UNIVERSIDADE FEDERAL DO PARANÁ

ELIAKIM CLEYTON MACHADO

THE CAUCHY PROBLEM FOR $p-$EVOLUTION OPERATORS WITH DATA IN GEVREY TYPE SPACES

CURITIBA

2025

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Tese apresentada como requisito parcial à obtenção do grau de Doutor em Matemática, no Curso de Pós-Graduação em Matemática, Setor de Ciências Exatas, da Univerisdade Federal do Paraná.

Orientador: Prof. Dr. Fernando de Ávila Silva.
Coorientador: Prof. Dr. Marco Cappiello.

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ATA DE SESSÃO PÚBLICA DE DEFESA DE DOUTORADO PARA A OBTENÇÃO DO GRAU DE DOUTOR EM MATEMÁTICA

No dia trinta de maio de dois mil e vinte e cinco às 10:00 horas, na sala 300, Departamento de Matemática, foram instaladas as atividades pertinentes ao rito de defesa de tese do doutorando **ELIAKIM CLEYTON MACHADO**, intitulada: **THE CAUCHY PROBLEM FOR p-EVOLUTION OPERATORS WITH DATA IN GEVREY TYPE SPACES**. A Banca Examinadora, designada pelo Colegiado do Programa de Pós-Graduação MATEMÁTICA da Universidade Federal do Paraná, foi constituída pelos seguintes Membros: ALEXANDRE KIRILOV (UNIVERSIDADE FEDERAL DO PARANÁ), MARCO CAPPIELLO (UNIVERSIDADE DE TORINO), MARCELO REMPEL EBERT (UNIVERSIDADE DE SÃO PAULO), ALESSIA ASCANELLI (UNIVERSIDADE DE FERRARA), CHIARA BOITI (UNIVERSIDADE DE FERRARA). A presidência iniciou os ritos definidos pelo Colegiado do Programa e, após exarados os pareceres dos membros do comitê examinador e da respectiva contra argumentação, ocorreu a leitura do parecer final da banca examinadora, que decidiu pela APROVAÇÃO. Este resultado deverá ser homologado pelo Colegiado do programa, mediante o atendimento de todas as indicações e correções solicitadas pela banca dentro dos prazos regimentais definidos pelo programa. A outorga de título de doutor está condicionada ao atendimento de todos os requisitos e prazos determinados no regimento do Programa de Pós-Graduação. Nada mais havendo a tratar a presidência deu por encerrada a sessão, da qual eu, ALEXANDRE KIRILOV, lavrei a presente ata, que vai assinada por mim e pelos demais membros da Comissão Examinadora.

CURITIBA, 30 de Maio de 2025.

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TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação MATEMÁTICA da Universidade Federal do Paraná foram convocados para realizar a arguição da tese de Doutorado de **ELIAKIM CLEYTON MACHADO**, intitulada: **THE CAUCHY PROBLEM FOR p-EVOLUTION OPERATORS WITH DATA IN GEVREY TYPE SPACES**, que após terem inquirido o aluno e realizada a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.

A outorga do título de doutor está sujeita à homologação pelo colegiado, ao atendimento de todas as indicações e correções solicitadas pela banca e ao pleno atendimento das demandas regimentais do Programa de Pós-Graduação.

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Dedico este trabalho à memória de meus pais, Ivaldir e Ana Matilde.

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De modo geral, agradeço a todos que estiveram ao meu lado nestes quatro anos de doutorado, um período bastante turbulento e desafiador. Sozinho, jamais teria chegado onde cheguei.

À minha esposa, Fernanda, que me acompanhou e me deu suporte desde a época do mestrado. Saiba que você é meu alicerce e o meu maior motivo para ser feliz.

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"Follow your steps and you will find
the unknown ways are on your mind..."

André Matos

RESUMO

Neste trabalho apresentamos o estudo do problema de Cauchy para operadores de $p-$evolução com coeficientes dependendo de tempo e espaço, com $p \geq 2$, dado por

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = g(x), & x \in \mathbb{R} \end{cases}$$

onde f e g pertencem a certos espaços de Gevrey ou Gelfand-Shilov, P é um operador da forma

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=1}^{p} a_{p-j}(t,x)D_x^{p-j},$$

$a_p(t) \in \mathbb{R}$, os coeficientes a_{p-j} assumem valores complexos e satisfazem condições de decaimento para $|x| \to \infty$. Para provar existência e unicidade de uma solução $u(t,x)$ para este problema em classes de funções ou distribuições convenientes, precisamos fazer uma conjugação em P usando algum operador pseudodiferencial de ordem infinita. Por fim, concluímos que existe uma única solução para este problema, satisfazendo uma estimativa de energia e, consequentemente, este problema é bem posto em alguma classe de Gevrey-Sobolev ou alguma classe de Gelfand-Shilov-Sobolev.

Palavras-chave: equações de $p-$evolução, espaços de Gevrey, espaços de Gelfand-Shilov, operadores pseudodiferenciais.

ABSTRACT

In this work we present the study of the Cauchy problem for $p-$evolution operators with time and space depending coefficients, with $p \geq 2$, given by

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = g(x), & x \in \mathbb{R} \end{cases} \tag{1}$$

where f and g are assumed to belong to some Gevrey or Gelfand-Shilov space, P is an operator of the form

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=1}^{p} a_{p-j}(t,x)D_x^{p-j},$$

$a_p(t) \in \mathbb{R}$, the coefficients a_{p-j} are complex-valued Gevrey regular and satisfy some decay conditions for $|x| \to \infty$. To prove existence and uniqueness of a solution $u(t,x)$ for this problem in suitable classes of functions or distributions, we need to perform a conjugation of P by some special pseudo-differential operator of infinite order. At the end, we conclude that there exists a unique solution for this problem, satisfying an energy estimate and, consequently, this problem is well-posed in a certain Gevrey-Sobolev space or Gelfand-Shilov-Sobolev space.

Keywords: $p-$evolution equations, Gevrey spaces, Gelfand-Shilov spaces, pseudo-differential operators.

