UNIVERSIDADE FEDERAL DO PARANÁ

LUANA DEMARCHI GRASSI

HIGHER AUSLANDER-REITEN SEQUENCES

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Orientador: Prof. Dr. Edson Ribeiro Alvares

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The perfect moment is a unicorn:

Don't wait for it.

RESUMO

Este texto tem como objetivo estudar a estrutura de categorias n-abelianas e n-conglomerados inclinados e suas n-sequências de Auslander Reiten. Estas são generalizações homológicas superiores de categorias abelianas e de módulos, bem como de sequências de Auslander Reiten. Estas generalizações foram iniciadas com o trabalho de Iyama (Iya07), definindo subcategorias n-conglomerados inclinados de categorias abelianas e introduzindo a teoria superior Auslander Reiten. Em seguida, Jasso (Jas16) introduziu a noção de categorias n-abelianas como categorias aditivas com n-núcleos e n-co-núcleos. Apresentamos algumas propriedades básicas e exemplos de categorias n-abelianas e subcategorias n-conglomerados inclinados, também, mostramos a relação entre elas. Exploramos a teoria superior de Auslander Reiten, que estuda as n-sequências de Auslander Reiten em categorias n-abelianas e n-conglomerados inclinados. Em particular, fornecemos uma caracterização das 2-sequências de Auslander Reiten e alguns exemplos.

Palavras-chave: Teoria de Auslander Reiten Superior. Categorias n-Abelianas. Subcategorias n-Conglomerados Inclinados. n-Sequências de Auslander Reiten.

ABSTRACT

This text aims to study the structure of n-abelian and n-cluster tilting categories and their n-Auslander Reiten sequences. These are higher homological generalizations of abelian and module categories, as well as Auslander Reiten sequences. These generalizations were started with Iyama's work (Iya07), by defining n-cluster tilting subcategories of abelian categories and introducing the higher Auslander Reiten theory. Then, Jasso (Jas16) introduced the notion of n-abelian categories as additive categories with n-kernels and n-cokernels. We present some basic properties and examples of n-abelian categories and n-cluster tilting subcategories, also, we show the relation between them. We explore higher Auslander Reiten theory, which studies the n-Auslander Reiten sequences in n-abelian and n-cluster tilting categories. In particular, we provide a characterization of 2-Auslander Reiten sequences and some examples.

Keywords: Higher Auslander Reiten Theory. n-Abelian Categories. n-Cluster Tilting Subcategories. n-Auslander Reiten Sequences.

Contents

INTRODUCTION

Abelian categories are categories that have enough structure to define kernels and cokernels, pushouts and pullbacks, and short exact sequences. They are well-known concepts in homological algebra. An important example of abelian categories are the categories of modules over Artin algebras.

Auslander-Reiten theory is a fundamental tool for the representation theory of Artin algebras, which are a generalization of finite-dimensional algebras over a field. It studies the structure of the category of modules over an algebra through outlining almost split sequences, also called Auslander Reiten sequences. These sequences carry important information about the module category because their morphisms are irreducible, and their end terms are indecomposable.

However, abelian categories are not sufficient to capture the homological behavior of some algebraic objects. These objects require higher homological methods, such as derived categories, triangulated categories and cluster tilting categories.

Cluster tilting subcategories are subcategories of the module category of an algebra that are functorially finite and self Ext^1-orthogonal. These subcategories are important in the study of cluster algebras and their categorifications.

The theory of n-cluster tilting subcategories and higher Auslander-Reiten theory, developed by Osamu Iyama (Iya07) in the early 2000s, is a generalization of the theory of cluster tilting subcategories and Auslander-Reiten theory that applies to more general classes of algebras. These theories use the concepts of n-cluster tilting objects and n-almost split sequences (also called n-Auslander Reiten sequences), which are generalizations of cluster tilting objects and almost split sequences, respectively.

Then, in 2013, Geiss, Keller and Oppermann (GKO13), introduced the idea of $n+2$-angulated categories, which are a generalization of triangulated categories. Inspired by that, Gustavo Jasso (Jas16) in 2016 introduced a generalization of the theory of abelian categories that incorporates the higher homological methods. An n-abelian category is an additive category that satisfies certain axioms involving n-kernels and n-cokernels. In this structure, it is possible to construct n-pushouts, n-pullbacks, and n-exact sequences, which are a generalization of the notions of pushouts, pullbacks, and short exact sequences.

In particular, n-cluster tilting subcategories are n-abelian, as proved by Jasso (Jas16). And a few years later, Kvamme (Kva22), and Ebrahimi and Nasr-Isfahani (EN22) proved that n-abelian categories are also equivalent to n-cluster tilting subcategories of abelian categories.

Therefore, the main topic of this dissertation is to understand the structure of n-abelian and n-cluster tilting categories and their n-Auslander Reiten sequences.

The text consists of four chapters, where the first one reviews some basic notions of categories and modules theories, quivers, and path algebras, and the classical Auslander-Reiten theory. Most proofs are not presented, by they are easily found in the bibliography cited, namely, (ASS06),(Ass97), (AC20), (ARS95).

The second chapter introduces the definitions and properties of n-abelian categories, such as n-kernels, n-cokernels, n-pushouts, n-pullbacks, and n-exact sequences. The reader may notice that many of the results presented in this chapter are higher analogues of the abelian case. This chapter main source was Jasso's work (Jas16).

The third chapter studies n-cluster tilting subcategories and their relation to n-abelian categories. Here, some examples are presented to illustrate the results and definitions. Also, the examples provide an idea of how to find these subcategories in module categories.

The fourth chapter explores higher Auslander-Reiten theory, exploring the definitions and properties of n-Auslander Reiten sequences. In particular, we provide a characterization of 2-Auslander Reiten sequences. Also, this chapter presents a series of examples of n-cluster tilting subcategories, the construction of n-Auslander Reiten sequences and n-Auslander Reiten quivers.

Chapter 1

BASIC THEORY

In this chapter we will establish many definitions, properties and results that will be used or generalized in the second and third chapters.

1.1 Categories, Modules and Complexes

The basic setting for this work will be module categories over algebras. More details about this can be found in the bibliography (Ass97), (AC20) and (ASS06). Throughout this text, K will be a field and Λ will be a K-algebra of finite dimension.

The category of right finitely generated modules over Λ, denoted by $mod\,\Lambda$, is an abelian category. To understand what an abelian category is, we start with the definitions of linear category, kernel, cokernel and isomorphism.

Definition 1.1.1. *A category $\mathcal{A}$ is called **K-linear** if*

(L_1) *For all $X, Y \in \mathcal{A}$, the set $Hom_{\mathcal{A}}(X, Y)$ is a K-module.*

(L_2) *The composition of $\mathcal{A}$ is compatible with the structure of K-module:*

$$g \circ (f_1\alpha_1 + f_2\alpha_2) = (g \circ f_1)\alpha_1 + (g \circ f_2)\alpha_2$$

$$(g_1\beta_1 + g_2\beta_2) \circ f = (g_1 \circ f)\beta_1 + (g_2 \circ f)\beta_2$$

for $f, f_1, f_2 : X \to Y$, $g, g_1, g_2 : Y \to Z$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in K$.

(L_3) *All finite families of objects of $\mathcal{A}$ admits direct product and sum in $\mathcal{A}$.*

When a category is $\mathbb{Z}$-linear, it is called an **additive category**. In this work, we will use the notation $add(M)$ to represent the additive subcategory of $mod\,\Lambda$ composed by all sums of direct summands of $M \in mod\,\Lambda$.

In order to simplify notations, we represent the morphism $Hom_{\mathcal{A}}(X, f)$ by $f \circ \_$ and the morphism $Hom_{\mathcal{A}}(f, X)$ by $\_ \circ f$.

Definition 1.1.2. *Let $f : A \to B$ be a morphism in a linear category $\mathcal{A}$. A **kernel** of f is a morphism $g : C \to A$, such that $fg = 0$ and for all $x : X \to A$, with $X \in \mathcal{A}$ and $fx = 0$, there is a unique morphism $p : X \to C$ with $gp = x$. Equivalently, g is a kernel of f, if for all $X \in \mathcal{A}$ the following sequence is exact:*

$$0 \longrightarrow Hom_{\mathcal{A}}(X, C) \xrightarrow{g\circ\_} Hom_{\mathcal{A}}(X, A) \xrightarrow{f\circ\_} Hom_{\mathcal{A}}(X, B).$$

*Dually, a **cokernel** of f is a morphism $h : B \to D$, such that $hf = 0$ and for all $y : B \to Y$, with $Y \in \mathcal{A}$ and $yf = 0$, there is a unique morphism $q : D \to Y$ with $y = qh$. Equivalently, for all $Y \in \mathcal{A}$, the following sequence is exact:*

$$0 \longrightarrow Hom_{\mathcal{A}}(D, Y) \xrightarrow{\_\circ h} Hom_{\mathcal{A}}(B, Y) \xrightarrow{\_\circ f} Hom_{\mathcal{A}}(A, Y).$$

In module categories, these definitions are equivalent to the classic definition of $ker\, f = \{x \in A | f(x) = 0\}$ and $coker\, f = B/Im\, f$.

Now, we define abelian categories.

Definition 1.1.3. *A linear category $\mathcal{A}$ is called an **Abelian Category** if it satisfies the following:*

(Ab_1) *All morphisms in $\mathcal{A}$ admit a kernel and a cokernel.*

(Ab_2) *For all morphism f of $\mathcal{A}$, there is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & Ker\, f \arrow[r] & A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[r] & Coker\, f \arrow[r] & 0 \\ & & Coim\, f \arrow[r, "\overline{f}"] & Im\, f \arrow[u, "i"] & & \end{tikzcd}$$

where $Coim\, f = Coker(Ker\, f)$, $Im\, f = Ker(Coker\, f)$ and $\overline{f}$ is an isomorphism.

A simple example of abelian category is $mod\,\Lambda$.

In the case of module categories, a frequent and important construction are projective and injective resolutions. See (Ass97).

Definition 1.1.4. *Given a Λ-module M, and its projective and injective resolutions*

$$\dots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

$$0 \longrightarrow M \xrightarrow{g_0} I_0 \xrightarrow{g_1} I_1 \xrightarrow{g_2} I_2 \xrightarrow{g_3} \dots,$$

*we denote by $\Omega^j M$ its j^{th}-**syzygy**, that is, the kernel of f_{j-1}. By $\Omega^{-j} M$, we denote its j^{th}-**cosyzygy**, which is the cokernel of g_{j-1}.*

We end this section with the important notion of a complex.

Definition 1.1.5. *Let Λ be an algebra. A **complex** $\mathbf{X} = (X_i, \alpha_i)_{i\in\mathbb{Z}}$ is defined by a sequence of Λ-modules and Λ-linear maps*

$$\mathbf{X} : \qquad \dots \xrightarrow{\alpha_{i-2}} X_{i-1} \xrightarrow{\alpha_{i-1}} X_i \xrightarrow{\alpha_i} X_{i+1} \xrightarrow{\alpha_{i+1}} \dots$$

where $\alpha_i \alpha_{i-1} = 0$, for all $i \in \mathbb{Z}$.

The condition that $\alpha_i\alpha_{i-1} = 0$ is equivalent to say that $Im\,\alpha_{i-1} \subseteq Ker\,\alpha_i$.

A **morphism of complexes**, is a map $\mathbf{f} = (f_i)_{i\in\mathbb{Z}}$, such that each f_i is a homomorphism in $mod\,\Lambda$, and the following diagram commutes:

$$\begin{tikzcd}
\mathbf{X}: \arrow[d, "\mathbf{f}"] & \ldots \arrow[r] & X_{i-1} \arrow[r, "\alpha_{i-1}"] \arrow[d, "f_{i-1}"'] & X_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"'] & X_{i+1} \arrow[r] \arrow[d, "f_{i+1}"'] & \ldots \\
\mathbf{Y}: & \ldots \arrow[r] & Y_{i-1} \arrow[r, "\beta_{i-1}"] & Y_i \arrow[r, "\beta_i"] & Y_{i+1} \arrow[r] & \ldots
\end{tikzcd}$$

We dennote by $Ch(\Lambda)$ or $Ch(mod\,\Lambda)$ the category of complexes, where the objects are complexes and the morphisms are morphisms of complexes.

We also define the cohomology of a complex.

Definition 1.1.6. *The* $\mathbf{k^{th}}$***-cohomology*** *of a complex* $\mathbf{X} = (X_i, \alpha_i)_{i\in\mathbb{Z}}$ *is*

$$\mathcal{H}^k(\mathbf{X}) = \mathcal{H}(X_k) = \frac{Ker(\alpha_k)}{Im(\alpha_{k-1})}.$$

Finally, we say a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is **null-homotopic** if there are morphisms $h_i : X_i \to Y_{i-1}$, such that $f_i = \beta_{i-1}h_i + h_{i+1}\alpha_i$, for all $i \in \mathbb{Z}$. That is, the following diagram is commutative.

$$\begin{tikzcd}
\mathbf{X}: \arrow[d, "\mathbf{f}"] & \ldots \arrow[r] & X_{i-1} \arrow[r, "\alpha_{i-1}"] \arrow[d, "f_{i-1}"'] \arrow[dl] & X_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"'] \arrow[dl, "h_i"'] & X_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] \arrow[dl, "h_{i+1}"'] & \ldots \arrow[dl] \\
\mathbf{Y}: & \ldots \arrow[r] & Y_{i-1} \arrow[r, "\beta_{i-1}"'] & Y_i \arrow[r, "\beta_i"'] & Y_{i+1} \arrow[r] & \ldots
\end{tikzcd}$$

In this case, $\mathbf{h}$ is called an **homotopy**. Also, two morphisms $\mathbf{f}, \mathbf{g}: \mathbf{X} \to \mathbf{Y}$ are called **homotopic equivalent**, if $\mathbf{f} - \mathbf{g}$ is null-homotopic. In this case, we denote it by $\mathbf{h}: \mathbf{f} \to \mathbf{g}$.

The last important definition is of the cone of a morphism of complexes.

Definition 1.1.7. *Given a morphism of complexes* $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$,

$$\begin{tikzcd}
\mathbf{X}: \arrow[d, "\mathbf{f}"] & \ldots \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & \ldots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[r] \arrow[d, "f_n"'] & \ldots \\
\mathbf{Y}: & \ldots \arrow[r] & Y_0 \arrow[r, "\beta_0"] & Y_1 \arrow[r, "\beta_1"] & \ldots \arrow[r, "\beta_{n-2}"] & Y_{n-1} \arrow[r, "\beta_{n-1}"] & Y_n \arrow[r] & \ldots
\end{tikzcd}$$

the ***cone of*** $\mathbf{f}$ *is the complex*

$$\mathbf{C(f)}: \qquad \ldots \longrightarrow X_0 \oplus Y_{-1} \xrightarrow{\gamma_{-1}} X_1 \oplus Y_0 \xrightarrow{\gamma_0} X_2 \oplus Y_1 \longrightarrow \ldots \longrightarrow X_n \oplus Y_{n-1} \xrightarrow{\gamma_{n-1}} Y_n \oplus X_{n+1} \longrightarrow \ldots$$

where

$$\gamma_k = \begin{pmatrix} -\alpha_{k+1} & 0 \\ f_{k+1} & \beta_k \end{pmatrix}, \text{ with } k \in \mathbb{Z}.$$

1.2 Quivers and Path Algebras

A **quiver** $Q = (Q_0, Q_1, s, t)$ is a quadruple, where Q_0 is a set, whose elements are called **vertices**, Q_1 is a set, with elements called **arrows**, s and t are maps $s, t : Q_1 \to Q_0$, called respectively, **source** and **target**. The quiver Q is called **finite** if Q_0 and Q_1 are finite sets.

For example, in the following quiver

$$^{1}\bullet \xrightarrow{\alpha} \bullet^{2}$$

we get $Q_0 = \{1, 2\}$, $Q_1 = \{\alpha\}$, $s(\alpha) = 1$ and $t(\alpha) = 2$.

A **path** of length r in a quiver is sequence of arrows $\alpha_1\alpha_2 \dots \alpha_r$ where $s(\alpha_j) = t(\alpha_{j-1})$ for all $2 \leq j \leq r$. We associate to each vertex x a stationary path, with length zero, that is denoted by ϵ_x.

A quiver is called **acyclic** if there is no path with length at least one with source and target at the same vertex.

A **walk** of length r in a quiver is a sequence $\alpha_1^{\nu_1}\alpha_2^{\nu_2} \dots \alpha_r^{\nu_r}$, where $\nu_j \in \{1, -1\}$, α_j^{-1} is the inverse of α_j, i.e., $s(\alpha_j^{-1}) = t(\alpha_j)$ and $t(\alpha_j^{-1}) = s(\alpha_j)$, and $\{s(\alpha_j), t(\alpha_j)\} \cap \{s(\alpha_{j+1}), t(\alpha_{j+1})\} \neq \emptyset$. A quiver is **connected** if for every two vertices, there is a walk connecting them.

Definition 1.2.1. *Let Q be a quiver. The **path algebra** KQ, is the algebra with underlying K-vector space with basis consisting of the set of all paths in Q, and multiplication set by*

$$(\alpha_1 \dots \alpha_r)(\beta_1 \dots \beta_p) = \begin{cases} \alpha_1 \dots \alpha_r\beta_1 \dots \beta_p, & \textit{if } t(\alpha_r) = s(\beta_1) \\ 0, & \textit{otherwise.} \end{cases}$$

The product is extended by distributivity for all other elements of KQ. The identity is $\sum_{x \in Q_0} \epsilon_x$.

For example, the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "a"] \end{tikzcd}$$

has path algebra KQ with basis $\{\epsilon_1, \alpha, \alpha^2, \alpha^3, \dots\}$. Thus, it is isomorphic to the polynomial algebra in one indeterminate.

We call KQ^+ the ideal generated by all arrows in Q. Then, $(KQ^+)^m$ is the ideal generated by all paths of length m. An ideal I of KQ is called an **admissible ideal**, if there is an integer $m \geq 2$, such that

$$KQ^{+m} \subseteq I \subseteq KQ^{+2}.$$

Then, the algebra KQ/I is called a **bound quiver algebra**.

We call an algebra Λ **basic** if Λ is finite dimensional K-algebra and $\Lambda/rad\,\Lambda$ is isomorphic to a finite product of copies of K, where $rad\,\Lambda$ is the radical of Λ.

Definition 1.2.2. *Let Λ be a basic algebra and $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents. The **ordinary quiver $\mathbf{Q_\Lambda}$** of Λ is defined so that $Q_{\Lambda 0} = \{1, \dots, n\}$ is in bijection with the idempotents $\{e_1, \dots, e_n\}$, for two vertices $x, y \in Q_{\Lambda 0}$, the number of arrows from x to y is equal to the dimension of the K-vector space $\epsilon_x \left(\dfrac{rad\,\Lambda}{rad^2\,\Lambda}\right) \epsilon_y$.*

The next Theorem establishes the relation between basic algebras and bound quiver algebras. It allows us to work with path algebras in the rest of this text.

Theorem 1.2.3. *(AC20)*

(a) *Let Λ be a basic K-algebra. There is a quiver Q and an algebra isomorphism $\phi : KQ_\Lambda/I \to \Lambda$, with I an admissible ideal.*

(b) *Let Q be a finite connected quiver and I an admissible ideal of KQ. Then, $\Lambda = KQ/I$ is a basic and connected finite dimensional algebra, and $rad\,\Lambda = KQ^+/I$.*

For convenience, we will denote the radical of a path algebra by J in the rest of the text.

For example, take the algebra

$$\Lambda = \begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ 0 & K & K \end{pmatrix}.$$

Then, e_{11}, e_{22}, e_{33} is a complete set of primitive orthogonal idempotents for Λ, and

$$rad\,\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ 0 & K & 0 \end{pmatrix} \text{ and } rad^2\,\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So,

$$e_{33}\left(\frac{rad\,\Lambda}{rad^2\,\Lambda}\right)e_{22} \text{ and } e_{22}\left(\frac{rad\,\Lambda}{rad^2\,\Lambda}\right)e_{11}$$

are one-dimensional and the rest of the $e_{ii}\left(\dfrac{rad\,\Lambda}{rad^2\,\Lambda}\right)e_{jj}$ are zero.

Which means that Q_Λ is like:

$$1 \xleftarrow[\beta]{} 2 \xleftarrow[\alpha]{} 3$$

And

$$\begin{array}{cccc} \psi : & KQ_\Lambda & \to & \Lambda \\ & \epsilon_j & \mapsto & e_{jj} \\ & \alpha & \mapsto & e_{32} \\ & \beta & \mapsto & e_{21} \end{array}$$

But, $\psi(\alpha\beta) = \psi(\alpha)\psi(\beta) = e_{32}e_{21} = e_{31} \notin \Lambda$, so, $\alpha\beta \in Ker\,\psi$. Therefore, $KQ_\Lambda/I \simeq \Lambda$, with $I = \langle\alpha\beta\rangle$.

The next Theorem gives a convenient description of all projective, injective and simple modules of a path algebra.

Theorem 1.2.4. *(AC20) Let $\Lambda \simeq KQ/I$. For each $x \in Q_0$,*

(i) *The indecomposable projective Λ-module P_x is generated, as K-vector space, by all paths starting at x, modulo I.*

(ii) *The indecomposable injective Λ-module I_x is generated, as K-vector space, by all paths ending at x, modulo I.*

(iii) *The simple Λ-module S_x is the one dimensional K-vector space generated by ϵ_x.*

In this work, we will use a representation of modules based on their radical filtration. This will become clearer with an Example:

Example 1.2.5. *Consider $\Lambda = KQ$, with Q the following quiver:*

$$\begin{tikzcd} & & 3 \arrow[dl, "\beta"] \\ 1 & 2 \arrow[l, "\alpha"'] & \\ & & 4 \arrow[ul, "\gamma"'] \end{tikzcd}$$

We will determine and present a notation for all projective and injective Λ-modules. According to the Theorem 1.2.4, for each $j \in Q_0$, the projective P_j has a space vector basis given by all paths starting at j. Then,

$$P_1 = \langle \epsilon_1 \rangle \qquad P_2 = \langle \epsilon_2, \alpha \rangle$$
$$P_3 = \langle \epsilon_3, \beta, \beta\alpha \rangle \qquad P_4 = \langle \epsilon_4, \gamma, \gamma\alpha \rangle.$$

In our notation, we will represent idempotents by their respective vertex and arrows by arrows or sobreposition:

$$P_1 = \quad 1 \qquad P_2 = \begin{tikzcd} 2 \arrow[d, "\alpha"] \\ 1 \end{tikzcd} = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$$

$$P_3 = \begin{tikzcd} 3 \arrow[d, "\beta"] \\ 2 \arrow[d, "\alpha"] \\ 1 \end{tikzcd} = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \qquad P_4 = \begin{tikzcd} 4 \arrow[d, "\gamma"] \\ 2 \arrow[d, "\alpha"] \\ 1 \end{tikzcd} = \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix}$$

On the other side, injective modules are given by all paths ending at each vertex, so,

$$I_4 = \langle \epsilon_4 \rangle \qquad I_3 = \langle \epsilon_3 \rangle$$
$$I_2 = \langle \epsilon_2, \beta, \gamma \rangle \qquad I_1 = \langle \epsilon_1, \alpha, \beta\alpha, \gamma\alpha \rangle,$$

which we represent by

$$I_4 = \quad 4 \qquad I_3 = \quad 3$$

$$I_2 = \begin{tikzcd} 3 \arrow[dr, "\beta"'] & & 4 \arrow[dl, "\gamma"] \\ & 2 & \end{tikzcd} = \begin{smallmatrix} 3 & & 4 \\ & 2 & \end{smallmatrix} \qquad I_1 = \begin{tikzcd} 3 \arrow[dr, "\beta"'] & & 4 \arrow[dl, "\gamma"] \\ & 2 \arrow[d, "\alpha"'] & \\ & 1 & \end{tikzcd} = \begin{smallmatrix} 3 & & 4 \\ & 2 & \\ & 1 & \end{smallmatrix}$$

This representation is called radical filtration because it helps us to see the powers of the radical of each module. For example,

$$rad(P_3) = rad\left(\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\right) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \text{ and } rad^2(P_3) = rad^2\left(\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\right) = rad\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = 1$$

then, the radical filtration of P_3 is

$$\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \supseteq \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \supseteq 1.$$

This notation is also useful to identify submodules: M is a submodule of N if the representation of M is a subdiagram of N, where all arrows enter and none leaves. For example, all isoclasses of submodules of I_1 are

$$\left\{ \begin{smallmatrix} 3 & & 4 \\ & 2 & \\ & 1 & \end{smallmatrix} , \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} , \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix} , \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} , 1 \right\}.$$

Dually, M is a quotient of N if M is a subdiagram of N where all arrows leave and none enters. For example,

$$P_4/rad^2(P_4) = \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} .$$

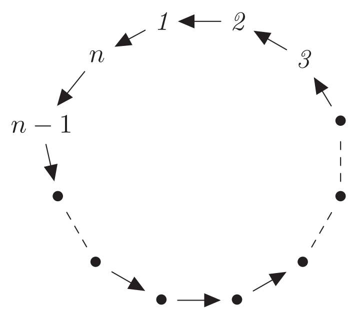
To end this section, we present Nakayama Algebras, which will be used several times in this text. To do that, we define a **composition series** of length l for a module M as a sequence of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$, such that for all $0 \leq j \leq l-1$, the quotient M_{j+1}/M_j is simple. We say a Λ-module is **uniserial** if it admits a unique composition series.

Definition 1.2.6. *An algebra Λ is called a **Nakayama Algebra** if all indecomposable projective and all indecomposable injective Λ-modules are uniserial, i.e. have a unique composition series.*

The following gives a characterization of Nakayama Algebras based on their ordinary quiver.

Theorem 1.2.7. *(AC20) A basic and connected algebra Λ is a Nakayama algebra, if and only if, its ordinary quiver Q_Λ is one of the two following quivers:*

$$\mathbb{A}_n : 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \ldots \longleftarrow n-1 \longleftarrow n,$$



$\tilde{\mathbb{A}}_n$:

Where the first type are called acyclic Nakayama algebras and the second are called cyclic Nakayama algebras.

An important property of Nakayama Algebras is the following classification of all their indecomposable modules.

Theorem 1.2.8. *(AC20) Let Λ be a Nakayama Algebra and M an indecomposable Λ-module. Then, there is an indecomposable projective Λ-module P, and $t \geq 0$, such that $M \simeq P/rad^t P$.*

1.3 Auslander Reiten Theory

In this section we will present some basic definitions and results from classical Auslander Reiten Theory. For more details, the reader can check (AC20), (ARS95), and (ASS06).

In this work, we will focus on module categories. And to study them, two definitions are very important: the radical and the irreducible morphisms.

Definition 1.3.1. *The* ***radical*** rad_Λ *of* $\mathrm{mod}\,\Lambda$ *is the unique ideal such that, if* $A, B \in \mathrm{mod}\,\Lambda$*, then* $rad_\Lambda(A, B)$ *consists of all morphisms* $f : A \to B$ *such that, for every* $g : B \to A$*,* $1_B - fg$ *or* $1_A - gf$ *is invertible. The morphisms in* rad_Λ *are called* ***radical morphisms****.*

An equivalent definition to $f : A \to B$ being a radical morphism is that, for every section $q : A' \to A$ and every retraction $p : B \to B'$, the composition $pfq : A' \to B'$ is not an isomorphism.

When we take A and B to be indecomposable, we have the following result:

Proposition 1.3.2. *(AC20) Let* A *and* B *be indecomposable* Λ*-modules.*

(i) If $A \not\simeq B$*, then* $rad_\Lambda(A, B) = Hom_\Lambda(A, B)$*.*

(ii) If $A \simeq B$*, then* $rad_\Lambda(A, B) \simeq rad\,End\,A$ *consists of all nonisomorphisms.*

Proposition 1.3.3. *(AC20) Let* $f : A \to B$ *be a morphism of* Λ*-modules.*

(i) If A *is indecomposable, then* f *is a radical morphism if and only if* f *is not a section.*

(ii) If B *is indecomposable, then* f *is a radical morphism if and only if* f *is not a retraction.*

Proposition 1.3.4. *(AC20) Let* A *and* B *be indecomposable* Λ*-modules. Then, every radical morphism* $f \in rad_\Lambda(A, B)$ *can be written as* $f = u + v$*, where* u *is a sum of compositions of irreducible morphisms and* $v \in rad_\Lambda^\infty$*.*

Now, we look at irreducible morphisms.

