}$ is equipped with the inner product defined by

$$(U_1, U_2)_{\mathcal{H}} = \int_0^L v^1 \overline{v^2} dx + \int_0^L \tilde{a}(\cdot) u_x^1 \overline{u_x^2} dx + \int_0^L \int_0^\infty a(\cdot) g(s) \eta_x^1(\cdot, s) \overline{\eta_x^2}(\cdot, s) ds dx, \tag{3.11}$$

for all $U_1 = (u^1, v^1, \eta^1(\cdot, s))$ and $U_2 = (u^2, v^2, \eta^2(\cdot, s))$ in $\mathcal{H}$. We use $\|U\|_{\mathcal{H}}$ to denote the corresponding norm. We define the unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, \eta(\cdot, s))^\top \in \mathcal{H} \mid v \in H_0^1(0, L), \\ \\ \eta_s(\cdot, s) \in \mathcal{G}_g, \eta(\cdot, 0) = 0 \text{ in } (0, L), \\ \\ \left(\tilde{a}(\cdot) u_x + \int_0^\infty a(\cdot) g(s) \eta_x(\cdot, s) ds + b(\cdot) v_x \right)_x \in L^2(0, L). \end{array} \right\}, \tag{3.12}$$

and

$$\mathcal{A} \begin{pmatrix} u \\ v \\ \eta(\cdot, s) \end{pmatrix} = \begin{pmatrix} v \\ \left(\tilde{a}(\cdot) u_x + \int_0^\infty a(\cdot) g(s) \eta_x(\cdot, s) ds + b(\cdot) v_x \right)_x \\ v - \eta_s(\cdot, s) \end{pmatrix},$$

for all $U = (u, v, \eta(\cdot, s))^\top \in D(\mathcal{A})$.
If $U = (u, u_t, \eta(\cdot, s))^\top$ is the state of system (3.6)-(3.9), then this system is transformed into the first order evolution problem on the Hilbert space $\mathcal{H}$ given by

$$U_t = \mathcal{A} U, \quad U(0) = U_0, \tag{3.13}$$

where $U_0 = (u_0, u_1, \eta_0(\cdot, s))^\top \in \mathcal{H}$. We have the following result on the well-posedness of system (3.13).

Proposition 3.2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $\mathcal{A}$ generates a C_0 semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$.*

Proof. Note that the $\mathcal{A}$ is a dissipative operator in the energy space $\mathcal{H}$. In fact, let $U = (u, v, \eta(\cdot, s))^\top \in D(\mathcal{A})$. Using the inner product in $\mathcal{H}$, integration by parts, and the boundary conditions (3.2), we have

$$\begin{aligned}
(\mathcal{A}U, U)_{\mathcal{H}} = &\left(\int_0^L \tilde{a}(\cdot) v_x \overline{u_x} dx - \int_0^L \tilde{a}(\cdot) u_x \overline{v_x} dx \right) - \int_0^L b(\cdot) |v_x|^2 dx \\
&+ \left(\int_0^L a(\cdot) \int_0^\infty g(s) \overline{\eta_x}(\cdot, s) v_x dx - \int_0^L a(\cdot) \int_0^\infty g(s) \eta_x(\cdot, s) \overline{v_x} dx \right) \\
&- \int_0^L a(\cdot) \int_0^\infty g(s) \eta_{xs}(\cdot, s) \overline{\eta_x}(\cdot, s) ds dx.
\end{aligned}$$

Using integration by parts with respect to s in the above equation, the hypotheses (3.4) and taking the real part, we get

$$\begin{aligned}
\mathrm{Re}(\mathcal{A}U, U)_{\mathcal{H}} &= -\mathrm{Re} \left\{ \int_0^L a(\cdot) \int_0^\infty g(s) \eta_{xs}(\cdot, s) \overline{\eta_x}(\cdot, s) ds dx \right\} - \int_0^L b(\cdot) |v_x|^2 dx \\
&= \frac{1}{2} \int_0^L a(\cdot) \int_0^\infty g'(s) |\eta_x(\cdot, s)|^2 ds dx - \int_0^L b(\cdot) |v_x|^2 dx \leq 0,
\end{aligned}$$

therefore, $\mathcal{A}$ is dissipative.

Now, we will prove $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. In fact, we will show that there is unique $U = (u, v, \eta(\cdot, s))^\top \in D(\mathcal{A})$ such that

$$-\mathcal{A}U = F, \tag{3.14}$$

for $F = (f^1, f^2, f^3(\cdot, s))^\top \in \mathcal{H}$.
Writing equation (3.14) in terms of its components we obtain the following system of differential equations

$$-v = f^1, \tag{3.15}$$

$$-\left[\tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot, s)ds + b(\cdot)v_x\right]_x = f^2, \tag{3.16}$$

$$\eta_s(\cdot, s) - v = f^3(\cdot, s), \tag{3.17}$$

with the boundary conditions

$$u(0) = u(L) = 0,\ \eta(\cdot, 0) = 0 \text{ in } (0, L) \text{ and } \eta(0, s) = \eta(L, s) = 0 \text{ in } (0, \infty). \tag{3.18}$$

First, using (3.15) and (3.18), we have

$$\eta(x, s) = \int_0^s f^3(x, \mu)d\mu - sf^1, \quad (x, s) \in (0, L) \times (0, \infty). \tag{3.19}$$

Note also that, from (3.15), (3.17) and (3.19), we get $\eta(\cdot, s) \in H_0^1(0, L)$ in $(0, \infty)$ and $\eta_s(\cdot, s) \in \mathcal{G}_g$, because $v = -f^1 \in H_0^1(0, L)$ and $f^3(\cdot, s) \in \mathcal{G}_g$. We will show

$$\int_0^\infty g(s)\|\eta_x(\cdot, s)\|^2_{L^2(0,L)}ds < \infty,$$

consequently, we will obtain $\eta(\cdot, s) \in \mathcal{G}_g$. By (3.4) and taking $y \in (0, +\infty)$, we have

$$\int_y^{1/y} g(s)\|\eta_x(\cdot, s)\|^2_{L^2(0,L)}ds \leq -\frac{1}{c_0}\int_y^{1/y} g'(s)\|\eta_x(\cdot, s)\|^2_{L^2(0,L)}ds.$$

Using that above equation and by using integration by parts with respect to s, we get

$$\begin{aligned} &\int_y^{1/y} g(s)\|\eta_x(\cdot, s)\|^2_{L^2(0,L)}ds \\ &\leq \frac{1}{c_0}\int_y^{1/y} g(s)\frac{d}{ds}\left(\|\eta_x(\cdot, s)\|^2_{L^2(0,L)}\right)ds \\ &\quad + \frac{1}{c_0}\left[g(y)\|\eta_x(\cdot, y)\|^2_{L^2(0,L)} - g(1/y)\|\eta_x(\cdot, 1/y)\|^2_{L^2(0,L)}\right]. \end{aligned} \tag{3.20}$$

Furthermore, using Young's inequality, we have

$$\begin{aligned} &\frac{1}{c_0}\int_y^{1/y} g(s)\frac{d}{ds}\left(\|\eta_x(\cdot, s)\|^2_{L^2(0,L)}ds\right) \\ &= \frac{2}{c_0}\int_y^{1/y} g(s)\mathrm{Re}\left\{\int_0^L \eta_x(\cdot, s)\overline{\eta_{sx}}(\cdot, s)dx\right\}ds \\ &\leq \frac{1}{2}\int_y^{1/y} g(s)\|\eta_x(\cdot, s)\|^2_{L^2(0,L)}ds + \frac{2}{c_0^2}\int_y^{1/y} g(s)\|\eta_{sx}(\cdot, s)\|^2_{L^2(0,L)}ds. \end{aligned} \tag{3.21}$$

From (3.20) and (3.21), we deduce

$$\begin{aligned}
&\int_y^{1/y} g(s)\|\eta_x(\cdot,s)\|^2_{L^2(0,L)}ds \\
&\leq \frac{4}{c_0^2}\int_y^{1/y} g(s)\|\eta_{sx}(\cdot,s)\|^2_{L^2(0,L)}ds \\
&\quad + \frac{2}{c_0}\left[g(y)\|\eta_x(\cdot,y)\|^2_{L^2(0,L)} - g(1/y)\|\eta_x(\cdot,1/y)\|^2_{L^2(0,L)}\right]. \qquad (3.22)
\end{aligned}$$

Hypoteses (3.4) implies that $g(s) \leq g_0 e^{-c_0 s}$, and using $\eta_s \in \mathcal{G}_g$, $\eta(\cdot,s) = 0$ in $(0,L)$, we get, as $y \to 0^+$, that

$$\int_0^\infty g(s)\|\eta_x(\cdot,s)\|^2_{L^2(0,L)}ds < \infty,$$

and therefore, $\eta(\cdot,s) \in \mathcal{G}_g$. Furthemore, from (3.19), we have

$$\eta_x(x,s) = \left(\int_0^s f^3(x,\mu)d\mu - sf^1\right)_x, \quad (x,s) \in (0,L)\times(0,\infty). \qquad (3.23)$$

Substituing the above inequality in (3.16), we have

$$-\left[\tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\left(\int_0^s f^3(x,\mu)d\mu - sf^1\right)_x ds + b(\cdot)v_x\right]_x = f^2. \qquad (3.24)$$

Multiplying (3.24) by $\overline{\phi} \in H_0^1(0,L)$, integrating over $(0,L)$ and using integration by parts, we get

$$\begin{aligned}
\int_0^L \tilde{a}(\cdot)u_x\overline{\phi_x}dx &= \int_0^L f^2\overline{\phi}dx - \int_0^\infty a(\cdot)g(s)\left(\int_0^s f^3(x,\mu)d\mu - sf^1\right)_x ds\overline{\phi_x}dx \\
&\quad - \int_0^L b(\cdot)f_x^1\overline{\phi_x}dx. \qquad (3.25)
\end{aligned}$$

Note that, from (3.25), we get

$$\Upsilon(u,\phi) = J(\phi), \quad \forall\phi \in H_0^1(0,L), \qquad (3.26)$$

where

$$\Upsilon(u,\phi) = \int_0^L \tilde{a}(\cdot)u_x\overline{\phi_x}dx, \quad \forall\phi \in H_0^1(0,L), \qquad (3.27)$$

and

$$\begin{aligned}
J(\phi) &= \int_0^L f^2\overline{\phi}dx - \int_0^\infty a(\cdot)g(s)\left(\int_0^s f^3(x,\mu)d\mu - sf^1\right)_x ds\overline{\phi_x}dx \\
&\quad - \int_0^L b(\cdot)f_x^1\overline{\phi_x}dx. \qquad (3.28)
\end{aligned}$$

Note that Υ is a sesquilinear form on $H_0^1(0,L)\times H_0^1(0,L)$. By Hölder's inequality and Poincaré's inequality, Υ is a continuous and coercive form on $H_0^1(0,L)\times H_0^1(0,L)$. On the other hand, J is an antilinear functional on $H_0^1(0,L)$ and since $\eta(x,s) = \int_0^s f^3(x,\mu)d\mu -$

$sf^1 \in \mathcal{G}_g$, then using Hölder's inequality, we get that J is a continuos functional on $H_0^1(0,L)$. Therefore, by Lax-Milgram theorem we have that (3.26) admits a unique solution $u \in H_0^1(0,L)$. By taking test function $\phi \in C_0^\infty(0,L)$, we deduce that

$$-\left[\tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds + b(\cdot)v_x\right]_x = f^2 \in L^2(0,L).$$

Therefore, $U \in D(\mathcal{A})$ is a unique solution of (3.14). This tells us that $\mathcal{A}$ is bijective and thererefore $\mathcal{A}^{-1}$ exists. So, to conclude that $0 \in \rho(\mathcal{A})$, we just need to show that $\mathcal{A}^{-1}$ is bounded. For this, as there is only one $U \in D(\mathcal{A})$ such that $-\mathcal{A}U = F$, we need to show $\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}}$. In fact, since $v = -f^1$, then by Poincaré's inequality, we have

$$\|v\|^2_{L^2(0,L)} \lesssim \|F\|^2_{L^2(0,L)}.$$

Now, by (3.4), we get

$$\begin{aligned}
\frac{c_0}{2}\|\eta(\cdot,s)\|^2_{\mathcal{G}_g} &= \frac{c_0}{2}\int_0^\infty g(s)\|\eta_x(\cdot,s)\|^2_{L^2(0,L)} \\
&\leq \frac{1}{2}\int_0^\infty -g'(s)\|\eta_x(\cdot,s)\|^2_{L^2(0,L)} \\
&\leq \frac{1}{2}\int_0^\infty -g'(s)\|\eta_x(\cdot,s)\|^2_{L^2(0,L)} + \int_0^L b(\cdot)|v_x|^2dx \\
&= \mathrm{Re}(-\mathcal{A}U, U)_{\mathcal{H}} \\
&= \mathrm{Re}(F, U)_{\mathcal{H}} \\
&\lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},
\end{aligned}$$

this is, $\|\eta(\cdot,s)\|^2_{\mathcal{G}_g} \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$. Thus, we have

$$\int_0^L\int_0^\infty a(\cdot)g(s)|\eta_x(\cdot,s)|^2dsdx \lesssim \|\eta(\cdot,s)\|^2_{\mathcal{G}_g} \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

On the other hand, multiplying (3.16) by $\overline{u}$, integrating over $(0,L)$, using integration by parts, using (3.15) and Hölder's inequality, we get

$$\int_0^L \tilde{a}(\cdot)|u_x|^2dx \lesssim \int_0^L\int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)\overline{u_x}dsdx + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.29}$$

Using Young's inequality in (3.29), we get

$$\int_0^L \tilde{a}(\cdot)|u_x|^2dx \lesssim \tilde{g}\|\eta(\cdot,s)\|^2_{\mathcal{G}_g} + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$$

Thus, we obtain

$$\begin{aligned}
\|U\|^2_{\mathcal{H}} &= \int_0^L |v|^2dx + \int_0^L \tilde{a}(\cdot)|u_x|^2dx + \int_0^L\int_0^\infty a(\cdot)g(s)|\eta_x(\cdot,s)|^2dsdx \\
&\lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}},
\end{aligned}$$

using Young's inequality, we get $\|U\|^2_{\mathcal{H}} \lesssim \|F\|^2_{\mathcal{H}}$, as we wanted to show.
Lastly, since $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, it follows from Lemma (1.4.3) that $D(\mathcal{A})$ is dense in $\mathcal{A}$. Therefore, the operator $\mathcal{A}$ satisfies the conditions of Lumer-Phillips Theorem (see Pazy [25]) and the result of the proposition follows.

□

The well-posedness of the problem (3.13) and therefore of problem (3.1)-(3.3) is a consequence of semigroup theory. From Preposition (3.2.1) we can state the following result.

Theorem 3.2.1. *Under the hypotheses (3.4), for all $U_0 \in \mathcal{H}$, there exists a unique solution to the system in the space (3.13) in the space*

$$U(x,s,t) = e^{t\mathcal{A}}U_0(x,s) \in C([0,\infty[;\mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the solutions belong to the following space:

$$U(x,s,t) = e^{t\mathcal{A}}U_0(x,s) \in C([0,\infty[;D(\mathcal{A})) \cap C^1([0,\infty[;\mathcal{H}).$$

3.3 Asymptotic behavior: Exponential Stability

We will study the asymptotic behavior of the semigroup $e^{t\mathcal{A}}$ associated to the system (3.6)-(3.8) under the hypotheses in (3.4). The results will be obtained using the spectral characterizations for exponential stability of semigroups (see [21] or [32]).

We will demonstrate the exponential stability of system (3.1)-(3.3) in the case that every elastic part of the string either connects only with the memory part or connects with both types of dampings. Since the proof of the decay rate is similar in all these cases, we focus on the proof considering the KEME model, which is given by (3.1)-(3.3) considering

$$b(x) = b_0\chi_{[0,L_1]}(x) \text{ and } a(x) = a_0\chi_{[L_2,L_3]}(x), \tag{3.30}$$

where $a_0, b_0 > 0$.

The main result of this section is Theorem 3.3.1 and to prove this theorem we will need to introduce some technical lemmas.

Let $\lambda \in \mathbb{R}$ and $F = (f^1, f^2, f^3(\cdot,s))^\top \in \mathcal{H}$. In what follows, the stationary problem

$$(i\lambda I - \mathcal{A})U = F, \tag{3.31}$$

will be considered several times. Note that $U = (u, v, \eta(\cdot,s))$ is a solution of this problem if the following equations are satisfied:

$$i\lambda u - v = f^1, \tag{3.32}$$

$$i\lambda v - \left(\tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds + b(\cdot)v_x\right)_x = f^2, \tag{3.33}$$

$$i\lambda\eta(\cdot,s) + \eta_s(\cdot,s) - v = f^3(\cdot,s), \tag{3.34}$$

with the following boundary conditions

$$u(0) = u(L) = v(0) = v(L) = 0, \eta(\cdot,0) = 0 \text{ in } (0,L) \text{ and } \eta(0,s) = 0 \text{ in } (0,\infty). \tag{3.35}$$

Note that

$$((i\lambda U - \mathcal{A}U), U)_{\mathcal{H}} = i\lambda\|U\|^2_{\mathcal{H}} - (\mathcal{A}U, U)_{\mathcal{H}},$$

so, we have

$$-\text{Re}(\mathcal{A}U, U)_{\mathcal{H}} \lesssim \|(i\lambda U - AU)\|_{\mathcal{H}}\|U\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Therefore, we get

$$-\int_0^L \int_0^\infty a(\cdot)g'(s)|\eta_x(\cdot,s)|^2 dsdx + \int_0^L b(\cdot)|v_x|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.36}$$

Thus, from the definitions of $a(x)$ and $b(x)$ defined in (3.30) and from (3.36), we obtain

$$\int_0^L b(x)|v_x|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{3.37}$$

and

$$\begin{aligned}\int_{L_2}^{L_3} \int_0^\infty g(s)|\eta_x(\cdot,s)|^2 dsdx &\leq -\frac{1}{c_0}\int_{L_2}^{L_3} \int_0^\infty g'(s)|\eta_x(\cdot,s)|^2 dsdx \\ &\lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.\end{aligned} \tag{3.38}$$

From (3.32) and (3.37), we obtain

$$\begin{aligned}\int_0^{L_1} \rho|u_x|^2 dx &\lesssim |\lambda|^{-2}\left(\int_0^{L_1} |v_x|^2 dx + \int_0^{L_1} |f_x^1|^2 dx\right) \\ &\lesssim |\lambda|^{-2}(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2).\end{aligned} \tag{3.39}$$

Furthermore, by Poincaré's inequality and (3.37), we have

$$\int_0^{L_1} |v|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2. \tag{3.40}$$

Then, from (3.39)-(3.40), for $|\lambda|$ large, we have

$$\int_0^{L_1} (|v|^2 + \varrho|u_x|^2)dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.41}$$

Lemma 3.3.1. *For $|\lambda|$ large, we have*

$$\int_{L_2}^{L_3} |u_x|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Proof. By substituting (3.32) into (3.34) and subsequently differentiating the combined equation with respect to x, we can derive:

$$i\lambda\eta_x(\cdot,s) + \eta_{sx}(\cdot,s) - i\lambda u_x = f_x^3(\cdot,s) - f_x^1. \tag{3.42}$$

After multiplying (3.42) by $\lambda^{-1}g(s)\overline{u_x}$, integrating over $(L_2, L_3)\times(0,\infty)$, then taking the imaginary part, we obtain

$$\begin{aligned}&\int_{L_2}^{L_3} \int_0^\infty g(s)|u_x|^2 dsdx \\ &= \mathrm{Im}\left\{i\int_{L_2}^{L_3} \int_0^\infty g(s)\eta_x(\cdot,s)\overline{u_x} dsdx\right\} + \mathrm{Im}\left\{\lambda^{-1}\int_{L_2}^{L_3} \int_0^\infty g(s)\eta_{sx}(\cdot,s)\overline{u_x} dsdx\right\} \\ &\quad - \mathrm{Im}\left\{\lambda^{-1}\int_{L_2}^{L_3} \int_0^\infty g(s)f_x^3(\cdot,s)\overline{u_x} dsdx\right\} + \mathrm{Im}\left\{\lambda^{-1}\int_{L_2}^{L_3} \int_0^\infty g(s)f_x^1\overline{u_x} dsdx\right\}.\end{aligned}$$

Using integration by parts with respect to s in the above equation and using the hypotheses (3.4), we get

$$\begin{aligned}
&\tilde{g}\int_{L_2}^{L_3}|u_x|^2dx\\
&=\operatorname{Im}\left\{i\int_{L_2}^{L_3}\int_0^\infty g(s)\eta_x(\cdot,s)\overline{u_x}dsdx\right\}+\operatorname{Im}\left\{\lambda^{-1}\int_{L_2}^{L_3}\int_0^\infty -g'(s)\eta_x(\cdot,s)\overline{u_x}dsdx\right\}\\
&\quad-\operatorname{Im}\left\{\lambda^{-1}\int_{L_2}^{L_3}\int_0^\infty g(s)f_x^3(\cdot,s)\overline{u_x}dsdx\right\}\\
&\quad+\operatorname{Im}\left\{\lambda^{-1}\int_{L_2}^{L_3}\int_0^\infty g(s)f_x^1\overline{u_x}dsdx\right\}. \qquad (3.43)
\end{aligned}$$

Using integration by parts with respect to s, hypotheses (3.4), Young's inequality for $\epsilon>0$, Poincaré inequality and Cauchy-Schwarz inequality, for $|\lambda|>0$ large and $\epsilon>0$ small, we get

$$\tilde{g}\int_{L_2}^{L_3}|u_x|^2dx\lesssim\epsilon\tilde{g}\int_{L_2}^{L_3}|u_x|^2dx+\left(C(\epsilon)\tilde{g}^{\frac{1}{2}}|\lambda|^{-1}+1\right)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Therefore, for $|\lambda|$ large and for $\epsilon>0$ small and suitable, we obtain

$$\int_{L_2}^{L_3}|u_x|^2dx\lesssim\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

□

Lemma 3.3.2. *For $|\lambda|$ large, we have*

$$\int_{L_2}^{L_3}\left|\tilde{a}(x)u_x+\int_0^\infty a(x)g(s)\eta_x(\cdot,s)ds\right|^2dx\lesssim\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Proof. Using (3.38) and Lemma 3.3.1, we get

$$\begin{aligned}
\int_{L_2}^{L_3}\left|\tilde{a}(x)u_x+a(x)\int_0^\infty g(s)\eta_x(\cdot,s)ds\right|^2dx&\lesssim\int_{L_2}^{L_3}|u_x|^2\,dx+\int_{L_2}^{L_3}\int_0^\infty g(s)|\eta_x(\cdot,s)|^2dsdx\\
&\lesssim\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.
\end{aligned}$$

□

Lemma 3.3.3. *Consider $\tilde{L}_2$ and $\tilde{L}_3$ such that $[\tilde{L}_2,\tilde{L}_3]\subsetneq(L_2,L_3)$. Then, for $|\lambda|$ large, we have*

$$\int_{\tilde{L}_2}^{\tilde{L}_3}|v|^2dx\lesssim\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \qquad (3.44)$$

Proof. Let us consider an auxiliary funtion q_1 which satisfies $q_1\in C^1([L_2,L_3])$, such that $0\le q_1(x)\le 1$, for all $x\in[L_2,L_3]$ with $q_1(x)=0$ if $x\in\{L_2,L_3\}$ and $q_1(x)=1$ if $x\in[\tilde{L}_2,\tilde{L}_3]$. Multiplying Equation (3.33) by $-q_1(x)\int_0^\infty g(s)\overline{\eta}(\cdot,s)ds$, integrating over

(L_2, L_3) and using integration by parts, since $q_1(L_2) = q_1(L_3) = 0$, we get

$$\begin{aligned}
&-i\lambda \int_{L_2}^{L_3} q_1(x) v \int_0^\infty g(s)\overline{\eta}(\cdot, s) ds dx \\
&\quad = \int_{L_2}^{L_3} \left((\varrho - a_0\tilde{g}) u_x + a_0 \int_0^\infty g(s)\eta_x(\cdot, s) ds \right) q_1'(x) \int_0^\infty g(s)\overline{\eta}(\cdot, s) ds dx \\
&\qquad + \int_{L_2}^{L_3} \left((\varrho - a_0\tilde{g}) u_x + a_0 \int_0^\infty g(s)\eta_x(\cdot, s) ds \right) q_1(x) \int_0^\infty g(s)\overline{\eta_x}(\cdot, s) ds dx \\
&\qquad - \int_{L_2}^{L_3} f^2 q_1(x) \int_0^\infty g(s)\overline{\eta}(\cdot, s) ds dx. \qquad (3.45)
\end{aligned}$$

From (3.34), we have

$$-i\lambda\overline{\eta}(\cdot, s) = \overline{v} + \overline{f^3}(\cdot, s) - \overline{\eta}_s(\cdot, s) \quad \text{in } (L_2, L_3) \times (0, \infty).$$

Inserting the above equation in the left-hand side of (3.45), we obtain

$$\begin{aligned}
&\tilde{g} \int_{L_2}^{L_3} q_1(x)|v|^2 dx \\
&\quad = \int_{L_2}^{L_3} q_1(x) v \int_0^\infty g(s)\overline{\eta}_s(\cdot, s) ds dx - \int_{L_2}^{L_3} q_1(x) v \int_0^\infty g(s)\overline{f^3}(\cdot, s) ds dx \\
&\qquad + \int_{L_2}^{L_3} \left((\varrho - a_0\tilde{g}) u_x + a_0 \int_0^\infty g(s)\eta_x(\cdot, s) ds \right) q_1'(x) \int_0^\infty g(s)\overline{\eta}(\cdot, s) ds dx \\
&\qquad + \int_{L_2}^{L_3} \left((\varrho - a_0\tilde{g}) u_x + a_0 \int_0^\infty g(s)\eta_x(\cdot, s) ds \right) q_1(x) \int_0^\infty g(s)\overline{\eta_x}(\cdot, s) ds dx \\
&\qquad - \int_{L_2}^{L_3} f^2 q_1(x) \int_0^\infty g(s)\overline{\eta}(\cdot, s) ds dx.
\end{aligned}$$

Using integration by parts with respect to s, hypotheses (3.4), Young's inequality for $\epsilon > 0$, Poincaré inequality, Cauchy-Schwarz inequality and Lemma 3.3.2, for $|\lambda| > 0$, we get

$$\int_{L_2}^{L_3} q_1(x)|v|^2 dx \lesssim \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}.$$

□

From Lemma 3.3.1 and Lemma 3.3.3, we have for $[\tilde{L}_2, \tilde{L}_3] \subsetneq (L_2, L_3)$ the following inequality

$$\int_{\tilde{L}_2}^{\tilde{L}_3} (|v|^2 + \tilde{a}(\cdot)|u_x|^2) dx \lesssim \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}, \qquad (3.46)$$

for $|\lambda|$ large.

The following lemma will be used in the next lemmas.

