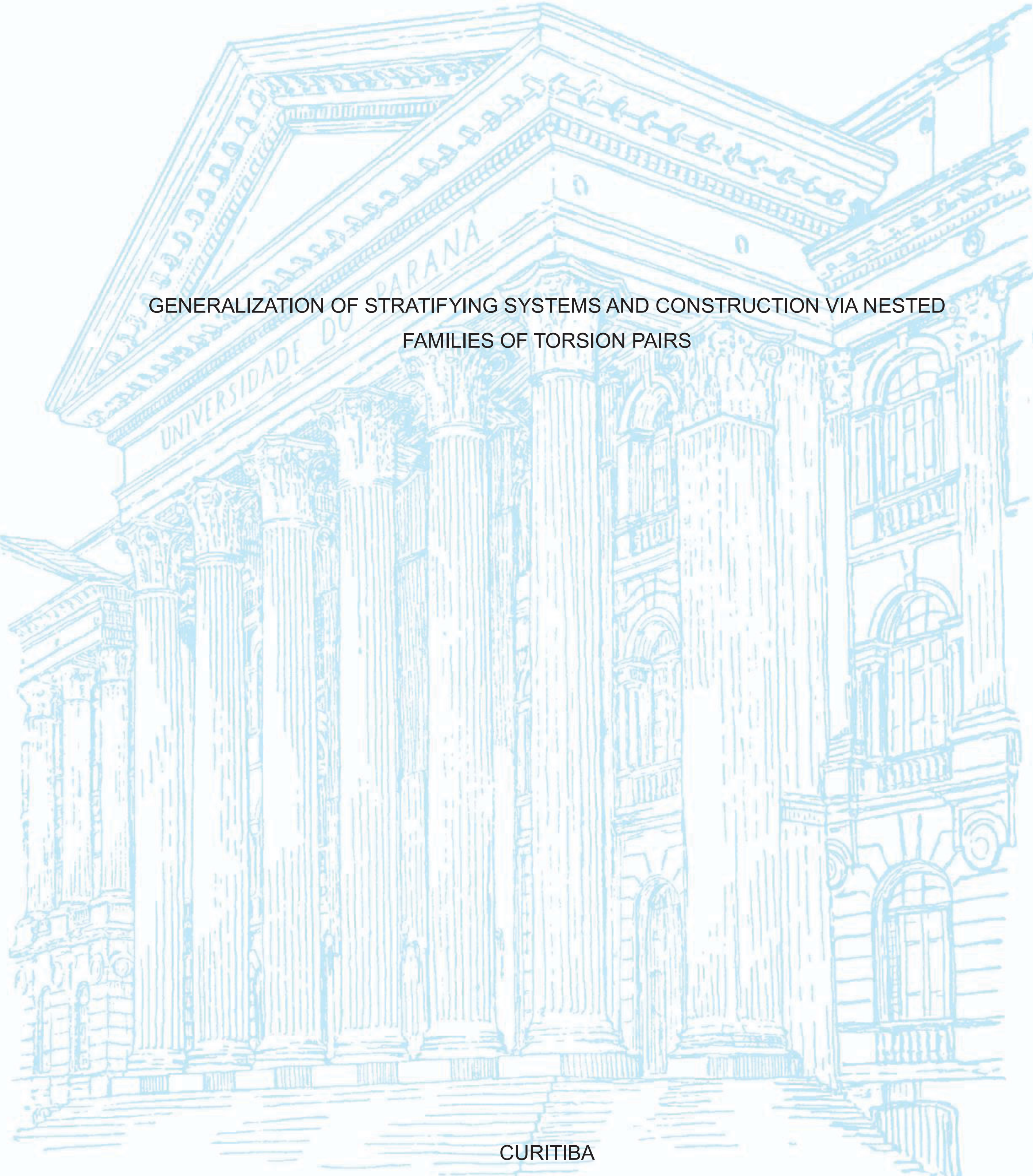
UNIVERSIDADE FEDERAL DO PARANÁ

MATHEUS VINICIUS DOS SANTOS

GENERALIZATION OF STRATIFYING SYSTEMS AND CONSTRUCTION VIA NESTED FAMILIES OF TORSION PAIRS



CURITIBA

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GENERALIZATION OF STRATIFYING SYSTEMS AND CONSTRUCTION VIA NESTED FAMILIES OF TORSION PAIRS

Dissertação apresentada como requisito parcial à obtenção do grau de Mestre em Matemática, no Curso de Pós-Graduação em Matemática, Setor de Ciências Exatas, da Universidade Federal do Paraná.

Orientador: Prof. Dr. Edson Ribeiro Alvares

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Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação MATEMÁTICA da Universidade Federal do Paraná foram convocados para realizar a arguição da Dissertação de Mestrado de **MATHEUS VINICIUS DOS SANTOS** intitulada: **GENERALIZATION OF STRATIFYING SYSTEMS AND CONSTRUCTION VIA NESTED FAMILIES OF TORSION PAIRS**, sob orientação do Prof. Dr. EDSON RIBEIRO ALVARES, que após terem inquirido o aluno e realizada a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.

A outorga do título de mestre está sujeita à homologação pelo colegiado, ao atendimento de todas as indicações e correções solicitadas pela banca e ao pleno atendimento das demandas regimentais do Programa de Pós-Graduação.

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I dedicate this master's thesis to my beloved Giovanna, my source of inspiration.

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RESUMO

Os sistemas estratificantes foram introduzidos por C. Sáenz e K. Erdmann como uma generalização dos módulos estandares e são conjuntos de módulos indecomponíveis que respeitam certas condições de ortogonalidade. Neste trabalho, introduzimos a teoria básica sobre sistemas estratificantes e apresentamos alguns resultados recentes sobre a construção de sistemas estratificantes a partir de módulos τ-rígidos. Em seguida, introduzimos a noção de família encaixante de pares de torção, que permitirá generalizar a construção de sistemas estratificantes para além daqueles obtidos por módulos τ-rígidos. Mostraremos que todo sistema estratificante pode ser obtido por meio dessa nova construção.
Palavras-chave: Sistemas Estratificantes, Teoria de Representações, Teoria de Torção, Álgebras Quase-hereditárias.

ABSTRACT

Stratifying systems were introduced by C. Sáenz and K. Erdmann as a generalization of standard modules and are sets of indecomposable modules that satisfy certain orthogonality conditions. In this text, we introduce the basic theory of stratifying systems and present some recent results on the construction of stratifying systems from τ-rigid modules. We then introduce the notion of nested families of torsion pairs, which will allow us to generalize the construction of stratifying systems beyond those obtained from τ-rigid modules. We will show that every stratifying system can be obtained through this new construction.
Keywords: Stratifying Systems, Representation Theory, Torsion Theory, Quasi-hereditary Algebras.

SUMMARY

Introduction

Stratifying systems were introduced by C. Sáenz and K. Erdmann as a generalization of standard modules, extending the notion of exceptional sequences. While the category of modules filtered by an exceptional sequence is equivalent to the subcategory of modules filtered by standard modules in the category of a quasi-hereditary algebra, the category of modules filtered by a stratifying system is likewise equivalent to the subcategory of modules filtered by standard modules in the category of modules over a standardly stratified algebra (25, Theorem 1.6).

Furthermore, the homological properties of these stratifying systems play an important role in the context of quasi-hereditary algebras and standardly stratified algebras (20; 23), attracting considerable interest from researchers in representation theory. Given a stratifying system, it is well known that the full subcategory consisting of the trivial module and all modules admitting a filtration by this system is closed under direct sums and direct summands. In this regard, the pioneering work of V. Dlab and C. Ringel (24) serves as a foundational reference. Moreover, for a novel characterization of quasi-hereditary algebras, we refer the reader to (3).

Additionally, in the context of the structural analysis of the subcategory of filtered modules, C. Ringel's work (38) establishes that this subcategory is functorially finite in the sense of Auslander-Smalø (9). Moreover, it is demonstrated that this subcategory contains almost split sequences, providing a deeper insight into its homological properties. Recently, R. Bautista, E. Pérez, and L. Salmerón have advanced this line of research (12), further exploring the structure of almost split sequences in these subcategories. They also proved that the category of filtered modules is either tame or wild, but not both. All of these results are clearly underpinned by the existence of a stratifying system.

However, given a set of modules, to verify whether it forms a stratifying system, it is often sufficient to directly check whether certain morphism properties and extension conditions hold. Nevertheless, an important question is how to find a stratifying system. Despite significant advances, with the exception of hereditary algebras, little is known about the existence of non-trivial stratifying systems (defined by the standard modules) in arbitrary algebras (18; 19). An important first step in this direction was made by O.

Mendoza and H. Treffinger (36), who demonstrated that it is possible to construct stratifying systems with indecomposable summands from a certain quotient of a τ-rigid module (36, Theorem 3.4). However, since a basic τ-rigid module M has at most $|A_A|$ direct summands (1, Proposition 1.3), the size of a stratifying system induced by M is bounded by $|A_A|$ (36, Theorem 3.4). This technique is noteworthy, as it forms the foundation for an idea we introduce in this text. Specifically, from these stratifying systems, we can generate what we call nested families of torsion pairs, with the stratifying system being intricately linked to this family in a special way.

It is also important to highlight another aspect that motivated us to present the more general definition of stratifying systems (see Definition 4.0.1). It is known that even in representation-finite algebras it is possible to find stratifying systems of size greater than $|A_A|$ (29, Remark 2.7). In (41), H. Treffinger presents a stratifying system indexed by ($\mathbb{N}$, $\leq$), where $\leq$ is the natural order, demonstrating that a stratifying system can indeed have infinite size. Since there does not always exist an order-preserving bijection between totally ordered sets $(G, \leq)$ and $(H, \leq)$ (even if G and H have the same cardinality), the possibility of stratifying systems indexed by infinite ordered sets other than $(\mathbb{N}, \leq)$ further justifies the need for this more general definition.

Our main result in this master's thesis is to show that the construction of stratifying systems can be done in a more general way by considering the concept of nested family of torsion pairs, which is a collection of torsion pairs $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ indexed by a totally ordered set $(G, \leq)$, satisfying $\mathcal{T}_k \supsetneq \mathcal{T}_l$ whenever $k < l$ (see Definition 3.1.1).

This concept arises naturally, as every stratifying system induces at least two nested families of torsion pairs (Corollary (4.2.3)). A stratifying system can belong to two types of nested families of torsion pairs, both of which are induced by modules possessing special properties. To facilitate understanding and exposition, we organize these modules into certain special subcategories, which we denote by $\mathcal{M}^*(\Gamma)$ and in $\mathcal{N}^*(\Gamma')$ (see Definition (3.1.3)).

The converse also holds: certain objects in these two subcategories induce stratifying systems. More precisely, we can formulate this as follows. We show that, given a nested family of torsion pairs $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$, if there exists a module $M \in \mathcal{M}^\dagger(\Gamma)$ (or in $\mathcal{N}^\dagger(\Gamma)$, see Definition 4.2.5), we can obtain at least one stratifying system by selecting indecomposable summands of each module $\mathfrak{F}_k(M)$ (or $\mathfrak{T}_k(M)$, respectively, see Definition 3.2.1), which is a quotient (or submodule, respectively) of M_k. Moreover, all stratifying systems can be obtained in this way, as demonstrated by the results of the following two theorems.

Theorem. *(5, Theorem 4.8) Let* $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ *be a nested family of torsion pairs and let* $M = \bigoplus_{k\in G} M_k$ *be a module.*

1. *If such a decomposition of M is compatible in $\mathcal{M}^{\dagger}(\Gamma)$, then M induces at least one stratifying system $\Theta = \{\Theta_k\}_{k\in G}$ such that Θ_k is a summand of $\mathfrak{F}_k(M)$.*

2. *If such a decomposition of M is compatible in $\mathcal{N}^{\dagger}(\Gamma)$, then M induces at least one stratifying system $\Theta = \{\Theta_k\}_{k\in G}$ such that Θ_k is a summand of $\mathfrak{T}_k(M)$.*

Theorem. *(5, Theorem 4.9) Let $\Theta = \{\Theta_k\}_{k\in G}$ be a stratifying system.*

1. *Then there exists a module $M \in \mathcal{M}^{\dagger}(A)$ such that Θ is induced by M.*

2. *Then there exists a module $N \in \mathcal{N}^{\dagger}(A)$ such that Θ is induced by N.*

The prerequisites for this text are basic knowledge of category theory (particularly abelian and module categories), including projective, injective, and simple objects, the notions of monomorphism and epimorphism, kernel, cokernel, as well as basic concepts of top, and radical.

In Chapter 1, we establish the notation and present six sections that will aid in understanding the text. The first and most comprehensive one is about finite dimensional k-algebras, where we introduce the reader to the main results and techniques in the area, such as the Auslander-Reiten quiver and the Auslander-Reiten translation, Gabriel's theorem, the category of representations, and the Jordan-Hölder theorem. We also provide a brief introduction to Torsion Theory, t-Tilting Theory, and n-representation infinite algebras. Finally, we include a section on filtered modules and an overview of quasi-hereditary algebras and standardly stratified algebras.

In Chapter 2, we present the definitions of Ext-injective stratifying system (EISS, for short), Ext-projective stratifying system (EPSS), and stratifying system (SS). We will show that the concepts of EISS and SS are equivalent, in addition to proving some results about the category of modules filtered by an SS and its relation to standardly stratified algebras. Next, we present a historical overview of the development of the theory of stratifying systems, where we chronologically outline the main articles and results on the topic. We conclude the chapter by presenting the results obtained in (36), which relate τ-rigid modules and stratifying systems.

In Chapter 3, we define the notion of a nested family of torsion pairs Γ (nested family, for short), a central idea that will allow us to generalize the results from (36). The definition of a nested family is strongly tied to the notion of a totally ordered set. Given a nested family, we introduce subcategories of $\operatorname{Mod} A$, denoted by $\mathcal{M}(\Gamma)$, $\mathcal{N}(\Gamma)$, $\mathcal{M}^*(\Gamma)$, and $\mathcal{N}^*(\Gamma)$, whose modules admit a certain ordered decomposition. We also define maps $\mathfrak{F} : \mathcal{M}(\Gamma) \longrightarrow \mathcal{M}^*(\Gamma)$ and $\mathfrak{T} : \mathcal{N}(\Gamma) \longrightarrow \mathcal{N}^*(\Gamma)$, called stratum and substratum, respectively, which will be used in the following chapter to obtain stratifying systems from certain modules.

In Chapter 4, we provide a more general definition of stratifying systems, opening the possibility for infinite stratifying systems not indexed by $\mathbb{N}$. We then show that the category of modules filtered by this new definition satisfies the Jordan-Hölder property. Next, we relate stratifying systems and nested families, showing that every stratifying system induces at least two distinct nested families and that every stratifying system can be induced by certain modules, thereby completing the results from (36). We conclude the chapter by introducing the notion of expansions in nested families, which will allow us to show that τ-rigid modules induce stratifying systems beyond those presented in (36), and we present a stratifying system that cannot be indexed by $(\mathbb{N}, \leq)$, where $\leq$ is the natural order.

Chapter 1

Preliminaries

Throughout this master's thesis, A denote a finite dimensional k-algebra over an algebraically closed field k. For a given algebra A we denote by $\operatorname{Mod} A$ the category of right A-modules and by $\operatorname{mod} A$ the category of finitely generated right A-modules. The number of pairwise non-isomorphic indecomposable direct summands of $M \in \operatorname{mod} A$ is denoted by $|M|$. If M does not have two isomorphic direct summands, we say that M is a **basic module**. The Auslander-Reiten translation is denoted by τ and $D = \operatorname{Hom}_k(-,k)$ is the duality functor. G will denote a set with a total order $\leq$, and G^{op} will denote the same set with the total order $\leq^{op}$, where $x \leq^{op} y$ in G^{op} if and only if $y \leq x$ in G. $\mathbb{I}_n$ will denote the set $\{1, \cdots, n\}$ with the natural order.

For a given class $\mathcal{C} \subseteq \operatorname{mod} A$, we define

$$\mathcal{C}^{\perp} := \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(-,M)|_{\mathcal{C}} = 0\}.$$
$$^{\perp}\mathcal{C} := \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M,-)|_{\mathcal{C}} = 0\}.$$

If $\mathcal{C} = \{M\}$ for some $M \in \operatorname{mod} A$, we will denote $\{M\}^{\perp}$ and $^{\perp}\{M\}$ simply by $M^{\perp}$ and $^{\perp}M$, respectively. A module $M \in \mathcal{C}$ is called **Ext-projective** in $\mathcal{C}$ if $\operatorname{Ext}^1_A(M,-)|_{\mathcal{C}} = 0$. Dually, it is called **Ext-injective** in $\mathcal{C}$ if $\operatorname{Ext}^1_A(-,M)|_{\mathcal{C}} = 0$.

For a given $M \in \operatorname{mod} A$ we define

$$\operatorname{Fac}(M) := \{X \in \operatorname{mod} A \mid \exists \text{ an epimorphism } M^{\oplus n} \to X \text{ for some } n \in \mathbb{N}\}.$$
$$\operatorname{Sub}(M) := \{X \in \operatorname{mod} A \mid \exists \text{ a monomorphism } X \to M^{\oplus n} \text{ for some } n \in \mathbb{N}\}.$$

Given a class $\mathcal{C} \subseteq \operatorname{mod} A$, we will denote by $\mathcal{P}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ the subclasses of $\mathcal{C}$ consisting of the Ext-projective and Ext-injective A-modules in $\mathcal{C}$, respectively. We denote by $P(\mathcal{C})$ (respectively, $I(\mathcal{C})$) the direct sum of a complete set of representatives of the isomorphism classes of indecomposable Ext-projective (respectively, Ext-injective) modules in $\mathcal{C}$.

Given a class $\mathcal{C}$, we define the **annihilator of** $\mathcal{C}$, denoted by $\operatorname{ann} \mathcal{C}$, as the subset of A

consisting of the elements that annihilate all the modules in $\mathcal{C}$. We say that $\mathcal{C}$ is **sincere** if there are no nonzero idempotents in $\operatorname{ann}\mathcal{C}$. We also say that $\mathcal{C}$ is **faithful** if $\operatorname{ann}\mathcal{C} = \{0\}$.

We say that $\mathcal{C}$ is

1. **Closed under direct sum** if $M, N \in \mathcal{C} \Longrightarrow M \bigoplus N \in \mathcal{C}$.

2. **Closed under summands** if $M \bigoplus N \in \mathcal{C} \Longrightarrow M, N \in \mathcal{C}$.

3. **Closed under extensions** if $M, N \in \mathcal{C}$ and

$$0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$$

implies $L \in \mathcal{C}$.

4. **Closed under submodules** if there exists a monomorphism $M \longrightarrow N$ with $N \in \mathcal{C}$, then $M \in \mathcal{C}$.

5. **Closed under quotients** if there exists an epimorphism $M \longrightarrow N$ with $M \in \mathcal{C}$, then $N \in \mathcal{C}$.

We have that $M^{\oplus n} := \bigoplus_{j=1}^{n} M$ and we will denote by $M'|M$ if M' is a direct summand of M, that is, if there exists a module M'' such that $M \cong M' \bigoplus M''$. For a given $M = M_1^{\oplus \alpha_1} \bigoplus \cdots \bigoplus M_t^{\oplus \alpha_t} \in \operatorname{mod} A$ with $M_1, \cdots, M_t$ indecomposable, we define

$$\operatorname{add}(M) = \{X \in \operatorname{mod} A \mid X \cong M_1^{\oplus n_1} \bigoplus \cdots \bigoplus M_t^{\oplus n_t} \text{ for some } n_1, \cdots, n_t \in \mathbb{N} \cup \{0\}\}.$$

If $S = \{M_1, M_2, \cdots, M_t\}$ is a set of indecomposable A-modules we also define

$$\operatorname{add}(S) = \{X \in \operatorname{mod} A \mid X \cong M_1^{\oplus n_1} \bigoplus \cdots \bigoplus M_t^{\oplus n_t} \text{ for some } n_1, \cdots, n_t \in \mathbb{N} \cup \{0\}\}.$$

Let $\mathcal{C} \subseteq \operatorname{mod} A$ be a full subcategory and $M \in \operatorname{mod} A$. A morphism $f : X \longrightarrow M$ in $\operatorname{mod} A$ is a **right $\mathcal{C}$-approximation** if $X \in \mathcal{C}$ and $\operatorname{Hom}_A(X', f) : \operatorname{Hom}_A(X', X) \longrightarrow \operatorname{Hom}_A(X', M)$ is surjective for any $X' \in \mathcal{C}$. The subcategory $\mathcal{C}$ is called **contravariantly finite** if any $M \in \operatorname{mod} A$ admits a right $\mathcal{C}$-approximation. Dually, a morphism $f : M \longrightarrow X$ in $\operatorname{mod} A$ is a **left $\mathcal{C}$-approximation** if $X \in \mathcal{C}$ and $\operatorname{Hom}_A(f, X') : \operatorname{Hom}_A(X, X') \longrightarrow \operatorname{Hom}_A(M, X')$ is surjective for any $X' \in \mathcal{C}$. The subcategory $\mathcal{C}$ is called **covariantly finite** if any $M \in \operatorname{mod} A$ admits a left $\mathcal{C}$-approximation. It is said that $\mathcal{C}$ is **functorially finite** if it is both contravariantly and covariantly finite.

1.1 Finite dimensional k-algebras

We are interested in studying categories of finitely generated modules $\operatorname{mod} A$, where A is a finite dimensional k-algebra and k is an algebraically closed field. In this section, we will present some useful tools and techniques for this purpose, many of which are used almost exclusively in this setting (such as the Auslander-Reiten translation and the duality functor). For a comprehensive and detailed reading, we recommend (7) and (8).

Finite dimensional, connected and basic k-algebras

Let A be an k-algebra, we say that A is a **finite dimensional k-algebra** if A_A is a finite dimensional k-vector space. We denote by $\operatorname{Mod} A$ the category of right A-modules and by $\operatorname{Mod} A^{op}$ the category of left A-modules, which can be viewed as the category of right A^{op}-modules, where A^{op} is the opposite algebra of A, having the same elements as A but with the product of $a, b \in A$ defined as follows: $a \times_{op} b = ba$ (where ba denotes the product of b, a in A). We are interested in the full subcategory of $\operatorname{Mod} A$ consisting of **finitely generated** A-modules, that is, right A-modules M such that there exists an epimorphism $f : A^{\oplus n} \longrightarrow M$ for some $n \in \mathbb{N}$. We will denote this subcategory by $\operatorname{mod} A$.

A A-module M is finitely generated if and only if it is a finite dimensional k-vector space. This result holds when we assume that A is a finite dimensional k-algebra, but it does not hold when A is an infinite dimensional k-algebra. For example, consider $A = \mathbb{C}[x]$. Clearly, A_A is an infinite dimensional $\mathbb{C}$-vector space, yet A_A is finitely generated over A.

We say that A is a **connected** k-algebra if $A \neq 0$ and if there exists a k-algebra isomorphism $f : A \longrightarrow A_1 \times A_2$, then $A_1 = 0$ or $A_2 = 0$. The following proposition is useful for determining whether an algebra is connected or not. We say that $e \in A$ is **idempotent** if $e^2 = e$, and we also say that an idempotent is **central** if $ea = ae$ for all $a \in A$.

Proposition 1.1.1. *An k-algebra is connected if and only if its only central idempotents are 0 and 1.*

The next proposition shows that to study categories of finitely generated modules, we can consider without loss of generality that A is a connected k-algebra.

Proposition 1.1.2. *Suppose that $A = A_1 \times A_2$. Then the category* $\operatorname{mod} A$ *is equivalent to the product category* $\operatorname{mod} A_1 \times \operatorname{mod} A_2$.

If A is a finite dimensional k-algebra, then $A_A = P(1)^{\oplus a_1} \bigoplus P(2)^{\oplus a_2} \bigoplus \cdots \bigoplus P(n)^{\oplus a_n}$ decomposes as a direct sum of a finite number of indecomposable projective modules, where $a_j \in \mathbb{N}$ for all $j = 1, 2, \cdots, n$ and $P(j) \ncong P(i)$ if $i \neq j$. We say that A is a **basic k-algebra** if $a_j = 1$ for all $j = 1, 2, \cdots, n$, that is, if two indecomposable factors of A are non-isomorphic.

Let A be a finite dimensional k-algebra such that $A_A = P(1)^{\oplus a_1} \bigoplus \cdots \bigoplus P(n)^{\oplus a_n}$, consider the A-module $M = P(1) \bigoplus P(2) \bigoplus \cdots \bigoplus P(n)$ and the k-algebra $B = \operatorname{End}_A(M)$, we have that B is a basic algebra. We also have that A and B are **Morita equivalent**, that is, there is an equivalence of categories between mod A and mod B. Therefore, to study the category of finitely generated modules mod A, we can assume without loss of generality that A is a finite dimensional k-algebra that is connected and basic.

Consider the A-module $A_A = P(1) \bigoplus P(2) \bigoplus \cdots \bigoplus P(n)$ where $P(j)$ is indecomposable for $j = 1, 2, \cdots, n$. There exists a set of idempotents $\{e_1, e_2, \cdots, e_n\}$ such that $P(j) = e_j A$, where the e_j are **orthogonal**, that is, $e_i e_j = 0$ if $i \neq j$, and the e_j are **primitive**, that is, if $e_j = e'_j + e''_j$ with e'_j, e''_j orthogonal idempotents, then $e'_j = 0$ or $e''_j = 0$. Furthermore, such a set is **complete**, that is, $1 = e_1 + e_2 + \cdots + e_n$.

Path algebras

We can characterize all finite dimensional k-algebras that are basic and connected. To do this, we start by defining a quiver.

A **quiver** is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is a set whose elements are called **vertices**, Q_1 is a set whose elements are called **arrows**, and $s, t : Q_1 \longrightarrow Q_0$ are two functions called **source** and **target**, respectively. Given an arrow $\alpha \in Q_1$, we say that the **start of α** is $s(\alpha)$ and the **end of α** is $t(\alpha)$. If $s(\alpha) = a$ and $t(\alpha) = b$, we will denote this situation by $\alpha : a \to b$.

Let $a, b \in Q_0$. A **path** γ of length $l(\gamma) \geq 1$ starting at a and ending at b is a sequence $(a|\alpha_1, \alpha_2, \cdots, \alpha_l|b)$ where $\alpha_k \in Q_1$ for $k = 1, 2, \cdots, l$, $s(\alpha_1) = a$, $t(\alpha_l) = b$, $t(\alpha_k) = s(\alpha_{k+1})$ for $k = 1, 2, \cdots, l-1$ and $l = l(\gamma)$. We will denote such a path by $\alpha_1 \alpha_2 \cdots \alpha_l$. Furthermore, for each vertex $a \in Q_0$, we define a path of length 0, $(a||a)$, denoted by ϵ_a. For a path $\gamma = (a|\alpha_1, \alpha_2, \cdots, \alpha_l|b)$ we define $s(\gamma) = a$ and $t(\gamma) = b$.

Given a quiver Q and a field k, we define the **path algebra** kQ as follows: kQ is a k-vector space with a basis given by all paths of length $l \geq 0$ in Q. The product is defined as follows: if $\gamma_1 = (a|\alpha_1, \alpha_2, \cdots, \alpha_l|b)$ and $\gamma_2 = (c|\beta_1, \beta_2, \cdots, \beta_k|d)$, then

$$\gamma_1 \gamma_2 = \begin{cases} (a|\alpha_1, \alpha_2, \cdots, \alpha_l, \beta_1, \cdots, \beta_k|d), & \text{if } b = c, \\ 0, & \text{otherwise.} \end{cases}$$

A **relation** in Q is a k-linear combination of paths of length $l \geq 2$ that have the same start and the same end. That is, a relation ρ is an element of kQ of the form $\rho = \sum_{j=1}^{n} \lambda_i \gamma_i$, where $\lambda_i \in k$ (not all zero), $s(\gamma_i) = s(\gamma_k)$, $t(\gamma_i) = t(\gamma_k)$, and $l(\gamma_k) \geq 2$ for all $i, k = 1, 2, \cdots, n$.

Given a quiver Q, we denote by R^n the ideal of kQ consisting of elements of length n (if $n = 1$, we can think of R as the ideal generated by the arrows). We say that an ideal I of kQ is **admissible** if $R^m \subseteq I \subseteq R^2$ for some $m \in \mathbb{N}$.

Note that if Q is a finite quiver (i.e., both Q_0 and Q_1 are finite sets) and I is an admissible ideal of kQ, then there exists a finite set of relations such that I is generated by this set of relations. The pair (Q, I) will be called a **quiver with relations**, and we associate with it the algebra kQ/I, called the **path algebra with relations**.

We can now state a characterization of finite dimensional k-algebras that are connected and basic.

Theorem 1.1.3 (Gabriel's Theorem). *Let A be a finite dimensional, connected, basic k-algebra, where k is an algebraically closed field. Then there exists a quiver Q and an admissible ideal I of kQ such that $A \cong kQ/I$.*

An interesting fact is that if $A = kQ/I$, then to obtain A^{op} it suffices to "reverse" the direction of the arrows in Q and the relations in I, that is, $A^{op} = kQ'/I'$, where Q' is such that $Q'_0 = Q_0$ and Q'_1 is such that $\alpha' : a \to b \in Q'_1$ if and only if $\alpha : b \to a \in Q_1$, and I' has the same relations as I but with all paths reversed in direction.

Representations of quivers

Given a quiver with relations (Q, I), we define a ***k*-linear representation *M* of (Q, I)** as follows:

1. For each vertex $a \in Q_0$, we associate a k-vector space M_a.
2. For each arrow $\alpha : a \to b$ in Q_1, we associate a k-linear map $\varphi_\alpha : M_a \longrightarrow M_b$ such that, if $\rho = \sum_{j=1}^{n} \lambda_j \gamma_j$ is a relation in I, then $\varphi_\rho := \sum_{j=1}^{n} \lambda_j \varphi_{\gamma_j} = 0$, where $\varphi_\gamma = \varphi_{\alpha_n} \circ \cdots \circ \varphi_{\alpha_2} \circ \varphi_{\alpha_1}$ if $\gamma = \alpha_1 \alpha_2 \cdots \alpha_n$.

Given two representations M and N, we define a **morphism of representations** as follows: for each $a \in Q_0$, we associate a k-linear map $f_a : M_a \longrightarrow N_a$ such that $\varphi'_\alpha f_a = f_b \varphi_\alpha$, where $\alpha : a \to b$ and φ_α, φ'_α are the maps defined in the representations M and N, respectively.

The composition of morphisms between representations is defined in the natural way: if $f : M \longrightarrow N$ and $g : N \longrightarrow L$ are morphisms of representations, then $g \circ f : M \longrightarrow L$ is given componentwise by $(g \circ f)_a = g_a \circ f_a$ for each $a \in Q_0$.

We say that a representation M is **finite dimensional** if $\bigoplus_{a \in Q_0} M_a$ is a finite dimensional k-vector space. It is easy to see that the class of k-linear representations of (Q, I) defines a category, denoted by $Rep_k(Q, I)$, where the morphisms are morphisms of representations. We denote by $rep_k(Q, I)$ the full subcategory of $Rep_k(Q, I)$ whose objects are finite dimensional representations.

We have the following theorem, which simplifies the study of the category $\operatorname{mod} A$.

Theorem 1.1.4. *Let $A = kQ/I$ be a finite dimensional, connected, basic k-algebra, where Q is a finite quiver, and I is an admissible ideal of kQ. Then there exists a k-linear* *equivalence of categories*

$$F : \operatorname{Mod} A \longrightarrow Rep_k(Q, I)$$

whose restriction defines an equivalence

$$F : \operatorname{mod} A \longrightarrow rep_k(Q, I).$$

Due to this equivalence, we will treat a A-module as synonymous with its k-linear representation via this equivalence.