Definition 1.3.5. *Let* $A, B \in \mathrm{mod}\,\Lambda$*, and* $f : A \to B$ *a morphism. Then,* f *is* ***irreducible*** *if* f *is neither a section nor a retraction, and whenever* $f = gh$*, then* h *is a section or* g *is a retraction.*

In the case A and B are both indecomposable, we have the following.

Proposition 1.3.6. *(ARS95) Let* $f : A \to B$ *be a morphism between indecomposable modules* A *and* B *in* $\mathrm{mod}\Lambda$*. Then* f *is irreducible if and only if* $f \in rad_\Lambda(A, B) \setminus rad_\Lambda^2(A, B)$*.*

Also, if f is an irreducible morphism, then it is a monomorphism or an epimorphism.

Now we present definitions for morphisms that will play an important role in the theory.

Definition 1.3.7. *Let* $f : A \to B$ *be a morphism of* Λ*-modules. Then* f *is called* *(i)* ***left almost split*** *if f is not a section and for every $u : A \to C$ that is not a section, there is a morphism $u' : B \to C$ such that $u = u'f$.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[dl, dashed, "\exists\, u'"] \\ C & \end{tikzcd}$$

(ii) ***right almost split*** *if f is not a retraction, and for every $v : D \to B$ that is not a retraction, there is a morphism $v' : D \to A$ such that $v = fv'$.*

$$\begin{tikzcd} & D \arrow[d, "v"] \arrow[dl, dashed, "\exists\, v'"'] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

(iii) ***left minimal*** *if, whenever there is $g \in End\, B$, such that $gf = f$, g is an automorphism.*

(iv) ***right minimal*** *if, whenever there is $h \in End\, A$, such that $fh = f$, h is an automorphism.*

An important remark is that every irreducible morphism is both left and right minimal.

Lemma 1.3.8. *(AC20) Let $f : A \to B$ be a morphism of Λ-modules. Then,*

(i) if f is left almost split, then there is a decomposition $B = B' \oplus B"$ such that $f = \begin{pmatrix} f' \\ 0 \end{pmatrix}$, with $f' : A \to B'$ left minimal almost split.

(ii) if f is right almost split, then there is a decomposition $A = A' \oplus A"$ such that $f = \begin{pmatrix} f' & 0 \end{pmatrix}$, with $f' : A' \to B$ right minimal almost split.

(iii) Every right or left minimal almost split morphism is irreducible.

Lemma 1.3.9. *(ASS06) Let $f : A \to B$ be a morphism.*

(i) If A is indecomposable, f is not a section if and only if $Im(\_ \circ f) \subseteq rad\, End\, A$.

(ii) If B is indecomposable, f is not a retraction if and only if $Im(f \circ \_) \subseteq rad\, End\, B$.

Now, we are ready to define the Auslander Reiten sequences.

Definition 1.3.10. *A short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in $mod\, \Lambda$ is called an ***Auslander Reiten sequence****, or almost split sequence, if f is left almost split and g is right almost split.*

The following theorem shows equivalent definitions for an Auslander Reiten sequence.

Theorem 1.3.11. *(ARS95) Consider a short exact sequence in $mod\, \Lambda$.*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

The following are equivalent:

(a) *The sequence is an Auslander Reiten sequence;*

(b) *A is indecomposable and g is right almost split;*

(c) *C is indecomposable and f is left almost split;*

(d) *f is minimal left almost split;*

(e) *g is minimal right almost split;*

(f) *A and C are indecomposable and f and g are irreducible.*

Similarly to how we defined the ideal rad_Λ, we will define the ideals $\mathcal{P}_\Lambda$ and $\mathcal{I}_\Lambda$ of the category $mod\,\Lambda$. Given modules $A, B \in mod\,\Lambda$, we denote by $\mathcal{P}_\Lambda(A,B)$ the set of all morphisms $f : A \to B$ that factor through a projective Λ-module. Dually, $\mathcal{I}_\Lambda(A,B)$ is the set of all morphisms $f : A \to B$ that factor through an injective Λ-module.

Definition 1.3.12. *We define the **projectively stable category** of $mod\,\Lambda$ as the quotient category*

$$\underline{mod\,\Lambda} = \frac{mod\,\Lambda}{\mathcal{P}_\Lambda}$$

*and the **injectively stable category** of $mod\,\Lambda$ by*

$$\overline{mod\,\Lambda} = \frac{mod\,\Lambda}{\mathcal{I}_\Lambda}.$$

Considering a minimal projective presentation of a Λ-module C

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} C$$

we define the **transpose** of C as

$$Tr(C) = Coker\left(Hom_\Lambda(P_0, \Lambda) \xrightarrow{\_\circ f_1} Hom_\Lambda(P_1, \Lambda)\right).$$

Then, the **Auslander Reiten translation** of C, and its inverse are given by

$$\tau(C) = D \circ Tr(C) \text{ and } \tau^-(C) = Tr \circ D(C),$$

where $D = Hom_K(\_, K)$.

Theorem 1.3.13. *(AC20) $\tau : \underline{mod\,\Lambda} \to \overline{mod\,\Lambda}$ and $\tau^- \overline{mod\,\Lambda} \to \underline{mod\,\Lambda}$ are equivalences of categories.*

This equivalence yields the following properties.

Proposition 1.3.14. *(AC20) Let A be an indecomposable Λ-module.*

(i) *If A is projective, then $\tau(A) = 0$. If A is not projective, then $\tau(A)$ is indecomposable, noninjective, and $\tau^-\tau(A) \simeq A$.*

(ii) If A is injective, then $\tau^-(A) = 0$. If A is not injective, then $\tau^-(A)$ is indecomposable, nonprojective, and $\tau\tau^-(A) \simeq A$.

Theorem 1.3.15. *(ARS95) Let Λ be an algebra.*

(i) If C is an indecomposable nonprojective module in $\mathrm{mod}\,\Lambda$, then there is an Auslander Reiten sequence in $\mathrm{mod}\,\Lambda$

$$0 \to \tau(C) \to B \to C \to 0.$$

(ii) If A is an indecomposable noninjective module in $\mathrm{mod}\,\Lambda$, then there is an Auslander Reiten sequence in $\mathrm{mod}\,\Lambda$

$$0 \to A \to B \to \tau^-(A) \to 0.$$

Finally, we present the **Auslander Reiten formulas**.

Theorem 1.3.16. *Let $A, B \in \mathrm{mod}\,\Lambda$. Then, there are functorial isomorphisms*

$$Ext^1_\Lambda(A,B) \simeq D\underline{Hom_\Lambda}(\tau^- B, A) \simeq \overline{Hom_\Lambda}(B, \tau(A))$$

To end this section, we will introduce the definition of Auslander Reiten quiver.

Definition 1.3.17. *The **Auslander Reiten quiver** $\Gamma(\mathrm{mod}\,\Lambda)$, or simply $\Gamma(\Lambda)$, of an algebra Λ is a quiver defined as*

(i) The vertices of $\Gamma(\Lambda)$ are the isoclasses of indecomposable Λ-modules.

(ii) For two vertices A and B, the arrows from A to B are in bijection with the vectors of a basis of the K-vector space of irreducible morphisms from A to B.

Notice that by this definition, there is an arrow from A to B in $\Gamma(\Lambda)$ if and only if there is an irreducible morphism from A to B. This is important, because in the case of finite representation type algebras, all nonisomorphisms between indecomposable modules are a sum of compositions of irreducible morphisms (Theorem 7.8, (ARS95)).

We end with an example of the construction and use of Auslander Reiten quivers.

Example 1.3.18. *Consider the quiver $\mathbb{A}_4$ and the Nakayama Algebra $\Lambda = K\mathbb{A}_4/J^3$.*

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4$$

We will construct the Auslander Reiten Quiver $\Gamma(\Lambda)$. To do that, we need all indecomposable nonisomorphic Λ-modules. By Theorem 1.2.8, we need to use projective modules and their radicals. The projective modules are

$$P_1 = 1 \quad P_2 = \begin{matrix} 2 \\ 1 \end{matrix} \quad P_3 = \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \quad P_4 = \begin{matrix} 4 \\ 3 \\ 2 \end{matrix}$$

And their radicals are

$$rad\,P_1 = 0 \quad rad\,P_2 = 1 \quad rad\,P_3 = \begin{smallmatrix}2\\1\end{smallmatrix} \quad rad\,P_4 = \begin{smallmatrix}3\\2\end{smallmatrix}$$

So, the indecomposable modules are

$$\left\{P_1, P_2, P_3, P_4, P_2/rad\,P_2 = 2, P_3/rad\,P_3 = 3, P_4/rad\,P_4 = 4, P_3/rad^2 P_3 = \begin{smallmatrix}3\\2\end{smallmatrix}, P_4/rad^2 P_4 = \begin{smallmatrix}4\\3\end{smallmatrix}\right\}$$

$$= \left\{1, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\end{smallmatrix}, 2, 3, 4, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}4\\3\end{smallmatrix}\right\}.$$

Now, we need all irreducible morphisms between indecomposable modules. Recall irreducible morphisms are monomorphisms or epimorphisms. The irreducible monomorphisms are

$$1 \to \begin{smallmatrix}2\\1\end{smallmatrix}, \quad 2 \to \begin{smallmatrix}3\\2\end{smallmatrix}, \quad 3 \to \begin{smallmatrix}4\\3\end{smallmatrix}, \quad \begin{smallmatrix}2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \quad \begin{smallmatrix}3\\2\end{smallmatrix} \to \begin{smallmatrix}4\\3\\2\end{smallmatrix}$$

And the irreducible epimorphisms are

$$\begin{smallmatrix}2\\1\end{smallmatrix} \to 2, \quad \begin{smallmatrix}3\\2\end{smallmatrix} \to 3, \quad \begin{smallmatrix}4\\3\end{smallmatrix} \to 4, \quad \begin{smallmatrix}3\\2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix}, \quad \begin{smallmatrix}4\\3\\2\end{smallmatrix} \to \begin{smallmatrix}4\\3\end{smallmatrix}$$

Now, we place all these morphisms together:

$$\begin{tikzcd}
 & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}4\\3\\2\end{smallmatrix} \arrow[dr] & & \\
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[dr] & \\
1 \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[ur] \arrow[rr, dotted, no head] & & 3 \arrow[ur] \arrow[rr, dotted, no head] & & 4
\end{tikzcd}$$

Notice that the arrows that are going "up" are the monomorphisms and the arrows that are going "down" are the epimorphisms. The dashed lines represent τ and τ^-. Therefore, we can construct all nonisomorphic Auslander Reiten sequences of $mod\,\Lambda$:

$$0 \to 1 \to \begin{smallmatrix}2\\1\end{smallmatrix} \to 2 \to 0,$$

$$0 \to 2 \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 3 \to 0,$$

$$0 \to 3 \to \begin{smallmatrix}4\\3\end{smallmatrix} \to 4 \to 0,$$

$$0 \to \begin{smallmatrix}2\\1\end{smallmatrix} \to 2 \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 0,$$

$$0 \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 3 \oplus \begin{smallmatrix}4\\3\\2\end{smallmatrix} \to \begin{smallmatrix}4\\3\end{smallmatrix} \to 0.$$

Finally, notice there are no Auslander Reiten sequences starting or ending in $\begin{smallmatrix}3\\2\\1\end{smallmatrix}$ and $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$, because they are projective and injective at the same time.

Chapter 2

n-ABELIAN CATEGORIES

In this chapter we will present an important structure that generalize abelian categories: n-abelian categories. An n-abelian category is a higher analog of an abelian category from the viewpoint of higher homological algebra. It is an additive category that has a notion of kernels and cokernels, but these are replaced by n-kernels and n-cokernels in the sense of homotopy theory. In the same way, there are n-pushouts and n-pullbacks in an n-abelian category.

Gustavo Jasso (Jas16) introduced the concept of n-abelian categories in 2016. Since then, these categories have been used in representation theory, commutative algebra, and non-commutative algebraic geometry.

2.1 n-Analogues

In his article, Jasso (Jas16) presented many definitions and results that allows us to understand this kind of structure. Here some of those results will be presented and proved with more details. Also, we will show an example to illustrate the definitions.

When studying abelian categories, some of the most basic definitions and constructions are kernels, cokernels, pullbacks, pushouts, and short exact sequences. We will start this section with the definitions of their analogues.

Definition 2.1.1. *Let $\mathcal{A}$ be an additive category and $f : A \to B$ a morphism in $\mathcal{A}$. A **weak kernel** of f is a morphism $g : C \to A$, such that for all $X \in \mathcal{A}$ we have that*

$$\operatorname{Hom}_{\mathcal{A}}(X, C) \xrightarrow{g\circ -} \operatorname{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f\circ -} \operatorname{Hom}_{\mathcal{A}}(X, B)$$

*is exact. Dually, a **weak cokernel** of f is a morphism $h : B \to D$, such that for all $X \in \mathcal{A}$ we have that*

$$\operatorname{Hom}_{\mathcal{A}}(D, X) \xrightarrow{-\circ h} \operatorname{Hom}_{\mathcal{A}}(B, X) \xrightarrow{-\circ f} \operatorname{Hom}_{\mathcal{A}}(A, X)$$

is exact.

When we compare definitions 1.1.2 and 2.1.1, we notice that their difference lies on whether $g \circ \_$ and $\_ \circ h$ are monomorphisms or not. For example, if $g \circ \_$ is a monomorphism, we guarantee that for all $x : X \to A$, which $fx = 0$ there exists only one morphism $p : X \to C$ such that $gp = x$. But if $g \circ \_$ is not a monomorphism, then we can't guarantee the uniqueness of p.

In conclusion, a weak kernel g of f is a kernel of f, if and only if, g is a monomorphism. Dually, a weak cokernel h of f is a cokernel of f, if and only if, f is an epimorphism.

Definition 2.1.2. *Let $\mathcal{A}$ be an additive category.*

- *A sequence of objects and morphisms in $\mathcal{A}$ of the form*

$$(\alpha_0, \ldots, \alpha_{n-1}) : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n$$

*is a n-**kernel** of a morphism $X_n \xrightarrow{\alpha_n} X_{n+1}$ in $\mathcal{A}$, if*

$$0 \to Hom_{\mathcal{A}}(A, X_0) \xrightarrow{\alpha_0 \circ \_} \cdots \xrightarrow{\alpha_{n-1} \circ \_} Hom_{\mathcal{A}}(A, X_n) \xrightarrow{\alpha_n \circ \_} Hom_{\mathcal{A}}(A, X_{n+1})$$

is an exact sequence for all $A \in \mathcal{A}$.

- *A sequence of objects and morphisms in $\mathcal{A}$ of the form*

$$(\alpha_1, \ldots, \alpha_n) : X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1}$$

*is a n-**cokernel** of a morphism $X_0 \xrightarrow{\alpha_0} X_1$ in $\mathcal{A}$, if*

$$0 \to Hom_{\mathcal{A}}(X_{n+1}, A) \xrightarrow{\_ \circ \alpha_n} \cdots \xrightarrow{\_ \circ \alpha_1} Hom_{\mathcal{A}}(X_1, A) \xrightarrow{\_ \circ \alpha_0} Hom_{\mathcal{A}}(X_0, A)$$

is an exact sequence for all $A \in \mathcal{A}$.

- *An n-**exact sequence** is a sequence of objects and morphisms in $\mathcal{A}$ of the form*

$$(\alpha_0, \ldots, \alpha_n) : 0 \to X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \to 0$$

such that $(\alpha_0, \ldots, \alpha_{n-1})$ is an n-kernel of α_n and $(\alpha_1, \ldots, \alpha_n)$ is an n-cokernel of α_0.

An alternative to these definitions is to use weak kernels and weak cokernels. In fact, the sequence $(\alpha_0, \ldots, \alpha_{n-1})$ is an n-kernel of α_n if, for all $2 \leq k \leq n$, α_{k-1} is a weak kernel of α_k and moreover, α_0 is a kernel of α_1. Dually, $(\alpha_1, \ldots, \alpha_n)$ is an n-cokernel of α_0 if, for all $1 \leq k \leq n-1$, α_k is a weak cokernel of α_{k-1} and, α_n is a cokernel of α_{n-1}.

One last definition is necessary before we define an n-abelian category.

Definition 2.1.3. *A category $\mathcal{A}$ is **idempotent complete** if for every idempotent $e = e^2 \in \mathrm{Hom}_{\mathcal{A}}(A, A)$, for all $A \in \mathcal{A}$, there exists an object $B \in \mathcal{A}$ and morphisms $p : A \to B$ and $i : B \to A$, such that $ip = e$ and $pi = 1_B$, in other words, every idempotent splits.*

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "p"'] & & A \\ & B \arrow[ur, "i"'] & \end{tikzcd}$$

A simple remark is that semisimple categories are idempotent complete. Also, abelian categories are always idempotent complete. Indeed, consider $e : A \to A$ an idempotent in an abelian category $\mathcal{A}$. From Definition 1.1.3, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Ker\, e \arrow[r] & A \arrow[r, "e"] \arrow[d, "p"] & A \arrow[r] & Coker\, e \arrow[r] & 0 \\
 & & Coim\, e \arrow[r, "\overline{e}"] & Im\, e \arrow[u, "i"] & &
\end{tikzcd}$$

where $Coim\, e = Coker(Ker\, e)$, $Im\, e = Ker(Coker\, e)$. Also, p is an epimorphism, i is a monomorphism and $\overline{e}$ is an isomorphism. Then, there is $\overline{e}^{-1} : Im\, e \to Coim\, e$ such that $\overline{e}^{-1} \circ \overline{e} = 1_{Coim\, e}$ and $\overline{e} \circ \overline{e}^{-1} = 1_{Im\, e}$. By taking $p' = \overline{e} \circ p$ an epimorphism and $X \cong Coime \cong Ime$, we have

$$\begin{tikzcd}
A \arrow[rr, "e"] \arrow[dr, "p'"'] & & A \\
 & X \arrow[ur, "i"'] &
\end{tikzcd}$$

with $e = ip'$ and since $e^2 = e$, we have $i\overline{e}p = i\overline{e}pi\overline{e}p$. Since i is a monomorphism and $\overline{e}p$ is an epimorphism, we conclude that $1_X = p'i$.

Now, we establish the higher analogue of an abelian category.

Definition 2.1.4. *An n-**abelian category** is an additive category $\mathcal{A}$ which satisfies the following:*

(A_0) The category $\mathcal{A}$ is idempotent complete.

(A_1) Every morphism in $\mathcal{A}$ has an n-kernel and an n-cokernel.

(A_2) For every monomorphism $\alpha_0 : X_0 \to X_1$ in $\mathcal{A}$ and for every n-cokernel $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of α_0, the following is an n-exact sequence:

$$0 \to X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \to 0$$

(A_2^{op}) For every epimorphism $\beta_n : X_n \to X_{n+1}$ in $\mathcal{A}$ and for every n-kernel $(\beta_0, \beta_1, \ldots, \beta_{n-1})$ of β_n, the following is an n-exact sequence:

$$0 \to X_0 \xrightarrow{\beta_0} X_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{n-1}} X_n \xrightarrow{\beta_n} X_n \to 0$$

Now, we will present an example to illustrate all of these definitions.

Example 2.1.5. *Consider the following quiver $\mathbb{A}_7$ and the Auslander Reiten quiver of the path algebra $K\mathbb{A}_7/J^2$:*

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5 \longleftarrow 6 \longleftarrow 7$$

$$\begin{tikzcd}[column sep=small]
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr, "f_2"] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr, "f_4"] & & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[dr, "f_6"] & & \begin{smallmatrix}5\\4\end{smallmatrix} \arrow[dr, "f_8"] & & \begin{smallmatrix}6\\5\end{smallmatrix} \arrow[dr, "f_{10}"] & & \begin{smallmatrix}7\\6\end{smallmatrix} \arrow[dr, "f_{12}"] & \\
1 \arrow[ur, "f_1"] \arrow[rr, dotted, no head] & & 2 \arrow[ur, "f_3"] \arrow[rr, dotted, no head] & & 3 \arrow[ur, "f_5"] \arrow[rr, dotted, no head] & & 4 \arrow[ur, "f_7"] \arrow[rr, dotted, no head] & & 5 \arrow[ur, "f_9"] \arrow[rr, dotted, no head] & & 6 \arrow[ur, "f_{11}"] \arrow[rr, dotted, no head] & & 7
\end{tikzcd}$$

Our objective is to present a 2-abelian category that is contained in the abelian category $\operatorname{mod} K\mathbb{A}_7/J^2$. *The construction of such subcategories will be developed in the following sessions, specially in Section 3.1.15. In this example, we will consider the subcategory* $\mathcal{C}_2 = add\left(1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus \begin{smallmatrix}7\\6\end{smallmatrix} \oplus 7 \oplus 3 \oplus 5\right)$ *and present an idea of the proof that it is a 2-abelian category. For convenience, the following quiver shows all indecomposable non-isomorphic objects and irreducible morphisms of* $\mathcal{C}_2$:

$$\begin{tikzcd}
 & \begin{matrix}2\\1\end{matrix} \arrow[r, "f_3f_2"] & \begin{matrix}3\\2\end{matrix} \arrow[dr, "f_4"] & & \begin{matrix}4\\3\end{matrix} \arrow[r, "f_7f_6"] & \begin{matrix}5\\4\end{matrix} \arrow[dr, "f_8"] & & \begin{matrix}6\\5\end{matrix} \arrow[r, "f_{11}f_{10}"] & \begin{matrix}7\\6\end{matrix} \arrow[dr, "f_{12}"] & \\
1 \arrow[ur, "f_1"] & & & 3 \arrow[ur, "f_5"] & & & 5 \arrow[ur, "f_9"] & & & 7
\end{tikzcd}$$

In this category, it is possible to see the 2-kernel and 2-cokernel of some morphisms and the 2-exact sequences.

(*i*) *Let* $f : \begin{smallmatrix}5\\4\end{smallmatrix} \to \begin{smallmatrix}6\\5\end{smallmatrix}$. *We will prove that its 2-kernel is given by* $0 \to 3 \to \begin{smallmatrix}4\\3\end{smallmatrix} \to \begin{smallmatrix}5\\4\end{smallmatrix}$.

We can prove it by noticing that $3 \xrightarrow{f_5} \begin{smallmatrix}4\\3\end{smallmatrix}$ *is the kernel of* $\begin{smallmatrix}4\\3\end{smallmatrix} \xrightarrow{f_7f_6} \begin{smallmatrix}5\\4\end{smallmatrix}$ *and* $\begin{smallmatrix}4\\3\end{smallmatrix} \xrightarrow{f_7f_6} \begin{smallmatrix}5\\4\end{smallmatrix}$ *is a weak kernel of* f. *Indeed, as the sequence*

$$0 \to 3 \xrightarrow{f_5} \begin{matrix}4\\3\end{matrix} \xrightarrow{f_6} 4 \to 0$$

is an Auslander Reiten sequence, $f_6f_5 = 0$, *so* $(f_7f_6)f_5 = 0$. *Now, let* $X \in \mathcal{C}_2$ *and* $X \xrightarrow{g} \begin{smallmatrix}4\\3\end{smallmatrix}$ *be a morphism such that* $(f_7f_6)g = 0$. *Since* f_7 *is a monomorphism, it follows that* $f_6g = 0$. *Then, as* f_5 *is the kernel of* f_6, *there is a unique* $h : X \to 3$, *such that* $f_5h = g$. *Therefore,* f_5 *is the kernel of* f_7f_6.

Now, we shall prove that f_7f_6 *is a weak kernel of* f. *Again, let* $X \in \mathcal{C}_2$ *and* $X \xrightarrow{g} \begin{smallmatrix}5\\4\end{smallmatrix}$ *be a morphism such that* $fg = (f_9f_8)g = 0$. *Since* f_9 *is a monomorphism, it follows that* $f_8g = 0$. *Since* f_7 *is the kernel of* f_8 *in* $\operatorname{mod} K\mathbb{A}_7/J^2$, *there is a unique* $h : X \to 4$ *such that* $f_7h = g$. *Then, as* f_6 *is minimal right almost split, there is a morphism* $r : X \to \begin{smallmatrix}4\\3\end{smallmatrix}$ *such that* $h = f_6r$. *Then,* $g = f_7h = f_7f_6r$. *In conclusion, if* $g \in Ker(f \circ \_)$, *then* $g \in Im(f_7f_6 \circ \_)$, *meaning that* f_7f_6 *is a weak kernel of* f.

(*ii*) *Similarly, the 2-cokernel of* f *is given by* $\begin{smallmatrix}6\\5\end{smallmatrix} \to \begin{smallmatrix}7\\6\end{smallmatrix} \to 7 \to 0$, *because* $\begin{smallmatrix}7\\6\end{smallmatrix} \xrightarrow{f_{12}} 7$ *is the cokernel of* $\begin{smallmatrix}6\\5\end{smallmatrix} \xrightarrow{f_{11}f_{10}} \begin{smallmatrix}7\\6\end{smallmatrix}$ *in* $\mathcal{C}_2$ *and* $\begin{smallmatrix}6\\5\end{smallmatrix} \xrightarrow{f_{11}f_{10}} \begin{smallmatrix}7\\6\end{smallmatrix}$ *is a weak cokernel of* f.

(*iii*) *The sequence*

$$0 \to 1 \to \begin{matrix}2\\1\end{matrix} \to \begin{matrix}3\\2\end{matrix} \to 3 \to 0$$

is a 2-exact sequence, because $1 \to \begin{smallmatrix}2\\1\end{smallmatrix}$ *is a monomorphism, and* $\begin{smallmatrix}2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 3$ *is its 2-cokernel.*

(*iv*) *Analogously, we can show all 2-exact sequences of* $\mathcal{C}_2$:

$$0 \to 1 \to \begin{matrix}2\\1\end{matrix} \to \begin{matrix}3\\2\end{matrix} \to 3 \to 0$$

$$0 \to 3 \to \begin{matrix}4\\3\end{matrix} \to \begin{matrix}5\\4\end{matrix} \to 5 \to 0$$

$$0 \to 5 \to \begin{matrix}6\\5\end{matrix} \to \begin{matrix}7\\6\end{matrix} \to 7 \to 0$$

2.2 Basic Properties

Now, we will present some properties that arise from the definitions of n-exact sequences, n-kernels, and n-cokernels. Some of them are analogues of the $n = 1$ case. A morphism of n-exact sequences is a morphism of complexes between two n-exact sequences.

To start, we will look at homotopic morphisms between n-exact sequences and conditions for an n-exact sequence to be split.

Proposition 2.2.1. *(Fed20) Let $\mathcal{A}$ be an n-abelian category and f a morphism of n-exact sequences*

$$\begin{tikzcd}
\mathbf{X}: \arrow[d, "f"] & 0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & \dots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & X_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & 0 \\
\mathbf{Y}: & 0 \arrow[r] & Y_0 \arrow[r, "\beta_0"'] & Y_1 \arrow[r, "\beta_1"'] & \dots \arrow[r, "\beta_{n-2}"'] & Y_{n-1} \arrow[r, "\beta_{n-1}"'] & Y_n \arrow[r, "\beta_n"'] & Y_{n+1} \arrow[r] & 0
\end{tikzcd}$$

the following are equivalent:

(a) There is a morphism $h_{n+1} : X_{n+1} \to Y_n$ such that $\beta_n \circ h_{n+1} = f_{n+1}$.

(b) There is a morphism $h_1 : X_1 \to Y_0$ such that $h_1 \circ \alpha_0 = f_0$.

(c) The morphism f is null-homotopic, that is, there are morphisms $h_i : X_i \to Y_{i-1}$ such that $f_i = \beta_{i-1}h_i + h_{i+1}\alpha_i$ for $i = 0, \dots, n+1$ and $Y_{-1} = X_{n+2} = 0$.