Lemma 3.3.4. *Let $L_1 \leq \alpha_0 < \beta_0 \leq L$ and $G_{u,\eta} = \tilde{a}(\cdot)u_x + \int_0^\infty g(s)a(\cdot)\eta_x(\cdot, s)ds$. For $|\lambda|$ large and $\epsilon > 0$ small, we have*

$$\begin{aligned}
&|v(\beta_0)|^2 + |v(\alpha_0)|^2 + |G_{u,\eta}(\beta_0)|^2 + |G_{u,\eta}(\alpha_0)|^2 \\
&\quad \lesssim \int_0^L \left(|v|^2 + |G_{u,\eta}|^2 \right) dx + \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2. \qquad (3.47)
\end{aligned}$$

Furthermore, the following inequalities are satisfied:

$$\int_{\alpha_0}^{\beta_0}(|v|^2+|G_{u,\eta}|^2)dx \\ \lesssim |v(\alpha_0)|^2+|G_{u,\eta}(\alpha_0)|^2+\epsilon\|U\|^2_{\mathcal{H}}+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|^2_{\mathcal{H}}, \tag{3.48}$$

$$\int_{\alpha_0}^{\beta_0}(|v|^2+|G_{u,\eta}|^2)dx \\ \lesssim |v(\beta_0)|^2+|G_{u,\eta}(\beta_0)|^2+\epsilon\|U\|^2_{\mathcal{H}}+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|^2_{\mathcal{H}}. \tag{3.49}$$

Proof. First, let us prove the inequality (3.47). From Equation(3.32) and Equation(3.34), we deduce that

$$i\lambda\overline{u_x}=-\overline{v_x}-\overline{f^1_x}, \tag{3.50}$$

$$i\lambda\overline{\eta}_x(\cdot,s)=\overline{\eta_{sx}}(\cdot,s)-\overline{v}_x-\overline{f^3_x}(\cdot,s), \quad \text{in} \quad (0,L)\times(0,\infty). \tag{3.51}$$

Let $G_{u,\eta}=\tilde{a}(\cdot)u_x+\int_0^\infty g(s)a(\cdot)\eta_x(\cdot,s)ds$. Multiplying (3.33) by $\left(x-\frac{\alpha_0+\beta_0}{2}\right)\overline{G}_{u,\eta}$ and integrating over (α_0,β_0), we get

$$i\lambda\int_{\alpha_0}^{\beta_0}\left(x-\frac{\alpha_0+\beta_0}{2}\right)v\overline{G}_{u,\eta}dx-\int_{\alpha_0}^{\beta_0}\left(x-\frac{\alpha_0+\beta_0}{2}\right)(G_{u,\eta})_x\overline{G}_{u,\eta}dx \\ =\int_{\alpha_0}^{\beta_0}\left(x-\frac{\alpha_0+\beta_0}{2}\right)f^2\overline{G}_{u,\eta}dx. \tag{3.52}$$

Using Equation(3.50) and Equation(3.51), we have

$$i\lambda\int_{\alpha_0}^{\beta_0}\left(x-\frac{\alpha_0+\beta_0}{2}\right)v\overline{G}_{u,\eta}dx \\ =-\int_{\alpha_0}^{\beta_0}\tilde{a}(\cdot)\left(x-\frac{\alpha_0+\beta_0}{2}\right)v\overline{v_x}dx-\tilde{g}\int_{\alpha_0}^{\beta_0}\left(x-\frac{\alpha_0+\beta_0}{2}\right)a(\cdot)v\overline{v_x}dx \\ +\int_{\alpha_0}^{\beta_0}\left(x-\frac{\alpha_0+\beta_0}{2}\right)v\int_0^\infty a(\cdot)g(s)\overline{\eta_{sx}(\cdot,s)}dsdx \\ -\int_{\alpha_0}^{\beta_0}\tilde{a}(\cdot)\left(x-\frac{\alpha_0+\beta_0}{2}\right)v\overline{f^1_x}dx \\ -\int_{\alpha_0}^{\beta_0}\left(x-\frac{\alpha_0+\beta_0}{2}\right)v\int_0^\infty a(\cdot)g(s)\overline{f^3_x(\cdot,s)}dsdx. \tag{3.53}$$

Note that

$$-\int_{\alpha_0}^{\beta_0}\tilde{a}(\cdot)\left(x-\frac{\alpha_0+\beta_0}{2}\right)v\overline{v_x}dx \\ =-\int_{\alpha_0}^{\beta_0}\varrho\left(x-\frac{\alpha_0+\beta_0}{2}\right)v\overline{v_x}dx+\tilde{g}\int_{\alpha_0}^{\beta_0}a(\cdot)\left(x-\frac{\alpha_0+\beta_0}{2}\right)v\overline{v_x}dx.$$

Then, from the above equation and from (3.53), we have

$$\begin{aligned}
i\lambda \int_{\alpha_0}^{\beta_0} &\left(x - \frac{\alpha_0+\beta_0}{2}\right) v\overline{G}_{u,\eta} dx \\
&= -\int_{\alpha_0}^{\beta_0} \varrho \left(x - \frac{\alpha_0+\beta_0}{2}\right) v\overline{v_x} dx - \int_{\alpha_0}^{\beta_0} \tilde{a}(\cdot) \left(x - \frac{\alpha_0+\beta_0}{2}\right) v\overline{f_x^1} dx \\
&\quad + \int_{\alpha_0}^{\beta_0} \left(x - \frac{\alpha_0+\beta_0}{2}\right) v \int_0^\infty a(\cdot) g(s) \overline{\eta_{sx}(\cdot,s)} ds dx \\
&\quad - \int_{\alpha_0}^{\beta_0} \left(x - \frac{\alpha_0+\beta_0}{2}\right) v \int_0^\infty a(\cdot) g(s) \overline{f_x^3(\cdot,s)} ds dx. \qquad (3.54)
\end{aligned}$$

Now, note that

$$\begin{aligned}
\mathrm{Re}&\left\{ \int_{\alpha_0}^{\beta_0} \varrho \left(x - \frac{\alpha_0+\beta_0}{2}\right) v\overline{v_x} dx \right\} \\
&= \varrho \frac{(\beta_0-\alpha_0)}{4} (|v(\beta_0)|^2 + |v(\alpha_0)|^2) - \frac{1}{2}\int_{\alpha_0}^{\beta_0} \varrho |v|^2 dx. \qquad (3.55)
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
\mathrm{Re}&\left\{ \int_{\alpha_0}^{\beta_0} \left(x - \frac{\alpha_0+\beta_0}{2}\right) (G_{u,\eta})_x \overline{G}_{u,\eta} dx \right\} \\
&= \frac{(\beta_0-\alpha_0)}{4} (|G_{u,\eta}(\beta_0)|^2 + |G_{u,\eta}(\alpha_0)|^2) - \frac{1}{2}\int_{\alpha_0}^{\beta_0} |G_{u,\eta}|^2 dx. \qquad (3.56)
\end{aligned}$$

Taking the real part in (3.52) and using (3.54)-(3.56), we obtain

$$\begin{aligned}
\varrho \frac{(\beta_0-\alpha_0)}{4} &(|v(\beta_0)|^2 + |v(\alpha_0)|^2) + \frac{(\beta_0-\alpha_0)}{4} (|G(\beta_0)|^2 + |G(\alpha_0)|^2) \\
&\lesssim \int_0^L \left(|v|^2 + |G_{u,\eta}|^2\right) dx - \mathrm{Re}\left\{ \int_{\alpha_0}^{\beta_0} \left(x - \frac{\alpha_0+\beta_0}{2}\right) f^2 \overline{G}_{u,\eta} dx \right\} \\
&\quad - \mathrm{Re}\left\{ \int_{\alpha_0}^{\beta_0} \tilde{a}(\cdot) \left(x - \frac{\alpha_0+\beta_0}{2}\right) v\overline{f_x^1} dx \right\} \\
&\quad + \mathrm{Re}\left\{ \int_{\alpha_0}^{\beta_0} \left(x - \frac{\alpha_0+\beta_0}{2}\right) v \int_0^\infty a(\cdot) g(s) \overline{\eta_{sx}(\cdot,s)} ds dx \right\} \\
&\quad - \mathrm{Re}\left\{ \int_{\alpha_0}^{\beta_0} \left(x - \frac{\alpha_0+\beta_0}{2}\right) v \int_0^\infty a(\cdot) g(s) \overline{f_x^3(\cdot,s)} ds dx \right\}.
\end{aligned}$$

To obtain the estimate of the terms on the right-hand side of the inequality above, simply proceed in the same way as was done in Lemma 3.3.3 and use that $\left|x - \frac{\alpha_0+\beta_0}{2}\right| \le \frac{\alpha_0+\beta_0}{2}$ for all $x \in [\alpha_0, \beta_0]$. Then, for $|\lambda| > 0$ large and $\epsilon > 0$ small, we get

$$\begin{aligned}
&|v(\beta_0)|^2 + |v(\alpha_0)|^2 + |G(\beta_0)|^2 + |G(\alpha_0)|^2 \\
&\lesssim \int_0^L \left(|v|^2 + |G_{u,\eta}|^2\right) dx + \epsilon \|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2.
\end{aligned}$$

To get inequalities (3.48) and (3.49) we multiply (3.33) by $(x-\beta_0)\overline{G}_{u,\eta}$ and $(x-\alpha_0)\overline{G}_{u,\eta}$, respectively. □

Lemma 3.3.5. *For $|\lambda|$ large and $\epsilon > 0$ small, we have*

$$\int_{L_1}^{\tilde{L}_2} (|v|^2 + |G_{u,\eta}|^2)dx \lesssim \epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.$$

Proof. From (3.46) and using Lemma 3.3.4 twice, we obtain

$$\begin{aligned}\int_{L_1}^{\tilde{L}_2} (|v|^2 + |G_{u,\eta}|^2)dx &\lesssim |u_x(\tilde{L}_2)|^2 + |v(\tilde{L}_2)|^2 + \epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.\\ &\lesssim \int_{\tilde{L}_2}^{\tilde{L}_3} (|v|^2 + |G_{u,\eta}|^2)dx + \epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.\\ &\lesssim \epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.\end{aligned}$$

for $|\lambda|$ large and $\epsilon > 0$ small. □

Lemma 3.3.6. *For $|\lambda|$ large and $\epsilon > 0$ small, we have*

$$\int_{\tilde{L}_2}^{L} (|v|^2 + |G_{u,\eta}|^2)dx \lesssim \epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.$$

Proof. From Lemma 3.3.5 and using Lemma 3.3.4 twice, we obtain

$$\begin{aligned}\int_{\tilde{L}_2}^{L} (|v|^2 + |G_{u,\eta}|^2)dx &\lesssim |u_x(\tilde{L}_2)|^2 + |v(\tilde{L}_2)|^2 + \epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.\\ &\lesssim \int_{L_1}^{\tilde{L}_2} (|v|^2 + |G_{u,\eta}|^2)dx + \epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.\\ &\lesssim \epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.\end{aligned}$$

for $|\lambda|$ large. □

The main result of this section is given by the following theorem.

Theorem 3.3.1. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, considering the conditions of the KEME model. Assume the hypotheses in (3.4). Then the system (3.13) is exponentially stable, that is, there exists a positive constant ϵ such that*

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \lesssim e^{-\epsilon t}\|U_0\|_{\mathcal{H}}, \quad \forall\, U_0 \in \mathcal{H}, \quad t > 0,$$

Proof. We apply Theorem 1.5.3 to prove the exponential stability. Pirst, let us prove that $i\mathbb{R} \subset \rho(\mathcal{A})$. For this, we will check

(1) $\mathrm{Ker}(i\lambda I - \mathcal{A}) = \{0\}, \quad \forall\lambda \in \mathbb{R}$;

(2) $R(i\lambda I - \mathcal{A}) = \mathcal{H}, \quad \forall\lambda \in \mathbb{R}$;

(3) $(i\lambda I - \mathcal{A})^{-1}$ is bounded, $\quad \forall\lambda \in \mathbb{R}$.

First, let's prove (1). From Proposition 3.2.1, we have $\mathrm{Ker}(-\mathcal{A}) = \{0\}$. We need to show the result for $\lambda \neq 0$. Suppose there is a real number non-zero λ and $U = (u, v, \eta(\cdot, s))^\top \in D(\mathcal{A})$ such that

$$\mathcal{A}U = i\lambda U,$$

this is, $F \equiv 0$ in (3.32)-(3.34). From (3.36), we get

$$-\int_0^L \int_0^\infty a(\cdot)g'(s)|\eta_x(\cdot,s)|^2 ds dx + \int_0^L b(\cdot)|v_x|^2 dx = 0;$$

we deduce that:

$$a(\cdot)\eta_x(\cdot,s) = 0 \text{ in } (0,L)\times(0,\infty) \quad \text{and} \quad b(\cdot)v_x = 0 \text{ in } (0,L). \tag{3.57}$$

So, from (3.57) and the fact that $\eta(0,s)=0$, we obtain

$$\eta(\cdot,s) = 0 \text{ in } (L_2,L_3)\times(0,\infty) \quad \text{and} \quad v_x = 0 \text{ in } (0,L_1). \tag{3.58}$$

Inserting (3.58) in (3.32)-(3.34), we get

$$u = 0 \text{ in } (L_2,L_3) \quad \text{and} \quad u_x = 0 \text{ in } (0,L_1). \tag{3.59}$$

Since $\tilde{a}(\cdot) = \varrho - a(\cdot)\tilde{g}$, from (3.58)-(3.59), we obtain

$$\tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds + b(\cdot)v_x = \varrho u_x, \quad \text{in } (0,L). \tag{3.60}$$

Therefore, from (3.60) and the fact that $U \in D(\mathcal{A})$, we get

$$\varrho u_{xx} = \left(\tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds + b(\cdot)v_x\right)_x \in L^2(0,L). \tag{3.61}$$

Thus, we obtain

$$u \in H^2(0,L) \text{ and consequently } u \in C^1([0,L]), \tag{3.62}$$

From (3.59) and (3.62), we have

$$u(L_1) = u_x(L_1) = u(L_2) = u_x(L_2) = u(L_3) = u_x(L_3) = u(L) = u_x(L) = 0 \tag{3.63}$$

Now, inserting (3.60) in (3.33)-(3.34), we get

$$\lambda^2 u + \varrho u_{xx} = 0, \quad \text{in } (0,L). \tag{3.64}$$

From (3.63)-(3.64), we obtain

$$u \equiv 0 \text{ in } (0,L). \tag{3.65}$$

Inserting (3.65) in (3.33)-(3.34), we get $U \equiv 0$, and (1) is proved.

At moment, let's prove (2). From Proposition 3.2.1, we have $R(-\mathcal{A}) = \mathcal{H}$. We need to show the result for $\lambda \neq 0$. For this aim, let $F = (f^1, f^2, f^3(\cdot,s)) \in \mathcal{H}$, we look for $U = (u, v, \eta(\cdot,s))^\top \in D(\mathcal{A})$ solution of (3.31), equivalently, of (3.32)-(3.35).
From (3.32),(3.34), and (3.35), we have

$$\begin{aligned}&\eta(x,s)\\ &= \frac{1}{i\lambda}(i\lambda u - f^1)(1-e^{-i\lambda s}) + \int_0^s f^3(x,\mu)e^{i\lambda(\mu-s)}d\mu, \quad \text{in } (0,L)\times(0,\infty).\end{aligned} \tag{3.66}$$

Inserting (3.66) in (3.33) and using (3.32) and that $\tilde{a}(x) = \varrho - a(x)\tilde{g}$, we get

$$\begin{aligned} F_0 = & -\lambda^2 u - \left\{ c(x)u_x + \frac{a(x)}{i\lambda} \int_0^\infty g(s)(1 - e^{-i\lambda s}) f_x^1 ds \right\}_x \\ & - \left\{ \int_0^\infty a(x)g(s) \int_0^s f_x^3(x,\mu) e^{i\lambda(\mu - s)} d\mu ds \right\}_x, \end{aligned} \tag{3.67}$$

where $c(\cdot) \doteq \varrho - a(\cdot) \int_0^\infty g(s)e^{-i\lambda s}ds + i\lambda b(\cdot)$ and $F_0 \doteq f^2 + i\lambda f^1 - b(\cdot) f_{xx}^1$.
Multiplying (3.67) by $\overline{\phi} \in H_0^1(0,L)$, integrating over $(0,L)$, then using integration by parts, we get

$$\begin{aligned} & -\lambda^2 \int_0^L u\overline{\phi} dx + \int_0^L c(\cdot) u_x \overline{\phi_x} dx + \frac{1}{i\lambda} \int_0^L \int_0^\infty a(\cdot) g(s)(1 - e^{-i\lambda s}) f_x^1 \overline{\phi_x} ds dx \\ & + \int_0^L a(\cdot) \int_0^\infty g(s) \int_0^s f_x^3(\cdot,\mu) e^{i\lambda(\mu - s)} \overline{\phi_x} d\mu ds dx = \int_0^L F_0 \overline{\phi} dx \end{aligned} \tag{3.68}$$

Note that, from (3.68), we obtain

$$\mathcal{G}(u,\phi) = J(\phi), \quad \forall \phi \in H_0^1(0,L), \tag{3.69}$$

where

$$\mathcal{G}(u,\phi) = \mathcal{G}_1(u,\phi) + \mathcal{G}_2(u,\phi) \tag{3.70}$$

with

$$\mathcal{G}_1(u,\phi) = \int_0^L c(\cdot) u_x \overline{\phi_x} dx, \qquad \mathcal{G}_2(u,\phi) = -\lambda^2 \int_0^L u\overline{\phi} dx, \tag{3.71}$$

and

$$\begin{aligned} J(\phi) = & \int_0^L F_0 \overline{\phi} dx - \frac{1}{i\lambda} \int_0^L \int_0^\infty a(\cdot) g(s)(1 - e^{-i\lambda s}) f_x^1 \overline{\phi_x} ds dx \\ & - \int_0^L a(\cdot) \int_0^\infty g(s) \int_0^s f_x^3(\cdot,\mu) e^{i\lambda(\mu - s)} \overline{\phi_x} d\mu ds dx \end{aligned} \tag{3.72}$$

Let us consider the following operators,

$$\left\{ \begin{array}{rcl} \mathbb{G} : H_0^1(0,L) & \longrightarrow & H^{-1}(0,L) \\ u & \longmapsto & \mathbb{G}(u) \end{array} \right. \qquad \left\{ \begin{array}{rcl} \mathbb{G}_1 : H_0^1(0,L) & \longrightarrow & H^{-1}(0,L) \\ u & \longmapsto & \mathbb{G}_1(u) \end{array} \right.$$

and

$$\left\{ \begin{array}{rcl} \mathbb{G}_2 : H_0^1(0,L) & \longrightarrow & H^{-1}(0,L) \\ u & \longmapsto & \mathbb{G}_2(u) \end{array} \right.$$

such that

$$\left\{ \begin{array}{rcll} \mathbb{G}(u,\phi) & = & \mathcal{G}(u,\phi), & \forall \phi \in H_0^1(0,L) \\ \mathbb{G}_1(u,\phi) & = & \mathcal{G}_1(u,\phi), & \forall \phi \in H_0^1(0,L) \\ \mathbb{G}_2(u,\phi) & = & \mathcal{G}_2(u,\phi), & \forall \phi \in H_0^1(0,L) \end{array} \right.$$

If $\mathbb{G}$ is an isomorphism and J is a antilinear on $H_0^1(0,L)$ and, furthemore, J is continuous from $H_0^1(0,L)$ to $\mathbb{C}$, then we get Equation (3.69) admits a unique solution $u \in H_0^1(0,L)$ and consequently (2) will be proven. In fact, if $u \in H_0^1(0,L)$, then $v = i\lambda u - f^1 \in L^2(0,L)$, because $f^1 \in L^2(0,L)$. Furthermore, from (3.33), we get

$$\left(\tilde{a}(\cdot)u_x + a(\cdot)\int_0^\infty g(s)\eta_x(\cdot,s)ds + b(\cdot)v_x\right)_x = i\lambda v - f^2 \in L^2(0,L).$$

Similarly as we did with (3.19), we can show that $\eta(\cdot,s)$ defined by (3.66) belongs to $\mathcal{G}_g$ and $\eta_s(\cdot,s) \in \mathcal{G}_g$, since $v \in H_0^1(0,L)$ and $f^3(\cdot,s) \in \mathcal{G}_g$.Thus, we can conclude that Equation(3.31) admits a unique solution $U \in D(A)$.

Therefore, our goal is to prove that $\mathbb{G}$ is an isomorphism operator and that and that J fulfills the conditions mentioned above. To do this, we will show:

$$\text{(i) Ker}\{\mathbb{G}\} = \{0\}; \quad \text{(ii) } \mathbb{G}_2 \text{ is compact}; \quad \text{(iii) } \mathbb{G}_1 \text{ is an isomorphism.}$$

Note that by proving the above items, we will be able to prove that $\mathbb{G}$ is an isomorphism. In fact, from (ii) and (iii), we get that the operator $\mathbb{G} = \mathbb{G}_1 + \mathbb{G}_2$ is a Fredholm operator. From (iii), we have that $\mathbb{G}_1$ is a Fredholm operator of index zero and from (i) we have $\dim N(\mathbb{G}) = 0$. Then, we get

$$0 = \text{ind}\mathbb{G}_1 = \text{ind}\mathbb{G} = \dim N(\mathbb{G}) - \text{codim}R(\mathbb{G});$$

this is, $\text{codim}R(\mathbb{G}) = 0$, how $R(\mathbb{G})$ is closed ($\mathbb{G}$ is Fredholm), we concluded $R(\mathbb{G}) = H^{-1}(0,L)$. Then, as $\mathbb{G}$ is injective, surjective and continuous ($\mathbb{G}$ is Fredholm), it follows by the closed graph theorem of Banach that $\mathbb{G}^{-1}$ is continuous, and therefore, $\mathbb{G}$ is an isomorphism.

With that in mind, let's now prove the three items mentioned.

Proof of (i): Let $u_0 \in \text{Ker}\{\mathbb{G}\}$, i.e.

$$\mathcal{G}(u_0,\phi) = 0, \quad \forall \phi \in H_0^1(0,L).$$

Equivalently, we have

$$\int_0^L c(\cdot)u_{0x}\overline{\phi_x}dx - \lambda^2\int_0^L u_0\overline{\phi}dx = 0, \quad \forall \phi \in H_0^1(0,L).$$

Therefore, taking $\phi \in C_0^\infty(0,L)$ and using $\overline{C_0^\infty(0,L)} = H_0^1(0,L)$, we get

$$-\lambda^2 u_0 - c(\cdot)(u_0)_{xx} = 0, \quad \text{with} \quad u_0(0) = u_0(L) = 0. \tag{3.73}$$

From (3.73) and using $u_0 \in H_0^1(0,L)$ and that $|1 - e^{-i\lambda s}| \le 2$, for $s \in (0,\infty)$, we get

$$U_0 = (u_0, i\lambda u_0, (1 - e^{-i\lambda s})u_0)^\top \in D(\mathcal{A}) \quad \text{and} \quad i\lambda U_0 - \mathcal{A}U_0 = 0. \tag{3.74}$$

Therefore, and by (1), we have $U_0 \in \text{Ker}(i\lambda I - \mathcal{A}) = \{0\}$, this is, $U_0 \equiv 0$, consequently, $u_0 = 0$ and $\text{Ker}\{\mathbb{G}\} = \{0\}$. Thus, (i) is proved.

Proof of (*ii*): By Holder's inequality, we have

$$|\mathcal{G}_2(u,\phi)| \lesssim C\|u\|_{L^2(0,L)}\|\phi\|_{L^2(0,L)}, \quad \forall \phi \in H_0^1(0,L).$$

Thus, we obtain

$$\|\mathbb{G}_2(u)\|_{H^{-1}(0,L)} \lesssim \|u\|_{L^2(0,L)}. \tag{3.75}$$

Now, consider $(u_n) \in H_0^1(0,L)$ bounded. From the compact embedding of $H_0^1(0,L)$ in $L^2(0,L)$, we have that (u_n) converges in $L^2(0,L)$ up to a subsequence. From (3.75), we get

$$\|\mathbb{G}_2(u_n)\|_{H^{-1}(0,L)} \lesssim \|u_n\|_{L^2(0,L)},$$

this is, $\mathbb{G}_2(u_n)$ is a Cauchy sequence in $H^{-1}(0,L)$, and therefore, converges in $H^{-1}(0,L)$ up to a subsequence. Thus, by definition, $\mathbb{G}_2$ is a compact operator. Thus, (ii) is proved.

Proof of (iii): note that by (3.4), we have $|c(\cdot)| \lesssim 1$ and using the Holder's inequality, we get

$$|\mathcal{G}_1(u,\phi)| \lesssim \|u\|_{H_0^1(0,L)}\|\phi\|_{H_0^1(0,L)},$$

this is, $\mathcal{G}_1$ is continuous. Now, using Poincaré's inequality, we get

$$\text{Re}\{\mathcal{G}_1(u,u)\} = \text{Re}\left\{\int_0^L c(\cdot)|u_x|^2dx\right\} \geq C\|u\|^2_{H_0^1(0,L)},$$

for some constante $C > 0$. Therefore, $\mathcal{G}_1$ is coercive. Furthermore, it is easy to see that $\mathcal{G}_1$ is sesquilinear form on $H_0^1(0,L)$. Then, by Lax-Milgram theorem, the operator $\mathbb{G}_1$ is an isomorphism. Thus, (iii) is proved.

It is easy to see that operator J is a antilinear on $H_0^1(0,L)$. Furthemore, note that, by Hölder's inequality, we have J is continuous from $H_0^1(0,L)$ to $\mathbb{C}$. Therefore, $\mathbb{G}$ is an isomorphism. Thus, (2) is proved. Thus, (3) and, therefore, $i\mathbb{R} \subset \rho(\mathcal{A})$, how we wanted to show.

Finally, let's prove (3). It is easy to verify the $(i\lambda - \mathcal{A})$ is closed for all $\lambda \in \mathbb{R}$, because $\mathcal{A}$ is closed. Furthermore, since $(i\lambda - \mathcal{A})$ is linear, injective (by (1)) and surjective (by (2)), we can to conclude that the graph of $(i\lambda I - \mathcal{A})^{-1}$ is closed. Consequently, by closed graph theorem of Banach we can deduce that $(i\lambda I - \mathcal{A})^{-1}$ is bounded for all $\lambda \in \mathbb{R}$.

Now, let $F \in \mathcal{H}$ and consider $U = (u, v, \eta(\cdot, s))$ solution of $(i\lambda I - \mathcal{A})U = F$, i.e, the system (3.32)-(3.35) is satisfied. To show the exponentially stability, it is sufficient to show That $\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}}$. In order to obtain this estimate, we use hypotheses (3.4) to deduce $\tilde{a}(\cdot) \geq \varrho - a_0\tilde{g}$. Recalling as well that $G_{u,\eta} = \tilde{a}(\cdot)u_x + \int_0^\infty g(s)a(\cdot)\eta_x(\cdot,s)ds$ we obtain

$$\begin{aligned}\int_{L_1}^L \left(|v|^2dx + \tilde{a}(\cdot)|u_x|^2\right)dx &\lesssim \int_{L_1}^L \left(|v|^2 + \frac{1}{\varrho - a_0\tilde{g}}\left|\tilde{a}(\cdot)u_x + \int_0^\infty g(s)a(\cdot)\eta_x(\cdot,s)ds\right|^2\right)dx \\ &\lesssim \int_{L_1}^L (|v|^2 + |G_{u,\eta}|^2)dx + \tilde{g}^{1/2}a_0\int_{L_2}^{L_3}\int_0^\infty g(s)|\eta_x(\cdot,s)|^2dsdx.\end{aligned} \tag{3.76}$$

So, from (3.38),(3.41), (3.76), Lemma 3.3.5 and Lemma 3.3.6 , we get

$$\begin{aligned}\|U\|^2_{\mathcal{H}} = &\int_0^{L_1}(|v|^2dx + \tilde{a}(\cdot)|u_x|^2)dx + \int_{L_1}^L(|v|^2dx + \tilde{a}(\cdot)|u_x|^2)dx \\ &+ a_0\int_{L_2}^{L_3}\int_0^\infty g(s)|\eta_x(\cdot,s)|^2dsdx \\ \lesssim\ &\epsilon\|U\|^2_{\mathcal{H}} + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}},\end{aligned}$$

for $|\lambda|$ large and $\epsilon > 0$ small. Thus, using Young's inequality, taking $|\lambda|$ large and a suitably small $\epsilon > 0$, we obtain

$$\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}},$$

as we desired to prove. □

3.4 Asymptotic behavior: Polynomial stability

To show the polynomial decay of the solution for the system (3.1)-(3.3) we use a result due to Borichev and Tomilov ([7]). Here we consider the cases where at least one of the elastic parts of the string (with no dissipation) connects with only the Kelvin-Voigt damping. We focus in investigating the stability of the EKEME model because the EKIME model and other cases are similar. We recall that the EKEME model is given by (3.1)-(3.3) considering

$$b(x) = b_0\chi_{[L_1,L_2]}(x) \text{ and } a(x) = a_0\chi_{[L_3,L_4]}(x), \quad a_0, b_0 > 0. \tag{3.77}$$

The main result of this section is Theorem 3.4.1 and to prove this theorem we will need to introduce some technical lemmas.

Analogously to what we did in (3.36), we have

$$-\int_{L_3}^{L_4}\int_0^\infty g'(s)|\eta_x(\cdot,s)|^2 dsdx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{3.78}$$

and

$$\int_{L_1}^{L_2} |v_x|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.79}$$

Moreover, from (3.4) and (3.78), we obtain

$$\begin{aligned}\int_{L_3}^{L_4}\int_0^\infty g(s)|\eta_x(\cdot,s)|^2 dsdx &\le -\frac{1}{c_0}\int_{L_3}^{L_4}\int_0^\infty g'(s)|\eta_x(\cdot,s)|^2 dsdx\\ &\lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.\end{aligned} \tag{3.80}$$

Now, from (3.32) and (3.79), we get

$$\begin{aligned}\int_{L_1}^{L_2} \varrho|u_x|^2 dx &\lesssim |\lambda|^{-2}\left(\int_{L_1}^{L_2}|v_x|^2dx + \int_{L_1}^{L_2}|f_x^1|^2dx\right)\\ &\lesssim |\lambda|^{-2}(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2).\end{aligned} \tag{3.81}$$

Proceeding in the same way as in the proof of Lemma 3.3.1, we get

$$\int_{L_3}^{L_4} \tilde{a}(\cdot)|u_x|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{3.82}$$

for $|\lambda|$ large.

Consider $\tilde{L}_3$ and $\tilde{L}_4$ such that $[\tilde{L}_3,\tilde{L}_4] \subsetneq (L_2, L_3)$. Then, using (3.82) and proceeding in the same way as in the proof of Lemma 3.3.3, we have

$$\int_{\tilde{L}_3}^{\tilde{L}_4} (|v|^2 + \tilde{a}(\cdot)|u_x|^2) dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{3.83}$$

for $|\lambda|$ large.