Projectives, Injectives and Simple modules

If A is a finite dimensional k-algebra that is connected and basic, then Gabriel's theorem states that there exists a quiver with relations (Q, I) such that $A \cong kQ/I$. It is possible to prove that Q_0 will be a finite set in this case. It is convenient to assume that $Q_0 = \mathbb{I}_n$, and in this case, we have that $|A_A| = n$.

For each $k \in \mathbb{I}_n$, there exists an associated indecomposable projective A-module, which we denote by $P(k)$. Conversely, each indecomposable projective A-module is associated with a vertex of Q. Since there exists a primitive idempotent e_k such that $P(k) \cong e_k A$, an enumeration of the vertices of the quiver corresponds to an enumeration of a complete set of pairwise orthogonal primitive idempotents. We also associate with each vertex the simple A-module $S(k) = \operatorname{top} P(k)$. All simple modules are of this form.

Consider the **duality functor** $D : \operatorname{mod} A \longrightarrow \operatorname{mod} A^{op}$ given by $D(-) = \operatorname{Hom}_k(-, k)$. The functor D establishes a contravariant equivalence between $\operatorname{mod} A$ and $\operatorname{mod} A^{op}$, whose quasi-inverse will again be denoted by $D : \operatorname{mod} A^{op} \longrightarrow \operatorname{mod} A$, where $D(-) = \operatorname{Hom}_k(-, k)$. The fact that the duality functor is a contravariant equivalence implies a (covariant) equivalence between $\operatorname{mod} A^{op}$ and $(\operatorname{mod} A)^{op}$. This equivalence is false in the general case, where, although $(\operatorname{mod} A)^{op}$ is an abelian category, it is not necessarily a category of modules.

Consider the projective A^{op}-module Ae_k. We have that $I(k) := D(Ae_k)$ is an indecomposable injective A-module. All injective modules are of this form, and we have that $\operatorname{soc} I(k) = S(k)$.

Jordan-Hölder Theorem

Given a nonzero $M \in \operatorname{mod} A$, there exists a finite chain

$$\eta : 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$$

such that M_i / M_{i-1} is isomorphic to a simple module $S(k_i)$ for some $k_i \in \mathbb{I}_n$, for all $i = 1, 2, \cdots, m$, where $n = |A_A|$. Such a chain is called a **composition series**. Given a composition series η of M, we denote by $[M : S(k)]_\eta$ the number of quotients isomorphic to $S(k)$ in η.

Theorem 1.1.5 (Jordan-Hölder theorem)**.** *Let A be a finite dimensional k-algebra and $M \in \operatorname{mod} A$ a nonzero module, η and η' two composition series for M. Then $[M : S(k)]_\eta = [M : S(k)]_{\eta'}$ for all $k = 1, 2, \cdots, n$.*

We denote $[M : S(k)]$ as the multiplicity $[M : S(k)]_\eta$ for any composition series of M, and we say that $S(k)$ is a **composition factor** of M if $[M : S(k)] \neq 0$.

It is immediate that the value $l(M) = \sum_{k=1}^{n} [M : S(k)]$ is well defined and does not depend on the choice of the composition series. We call this value the **length of M** and say that M has finite length. It is convenient to define $l(0) = 0$.

Length generalizes the notion of dimension. In fact, the rank-nullity theorem implies that if

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is an exact sequence of k-vector spaces, then $\dim M = \dim L + \dim N$. Similarly, if such a sequence is an exact sequence of A-modules, then $l(M) = l(L) + l(N)$. Furthermore, we have $[M : S(k)] = [L : S(k)] + [N : S(k)]$ for all $k = 1, 2, \cdots, n$.

Another important fact is that if $M \in \operatorname{mod} A$, then M is both artinian and noetherian. We say that M is **artinian** (respectively, **noetherian**) if every set of submodules of M has a minimal (respectively, maximal) element with respect to inclusion.

Homology

A morphism $f : N \longrightarrow M$ is called **left minimal** if, for every $h : M \longrightarrow M$ such that $hf = f$, h is an automorphism. Dually, such a morphism is called **right minimal** if, for every $g : N \longrightarrow N$ such that $fg = f$, g is an automorphism.

Given a module $M \in \operatorname{mod} A$, there exist long exact sequences of the form

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \twoheadrightarrow M \longrightarrow 0$$

where P_j is a projective module for every j. Such a sequence is called a **projective resolution**. If all morphisms in a projective resolution are right minimal, we will refer to it as a **minimal projective resolution**. It can be proven that the minimal projective resolution always exists and is unique up to isomorphism.

We define the **projective dimension** of M, denoted by $\operatorname{pd}(M)$, as:

$$\operatorname{pd}(M) = \sup\{m \mid P_m \neq 0\},$$

where the P_m are obtained from the minimal projective resolution. It is possible that $\operatorname{pd}(M) = \infty$.

We also define the **global dimension** of the algebra, denoted by $\operatorname{gl.dim}(A)$, as:

$$\operatorname{gl.dim}(A) = \sup\{\operatorname{pd}(M) \mid M \in \operatorname{mod} A\}.$$

We will say that A has **finite global dimension** if $\operatorname{gl.dim}(A) < \infty$, and we will say that A has **infinite global dimension** otherwise.

Much can be said about the module category of an algebra when its global dimension is known. For example, if $\operatorname{gl.dim}(A) = m$, then both $\operatorname{Ext}^j_A(-,M)$ and $\operatorname{Ext}^j_A(M,-)$ are equal to 0 for all $M \in \operatorname{mod} A$ and $j > m$.

A particular case of strong interest is that of hereditary algebras. We say that A is a **hereditary algebra** if $\operatorname{gl.dim}(A) \leq 1$, and, in particular, we say that A is a **semisimple algebra** if $\operatorname{gl.dim}(A) = 0$.

We can characterize hereditary algebras as those where every submodule of a projective module is projective, and dually, where every quotient of an injective module is injective.

If $A = kQ/I$, then A is hereditary if and only if $I = 0$, that is, $A = kQ$. Furthermore, A is semisimple if and only if $A = kQ$, where Q is a finite quiver with $Q_1 = \emptyset$.

Short exact sequences

Short exact sequences play a central role in the study of module categories and abelian categories in general, being one of the most important tools.

Consider the following diagram:

$$\begin{array}{ccccccccc} & & & & & & N' & & \\ & & & & & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

where the bottom row is exact. It is possible to complete the diagram above to obtain the **pull-back diagram**

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_L} & & \downarrow & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

where the top row is exact, and the squares commute. If h is an epimorphism, then the diagram can be completed as follows:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & \xrightarrow{1_K} & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_L} & & \downarrow & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where both rows and columns are exact, and all squares commute.

Analogously, consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h'} & & & & & & \\
 & & L' & & & & & &
\end{array}
$$

where the top row is exact. It can be completed to the **push-out diagram**

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h'} & & \downarrow & & \downarrow{\scriptstyle 1_N} & & \\
0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0
\end{array}
$$

where both rows are exact, and the squares commute. If h' is a monomorphism, then we obtain:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h'} & & \downarrow & & \downarrow{\scriptstyle 1_N} & & \\
0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Q & \xrightarrow{1_Q} & Q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where both rows and columns are exact, and all squares commute.

We also have the theorem known as the Snake Lemma.

Theorem 1.1.6 (Snake Lemma)**.** *Let $L, M, N, L', M', N' \in \operatorname{mod} A$ and consider the following commutative diagram in which the two middle rows are exact.*

$$
\begin{array}{ccccccccc}
 & & \ker f & \dashrightarrow & \ker g & \dashrightarrow & \ker h & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker} f & \dashrightarrow & \operatorname{coker} g & \dashrightarrow & \operatorname{coker} h & &
\end{array}
$$

Then there exists a morphism $\delta : \ker h \longrightarrow \operatorname{coker} f$ that makes the following sequence exact:

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \longrightarrow \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h.$$

Consider two exact sequences ϵ and ϵ' with the same endpoints. We say that **ϵ and ϵ' are equivalent** if there exists an isomorphism $h : M \longrightarrow M'$ such that the following diagram commutes:

$$
\begin{array}{lccccccccc}
\epsilon : & 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & & \downarrow 1_L & & \downarrow h & & \downarrow 1_N & & \\
\epsilon' : & 0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0
\end{array}
$$

We denote by $E(N, L)$ the set of equivalence classes of exact sequences that begin with L and end with N, and we denote by $[\epsilon]$ the equivalence class containing ϵ. It is possible to endow the set $E(N, L)$ with a k-vector space structure using pull-back and push-out diagrams. For details on the construction of this structure, we recommend (10, Chapter I, Section 5) or (37, Chapter VII). The zero element of the vector space $E(N, L)$ is the class of the **split exact sequences**, one of whose representatives is given by:

$$0 \longrightarrow L \xrightarrow{\ i\ } L \bigoplus N \xrightarrow{\ \pi\ } N \longrightarrow 0$$

where i is the inclusion into the first coordinate, and π is the projection onto the second.

It is known that there exists a k-vector space isomorphism between $E(N, L)$ and $\operatorname{Ext}^1_A(N, L)$, such that $\operatorname{Ext}^1_A(N, L) = 0$ if and only if every exact sequence that begins with L and ends with N is split.

Consider a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and a module $M' \in \operatorname{mod} A$. The application of the functor $\operatorname{Hom}_A(M',-)$ to the above exact sequence induces exact sequences of k-vector spaces:

$$0 \longrightarrow \operatorname{Hom}_A(M',L) \longrightarrow \operatorname{Hom}_A(M',M) \longrightarrow \operatorname{Hom}_A(M',N)$$
$$\operatorname{Ext}^j_A(M',L) \longrightarrow \operatorname{Ext}^j_A(M',M) \longrightarrow \operatorname{Ext}^j_A(M',N)$$

for every $j \geq 1$. There also exist morphisms $\delta : \operatorname{Hom}(M',N) \longrightarrow \operatorname{Ext}^1_A(M',L)$ and $\delta_j : \operatorname{Ext}^j_A(M',N) \longrightarrow \operatorname{Ext}^{j+1}_A(M',L)$ for every $j \geq 1$ that "connect" all the above sequences into a long exact sequence. These linear transformations are known as **connecting morphisms** (it is possible to dualize the above result by applying the functor $\operatorname{Hom}_A(-,M')$).

There exists an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N^{\oplus m} \longrightarrow 0$$

such that the connecting morphism $\delta : \operatorname{Hom}(N,N^{\oplus m}) \longrightarrow \operatorname{Ext}^1_A(N,L)$ is surjective. Such a sequence is called a **universal extension**. The construction of the universal extension proceeds as follows: we fix $m = \dim \operatorname{Ext}^1_A(N,L)$, take exact sequences $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$ such that $[\epsilon_1], [\epsilon_2], \cdots, [\epsilon_m]$ form a basis of $E(N,L)$. Taking the direct sum of the m exact sequences, we obtain the following exact sequence:

$$0 \longrightarrow L^{\oplus m} \longrightarrow E \longrightarrow N^{\oplus m} \longrightarrow 0$$

Defining $h : L^{\oplus m} \longrightarrow L$ as the map that takes the vector $(l_1, l_2, \cdots, l_m)^T$ to $l_1 + \cdots + l_m$, we obtain the following push-out diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^{\oplus m} & \longrightarrow & E & \longrightarrow & N^{\oplus m} & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N^{\oplus m} & \longrightarrow & 0 \end{array}$$

The second row constitutes the universal extension we were looking for.

The Auslander-Reiten translation

We define the **stable category modulo projectives** $\underline{\operatorname{mod}} A$, whose objects are the same as those of $\operatorname{mod} A$, and whose morphisms are the same as in $\operatorname{mod} A$, modulo the following relation: we say that $f, g : M \longrightarrow N$ in $\operatorname{mod} A$ are congruent if their difference factors through a projective module, that is, if there exists a projective module P and morphisms $h_1 : M \longrightarrow P$ and $h_2 : P \longrightarrow N$ such that $f - g = h_2 h_1$. In this case, we denote $f \cong g$. Notice that every projective module P becomes isomorphic to 0 in $\underline{\operatorname{mod}} A$ since if $f : P \longrightarrow M$ and $g : M \longrightarrow P$, then $f = f \circ 1_P$ and $g = 1_P \circ g$. Dually, we define the **stable category modulo injectives** $\overline{\operatorname{mod}} A$.

Consider the functor $(-)^* := \mathrm{Hom}_A(-,A) : \mathrm{proj}\, A \longrightarrow \mathrm{proj}\, A^{op}$, where $\mathrm{proj}\, A$ denotes the full subcategory of $\mathrm{mod}\, A$ whose objects are projective A-modules, and consider the **minimal projective presentation** of $M \in \mathrm{mod}\, A$

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

where d_0 and d_1 are minimal right morphisms. Applying the functor $(-)^*$ to d_1 and taking the cokernel, we obtain the **transpose of M**, denoted by $\mathrm{Tr}\, M$

$$P_0^* \xrightarrow{d_1^*} P_1^* \longrightarrow \mathrm{Tr}\, M \longrightarrow 0.$$

We define the **Nakayama functor** $\nu : D\circ(-)^* : \mathrm{proj}\, A \longrightarrow \mathrm{inj}\, A$, whose quasi-inverse is given by $\nu^{-1} : (-)^*\circ D : \mathrm{inj}\, A \longrightarrow \mathrm{proj}\, A$, where $\mathrm{inj}\, A$ denotes the full subcategory of $\mathrm{mod}\, A$ whose objects are injective A-modules. Applying the functor ν to d_1 and taking the kernel, we obtain the **Auslander-Reiten translation** $\tau : \underline{\mathrm{mod}} A \longrightarrow \overline{\mathrm{mod}} A$

$$0 \longrightarrow \tau(M) \longrightarrow \nu(P_1) \xrightarrow{\nu(d_1)} \nu(P_0).$$

Notice that $\tau = D\circ\mathrm{Tr}$. The Auslander-Reiten translation is a covariant functor that defines an equivalence between the categories $\underline{\mathrm{mod}} A$ and $\overline{\mathrm{mod}} A$, whose quasi-inverse is given by $\tau^{-1} = \mathrm{Tr}\circ D$. In particular, $\tau(M) = 0$ if and only if M is a projective A-module. Furthermore, if $M, N \notin \mathrm{proj}\, A$ with $M \ncong N$ both indecomposable, then $\tau(M) \ncong \tau(N)$.

If $M \notin \mathrm{proj}\, A$ is indecomposable, then $\tau(M)$ is also indecomposable. Moreover, we have that $\tau(M \bigoplus N) = \tau(M) \bigoplus \tau(N)$.

The Auslander-Reiten translation of M establishes connections between exact sequences involving M and morphisms involving $\tau(M)$. For example, if $M \notin \mathrm{proj}\, A$, then $\mathrm{Ext}_A^1(M, \tau(M)) \neq 0$. In particular, there exists at least one non-split exact sequence

$$0 \longrightarrow \tau(M) \longrightarrow N \longrightarrow M \longrightarrow 0.$$

Similarly, if $M \notin \mathrm{inj}\, A$, then $\mathrm{Ext}_A^1(\tau^{-1}(M), M) \neq 0$, and there exists at least one non-split exact sequence that begin with M and end with $\tau^{-1}(M)$.

We also have the following formulas known as the **Auslander-Reiten duality**

$$\mathrm{Ext}_A^1(M, N) \cong D\underline{\mathrm{Hom}}_A(\tau^{-1}(N), M) \cong D\overline{\mathrm{Hom}}_A(N, \tau(M))$$

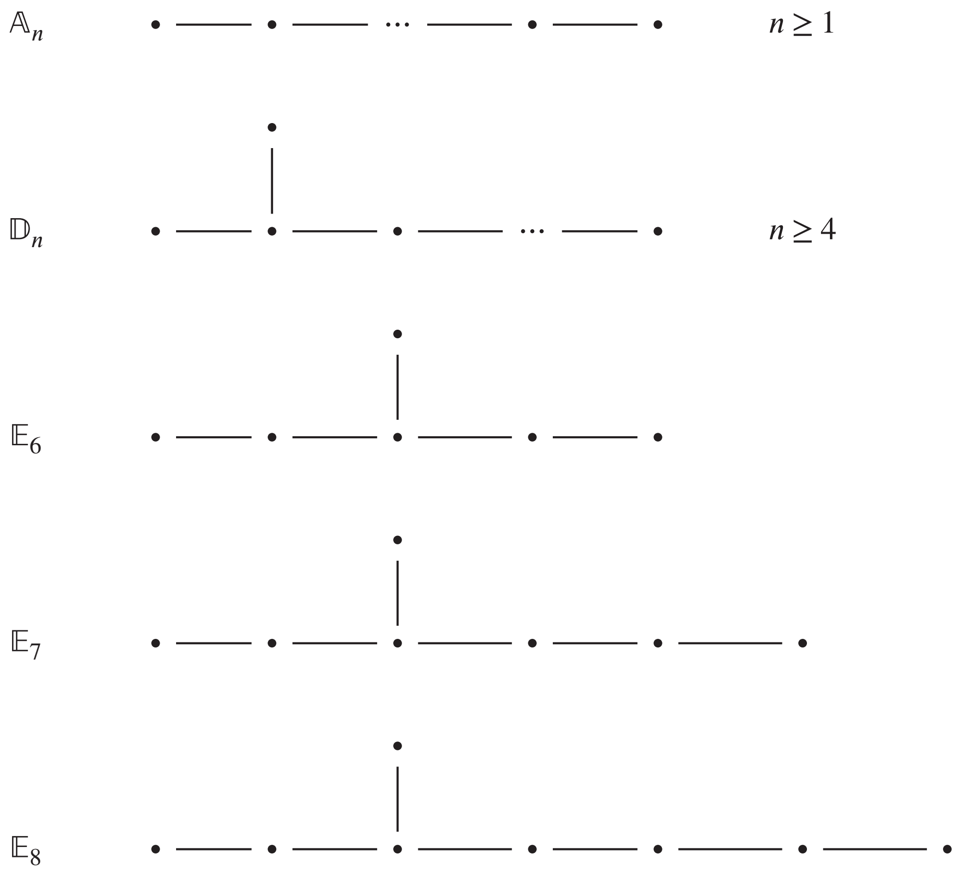
where $\underline{\mathrm{Hom}}_A(M, N)$ and $\overline{\mathrm{Hom}}_A(M, N)$ denote the set of morphisms between M and N in the categories $\underline{\mathrm{mod}} A$ and $\overline{\mathrm{mod}} A$, respectively. In particular, if A is a hereditary algebra, the Auslander-Reiten duality simplifies to

$$\mathrm{Ext}_A^1(M, N) \cong D\,\mathrm{Hom}_A(\tau^{-1}(N), M) \cong D\,\mathrm{Hom}_A(N, \tau(M)).$$

The Auslander-Reiten quiver

We say that A is a **representation-finite algebra** if $\mathrm{Ind}(A)$ is a finite set, where $\mathrm{Ind}(A)$ denotes a complete set of representatives of the isomorphism classes of indecomposable A-modules in $\mathrm{mod}\,A$. Otherwise, we say that A is a **representation-infinite algebra**.

We can characterize representation-finite hereditary algebras as those of the form $A = kQ$, where the underlying graph of Q is one of the following cases, known as **Dynkin diagrams**:



The underlying graph of a quiver Q, denoted by $\bar{Q}$, is defined as follows: the vertices of $\bar{Q}$ coincide with the set Q_0, and the edges from a to b (with $a, b \in Q_0$) are in bijection with the arrows $\alpha : a \to b$ and $\beta : b \to a$. We can imagine obtaining the underlying graph by "forgetting" the directions of the arrows in Q.

Let $X, Y \in \mathrm{mod}\,A$ be two indecomposable modules. We define $\mathbf{rad}_A(X, Y)$ as the k-vector space of non-invertible morphisms $f : X \longrightarrow Y$. For a fixed $n > 1$, we also define $\mathrm{rad}^n_A(X, Y)$ as the k-vector space of morphisms in $\mathrm{mod}\,A$ of the form $f_n \cdots f_2 f_1$, where $f_k : Z_k \longrightarrow Z_{k+1}$ with $Z_1 = X$, $Z_{n+1} = Y$, and $f_k \in \mathrm{rad}_A(Z_k, Z_{k+1})$ for all $k = 1, 2, \cdots, n$. Finally, we define $\mathrm{Irr}(X, Y) = \mathrm{rad}_A(X, Y) / \mathrm{rad}^2_A(X, Y)$, called the **space of irreducible morphisms** from X to Y.

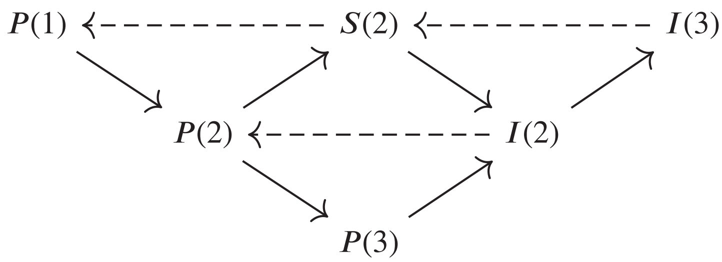
Given a finite dimensional k-algebra A, we define the **Auslander-Reiten quiver** $\Gamma(\operatorname{mod} A)$ as follows: the set of vertices is given by $Q_0 = \operatorname{Ind}(A)$. The number of arrows between $M, N \in \operatorname{Ind} A$ corresponds to the dimension of the k-vector space $\operatorname{Irr}(M, N)$.

Note that the Auslander-Reiten quiver of an algebra is finite if and only if A is a representation-finite algebra. If this is the case and furthermore A is a connected algebra, then the Auslander-Reiten quiver will be connected.

As an example, consider $A = kQ$, where Q is the following quiver:

$$1 \longleftarrow 2 \longleftarrow 3$$

We have that $\#\operatorname{Ind}(A) = 6$. The Auslander-Reiten quiver has only one component and is given by:



where the solid lines represent linearly independent irreducible morphisms. We have that $P(1) = S(1)$, $P(3) = I(1)$, and $I(3) = S(3)$.

It is convenient, but not mandatory, to represent the Auslander-Reiten translates of the indecomposable modules in the Auslander-Reiten quiver. The dashed arrows point to the respective translates. Notice that no dashed arrow originates from a projective module, just as none ends at an injective module.

Another interesting example is the **Kronecker algebra**, given by $A = kQ$, where Q is the following quiver:

$$1 \leftleftarrows 2$$

Since A is a hereditary algebra and $\bar{Q}$ is not a Dynkin diagram, we have that A is a representation-infinite algebra, and consequently, the Auslander-Reiten quiver of A will be infinite.

1.2 Torsion Theory

Torsion Theory in abelian categories was introduced by Dickson in (22) as a categorical generalization of the concept of torsion abelian groups. The results presented here can be found in (8, Chapter VI).

Definition 1.2.1. *A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of* $\operatorname{mod} A$ *is called a* ***torsion pair*** *if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. $\mathcal{T}$ and $\mathcal{F}$ are called the* ***torsion class*** *and* ***torsion-free class****, respectively.*

The next proposition provides a characterization of torsion classes and torsion-free classes.

Proposition 1.2.2. *1. Let $\mathcal{T}$ be a full subcategory of* mod A*. The following conditions are equivalent:*

a) $\mathcal{T}$ is the torsion class of some torsion pair $(\mathcal{T},\mathcal{F})$ in mod A*.*

b) $\mathcal{T}$ is closed under quotients, direct sums and extensions.

2. Let $\mathcal{F}$ be a full subcategory of mod A*. The following conditions are equivalent:*

a) $\mathcal{F}$ is the torsion-free class of some torsion pair $(\mathcal{T},\mathcal{F})$ in mod A*.*

b) $\mathcal{F}$ is closed under submodules, direct products and extensions.

For every $M \in \operatorname{mod} A$, there exists a short exact sequence (unique, up to isomorphism of exact sequences)

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow f(M) \longrightarrow 0$$

such that $t(M)$ is the maximal submodule of M contained in $\mathcal{T}$ and $f(M) \in \mathcal{F}$. This exact sequence, called **canonical short exact sequence** defines covariant functors $t(-) : \operatorname{mod} A \longrightarrow \operatorname{mod} A$ and $f(-) : \operatorname{mod} A \longrightarrow \operatorname{mod} A$, called the **torsion functor** and **torsion-free functor**, respectively. We have that $t(M \bigoplus N) = t(M) \bigoplus t(N)$ and $f(M \bigoplus N) = f(M) \bigoplus f(N)$.

We have that $t(M) \longrightarrow M$ is a $\mathcal{T}$-right approximation of M and $M \longrightarrow f(M)$ is a $\mathcal{F}$-left approximation of M. In particular, $\mathcal{T}$ is always contravariantly finite and $\mathcal{F}$ is always covariantly finite.

A torsion pair $(\mathcal{T},\mathcal{F})$ such that each indecomposable A-module lies either in $\mathcal{T}$ or in $\mathcal{F}$ is called **splitting**. The following proposition characterizes splitting torsion pairs:

Proposition 1.2.3. *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in* mod A*. The following conditions are equivalent:*

1. *$(\mathcal{T},\mathcal{F})$ is splitting.*
2. *For each A-module M, the canonical sequence for M splits.*
3. $\operatorname{Ext}^1_A(N,M) = 0$ *for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$.*
4. *If $M \in \mathcal{T}$, then $\tau^{-1}(M) \in \mathcal{T}$.*
5. *If $N \in \mathcal{F}$, then $\tau(N) \in \mathcal{F}$.*

Proposition 1.2.4. *1. Assume that M is Ext-projective in* $\operatorname{Fac}(M)$*, then* $(\operatorname{Fac}(M), M^{\perp})$ *is a torsion pair.*

2. Assume that M is Ext-injective in $\operatorname{Sub}(M)$*, then* $({}^{\perp}M, \operatorname{Sub}(M))$ *is a torsion pair.*

It is known that if $\mathcal{T} = \operatorname{Fac}(M)$ is a torsion class for some $M \in \operatorname{mod} A$, then $\mathcal{F} = M^{\perp}$; similarly, if $\mathcal{F} = \operatorname{Sub}(M)$ is a torsion-free class, then $\mathcal{T} = {}^{\perp}M$. If C is a class in $\operatorname{mod} A$, then the smallest torsion class containing C is given by $(\mathcal{T}, \mathcal{F})$, where $\mathcal{F} = C^{\perp}$ and $\mathcal{T} = {}^{\perp}\mathcal{F}$. We will denote the smallest torsion class containing C as $\mathsf{T}(C)$. Analogously, the smallest torsion-free class containing C is given by $(\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = {}^{\perp}C$ and $\mathcal{F} = \mathcal{T}^{\perp}$. We will denote the smallest torsion-free class containing C as $\mathsf{F}(C)$. It is known that if $\mathcal{T}$ is a torsion class containing C, then $\mathsf{T}(C) \subseteq \mathcal{T}$. Similarly, if $\mathcal{F}$ is a torsion-free class containing C, then $\mathsf{F}(C) \subseteq \mathcal{F}$. Proposition 1.3.4 provides a characterization of torsion classes of the form Fac and torsion-free classes of the form Sub.

1.3 τ-Tilting Theory

τ-Tilting Theory was introduced by T. Adachi, O. Iyama, and I. Reiten in (1) as a generalization of Tilting Theory and is strongly related to Torsion Theory. We can think of τ-Tilting Theory as “filling the gaps” in Tilting Theory in the non-hereditary case, as when A is a hereditary algebra, the theories coincide. For a review of Tilting Theory, we recommend Chapter VI of (8). We begin by defining tilting and τ-tilting modules.

Definition 1.3.1. *Let A be an k-algebra. A A-module T is called a* **tilting module***if the following conditions are satisfied:*

1. *$pd\, T \leq 1$.*
2. $\operatorname{Ext}^1_A(T, T) = 0$.
3. *There exists a short exact sequence*

$$0 \longrightarrow A_A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

with $T_0, T_1 \in \operatorname{add}(T)$.

A module satisfying conditions 1 and 2 of the definition above is called a **partial tilting module**. It is possible to prove that if T is a tilting module, then $|T| = |A_A|$. Moreover, if T is a partial tilting module, then there exists a tilting module U such that $T|U$ and $\operatorname{Fac}(U) = \{M \mid \operatorname{Ext}^1_A(T, M) = 0\}$. Such a module is called the **Bongartz completion** of T. Dually, one can define cotilting and partial cotilting modules.

Definition 1.3.2. *1. We call M in* $\operatorname{mod} A$ ***τ-rigid*** *if* $\operatorname{Hom}_A(M, \tau M) = 0$.

2. *We call M in* $\operatorname{mod} A$ ***τ-tilting*** *(respectively,* ***almost complete τ-tilting****) if M is τ-rigid and $|M| = |A_A|$ (respectively, $|M| = |A_A| - 1$).*

3. *We call M in* $\operatorname{mod} A$ ***support τ-tilting*** *if there exists an idempotent e of A such that M is a τ-tilting $(A/\langle e \rangle)$-module.*

Dually, M is said to be **τ^{-}-rigid** if $\operatorname{Hom}_A(\tau^{-1} M, M) = 0$. All the above definitions and subsequent results can be dualized for τ^{-}-rigid modules. Just like tilting modules, every τ-rigid module is a direct summand of a τ-tilting module. Furthermore, we have that if M is a τ-rigid module, then $|M| \leq |A_A|$.

Recall that a A-module M is said to be **sincere** if no nonzero idempotent annihilates M, which is equivalent to every simple module appearing as a composition factor of M. A module M is said to be **faithful** if $\operatorname{ann} M = 0$. The next proposition relates tilting and τ-rigid modules.

Proposition 1.3.3. *(1, Proposition 2.2)*

1. *τ-tilting modules are precisely sincere support τ-tilting modules.*

2. *Tilting modules are precisely faithful support τ-tilting modules.*

3. *Any τ-tilting (respectively, τ-rigid) A-module T is a tilting (respectively, partial tilting)* $(A/\operatorname{ann} T)$*-module.*

Proposition 1.3.4. *(1, Proposition 1.1) Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* $\operatorname{mod} A$. *Then the following conditions are equivalent.*

1. *$\mathcal{T}$ is functorially finite.*

2. *$\mathcal{F}$ is functorially finite.*

3. $\mathcal{T} = \operatorname{Fac}(X)$ *for some X in* $\operatorname{mod} A$.

4. $\mathcal{F} = \operatorname{Sub}(Y)$ *for some Y in* $\operatorname{mod} A$.

5. *$P(\mathcal{T})$ is a tilting $(A/ann\ \mathcal{T})$-module.*

6. *$I(\mathcal{F})$ is a cotilting $(A/ann\ \mathcal{F})$-module.*

7. $\mathcal{T} = \operatorname{Fac} P(\mathcal{T})$.

8. $\mathcal{F} = \operatorname{Sub} I(\mathcal{F})$.

Proposition 1.3.5. *(11, Proposition 5.8) The following statements are equivalent for a pair of modules* $M, N \in \operatorname{mod} A$.

1. $\operatorname{Ext}^1_A(M, N'') = 0$ *if* $N'' \in \operatorname{Fac}(N)$.
2. $\operatorname{Hom}_A(N, \tau M) = 0$.

Theorem 1.3.6. *(11, Theorem 5.10) If* M *is* τ*-rigid, then* $\operatorname{Fac}(M)$ *is a functorially finite torsion class and* M *is Ext-projective in* $\operatorname{Fac}(M)$.

Lemma 1.3.7. *(15, Lemma 4.6) If* X *is an indecomposable* A*-module such that* $X \bigoplus U$ *is* τ*-rigid, then either* $f(X)$ *is indecomposable or* $f(X) = 0$*, where* f *is the torsion-free functor with respect to the torsion pairs* $(\operatorname{Fac}(U), U^{\perp})$*. We have* $f(X) = 0$ *if and only if* X *is in* $\operatorname{Fac}(U)$.