CONTENTS

INTRODUCTION

The main concern in this PhD thesis is the study of the Cauchy problem for linear $p-$evolution equations with initial data in Gevrey and in Gelfand-Shilov spaces. Linear $p-$evolution equations are a wide class of PDE introduced by Mizohata in [39] for which the analysis of the Cauchy problem in the Gevrey setting is still, for several aspects, a challenging and open problem. In this thesis we shall focus on equations of order 1 in the time variable and p in the space variables, $p \geq 2$ being a fixed integer, and with real characteristics. Namely, we shall consider equations of the form

$$P(t,x,D_t,D_x)u := D_t u + \sum_{j=0}^{p} \sum_{|\alpha|=j} a_{j,\alpha}(t,x) D_x^\alpha u = f(t,x), \quad t \in [0,T], \; x \in \mathbb{R}^n, \tag{2}$$

where $D := -i\partial$, whose principal symbol in the sense of Petrowski $\tau + \sum_{|\alpha|=p} a_{p,\alpha}\xi^\alpha = 0$ admits a real root. The coefficients $a_{j,\alpha}$ depend in general on (t,x). We notice that for $p = 1$ the equation (2) is strictly hyperbolic, whereas $2-$evolution equations are also called Schrödinger type equations. In fact, the Schrödinger equation $i\hbar\partial_t u = -\frac{\hbar^2}{2m}\Delta u + Vu$, $t \in [0,T]$, $x \in \mathbb{R}^n$, which models the evolution in time of the wave function's state $u(t,x)$ of a quantic particle with mass equals to m, is the most relevant example of $2-$evolution equation. For higher values of p, the importance of $p-$evolution equations is mainly related to the fact that they can be viewed as linearizations of several non-linear equations of physical interest. Namely, assuming for a moment that the coefficients of (3) may depend also on the unknown u, an example of non-linear $3-$evolution equation, for $n = 1$, is the *KdV equation*, which is given by

$$D_t u - \frac{1}{2}\sqrt{\frac{g}{h}}\sigma D_x^3 u + \sqrt{\frac{g}{h}}\left(\alpha + \frac{3}{2}u\right) D_x u = 0, \quad t \in \mathbb{R}, \; x \in \mathbb{R},$$

and describes the wave motion in shallow waters, where $u(t,x)$ stands for the wave elevation with respect to the water level h. Another famous non-linear $3-$evolution equation is the *KdV-Burgers equation*

$$D_t u + 2auD_x u + 5ibD_x u - cD_x^3 u = 0,$$

with $a, b, c \in \mathbb{R}$, which models both the flow of liquids containing gas bubbles and the propagation of waves in an elastic tube containing viscous fluid in [31]. For $p = 5$, we mention the *Kawahara equation*

$$D_t u + uD_x u - aD_x^3 u - bD_x^5 u = 0, \quad a, b > 0,$$

describing magneto-acoustic waves in plasma and long water waves under ice cover (see [36, 38]).

Notice that, in all examples mentioned above, the coefficients are independent of time and space variables t and x, respectively. Actually, in a first approach the coefficients might depend on the variables t and x, but they are approximated by their main value in order to simplify the equations.

Although non-linear equations are out of the goals of this thesis, since linearization is often the first step in the analysis of a non-linear problem, we stress the fact that the results presented here may serve also for future applications in this direction.

Let us now go back to (2). If the dependence is only with the respect to the time variable, the way to deal with the equation is via Fourier transform, while if the coefficients depend on the space variable must be employed some micro-local analysis technique, in particular the pseudo-differential calculus. Another important fact that can be observed about the coefficients in the examples here above is that the lower order terms may appear and they can be either real or complex-valued. As we can see in [16, 23], complex-valued coefficients naturally arise in the study of higher order (in t) evolution equations. Finally, we stress the fact that, except for the case $p = 2$, all the existing results on $p-$evolution equations concern the one space dimensional case, that is $x \in \mathbb{R}$. The general case is still totally unexplored and more involved. For simplicity we also focus here on equations in space dimension 1, which nevertheless is sufficiently motivated by the presence of the above mentioned equations from Mathematical Physics.

After all the previous considerations, we are motivated to study equations of the form $Pu = f$ where

$$P(t, x, D_t, D_x) = D_t + a_p(t)D_x^p + \sum_{j=1}^{p} a_{p-j}(t, x)D_x^{p-j}, \quad (t, x) \in [0, T] \times \mathbb{R}, \tag{3}$$

under the assumption that the leading coefficient a_p is continuous on $[0, T]$, it is real-valued and does not vanish on $[0, T]$, and the lower order coefficients a_{p-j}, for $j = 1, ..., p-1$, belong to the space $C([0, T]; \mathcal{B}^\infty(\mathbb{R}; \mathbb{C}))$, where $\mathcal{B}^\infty(\mathbb{R}; \mathbb{C})$ is the space of all smooth functions on $\mathbb{R}$ which are bounded with all their derivatives. Our goal in this work is to study the Cauchy problem associated with (3), namely

$$\begin{cases} P(t, x, D_t, D_x)u(t, x) = f(t, x), & (t, x) \in [0, T] \times \mathbb{R} \\ u(0, x) = g(x), & x \in \mathbb{R} \end{cases}. \tag{4}$$

More precisely, we intend to study the well-posedness of the problem (4), that is, the existence of a unique solution and the continuous dependence of the solution on the Cauchy data, in suitable classes of functions (which will be the Gevrey and Gelfand-Shilov spaces).

The choice of assuming the coefficient $a_p(t)$ of the leading term independent of x is due to some technical difficulties which will be clarified later, cf. Remark 3.5. Since a_p is real-valued, $\tau = -a_p(t)\xi^p$ is the real root of the principal symbol of P, and the necessary condition $\mathbf{Im}\, a_p(t) \leq 0$, for all $t \in [0, T]$, for well-posedness in $H^\infty(\mathbb{R}) := \cap_{s\in\mathbb{R}} H^s(\mathbb{R})$ given in [40] is satisfied. Thus, if the well-posedness in H^∞ fails, this is not caused by the principal part but by the lower-order terms. It is known that if the lower-order coefficients are all real-valued, then the Cauchy problem (4) is well-posed in all the classical functional settings (L^2, Sobolev spaces, Gevrey type spaces) under suitable decay assumptions on the x-derivatives of a_{p-j} for $|x| \to \infty$.