Lemma 3.4.1. *For $|\lambda|$ large, we have*

$$\int_{L_1}^{L_2} |\tilde{a}(x)u_x + b(x)v_x|^2\, dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2,$$

and

$$\int_{L_3}^{L_4} \left|\tilde{a}(x)u_x + \int_0^\infty a(x)g(s)\eta_x(\cdot,s)ds\right|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.84}$$

Proof. Using the definitions of a and b given in (3.77), as well as the definition of $\tilde{a}$ and the estimates in equations (3.79) and (3.81), we get

$$\int_{L_1}^{L_2} |\tilde{a}(x)u_x + b(x)v_x|^2\,dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large. The equation (3.84) is obtained in the same way as was done in Lemma 3.3.2. ☐

Lemma 3.4.2. *For $|\lambda|$ large, we have*

$$|\lambda|\|v\|_{L^2(L_1,L_2)}^2 \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2} + \|F\|_{\mathcal{H}}^2.$$

Proof. From Equations(3.33), (3.79), and (3.81), we get

$$\begin{aligned} |\lambda|\|v\|_{H^{-1}(L_1,L_2)} &\lesssim \|u_x\|_{L^2(L_1,L_2)} + \|v_x\|_{L^2(L_1,L_2)} + \|f^2\|_{L^2(L_1,L_2)} \\ &\lesssim \|F\|_{\mathcal{H}}^{\frac{1}{2}}\|U\|_{\mathcal{H}}^{\frac{1}{2}} + \|F\|_{\mathcal{H}} \end{aligned} \tag{3.85}$$

Using Interpolation, inequalities (3.79) and (3.85) and Young' inequality, we get

$$\begin{aligned} \|v\|_{L^2(L_1,L_2)}^2 &\lesssim \|v\|_{H^{-1}(L_1,L_2)}\|v\|_{H^1(L_1,L_2)} \\ &\lesssim \frac{1}{|\lambda|}\left(\|F\|_{\mathcal{H}}^{\frac{1}{2}}\|U\|_{\mathcal{H}}^{\frac{1}{2}} + \|F\|_{\mathcal{H}}\right)\left(\|v_x\|_{L^2(L_1,L_2)} + \|v\|_{L^2(L_1,L_2)}\right) \\ &\lesssim \frac{1}{|\lambda|}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{\frac{3}{2}}\|U\|_{\mathcal{H}}^{\frac{1}{2}}\right) \\ &\quad + \frac{C(\epsilon)}{|\lambda|^2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\right) + \epsilon\|v\|_{L^2(L_1,L_2)}^2, \end{aligned}$$

for $|\lambda|$ large and $\epsilon > 0$. Considering an appropriate $\epsilon > 0$ small enough, we obtain

$$|\lambda|\|v\|_{L^2(L_1,L_2)}^2 \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{\frac{3}{2}}\|U\|_{\mathcal{H}}^{\frac{1}{2}} + \|F\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large. ☐

Lemma 3.4.3. *We have*

$$\int_0^{L_1} (\varrho|u_x|^2 + |v|^2)dx \lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{\frac{5}{4}}\|U\|_{\mathcal{H}}^{\frac{3}{2}} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right) + \|F\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large.

Proof. Using integration by parts, we deduce that

$$\frac{1}{2}\int_0^{L_1} \varrho|u_x|^2 dx = -\mathrm{Re}\left\{\int_0^{L_1} \varrho x\overline{u_x}u_{xx}dx\right\} + \frac{L_1}{2}\varrho|u_x(L_1)|^2. \tag{3.86}$$

In order to estimate the firs term on the right of the previous equation, we multiply Equation(3.33) by $x\overline{u_x}$ and take the real part to get

$$-\mathrm{Re}\left\{\int_0^{L_1} \varrho x\overline{u_x}u_{xx}dx\right\} = \mathrm{Re}\left\{i\lambda\int_0^{L_1} xv\overline{u_x}dx\right\} - \mathrm{Re}\left\{\int_0^{L_1} xf^2\overline{u_x}dx\right\}. \tag{3.87}$$

Using once again integration by parts, it follows that

$$\int_0^{L_1} |v|^2 dx = \frac{L_1}{2}|v(L_1)|^2 - \text{Re}\left\{\varrho \int_0^{L_1} xv\overline{v_x}dx\right\} \tag{3.88}$$

We wish to estimate the last term, for which we use Equation(3.32) to deduce

$$\text{Re}\left\{\int_0^{L_1} xv\overline{v_x}dx\right\} = -\text{Re}\left\{i\lambda \int_0^{L_1} xv\overline{u_x}dx\right\} - \text{Re}\left\{\int_0^{L_1} xv\overline{f_x^1}dx\right\} \tag{3.89}$$

Substituting (3.87) in (3.86) and (3.89) in (3.88) and taking the sum, using as well Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^{L_1} (\varrho|u_x|^2 + |v|^2)dx \\
&= \frac{L_1}{2}\left[\varrho|u_x(L_1)|^2 + |v(L_1)|^2\right] - \text{Re}\left\{\int_0^{L_1} xv\overline{f_x^1}dx\right\} - \text{Re}\left\{\int_0^{L_1} xf^2\overline{u_x}dx\right\} \\
&\lesssim I_u + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}
\end{aligned} \tag{3.90}$$

where I_u denotes

$$I_u = \frac{(L_2 - L_1)}{2}\left[\varrho|v(L_1)|^2 + |\varrho u_x(L_1) + b_0 v_x(L_1)|^2\right].$$

We use integration by parts to estimate the first term in I_u

$$\frac{\varrho(L_2 - L_1)}{2}|v(L_1)|^2 = \text{Re}\left\{\varrho \int_{L_1}^{L_2} (L_2 - x)v\overline{v_x}dx\right\} + \frac{\varrho}{2}\int_{L_1}^{L_2} |v|^2 dx \tag{3.91}$$

By Equation(3.32), we deduce that $\overline{i\lambda u_x} - \overline{v_x} = \overline{f_x^1}$ and therefore multiplying by $\varrho(L_2 - x)v$, integrating in (L_1, L_2) and taking the real part we obtain

$$\begin{aligned}
&\text{Re}\left\{\varrho \int_{L_1}^{L_2} (L_2 - x)v\overline{v_x}dx\right\} \\
&= -\text{Re}\left\{\varrho \int_{L_1}^{L_2} (L_2 - x)v\overline{(-i\lambda u_x)}dx\right\} - \text{Re}\left\{\varrho \int_{L_1}^{L_2} (L_2 - x)v\overline{f_x^1}dx\right\}
\end{aligned} \tag{3.92}$$

We use also integration by parts to estimate the second term in I_u

$$\begin{aligned}
\frac{1}{2}(L_2 - L_1)|\varrho u_x(L_1) + b_0 v_x(L_1)|^2 = &-\text{Re}\left\{\frac{1}{2}\int_{L_1}^{L_2} (L_2 - x)\frac{d}{dx}|\varrho u_x + b_0 v_x|^2 dx\right\} \\
&+ \frac{1}{2}\int_{L_1}^{L_2} |\varrho u_x + b_0 v_x|^2 dx.
\end{aligned} \tag{3.93}$$

Multiplying Equation(3.33) by $(L_2 - x)\overline{(\varrho u_x + b_0 v_x)}$, integrating in (L_1, L_2) and taking the real part, we arrive at

$$\begin{aligned}
\text{Re}\left\{\frac{1}{2}\int_{L_1}^{L_2} (L_2 - x)\frac{d}{dx}|\varrho u_x + b_0 v_x|^2 dx\right\} = &\ \text{Re}\left\{\int_{L_1}^{L_2} i\lambda(L_2 - x)v\overline{(\varrho u_x + b_0 v_x)}dx\right\} \\
&- \text{Re}\left\{\int_{L_1}^{L_2} (L_2 - x)f^2\overline{(\varrho u_x + b_0 v_x)}dx\right\}.
\end{aligned} \tag{3.94}$$

On the other hand, by Lemma 3.4.2 and Equation (3.79), we get

$$
\begin{aligned}
\mathrm{Re}&\left\{ b_0 \int_{L_1}^{L_2} i\lambda(L_2-x)v\overline{v_x}dx \right\}\\
&\lesssim b_0|\lambda|^{1/2}\int_{L_1}^{L_2}|v_x|(|\lambda|^{1/2}|v|)dx\\
&\lesssim |\lambda|^{1/2}\left(\int_{L_1}^{L_2}|v_x|^2dx\right)^{1/2}\left(|\lambda|\int_{L_1}^{L_2}|v|^2dx\right)^{1/2}\\
&\lesssim |\lambda|^{1/2}\|F\|_{\mathcal{H}}^{1/2}\|U\|_{\mathcal{H}}^{1/2}\left(\|F\|_{\mathcal{H}}^{1/2}\|U\|_{\mathcal{H}}^{1/2}+\|F\|_{\mathcal{H}}^{3/4}\|U\|_{\mathcal{H}}^{1/4}+\|F\|_{\mathcal{H}}\right)\\
&\lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right). \qquad (3.95)
\end{aligned}
$$

So, from (3.91)-(3.95) and using Lemma 3.4.1 and Lemma 3.4.2, we get

$$
\begin{aligned}
I_u =&\frac{\varrho}{2}\int_{L_1}^{L_2}|v|^2dx+\frac{1}{2}\int_{L_1}^{L_2}|\varrho u_x+b_0v_x|^2dx+\mathrm{Re}\left\{\varrho\int_{L_1}^{L_2}(L_2-x)v\overline{f_x^1}dx\right\}\\
&+\mathrm{Re}\left\{\int_{L_1}^{L_2}(L_2-x)f^2\overline{(\varrho u_x+b_0v_x)}dx\right\}\\
&-\mathrm{Re}\left\{b_0\int_{L_1}^{L_2}i\lambda(L_2-x)v\overline{v_x}dx\right\}\\
\lesssim& |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right)+\|F\|_{\mathcal{H}}^2, \qquad (3.96)
\end{aligned}
$$

for $|\lambda|$ large.
Therefore, from (3.90) and (3.96), we have

$$
\begin{aligned}
&\int_0^{L_1}(\varrho|u_x|^2+|v|^2)dx\\
&=\frac{L_1}{2}\left[\varrho|u_x(L_1)|^2+|v(L_1)|^2\right]-\mathrm{Re}\left\{\int_0^{L_1}xv\overline{f_x^1}dx\right\}-\mathrm{Re}\left\{\int_0^{L_1}xf^2\overline{u_x}dx\right\}\\
&\lesssim I_u+\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\\
&\lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right)+\|F\|_{\mathcal{H}}^2,
\end{aligned}
$$

for $|\lambda|$ large, as we desired to prove. □

The following lemma will be used in the next lemmas.

Lemma 3.4.4. *Let $q\in C^1([0,L])$ be a function with $q(0)=q(L)=0$. Then, we obtain for $\epsilon>0$,*

$$
\begin{aligned}
\int_0^L &q'(x)\left(|v|^2+|W(u,\eta)|^2\right)dx\\
&\lesssim Re\left\{\int_0^L q(x)f^2\overline{W}(u,\eta)dx\right\}-Re\left\{i\lambda\int_0^L b(\cdot)q(x)v\overline{v_x}dx\right\}\\
&\quad+\epsilon\|U\|_{\mathcal{H}}^2+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.
\end{aligned}
$$

where $W(u,\eta)\doteq\tilde{a}(\cdot)u_x+\int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds+b(\cdot)v_x$.

Proof. First, from Equation(3.32) and Equation(3.34), we deduce that

$$i\lambda\overline{u_x} = -\overline{v_x} - \overline{f_x^1}, \tag{3.97}$$

$$i\lambda\overline{\eta}_x(\cdot,s) = \overline{\eta_{sx}}(\cdot,s) - \overline{v_x} - \overline{f_x^3}(\cdot,s), \quad \text{in} \quad (0,L)\times(0,\infty). \tag{3.98}$$

Multiplying Equation(3.33) by $q(x)\overline{W}(u,\eta)$ and integrating over $(0,L)$, we get

$$i\lambda\int_0^L q(x)v\overline{W}(u,\eta)dx - \int_0^L q(x)W_x(u,\eta)\overline{W}(u,\eta)dx = \int_0^L q(x)f^2\overline{W}(u,\eta)dx. \tag{3.99}$$

Using (3.97) and (3.98), we have

$$\begin{aligned}
&i\lambda\int_0^L q(x)v\overline{W}(u,\eta)dx\\
&\quad= i\lambda\int_0^L \tilde{a}(\cdot)q(x)v\overline{u_x}dx + i\lambda\int_0^L q(x)v\int_0^\infty a(\cdot)g(s)\overline{\eta_x(\cdot,s)}dsdx\\
&\qquad + i\lambda\int_0^L q(x)b(\cdot)v\overline{v_x}dx\\
&\quad= \int_0^L q(x)v\int_0^\infty a(\cdot)g(s)\overline{\eta_{sx}(\cdot,s)}dsdx - \int_0^L \tilde{a}(\cdot)q(x)v\overline{v_x}dx\\
&\qquad - \tilde{g}\int_0^L a(\cdot)q(x)v\overline{v_x}dx - \int_0^L \tilde{a}(\cdot)q(x)v\overline{f_x^1}dx\\
&\qquad - \int_0^L q(x)v\int_0^\infty a(\cdot)g(s)\overline{f_x^3(\cdot,s)}dsdx + i\lambda\int_0^L q(x)b(\cdot)v\overline{v_x}dx.
\end{aligned}$$

Since $\tilde{a}(\cdot) = \varrho - a(\cdot)\tilde{g}$, note that

$$-\int_0^L \tilde{a}(\cdot)q(x)v\overline{v_x}dx = -\varrho\int_0^L q(x)v\overline{v_x}dx + \tilde{g}\int_0^L a(\cdot)q(x)v\overline{v_x}dx.$$

So, we have

$$\begin{aligned}
&i\lambda\int_0^L q(x)v\overline{W}(u,\eta)dx\\
&\quad= \int_0^L q(x)v\int_0^\infty a(\cdot)g(s)\overline{\eta_{sx}(\cdot,s)}dsdx - \varrho\int_0^L q(x)v\overline{v_x}dx-\\
&\qquad - \int_0^L (\varrho - a(\cdot)\tilde{g})q(x)v\overline{f_x^1}dx + i\lambda\int_0^L q(x)b(\cdot)v\overline{v_x}dx.\\
&\qquad - \int_0^L q(x)v\int_0^\infty a(\cdot)g(s)\overline{f_x^3(\cdot,s)}dsdx.
\end{aligned} \tag{3.100}$$

Since $q(0) = q(L) = 0$, we have

$$-\mathrm{Re}\left\{\varrho\int_0^L q(x)v\overline{v_x}dx\right\} = \varrho\int_0^L \frac{q'(x)}{2}|v|^2dx, \tag{3.101}$$

and

$$-\mathrm{Re}\left\{\int_0^L q(x)W_x(u,\eta)\overline{W}(u,\eta)dx\right\} = \int_0^L \frac{q'(x)}{2}|W(u,\eta)|^2dx. \tag{3.102}$$

Taking the real part in (3.99) and using (3.100)-(3.102), we obtain

$$\begin{aligned}
&\int_0^L q'(x)\left(|v|^2+|W(u,\eta)|^2\right)dx\\
&\lesssim \text{Re}\left\{\int_0^L q(x)f^2\overline{W}(u,\eta)dx\right\}-\text{Re}\left\{i\lambda\int_0^L b(\cdot)q(x)v\overline{v_x}dx\right\}\\
&\quad-\text{Re}\left\{\int_0^L q(x)v\int_0^\infty a(\cdot)g(s)\overline{\eta_{sx}(\cdot,s)}dsdx\right\}+\text{Re}\left\{\int_0^L(\varrho-a(\cdot)\tilde{g})q(x)v\overline{f_x^1}dx\right\}\\
&\quad+\text{Re}\left\{\int_0^L q(x)v\int_0^\infty a(\cdot)g(s)\overline{f_x^3(\cdot,s)}dsdx\right\}.
\end{aligned}$$

Next, we will estimate the terms on the right hand side of the above equation. To do this, we will utilize integration by parts with respect to s with the help that $\eta(\cdot,0)=0$, the Young's inequality, Poincaré inequality, Cauchy-Schwarz inequality. Note that

$$\begin{aligned}
&\text{Re}\left\{\int_0^L q(x)v\int_0^\infty a(\cdot)g(s)\overline{\eta_{sx}(\cdot,s)}dsdx\right\}\\
&\lesssim\left|\int_{L_3}^{L_4}a_0q(x)v\int_0^\infty g(s)\overline{\eta_{sx}(\cdot,s)}dsdx\right|\\
&=\left|\int_{L_3}^{L_4}a_0q(x)v\int_0^\infty -g'(s)\overline{\eta_x(\cdot,s)}dsdx\right|\\
&\lesssim\int_{L_3}^{L_4}a_0q(x)|v|\left(\int_0^\infty -g'(s)ds\right)^{\frac{1}{2}}\left(\int_0^\infty -g'(s)|\eta_x(\cdot,s)|^2ds\right)^{\frac{1}{2}}dx\\
&\lesssim[\max|q(x)|]g_0\left(\epsilon\int_{L_3}^{L_4}a_0|v|^2dx+C(\epsilon)\int_{L_3}^{L_4}a_0\int_0^\infty -g'(s)|\eta_x(\cdot,s)|^2dsdx\right)\\
&\lesssim\epsilon\int_0^L|v|^2dx+C(\epsilon)\int_{L_3}^{L_4}\int_0^\infty -g'(s)|\eta_x(\cdot,s)|^2dsdx\\
&\lesssim\epsilon\|U\|_{\mathcal{H}}^2+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},
\end{aligned}$$

where in the latest inequality we utilize (3.78). The other terms can be estimated analogously as in the proof of lemmas 3.3.1 and 3.3.3. Therefore, we obtain

$$\begin{aligned}
&\int_0^L q'(x)\left(|v|^2+|W(u,\eta)|^2\right)dx\\
&\lesssim\text{Re}\left\{\int_0^L q(x)f^2\overline{W}(u,\eta)dx\right\}-\text{Re}\left\{i\lambda\int_0^L b(\cdot)q(x)v\overline{v_x}dx\right\}\\
&\quad+\epsilon\|U\|_{\mathcal{H}}^2+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.
\end{aligned}$$

□

To address the next lemmas, let us consider $\tilde{L}_1,\tilde{L}_2,\tilde{L}_3$ and $\tilde{L}_4$ such that $[\tilde{L}_1,\tilde{L}_2]\subset(L_1,L_2)$ and $[\tilde{L}_3,\tilde{L}_4]\subset(L_3,L_4)$. Also, we consider a function q_3 satisfying $q_3\in C^1([0,L])$, where $q_3(x)=0$ for all $x\in[0,\tilde{L}_2]\cup[\tilde{L}_4,L]$ and $q_3'(x)=q_0>0$ for all $x\in[L_2,\tilde{L}_3]$.

Lemma 3.4.5. *For $|\lambda|$ large and $\epsilon > 0$ small, we have*

$$\int_{L_2}^{\tilde{L}_3} \left(|v|^2 + \left| \tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds \right|^2 \right) dx$$
$$\lesssim |\lambda|^{1/2} \left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2} \right)$$
$$+ \epsilon\|U\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}^2.$$

Proof. Using the result of Lemma 3.4.4 with $q_3(x)$ and definition of $W(u,\eta)$, we get

$$\int_{L_2}^{\tilde{L}_3} q_3'(x) \left(|v|^2 + \left| \tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds \right|^2 \right) dx$$
$$\lesssim -\int_{\tilde{L}_2}^{L_2} q_3'(x) \left(|v|^2 + |\varrho u_x + b_0 v_x|^2 \right) dx$$
$$- \int_{\tilde{L}_3}^{\tilde{L}_4} q_3'(x) \left(|v|^2 + \left| \tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds \right|^2 \right) dx$$
$$+ \mathrm{Re}\left\{ \int_{\tilde{L}_2}^{L_2} q_3(x) f^2 \overline{(\varrho u_x + b_0 v_x)} dx \right\} + \mathrm{Re}\left\{ \int_{L_2}^{L_3} \varrho q_3(x) f^2 \overline{u_x} dx \right\}$$
$$+ \mathrm{Re}\left\{ \int_{L_3}^{\tilde{L}_4} q_3(x) f^2 \overline{\left(\tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds \right)} dx \right\}$$
$$- \mathrm{Re}\left\{ i\lambda \int_{\tilde{L}_2}^{L_2} b_0 q_3(x) v \overline{v_x} dx \right\} + \epsilon\|U\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Using Cauchy-Schwarz inequality and Poincaré inequality in the above equation, we obtain

$$\int_{L_2}^{\tilde{L}_3} q_3'(x) \left(|v|^2 + \left| \tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds \right|^2 \right) dx$$
$$\lesssim \int_{\tilde{L}_2}^{L_2} \left(|v|^2 + |\varrho u_x + b_0 v_x|^2 \right) dx + \int_{\tilde{L}_3}^{\tilde{L}_4} \left(|v|^2 + \left| \tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds \right|^2 \right) dx$$
$$+ \int_{\tilde{L}_2}^{L_2} |f^2|^2 dx + \int_{\tilde{L}_2}^{L_2} |\varrho u_x + b_0 v_x|^2 dx + \left| i\lambda \int_{\tilde{L}_2}^{L_2} b_0 q_3(x) v \overline{v_x} dx \right|$$
$$+ \int_{L_3}^{\tilde{L}_4} |f^2|^2 dx + \int_{L_3}^{\tilde{L}_4} \left| \tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds \right|^2 dx + \epsilon\|U\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Following a similar approach to Rivera et al. ([3]), we have the following estimate

$$\left| i\lambda \int_{\tilde{L}_2}^{L_2} b_0 q_3(x) v \overline{v_x} dx \right| \lesssim |\lambda|^{1/2} \left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2} \right).$$

Therefore, from the above inequality and using (3.83) and the Lemmas 3.4.1-3.4.2, we obtain

$$\int_{L_2}^{\tilde{L}_3} \left(|v|^2 + \left| \tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot,s)ds \right|^2 \right) dx$$
$$\lesssim |\lambda|^{1/2} \left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2} \right) + \|F\|_{\mathcal{H}}^2$$
$$+ \epsilon\|U\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large and $\epsilon > 0$ small. □

To address the next lemmas, let us consider $[\tilde{L}_3, \tilde{L}_4] \subset (L_3, L_4)$. Also, we fix $q_4 \in C^1([0,L])$, where $q_4(x) = 0$ for all $x \in [0, L_3] \cup \{L\}$ and $q_4'(x) = q_1 > 0$ for all $x \in [\tilde{L}_3, L]$.

Lemma 3.4.6. *For $|\lambda|$ large and $\epsilon > 0$ small, we have*

$$\int_{\tilde{L}_3}^{L} \left(|v|^2 + \left| \tilde{a}(\cdot) u_x + \int_0^\infty a(\cdot) g(s) \eta_x(\cdot, s) ds \right|^2 \right) dx$$
$$\lesssim |\lambda|^{1/2} \left(\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4} \|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2} \|U\|_{\mathcal{H}}^{1/2} \right) + \|F\|_{\mathcal{H}}^2$$
$$+ \epsilon \|U\|_{\mathcal{H}}^2.$$

Proof. Using the result of Lemma 3.4.4 with $q(x) = q_4(x)$ and definition of $W(u, \eta)$, we get

$$\int_{\tilde{L}_3}^{L} q_4'(x) \left(|v|^2 + \left| \tilde{a}(\cdot) u_x + \int_0^\infty a(\cdot) g(s) \eta_x(\cdot, s) ds \right|^2 \right) dx$$
$$\lesssim - \int_{L_3}^{\tilde{L}_3} q_4'(x) \left(|v|^2 + \left| \tilde{a}(\cdot) u_x + \int_0^\infty a(\cdot) g(s) \eta_x(\cdot, s) ds \right|^2 \right) dx$$
$$+ \operatorname{Re} \left\{ \int_{L_3}^{L} q_4(x) f^2 \overline{\left(\tilde{a}(\cdot) u_x + \int_0^\infty a(\cdot) g(s) \eta_x(\cdot, s) ds \right)} dx \right\}$$
$$+ \epsilon \|U\|_{\mathcal{H}}^2 + C(\epsilon) \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \tag{3.103}$$

Using Cauchy-Schwarz inequality, Holder's inequality and Lemma 3.4.1, we have

$$\left| \operatorname{Re} \left\{ \int_{L_3}^{L} q_4(x) f^2 \overline{\left(\tilde{a}(\cdot) u_x + \int_0^\infty a(\cdot) g(s) \eta_x(\cdot, s) ds \right)} dx \right\} \right|$$
$$\lesssim \left| \int_{L_3}^{L_4} q_4(x) f^2 \overline{\left(\tilde{a}(\cdot) u_x + \int_0^\infty a(\cdot) g(s) \eta_x(\cdot, s) ds \right)} dx \right|$$
$$+ \left| \int_{L_4}^{L} \varrho q_4(x) f^2 \overline{u_x} dx \right|$$
$$\lesssim \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2. \tag{3.104}$$

Using (3.104) and Lemma 3.4.5 in inequality (3.103), we get

$$\int_{\tilde{L}_3}^{L} \left(|v|^2 + \left| \tilde{a}(\cdot) u_x + \int_0^\infty a(\cdot) g(s) \eta_x(\cdot, s) \right|^2 ds \right) dx$$
$$\lesssim |\lambda|^{1/2} \left(\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4} \|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2} \|U\|_{\mathcal{H}}^{1/2} \right) + \|F\|_{\mathcal{H}}^2$$
$$+ \epsilon \|U\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large and $\epsilon > 0$ small. $\square$

The main result of this section is given by the following theorem.

Theorem 3.4.1. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before and assume the hypotheses in* (3.4). *Then, the EKEME model is not exponentially stable. Moreover, the semigroup $e^{t\mathcal{A}}$ of system* (3.13) *decays polynomially with the rate t^{-2}, that is*

$$\|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} \lesssim t^{-2} \|U_0\|_{D(\mathcal{A})}, \quad \forall\, U_0 \in D(\mathcal{A}), \quad t \geq 1,$$

Proof. The proof is based on Theorem 1.5.4. Using the same arguments as in the proof Theorem 3.3.1 we can show that $i\mathbb{R} \subset \rho(\mathcal{A})$.

Now, let $F \in \mathcal{H}$, consider $U = (u, v, \eta(\cdot, s))$ solution of $(i\lambda I - \mathcal{A})U = F$, i.e, the system (3.32)-(3.35) is satisfied. To show the polynomial decay with the rate t^{-2}, according to Borichev and Tomilov's Theorem (see Borichev and Tomilov [5]), is sufficient to show

$$\|U\|_{\mathcal{H}} \lesssim |\lambda|^{1/2}\|F\|_{\mathcal{H}},$$

for $|\lambda|$ large.
Using the assumptions in (3.4) as was done in (3.76), we have

$$\begin{aligned}
&\int_{L_2}^{L} (|v|^2 dx + \tilde{a}(\cdot)|u_x|^2) dx \\
&\lesssim \int_{L_2}^{L} \left(|v|^2 + \left| \tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot, s)ds \right|^2 \right) dx \\
&\quad + \tilde{g}^{1/2} a_0 \int_{L_3}^{L_4} \int_0^\infty g(s)|\eta_x(\cdot, s)|^2 ds dx. \qquad (3.105)
\end{aligned}$$

Therefore, from the above equation, Lemmas 3.4.2-3.4.3, Lemmas 3.4.5-3.4.6 and Equations(3.80)-(3.81), we obtain

$$\begin{aligned}
\|U\|_{\mathcal{H}}^2 &= \int_0^{L_2} \left(|v|^2 + \tilde{a}(\cdot)|u_x|^2\right) dx + \int_{L_2}^{L} \left(|v|^2 + \tilde{a}(\cdot)|u_x|^2\right) dx + \int_0^L \int_0^\infty a(\cdot)g(s)|\eta_x(\cdot, s)|^2 ds dx \\
&\lesssim |\lambda|^{1/2} \left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2} \right) + \|F\|_{\mathcal{H}}^2,
\end{aligned}$$

for $|\lambda|$ large. Thus, using Young's inequality, we obtain

$$\|U\|_{\mathcal{H}}^2 \lesssim \left(|\lambda| + |\lambda|^{4/5} + |\lambda|^{2/3} + 1\right) \|F\|_{\mathcal{H}}^2 \lesssim |\lambda| \|F\|_{\mathcal{H}}^2$$

for $|\lambda|$ large. Therefore, we get

$$\|U\|_{\mathcal{H}} \lesssim |\lambda|^{1/2}\|F\|_{\mathcal{H}},$$

for $|\lambda|$ large. □

3.5 Optimality of the decay rates

In this section we show that the decay rate obtained in Theorem 3.4.1 is the best. We recall that Theorem 3.4.1 was proved for the EKEME model, but the proof is similar for the cases where at least one of the elastic parts of the string is connected only with the Kelvin-Voigt damping leading to the same decay rate. In order to prove the optimality it will be necessary to separate in two cases. The first case is the EKEME model, already introduced, considering in problem (3.1)-(3.3) that

$$b(x) = b_0 \chi_{[L_1, L_2](x)} \text{ and } a(x) = a_0 \chi_{[L_3, L_4](x)}, \quad a_0, b_0 > 0.$$

and therefore we assume that the supports of the viscoelastic dampings do not intersect. In the second case, we consider

$$b(x) = b_0 \chi_{[L_1, L_3]}(x) \text{ and } a(x) = a_0 \chi_{[L_2, L_4]}(x), \quad a_0, b_0 > 0.$$

Thus, both the Kelvin-Voigt and memory type dampings are effective in the interval (L_2, L_3). We refer to this model as EKIME model to emphasize the intersection of the supports of the viscoelastic dampings. The EKEME and EKIME model are shown in figure 3.1.