We denote by sτ-tilt A (respectively, τ-tilt and tilt A) the set of isomorphism classes of basic support τ-tilting A-modules (respectively, τ-tilting and tilting A-modules). We also denote by f-tors A (respectively, sf-tors A and ff-tors A) the set of functorially finite (respectively, sincere functorially finite and faithful functorially finite) torsion classes in $\operatorname{mod} A$.

Theorem 1.3.8. *(1, Theorem 2.7, Corollary 2.8) There are bijections*

1. *sτ-tilt $A \longleftrightarrow$ f-tors A.*
2. *τ-tilt $A \longleftrightarrow$ sf-tors A.*
3. *tilt $A \longleftrightarrow$ ff-tors A.*

In all cases, the bijection is given by $T \mapsto \operatorname{Fac}(T)$ and $\mathcal{T} \mapsto P(\mathcal{T})$.

Proposition 1.3.9. *(1, Proposition 2.9) Let* $\mathcal{T}$ *be a functorially finite torsion class and* U *a* τ*-rigid* A*-module. Then* $U \in \operatorname{add}(P(\mathcal{T}))$ *if and only if* $\operatorname{Fac}(U) \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)$.

Theorem 1.3.10 (Bongartz Completion)**.** *(1, Theorem 2.10) Let* T *be a* τ*-rigid* A*-module. Then* $\mathcal{T} = {}^{\perp}(\tau T)$ *is a sincere functorially finite torsion class and* $P(\mathcal{T})$ *is a* τ*-tilting* A*-module satisfying* $T \in \operatorname{add}(P(\mathcal{T}))$ *and* $\operatorname{Fac}(P(\mathcal{T})) = {}^{\perp}(\tau P(\mathcal{T}))$.

We call $\mathcal{B}_T := P({}^{\perp}(\tau T))$ the **Bongartz completion of T**.

Theorem 1.3.11. *(1, Theorem 2.12) The following are equivalent for a* τ*-rigid* A*-module* M.

1. M *is* τ*-tilting.*

2. $^{\perp}(\tau M) = \text{Fac}(M)$.

Definition 1.3.12. *Let (M,P) be a pair with $M \in \text{mod}\, A$ and $P \in$ proj A.*

1. *We call (M,P) a **τ-rigid pair** if M is τ-rigid and $\text{Hom}_A(P,M) = 0$.*

2. *We call (M,P) a **support τ-tilting pair** (respectively, **almost complete support τ-tilting pair**) if (M,P) is a τ-rigid pair and $|M| + |P| = |A_A|$ (respectively, $|M| + |P| = |A_A| - 1$).*

Proposition 1.3.13. *(1, Proposition 2.17) Let T be a basic τ-rigid module which is not τ-tilting. Then there are at least two basic support τ-tilting modules which have T as a direct summand.*

We will say that the τ-rigid pair (U,Q) is a direct summand of (T,P) if $U|T$ and $Q|P$.

Theorem 1.3.14. *(1, Theorem 2.18) Any basic almost complete support τ-tilting pair (U,Q) is a direct summand of exactly two basic support τ-tilting pairs (T,P) and (T',P'). Moreover we have $\{\text{Fac}(T), \text{Fac}(T')\} = \{\text{Fac}(U), {}^{\perp}(\tau U) \cap Q^{\perp}\}$.*

Definition 1.3.15 (Mutation)**.** *Two basic support τ-tilting pairs (T,P) and (T',P') are said to be **mutations** of each other if there exists a basic almost complete support τ-tilting pair (U,Q) which is a direct summand of (T,P) and (T',P'). In this case we write $(T',P') = \mu_X(T,P)$ or simply $T' = \mu_X(T)$ if X is an indecomposable A-module satisfying either $T = U \bigoplus X$ or $P = Q \bigoplus X$.*

Let (T,P) be a basic support τ-tilting pair and X an indecomposable direct summand of either T or P.

1. If X is a direct summand of T, precisely one the following holds.

   a) There exists an indecomposable A-module Y such that $X \ncong Y$ and $\mu_X(T,P) = (T/X \bigoplus Y, P)$ is a basic support τ-tilting pair.

   b) There exists an indecomposable projective A-module Y such that $\mu_X(T,P) = (T/X, P \bigoplus Y)$ is a basic support τ-tilting pair.

2. If X is a direct summand of P, there exists an indecomposable A-module Y such that $\mu_X(T,P) = (T \bigoplus Y, P/X)$ is a basic support τ-tilting pair.

Proposition 1.3.16. *(1, Proposition 2.22) Let $T = X \bigoplus U$ be a basic τ-tilting A-module, with X indecomposable. Then exactly one of ${}^{\perp}(\tau U) \subseteq {}^{\perp}(\tau X)$ and $X \in \text{Fac}(U)$ holds.*

Proposition 1.3.17. *(1, Proposition 2.28) Let $T = X \bigoplus U$ be a basic τ-tilting A-module, with X indecomposable. Then the following conditions are equivalent:*

1. ${}^{\perp}(\tau U) \subseteq {}^{\perp}(\tau X)$.

2. *T is the Bongartz completion of U.*

Theorem 1.3.18. *(21, Theorem 3.1) Let A be a finite dimensional algebra and M a support τ-tilting A-module. Then, the following statements hold:*

1. *Let $\mathcal{T}$ be a torsion class in* $\operatorname{mod} A$ *such that* $\mathcal{T} \subsetneq \operatorname{Fac}(M)$. *Then, there exists $N \in$ $s\tau$-tilt A-module satisfying the following conditions:*

    a) *The support τ-tilting A-modules M and N are mutation of each other.*

    b) *We have* $\mathcal{T} \subseteq \operatorname{Fac}(N) \subsetneq \operatorname{Fac}(M)$.

2. *Let $\mathcal{T}$ be a torsion class in* $\operatorname{mod} A$ *such that* $\operatorname{Fac}(M) \subsetneq \mathcal{T}$. *Then, there exists $L \in$ $s\tau$-tilt A-module satisfying the following conditions:*

    a) *The support τ-tilting A-modules M and L are mutation of each other.*

    b) *We have* $\operatorname{Fac}(M) \subsetneq \operatorname{Fac}(L) \subseteq \mathcal{T}$.

1.4 n-Representation infinite algebras

n-representation infinite algebras were introduced by M. Herschend, O. Iyama, and S. Oppermann in (26). One of the central motivations in higher Auslander-Reiten theory is to generalize the classical notion of representation-finite algebras. In this context, the class of **n-representation finite algebras** arises naturally as a higher analog of representation-finite hereditary algebras. An algebra A of global dimension n is called n-representation finite if for every indecomposable projective module P, there exists an integer $l_P \geq 0$ such that $\nu_n^{-l_P}(P)$ is an indecomposable injective module. Here, $\nu_n := \nu \circ [-n]$ is defined as the composition of the classical Nakayama functor $\nu = D\operatorname{Hom}_A(-, A)$ and the $[-n]$ shift in the bounded derived category $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Thus for $n = 1$ the notion of 1-representation finite algebras coincides with the classical notion of representation finite hereditary algebras.

In analogy with the classical case, it is natural to ask what should play the role of representation-infinite algebras in higher dimensions.

Let n be a positive integer, and let A be a connected, finite dimensional k-algebra which has global dimension at most n and $\mathcal{D}^b(\operatorname{mod} A)$ it's bounded derived category (for an introduction to derived categories, see (42, Chapter III)). The Nakayama functors are defined as follows

$$\nu := DR\operatorname{Hom}_A(-, A) : \mathcal{D}^b(\operatorname{mod} A) \longrightarrow \mathcal{D}^b(\operatorname{mod} A).$$
$$\nu^{-1} := R\operatorname{Hom}_{A^{op}}(D-, A) : \mathcal{D}^b(\operatorname{mod} A) \longrightarrow \mathcal{D}^b(\operatorname{mod} A).$$

They are quase-inverse each other. We also define the functors

$$\nu_n := \nu \circ [-n] : \mathcal{D}^b(\operatorname{mod} A) \longrightarrow \mathcal{D}^b(\operatorname{mod} A).$$
$$\nu_n^{-1} := [n] \circ \nu^{-1} : \mathcal{D}^b(\operatorname{mod} A) \longrightarrow \mathcal{D}^b(\operatorname{mod} A).$$

Definition 1.4.1. *Let n be a positive integer. A finite dimensional algebra of global dimension at most n is called* ***n-representation infinite*** *if any indecomposable projective A-module P satisfies that* $\nu_n^{-j}(P)$ *is a module (i. e. concentrad in degree 0) for all* $j \geq 0$.

Proposition 1.4.2. *(26, Proposition 2.3 (b)) If A is an n-representation infinite algebra, then* $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\nu_n^i(A), \nu_n^j(A)) = 0$ *if* $i < j$.

Proposition 1.4.2 is equivalent to $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\nu_n^i(DA), \nu_n^j(DA)) = 0$ if $i > j$. Note that $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\nu_n^i(A), \nu_n^j(A)) = 0$ implies that $\operatorname{Hom}_A(\nu_n^i(A), \nu_n^j(A)) = 0$.

Proposition 1.4.3. *(26, Proposition 2.9) The following conditions are equivalent.*

(a) A is an n-representation infinite algebra.

(b) $\nu_n^{-i}(A) \in \operatorname{mod} A$ *for any* $i \geq 0$ *and A has global dimension at most n.*

(c) $\nu_n^{i}(DA) \in \operatorname{mod} A$ *for any* $i \geq 0$ *and A has global dimension at most n.*

Let $\mathscr{P} := add\{\nu_n^{-i}(A) \mid i \geq 0\}$ and $\mathscr{I} := add\{\nu_n^{i}(D(A)) \mid i \geq 0\}$ be two full subcategories of $\operatorname{mod} A$. We call modules in $\mathscr{P}$ (respectively, $\mathscr{I}$) ***n*-preprojective** (respectively, ***n*-preinjective**) modules.

Proposition 1.4.4. *(26, Proposition 4.10) Let A be an n-representation infinite algebra, with* $A_A = \bigoplus_{j=1}^{N} P(j)$*. The following assertions hold.*

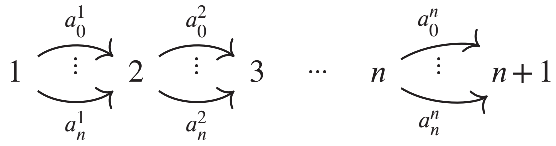
(a) We have a bijection from $\mathbb{I}_N \times \mathbb{Z}_{\geq 0}$ *to the set of indecomposable modules in* $\mathscr{P}$ *given by* $(j,i) \mapsto \nu_n^{-i}(P(j))$.

(b) We have a bijection from $\mathbb{I}_N \times \mathbb{Z}_{\geq 0}$ *to the set of indecomposable modules in* $\mathscr{I}$ *given by* $(j,i) \to \nu_n^{i}(D(P(j)))$.

(c) $\operatorname{Hom}_A(\mathscr{I}, \mathscr{P}) = 0$.

(d) $\operatorname{Ext}_A^j(\mathscr{P} \vee \mathscr{I}, \mathscr{P} \vee \mathscr{I}) = 0$ *for any j with* $0 < j < n$.

An example of an *n*-representation infinite algebra is the Beilinson algebra $A = kQ/I$, where Q is the following quiver:



and I is the ideal generated by $a_i^k a_j^{k+1} - a_j^k a_i^{k+1}$ for all $k \in \mathbb{I}_{n-1}$ and $i, j \in \mathbb{I}_n \cup \{0\}$ (26, Example 2.15). Note that $|A_A| = N = n+1$.

1.5 Filtered modules

Given a class $\mathcal{C}$ of modules, we denote by $\mathcal{F}(\mathcal{C})$ the full subcategory of $\operatorname{mod} A$ that contains the zero module and the nonzero modules M such that there is a finite chain

$$\eta: \ 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

of submodules of M such that M_i/M_{i-1} is isomorphic to a module in $\mathcal{C}$ for all $i = 1, 2, \cdots, n$. We say that a nonzero module in $\mathcal{F}(\mathcal{C})$ is a **$\mathcal{C}$-filtered module** and that η is a **$\mathcal{C}$-filtration of M**.

We have that $\mathcal{F}(\mathcal{C})$ is closed under extensions. Indeed, consider the short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

with $L, N \in \mathcal{F}(\mathcal{C})$. In fact, if

$$\eta_L: \ 0 = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_t$$

and

$$\eta_N: \ 0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_s$$

are $\mathcal{C}$-filtrations of L and N, respectively, then

$$\eta_M: \ 0 \subseteq f(L_1) \subseteq f(L_2) \subseteq \cdots \subseteq f(L_t) \subseteq g^{-1}(N_1) \subseteq \cdots \subseteq g^{-1}(N_s) = M \tag{1.1}$$

is a $\mathcal{C}$-filtration of M, where $g^{-1}(N_k)$ denotes the preimage of N_k in M. In particular, $\mathcal{F}(\mathcal{C})$ is closed under direct sums.

Let $\Theta \in \mathcal{C}$ and let η be a $\mathcal{C}$-filtration of $N \in \mathcal{F}(\mathcal{C})$. We define $[M : \Theta]_\eta$ as the number of quotients isomorphic to Θ in the filtration η. We say that $\mathcal{F}(\mathcal{C})$ is a **Jordan-Hölder category** if, given $N \in \mathcal{F}(\mathcal{C})$, the multiplicity $[M : \Theta]_\eta$ does not depend on the filtration η, i.e., if η and η' are two filtrations for N and $\Theta \in \mathcal{C}$, then $[M : \Theta]_\eta = [M : \Theta]_{\eta'} := [M : \Theta]$ (see (13, Definition 3.1) for a more general definition).

Note that whether a category is Jordan-Hölder or not depends not only on $\mathcal{F}(\mathcal{C})$ but necessarily on $\mathcal{C}$. For example, if $\mathcal{C} = \{S(1), \cdots, S(n)\}$ is the set of simple modules up to isomorphism, then $\mathcal{F}(C) = \text{mod}\, A$ is a Jordan-Hölder category. On the other hand, if A is an algebra that is not semisimple, then there exists a indecomposable module M that is not simple. Considering $C' = \{S(1), \cdots, S(n), M\}$, we have that $\mathcal{F}(C') = \text{mod}\, A$ will not be a Jordan-Hölder category, since the multiplicity $[M : M]_\eta$ will depend on the filtration.

It is easy to see that if $\mathcal{F}(C)$ is a Jordan-Hölder category, then the value

$$l_C(M) = \sum_{\Theta \in C} [M : \Theta]$$

is well defined. This value will be called the **C-length of** N. Furthermore, the C-filtration (1.1) shows that $[M : \Theta] = [L : \Theta] + [N : \Theta]$ and consequently $l_C(M) = l_C(L) + l_C(N)$.

The Jordan-Hölder theorem ensures that if $C = \{S(1), S(2), \cdots, S(n)\}$ is the class of non-isomorphic simple modules, then $\mathcal{F}(C) = \text{mod}\, A$ is a Jordan-Hölder category (see Theorem 1.1.5). It is clear that the modules of length 1 are precisely the simple modules. Similarly, if $\mathcal{F}(C)$ is any Jordan-Hölder category, then the modules with C-length 1 are exactly the modules in C. This fact motivates us to call the modules in C **relatively simple in $\mathcal{F}(C)$**.

1.6 Quasi-hereditary algebras and standardly stratified algebras

Quasi-hereditary algebras and standardly stratified algebras were introduced by E. Cline, B. Parshall, and L. Scott in the context of the algebraic theory of groups and highest weight categories in the representation theory of finite-dimensional complex semisimple Lie algebras. This section follows the exposition in the master's thesis of P. Cadavid (16).

We now define the concept of a standard module. Let A be a k-algebra of finite dimension over an algebraically closed field k with rank n (that is, $|A_A| = n$) that is basic and connected. Under these conditions, Gabriel's theorem guarantees that there exists a k-algebra isomorphism between A and kQ/I, where Q is a finite quiver and I is an admissible ideal.

An enumeration of the vertices of Q is equivalent to an enumeration of the isomorphism classes of simple modules, which is in turn equivalent to an enumeration of the isomorphism classes of indecomposable projective modules. Since each indecomposable projective module is of the form eA, where e is a primitive idempotent, we obtain an enumeration of a complete list of pairwise orthogonal primitive idempotents $(e_1, e_2, \ldots, e_n)$.

We also consider the idempotents $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \epsilon_{n+1})$, where $\epsilon_j = e_j + e_{j+1} + \cdots + e_n$ for $j = 1, 2, \ldots, n$, and $\epsilon_{n+1} = 0$. Note that $\epsilon_1 = 1$.

Given two modules M and N, we define the **trace of M in N**, denoted by $\tau_M(N)$, as the submodule of N generated by the sums of the images of morphisms from M to N.

Definition 1.6.1 (Standard modules)**.** *Let $e = (e_1, e_2, \dots, e_n)$ be a fixed ordering of a complete set of pairwise orthogonal primitive idempotents of A. The sequence $\Delta = \{\Delta(1), \Delta(2), \dots, \Delta(n)\}$ of **standard modules**, with respect to the order e, is given by*

$$\Delta(i) = \frac{P(i)}{\tau_{\epsilon_{i+1}A}(P(i))}$$

where $P(i) = e_i A$.

Remark 1.6.2. *1. Note that $\Delta(n) = P(n)$.*

2. Note that in the above definition, for simplicity, we assumed that the order of the idempotents indexed in the set $\mathbb{I}_n$ is the natural order. This is possible without loss of generality, as we can always reindex the vertices in this way. In Proposition 2.1.9, it will be convenient to take the above definition with the opposite order to the natural order of $\mathbb{I}_n$.

Observe that the standard modules are necessarily indecomposable. Indeed, since they are quotients of an indecomposable projective module, they have a simple top.

Fixing an order $e = (e_1, e_2, \dots, e_n)$ for a complete set of pairwise orthogonal primitive idempotents, we can characterize the standard modules through the following lemma:

Lemma 1.6.3. *For each $i = 1, 2, \dots, n$, $\Delta(i)$ is the maximal quotient of $P(i)$ whose composition factors belong to the set $\{S(1), S(2), \dots, S(i)\}$.*

Standard modules satisfy properties that will also hold for modules in a stratifying system.

Proposition 1.6.4. *Let $\Delta = \{\Delta(1), \Delta(2), \dots, \Delta(n)\}$ be a set of standard modules over an algebra A. Then:*

1. $\mathrm{Hom}_A(\Delta(i), \Delta(j)) = 0$ *if $i > j$.*

2. $\mathrm{Ext}^1_A(\Delta(i), \Delta(j)) = 0$ *if $i \geq j$.*

We consider the category $\mathcal{F}(\Delta)$ of Δ-filtered modules, where such modules are called **Δ-good modules**.

The next proposition presents the main results about the category of Δ-filtered modules when Δ is a set of standard modules.

Proposition 1.6.5. *Let Δ be a set of standard modules over an algebra A. Then the following statements hold:*

1. $\mathcal{F}(\Delta)$ is closed under extensions, direct sums, and direct summands.

2. $\mathcal{F}(\Delta)$ *is functorially finite in* $\operatorname{mod} A$.

3. $\mathcal{F}(\Delta)$ *is a Jordan-Hölder subcategory.*

4. $\mathcal{F}(\Delta)$ *is closed under kernels of epimorphisms; that is, if*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $M, N \in \mathcal{F}(\Delta)$, *then* $K \in \mathcal{F}(\Delta)$.

Finally, we present the main definitions of this section: standardly stratified and quasi-hereditary algebras.

Definition 1.6.6 (Standardly stratified and quasi-hereditary algebras)**.** *Let A be a k-algebra and Δ a set of standard modules relative to an order $e = (e_1, e_2, \cdots, e_n)$ of pairwise orthogonal primitive idempotents. We say that A is a* ***standardly stratified algebra*** *if $A_A \in \mathcal{F}(\Delta)$. Furthermore, we say that a standardly stratified algebra is* ***quasi-hereditary*** *if* $\operatorname{End}_A(\Delta(i))$ *is a division ring for $i = 1, 2, \cdots, n$.*

It is straightforward to see from the definition that A is a standardly stratified algebra with respect to an order of pairwise orthogonal primitive idempotents if and only if $P(i) \in \mathcal{F}(\Delta)$ for every $i = 1, 2, \cdots, n$.

The following example shows that whether an algebra is quasi-hereditary or standardly stratified is strongly tied to the chosen order of the idempotents.

Example 1.6.7. *Let k be an algebraically closed field, and consider $A = kQ/rad^2 A$, where Q is the quiver*

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} 3 \longleftarrow \cdots \longleftarrow (n-1) \xleftarrow{\alpha_n} n$$

Considering the order $e_1 = (n, n-1, \cdots, 2, 1)$, we have $\Delta(i) = S(i)$, hence $\mathcal{F}(\Delta) =$ $\operatorname{mod} A$ by the Jordan-Hölder theorem. In particular, $A_A \in \mathcal{F}(\Delta)$, so A is standardly stratified. Moreover, since $\operatorname{End}_A(S(i)) = k$ for all $i = 1, 2, \cdots, n$, we conclude that A is quasi-hereditary.

On the other hand, if we consider the order $e_2 = (1, n, n-1, \cdots, 3, 2)$, then $\Delta(1) = S(1)$, $\Delta(2) = P(2)$, and $\Delta(i) = S(i)$ for $i = 3, 4, \cdots, n$.

Notice that the only composition series for $P(3)$ is given by

$$0 \subseteq S(2) \subseteq P(3).$$

However, $S(2) \notin \Delta$, so $P(3) \notin \mathcal{F}(\Delta)$, and we conclude that in this order, A is not even standardly stratified.

It can be proven that an algebra is hereditary if and only if it is quasi-hereditary for any order of the idempotents. Moreover, F. Advincula and E. Marcos showed that the algebras that are standardly stratifying for any ordering of a set of primitive orthogonal idempotents are precisely those whose idempotent ideals are projective modules (2).

Modules in $\mathcal{F}(\Delta)$ exhibit "well-behaved" homological dimensions when Δ is a set of standard modules over a quasi-hereditary algebra A, as shown in the following proposition.

Theorem 1.6.8. *Let A be a quasi-hereditary k-algebra of rank n, and let $\Delta = \{\Delta(1), \cdots, \Delta(n)\}$ be its set of standard modules. Then, for all $i = 1, 2, \cdots, n$, the following hold:*

1. $\operatorname{pd} \Delta(i) \leq n - i$.

2. $\operatorname{pd} S(i) \leq n + i - 2$.

3. $\operatorname{pd} M \leq n - 1$ *for all* $M \in \mathcal{F}(\Delta)$.

4. $\operatorname{gl.dim} A \leq 2(n-1)$.

The third item in the theorem above holds for standardly stratified algebras in general. It is possible to prove that the upper bound for the global dimension of A is optimal.

We conclude this section by presenting a theorem that relates standardly stratified algebras and generalized tilting modules.

Definition 1.6.9. *Let A be a k-algebra. We say that a module $T \in \operatorname{mod} A$ is a* ***generalized tilting module*** *if it satisfies the following conditions:*

1. $pd\, T < \infty$.

2. $Ext^i_A(T, T) = 0$ *for all* $i \geq 1$.

3. *There exists an exact sequence*

$$0 \longrightarrow A_A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_s \longrightarrow 0$$

with $T_j \in \operatorname{add}(T)$ *for all* $j = 0, 1, \cdots, s$.

We denote by $\mathcal{I}(\Delta)$ the full subcategory of $\mathcal{F}(\Delta)$ whose objects are Ext-injective modules in $\mathcal{F}(\Delta)$. We have the following theorem.

Theorem 1.6.10. *Let A be a standardly stratified algebra. Then there exists, up to isomorphism, a unique basic generalized tilting module T such that $\mathcal{I}(\Delta) = \operatorname{add}(T)$.*

The module T in the above theorem is called the **characteristic tilting module associated with $\boldsymbol{\Delta}$**.

It is possible to prove that $B^{op} = \mathrm{End}_A(T)^{op}$ is a standardly stratified algebra. Thus, by applying the above theorem again to B^{op}, we obtain a B^{op} generalized tilting module T' such that $\mathrm{add}(T') = \mathcal{I}(\Delta')$, where Δ' is the set of standard modules of B^{op}. Again, we have that $C = \mathrm{End}_{B^{op}}(T')$ is a standardly stratified algebra. The next theorem relates the module categories of A and C.

Theorem 1.6.11. *Let A be a standardly stratified k-algebra, and let T be its characteristic A-tilting module associated with it. Let T' be the characteristic B^{op}-tilting module associated with the standardly stratified algebra $B^{op} = \mathrm{End}_A(T)^{op}$. Then, $\mathrm{End}_{B^{op}}(T')$ is Morita equivalent to A.*

Chapter 2

Stratifying Systems

Stratifying systems were introduced by K. Erdmann and C. Sáenz in (25) as a generalization of standard modules, which play a central role in quasi-hereditary algebras and standardly stratified algebras. This initial definition of a stratifying system is known in the literature as Ext-injective stratifying system (EISS, for short). Later, E. Marcos, O. Mendoza, and C. Sáenz defined in (29) and (30) stratifying systems in two equivalent ways, known in the literature as Ext-projective stratifying systems (EPSS) and simply stratifying systems (SS), with the latter being the most commonly used definition. In this context, a stratifying system yields a module Y whose endomorphism ring $B = \mathrm{End}_A(Y)$ is left standardly stratified, and the category of modules filtered by the stratifying system can be naturally identified inside the category $\mathrm{mod}\, B^{op}$.

In the first section, we will define EISS and prove some basic results relating a stratifying system to a set of standard modules over an appropriate algebra. We will also show that the definitions of EISS and SS are equivalent. In the second section, we will briefly survey the history of stratifying systems from their inception to the present day. In the third section, we will detail some results from (36) that establish connections between τ-rigid modules and stratifying systems.

2.1 Stratifying systems

In this section, we will define the Ext-injective stratifying system and the stratifying system, and prove that the two definitions are equivalent. This discussion is based on (25) and (29).

Definition 2.1.1 (Ext-Injective Stratifying System (25))**.** *Let A be an k-algebra and let $\Theta = \{\Theta(1), \Theta(2), \cdots, \Theta(n)\}$ be a fixed ordered set of A-modules. Moreover, suppose $\mathbb{Y} = \{Y(1), Y(2), \cdots, Y(n)\}$ is a ordered set of indecomposable A-modules. We call $(\Theta, \mathbb{Y}, \leq)$ an* ***Ext-injective stratifying system of size n*** *(EISS, for short) if the following conditions hold:*

1. $\operatorname{Hom}_A(\Theta(k), \Theta(j)) = 0$ *if* $k > j$.

2. *For all* $i \in \mathbb{I}_n$, *there exists an exact sequence*

$$0 \longrightarrow \Theta(i) \longrightarrow Y(i) \longrightarrow Z(i) \longrightarrow 0,$$

   where $Z(i)$ *is filtered by* $\Theta(j)$ *with* $j < i$; *that is,* $Z(i) \in \mathcal{F}(\{\Theta(j) \mid j < i\})$ *(see Section 1.5).*

3. $\operatorname{Ext}^1_A(-, Y)|_{\mathcal{F}(\Theta)} = 0$, *where* $Y = \bigoplus_{j=1}^{n} Y(j)$.

Remark 2.1.2. 1. *The condition 3 shows that the module* Y *is Ext-injective in* $\mathcal{F}(\Theta)$.

2. *In the above definition,* $\leq$ *is a total order on the set* $\mathbb{I}_n = \{1, \ldots, n\}$. *We will assume throughout the text, without loss of generality, that this order is the natural order on* $\mathbb{I}_n$.

3. *Condition 2 implies that* $\Theta(1) = Y(1)$.

We can define a stratifying system by presenting only the set of modules Θ.

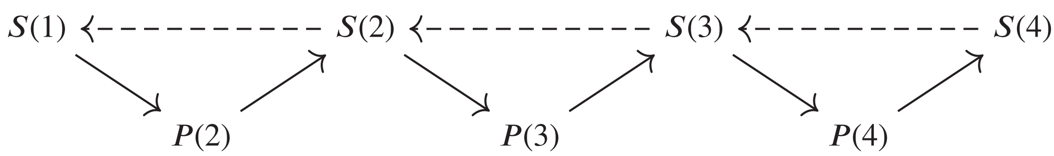
Definition 2.1.3 (Stratifying System (29))**.** *Let* A *be an* k*-algebra and let* $(\Theta, \leq)$ *be a fixed ordered set of* A*-modules* $\Theta = \{\Theta(1), \cdots, \Theta(n)\}$. *We call* $(\Theta, \leq)$ *a* ***stratifying system of size*** n *(SS, for short) if the following conditions hold:*

1. $\Theta(j)$ *is indecomposable for all* $j = 1, 2, \cdots, n$.

2. $\operatorname{Hom}_A(\Theta(k), \Theta(j)) = 0$ *if* $k > j$.

3. $\operatorname{Ext}^1_A(\Theta(k), \Theta(j)) = 0$ *if* $k \geq j$.

Example 2.1.4. *Let* $A = kQ/I$, *where* Q *is the quiver* $1 \xleftarrow{\gamma} 2 \xleftarrow{\beta} 3 \xleftarrow{\alpha} 4$ *and* $I = rad^2 A$. *The Auslander-Reiten quiver of the algebra is given by:*



We will show that $(\Theta, \mathbb{Y}, \leq)$ *is an EISS of size 3, where* $\Theta = \{S(3), S(2), S(1)\}$ *and* $\mathbb{Y} = \{S(3), P(3), P(2)\}$. *Throughout the text, the elements inside such sets will always be listed from left to right according to the chosen order; for instance, in* Θ *we have* $S(3)$ *as the first element,* $S(2)$ *as the second, and* $S(1)$ *as the third.*

Since the modules in Θ *are simple, we can easily determine the category of* Θ*-filtered* A*-modules as the full subcategory of* $\operatorname{mod} A$ *consisting of modules whose composition factors are in* $\{S(1), S(2), S(3)\}$*, that is,*

$$\mathcal{F}(\Theta) = \operatorname{add}(S(1) \bigoplus P(2) \bigoplus P(3) \bigoplus S(2) \bigoplus S(3)).$$

As the modules in Θ *are simple, we have that* $\operatorname{Hom}_A(\Theta(k), \Theta(j)) = 0$ *for all* $j, k \in \mathbb{I}_3$ *with* $j \neq k$*, in particular when* $k > j$*, so condition 1 is satisfied.*

The following exact sequences show that condition 2 is also satisfied:

$$\begin{aligned} 0 \longrightarrow S(3) \longrightarrow S(3) \longrightarrow \quad 0 \quad \longrightarrow 0 \\ 0 \longrightarrow S(2) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0 \\ 0 \longrightarrow S(1) \longrightarrow P(2) \longrightarrow S(2) \longrightarrow 0. \end{aligned}$$

Finally, it is clear from the Auslander-Reiten quiver that $P(3), P(2)$*, and* $S(3)$ *are Ext-injective in* $\mathcal{F}(\Theta)$*. Therefore,* $Y = P(3) \bigoplus P(2) \bigoplus S(3)$ *is Ext-injective in* $\mathcal{F}(\Theta)$*, and condition 3 is satisfied.*

We can also see that $(\Theta, \leq)$ *is an SS. Conditions 1 and 2 are satisfied since* Θ *is a set of simple modules, and condition 3 holds since* $\operatorname{Ext}^1_A(S(k), S(j)) = 0$ *if* $k \leq j$ *(hence* $\operatorname{Ext}^1_A(\Theta(k), \Theta(j)) = 0$ *if* $k \geq j$*).*

The example above suggests that $(\Theta, \leq)$ is an SS whenever $(\Theta, \mathbb{Y}, \leq)$ is an EISS. We will show that this is always true. Furthermore, we will demonstrate that if $(\Theta, \leq)$ is an SS, then there exists a unique set of indecomposable A-modules $\mathbb{Y}$ (up to isomorphism) such that $(\Theta, \mathbb{Y}, \leq)$ forms an EISS.