If some of the lower order terms $a_{p-j}, j = 1, \dots, p-1$, are complex-valued, the problem becomes more challenging. For instance, in the case $p = 2$, a simple computation leads us to

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_{L^2}^2 &= 2\mathbf{Re}\ \langle \partial_t u(t), u(t)\rangle_{L^2} \\
&= 2\mathbf{Re}\ \langle iPu(t), u(t)\rangle_{L^2} \\
&\quad +2\mathbf{Im}\ \left\langle \{a_2(t)D_x^2 + a_1(t,x)D_x + a_0(t,x)\}u(t), u(t)\right\rangle_{L^2} \\
&\leq \|Pu(t)\|_{L^2}^2 + C\|u\|_{L^2}^2 + 2\mathbf{Im}\ \langle a_1(t,x)D_x u(t), u(t)\rangle_{L^2}.
\end{aligned}$$

Once $\mathbf{Im}\, a_1(t,x) \neq 0$, the last term in the right-hand side of the inequality above does not allow to derive an L^2 energy estimate in a straightforward way. Besides, some control on the behaviour of $\mathbf{Im}\ a_1(t,x)$ for $|x| \to \infty$ is necessary. Moreover, it becomes necessary to introduce a suitable change of variable, which will transform the Cauchy problem (4) into an auxiliary Cauchy problem for a new operator of the same form as (3) but with lower-order terms given by positive operators whose contribution can be ignored in the application of the energy method. This technique has been used for the first time in [32] and adapted and generalized in [4, 6, 13, 15, 24]. Of course, there is a price to be paid when we use the mentioned change of variable, that is, some technical difficulties arise, especially in the Gevrey setting, where it requires the use of pseudo-differential operators of infinite order. Before moving on with more details of our work, let us give an overview of results that can be found in the literature about the problem (4) with P as in (3).

About L^2 and H^∞ theory: for $p = 2$, necessary and sufficient conditions for well-posedness in the Sobolev setting have been given in [29, 30, 32, 40]. For the general case $p \geq 2$, in [13, 14] the authors have studied the problem. The semi-linear case is treated in [12]. In particular, by [14], we know that a necessary condition for well-posedness of (4) in $H^\infty(\mathbb{R})$ is the existence of constants $M, N > 0$ such that:

$$\sup_{x\in\mathbb{R}} \min_{0\leq\tau\leq t\leq T} \int_{-\varrho}^{\varrho} \mathbf{Im}\, a_{p-1}(t, x + pa_p(\tau)\theta)d\theta \leq M\log(1+\varrho) + N, \qquad \forall \varrho > 0. \tag{5}$$

In the Gevrey setting, the well-posedness is studied by introducing a suitable scale of Gevrey-Sobolev spaces, namely, for $m \in \mathbb{R}$, $\rho > 0$ and $\theta \geq 1$ the *Gevrey-Sobolev* space $H^m_{\rho;\theta}(\mathbb{R}^n)$ is

$$H^m_{\rho;\theta}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \colon \langle D_x\rangle^m e^{\rho\langle D_x\rangle^{1/\theta}} u \in L^2(\mathbb{R}^n)\},$$

where $\langle D_x\rangle^m$ is the pseudo-differential operator given by the symbol $\langle\xi\rangle^m := (1+|\xi|^2)^{\frac{m}{2}}$ and $e^{\rho\langle D_x\rangle^{1/\theta}}$ is the pseudo-differential operator given by the symbol $e^{\rho\langle\xi\rangle^{1/\theta}}$. Moreover, we set $H^\infty_\theta(\mathbb{R}^n) = \bigcap_{\rho>0} H^m_{\rho;\theta}(\mathbb{R}^n)$ and $\mathcal{H}^\infty_\theta(\mathbb{R}^n) = \bigcup_{\rho>0} H^m_{\rho;\theta}(\mathbb{R}^n)$. Sufficient conditions for well-posedness in $\mathcal{H}^\infty_\theta(\mathbb{R}^n)$ have been studied only in the cases $p = 2$ in [17, 24, 32] and recently in [6] for 3-evolution equations. Concerning necessary conditions in the Gevrey setting, the authors in [7] have proved the following result for the operator (3):

Theorem. *If the Cauchy problem (4) is well-posed in $\mathcal{H}^\infty_\theta(\mathbb{R})$, $\theta > 1$, and:*

(i) *there exist* $R, A > 0$ *and* $\sigma_{p-j} \in [0,1]$, $j = 1, \ldots, p-1$, *such that*

$$\mathbf{Im}\, a_{p-j}(t,x) \geq A\langle x\rangle^{-\sigma_{p-j}}, \quad x > R\,(or\ x < -R),\, t \in [0,T],\, j = 1, \ldots, p-1;$$

(ii) *there exists* $C > 0$ *such that for every* $\beta \in \mathbb{N}$*:*

$$|\partial_x^\beta a_{p-j}(t,x)| \leq C^{\beta+1}\beta!\langle x\rangle^{-\beta}, \quad x \in \mathbb{R},\, t \in [0,T], \quad j = 1, \ldots, p,$$

then

$$\Xi := \max_{j=1,\ldots,p-1}\{(p-1)(1-\sigma_{p-j}) - j + 1\} \leq \frac{1}{\theta}. \tag{6}$$

The previous result gives some necessary conditions on the decay rates of the coefficients for the Gevrey well-posedness. Due to the fact that $\Xi \geq 0$ and $\frac{1}{\theta} < 1$, the following sentences are consequences of (6):

- if $\sigma_{p-j} \leq \frac{p-1-j}{p-1}$ for some $j = 1, \ldots, p-1$, the Cauchy problem is not well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$;
- if $\sigma_{p-j} \in \left(\frac{p-1-j}{p-1}, \frac{p-j}{p-1}\right)$ for some $j = 1, \ldots, p-1$, then the power σ_{p-j} imposes the restriction $(p-1)(1-\sigma_{p-j}) - j + 1 \leq \frac{1}{\theta}$ for the indices θ where $\mathcal{H}_\theta^\infty(\mathbb{R})$ well-posedness can be found;
- if $\sigma_{p-j} \geq \frac{p-j}{p-1}$ for some $j = 1, \ldots, p-1$, then the power σ_{p-j} has no effect on the $\mathcal{H}_\theta^\infty(\mathbb{R})$ well-posedness.