Theorem 3.5.1. *Assume that the kernel $g(t) = g_0 e^{-c_0 t}$ satisfies the hypotheses in (3.4). Then, the polynomial decay rate obtained in Theorem 3.4.1 for the EKEME model, which is also valid for the EKIME model, is optimal in the sense that the semigroup does not decay with the rate t^{-s} for $s > 2$.*

To prove the above theorem, we need to state and prove some important results.
Let $\lambda \in \mathbb{R}$ and $F = (f^1, f^2, f^3(\cdot, s))^\top \in \mathcal{H}$. In what follows, the stationary problem

$$(i\lambda I - \mathcal{A})U = F, \tag{3.106}$$

will be considered several times. Note that $U = (u, v, \eta(\cdot, s))$ is a solution of this problem if the following equations are satisfied:

$$i\lambda u - v = f^1, \tag{3.107}$$

$$i\lambda v - \left(\tilde{a}(\cdot)u_x + \int_0^\infty a(\cdot)g(s)\eta_x(\cdot, s)ds) + b(\cdot)v_x\right)_x = f^2, \tag{3.108}$$

$$i\lambda\eta(\cdot, s) + \eta_s(\cdot, s) - v = f^3(\cdot, s), \tag{3.109}$$

with the following boundary conditions

$$u(0) = u(L) = v(0) = v(L) = 0, \eta(\cdot, 0) = 0 \text{ in } (0, L),\ \eta(0, s) = 0 \text{ in } (0, \infty). \tag{3.110}$$

And

$$b(x) = b_0 \chi_{[L_1, L_2](x)} \text{ and } a(x) = a_0 \chi_{[L_3, L_4](x)},$$

Here, consider the following relaxation function g that satisfies the conditions in (3.4)

$$g(s) = g_0 e^{-c_0 s}, \quad \forall s \geq 0. \tag{3.111}$$

For $i = 1, \ldots, 5$, we consider

$$u_i = u\chi_{[L_{i-1}, L_i]}, \quad L_0 = 0, \quad L_5 = L.$$

We have the following lemma

Lemma 3.5.1. *Consider $F = (0, (\varrho h)\chi_{(0,L_1)}(x), 0)^T$, in system (3.106), for a function h to be chosen later. Then, the system (3.106) can be written as*

$$\begin{array}{ll} (u_1)_{xx} + \ell_1^2 u_1 = -h(x), & x \in (L_0, L_1), \\ (u_i)_{xx} + \ell_i^2 u_i = 0, & x \in (L_{i-1}, L_i),\ i = 2, \ldots, 5 \end{array}$$

with boundary conditions

$$u(0) = u(L) = 0, \quad u_i(L_i) = u_{i+1}(L_i), \quad \sigma_i(u_i)_x(L_i) = \sigma_{i+1}(u_{i+1})_x(L_i), \quad i = 1 \ldots 4.$$

where

$$\ell_1^2 = \ell_5^2 = \frac{\lambda^2}{\varrho}, \quad \ell_2^2 = \frac{\lambda^2}{\varrho + b_0 i\lambda}, \quad \ell_4^2 = \frac{\lambda^2}{\varrho - \frac{a_0 g(0)}{c_0 + i\lambda}};$$

$$\sigma_1 = \sigma_5 = \varrho, \quad \sigma_2 = \varrho + b_0 i\lambda, \quad \sigma_4 = \varrho - \frac{a_0 g(0)}{c_0 + i\lambda}.$$

For case 1:

$$\ell_3^2 = \frac{\lambda^2}{\varrho} \quad \textit{and} \quad \sigma_3 = \varrho.$$

For case 2:

$$\ell_3^2 = \frac{\lambda^2}{\varrho - \frac{a_0 g(0)}{c_0 + i\lambda} + ib_0\lambda} \quad \textit{and} \quad \sigma_3 = \varrho - \frac{a_0 g(0)}{c_0 + i\lambda} + ib_0\lambda.$$

Proof. We will show the proof only for case 1, since case 2 follows in a similar way. Given that $f^1 = f^3(\cdot, s) = 0$ and $f^2 \doteq (\varrho h)\chi_{(0,L_1)}(x)$, where h will be chosen later, then, from (3.109), we deduce

$$\eta(\cdot, s) = u(1 - e^{-i\lambda s}). \tag{3.112}$$

Using (3.112) and that $\tilde{a}(x) = \varrho - a(x)\int_0^\infty g(s)ds,$ for $x \in (L_3, L_4)$, we get

$$\tilde{a}(x)u + a(x)\int_0^\infty g(s)\eta(x,s)ds = (\varrho - a_0 g_\lambda)u, \quad \text{for } x \in (L_3, L_4),$$

where $g_\lambda = \int_0^\infty g(s)e^{-i\lambda s}$. Furthermore, from (3.111), note that

$$\varrho - a_0 g_\lambda = \varrho - \int_0^\infty g(s)e^{-i\lambda s}ds = \varrho - \int_0^\infty g(0)e^{-(c_0+i\lambda)s}ds = \varrho - \frac{a_0 g(0)}{c_0 + i\lambda}.$$

From (3.107) - (3.109), we have $v = i\lambda u$ and

$$\begin{cases} (u_1)_{xx} + \frac{\lambda^2}{\varrho}u_1 = -h, & \text{in} \quad (0, L_1); \\ (u_2)_{xx} + \frac{\lambda^2}{\varrho + ib_0\lambda}u_2 = 0, & \text{in} \quad (L_1, L_2); \\ (u_3)_{xx} + \frac{\lambda^2}{\varrho}u_3 = 0, & \text{in} \quad (L_2, L_3); \\ (u_4)_{xx} + \frac{\lambda^2}{\varrho - a_0 g_\lambda}u_4 = 0, & \text{in} \quad (L_3, L_4); \\ (u_5)_{xx} + \frac{\lambda^2}{\varrho}u_5 = 0, & \text{in} \quad (L_4, L). \end{cases}$$

with transmission condition

$$\begin{aligned} &u_1(L_1) = u_2(L_1), && \varrho u_{1x}(L_1) = \varrho u_{2x}(L_1) + b_0 v_{2x}(L_1), \\ &u_2(L_2) = u_3(L_2), && \varrho u_{2x}(L_2) + b_0 v_{2x}(L_2) = \varrho u_{3x}(L_2), \\ &u_3(L_3) = u_4(L_3), && \varrho u_{3x}(L_3) = \tilde{a}(\cdot)u_{4x}(L_3) + a_0\textstyle\int_0^\infty g(s)\eta_x(\cdot, s)ds, \\ &u_4(L_4) = u_5(L_4), && \tilde{a}(\cdot)u_{4x}(L_4) + a_0\textstyle\int_0^\infty g(s)\eta_x(\cdot, s)ds = \varrho u_{5x}(L_4), \end{aligned}$$

and boudary condition $u_1(0) = 0$, $u_5(L) = 0$. □

The following result will help us find an estimate for the system in Lemma 3.5.1.

Lemma 3.5.2. *Let $h \in L^2(0, L_1)$. For $i = 1, \dots, 5$, let us consider the following system*

$$\begin{array}{ll} (u_1)_{xx} + \ell_1^2 u_1 = -h(x), & x \in (L_0, L_1), \\ (u_i)_{xx} + \ell_i^2 u_i = 0, & x \in (L_{i-1}, L_i),\ i = 2, \dots, 5 \end{array}$$

with boundary conditions

$$u(0) = u(L) = 0, \quad u_i(L_i) = u_{i+1}(L_i), \quad \sigma_i(u_i)_x(L_i) = \sigma_{i+1}(u_{i+1})_x(L_i), \quad i = 1 \dots 4.$$

We have

$$\ell_1^2 = \ell_5^2 = \frac{\lambda^2}{\varrho}, \quad \ell_2^2 = \frac{\lambda^2}{\varrho + b_0 i\lambda}, \quad \ell_4^2 = \frac{\lambda^2}{\varrho - \frac{a_0 g(0)}{c_0 + i\lambda}}; \tag{3.113}$$

$$\sigma_1 = \sigma_5 = \varrho, \quad \sigma_2 = \varrho + b_0 i\lambda, \quad \sigma_4 = \varrho - \frac{a_0 g(0)}{c_0 + i\lambda}. \tag{3.114}$$

For case 1:

$$\ell_3^2 = \frac{\lambda^2}{\varrho} \quad \text{and} \quad \sigma_3 = \varrho.$$

For case 2:

$$\ell_3^2 = \frac{\lambda^2}{\varrho - \frac{a_0 g(0)}{c_0 + i\lambda} + ib_0\lambda} \quad \text{and} \quad \sigma_3 = \varrho - \frac{a_0 g(0)}{c_0 + i\lambda} + ib_0\lambda.$$

The solution of the system is given by

$$u_1(x) = u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} + \frac{1}{\ell_1}\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} H_2(L_1) - \frac{1}{\ell_1} H_2(x). \tag{3.115}$$

where we have the estimate

$$|u(L_1)| \approx \frac{|\sigma_1 H_1(L_1)|}{|W_1 \sin(\ell_1 L_1)|}, \tag{3.116}$$

with $W_1 = \sigma_1 \ell_1 \cot(\ell_1 L_1) - \sigma_2 \ell_2 \cot(\ell_2(L_1 - L_2))$ and

$$H_1(x) = \int_0^x \sin(\ell_1 s) h(s)\, ds, \quad H_2(x) = \int_0^x \sin(\ell_1(x - s)) h(s)\, ds.$$

Proof. Solving the stationary differential equations with boundary conditions $u(0) = u(L) = 0$, we have

$$\begin{aligned}
u_1(x) &= u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} + \frac{1}{\ell_1}\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} H_2(L_1) - \frac{1}{\ell_1} H_2(x), \\
u_2(x) &= u(L_1)\frac{\sin(\ell_2(x - L_2))}{\sin(\ell_2(L_1 - L_2))} + u(L_2)\frac{\sin(\ell_2(x - L_1))}{\sin(\ell_2(L_2 - L_1))}, \\
u_3(x) &= u(L_2)\frac{\sin(\ell_3(x - L_3))}{\sin(\ell_3(L_2 - L_3))} + u(L_3)\frac{\sin(\ell_3(x - L_2))}{\sin(\ell_3(L_3 - L_2))}, \\
u_4(x) &= u(L_3)\frac{\sin(\ell_4(x - L_4))}{\sin(\ell_4(L_3 - L_4))} + u(L_4)\frac{\sin(\ell_4(x - L_3))}{\sin(\ell_3(L_3 - L_4))}, \\
u_5(x) &= u(L_4)\frac{\sin(\ell_5(x - L_5))}{\sin(\ell_5(L_4 - L_5))}.
\end{aligned}$$

Using the transmission conditions $\sigma_1 u_1'(L_1) = \sigma_2 u_2'(L_1)$ we have

$$\sigma_1\ell_1 u(L_1)\cot(\ell_1 L_1) + \sigma_1\left\{\cot(\ell_1 L_1)H_2(L_1) - \frac{1}{\ell_1}H_2'(L_1)\right\}$$
$$= \sigma_2\ell_2 u(L_1)\cot(\ell_2(L_1-L_2)) + \frac{\sigma_2\ell_2 u(L_2)}{\sin(\ell_2(L_2-L_1))}.$$

Since

$$\cot(\ell_1 x)H_2(x) - \frac{1}{\ell_1}H_2'(x) = -\frac{1}{\sin(\ell_1 x)}H_1(x),$$

we obtain

$$\sigma_1\ell_1 u(L_1)\cot(\ell_1 L_1) - \frac{\sigma_1 H_1(L_1)}{\sin(\ell_1 L_1)} = \sigma_2\ell_2 u(L_1)\cot(\ell_2(L_1-L_2)) + \frac{\sigma_2\ell_2 u(L_2)}{\sin(\ell_2(L_2-L_1))},$$

this is

$$W_1 u(L_1) + U_2 u(L_2) = \frac{\sigma_1 H_1(L_1)}{\sin(\ell_1 L_1)}, \tag{3.117}$$

where

$$W_1 = \sigma_1\ell_1\cot(\ell_1 L_1) - \sigma_2\ell_2\cot(\ell_2(L_1-L_2)), \quad U_2 = \frac{\sigma_2\ell_2}{\sin(\ell_2(L_2-L_1))}.$$

Using the transmission conditions $\sigma_2 u_2'(L_2) = \sigma_3 u_3'(L_2)$, we get

$$\frac{\sigma_2\ell_2 u(L_1)}{\sin(\ell_2(L_1-L_2))} + \sigma_2\ell_2 u(L_2)\cot(\ell_2(L_2-L_1))$$
$$= \sigma_3\ell_3 u(L_2)\cot(\ell_3(L_2-L_3)) + \frac{\sigma_3\ell_3 u(L_3)}{\sin(\ell_3(L_3-L_2))}.$$

from where follows that

$$U_2 u(L_1) + W_2 u(L_2) + U_3 u(L_3) = 0, \tag{3.118}$$

where

$$W_2 = \sigma_2\ell_2\cot(\ell_2(L_2-L_1)) - \sigma_3\ell_3\cot(\ell_3(L_2-L_3)), \quad U_3 = \frac{\sigma_3\ell_3}{\sin(\ell_3(L_3-L_2))}.$$

Using the transmission conditions $\sigma_3 u_3'(L_3) = \sigma_4 u_4'(L_3)$ we get

$$\frac{\sigma_3\ell_3 u(L_2)}{\sin(\ell_3(L_2-L_3))} + \sigma_3\ell_3 u(L_3)\cot(\ell_3(L_3-L_2))$$
$$= \sigma_4\ell_4 u(L_3)\cot(\ell_4(L_3-L_4)) + \frac{\sigma_4\ell_4 u(L_4)}{\sin(\ell_4(L_4-L_3))},$$

from where follows that

$$U_3 u(L_2) + W_3 u(L_3) + U_4 u(L_4) = 0, \tag{3.119}$$

where

$$W_3 = \sigma_3 \ell_3 \cot(\ell_3(L_3 - L_2)) - \sigma_4 \ell_4 \cot(\ell_4(L_3 - L_4)), \quad U_4 = \frac{\sigma_4 \ell_4}{\sin(\ell_4(L_4 - L_3))}.$$

Using the transmission conditions $\sigma_4 u_4'(L_4) = \sigma_5 u_5'(L_4)$, we get

$$\frac{\sigma_4 \ell_4 u(L_3)}{\sin(\ell_4(L_3 - L_4))} + \sigma_4 \ell_4 u(L_4) \cot(\ell_4(L_4 - L_3)) = \sigma_5 \ell_5 u(L_4) \cot(\ell_5(L_4 - L_5)),$$

from where follows that

$$U_4 u(L_3) + W_4 u(L_4) = 0, \tag{3.120}$$

where

$$W_4 = \sigma_4 \ell_4 \cot(\ell_4(L_4 - L_3)) - \sigma_5 \ell_5 \cot(\ell_5(L_4 - L_5)).$$

Solving $u(L_3)$ (from (3.119)-(3.120)) in terms of $u(L_2)$, we get

$$\begin{bmatrix} W_3 & U_4 \\ U_4 & W_4 \end{bmatrix} \begin{bmatrix} u(L_3) \\ u(L_4) \end{bmatrix} = \begin{bmatrix} -U_3 u(L_2) \\ 0 \end{bmatrix}$$

this is

$$u(L_3) = \frac{-U_3 W_4}{W_3 W_4 - U_4^2} u(L_2).$$

Solving $u(L_1)$ (from (3.117)-(3.118)), we get

$$\begin{bmatrix} W_1 & U_2 \\ U_2 & W_2 - \frac{U_3^2 W_4}{W_3 W_4 - U_4^2} \end{bmatrix} \begin{bmatrix} u(L_1) \\ u(L_2) \end{bmatrix} = \begin{bmatrix} H_0 \\ 0 \end{bmatrix}, \quad H_0 := \frac{\sigma_1 H_1(L_1)}{\sin(\ell_1 L_1)},$$

this is

$$u(L_1) = \frac{H_0 W_0}{W_1 W_0 - U_2^2}, \quad \text{with} \quad W_0 := W_2 - \frac{U_3^2 W_4}{W_3 W_4 - U_4^2}.$$

We consider $a_0 > 0$, such that

$$\frac{a_i}{\pi} \left(2\pi n + \frac{a_0}{\sqrt{n}} \right) \notin \mathbb{Z}, \quad i = 1, 2, \tag{3.121}$$

where

$$a_1 = \frac{L_3 - L_2}{L_1}, \quad a_2 = \frac{L_5 - L_4}{L_1}.$$

We take $\lambda = \lambda_n$, where

$$\lambda_n := \frac{\sqrt{\varrho}}{L_1} \left(2\pi n + \frac{a_0}{\sqrt{n}} \right) \quad \Rightarrow \quad \lambda_n \approx n.$$

From (3.113)-(3.114), note that

$$\sigma_i \ell_i = \frac{\lambda_n^2}{\ell_i}, \quad i = 1, \ldots, 5.$$

• Our objective will now be to estimate the terms that involve: ℓ_1 and ℓ_5. Note that

$$\ell_1 \approx \ell_5 \approx \lambda_n.$$

Furthermore, $\ell_1 L_1 = 2\pi n + \frac{a_0}{\sqrt{n}}$, we have

$$\sin(\ell_1 L_1) = \sin\left(\frac{a_0}{\sqrt{n}}\right) \approx \lambda_n^{-1/2} \quad \Rightarrow \quad \sigma_1 \ell_1 \cot(\ell_1 L_1) \approx \lambda_n^{3/2}.$$

• Our objective will now be to estimate the terms that involve: ℓ_2 e ℓ_4 .
Note that, if

$$\ell = x + iy \quad \text{and} \quad \ell^2 = A + iB = x^2 - y^2 + i2xy,$$

then

$$x^2 = \frac{A + \sqrt{A^2 + B^2}}{2}, \quad y^2 = \frac{-A + \sqrt{A^2 + B^2}}{2} \overset{A \geq 0}{=} \frac{B^2}{2(A + \sqrt{A^2 + B^2})}.$$

Since

$$\ell_2^2 = \frac{\lambda_n^2}{\varrho + b_0 i \lambda_n} = \frac{\lambda_n^2(\varrho - b_0 i \lambda_n)}{\varrho^2 + (b_0 \lambda_n)^2} = A_2 + iB_2,$$

we have $A_2 \approx 1$ and $B_2 \approx \lambda_n$. Then

$$\text{Re}\ell_2 \approx \lambda_n^{1/2}, \quad \text{Im}\ell_2 \approx \lambda_n^{1/2}. \tag{3.122}$$

Since

$$\begin{aligned}
\ell_4^2 = \frac{\lambda_n^2}{\varrho - \frac{a_0 g(0)}{c_0 + i\lambda_n}} &= \frac{\lambda_n^2(c_0 + i\lambda_n)}{(\varrho c_0 - a_0 g(0)) + i\varrho\lambda_n} \left[\frac{(\varrho c_0 - a_0 g(0)) - i\beta\lambda_n}{(\varrho c_0 - a_0 g(0)) - i\varrho\lambda_n}\right] \\
&= \frac{\lambda_n^2(c_0(\varrho c_0 - a_0 g(0)) + (\varrho\lambda_n)^2 - i a_0 g(0)\lambda_n)}{(\varrho c_0 - a_0 g(0))^2 + (\varrho\lambda_n)^2} \\
&= A_4 + iB_4,
\end{aligned}$$

we have that $A_4 \approx \lambda_n^2$ and $B_4 \approx \lambda_n$. Then

$$\text{Re}\ell_4 \approx \lambda_n, \quad \text{Im}\ell_4 \approx 1.$$

On the other hand, since $\sigma_i \ell_i = \frac{\lambda_n^2}{\ell_i}$, $i = 2, 4$, from the previous estimates we have

$$\sigma_2 \ell_2 \approx \lambda_n^{3/2}(1 + i), \quad \sigma_4 \ell_4 \approx \lambda_n + i.$$

In that follows, we use the identities

$$\begin{aligned}
|\sin(\mu + i\eta)|^2 &= \sin^2(\mu) + \sinh^2(\eta), \\
\cot(\mu + i\eta) &= \frac{\cos(\mu)\sin(\mu) - i\cosh(\eta)\sinh(\eta)}{\cosh^2(\eta) - \cos^2(\mu)}.
\end{aligned} \tag{3.123}$$

Since $\ell_2 \approx \lambda_n^{1/2}(1+i)$ we can write $\ell_2(L_2 - L_1) = \lambda_n^{1/2}(\mu_{2,n} + i\eta_{2,n})$ with $\mu_{2,n} \approx 1 \approx \eta_{2,n}$. Then, we have

$$|\sin(\ell_2(L_2 - L_1))| \gtrsim |\sinh(\lambda_n^{1/2}\eta_{2,n})| \quad \Rightarrow \quad |U_2| \lesssim \frac{1}{|\sinh(\lambda_n^{1/2}\eta_{2,n})|} \to 0.$$

Moreover, we also have

$$\cot(\ell_2(L_2 - L_1)) \approx i \quad \Longrightarrow \quad \sigma_2\ell_2 \cot(\ell_2(L_2 - L_1)) \approx \lambda_n^{3/2}(1+i). \tag{3.124}$$

Since $\ell_4 \approx \lambda_n + i$ we can write $\ell_4(L_4 - L_3) = \lambda_n\mu_{4,n} + i\eta_{4,n}$ with $\mu_{4,n} \approx 1 \approx \eta_{4,n}$. Then

$$|\sin(\ell_4(L_4 - L_3))| \approx 1 \quad \Rightarrow \quad |U_4| \approx \lambda_n.$$

Writing $\cot(\ell_4(L_4 - L_3)) = p_n + iq_n$, we have

$$|p_n| \lesssim 1, \quad q_n \approx 1.$$

Since $\sigma_4\ell_4 \approx \lambda_n + i$, we can write $\sigma_4\ell_4 = r_n\lambda_n + is_n$ where $r_n \approx 1 \approx s_n$, then

$$\begin{aligned}\sigma_4\ell_4 \cot(\ell_4(L_4 - L_3)) &= (r_n\lambda_n + is_n)(p_n + iq_n)\\ &= \lambda_n r_n p_n - s_n q_n + i(\lambda_n r_n q_n + s_n p_n).\end{aligned}$$

From where follows

$$|\mathrm{Re}\,\{\sigma_4\ell_4 \cot(\ell_4(L_4 - L_3))\}\,| \lesssim \lambda_n, \quad |\mathrm{Im}\,\{\sigma_4\ell_4 \cot(\ell_4(L_4 - L_3))\}\,| \approx \lambda_n$$

Therefore

$$|\sigma_4\ell_4 \cot(\ell_4(L_4 - L_3))| \approx \lambda_n.$$

• Our objective will now be to estimate the terms that involve ℓ_3.
As ℓ_3 is different for cases 1 and 2, we need to analyze these cases and the terms that depend on it separately.
Case 1: we have $\ell_3 \approx \lambda_n$. Since $\ell_3(L_3 - L_2) = a_1\left(2\pi n + \frac{a_0}{\sqrt{n}}\right)$ from the condition (3.121) we infer the sequence $\sin(\ell_3(L_3 - L_2))$ is never zero. We can assume that $\sin(2a_1\pi n) \to 0$, otherwise we consider a subsequence. Then, from the identity

$$\sin\left(2a_1\pi n + \frac{a_1 a_0}{\sqrt{n}}\right) = \sin\left(2a_1\pi n\right)\cos\left(\frac{a_1 a_0}{\sqrt{n}}\right) + \cos(2a_1\pi n)\sin\left(\frac{a_1 a_0}{\sqrt{n}}\right),$$

we can conclude that

$$\Lambda_{1,n} := \sin(\ell_3(L_3 - L_2)) \approx \frac{1}{\sqrt{\lambda_n}} \quad \Rightarrow \quad |U_3| \approx \frac{\lambda_n}{|\Lambda_{1,n}|}.$$

Furthermore, note that

$$|\sigma_3\ell_3 \cot(\ell_3(L_3 - L_2))| \approx \frac{\lambda_n}{|\Lambda_{1,n}|} \approx \lambda_n^{3/2}.$$

Case 2: note that

$$
\begin{aligned}
\ell_3^2 &= \frac{\lambda_n^2}{\varrho - \frac{a_0 g(0)}{c_0 + i\lambda_n} + ib_0\lambda_n} \\
&= \frac{\lambda_n^2(c_0 + i\lambda_n)}{[\varrho c_0 - \lambda_n^2 b_0 - a_0 g(0)] + i\lambda_n[c_0 b_0 + \varrho]} \\
&= \frac{\lambda_n^2\{c_0[\varrho c_0 - \lambda_n^2 b_0 - a_0 g(0)] + \lambda_n^2[c_0 b_0 + \varrho]\}}{[\varrho c_0 - \lambda_n^2 b_0 - a_0 g(0)]^2 + [\lambda_n(c_0 b_0 + \varrho)]^2} \\
&\quad + i\left(\frac{\lambda_n^2\{-c_0\lambda_n(c_0 b_0 + \varrho) + \lambda_n[\varrho c_0 - \lambda_n^2 b_0 - a_0 g(0)]\}}{[\varrho c_0 - \lambda_n^2 b_0 - a_0 g(0)]^2 + [\lambda_n(c_0 b_0 + \varrho)]^2}\right) \\
&= \frac{\lambda_n^2\{c_0[\varrho c_0 - a_0 g(0)] + \lambda_n^2 \varrho\}}{[\varrho c_0 - \lambda_n^2 b_0 - a_0 g(0)]^2 + [\lambda_n(c_0 b_0 + \varrho)]^2} \\
&\quad + i\left(\frac{\lambda_n^2\{-c_0\lambda_n(c_0 b_0 + \varrho) + \lambda_n[\varrho c_0 - \lambda_n^2 b_0 - a_0 g(0)]\}}{[\varrho c_0 - \lambda_n^2 b_0 - a_0 g(0)]^2 + [\lambda_n(c_0 b_0 + \varrho)]^2}\right),
\end{aligned}
$$

then $\text{Re}(\ell_3^2) \approx 1$ and $\text{Im}(\ell_3^2) \approx \lambda_n$. Then

$$
\text{Re}(\ell_3) \approx \lambda_n^{1/2}, \quad \text{Im}(\ell_3) \approx \lambda_n^{1/2}.
$$

On the other hand, since $\sigma_3\ell_3 = \frac{\lambda_n^2}{\ell_3}$, from the previous estimates we have

$$
\cot(\ell_3(L_3 - L_2)) \approx i \quad \Longrightarrow \quad \sigma_3\ell_3\cot(\ell_3(L_3 - L_3)) \approx \lambda_n^{3/2}(1 + i).
$$

In that follows, we use the identities (3.123).
Since $\ell_3 \approx \lambda_n^{1/2}(1 + i)$ we can write $\ell_3(L_3 - L_2) = \lambda_n^{1/2}(\mu_{3,n} + i\eta_{3,n})$ with $\mu_{3,n} \approx 1 \approx \eta_{3,n}$. Then we have

$$
|\sin(\ell_3(L_3 - L_2))| \gtrsim |\sinh(\lambda_n^{1/2}\eta_{3,n})| \quad \Rightarrow \quad |U_3| \lesssim \frac{\lambda_n^{3/2}}{|\sinh(\lambda_n^{1/2}\eta_{3,n})|} \to 0,
$$

• Our objective will now be to estimate the terms that involve: ℓ_1 and ℓ_5. Note that

$$
\ell_1 \approx \ell_5 \approx \lambda_n.
$$

Furthermore, $\ell_1 L_1 = 2\pi n + \frac{a_0}{\sqrt{n}}$, we have

$$
\sin(\ell_1 L_1) = \sin\left(\frac{a_0}{\sqrt{n}}\right) \approx \lambda_n^{-1/2} \quad \Rightarrow \quad \sigma_1\ell_1\cot(\ell_1 L_1) \approx \lambda_n^{3/2}. \tag{3.125}
$$

Now, since $\ell_5(L_5 - L_4) = a_2\left(2\pi n + \frac{a_0}{\sqrt{n}}\right)$, we have

$$
\sin\left(2a_2\pi n + \frac{a_2 a_0}{\sqrt{n}}\right) = \sin(2a_2\pi n)\cos\left(\frac{a_2 a_0}{\sqrt{n}}\right) + \cos(2a_2\pi n)\sin\left(\frac{a_2 a_0}{\sqrt{n}}\right),
$$

From the condition (3.121), note that the sequence

$$
\Lambda_{2,n} := \sin(\ell_5(L_5 - L_4)),
$$

is a non-zero bounded sequence (note that maybe it could converge to zero).