$\mathcal{F}(\Theta)$ is a Jordan-Hölder category

Lemma 2.1.5. *Let* $(\Theta, \mathbb{Y}, \leq)$ *be an EISS of size n. Then the following statements hold true*

1. $\operatorname{Hom}_A(\Theta(j), Z(i)) = 0$ *if* $j \geq i$*.*
2. $\operatorname{Hom}_A(\Theta(j), Y(i)) \cong \operatorname{Hom}_A(\Theta(j), \Theta(i))$ *if* $j \geq i$*. In particular,* $\operatorname{Hom}_A(\Theta(j), Y(i)) = 0$ *if* $j > i$*.*

Proof. 1) Suppose that $\operatorname{Hom}_A(\Theta(j), Z(i)) \neq 0$ with $j \geq i$. Since $Z(i) \in \mathcal{F}(\{\Theta(k) \mid k < i\})$, there exists a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = Z(i)$$

of submodules of $Z(i)$ such that M_l/M_{l-1} is isomorphic to a module in $\{\Theta(1), \cdots, \Theta(i-1)\}$ for all $l = 1, 2, \cdots, t$. In particular, there exists an epimorphism

$$\pi : Z(i) \longrightarrow Z(i)/M_{t-1} \cong \Theta(k),$$

for some $1 \leq k < i$.

If $f \in \mathrm{Hom}_A(\Theta(j), Z(i))$ is a nonzero morphism, then $\pi \circ f : \Theta(j) \longrightarrow \Theta(k)$ is a morphism that must be zero since $\mathrm{Hom}_A(\Theta(j), \Theta(k)) = 0$. Hence, the image of f is contained in $\ker \pi = M_{t-1}$, and there exists a nonzero morphism $f' : \Theta(j) \longrightarrow M_{t-1}$ such that $f = i_{M_{t-1}} \circ f'$, where $i_{M_{t-1}} : M_{t-1} \longrightarrow Z(i)$ is the inclusion.

Similarly, there exists an epimorphism

$$\pi' : M_{t-1} \longrightarrow M_{t-1}/M_{t-2} \cong \Theta(k')$$

for some $1 \leq k' < i$. Again, the image of f' is contained in $\ker \pi' = M_{t-2}$ and there exists a nonzero morphism $f'' : \Theta(j) \longrightarrow M_{t-2}$ such that $f' = i_{M_{t-2}} \circ f''$, where $i_{M_{t-2}} : M_{t-2} \longrightarrow M_{t-1}$ is the inclusion. Since the chain is finite, by the well-ordering principle, we conclude that $\mathrm{Hom}_A(\Theta(j), \Theta(l)) \neq 0$ for some $1 \leq l < i$, a contradiction. Therefore, $\mathrm{Hom}_A(\Theta(j), Z(i)) = 0$.

2) Suppose that $j \geq i$ and consider the exact sequence

$$0 \longrightarrow \Theta(i) \longrightarrow Y(i) \longrightarrow Z(i) \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_A(\Theta(j), -)$ to this sequence, we obtain

$$0 \longrightarrow \mathrm{Hom}_A(\Theta(j), \Theta(i)) \longrightarrow \mathrm{Hom}_A(\Theta(j), Y(i)) \longrightarrow \mathrm{Hom}_A(\Theta(j), Z(i)),$$

where item 1) ensures that $\mathrm{Hom}_A(\Theta(j), Z(i)) = 0$, and the result follows. ■

Theorem 2.1.6. *Let $(\Theta, \mathbb{Y}, \leq)$ be an EISS of size n. Then $\mathcal{F}(\Theta)$ is a Jordan-Hölder category.*

Proof. Consider the matrix $D = [d_{ij}]$, where $d_{ij} = \dim \mathrm{Hom}_A(\Theta(i), Y(j))$. By Lemma 2.1.5, we have that D is an upper triangular matrix with $\det D \neq 0$, since $\mathrm{Hom}_A(\Theta(i), Y(j)) = 0$ if $i > j$ and $\mathrm{Hom}_A(\Theta(i), Y(i)) \neq 0$ for all $i = 1, 2, \cdots, n$.

Let $M \in \mathcal{F}(\Theta)$ be a nonzero module. It is clear that for all $i \in \mathbb{I}_n$, the dimension of $\mathrm{Hom}_A(M, Y(i))$ is independent of the choice of filtration of M.

Consider an arbitrary filtration of M:

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_m = M,$$

then we have the following exact sequences:

$$\begin{array}{ccccccccc}
0 \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & \Theta(j_1) & \longrightarrow 0 \\
0 \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & \Theta(j_2) & \longrightarrow 0 \\
 & & & \vdots & & & \\
0 \longrightarrow & M_{m-1} & \longrightarrow & M_m & \longrightarrow & \Theta(j_m) & \longrightarrow 0
\end{array}$$

where $j_k \in \{1,2,\cdots,n\}$ for $k = 1,2,\cdots,m$.

Applying the functor $\mathrm{Hom}_A(-,Y(i))$, which is exact in $\mathcal{F}(\Theta)$, we obtain

$$\begin{array}{c}0 \longrightarrow \mathrm{Hom}_A(\Theta(j_1),Y(i)) \longrightarrow \mathrm{Hom}_A(M_1,Y(i)) \longrightarrow \mathrm{Hom}_A(M_0,Y(i)) \longrightarrow 0\\ 0 \longrightarrow \mathrm{Hom}_A(\Theta(j_2),Y(i)) \longrightarrow \mathrm{Hom}_A(M_2,Y(i)) \longrightarrow \mathrm{Hom}_A(M_1,Y(i)) \longrightarrow 0\\ \vdots \\ 0 \longrightarrow \mathrm{Hom}_A(\Theta(j_m),Y(i)) \longrightarrow \mathrm{Hom}_A(M_m,Y(i)) \longrightarrow \mathrm{Hom}_A(M_{m-1},Y(i)) \longrightarrow 0.\end{array}$$

In an exact sequence of k-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ we have that $\dim B = \dim A + \dim C$, so we obtain

$$\sum_{k=1}^{m} \dim \mathrm{Hom}_A(M_k,Y(i)) = \sum_{k=1}^{m} \dim \mathrm{Hom}_A(M_{k-1},Y(i)) + \sum_{k=1}^{m} \dim \mathrm{Hom}_A(\Theta(j_k),Y(i))$$

which can be simplified if we consider m_j as the number of quotients isomorphic to $\Theta(j)$ in such a filtration (note that $\mathrm{Hom}_A(M_0,Y(i)) = 0$)

$$\dim \mathrm{Hom}_A(M,Y(i)) = \sum_{j=1}^{n} m_j d_{ji}.$$

If $i = 1$, then $m_1 = \dim \mathrm{Hom}_A(M,Y(1))/d_{11}$ is defined independently of the choice of filtration. Recursively, we have that $m_i = (\dim \mathrm{Hom}_A(M,Y(i)) - \sum_{k=1}^{i-1} d_{ki} m_k)/d_{ii}$ is defined independently of the choice of filtration, which proves the lemma. ■

Let $(\Theta,\mathbb{Y},\leq)$ be an EISS of size n and let $M \in \mathcal{F}(\Theta)$. For every $\Theta(i) \in \Theta$, we define $[M : \Theta(i)]$ as the number of composition factors isomorphic to $\Theta(i)$ that appear in some Θ-filtration of M (here, $[M : \Theta(i)] = 0$ for all i if $M = 0$). The theorem above ensures that this number is well-defined and independent of the choice of such a filtration. We define the Θ-length of M as $l_\Theta(M) := \sum_{i=1}^{n} [M : \Theta(i)]$.

Corollary 2.1.7. *Let $(\Theta,\mathbb{Y},\leq)$ be an EISS of size n. If $L,N \in \mathcal{F}(\Theta)$ and the following sequence is exact*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0,$$

then $M \in \mathcal{F}(\Theta)$ and $[M : \Theta(i)] = [L : \Theta(i)] + [N : \Theta(i)]$ for all $i \in \mathbb{I}_n$. Furthermore, $l_\Theta(M) = l_\Theta(L) + l_\Theta(N)$.

Proof. It suffices to note that if

$$\eta_L : \ 0 = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_t$$

and

$$\eta_N : \ 0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_s$$

are Θ-filtrations of L and N, respectively, then

$$\eta_M : 0 \subseteq f(L_1) \subseteq f(L_2) \subseteq \cdots \subseteq f(L_t) \subseteq g^{-1}(N_1) \subseteq \cdots \subseteq g^{-1}(N_s) = M$$

is a Θ-filtration of M ($g^{-1}(N_k)$ denotes the preimage of N_k in M), since

$$f(L_k)/f(L_{k-1}) \cong L_k/L_{k-1}$$

for $k = 1, 2, \cdots, t$ and

$$g^{-1}(N_k)/g^{-1}(N_{k-1}) \cong N_k/N_{k-1}$$

for $k = 1, 2, \cdots, s$ (note that $g^{-1}(N_0) = \ker g = L = f(L_t)$). ■

Relationship Between Stratifying Systems and Standardly Stratified Algebras

Lemma 2.1.8. *Let* $(\Theta, \mathbb{Y}, \leq)$ *be an EISS, where* $\Theta = \{\Theta(1), \Theta(2), \cdots, \Theta(n)\}$. *Then, we have that* $\operatorname{Ext}^1_A(\Theta(j), \Theta(i)) = 0$ *if* $j \geq i$.

Proof. Assume $j \geq i$ and consider the exact sequence

$$0 \longrightarrow \Theta(i) \longrightarrow Y(i) \longrightarrow Z(i) \longrightarrow 0.$$

Applying the functor $\operatorname{Hom}_A(\Theta(j), -)$, we obtain

$$\operatorname{Hom}_A(\Theta(j), Z(i)) \longrightarrow \operatorname{Ext}^1_A(\Theta(j), \Theta(i)) \longrightarrow \operatorname{Ext}^1_A(\Theta(j), Y(i)).$$

We have $\operatorname{Ext}^1_A(\Theta(j), Y(i)) = 0$ since $\Theta(j) \in \mathcal{F}(\Theta)$ and $\operatorname{Hom}_A(\Theta(j), Z(i)) = 0$ by Lemma 2.1.5, consequently $\operatorname{Ext}^1_A(\Theta(j), \Theta(i)) = 0$. ■

The next proposition shows that there is a strong connection between Ext-injectives stratifying systems and standardly stratified algebras. In what follows, given $(\Theta, \mathbb{Y}, \leq)$ a stratifying system over an algebra A, we define $B = \operatorname{End}_A(Y)$. We consider the functor $F : \operatorname{mod} A \longrightarrow \operatorname{mod} B^{op}$ defined by $F = \operatorname{Hom}_A(-, Y)$. Observe that F is an exact functor on $\mathcal{F}(\Theta)$, as Y is Ext-injective in this category. We have that $P(i) = \operatorname{Hom}_A(Y(i), Y)$ is the i-th indecomposable projective left B-module, and we denote by $S(i)$ the simple top of $P(i)$. In what follows, by a "left standardly stratified algebra" we mean that B, regarded as a left module over itself, is filtered by left standard modules.

Proposition 2.1.9. *Suppose* $(\Theta, \mathbb{Y}, \leq)$ *is an EISS of size n. Then* $B = \operatorname{End}_A(Y)$ *is a left standardly stratified algebra, with* $\Delta(i) = \operatorname{Hom}_A(\Theta(i), Y)$, *with respect to* $(\mathbb{I}_n, \leq^{op})$ *where* $\leq^{op}$ *is the opposite of the natural order.*

Proof. Let $P(i) = \mathrm{Hom}_A(Y(i), Y)$ be the i-th indecomposable B-projective left module and $F(\Theta(i)) = \mathrm{Hom}_A(\Theta(i), Y)$. We need to show that $F(\Theta(i))$ is the maximal quotient of $P(i)$ with composition factors $S(j)$ for $j \geq i$ (not $j \leq i$, since the order on $\mathbb{I}_n$ is the opposite) and that ${}_BB \in \mathcal{F}(\{F(\Theta(j)) \mid j \in \mathbb{I}_n\})$.

Consider the sequence

$$0 \longrightarrow \Theta(i) \xrightarrow{\beta_i} Y(i) \longrightarrow Z(i) \longrightarrow 0. \tag{2.1}$$

Applying the functor $F(-) = \mathrm{Hom}_A(-, Y)$ to the sequence, we obtain

$$0 \longrightarrow F(Z(i)) \longrightarrow P(i) \xrightarrow{F(\beta_i)} F(\Theta(i)) \longrightarrow 0. \tag{2.2}$$

where the exactness on the right is obtained since $Z(i) \in \mathcal{F}(\Theta)$.

We will show that $F(Z(i))$ is filtered by $F(\Theta(1)), F(\Theta(2)), \cdots, F(\Theta(i-1))$. Since $Z(i)$ is filtered by $\{\Theta(1), \Theta(2), \cdots, \Theta(i-1)\}$, there exists a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = Z(i)$$

of submodules of $Z(i)$ such that

$$\begin{array}{c} 0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \Theta(j_1) \longrightarrow 0 \\ 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \Theta(j_2) \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow M_{t-1} \longrightarrow M_t \longrightarrow \Theta(j_t) \longrightarrow 0 \end{array}$$

where $j_k \in \{1, 2, \cdots, i-1\}$ for all $k = 1, 2, \cdots, t$. Applying the functor $F(-)$ we obtain that

$$\begin{array}{c} 0 \longrightarrow F(\Theta(j_1)) \longrightarrow F(M_1) \longrightarrow F(M_0) \longrightarrow 0 \\ 0 \longrightarrow F(\Theta(j_2)) \longrightarrow F(M_2) \longrightarrow F(M_1) \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow F(\Theta(j_t)) \longrightarrow F(M_t) \longrightarrow F(M_{t-1}) \longrightarrow 0. \end{array}$$

From the first exact sequence, we conclude that $F(M_1)$ is filtered by $F(\Theta(j_1))$ (in fact, we have $F(M_1) = F(\Theta(j_1))$). By induction, assuming that $F(M_{k-1})$ is filtered by $F(\Theta(j_1)), F(\Theta(j_2)), \cdots, F(\Theta(j_{k-1}))$, from the sequence

$$0 \longrightarrow F(\Theta(j_k)) \longrightarrow F(M_k) \longrightarrow F(M_{k-1}) \longrightarrow 0$$

we conclude that $F(M_k)$ is filtered by $F(\Theta(j_1)), F(\Theta(j_2)), \cdots, F(\Theta(j_{k-1})), F(\Theta(j_k))$. Therefore,

$$F(Z(i)) \in \mathcal{F}(\{F(\Theta(j_k)) \mid k = 1, 2, \cdots, t\}) \subseteq \mathcal{F}(\{F(\Theta(j)) \mid 1 \leq j < i\}).$$

From the exact sequence (2.2), we conclude that $P(i)$ is filtered by $F(\Theta(j))$ for $j = 1, 2, \cdots, i$ and consequently ${}_B B \in \mathcal{F}(\{F(\Theta(j)) \mid j \in \mathbb{I}_n\})$ (Note that $[P(i), F(\Theta(i))] = 1$ and $[P(i), F(\Theta(j))] = 0$ if $j > i$).

We will show that $F(\Theta(i)) = \Delta(i)$, that is, $F(\Theta(i))$ is the maximal quotient of $P(i)$ with composition factors $S(j)$ for $j \geq i$. Let $U(i)$ be the sum of the images of the morphisms $\phi : P(j) \longrightarrow P(i)$ with $j < i$. We have that $U(i)$ is a submodule of $P(i)$ and that there exists a short exact sequence

$$0 \longrightarrow U(i) \longrightarrow P(i) \longrightarrow \Delta(i) \longrightarrow 0.$$

We will show that $U(i)$ is a submodule of $F(Z(i)) = \mathrm{Hom}_A(Z(i), Y)$. Consider $g : P(j) \longrightarrow P(i)$ with $j < i$. We have the well-known isomorphism

$$\mathrm{Hom}_A(Y(i), Y(j)) \cong \mathrm{Hom}_B(\mathrm{Hom}_A(Y(j), Y), \mathrm{Hom}_A(Y(i), Y))$$

which maps $h : Y(i) \longrightarrow Y(j)$ to $g = F(h)$. Then, there exists a morphism $h : Y(i) \longrightarrow Y(j)$ such that $F(h) = g$.

We have that $\mathrm{Hom}_A(\Theta(i), Y(j)) = 0$ if $i > j$ by Lemma 2.1.5, therefore, $h \circ \beta_i = 0$, where β_i is the first morphism in the short exact sequence (2.1), so we have

$$0 = F(h \circ \beta_i) = F(\beta_i) \circ F(h) = F(\beta_i) \circ g.$$

This implies that $\mathrm{Im}\ g \subseteq \ker F(\beta_i) = \mathrm{Hom}_A(Z(i), Y) = F(Z(i))$. Consequently, $U(i) \subseteq F(Z(i))$ and $F(\Theta(i))$ is a quotient of $\Delta(i)$.

Finally, $P(i)$ is filtered by $F(\Theta(t))$ with $t \leq i$. Moreover, $F(\Theta(j))$ has simple top $S(j)$ since it is a quotient of $P(j)$. Therefore, top $F(Z(i))$ is isomorphic to the sum of copies of $S(j)$ with $j < i$. Indeed, suppose that the simple module $S(k)$ satisfies $S(k) | \mathrm{top}\ F(Z(i))$ with $k \geq i$, and consider

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_t = F(Z(i))$$

a filtration of $F(Z(i))$ in the set $\{F(\Theta(j)) \mid 1 \leq j < i\}$. Then we have an exact sequence

$$0 \longrightarrow L_{t-1} \longrightarrow F(Z(i)) \longrightarrow F(\Theta(t)) \longrightarrow 0$$

where $t \in \{1, 2, \cdots, i-1\}$. Such a sequence induces the exact sequence (observe that the first morphism in the following sequence need not be a monomorphism).

$$\mathrm{top}\ L_{t-1} \longrightarrow \mathrm{top}\ F(Z(i)) \longrightarrow \mathrm{top}\ F(\Theta(t)) \longrightarrow 0.$$

Since top $F(\Theta(t))$ is simple and different from $S(k)$, we have $S(k) | \mathrm{top}\ L_{t-1}$. Now consider the exact sequence

$$0 \longrightarrow L_{t-2} \longrightarrow L_{t-1} \longrightarrow F(\Theta(t')) \longrightarrow 0$$

where $t' \in \{1, 2, \cdots, i-1\}$. This exact sequence induces

$$\text{top } L_{t-2} \longrightarrow \text{top } L_{t-1} \longrightarrow \text{top } F(\Theta(t')) \longrightarrow 0.$$

Again, since top $F(\Theta(t'))$ is simple and different from $S(k)$, we conclude $S(k)|\text{top } L_{t-2}$, and proceeding recursively we obtain a contradiction. Since top $F(Z(i))$ is isomorphic to the sum of copies of $S(j)$ with $j < i$, we have that $[F(\Theta(i)), S(k)] \geq [\Delta(i), S(k)]$ for $k \geq i$. Since $\Delta(i)$ is the maximal quotient of $P(i)$ whose composition factors are $S(j)$ with $j \geq i$, it follows that $\Delta(i) = F(\Theta(i))$. ■

Lemma 2.1.10. *Let* $(\Theta, \mathbb{Y}, \leq)$ *be an EISS of size n and* $M \in \mathcal{F}(\Theta)$*. Then there exists an exact sequence*

$$0 \longrightarrow M \xrightarrow{f_{-1}} Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \cdots \longrightarrow Y_{k-1} \xrightarrow{f_{k-1}} Y_k \longrightarrow 0$$

where $Y_r \in \text{add}(Y)$ *for* $r = 0, 1, \cdots, k$ *with* $k < n$*. Moreover, we have that* $Imf_r \in \mathcal{F}(\Theta)$ *for* $r = -1, 0, \cdots, k-1$.

Proof. If $M = 0$ it is obvious. If $M \neq 0$, define $\delta(M) = \max\{j \mid [M : \Theta(j)] \neq 0\}$. We will prove by induction on $i = \delta(M)$ that there exist short exact sequences

$$0 \longrightarrow M_t \longrightarrow Y_t \longrightarrow M_{t+1} \longrightarrow 0$$

with $Y_t \in \text{add}(Y)$ and $M_t \in \mathcal{F}(\Theta)$ for $t = 0, 1, \cdots, k$, such that $M_0 = M$, $M_{k+1} = 0$, and $\delta(M_t) > \delta(M_{t+1})$ for $t = 0, 1, \cdots, k$. In this way, we will construct the exact sequence of the statement through the following composition:

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & M & \longrightarrow & Y_0 & & \longrightarrow & & Y_1 & & \longrightarrow & \cdots & \longrightarrow & Y_k & \longrightarrow 0. \\
 & & & & & \searrow & & \nearrow & & \searrow & & \nearrow \; \searrow & & \nearrow & \\
 & & & & & & M_1 & & & & M_2 & & M_k & &
\end{array}$$

If $\delta(M) = 1$, then M has only quotients isomorphic to $\Theta(1)$. Thus, consider a filtration of M

$$0 \subseteq \Theta(1) \subseteq M_2 \subseteq \cdots \subseteq M_t = M.$$

Since $\text{Ext}^1_A(\Theta(1), \Theta(1)) = 0$ by Lemma 2.1.8, we have that the sequence

$$0 \longrightarrow \Theta(1) \longrightarrow M_2 \longrightarrow \Theta(1) \longrightarrow 0$$

is split, so $M_2 \cong \Theta(1)^{\oplus 2}$. Consider the short exact sequence

$$0 \longrightarrow M_k \longrightarrow M_{k+1} \longrightarrow \Theta(1) \longrightarrow 0.$$

By induction, we have $M_k \cong \Theta(1)^{\oplus k}$, so $M_{k+1} \cong \Theta(1)^{\oplus k+1}$, and consequently $M \cong \Theta(1)^{\oplus t}$. Since $\Theta(1) = Y(1)$, we set $Y_0 = \Theta(1)^{\oplus t}$ and the lemma holds if $\delta(M) = 1$.

Suppose that $\delta(M) = i$ and that the lemma holds for all $N \in \mathcal{F}(\Theta)$ with $\delta(N) < i$. We will show that there exists a short exact sequence

$$0 \longrightarrow \Theta(i)^{\oplus a} \longrightarrow M \longrightarrow N \longrightarrow 0$$

where $a \in \mathbb{N}$ and $N \in \mathcal{F}(\Theta)$ with $\delta(N) < i$. Let

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$$

be a filtration of M and consider the minimal $k \in \{1, 2, \cdots, t\}$ such that $M_k/M_{k-1} \cong \Theta(i)$. We have the following short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & \Theta(j_1) & \longrightarrow & 0 \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & \Theta(j_2) & \longrightarrow & 0 \\ & & & & \vdots & & & & \\ 0 & \longrightarrow & M_{k-1} & \longrightarrow & M_k & \longrightarrow & \Theta(i) & \longrightarrow & 0 \end{array}$$

where $j_l < i$ for every $l = 1, 2, \cdots, k-1$. Applying the functor $\mathrm{Hom}_A(\Theta(i), -)$ yields the following exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_A^1(\Theta(i), M_0) & \longrightarrow & \mathrm{Ext}_A^1(\Theta(i), M_1) & \longrightarrow & \mathrm{Ext}_A^1(\Theta(i), \Theta(j_1)) \\ 0 & \longrightarrow & \mathrm{Ext}_A^1(\Theta(i), M_1) & \longrightarrow & \mathrm{Ext}_A^1(\Theta(i), M_2) & \longrightarrow & \mathrm{Ext}_A^1(\Theta(i), \Theta(j_2)) \\ & & & & \vdots & & \\ 0 & \longrightarrow & \mathrm{Ext}_A^1(\Theta(i), M_{k-1}) & \longrightarrow & \mathrm{Ext}_A^1(\Theta(i), M_k) & \longrightarrow & \mathrm{Ext}_A^1(\Theta(i), \Theta(i)) \end{array}$$

where exactness on the left is guaranteed by the fact that $\mathrm{Hom}_A(\Theta(i), \Theta(j_l)) = 0$ for $l = 1, 2, \cdots, k-1$. Proceeding recursively, since $M_0 = 0$ and Lemma 2.1.8 ensures that $\mathrm{Ext}_A^1(\Theta(i), \Theta(j_l)) = 0$ for $l = 1, 2, \cdots, k-1$, we obtain that $\mathrm{Ext}_A^1(\Theta(i), M_l) = 0$, hence $\mathrm{Ext}_A^1(\Theta(i), M_k) = 0$. Consequently, the sequence

$$0 \longrightarrow M_{k-1} \longrightarrow M_k \longrightarrow \Theta(i) \longrightarrow 0$$

splits and $M_k \cong \Theta(i) \bigoplus M_{k-1}$. Now observe that

$$0 \subseteq \Theta(i) \subseteq \Theta(i) \bigoplus M_1 \subseteq \Theta(i) \bigoplus M_2 \subseteq \cdots \subseteq \Theta(i) \bigoplus M_{k-1} \subseteq M_{k+1} \subseteq \cdots \subseteq M$$

is a Θ-filtration of M. Repeating the process if necessary, we obtain a Θ-filtration of M of the form

$$0 = L_0 \subseteq L_1 \subseteq L_2 \cdots \subseteq L_a \subseteq N_{a+1} \subseteq N_{a+2} \subseteq \cdots \subseteq N_t = M$$

where $L_j/L_{j-1} \cong \Theta(i)$ for $j = 1,2,\cdots,a$, and $N_j/N_{j-1} \cong \Theta(l_j)$ for $j = a+1, a+2, \cdots, t$ with $l_j \in \{1,2,\cdots,i-1\}$ (here we set $N_a = L_a$).

Since $\mathrm{Ext}^1_A(\Theta(i),\Theta(i)) = 0$, $L_a = \Theta(i)^{\oplus a}$ and we obtain a short exact sequence

$$0 \longrightarrow \Theta(i)^{\oplus a} \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{2.3}$$

where $N = M/\Theta(i)^{\oplus a}$, which is filtered by $\{\Theta(1),\Theta(2),\cdots,\Theta(i-1)\}$ as it admits the following composition series:

$$0 \subseteq N_{a+1}/\Theta(i)^{\oplus a} \subseteq N_{a+2}/\Theta(i)^{\oplus a} \subseteq \cdots \subseteq N_t/\Theta(i)^{\oplus a} = N$$

(here we use the fact that if $L \subseteq N \subseteq M$, then $\dfrac{M/L}{N/L} \cong M/N$).

Furthermore, the induction hypothesis states that there exists a short exact sequence

$$0 \longrightarrow N \longrightarrow \bar{Y} \longrightarrow \bar{N} \longrightarrow 0$$

with $\bar{Y} \in \mathrm{add}(Y)$ and $\bar{N} \in \mathcal{F}(\Theta)$ such that $\delta(\bar{N}) < \delta(N)$.

Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Theta(i)^{\oplus a} & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_i^{\oplus a}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y(i)^{\oplus a} & \longrightarrow & W & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Z(i)^{\oplus a} & \longrightarrow & Z(i)^{\oplus a} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the second row and the second column are obtained as the push-out of the morphisms $\beta_i^{\oplus a} : \Theta(i)^{\oplus a} \longrightarrow Y(i)^{\oplus a}$ and the inclusion $\bar{i} : \Theta(i)^{\oplus a} \longrightarrow M$ given in the sequence (2.3).

Since $N \in \mathcal{F}(\Theta)$, the second row splits, so $W \cong Y(i)^{\oplus a} \bigoplus N$. Composing the inclusion $M \longrightarrow W$ with the inclusion $W \cong Y(i)^{\oplus a} \bigoplus N \longrightarrow Y(i)^{\oplus a} \bigoplus \bar{Y}$, we obtain

$$0 \longrightarrow M \longrightarrow \bar{Y} \bigoplus Y(i)^{\oplus a} \longrightarrow \frac{\bar{Y} \bigoplus Y(i)^{\oplus a}}{M} \longrightarrow 0 \tag{2.4}$$

which is the inclusion of M into a module in $\mathrm{add}(Y)$. Note that the quotient of such inclusion fits into the exact sequence

$$0 \longrightarrow Z(i)^{\oplus a} \cong \frac{N \bigoplus Y(i)^{\oplus a}}{M} \longrightarrow \frac{\bar{Y} \bigoplus Y(i)^{\oplus a}}{M} \longrightarrow \frac{\bar{Y}}{N} \cong \bar{N} \longrightarrow 0.$$

Since $Z(i)^{\oplus a}$ and $\bar{N} \in \mathcal{F}(\Theta)$, we have that $(\bar{Y} \bigoplus Y(i)^{\oplus a})/M \in \mathcal{F}(\Theta)$. Moreover, as $\delta(Z(i)^{\oplus a}) < i$ and $\delta(\bar{N}) < \delta(N) < i$, we conclude that $\delta\left((\bar{Y} \bigoplus Y(i)^{\oplus a})/M\right) < i$. Therefore, the exact sequence (2.4) is the desired sequence. ■

For the next corollary, we define the evaluation map $\epsilon_X : \operatorname{mod} A \longrightarrow \operatorname{mod} A$ as $\epsilon_X : X \longrightarrow \operatorname{Hom}_B(\operatorname{Hom}_A(X,Y),Y)$. Note that Y is an B-A-bimodule.

Corollary 2.1.11. *The evaluation map ϵ_X is an isomorphism for any $X \in \mathcal{F}(\Theta)$.*

Proof. If $X = Y$, then $\operatorname{Hom}_B(\operatorname{Hom}_A(Y,Y),Y) = \operatorname{Hom}_B(B,Y) \cong Y$; consequently, ϵ_X is an isomorphism for $X \in \operatorname{add} Y$. If $X \in \mathcal{F}(\Theta)$ is an arbitrary module, by the previous lemma, there exists an exact sequence

$$0 \longrightarrow X \xrightarrow{f_{-1}} Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \cdots \longrightarrow Y_{s-1} \xrightarrow{f_{s-1}} Y_s \longrightarrow 0$$

such that $X_{i-1} := \operatorname{im} f_{i-1} \in \mathcal{F}(\Theta)$ and $Y_i \in \operatorname{add}(Y)$ for $i = 0,1,2,\cdots,s$. Consider the exact sequences

$$\gamma_i : 0 \longrightarrow X_{i-1} \longrightarrow Y_i \longrightarrow X_i \longrightarrow 0$$

with $0 \le i \le s-1$, where we set $X_{-1} = X$ and $X_{s-1} = Y_s$. Since the functor $F(-) = \operatorname{Hom}_A(-,Y)$ is exact on $\mathcal{F}(\Theta)$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_{s-2} & \longrightarrow & Y_{s-1} & \longrightarrow & Y_s & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \epsilon_{X_{s-2}}} & & \downarrow{\scriptstyle \epsilon_{Y_{s-1}}} & & \downarrow{\scriptstyle \epsilon_{Y_s}} & & \\ 0 & \longrightarrow & E(X_{s-2}) & \longrightarrow & E(Y_{s-1}) & \longrightarrow & E(Y_s) & & \end{array}$$

where $E(X) = \operatorname{Hom}_B(\operatorname{Hom}_A(X,Y),Y)$. Since $\epsilon_{Y_{s-1}}$ and ϵ_{Y_s} are isomorphisms, it follows that $\epsilon_{X_{s-2}}$ is also an isomorphism. By induction, we show that $\epsilon_{X_{-1}} = \epsilon_X$ is an isomorphism. ■

Theorem 2.1.12. *The category $\mathcal{F}(\Theta) \subseteq \operatorname{mod} A$ is contravariantly equivalent to $\mathcal{F}(\Delta) \subseteq \operatorname{mod} B^{op}$.*

Proof. Since the functor $F(-) = \operatorname{Hom}_A(-,Y)$ is exact on $\mathcal{F}(\Theta)$, it induces an equivalence between $\mathcal{F}(\Theta)$ and its image. We will show that F is faithful and full, that is, given $W,V \in \mathcal{F}(\Theta)$, there is a functorial isomorphism

$$\operatorname{Hom}_A(W,V) \cong \operatorname{Hom}_B(F(V),F(W)).$$

We have that

$$\operatorname{Hom}_B(F(V),F(W)) = \operatorname{Hom}_B(F(V),\operatorname{Hom}_A(W,Y)) \cong \operatorname{Hom}_A(W,\operatorname{Hom}_B(F(V),Y)).$$

The previous corollary ensures that $\operatorname{Hom}_B(F(V),Y) = \operatorname{Hom}_B(\operatorname{Hom}_A(V,Y),Y) \cong V$, thus $\operatorname{Hom}_A(W,V) \cong \operatorname{Hom}_B(F(V),F(W))$.