Proof. Clearly, (c) implies (a) and (b). Now, suppose (a) holds. By applying $Hom_{\mathcal{A}}(X_n, \_)$ to $\mathbf{Y}$, we get the sequence

$$\cdots \longrightarrow Hom_{\mathcal{A}}(X_n, Y_{n-1}) \xrightarrow{\beta_{n-1}\circ\_} Hom_{\mathcal{A}}(X_n, Y_n) \xrightarrow{\beta_n\circ\_} Hom_{\mathcal{A}}(X_n, Y_{n+1})$$

which is exact because $\mathbf{Y}$ is n-exact. Note that

$$\beta_n (f_n - h_{n+1}\alpha_n) = \beta_n f_n - \beta_n h_{n+1}\alpha_n = \beta_n f_n - f_{n+1}\alpha_n = 0$$

and as $(f_n - h_{n+1}\alpha_n) : X_n \to Y_n$, $(f_n - h_{n+1}\alpha_n) \in Ker\,(\beta_n \circ \_) = Im\,(\beta_{n-1} \circ \_)$. This means that there is a morphism $h_n : X_n \to Y_{n-1}$ such that $\beta_{n-1}h_n = f_n - h_{n+1}\alpha_n$, equivalently, $f_n = \beta_{n-1}h_n + h_{n+1}\alpha_n$.

Notice that $\beta_{n-1}(f_{n-1} - h_n\alpha_{n-1}) = 0$, then, inductively, by applying $Hom_{\mathcal{A}}(X_i, \_)$ to $\mathbf{Y}$ and taking by hypothesis that $\beta_i\,(f_i - h_{i+1}\alpha_i) = 0$, and the fact that $Ker(\beta_i \circ \_) = Im(\beta_{i-1} \circ \_)$, it is possible to construct $h_i : X_i \to Y_{i-1}$, for all $i = n-1, n-2, \dots, 1$, such that $f_i = \beta_{i-1}h_i + h_{i+1}\alpha_i$.

Finally, as $\beta_0 h_1 = f_1 - h_2\alpha_1$,

$$\begin{aligned}
\beta_0 h_1 \alpha_0 &= f_1\alpha_0 - h_2\alpha_1\alpha_0 \\
&= f_1\alpha_0 \\
&= \beta_0 f_0
\end{aligned}$$

and β_0 is a monomorphism, it follows that $h_1\alpha_0 = f_0$. This proves $(a) \Rightarrow (c)$.

Now, suppose (b) holds. Analogously, apply $Hom_{\mathcal{A}}(\_, Y_1)$ to $\mathbf{X}$:

$$\cdots \longrightarrow Hom_{\mathcal{A}}(X_2, Y_1) \xrightarrow{\_\circ\alpha_1} Hom_{\mathcal{A}}(X_1, Y_1) \xrightarrow{\_\circ\alpha_0} Hom_{\mathcal{A}}(X_0, Y_1)$$

Again, we notice that

$$(f_1 - \beta_0 h_1)\,\alpha_0 = f_1\alpha_0 - \beta_0 h_1 \alpha_0 = \beta_0 f_0 - \beta_0 f_0 = 0$$

Then, $(f_1 - \beta_0 h_1) \in Ker\,(\_ \circ \alpha_0) = Im\,(\_ \circ \alpha_1)$. Hence, there is a morphism $h_2 : X_2 \to Y_1$ such that $h_2\alpha_1 = f_1 - \beta_0 h_1$, or equivalently, $f_1 = h_2\alpha_1 + \beta_0 h_1$.

Inductively, we apply $Hom_{\mathcal{A}}(\_, Y_i)$ to $\mathbf{X}$ and check that $f_i - \beta_{i-1}h_i \in Ker\,(\_ \circ \alpha_{i-1})$ and construct h_{i+1} for all $i = 2, \ldots, n+1$. In conclusion, this gives that $(b) \Rightarrow (c)$. $\square$

An easy corollary follows when we take $\mathbf{Y}$=$\mathbf{X}$ and $f = Id_X$:

Corollary 2.2.2. *(Fed20) Let $\mathcal{A}$ be an n-abelian category and $\boldsymbol{X}$ an n-exact sequence:*

$$\mathbf{X}: \qquad 0 \longrightarrow X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow 0.$$

The following are equivallent:

(a) α_0 is a split monomorphism.

(b) α_n is a split epimorphism.

(c) the identity on $\boldsymbol{X}$ is null-homotopic.

Definition 2.2.3. *An n-exact sequence is called a **split n-exact sequence** if it satisfies any (therefore all) of the conditions from Corollary 2.2.2.*

The following results present many interesting properties of n-exact sequences and their morphisms in n-abelian categories.

Lemma 2.2.4. *(Fed20) Let $\mathcal{A}$ be an n-abelian category and $\mathbf{X}$ and $\mathbf{Y}$ two n-exact sequences. Suppose, for some $0 \leq i < j \leq n+1$ there are morphisms $f_i, f_{i+1}, \ldots, f_j$ such that the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] & X_1 \arrow[r, "\alpha_1"] & \ldots \arrow[r] & X_i \arrow[r] \arrow[d, "f_i"'] & \ldots \arrow[r] & X_j \arrow[r] \arrow[d, "f_j"'] & \ldots \arrow[r] & X_n \arrow[r, "\alpha_n"] & X_{n+1} \arrow[r] & 0 \\
0 \arrow[r] & Y_0 \arrow[r, "\beta_0"'] & Y_1 \arrow[r, "\beta_1"'] & \ldots \arrow[r] & Y_i \arrow[r] & \ldots \arrow[r] & Y_j \arrow[r] & \ldots \arrow[r] & Y_n \arrow[r, "\beta_n"'] & Y_{n+1} \arrow[r] & 0
\end{tikzcd}$$

Then, there are morphisms f_k, $k \in \{0, \ldots, i-1, j+1, \ldots, n+1\}$ completing f to a morphism of n-exact sequences.

Proof. First, let's consider this part of the diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & X_{i-1} \arrow[r, "\alpha_{i-1}"] & X_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"'] & X_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & \ldots \\
\ldots \arrow[r] & Y_{i-1} \arrow[r, "\beta_{i-1}"] & Y_i \arrow[r, "\beta_i"] & Y_{i+1} \arrow[r] & \ldots
\end{tikzcd}$$

Here, $f_{i+1}\alpha_i = \beta_i f_i$. Then, $\beta_i f_i \alpha_{i-1} = 0$. But, β_{i-1} is a weak kernel of β_i, which means there is a morphism $f_{i-1} : X_{i-1} \to Y_{i-1}$ such that $f_i\alpha_{i-1} = \beta_{i-1}f_{i-1}$. Recursively, all morphisms $f_0, \ldots, f_{i-1}$ are constructed.

Analogously, consider the other side of the diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & X_{j-1} & \xrightarrow{\alpha_{j-1}} & X_j & \xrightarrow{\alpha_j} & X_{j+1} & \longrightarrow & \dots \\ & & {\scriptstyle f_{j-1}}\downarrow & & {\scriptstyle f_j}\downarrow & & & & \\ \dots & \longrightarrow & Y_{j-1} & \xrightarrow{\beta_{j-1}} & Y_j & \xrightarrow{\beta_j} & Y_{j+1} & \longrightarrow & \dots \end{array}$$

As $f_j\alpha_{j-1} = \beta_{j-1}f_{j-1}$, we get that $\beta_j f_j \alpha_{j-1} = 0$. Since α_j is a weak cokernel of α_{j-1}, there is a morphism $f_{j+1} : X_{j+1} \to Y_{j+1}$ such that $f_{j+1}\alpha_j = \beta_j f_j$. Again, proceeding recursively gives us all morphisms $f_{j+1}, \dots, f_{n+1}$. ☐

An important remark about Lemma 2.2.4, is that it is possible to take $i = 0$ and $j = 1$ or $i = n$ and $j = n + 1$. In fact, this will usually be the case when we use it.

Lemma 2.2.5. *(Fed20) Let* $\mathbf{X}$ *be an* n*-exact sequence in an* n*-abelian category* $\mathcal{A}$*.*

$$0 \longrightarrow X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow 0.$$

Then, for all $1 \leq i \leq n$, α_i *is right minimal, if and only if,* $\alpha_{i-1} \in rad_{\mathcal{A}}$*.*

Proof. If α_j is right minimal, then, for all $f : X_j \to X_{j-1}$, we have

$$\alpha_j(1_{X_j} - \alpha_{j-1}f) = \alpha_j - \alpha_j\alpha_{j-1}f = \alpha_j.$$

Since α_j is right minimal, $1_{X_j} - \alpha_{j-1}f$ is invertible, and consequently, $\alpha_{j-1} \in rad_{\mathcal{A}}$.

Consider now $\alpha_{j-1} \in rad_{\mathcal{A}}$, and let $h : X_j \to X_j$ be a morphism such that $\alpha_j h = \alpha_j$. Then, $\alpha_j(h - 1_{X_j}) = 0$.

Since $(\alpha_0, \dots, \alpha_{n-1})$ is an n-kernel of α_n, α_{j-1} is a weak kernel of α_j. Then, there is a morphism $g : X_j \to X_{j-1}$, such that $h - 1_{X_j} = \alpha_{j-1}g$, meaning that $h = 1_{X_j} + \alpha_{j-1}g$. As $\alpha_{j-1} \in rad_{\mathcal{A}}$, h is invertible and α_j is right minimal. ☐

Lemma 2.2.6. *(Fed20) Let* $\mathbf{X}$ *be an* n*-exact sequence in an* n*-abelian category* $\mathcal{A}$*, with* $\alpha_0, \dots, \alpha_{n-1} \in rad_{\mathcal{A}}$ *and consider* $f : \mathbf{X} \to \mathbf{X}$ *a morphism of* n*-exact sequences, with* f_n *an isomorphism.*

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \longrightarrow & 0 \\ & & {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & & & {\scriptstyle f_{n-1}}\downarrow & & {\scriptstyle f_n}\downarrow & & \| & & \\ 0 & \longrightarrow & X_0 & \xrightarrow[\alpha_0]{} & X_1 & \xrightarrow[\alpha_1]{} & \dots & \xrightarrow[\alpha_{n-2}]{} & X_{n-1} & \xrightarrow[\alpha_{n-1}]{} & X_n & \xrightarrow[\alpha_n]{} & X_{n+1} & \longrightarrow & 0 \end{array}$$

Then, $f_0, \dots, f_{n-1}$ *are all isomorphisms.*

Proof. By Lemma 2.2.5, $\alpha_1, \dots, \alpha_n$ are right minimal. Also, since f_n is invertible, $\alpha_n f_n = \alpha_n$ and $\alpha_n = \alpha_n f_n^{-1}$, then, using Lemma 2.2.4, we have a commutative diagram:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \longrightarrow & 0 \\ & & {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & & & {\scriptstyle f_{n-1}}\downarrow & & {\scriptstyle f_n}\downarrow & & \| & & \\ 0 & \longrightarrow & X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & & & \downarrow{\scriptstyle g_{n-1}} & & \downarrow{\scriptstyle f_n^{-1}=g_n} & & \| & & \\ 0 & \longrightarrow & X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & X_{n+1} & \longrightarrow & 0 \end{array}$$

So, $\alpha_{n-1} = \alpha_{n-1}g_{n-1}f_{n-1}$. As α_{n-1} is right minimal, $g_{n-1}f_{n-1}$ is an isomorphism. Analogously, $f_{n-1}g_{n-1}$ is an isomorphism. Therefore, f_{n-1} is an isomorphism.

Take $h_{n-1} = (g_{n-1}f_{n-1})^{-1}$, and again, applying Lemma 2.2.4, we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "g_0f_0"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "g_1f_1"'] & \dots \arrow[r] & X_{n-2} \arrow[r, "\alpha_{n-2}"] \arrow[d, "g_{n-2}f_{n-2}"'] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "g_{n-1}f_{n-1}"'] & X_n \arrow[r, "\alpha_n"] \arrow[d, equal] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "h_0"] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "h_1"] & \dots \arrow[r] & X_{n-2} \arrow[r, "\alpha_{n-2}"] \arrow[d, "h_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "h_{n-1}"] & X_n \arrow[r, "\alpha_n"] \arrow[d, equal] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] & X_1 \arrow[r, "\alpha_1"] & \dots \arrow[r] & X_{n-2} \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] & X_n \arrow[r, "\alpha_n"] & X_{n+1} \arrow[r] & 0
\end{tikzcd}$$

Here, $\alpha_{n-2} = h_{n-1}g_{n-1}f_{n-1}\alpha_{n-2} = \alpha_{n-2}h_{n-2}g_{n-2}f_{n-2}$.

Again, since α_{n-2} is right minimal, $h_{n-2}g_{n-2}f_{n-2}$ is an isomorphism. Dually, $g_{n-2}f_{n-2}h_{n-2}$ is an isomorphism, which implies that $g_{n-2}f_{n-2}$ is an isomorphism. If we construct the diagram for $\mathbf{fg}$ instead of $\mathbf{gf}$, we would get that $f_{n-2}g_{n-2}$ is an isomorphism, and thus, f_{n-2} is an isomorphism.

Proceeding recursively, $f_1, \dots, f_{n-2}$ are all isomorphisms. Finally, for f_0, we have the diagram bellow, where $\mathbf{r}$ was constructed so that $f_1^{-1} = r_1$:

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"] & \dots \arrow[r] & X_{n-1} \arrow[r] \arrow[d, equal] & X_n \arrow[r] \arrow[d, equal] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "r_0"'] & X_1 \arrow[r] \arrow[d, "r_1=f_1^{-1}"] & \dots \arrow[r] & X_{n-1} \arrow[r] \arrow[d, equal] & X_n \arrow[r] \arrow[d, equal] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] & X_1 \arrow[r] & \dots \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[r] & X_{n+1} \arrow[r] & 0
\end{tikzcd}$$

Then, $\alpha_0 = \alpha_0 r_0 f_0$, since α_0 is a monomorphism, $r_0 f_0 = 1_{X_0}$. If we constructed the diagram for $\mathbf{fr}$, we would get $\alpha_0 = \alpha_0 f_0 r_0$, and again, $f_0 r_0 = 1_{X_0}$. In conclusion, f_0 is an isomorphism. □

2.3 n-Pushouts and n-Pullbacks

Finally, we define n-pullbacks and n-pushouts.

Definition 2.3.1. *Let $\mathcal{A}$ be an additive category, a complex $\mathbf{X}$ and a morphism $f_0 : X_0 \to Y_0 \in \mathcal{A}$ of the form*

$$\begin{tikzcd}
\mathbf{X}: & X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] & \dots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] & X_n \\
& Y_0 & & & &
\end{tikzcd}$$

*An n-**pushout of** $\mathbf{X}$ **along** f_0 is a morphism of complexes*

$$\begin{tikzcd}
\mathbf{X}: \arrow[d, "f"'] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & \dots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[d, "f_n"'] \\
\mathbf{Y}: & Y_0 \arrow[r, "\beta_0"'] & Y_1 \arrow[r, "\beta_1"'] & \dots \arrow[r, "\beta_{n-2}"'] & Y_{n-1} \arrow[r, "\beta_{n-1}"'] & Y_n
\end{tikzcd} \tag{2.1}$$

such that in the mapping cone $C(f)$

$$X_0 \xrightarrow{\gamma_{-1}} X_1 \oplus Y_0 \xrightarrow{\gamma_0} X_2 \oplus Y_1 \xrightarrow{\gamma_1} \dots \xrightarrow{\gamma_{n-2}} X_n \oplus Y_{n-1} \xrightarrow{\gamma_{n-1}} Y_n,$$

the sequence $(\gamma_0, \dots, \gamma_{n-1})$ *is an* n*-cokernel of* γ_{-1}*, where*

$$\gamma_{-1} = \begin{bmatrix} -\alpha_0 \\ f_0 \end{bmatrix}, \quad \gamma_{n-1} = \begin{bmatrix} f_n & \beta_{n-1} \end{bmatrix} \text{ and } \gamma_k = \begin{bmatrix} -\alpha_{k+1} & 0 \\ f_{k+1} & \beta_k \end{bmatrix}, \quad \text{for all } 0 \leq k \leq n-2.$$

Dually, given a complex $\mathbf{Y}$ *and a morphism* $f_n : X_n \to Y_n$*, an* n***-pullback of* $\mathbf{Y}$ *along*** f_n *is a morphism of complexes like (2.1) and in the mapping cone* $C(f)$*, the sequence* $(\gamma_{-1}, \dots, \gamma_{n-2})$ *is an* n*-kernel of* γ_{n-1}*.*

An important remark is that in the construction of an n-pushout, the last square of (2.1) is a pushout in the usual sense.

The next three results will be used to prove the existence of n-pushouts and n-pullbacks in n-abelian categories.

Lemma 2.3.2. *(Jas16)* ***Comparison Lemma****. Let* $\mathcal{A}$ *be an additive category and* $\mathbf{X}$

$$\mathbf{X}: \qquad X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \longrightarrow \dots \longrightarrow X_n \xrightarrow{\alpha_n} X_{n+1}$$

such that for all $0 \leq k \leq n$*,* α_{j+1} *is a weak cokernel of* α_j*. If* $f : \mathbf{X} \to \mathbf{Y}$ *and* $g : \mathbf{X} \to \mathbf{Y}$ *are morphisms of complexes such that* $f_0 = g_0$*, then there is an homotopy* $h : f \to g$*, such that* h_1 *is the zero morphism.*

Lemma 2.3.3. *(Jas16) Let* $\mathcal{A}$ *be an idempotent complete additive category and consider a sequence of morphisms in* $\mathcal{A}$ *as the following*

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z.$$

If g *is a weak cokernel of* f*, and* h *is both a split epimorphism and a cokernel of* g*, then* f *admits a cokernel in* $\mathcal{A}$*.*

Proof. If h is a split epimorphism, there is a morphism $h' : Z \to Y$ such that $hh' = 1_Z$. Then, $e = 1_Y - h'h$ is an idempotent, since

$$\begin{aligned} e^2 &= 1_Y - h'h - h'h + h'hh'h \\ &= 1_Y - 2h'h + h'1_Z h \\ &= e. \end{aligned}$$

By hypothesis, e splits, which means there are morphisms r and s, such that $sr = e$ and $rs = 1_N$, for some $N \in \mathcal{A}$.

Notice that $hs = 0$, because

$$\begin{aligned} hsr &= he \\ &= h(1_Y - h'h) \\ &= h - hh'h \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} hs &= hsrs \\ &= 0s \\ &= 0. \end{aligned}$$

We claim that rg is a cokernel of f. Indeed, $rg(f) = r(gf) = r0 = 0$, and given a morphism $u : X \to M$ such that $uf = 0$, there is a morphism $v : Y \to M$, with $vg = u$, because g is a weak cokernel of f.

$$\begin{tikzcd} W \arrow[r, "f"] & X \arrow[r, "g"] \arrow[d, "u"'] & Y \arrow[r, "h"] \arrow[dl, "v"'] \arrow[d, bend right, "r"'] & Z \\ & M & N \arrow[u, bend right, "s"'] & \end{tikzcd}$$

Then,

$$\begin{aligned} vsrg &= v(e)g \\ &= v(1_Y - h^{'}h)g \\ &= vg - vh^{'}hg \\ &= vg - vh^{'}0 \\ &= vg = u. \end{aligned}$$

In other words, $u = (vs)(rg)$. Now, we need to show that (vs) is unique. For that, we claim rg is an epimorphism. To prove that, suppose there are morphisms $\alpha, \beta : N \to Q$ such that $\alpha rg = \beta rg$. Then, $(\alpha r - \beta r)g = 0$. As h is the cokernel of g, there is a unique morphism $w : Z \to Q$ with $\alpha r - \beta r = wh$.

$$\begin{tikzcd} W \arrow[r, "f"] & X \arrow[r, "g"] & Y \arrow[r, "h"] \arrow[d, "\alpha r - \beta r"'] & Z \arrow[dl, dashed, "\exists!\, w"] \\ & & Q & \end{tikzcd}$$

Now,

$$\begin{aligned} whs &= \alpha rs - \beta rs \\ w0 &= \alpha(1_N) - \beta(1_N) \\ 0 &= \alpha - \beta, \end{aligned}$$

proving that rg is an epimorphism.

With that, we conclude that vs is unique, because if there exists a morphism γ with $u = \gamma rg$ and $u = (vs)(rg)$, as rg is an epimorphism, we can conclude that $\gamma = vs$. Which proves that rg is the cokernel of f. $\square$

Proposition 2.3.4. *(Jas16) Let $\mathcal{A}$ be an additive category that satisfies axioms (A_0) and (A_1) from Definition 2.1.4 and $\mathbf{X}$ a complex*

$$X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n.$$

If for all $1 \leq k \leq n-1$ the morphism α_k is a weak cokernel of α_{k-1}, then α_{n-1} admits a cokernel in $\mathcal{A}$.

Proof. We prove this result by induction on n.

For $n = 1$, $X_0 \xrightarrow{\alpha_0} X_1$ has (1-) cokernel by hypothesis, since $\mathcal{A}$ satisfies axiom (A_1).

For $n \geq 2$, by hypothesis, the morphisms α_0 and α_{n-1} have an n-cokernel, respectively, $X_1 = Y_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} Y_n \xrightarrow{\beta_n} Y_{n+1}$ and $X_n \xrightarrow{\alpha_n} X_{n+1} \xrightarrow{\alpha_{n+1}} \dots \xrightarrow{\alpha_{2n-2}} X_{2n-1} \xrightarrow{\alpha_{2n-1}} X_{2n}$. By placing them all together, we have the following diagram

$$\begin{tikzcd}
X_0 \arrow[r, "\alpha_0"] \arrow[d, equal] & X_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & X_2 \arrow[r] & \dots \arrow[r] & X_{n-1} \arrow[r, "\alpha_{n-1}"] & X_n \arrow[r, "\alpha_n"] & X_{n+1} \arrow[r, "\alpha_{n+1}"] & X_{n+2} \arrow[r] \arrow[d, "0"] & \dots \arrow[r] & X_{2n-1} \arrow[r, "\alpha_{2n-1}"] \arrow[d, "0"] & X_{2n} \arrow[d, "0"] \\
X_0 \arrow[r, "\alpha_0"] \arrow[d, equal] & X_1 \arrow[r, "\beta_1"] \arrow[d, equal] & Y_2 \arrow[r] & \dots \arrow[r] & Y_{n-1} \arrow[r, "\beta_{n-1}"] & Y_n \arrow[r, "\beta_n"] & Y_{n+1} \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & \dots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[d, "0"] \\
X_0 \arrow[r, "\alpha_0"] & X_1 \arrow[r, "\alpha_1"] & X_2 \arrow[r] & \dots \arrow[r] & X_{n-1} \arrow[r, "\alpha_{n-1}"] & X_n \arrow[r, "\alpha_n"] & X_{n+1} \arrow[r, "\alpha_{n+1}"] & X_{n+2} \arrow[r] & \dots \arrow[r] & X_{2n-1} \arrow[r, "\alpha_{2n-1}"] & X_{2n}
\end{tikzcd}$$

As α_1 is a weak cokernel of α_0, and $\beta_1\alpha_0 = 0$, there is a morphism $f_2 : X_2 \to Y_2$ with $f_2\alpha_1 = \beta_1$. Recursively, as α_k is a weak cokernel of α_{k-1}, and $\beta_k f_{k-1}\alpha_{k-1} = \beta_k\beta_{k-1}f_{k-2} = 0$, there is a morphism $f_k : X_k \to Y_k$ with $f_k\alpha_{k-1} = \beta_k f_{k-1}$, for all $2 \leq k \leq n+1$.

Analogously, as β_k is a weak cokernel of β_{k-1}, and $\alpha_k g_{k-1}\beta_{k-1} = \alpha_k\alpha_{k-1}g_{k-2} = 0$, there is a morphism $g_k : Y_k \to X_k$ with $g_k\beta_{k-1} = \alpha_k g_{k-1}$, for all $1 \leq k \leq n+1$.

$$\begin{tikzcd}
X_0 \arrow[r, "\alpha_0"] \arrow[d, equal] & X_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & X_2 \arrow[r] \arrow[d, "f_2"'] & \dots \arrow[r] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & X_{n+1} \arrow[r, "\alpha_{n+1}"] \arrow[d, "f_{n+1}"'] & X_{n+2} \arrow[r] \arrow[d, "0"] & \dots \arrow[r] & X_{2n-1} \arrow[r, "\alpha_{2n-1}"] \arrow[d, "0"] & X_{2n} \arrow[d, "0"] \\
X_0 \arrow[r, "\alpha_0"] \arrow[d, equal] & X_1 \arrow[r, "\beta_1"] \arrow[d, equal] & Y_2 \arrow[r] \arrow[d, "g_2"'] & \dots \arrow[r] & Y_{n-1} \arrow[r, "\beta_{n-1}"] \arrow[d, "g_{n-1}"'] & Y_n \arrow[r, "\beta_n"] \arrow[d, "g_n"'] & Y_{n+1} \arrow[r] \arrow[d, "g_{n+1}"'] & 0 \arrow[r] \arrow[d, "0"] & \dots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[d, "0"] \\
X_0 \arrow[r, "\alpha_0"] & X_1 \arrow[r, "\alpha_1"] & X_2 \arrow[r] & \dots \arrow[r] & X_{n-1} \arrow[r, "\alpha_{n-1}"] & X_n \arrow[r, "\alpha_n"] & X_{n+1} \arrow[r, "\alpha_{n+1}"] & X_{n+2} \arrow[r] & \dots \arrow[r] & X_{2n-1} \arrow[r, "\alpha_{2n-1}"] & X_{2n}
\end{tikzcd}$$

Now, we apply Lemma 2.3.2, which garantees that there is an homotopy $h : Id_X \to gf$, with $h_{2n} : X_{2n} \to X_{2n-1}$ such that $1_{X_{2n}} = \alpha_{2n-1}h_{2n}$.

This allows us to apply Lemma 2.3.3: α_{2n-3} has a cokernel in $\mathcal{A}$. This allows us to reduce the n-cokernel of α_{n-1} by one morphism. Proceeding inductively, we conclude that α_{n-1} has a cokernel in $\mathcal{A}$. $\square$

Theorem 2.3.5 ((Jas16), **Existence of n-pushouts**). *Let $\mathcal{A}$ be an additive category that satisfies axioms (A_0) and (A_1) from Definition 2.1.4 and consider the complex $\mathbf{X}$ and the morphism f_0 as follows:*

$$\begin{tikzcd}
X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] & \dots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] & X_n \\
Y_0
\end{tikzcd}$$

Then,

(i) There exists an n-pushout diagram

$$\begin{tikzcd}
X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & \dots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[d, "f_n"] \\
Y_0 \arrow[r, "\beta_0"'] & Y_1 \arrow[r, "\beta_1"'] & \dots \arrow[r, "\beta_{n-2}"'] & Y_{n-1} \arrow[r, "\beta_{n-1}"'] & Y_n
\end{tikzcd}$$

(ii) If, moreover, $\mathcal{A}$ is n-abelian and α_0 is a monomorphism, then β_0 is a monomorphism.

Proof. (i) We will proceed by induction on n. We start by setting

$$X_0 \xrightarrow{\gamma_{-1}=\begin{pmatrix} -\alpha_0 \\ f_0 \end{pmatrix}} X_1 \oplus Y_0$$

Now, we suppose that for all $0 \leq k \leq n-2$ and $l \leq k$ we have constructed Y_l, $f_l : X_l \to Y_l$ and $\beta_{l-1} : Y_{l-1} \to Y_l$ with $\gamma_{l-1} \circ \gamma_{l-2} = 0$ and $\gamma_{l-1} = \begin{bmatrix} -\alpha_l & 0 \\ f_l & \beta_{l-1} \end{bmatrix}$.

By hypothesis (A_1), γ_{k-1} has weak cokernel $g_k = [\ f_{k+1} \quad \beta_k\ ]$:

$$X_k \oplus Y_{k-1} \xrightarrow{\gamma_{k-1}} X_{k+1} \oplus Y_k \xrightarrow{g_k} Y_{k+1}$$

which means that $-f_{k+1}\alpha_k + \beta_k f_k = 0$ and $\beta_k \beta_{k-1} = 0$.