Note that, if $\Lambda_{2,n} \to 0$, then $\cos(2a_2\pi n) \nrightarrow 0$ and we get

$$|\sigma_5\ell_5 \cot(\ell_5(L_5 - L_4))| \approx \frac{\lambda_n}{|\Lambda_{2,n}|}.$$

On the other hand, if $\Lambda_{2,n} \nrightarrow 0$, then

$$|\sigma_5\ell_5 \cot(\ell_5(L_5 - L_4))| \approx \frac{\lambda_n |\cos(2a_2\pi n)|}{|\Lambda_{2,n}|}.$$

• Our objective will now be to estimate the terms: W_1 and W_4.
From the previous estimates, we immediately obtain that $|W_1| \approx \lambda_n^{3/2}$.
If $\Lambda_{2,n} \to 0$ and since $|\Lambda_{2,n}| = |\sin(\ell_5(L_5 - L_4))| \lesssim 1$, we obtain

$$|W_4| = |\sigma_4\ell_4 \cot(\ell_4(L_4 - L_3)) - \sigma_5\ell_5 \cot(\ell_5(L_4 - L_5))| \approx \frac{\lambda_n}{|\Lambda_{2,n}|}.$$

On the other hand, if $\Lambda_{2,n} \nrightarrow 0$, we have

$$\begin{aligned} |W_4| =& |\sigma_4\ell_4 \cot(\ell_4(L_4 - L_3)) - \sigma_5\ell_5 \cot(\ell_5(L_4 - L_5))| \\ \lesssim& \lambda_n\left(1 + \frac{|\cos(2a_2\pi n)|}{|\Lambda_{2,n}|}\right) \\ \lesssim& \lambda_n. \end{aligned}$$

and

$$\begin{aligned} |W_4| =& |\sigma_4\ell_4 \cot(\ell_4(L_4 - L_3)) - \sigma_5\ell_5 \cot(\ell_5(L_4 - L_5))| \\ \gtrsim& \lambda_n\left(1 - \frac{|\cos(2a_2\pi n)|}{|\Lambda_{2,n}|}\right) \\ \gtrsim& \lambda_n. \end{aligned}$$

It is important to note that this last inequality cannot occur if $|W_4| \approx 0$, but in this case, the proof to estimate $|u(L_1)|$ follows in an analogous way.
Then, if $\Lambda_{2,n} \nrightarrow 0$, we have $|W_4| \approx \lambda_n$. So, we get

$$|W_4| \approx \frac{\lambda_n}{|\Lambda_{2,n}|}, \text{ if } \quad \Lambda_{2,n} \to 0 \quad \text{and} \quad |W_4| \approx \lambda_n, \text{ if } \quad \Lambda_{2,n} \nrightarrow 0.$$

• We obtain estimates for W_2 and W_3.
For case 1, we have

$$|W_3| \approx \frac{\lambda_n}{|\Lambda_{1,n}|} \quad \text{and} \quad |W_2| \lesssim \frac{\lambda_n}{|\Lambda_{1,n}|}.$$

For case 2, we have

$$|W_3| \approx \lambda_n^{3/2} \quad \text{and} \quad |W_2| \lesssim \lambda_n^{3/2}.$$

• Now let us estimate the term $u(L_1)$.
For $\Lambda_{2,n} \to 0$. We need to estimate for both cases.
Case 1: Let us first estimate the term

$$W_0 = W_2 - \frac{U_3^2 W_4}{W_3W_4 - U_4^2} = \frac{W_4(W_2W_3 - U_3^2) - W_2U_4^2}{W_3W_4 - U_4^2}. \tag{3.126}$$

Since
$$|W_3||W_4| \approx \frac{\lambda_n^2}{|\Lambda_{1,n}\Lambda_{2,n}|} \quad \text{and} \quad |U_4|^2 \approx \lambda_n^2,$$
We have
$$|W_3W_4 - U_4^2| \approx \frac{\lambda_n^2}{|\Lambda_{1,n}\Lambda_{2,n}|}.$$
On the other hand, note that
$$\begin{aligned}\text{Im}(W_2W_3 - U_3^2) &= \text{Im}(W_2W_3)\\ &= \text{Im}\{AB + (A+B)C + B^2)\}\\ &= \text{Im}\{AB + (A+B)C)\},\end{aligned}$$
where
$$A = \sigma_2\ell_2\cot(\ell_2(L_2 - L_1)), \quad B = \sigma_3\ell_3\cot(\ell_3(L_3 - L_2)), \quad C = \sigma_4\ell_4\cot(\ell_4(L_4 - L_3)),$$
Provided that
$$|\text{Im}(AB)| = |\text{Im}(A)B| \approx \frac{\lambda_n^{5/2}}{|\Lambda_{1,n}|},$$

$$|\text{Im}(A+B)C| \leq |A||C| + |B||C| \lesssim \lambda_n^{5/2} + \frac{\lambda_n^2}{|\Lambda_{1,n}|}$$
From where follows that
$$|W_2W_3 - U_3^2| \gtrsim |\text{Im}(W_2W_3 - U_3^2)| \approx |\text{Im}(A)B| \approx \frac{\lambda_n^{5/2}}{|\Lambda_{1,n}|}.$$
Since
$$|W_4||W_2W_3 - U_3^2| \gtrsim \frac{\lambda_n^{7/2}}{|\Lambda_{1,n}\Lambda_{2,n}|} \quad \text{and} \quad |W_2||U_4|^2 \lesssim \frac{\lambda_n^3}{|\Lambda_{1,n}|},$$
follows
$$|W_4(W_2W_3 - U_3^2) - W_2U_4^2| \ \gtrsim \ \frac{\lambda_n^{7/2}}{|\Lambda_{1,n}\Lambda_{2,n}|},$$
Using all previous estimates we have
$$|W_0| = \frac{|W_4(W_2W_3 - U_3^2) - W_2U_4^2|}{|W_3W_4 - U_4^2|} \gtrsim \lambda_n^{3/2}.$$
Note that
$$u(L_1) = \frac{H_0W_0}{W_1W_0 - U_2^2} = \frac{H_0}{W_1 - \frac{U_2^2}{W_0}}.$$
Since
$$|W_1| \approx \lambda_n^{3/2} \quad \text{and} \quad \frac{|U_2^2|}{|W_0|} \to 0,$$

then we can conclude

$$|u(L_1)| \approx \frac{|H_0|}{|W_1|}.$$

Case 2: Using (3.126) and

$$|W_3||W_4| \approx \frac{\lambda_n^{5/2}}{|\Lambda_{2,n}|} \quad \text{and} \quad |U_4|^2 \approx \lambda_n^2,$$

we have

$$|W_3W_4 - U_4^2| \approx \frac{\lambda_n^{5/2}}{|\Lambda_{2,n}|}.$$

On the other hand, since $|U_3| \to 0$, note that

$$\begin{aligned} \text{Im}(W_2W_3 - U_3^2) &\approx \text{Im}(W_2W_3) \\ &= \text{Im}\{AB + (A+B)C + B^2 - U_3^2)\} \\ &= \text{Im}\{AB + (A+B)C - U_3^2)\}, \end{aligned}$$

where

$$A = \sigma_2\ell_2 \cot(\ell_2(L_2 - L_1)), \quad B = \sigma_3\ell_3 \cot(\ell_3(L_3 - L_2)), \quad C = \sigma_4\ell_4 \cot(\ell_4(L_4 - L_3)),$$

Since

$$|\text{Im}(AB)| = |\text{Re}(A)\text{Im}(B) + \text{Re}(B)\text{Im}(A)| \approx \lambda_n^3,$$

and

$$|\text{Im}(A+B)C| \leq |A||C| + |B||C| \lesssim \lambda_n^{5/2},$$

from where follows that

$$|W_2W_3 - U_3^2| \gtrsim |\text{Im}(W_2W_3 - U_3^2)| \gtrsim |\text{Im}(AB)| \approx \lambda_n^3.$$

Provided that

$$|W_4||W_2W_3 - U_3^2| \gtrsim \frac{\lambda_n^4}{|\Lambda_{2,n}|} \quad \text{and} \quad |W_2||U_4|^2 \approx \lambda_n^{7/2},$$

follows

$$|W_4(W_2W_3 - U_3^2) - W_2U_4^2| \gtrsim \frac{\lambda_n^4}{|\Lambda_{2,n}|},$$

Using all previous estimates we have

$$|W_0| = \frac{|W_4(W_2W_3 - U_3^2) - W_2U_4^2|}{|W_3W_4 - U_4^2|} \gtrsim \lambda_n^{3/2}.$$

Therefore,

$$\frac{|U_2^2|}{|W_0|} \to 0.$$

For $\Lambda_{2,n} \nrightarrow 0$. We need to estimate for both cases.
Caso 1: Again, note that

$$W_0 = W_2 - \frac{U_3^2 W_4}{W_3 W_4 - U_4^2} = \frac{W_4(W_2 W_3 - U_3^2) - W_2 U_4^2}{W_3 W_4 - U_4^2}.$$

Since

$$|W_3||W_4| \approx \frac{\lambda_n^2}{|\Lambda_{1,n}|} \quad \text{and} \quad |U_4|^2 \approx \lambda_n^2,$$

We have

$$|W_3 W_4 - U_4^2| \approx \frac{\lambda_n^2}{|\Lambda_{1,n}|}.$$

We saw that

$$|W_2 W_3 - U_3^2| \gtrsim \frac{\lambda_n^{5/2}}{|\Lambda_{1,n}|}.$$

Provided that

$$|W_4||W_2 W_3 - U_3^2| \gtrsim \frac{\lambda_n^{7/2}}{|\Lambda_{1,n}|} \quad \text{and} \quad |W_2||U_4|^2 \lesssim \frac{\lambda_n^3}{|\Lambda_{1,n}|},$$

follows

$$|W_4(W_2 W_3 - U_3^2) - W_2 U_4^2| \gtrsim \frac{\lambda_n^{7/2}}{|\Lambda_{1,n}|}.$$

Using all previous estimates we have

$$|W_0| = \frac{|W_4(W_2 W_3 - U_3^2) - W_2 U_4^2|}{|W_3 W_4 - U_4^2|} \gtrsim \lambda_n^{3/2}.$$

And the conclusion follows as before.
Case 2: Using (3.126) and

$$|W_3||W_4| \approx \lambda_n^{5/2} \quad \text{and} \quad |U_4|^2 \approx \lambda_n^2,$$

we have

$$|W_3 W_4 - U_4^2| \approx \lambda_n^{5/2}.$$

On the other hand, since $|U_3| \to 0$, note that

$$\begin{aligned} \operatorname{Im}(W_2 W_3 - U_3^2) &\approx \operatorname{Im}(W_2 W_3) \\ &= \operatorname{Im}\{AB + (A+B)C + B^2 - U_3^2)\} \\ &= \operatorname{Im}\{AB + (A+B)C - U_3^2)\}, \end{aligned}$$

where

$$A = \sigma_2 \ell_2 \cot(\ell_2(L_2 - L_1)), \quad B = \sigma_3 \ell_3 \cot(\ell_3(L_3 - L_2)), \quad C = \sigma_4 \ell_4 \cot(\ell_4(L_4 - L_3)),$$

Since

$$|\text{Im}(AB)| = |\text{Re}(A)\text{Im}(B) + \text{Re}(B)\text{Im}(A)| \approx \lambda_n^3,$$

and

$$|\text{Im}(A+B)C| \leq |A||C| + |B||C| \lesssim \lambda_n^{5/2},$$

from where follows that

$$|W_2W_3 - U_3^2| \gtrsim |\text{Im}(W_2W_3 - U_3^2)| \approx |\text{Im}(AB)| \approx \lambda_n^3.$$

Provided that

$$|W_4||W_2W_3 - U_3^2| \gtrsim \lambda_n^4 \quad \text{and} \quad |W_2||U_4|^2 \approx \lambda_n^{7/2},$$

follows

$$|W_4(W_2W_3 - U_3^2) - W_2U_4^2| \ \gtrsim \ \lambda_n^4,$$

Using all previous estimates we have

$$|W_0| = \frac{|W_4(W_2W_3 - U_3^2) - W_2U_4^2|}{|W_3W_4 - U_4^2|} \gtrsim \lambda_n^{3/2}.$$

Therefore,

$$\frac{|U_2^2|}{|W_0|} \to 0.$$

□

Lemma 3.5.3. *Take* $h(s) = \sin(\ell s)$ *in the previous Lemma 3.5.2. If* $\ell \in \mathbb{R}$ *is such that* $\sin(\ell L_1) = 0$ *and* $\cos(\ell L_1) = 1$*, then the solution in (3.115) satisfies*

$$u_1(x) = u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} - \frac{1}{\ell_1^2 - \ell^2}\sin(\ell x).$$

Proof. Take $h(s) = \sin(\ell s)$, we have

$$H_1(x) = \int_0^x \sin(\ell_1 s)\sin(\ell s)\ ds, \quad H_2(x) = \int_0^x \sin(\ell_1(x-s))\sin(\ell s)\ ds.$$

Since

$$\sin(\ell s)\sin(\ell_1(x-s)) = \sin(\ell_1 x)\cos(\ell_1 s)\sin(\ell s) - \cos(\ell_1 x)\sin(\ell_1 s)\sin(\ell s)],$$

$$\sin(\ell_1 x)\int_0^x \cos(\ell_1 s)\sin(\ell s)ds = \sin(\ell_1 x)\left[\frac{\ell_1\sin(\ell_1 x)\sin(\ell x) + \ell\cos(\ell_1 x)\cos(\ell x) - \ell}{\ell_1^2 - \ell^2}\right],$$

$$\cos(\ell_1 x)\int_0^x \sin(\ell_1 s)\sin(\ell s)ds = \cos(\ell_1 x)\left[\frac{\ell\sin(\ell_1 x)\cos(\ell x) - \ell_1\cos(\ell_1 x)\sin(\ell x)}{\ell_1^2 - \ell^2}\right].$$

Using the hypothesis, we get

$$H_1(L_1) = \frac{\ell\sin(\ell_1 L_1)\cos(\ell L_1) - \ell_1\cos(\ell_1 L_1)\sin(\ell L_1)}{\ell_1^2 - \ell^2} = \frac{\ell\sin(\ell_1 L_1)}{\ell_1^2 - \ell^2},$$

and

$$\begin{aligned}
H_2(x) =& \frac{\ell_1 \sin(\ell x)[\sin^2(\ell_1 x) + \cos^2(\ell_1 x)]}{\ell_1^2 - \ell^2} \\
& - \frac{\ell \sin(\ell_1 x)[\cos(\ell_1 x)\cos(\ell x) - \cos(\ell_1 x)\cos(\ell x)]}{\ell_1^2 - \ell^2} - \frac{\ell \sin(\ell_1 x)}{\ell_1^2 - \ell^2} \\
=& \frac{\ell_1 \sin(\ell x) - \ell \sin(\ell_1 x)}{\ell_1^2 - \ell^2}.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{\ell_1} & \frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} H_2(L_1) - \frac{1}{\ell_1} H_2(x) \\
&= \frac{1}{\ell_1} \frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} \left[\frac{-\ell \sin(\ell_1 L_1)}{\ell_1^2 - \ell^2} \right] - \frac{1}{\ell_1} \left[\frac{\ell_1 \sin(\ell x) - \ell \sin(\ell_1 x)}{\ell_1^2 - \ell^2} \right] \\
&= -\frac{1}{\ell_1^2 - \ell^2} \sin(\ell x).
\end{aligned}$$

Thereby, we have

$$\begin{aligned}
u_1(x) &= u(L_1) \frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} + \frac{1}{\ell_1} \frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} H_2(L_1) - \frac{1}{\ell_1} H_2(x) \\
&= u(L_1) \frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)} - \frac{1}{\ell_1^2 - \ell^2} \sin(\ell x).
\end{aligned}$$

□

Lemma 3.5.4. *Assuming the same conditions as the Lemma 3.5.3 and considering $\ell = \ell_n = \frac{2n\pi}{L_1}$, we have*

$$Im(u(L_1)) \approx Im \left(\frac{H_0}{W_1} \right).$$

Proof. We know

$$u(L_1) = \frac{H_0}{W_1 - \frac{U_2^2}{W_0}}, \quad \text{with} \quad \left| \frac{U_2^2}{W_0} \right| \to 0.$$

We remember $H_0 := \frac{\sigma_1 H_1(L_1)}{\sin(\ell_1 L_1)}$, where $H_1(L_1) = \frac{\ell \sin(\ell_1 L_1)}{\ell_1^2 - \ell^2}$. Then, from $\ell_1 \approx \ell \approx \lambda_n \approx n$, we have

$$H_0 = \frac{\ell \sigma_1}{\ell_1^2 - \ell^2} \approx \lambda_n^{1/2}.$$

Since $W_1 = \sigma_1 \ell_1 \cot(\ell_1 L_1) - \sigma_2 \ell_2 \cot(\ell_2 (L_1 - L_2))$, then

$$|\text{Im}(W_1)| = |\text{Im}(\sigma_2 \ell_2 \cot(\ell_2 (L_1 - L_2)))| \approx \lambda_n^{3/2}.$$

Let $\tilde{h}_n \doteq H_0$, $W_1 = \mu_n + i\eta_n$, with $\eta_n \to \infty$ as $n \to \infty$ and $\frac{U_2^2}{W_0} = \jmath_{1,n} + i\jmath_{2,n}$ where $\jmath_{1,n}, \jmath_{2,n} \to 0$ as $n \to \infty$. Note that $\tilde{h}_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. From

$$\begin{aligned}
u(L_1) =& \frac{\tilde{h}_n}{(\mu_n + i\eta_n) - (\jmath_{1,n} + i\jmath_{2,n})} \\
=& \frac{\tilde{h}_n}{(\mu_n - \jmath_{1,n}) + i(\eta_n - \jmath_{2,n})} \\
=& \frac{\tilde{h}_n((\mu_n - \jmath_{1,n}) - i(\eta_n - \jmath_{2,n}))}{(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2},
\end{aligned}$$

we get

$$\mathrm{Im}(u_1(L_1)) = \frac{-\tilde{h}_n(\eta_n - \jmath_{2,n})}{(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2}.$$

Note that

$$\frac{-\tilde{h}_n(\eta_n - \jmath_{2,n})}{-h_n\eta_n} = 1 - \frac{\jmath_{2,n}}{\eta_n} \approx 1 \quad \Rightarrow \quad -\tilde{h}_n(\eta_n - \jmath_{2,n}) \approx -\tilde{h}_n\eta_n.$$

and

$$\begin{aligned}\frac{(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2}{\mu_n^2 + \eta_n^2} &= \frac{\mu_n^2 - 2\mu_n\jmath_{1,n} + \jmath_{1,n}^2 + \eta_n^2 - 2\eta_n\jmath_{2,n} + \jmath_{2,n}^2}{\mu_n^2 + \eta_n^2} \\ &= 1 + \frac{-2\mu_n\jmath_{1,n} + \jmath_{1,n}^2 - 2\eta_n\jmath_{2,n} + \jmath_{2,n}^2}{\mu_n^2 + \eta_n^2} \approx 1,\end{aligned}$$

this is,

$$(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2 \approx \mu_n^2 + \eta_n^2.$$

Therefore, we get

$$\mathrm{Im}(u_1(L_1) = \frac{-\tilde{h}_n(\eta_n - \jmath_{2,n})}{(\mu_n - \jmath_{1,n})^2 + (\eta_n - \jmath_{2,n})^2} \approx \frac{-\tilde{h}_n\eta_n}{\mu_n^2 + \eta_n^2} = \mathrm{Im}\left(\frac{H_0}{W_1}\right).$$

□

Lemma 3.5.5. *Let $\ell_1 L_1 = 2n\pi + \frac{a_0}{\sqrt{n}}$. We have*

$$Im\left(\frac{1}{W_1}\right) \approx \lambda_n^{-3/2}.$$

Proof. Let $\ell_1 L_1 = 2n\pi + \frac{a_0}{\sqrt{n}}$. We know that $W_1 = \sigma_1\ell_1\cot(\ell_1 L_1) + \sigma_2\ell_2\cot(\ell_2(L_2 - L_1))$ and we saw in (3.125) and (3.124) that $\sigma_1\ell_1\cot(\ell_1 L_1) \approx \lambda_n^{3/2}$ and $\sigma_2\ell_2\cot(\ell_2(L_2 - L_1)) \approx \lambda_n^{3/2}(1+i)$, then

$$\mathrm{Im}(W_1) \approx \mathrm{Im}(\sigma_2\ell_2\cot(\ell_2(L_2 - L_1))) \approx \lambda_n^{3/2} \quad \text{and} \quad \mathrm{Re}(W_1) \approx \lambda_n^{3/2}.$$

Therefore, we get

$$\mathrm{Im}\left(\frac{1}{W_1}\right) = \frac{-\mathrm{Im}(W_1)}{\mathrm{Re}(W_1)^2 + \mathrm{Im}(W_1)^2} \approx \lambda_n^{-3/2}.$$

□

Lemma 3.5.6. *There exists $(\lambda_n)_n \in \mathbb{R}_+^*$ with $\lambda_n \to \infty$, $(U_n)_n \subset D(\mathcal{A})$ and $(F_n)_n \in \mathcal{H}$ such that $(i\lambda_n I - \mathcal{A})U_n = F_n$ is bounded in $\mathcal{H}$ with*

$$\lambda_n \|U_n\|_{\mathcal{H}}^2 \gtrsim 1, \quad \textit{for } n \textit{ large.}$$

Proof. For $n \in \mathbb{N}$, we consider $\lambda = \lambda_n = \dfrac{\sqrt{\varrho}}{L_1}\left(2n\pi + \dfrac{1}{\sqrt{n}}\right)$, $F_n = (0, (\varrho h_n)\mathbb{1}_{(0,L_1)}, 0) \in \mathcal{H}$, $h_n(s) = \sin(\ell_n s)$, $s \in (0, L_1)$ with $\ell_n = \frac{2n\pi}{L_1}$ and $U_n = (u_n, v_n, \eta_n(\cdot, s)) \in D(\mathcal{A})$. Therefore $v_n = i\lambda_n u_n$ and $\eta_n(\cdot, s) = u_n(1 - e^{-i\lambda_n s})$. In what follows, we will avoid putting the subindex n in some variables that depend on n. Thus, from the lemma

3.5.1, the coordinate u of solution $U = (u, v, \eta(\cdot, s))$ satisfies the hypotheses of Lemma 3.5.2. Moreover, note that λ satisfies the conditions of Lemma 3.5.2. Hence, by Lemmas 3.5.3 and 3.5.4, u_1 satisfies the following estimate

$$|\lambda_n u_1(x)| \\ \gtrsim |\mathrm{Im}(\lambda_n u_1(x))| = \left|\mathrm{Im}\left(\lambda_n u(L_1)\frac{\sin(\ell_1 x)}{\sin(\ell_1 L_1)}\right)\right| = \left|\mathrm{Im}\left(\frac{1}{W_1}\right)\frac{\lambda_n H_0 \sin(\ell_1 x)}{\sin(\ell_1 L_1)}\right|, \quad (3.127)$$

On the other hand

$$\|F_n\|_{\mathcal{H}}^2 = \varrho^2 \|h_n\|_{L^2(0,L_1)}^2 = \varrho^2 \left(\frac{L_1}{2} - \frac{\sin(2\ell_n L_1)}{4\ell_n}\right) \lesssim \varrho^2 \left(\frac{L_1}{2} + \frac{1}{4\ell_n}\right) \approx \frac{\varrho^2 L_1}{2},$$

which implies that F_n is bounded.
Now, note that $\ell_1 \approx \ell \approx \lambda_n \approx n$, and as $\ell_1 L_1 = 2n\pi + \frac{a_0}{\sqrt{n}}$, then

$$\sin(\ell_1 L_1) = \sin\left(\frac{a_0}{\sqrt{n}}\right) \approx \frac{a_0}{\sqrt{n}} \approx \lambda_n^{-1/2},$$

Using in Equation (3.127) the previous estimate, the estimate obtained in Lemma 3.5.5 and the fact that we already deduced in Lemma 3.5.4 that $H_0 \approx \lambda_n^{1/2}$, we get

$$|\lambda_n u_1(x)| \gtrsim \left|\mathrm{Im}\left(\frac{1}{W_1}\right)\frac{\lambda_n H_0 \sin(\ell_1 x)}{\sin(\ell_1 L_1)}\right| \approx \lambda_n^{1/2} |\sin(\ell_1 x)|. \quad (3.128)$$

Since $\|\sin(\ell_1 \cdot)\|_{L^2} \gtrsim \sqrt{L_1}/2$, from Equation (3.128), we conclude

$$\|U_n\|_{\mathcal{H}} \gtrsim \|\lambda_n u_1(x)\|_{L^2} \gtrsim \lambda_n^{1/2},$$

□

Finally, let us prove the main result of the section.

Proof of Theorem 3.5.1. Suppose that the rate t^{-2} is not optimal. Then, we can improve the decay rate, say to $t^{\frac{-2}{1-\epsilon}}$. By Borichev-Tomilov theorem, subsequently $\|U_n\|_{\mathcal{H}} \lesssim |\lambda_n|^{\frac{1-\epsilon}{2}} \|F_n\|_{\mathcal{H}}$. Thus, for n large, we obtain

$$\|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \lesssim |\lambda_n|^{\frac{1-\epsilon}{2}}. \quad (3.129)$$

On the other hand, by Lemma 3.5.6, there exists $(\lambda_n)_n \in \mathbb{R}_+^*$ with $\lambda_n \to \infty$, $(U_n)_n \subset D(\mathcal{A})$ and $(F_n)_n \in \mathcal{H}$ such that $(i\lambda_n I - \mathcal{A})U_n = F_n$ is bounded in $\mathcal{H}$ and

$$\frac{1}{\lambda_n}\|U_n\|_{\mathcal{H}}^2 \gtrsim 1 \Rightarrow \frac{1}{\lambda_n^{1-\epsilon}}\|U_n\|_{\mathcal{H}}^2 \gtrsim \lambda_n^{\epsilon} \Rightarrow \frac{1}{\lambda_n^{1-\epsilon}}\|(i\lambda_n I - \mathcal{A})^1 F_n\|_{\mathcal{H}}^2 \gtrsim \lambda_n^{\epsilon},$$

we get

$$\frac{1}{\lambda_n^{\frac{1-\epsilon}{2}}}\|(i\lambda_n I - \mathcal{A})^1 F_n\|_{\mathcal{H}} \gtrsim \lambda_n^{\frac{\epsilon}{2}} \to +\infty \text{ as } n \to \infty,$$

what is a contradiction with (3.129). Note that this inequality also implies the lack of exponential stability. □

Chapter 4

Asymptotic behavior for a Thermoelastic transmission problem with Kelvin-Voigt damping

4.1 Introduction to the problem

In recent years, there has been a growing interest in studying the dynamical behavior of various thermoelastic problems to better understand the thermo-mechanical interactions in elastic materials (see [12, 22, 28]). Initially, research focused primarily on the dynamical aspects of classical thermoelastic systems, whose one-dimensional linear model is given by:

$$u_{tt} - u_{xx} + b\theta_x = 0, \quad x \in (0, L), \quad t > 0,$$

$$\theta_t - \theta_{xx} + bu_{xt} = 0, \quad x \in (0, L), \quad t > 0,$$

where $u(x, t)$ represents the displacement of the rod at time t, and $\theta(x, t)$ denotes the temperature variation relative to a fixed reference temperature. In the 1960s, Dafermos [13] investigated the existence of solutions for the classical thermoelastic system and demonstrated its.

In this paper, we consider the asymptotic behavior of beams composed of three distinct regions: one made of purely elastic material, another of thermoelastic material with Kelvin–Voigt-type dissipation, and a third composed of thermoelastic material without dissipation.

Due to the presence of three different materials, the density of the beam is, in general, not a continuous function. Moreover, since the stress–strain relationship varies across the regions—for instance, from the thermoelastic part to the purely elastic one, the resulting model is not continuous in the classical sense.

The mathematical problem that describes this situation is known as a transmission problem. From a mathematical point of view, it is modeled by a system of partial differential equations with discontinuous coefficients, requiring appropriate coupling conditions at the interfaces between the different materials.

When thermoelastic dissipation is effective throughout the entire domain of a body, it is sufficiently strong to guarantee an exponential decay rate of the solutions to zero for one-dimensional bodies or plates as time approaches infinity. Examples of this situation can be found in ([23, 35]). For nonlinear problems in one-dimensional thermoelasticity, see also ([34, 37]).

Rivera et al. ([39]) studied the asymptotic stability of a transmission system involving two materials: one thermoelastic and the other insensitive to temperature changes, characterizing a system with localized damping. Due to differences in densities and elastic coefficients, the model involves discontinuous coefficients. The authors showed that, even when thermal dissipation acts only on part of the domain, the solution of the semilinear problem decays exponentially to zero. See also ([10, 12, 35]).

In studies of vibrating systems modeled by wave equations, bars or plates, it is known that Kelvin-Voigt damping mechanisms, when distributed globally, stabilize the solutions of these systems exponentially. Furthermore, this damping mechanism is so strong that it tends to regularize the solutions. The situation may be completely different if this type of damping acts only on a part of the body as was shown by K. Liu and Z. Liu in [24] (see also [11]). These authors proved that if Kelvin-Voigt damping acts locally in a wave equation with discontinuous coefficient then the solutions of the equation are not exponentially stable.

Later, Alves et al [3], studied the stabilizing force that Kelvin-Voigt damping exerts on a transmission problem. This time, two dissipative mechanisms act on different parts of the body. In one part, Kelvin-Voigt damping and in the other, frictional damping. Even with the collaboration of frictional damping, the authors showed that Kelvin-Voigt damping can predominate in the decay of the solutions, not allowing the exponential decay of the solutions. However, the authors showed that the solutions decay polynomially with the optimal decay rate $t^{-1/2}$. In the literature we have not found a study on the behavior of solutions where Kelvin-Voigt damping acts collaboratively with thermoelastic damping and this was what motivated this work.

Problems with localized Kelvin-Voigt damping have aroused the interest of several researchers in the last two decades and several results have been obtained. The problem

$$u_{tt}(x,t) - u_{xx}(x,t) - \big(b(x)u_{xt}(x,t)\big)_x = 0,$$

was studied by Liu and Zhang [27] in the interval $(-1,1)$ (see also [43]). They showed that if the coefficient $b(x)$ is zero in $(-1,0]$, positive in $(0,1)$ and has a behavior like x around zero then the solution of this problem is exponentially stable. Also, if the behavior of $b(x)$ around zero is x^α, $\alpha > 1$, the solution is polynomially stable with a decay rate depending on α. A result with sharp stability $t^{-\frac{2-\alpha}{1-\alpha}}$ were obtained by Han et al in [19] (see also [18, 27]).

When the coefficient $b(x)$ is discontinuous, Liu et al. [24] had shown the solution does not decay exponentially. A few years later, this same problem was studied by Rivera et al. [2] where they showed that the solutions of the system decay polynomially with the optimal rate t^{-2} (see also [17, 20, 29, 40]).

Taking into account the results mentioned above, it is interesting to study the transmission problem with localized viscoelasticity of Kelvin–Voigt type. Here we consider a bar composed of three different components, one of viscoelastic type, one of only an elastic part, and the third of type thermoelastic. The main result of this work is that the position of this component plays an important role in the study of the stabilization. We consider the models shown in Figure 1.1.

In the first case, we consider, we consider the viscolestatic part of Kelvin-Voigt type in $(0, L_1)$. Therefore, in this case, we define

$$a(x) = a_0\chi_{[0,L_1]}(x), \quad a_0 > 0.$$

We refer to this model as the KET model. We show that there is exponential stability for the KET model.