We can use this result for a better understanding of the sufficient conditions given in [6, 32] for the cases $p = 2, 3$ and in this thesis for a generic p. In short, in the case $p = 2$ in [32], the assumption over the coefficient a_1 is that it is Gevrey regular of order $\theta_0 > 1$ and the decay at infinity goes like $|x|^{-\sigma}$ for some $\sigma \in (0,1)$, and well-posedness in $\mathcal{H}_\theta^\infty(\mathbb{R})$ is achieved for $\theta_0 \leq \theta < (1-\sigma)^{-1}$. On other hand, for the case $p = 3$, in [6], assuming $\sigma \in \left(\frac{1}{2}, 1\right), \theta_0 < \frac{1}{2(1-\sigma)}$ and

(i) $a_3 \in C([0,T];\mathbb{R})$ and $\exists C_{a_3} > 0$ such that $|a_3(t)| \geq C_{a_3}$ $\forall t \in [0,T]$,

(ii) $a_{p-j} \in C\left([0,T]; G^{\theta_0}(\mathbb{R})\right)$, $\theta_0 > 1$, for $j = 1, 2, 3$,

(iii) there exists $C_{a_2} > 0$ such that

$$|\partial_x^\beta a_2(t,x)| \leq C_{a_2}^{\beta+1}\beta!^{\theta_0}\langle x\rangle^{-\sigma}, \quad \forall t \in [0,T],\, x \in \mathbb{R}, \beta \in \mathbb{N}_0,$$

(iv) there exists C_{a_1} such that $|\mathbf{Im}\, a_1(t,x)| \leq C_{a_1}\langle x\rangle^{-\frac{\sigma}{2}}$ for every $t \in [0,T]$, $x \in \mathbb{R}$,

then well-posedness in $\mathcal{H}_\theta^\infty(\mathbb{R})$, for $\theta \in \left[\theta_0, \frac{1}{2(1-\sigma)}\right)$, holds. A similar analysis has been developed for $H_\theta^\infty(\mathbb{R})$, cf. [5] and in the Gelfand-Shilov spaces $\mathcal{S}_s^\theta(\mathbb{R})$, cf. [3, 4, 15]. The space $\mathcal{S}_s^\theta(\mathbb{R})$, $s > 1$, $\theta > 1$, is defined as the space of all $f \in C^\infty(\mathbb{R})$ such that

$$|\partial^\beta f(x)| \leq C^{|\beta|+1}\beta!^\theta \exp\left(-c|x|^{\frac{1}{s}}\right), \quad x \in \mathbb{R},$$

for all $\beta \in \mathbb{N}_0$ and for some constants $C, c > 0$ independent of β. These spaces introduced in [26] represent a global counterpart of Gevrey spaces and acquired strong importance in the last decades as a functional setting to micro-local analysis and PDE, cf. [3, 6, 4, 9, 15, 20, 21, 22, 41, 43].

Taking into account the above mentioned results, the aim of this thesis is then to extend the analysis of the Cauchy problem with data in Gevrey and in Gelfand-Shilov spaces to the case of $p-$evolution equations of arbitrary order p. We are now ready to describe in short how the chapters are presented and divided. As a matter of fact, it can be informed in advance that, the most important results of this thesis are:

- Theorems 3.1 and 3.3 in Chapter 3
- Theorem 4.1 and Theorem 4.2 in Chapter 4.

The Chapters 1 and 2 are dedicated to background theory. To be more specific, we devote Chapter 1 to introduce notations, formulas, inequalities and some functional spaces. In Chapter 2, the symbol classes are presented together with the associated pseudo-differential operators and we also collect several results concerning pseudo-differential calculus.

In Chapter 3, our goal is to prove Theorems 3.1 and 3.3. In short, both theorems aim to study well-posedness for the Cauchy problem (4); the only difference is that in Theorem 3.1 we are interested in studying the problem for the inductive Gevrey-Sobolev space $\mathcal{H}^{\infty}_{\theta}(\mathbb{R})$, while in Theorem 3.3 we are interested in the projective Gevrey-Sobolev space $H^{\infty}_{\theta}(\mathbb{R})$. In order to obtain these results, we devote most part of the chapter to find a way to conjugate our $p-$evolution operator to get an equivalent Cauchy problem which allows to employ well known theory and obtain the desired conclusions. Part of the content of this chapter, namely Theorem 3.1, appeared in the recent paper [8].

Finally, in Chapter 4, in the first part, we study well-posedness for the Cauchy problem (4), in the Gelfand-Shilov setting, that is, the well-posedness is studied with respect to the space $\mathcal{S}^{\theta}_{s}(\mathbb{R})$. Namely, in Theorem 4.1, under the same assumptions of Theorem 3.1 on $P(t, x, D_t, D_x)$ we prove a well-posedness result in $\mathcal{S}^{\theta}_{s}(\mathbb{R})$ for $\theta < \min\left\{\frac{1}{(p-1)(1-\sigma)}, \frac{s}{p-1}\right\}$. In the second part of the chapter we prove that this upper bound for θ is indeed sharp, i.e. we prove that if $(p-1)\theta > \min\left\{\frac{1}{1-\sigma}, s\right\}$, then the Cauchy problem (4) is not well-posed in general for an operator satisfying assumptions of Theorem 4.1. This last part is the content of Theorem 4.2.