We claim that $\gamma_k := \begin{bmatrix} -\alpha_{k+1} & 0 \\ f_{k+1} & \beta_k \end{bmatrix} : X_{k+1} \oplus Y_k \to X_{k+2} \oplus Y_{k+1}$ is a weak cokernel of γ_{k-1}. Indeed, we have that

$$\gamma_k \circ \gamma_{k-1} = \begin{bmatrix} \alpha_{k+1}\alpha_k & 0 \\ -f_{k+1}\alpha_k + \beta_k f_k & \beta_k \beta_{k-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Moreover, suppose there is a morphism $u : X_{k+1} \oplus Y_k \to M$ such that $u\gamma_{k-1} = 0$. As g_k is a weak cokernel of γ_{k-1}, there is a morphism $v : Y_{k+1} \to M$, with $v \circ g_k = u$:

$$\begin{tikzcd} X_k \oplus Y_{k-1} \arrow[r, "\gamma_{k-1}"] & X_{k+1} \oplus Y_k \arrow[r, "g_k"] \arrow[d, "u"'] & Y_{k+1} \arrow[dl, "v"] \\ & M & \end{tikzcd}$$

By taking $\begin{bmatrix} 0 & v \end{bmatrix} : X_{k+2} \oplus Y_{k+1} \to M$, we also verify that $\begin{bmatrix} 0 & v \end{bmatrix} \circ \gamma_k = \begin{bmatrix} vf_{k+1} & v\beta_k \end{bmatrix} = vg_k = u$, so, γ_k is a weak cokernel of γ_{k-1}.

Then, by Proposition 2.3.4, the last morphism of the cone $C(f)$ is a cokernel. In other words, $X_1 \oplus Y_0 \xrightarrow{\gamma_0} X_2 \oplus Y_1 \longrightarrow \ldots \longrightarrow X_{n-1} \oplus Y_{n-2} \xrightarrow{\gamma_{n-2}} X_n \oplus Y_{n-1} \xrightarrow{\gamma_{n-1}} Y_n$ is an n-cokernel of γ_{-1}, and consequently, there is an n-pushout diagram for $\mathbf{X}$ along f_0.

(ii) To prove this, we first claim that γ_{-1} is a monomorphism.

Suppose there is a morphism $u : M \to X_0$ with $\gamma_{-1} \circ u = 0$. Then, $-\alpha_0 u = 0$ and $f_0 u = 0$. As α_0 is a monomorphism, we get that $u = 0$. Which yields that γ_{-1} is a monomorphism.

More than that, the cone $C(f)$ is n-exact, because γ_{-1} is a monomorphism and $(\gamma_0, \ldots, \gamma_{n-1})$ is an n-cokernel of γ_{-1}.

Now, suppose there is a morphism $v : M \to Y_0$ such that $\beta_0 v = 0$. Then we have the diagram bellow:

$$\begin{tikzcd} & M \arrow[d, "\begin{bmatrix} 0 \\ v \end{bmatrix}"'] \arrow[dr, "0"] & \\ X_0 \arrow[r, "\begin{bmatrix} -\alpha_0 \\ f_0 \end{bmatrix}"'] & X_1 \oplus Y_0 \arrow[r, "\begin{bmatrix} -\alpha_1 & 0 \\ f_1 & \beta_0 \end{bmatrix}"'] & X_2 \oplus Y_1 \end{tikzcd}$$

Since the cone $C(f)$ is n-exact, γ_{-1} is the kernel of γ_0. This means that there is a morphism $w : M \to X_0$ with $\gamma_{-1} \circ w = \begin{bmatrix} 0 \\ v \end{bmatrix}$, then

$$-\alpha_0 w = 0 \quad \Rightarrow \quad w = 0$$
$$f_0 w = v \quad \Rightarrow \quad v = 0$$

concluding that β_0 is a monomorphism.

$\square$

The last result of this section will show an interesting and useful property of n-exact sequences and n-pushout diagrams. To prove it, we will need a Lemma first.

Lemma 2.3.6. *Let $\mathcal{A}$ be an n-abelian category, and consider the commutative diagram bellow in $\mathcal{A}$, where $\mathbf{X}$ is n-exact, $\mathbf{Y}$ is the n-pushout of $\mathbf{X}$ along f_0, $C(f)$ is the cone of f, and for every $j \in \{0, \dots, n-1\}$, $0 \to Y_j \xrightarrow{i_j} X_{j+1} \oplus Y_j \xrightarrow{p_{j+1}} X_{j+1} \to 0$ is a short split exact sequence:*

$$\begin{tikzcd}
\mathbf{X}: \arrow[d] & & 0 \arrow[r] \arrow[d] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"] & \dots \arrow[r] & X_{n-2} \arrow[r, "\alpha_{n-2}"] \arrow[d, "f_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"] & X_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[r] & 0 \\
\mathbf{Y}: \arrow[d] & & 0 \arrow[r] \arrow[d] & Y_0 \arrow[r, "\beta_0"] \arrow[d, "i_0"'] & Y_1 \arrow[r] \arrow[d, "i_1"] & \dots \arrow[r] & Y_{n-2} \arrow[r, "\beta_{n-2}"] \arrow[d, "i_{n-2}"] & Y_{n-1} \arrow[r, "\beta_{n-1}"] \arrow[d, "i_{n-1}"'] & Y_n \arrow[d, equal] & & \\
C(f): \arrow[d] & 0 \arrow[r] & X_0 \arrow[r, "\gamma_{-1}"] \arrow[d, equal] & X_1 \oplus Y_0 \arrow[r, "\gamma_0"] \arrow[d, "p_1"'] & X_2 \oplus Y_1 \arrow[r] \arrow[d, "p_2"] & \dots \arrow[r] & X_{n-1} \oplus Y_{n-2} \arrow[r] \arrow[d, "p_{n-1}"] & X_n \oplus Y_{n-1} \arrow[r, "\gamma_{n-1}"] \arrow[d, "p_n"'] & Y_n \arrow[r] \arrow[d] & 0 & \\
\mathbf{X}[1]: & 0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] & X_1 \arrow[r, "\alpha_1"] & X_2 \arrow[r] & \dots \arrow[r] & X_{n-1} \arrow[r, "\alpha_{n-1}"] & X_n \arrow[r] & 0 & &
\end{tikzcd}$$

Then, when we apply $Hom_{\mathcal{A}}(\_, M)$ to the the 3 bottom lines of the diagram, with $M \in \mathcal{A}$, the following commutative diagram has exact lines, where $(\_, \_) := Hom_{\mathcal{A}}(\_, \_)$:

$$\begin{tikzcd}
& & (X_n, M) \arrow[r, "_\circ\alpha_{n-1}"] \arrow[d, "_\circ p_n"'] & (X_{n-1}, M) \arrow[r] \arrow[d] & \dots \arrow[r] & (X_2, M) \arrow[r] \arrow[d, "_\circ p_2"] & (X_1, M) \arrow[r, "_\circ\alpha_0"] \arrow[d, "_\circ p_1"] & (X_0, M) \arrow[d, equal] \\
0 \arrow[r] & (Y_n, M) \arrow[r, "_\circ\gamma_{n-1}"] \arrow[d, equal] & (X_n \oplus Y_{n-1}, M) \arrow[r] \arrow[d, "_\circ i_{n-1}"] & (X_{n-1} \oplus Y_{n-2}, M) \arrow[r] \arrow[d] & \dots \arrow[r] & (X_2 \oplus Y_1, M) \arrow[r] \arrow[d, "_\circ i_1"] & (X_1 \oplus Y_0, M) \arrow[r, "_\circ\gamma_{-1}"] \arrow[d, "_\circ i_0"] & (X_0, M) \\
& (Y_n, M) \arrow[r, "_\circ\beta_{n-1}"] & (Y_{n-1}, M) \arrow[r] & (Y_{n-2}, M) \arrow[r] & \dots \arrow[r] & (Y_1, M) \arrow[r, "_\circ\beta_0"] & (Y_0, M) \arrow[r] & 0
\end{tikzcd}$$

Proof. The top line is exact because $\mathbf{X}$ is n-exact, and thus, $(\alpha_1, \dots, \alpha_n)$ is an n-cokernel of α_0, meaning that the sequence

$$0 \longrightarrow (X_{n+1}, M) \xrightarrow{\_\circ\alpha_n} (X_n, M) \xrightarrow{\_\circ\alpha_{n-1}} (X_{n-1}, M) \longrightarrow \dots \longrightarrow (X_2, M) \longrightarrow (X_1, M) \xrightarrow{\_\circ\alpha_0} (X_0, M)$$

is exact, and by "forgetting" the first morphism, we get the first line is exact.

The middle line is exact because the cone is an n-cokernel. And each column is exact. To prove that the last line is exact, we will show that $Ker(\_ \circ \beta_{j-1}) \subset Im(\_ \circ \beta_j)$, for all $1 \leq j \leq n-1$.

For each $1 \leq j \leq n-1$, let $f \in Ker\ \_ \circ \beta_{j-1}$, meaning $f \circ \beta_{j-1} = 0$. Since $\_ \circ i_j$ is an epimorphism, there is $g \in (X_{j+1} \oplus Y_j, M)$, such that $g \circ i_j = f$. So, $(g \circ \gamma_{j-1}) \circ i_{j-1} = (g \circ i_j) \circ \beta_{j-1} = 0$. Thus, $g \circ \gamma_{j-1} \in Ker\ \_ \circ i_{j-1} = Im\ \_ \circ p_j$. Then, there is $h \in (X_j, M)$ such that $h \circ p_j = g \circ \gamma_{j-1}$.

$$\begin{tikzcd}
& (X_{j+1}, M) \arrow[r, "_\circ\alpha_j"] \arrow[d, "_\circ p_{j+1}"] & (X_j, M) \arrow[r, "_\circ\alpha_{j-1}"] \arrow[d, "_\circ p_j"] & (X_{j-1}, M) \arrow[d, "_\circ p_{j-1}"] \\
(X_{j+2} \oplus Y_{j+1}, M) \arrow[r, "_\circ\gamma_j"] \arrow[d, "_\circ i_{j+1}"] & (X_{j+1} \oplus Y_j, M) \arrow[r, "_\circ\gamma_{j-1}"] \arrow[d, "_\circ i_j"] & (X_j \oplus Y_{j-1}, M) \arrow[r, "_\circ\gamma_{j-2}"] \arrow[d, "_\circ i_{j-1}"] & (X_{j-1} \oplus Y_{j-2}, M) \\
(Y_{j+1}, M) \arrow[r, "_\circ\beta_j"] & (Y_j, M) \arrow[r, "_\circ\beta_{j-1}"] & (Y_{j-1}, M) &
\end{tikzcd}$$

If $h = 0$, then, $g \circ \gamma_{j-1} = 0$ and $g \in Ker\ \_ \circ \gamma_{j-1} = Im\ \_ \circ \gamma_j$. So, there is $r \in (X_{j+2} \oplus Y_{j+1})$ with $r \circ \gamma_j = g$. Therefore, $(r \circ i_{j+1}) \circ \beta_j = f$.

If $h \neq 0$, and $h \in Ker\ \_\circ\alpha_{j-1} = Im\ \_\circ\alpha_j$, there is $r \in (X_{j+1}, M)$ with $r\circ\alpha_j = h$. So, $(r\circ p_{j+1})\circ\gamma_{j-1} = h\circ p_j = g\circ\gamma_{j-1}$, and $g-(r\circ p_{j+1}) \in Ker\ \_\circ\gamma_{j-1} = Im\ \_\circ\gamma_j$. Then, there is $s \in (X_{j+2}\oplus Y_{j+1}, M)$ such that $s\circ\gamma_j = g-(r\circ p_{j+1})$. On the other side, $(s\circ i_{j+1})\circ\beta_j = (s\circ\gamma_j)\circ i_j = (g-(r\circ p_{j+1}))\circ i_j = f-0 = f$.

Finally, if $0 \neq h \notin Ker\ \_\circ\alpha_{j-1}$, there is $0 \neq t \in (X_{j-1}, M)$ such that $h\circ\alpha_{j-1} = t$. But, $t\circ p_{j-1} = (h\circ\alpha_{j-1})\circ p_{j-1} = (h\circ p_j)\circ\gamma_{j-2} = (g\circ\gamma_{j-1})\circ\gamma_{j-2} = 0$, which means $t \in Ker\ \_\circ p_{j-1}$, and since $\_\circ p_{j-1}$ is a monomorphism, $t = 0$, a contradiction. With this, we conclude that the last line is exact. $\square$

Theorem 2.3.7. *(Fed20) Consider $\mathcal{A}$ an n-abelian category, $\mathbf{X}$ an n-exact sequence, and a morphism f_0 as follows:*

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] & \dots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] & X_n \arrow[r, "\alpha_n"] & X_{n+1} \arrow[r] & 0 \\
& Y_0 & & & & & &
\end{tikzcd}$$

Then, the n-pushout of $X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n$ along f_0 can be extended to a morphism of n-exact sequences of the form:

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & \dots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y_0 \arrow[r, "\beta_0"'] & Y_1 \arrow[r, "\beta_1"'] & \dots \arrow[r, "\beta_{n-2}"'] & Y_{n-1} \arrow[r, "\beta_{n-1}"'] & Y_n \arrow[r, "\beta_n"'] & X_{n+1} \arrow[r] & 0
\end{tikzcd}$$

Proof. From Theorem 2.3.5, we know that there is an n-pushout diagram,

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & \dots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & X_{n+1} \arrow[r] & 0 \\
0 \arrow[r] & Y_0 \arrow[r, "\beta_0"'] & Y_1 \arrow[r, "\beta_1"'] & \dots \arrow[r, "\beta_{n-2}"'] & Y_{n-1} \arrow[r, "\beta_{n-1}"'] & Y_n & &
\end{tikzcd}$$

such that β_0 is a monomorphism, and in the cone $C(f)$, the morphisms $(\gamma_0, \dots \gamma_{n-1})$ are an n-cokernel of γ_{-1}. With that, we can use Lemma 2.3.6, and the diagram bellow has exact lines:

$$\begin{tikzcd}
& (X_n, M) \arrow[r, "_\circ\alpha_{n-1}"] \arrow[d, "_\circ p_n"'] & (X_{n-1}, M) \arrow[r] \arrow[d] & \dots \arrow[r] & (X_2, M) \arrow[r] \arrow[d, "_\circ p_2"] & (X_1, M) \arrow[r, "_\circ\alpha_0"] \arrow[d, "_\circ p_1"] & (X_0, M) \arrow[d, equal] \\
0 \arrow[r] (Y_n, M) \arrow[r, "_\circ\gamma_{n-1}"] \arrow[d, equal] & (X_n \oplus Y_{n-1}, M) \arrow[r] \arrow[d, "_\circ i_{n-1}"] & (X_{n-1} \oplus Y_{n-2}, M) \arrow[r] \arrow[d] & \dots \arrow[r] & (X_2 \oplus Y_1, M) \arrow[r] \arrow[d, "_\circ i_1"] & (X_1 \oplus Y_0, M) \arrow[r, "_\circ\gamma_{-1}"] \arrow[d, "_\circ i_0"] & (X_0, M) \\
(Y_n, M) \arrow[r, "_\circ\beta_{n-1}"] & (Y_{n-1}, M) \arrow[r] & (Y_{n-2}, M) \arrow[r] & \dots \arrow[r] & (Y_1, M) \arrow[r, "_\circ\beta_0"] & (Y_0, M) \arrow[r] & 0
\end{tikzcd}$$

Consider now the diagram

$$\begin{tikzcd}
X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[d, "f_n"] \arrow[ddr, bend left, "\alpha_n"] & \\
Y_{n-1} \arrow[r, "\beta_{n-1}"] \arrow[drr, bend right, "0"'] & Y_n & \\
& & X_{n+1}
\end{tikzcd}$$

In the cone $C(f)$, the morphism $\gamma_{n-1} = \begin{pmatrix} f_n & \beta_{n-1} \end{pmatrix}$ is the cokernel of the morphism $\gamma_{n-2} = \begin{pmatrix} -\alpha_{n-1} & 0 \\ f_{n-1} & \beta_{n-2} \end{pmatrix}$. Also, notice that $\begin{pmatrix} \alpha_n & 0 \end{pmatrix}$ is such that

$$\begin{pmatrix} \alpha_n & 0 \end{pmatrix} \begin{pmatrix} -\alpha_{n-1} & 0 \\ f_{n-1} & \beta_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

so, there is a morphism $\beta_n : Y_n \to X_{n+1}$ such that

$$\beta_n \begin{pmatrix} f_n & \beta_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha_n & 0 \end{pmatrix}.$$

Then, the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\alpha_{n-2}"] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r, "\beta_{n-2}"'] & Y_{n-1} \arrow[r, "\beta_{n-1}"'] & Y_n \arrow[r, "\beta_n"'] & X_{n+1} &
\end{tikzcd}$$

commutes and, as α_n is an epimorphism and $\alpha_n = \beta_n f_n$, it follows that β_n is also an epimorphism. All is left to do, is to prove that β_n is the cokernel of β_{n-1}.

Let $g : Y_n \to M$ be a morphism such that $g\beta_{n-1} = 0$.

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"'] & X_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & Y_{n-1} \arrow[r, "\beta_{n-1}"] \arrow[dr, "0"'] & Y_n \arrow[r, "\beta_n"] \arrow[d, "g"] & X_{n+1} \arrow[r] & 0 \\
& & M & &
\end{tikzcd}$$

In particular, $g\beta_{n-1}f_{n-1} = 0$ and $gf_n\alpha_{n-1} = 0$. As α_n is the cokernel of α_{n-1}, there is a unique morphism $u : X_{n+1} \to M$, such that $u\alpha_n = gf_n$.

Recall that $\begin{pmatrix} f_n & \beta_{n-1} \end{pmatrix}$ is an epimorphism. So, as

$$\begin{aligned}
g \begin{pmatrix} f_n & \beta_{n-1} \end{pmatrix} &= \begin{pmatrix} gf_n & g\beta_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} gf_n & 0 \end{pmatrix} \\
&= \begin{pmatrix} u\beta_n f_n & 0 \end{pmatrix} \\
&= \begin{pmatrix} u\beta_n f_n & u\beta_n\beta_{n-1} \end{pmatrix} \\
&= u\beta_n \begin{pmatrix} f_n & \beta_{n-1} \end{pmatrix},
\end{aligned}$$

we have that $g = u\beta_n$, and consequently, β_n is the cokernel of β_{n-1}.

Finally, since α_0 is a monomorphism, by Theorem 2.3.5, β_0 is a monomorphism. Also, by Lemma 2.3.6 and the calculations above, $(\beta_1, \ldots, \beta_n)$ is an n-cokernel of β_0. Therefore,

$$0 \to Y_0 \xrightarrow{\beta_0} Y_1 \to \cdots \to Y_{n-1} \xrightarrow{\beta_n} X_{n+1} \to 0$$

is an n-exact sequence. □

We present an example of the construction 2-pushout and the application of Theorem 2.3.7 in Example 4.2.2.

Chapter 3

n-CLUSTER TILTING SUBCATEGORIES

In the previous chapter, we presented n-abelian categories, which are a generalization of abelian categories. Like module categories are examples of abelian categories, we now present n-cluster tilting categories, which are examples of n-abelian categories, as stated in Theorem 3.2.3. Even more, all n-abelian categories are equivalent to an n-cluster tilting subcategory of an abelian category.

3.1 n-Cluster Tilting Subcategories

Iyama (Iya07) was the first to introduce n-cluster tilting categories. His work was motivated by the study of preprojective algebras and cluster categories. In particular, he was interested in generalize the notion of cluster tilting objects. By extending this notion to n-cluster tilting objects, Iyama was able to create a new class of categories that have many interesting properties and applications.

An n-cluster tilting subcategory is a subcategory of an abelian category, that is closed under direct summands and satisfies certain homological properties, as stated in Definition 3.1.3.

For this work, we will focus on n-cluster tilting subcategories of module categories. In the following, Λ is an algebra and $\mathcal{C}$ is a full subcategory of $mod\,\Lambda$.

Definition 3.1.1. *Let $f : A \to X$ and $g : Y \to A$ be morphisms, such that $X, Y \in \mathcal{C}$ and $A \in mod\,\Lambda$. Then,*

(*i*) *f is called a* ***left $\mathcal{C}$-approximation*** *of A, if for every morphism $u : A \to X'$, with $X' \in \mathcal{C}$, there is a morphism $v : X \to X'$ such that $u = vf$.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & X \arrow[dl, dashed, "\exists\, v"] \\ X' & \end{tikzcd}$$

(ii) *g is called **right $\mathcal{C}$-approximation** of A, if for every morphism $w : Y' \to A$, with $Y' \in \mathcal{C}$, there is a morphism $z : Y' \to Y$, such that $w = gz$.*

$$\begin{tikzcd} Y \arrow[r, "g"] & A \\ Y' \arrow[u, dashed, "\exists z"] \arrow[ur, "w"'] & \end{tikzcd}$$

*If for any $A \in mod\,\Lambda$ there exists a left $\mathcal{C}$-approximation and a right $\mathcal{C}$-approximation we say that $\mathcal{C}$ is **functorially finite**.*

Example 3.1.2. *Consider the subcategory $\mathcal{C}_2 = add\left(1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus \begin{smallmatrix}7\\6\end{smallmatrix} \oplus 7 \oplus 3 \oplus 5\right)$ of $mod\,K\mathbb{A}_7/J^2$ presented in Example 2.1.5. This category is functorially finite. We shall prove this by presenting a left and a right $\mathcal{C}_2$-approximation for each $M \in mod\,K\mathbb{A}_7/J^2$.*

When $M \in \mathcal{C}_2$, we take $M \xrightarrow{1_M} M$ as both left and right $\mathcal{C}_2$-approximation. Also, in the case of decomposable modules, the components of their approximations are the approximations of their indecomposable summands. So, we will only present the $\mathcal{C}_2$-approximations for indecomposable $mod\,K\mathbb{A}_7/J^2$-modules that are not in $\mathcal{C}_2$, namely, 2, 4 and 6.

For the module 2, its left $\mathcal{C}_2$-approximation is the morphism $2 \to \begin{smallmatrix}3\\2\end{smallmatrix}$, because for all other modules X in $\mathcal{C}_2$, if there is a non zero morphism $f : 2 \to X$, f factors through $\begin{smallmatrix}3\\2\end{smallmatrix}$, as it is possible to see in the Auslander Reiten quiver of $mod\,K\mathbb{A}_7/J^2$. Similarly, the right $\mathcal{C}_2$-approximation of 2 is the morphism $\begin{smallmatrix}2\\1\end{smallmatrix} \to 2$.

For the module 4, its left $\mathcal{C}_2$-approximation is the morphism $4 \to \begin{smallmatrix}5\\4\end{smallmatrix}$ and its right $\mathcal{C}_2$-approximation is the morphism $\begin{smallmatrix}4\\3\end{smallmatrix} \to 4$. And for 6, the morphism $6 \to \begin{smallmatrix}7\\6\end{smallmatrix}$ is its left $\mathcal{C}_2$-approximation, and $\begin{smallmatrix}6\\5\end{smallmatrix} \to 6$ is its right $\mathcal{C}_2$-approximation.

Definition 3.1.3. *A subcategory $\mathcal{C} \subset mod\,\Lambda$ is called an n-**cluster tilting subcategory** if it is functorially finite and*

$$\mathcal{C} = {}^{\perp_n}\mathcal{C} = \mathcal{C}^{\perp_n},$$

where

$${}^{\perp_n}\mathcal{C} = \{X \in mod\,\Lambda \mid Ext^i_\Lambda(X, \mathcal{C}) = 0 \text{ for all } 0 < i < n\} \text{ and}$$

$$\mathcal{C}^{\perp_n} = \{X \in mod\,\Lambda \mid Ext^i_\Lambda(\mathcal{C}, X) = 0 \text{ for all } 0 < i < n.\}$$

Remark 3.1.4. (i) *$mod\,\Lambda$ is always the unique 1-cluster tilting subcategory of $mod\,\Lambda$. For $n \geq 2$, the existence of an n-cluster tilting subcategory is not always guaranteed.*

(ii) *Some authors ((Jas16), for example) define an n-cluster tilting subcategory $\mathcal{C}$ from any abelian category $\mathcal{A}$. In that case, an additional hypothesis is necessary: $\mathcal{C}$ needs to be generating and cogenerating in $\mathcal{A}$.*

*A full subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is called **generating** in $\mathcal{A}$, if for all $M \in \mathcal{A}$, there is $Y \in \mathcal{C}$ and an epimorphism $Y \to M$. Dually, $\mathcal{C}$ is called **cogenerating** in $\mathcal{A}$, if for all $M \in \mathcal{A}$, there is $X \in \mathcal{C}$ and a monomorphism $M \to X$.*

In fact, when $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is n-cluster tilting, it is generating and cogenerating. To prove that, we take the projective presentation of M, $P \to M \to 0$. This is an epimorphism and $P \in \mathcal{C}$, because $P \in^{\perp_n} \mathcal{C}$, for any $\mathcal{C} \subset \operatorname{mod}\Lambda$. Similarly, taking the injective hull of M, $0 \to M \to I$, we have a monomorphism, and $I \in \mathcal{C}$, because $I \in \mathcal{C}^{\perp_n}$, for any $\mathcal{C} \subset \operatorname{mod}\Lambda$.

Because of this, each left $\mathcal{C}$-approximation of M is a monomorphism and each right $\mathcal{C}$-approximation of M is an epimorphism.

(iii) *If there is an n-cluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$, then, the subcategory of all projective modules $\mathcal{P}$ and the subcategory of all injective modules $\mathcal{I}$ are contained in $\mathcal{C}$, because for any $n \geq 1$, $\mathcal{P} \subseteq^{\perp_n} \mathcal{C}$ and $\mathcal{I} \subseteq \mathcal{C}^{\perp_n}$.*

The next example will show how to find an n-cluster tilting subcategory using Definition 3.1.3 and Remark 3.1.4.

Example 3.1.5. *Consider the path algebra $\Lambda = K\mathbb{A}_5/J^2$, whose Auslander Reiten quiver is represented bellow with projective modules in blue, injective modules in red and projective-injective modules in purple.*

$$\begin{tikzcd}[column sep=small]
 & {\begin{smallmatrix}2\\1\end{smallmatrix}} \arrow[dr] & & {\begin{smallmatrix}3\\2\end{smallmatrix}} \arrow[dr] & & {\begin{smallmatrix}4\\3\end{smallmatrix}} \arrow[dr] & & {\begin{smallmatrix}5\\4\end{smallmatrix}} \arrow[dr] & \\
1 \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[ur] \arrow[rr, dotted, no head] & & 3 \arrow[ur] \arrow[rr, dotted, no head] & & 4 \arrow[ur] \arrow[rr, dotted, no head] & & 5
\end{tikzcd}$$

We want to find a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ such that $\mathcal{C}$ is 2-cluster tilting. Considering the Definition 3.1.3, we need to find ${}^{\perp_2}\mathcal{C}$ and $\mathcal{C}^{\perp_2}$.

First, notice that all projective modules are in ${}^{\perp_2}\mathcal{C}$ and all injective modules are in $\mathcal{C}^{\perp_2}$. Since we want $\mathcal{C} =^{\perp_2} \mathcal{C} = \mathcal{C}^{\perp_2}$, our next step is to determine ${}^{\perp_2}\mathcal{P}$ and $\mathcal{I}^{\perp_2}$, where $\mathcal{P}$ is the subcategory of all projective modules, and $\mathcal{I}$ is the subcategory of all injective modules.

As $\begin{smallmatrix}2\\1\end{smallmatrix}$, $\begin{smallmatrix}3\\2\end{smallmatrix}$, $\begin{smallmatrix}4\\3\end{smallmatrix}$, $\begin{smallmatrix}5\\4\end{smallmatrix}$ are projective and injective at the same time, they are in ${}^{\perp_2}\mathcal{P}$ and $\mathcal{I}^{\perp_2}$. So, ${}^{\perp_2}\mathcal{P} =^{\perp_2} 1$ and $\mathcal{I}^{\perp_2} = 5^{\perp_2}$. Then,

$${}^{\perp_2}1 = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(X,1) = 0\},$$

$$5^{\perp_2} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(5,X) = 0\}$$

and by the Auslander Reiten formulas,

$${}^{\perp_2}1 = \{X \in \operatorname{mod}\Lambda \mid D \circ \overline{Hom}_\Lambda(\tau^{-}1, X) = 0\} = \{X \in \operatorname{mod}\Lambda \mid D \circ \underline{Hom}_\Lambda(2, X) = 0\},$$

$$5^{\perp_2} = \{X \in \operatorname{mod}\Lambda \mid D \circ \overline{Hom}_\Lambda(X, \tau 5) = 0\} = \{X \in \operatorname{mod}\Lambda \mid D \circ \overline{Hom}_\Lambda(X, 4) = 0\}.$$

Then, looking at the Auslander Reiten quiver of $\operatorname{mod}\Lambda$, we get that

$${}^{\perp_2}1 = \{X \in \operatorname{mod}\Lambda \mid D \circ \underline{Hom}_\Lambda(2, X) = 0\} = add\left(1 \oplus 3 \oplus 4 \oplus 5 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix}\right),$$

$$5^{\perp_2} = \{X \in \operatorname{mod}\Lambda \mid D \circ \overline{Hom}_\Lambda(X, 4) = 0\} = add\left(1 \oplus 2 \oplus 3 \oplus 5 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix}\right).$$

If there is a 2-cluster tilting subcategory, then $\mathcal{C} \subseteq^{\perp_2} \mathcal{P}$, because $\mathcal{P} \subseteq \mathcal{C}$, and $\mathcal{C} \subseteq \mathcal{I}^{\perp_2}$, because $\mathcal{I} \subseteq \mathcal{C}$. Then, when we look at the intersection $\mathcal{C} = ^{\perp_2} \mathcal{P} \cap \mathcal{I}^{\perp_2} = add\left(1 \oplus 3 \oplus 5 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix}\right)$, we get that $\mathcal{C}$ is a 2-cluster tilting subcategory, because it is functorially finite and $\mathcal{C} = ^{\perp_2} \mathcal{C} = \mathcal{C}^{\perp_2}$.