Since $\Delta(i) = F(\Theta(i))$, the image of $\mathcal{F}(\Theta)$ under the functor F is contained in $\mathcal{F}(\Delta)$. We will show that the image is precisely $\mathcal{F}(\Delta)$. We start by showing that the functor $G : \operatorname{mod} B^{op} \longrightarrow \operatorname{mod} A$ defined as $G(-) := \operatorname{Hom}_B(-, Y)$ is exact on $\mathcal{F}(\Delta)$. Applying the functor F to the exact sequence given in condition 2 of Definition (2.1.1), we obtain

$$0 \longrightarrow F(Z(i)) \longrightarrow P(i) \longrightarrow \Delta(i) \longrightarrow 0.$$

Applying the functor G, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta(i) & \longrightarrow & Y(i) & \longrightarrow & Z(i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G(\Delta(i)) & \longrightarrow & G(P(i)) & \longrightarrow & G(F(Z(i))) & \longrightarrow & \operatorname{Ext}^1_B(\Delta(i), Y) \longrightarrow 0
\end{array}$$

Since the vertical morphisms are isomorphisms by the previous corollary, we have that $\operatorname{Ext}^1_B(\Delta(i), Y) = 0$ for all $i = 1, 2, \cdots, n$, and consequently, the functor G is exact on $\mathcal{F}(\Delta)$. Given $W \in \mathcal{F}(\Delta)$, we will show by induction on the Δ-length $k = l_\Delta(W) = \sum_{j=1}^{n} [W : \Delta(j)]$ that $W = F(V)$ for some $V \in \mathcal{F}(\Theta)$.

If $W = F(X)$ where $X \in \mathcal{F}(\Theta)$, then W is isomorphic to $F(G(W))$. Indeed,

$$F(G(W)) = \operatorname{Hom}_A(\operatorname{Hom}_B(\operatorname{Hom}_A(X, Y), Y), Y),$$

but the previous corollary ensures that $\operatorname{Hom}_B(\operatorname{Hom}_A(W, Y), Y) \cong X$, hence $F(G(W)) \cong \operatorname{Hom}_A(X, Y) = W$. In particular, we have an isomorphism between $\Delta(i) = F(\Theta(i))$ and $F(G(\Delta(i)))$.

Let $W \in \mathcal{F}(\Delta)$ such that $l_\Delta(W) = k$. There exists a Δ-filtration of length k

$$0 = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{k-1} \subseteq W_k = W$$

such that W_j / W_{j-1} is isomorphic to a module in Δ for $j = 1, 2, \cdots, k$. By the induction hypothesis, since $l_\Delta(W_{k-1}) = k - 1$, we have $W_{k-1} = F(V')$ for some $V' \in \mathcal{F}(\Theta)$. In particular, there is an exact sequence

$$0 \longrightarrow W_{k-1} \longrightarrow W \longrightarrow \Delta(i) \longrightarrow 0$$

for some $i \in \mathbb{I}_n$. Applying the functor G and then the functor F, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_{k-1} & \longrightarrow & W & \longrightarrow & \Delta(i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(G(W_{k-1})) & \longrightarrow & F(G(W)) & \longrightarrow & F(G(\Delta(i))) & \longrightarrow & 0
\end{array}$$

where the vertical maps are given by evaluation. Since the vertical morphisms on the left and right are isomorphisms, the central morphism is an isomorphism, which proves the theorem. ■

Equivalence Between EISS and SS

Theorem 2.1.13. *Let* $(\Theta, \mathbb{Y}, \leq)$ *be an EISS of size n, where* $\Theta = \{\Theta(1), \Theta(2), \cdots, \Theta(n)\}$. *Then the modules in* Θ *satisfy:*

1. $\mathrm{Hom}_A(\Theta(j), \Theta(i)) = 0$ *if* $j > i$.
2. $\mathrm{Ext}^1_A(\Theta(j), \Theta(i)) = 0$ *if* $j \geq i$.
3. $\Theta(i)$ *is an indecomposable module for* $i = 1, 2, \cdots, n$.

Proof. The first item follows from the definition, while the second was proven in Lemma 2.1.8. To show that $\Theta(i)$ is indecomposable, note that under the conditions of the previous theorem, the contravariantly equivalence between $\mathcal{F}(\Theta) \subseteq \mathrm{mod}\, A$ and $\mathcal{F}(\Delta) \subseteq \mathrm{mod}\, B^{op}$ implies $\mathrm{End}_A(\Theta(i)) \cong \mathrm{End}_{B^{op}}(\Delta(i))$. Since $\Delta(i)$ is the quotient of an indecomposable projective module, $\Delta(i)$ has a simple top, hence $\Delta(i)$ is indecomposable. Consequently, $\mathrm{End}_{B^{op}}(\Delta(i))$ is a local ring, which implies that $\mathrm{End}_A(\Theta(i))$ is also a local ring, and thus $\Theta(i)$ is indecomposable. ■

The theorem above provides necessary conditions for a finite set of modules Θ to be part of an EISS $(\Theta, \mathbb{Y}, \leq)$. We will show that these conditions are, in fact, sufficient. To do this, we will construct the set of indecomposable modules $\mathbb{Y}$ such that $(\Theta, \mathbb{Y}, \leq)$ forms an EISS. We will start with a lemma.

Lemma 2.1.14. *Let* N, M *be modules such that* $\mathrm{Hom}_A(N, M) = 0$ *and* $\mathrm{Ext}^1_A(M, N) \neq 0$ *and* N *is indecomposable. Consider a non-split exact sequence*

$$\beta: \qquad 0 \longrightarrow N \xrightarrow{f} W \xrightarrow{g} M \longrightarrow 0.$$

Then β *is isomorphic to the direct sum of a split sequence and a non-split sequence*

$$\beta': \qquad 0 \longrightarrow N \longrightarrow W' \longrightarrow M' \longrightarrow 0,$$

with $W'|W$, $M'|M$ *and* W' *indecomposable.*

Proof. We may assume without loss of generality that $N \longrightarrow W$ is an inclusion, so that N is a submodule of W. Suppose $W = W_1 \bigoplus W_2$, then we have $g = [g_1\ g_2]$ with $g_1 : W_1 \longrightarrow M$ and $g_2 : W_2 \longrightarrow M$.

We will show that $N = \ker(g_1) \bigoplus \ker(g_2)$. Since $\ker(g_1) \subseteq W_1$ and $\ker(g_2) \subseteq W_2$, it follows that $\ker(g_1) \bigoplus \ker(g_2) \subseteq W$. Furthermore, since $g(\ker(g_1) \bigoplus \ker(g_2)) = 0$, we have $\ker(g_1) \bigoplus \ker(g_2) \subseteq \ker(g) = N$. Conversely, let $n = (n_1, n_2)^T \in N$ (where $n_1 \in W_1$ and $n_2 \in W_2$). Then $h_i : N \longrightarrow M$, defined by $h_i(n) = g_i(n_i)$ for $i = 1, 2$, is a morphism in $\mathrm{Hom}_A(N, M)$. By the hypothesis $\mathrm{Hom}_A(N, M) = 0$, it follows that $h_i = 0$ for $i = 1, 2$, and thus $n = (n_1, n_2)^T \in \ker(g_1) \bigoplus \ker(g_2)$. Therefore, $\ker(g_1) \bigoplus \ker(g_2) = N$.

For the remaining part, using the isomorphism theorem, we obtain

$$M \cong \frac{W}{N} \cong \frac{W_1 \bigoplus W_2}{\ker(g_1) \bigoplus \ker(g_2)} \cong \frac{W_1}{\ker(g_1)} \bigoplus \frac{W_2}{\ker(g_2)}.$$

Thus, $\beta = \beta_1 \bigoplus \beta_2$, where

$$\beta_i : 0 \longrightarrow \ker(g_i) \longrightarrow W_i \longrightarrow \frac{W_i}{\ker(g_i)} \longrightarrow 0.$$

However, by hypothesis, N is indecomposable, so $\ker(g_2) = 0$ (without loss of generality), and β_2 is split. We can repeat the process on β_1 and "remove" split exact sequences from β iteratively until obtaining β'. The number of iterations is finite since W is a module of finite length, and in such a construction, we have $l(W) > l(W_1)$. ■

Theorem 2.1.15. *Suppose that* $(\Theta, \leq)$ *is an SS of size n. Then there exists a set of indecomposable modules* $\mathbb{Y} = \{Y(1), Y(2), \cdots, Y(n)\}$ *such that* $(\Theta, \mathbb{Y}, \leq)$ *is an EISS.*

Proof. We take $Y(1) = \Theta(1)$. Fixing $i = 2, \cdots, n$, for $k = 1, 2, \cdots, i-1$ we will construct exact sequences

$$\gamma_k : 0 \longrightarrow \Theta(i) \longrightarrow U_k \longrightarrow V_k \longrightarrow 0$$

with $V_k \in \mathcal{F}(\{\Theta(i-1), \cdots, \Theta(i-k)\})$ and U_k indecomposable satisfying $\mathrm{Ext}^1_A(\Theta(j), U_k) = 0$ for $i-k \leq j \leq n$. Under these conditions, we take $Y(i) = U_{i-1}$ and $Z(i) = V_{i-1}$. The exact sequence γ_{i-1} is the exact sequence given in condition 2 of Definition 2.1.1, where $V_{i-1} \in \mathcal{F}(\{\Theta(j) \mid j < i\})$. Condition 3 is also satisfied since $\mathrm{Ext}^1_A(\mathcal{F}(\Theta), Y) = 0$ if, and only if, $\mathrm{Ext}^1_A(\mathcal{F}(\Theta), Y(j)) = 0$ for all $j = 1, 2, \cdots, n$, where $Y = \bigoplus_{j=1}^{n} Y(j)$.

We start by constructing γ_1. If $\mathrm{Ext}^1_A(\Theta(i-1), \Theta(i)) = 0$, take $U_1 = \Theta(i)$ and $V_1 = 0$. Otherwise, there exists a non-split exact sequence

$$0 \longrightarrow \Theta(i) \longrightarrow U \longrightarrow \Theta(i-1)^{\oplus m} \longrightarrow 0 \qquad (2.5)$$

given by the universal extension, where $m = \dim \mathrm{Ext}^1_A(\Theta(i-1), \Theta(i))$. Applying the functor $\mathrm{Hom}_A(\Theta(i-1), -)$, we obtain

$$\mathrm{Ext}^1_A(\Theta(i-1), \Theta(i)) \xrightarrow{\ 0\ } \mathrm{Ext}^1_A(\Theta(i-1), U) \longrightarrow \mathrm{Ext}^1_A(\Theta(i-1), \Theta(i-1)^{\oplus m})$$

where the first morphism is 0 since the connecting morphism

$$\delta : \mathrm{Hom}_A(\Theta(i-1), \Theta(i-1)^{\oplus m}) \longrightarrow \mathrm{Ext}^1_A(\Theta(i-1), \Theta(i))$$

is surjective. Moreover, we have $\mathrm{Ext}^1_A(\Theta(i-1), \Theta(i-1)^{\oplus m}) = \bigoplus_{t=1}^{m} \mathrm{Ext}^1_A(\Theta(i-1), \Theta(i-1)) = 0$ by hypothesis, and thus $\mathrm{Ext}^1_A(\Theta(i-1), U) = 0$.

Applying now the functor $\mathrm{Hom}_A(\Theta(j),-)$ for $j \geq i$, we obtain

$$\mathrm{Ext}^1_A(\Theta(j),\Theta(i)) \longrightarrow \mathrm{Ext}^1_A(\Theta(j),U) \longrightarrow \mathrm{Ext}^1_A(\Theta(j),\Theta(i-1)^{\oplus m}).$$

The end terms of the exact sequence above are 0 by hypothesis, and thus $\mathrm{Ext}^1_A(\Theta(i),U)=0$. By Lemma 2.1.14, removing split summands from (2.5), we obtain

$$\gamma_1 : 0 \longrightarrow \Theta(i) \longrightarrow U_1 \longrightarrow \Theta(i-1)^{\oplus m'} \longrightarrow 0,$$

with $U_1|U$ indecomposable and $m' \leq m$. This is the desired sequence.

Now, suppose γ_k has been constructed, with $k < i-1$. We will construct γ_{k+1}. If $\mathrm{Ext}^1_A(\Theta(i-k-1),U_k)=0$, we set $\gamma_{k+1}=\gamma_k$. Otherwise, constructing the universal extension,

$$0 \longrightarrow U_k \longrightarrow U \longrightarrow \Theta(i-k-1)^{\oplus a} \longrightarrow 0. \tag{2.6}$$

Similarly to the base case, applying the functor $\mathrm{Hom}_A(\Theta(i-k-1),-)$, we obtain that $\mathrm{Ext}^1_A(\Theta(i-k-1),U)=0$, since $\mathrm{Ext}^1_A(\Theta(i-k-1),\Theta(i-k-1)^{\oplus a})=0$, and the connecting morphism $\delta' : \mathrm{Hom}_A(\Theta(i-k-1),\Theta(i-k-1)^{\oplus a}) \longrightarrow \mathrm{Ext}^1_A(\Theta(i-k-1),U_k)$ is surjective.

Applying the functor $\mathrm{Hom}_A(\Theta(j),-)$ with $i-k \leq j \leq n$, we obtain

$$\mathrm{Ext}^1_A(\Theta(j),U_k) \longrightarrow \mathrm{Ext}^1_A(\Theta(j),U) \longrightarrow \mathrm{Ext}^1_A(\Theta(j),\Theta(i-k-1)^{\oplus a}).$$

By the induction hypothesis, we have $\mathrm{Ext}^1_A(\Theta(j),U_k)=0$ and $\mathrm{Ext}^1_A(\Theta(j),\Theta(i-k-1)^{\oplus a})=0$, since $j > i-k-1$. Thus, $\mathrm{Ext}^1_A(\Theta(j),U)=0$.

Using Lemma 2.1.14 on the exact sequence (2.6), we obtain the sequence (where $a' \leq a$)

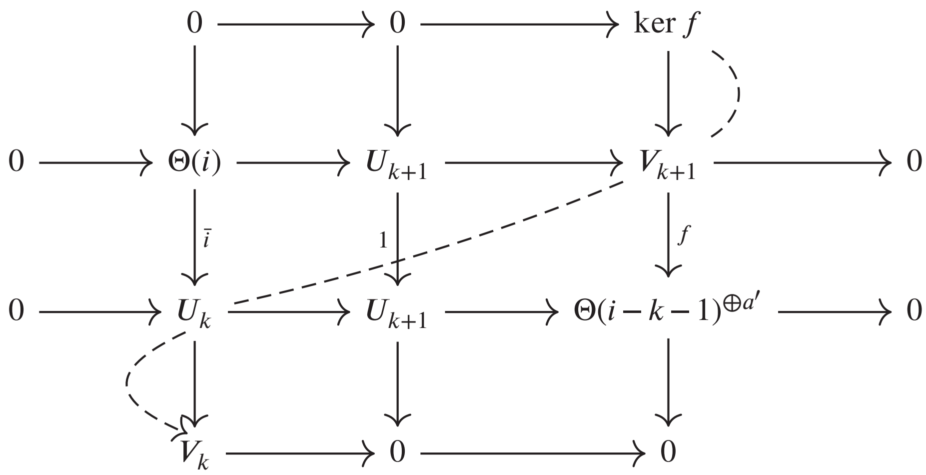
$$0 \longrightarrow U_k \longrightarrow U_{k+1} \longrightarrow \Theta(i-k-1)^{\oplus a'} \longrightarrow 0,$$

with U_{k+1} indecomposable. Composing the inclusion $\bar{i} : \Theta(i) \longrightarrow U_k$ with $U_k \longrightarrow U_{k+1}$, we obtain

$$\gamma_{k+1} : 0 \longrightarrow \Theta(i) \longrightarrow U_{k+1} \longrightarrow V_{k+1} \longrightarrow 0,$$

which is the desired sequence. Indeed, we have that $\mathrm{Ext}^1_A(\Theta(j),U_k)=0$ for $i-k-1 \leq j \leq n$, as $U_k|U$, and U_{k+1} is indecomposable.

To see that V_{k+1} is filtered by $\{\Theta(i-1),\cdots,\Theta(i-k-1)\}$, consider the following diagram:



where f is the unique morphism that completes the diagram. Since $\bar{i}$ is a monomorphism and the second morphism is the identity, we have that f is an epimorphism (see (6, Chapter 2, Theorem 4.4)). Moreover, the Snake Lemma ensures that $\ker f \cong V_k$, and thus the following sequence is exact

$$0 \longrightarrow V_k \longrightarrow V_{k+1} \longrightarrow \Theta(i-k-1)^{\oplus a'} \longrightarrow 0$$

shows that $V_{k+1} \in \mathcal{F}(\{\Theta(i-1), \cdots, \Theta(i-k-1)\})$. ■

Combining Theorems 2.1.13 and 2.1.15, we have the following theorem:

Theorem 2.1.16. *Let $\Theta = \{\Theta(1), \Theta(2), \cdots, \Theta(n)\}$ be a totally ordered set of nonzero modules. The following are equivalent:*

1. *There exists a set of indecomposable modules $\mathbb{Y} = \{Y(1), Y(2), \cdots, Y(n)\}$ such that $(\Theta, \mathbb{Y}, \leq)$ is an EISS.*
2. *$(\Theta, \leq)$ is an SS.*

■

To characterize a stratifying system by the set of relatively simple modules, we must prove that, given $\Theta = \{\Theta(1), \cdots, \Theta(n)\}$ satisfying the conditions of Theorem 2.1.16, there exists a unique set of modules $\mathbb{Y} = \{Y(1), \cdots, Y(n)\}$, up to isomorphism, such that $(\Theta, \mathbb{Y}, \leq)$ is an Ext-injective stratifying system.

For what follows, we define the following subcategories of $\mathcal{F}(\Theta)$:

$$\mathcal{I}(\Theta) = \{M \in \mathcal{F}(\Theta) \mid \mathrm{Ext}^1_A(-, M)|_{\mathcal{F}(\Theta)} = 0\}.$$
$$\mathcal{P}(\Theta) = \{M \in \mathcal{F}(\Theta) \mid \mathrm{Ext}^1_A(M, -)|_{\mathcal{F}(\Theta)} = 0\}.$$

Lemma 2.1.17. *Let $\alpha_i : \Theta(i) \longrightarrow Y(i)$ be the morphism given in Definition 2.1.1. Then α_i is the minimal $\mathcal{I}(\Theta)$-left approximation of $\Theta(i)$.*

Proof. Consider the exact sequence

$$0 \longrightarrow \Theta(i) \xrightarrow{\alpha_i} Y(i) \longrightarrow Z(i) \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_A(-,I)$ with $I \in \mathcal{I}(\Theta)$, we obtain

$$\mathrm{Hom}_A(Y(i),I) \xrightarrow{\alpha_i^*} \mathrm{Hom}_A(\Theta(i),I) \longrightarrow 0.$$

Thus, α_i is an $\mathcal{I}(\Theta)$-left approximation. This approximation is minimal since $Y(i)$ is indecomposable. ■

Proposition 2.1.18. *Let $(\Theta, \mathbb{Y}, \leq)$ and $(\Theta, \mathbb{Y}' \leq)$ be two EISS of size n. Then for each $i = 1,2,\cdots,n$, there exists an isomorphism $f_i : Y(i) \longrightarrow Y'(i)$, where $Y(i) \in \mathbb{Y}$ and $Y'(i) \in \mathbb{Y}'$.*

Proof. Condition 3 of Definition 2.1.1 states that $Y(i)$ and $Y'(i)$ are in $\mathcal{I}(\Theta)$ for all $i = 1,2,\cdots,n$. Consider the minimal left approximations $\alpha_i : \Theta(i) \longrightarrow Y(i)$ and $\alpha_i' : \Theta(i) \longrightarrow Y'(i)$ of the module $\Theta(i)$. There exist morphisms $f_i : Y(i) \longrightarrow Y'(i)$ and $g_i : Y'(i) \longrightarrow Y(i)$ such that $\alpha_i' = f_i \circ \alpha_i$ and $\alpha_i = g_i \circ \alpha_i'$. Hence, $\alpha_i = (f_i \circ g_i) \circ \alpha_i$ and $\alpha_i' = (g_i \circ f_i) \circ \alpha_i'$, which implies that $f_i \circ g_i$ and $g_i \circ f_i$ are isomorphisms. Thus, f_i and g_i are isomorphisms. ■

It is also possible to provide a third characterization of a stratifying system (see (30)).

Definition 2.1.19 (Ext-Projective Stratifying System)**.** *Let A be an algebra and let $\Theta = \{\Theta(1), \Theta(2), \cdots, \Theta(n)\}$ be a fixed ordered set of A-modules. Moreover, suppose $\mathbb{W} = \{W(1), W(2), \cdots, W(n)\}$ is a set of indecomposable A-modules. We call $(\Theta, \mathbb{W}, \leq)$ an* ***Ext-projective stratifying system of size n*** *(EPSS, for short) if the following conditions hold:*

1. $\mathrm{Hom}_A(\Theta(k), \Theta(j)) = 0$ *if $k > j$.*
2. *For all $i \in \mathbb{I}_n$, there is an exact sequence*

$$0 \longrightarrow K(i) \longrightarrow W(i) \longrightarrow \Theta(i) \longrightarrow 0;$$

and $K(i)$ is filtered by $\Theta(j)$ with $j > i$, that is, $K(i) \in \mathcal{F}(\{\Theta(j) \mid j > i\})$.

3. $\mathrm{Ext}^1_A(W,-)|_{\mathcal{F}(\Theta)} = 0$*, where* $W = \bigoplus_{j=1}^{n} W(j)$.

All the theorems presented in this section can be dualized for EPSS.

It is possible to use the equivalence of Theorem 2.1.12 to prove that, if $(\Theta, \mathbb{Y}, \leq)$ and $(\Theta', \mathbb{Y}, \leq)$ are two EISS (or EPSS) modules of size n, then there exist isomorphisms $f_i : \Theta(i) \longrightarrow \Theta'(i)$ for all $i = 1,2,\cdots,n$. Moreover, it can be shown that $\mathcal{F}(\Theta)$ is a functorially finite subcategory of $\mathrm{mod}\, A$ closed under direct summands, sums, and extensions. Additionally, it can be proven that $\mathcal{I}(\Theta) = \mathrm{add}\, Y$ and $\mathcal{P}(\Theta) = \mathrm{add}\, W$.

Let Δ be a set of standard modules. Then Δ is a stratifying system. Consequently, there exist $(\Delta, \mathbb{Y}, \leq)$ and $(\Delta, \mathbb{W}, \leq)$, EISS and EPSS, respectively, where $\mathbb{Y}$ and $\mathbb{W}$ are uniquely defined up to isomorphism. If $A_A \in \mathcal{F}(\Delta)$ (i.e., if A is a standardly stratified algebra), it is possible to identify $\mathbb{W}$ as the set of indecomposable projective modules and $\mathbb{Y}$ as the set of indecomposable summands of Y, the characteristic generalized tilting module relative to Δ.

It is easy to see that if $(\Theta, \leq)$ is a stratifying system of size n and $\Theta' \subseteq \Theta$ is a non-empty subset, then Θ' with the induced order from Θ is again a stratifying system. We conclude this section with an example that illustrates the relationship between Θ' and the set $\mathbb{Y}'$ such that $(\Theta', \mathbb{Y}', \leq)$ is an EISS.

Example 2.1.20. *Consider $(\Theta, \leq)$ as the same stratifying system from Example 2.1.4. We have that $\Theta' = \{S(3), S(1)\}$ is a stratifying system. In this case,*

$$\mathcal{F}(\Theta') = \text{add}(S(1) \bigoplus S(3)),$$

and the sets of relatively injective modules are now given by $\mathbb{Y}' = \Theta'$. Note that $\mathbb{Y}' \subsetneq \mathbb{Y}$ is different from the expected $\{S(3), P(2)\}$, where we only exclude the Ext-injective associated to $\Theta(2) = S(2)$.

To justify this, consider the canonical sequence associated to the SS Θ:

$$0 \longrightarrow S(1) \longrightarrow P(2) \longrightarrow S(2) \longrightarrow 0.$$

Here, $S(2) = Z(3)$ is filtered by $\Theta(2) \in \mathcal{F}(\Theta(1) \bigoplus \Theta(2))$, but $S(2) \notin \mathcal{F}(\Theta(1))$, hence $P(2)$ cannot be the Ext-injective associated to $S(1)$ in Θ'.

2.2 Historical overview of stratifying systems

In this section, we present a historical overview of the development of the theory of stratifying systems through a selection of relevant articles, arranged chronologically. Our goal is not to offer an exhaustive or comprehensive review of the literature but rather to highlight some works that played a significant role in advancing this topic. For each selected article, we provide a brief description, emphasizing the main results, contributions, and potential innovations introduced.

This approach aims to provide the reader with a broad perspective on the evolution of the central ideas of the theory, as well as to establish a context for the concepts and results we will explore in the subsequent sections.

(2003) - On Standardly Stratified Algebras (25)

This was the first article on the topic. In this paper, K. Erdmann and C. Sáenz define Ext-injective stratifying systems. In addition to proving the results presented in the previous section, they constructed some stratifying systems in special biserial self-injective algebras.

(2004-2005) - Stratifying Systems via Relative Simple (Projective) Modules (29; 30)

In a series of two papers, E. Marcos, O. Mendoza, and C. Sáenz presented two equivalent definitions of stratifying systems, via relatively projective and simple modules, with the latter being the most accepted due to its simplicity. Moreover, they presented various results about the category $\mathcal{F}(\Theta)$ and homological results involving standard stratifying systems, for instance:

Theorem 2.2.1. *The algebra A is quasi-hereditary if and only if gl.dim $A < \infty$ and there is a stratifying system $(\Theta, \leq)$ such that $A_A \in \mathcal{F}(\Theta)$ and* $\mathrm{Ext}^2_A(\mathcal{F}(\Theta), \mathcal{I}(\Theta)) = 0$.

(2006) - Applications of Stratifying Systems to the Finitistic Dimension (31)

One of the main tools in the study of homological algebra is homological dimensions, which, in some sense, "measure the deviation" of a module category from being semisimple. One such homological dimension is the finitistic dimension, defined as

$$\mathrm{pfd}(A) = \sup\{\mathrm{pd}\ M \mid M \in \mathrm{mod}\, A \text{ and } \mathrm{pd}(M) < \infty\}.$$

H. Bass conjectured in the 60s that the finitistic dimension is finite whenever A is a finite dimensional k-algebra over an algebraically closed field. Since then, significant progress has been made, and partial results have been obtained (34).

In this article, E. Marcos, O. Mendoza, and C. Sáenz proved that, given an Ext-injective stratifying system $(\Theta, \mathbb{Y}, \leq)$ such that $\mathrm{pd}\, Y \leq \infty$, where $Y = \bigoplus_{i=1}^{t} Y(i)$, then

$$\mathrm{pfd}(A) = \mathrm{pfd}(\mathcal{I}(\Theta)).$$

(2006) - Quadratic Forms Associated to Stratifying Systems (33)

To determine when $\mathcal{F}(\Theta)$ has a finite number of indecomposable modules, under the condition that $(\Theta, \leq)$ is a stratifying system satisfying certain properties, E. Marcos, O. Mendoza, C. Sáenz, and R. Zuazua defined quadratic forms involving Θ. These forms address the problem in the case where $\mathcal{F}(\Theta)$ is Θ-directing, that is, for any $M \in \mathcal{F}(\Theta)$, there does not exist a finite sequence of modules $X_0, X_1, \cdots, X_m$ with $X_0 \cong X_m \cong M$ such that $\mathrm{rad}_A(X_{i-1}, X_i) \neq 0$ for all $1 \leq i \leq m$.

Definition 2.2.2. *Given a stratifying system $(\Theta, \leq)$ of size t, we define two quadratic forms:*

1. *The Tits form $q_\Theta : \mathbb{Z}^t \longrightarrow \mathbb{Z}$, given by the equation*

$$q_\Theta(x) = \sum_{l=0}^{2} \sum_{i=1}^{t} \sum_{j=1}^{t} (-1)^l \dim_k \mathrm{Ext}^l_A(\Theta(i), \Theta(j)) x(i) x(j),$$

where $x = (x(1), x(2), \cdots, x(t)) \in \mathbb{Z}^t$.

2. *If* $\mathrm{pd}_{\mathcal{F}(\Theta)}(\Theta) < \infty$*, we define a bilinear form* $\langle -,- \rangle_\Theta$ *on* $\mathbb{Z}^t$ *by*

$$\langle x,y\rangle_\Theta = \sum_{l=0}^{\infty}\sum_{i=1}^{t}\sum_{j=1}^{t}(-1)^l \dim_k \mathrm{Ext}_A^l(\Theta(i),\Theta(j))x(i)y(j).$$

In this case, the Euler quadratic form $\eta_\Theta : \mathbb{Z}^t \longrightarrow \mathbb{Z}$ *is defined as*

$$\eta_\Theta(x) = \langle x,x\rangle_\Theta.$$

Note that the summation in the second item is finite. Using these quadratic forms, the following theorem was established:

Theorem 2.2.3. *Let A be an algebra, and* $(\Theta,\leq)$ *a stratifying system. If* $\mathcal{F}(\Theta)$ *is* Θ*-directing, then* $\mathcal{F}(\Theta)$ *has a finite number of indecomposable modules.*

(2008) - An Approach to The Finitistic Dimension Conjecture (27)

In “Finitistic dimension of standardly stratified algebras” (4), it was proven that if A is a standardly stratified algebra for some ordering of the projectives, then

$$\mathrm{pfd}(A) \leq 2n-2,$$

where $n = |A_A|$.

In this article, F. Huard, M. Lanzilotta, and O. Mendoza consider the following conjecture:

Conjecture 2.2.4. *Let A be any finite dimensional k-algebra. Then, for any stratifying system* $(\Theta,\leq)$ *in* $\mathrm{mod}\,A$*, we have that*

$$pfd\ \mathcal{F}(\Theta) := sup\{pd\ M \mid M \in \mathcal{F}(\Theta) \text{ and } pd(M) < \infty\} < \infty.$$

In the same article, a proof of this conjecture was provided for some particular cases. For instance, if $(\Theta,\leq)$ is a stratifying system such that pd $\Theta(j) < \infty$ for all $j \neq i$, except for exactly one index i where pd $\Theta(i) = \infty$, then

$$\mathrm{pfd}\ \mathcal{F}(\Theta) \leq \max\{\mathrm{pd}\ \Theta(j) \mid j \neq i\} < \infty.$$

They also proved an analogous result in the case where Θ has exactly two or three modules with pd $\Theta(i) = \infty$.

(2014) - Split-by-Nilpotent Extensions Algebras and Stratifying Systems (28)

Let A and B be two algebras such that B is a split-by-nilpotent extension of A, that is, there exists a split surjective algebra morphism $f : B \longrightarrow A$ whose kernel K is a nilpotent ideal of B, and let $G = B \otimes_A -$ and $F = A \otimes_B -$ be the change-of-rings functors. In this article, M. Lanzilotta, O. Mendoza, and C. Sáenz provided conditions for when a stratifying system $(\Theta, \leq)$ in $\operatorname{mod} A$ induces a stratifying system $(G(\Theta), \leq)$ in $\operatorname{mod} B$ and when a stratifying system $(\Theta', \leq)$ in $\operatorname{mod} B$ restricts to a stratifying system in $\operatorname{mod} A$ through the functor F.