The thesis is concluded by some final remarks on open problems and possible improvements of the results presented.

Chapter 1

PRELIMINARIES

As usual, we dedicate this chapter to introduce some basic notations and concepts that we will need for the whole work.

1.1 Multi-indexes and some differential formulas

By $\mathbb{N}_0$ we denote the union of the sets $\{0\}$ and $\mathbb{N} = \{1, 2, ...\}$. Any element of the set $\mathbb{N}_0^n = \underbrace{\mathbb{N}_0 \times \cdots \times \mathbb{N}_0}_{n-\text{times}}$, $n \in \mathbb{N}_0$, is called a *multi-index* and we can represent $\alpha \in \mathbb{N}_0^n$ as $\alpha := (\alpha_1, ..., \alpha_n)$. The *length* of a multi-index is defined as the number

$$|\alpha| := \alpha_1 + \cdots + \alpha_n.$$

For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ we define

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

In $\mathbb{R}^n$ the j-th partial derivative operator, $j = 1, ..., n$, is represented by $\frac{\partial}{\partial x_j}$ (when we have no doubt about the variable, we can write it just as ∂_{x_j}). Another important notation is $D_{x_j} = -i\partial_{x_j}$, where $i = \sqrt{-1}$. If $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, then we may define

$$\partial^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \qquad \text{and} \qquad D^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}.$$

The above notation also can be viewed as $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ and

$$D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} = (-i\partial_{x_1})^{\alpha_1} \cdots (-i\partial_{x_n})^{\alpha_n} = (-i)^{|\alpha|}(\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}) = (-i)^{|\alpha|}\partial^\alpha.$$

If $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ are multi-indexes, we say that α is less than or equal to β ($\alpha \leq \beta$) if and only if

$$\alpha_j \leq \beta_j, \ \forall j = 1, ..., n.$$

The *factorial* of $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ is defined by

$$\alpha! := \alpha_1! \cdots \alpha_n!.$$

Given $\alpha, \beta \in \mathbb{N}_0$, with $\beta \leq \alpha$, we define the binomial coefficient

$$\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha - \beta)!}$$

with $\alpha - \beta := (\alpha_1 - \beta_1, ..., \alpha_n - \beta_n)$.

Now, by using these ideas about multi-indexes, let us present some important formulas whose notations will depend of it.

- **The Taylor formula**: The set of all infinite differentiable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ will be denoted by $C^\infty(\mathbb{R}^n)$. If $f \in C^\infty(\mathbb{R}^n)$, its *Taylor formula* is given by the following: given $N \in \mathbb{N}$,

$$f(x+\xi) = \sum_{|\alpha|<N} \frac{\xi^\alpha}{\alpha!}(\partial^\alpha f)(x) + N \sum_{|\alpha|=N} \frac{\xi^\alpha}{\alpha!} \int_0^1 (1-\theta)^{N-1}(\partial^\alpha f)(x+\theta\xi)d\theta, \quad x, \xi \in \mathbb{R}^n.$$

- **Leibniz Rule**: Another formula that will be often used is the well known *Leibniz rule*:

$$D^\alpha(f \cdot g) = \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} D^{\alpha_1} f D^{\alpha_2} g = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g.$$

- **Faà di Bruno formula**: If $f\colon \mathbb{R} \to \mathbb{C}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$, then

$$\partial_x^\alpha (f \circ g)(x) = \sum_{j=1}^{|\alpha|} \frac{f^{(j)}(g(x))}{j!} \sum_{\substack{\alpha_1+\cdots+\alpha_j=\alpha \\ |\alpha_\ell|\geq 1}} \frac{\alpha!}{\alpha_1!\cdots\alpha_\ell!} \prod_{\ell=1}^{j} \partial_x^{\alpha_\ell} g(x).$$

1.2 Useful identities and inequalities

Now we intend to establish some identities and inequalities that will be used frequently during our work to perform very important estimates which characterize the spaces we are interested to work with. Given $N \in \mathbb{N}$ and $t_1, ..., t_n \in \mathbb{R}$, the *generalized Newton formula* is given by

$$(t_1 + \cdots + t_n)^N = \sum_{\substack{|\alpha|=N \\ \alpha\in\mathbb{N}_0}} \frac{N!}{\alpha!} t^\alpha \qquad \text{or} \qquad (t_1 + \cdots + t_n)^N = \sum_{\substack{|\alpha|=N \\ \alpha\in\mathbb{N}_0}} \frac{N!}{\alpha_1!\cdots\alpha_n!} \prod_{i=1}^{n} t_i^{\alpha_i}.$$

In particular, if $t_1 = \cdots = t_n = 1$, if follows that

$$n^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!}.$$

From the above identity, we can derive some inequalities and identities. In fact:

- If $N = |\alpha|$, then

$$|\alpha|! \leq n^{|\alpha|}\alpha!, \quad \alpha \in \mathbb{N}_0^n.$$

- If $n = 2$, we obtain

$$2^N = \sum_{j+k=N} \frac{N!}{j!k!}. \tag{1.1}$$

  From this, with $N = j + k$, if follows that

$$\frac{(j+k)!}{j!k!} \leq 2^{j+k} \quad \Rightarrow \quad (j+k)! \leq 2^{j+k} j!k!.$$

Obviously, $(j+k)! \leq 2^{j+k} j!k!$ implies that

$$(\alpha+\beta)! \leq 2^{|\alpha+\beta|}\alpha!\beta!.$$

It is easy to notice that (1.1) can be written as

$$\sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} = 2^{|\alpha|}.$$

Another inequality that will be often used is

$$\alpha!\beta! \leq (\alpha+\beta)!.$$

Now, by considering the Taylor expansion of the exponential $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$, $t \geq 0$, we can obtain

$$t^N \leq N!e^N, \quad N \in \mathbb{N}_0,\ t \geq 0.$$

For the next inequalities, let us consider the following notations: if $x, \xi \in \mathbb{R}^n$, we set

$$x\xi := x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j, \qquad |x|^2 := \sum_{j=1}^{n} x_j^2 \qquad \text{and} \qquad \langle x \rangle := \sqrt{1+|x|^2}.$$