On the other hand, the following is an example of a category that has no n-cluster tilting subcategories.

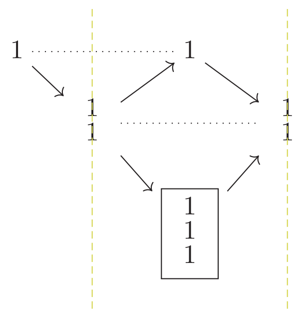
Example 3.1.6. *By taking Q as the following quiver*

$$\begin{tikzcd} 1 \arrow[loop above, "a"] \end{tikzcd}$$

It is possible to define an isomorphism of algebras:

$$\begin{array}{cccc} \varphi: & KQ & \to & K[x] \\ & \epsilon_1 & \mapsto & 1 \\ & a & \mapsto & x \end{array}$$

Therefore, when we consider KQ/J^q, we are actually considering $K[x]/\langle x^q \rangle$. Taking $q = 3$, we get the following Auslander Reiten quiver, where the boxed module is projective and injective and the yellow line shows a gluing of the quiver.



We will prove that this category does not have any n-cluster tilting subcategories.

By Remark 3.1.4, if there exists an n-cluster tilting subcategory $\mathcal{C}$, then $\begin{smallmatrix}1\\1\\1\end{smallmatrix} \in \mathcal{C}$, since it is projective and injective. But, by Definition 3.1.3, it is necessary that $\mathcal{C} = ^{\perp_n} \mathcal{C} = \mathcal{C}^{\perp_n}$. As

$$^{\perp_n} add \begin{pmatrix}1\\1\\1\end{pmatrix} = add \begin{pmatrix}1\\1\\1\end{pmatrix}^{\perp_n} = mod\, KQ/J^3 \neq add \begin{pmatrix}1\\1\\1\end{pmatrix}$$

we conclude that $add \begin{pmatrix}1\\1\\1\end{pmatrix}$ is not an n-cluster tilting subcategory for any n.

Our next step is to try to include more modules in this possible n-cluster tilting. The problem is that $Ext^1(1,1) \neq 0$, because

$$0 \to 1 \to \begin{smallmatrix}1\\1\end{smallmatrix} \to 1 \to 0$$

is an Auslander Reiten sequence, and $Ext^1\left(\begin{smallmatrix}1\\1\end{smallmatrix}, \begin{smallmatrix}1\\1\end{smallmatrix}\right) \neq 0$, since

$$0 \to \begin{smallmatrix}1\\1\end{smallmatrix} \to \begin{smallmatrix}1\\1\\1\end{smallmatrix} \oplus 1 \to \begin{smallmatrix}1\\1\end{smallmatrix} \to 0$$

is also an Auslander Reiten sequence. Hence, ${}^{\perp_n}add\left(\begin{smallmatrix}1\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) \neq add\left(\begin{smallmatrix}1\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$ *and* ${}^{\perp_n}add\left(1 \oplus \begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) \neq add\left(1 \oplus \begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$. *The result is that there is no n-cluster tilting subcategory for this algebra.*

In the same way as the classical Auslander-Reiten theory helps us to understand module categories by specifying all irreducible morphisms between indecomposable modules, its higher analogue - Higher Auslander-Reiten theory - helps us to understand all irreducible morphisms in n-cluster tilting subcategories. By introducing this theory, Iyama (Iya07) has presented the following definition:

Definition 3.1.7. *Given a* Λ *module* M*, we define the* ***n-Auslander-Reiten translations*** τ_n *and* τ_n^- *by*

$$\tau_n M = \tau(\Omega^{n-1}M) \text{ and } \tau_n^- M = \tau^-(\Omega^{-(n-1)}M).$$

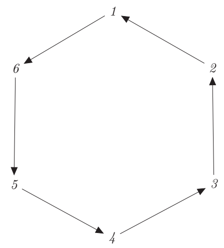
Remark 3.1.8. *Along with the definition, Iyama (Iya11) also proved that for any n-cluster tilting subcategory* $\mathcal{C}$*, the functors* τ_n *and* τ_n^- *induce mutually inverse bijections between isoclasses of indecomposable non-projective objects in* $\mathcal{C}$ *and isoclasses of indecomposable non-injective objects in* $\mathcal{C}$*.*

From that, it is possible to conclude that $\mathcal{M} = add\{\tau_n^i(D\Lambda) \mid i \geq 0\}$ *is contained in any n-cluster tilting subcategory of* $\mathrm{mod}\,\Lambda$*. Moreover, when* $gl.dim(\Lambda) < \infty$*, we have te following theorem.*

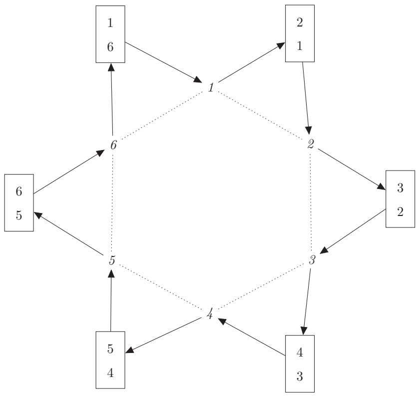
Theorem 3.1.9. *(Iya11) Let* Λ *be an algebra of finite global dimension, and* $\mathcal{C}$ *an n-cluster tilting subcategory of* $\mathrm{mod}\,\Lambda$*. Then* $\mathcal{C}$ *is the unique n-cluster tilting subcategory of* $\mathrm{mod}\,\Lambda$*. In particular,* $\mathcal{C} = \mathcal{M}$*.*

The following Example shows the necessity of the finite global dimension for an algebra to have a unique n-cluster tilting subcategory.

Example 3.1.10. *The algebra* $K\tilde{A}_6/J^2$ *is the path algebra over the quiver bellow, quotiented by the square of its Jacobson radical.*



Its Auslander Reiten quiver is the following, where boxed modules are projective and injective at the same time.



Notice that this algebra has infinite global dimension. It is interesting to notice that mod $K\tilde{A}_6/J^2$ has two different 2-cluster tilting subcategories, three different 3-cluster tilting subcategories, and six different 6-cluster tilting categories. These categories are obtained by using Definition 3.1.3. More details in this procedure are explained in Example 3.1.6.

$$\mathcal{C}_2^1 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 1 \oplus 3 \oplus 5\right),$$

$$\mathcal{C}_2^2 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 2 \oplus 4 \oplus 6\right),$$

$$\mathcal{C}_3^1 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 1 \oplus 4\right),$$

$$\mathcal{C}_3^2 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 2 \oplus 5\right),$$

$$\mathcal{C}_3^3 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 3 \oplus 6\right),$$

$$\mathcal{C}_6^1 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 1\right),$$

$$\mathcal{C}_6^2 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 2\right),$$

$$\mathcal{C}_6^3 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 3\right),$$

$$\mathcal{C}_6^4 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 4\right),$$

$$\mathcal{C}_6^5 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 5\right),$$

$$\mathcal{C}_6^6 = add\left(\begin{matrix}1\\6\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\2\end{matrix} \oplus \begin{matrix}4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \oplus \begin{matrix}6\\5\end{matrix} \oplus 6\right).$$

As presented in (DI20), Proposition 5.3, these n-cluster tilting subcategories are related to $(n+1)$-angulations of an hexagon (which is the shape of the quiver $\tilde{A}_6$). This will be more visible in Example 4.1.13

The existence and the constructions of n-cluster tilting subcategories is not a completely solved problem. In general, it is hard to determine when there are n-cluster tilting subcategories. There are many articles showing necessary and sufficient conditions for an algebra to have such subcategories. In this work, we explore some of the results presented by Vaso (Vas19), to construct examples and comprehend better the structure and proprieties of this subcategories. In his work, Vaso worked with representation directed algebras.

Definition 3.1.11. *A **path** of Λ-modules M_0 to M_l is a sequence of nonzero nonisomorphisms $f_k : M_k \to M_{k+1}$, with $0 \leq k \leq l-1$. Then, Λ is called a **representation-directed algebra** if there is no path from M to N in $\operatorname{mod}\Lambda$ with $M \simeq N$.*

An alternative characterization for representation directed algebras is given in the following proposition.

Proposition 3.1.12. *(Rin84) A representation-finite algebra is a directed algebra, if and only if, its Auslander Reiten quiver does not contain cyclic paths.*

We will need a Lemma to prove the next theorem.

Lemma 3.1.13. *Let Λ be a finite dimensional algebra.*

(i) Let $M \in \operatorname{mod}\Lambda$ be an indecomposable and non projective module, and P its projective cover. If ΩM is decomposable, then $Ext^1_\Lambda(M, P) \neq 0$.

(ii) Let $N \in \operatorname{mod}\Lambda$ be an indecomposable and non injective module, and I its injective hull. If $\Omega^- N$ is decomposable, then $Ext^1_\Lambda(I, N) \neq 0$.

Proof. We will prove (i), as (ii) follows dually. Assume, by absurd, $\Omega M = X_1 \oplus X_2$ and $Ext^1_\Lambda(M, P) = 0$. Consider the exact sequence

$$0 \to \Omega M \xrightarrow{i} P \xrightarrow{p} M \to 0$$

and apply $Hom_\Lambda(\_, P)$ to it:

$$0 \to Hom_\Lambda(M, P) \xrightarrow{\_\circ p} Hom_\Lambda(P, P) \xrightarrow{\_\circ i} Hom_\Lambda(\Omega M, P) \to Ext^1_\Lambda(M, P) \to \dots$$

If $Ext^1_\Lambda(M, P) = 0$, then $\_ \circ i$ is surjective. Then, i is a left $add(P)$-approximation. Even more, it is left minimal, because if $P = P_1 \oplus P_2$ such that $i \simeq \Omega M \xrightarrow{\begin{pmatrix} i_1 \\ 0 \end{pmatrix}} P_1 \oplus P_2$, then P_2 would be a summand of M, and since M is non projective and indecomposable, $P_2 = 0$.

Now, let $f_1 : X_1 \to P'$ and $f_2 : X_2 \to P$" be $add(P)$-minimal left approximations of X_1 and X_2, respectively. Then, $f_1 \oplus f_2$ is a $add(P)$-minimal left approximation of $X_1 \oplus X_2$ and therefore, it is isomorphic to i as a map.

Since P is the projective cover of M, f_1 and f_2 are monomorphisms, but they are not isomorphisms. Thus, $coker\, f_1 \neq 0$ and $coker\, f_2 \neq 0$. But, $M = coker\, f_1 \oplus coker\, f_2$, a contradiction, since M is indecomposable. $\square$

Theorem 3.1.14. *(Vas19) Assume $\mathcal{C}$ is a full subcategory of $\operatorname{mod}\Lambda$, closed under direct sums and summands. Denote by $\mathcal{C}_P$ and $\mathcal{C}_I$ the sets of isomorphisms classes of indecomposable nonprojective respectively noninjective Λ-modules in $\mathcal{C}$. Then, if $\mathcal{C}$ is an n-cluster tilting subcategory, the following hold:*

(i) $\Lambda \in \mathcal{C}$

(ii) τ_n *and* τ_n^- *induce mutually inverse bijections*

$$\begin{tikzcd} \mathcal{C}_P \arrow[r, bend left, "\tau_n"] & \mathcal{C}_I \arrow[l, bend left, "\tau_n^-"] \end{tikzcd}$$

(iii) $\Omega^i M$ *is indecomposable for all* $M \in \mathcal{C}_P$ *and* $0 < i < n$

(iv) $\Omega^{-i} N$ *is indecomposable for all* $N \in \mathcal{C}_I$ *and* $0 < i < n$

Also, if Λ *is a representation directed algebra, then the reverse implication holds.*

Proof. As stated in Remark 3.1.8, items (i) and (ii) have been proved by Iyama (Iya11). We will prove item (iii) and item (iv) is proved dually. The reverse implication is presented by Vaso (Vas19).

Let $M \in \mathcal{C}_P$ and suppose by absurd, there is a minimal $k < n$ such that $\Omega^k M$ is decomposable. Then, $k \geq 1$ and $\Omega^{k-1} M$ is indecomposable. Also, $\Omega^{k-1} M$ is not projective, otherwise $\Omega^k M = 0$. And for $k = 1$, $\Omega^{k-1} M = M$ and M is not projective.

Let P be the projective cover of $\Omega^{k-1} M$. By Lemma 3.1.13, we have that $Ext^1_\Lambda(\Omega^{k-1} M, P) \neq 0$. But that means that $Ext^k_\Lambda(M, P) \neq 0$, which is a contradiction to the fact that $M, P \in \mathcal{C}$, an n-cluster tilting subcategory. Therefore, $\Omega^i M$ is indecomposable for all $0 < i < n$. $\square$

Now, we show how to use the Theorem 3.1.14 to identify n-cluster tilting subcategories for representation directed algebras.

Example 3.1.15. *Given the algebra* $K\mathbb{A}_7$ *quotiented by the square of its radical, we get the following Auslander-Reiten quiver, where the blue are the projective modules, the red are the injective modules and the purple are both projective and injective.*

$$\begin{tikzcd}[column sep=small] & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}5\\4\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}6\\5\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}7\\6\end{smallmatrix} \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[ur] \arrow[rr, dotted, no head] & & 3 \arrow[ur] \arrow[rr, dotted, no head] & & 4 \arrow[ur] \arrow[rr, dotted, no head] & & 5 \arrow[ur] \arrow[rr, dotted, no head] & & 6 \arrow[ur] \arrow[rr, dotted, no head] & & 7 \end{tikzcd}$$

This category has a 2*-cluster tilting, a* 3*-cluster tilting and a* 6*-cluster tilting subcategories:*

$$\mathcal{C}_2 = add\left(1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus \begin{smallmatrix}7\\6\end{smallmatrix} \oplus 7 \oplus 3 \oplus 5\right),$$

$$\mathcal{C}_3 = add\left(1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus \begin{smallmatrix}7\\6\end{smallmatrix} \oplus 7 \oplus 4\right),$$

$$\mathcal{C}_6 = add\left(1 \oplus \begin{matrix}2\\ \\1\end{matrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus \begin{smallmatrix}7\\6\end{smallmatrix} \oplus 7\right).$$

The construction of these categories follows directly from the conditions of Theorem 3.1.14. This algebra is representation-directed because it is representation-finite and its Auslander Reiten quiver does not contain cyclic paths (See Proposition 3.1.12).

(iii) **and** *(iv) All 2^{nd} syzygys and cosyzygys should be indecomposable, this is visible in the Auslander Reiten quiver and is valid for all $K\mathbb{A}_7$-modules, including those which will be in the 2-cluster tiltitng category.*

(i) All projective modules should be objects in an n-cluster tilting category. So, we start with

$$\mathcal{C}' = add\left(1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus \begin{smallmatrix}7\\6\end{smallmatrix}\right)$$

(ii) τ_n and τ_n^- should induce mutually inverse bijections between nonprojective and noninjective modules in the category. As 1 is the only noninjective in the category so far, we only need to calculate $\tau_2^- 1$.

$$\tau_2^- 1 = \tau^-(\Omega^- 1) = \tau^- 2 = 3,$$

$$\mathcal{C}" = add\left(1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus \begin{smallmatrix}7\\6\end{smallmatrix} \oplus 3\right).$$

Then, as 3 is not injective, we also need to find $\tau_2^- 3$:

$$\tau_2^- 3 = \tau^-(\Omega^- 3) = \tau^- 4 = 5,$$

$$\mathcal{C}''' = add\left(1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus \begin{smallmatrix}7\\6\end{smallmatrix} \oplus 3 \oplus 5\right).$$

Finally, as 5 is not injective, we find $\tau_2^- 5 = 7$ and conclude that

$$\mathcal{C}_2 = add\left(1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus \begin{smallmatrix}7\\6\end{smallmatrix} \oplus 3 \oplus 5 \oplus 7\right).$$

But one might ask if it is possible to include other modules, like 2 in $\mathcal{C}_2$. This is why it is not possible: 2 is nonprojective and noninjective, so calculating τ_2 and τ_2^-, we have

$$\tau_2(2) = \tau(\Omega 2) = \tau(1).$$

But as 1 is a projective module, it does not have an Auslander Reiten translate.

The next proposition shows a criterion to discard some algebras, based on their ordinary quivers, when looking for n-cluster tilting subcategories.

Proposition 3.1.16. *(Vas19) Let Q be a connected quiver with m vertices, $\Lambda = KQ/I$ where I is an admissible ideal and $n \geq 2$. Let k be a vertex in Q_0, which is a sink or a source such that the full subquiver of Q with vertex set $Q_0 \setminus \{k\}$ is disconnected. Then Λ admits no n-cluster tilting subcategory.*

Proof. Let k be a sink and consider a possible renaming of the vertices of Q, so that $Q_0' = Q_0 \setminus \{k\} = Q_{A_0} \sqcup Q_{B_0}$, with $Q_{A_0} = \{1, \ldots, k-1\}$ and $Q_{B_0} = \{k+1, \ldots, m\}$.

The dimension vector of the projective module P_k is

$$dim\, P_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

P_k is not injective, and its injective hull I_k has its k^{th} entry with dimension one and at least two other non zero entries, since k is a sink. So, considering the exact sequence

$$0 \to P_k \to I_k \to \Omega^- P_k \to 0$$

we have the dimension vectors $dim\, I_k = \begin{pmatrix} a_1 \\ \vdots \\ a_{k-1} \\ 1 \\ b_{k+1} \\ \vdots \\ b_m \end{pmatrix}$ and $dim\, \Omega^- P_k = \begin{pmatrix} a_1 \\ \vdots \\ a_{k-1} \\ 0 \\ b_{k+1} \\ \vdots \\ b_m \end{pmatrix}$.

Let $\Omega^- P_k = (M_i, \phi_\alpha)_{\substack{i \in Q_0 \\ \alpha \in Q_1}}$ and $f = (f_i)_{i \in Q_0}$, with $f_i : M_i \to M_i$ the identity if $i < k$ and zero otherwise. With this definition, f is not zero and it is not the identity. We will prove that f is an endomorphism of $\Omega^- P_k$.

Let $\alpha : a \to b$ be an arrow in Q. Notice that we can not have $a < k < b$ or $b < k < a$, since Q' is disconnected, also, since k is a sink, $a \neq k$. Our aim is to show $\phi_\alpha f_a = f_b \phi_\alpha$. We look at three possible cases:

(i) If $a, b < k$, then, $f_a = f_b = Id$, and the equality holds $\phi_\alpha f_a = f_b \phi_\alpha$;

(ii) If $k < a, b$, then, $f_a = f_b = 0$, and the equality also holds;

(iii) If $b = k$, since $M_k = 0$, we get $\phi_\alpha = 0$, and the equality $\phi_\alpha f_a = f_b \phi_\alpha$ holds.

In all of them, $f \in End(\Omega^- P_k)$. As $f^2 = f$ and $0 \neq f \neq Id$, we have that $End(\Omega^- P_k)$ is not local, which means that $\Omega^- P_k$ is not indecomposable.

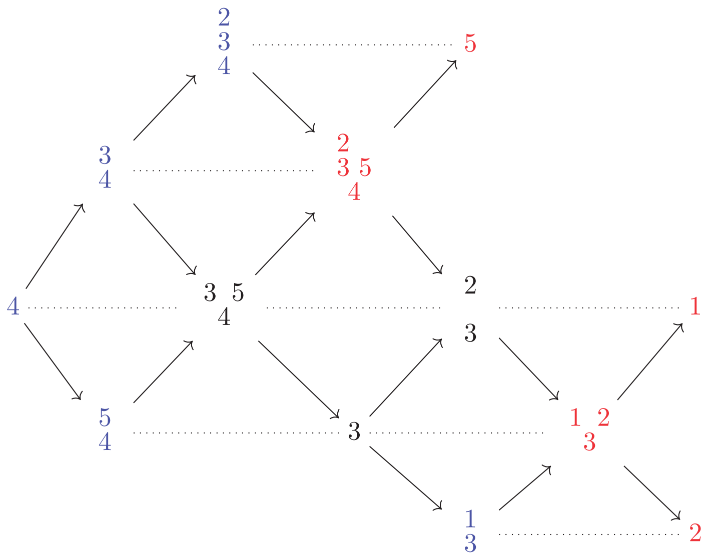
By Theorem 3.1.14, this means $P_k \notin \mathcal{C}$ for any n-cluster tilting subcategory $\mathcal{C}$ of $mod\, \Lambda$. Since P_k is projective, and is always in an n-cluster tilting subcategory, this means there are no n-cluster tilting subcategories for $mod\, \Lambda$. □

Let us show this condition in an example.

Example 3.1.17. *Consider the following quiver Q:*

$$\begin{tikzcd} 1 \arrow[dr, "\alpha"] & & & \\ & 3 \arrow[r, "\gamma"'] & 4 & 5 \arrow[l, "\delta"] \\ 2 \arrow[ur, "\beta"'] & & & \end{tikzcd}$$

Now, let $I = \langle \alpha\gamma \rangle$, and consider the Auslander Reiten quiver of $mod\, \Lambda = mod\, KQ/I$, where projective *modules are in blue and injective modules are in red.*



Notice that in Q, the vertex 4 is a sink, and if we remove it, $Q \setminus \{4\}$ would be a disconnected quiver.

Also, $P_4 = 4$ is a simple module, with dimension 1 as a vector space. Its injective hull is $I_4 = \begin{smallmatrix} 2 & \\ 3 & 5 \\ & 4 \end{smallmatrix}$, and therefore, its cosyzygy is

$$\Omega^{-}(4) = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 5$$

which is decomposable. So, $\operatorname{mod}\Lambda$ has no n-cluster tilting subcategories.

This motivates the study of Nakayama algebras as a source of examples of n-cluster tilting subcategories, because there are no sinks or sources in their ordinary quivers. The following two theorems establish simple and useful relations to determine when a Nakayama algebra quotiented by a power of its radical has an n-cluster tilting subcategory.

Theorem 3.1.18. *(Vas19) Let $\Lambda = K\mathbb{A}_m/J^l$ be an acyclic Nakayama algebra. Then, Λ admits an n-cluster tilting subcategory, if and only if, $l = 2$ and $m = nk+1$ with $k \geq 0$, or n is even and $m = \frac{n}{2}l+1+k(nl-l+2)$, with $k \geq 0$.*

For the case of cyclic Nakayama algebras, the following result shows when it is possible to obtain n-cluster tilting subcategories.

Theorem 3.1.19. *(DI20) Let $\Lambda = K\tilde{\mathbb{A}}_m/J^l$ be a self-injective Nakayama algebra. Then Λ has an n-cluster tilting subcategory if and only if $l(n-1)+2 \mid 2m$ or $l(n-1)+2 \mid tm$, where $t = gdc(n+1, 2l-1)$.*

Notice that for both theorems, the relations between m, l and n are simple divisibility conditions. This is due to the shape of the Auslander Reiten quiver of Nakayama algebras with homogeneous relations, and the position of projectives and injetives on it. More details are on (Vas19).

Those conditions are checked in our previous examples. In Example 3.1.5, we have $n = 2, m = 5, l = 2$ and an acyclic Nakayama algebra. Then, the condition "$l = 2$ and $m = nk+1$" is satisfied, because $5 = 2\dot{2}+1$.

In Example 3.1.10, we have a cyclic Nakayama algebra, $l = 2, m = 6$ and $n \in \{2, 3, 6\}$. Then, $2(2-1)+2 \mid 2\dot{6}$, $2(3-1)+2 \mid 2\dot{6}$, and $2(6-1)+2 \mid 2\dot{6}$, i.e., the condition holds for all cases.

3.2 n-Cluster Tilting Subcategories and n-Abelian Categories

In this section, we will present the close relation between n-cluster tilting categories and n-abelian categories. Two articles, (Kva22) and (EN22), prove that any n-abelian category is equivalent to an n-cluster tilting subcategory of an abelian category. Those proofs are more technical and therefore will not be presented in this work.

The following lemmas will be necessary to prove that n-cluster tilting subcategories are n-abelian.

Lemma 3.2.1. *(Jas16) Let $\mathcal{C}$ be an n-cluster tilting subcategory of $mod\,\Lambda$. Then, for all $A \in mod\,\Lambda$, there is an exact sequence in $mod\,\Lambda$.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[rr, "\alpha_0"] \arrow[dr, equal] & & X_1 \arrow[rr, "\alpha_1"] \arrow[dr, "\beta_1"] & & \dots \arrow[rr, "\alpha_{n-2}"] \arrow[dr, "\beta_{n-2}"] & & X_{n-1} \arrow[rr, "\alpha_{n-1}"] \arrow[dr, "\beta_{n-1}"] & & X_n \arrow[r] & 0 \\
& & C_1 \arrow[ur, "\gamma_1"'] & & C_2 \arrow[ur, "\gamma_2"'] & & C_{n-1} \arrow[ur, "\gamma_{n-1}"'] & & C_n \arrow[ur, equal] & &
\end{tikzcd}$$

satisfying the following:

(i) For all $1 \leq k \leq n$, $X_k \in \mathcal{C}$.

(ii) For all $1 \leq k \leq n-1$ the morphism $\gamma_k : C_k \to X_k$ is a left $\mathcal{C}$-approximation.

(iii) For all $1 \leq k \leq n-1$ the morphism $\beta_k : X_k \to C_{k+1}$ is a cokernel of γ_k.

(iv) For all $X \in \mathcal{C}$, the induced sequence of abelian groups

$$0 \longrightarrow Hom_\Lambda(X_n, X) \longrightarrow \dots \longrightarrow Hom_\Lambda(X_1, X) \longrightarrow Hom_\Lambda(A, X) \longrightarrow 0$$

is exact.

Proof. Since $\mathcal{C}$ is functorially finite, there is a left $\mathcal{C}$-approximation $\alpha_0 = \gamma_1 : A = C_1 \to X_1$. Also, α_0 is a monomorphism, because $\mathcal{C}$ is generating and cogenerating, see 3.1.4.

Since $mod\,\Lambda$ is abelian, there is a cokernel C_2 of γ_1, and thus, we have an exact sequence

$$0 \to A = C_1 \xrightarrow{\gamma_1} X_1 \xrightarrow{\beta_1} C_2 \to 0.$$

Although C_2 may not be in $\mathcal{C}$, there is a left $\mathcal{C}$-approximation γ_2. Following this construction, conditions $(i), (ii)$ and (iii) are satisfied.

Here, each $\gamma_i : C_i \to X_i$ is a left-$\mathcal{C}$ approximation and thus, is a monomorphism, and each $\beta_i : X_i \to C_{i+1}$ is a a cokernel and therefore, an epimorphism. Since

$$0 \to C_i \xrightarrow{\gamma_i} X_i \xrightarrow{\beta_i} C_{i+1} \to 0$$

is an exact sequence in $mod\,\Lambda$, and as $\alpha_i = \gamma_{i+1}\beta_i$, we have that $Ker\,\alpha_i = \{x \mid \gamma_{i+1}\beta_i(x) = 0\} = Ker\,\beta_i = Im\,\gamma_i$, because the sequence is exact. Because β_{i-1} is an epimorphism, $Im\,\alpha_{i-1} = Im\,\gamma_i$, and thus $Ker\,\alpha_i = Im\,\alpha_{i-1}$. Therefore, consider the exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_2} \quad \dots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n \longrightarrow 0$$

We will prove that for all $X \in \mathcal{C}$, the induced sequence of abelian groups

$$0 \longrightarrow Hom_\Lambda(X_n, X) \xrightarrow{\_\circ\alpha_{n-1}} \dots \xrightarrow{\_\circ\alpha_1} Hom_\Lambda(X_1, X) \xrightarrow{\_\circ\alpha_0} Hom_\Lambda(A, X) \longrightarrow 0$$

is exact.