In the second case, we consider that the elastic part of the bar (with no dissipation) connects with only the Kelvin-Voigt damping. In this case, we consider a Kelvin-Voigt damping (K) defined in (L_1, L_2), in this case, we define

$$a(x) = a_0\chi_{[L_1,L_2](x)}, \quad a_0 > 0.$$

We refer to this model as EKT. We show that the solutions of this models dacays to zero polynomially as t^{-2}.

The KET model is

$$\begin{aligned} \delta_1 u^1_{tt} - [\beta_1 u^1_x + a(x)u^1_{xt}]_x &= 0, \quad \text{in} \quad (0, L_1) \times \mathbb{R}^+, \\ \delta_2 u^2_{tt} - \beta_2 u^2_{xx} &= 0, \quad \text{in} \quad (L_1, L_2) \times \mathbb{R}^+, \\ \delta_3 u^3_{tt} - \beta_3 u^3_{xx} + k_0\theta_x &= 0, \quad \text{in} \quad (L_2, L) \times \mathbb{R}^+, \\ \delta_4\theta_t - \beta_4\theta_{xx} + k_0 u^3_{xt} &= 0, \quad \text{in} \quad (L_2, L) \times \mathbb{R}^+, \end{aligned} \tag{4.1}$$

The coefficients δ_i, β_i, k_0 are positive for $i = 1, \cdots, 4$ and $a(x) = a_0\chi_{[0,L_1]}$, where $a_0 > 0$. The transmission conditions are given by

$$\begin{aligned} &u^1(L_1,t) = u^2(L_1,t), \quad (\beta_1 u^1_x + a(x)u^1_{xt})(L_1,t) = \beta_2 u^2_x(L_1,t), \quad t \geq 0, \\ &u^2(L_2,t) = u^3(L_2,t), \quad \beta_2 u^2_x(L_2,t) = \beta_3 u^3_x(L_2,t), \quad t \geq 0. \end{aligned} \tag{4.2}$$

The boundary conditions are

$$u^1(0,t) = 0, \quad u^3(L,t) = 0, \quad \theta(L_2,t) = \theta(L,t) = 0, \quad t \geq 0, \tag{4.3}$$

and the initial data are

$$\begin{aligned} &u^1(x,0) = u^1_0(x), \quad u^1_t(x,0) = u^1_1(x), \quad \text{in} \quad (0, L_1), \\ &u^2(x,0) = u^2_0(x), \quad u^2_t(x,0) = u^2_1(x), \quad \text{in} \quad (L_1, L_2), \\ &u^3(x,0) = u^3_0(x), \quad u^3_t(x,0) = u^3_1(x), \quad \text{in} \quad (L_2, L), \\ &\theta(x,0) = \theta_0(x), \qquad\qquad\qquad\qquad\quad \text{in} \quad (L_2, L). \end{aligned} \tag{4.4}$$

The natural energy of (u^1, u^2, u^3, θ) solution (4.1)-(4.4) and instant $t > 0$ is given by

$$\begin{aligned} 2E_1(t) = &\int_0^{L_1} (\delta_1|u^1_t|^2 + \beta_1|u^1_x|^2)dx + \int_{L_1}^{L_2} (\delta_2|u^2_t|^2 + \beta_2|u^2_x|^2)dx \\ &+ \int_{L_2}^{L} (\delta_3|u^3_t|^2 + \beta_3|u^3_x|^2 + \delta_4|\theta|^2)dx, \end{aligned}$$

and

$$E_1'(t) = -\int_0^{L_1} a_0|u^2_{tx}|^2dx - \int_{L_2}^{L} \beta_4|\theta_x|^2dx. \quad \forall\, t > 0. \tag{4.5}$$

The EKT model is

$$\begin{aligned}
\delta_1 u^1_{tt} - \beta_1 u^1_{xx} &= 0, \quad \text{in} \quad (0, L_1) \times \mathbb{R}^+, \\
\delta_2 u^2_{tt} - [\beta_2 u^2_x + a(x) u^2_{xt}]_x &= 0, \quad \text{in} \quad (L_1, L_2) \times \mathbb{R}^+, \\
\delta_3 u^3_{tt} - \beta_3 u^3_{xx} + k_0 \theta_x &= 0, \quad \text{in} \quad (L_2, L) \times \mathbb{R}^+, \\
\delta_4 \theta_t - \beta_4 \theta_{xx} + k_0 u^3_{xt} &= 0, \quad \text{in} \quad (L_2, L) \times \mathbb{R}^+,
\end{aligned} \tag{4.6}$$

The coefficients δ_i, β_i, k_0 are positive for $i = 1, \cdots, 4$ and $a(x) = a_0 \chi_{[L_1, L_2]}$, where $a_0 > 0$. The transmission conditions are given by

$$\begin{aligned}
u^1(L_1, t) = u^2(L_1, t), &\quad (\beta_2 u^2_x + a(x) u^2_{xt})(L_1, t) = \beta_1 u^1_x(L_1, t), \quad t \geq 0, \\
u^2(L_2, t) = u^3(L_2, t), &\quad (\beta_2 u^2_x + a(x) u^2_{xt})(L_2, t) = \beta_3 u^3_x(L_2, t), \quad t \geq 0.
\end{aligned} \tag{4.7}$$

The boundary conditions are

$$u^1(0, t) = 0, \quad u^3(L, t) = 0, \quad \theta(L_2, t) = \theta(L, t) = 0, \quad t \geq 0, \tag{4.8}$$

and the initial data are

$$\begin{aligned}
&u^1(x, 0) = u^1_0(x), \quad u^1_t(x, 0) = u^1_1(x), && \text{in} \quad (0, L_1), \\
&u^2(x, 0) = u^2_0(x), \quad u^2_t(x, 0) = u^2_1(x), && \text{in} \quad (L_1, L_2), \\
&u^3(x, 0) = u^3_0(x), \quad u^3_t(x, 0) = u^3_1(x), && \text{in} \quad (L_2, L), \\
&\theta(x, 0) = \theta_0(x), && \text{in} \quad (L_2, L).
\end{aligned} \tag{4.9}$$

The natural energy of (u^1, u^2, u^3, θ) solution (4.6)-(4.9) and instant $t > 0$ is given by

$$\begin{aligned}
2E_2(t) = &\int_0^{L_1} (\delta_1 |u^1_t|^2 + \beta_1 |u^1_x|^2) dx + \int_{L_1}^{L_2} (\delta_2 |u^2_t|^2 + \beta_2 |u^2_x|^2) dx \\
&+ \int_{L_2}^{L} (\delta_3 |u^3_t|^2 + \beta_3 |u^3_x|^2 + \delta_4 |\theta|^2) dx,
\end{aligned}$$

and

$$E_2'(t) = -\int_{L_1}^{L_2} a_0 |u^2_{tx}|^2 dx - \int_{L_2}^{L} \beta_4 |\theta_x|^2 dx. \quad \forall\, t > 0. \tag{4.10}$$

Therefore, it is worth highlighting that the energy is a non-increasing function of the time variable t.

The remainder of this chapter is organized as follows: In Section 4.2, we establish the well-posedness of the corresponding models. In Section 4.3, we demonstrate that the associated semigroup is exponentially stable, provided the viscous component is not located at the midpoint of the beam. Finally, in Section 4.4, we prove that, in the absence of exponential stability, the semigroup decays polynomially to zero at the rate of t^{-2}.

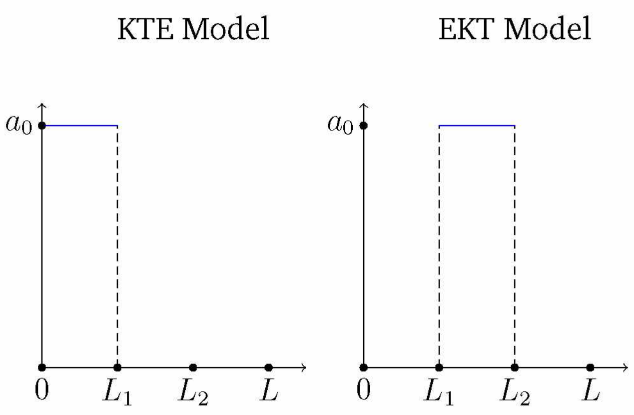

Figure 4.1: Geometric description of the function a(x)

4.2 Existence of solutions

In this section, we use the semigroup approach to show the well-posedness of systems (4.6)-(4.9) and (4.1)-(4.4).
Let us define

$$\mathbb{H}^m = H^m(0,L_1)\times H^m(L_1,L_2)\times H^m(L_2,L), \mathbb{L}^2 = L^2(0,L_1)\times L^2(L_1,L_2)\times (L^2(L_2,L))^2,$$
$$\mathbb{H}^1_L = \{(u^1,u^2,u^3)\in \mathbb{H}^1 : u^1(0)=u^3(L)=0, \quad u^1(L_1)=u^2(L_1), \quad u^2(L_2)=u^3(L_2)\}.$$

where $m=1,2$. Under the above conditions, we have that the phase space is given by

$$\mathcal{H} = \mathbb{H}^1_L \times \mathbb{L}^2.$$

The Hilbert space $\mathcal{H}$ is equipped with the inner product defined by

$$\begin{aligned}(U_1,U_2)_{\mathcal{H}} =& (\sqrt{\delta_1}v_1^1, \sqrt{\delta_1}v_2^1) + (\sqrt{\delta_2}v_1^2, \sqrt{\delta_2}v_2^2) + (\sqrt{\delta_3}v_1^3, \sqrt{\delta_3}v_2^3) + (\sqrt{\beta_1}u^1_{1,x}, \sqrt{\beta_1}u^1_{2,x})\\ &+ (\sqrt{\beta_2}u^2_{1,x}, \sqrt{\beta_2}u^2_{2,x}) + (\sqrt{\beta_3}u^3_{1,x}, \sqrt{\beta_3}u^3_{2,x}) + (\sqrt{\beta_4}\theta_1, \sqrt{\beta_4}\theta_2),\end{aligned}$$

for $U_i = (u_i^1,u_i^2,u_i^3,v_i^1,v_i^2,v_i^3,\theta_i)\in\mathcal{H},\ i=1,2$. The $(\cdot,\cdot)$ denotes the inner product in L^2.
We use $\|U\|_{\mathcal{H}}$ to denote the corresponding norm.

We define the unbounded linear operator $\mathcal{A}_i : D(\mathcal{A}_i)\subset\mathcal{H}\to\mathcal{H}$, for $i=1,2$, by

$$\mathcal{A}_1(U) = \begin{pmatrix} v^1\\ v^2\\ v^3\\ \frac{1}{\delta_1}[\beta_1 u^1_x + a(x)v^1_x]_x\\ \frac{\beta_2}{\delta_2}u^2_{xx}\\ \frac{1}{\delta_3}(\beta_3 u^3_{xx} - k_0\theta_x)\\ \frac{1}{\delta_4}(\beta_4\theta_{xx} - k_0 v^3_x)\end{pmatrix} \quad\text{and}\quad \mathcal{A}_2(U) = \begin{pmatrix} v^1\\ v^2\\ v^3\\ \frac{\beta_1}{\delta_1}u^1_{xx}\\ \frac{1}{\delta_2}[\beta_2 u^2_x + a(x)v^2_x]_x\\ \frac{1}{\delta_3}(\beta_3 u^3_{xx} - k_0\theta_x)\\ \frac{1}{\delta_4}(\beta_4\theta_{xx} - k_0 v^3_x)\end{pmatrix}$$

on its domain

$$D(\mathcal{A}_1) = \left\{\begin{array}{l} U = (u^1,u^2,u^3,v^1,v^2,v^3,\theta)^T\in\mathcal{H} \mid (v^1,v^2,v^3)\in\mathbb{H}^1_L,\\ \theta\in H^2(L_2,L)\cap H^1_0(L_2,L),\\ (\beta_1 u^1 + a(x)v^1), u^2, u^3)\in\mathbb{H}^2.\end{array}\right\}$$

and

$$D(\mathcal{A}_2)=\left\{\begin{array}{l} U=(u^1,u^2,u^3,v^1,v^2,v^3,\theta)^T\in\mathcal{H}\mid(v^1,v^2,v^3)\in\mathbb{H}^1_L,\\ \theta\in H^2(L_2,L)\cap H^1_0(L_2,L),\\ (u^1,(\beta_2u^2+a(x)v^2),u^3)\in\mathbb{H}^2.\end{array}\right\}$$

for all $U=(u^1,u^2,u^3,v^1,v^2,v^3,\theta)^T\in D(\mathcal{A}_i)$, for $i=1,2$.

Remark 4.2.1. *In order to verify this equality, note that $\mathcal{A}_1U\in D(\mathcal{A}_1)$ implies $(v^1,v^2,v^3)\in\mathbb{H}^1_L$, $u^2\in H^2(L_1,L_2)$, $(\beta_1u^2+a(x)v^1)\in H^2(0,L_1)$. Moreover, $(\beta_3u^3_x-k_0\theta)=:z_0\in H^1(L_2,L)$, hence $\theta=k_0^{-1}(z-\beta_3u^3_x)\in H^1(L_2,L)$. Now, one can read from $\mathcal{A}_1U\in D(\mathcal{A}_1)$ that $v^3\in H^1(L_2,L)$ and $(\beta_4\theta_x-k_0v^3)=:z_1\in H^1(L_2,L)$ which imply together that $\theta_x=\beta_4^{-1}(k_0v^3+z_1)\in H^1(L_2,L)$, then $\theta\in H^2(L_2,L)$. Finally, one can directly see that $u^3\in H^2(L_2,L)$. Similarly, we can check $D(\mathcal{A}_2)$.*

When the subindex i is removed $\mathcal{A}$ stands for any of the operators $\mathcal{A}_i$. The system (4.6)-(4.9) and (4.1)-(4.4) can be rewritten as an evolution equation in $\mathcal{H}$:

$$U_t=\mathcal{A}U,\ \ U(0)=U_0\ \ t>0, \tag{4.11}$$

where $U(t)=(u^1,u^2,u^3,u^1_t,u^2_t,u^3_t,\theta_t)^T$ and $U(0)=(u^1_0,u^2_0,u^3_0,u^1_1,u^2_1,u^3_1,\theta_0)^T\in\mathcal{H}$.

Proposition 4.2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $\mathcal{A}$ generates a C_0 semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$.*

Proof. Note that the $\mathcal{A}$ is a dissipative operator in the energy space $\mathcal{H}$. In fact, let $U=(u^1,u^2,u^3,v^1,v^2,v^3,\theta)^\top\in D(\mathcal{A})$. Using the inner product in $\mathcal{H}$, integration by parts, the transmission conditions (4.7) and the boundary conditions (4.8), we have

$$\begin{aligned}(\mathcal{A}_2U{,}U)_{\mathcal{H}}&\\ =&\left(\beta_1u^1_{xx},v^1\right)+\left((\beta_2u^2_x+a(x)v^2_x)_x,v^2\right)+\left(\beta_3u^3_{xx}-k_0\theta_x,v^3\right)+\left(\sqrt{\beta_1}v^1_x,\sqrt{\beta_1}u^1_x\right)\\ &+\left(\sqrt{\beta_2}v^2_x,\sqrt{\beta_2}u^2_x\right)+\left(\sqrt{\beta_3}v^3_x,\sqrt{\beta_3}u^3_x\right)+\left(\beta_4\theta_{xx}-k_0v^3_x,\theta\right)\\ =&\left(\sqrt{\beta_1}v^1_x,\sqrt{\beta_1}u^1_x\right)-\overline{\left(\sqrt{\beta_1}v^1_x,\sqrt{\beta_1}u^1_x\right)}+\left(\sqrt{\beta_2}v^2_x,\sqrt{\beta_2}u^2_x\right)-\overline{\left(\sqrt{\beta_2}v^2_x,\sqrt{\beta_2}u^2_x\right)}\\ &+\left(\sqrt{\beta_3}v^3_x,\sqrt{\beta_3}u^3_x\right)-\overline{(v^3_x,\beta_3u^3_x)}+\left(k_0v^3,\theta_x\right)-\overline{(k_0v^3,\theta_x)}\\ &-\left(\sqrt{a_0}v^2_x,\sqrt{a_0}v^2_x\right)-\left(\sqrt{\beta_4}\theta_x,\sqrt{\beta_4}\theta_x\right).\end{aligned}$$

Using $-(z,w)_{L^2}+\overline{(z,w)}_{L^2}=-2\mathrm{Im}(z,w)_{L^2}$, for $z,w\in L^2$, and taking the real part, we get

$$\mathrm{Re}(\mathcal{A}_2U,U)_{\mathcal{H}}=-\left(\sqrt{a_0}v^2_x,\sqrt{a_0}v^2_x\right)-\left(\sqrt{\beta_4}\theta_x,\sqrt{\beta_4}\theta_x\right)\leq 0. \tag{4.12}$$

Similarly, we have that

$$\mathrm{Re}(\mathcal{A}_1U,U)_{\mathcal{H}}=-\left(\sqrt{a_0}v^1_x,\sqrt{a_0}v^1_x\right)-\left(\sqrt{\beta_4}\theta_x,\sqrt{\beta_4}\theta_x\right)\leq 0. \tag{4.13}$$

Therefore, $\mathcal{A}$ is dissipative.

Let us prove $0\in\rho(A_2)$. The case $0\in\rho(A_1)$ follows similarly.

Let $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7)^\top$ in $\mathcal{H}$. We will show that there is unique $U = (u^1, u^2, u^3, v^1, v^2, v^3, \theta)^\top \in D(\mathcal{A}_2)$, such that

$$-\mathcal{A}_2 U = F. \tag{4.14}$$

The solver equation (4.14) in terms of its components is equivalent to the following system of differential equations

$$\begin{aligned}
-v^1 &= f^1, &(4.15)\\
-v^2 &= f^2, &(4.16)\\
-v^3 &= f^3, &(4.17)\\
-\beta_1 u^1_{xx} &= \delta_1 f^4, &(4.18)\\
-(\beta_2 u^2_x + a(x) v^2_x)_x &= \delta_2 f^5, &(4.19)\\
-\left(\beta_3 u^3_{xx} - k_0 \theta_x\right) &= \delta_3 f^6, &(4.20)\\
-\left(\beta_4 \theta_{xx} - k_0 v^3_x\right) &= \delta_4 f^7, &(4.21)
\end{aligned}$$

with the transmission conditions (4.7) and the boundary conditions (4.8).
Thus, it follows from equations (4.15)-(4.17) that we can consider

$$(v^1, v^2, v^3) = (-f^1, -f^2, -f^3) \in \mathbb{H}^1_L.$$

Moreover, using $v^3 = -f^3 \in H^1(L_2, L)$ and the (4.21), we have

$$\theta_{xx} = g^1, \quad \text{with} \quad \theta(L_2) = \theta(L) = 0, \tag{4.22}$$

where $g^1 := -\frac{1}{\beta_4}(k_0 f^3_x + \delta_4 f^7) \in L^2(L_2, L)$. We already know that the bounded problem above has a unique solution $\theta \in H^2(L_2, L) \cap H^1_0(L_2, L)$. Now, from (4.18)-(4.20), we have

$$\begin{aligned}
\beta_1 u^1_{xx} &= g^2, &(4.23)\\
(\beta_2 u^2_x + a(x) f^2_x)_x &= g^3, &(4.24)\\
\beta_3 u^3_{xx} &= g^4, &(4.25)
\end{aligned}$$

with $g^2 := -\delta_1 f^4 \in L^2(0, L_1), g^3 := -\delta_2 f^5 \in L^2(L_1, L_2)$ and $g^4 := (-\delta_3 f^6 + k_0 \theta_x) \in L^2(L_2, L)$.

The objective is to show that the above system has a unique solution $(u^1, u^2, u^2) \in \mathbb{H}^1_L$ and $(u^1, (\beta_2 u^2 + a(x) v^2), u^3) \in \mathbb{H}^2$. To do this, the Lax–Milgram theorem will be used. Define $\mathcal{B} : \mathbb{H}^1_L \times \mathbb{H}^1_L \to \mathbb{C}$ and $F : \mathbb{H}^1_L \to \mathbb{C}$, such that

$$\begin{aligned}
\mathcal{B}(Y_1, Y_2) := (Y_1, Y_2)_{\mathbb{H}^1_L} &= \int_0^{L_1} \beta_1 u^1_x \tilde{u}^1_x dx + \int_{L_1}^{L_2} \beta_2 u^2_x \tilde{u}^2_x dx + \int_{L_2}^{L} \beta_3 u^3_x \tilde{u}^3_x dx,\\
J(Y_2) &= \int_0^{L_1} g^2 \tilde{u}^1 dx + \int_{L_1}^{L_2} (g^3 \tilde{u}^2 - a(x) f^2_x \tilde{u}^2_x) dx + \int_{L_2}^{L} g^4 \tilde{u}^3 dx,
\end{aligned}$$

for all $Y_1 = (u^1, u^2, u^2), Y_2 = (\tilde{u}^1, \tilde{u}^2, \tilde{u}^3) \in \mathbb{H}^1_L$. Note that $\mathcal{B}$ is a sesquilinear form on $\mathbb{H}^1_L \times \mathbb{H}^1_L$. Moreover, $\mathcal{B}$ is a continuous and coercive form on $\mathbb{H}^1_L \times \mathbb{H}^1_L$, because

$$\mathcal{B}(Y_1, Y_2) := \left| (Y_1, Y_2)_{\mathbb{H}^1_L} \right| \lesssim \|Y_1\|_{\mathbb{H}^1_L} \|Y_2\|_{\mathbb{H}^1_L}, \quad \forall Y_1, Y_2 \in \mathbb{H}^1_L,$$

and

$$\mathcal{B}(Y,Y) := (Y,Y)_{\mathbb{H}^1_L} = \|Y\|^2_{\mathbb{H}^1_L}, \quad \forall Y \in \mathbb{H}^1_L.$$

On the other hand, J is an antilinear functional in $\mathbb{H}^1_L$ and using Hölder's inequality, we get J is a continuous functional in $H^1_0(0,L)$. Therefore, by Lax-Milgram theorem we have that

$$\mathcal{B}(Y,\tilde{Y}) = J(\tilde{Y}), \quad \forall\, \tilde{Y} = (\tilde{u}^1,\tilde{u}^2,\tilde{u}^3) \in \mathbb{H}^1_L. \tag{4.26}$$

admits a unique solution $Y = (u^1,u^2,u^3) \in \mathbb{H}^1_L$. In particular, taking $\tilde{Y}_1 = (\tilde{u}^1,0,0), \tilde{Y}_2 = (0,\tilde{u}^2,0)$ and $\tilde{Y}_3 = (0,0,\tilde{u}^3)$ in (4.26) with $\tilde{u}^1 \in C^\infty_0(0,L_1)$, $\tilde{u}^2 \in C^\infty_0(L_1,L_2)$ and $\tilde{u}^3 \in C^\infty_0(L_2,L)$, we obtain $(u^1,(\beta_2 u^2 + a(x)v^2),u^3) \in \mathbb{H}^2$, how we wanted to show. Therefore, there is unique $U = (u^1,u^2,u^3,v^1,v^2,v^3,\theta)^\top \in D(\mathcal{A}_2)$ such that $-\mathcal{A}_2 U = F$. This tells us that $\mathcal{A}_2$ is bijective and there is $\mathcal{A}_2^{-1}$. So, to conclude that $0 \in \rho(\mathcal{A}_2)$, we just need to show that $\mathcal{A}_2^{-1}$ is bounded. For this, as there is only one $U \in D(\mathcal{A}_2)$ such that $-\mathcal{A}_2 U = F$, we need to show $\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}}$. In fact, since $v^1 = -f^1$, $v^2 = -f^2$ and $v^3 = -f^3$, then by Poincaré's inequality, we have

$$\|v^1\|^2_{L^2(0,L_1)} \lesssim \|F\|^2_{\mathcal{H}}, \|v^2\|^2_{L^2(L_1,L_2)} \lesssim \|F\|^2_{\mathcal{H}}, \|v^3\|^2_{L^2(L_2,L)} \lesssim \|F\|^2_{\mathcal{H}}. \tag{4.27}$$

Multiplying (4.23),(4.24),(4.25) by $\overline{u}^1,\overline{u}^2,\overline{u}^3$ respectively, using integration by parts, the transmission and boundary conditions (4.7) -(4.8), Hölder's inequality, Poicanré's inequality, Young's inequality, we get

$$\begin{aligned} \beta_1\|u^1_x\|^2_{L^2(0,L_1)} &\lesssim \|f^4\|^2_{L^2(0,L_1)} \lesssim \|F\|^2_{\mathcal{H}}, \\ \beta_3\|u^3_x\|^2_{L^2(L_2,L)} &\lesssim \|f^6\|^2_{L^2(L_2,L)} + \|\theta\|^2_{L^2(L_2,L)} \lesssim \|F\|^2_{\mathcal{H}}, \end{aligned}$$

and

$$\begin{aligned} \beta_2\|u^2_x\|^2_{L^2(L_1,L_2)} &\lesssim \frac{\beta_1}{2}\|u^1_x\|^2_{L^2(0,L_1)} + \|f^2_x\|^2_{L^2(L_1,L_2)} + \|f^5\|^2_{L^2(L_1,L_2)} \\ &\lesssim \frac{\beta_1}{2}\|u^1_x\|^2_{L^2(0,L_1)} + \|F\|^2_{\mathcal{H}}. \end{aligned}$$

Moreover, from (4.22), we have

$$\beta_4\|\theta\|^2_{L^2(L_2,L)} \lesssim \|g^1\|^2_{L^2(L_1,L_2)} \lesssim \|f^3_x\|^2_{L^2(L_1,L_2)} + \|f^7\|^2_{L^2(L_1,L_2)} \lesssim \|F\|^2_{\mathcal{H}}. \tag{4.28}$$

Therefore, from (4.27)-(4.28), we get $\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}}$.

Lastly, since $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, then $D(\mathcal{A})$ is dense in $\mathcal{A}$. Therefore, the operator $\mathcal{A}$ satisfies the conditions of the Lumer-Phillips Theorem (see Pazy [25]) and the result of the proposition follows. □

The well-posedness of the problem (4.11) is a consequence of the semigroup theory whose result we enunciate to follow.

Theorem 4.2.1. *For $U(0) = (u^1_0,u^2_0,u^3_0,u^1_1,u^2_1,u^3_1,\theta_0) \in \mathcal{H}$ there exists an unique solution of the system (4.6)-(4.9) and (4.1)-(4.4) in the space*

$$U = (u^1,u^2,u^3,u^1_t,u^2_t,u^3_t,\theta) \in C([0,\infty[;\mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A}_i)$, then the solutions belong to the following space

$$U \in C([0,\infty[;D(\mathcal{A}_i)) \cap C^1([0,\infty[;\mathcal{H}).$$

4.3 Asymptotic behavior: Exponential Stability

We will study the asymptotic behavior of the semigroup $e^{t\mathcal{A}}$ associated to the system (4.1)-(4.4). The results will be obtained using the spectral characterizations for exponential stability of semigroups (see [21] or [32]).

It is important to highlight that we will demonstrate the exponential stability of system (4.1)-(4.4) in the case that every elastic part of the string either connects only with the frictional part or connects with both types of dampings. Since the proof of the decay rate is similar in all these cases, we focus on the proof considering the KET model, which is given by (4.1)-(4.4) considering

$$a(x) = a_0\chi_{[0,L_1]}(x), \tag{4.29}$$

where $a_0 > 0$.