We say that a stratifying system $(\Theta, \leq)$ is compatible with the ideal I if it satisfies the following conditions:

1. $\operatorname{Hom}_A(\Theta(j), I \otimes_A \Theta(i)) = 0$ for $j > i$.

2. $\operatorname{Ext}^1_A(\Theta(j), I \otimes_A \Theta(i)) = 0$ for $j \geq i$.

Proposition 2.2.5. *Let Θ be an ordered set of modules in* $\operatorname{mod} A$ *and $\leq$ be a total order. If I is projective, then the following conditions are equivalent:*

1. $(\Theta, \leq)$ is a stratifying system in $\operatorname{mod} A$*, which is compatible with the ideal K.*

2. $(G(\Theta), \leq)$ is a stratifying system in $\operatorname{mod} B$.

(2015) - Stratifying Systems Over Hereditary Algebras (18)

In this paper, P. Cadavid and E. Marcos show that the size of a stratifying system is at most $|A_A|$ if A is a hereditary algebra. Furthermore, they construct all complete stratifying systems in generalized Kronecker algebras as shown in the proposition below (A stratifying system is said to be complete if it is not a proper subset of another stratifying system):

Proposition 2.2.6. *Let $A = kQ$, where Q is the quiver*

$$1 \underset{\alpha_m}{\overset{\alpha_1}{\underset{\cdots}{\leftleftarrows}}} 2$$

with $m \geq 2$. Then the list of all complete stratifying systems over A is the following:

1. $\{I(2), P(1)\}$.

2. $\{\tau^{-i}(P(1)), \tau^{-i}(P(2))\}$ with $i \geq 0$.

3. $\{\tau^{-i}(P(2)), \tau^{-i-1}(P(1))\}$ with $i \geq 0$.

4. $\{\tau^{i}(I(1)), \tau^{i}(I(2))\}$ with $i \geq 0$.

5. $\{\tau^{i+1}(I(2)), \tau^{i}(I(1))\}$ with $i \geq 0$.

The authors also showed that if A is of Euclidean type, a stratifying system over A has at most $n-2$ regular modules. A regular module is a module that is neither of the form $\tau^{-k}(P(i))$ nor of the form $\tau^{k}(I(i))$ for any i, k. For algebras of Euclidean type, we recommend Chapter VII.2 of (8).

(2016) - Stratifying Systems Over the Hereditary Path Algebra with Quiver $\mathbb{A}_{p,q}$ (19)

Continuing their previous work, P. Cadavid and E. Marcos construct some complete stratifying systems with the maximum number of regular modules in algebras of type $\mathbb{A}_{p,q}$. The construction of such stratifying systems involves modular arithmetic with the numbers p, q and the powers of the Auslander-Reiten translation of certain modules.

(2019) - Cokernels of the Cartan Matrix and Stratifying Systems (32)

Let A be an algebra. It is well-known that the Cartan matrix C_A of such an algebra is an important tool used in various contexts. It is known that $\det C_A = \pm 1$ for any k-algebra of finite global dimension, and it is conjectured that $\det C_A = 1$ in this case. Considering the Cartan matrix as a homomorphism $C_A : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$, where $n = |A_A|$, one can define the abelian group $G_A := \operatorname{coker}(C_A)$.

Considering an EPSS $(\Theta, \mathbb{W}, \leq)$ over an algebra A, E. Marcos, O. Mendoza, and C. Sáenz studied properties of the Cartan matrix C_B and the group G_B when $B = \operatorname{End}_A(W)^{\mathrm{op}}$, with $W = \bigoplus_{j=1}^{t} W(j)$, yielding results such as the following:

Theorem 2.2.7. *Let* $(\Theta, \mathbb{W}, \leq)$ *be an Ext-projective stratifying system of size t in* mod A*, and* $B = \operatorname{End}_A(W)^{op}$*. Then, the following statements hold true:*

1. $G_B \cong \operatorname{coker}(C_\Theta)$ *and* $|G_B| = \prod_{j=1}^{t} \dim \operatorname{End}_A(\Theta(j))$.
2. *The exponent of* G_B *is a multiple of* $\dim \operatorname{End}_A(\Theta(j))$*, for any* $j \in \mathbb{I}_t$*.*

Corollary 2.2.8. *Let* $(A, \leq)$ *be a standardly stratified k-algebra. Then, the following statements hold true:*

1. $G_A \cong \operatorname{coker}(C_\Delta)$ *and* $|G_A| = \prod_{j=1}^{t} \dim \operatorname{End}_A(\Delta(j))$.
2. *If* $(A, \leq)$ *is weakly triangular, then* $C_\Delta = diag(d_1, \cdots, d_n)$ *and* $G_A = \bigoplus_{j=1}^{n} \mathbb{Z}/d_j\mathbb{Z}$*, where* $d_j = \dim(\Delta(j))$.
3. $(A, \leq)$ *is quasi-hereditary if and only if* $G_A = 0$.

(2020) - Stratifying Systems Through τ-Tilting Theory (36)

In this work, O. Mendoza and H. Treffinger show that, given a basic τ-rigid module M such that $|M| = t$, there exists at least one stratifying system of size t induced by M. The construction of the stratifying system depends on a specific order of the summands of M and, in some sense, generalizes the notion of standard modules since we can follow the construction presented in this paper with the τ-rigid module A_A. In the next section, we will present this construction in detail.

(2022) - Stratifying Systems and Jordan-Hölder Extriangulated Categories (13)

In this paper, T. Brüstle, S. Hassoun, A. Shah, and A. Tattar define stratifying systems in extriangulated categories, which generalize both abelian categories and triangulated categories. Furthermore, they studied the subcategory of objects filtered by a stratifying system and provided a sufficient condition for such a subcategory to be Jordan-Hölder.

(2023) - The Size of a Stratifying System Can Be Arbitrarily Large (41)

It was conjectured that the size of a stratifying system was somehow limited by the rank $|A_A|$ of the algebra. In this paper, H. Treffinger shows that the conjecture is false by constructing counterexamples in d-Auslander algebras of $\mathbb{A}_n$.

Theorem 2.2.9. *For every positive integer m, there exists an algebra A_m and a stratifying system $(\Theta_m, \leq)$ of size s_m in* $\operatorname{mod} A_m$ *such that $s_m > m|A_m|$.*

Furthermore, he showed how to construct a stratifying system of infinite size indexed by $(\mathbb{N}, \leq)$ in n-representation infinite algebras, where $\leq$ is the natural order.

(2023) - Stratifying Systems and g-Vectors (35)

Following a similar approach to "Cokernels of the Cartan Matrix and Stratifying Systems", O. Mendoza, C. Sáenz, and H. Treffinger studied properties of the Cartan matrix C_{Θ_M} and the group G_{Θ_M} when Θ_M is a stratifying system induced by a basic τ-rigid module. As an example, we present one of the main results:

Theorem 2.2.10. *For a basic τ-tilting module $M \in \operatorname{mod} A$ with a TF-admissible decomposition (see Definition 2.3.1) $M = \bigoplus_{i=1}^{t} M_i$ such that $M \in \mathcal{F}(\Theta_M)$, the following statements are equivalent:*

1. *The matrix C_{Θ_M} is diagonal.*
2. $\operatorname{Hom}_A(M_i, M_j) = 0$ *for $i < j$.*
3. $\operatorname{Hom}_A(M_i, \Theta_M(j)) = 0$ *for $i < j$.*

Moreover, if one of the above conditions holds true, then $M \cong A_A$ and A is a weakly triangular algebra.

2.3 Stratifying systems through τ-Tilting Theory

Except for standard module sets, no general method for constructing a stratifying system was previously known. In other words, there was no method that could describe the construction of a stratifying system for any given algebra A. In 2020, H. Treffinger and O. Mendoza demonstrated how to construct a stratifying system starting from a basic nonzero τ-rigid module. The method can be summarized as follows:

- Given a basic nonzero τ-rigid module M such that $|M| = t$, we write M as an ordered decomposition into indecomposables:

$$M = M_1 \bigoplus M_2 \bigoplus \cdots \bigoplus M_t,$$

  such that $M_k \notin \operatorname{Fac}\left(\bigoplus_{j>k} M_j\right)$. There always exists at least one decomposition as above, called torsion-free admissible. Such a decomposition can be obtained using mutation methods for τ-rigid modules.

- Define $\Theta(k) = f_{k+1}(M_k)$ for $k = 1, 2, \cdots, t$, where f_k is the torsion-free functor associated with the pair $\left(\operatorname{Fac}\left(\bigoplus_{j\geq k} M_j\right), \left(\bigoplus_{j\geq k} M_j\right)^{\perp}\right)$. The ordered set $\Theta_M = \{\Theta(1), \cdots, \Theta(t)\}$ will be a stratifying system. Note that $f_{t+1}(M_t) = M_t$.

If $t < n = |A_A|$, it is possible to extend the stratifying system as follows: Let $\mathcal{B}_M$ denote the Bongartz completion of M (see Theorem 1.3.10), so that $\mathcal{B}_M = M \bigoplus N$ for some τ-rigid module N with $|N| = n - t$. Consider the following decomposition of $\mathcal{B}_M$:

$$\mathcal{B}_M = (N_1 \bigoplus \cdots \bigoplus N_{n-t}) \bigoplus (M_1 \bigoplus \cdots \bigoplus M_t),$$

where $(N_1 \bigoplus \cdots \bigoplus N_{n-t})$ is any ordered decomposition of N, and $(M_1 \bigoplus \cdots \bigoplus M_t)$ is a TF-admissible decomposition of M. This decomposition of $\mathcal{B}_M$ will also be TF-admissible. Following the previous steps, we obtain a stratifying system $\Theta'_{\mathcal{B}_M}$ of size n that contains Θ_M.

The results presented below can be dualized for τ^--rigid modules.

Definition 2.3.1. *Let A be an algebra and $M \in \operatorname{mod} A$ be a basic nonzero τ-rigid A-module. We say that a decomposition of $M = \bigoplus_{i=1}^{t} M_i$ as the direct sum of indecomposable A-modules is **torsion-free admissible** (TF-admissible, for short) if $M_i \notin \operatorname{Fac}(\bigoplus_{j>i} M_j)$, for every $i \in \mathbb{I}_t$.*

Proposition 2.3.2. *For an algebra A, every basic nonzero τ-rigid module in $\operatorname{mod} A$ admits a TF-admissible decomposition.*

Proof. We will prove this by induction on the number of summands of M. If $|M| = 1$, the result follows immediately.

Assume that $|M| = t \geq 2$. We have that $\operatorname{Fac}(M)$ is a functorially finite torsion class by Proposition 1.3.6, so by Theorem 1.3.8, there exists a basic support τ-tilting module T such that $\operatorname{Fac}(M) = \operatorname{Fac}(T)$ (if M is support τ-tilting, then $T = M$).

Since $\operatorname{Fac}(M) \subseteq \operatorname{Fac}(T) \subseteq {}^{\perp}(\tau M)$ (the second inclusion follows from the fact that $\operatorname{Fac}(M) = \operatorname{Fac}(T)$ and we always have $\operatorname{Fac}(M) \subseteq {}^{\perp}(\tau M)$ by Proposition 1.3.9), Proposition

1.3.9 ensures that $M \in \operatorname{add} P(\operatorname{Fac}(T)) = T$, so $T = M \bigoplus M'$ for some basic τ-rigid module M' (M' will be zero in the case where $M = T$).

Since M is nonzero, we have $\{0\} \subsetneq \operatorname{Fac}(T)$. Proposition 1.3.18 states that there exists a basic τ-rigid module N, which is a mutation of T over a summand M_1, such that

$$\{0\} \subseteq \operatorname{Fac}(N) \subsetneq \operatorname{Fac}(T).$$

We claim that M_1 is a summand of M. Suppose otherwise. Then $N = M \bigoplus N'$, so $\operatorname{Fac}(N) = \operatorname{Fac}(M \bigoplus N') \supset \operatorname{Fac}(M)$, a contradiction. Therefore, $M \cong M_1 \bigoplus M''$, where M'' is a basic τ-rigid module with $t-1$ summands. By the induction hypothesis, there exists a TF-admissible decomposition $M'' = \bigoplus_{i=2}^{t} M_i$, that is, $M_i \notin \operatorname{Fac}\left(\bigoplus_{j>i} M_j\right)$ for $2 \leq i \leq t$.

For the remaining part, since $N = M/M_1 \bigoplus Q = M'' \bigoplus Q$, where Q is either zero or an indecomposable τ-rigid module, we have $\operatorname{Fac}(M'') \subseteq \operatorname{Fac}(N)$. As $M_1 \notin \operatorname{Fac}(N)$, we conclude that $M_1 \notin \operatorname{Fac}(M'') = \operatorname{Fac}\left(\bigoplus_{j>1} M_j\right)$. Hence, we obtain a TF-admissible decomposition for M. ■

Proposition 2.3.3. *Let A be an algebra, $n = |A_A|$ and M be a basic nonzero τ-rigid A-module with TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$. Then every decomposition $\mathcal{B}_M = \bigoplus_{j=1}^{n} N_j$ into indecomposable modules of the Bongartz completion of M such that $M_i \cong N_{n-t+i} \forall i \in \mathbb{I}_t$ is a TF-admissible decomposition of $\mathcal{B}_M$.*

Proof. We have that $\mathcal{B}_M = M' \bigoplus M$ for some τ-rigid basic M' (where $M' = 0$ if $|M| = n$).

Consider $X | M'$ indecomposable, $U = M \bigoplus M'/X$, and T the mutation of $\mathcal{B}_M$ over X. We have that $\mathcal{B}_U = \mathcal{B}_M$. In fact, since $U | \mathcal{B}_M$, we have ${}^{\perp}(\tau\mathcal{B}_M) \subseteq {}^{\perp}(\tau U)$, hence Proposition 1.3.9 and Proposition 1.3.10 ensure that

$$\operatorname{Fac}(U) \subseteq \operatorname{Fac}(\mathcal{B}_M) = {}^{\perp}(\tau\mathcal{B}_M) \subseteq {}^{\perp}(\tau U) = \operatorname{Fac}(\mathcal{B}_U)$$

and consequently $\mathcal{B}_M | \mathcal{B}_U$. Since $\mathcal{B}_M$ is τ-tilting, we have $|\mathcal{B}_M| = n$, and thus $\mathcal{B}_M = \mathcal{B}_U$.

We have that $(U, 0)$ is an almost complete support τ-tilting pair; therefore, Theorem 1.3.14 states that:

$$\{\operatorname{Fac}(\mathcal{B}_M), \operatorname{Fac}(T)\} = \{\operatorname{Fac}(U), {}^{\perp}(\tau U)\}.$$

We have that $\operatorname{Fac}(T) = \operatorname{Fac}(U)$. Indeed, suppose $\operatorname{Fac}(\mathcal{B}_M) = \operatorname{Fac}(U)$. Since $X | \mathcal{B}_M$, we would have $X \in \operatorname{Fac}(U)$. Proposition 1.3.16 asserts that ${}^{\perp}(\tau U) \nsubseteq {}^{\perp}(\tau X)$, and Proposition 1.3.17 guarantees that $\mathcal{B}_M$ is not the Bongartz complement of U, which is a contradiction. Therefore, $\operatorname{Fac}(T) = \operatorname{Fac}(U) \subsetneq \operatorname{Fac}(\mathcal{B}_M) = {}^{\perp}(\tau U)$.

Suppose that $X \in \mathrm{Fac}(T)$. Since $T = U \bigoplus Y$, where $Y = 0$ or Y is indecomposable, we would have $\mathcal{B}_M \in \mathrm{Fac}(T)$, a contradiction. Therefore, $X \notin \mathrm{Fac}(T)$ and, in particular, $X \notin \mathrm{Fac}(M \bigoplus M'/X)$.

Consider the decomposition

$$\mathcal{B}_M = (N_1 \bigoplus \cdots \bigoplus N_{n-t}) \bigoplus (M_1 \bigoplus \cdots \bigoplus M_t),$$

where $(M_1 \bigoplus \cdots \bigoplus M_t)$ is a TF-admissible decomposition of M. If $i = 1, 2, \cdots, n-t$, we have shown that $N_i \notin \mathrm{Fac}(\mathcal{B}_M/N_i)$, and, in particular, $N_i \notin \mathrm{Fac}(\bigoplus_{j>i} N_j)$. If $i = n-t+1, \cdots, n$, we have that $N_i \notin \mathrm{Fac}(\bigoplus_{j>i} N_j)$ by hypothesis, which proves the proposition. ■

Theorem 2.3.4. *Let A be an algebra such that $n = |A_A|$ and M be a basic nonzero τ-rigid module with a TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$. If f_k is the torsion-free functor associated to the torsion pair $\left(\mathrm{Fac}\left(\bigoplus_{j \geq k} M_j\right), \left(\bigoplus_{j \geq k} M_j\right)^{\perp}\right)$, then the following statements hold true.*

1. *The family $\Theta_M = \{\Theta(i) = f_{i+1}(M_i) \mid i \in \mathbb{I}_t\}$ and the natural order on $\mathbb{I}_t$ form a stratifying system of size t in* mod *A.*
2. *There exists at least one stratifying system Θ'_M of size n in* mod *A such that $\Theta_M \subseteq \Theta'_M$.*

Proof. 1) We will denote $\mathcal{T}_k = \mathrm{Fac}\left(\bigoplus_{j \geq k} M_j\right)$ and $\mathcal{F}_k = \left(\bigoplus_{j \geq k} M_j\right)^{\perp}$.

We start by showing that $\mathrm{Hom}_A(\Theta(k), \Theta(j)) = 0$ if $k > j$. Since $M_k \in \mathcal{T}_k$ and $\mathcal{T}_k$ is closed under quotients, we have that $\Theta(k) \in \mathcal{T}_k$. On the other hand, we have that $\Theta(j) \in \mathcal{F}_j \subseteq \mathcal{F}_k$, hence $\mathrm{Hom}_A(\Theta(k), \Theta(j)) = 0$ if $k > j$ since $\mathrm{Hom}_A(\mathcal{T}_k, \mathcal{F}_k) = 0$.

We have that $\mathrm{Ext}^1_A(\Theta(k), \Theta(j)) = 0$ if $k \geq j$. Consider the canonical exact sequence

$$0 \longrightarrow t_{k+1}(M_k) \longrightarrow M_k \longrightarrow \Theta(k) \longrightarrow 0,$$

applying the functor $\mathrm{Hom}_A(-, \Theta(j))$ we obtain

$$\mathrm{Hom}_A(t_{k+1}(M_k), \Theta(j)) \longrightarrow \mathrm{Ext}^1_A(\Theta(k), \Theta(j)) \longrightarrow \mathrm{Ext}^1_A(M_k, \Theta(j)).$$

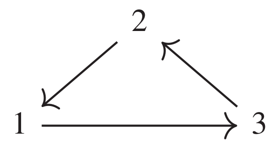
We have that $\mathrm{Hom}_A(t_{k+1}(M_k), \Theta(j)) = 0$ because $t_{k+1}(M_k) \in \mathcal{T}_{k+1}$ and $\Theta(j) \in \mathcal{F}_j \subseteq \mathcal{F}_{k+1}$. We also have that $\mathrm{Ext}^1_A(M_k, \Theta(j)) = 0$ because $\Theta(j) \in \mathcal{T}_j \subseteq \mathrm{Fac}(M)$ (since $\Theta(j)$ is a quotient of M_j) and M_k is Ext-projective in $\mathrm{Fac}(M)$ by Theorem 1.3.6. Therefore, $\mathrm{Ext}^1_A(\Theta(k), \Theta(j)) = 0$ if $k \geq j$.

Finally, Lemma 1.3.7 ensures that $\Theta(k) = f_{k+1}(M_k)$ is indecomposable since $M_k \notin \mathrm{Fac}(\bigoplus_{j>k} M_j)$.

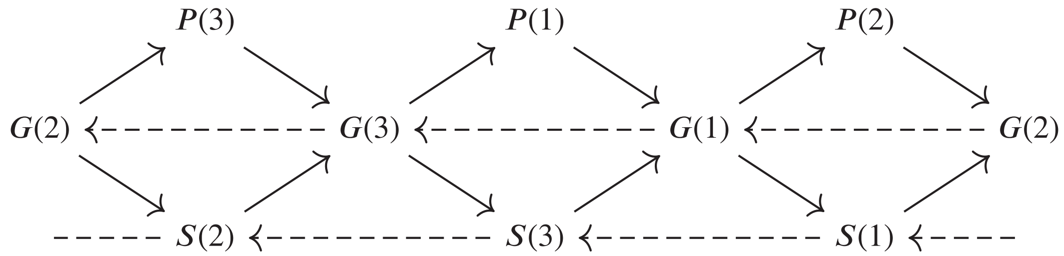
2) This follows directly from Proposition 2.3.3. ■

We conclude this chapter with an example where we apply the result of the theorem above to obtain a stratifying system through a τ-rigid module.

Example 2.3.5. *Let $A = kQ/\mathrm{rad}^3 A$, where Q is the following quiver:*



The Auslander-Reiten quiver of the algebra is given by:



In the quiver above, we denote $G(k) = P(k)/\mathrm{rad}^2 P(k)$ (we must identify the two copies of $G(2)$, so the quiver takes the shape of a cylinder. Notice that $\tau(S(2)) = S(1)$).

Consider the module $M = P(1) \bigoplus P(2) \bigoplus S(1)$. We have $\tau(M) = S(3)$, and from the Auslander-Reiten quiver, it is clear that $\mathrm{Hom}_A(M, \tau(M)) = 0$.

The ordered decomposition $M = P(1) \bigoplus P(2) \bigoplus S(1)$ is TF-admissible (it can also be shown that $M = P(2) \bigoplus P(1) \bigoplus S(1)$ is another TF-admissible decomposition). Indeed, we have:

$$\mathrm{Fac}(P(2) \bigoplus S(1)) = \mathrm{add}\{P(2), G(2), S(2), S(1)\} \qquad (P(2) \bigoplus S(1))^{\perp} = \mathrm{add}\{G(1), S(3)\}$$
$$\mathrm{Fac}(S(1)) = \mathrm{add}\{S(1)\} \qquad (S(1))^{\perp} = \mathrm{add}\{S(2), S(3), P(1), P(2), G(3), G(1)\}.$$

It is clear that $P(1) \notin \mathrm{Fac}(P(2) \bigoplus S(1))$ and $P(2) \notin \mathrm{Fac}(S(1))$.

To obtain the desired stratifying system, we start by finding the torsion-free part of $P(1)$ with respect to the torsion pair $(\mathrm{Fac}(P(2) \bigoplus S(1)), (P(2) \bigoplus S(1))^{\perp})$. We have the following exact sequence:

$$0 \longrightarrow S(2) \longrightarrow P(1) \longrightarrow G(1) \longrightarrow 0.$$

Since $S(2) \in \mathrm{Fac}(P(2) \bigoplus S(1))$ and $G(1) \in (P(2) \bigoplus S(1))^{\perp}$, we conclude that the torsion-free part of $P(1)$ with respect to this pair is $G(1)$. Therefore, $\Theta(1) = G(1)$.

Similarly, we determine the torsion-free part of $P(2)$ with respect to the torsion pair $(\mathrm{Fac}(S(1)), (S(1))^{\perp})$. Since $P(2) \in (S(1))^{\perp}$, we have $\Theta(2) = P(2)$. Finally, we set $\Theta(3) = S(1)$ (we can think of $\Theta(3)$ as the torsion-free part of $S(1)$ with respect to the torsion pair $(\{0\}, \mathrm{mod}\, A)$).

Theorem 2.3.4 ensures that $\Theta_M = \{G(1), P(2), S(1)\}$ is a stratifying system.

Stratifying systems induced by τ-rigid modules via the method described above generalize standard modules. In fact, since $\tau(A_A) = 0$, we have that A_A is τ-rigid.

Consider $A_A = P(1) \bigoplus \cdots \bigoplus P(n)$ an arbitrary ordered decomposition into indecomposable summands of A_A. Such a decomposition will necessarily be TF-admissible. Indeed, if $P(k)$ belonged to $\mathrm{Fac}(\bigoplus_{j>k} P(j))$, we would have an epimorphism $f : (\bigoplus_{j>k} P(j))^l \longrightarrow P(k)$ for some $l \in \mathbb{N}$. Since $P(k)$ is projective, this epimorphism would split, and $P(k)$ would be a summand of $\bigoplus_{j>k} P(j)$, which is a contradiction.

Consider the torsion pair $(\mathrm{Fac}(\bigoplus_{j>k} P(j)), (\bigoplus_{j>k} P(j))^{\perp})$. The canonical exact sequence of $P(k)$ with respect to this torsion pair is given by:

$$0 \longrightarrow t_{k+1}(P(k)) \longrightarrow P(k) \longrightarrow f_{k+1}(P(k)) \longrightarrow 0.$$

Since every torsion class is contravariantly finite, we have that $t(P(k))$ coincides with the sum of the images of the morphisms $f : \bigoplus_{j>k} P(j) \longrightarrow P(k)$. Therefore, $f_{k+1}(P(k)) = \Delta(k)$, and the induced stratifying system is $\Theta_A = \{\Delta(1), \cdots, \Delta(n)\}$.

Although this new method constructs stratifying systems beyond those given by standard modules, it has the limitation that the size of the stratifying system induced by a τ-rigid module M satisfies $|M| \leq |A_A|$. It is known that there exist stratifying systems of size greater than $|A_A|$, for example:

Let $A = kQ/I$, where Q is the following quiver:

$$3 \xrightarrow{\alpha} 1 \xleftarrow{\beta} 2 \xleftarrow{\gamma} 4$$

and I is the ideal generated by $\gamma\beta$. We have that $\Theta = \{P(1), P(2), P(3), P(4), S(4)\}$ is a stratifying system (we will prove this in Example 4.2.8).

Chapter 3

Nested Families of torsion pairs

In the previous chapter, we presented a recent result on stratifying systems showing that a τ-rigid module M induces a stratifying system Θ_M of size $t = |M|$. We know that $|M| \leq |A_A|$ and that, in general, there exist stratifying systems of size $s > |A_A|$, so this method cannot generate all stratifying systems. With this in mind, we aim to present a generalization that makes it possible to generate all stratifying systems through a certain module.

In this chapter, we introduce the concept of nested families of torsion pairs, which will play a central role in our approach. This construction not only provides a clear organizational structure for the study of stratifying systems but also enables the formulation of new results with greater generality.

We begin with the formal definition of nested families of torsion pairs, followed by illustrative examples and some fundamental properties that will be essential for the subsequent development. The present and the next chapter follow the development presented in (5).

3.1 Nested families of torsion pairs

Definition 3.1.1 (Nested family of torsion pairs)**.** *A **nested family of torsion pairs** (nested family, for short) is a set* $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k \in G}$ *of torsion pairs in* $\operatorname{mod} A$ *indexed by* $(G, \leq)$ *satisfying* $\mathcal{T}_k \supsetneq \mathcal{T}_l$ *if* $k < l$*, or equivalently,* $\mathcal{F}_k \subsetneq \mathcal{F}_l$ *if* $k < l$*.*

We will denote by t_k *and* f_k *the torsion and torsion-free functors with respect to the torsion pair* $(\mathcal{T}_k, \mathcal{F}_k)$*, respectively.*

Example 3.1.2. *We will show that there exist nested families indexed by any totally ordered set* G *in a suitable algebra.*

We initially consider G *as an infinite totally ordered set. If card* $G =$ *card* $\mathbb{N}$*, take* $K = \overline{\mathbb{Q}}$*, where* $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ *denotes the field of algebraic numbers, that is, the set of roots of* *nonzero polynomials with integer coefficients; if card $G >$ card $\mathbb{N}$, consider the integral domain $\overline{\mathbb{Q}}[G]$. We have card $\overline{\mathbb{Q}}[G] =$ card G. We construct the field of fractions frac $\overline{\mathbb{Q}}[G]$ and take its algebraic closure K. Both constructions preserve the cardinality of G, so that card $K =$ card G.*

Since G and K have the same cardinality, there exists a bijection $\varphi : G \longrightarrow K$. Through this bijection, we can transfer the total order of G to K, defining $\lambda_1 < \lambda_2$ in K if and only if $\varphi^{-1}(\lambda_1) < \varphi^{-1}(\lambda_2)$ in G.

Let $A = KQ$ be the Kronecker algebra. For every $\lambda \in K$, consider the module M_λ whose representation is given by

$$K \underset{1}{\overset{\lambda}{\leftleftarrows}} K$$

and consider for every $i \in G$ the torsion pairs $(\mathcal{T}_i, \mathcal{F}_i)$

$$\mathcal{T}_i = T(\{M_\lambda \mid \lambda \geq \varphi(i)\})$$
$$\mathcal{F}_i = (\{M_\lambda \mid \lambda \geq \varphi(i)\})^\perp,$$

We have that $\Gamma = \{(\mathcal{T}_i, \mathcal{F}_i)\}_{i\in G}$ is a nested family. Indeed, it is immediate from the definition that if $j < k$ then $\mathcal{T}_j \supseteq \mathcal{T}_k$. To show that the inclusion is proper, we just notice that $M_\lambda \in \mathcal{T}_{\varphi^{-1}(\lambda)}$. However, since $\mathrm{Hom}_A(M_\rho, M_\lambda) = 0$ if $\lambda \neq \rho$, we have that $M_\lambda \in \mathcal{F}_{\varphi^{-1}(\rho)} \Longrightarrow M_\lambda \notin \mathcal{T}_{\varphi^{-1}(\rho)}$ if $\rho > \lambda$. Therefore, $\mathcal{T}_j \supsetneq \mathcal{T}_k$.

If card $G = n$, we can follow the same construction over a finite subset of $\overline{\mathbb{Q}}$.

Nested families of torsion pairs in an abelian category $\mathcal{A}$ are defined analogously. Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ be a nested family in the category $\mathcal{A}$. An easy calculation shows that if $(\mathcal{T}_k, \mathcal{F}_k)$ is a torsion pair in $\mathcal{A}$, so $(\mathcal{F}_k^{op}, \mathcal{T}_k^{op})$ is a torsion pair in $\mathcal{A}^{op}$, and consequently $\Gamma^{op} := \{(\mathcal{F}_k^{op}, \mathcal{T}_k^{op})\}_{k\in G^{op}}$ is a nested family in $\mathcal{A}^{op}$. In particular, if A is a finite dimensional k-algebra, then $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ is a nested family in mod A if and only if $\Gamma^{op} := \{(\mathcal{F}_k^{op}, \mathcal{T}_k^{op})\}_{k\in G^{op}}$ is a nested family in $(\mathrm{mod}\, A)^{op}$.

The definition of a nested family of torsion pairs is similar to, but distinct from, the definition of a maximal green sequence (14, Definition 4.8), a chain of torsion classes (40, Definition 1.1), and twin torsion pairs (39, Section 3).

Given a nested family of torsion pairs Γ, the modules M in Mod A that admit a certain ordered decomposition relative to the torsion (or torsion-free) classes of Γ will play an important role.