First, let $g \in Ker(\_\circ\alpha_i) \subset Hom_\Lambda(X_{i+1}, X)$, that is $g\alpha_i = 0 \Leftrightarrow g\gamma_{i+1}\beta_i = 0$. Since β_i is a cokernel, thus an epimorphism, it follows that $g\gamma_{i+1} = 0$.

$$\begin{tikzcd}
 & X & \\
 & X_{i+1} \arrow[u, "g"'] \arrow[dr, "\beta_{i+1}"] & \\
C_{i+1} \arrow[uur, "0"] \arrow[ur, "\gamma_{i+1}"'] & & C_{i+2}
\end{tikzcd}$$

Since β_{i+1} is a cokernel of γ_{i+1}, there is a unique morphism $g' : C_{i+2} \to X$ such that $g'\beta_{i+1} = g$. Then, as $\gamma_{i+2} : C_{i+2} \to X_{i+2}$ is a left $\mathcal{C}$-approximation and $X \in \mathcal{C}$, there is a morphism $\alpha : X_{i+2} \to X$ such that $g' = \alpha\gamma_{i+2}$. Therefore, $g = g'\beta_{i+1} = \alpha\gamma_{i+2}\beta_{i+1} = \alpha\alpha_{i+1}$, meaning $g \in Im(\_\circ\alpha_{i+1})$.

It is simple to see that $\_\circ\alpha_{n-1}$ is injective: Let $g \in Hom_\Lambda(X_n, X)$ such that $g \circ \alpha_{n-1} = 0$. Then, $g \circ \beta_{n-1} = 0$ and as β_{n-1} is an epimorphism, $g = 0$.

Next, we check that $\_\circ\alpha_0$ is surjective: let $g \in Hom_\Lambda(A, X)$. Since α_0 is a left $\mathcal{C}$-approximation, there is a morphism $g' : X_1 \to X$ such that $g = g' \circ \alpha_0$.

Finally, we need to prove that $C_n \in \mathcal{C}$. In order to do that, we claim that for all $X \in \mathcal{C}$ and for all $k \in \{2, \dots, n\}$, $Ext^i_\Lambda(C_k, X) = 0$, for all $1 \le i \le k-1$.

We prove the claim by induction in k. For $k = 2$, we apply $Hom_\Lambda(\_, X)$ to $0 \longrightarrow A \xrightarrow{f_0} X_1 \xrightarrow{g_1} C_2 \longrightarrow 0$:

$$0 \longrightarrow Hom_\Lambda(C_2, X) \xrightarrow{\_\circ g_1} Hom_\Lambda(X_1, X) \xrightarrow{\_\circ f_0} Hom_\Lambda(A, X) \longrightarrow Ext^1_\Lambda(C_2, X) \longrightarrow Ext^1_\Lambda(X_1, X) = 0$$

because $X_1, X \in \mathcal{C}$. Since $\_\circ\alpha_0$ is also an epimorphism, $Ext^1_\Lambda(C_2, X) = 0$. Suppose now that the claim holds for $l \le k$ and $2 \le k \le n-1$. We apply $Hom_\Lambda(\_, X)$ to $0 \longrightarrow C_k \xrightarrow{\gamma_k} X_k \xrightarrow{\beta_k} C_{k+1} \longrightarrow 0$:

$$\begin{tikzcd}
\dots \arrow[r] & Ext^{i-1}_\Lambda(C_k, X) \arrow[r] \arrow[d, equal] & Ext^i_\Lambda(C_{k+1}, X) \arrow[r] & Ext^i_\Lambda(X_k, X) \arrow[d, equal] \\
 & 0 & & 0
\end{tikzcd}$$

Here, the first is equal to zero by hypothesis and the last because $X_k, X \in \mathcal{C}$. Then, $Ext^i_\Lambda(C_{k+1}, X) = 0$, for all $2 \le i \le k$.

Also, applying $Hom_\Lambda(\_, X)$ to $0 \longrightarrow C_k \xrightarrow{\gamma_k} X_k \xrightarrow{\beta_k} C_{k+1} \longrightarrow 0$, gives

$$0 \longrightarrow Hom_\Lambda(X_k, X) \xrightarrow{\_\circ h_k} Hom_\Lambda(C_k, X) \longrightarrow Ext^1_\Lambda(C_{k+1}, X) \longrightarrow Ext^1_\Lambda(X_k, X) = 0.$$

Since $\_ \circ \gamma_k$ is an epimorphism, it follows that $Ext^1_\Lambda(C_{k+1}, X) = 0$.

In conclusion, for all $X \in \mathcal{C}$, $Ext^i_\Lambda(C_n, X) = 0$, for all $1 \leq i \leq n-1$, meaning that $C_n \in \mathcal{C}$. $\square$

Lemma 3.2.2. *(Jas16) Let $M \in mod\,\Lambda$ and $\mathcal{A}$ be a subcategory of $mod\,\Lambda$, such that $Ext^i_\Lambda(\mathcal{A}, M) = 0$ for all $0 \leq i \leq n-1$. Consider an exact sequence*

$$X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \dots \longrightarrow X_{n-1} \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} N \longrightarrow 0$$

in $mod\,\Lambda$ such that $X_j \in \mathcal{A}$, for all $1 \leq j \leq n$. Then, for each $k \in \{0, \dots, n-1\}$, there is an isomorphism between $Ext^k_\Lambda(N, M)$ and the homology of the induced complex

$$Hom_\Lambda(X_n, M) \longrightarrow Hom_\Lambda(X_{n-1}, M) \longrightarrow \dots \longrightarrow Hom_\Lambda(X_1, M) \longrightarrow Hom_\Lambda(X_0, M) \tag{3.1}$$

at $Hom_\Lambda(X_{n-k}, M)$.

Proof. Let $C_j := coker\,\alpha_{j-1}$, where $\alpha_{j-1} : X_{j-1} \to X_j$, $1 \leq j \leq n$, notice $C_n = coker\,\alpha_{n-1} = N$. Since the sequence is exact, $\alpha_j\alpha_{j-1} = 0$, then, there is a unique $i_j : C_j \to X_{j+1}$, with $\alpha_j = i_j \circ p_j$, where $p_j : X_j \to C_j$ is the projection on the cokernel. We will prove the lemma by induction on k. For $k = 0$, we have $Ext^0_\Lambda(N, M) = Hom_\Lambda(N, M)$. And, $\mathcal{H}(Hom_\Lambda(X_n, M)) = Ker(\_ \circ \alpha_{n-1})$. Since the sequence

$$0 \longrightarrow Hom_\Lambda(N, M) \xrightarrow{\_ \circ \alpha_n} Hom_\Lambda(X_n, M) \xrightarrow{\_ \circ \alpha_{n-1}} Hom_\Lambda(X_{n-1}, M)$$

is exact, we have $Ker(\_ \circ \alpha_{n-1}) = Im(\_ \circ \alpha_n) \simeq Hom_\Lambda(N, M)$.

Now, we claim that, for all $k \in \{1, \dots, n-1\}$, $Ext^k_\Lambda(C_n, M) \simeq Ext^{k-1}_\Lambda(C_{n-1}, M) \simeq \dots \simeq Ext^1_\Lambda(C_{n-k+1}, M)$. Indeed, for $k = 1$, we get $Ext^1_\Lambda(C_n, M) = Ext^1_\Lambda(C_{n-1+1}, M) = Ext^1_\Lambda(C_n, M)$. Now suppose the claim holds for all $2 \leq l \leq k$. Then, we apply $Hom_\Lambda(\_, M)$ to the sequence

$$0 \longrightarrow C_{n-k+l-1} \xrightarrow{i_{n-k+l-1}} X_{n-k+l} \xrightarrow{p_{n-k+l}} C_{n-k+l} \longrightarrow 0$$

where $i_{n-k+l-1}$ is a inclusion and p_{n-k+l} is a projection. We get

$$\dots \longrightarrow Ext^{l-1}_\Lambda(X_{n-k+l}, M) \longrightarrow Ext^{l-1}_\Lambda(C_{n-k+l-1}, M) \longrightarrow Ext^l_\Lambda(C_{n-k+l}, M) \longrightarrow Ext^l_\Lambda(X_{n-k+l}, M) \tag{3.2}$$

By hypothesis, $Ext^i(\mathcal{A}, M) = 0$ for all $0 \leq i \leq n-1$, and $X_j \in \mathcal{A}$ for $j \in \{1, \dots, n\}$. Then, $Ext^{l-1}_\Lambda(X_{n-k+l}, M) = Ext^l_\Lambda(X_{n-k+l}, M) = 0$ because $X_{n-k+l} \in \mathcal{A}$. Meaning that $Ext^{l-1}_\Lambda(C_{n-k+l-1}, M) \simeq Ext^l_\Lambda(C_{n-k+l}, M)$, as claimed.

Our second claim is that $Ext^1_\Lambda(C_{n-k+1}, M)$ is isomorphic to the homology of the complex at $Hom_\Lambda(X_{n-k}, M)$. We have the following diagram, obtained by putting together 3.1 and 3.2:

$$\begin{tikzcd}
Hom_\Lambda(X_{n-k+1}, M) \arrow[rr, "_ \circ \alpha_{n-k}"] \arrow[dr, "_ \circ i_{n-k}"'] & & Hom_\Lambda(X_{n-k}, M) \arrow[r, "_ \circ \alpha_{n-k-1}"] & Hom_\Lambda(X_{n-k-1}, M) \\
& Hom_\Lambda(C_{n-k}, M) \arrow[ur, "_ \circ p_{n-k}"] \arrow[dr, "\delta"] & & \\
0 \arrow[ur] & & Ext^1_\Lambda(C_{n-k+1}, M) \arrow[dr] & \\
& & & Ext^1_\Lambda(X_{n-k+1}, M) = 0
\end{tikzcd}$$

Then, $\mathcal{H}(Hom_\Lambda(C_{n-k}, M)) = \dfrac{Ker(\_ \circ \alpha_{n-k-1})}{Im(\_ \circ \alpha_{n-k})}$.

Also, $Ext^1_\Lambda(C_{n-k+1}, M) = \delta Hom_\Lambda(C_{n-k}, M)$, which gives that

$$Ext^1_\Lambda(C_{n-k+1}, M) \simeq \frac{Hom_\Lambda(C_{n-k}, M)}{Ker\,\delta} \simeq \frac{Hom_\Lambda(C_{n-k}, M)}{Im(\_ \circ i_{n-k})} \simeq \frac{Ker(\_ \circ \alpha_{n-k-1})}{Im(\_ \circ \alpha_{n-k})}.$$

Here, $Im(\_ \circ i_{n-k}) \simeq Im(\_ \circ \alpha_{n-k})$, because $\_ \circ \alpha_{n-k} = \_ \circ i_{n-k} \circ p_{n-k}$, since $\alpha_{n-k} = i_{n-k} \circ p_{n-k}$, and $\_ \circ p_{n-k}$ is a monomorphism.

By uniting the two claims, we get that $Ext^k_\Lambda(N, M) \simeq Ext^1_\Lambda(C_{n-k+1}, M) \simeq \mathcal{H}(Hom_\Lambda(X_{n-k}, M))$, as stated. $\square$

Theorem 3.2.3. *(Jas16) Let Λ be a finite dimensional algebra. If $\mathcal{C} \subset mod\,\Lambda$ is an n-cluster tilting subcategory, then, $\mathcal{C}$ is an n-abelian category.*

Proof. We will prove that an n-cluster tilting categories is n-abelian by showing all axioms from Definition 2.1.4.

To check Axiom (A_0), recall that $mod\,\Lambda$ is an abelian category, and consequently, it is idempotent complete. Let $A \in \mathcal{C} \subset mod\,\Lambda$ and $e : A \to A$ be an idempotent. Then, there is a module $B \in mod\,\Lambda$, an epimorphism $r : A \to B$ and a monomorphism $s : B \to A$, such that $e = sr$ and $1_B = rs$. Our aim is to show that $B \in \mathcal{C}$. Since $mod\,\Lambda$ is abelian, r has a kernel K and s has a cokernel C:

$$0 \longrightarrow K \longrightarrow A \xrightarrow{r} B \longrightarrow 0$$

$$0 \longrightarrow B \xrightarrow{s} A \longrightarrow C \longrightarrow 0$$

As $rs = 1_B$, s is a section and r is a retraction, which means that the sequences above split. In particular, it means that $A \simeq B \oplus C$. Since $A \in \mathcal{C}$ and Ext^k_Λ is an additive functor,

$$0 = Ext^k_\Lambda(A, \mathcal{C}) = Ext^k_\Lambda(B, \mathcal{C}) \oplus Ext^k_\Lambda(C, \mathcal{C}),$$

$$0 = Ext^k_\Lambda(\mathcal{C}, A) = Ext^k_\Lambda(\mathcal{C}, B) \oplus Ext^k_\Lambda(\mathcal{C}, C).$$

Therefore, $0 = Ext^k_\Lambda(B, \mathcal{C}) = Ext^k_\Lambda(\mathcal{C}, B)$, and $B \in \mathcal{C}$, for $1 \leq k \leq n-1$.

To prove axiom (A_1), we need to present an n-kernel and n-cokernel for all morphisms in $\mathcal{C}$. Let $f : A \to B \in \mathcal{C}$. Since $mod\,\Lambda$ is abelian, there is a cokernel $p : B \to C$ of f in $mod\,\Lambda$. We apply Lemma 3.2.1 and obtain an n-cokernel $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[rr, "\alpha_0"] \arrow[dr, "p"] & & X_1 \arrow[rr, "\alpha_1"] \arrow[dr] & & X_2 \arrow[r] \arrow[dr] & & \ldots \arrow[rr, "\alpha_{n-1}"] \arrow[dr] & & X_n \\
& & C \arrow[ur] & & C_1 \arrow[ur] & & \ldots \arrow[ur] & & C_n \arrow[ur, equal] &
\end{tikzcd}$$

Dually, it is possible to obtain an n-kernel.

Finally, we prove axiom (A_2): let $\alpha_n : X_n \to X_{n+1}$ be an epimorphism in $\mathcal{C}$ and $(\alpha_0, \ldots, \alpha_{n-1})$ its n-kernel obtained using axiom (A_1). Then, we have the following sequence:

$$X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow 0.$$

We apply Lemma 3.2.2 to this sequence, and get that $\mathcal{H}(Hom_\Lambda(X_{n-k}, M)) = Ext^k_\Lambda(X_{n+1}, M) = 0$, for all $M \in \mathcal{C}$ and for all $1 \leq k \leq n-1$. So,

$$0 \longrightarrow Hom_\Lambda(X_{n+1}, M) \longrightarrow Hom_\Lambda(X_n, M) \longrightarrow \ldots \longrightarrow Hom_\Lambda(X_1, M) \longrightarrow Hom_\Lambda(X_0, M) \longrightarrow 0$$

is exact, which guarantees that $(\alpha_0, \ldots, \alpha_n)$ is an n-exact sequence and that $(\alpha_1, \ldots, \alpha_n)$ is an n-cokernel of α_0. The axiom (A_2^{op}) is proved dually. $\square$

With this, we can conclude that all of our previous examples were of n-abelian categories. With that structure, we want to understand better the n-exact sequences of this subcategories. In order to do that, we will study Higher Auslander Reiten Theory in the next chapter.

Chapter 4

HIGHER AUSLANDER REITEN THEORY AND EXAMPLES

In the classical case, Auslander Reiten theory is a fundamental tool for the study of representation theory of Artin algebras. It was introduced by Maurice Auslander and Idun Reiten (ARS95), in the 1970s and has since been developed by many mathematicians. The theory studies the representation theory of artinian rings using techniques such as Auslander-Reiten sequences and Auslander-Reiten quivers.

The Higher Auslander Reiten theory was introduced by Iyama (Iya07), in the early 2000s and has since been used to study categories that appear in representation theory, such as n-cluster tilting subcategories.

In this last chapter we present some known results about higher Auslander Reiten theory, some examples and give a description to 2-Auslander Reiten sequences of 2-cluster tilting subcategories of Nakayama algebras.

4.1 Higher Auslander Reiten Theory

Higher Auslander Reiten theory is a generalization of the classical Auslander-Reiten theory, in the sense that it studies n-almost split sequences in n-cluster tilting subcategories.

In this section, although not necessary in general, we will consider Λ a finite representation type algebra, and $\mathcal{C}$ is an n-cluster tilting subcategory of $mod\,\Lambda$. We start with two interesting properties of n-exact sequences.

Proposition 4.1.1. *(Fed20) Let* $\mathbf{X}$ *be a sequence in* $\mathcal{C}$

$$\mathbf{X}: \qquad 0 \longrightarrow X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow 0$$

then, $\mathbf{X}$ *is* n*-exact in* $\mathcal{C}$*, if and only if,* $\mathbf{X}$ *is exact in* $mod\,\Lambda$*.*

Proof. First, consider $\mathbf{X}$ an exact sequence in $mod\,\Lambda$. Then, for all $M \in \mathcal{C}$, we will prove that the sequence

$$0 \longrightarrow Hom_\Lambda(M, X_0) \longrightarrow \dots \longrightarrow Hom_\Lambda(M, X_{n+1}) \tag{4.1}$$

is exact. Indeed, first, we need to prove that $Ker\,\alpha_1 \circ \_ = Im\,\alpha_0 \circ \_$. This follows from the exact sequence

$$0 \to X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2$$

that yields the exact sequence

$$0 \longrightarrow Hom_\Lambda(M, X_0) \xrightarrow{\alpha_0 \circ \_} Hom_\Lambda(M, X_1) \xrightarrow{\alpha_1 \circ \_} Hom_\Lambda(M, X_2)$$

Now, consider the following diagram

$$\begin{tikzcd}
\mathbf{X}: & 0 \arrow[r] & X_0 \arrow[rr, "\alpha_0"] \arrow[dr, "p_0"'] & & X_1 \arrow[rr, "\alpha_1"] \arrow[dr, "p_1"] & & X_2 \arrow[r] & \dots \arrow[r] & X_n \arrow[rr, "\alpha_n"] \arrow[dr, "p_n"'] & & X_{n+1} \arrow[r] & 0 \\
& & & C_0 \arrow[ur, "i_0"'] & & C_1 \arrow[ur, "i_1"'] & & & & C_n \arrow[ur, "i_n"'] & &
\end{tikzcd}$$

where $C_j = Im\,\alpha_j = Ker\,\alpha_{j+1}$, for $0 \leq j \leq n$, and $\alpha_j = i_j p_j$.

From the exact sequence

$$0 \to C_1 \xrightarrow{i_1} X_2 \xrightarrow{\alpha_2} X_3$$

we get the exact sequence

$$0 \longrightarrow Hom_\Lambda(M, C_1) \xrightarrow{i_1 \circ \_} Hom_\Lambda(M, X_2) \xrightarrow{\alpha_2 \circ \_} Hom_\Lambda(M, X_3),$$

And, therefore, $Ker\,\alpha_2 \circ \_ = Im\,i_1 \circ \_$.

Then, for the exact sequence

$$0 \to X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{p_1} C_1 \to 0$$

we apply $Hom_\Lambda(M, \_)$, and get

$$0 \longrightarrow Hom_\Lambda(M, X_0) \xrightarrow{\alpha_0 \circ \_} Hom_\Lambda(M, X_1) \xrightarrow{p_1 \circ \_} Hom_\Lambda(M, C_1) \longrightarrow Ext^1_\Lambda(M, X_0) = 0,$$

because $X_0, M \in mod\,\mathcal{C}$. Which means $p_1 \circ \_$ is an epimorphism. So, for all $f \in Hom_\Lambda(M, C_1)$, there is $g \in Hom_\Lambda(M, X_1)$, such that $f = p_1 g$.

Now, a morphism $u \in Hom_\Lambda(M, X_2)$ is in $Im\,i_1 \circ \_$ if and only if $u = i_1 \circ f$, with $f \in Hom_\Lambda(M, C_1)$, if and only if $u = i_1 p_1 g$, $g \in Hom_\Lambda(M, X_1)$. Since $i_1 p_1 = \alpha_1$, $u \in Im\,\alpha_1 \circ \_$, and it follows that $Im\,i_1 \circ \_ \subseteq Im\,\alpha_1 \circ \_$.

On the other side, since $\alpha_1 \circ \_ = (i_1 \circ \_) \circ (p_1 \circ \_)$, it follows that $Im\,\alpha_1 \circ \_ \subseteq Im\,i_1 \circ \_$. Then, $Im\,\alpha_1 \circ \_ = Im\,i_1 \circ \_ = Ker\,\alpha_2 \circ \_$.

Recursively, we prove that $Im\,\alpha_j \circ \_ = Im\,i_j \circ \_ = Ker\,\alpha_{j+1} \circ \_$, for all $j \in \{0, \dots, n-1\}$, and 4.1 is an exact sequence. This means that $(\alpha_0, \dots, \alpha_{n-1})$ is an n-kernel of α_n. And dually, since the sequence

$$0 \longrightarrow Hom_\Lambda(X_{n+1}, M) \longrightarrow \dots \longrightarrow Hom_\Lambda(X_0, M)$$

is exact, $(\alpha_1, \dots, \alpha_n)$ is an n-cokernel of α_0. Therefore, $\mathbf{X}$ is n-exact in $\mathcal{C}$.

Now, consider $\mathbf{X}$ an n-exact sequence in $\mathcal{C}$. We will prove that $\mathbf{X}$ is exact in $mod\,\Lambda$, that is, $Ker\,\alpha_j = Im\,\alpha_{j-1}$, for all $1 \leq j \leq n$.

Since $\mathbf{X}$ is n-exact, we already have that $\alpha_{j-1} \circ \alpha_j = 0$ for all $1 \leq j \leq n$, meaning that $Im\, \alpha_{j-1} \subseteq Ker\, \alpha_j$. Next, take $M \in \mathcal{C}$ and an epimorphism $u : M \to Ker\, \alpha_j$. This epimorphism exists because $\mathcal{C}$ is functorially finite. Then, we have an exact sequence

$$M \xrightarrow{v_j} X_j \xrightarrow{\alpha_j} X_{j+1}$$

in $mod\, \Lambda$, where v_j is the composition of u and the inclusion $Ker\, \alpha_j \to X_j$. Then, $Ker\, \alpha_j = Im\, u = Im\, v_j$ and, because $\mathbf{X}$ is n-exact, the following is exact

$$Hom_\Lambda(M, X_{j-1}) \xrightarrow{\alpha_{j-1} \circ \_} Hom_\Lambda(M, X_j) \xrightarrow{\alpha_j \circ \_} Hom_\Lambda(M, X_{j+1}).$$

Also, $\alpha_j \circ v = 0$, because $Im\, v_j = Ker\, \alpha_j$, then there is a morphism $w : M \to X_{j-1}$ such that $v = \alpha_{j-1} \circ w$. In other words, $Ker\, \alpha_j = Im\, v_j = Im\, (\alpha_{j-1} \circ w) \subseteq Im\, \alpha_{j-1}$.

Finally, by definition of n-kernel and n-cokernel, we know that α_0 is the kernel of α_1 and α_n is the cokernel of α_{n-1}. So, α_0 is a monomorphism and α_n is an epimorphism. In conclusion, $\mathbf{X}$ is exact in $\mod \Lambda$. $\square$

Lemma 4.1.2. *Let $\mathbf{X}$ an n-exact sequence in $\mathcal{C}$*

$$0 \longrightarrow X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow 0$$

and denote $K_j = Ker\, \alpha_j$, for $1 \leq j \leq n$. Then, for all $Y \in \mathcal{C}$,

$$Ext^{n-1}_\Lambda(Y, K_2) \simeq \cdots \simeq Ext^1_\Lambda(Y, K_n).$$

Proof. First, since $Ker\, \alpha_j = Im\, \alpha_{j-1}$, consider the following induced exact sequence

$$0 \longrightarrow K_{n-1} \xrightarrow{i_{n-1}} X_{n-1} \xrightarrow{\overline{\alpha_{n-1}}} K_n \longrightarrow 0.$$

Then, applying $Hom_\Lambda(Y, \_)$ to it, we get

$$\begin{array}{l} 0 \longrightarrow Hom_\Lambda(Y, K_{n-1}) \longrightarrow Hom_\Lambda(Y, X_{n-1}) \longrightarrow Hom_\Lambda(Y, K_n) \longrightarrow Ext^1_\Lambda(Y, K_{n-1}) \\ \longrightarrow \underset{\underset{0}{\|}}{Ext^1_\Lambda(Y, X_{n-1})} \longrightarrow Ext^1_\Lambda(Y, K_n) \longrightarrow Ext^2_\Lambda(Y, K_{n-1}) \longrightarrow \underset{\underset{0}{\|}}{Ext^2_\Lambda(Y, X_{n-1})} \end{array}$$

Then, $Ext^1_\Lambda(Y, K_n) \simeq Ext^2_\Lambda(Y, K_{n-1})$. The result follows recursively. $\square$

Proposition 4.1.3. *(Fed20) Let $\mathbf{X}$ an n-exact sequence in $\mathcal{C}$.*

$$0 \longrightarrow X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow 0.$$

Then, $\mathbf{X}$ induces exact sequences

$$0 \longrightarrow Hom_\mathcal{C}(Y, X_0) \longrightarrow \dots \longrightarrow Hom_\mathcal{C}(Y, X_n) \longrightarrow Hom_\mathcal{C}(Y, X_{n+1}) \longrightarrow Ext^n_\Lambda(Y, X_0) \longrightarrow Ext^n_\Lambda(Y, X_1)$$

$$0 \longrightarrow Hom_\mathcal{C}(X_{n+1}, Y) \longrightarrow \dots \longrightarrow Hom_\mathcal{C}(X_1, Y) \longrightarrow Hom_\mathcal{C}(X_0, Y) \longrightarrow Ext^n_\Lambda(X_{n+1}, Y) \longrightarrow Ext^n_\Lambda(X_n, Y)$$

for any $Y \in \mathcal{C}$.

Proof. We apply $Hom_\Lambda(Y, \_)$ to $\mathbf{X}$:

$$0 \longrightarrow Hom_\Lambda(Y, X_0) \longrightarrow \ldots \longrightarrow Hom_\Lambda(Y, X_n) \longrightarrow Hom_\Lambda(Y, X_{n+1}). \tag{4.2}$$

Since $\mathcal{C}$ is n-cluster tilting, $Ext^j_\Lambda(Y, X_k) = 0$, for all $1 \leq j \leq n-1$ and all $0 \leq k \leq n+1$.

Let $K = Ker\,\alpha_n$, and consider the exact sequence $0 \to K \to X_n \to X_{n+1} \to 0$. We apply $Hom_\Lambda(Y, \_)$ to it:

$$0 \longrightarrow Hom_\Lambda(Y, K) \longrightarrow Hom_\Lambda(Y, X_n) \longrightarrow Hom_\Lambda(Y, X_{n+1}) \longrightarrow Ext^1_\Lambda(Y, K) \longrightarrow 0 \tag{4.3}$$

Analogously, we apply $Hom_\Lambda(Y, \_)$ to the sequence $0 \to X_0 \to X_1 \to C \to 0$, where $C = Coker\,\alpha_0$, and get

$$\begin{array}{l} 0 \longrightarrow Hom_\Lambda(Y, X_0) \longrightarrow Hom_\Lambda(Y, X_1) \longrightarrow Hom_\Lambda(Y, C) \longrightarrow \ldots \\ \longrightarrow Ext^{n-1}_\Lambda(Y, C) \longrightarrow Ext^n_\Lambda(Y, X_0) \longrightarrow Ext^n_\Lambda(Y, X_1). \end{array} \tag{4.4}$$

We claim $Ext^{n-1}_\Lambda(Y, C) \simeq Ext^1_\Lambda(Y, K)$. Indeed, by Lemma 4.1.2, we know that $Ext^{n-1}_\Lambda(Y, K_2) \simeq Ext^1_\Lambda(Y, K)$, where $K_2 = Ker\,\alpha_2$. Then, we need to prove that $K_2 \simeq C$, which is very simple considering the isomorphism theorem:

$$C \simeq \frac{X_1}{Im\,\alpha_0} \simeq \frac{X_1}{Ker\,\alpha_1} \simeq Im\,\alpha_1 \simeq Ker\,\alpha_2 = K_2.$$

By connecting 4.2, a part of 4.3 and the second part of 4.4 using the isomorphism, we get

$$\begin{tikzcd} \ldots \arrow[r] & Hom_\Lambda(Y, X_n) \arrow[r] & Hom_\Lambda(Y, X_{n+1}) \arrow[rr] \arrow[dr] & & Ext^n_\Lambda(Y, X_0) \arrow[r] & Ext^n_\Lambda(Y, X_1) \\ & & & Ext^{n-1}_\Lambda(Y, C) \arrow[ur] & & \end{tikzcd}$$

Dually, we construct the second exact sequence. $\square$

Now, we establish the higher analogue of Auslander-Reiten sequences. This notion was first introduced by Iyama (Iya07).