The symbol $\langle x \rangle$ is called the *japanese bracket* of x. Besides, $\langle \cdot \rangle$ is a smooth function with asymptotic behaviour equivalent to $1+|x|$ for $|x| \to \infty$ since

$$\langle x \rangle \leq 1+|x| \leq \sqrt{2}\langle x \rangle, \quad x \in \mathbb{R}^n$$

and

$$|\partial^\beta \langle x \rangle^m| \leq C^{|\beta|}\beta!\langle x \rangle^{m-|\beta|},$$

for some constant $C > 0$ independent of β. Finally, for any $s \in \mathbb{R}$ there exists $c_s > 0$ such that

$$\langle x+\xi \rangle^s \leq c_s \langle x \rangle^{|s|} \langle \xi \rangle^s, \quad x, \xi \in \mathbb{R}^n,$$

and this inequality is known as *Peetre's* inequality. Another useful notation we shall use frequently is the bracket $\langle \xi \rangle_h$, where 1 is replaced by some constant $h > 1$, that is,

$$\langle \xi \rangle_h := \sqrt{h^2+|\xi|^2}.$$

1.3 Fourier transform and inversion formula

Our intention from now until the end of this chapter is to define several spaces of functions and/or distributions which will be used as domain of pseudo-differential operators or simply the universe where we are considering the data of a Cauchy problem.

As mentioned before, the space of all infinitely differentiable functions $f\colon \mathbb{R}^n \to \mathbb{C}$, or simply the space of *smooth functions*, is denoted by $C^\infty(\mathbb{R}^n)$.

The space $L^2(\mathbb{R}^n)$ is the set of all measurable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\|f\|_{L^2} := \left[\int_{\mathbb{R}^n} |f(x)|^2 dx\right]^{\frac{1}{2}} < +\infty.$$

We point that $\|\cdot\|_{L^2}$ is a norm in $L^2(\mathbb{R}^n)$ induced by the inner product

$$(f,g)_{L^2} := \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx, \quad f,g \in L^2(\mathbb{R}^n)$$

and $\big(L^2(\mathbb{R}^n), (\cdot,\cdot)_{L^2}\big)$ is a Hilbert space.

The space of all smooth rapidly decreasing functions, also known as the *Schwartz space*, will be denoted by $\mathscr{S}(\mathbb{R}^n)$ and it is characterized as

$$\varphi \in \mathscr{S}(\mathbb{R}^n) \quad \Leftrightarrow \quad \varphi \in C^\infty(\mathbb{R}^n) \text{ and satisfies } \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta \varphi(x)| < +\infty, \quad \alpha,\beta \in \mathbb{N}_0^n.$$

The Schwartz space is a Fréchet space when equipped with the usual semi-norms

$$\|\varphi\|_{\ell,\mathscr{S}} := \max_{|\alpha+\beta|\le \ell} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta \varphi(x)|, \quad \ell \in \mathbb{N}_0^n,\ \varphi \in \mathscr{S}(\mathbb{R}^n).$$

The topological dual of $\mathscr{S}(\mathbb{R}^n)$, denoted by $\mathscr{S}'(\mathbb{R}^n)$, is called the space of *tempered distributions*.

Now we are able to talk about the Fourier transform. We define the *Fourier transform* of a function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ as

$$\mathscr{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-i\xi x}\varphi(x)dx, \quad \xi \in \mathbb{R}^n.$$

The map $\mathscr{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is an isomorphism which can be extended to an isomorphism $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The *inverse Fourier transform* of $\varphi \in \mathscr{S}(\mathbb{R}^n)$ is defined by

$$\mathscr{F}^{-1}(\varphi)(x) := (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi x}\varphi(\xi)d\xi, \quad x \in \mathbb{R}^n.$$

We conclude this section with two very important formulas involving the Fourier transform:

- **Parseval formula**: $\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = (2\pi)^{-n}\int_{\mathbb{R}^n} \mathscr{F}(f)(\xi)\overline{\mathscr{F}(g)(\xi)}d\xi$.
- **Plancherel formula**: $\|f\|_{L^2}^2 = (2\pi)^{-n}\|\mathscr{F}(f)\|_{L^2}^2$.

1.4 Sobolev, Gevrey and Gelfand-Shilov spaces

The standard *Sobolev* space $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n) :\ \langle D\rangle^s u \in L^2(\mathbb{R}^n)\}, \quad s \in \mathbb{R},$$

where $\langle D\rangle^s$ is the Fourier multiplier defined by $\langle D\rangle^s u = \mathscr{F}^{-1}(\langle\cdot\rangle^s \widehat{u}(\cdot))$. For $s = (s_1, s_2) \in \mathbb{R}^2$ we also define the weighted Sobolev space

$$H^s(\mathbb{R}^n) = H^{s_1,s_2}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) \colon \langle x\rangle^{s_2}\langle D\rangle^{s_1} u \in L^2(\mathbb{R}^n)\}.$$

Notice that

$$\bigcap_{s\in\mathbb{R}^2} H^s(\mathbb{R}^n) = \mathscr{S}(\mathbb{R}^n), \qquad \bigcup_{s\in\mathbb{R}^2} H^s(\mathbb{R}^n) = \mathscr{S}'(\mathbb{R}^n).$$

One of the most important spaces that we need to consider for our work is the space of functions of Gevrey type. Namely, some of the functions involved in the main result of this work are functions of this type. In the book [44], L. Rodino presents a very well detailed development of the Gevrey theory.