Definition 3.1.3 (Γ-compatibility). *Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ be a nested family of torsion pairs. We define $\mathcal{M}(\Gamma)$ as the full subcategory of* Mod A *that contains 0 and the modules M that admit a direct sum decomposition $M = \bigoplus_{k\in G} M_k$ satisfying the following conditions for all $k \in G$:*

t1. $M_k \in \operatorname{mod} A$ *is a nonzero module;*

t2. $M_k \in \mathcal{T}_k$*;*

t3. If $X | M_k$ *with* $X \neq 0$*, then* $X \notin \mathcal{T}_j$ *if* $j > k$*.*

We will say that such a decomposition is ***compatible in*** $\boldsymbol{\mathcal{M}(\Gamma)}$*. Furthermore, we define* $\mathcal{M}^*(\Gamma)$ *as the full subcategory of* $\mathcal{M}(\Gamma)$ *that contains 0 and the modules* M *whose compatible decomposition in* $\mathcal{M}(\Gamma)$ *satisfies* $M_k \in \mathcal{F}_j$ *if* $j > k$ *for all* $k \in G$*. We will say that such a decomposition is* ***compatible in*** $\boldsymbol{\mathcal{M}^*(\Gamma)}$*.*

Analogously, we define $\mathcal{N}(\Gamma)$ *as the full subcategory of* $\operatorname{Mod} A$ *that contains 0 and the modules* N *that admit a direct sum decomposition* $N = \bigoplus_{k \in G} N_k$ *satisfying the following conditions for all* $k \in G$*:*

f1. $N_k \in \operatorname{mod} A$ *is a nonzero module;*

f2. $N_k \in \mathcal{F}_k$*;*

f3. If $X | N_k$ *with* $X \neq 0$*, then* $X \notin \mathcal{F}_j$ *if* $j < k$*.*

We will say that such a decomposition is ***compatible in*** $\boldsymbol{\mathcal{N}(\Gamma)}$*. Additionally, we define* $\mathcal{N}^*(\Gamma)$ *as the full subcategory of* $\mathcal{N}(\Gamma)$ *that contains 0 and the modules* N *whose compatible decomposition in* $\mathcal{N}(\Gamma)$ *satisfies* $N_k \in \mathcal{T}_j$ *if* $j < k$ *for all* $k \in G$*. We will say that such a decomposition is* ***compatible in*** $\boldsymbol{\mathcal{N}^*(\Gamma)}$*.*

Note that M_k is not necessarily indecomposable. The next proposition shows that if $M \in \mathcal{M}(\Gamma)$ (or $M \in \mathcal{N}(\Gamma)$), then M admits a unique compatible decomposition in $\mathcal{M}(\Gamma)$ (or in $\mathcal{N}(\Gamma)$, respectively) up to isomorphisms.

Proposition 3.1.4. *Let* $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k \in G}$ *be a nested family. Then the following hold:*

1. *If* $M = \bigoplus_{k \in G} M_k = \bigoplus_{k \in G} M'_k$ *are two compatible decompositions in* $\mathcal{M}(\Gamma)$*, then* $M_k \cong M'_k$ *for all* $k \in G$*.*

2. *If* $N = \bigoplus_{k \in G} N_k = \bigoplus_{k \in G} N'_k$ *are two compatible decompositions in* $\mathcal{N}(\Gamma)$*, then* $N_k \cong N'_k$ *for all* $k \in G$*.*

Proof. We will prove the first item, with the proof of the second being analogous. Consider the summand M_k. By definition of $\mathcal{M}(\Gamma)$, M_k has no direct summands in common with M_j for any $j \neq k$. We will show that M_k is a direct summand of M'_k. Since M_k is a direct summand of M, there exist $f : M_k \longrightarrow M$ a split monomorphism and $g : M \longrightarrow M_k$ a split epimorphism such that $gf = Id_{M_k}$. We have that

$$f \in \operatorname{Hom}_A(M_k, M) = \operatorname{Hom}_A\left(M_k, \bigoplus_{j\in G} M'_j\right) = \bigoplus_{j\in G} \operatorname{Hom}_A(M_k, M'_j)$$

since M_k is a module of finite presentation. Thus, there exist $\phi(1), \cdots, \phi(t) \in G$ such that $f' : M_k \longrightarrow M'_{\phi(1)} \bigoplus \cdots \bigoplus M'_{\phi(t)}$ is a split monomorphism (f' is the co-restriction of f), which implies that M_k is a direct summand of $M'_{\phi(1)} \bigoplus \cdots \bigoplus M'_{\phi(t)}$. Therefore, M_k is a direct summand of M'_k since Γ is a nested family. Similarly, M'_k is a direct summand of M_k, and $M_k \cong M'_k$ for all $k \in G$. ■

Furthermore, $M \ncong 0$ in $\mathcal{M}(\Gamma)$ is finitely generated if and only if G is a finite set. Moreover, if G has a maximal element m (respectively, a minimal element n) and $\mathcal{T}_m = 0$ (resp. $\mathcal{F}_n = 0$), then

$$\mathcal{M}(\Gamma) = \{0\} \quad (\text{resp. } \mathcal{N}(\Gamma) = \{0\}).$$

We have that the category $\mathcal{M}(\Gamma)$ is an additive subcategory of $\operatorname{Mod} A$ (or $\operatorname{mod} A$ if G is finite). In fact, if $M, N \in \mathcal{M}(\Gamma)$, then there exist decompositions $M = \bigoplus_{k\in G} M_k$ and $N = \bigoplus_{k\in G} N_k$ compatible in $\mathcal{M}(\Gamma)$. It is easy to see that the decomposition $M \bigoplus N = \bigoplus_{k\in G} (M_k \bigoplus N_k)$ is compatible in $\mathcal{M}(\Gamma)$, hence $M \bigoplus N \in \mathcal{M}(\Gamma)$. The above comments also apply to the categories $\mathcal{M}^*(\Gamma)$, $\mathcal{N}(\Gamma)$, and $\mathcal{N}^*(\Gamma)$.

Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ be a nested family in the abelian category $\mathcal{A}$. The following pairs of categories form contravariant equivalences (or dualities):

$$(\mathcal{M}(\Gamma), \mathcal{N}(\Gamma^{op})), (\mathcal{N}(\Gamma), \mathcal{M}(\Gamma^{op})), (\mathcal{M}^*(\Gamma), \mathcal{N}^*(\Gamma^{op})), (\mathcal{N}^*(\Gamma), \mathcal{M}^*(\Gamma^{op})).$$

Indeed, let $M = \bigoplus_{k\in G} M_k$ be the compatible decomposition in $\mathcal{M}(\Gamma)$. We have that $M_k \notin \mathcal{T}_j$ if $j > k$, which is equivalent to $M_k \notin \mathcal{T}_j^{op}$ if $j <^{op} k$. This shows that such a decomposition of M is compatible in $\mathcal{N}(\Gamma^{op})$. (Note that $\mathcal{T}_j^{op}$ is the torsion-free part of the pair $(\mathcal{F}_j^{op}, \mathcal{T}_j^{op})$ in $\mathcal{A}^{op}$). Moreover, we have that $\mathcal{M}(\Gamma) \subseteq \mathcal{N}(\Gamma^{op})$; similarly, $\mathcal{N}(\Gamma^{op}) \subseteq \mathcal{M}(\Gamma)$ can be shown. The other cases follow analogously.

The categories $\mathcal{M}(\Gamma)$ and $\mathcal{N}(\Gamma)$ will play a dual role in the development of this master's thesis. For the sake of completeness, we will state both versions of each result; however, we will only prove the ones related to $\mathcal{M}(\Gamma)$, as the other case is analogous.

3.2 Stratum and Substratum

For the next definition, let $C = \{M_k\}_{k\in G}$ be a collection of submodules of M in $\operatorname{mod} A$ such that $M_j \subseteq M_k$ if $j \geq k$. We define the smallest and largest elements of C with respect to the order given by inclusion. Let $\mathcal{D} = \{M/M_k\}_{k\in G}$ be a collection of quotients of M. We define the smallest quotient in $\mathcal{D}$ as the element M/M_{max}, where M_{max} is the largest submodule in C with respect to the order given by inclusion.

Definition 3.2.1 (Stratum and Substratum)**.** *Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ be a nested family of torsion pairs and let M be a module.*

*If $M = \bigoplus_{k\in G} M_k$ is a compatible decomposition in $\mathcal{M}(\Gamma)$, for every $k \in G$ we define $\mathfrak{F}_k(M)$ as the smallest quotient of the family $\{f_j(M_k) \mid j > k\} \cup \{M_k\}$. We define the **stratum of M** as the module $\mathfrak{F}(M) = \bigoplus_{k\in G} \mathfrak{F}_k(M)$.*

*If $N = \bigoplus_{k\in G} N_k$ is a compatible decomposition in $\mathcal{N}(\Gamma)$, for every $k \in G$ we define $\mathfrak{T}_k(N)$ as the smallest submodule of the family $\{t_j(N_k) \mid j < k\} \cup \{N_k\}$. We define the **substratum of N** as the module $\mathfrak{T}(N) = \bigoplus_{k\in G} \mathfrak{T}_k(N)$.*

Remark 3.2.2. *(a) $\mathfrak{F}_k(M)$ and $\mathfrak{T}_k(M)$ are well-defined since M_k is both artinian and noetherian. Moreover, if $k < j$ then $\mathcal{T}_k \supsetneq \mathcal{T}_j$ implies that $t_k(M) \supseteq t_j(M)$.*

(b) If $G = \mathbb{I}_n$, then $\mathfrak{T}_k(N) = t_{k-1}(N_k)$ and $\mathfrak{F}_k(M) = f_{k+1}(M_k)$, where we set $t_0(N_1) = N_1$ and $f_{n+1}(M_n) = M_n$.

It is easy to see that if $M = \bigoplus_{k\in G} M_k$ is a compatible decomposition in $\mathcal{M}(\Gamma)$, then if k is not a maximal element of G, we can take $\mathfrak{F}_k(M) = f_m(M)$, where m is such that, if $k < l < m$, then $f_l(M) = f_m(M)$. If k is the greatest element of G (if it exists), then $\mathfrak{F}_k(M) = M_k$. Similarly, if $N = \bigoplus_{k\in G} N_k$ is a compatible decomposition in $\mathcal{N}(\Gamma)$, then if k is not a minimal element of G, we can take $\mathfrak{T}_k(N) = t_m(N)$, where m is such that, if $m < l < k$, then $t_l(N) = t_m(N)$. If k is the minimal element of G (if it exists), then $\mathfrak{T}_k(N) = N_k$.

Lemma 3.2.3. *Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ be a nested family of torsion pairs and let $M = \bigoplus_{k\in G} M_k$ be a non-zero module.*

1. *If such a decomposition of M is compatible in $\mathcal{M}(\Gamma)$, then $\mathfrak{F}(M) \in \mathcal{M}^*(\Gamma)$ and $\mathfrak{F}(M) = M$ if and only if $M \in \mathcal{M}^*(\Gamma)$.*

2. *If such a decomposition of M is compatible in $\mathcal{N}(\Gamma)$, then $\mathfrak{T}(M) \in \mathcal{N}^*(\Gamma)$ and $\mathfrak{T}(M) = M$ if and only if $M \in \mathcal{N}^*(\Gamma)$.*

Proof. We will prove the first item, with the proof of the second being analogous.

We have that $\mathfrak{F}_k(M) \neq 0$. Indeed, if k is a maximal element of G, then $\mathfrak{F}_k(M) = M_k \neq 0$. Suppose otherwise, then by definition there exists $m > k$ such that $\mathfrak{F}_k(M) = f_m(M_k)$. Since $M_k \notin \mathcal{T}_m$, then $\mathfrak{F}_k(M) = f_m(M_k) \neq 0$.

We will show that $\mathfrak{F}_k(M) \in \mathcal{T}_j$ if $j \le k$. If k is a maximal element of G then $\mathfrak{F}_k(M) = M_k \in \mathcal{T}_k$ and then $\mathfrak{F}_k(M) \in \mathcal{T}_j$ for all $j \in G$ since the family of torsion pairs is nested. Suppose otherwise, by definition there exists $m > k$ such that $\mathfrak{F}_k(M) = f_m(M_k)$. Consider the short exact sequence

$$0 \longrightarrow t_m(M_k) \longrightarrow M_k \longrightarrow \mathfrak{F}_k(M) \longrightarrow 0.$$

Since $\mathfrak{F}_k(M)$ is a quotient of M_k and $M_k \in \mathcal{T}_k$, then $\mathfrak{F}_k(M) \in \mathcal{T}_j$ for $j \le k \in G$ since the family of torsion pairs is nested.

We have that $\mathfrak{F}_k(M) = f_m(M_k) \in \mathcal{F}_j$ for $j \ge m$. Consider $k < l < m$, we have that $f_l(M_k) = f_m(M_k)$ since $t_m(M) \subseteq t_l(M)$ and $f_m(M_k)$ is the smallest quotient of

$$\{f_j(M_k) \mid j > k\} \cup \{M_k\}.$$

This implies that $\mathfrak{F}_k(M) \cong f_l(M_k) \in \mathcal{F}_l$. This proves that the decomposition $\mathfrak{F}(M) = \bigoplus_{k \in G} \mathfrak{F}_k(M)$ is compatible in $\mathcal{M}^*(\Gamma)$.

Finally, since $\mathfrak{F}(M) \in \mathcal{M}^*(\Gamma)$, we have that $\mathfrak{F}(M) = M$ implies $M \in \mathcal{M}^*(\Gamma)$. Suppose $M \in \mathcal{M}^*(\Gamma)$, then $\{f_j(M_k) \mid j > k\} \cup \{M_k\} = \{M_k\}$ and $\mathfrak{F}(M) = M$. ■

Corollary 3.2.4. *Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k \in G}$ be a nested family of torsion pairs.*

1. *If $M \neq 0$ is in $\mathcal{M}(\Gamma)$, then*

   a. $\mathfrak{F}_k(M) \neq 0$.

   b. $\mathfrak{F}_k(M) \in \mathcal{T}_j$ *if* $j \le k$.

   c. $\mathfrak{F}_k(M) \in \mathcal{F}_j$ *if* $j > k$.

2. *If $N \neq 0$ is in $\mathcal{N}(\Gamma)$, then*

   a'. $\mathfrak{T}_k(N) \neq 0$.

   b'. $\mathfrak{T}_k(N) \in \mathcal{F}_j$ *if* $j \ge k$.

   c'. $\mathfrak{T}_k(N) \in \mathcal{T}_j$ *if* $j < k$.

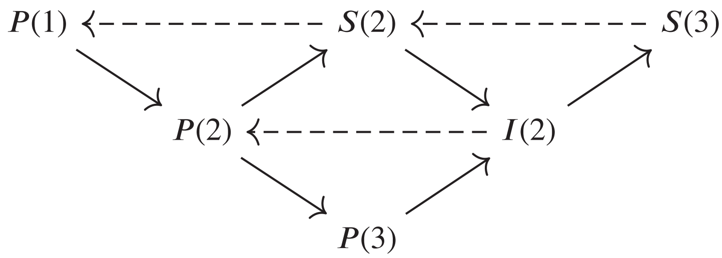
■

We conclude this section with an example of how to construct modules in the category $\mathcal{M}(\Gamma)$ and their stratums.

Example 3.2.5. *Let $A = kQ$, where Q is the following quiver:*

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$$

The Auslander-Reiten quiver is given by:



Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k \in \mathbb{I}_2}$ *be a nested family of torsion pairs, where:*

$$\mathcal{T}_1 = \operatorname{add}\{P(1), P(2), S(2)\} \qquad \mathcal{F}_1 = \operatorname{add}\{S(3)\}$$
$$\mathcal{T}_2 = \operatorname{add}\{P(1)\} \qquad \mathcal{F}_2 = \operatorname{add}\{S(2), S(3), I(2)\}.$$

We can easily find the nonzero modules in $\mathcal{M}(\Gamma)$*. Let* $M \in \mathcal{M}(\Gamma)$ *be a nonzero module, then* M *admits a decomposition* $M_1 \bigoplus M_2$*, where* $M_k \in \mathcal{T}_k$ *and no nonzero direct summand of* M_k *belongs to* $\mathcal{T}_j$ *if* $j > k$*. It is easy to see that* $M_1 = P(2)^{\oplus n_1} \bigoplus S(2)^{\oplus n_2}$ *and* $M_2 = P(1)^{\oplus n_3}$*, with* $n_1 + n_2 \neq 0$ *and* $n_3 \neq 0$*, since* M_1 *and* M_2 *are nonzero modules. Similarly, it can be shown that the nonzero modules in* $\mathcal{M}^*(\Gamma)$ *are of the form* $M = S(2)^{\oplus n_1} \bigoplus P(1)^{\oplus n_2}$*, with* $n_1, n_2 \neq 0$*.*

We will find the stratum $\mathfrak{F}(M)$ *of* M*. We have that*

$$\mathfrak{F}_1(M) = f_2(M_1) = f_2(P(2)^{\oplus n_1} \bigoplus S(2)^{\oplus n_2}) = S(2)^{\oplus (n_1+n_2)}$$

and

$$\mathfrak{F}_2(M) = f_3(M_2) = f_3(P(1)^{\oplus n_3}) = P(1)^{\oplus n_3},$$

so

$$\mathfrak{F}(P(2)^{\oplus n_1} \bigoplus S(2)^{\oplus n_2} \bigoplus P(1)^{\oplus n_3}) = S(2)^{\oplus (n_1+n_2)} \bigoplus P(1)^{\oplus n_3}.$$

Chapter 4

Conditions for the existence of a stratifying system

In (41), H. Treffinger demonstrated the existence of stratifying systems of infinite size indexed by $\mathbb{N}$ in n-representation infinite algebras. Since there does not always exist an order-preserving bijection between totally ordered sets $(G, \leq)$ and $(H, \leq)$, even if G and H have the same cardinality, we will consider a more general definition of stratifying systems, maintaining the same philosophy that a stratifying system is a set of modules that satisfies certain orthogonality conditions.

In this chapter, we will show that a stratifying system $\Theta = \{\Theta_k\}_{k\in G}$ induces two distinct nested families of torsion pairs Γ and Γ', both indexed by G, such that the characteristic module M_Θ of Θ is in $\mathcal{M}^*(\Gamma)$ and in $\mathcal{N}^*(\Gamma')$ (see Definition 4.2.1). Conversely, let Γ be a nested family of torsion pairs indexed by G. If M is a module in $\mathcal{M}^\dagger(\Gamma)$ (or $\mathcal{N}^\dagger(\Gamma)$, see Definition 4.2.5), then M induces at least one stratifying system Θ such that M_Θ is a quotient of M (or a submodule of M, respectively). We will also show that every stratifying system can be obtained in this way.

Definition 4.0.1 (Stratifying System)**.** *Let G be a set with total order $\leq$. A* ***stratifying system indexed by G*** *is a set $\Theta = \{\Theta_k\}_{k\in G}$ of indecomposable modules in* mod A *satisfying the following conditions:*

1. $\operatorname{Hom}_A(\Theta_k, \Theta_j) = 0$ *if* $k > j$.

2. $\operatorname{Ext}^1_A(\Theta_k, \Theta_j) = 0$ *if* $k \geq j$.

4.1 Filtered modules

Given a stratifying system Θ, we denote by $\mathcal{F}(\Theta)$ the full subcategory of mod A containing the zero module and all modules which are filtered by modules in Θ. That is, a nonzero module M belongs to $\mathcal{F}(\Theta)$ if there is a finite chain

$$\eta : 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

of submodules of M such that M_i/M_{i-1} is isomorphic to a module in Θ for all $i = 1, 2, \cdots, n$. Such a decomposition is called a **Θ-filtration**.

K. Erdmann and C. Sáenz proved that $\mathcal{F}(\Theta)$ is a Jordan-Hölder category if Θ is a finite stratifying system (25, Lemma 1.4) (see Lemma 2.1.6). Using this result, we can show that $\mathcal{F}(\Theta)$ is a Jordan-Hölder category even if Θ is a stratifying system of infinite size.

Theorem 4.1.1. *Let $\Theta = \{\Theta_k\}_{k \in G}$ be a stratifying system, then $\mathcal{F}(\Theta)$ is a Jordan-Hölder category.*

Proof. Let $M \in \mathcal{F}(\Theta)$ and consider two Θ-filtrations of M:

$$\eta : 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

and

$$\eta' : 0 = M'_0 \subseteq M'_1 \subseteq \cdots \subseteq M'_m = M.$$

Consider the set $\mathcal{S} = \{M_i/M_{i-1} \mid i = 1, 2, \cdots, n\} \cup \{M'_i/M'_{i-1} \mid i = 1, 2, \cdots, m\}$. We have that $\mathcal{S}$ is a finite subset of Θ, thus $\mathcal{S}$ is a finite stratifying system with the order inherited from Θ. Furthermore, η and η' are $\mathcal{S}$-filtrations. Therefore, $n = m$ and $[M : \Theta_k]_\eta = [M : \Theta_k]_{\eta'}$ by Lemma 2.1.6. ■

The modules in Θ are called **relative simple modules of $\mathcal{F}(\Theta)$** because they admit a unique Θ-filtration $\eta : 0 \subseteq \Theta_k$.

4.2 Nested families and stratifying systems

Given a stratifying system Θ indexed by G, we can define a module M_Θ that will be used to connect the stratifying system to a nested family; this is the characteristic module of Θ.

Definition 4.2.1 (Characteristic module of Θ)**.** *Let $\Theta = \{\Theta_k\}_{k \in G}$ be a stratifying system. We will call the module $M_\Theta := \bigoplus_{k \in G} \Theta_k$ the **characteristic module of the stratifying system Θ**.*

Let $M = \bigoplus_{k \in G} M_k$ be a module in $\operatorname{Mod} A$ such that M_k is a nonzero finitely generated module for every $k \in G$. We will denote by $\mathsf{T}(\bigoplus_{k \in G} M_k)$ the smallest torsion class in $\operatorname{mod} A$ containing the summands of M, that is, $\mathsf{T}(\bigoplus_{k \in G} M_k) := \mathsf{T}(\mathcal{C})$, where $\mathcal{C} = \{M_k \mid k \in G\}$. We define $\mathsf{F}(\bigoplus_{k \in G} M_k)$ analogously.

Let $M \in \operatorname{Mod} A$ and Γ a nested family of torsion pair. We will say that **M induces the nested family Γ** if there exists a decomposition $M = \bigoplus_{k \in G} M_k$ with M_k nonzero and finitely generated such that $\Gamma = \{(\mathsf{T}(\bigoplus_{j \geq k} M_j), (\bigoplus_{j \geq k} M_j)^{\perp})\}_{k \in G}$ or $\Gamma = \{({}^{\perp}(\bigoplus_{j \geq k} M_j), \mathsf{F}(\bigoplus_{j \geq k} M_j))\}_{k \in G}$. We will also say that a stratifying system Θ induces the nested family of torsion pairs Γ if M_Θ induces Γ.

The following theorem shows how to construct two distinct nested families from a module $M = \bigoplus_{k \in G} M_k$ such that $M_k \neq 0$ and $\operatorname{Hom}_A(M_j, M_i) = 0$ if $j > i$. In particular, every stratifying system induces two distinct nested families of torsion pairs through its characteristic module M_Θ.

Theorem 4.2.2. *Let $M = \bigoplus_{k \in G} M_k$ be a decomposition of M with $M_k \in \operatorname{mod} A$ being nonzero modules such that $\operatorname{Hom}_A(M_j, M_i) = 0$ if $j > i$. Then the following statements hold true.*

1. $\Gamma = \{(\mathsf{T}(\bigoplus_{j \geq k} M_j), (\bigoplus_{j \geq k} M_j)^{\perp})\}_{k \in G}$ *and* $\Gamma' = \{({}^{\perp}(\bigoplus_{j \geq k} M_j), \mathsf{F}(\bigoplus_{j \geq k} M_j))\}_{k \in G}$ *are distinct nested families of torsion pairs.*

2. *The decomposition of $M = \bigoplus_{k \in G} M_k$ is compatible in $\mathcal{M}^*(\Gamma)$ and in $\mathcal{N}^*(\Gamma')$.*

Proof. We will prove only one side of each item, the other side is analogous.

1) and 2) Since $\bigoplus_{j \geq k} M_j \subsetneq \bigoplus_{j \geq i} M_j$ if $i < k$, then $\mathsf{T}(\bigoplus_{j \geq k} M_j) \subseteq \mathsf{T}(\bigoplus_{j \geq i} M_j)$. $\mathsf{T}(\bigoplus_{j \geq k} M_j)$ is defined as the smallest torsion class that contains $\{M_j \mid j \geq k\}$, so we have that $M_k \in \mathsf{T}(\bigoplus_{j \geq k} M_j)$. On the other hand, we have that $M_k \in (\bigoplus_{j \geq i} M_j)^{\perp}$ if $i > k$. Since $M_k \in (\bigoplus_{j \geq i} M_j)^{\perp}$ for all $i > k$, we have that $M_k \notin \mathsf{T}(\bigoplus_{j \geq i} M_j)$ if $i > k$, then Γ is a nested family of torsion pairs and the decomposition $M = \bigoplus_{k \in G} M_k$ is compatible in $\mathcal{M}(\Gamma)$. We have that Γ and Γ' are distinct because $M_k \in \mathsf{T}(\bigoplus_{j \geq k} M_j)$ and $M_k \notin {}^{\perp}(\bigoplus_{j \geq k} M_j)$.

Finally, we have that $M_k \in \mathcal{F}_j$ if $j > k$, then $\{f_j(M_k) \mid j > k\} \cup \{M_k\} = \{M_k\}$. Therefore, $\mathfrak{F}(M) = M$ and Lemma 3.2.3 ensures that $M \in \mathcal{M}^*(\Gamma)$. ■

Corollary 4.2.3. *Let $\Theta = \{\Theta_k\}_{k \in G}$ be a stratifying system. Then Θ induces two distinct nested families of torsion pairs Γ and Γ', both indexed by G, such that M_Θ is in $\mathcal{M}^*(\Gamma)$ and in $\mathcal{N}^*(\Gamma')$.*

■

The following example illustrates how to obtain a non-enumerable nested family that cannot be induced from a stratifying system.

Example 4.2.4. *Let $A = \mathbb{C}Q$ be the Kronecker algebra, and let $(\mathbb{C}, \leq)$ be the field of complex numbers with the lexicographic order, that is, $x + yi > x' + y'i$ if $x > x'$ or $x = x'$ and $y > y'$. For every $\lambda \in \mathbb{C}$, consider the module M_λ whose representation is given by*

$$\mathbb{C} \underset{1}{\overset{\lambda}{\Longleftarrow}} \mathbb{C}$$

We have that the module $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ satisfies the conditions of the Theorem 4.2.2 (since $\operatorname{Hom}_A(M_\lambda, M_\rho) = 0$ if $\lambda \neq \rho$), thus inducing an uncountable nested family of torsion pairs. However, there do not exist stratifying systems of size greater than 2 in the Kronecker algebra (17, Theorem 2.3.18).

Let Γ be a nested family and let $M = \bigoplus_{k \in G} M_k$ be a compatible decomposition in $\mathcal{M}(\Gamma)$. Corollary 3.2.4 ensures that $\mathfrak{F}(M)$ satisfies $\operatorname{Hom}_A(\mathfrak{F}_k(M), \mathfrak{F}_j(M)) = 0$ whenever $k > j$, with $\mathfrak{F}_k(M) \neq 0$ for all $k \in G$. If it is possible to ensure that $\operatorname{Ext}^1_A(\mathfrak{F}_k(M), \mathfrak{F}_j(M)) = 0$ if $k \geq j$, then, by choosing an indecomposable summand Θ_k of $\mathfrak{F}_k(M)$ for each k, we obtain a stratifying system $\Theta = \{\Theta_k\}_{k \in G}$. This observation motivates the following definition.

Definition 4.2.5. *Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k \in G}$ be a nested family of torsion pairs. We define $\mathcal{M}^\dagger(\Gamma)$ as the subcategory of $\mathcal{M}(\Gamma)$ that contains the modules M that admit a compatible decomposition $M = \bigoplus_{k \in G} M_k$ in $\mathcal{M}(\Gamma)$ satisfying $\operatorname{Ext}^1_A(M_k, \mathfrak{F}_j(M)) = 0$ if $k \geq j$. Such a decomposition is called* ***compatible in $\mathcal{M}^\dagger(\Gamma)$****.*

We also define $\mathcal{N}^\dagger(\Gamma)$ as the subcategory of $\mathcal{N}(\Gamma)$ that contains the modules N that admit a compatible decomposition $N = \bigoplus_{k \in G} N_k$ in $\mathcal{N}(\Gamma)$ satisfying $\operatorname{Ext}^1_A(\mathfrak{T}_k(N), N_j) = 0$ if $k \geq j$. Such a decomposition is called ***compatible in $\mathcal{N}^\dagger(\Gamma)$****.*

Let M be in $\mathcal{M}^\dagger(\Gamma)$. We say that ***$M$* induces the stratifying system $\Theta = \{\Theta_k\}_{k \in G}$ as a quotient** if M admits a compatible decomposition $M = \bigoplus_{k \in G} M_k$ in $\mathcal{M}^\dagger(\Gamma)$ such that Θ_k is a direct summand of $\mathfrak{F}_k(M)$. Similarly, let N be in $\mathcal{N}^\dagger(\Gamma)$. We say that ***$N$* induces the stratifying system $\Theta = \{\Theta_k\}_{k \in G}$ as a submodule** if N admits a compatible decomposition $N = \bigoplus_{k \in G} N_k$ in $\mathcal{N}^\dagger(\Gamma)$ such that Θ_k is a direct summand of $\mathfrak{T}_k(N)$. The following theorem shows that modules in $\mathcal{M}^\dagger(\Gamma)$ and in $\mathcal{N}^\dagger(\Gamma)$ induce at least one stratifying system.

Theorem 4.2.6. *Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k \in G}$ be a nested family of torsion pairs and let $M = \bigoplus_{k \in G} M_k$ be a module.*

1. *If such a decomposition of M is compatible in $\mathcal{M}^\dagger(\Gamma)$, then M induces at least one stratifying system $\Theta = \{\Theta_k\}_{k \in G}$ as a quotient such that Θ_k is a summand of $\mathfrak{F}_k(M)$.*

2. *If such a decomposition of M is compatible in $\mathcal{N}^{\dagger}(\Gamma)$, then M induces at least one stratifying system $\Theta = \{\Theta_k\}_{k\in G}$ as a submodule such that Θ_k is a summand of $\mathfrak{T}_k(M)$.*

Proof. We will prove the first item, with the proof of the second being analogous.

1) Suppose first that $i > j$, we have that $\mathfrak{F}_i(M) \in \mathcal{T}_i$ and $\mathfrak{F}_j(M) \in \mathcal{F}_i$ by Corollary 3.2.4, so $\mathrm{Hom}_A(\mathfrak{F}_i(M), \mathfrak{F}_j(M)) = 0$.

Suppose now that $i \geq j$ and that i is not the maximal element of G (if it exists). In this case, there exists $m > i \geq j$ and a short exact sequence such that

$$0 \longrightarrow t_m(M_i) \longrightarrow M_i \longrightarrow \mathfrak{F}_i(M) \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_A(-, \mathfrak{F}_j(M))$ we obtain

$$\mathrm{Hom}_A(t_m(M_i), \mathfrak{F}_j(M)) \longrightarrow \mathrm{Ext}^1_A(\mathfrak{F}_i(M), \mathfrak{F}_j(M)) \longrightarrow \mathrm{Ext}^1_A(M_i, \mathfrak{F}_j(M)).$$

We have $\mathrm{Hom}_A(t_m(M_i), \mathfrak{F}_j(M)) = 0$ since $t_m(M_i) \in \mathcal{T}_m$ and $\mathfrak{F}_j(M) \in \mathcal{F}_m$, as stated in Corollary 3.2.4. We also have that $\mathrm{Ext}^1_A(M_i, \mathfrak{F}_j(M)) = 0$ since the decomposition of M is compatible in $\mathcal{M}^{\dagger}(\Gamma)$, then $\mathrm{Ext}^1_A(\mathfrak{F}_i(M), \mathfrak{F}_j(M)) = 0$. If i is the maximal element of G, the result follows using the same argument, starting from the following short exact sequence:

$$0 \longrightarrow 0 \longrightarrow M_i \longrightarrow \mathfrak{F}_i(M) \longrightarrow 0.$$

Finally, since $\mathrm{mod}\, A$ is a Krull-Schmidt category, we have that $\mathfrak{F}_k(M)$ decomposes uniquely (up to isomorphism) as a finite direct sum of indecomposable modules. By choosing an indecomposable summand Θ_k of $\mathfrak{F}_k(M)$, we obtain that $\Theta = \{\Theta_k\}_{k\in G}$ is a stratifying system. ■

In the above theorem, the module M induces $\prod_{k\in G} |\mathfrak{F}_k(M)|$ (or $\prod_{k\in G} |\mathfrak{T}_k(M)|$) stratifying systems indexed by G, where the product should be interpreted as ∞ if there are infinitely many indices k such that $|\mathfrak{F}_k(M)| > 1$ (or $|\mathfrak{T}_k(M)| > 1$, respectively).