Definition 4.1.4. *An n-exact sequence*

$$\mathbf{X}: \qquad 0 \longrightarrow X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow 0$$

in $\mathcal{C}$ is called **n-*Auslander Reiten sequence***, *or n-almost split sequence, if α_0 is left almost split in $\mathcal{C}$, α_n is right almost split in $\mathcal{C}$, and $\alpha_k \in rad_\mathcal{C}$ for $k \in \{1, \ldots, n-1\}$.*

Remark 4.1.5. *We will prove that when $\mathbf{X}$ as above is an n-Auslander Reiten sequence, then X_0 and X_{n+1} are indecomposable. Indeed, assume towards a contradiction $X_0 = X' \oplus X''$ and consider $p' : X_0 \to X'$ and $p'' : X_0 \to X''$ be the canonical projections. Since X_0 is decomposable, p' and p'' are not sections, otherwise they would be isomorphisms. Then, as α_0 is left almost split, there are morphisms $u' : X_1 \to X'$ and $u'' : X_1 \to X''$, such that $u'\alpha_0 = p'$ and $u''\alpha_0 = p''$. But then,*

$$u = \begin{pmatrix} u' \\ u'' \end{pmatrix} : X_1 \to X_0$$

satisfies $u\alpha_0 = 1_{X_0}$, which is a contradiction, as α_0 is not a section.

Similarly, we prove that X_{n+1} is indecomposable.

Thus, $End_\Lambda(X_0)$ and $End_\Lambda(X_{n+1})$ are local. Now, for any $\beta : X_1 \to X_0$, $\beta\alpha_0 \in End_\Lambda(X_0)$. As $End_\Lambda(X_0)$ is local, $\beta\alpha_0$ or $1-\beta\alpha_0$ is invertible. Since α_0 is not a section, $\beta\alpha_0$ is not invertible, so $1-\beta\alpha_0$ is invertible. By definition of radical, $\alpha_0 \in rad_{\mathcal{C}}$. Dually, $\alpha_n \in rad_{\mathcal{C}}$.

The next result helps us to identify these sequences, by giving alternative characterizations.

Lemma 4.1.6. *(Fed20) For an n-exact sequence in $\mathcal{C}$*

$$\mathbf{X}: \qquad 0 \longrightarrow X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow 0,$$

the following are equivalent:

(a) $\mathbf{X}$ is an n-Auslander Reiten sequence in $\mathcal{C}$

(b) $\alpha_0, \alpha_1, \dots, \alpha_{n-1} \in rad_{\mathcal{C}}$ and α_n is right almost split in $\mathcal{C}$

(c) $\alpha_1, \dots, \alpha_{n-1}, \alpha_n \in rad_{\mathcal{C}}$ and α_0 is left almost split in $\mathcal{C}$.

Proof. By definition of n-Auslander Reiten sequence and by Remark 4.1.5, we have $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$.

Suppose (b) holds. To conclude that $\mathbf{X}$ is an n-Auslander Reiten sequence, we need to prove that α_0 is left almost split.

Since α_n is right almost split, α_n is not a split epimorphism, which means, by Corollary 2.2.2, that α_0 is not a split monomorphism.

Let $u : X_0 \to W$ be a morphism in $\mathcal{C}$, such that $u \neq v \circ \alpha_0$, for all $v : X_1 \to W$ in $\mathcal{C}$. We will prove that u is a split monomorphism. Indeed, consider the following diagram, obtained by the n-pushout of $\mathbf{X}$ along u and completed as in Theorem 2.3.7:

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "u"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "u_1"'] & \dots \arrow[r, "\alpha_{n-1}"] & X_n \arrow[r, "\alpha_n"] \arrow[d, "u_n"'] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W \arrow[r, "g_0"] & W_1 \arrow[r, "g_1"] & \dots \arrow[r, "g_{n-1}"] & W_n \arrow[r, "g_n"] & X_{n+1} \arrow[r] & 0
\end{tikzcd}$$

The morphism g_0 is not a split monomorphism, because if it was, there would exist a morphism $v : X_1 \to W$ such that $v = (g_0)^{-1}u_1$ and then, $u = v \circ \alpha_0$. So, g_n is not a split epimorphism. Since α_n is right almost split, there is a morphism $h_n : W_n \to X_n$ such that $g_n = \alpha_n h_n$. By Lemma 2.2.4, h_n can be extended to a morphism of n-exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] \arrow[d, "u"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "u_1"'] & \dots \arrow[r, "\alpha_{n-1}"] & X_n \arrow[r, "\alpha_n"] \arrow[d, "u_n"'] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W \arrow[r, "g_0"] \arrow[d, "h_0"] & W_1 \arrow[r, "g_1"] \arrow[d, "h_1"] & \dots \arrow[r, "g_{n-1}"] & W_n \arrow[r, "g_n"] \arrow[d, "h_n"] & X_{n+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_0 \arrow[r, "\alpha_0"] & X_1 \arrow[r, "\alpha_1"] & \dots \arrow[r, "\alpha_{n-1}"] & X_n \arrow[r, "\alpha_n"] & X_{n+1} \arrow[r] & 0
\end{tikzcd}$$

Note that $\alpha_n h_n u_n = \alpha_n$. Since α_n is right minimal, it follows that $h_n u_n$ is an isomorphism. Then, by Lemma 2.2.6, $h_0 u$ is an isomorphism, and consequently, u is a split monomorphism.

Hence, α_0 is left almost split, and we proved that $(b) \Rightarrow (a)$. Dually, it is possible to prove that $(c) \Rightarrow (a)$. □

The next theorem guarantees the existence of n-Auslander Reiten sequences in n-cluster tilting categories.

Theorem 4.1.7. *(Iya07) Let $\mathcal{C}$ be an n-cluster tilting subcategory of $\mathcal{C}$. Then,*

(i) For any non-projective $X_{n+1} \in ind\,\mathcal{C}$, there exists an n-Auslander Reiten sequence $0 \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow 0$ in $\mathcal{C}$.

(ii) For any non-injective $X_0 \in ind\,\mathcal{C}$, there exists an n-Auslander Reiten sequence $0 \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow 0$ in $\mathcal{C}$.

(iii) Any n-Auslander Reiten sequence $0 \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow 0$ satisfies $X_0 \simeq \tau_n X_{n+1}$ and $X_{n+1} \simeq \tau_n^- X_0$.

Analogously to the $n = 1$ case, there is an n-Auslander Reiten duality.

Theorem 4.1.8 ((Iya07), (Fed20)). *For all $A, B \in \mathcal{C}$ there are functorial isomorphisms*

$$Ext^n_\Lambda(A, B) \simeq D\underline{Hom}_\Lambda(\tau_n^{-1} N, M) \simeq D\overline{Hom}_\Lambda(N, \tau_n M).$$

In the next example, we will use these isomorphisms in the construction of n-cluster tilting subcategories.

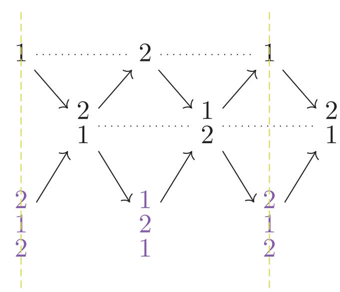
Example 4.1.9. *It is possible to prove that the group algebra KS_3, where S_3 is the symmetric group and $char(K) = 3$ is isomorphic to the path algebra of the quiver Q (shown bellow) module the ideal generated by $\{\alpha\beta\alpha, \beta\alpha\beta\}$.*

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

This isomorphism is given by

$$\begin{array}{cccc} \phi: & KQ/I & \to & KS_3 \\ & \epsilon_1 & \mapsto & (-1-s) \\ & \epsilon_2 & \mapsto & (-1+s) \\ & \alpha & \mapsto & (1+s-r-sr^2) \\ & \beta & \mapsto & (1+s-r^2-sr) \end{array}$$

where $s = \begin{pmatrix} 1 & 2 \end{pmatrix}$ *and* $r = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \in S_3$. *Also, its Auslander-Reiten quiver is given by*



where the purple modules are projective and injective and the yellow lines show the gluing of the quiver.

This category has no 2-cluster tilting subcategories, and we will prove it by looking at $Ext^1_\Lambda(A,B)$ *and the formula from Theorem 4.1.8. First notice that*

$$Ext^1_\Lambda\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix},M\right)=Ext^1_\Lambda\left(M,\begin{smallmatrix}2\\1\\2\end{smallmatrix}\right)=Ext^1_\Lambda\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix},M\right)=Ext^1_\Lambda\left(M,\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)=0$$

for all $M\in mod\,\Lambda$*, since* $\begin{smallmatrix}2\\1\\2\end{smallmatrix}$ *and* $\begin{smallmatrix}1\\2\\1\end{smallmatrix}$ *are injective and projective. Now, we look at the other modules in the category:*

$$Ext^1_\Lambda(1,1)\simeq D\overline{Hom_\Lambda}(1,\tau 1)\simeq D\overline{Hom_\Lambda}(1,2)=0$$

Since there are no nonzero morphisms $1\to 2$*. Similarly,*

$$Ext^1_\Lambda(2,2)\simeq D\overline{Hom_\Lambda}(2,\tau 2)\simeq D\overline{Hom_\Lambda}(2,1)=0$$

$$Ext^1_\Lambda\left(\begin{smallmatrix}1\\2\end{smallmatrix},\begin{smallmatrix}1\\2\end{smallmatrix}\right)\simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}1\\2\end{smallmatrix},\tau\begin{smallmatrix}1\\2\end{smallmatrix}\right)\simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}1\\2\end{smallmatrix},\begin{smallmatrix}2\\1\end{smallmatrix}\right)=0$$

$$Ext^1_\Lambda\left(\begin{smallmatrix}2\\1\end{smallmatrix},\begin{smallmatrix}2\\1\end{smallmatrix}\right)\simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}2\\1\end{smallmatrix},\tau\begin{smallmatrix}2\\1\end{smallmatrix}\right)\simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}2\\1\end{smallmatrix},\begin{smallmatrix}1\\2\end{smallmatrix}\right)=0$$

On the other hand,

$$Ext^1_\Lambda(1,2)\simeq D\overline{Hom_\Lambda}(2,\tau 1)\simeq D\overline{Hom_\Lambda}(2,2)\neq 0$$

since $2\xrightarrow{id}2$ *does not factor through an injective module and is not zero. In the same way,*

$$Ext^1_\Lambda(2,1)\simeq D\overline{Hom_\Lambda}(1,1)\neq 0$$

$$Ext^1_\Lambda\left(\begin{smallmatrix}2\\1\end{smallmatrix},\begin{smallmatrix}1\\2\end{smallmatrix}\right)\simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}1\\2\end{smallmatrix},\begin{smallmatrix}1\\2\end{smallmatrix}\right)\neq 0$$

$$Ext^1_\Lambda\left(\begin{smallmatrix}1\\2\end{smallmatrix},\begin{smallmatrix}2\\1\end{smallmatrix}\right)\simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}2\\1\end{smallmatrix},\begin{smallmatrix}2\\1\end{smallmatrix}\right)\neq 0$$

And

$$Ext^1_\Lambda\left(1,\begin{smallmatrix}1\\2\end{smallmatrix}\right)\simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}1\\2\end{smallmatrix},2\right)=0$$

$$Ext^1_\Lambda\left(1,\begin{smallmatrix}2\\1\end{smallmatrix}\right)\simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}2\\1\end{smallmatrix},2\right)\neq 0$$

And following this procedure,

$$\begin{array}{ll} Ext^1_\Lambda\left(2,\begin{smallmatrix}1\\2\end{smallmatrix}\right)\neq 0 & Ext^1_\Lambda\left(2,\begin{smallmatrix}2\\1\end{smallmatrix}\right)=0\\ Ext^1_\Lambda\left(\begin{smallmatrix}1\\2\end{smallmatrix},1\right)\neq 0 & Ext^1_\Lambda\left(\begin{smallmatrix}1\\2\end{smallmatrix},2\right)=0\\ Ext^1_\Lambda\left(\begin{smallmatrix}2\\1\end{smallmatrix},1\right)=0 & Ext^1_\Lambda\left(\begin{smallmatrix}2\\1\end{smallmatrix},2\right)\neq 0 \end{array}$$

So, we can conclude that

$$\begin{array}{ll} {}^{\perp_2}1=add\left(1\oplus\begin{smallmatrix}2\\1\end{smallmatrix}\oplus\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right) & 1^{\perp_2}=add\left(1\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)\\ {}^{\perp_2}2=add\left(2\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right) & 2^{\perp_2}=add\left(2\oplus\begin{smallmatrix}2\\1\end{smallmatrix}\oplus\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)\\ {}^{\perp_2}\begin{smallmatrix}1\\2\end{smallmatrix}=add\left(\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1\oplus\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right) & \begin{smallmatrix}1\\2\end{smallmatrix}^{\perp_2}=add\left(\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)\\ {}^{\perp_2}\begin{smallmatrix}2\\1\end{smallmatrix}=add\left(\begin{smallmatrix}2\\1\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right) & \begin{smallmatrix}2\\1\end{smallmatrix}^{\perp_2}=add\left(\begin{smallmatrix}2\\1\end{smallmatrix}\oplus 1\oplus\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right) \end{array}$$

In other words, there is no subcategory $\mathcal{C}$, such that ${}^{\perp_2}\mathcal{C} = \mathcal{C}^{\perp_2}$. Therefore, there are no 2-cluster tilting subcategories.

But, $mod\,\Lambda$ has four different 3-cluster tilting subcategories. To find them, we will use $Ext^2_\Lambda(A,B)$:

$$Ext^2_\Lambda(1,1) \simeq D\overline{Hom_\Lambda}(1,\tau_2 1) \simeq D\overline{Hom_\Lambda}(1,\tau\Omega 1) \simeq D\overline{Hom_\Lambda}\left(1,\tau\ \begin{smallmatrix}2\\1\end{smallmatrix}\right) \simeq D\overline{Hom_\Lambda}\left(1,\ \begin{smallmatrix}1\\2\end{smallmatrix}\right) = 0$$

$$Ext^2_\Lambda\left(1,\ \begin{smallmatrix}1\\2\end{smallmatrix}\right) \simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}1\\2\end{smallmatrix},\tau_2 1\right) \simeq D\overline{Hom_\Lambda}\left(\begin{smallmatrix}1\\2\end{smallmatrix},\ \begin{smallmatrix}1\\2\end{smallmatrix}\right) \neq 0$$

$$Ext^2_\Lambda\left(\begin{smallmatrix}2\\1\end{smallmatrix},1\right) \simeq D\overline{Hom_\Lambda}\left(1,\tau_2\ \begin{smallmatrix}2\\1\end{smallmatrix}\right) \simeq D\overline{Hom_\Lambda}\,(1,1) \neq 0$$

So, ${}^{\perp_3}1 = 1^{\perp_3} = add\left(1 \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)$*, which means,*

$$\mathcal{C}^1_3 = add\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus 1\right)$$

is a 3-cluster tilting subcategory of $mod\,\Lambda$. Doing the same for the other modules, we find the four different 3-cluster tilting subcategories:

$$\mathcal{C}^1_3 = add\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus 1\right) \quad \mathcal{C}^2_3 = add\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus 2\right)$$
$$\mathcal{C}^3_3 = add\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}\right) \quad \mathcal{C}^4_3 = add\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}\right)$$

It also does not have any other n-cluster tilting subcategory, because for all n,

$${}^{\perp_n}add\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix}\right) = add\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix}\right)^{\perp_n} = mod\,KQ/I \neq add\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix}\right)$$

and for $M \in \left\{1, 2,\ \begin{smallmatrix}1\\2\end{smallmatrix},\ \begin{smallmatrix}2\\1\end{smallmatrix}\right\}$*, and $q \geq 3$, $Ext^q(M,M) \neq 0$, because*

$$Ext^3(1,1) \simeq D\overline{Hom_\Lambda}(1,\tau_3 1) \simeq D\overline{Hom_\Lambda}(1,1) \neq 0$$

And for all other M, the same happens. That means M is not in a n-cluster tilting subcategory, with $n \geq 4$.

To study n-Auslander Reiten sequences, it is necessary to understand the morphisms that are in the radical of an n-cluster tilting category. For that, we present two results that characterize them.

Theorem 4.1.10. *(DN22) Let $\mathcal{C}$ be an n-cluster tilting subcategory of $mod\Lambda$, with Λ a finite-representation algebra. Then, $rad^\infty_{\mathcal{C}} = 0$.*

Notice that, since $\mathcal{C}$ is a full subcategory of $mod\,\Lambda$, for any two modules $A, B \in \mathcal{C}$, we have $rad_{\mathcal{C}}(A,B) = rad_\Lambda(A,B)$.

Theorem 4.1.11. *(DN22) Let $\mathcal{C}$ be an n-cluster tilting subcategory of $mod\Lambda$ and $f \in rad_{\mathcal{C}}(A,B)$ be a nonzero morphism with A and B indecomposable modules in $\mathcal{C}$. Then f is a sum of compositions of irreducible morphisms in $\mathcal{C}$ between indecomposable Λ-modules.*

With this idea in mind, we define a higher analogue of the Auslander Reiten quiver.

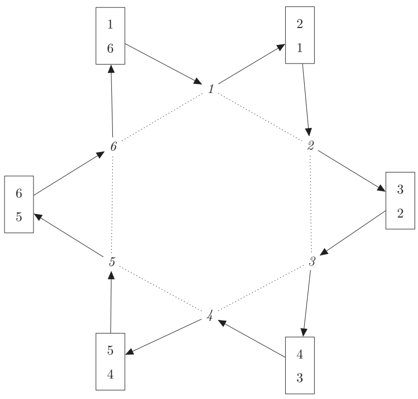
Definition 4.1.12. *We define the* ***n-Auslander Reiten quiver*** $\Gamma(\mathcal{C})$ *of an n-cluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ as follows:*

- *The vertices are the indecomposable and non-isomorphic Λ-modules that are in $\mathcal{C}$.*
- *For M and N, vertices in $\Gamma(\mathcal{C})$, there is an arrow $M \to N$, if there is an irreducible morphism $f : M \to N$ in $\mathcal{C}$.*

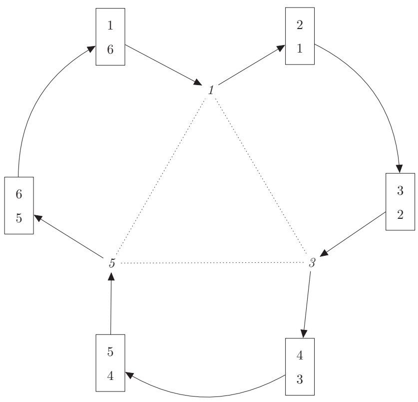
Notice that if there is an arrow $M \to N$ in $\Gamma(\mathcal{C})$, then either there is an arrow from M to N in $\Gamma(\Lambda)$, or there is an irreducible morphism $f : M \to N$ in $\mathcal{C}$, such that $f \neq 0$ and $f = gh$, with $h : M \to L$, $g : L \to N$, and $L \notin \mathcal{C}$.

Similarly to the classical Auslander Reiten quiver, we will place dotted lines to represent τ_n and τ_n^-.

Example 4.1.13. *Recall Example 3.1.10, and the Auslander Reiten quiver of $\operatorname{mod} K\tilde{\mathbb{A}}_6/J^2$:*



Here, we present the 2-Auslander Reiten quiver of $\mathcal{C}_2^1 = add\left(\begin{smallmatrix} 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 5 \end{smallmatrix} \oplus 1 \oplus 3 \oplus 5 \right)$*:*



Notice the vertices are the indecomposable modules of $\mathcal{C}_2^1$. While some arrows are the same as in $\Gamma(\Lambda)$, others are new, like

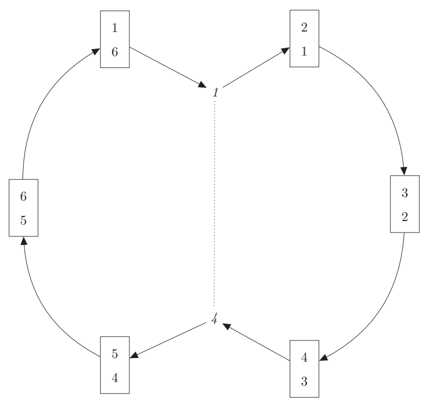
$$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} .$$

This arrow represents an irreducible morphism f in $\mathcal{C}_2^1$, since f does not factor through any other morphism of $\mathcal{C}_2^1$. But in $\operatorname{mod}\Lambda$*, f was not irreducible, as it was a composition*

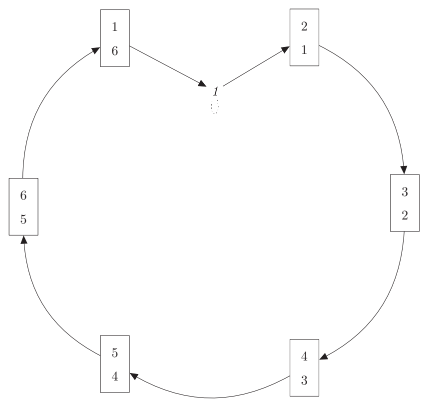
$$\begin{smallmatrix}2\\1\end{smallmatrix} \to 2 \to \begin{smallmatrix}3\\2\end{smallmatrix}.$$

Also, the dotted lines represent τ_2 and τ_2^-, as $\tau_2(5) = 3$, $\tau_2(1) = 5$ and $\tau_2(3) = 1$.

The 3-Auslander Reiten quiver of $\mathcal{C}_3^1 = add\Big(\begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus 1 \oplus 4\Big)$ *is given by*



and the 6-Auslander Reiten quiver of $\mathcal{C}_6^1 = add\Big(\begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\end{smallmatrix} \oplus 1\Big)$*:*



In the special case of Nakayama algebras, and inspired by the study of 2-cluster tilting subcategories of $K\mathbb{A}_m/J^l$ and $K\tilde{\mathbb{A}}_m/J^l$, we give the following result.

Theorem 4.1.14. *Let $\Lambda = K\mathbb{A}_m/J^l$, or $\Lambda = K\tilde{\mathbb{A}}_m/J^l$ such that there is a 2-cluster tilting subcategory $\mathcal{C}$ of* $\operatorname{mod}\Lambda$*. Then,*

$$0 \longrightarrow M \xrightarrow{f} X_1 \xrightarrow{g} X_2 \xrightarrow{h} N \longrightarrow 0$$

is a 2-Auslander Reiten sequence of $\mathcal{C}$, if and only if

(i) $M \simeq \tau_2 N$ *and* $N \simeq \tau_2^- M$*;*

(ii) M *is not injective and* N *is not projective;*

(iii) *The components of* $f : M \to X_1$ *are all the irreducible morphisms of* $\mathcal{C}$ *starting at* M.

(iv) *The components of* $h : X_2 \to N$ *are all the irreducible morphisms of* $\mathcal{C}$ *ending at* N.

(v) *The components of* $g : X_1 \to X_2$ *are sums of compositions of irreducible morphisms of* $\mathcal{C}$.

Proof. Suppose

$$0 \longrightarrow M \xrightarrow{f} X_1 \xrightarrow{g} X_2 \xrightarrow{h} N \longrightarrow 0$$

is a 2-Auslander Reiten sequence. Then, by Theorem 4.1.7, (i), holds. If M was injective, then $M \simeq \tau_2 N = \tau\Omega N$ would imply $0 = \tau^- M \simeq \Omega N$, and then, N would be projective and $\tau\Omega N = \tau 0 = 0$. That is a contradiction, since $M \simeq \tau_2 N$. So, M is not injective. Similarly, N is not projective, which proves (ii).

From Theorem 4.1.11, since $g \in rad_{\mathcal{C}}$, we know the components of g are sums of compositions of irreducible morphisms in $\mathcal{C}$, so, (v) holds.

Finally, we need to prove that the components of the left almost split morphism f are all the irreducible morphisms of $\mathcal{C}$ that start at M.

First, suppose

$$f = \begin{pmatrix} f' \\ f" \end{pmatrix} : M \to X_1 = Y_1 \oplus Y_2,$$

where f' was not irreducible. That means f' was a section or a retraction, or $f' = f_1 f_2$ with f_2 not a section and f_1 not a retraction.

If f is a section, then there is a morphism $u : Y_1 \to M$ such that $uf' = 1_M$, and then,

$$\begin{pmatrix} u & 0 \end{pmatrix} \begin{pmatrix} f' \\ f" \end{pmatrix} = 1_M$$

which means f is a section, a contradiction. If f' is a retraction, then the sequence

$$0 \to Ker\, f' \to M \xrightarrow{f'} Y_1 \to 0$$

splits, and then $M \simeq Y_1 \oplus Ker\, f'$. Since M is indecomposable, $Ker\, f' = 0$, and f' would be a isomorphism. That would contradict the fact that f is left minimal almost split. Finally, if $f' = f_1 f_2$ with f_2 not a section, we have

$$\begin{pmatrix} f' \\ f" \end{pmatrix} = \begin{pmatrix} f_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_2 \\ f" \end{pmatrix}$$

Since f_2 is not a section, by Lemma 1.3.9, $Im\,(\_ \circ f_2) \subseteq rad\, End\, M$, and then, by the same Lemma, $\begin{pmatrix} f_2 \\ f" \end{pmatrix}$ is not a section. As f is irreducible, $\begin{pmatrix} f_1 & 0 \\ 0 & 1 \end{pmatrix}$ is a retraction, thus, f_1 is a retraction. So, f' is irreducible.

Suppose there is a morphism $\phi : M \to U$ in $\mathcal{C}$ that is irreducible. Since ϕ is not a section and f is left almost split, there is a morphism $u : X_1 \to U$, with $\phi = uf$. As ϕ is irreducible and f is not a retraction, it follows that u is a section. Therefore, ϕ is isomorphic to a component of f.

That proves (iii). Dually, (iv) is proven.

For the reverse implication, by Definition 4.1.4, we need to prove that f is left almost split, h is right almost split and $g \in rad_{\mathcal{C}}$.

From Proposition 1.3.6, we get that a composition of p irreducible morphisms is in $rad^p_{\mathcal{C}} \subseteq rad_{\mathcal{C}}$. As g is a composition of irreducible modules, $g \in rad_{\mathcal{C}}$.

Consider now, a morphism $u : M \to U$ in $\mathcal{C}$ that is not a section, where U is indecomposable. Then, since M and U are indecomposable, u is irreducible if and only if $u \in rad_{\mathcal{C}}(M, U) \setminus rad^2_{\mathcal{C}}(M, U)$. Also, $rad_{\mathcal{C}}(M, U) = Hom_{\mathcal{C}}(M, U)$, which means, $u \in rad_{\mathcal{C}}(M, U)$.