Given $\theta \geq 1$ and $h > 0$, the set $G^\theta(\mathbb{R}^n; h)$ is defined as follows:

$$\varphi \in G^\theta(\mathbb{R}^n; h) \quad \Leftrightarrow \quad \varphi \in C^\infty(\mathbb{R}^n) \text{ and } \|\varphi\|_{G^\theta(\mathbb{R}^n;h)} < +\infty,$$

where

$$\|\varphi\|_{G^\theta(\mathbb{R}^n;h)} := \sup_{x\in\mathbb{R}^n} \sup_{\alpha\in\mathbb{N}_0^n} |\partial_x^\alpha \varphi(x)| h^{-|\alpha|} \alpha!^{-\theta}.$$

The norm $\|\cdot\|_{G^\theta(\mathbb{R}^n;h)}$ turns $G^\theta(\mathbb{R}^n; h)$ into a Banach space. Observe that $G^\theta(\mathbb{R}^n; h) \subset G^\theta(\mathbb{R}^n; h')$ if $h < h'$. Then we can define

$$G^\theta(\mathbb{R}^n) := \bigcup_{h>0} G^\theta(\mathbb{R}^n; h),$$

endowed with the inductive limit topology. By $G_0^\theta(\mathbb{R}^n)$ we denote the set of all compactly supported functions in $G^\theta(\mathbb{R}^n)$. Still in the Gevrey setting, we also define the space $\gamma^\theta(\mathbb{R}^n)$ as the space of all smooth functions such that

$$\sup_{\alpha\in\mathbb{N}_0^n} \sup_{x\in\mathbb{R}^n} h^{-|\alpha|}(\alpha!)^{-\theta}|\partial^\alpha f(x)| < \infty$$

for every $h > 0$, and $\gamma_0^\theta(\mathbb{R}^n)$ stands for the space of all compactly supported functions in $\gamma^\theta(\mathbb{R}^n)$.

In order to obtain energy estimates for our $p-$evolution equations we need to work with Hilbert spaces of Gevrey regular functions. With this purpose, for $\theta \geq 1$, $m, \rho \in \mathbb{R}$, we set

$$H^m_{\rho;\theta}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \ \langle D\rangle^m e^{\rho\langle D\rangle^{1/\theta}} u \in L^2(\mathbb{R}^n)\},$$

where $\langle D\rangle^m$ and $e^{\rho\langle D\rangle^{1/\theta}}$ are the Fourier multipliers with symbols $\langle\xi\rangle^m$ and $e^{\rho\langle\xi\rangle^{1/\theta}}$, respectively. We call this kind of space a *Gevrey-Sobolev* space. It is a Hilbert space equipped with the inner product

$$(u,v)_{H^m_{\rho;\theta}} = \left(\langle D\rangle^m e^{\rho\langle D\rangle^{1/\theta}} u, \langle D\rangle^m e^{\rho\langle D\rangle^{1/\theta}} v\right)_{L^2}, \quad u, v \in H^m_{\rho;\theta}(\mathbb{R}^n).$$

For $\rho = 0$ we recover the standard Sobolev spaces $H^m(\mathbb{R}^n)$. We set

$$\mathcal{H}^\infty_\theta(\mathbb{R}^n) := \bigcup_{\rho>0} H^m_{\rho;\theta}(\mathbb{R}^n) \quad \text{and} \quad H^\infty_\theta(\mathbb{R}^n) := \bigcap_{\rho>0} H^m_{\rho;\theta}(\mathbb{R}^n).$$

Remark 1.1. *Notice that* $H^{m'}_{\rho';\theta}(\mathbb{R}^n) \subseteq H^m_{\rho;\theta}(\mathbb{R}^n)$ *whenever* $m' \geq m$ *and* $\rho' \geq \rho$*. This is a consequence of Plancherel formula, since we have*

$$\|f\|_{H^m_{\rho;\theta}} \leq \|f\|_{H^{m'}_{\rho';\theta}}, \quad f \in H^{m'}_{\rho';\theta}(\mathbb{R}^n).$$

The previous Gevrey spaces are related with $\mathcal{H}^\infty_\theta(\mathbb{R}^n)$ and $H^\infty_\theta(\mathbb{R}^n)$ through the inclusions

$$G^\theta_0(\mathbb{R}^n) \subset \mathcal{H}^\infty_\theta(\mathbb{R}^n) \subset G^\theta(\mathbb{R}^n)$$

and

$$\gamma^\theta_0(\mathbb{R}^n) \subset H^\infty_\theta(\mathbb{R}^n) \subset \gamma^\theta(\mathbb{R}^n).$$

Next, let us introduce some definitions and properties concerning the so called *Gelfand-Shilov spaces*. The reader can find a deeper approach to this subject in [22, 26, 39, 43]. If $s \geq 1$, $\theta \geq 1$, $A > 0$ and $B > 0$, we say that a function f belongs to the set $\mathcal{S}^{\theta,A}_{s,B}(\mathbb{R}^n)$ if f is smooth and there exists $C > 0$ such that

$$|x^\beta \partial^\alpha_x f(x)| \leq C A^{|\alpha|} B^{|\beta|} (\alpha!)^\theta (\beta!)^s,$$

for each $\alpha, \beta \in \mathbb{N}^n_0$ and $x \in \mathbb{R}^n$. The function $\|\cdot\|_{\theta,s,A,B} : \mathcal{S}^{\theta,A}_{s,B}(\mathbb{R}^n) \to \mathbb{R}$ given by

$$\|f\|_{\theta,s,A,B} := \sup_{x\in\mathbb{R}^n} \sup_{\alpha,\beta\in\mathbb{N}^n_0} |x^\beta \partial^\alpha_x f(x)| A^{-|\alpha|} B^{-|\beta|} (\alpha!)^{-\theta} (\beta!)^{-s},$$

is a norm in $\mathcal{S}^{\theta,A}_{s,B}(\mathbb{R}^n)$ which turns it into a Banach space. As usual, we can define the spaces

$$\mathcal{S}^\theta_s(\mathbb{R}^n) := \bigcup_{A,B>0} \mathcal{S}^{\theta,A}_{s,B}(\mathbb{R}^n) \quad \text{and} \quad \Sigma^\theta_s(\mathbb{R}^n) := \bigcap_{A,B>0} \mathcal{S}^{\theta,A}_{s,B}(\mathbb{R}^n),$$

respectively, with the inductive and projective limit topology. When $s = \theta$, the notations $\mathcal{S}^\theta_\theta(\mathbb{R}^n)$ and $\Sigma^\theta_\theta(\mathbb{R}^n)$ can be simplified as $\mathcal{S}_\theta(\mathbb{R}^n)$ and $\Sigma_\theta(\mathbb{R}^n)$, respectively.