Example 4.2.4 shows that $\mathcal{M}^{\dagger}(\Gamma)$ may be empty; otherwise, it would be possible to obtain a stratifying system of infinite size in the Kronecker algebra.

We say that a A-module M is **stratified** (or **substratified**) if there exists a nested family of torsion pairs Γ and a decomposition $M = \bigoplus_{k\in G} M_k$ (not necessarily into indecomposable summands) such that this decomposition is compatible in $\mathcal{M}^{\dagger}(\Gamma)$ (or $\mathcal{N}^{\dagger}(\Gamma)$, respectively). We will denote by $\mathcal{M}^{\dagger}(A)$ and $\mathcal{N}^{\dagger}(A)$ the class of A-modules that are stratified and substratified, respectively.

The following theorem shows that every stratifying system can be induced by a stratified module or by a substratified module.

Theorem 4.2.7. *Let $\Theta = \{\Theta_k\}_{k\in G}$ be a stratifying system.*

1. *Then there exists a module $M \in \mathcal{M}^{\dagger}(A)$ such that Θ is induced as a quotient by M.*

2. *Then there exists a module $N \in \mathcal{N}^{\dagger}(A)$ such that Θ is induced as a submodule by N.*

Proof. We will prove the first item, with the proof of the second being analogous.

Suppose that $\Theta = \{\Theta_k\}_{k \in G}$ is a stratifying system and let $M_{\Theta} = \bigoplus_{k \in G} \Theta_k$ be its characteristic module. Theorem 4.2.2 ensures that $\Gamma = \{(\mathsf{T}(\bigoplus_{j \geq k} \Theta_j), (\bigoplus_{j \geq k} \Theta_j)^{\perp})\}_{k \in G}$ is a nested family of torsion pairs and that this decomposition of M_{Θ} is compatible in $\mathcal{M}^*(\Gamma)$.

We will show that this decomposition is, in fact, compatible in $\mathcal{M}^{\dagger}(\Gamma)$. Theorem 4.2.2 ensures that $\mathfrak{F}_k(M) = \Theta_k$. Therefore, $\mathrm{Ext}^1_A(\Theta_j, \mathfrak{F}_k(M)) = \mathrm{Ext}^1_A(\Theta_j, \Theta_k) = 0$ if $j \geq k$ since Θ is a stratifying system. Applying Theorem 4.2.6, we have that the family Γ and the module M_{Θ} induce the stratifying system Θ as a quotient. ■

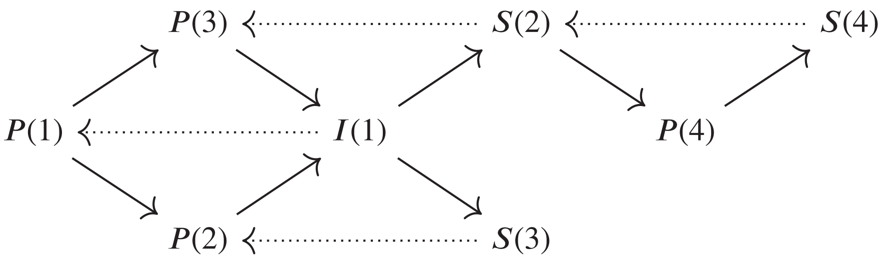
Note that we do not use the fact that the modules are indecomposable.

We conclude this section with a detailed example of applications of Theorem 4.2.6. In this example, we will show how the stratifying system of size 5 presented in (29, Remark 2.7.) can be induced from a module $N \in \mathcal{N}^{\dagger}(A)$.

Example 4.2.8. *Let $A = kQ/I$, where Q is the following quiver:*

$$3 \xrightarrow{\alpha} 1 \xleftarrow{\beta} 2 \xleftarrow{\gamma} 4$$

and I is the ideal generated by $\gamma\beta$. The Auslander-Reiten quiver of the algebra is given by:



Consider $N = \bigoplus_{k=1}^{5} N_k$, where N_k is $P(1), P(2), I(1), P(4)$, and $S(4)$ for $k = 1,2,3,4$, and 5, respectively. Note that N is not τ^--rigid, since there exists a morphism $N \to \tau^{-1}(N)$, where $\tau^{-1}(N) = I(1) \bigoplus S(3)$.

Consider the nested family of torsion pairs $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k \in \mathbb{I}_5}$, where

$\mathcal{T}_1 = \mathrm{add}\{P(2), P(3), P(4), I(1), S(2), S(3), S(4)\}$ $\qquad \mathcal{F}_1 = \mathrm{add}\{P(1)\}$

$\mathcal{T}_2 = \mathrm{add}\{P(3), P(4), S(3), S(4)\}$ $\qquad \mathcal{F}_2 = \mathrm{add}\{P(1), P(2), S(2)\}$

$\mathcal{T}_3 = \mathrm{add}\{P(4), S(3), S(4)\}$ $\qquad \mathcal{F}_3 = \mathrm{add}\{P(1), P(2), P(3), I(1), S(2)\}$

$\mathcal{T}_4 = \mathrm{add}\{S(4)\}$ $\qquad \mathcal{F}_4 = \mathrm{add}\{P(1), P(2), P(3), P(4), I(1), S(2), S(3)\}$

$\mathcal{T}_5 = \mathrm{add}\{0\}$ $\qquad \mathcal{F}_5 = \mathrm{add}\{P(1), P(3), P(2), P(4), I(1), S(2), S(3), S(4)\}.$

It is easy to see that such a decomposition of N is compatible in $\mathcal{N}(\Gamma)$.

The substratum of N is given by $\mathfrak{T}(N) = \bigoplus_{k=1}^{5} t_{k-1}(N_k)$. Since $(\mathcal{T}_k, \mathcal{F}_k)$ is splitting if $k \neq 2$ we have $t_{k-1}(N_k) = N_k$ for $k \neq 3$.

Note that there exists an exact sequence

$$0 \longrightarrow P(3) \longrightarrow I(1) \longrightarrow S(2) \longrightarrow 0,$$

where $P(3) \in \mathcal{T}_2$ and $S(2) \in \mathcal{F}_2$, hence $t_2(N_3) = P(3)$.

It is also easy to see that such a decomposition is compatible in $\mathcal{N}^{\dagger}(\Gamma)$. Indeed, we have that $\operatorname{Ext}_A^1(\mathfrak{T}_k(N), N_j) = \operatorname{Ext}_A^1(t_{k-1}(N_k), N_j) = 0$ if $j \leq k$ since $t_{k-1}(N_k)$ is projective if $k \neq 5$, and the only exact sequence ending at $S(4)$ is given by:

$$0 \longrightarrow S(2) \longrightarrow P(4) \longrightarrow S(4) \longrightarrow 0.$$

Theorem 4.2.6 ensures that $\Theta = \{P(1), P(2), P(3), P(4), S(4)\}$ is a stratifying system.

4.3 Stratifying system via τ-rigid modules

In Theorem 2.3.4, H. Treffinger and O. Mendoza showed that every nonzero basic τ-rigid module induces a stratifying system as a quotient. In this section, we will demonstrate that this result can be obtained as a corollary of Theorem 4.2.6.

We begin with a lemma that provides a sufficient condition for a compatible decomposition in $\mathcal{M}(\Gamma)$ to be compatible decomposition in $\mathcal{M}^{\dagger}(\Gamma)$.

Lemma 4.3.1. *Let $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k \in \mathbb{I}_n}$ be a nested family of torsion pairs and let $M = \bigoplus_{k=1}^{n} M_k$ be a decomposition of M.*

1. *If M is Ext-projective in $\mathcal{T}_1$ and such a decomposition is compatible in $\mathcal{M}(\Gamma)$, then such decompositon is compatible in $\mathcal{M}^{\dagger}(\Gamma)$.*

2. *If M is Ext-injective in $\mathcal{F}_n$ and such a decomposition is compatible in $\mathcal{N}(\Gamma)$, then such decompositon is compatible in $\mathcal{N}^{\dagger}(\Gamma)$.*

Proof. We will prove the first item, with the proof of the second being analogous.

Since M_k is a summand of M, we have that M_k is Ext-projective in $\mathcal{T}_1$. On the other hand, Lemma 3.2.3 ensures that $\mathfrak{F}_j(M) \in \mathcal{T}_1$. Thus, $\operatorname{Ext}_A^1(M_k, \mathfrak{F}_j(M)) = 0$ for all $j \in \mathbb{I}_n$. In particular, $\operatorname{Ext}_A^1(M_k, \mathfrak{F}_j(M)) = 0$ if $k \geq j$ and $M \in \mathcal{M}^{\dagger}(\Gamma)$. ■

Theorem 4.2.2 demonstrates how to obtain nested families of torsion pairs from a class of modules that admit a certain decomposition. The following theorem shows how to induce a nested family of torsion pairs from another class of modules.

Theorem 4.3.2. *Let $M = \bigoplus_{k \in G} M_k$ be a decomposition of M with $M_k \in \operatorname{mod} A$ a nonzero module for each $k \in G$.*

1. *If for every k, we have $M_k \notin \mathsf{T}(\bigoplus_{j>k} M_j)$, then $\Gamma = \{(\mathsf{T}(\bigoplus_{j\geq k} M_j),(\bigoplus_{j\geq k} M_j)^{\perp})\}_{k\in G}$ is a nested family of torsion pairs, and this decomposition of M is compatible in $\mathcal{M}(\Gamma)$.*
2. *If for every k, we have $M_k \notin \mathsf{F}(\bigoplus_{j<k} M_j)$, then $\Gamma = \{({}^{\perp}(\bigoplus_{j\leq k} M_j),\mathsf{F}(\bigoplus_{j\leq k} M_j))\}_{k\in G}$ is a nested family of torsion pairs, and this decomposition of M is compatible in $\mathcal{N}(\Gamma)$.*

Proof. 1) Clearly, $\mathsf{T}(\bigoplus_{j>k} M_j) \subseteq \mathsf{T}(\bigoplus_{j\geq k} M_j)$ and $M_k \in \mathsf{T}(\bigoplus_{j\geq k} M_j)$. Since $M_k \notin \mathsf{T}(\bigoplus_{j>k} M_j)$, it follows that Γ is a nested family of torsion pairs, and this decomposition of M is compatible in $\mathcal{M}(\Gamma)$. The second item is analogous. ■

Such a decomposition of M, as described in the lemma above, generalizes the concept of a **torsion-free admissible decomposition** described in Definition 2.3.1. Note that Theorem 4.2.2 can be seen as a corollary of Theorem 4.3.2. Unlike in Theorem 4.2.2, it is not necessarily the case that M is in $\mathcal{M}^*(\Gamma)$ (or in $\mathcal{N}^*(\Gamma)$) if M induces Γ as in the lemma above.

Example 4.3.3. *Let $P(1), P(2), \cdots, P(n)$ be an enumeration of the indecomposable projective modules of $\operatorname{mod} A$. Given $\sigma : \mathbb{I}_n \longrightarrow \mathbb{I}_n$ a permutation of $\mathbb{I}_n$ and given $\{k_2, k_3, \cdots, k_t\} \in 2^{\mathbb{I}_{n-1}}$ a set with $t-1$ elements, we define a decomposition of $A_A = \bigoplus_{j=1}^{t} M_j$ where $M_j = \bigoplus_{i=k_j+1}^{k_{j+1}} P(\sigma(i))$ for $1 \leq j < t$ and $M_t = \bigoplus_{i=k_t+1}^{n} P(\sigma(i))$ (here, we fix $k_1 = 0$ and $\mathbb{I}_0 = \{\}$).*

It is easy to see that $M_j \notin \operatorname{Fac}\left(\bigoplus_{k>j}^{t} M_k\right)$ since every epimorphism onto M_j splits, then such a decomposition induces a nested family indexed by $\mathbb{I}_t$. This shows that any decomposition of A_A is torsion-free admissible and that A_A induces $2^{n-1}n!$ distinct nested families. A similar reasoning shows that $D({}_AA)$ induces $2^{n-1}n!$ distinct nested families.

Let M be a τ-rigid module. Proposition 1.3.6 ensures that $\mathsf{T}(M) = \operatorname{Fac}(M)$ since it always holds that $\operatorname{Fac}(M) \subseteq \mathsf{T}(M)$. Using mutation techniques from τ-Tilting Theory, H. Treffinger and O. Mendoza proved Proposition 2.3.2, which states that every basic nonzero τ-rigid module admits at least one torsion-free admissible indecomposable decomposition. Assuming the existence of such a decomposition, we will show that Theorem 2.3.4 can be seen as a corollary of Theorem 4.2.6.

Corollary 4.3.4. *(36, Theorem 3.4) Let M be a nonzero, basic, and τ-rigid module with a torsion-free decomposition $M = \bigoplus_{j=1}^{t} M_j$. If f_k denotes the torsion-free functor* *associated to the pair* $(\mathcal{T}_k, \mathcal{F}_k)$*, where* $\mathcal{T}_k = \mathrm{Fac}(\bigoplus_{j\geq k} M_j)$ *and* $\mathcal{F}_k = (\bigoplus_{j\geq k} M_j)^{\perp}$*, then* $\Theta = \{\Theta_k := f_{k+1}(M_k)\}_{k\in\mathbb{I}_t}$ *is a stratifying system of size t.*

Proof. Since the decomposition of M is torsion-free admissible, Theorem 4.3.2 ensures that $\Gamma = \{(\mathrm{Fac}(\bigoplus_{j\geq k} M_j), (\bigoplus_{j\geq k} M_j)^{\perp})\}_{k\in\mathbb{I}_t}$ is a nested family of torsion pairs and that this decomposition of M is compatible in $\mathcal{M}(\Gamma)$. On the other hand, Proposition 1.3.5 and Lemma 4.3.1 guarantee that such a decomposition of M is compatible in $\mathcal{M}^{\dagger}(\Gamma)$.

In accordance with Theorem 4.2.6, it is possible to choose an indecomposable summand for each module of $\{\mathfrak{F}_k(M) = f_{k+1}(M_k)\}_{k\in\mathbb{I}_t}$ and construct a stratifying system.

Lemma 1.3.7 ensures that $f_{i+1}(M_i)$ is indecomposable since $M_i \bigoplus \left(\bigoplus_{k>i} M_k\right)$ is τ-rigid and $M_i \notin \mathrm{Fac}\left(\bigoplus_{k>i} M_k\right)$, which proves the corollary. ■

Corollary 4.3.4 and Theorem 4.2.7 ensure that $\mathcal{M}^{\dagger}(A)$ contains all τ-rigid modules (such as projective and tilting modules) and the characteristic modules M_{Θ} of all stratifying systems Θ. It is possible to prove the dual of Corollary 4.3.4 and show that $\mathcal{N}^{\dagger}(A)$ contains all τ^{-}-rigid modules (such as injective modules) and the characteristic modules M_{Θ} of all stratifying systems Θ.

4.4 Expansions in nested families

Let $M = \bigoplus_{j=1}^{t} M_j$ be a torsion-free admissible decomposition into indecomposable summands of the basic τ-rigid module M. Corollary 4.3.4 shows that such a decomposition of M is compatible in $\mathcal{M}^{\dagger}(\Gamma)$, where $\Gamma = \{(\mathrm{Fac}(\bigoplus_{j\geq k} M_j), (\bigoplus_{j\geq k} M_j)^{\perp})\}_{k\in\mathbb{I}_t}$. However, it is possible that the same decomposition is compatible in $\mathcal{M}^{\dagger}(\Gamma')$ for $\Gamma' \neq \Gamma$.

In this section, we will study the relationship between the stratifying systems induced by the same module when the nested family is changed. We begin with a definition.

Definition 4.4.1. *(Expansion) Let* $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ *and* $\Gamma' = \{(\mathcal{T}'_k, \mathcal{F}'_k)\}_{k\in G}$ *be nested families of torsion pairs. We say that* $\boldsymbol{\Gamma}$ ***expands*** $\boldsymbol{\Gamma'}$ *if* $\mathcal{T}'_k \subseteq \mathcal{T}_k$ *for all* $k \in G$*.*

We write $\Gamma' \leq \Gamma$ to indicate that Γ expands Γ'. We will say that Γ' is **tighter** than Γ and that Γ is **looser** than Γ'. Note that, by definition, every nested family Γ expands itself.

The following proposition shows the relationship between the nested families induced by M and a nested families Γ' such that $M \in \mathcal{M}(\Gamma')$.

Proposition 4.4.2. *Let* $M = \bigoplus_{k\in G} M_k$ *be a decomposition of* M *with* $M_k \in \mathrm{mod}\, A$ *being a nonzero module for each* $k \in G$*, and let* $\Gamma' = \{(\mathcal{T}'_k, \mathcal{F}'_k)\}_{k\in G}$ *be a nested family of torsion pairs.*

1. *If such a decomposition induces the nested family* $\Gamma = \{(\mathsf{T}(\bigoplus_{j\geq k} M_j), (\bigoplus_{j\geq k} M_j)^{\perp})\}_{k\in G}$, *then* Γ *is the* ***tightest*** *nested family such that* $M \in \mathcal{M}(\Gamma)$*, that is, if* $M \in \mathcal{M}(\Gamma')$*, then* $\Gamma \leq \Gamma'$.

2. *If such a decomposition induces the nested family* $\Gamma = \{({}^{\perp}(\bigoplus_{j\leq k} M_j), \mathsf{F}(\bigoplus_{j\leq k} M_j))\}_{k\in G}$, *then* Γ *is the* ***loosest*** *nested family such that* $M \in \mathcal{N}(\Gamma)$*, that is, if* $M \in \mathcal{N}(\Gamma')$*, then* $\Gamma' \leq \Gamma$.

Proof. 1) Suppose that $M = \bigoplus_{k\in G} M_k$ belongs to $\mathcal{M}(\Gamma')$, then $\{M_j \mid j \geq k\} \in \mathcal{T}'_k$ for all $k \in G$. Since $\mathsf{T}(\bigoplus_{j\geq k} M_j)$ is the smallest torsion class containing $\{M_j \mid j \geq k\}$, we have $\mathsf{T}(\bigoplus_{j\geq k} M_j) \subseteq \mathcal{T}'_k$, hence $\Gamma \leq \Gamma'$. The second item is analogous. ■

Let $M \neq 0$ be a basic τ-rigid module with a torsion-free decomposition $M = \bigoplus_{j=1}^{t} M_j$. Corollary 4.3.4 shows that the stratum of the torsion-free admissible decomposition of M is a basic module when considering the tightest nested family such that $M \in \mathcal{M}(\Gamma)$. The following proposition shows the relationship between the stratums of M when changing the family Γ to a looser one.

Proposition 4.4.3. *Let* $\Gamma = \{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in G}$ *and* $\Gamma' = \{(\mathcal{T}'_k, \mathcal{F}'_k)\}_{k\in G}$ *be nested families of torsion pairs such that* $\Gamma \leq \Gamma'$ *and* $M = \bigoplus_{k\in G} M_k$ *a decomposition of* M.

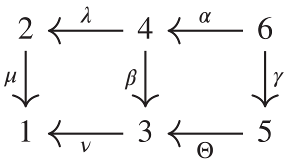
1. *If such a decomposition of* M *is compatible in* $\mathcal{M}(\Gamma)$ *and in* $\mathcal{M}(\Gamma')$*, then* $\mathfrak{F}'(M)$ *is a quotient of* $\mathfrak{F}(M)$*, where* $\mathfrak{F}(M)$ *and* $\mathfrak{F}'(M)$ *are the stratum of* M *with respect to* Γ *and* Γ'*, respectively.*

2. *If such a decomposition of* M *is compatible in* $\mathcal{N}(\Gamma)$ *and in* $\mathcal{N}(\Gamma')$*, then* $\mathfrak{T}(M)$ *is a submodule of* $\mathfrak{T}'(M)$*, where* $\mathfrak{T}(M)$ *and* $\mathfrak{T}'(M)$ *are the substratum of* M *with respect to* Γ *and* Γ'*, respectively.*

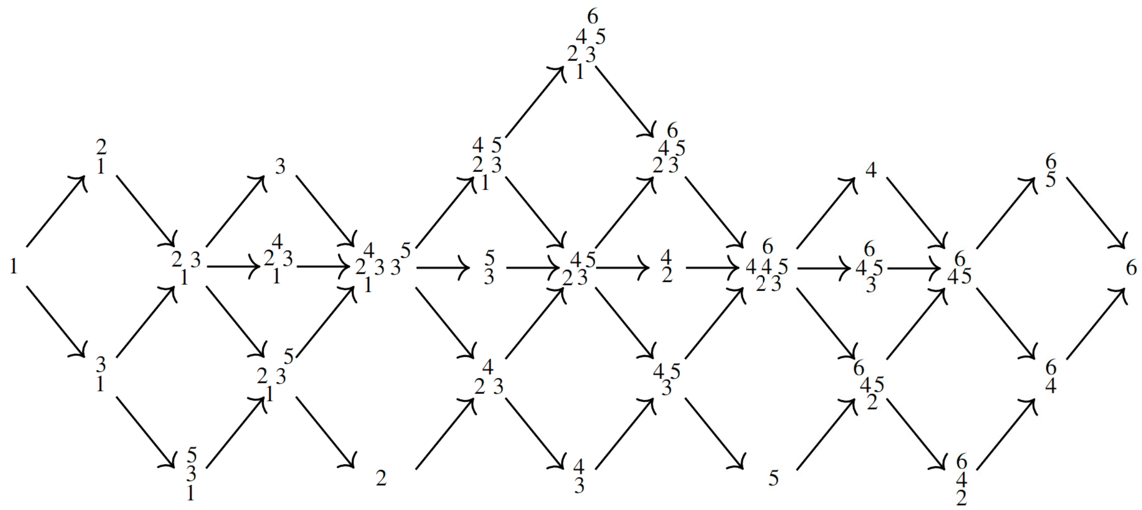
Proof. 1) We have that $\mathfrak{F}_k(M)$ and $\mathfrak{F}'_k(M)$ are the smallest quotients of the families $\{f_j(M_k) \mid j > k\} \cup \{M_k\}$ and $\{f'_j(M_k) \mid j > k\} \cup \{M_k\}$, respectively. Since $\mathcal{T}_j \subseteq \mathcal{T}'_j$, we have that $t_j(M_k) \subseteq t'_j(M_k)$ implies that $f'_j(M_k)$ is a quotient of $f_j(M_k)$. Therefore, $\mathfrak{F}'_k(M)$ is a quotient of $\mathfrak{F}_k(M)$, and $\mathfrak{F}'(M)$ is a quotient of $\mathfrak{F}(M)$. The second item is analogous. ■

The next two examples explore the consequences of Proposition 4.4.3.

Example 4.4.4. *Let $A = kQ/I$, where Q is the quiver*



and I is the ideal generated by $\alpha\beta - \gamma\Theta$ and $\lambda\mu - \beta\nu$. The Auslander-Reiten quiver of the algebra is given by:



Consider the τ-rigid $M = M_1 \bigoplus M_2$, where $M_1 = {}^{2}{}_{1}{}^{3}$ and $M_2 = 6$ (see (7, Example I.2.18) for a brief explanation of the notation). Such a decomposition of the module is torsion-free admissible.

Consider the nested family

$$\Gamma = \{(\mathrm{Fac}(M_1 \bigoplus M_2), (M_1 \bigoplus M_2)^{\perp}); (\mathrm{Fac}(M_2), M_2^{\perp})\}.$$

We have that $\mathfrak{F}_1(M) = M_1$ and $\mathfrak{F}_2(M) = M_2$. Corollary 4.3.4 ensures that $\Theta = \{{}^{2}{}_{1}{}^{3}, 6\}$ is a stratifying system.

On the other hand, consider the nested family

$$\Gamma' = \{(\mathrm{mod}\, A, \{0\}); (\mathrm{Fac}(1 \bigoplus 6), (1 \bigoplus 6)^{\perp})\}.$$

Note that $\Gamma \leq \Gamma'$ and that such a decomposition of M is compatible in $\mathcal{M}^{\dagger}(\Gamma')$. We have $\mathfrak{F}_1(M) = 2 \bigoplus 3$ and $\mathfrak{F}_2(M) = 6$. Theorem 4.2.6 ensures that $\Theta' = \{2, 6\}$ and $\Theta'' = \{3, 6\}$ are two stratifying systems induced as a quotient by M.

The above example suggests that a module $M \in \mathcal{M}^{\dagger}(A)$ induces a unique stratifying system if Γ is a "sufficiently tight" nested family. The next example shows that this is not the case.

Example 4.4.5. *Let A be the same algebra as in Example 4.4.4 and consider also the module $M = M_1 \bigoplus M_2$, where $M_1 = {}^{2}{}_{1}{}^{3}$ and $M_2 = 1$. We have that M is not τ-rigid since $\tau(M) = M_2$.*

Let $\Gamma = \{(\mathsf{T}(M), M^{\perp}), (\mathsf{T}(1), 1^{\perp})\}$ be the tightest nested family such that $M \in \mathcal{M}(\Gamma)$. We have that $M \in \mathcal{M}^{\dagger}(\Gamma)$ and we have the exact sequence

$$0 \longrightarrow 1 \longrightarrow {}^{2}{}_{1}{}^{3} \longrightarrow 2 \bigoplus 3 \longrightarrow 0.$$

Since $1 \in \mathcal{T}_2$ and $2 \bigoplus 3 \in \mathcal{F}_2$, we have that $\mathfrak{F}_1(M) = f_2(M_1) = 2 \bigoplus 3$. Theorem 4.2.6 ensures that $\Theta = \{3, 1\}$ and $\Theta' = \{2, 1\}$ are stratifying systems induced by M.

4.5 A stratifying system of infinite size that cannot be indexed by $(\mathbb{N}, \leq)$

In (41), H. Treffinger demonstrated the existence of stratifying systems of infinite size indexed by $(\mathbb{N}, \leq)$ in n-representation infinite algebras. We conclude this master's thesis by showing that, using the formalism of a nested family of torsion pairs, it is possible to obtain, all at once, infinite stratifying systems of infinite size, including those obtained in (41). Moreover, we obtain stratifying systems that cannot be indexed by $(\mathbb{N}, \leq)$, where $\leq$ is the natural order, as shown below.

Theorem 4.5.1. *Let A be an n-representation infinite algebra with $n > 1$. Then, there exists a stratifying system of infinite size that cannot be indexed by $(\mathbb{N}, \leq)$, where $\leq$ is the natural order.*

Proof. Let A be an n-representation infinite algebra such that $A_A = \bigoplus_{j=1}^{N} P(j)$, with $P(j)$ indecomposable for all $j \in \mathbb{I}_N$, and let $G = \{-1, 1\} \times \mathbb{N}$ be a totally ordered set with the lexicographic order, i.e., with the $\leq$ order such that $(y, m) > (x, l)$ if $y > x$ or $x = y$ and $m > l$. Define $M_{(x,l)} = \nu_n^{xl}(D^{\frac{x+1}{2}}(A))$ and $M = \bigoplus_{(x,l) \in G} M_{(x,l)}$ (where we set $D^0 A = A$).

Let $\mathcal{T}_{(y,m)} = \mathsf{T}\left(\bigoplus_{(x,l) \geq (y,m)} M_{(x,l)}\right)$ and $\mathcal{F}_{(y,m)} = \left(\bigoplus_{(x,l) \geq (y,m)} M_j\right)^{\perp}$. Since such a decomposition of M is such that $\mathrm{Hom}_A(M_{(y,m)}, M_{(x,l)}) = 0$ if $(y, m) > (x, l)$ by Proposition 1.4.2 and Proposition 1.4.4 (c), Theorem 4.2.2 ensures that $\Gamma = \{(\mathcal{T}_{(y,m)}, \mathcal{F}_{(y,m)})\}_{(y,m) \in G}$ is a nested family of torsion pairs, such a decomposition of M is compatible in $\mathcal{M}^*(\Gamma)$ and $\mathfrak{F}(M) = M$.

Since $n > 1$, we have that $\mathrm{Ext}_A^1(M_{(y,m)}, \mathfrak{F}_{(x,l)}(M)) = \mathrm{Ext}_A^1(M_{(y,m)}, M_{(x,l)}) = 0$ by Proposition 1.4.4 (d), in particular, if $(y, m) \geq (x, l)$. Therefore, the decomposition of M is also compatible in $\mathcal{M}^{\dagger}(\Gamma)$.

According to Theorem 4.2.6, for each $(x, l) \in G$ and for each indecomposable summand $\Theta(x, l)$ of $M_{(x,l)}$, we have that $\Theta = \{\Theta(x, l)\}_{(x,l)\in G}$ is a stratifying system indexed by G.

Such a stratifying system cannot be indexed by $(\mathbb{N}, \leq)$. Indeed, there does not exist an order-preserving bijection $\sigma : G \longrightarrow \mathbb{N}$; since $\{(x, l) < (1, 1) \mid (x, l) \in G\}$ is an infinite set, which, in turn, would imply that $\{n < \sigma(1, 1) \mid n \in \mathbb{N}\}$ is infinite, which is absurd. ■

We can explicitly present all the stratifying systems constructed in the theorem above. We have that $\Theta(x, l)$ is an indecomposable summand of $M_{(x,l)} = \bigoplus_{j=1}^{N} \nu_n^{xl}(D^{\frac{x+1}{2}}(P(j)))$. Proposition 1.4.4 (a) and (b) ensures that $\nu_n^{xl}(D^{\frac{x+1}{2}}(P(j)))$ is an indecomposable module for every $j \in \mathbb{I}_N$, which implies that $\Theta = \{\nu_n^{xl}(D^{\frac{x+1}{2}}(P(f(x, l))))\}_{(x,l)\in G}$ is a stratifying system for any function $f : G \longrightarrow \mathbb{I}_N$. In particular, the number of induced stratifying systems is in bijection with the number of functions $f : G \longrightarrow \mathbb{I}_N$.

Corollary 4.5.2. *Let A be an n-representation infinite algebra with $n > 1$ and $|A_A| > 1$. Then, there exist infinitely many stratifying systems of infinite size that cannot be indexed by $(\mathbb{N}, \leq)$, where $\leq$ is the natural order.*

■

It is important to note that the order plays a central role in the definition of a stratifying system. Observe that $\Delta = \{P(2), P(3)\}$ and $\Delta' = \{P(3), P(2)\}$ are distinct stratifying systems, despite defining the same set, where $P(2)$ and $P(3)$ are defined in the Example 4.2.8.

Since G is a countable set, there exists a bijection $\sigma : G \longrightarrow (\mathbb{N}, \leq)$ (which does not preserve the order) such that $M = \bigoplus_{(x,l)\in G} M_{(x,l)} = \bigoplus_{\sigma(x,l)\in\mathbb{N}} M_{\sigma(x,l)}$. It is not possible to follow the proof of Theorem 4.5.1 to construct a stratifying system indexed by $(\mathbb{N}, \leq)$ induced by M. In the proof, we heavily used the fact that $\mathrm{Hom}_A(\nu_n^l(DA), \nu_n^{-k}(A)) = 0$ for all $l, k \geq 0$. On the other hand, indexing the summands of M by $(\mathbb{N}, \leq)$, there will exist infinitely many pairs of indices $(i, j) \in (\mathbb{N}, \leq)$ with $i > j$ such that $M_i = \nu_n^{-k}(A)$ and $M_j = \nu_n^l(DA)$, with $k, l \geq 0$. In general, we do not have that $\mathrm{Hom}_A(\nu_n^{-k}(A), \nu_n^l(DA)) = 0$.

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