The main result of this section is Theorem 4.3.1 and to prove this theorem we will need to introduce some technical lemmas.
Let $\lambda \in \mathbb{R}$ and $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7) \in \mathcal{H}$. In what follows, the stationary problem

$$(i\lambda I - \mathcal{A}_1)U = F, \tag{4.30}$$

will be considered several times. Note that $U = (u^1, u^1, u^3, v^1, v^2, v^3, \theta)$ is a solution of this problem if the following equations are satisfied:

$$i\lambda u^1 - v^1 = f^1, \tag{4.31}$$
$$i\lambda u^2 - v^2 = f^2, \tag{4.32}$$
$$i\lambda u^3 - v^3 = f^3, \tag{4.33}$$
$$i\lambda\delta_1 v^1 - (\beta_1 u^1_x + a(x)v^1_x)_x = \delta_1 f^4, \tag{4.34}$$
$$i\lambda\delta_2 v^2 - \beta_2 u^2_{xx} = \delta_2 f^5, \tag{4.35}$$
$$i\lambda\delta_3 v^3 - \left(\beta_3 u^3_{xx} - k_0\theta_x\right) = \delta_3 f^6, \tag{4.36}$$
$$i\lambda\delta_4\theta - \beta_4\theta_{xx} + k_0 v^3_x = \delta_4 f^7, \tag{4.37}$$

with the transmission conditions (4.2) and the boundary conditions (4.3).
Note that

$$((i\lambda U - \mathcal{A}U), U)_{\mathcal{H}} = i\lambda\|U\|^2_{\mathcal{H}} - (\mathcal{A}U, U)_{\mathcal{H}},$$

so, we have

$$-\text{Re}(\mathcal{A}U, U)_{\mathcal{H}} \lesssim \|(i\lambda U - AU)\|_{\mathcal{H}}\|U\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Therefore, from (4.13), we get

$$\int_0^{L_1} \delta_1 a_0|v^1_x|^2 dx + \int_{L_2}^{L} \beta_4|\theta_x|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.38}$$

Using the above equation and Poicanré's inequality, we have

$$\int_0^{L_1} \delta_1 a_0|v^1|^2 dx \lesssim \int_0^{L_1} \delta_1 a_0|v^1_x|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{4.39}$$

and

$$\int_{L_2}^{L} \delta_4|\theta|^2dx \lesssim \frac{\delta_4}{\beta_4}\int_{L_2}^{L} \beta_4|\theta_x|^2dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.40}$$

From (4.32) and (4.38), we have

$$\int_0^{L_1} \beta_1 a_0|u_x^1|^2dx \lesssim |\lambda|^{-2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\right). \tag{4.41}$$

Lemma 4.3.1. *We have*

$$\int_{L_1}^{L_2} (\delta_2|v^2|^2 + \beta_2|u_x^2|^2)dx \lesssim |u_x^2(L_2)|^2 + |v^2(L_2)|^2 + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Proof. Multiplying (4.35) by $(x-\beta_0)\overline{u_x^2}$ and using (4.32), we have

$$\begin{aligned} -(x-L_2)\delta_2\overline{v_x^2}v^2 - &(x-L_2)\beta_2\overline{u_x^2}u_{xx}^2 \\ &= (x-L_2)\delta_2\overline{u_x^2}f^5 - (x-L_2)\delta_2\overline{f_x^2}v^2, \quad \text{in } [L_1,L_2]. \end{aligned}$$

Using integration by parts in $[L_1,L_2]$, taking real part, we get

$$\begin{aligned} \int_{L_1}^{L_2} (\delta_2|v^2|^2 + \beta_2|u_x^2|^2)dx \lesssim |u_x^2(L_2)|^2 + |v^2(L_2)|^2 &+ \mathrm{Re}\left\{\int_{L_1}^{L_2}(x-L_2)\delta_2\overline{u_x^2}f^5dx\right\} \\ &- \mathrm{Re}\left\{\int_{L_1}^{L_2}(x-L_2)\delta_2\overline{f_x^2}v^2dx\right\}. \end{aligned}$$

Note that $|x-L_2| \leq L_2$, for all $x \in [L_1,L_2]$, using Holder's inequality in the above equation, we get

$$\int_{L_1}^{L_2} (\delta_2|v^2|^2 + \beta_2|u_x^2|^2)dx \lesssim |u_x^2(L_2)|^2 + |v^2(L_2)|^2 + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

□

Lemma 4.3.2. *For $\epsilon > 0$ small, we have*

$$|u_x^3(L_2)|^2 + |u_x^3(L)|^2 + |v^3(L_2)|^2 \lesssim \int_{L_2}^{L} (\delta_3|v^3|^2 + \beta_3|u_x^3|^2)dx + \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Proof. Multiplying (4.36) by $\left(x-\frac{L_2+L}{2}\right)\overline{u_x^3}$ and using (4.33), we have

$$\begin{aligned} -\left(x-\frac{L_2+L}{2}\right)\delta_3\overline{v_x^3}v^3 - \left(x-\frac{L_2+L}{2}\right)\beta_3\overline{u_x^3}u_{xx}^3 + \left(x-\frac{L_2+L}{2}\right)k_0\overline{u_x^3}\theta_x \\ = \left(x-\frac{L_2+L}{2}\right)\delta_3\overline{u_x^3}f^6 - \left(x-\frac{L_2+L}{2}\right)\delta_3\overline{f_x^3}v^3, \quad \text{in } [L_2,L]. \end{aligned}$$

Using integration by parts in $[L_2,L]$, $v^3(L)=0$ and taking real part, we obtain

$$\begin{aligned} &|u_x^3(L_2)|^2 + |u_x^3(L)|^2 + |v^3(L_2)|^2 \\ &\quad \lesssim \int_{L_2}^{L}(\delta_3|v^3|^2 + \beta_3|u_x^3|^2)dx - \mathrm{Re}\left\{\int_{L_2}^{L} k_0\left(x-\frac{L_2+L}{2}\right)\overline{u_x^3}\theta_xdx\right\} \\ &\quad + \mathrm{Re}\left\{\int_{L_2}^{L}\delta_3\left(x-\frac{L_2+L}{2}\right)\overline{u_x^3}f^6dx\right\} \\ &\quad - \mathrm{Re}\left\{\int_{L_2}^{L}\delta_3\left(x-\frac{L_2+L}{2}\right)\overline{f_x^3}v^3dx\right\}. \end{aligned} \tag{4.42}$$

Note that $\left|x - \frac{L_2+L}{2}\right| \le \frac{L-L_2}{2}$ for all $x \in [L_2, L]$ and using Holder's inequality, Young's inequality and (4.38), we get

$$\begin{aligned}\text{Re}\left\{\int_{L_2}^{L} k_0\left(x - \frac{L_2+L}{2}\right)\overline{u_x^3}\theta_x dx\right\} &\lesssim \frac{L-L_2}{2}\int_{L_2}^{L} k_0|\overline{u_x^3}\theta_x|dx\\ &\lesssim \epsilon\int_{L_2}^{L}|u_x^3|^2dx + C(\epsilon)\int_{L_2}^{L}|\theta_x|^2dx\\ &\lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \end{aligned} \tag{4.43}$$

for $\epsilon > 0$ small.

Therefore, using (4.43) and Holder's inequality to estimate the other terms on the right-hand side of inequality (4.42), we obtain

$$\begin{aligned}&|u_x^3(L_2)|^2 + |u_x^3(L)|^2 + |v^3(L_2)|^2\\ &\qquad \lesssim \int_{L_2}^{L}(\delta_3|v^3|^2 + \beta_3|u_x^3|^2)dx + \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},\end{aligned}$$

for $\epsilon > 0$ small. □

Lemma 4.3.3. *For $\epsilon > 0$ small, we have*

$$\int_{L_1}^{L_2}(\delta_2|v^2|^2 + \beta_2|u_x^2|^2)dx \lesssim \int_{L_2}^{L}(\delta_3|v^3|^2 + \beta_3|u_x^3|^2)dx + \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Proof. Using Lemmas 4.3.1 and (4.3.2) and the transmission conditions, we obtain

$$\begin{aligned}\int_{L_1}^{L_2}(\delta_2|v^2|^2 + \beta_2|u_x^2|^2)dx &\lesssim |u_x^2(L_2)|^2 + |v^2(L_2)|^2 + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\\ &\lesssim |u_x^3(L_2)|^2 + |v^3(L_2)|^2 + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\\ &\lesssim \int_{L_2}^{L}(\delta_3|v^3|^2 + \beta_3|u_x^3|^2)dx + \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},\end{aligned}$$

for $\epsilon > 0$ small. □

Lemma 4.3.4. *Let $[\tilde{\ell}_2, \tilde{\ell}] \subset (L_2, L)$ and $\epsilon > 0$ small . We have*

$$\int_{\tilde{\ell}_2}^{\tilde{\ell}}\left(\beta_3|u_x^3|^2 + \beta_4|v^3|^2\right)dx \lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2.$$

Proof. Let $q_1 \in C^1([L_2, L])$ and $[\tilde{\ell}_2, \tilde{\ell}] \subset (\ell_2, \ell) \subset (L_2, L)$, such that $q_1 \ge 0$, for all $x \in [L_2, L]$ with supp$(q_1) \subset (\ell_2, \ell)$,$|q_1'|^2 \lesssim |q_1|$ and $q_1(x) \ge c_0 > 0$, for all $x \in [\tilde{\ell}_2, \tilde{\ell}]$.
Inserting (4.33) in (4.37), we have

$$i\lambda\delta_4\theta - \beta_4\theta_{xx} + ik_0\lambda u_x^3 = \delta_4 f^7 + k_0 f_x^3 \tag{4.44}$$

Multiplying (4.44) by $q_1\overline{u_x^3}$ and integrating over (L_2, L), we get

$$\begin{aligned}&i\lambda k_0\int_{L_2}^{L} q_1(x)|u_x^3|^2\\ &\quad = \beta_4\int_{L_2}^{L} q_1(x)\theta_{xx}\overline{u_x^3}dx - i\lambda\delta_4\int_{L_2}^{L} q_1(x)\theta\overline{u_x^3}dx\\ &\qquad + \int_{L_2}^{L}(\delta_4 f^7 + k_0 f_x^3)q_1(x)\overline{u_x^3}dx.\end{aligned} \tag{4.45}$$

Using integration by parts in (4.45), we obtain

$$
\begin{aligned}
i\lambda k_0 &\int_{L_2}^{L} q_1(x)|u_x^3|^2 \\
&= -\beta_4 \int_{L_2}^{L} q_1'(x)\theta_x \overline{u_x^3} dx - \beta_4 \int_{L_2}^{L} q_1(x)\theta_x \overline{u_{xx}^3} dx - i\lambda\delta_4 \int_{L_2}^{L} q_1(x)\theta \overline{u_x^3} dx \\
&\quad + \int_{L_2}^{L} (\delta_4 f^7 + k_0 f_x^3) q_1(x) \overline{u_x^3} dx. \qquad (4.46)
\end{aligned}
$$

From (4.37), we have

$$
-\beta_3 \overline{u_{xx}^3} = i\lambda\delta_3 \overline{v^3} - k_0 \overline{\theta_x} + \delta_3 \overline{f^6}. \qquad (4.47)
$$

Multiplying (4.47) by $\frac{\beta_4}{\beta_3} q_1 \theta_x$ and integrating over (L_2, L), we obtain

$$
\begin{aligned}
-\beta_4 &\int_{L_2}^{L} q_1(x)\theta_x \overline{u_{xx}^3} dx \\
&= i\frac{\lambda\beta_4\delta_3}{\beta_3} \int_{L_2}^{L} q_1(x)\theta_x \overline{v^3} dx - \frac{\beta_4 k_0}{\beta_3} \int_{L_2}^{L} q_1(x)|\theta_x|^2 dx + \frac{\beta_4\delta_3}{\beta_3} \int_{L_2}^{L} q_1(x)\theta_x \overline{f^6} dx. \qquad (4.48)
\end{aligned}
$$

Inserting (4.48) in (4.46) and using $|q_1'|^2 \lesssim |q_1|$, Young's inequality, we get

$$
\begin{aligned}
&\int_{L_2}^{L} q_1(x)|u_x^3|^2 \\
&\lesssim \frac{\epsilon}{|\lambda|} \int_{L_2}^{L} q_1(x)|u_x^3|^2 dx + \frac{C(\epsilon)}{|\lambda|} \int_{L_2}^{L} |\theta_x|^2 dx + \frac{\epsilon}{|\lambda|} \int_{L_2}^{L} q_1(x)|v^3|^2 dx + \frac{1}{|\lambda|} \int_{L_2}^{L} |\theta_x|^2 dx \\
&\quad + \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \frac{1}{|\lambda|} \|F\|_{\mathcal{H}}^2. \qquad (4.49)
\end{aligned}
$$

Using Poincaré's inequality, the Equation (4.38) and taking $|\lambda|$ large in (4.49), we obtain

$$
\int_{L_2}^{L} \beta_4 q_1(x)|u_x^3|^2 \lesssim \epsilon \int_{L_2}^{L} \delta_4 q_1(x)|v^3|^2 dx + C(\epsilon)\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2, \qquad (4.50)
$$

for $\epsilon > 0$ small.
Multiplying (4.36) by $q_1 \overline{u^3}$, integrating over (L_2, L) and using (4.33), we have

$$
\begin{aligned}
-\int_{L_2}^{L} \delta_3 q_1(x)|v^3|^2 dx - \int_{L_2}^{L} \beta_3 q_1(x) u_{xx}^3 \overline{u^3} dx + \int_{L_2}^{L} k_0 q_3(x)\theta_x \overline{u^3} \\
= \int_{L_2}^{L} \delta_3 q_1(x) \left(f^6 \overline{u^3} + \overline{f^3} v^3 \right) dx.
\end{aligned}
$$

Integration by parts and using $q_1(L_2) = q_1(L) = 0$, we have

$$
\begin{aligned}
\int_{L_2}^{L} \delta_3 q_1(x)|v^3|^2 dx = &- \int_{L_2}^{L} \beta_3 q_1'(x) u_x^3 \overline{u^3} dx - \int_{L_2}^{L} \beta_3 q_1(x)|u_x^3|^2 dx + \int_{L_2}^{L} k_0 q_1(x)\theta_x \overline{u^3} dx \\
&+ \int_{L_2}^{L} \delta_3 q_1(x) \left(f^6 \overline{u^3} + \overline{f^3} v^3 \right) dx. \qquad (4.51)
\end{aligned}
$$

Since supp$(q_1) \subset (\ell_2, \ell)$ and $|q'|^2 \lesssim |q|$, by Young's inequality, we have

$$\left|\int_{L_2}^{L} \beta_3 q_1'(x) u_x^3 \overline{u^3} dx\right| \leq \epsilon \int_{\ell_2}^{\ell} \beta_3 |u_x^3|^2 dx + C(\epsilon) \int_{L_2}^{L} \beta_3 |q_1'(x)|^2 ||u_x^3|^2 dx$$
$$\lesssim \epsilon \int_{\ell_2}^{\ell} \beta_3 |u_x^3|^2 dx + C(\epsilon) \int_{L_2}^{L} \beta_3 q_1(x) ||u_x^3|^2 dx, \tag{4.52}$$

for $\epsilon > 0$ small. Using Young's inequality, Poincaré's inequality, the equation (4.38) and (4.52) in (4.51), we get

$$\int_{L_2}^{L} \delta_3 q_1(x) |v^3|^2 dx \lesssim \epsilon \|U\|_{\mathcal{H}}^2 + C(\epsilon) \int_{L_2}^{L} \beta_3 q_1(x) |u_x^3|^2 dx + \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}, \tag{4.53}$$

for $\epsilon > 0$ small. Therefore, from (4.50) and (4.53), we obtain

$$\int_{L_2}^{L} q_1(x) \left(\beta_3 |u_x^3|^2 + \delta_3 |v^3|^2\right) dx \lesssim \epsilon \|U\|_{\mathcal{H}}^2 + C(\epsilon) \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2,$$

for $\epsilon > 0$ small. $\square$

Lemma 4.3.5. *Let $\epsilon > 0$ small and $q_3 \in C^1([L_2, L])$, such that $q_3(x) \geq 0$, for all $x \in [L_2, L]$ with $q_3(L_2) = q_3(L) = 0$ and $q_3'(x) = 1$ for all $x \in [\tilde{\ell}_2, \tilde{\ell}]^c$. We have*

$$\int_{[\tilde{\ell}_2, \tilde{\ell}]^c} \left(\beta_3 |u_x^3|^2 + \delta_3 |v^3|^2\right) dx \lesssim \epsilon \|U\|_{\mathcal{H}}^2 + C(\epsilon) \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2.$$

Proof. Multiplying (4.36) by $q_3 \overline{u_x^3}$, integrating over (L_2, L) and using (4.33), we have

$$-\int_{L_2}^{L} \delta_3 q_3(x) v^3 \overline{v_x^3} dx - \int_{L_2}^{L} \beta_3 q_3(x) u_{xx}^3 \overline{u_x^3} dx + \int_{L_2}^{L} k_0 q_3(x) \theta_x \overline{u_x^3}$$
$$= \int_{L_2}^{L} \delta_3 q_3(x) \left(f^6 \overline{u_x^3} + \overline{f_x^3} v^3\right) dx. \tag{4.54}$$

Integration by parts and using $q_3(L_2) = q_3(L) = 0$, we have

$$-\beta_3 2\text{Re} \int_{L_2}^{L} q_1(x) u_{xx}^3 \overline{u_x^3} dx = \beta_3 \int_{L_2}^{L} q_1'(x) |u_x^3|^2 dx. \tag{4.55}$$

On the other hand, similarly, using integration by parts, we get

$$-\delta_3 2\text{Re} \int_{L_2}^{L} \delta_3 q_3(x) v^3 \overline{v_x^3} dx = \delta_3 \int_{L_2}^{L} q_3'(x) |v^3|^2 dx. \tag{4.56}$$

Taking the real part in (4.54) and using (4.55)-(4.56), we obtain

$$\frac{1}{2} \int_{L_2}^{L} q_3'(x) \left(\beta_3 |u_x^3|^2 + \delta_3 |v^3|^2\right) dx$$
$$= \text{Re} \int_{L_2}^{L} \delta_3 q_3(x) \left(f^6 \overline{u_x^3} + \overline{f_x^3} v^3\right) dx - \text{Re} \int_{L_2}^{L} k_0 q_3(x) \theta_x \overline{u_x^3} \tag{4.57}$$

Therefore, using Lemma 4.3.4, the Equation (4.38) and Young's inequality in (4.57), we get

$$\begin{aligned}
&\int_{[\tilde{\ell}_2,\tilde{\ell}]^c} q_3'(x)\left(\beta_3|u_x^3|^2+\delta_3|v^3|^2\right)dx \\
&\quad\lesssim \epsilon\int_{L_2}^{L} q_3(x)|u_x^3|^2dx+\max_{x\in[L_2,L]}|q_3'(x)|\int_{[\tilde{\ell}_2,\tilde{\ell}]}\left(\beta_3|u_x^3|^2+\delta_3|v^3|^2\right)dx+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \\
&\quad\lesssim \epsilon\|U\|_{\mathcal{H}}^2+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^2,
\end{aligned}$$

for $\epsilon>0$ small. □

Lemma 4.3.6. *For $\epsilon>0$ small, we have*

$$\int_{L_2}^{L}\left(\beta_3|u_x^3|^2+\beta_4|v^3|^2\right)dx\lesssim \epsilon\|U\|_{\mathcal{H}}^2+C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^2.$$

Proof. Follow from Lemma 4.3.4 and from Lemma 4.3.5. □

The main result of this section is given by the following theorem.

Theorem 4.3.1. *Let $\mathcal{H}$ and $\mathcal{A}_1$ be defined as before, considering the conditions of the KET model. Then the semigroup $e^{t\mathcal{A}_1}$ of system* (4.11) *is exponentially stable, that is, there exists a positive constant ϵ such that*

$$\|e^{t\mathcal{A}_1}U_0\|_{\mathcal{H}}\lesssim e^{-\epsilon t}\|U_0\|_{\mathcal{H}},\quad \forall\, U_0\in\mathcal{H},\quad t>0,$$

Proof. The proof is based on using Theorem 1.5.3. First, let us prove that $i\mathbb{R}\subset\rho(\mathcal{A}_1)$. For this, we will check

(1) $\mathrm{Ker}(i\lambda I-\mathcal{A}_1)=\{0\},\quad \forall\lambda\in\mathbb{R}$;

(2) $\mathrm{R}(i\lambda I-\mathcal{A}_1)=\mathcal{H},\quad \forall\lambda\in\mathbb{R}$;

(3) $(i\lambda I-\mathcal{A}_1)^{-1}$ is bounded, $\quad\forall\lambda\in\mathbb{R}$.

First, let's prove (1). In fact, from Theorem 4.2.1, we have $\mathrm{Ker}(-\mathcal{A}_1)=\{0\}$. We need to show the result for $\lambda\neq 0$. Suppose that there is a real number non-zero λ and $U=(u^1,u^2,u^3,v^1,v^2,v^3,\theta)\in D(\mathcal{A}_1)$, such that

$$-\mathcal{A}_1U=i\lambda U,$$

this is, $F\equiv 0$ in (4.31)-(4.37). So, from (4.13), a direct computation gives

$$0=\mathrm{Re}(i\lambda U,U)_{\mathcal{H}}=\mathrm{Re}(-A_1U,U)_{\mathcal{H}}=\int_0^{L_1}a_0|v_x^1|^2dx+\int_{L_2}^{L}\beta_4|\theta_x|^2dx.$$

consequently, we deduce that

$$v_x^1=0,\quad\text{in}\quad(0,L_1)\quad\text{and}\quad\theta_x=0\quad\text{in}\quad(L_2,L).\tag{4.58}$$

Since $\theta\in H^2(L_2,L)\cap H_0^1(L_2,L)$, because $U\in D(\mathcal{A}_1)$, then $\theta\in C^1([L_2,L])$ and consequently, from (4.58), we have

$$\theta=0\quad\text{in}\quad[L_2,L],$$

it follows, from (4.33) and (4.37), that

$$u_x^3 = v_x^3 = 0, \quad \text{in} \quad (L_2, L). \tag{4.59}$$

Since $u^3 \in H^2(L_2, L)$, because $U \in D(\mathcal{A}_1)$, then $u^3 \in C^1([L_2, L])$ and consequently, from (4.59), we have

$$u^3 = 0 \quad \text{in} \quad [L_2, L], \tag{4.60}$$

and, therefore, from (4.33), we get

$$v^3 = 0 \quad \text{in} \quad [L_2, L]. \tag{4.61}$$

On the other hand, Inserting (4.32) in (4.35), we get

$$u_{xx}^2 + \frac{\delta_2}{\beta_2}\lambda^2 u^2 = 0, \quad \text{in} \quad (L_1, L_2). \tag{4.62}$$

Moreover, since $u_x^2(L_2) = u_x^3(L_2) = 0$ and $u^2(L_2) = u^3(L_2) = 0$, from (4.62) and using $u^2 \in H^2(L_1, L_2)$, we obtain

$$u^2 = 0, \quad \text{in} \quad [L_1, L_2],$$

and, hence, from (4.32), we get

$$v^2 = 0, \quad \text{in} \quad (0, L_2).$$

Since $(\beta u_x^1 + a(x)v_x^1) \in H^1(0, L_1)$, because $U \in D(\mathcal{A}_1)$, then, from (4.58), we have

$$u^1 \in C^1([0, L_1]) \quad \text{and consequently,} \quad u_x^1 = 0 \quad \text{in} \quad [0, L_1).$$

Since $u^1(0) = 0$ and $u^1(L_1) = u^2(L_1) = 0$, by Poincaré's inequality,

$$u^1 = 0, \quad \text{in} \quad [0, L_1],$$

and, therefore, from (4.31), we get

$$v^1 = 0, \quad \text{in} \quad [0, L_1].$$

Therefore, $U = 0$ and the proof is complete.

At moment, let's prove (2). From Theorem 4.2.1, we have $\mathrm{R}(-\mathcal{A}_1) = \mathcal{H}$. We will need to show the result for $\lambda \neq 0$. Set $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7) \in \mathcal{H}$, we look for $U = (u^1, u^2, u^3, v^1, v^2, v^3, \theta) \in D(\mathcal{A}_1)$ solution (4.30), equivalently, of (4.31)-(4.37).
Let $(\varphi^1, \varphi^2, \varphi^3, \phi) \in \mathbb{H}_L^1 \times H_0^1(L_2, L)$, multiplying Equations (4.34)-(4.37) by $\overline{\varphi}^1, \overline{\varphi}^2, \overline{\varphi}^3$, and $\overline{\phi}$ respectively and using integration by parts, we get

$$\int_0^{L_1} i\lambda\delta_1 v^1\overline{\varphi}^1 dx + \int_0^{L_1} \left(\beta_1 u_x^1 + a(x)v_x^1\right)\overline{\varphi}_x^1 dx - \left(\beta_1 u_x^1 + a(x)v_x^1\right)\overline{\varphi}^1\Big|_0^{L_1} = \int_0^{L_1} \delta_1 f^4\overline{\varphi}^1 dx, \tag{4.63}$$

$$\int_{L_1}^{L_2} i\lambda\delta_2 v^2\overline{\varphi}^1 dx + \int_{L_1}^{L_2} \beta_2 u_x^2\overline{\varphi}_x^2 dx - \beta_2 u^2\overline{\varphi}^2\Big|_{L_1}^{L_2} = \int_{L_1}^{L_2} \delta_2 f^5\overline{\varphi}^2 dx,$$

$$\int_{L_2}^{L} i\lambda\delta_3 v^3\overline{\varphi}^3 dx + \int_{L_2}^{L} \beta_3 u_x^3\overline{\varphi}_x^3 dx - \int_{L_2}^{L} k_0\theta\overline{\varphi}_x^3 dx + \beta_3 u_x^3(L_2)\overline{\varphi}^3(L_2) = \int_{L_2}^{L} \delta_3 f^6\overline{\varphi}^3 dx,$$

$$\int_{L_2}^{L} i\lambda\delta_4\theta\overline{\phi} dx + \int_{L_2}^{L} \beta_4\theta_x\overline{\phi}_x dx + \int_{L_2}^{L} k_0 v_x^3\overline{\phi} dx = \int_{L_2}^{L} \delta_4 f^7\overline{\phi} dx. \tag{4.64}$$

Substituing v^1, v^2, v^3 by $i\lambda u^1 - f^1, i\lambda u^2 - f^2$ and $i\lambda u^3 - f^3$ respectively in (4.63)-(4.64), we obtain

$$-\lambda^2 \int_0^{L_1} \delta_1 u^1\overline{\varphi}^1 dx + \int_0^{L_1} \left(\beta_1 u_x^1 + a(x)(i\lambda u_x^1 - f_x^2)\right)\overline{\varphi}_x^1 dx - \left(\beta_1 u_x^1 + a(x)v_x^1\right)\overline{\varphi}^1\Big|_{L_1}^{L_2}$$
$$= \delta_1 \int_0^{L_1} \left(f^4 + i\lambda f^2\right)\overline{\varphi}^1 dx, \tag{4.65}$$

$$-\lambda^2 \int_{L_1}^{L_2} \delta_2 u^2\overline{\varphi}^2 dx + \int_{L_1}^{L_2} \beta_2 u_x^2\overline{\varphi}_x^2 dx - \beta_2 u^2\overline{\varphi}^1\Big|_{L_1}^{L_2} = \delta_2 \int_{L_1}^{L_2} \left(f^5 + i\lambda f^1\right)\overline{\varphi}^2 dx,$$

$$-\lambda^2 \int_{L_2}^{L} \delta_3 u^3\overline{\varphi}^3 dx + \int_{L_2}^{L} \beta_3 u_x^3\overline{\varphi}_x^3 dx - \int_{L_2}^{L} k_0\theta\overline{\varphi}_x^3 dx + \beta_3 u_x^3(L_2)\overline{\varphi}^3(L_2)$$
$$= \delta_3 \int_{L_2}^{L} \left(f^6 + i\lambda f^3\right)\overline{\varphi}^3 dx,$$

$$\int_{L_2}^{L} i\lambda\delta_4\theta\overline{\phi} dx + \int_{L_2}^{L} \beta_4\theta_x\overline{\phi}_x dx + \int_{L_2}^{L} i\lambda k_0 u_x^3\overline{\phi} dx = \int_{L_2}^{L} \left(\delta_4 f^7 + k_0 f_x^3\right)\overline{\phi} dx. \tag{4.66}$$

Adding the equations (4.65)-(4.66), using the transmission conditions (4.2) and the boundary conditions (4.3), we obtain

$$\mathcal{B}(\mu,\nu) = \mathcal{F}(\nu), \quad \forall \nu = (\varphi^1, \varphi^2, \varphi^3, \phi) \in \mathbb{H}_L^1 \times H_0^1(L_2, L). \tag{4.67}$$

and $\mu = (u^1, u^2, u^3, \theta) \in \mathbb{H}_L^1 \times H_0^1(L_2, L)$, where

$$\mathcal{B}(\mu,\nu) = \mathcal{B}_1(\mu,\nu) + \mathcal{B}_2(\mu,\nu),$$

with

$$\mathcal{B}_1(\mu,\nu) = -\lambda^2 \int_0^{L_1} \delta_1 u^1\overline{\varphi}^1 dx - \lambda^2 \int_{L_1}^{L_2} \delta_2 u^2\overline{\varphi}^2 dx - \lambda^2 \int_{L_2}^{L} \delta_3 u^3\overline{\varphi}^3 dx + \int_{L_2}^{L} i\lambda\delta_4\theta\overline{\phi} dx$$
$$- \int_{L_2}^{L} k_0\theta\overline{\varphi}_x^3 dx,$$

$$\mathcal{B}_2(\mu,\nu) = \int_0^{L_1} \left(\beta_1 + i\lambda a(x)\right) u_x^1\overline{\varphi}_x^1 dx + \int_{L_1}^{L_2} \beta_2 u_x^2\overline{\varphi}_x^2 dx + \int_{L_2}^{L} \beta_3 u_x^3\overline{\varphi}_x^3 dx + \int_{L_2}^{L} \beta_4\theta_x\overline{\phi}_x dx$$
$$+ \int_{L_2}^{L} i\lambda k_0 u_x^3\overline{\phi} dx,$$

and

$$\mathcal{F}(\nu) = \delta_1 \int_0^{L_1} \left(f^4 + i\lambda f^1\right)\overline{\varphi}^1 dx + \delta_2 \int_{L_1}^{L_2} \left(f^5 + i\lambda f^2\right)\overline{\varphi}^2 dx + \int_0^{L_1} a(x) f_x^2\overline{\varphi}_x^1 dx$$
$$+ \delta_3 \int_{L_2}^{L} \left(f^6 + i\lambda f^3\right)\overline{\varphi}^3 dx + \int_{L_2}^{L} \left(\delta_4 f^7 - k_0 f_x^3\right)\overline{\phi} dx.$$

Consider $\mathbb{M} := \mathbb{H}^1_L \times H^1_0(L_2, L)$, $\mathbb{H}^{-1}_L$ the dual space of $\mathbb{H}^1_L$ and $\mathbb{M}^\star := \mathbb{H}^{-1}_L \times H^{-1}(L_2, L)$ the dual of $\mathbb{M}$. Let us consider the following operators,

$$\left\{\begin{array}{rcl} \mathbb{B}: \mathbb{M} & \longrightarrow & \mathbb{M}^\star \\ \mu & \longmapsto & \mathbb{B}(\mu) \end{array}\right. \quad \left\{\begin{array}{rcl} \mathbb{B}_1: \mathbb{M} & \longrightarrow & \mathbb{M}^\star \\ \mu & \longmapsto & \mathbb{B}_1(\mu) \end{array}\right. \quad \left\{\begin{array}{rcl} \mathbb{B}_2: \mathbb{M} & \longrightarrow & \mathbb{M}^\star \\ \mu & \longmapsto & \mathbb{B}_2(\mu) \end{array}\right.$$

such that

$$\mathbb{B}(\mu,\nu) = \mathcal{B}(\mu,\nu), \quad \mathbb{B}_1(\mu,\nu) = \mathcal{B}_1(\mu,\nu) \quad \text{and} \quad \mathbb{B}_2(\mu,\nu) = \mathcal{B}_2(\mu,\nu), \quad \forall \nu \in \mathbb{M}.$$

If $\mathbb{B}$ is an isomorphism and $\mathcal{F}$ is a antilinear on $\mathbb{M}$ and, furthemore, $\mathcal{F}$ is continuous from $\mathbb{M}$ to $\mathbb{C}$, then we get Equation (4.67) admits a unique solution $\mu \in \mathbb{M}$ and consequently (2) will be proven. In fact, if $\mu \in \mathbb{M} \subset \mathcal{H}$, then $(v^1, v^2, v^3) \in \mathbb{H}^1_L$, because $v^i = i\lambda u^i - f^i$ for $i = 1, 2, 3$ and $(f^1, f^2, f^3) \in \mathbb{H}^1_L$. Furthermore, using the classical regularity arguments (similarly when we proved that $0 \in \rho(\mathcal{A}_1)$), we concluded that Equation (4.30) admits a unique solution $U \in D(\mathcal{A}_1)$.