If $u \notin rad^2_{\mathcal{C}}(M, U)$, u is irreducible and thus, is a component of f. Then, $u = fu'$, where $u' : X_1 = U \oplus X'_1 \to U$ is such that $u' = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

If $u \in rad^2_{\mathcal{C}}(M, U)$, there are indecomposable modules $V_1, \dots, V_r \in \mathcal{C}$ and morphisms $v_j : M \to V_j$, $w_j : V_j \to U$, such that $u = \sum_{j=1}^{r} w_j v_j$, $v_j \in rad_{\mathcal{C}}(M, V_j)$, and w_j is a sum of composition of irreducible morphisms. Also, by Proposition 1.3.4, each v_j is a sum $v_j = \alpha_j + \beta_j$, with $\beta_j \in rad^{\infty}_{\mathcal{C}}$, and $\alpha_j = \sum_{k=1}^{t_j} d_{j_k}$, where each $d_{j_k} = d^1_{j_k} \dots d^{s_{jk}}_{j_k}$ is a composition of irreducible morphisms. But, from Theorem 4.1.10, $rad^{\infty}_{\mathcal{C}} = 0$. So, $v_j = \alpha_j$ is a sum of compositions of irreducible morphisms. Also, for each j, and each k, the morphism $d^{s_{jk}}_{j_k}$ is a irreducible morphism that starts at M, and therefore, is a component of f. Since $u = \sum_{j=1}^{r} w_j v_j$, it follows that there is a morphism $u' : X_1 \to U$ such that $u = u'f$.

This concludes that f is left almost split. Dually, we prove that h is right almost split. $\square$

4.2 Examples

In this section, we will present some examples that illustrate many of the definitions and results approached during this work, such as the construction of n-cluster tilting subcategories, n-Auslander Reiten sequences, n-pushouts, n-Auslander Reiten quivers and 2-Auslander Reiten sequences of some Nakayama algebras.

To start, we conclude this example that was recurrent throughout the text.

Example 4.2.1. *Concluding Example 3.1.15, we highlight the* 2*-Auslander Reiten sequences of* $\mathcal{C}_2 \subseteq mod\, K\mathbb{A}_7/J^2$. *From 4.1.14, we can see that the first morphism of each of them is the original irreducible morphism from the module category. The last morphism is given in the same way. The middle morphism is a connection morphism given by composing irreducible morphisms.*

$$0 \to 1 \to \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \to 3 \to 0,$$

$$0 \to 3 \to \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \to \begin{smallmatrix} 5 \\ 4 \end{smallmatrix} \to 5 \to 0,$$

$$0 \to 5 \to \begin{smallmatrix} 6 \\ 5 \end{smallmatrix} \to \begin{smallmatrix} 7 \\ 6 \end{smallmatrix} \to 7 \to 0.$$

To properly prove these sequences are 2*-Auslander Reiten sequences, it is necessary to consider Definition 4.1.4. In other words, the first morphism needs to be left almost split in* $\mathcal{C}_2$*, the last morphism needs to be right almost split in* $\mathcal{C}_2$ *and the middle morphism needs to be in the radical of* $\mathcal{C}_2$.

We will analyse only the first sequence, as the others are analogues.

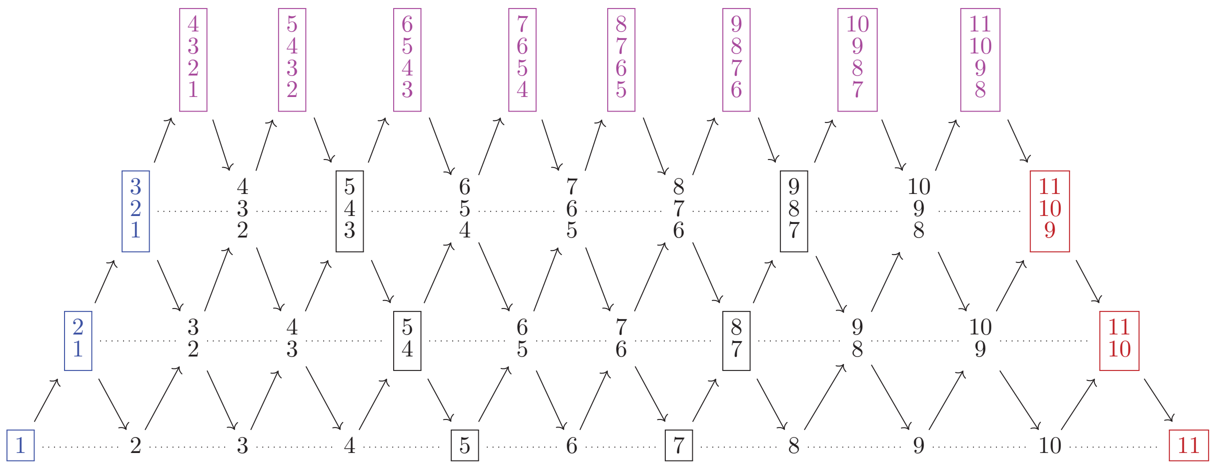
- *The morphism* $1 \to \begin{smallmatrix}2\\1\end{smallmatrix}$ *is left almost split in* $\mathcal{C}_2$ *because it already is so in* $\operatorname{mod} K\mathbb{A}_7/J^2$. *Moreover, it is not a split monomorphism and for all* $M \in \mathcal{C}_2$, *every morphism* $\alpha : 1 \to M$ *which is not a split monomorphism factors through* $1 \to \begin{smallmatrix}2\\1\end{smallmatrix}$, *which is visible from the quiver, as this is the only irreducible morphism starting at* 1.

- *In the same way,* $\begin{smallmatrix}3\\2\end{smallmatrix} \to 3$ *is right almost split in* $\mathcal{C}_2$, *because it is not a split epimorphism and for all* $M \in \mathcal{C}_2$, *every morphism* $\beta : M \to 3$ *that is not a split epimorphism factors through* $\begin{smallmatrix}3\\2\end{smallmatrix} \to 3$.

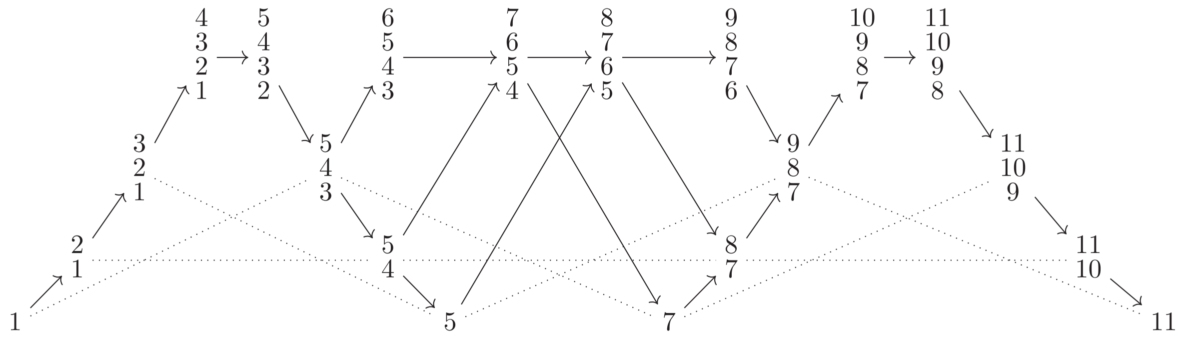
- *Finally, the morphism* $\begin{smallmatrix}2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix}$ *is in the radical of* $\mathcal{C}_2$. *To prove that,we remember that (Proposition 1.3.6) in* $\operatorname{mod} K\mathbb{A}_7/J^2$, *all irreducible morphisms are in* $\operatorname{rad}\left(\operatorname{mod} K\mathbb{A}_7/J^2\right) \setminus \operatorname{rad}^2\left(\operatorname{mod} K\mathbb{A}_7/J^2\right)$, *meaning that a composite of* n *irreducible morphisms is in* $\operatorname{rad}^n\left(\operatorname{mod} K\mathbb{A}_7/J^2\right) \subset \operatorname{rad}\left(\operatorname{mod} K\mathbb{A}_7/J^2\right)$. *Moreover, as stated in Proposition 1.3.3, since* $\begin{smallmatrix}2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix}$ *is not a retraction or a section and* $\begin{smallmatrix}2\\1\end{smallmatrix}$ *and* $\begin{smallmatrix}3\\2\end{smallmatrix}$ *are indecomposable modules, we have that* $\begin{smallmatrix}2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix} \in \operatorname{rad}(\mathcal{C}_2)$.

The next example shows the use of the n-Auslander Reiten quiver to visualize the 2-Auslander Reiten sequences of a subcategory of a Nakayama algebra. In this case, the morphisms may have more than one component.

Example 4.2.2. *Given the algebra* KA_{11} *quotiented by the* 4^{th} *power of its radical, we get the following Auslander-Reiten quiver, where the blue are the projective modules, the red are the injective modules and the purple are both projective and injective.*



The category $\mathcal{C}_2$ *generated by all the boxed modules, their sums and summands is a* 2*-cluster-tilting subcategory. It is obtained in the same way as the one presented before and it allows us to better understand* 2*-Auslander Reiten sequences for Nakayama algebras, as shown in Proposition 4.1.14. In particular,* $\mathcal{C}_2$ *can be seen by its* 2*-Auslander Reiten quiver, where the vertices are the indecomposable nonisomorphic modules, the arrows are the irreducible morphisms of* $\mathcal{C}_2$ *and the dashed lines are* τ_2 *and* τ_2^-:



These are all the 2-Auslander Reiten sequences for $\mathcal{C}_2$:

$$0 \to 1 \to \begin{matrix}2\\1\end{matrix} \to \begin{matrix}5\\4\\3\\2\end{matrix} \to \begin{matrix}5\\4\\3\end{matrix} \to 0,$$

$$0 \to \begin{matrix}2\\1\end{matrix} \to \begin{matrix}3\\2\\1\end{matrix} \to \begin{matrix}5\\4\\3\end{matrix} \to \begin{matrix}5\\4\end{matrix} \to 0,$$

$$0 \to \begin{matrix}3\\2\\1\end{matrix} \to \begin{matrix}4\\3\\2\\1\end{matrix} \to \begin{matrix}5\\4\end{matrix} \to 5 \to 0,$$

$$0 \to \begin{matrix}5\\4\\3\end{matrix} \to \begin{matrix}6\\5\\4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \to \begin{matrix}7\\6\\5\\4\end{matrix} \to 7 \to 0,$$

$$0 \to \begin{matrix}5\\4\end{matrix} \to \begin{matrix}7\\6\\5\\4\end{matrix} \oplus 5 \to \begin{matrix}8\\7\\6\\5\end{matrix} \oplus 7 \to \begin{matrix}8\\7\end{matrix} \to 0,$$

$$0 \to 5 \to \begin{matrix}8\\7\\6\\5\end{matrix} \to \begin{matrix}8\\7\end{matrix} \oplus \begin{matrix}9\\8\\7\\6\end{matrix} \to \begin{matrix}9\\8\\7\end{matrix} \to 0,$$

$$0 \to 7 \to \begin{matrix}8\\7\end{matrix} \to \begin{matrix}11\\10\\9\\8\end{matrix} \to \begin{matrix}11\\10\\9\end{matrix} \to 0,$$

$$0 \to \begin{matrix}8\\7\end{matrix} \to \begin{matrix}9\\8\\7\end{matrix} \to \begin{matrix}11\\10\\9\end{matrix} \to \begin{matrix}11\\10\end{matrix} \to 0,$$

$$0 \to \begin{matrix}9\\8\\7\end{matrix} \to \begin{matrix}10\\9\\8\\7\end{matrix} \to \begin{matrix}11\\10\end{matrix} \to 11 \to 0.$$

It is possible to see that the first and the last morphisms of each sequence are the sum of the irreducible morphisms of $\mathcal{C}_2$ that appear starting or ending at the modules. Also, the middle morphism is a composition of irreducible morphisms, resulting that it lies in the category's radical.

In this example, it is possible to construct a 2-pushout diagram, like Theorem 2.3.7. To start, consider the following 2-exact sequence. Notice this sequence is not a 2-Auslander Reiten sequence.

$$0 \to \begin{matrix}5\\4\\3\end{matrix} \xrightarrow{\alpha_0} \begin{matrix}6\\5\\4\\3\end{matrix} \xrightarrow{\alpha_1} \begin{matrix}9\\8\\7\\6\end{matrix} \xrightarrow{\alpha_2} \begin{matrix}9\\8\\7\end{matrix} \to 0.$$

And let f_0 be the morphism $\begin{matrix}5\\4\\3\end{matrix} \to \begin{matrix}5\\4\end{matrix}$.

We will construct a 2-pushout of $\{\alpha_0, \alpha_1\}$ along f_0. By Definition 2.3.1, we need a 2-cokernel in the mapping cone.

$$\begin{matrix}5\\4\\3\end{matrix} \xrightarrow{\quad\gamma_{-1}\quad} \begin{matrix}6\\5\\4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \xrightarrow{\quad\gamma_0\quad} \begin{matrix}9\\8\\7\\6\end{matrix} \oplus X \xrightarrow{\quad\gamma_1\quad} Y.$$

First, we look at the morphism

$$\begin{matrix}5\\4\\3\end{matrix} \xrightarrow{\gamma_{-1}} \begin{matrix}6\\5\\4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix}.$$

In $\operatorname{mod}\Lambda$, its cokernel would be $\begin{matrix}6\\5\\4\end{matrix}$, but since this module is not in $\mathcal{C}$, we take its left $\mathcal{C}$-approximation: $\begin{matrix}6\\5\\4\end{matrix} \to \begin{matrix}7\\6\\5\\4\end{matrix}$. So, $X = \begin{matrix}7\\6\\5\\4\end{matrix}$, this way, γ_0 is a weak cokernel of γ_{-1}.

Now, we look at γ_0:

$$\begin{matrix}5\\4\\3\end{matrix} \xrightarrow{\quad\gamma_{-1}\quad} \begin{matrix}6\\5\\4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \xrightarrow{\quad\gamma_0\quad} \begin{matrix}9\\8\\7\\6\end{matrix} \oplus \begin{matrix}7\\6\\5\\4\end{matrix}.$$

Its cokernel is $7 \oplus \begin{matrix}9\\8\\7\\6\end{matrix}$, which is in $\mathcal{C}$. So, $Y = 7 \oplus \begin{matrix}9\\8\\7\\6\end{matrix}$. Since γ_1 is the cokernel of γ_0 and γ_0 is a weak cokernel of γ_{-1} it follows that (γ_0, γ_1) is a 2-cokernel of γ_{-1}:

$$\begin{matrix}5\\4\\3\end{matrix} \xrightarrow{\quad\gamma_{-1}\quad} \begin{matrix}6\\5\\4\\3\end{matrix} \oplus \begin{matrix}5\\4\end{matrix} \xrightarrow{\quad\gamma_0\quad} \begin{matrix}9\\8\\7\\6\end{matrix} \oplus \begin{matrix}7\\6\\5\\4\end{matrix} \xrightarrow{\quad\gamma_1\quad} 7 \oplus \begin{matrix}9\\8\\7\\6\end{matrix}.$$

Then, this is a 2-pushout diagram:

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix}5\\4\\3\end{matrix} \arrow[r, "\alpha_0"] \arrow[d, "f_0"] & \begin{matrix}6\\5\\4\\3\end{matrix} \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & \begin{matrix}9\\8\\7\\6\end{matrix} \arrow[d, "f_2"] \\
0 \arrow[r] & \begin{matrix}5\\4\end{matrix} \arrow[r, "\beta_0"'] & \begin{matrix}7\\6\\5\\4\end{matrix} \arrow[r, "\beta_1"'] & 7 \oplus \begin{matrix}9\\8\\7\\6\end{matrix}
\end{tikzcd}$$

Finally, acording to Theorem 2.3.7, it is possible to complete the bottom row, such that f_3 is the identity and the bottom row is 2-exact.

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix}5\\4\\3\end{matrix} \arrow[r, "\alpha_0"] \arrow[d, "f_0"] & \begin{matrix}6\\5\\4\\3\end{matrix} \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & \begin{matrix}9\\8\\7\\6\end{matrix} \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & \begin{matrix}9\\8\\7\end{matrix} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \begin{matrix}5\\4\end{matrix} \arrow[r, "\beta_0"'] & \begin{matrix}7\\6\\5\\4\end{matrix} \arrow[r, "\beta_1"'] & 7 \oplus \begin{matrix}9\\8\\7\\6\end{matrix} \arrow[r, "\beta_2"'] & \begin{matrix}9\\8\\7\end{matrix} \arrow[r] & 0
\end{tikzcd}$$

The following examples explore n-cluster tilting categories in cyclic Nakayama algebras.

Example 4.2.3. *Considering $\tilde{A}_2$ the following quiver*

$$\begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] \end{tikzcd}$$

And $\Lambda = K\tilde{A}_2/J^2$, we get $\Gamma(\Lambda)$ as

$$\begin{tikzcd}[column sep=small] & \boxed{\begin{smallmatrix}2\\1\end{smallmatrix}} \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[dl] \\ & \boxed{\begin{smallmatrix}1\\2\end{smallmatrix}} \arrow[ul] & \end{tikzcd}$$

The interesting thing about the n-cluster tilting subcategories of $\operatorname{mod}\Lambda$ is that they have τ_2-periodic modules. By τ_2-periodic module, we mean that there exists a module M, which has $M = \tau_2(M)$. We see this looking at the 2-cluster tilting subcategories and the 2-Auslander Reiten sequences:

$\mathcal{C}_2^1 = add\left(1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}\right)$, *has the 2-exact sequence:*

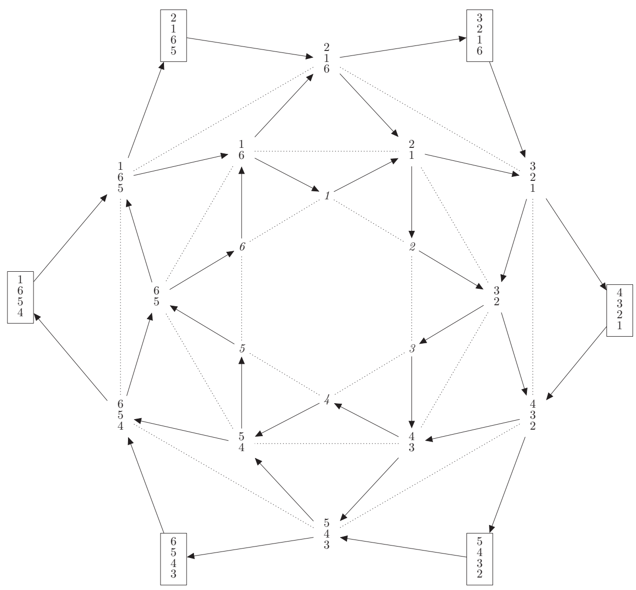
$$0 \to 1 \to \begin{matrix}2\\1\end{matrix} \to \begin{matrix}1\\2\end{matrix} \to 1 \to 0$$

And $\mathcal{C}_2^2 = add\left(2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}\right)$, has the 2-exact sequence

$$0 \to 2 \to \begin{matrix}1\\2\end{matrix} \to \begin{matrix}2\\1\end{matrix} \to 2 \to 0$$

Notice that since Λ is a Nakayama algebra, the 2-exact sequences are in the form of Theorem 4.1.14.

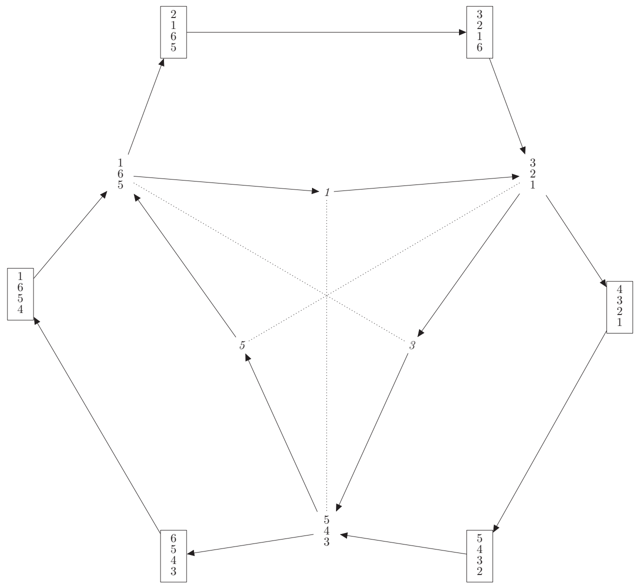
Example 4.2.4. *Let $\Lambda = K\tilde{\mathbb{A}}_6/J^4$ and consider its Auslander Reiten quiver:*



The subcategory

$$\mathcal{C}_2^1 = add\left(\begin{matrix}1\\6\\5\\4\end{matrix} \oplus \begin{matrix}2\\1\\6\\5\end{matrix} \oplus \begin{matrix}3\\2\\1\\6\end{matrix} \oplus \begin{matrix}4\\3\\2\\1\end{matrix} \oplus \begin{matrix}5\\4\\3\\2\end{matrix} \oplus \begin{matrix}6\\5\\4\\3\end{matrix} \oplus \begin{matrix}1\\6\\5\end{matrix} \oplus \begin{matrix}3\\2\\1\end{matrix} \oplus \begin{matrix}5\\4\\3\end{matrix} \oplus 1 \oplus 3 \oplus 5\right)$$

is a 2*-cluster tilting subcategory. Here, we show its* 2*-Auslander Reiten quiver:*



And all of its 2*-Auslander Reiten sequences:*

$$0 \to 1 \to \begin{matrix}3\\2\\1\end{matrix} \to \begin{matrix}5\\4\\3\\2\end{matrix} \oplus 3 \to \begin{matrix}5\\4\\3\end{matrix} \to 0$$

$$0 \to 3 \to \begin{matrix}5\\4\\3\end{matrix} \to \begin{matrix}1\\6\\5\\4\end{matrix} \oplus 5 \to \begin{matrix}1\\6\\5\end{matrix} \to 0$$

$$0 \to 5 \to \begin{matrix}1\\6\\5\end{matrix} \to \begin{matrix}3\\2\\1\\6\end{matrix} \oplus 1 \to \begin{matrix}3\\2\\1\end{matrix} \to 0$$

$$0 \to \begin{matrix}3\\2\\1\end{matrix} \to \begin{matrix}4\\3\\2\\1\end{matrix} \oplus 3 \to \begin{matrix}5\\4\\3\end{matrix} \to 5 \to 0$$

$$0 \to \begin{matrix}5\\4\\3\end{matrix} \to \begin{matrix}6\\5\\4\\3\end{matrix} \oplus 5 \to \begin{matrix}1\\6\\5\end{matrix} \to 1 \to 0$$

$$0 \to \begin{matrix}1\\6\\5\end{matrix} \to \begin{matrix}2\\1\\6\\5\end{matrix} \oplus 1 \to \begin{matrix}3\\2\\1\end{matrix} \to 3 \to 0$$

This is not the unique 2*-cluster tilting subcategory of* $\mathrm{mod}\,\Lambda$*. In fact,* $\mathrm{mod}\,\Lambda$ *has 14 different* 2*-cluster tilting subcategories and 6 different* 5*-cluster tilting subcategories.*

Example 4.2.5. *Although the following quiver is type $\tilde{A}_3$, it sets apart from the examples of other Nakayama algebras because of its quotient. In this example, the admissible ideal is not a power of the radical.*

$$\begin{tikzcd}
 & 1 \arrow[dl, "a"'] & \\
2 \arrow[rr, "b"] & & 3 \arrow[ul, "c"']
\end{tikzcd}$$

Let $\Lambda = K\tilde{A}_3/I$, where $I = \langle cab, bc\rangle$. This is its Auslander Reiten quiver, where blue modules are projective, red modules are injective, purple modules are injective and projective and the yellow lines show the gluing of the quiver.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & {\color{purple}\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \arrow[dr] & & {\color{purple}\begin{smallmatrix}3\\1\\2\end{smallmatrix}} \arrow[dr] & & \\
 & {\color{blue}\begin{smallmatrix}2\\3\end{smallmatrix}} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & {\color{red}\begin{smallmatrix}3\\1\end{smallmatrix}} \arrow[dr] & \\
3 \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[ur] \arrow[rr, dotted, no head] & & 1 \arrow[ur] \arrow[rr, dotted, no head] & & 3
\end{tikzcd}$$

We will apply Definition 3.1.3 to find all possible n-cluster tilting subcategories. From Remark 3.1.4, we know that $add\, M := add\left(\begin{smallmatrix}3\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}\right) \subset \mathcal{C}$. *Now, looking at the definition, we find*

$$^{\perp_2}add\, M = add\,(3 \oplus 2 \oplus M)$$

$$add\, M^{\perp_2} = add\left(1 \oplus 3 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus M\right)$$

As $^{\perp_2}add\, M \neq add\, M^{\perp_2} \neq add\, M$, *we need to include more modules in $\mathcal{C}$. The obvious candidate is 3, because* $3 \in^{\perp_2} add\, M \cap add\, M^{\perp_2}$. *Let $N := 3 \oplus M$. Then,*

$$^{\perp_2}add\, N = add\,(N)$$

$$add\, N^{\perp_2} = add\,(N)$$

Which shows us that $\mathcal{C} = add\, N$ is a 2-cluster tilting subcategory. In fact, we also have that

$$\tau_2^-\left(\begin{smallmatrix}2\\3\end{smallmatrix}\right) = 3 \quad \tau_2^-(3) = \begin{smallmatrix}3\\1\end{smallmatrix}$$
$$\tau_2\left(\begin{smallmatrix}3\\1\end{smallmatrix}\right) = 3 \quad \tau_2(3) = \begin{smallmatrix}2\\3\end{smallmatrix}$$

which determines the bijection between isoclasses of indecomposable nonprojectives and indecomposable non-injectives. This is the 2-Auslander Reiten quiver of $\mathcal{C}$:

$$\begin{tikzcd}
 & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[rr] & & \begin{smallmatrix}3\\1\\2\end{smallmatrix} \arrow[dr] & \\
\begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[drr, bend right, dotted, no head] & & & & \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[dll] \\
 & & 3 \arrow[ull] \arrow[urr, bend right, dotted, no head] & &
\end{tikzcd}$$

With that, the 2-Auslander Reiten sequences of $\mathcal{C}$ are

$$0 \to 3 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}3\\1\\2\end{smallmatrix} \to \begin{smallmatrix}3\\1\end{smallmatrix} \to 0$$

$$0 \to \begin{matrix}2\\3\end{matrix} \to \begin{matrix}1\\2\\3\end{matrix} \to \begin{matrix}3\\1\end{matrix} \to 3 \to 0$$

Finally, one might ask if there is another n, for which there exists an n-cluster tilting subcategory. The answer is affirmative and found within some calculations. Again, $M := \begin{matrix}3\\1\end{matrix} \oplus \begin{matrix}1\\2\\3\end{matrix} \oplus \begin{matrix}3\\1\\2\end{matrix} \oplus \begin{matrix}2\\3\end{matrix}$.

$$^{\perp_3} add\, M = add\,(3 \oplus 2 \oplus M)$$

$$add\, M^{\perp_3} = add\left(1 \oplus \begin{matrix}1\\2\end{matrix} \oplus M\right)$$

As $^{\perp_2} add\, M \neq add\, M^{\perp_2}$ *and* $^{\perp_3} add\, M \cap add\, M^{\perp_3} = add\, M$, $add\, M$ *is not a* 3*-cluster tilting subcategory and there are no candidates to be included. But,*

$$^{\perp_4} add\, M = add\, M,$$

$$add\, M^{\perp_4} = add\, M.$$

Which means that $\mathcal{C}^{'} = add\, M$ *is a* 4*-cluster tilting subcategory.*

$$\begin{tikzcd}
 & \begin{matrix}1\\2\\3\end{matrix} \arrow[r] & \begin{matrix}3\\1\\2\end{matrix} \arrow[dr] & \\
\begin{matrix}2\\3\end{matrix} \arrow[ur] \arrow[rrr, dotted, dash, bend right] & & & \begin{matrix}3\\1\end{matrix} \arrow[lll]
\end{tikzcd}$$

Also, $\tau_4\left(\begin{matrix}3\\1\end{matrix}\right) = \begin{matrix}2\\3\end{matrix}$ *and* $\tau_4^{-}\left(\begin{matrix}2\\3\end{matrix}\right) = \begin{matrix}3\\1\end{matrix}$. *So, the unique* 4*-Auslander Reiten sequence of* $\mathcal{C}'$ *is*

$$0 \to \begin{matrix}2\\3\end{matrix} \to \begin{matrix}1\\2\\3\end{matrix} \to \begin{matrix}3\\1\end{matrix} \to \begin{matrix}2\\3\end{matrix} \to \begin{matrix}3\\1\\2\end{matrix} \to \begin{matrix}3\\1\end{matrix} \to 0.$$

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