Therefore, our goal is to prove that $\mathbb{B}$ is an isomorphism operator and that and that $\mathcal{F}$ fulfills the conditions mentioned above. To do this, we will show:

(i) $\text{Ker}\{\mathbb{B}\} = \{0\}$; (ii) $\mathbb{B}_2$ is compact; (iii) $\mathbb{B}_1$ is an isomorphism.

Note that by proving the above items, we will be able to prove that $\mathbb{B}$ is an isomorphism. In fact, from (ii) and (iii), we get that the operator $\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2$ is a Fredholm operator. From (iii), we have that $\mathbb{B}_1$ is a Fredholm operator of index zero and from (i) we have $\dim N(\mathbb{B}) = 0$. Then, we get

$$0 = \text{ind}\mathbb{B}_1 = \text{ind}\mathbb{B} = \dim N(\mathbb{B}) - \text{codim} R(\mathbb{B});$$

this is, $\text{codim} R(\mathbb{B}) = 0$, how $R(\mathbb{B})$ is closed ($\mathbb{B}$ is Fredholm), we concluded $R(\mathbb{B}) = \mathbb{M}^*$. Then, as $\mathbb{B}$ is injective, surjective and continuous ($\mathbb{B}$ is Fredholm), it follows by the closed graph theorem of Banach that $\mathbb{B}^{-1}$ is continuous, and therefore, $\mathbb{B}$ is an isomorphism.
With that in mind, let's now prove the three items mentioned.
(i) We prove that $\ker\{\mathbb{B}\} = \{0\}$. For this aim, let $\tilde{\mu} \in \ker\{\mathbb{B}\}$, i.e.

$$\mathcal{B}(\tilde{\mu}, \nu) = 0, \quad \forall \nu \in \mathbb{M}.$$

Equivalently, we have

$$\begin{aligned} &-\lambda^2 \int_0^{L_1} \delta_1 \tilde{u}^1 \overline{\varphi}^1 dx - \lambda^2 \int_{L_1}^{L_2} \delta_2 \tilde{u}^2 \overline{\varphi}^2 dx - \lambda^2 \int_{L_2}^{L} \delta_3 \tilde{u}^3 \overline{\varphi}^3 dx + \int_{L_2}^{L} i\lambda\delta_4 \tilde{\theta}\overline{\phi} dx \\ &- \int_{L_2}^{L} k_0 \tilde{\theta} \overline{\varphi}^3_x dx + \int_{L_1}^{L_2} \beta_2 \tilde{u}^2_x \overline{\varphi}^2_x dx + \int_0^{L_1} (\beta_1 + i\lambda a(x))\, \tilde{u}^1_x \overline{\varphi}^1_x dx + \int_{L_2}^{L} \beta_3 \tilde{u}^3_x \overline{\varphi}^3_x dx \\ &+ \int_{L_2}^{L} \beta_4 \tilde{\theta}_x \overline{\phi}_x dx + \int_{L_2}^{L} i\lambda k_0 \tilde{u}^3_x \overline{\phi} dx = 0. \end{aligned} \tag{4.68}$$

Taking $\nu = (0, 0, i\lambda\tilde{u}^3, \tilde{\theta})$ in (4.68), we obtain

$$i\lambda\delta_4 \int_{L_2}^{L} |\tilde{\theta}|^2 dx - i2k_0\lambda \text{Im} \int_{L_2}^{L} \tilde{u}^3_x \overline{\tilde{\theta}} dx + \int_{L_2}^{L} \beta_4 |\tilde{\theta}_x|^2 dx = 0. \tag{4.69}$$

Taking the real part of (4.69) and using $\tilde{\theta} \in H_0^1(L_2, L)$, we get

$$\tilde{\theta} = 0, \quad \text{in} \quad (L_2, L). \tag{4.70}$$

Moreover, using (4.68) and (4.70), we find that

$$\begin{aligned}
\lambda^2\delta_1\tilde{u}^1 + (\beta_1 + i\lambda a(x))\tilde{u}^1_{xx} &= 0, \quad \text{in} \quad (0, L_1) \\
\lambda^2\delta_2\tilde{u}^2 + \beta_2\tilde{u}^2_{xx} &= 0, \quad \text{in} \quad (L_1, L_2) \\
\lambda^2\delta_3\tilde{u}^3 + \beta_2\tilde{u}^3_{xx} &= 0, \quad \text{in} \quad (L_2, L) \\
i\lambda k_0\tilde{u}^3_x &= 0. \quad \text{in} \quad (L_2, L)
\end{aligned}$$

Therefore, the vector $\tilde{U}$ defined by

$$\tilde{U} = (\tilde{u}^1, \tilde{u}^2, \tilde{u}^3, i\lambda\tilde{u}^1, i\lambda\tilde{u}^2, i\lambda\tilde{u}^3, 0)$$

belongs to $D(\mathcal{A}_2)$ and we have

$$i\lambda\tilde{U} - \mathcal{A}_1 U = 0.$$

Hence, $\tilde{U} \in \text{Ker}(i\lambda I - \mathcal{A}_1)$, then we get $\tilde{U} = 0$, this implies that $\tilde{u}^1 = \tilde{u}^2 = \tilde{u}^3 = 0$. Consequently, by (4.70), $\ker\{\mathbb{B}\} = \{0\}$. Since $\tilde{u} = \tilde{v} = 0$, we have $\ker\{\mathbb{B}\} = \{0\}$, how we wanted to show.

(ii) We prove that the operator $\mathbb{B}_1$ is compact. For this goal, note that, by Hölder's inequality, we have

$$|\mathcal{B}_1(\mu, \nu)| \lesssim \|\mu\|_{\mathbb{L}^2}\|\nu\|_{\mathbb{L}^2}, \quad \forall \nu \in \mathbb{M}.$$

Or yet, using Poincaré's inequality, we get

$$\sup_{\|\nu\|_{\mathbb{M}} \neq 0} \frac{|\mathcal{B}_1(\mu, \nu)|}{\|\nu\|_{\mathbb{M}}} \lesssim \|\mu\|_{\mathbb{L}^2}, \quad \forall \nu \in \mathbb{M}.$$

Soon, by definition, we have

$$\|\mathbb{B}_1(\mu)\|_{\mathbb{M}^\star} \lesssim \|\mu\|_{\mathbb{L}^2}. \tag{4.71}$$

Now, consider $\mu_n \in \mathbb{M}$ bounded. From the compact embedding of $\mathbb{M}$ in $\mathbb{L}^2$, since H^1 is compactly embedded in L^2, we have that μ_n converges in $\mathbb{L}^2$ up to a subsequence. So, from Equation (4.71), we get

$$\|\mathbb{B}_1(\mu_n)\|_{\mathbb{M}^\star} \lesssim \|\mu_n\|_{\mathbb{L}^2},$$

This is, $\mathbb{B}_1(\mu_n)$ is a Cauchy sequence in $\mathbb{M}^\star$, and therefore, converges in $\mathbb{M}^\star$ up to a subsequence. Thus, by definition, $\mathbb{B}_1$ is compact.

(iii) We prove that the operator $\mathbb{B}_2$ is an isomorphism. For this goal, note that by Hölder's inequality, we have

$$|\mathcal{B}_2(\mu, \nu)| \lesssim \|\mu\|_{\mathbb{M}}\|\nu\|_{\mathbb{M}},$$

this is, $\mathcal{B}_2$ is continuous. Now, using Poincaré's inequality, we have

$$\text{Re}\mathcal{B}_2(\mu, \mu) = \int_0^{L_1} (\beta_1|u^1_x|^2 dx + \int_{L_1}^{L_2} (\beta_2|u^2_x|^2 dx + \int_{L_2}^{L} \left(\beta_3|u^3_x|^2 + \beta_4|\theta|^2\right) dx \gtrsim \|\mu\|^2_{\mathbb{M}}.$$

Therefore, $\mathcal{B}_2$ is coercive. Furthermore, it is easy to see that $\mathcal{B}_2$ is a sesquilinear form on $\mathbb{M}$. Then, by Lax-Milgram Lemma, the operator $\mathbb{B}_2$ is an isomorphism. Finally, from **(iii)** and Fredholm alternative, we deduce that the operator $\mathbb{B}$ is isomorphism. It is easy to see that the operator $\mathcal{F}$ is a antilinear on $\mathbb{M}$. Moreover, by Hölder's inequality, we have $\mathcal{F}$ is continuous from $\mathbb{M}$ to $\mathbb{C}$. Therefore, (2) is proven.

Finally, let's prove (3). It is easy to verify that $(i\lambda - \mathcal{A}_1)$ is closed for all $\lambda \in \mathbb{R}$, because $\mathcal{A}_2$ is closed. Furthermore, since $(i\lambda - \mathcal{A}_1)$ is linear, injective and surjective, we can apply Lemma 4.4.3 to conclude that the graph of $(i\lambda I - A_1)^{-1}$ is closed. Consequently, by closed graph theorem of Banach we can deduce that $(i\lambda I - \mathcal{A}_1)^{-1}$ is bounded for all $\lambda \in \mathbb{R}$. Therefore, $i\mathbb{R} \subset \rho(\mathcal{A}_1)$, how we wanted to show.

Now, let $F \in \mathcal{H}$, consider $U = (u^1, u^1, u^3, v^1, v^2, v^3, \theta)$ solution of $(i\lambda I - \mathcal{A})U = F$, i.e, the system (4.31)-(4.37) is satisfied. To show the exponential decay, according Theorem 1.5.3 is sufficient to show $\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}}$, for $|\lambda|$ large. In fact, from (4.39) -(4.40) and from Lemmas 4.3.3 and 4.3.6, we get

$$
\begin{aligned}
\|U\|_{\mathcal{H}}^2 &= \int_0^{L_1} (\delta_1|v^1|^2 + \beta_1|u_x^1|^2)dx + \int_{L_1}^{L_2} (\delta_2|v^2|^2 + \beta_2|u_x^2|^2)dx \\
&+ \int_{L_2}^{L} (\delta_3|v^3|^2 + \beta_3|u_x^3|^2 + \delta_4|\theta|^2)dx \\
&\lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2,
\end{aligned}
$$

for $|\lambda|$ large and $\epsilon > 0$ small. Thus, using Young's inequality, taking $|\lambda|$ large and a suitably small $\epsilon > 0$, we obtain

$$
\|U\|_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}},
$$

as we desired to prove. □

4.4 Asymptotic behavior: Polynomial stability

To show the polynomial decay of the solution for the system (4.6)-(4.9) we use a result due to Borichev and Tomilov ([7]).

In this subsection, we will prove the polynomial stablity, which is valid for the cases where the elastic part of the string (with no dissipation) connects with only the Kelvin-Voigt damping. As the proof is similar for all these cases, we focus in investigating the stability of the EKT model. We recall that the EKT model is given by (4.6)-(4.9) considering

$$
a(x) = a_0\chi_{[L_1,L_2](x)}, \quad a_0 > 0.
$$

The main result of this section is Theorem 4.4.1 and to prove this theorem we will need to introduce some technical lemmas.
Let $\lambda \in \mathbb{R}$ and $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7) \in \mathcal{H}$. In what follows, the stationary problem

$$
(i\lambda I - \mathcal{A}_2)U = F, \tag{4.72}
$$

will be considered several times. Note that $U = (u^1, u^1, u^3, v^1, v^2, v^3, \theta)$ is a solution of this problem if the following equations are satisfied:

$$i\lambda u^1 - v^1 = f^1, \tag{4.73}$$
$$i\lambda u^2 - v^2 = f^2, \tag{4.74}$$
$$i\lambda u^3 - v^3 = f^3, \tag{4.75}$$
$$i\lambda\delta_1 v^1 - \beta_1 u^1_{xx} = \delta_1 f^4, \tag{4.76}$$
$$i\lambda\delta_2 v^2 - (\beta_2 u^2_x + a(x)v^2_x)_x = \delta_2 f^5, \tag{4.77}$$
$$i\lambda\delta_3 v^3 - \left(\beta_3 u^3_{xx} - k_0\theta_x\right) = \delta_3 f^6, \tag{4.78}$$
$$i\lambda\delta_4\theta - \beta_4\theta_{xx} + k_0 v^3_x = \delta_4 f^7, \tag{4.79}$$

with the transmission conditions (4.7) and the boundary conditions (4.8).
Note that inequality (4.12) implies

$$\int_{L_1}^{L_2} a_0\delta_2|v^2_x|^2dx + \int_{L_2}^{L} \beta_4|\theta_x|^2dx = \mathrm{Re}((i\lambda I - \mathcal{A}_2)U, U)_{\mathcal{H}} \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.80}$$

From (4.74) and (4.80), we have

$$\int_{L_1}^{L_2} a_0\beta_2|u^2_x|^2dx \lesssim |\lambda|^{-2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}\right). \tag{4.81}$$

From (4.80) e using the Poicanré's inequality, we have

$$\int_{L_2}^{L} \delta_4|\theta|^2dx \lesssim \frac{\delta_4}{\beta_4}\int_{L_2}^{L} \beta_4|\theta_x|^2dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.82}$$

Lemma 4.4.1. *For $|\lambda|$ large, we have*

$$\int_{L_1}^{L_2} \left|\beta_2 u^2_x + a_0 v^2_x\right|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.$$

Proof. Using (4.80) and (4.81), we get

$$\int_{L_1}^{L_2} \left|\beta_2 u^2_x + a_0 v^2_x\right|^2 dx \lesssim \int_{L_1}^{L_2} \beta_2\left|u^2_x\right|^2 dx + \int_{L_1}^{L_2} a_0\left|v^2_x\right|^2 dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}},$$

for $|\lambda|$ large. □

Lemma 4.4.2. *For $|\lambda|$ large, we have*

$$|\lambda|\|v^2\|^2_{L^2(L_1,L_2)} \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^{3/2}_{\mathcal{H}}\|U\|^{1/2}_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}.$$

Proof. Consider $\mathbb{H}^1_\star$ the dual space of $\mathbb{H}^1(L_1, L_2)$. From (4.77),(4.80) and (4.81), we get

$$\begin{aligned} |\lambda|\|v^2\|_{\mathbb{H}^1_\star} &\lesssim \|u^2_x\|_{L^2(L_1,L_2)} + \|a(x)^{\frac{1}{2}}v^2_x\|_{L^2(L_1,L_2)} + \|f^5\|_{L^2(L_1,L_2)} \\ &\lesssim \|F\|^{\frac{1}{2}}_{\mathcal{H}}\|U\|^{\frac{1}{2}}_{\mathcal{H}} + \|F\|_{\mathcal{H}} \end{aligned} \tag{4.83}$$

Using Interpolation and inequalities (4.77) and (4.83), we obtain

$$\begin{aligned} |\lambda|\|v^2\|^2_{L^2(L_1,L_2)} &\lesssim |\lambda|\|v^2\|_{\mathbb{H}^1_\star}\|v^2\|_{\mathbb{H}^1(L_1,L_2)} \\ &\lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|^{3/2}_{\mathcal{H}}\|U\|^{1/2}_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}. \end{aligned}$$

for $|\lambda|$ large. □

Note that from Equation (4.81) and from Lemma 4.83, we have

$$\int_{L_1}^{L_2}(\beta_2|u_x^2|^2+\delta_2|v^2|^2)dx \lesssim |\lambda|^{-1}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}+\|F\|_{\mathcal{H}}^2\right). \tag{4.84}$$

Lemma 4.4.3. *We have*

$$\begin{aligned}&\int_0^{L_1}(\beta_1|u_x^1|^2+\delta_1|v^1|^2)dx\\ &\qquad\lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right)+\|F\|_{\mathcal{H}}^2,\end{aligned}$$

for $|\lambda|$ large.

Proof. Multiplying (4.77) by $(L_2-x)\overline{(\beta_2u_x^2+a_0v_x^2)}$ and taking the real part, we arrive at

$$\begin{aligned}&\text{Re}\left\{\int_{L_1}^{L_2} i\lambda\delta_2(L_2-x)v^2\overline{(\beta_2u_x^2+a_0v_x^2)}dx\right\}-\text{Re}\left\{\frac{1}{2}\int_{L_1}^{L_2}(L_2-x)\frac{d}{dx}|\beta_2u_x^2+a_0v_x^2|^2dx\right\}\\ &=\text{Re}\left\{\int_{L_1}^{L_2}\delta_2(L_2-x)f^5\overline{(\beta_2u_x^2+a_0v_x^2)}dx\right\}.\end{aligned}$$

Note that the above equation can be rewritten as follows

$$\begin{aligned}&\text{Re}\left\{\int_{L_1}^{L_2} i\lambda\delta_2\beta_2(L_2-x)v^2\overline{u_x^2}dx\right\}-\text{Re}\left\{\frac{1}{2}\int_{L_1}^{L_2}(L_2-x)\frac{d}{dx}|\beta_2u_x^2+a_0v_x^2|^2dx\right\}\\ &=\text{Re}\left\{\int_{L_1}^{L_2}\delta_2(L_2-x)f^5\overline{(\beta_2u_x^2+a_0v_x^2)}dx\right\}\\ &\quad-\text{Re}\left\{\int_{L_1}^{L_2} i\lambda\delta_2a_0(L_2-x)v^2\overline{v_x^2}dx\right\}\end{aligned} \tag{4.85}$$

Next, let us rewrite the terms on the left side of the above equation.
By (4.74), we have

$$\begin{aligned}&\text{Re}\left\{\beta_2\int_{L_1}^{L_2}\delta_2(L_2-x)v^2\overline{(-i\lambda u_x^2)}dx\right\}\\ &=-\text{Re}\left\{\beta_2\int_{L_1}^{L_2}\delta_2(L_2-x)v^2\overline{v_x^2}dx\right\}-\text{Re}\left\{\beta\int_{L_1}^{L_2}\delta_2(L_2-x)v^2\overline{f_x^2}dx\right\}\\ &=\frac{\beta_2\delta_2(L_2-L_1)}{2}|v(L_1)|^2-\frac{\beta_2}{2}\int_{L_1}^{L_2}\delta_2|v^2|^2dx\\ &\quad-\text{Re}\left\{\beta_2\int_{L_1}^{L_2}\delta_2(L_2-x)v^2\overline{f_x^2}dx\right\},\end{aligned} \tag{4.86}$$

and

$$\begin{aligned}&\text{Re}\left\{\frac{1}{2}\int_{L_1}^{L_2}(L_2-x)\ \frac{d}{dx}|\beta_2u_x^2+a_0v_x^2|^2dx\right\}\\ &=\frac{1}{2}(L_2-L_1)|\beta_2u_x^2(L_1)+a_0v_x^2(L_1)|^2-\frac{1}{2}\int_{L_1}^{L_2}|\beta_2u_x^2+a_0v_x^2|^2dx.\end{aligned} \tag{4.87}$$

Thus, substituting (4.86) and (4.87) into equation (4.85) yields:

$$\begin{aligned}&\frac{(L_2-L_1)}{2}\left[\beta_2\delta_2|v^2(L_1)|^2+|\beta_2u_x^2(L_1)+a_0v_x^2(L_1)|^2\right]\\ =&\frac{\beta_2\delta_2}{2}\int_{L_1}^{L_2}|v^2|^2dx+\frac{1}{2}\int_{L_1}^{L_2}|\beta_2u_x^2+a_0v_x^2|^2dx+\text{Re}\left\{\beta_2\int_{L_1}^{L_2}\delta_2(L_2-x)v^2\overline{f_x^2}dx\right\}\\ &+\text{Re}\left\{\int_{L_1}^{L_2}\delta_2(L_2-x)f^5\overline{(\beta_2u_x^2+a_0v_x^2)}dx\right\}-\text{Re}\left\{a_0\int_{L_1}^{L_2}i\lambda\delta_2(L_2-x)v^2\overline{v_x^2}dx\right\}.\end{aligned} \tag{4.88}$$

We define the functional

$$I_u=\frac{(L_2-L_1)}{2}\left[\beta_2\delta_2|v^2(L_1)|^2+|\beta_2u_x^2(L_1)+a_0v_x^2(L_1)|^2\right].$$

Thus, from (4.88), we have

$$\begin{aligned}I_u=&\frac{\beta_2\delta_2}{2}\int_{L_1}^{L_2}|v^2|^2dx+\frac{1}{2}\int_{L_1}^{L_2}|\beta_2u_x^2+a_0v_x^2|^2dx+\text{Re}\left\{\beta_2\int_{L_1}^{L_2}\delta_2(L_2-x)v^2\overline{f_x^2}dx\right\}\\ &+\text{Re}\left\{\int_{L_1}^{L_2}\delta_2(L_2-x)f^5\overline{(\beta_2u_x^2+a_0v_x^2)}dx\right\}-\text{Re}\left\{a_0\int_{L_1}^{L_2}i\lambda\delta_2(L_2-x)v^2\overline{v_x^2}dx\right\}.\end{aligned}$$

We now need to estimate the terms on the right-hand side of the equation above. To this end, we note that by Lemma 4.4.2 and using (4.80), we get

$$\begin{aligned}&\text{Re}\left\{a_0\delta_2\int_{L_1}^{L_2}i\lambda(L_2-x)v^2\overline{v_x^2}dx\right\}\\ &\lesssim a_0|\lambda|^{1/2}\int_{L_1}^{L_2}\delta_2a(x)|v_x^2|(|\lambda|^{1/2}|v^2|)dx\\ &\lesssim|\lambda|^{1/2}\left(\int_{L_1}^{L_2}a(x)|v_x^2|^2dx\right)^{1/2}\left(|\lambda|\int_{L_1}^{L_2}|v^2|^2dx\right)^{1/2}\\ &\lesssim|\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right)+\|F\|_{\mathcal{H}}^2.\end{aligned}$$

The estimates for the remaining terms follow from Lemmas 4.4.1 and 4.4.2, and from Hölder's inequality. Therefore, we obtain

$$I_u\lesssim|\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4}+\|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right)+\|F\|_{\mathcal{H}}^2 \tag{4.89}$$

On the other hand, multiplying Equation(4.76) by $x\overline{u_x^1}$ and taking the real part, we get

$$\text{Re}\left\{i\lambda\delta_1\int_0^{L_1}xv^1\overline{u_x^1}dx\right\}-\text{Re}\left\{\int_0^{L_1}\beta_1x\overline{u_x^1}u_{xx}^1dx\right\}=\text{Re}\left\{\int_0^{L_1}\delta_1xf^4\overline{u_x^1}dx\right\}. \tag{4.90}$$

Using Equation(4.73), we get

$$\begin{aligned}\text{Re}\left\{i\lambda\delta_1\int_0^{L_1}xv\overline{u_x^1}dx\right\}&=-\text{Re}\left\{\delta_1\int_0^{L_1}xv^1\overline{v_x^1}dx\right\}-\text{Re}\left\{\delta_1\int_0^{L_1}xv^1\overline{f_x^1}dx\right\}\\ &=\frac{L_1\delta_1}{2}|v^1(L_1)|^2-\int_0^{L_1}\delta_1|v^1|^2dx-\text{Re}\left\{\delta_1\int_0^{L_1}xv^1\overline{f_x^1}dx\right\}.\end{aligned} \tag{4.91}$$

Also, we have

$$-\text{Re}\left\{\int_0^{L_1} \beta_1 x\overline{u_x^1}\partial_{xx}u^1 dx\right\} = \frac{L_1}{2}\beta_1|u_x^1(L_1)|^2 - \int_0^{L_1} \beta_1|u_x^1|^2 dx. \tag{4.92}$$

Therefore, from(4.89)-(4.92), we have

$$\begin{aligned}
&\int_0^{L_1} (\beta_1|u_x^1|^2 + \delta_1|v^1|^2)dx \\
&\quad = \frac{L_1}{2}\left[\beta_1|u_x^1(L_1)|^2 + \delta_1|v^1(L_1)|^2\right] - \text{Re}\left\{\int_0^{L_1} xv^1\overline{f_x^1}dx\right\} - \text{Re}\left\{\int_0^{L_1} xf^4\overline{u_x^1}dx\right\} \\
&\quad \lesssim I_u + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \\
&\quad \lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right) + \|F\|_{\mathcal{H}}^2,
\end{aligned}$$

for $|\lambda|$ large. □

Proceeding in a manner similar to that in the proof of Lemma 4.3.6, we obtain the following result:

$$\int_{L_2}^{L} \left(\beta_3|u_x^3|^2 + \beta_4|v^3|^2\right) dx \lesssim \epsilon\|U\|_{\mathcal{H}}^2 + C(\epsilon)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2, \tag{4.93}$$

for $\epsilon > 0$ small.

Theorem 4.4.1. *Let $\mathcal{H}$ and $\mathcal{A}_2$ be defined as before. Then, the EKT model is not exponentially stable. Moreover, the semigroup $e^{t\mathcal{A}}$ of system* (4.11) *decays polynomially with the rate t^{-2}, that is*

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \lesssim t^{-2}\|U_0\|_{D(\mathcal{A}_2)}, \quad \forall\, U_0 \in D(\mathcal{A}_2), \quad t \geq 1,$$

Proof. The proof is based on Theorem 1.5.4. Similarly, as we did in the proof of the Theorem 4.3.1, we obtain that $i\mathbb{R} \subset \rho(\mathcal{A})$.

Now, let $F \in \mathcal{H}$, consider $U = (u^1, u^1, u^3, v^1, v^2, v^3, \theta)$ solution of $(i\lambda I - \mathcal{A})U = F$, i.e, the system (4.6)-(4.9) is satisfied. To show the polynomial decay with the rate t^{-2}, according to Borichev and Tomilov's Theorem (see Borichev and Tomilov [5]), is sufficient to show

$$\|U\|_{\mathcal{H}} \lesssim |\lambda|^{1/2}\|F\|_{\mathcal{H}},$$

for $|\lambda|$ large.
Therefore, from Equations (4.82), (4.84) and (4.93) and Lemma 4.4.3, we get

$$\begin{aligned}
\|U\|_{\mathcal{H}}^2 &= \int_0^{L_1} \left(\delta_1|v^1|^2 + \beta_1|u_x^1|^2\right) dx + \int_{L_1}^{L_2} \left(\delta_2|v^2|^2 + \beta_2|u_x^2|^2\right) dx \\
&\quad + \int_{L_2}^{L} \left(\delta_3|v^3|^2 + \beta_3|u_x^3|^2\right) dx \\
&\lesssim |\lambda|^{1/2}\left(\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^{5/4}\|U\|_{\mathcal{H}}^{3/4} + \|F\|_{\mathcal{H}}^{3/2}\|U\|_{\mathcal{H}}^{1/2}\right) + \|F\|_{\mathcal{H}}^2,
\end{aligned}$$

for $|\lambda|$ large. Thus, using Young's inequality, we obtain

$$\|U\|_{\mathcal{H}}^2 \lesssim \left(|\lambda| + |\lambda|^{4/5} + |\lambda|^{2/3} + 1\right)\|F\|_{\mathcal{H}}^2 \lesssim |\lambda|\|F\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large. Therefore, we get

$$\|U\|_{\mathcal{H}} \lesssim |\lambda|^{1/2}\|F\|_{\mathcal{H}},$$

for $|\lambda|$ large. □

Conclusion

In this thesis, we analyze the asymptotic behavior of solutions to three elastic problems related to wave equations with localized Kelvin-Voigt damping, in combination with other dissipative mechanisms—namely, frictional, memory, and thermoelastic damping. Considering systems composed of different elastic and dissipative regions, we demonstrate that the spatial distribution of these mechanisms plays a decisive role in the system's energy decay rate.

In the first two problems, we study strings composed of three main regions: one viscoelastic, one purely elastic, and one with frictional or memory damping. Through the application of Semigroup Theory, we established the correct mathematical formulation for each model and obtained precise conditions under which exponential energy decay occurs. When any part of the elastic region is connected to the frictional component (in the first problem) or the memory component (in the second), the semigroup is exponentially stable. On the other hand, when there is a portion of the elastic region that is connected only to the viscoelastic dissipative region, the system ceases to be exponentially stable, and we show that the associated semigroup decays polynomially at the rate t^{-2}—a result that was also proven to be optimal.

In the third problem, involving a rod with thermoelastic damping, we again verify that exponential stability depends on the position of the dissipative mechanism. When the viscoelastic component is located at the center of the rod, exponential stability is lost. In this configuration, we demonstrate that the semigroup decays polynomially at the rate t^{-2}; however, the optimality of this rate remains an open problem.

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