Remark 1.2. *The spaces* $\mathcal{S}^\theta_s(\mathbb{R}^n)$ *and* $\Sigma^\theta_s(\mathbb{R}^n)$ *may also be characterized in a different way. Let* $C > 0$ *and* $\epsilon > 0$*. The space* $\mathcal{S}^{\theta,C}_{s,\epsilon}(\mathbb{R}^n)$ *is defined as the set of all smooth functions* f *such that*

$$\|f\|^{\epsilon,C}_{s,\theta} < +\infty,$$

where

$$\|f\|^{\epsilon,C}_{s,\theta} := \sup_{x\in\mathbb{R}^n} \sup_{\alpha\in\mathbb{N}^n_0} C^{-|\alpha|} (\alpha!)^{-\theta} e^{\epsilon|x|^{\frac{1}{s}}} |\partial^\alpha_x f(x)|$$

is a norm which turns $\mathcal{S}^{\theta,C}_{s,\epsilon}(\mathbb{R}^n)$ *into a Banach space. Then, we have*

$$\mathcal{S}^\theta_s(\mathbb{R}^n) = \bigcup_{C,\epsilon>0} \mathcal{S}^{\theta,C}_{s,\epsilon}(\mathbb{R}^n) \quad \textit{and} \quad \Sigma^\theta_s(\mathbb{R}^n) = \bigcap_{C,\epsilon>0} \mathcal{S}^{\theta,C}_{s,\epsilon}(\mathbb{R}^n)$$

with equivalent topologies to the ones described above.

For $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$, $\rho_1, \rho_2 > 0$, and $m = (m_1, m_2) \in \mathbb{R}^2$, we consider

$$H^m_{\rho;s,\theta}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) \ : \ \langle x\rangle^{m_2}\langle D\rangle^{m_1} e^{\rho_2\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}} u \in L^2(\mathbb{R}^n)\}.$$

which is called *Gelfand-Shilov-Sobolev* space. By endowing $H^m_{\rho;s,\theta}(\mathbb{R}^n)$ with the inner product

$$(u,v)_{H^m_{\rho;s,\theta}} := \left(\langle x\rangle^{m_2}\langle D\rangle^{m_1} e^{\rho_2\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}} u, \langle x\rangle^{m_2}\langle D\rangle^{m_1} e^{\rho_2\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}} v\right)_{L^2(\mathbb{R}^n)}, u, v \in H^m_{\rho;s,\theta}(\mathbb{R}^n),$$

it becomes a Hilbert space and the induced norm is denoted by $\|\cdot\|_{H^m_{\rho;s,\theta}}$. For $s \leq s'$, $\theta \leq \theta'$, $m_j \geq m'_j$ and $\rho_j \geq \rho'_j$, $j = 1, 2$, the following inclusion holds

$$H^m_{\rho;s,\theta}(\mathbb{R}^n) \subset H^{m'}_{\rho';s',\theta'}(\mathbb{R}^n).$$

Then, the Gelfand-Shilov classes can be expressed as

$$\mathcal{S}^\theta_s(\mathbb{R}^n) = \bigcup_{\rho_1,\rho_2>0} H^0_{\rho;s,\theta}(\mathbb{R}^n) \quad \text{and} \quad \Sigma^\theta_s(\mathbb{R}^n) = \bigcap_{\rho_1,\rho_2>0} H^0_{\rho;s,\theta}(\mathbb{R}^n)$$

As usual, the dual spaces of $\mathcal{S}^\theta_s(\mathbb{R}^n)$ and $\Sigma^\theta_s(\mathbb{R}^n)$ will be denoted as $(\mathcal{S}^\theta_s)'(\mathbb{R}^n)$ and $(\Sigma^\theta_s)'(\mathbb{R}^n)$, respectively.

Remark 1.3. *The following inclusions are obviously continuous*

$$\Sigma^\theta_s(\mathbb{R}^n) \subset \mathcal{S}^\theta_s(\mathbb{R}^n) \subset \Sigma^{s+\epsilon}_{s+\epsilon}(\mathbb{R}^n),$$

for each $\epsilon > 0$.

Concerning the Fourier transform, the Gelfand-Shilov type spaces have the following properties:

$$\mathscr{F} : \Sigma^\theta_s(\mathbb{R}^n) \to \Sigma^s_\theta(\mathbb{R}^n), \quad \mathscr{F} : \mathcal{S}^\theta_s(\mathbb{R}^n) \to \mathcal{S}^s_\theta(\mathbb{R}^n),$$

$$\mathscr{F} : (\Sigma^\theta_s)'(\mathbb{R}^n) \to (\Sigma^s_\theta)'(\mathbb{R}^n), \quad \mathscr{F} : (\mathcal{S}^\theta_s)'(\mathbb{R}^n) \to (\mathcal{S}^s_\theta)'(\mathbb{R}^n)$$

are all isomorphisms.

Chapter 2

PSEUDO-DIFFERENTIAL OPERATORS AND SYMBOLIC CALCULUS

This chapter is dedicated to introduce several classes of symbols and the related pseudo-differential operators we shall work with and a suitable version of the sharp Gårding inequality. The pseudo-differential operator theory developed here is based mostly in the work [2]. In the frame of pseudo-differential operators, we are also using results and concepts which can be found in [1, 18, 19, 20, 21, 25, 28, 37, 41, 42, 44, 45, 46].

Let us consider the set

$$\mathcal{B}^{\infty}(\mathbb{R}^n) = \{\varphi \in C^{\infty}(\mathbb{R}^n;\mathbb{C}) : \sup_{x\in\mathbb{R}^n} |\partial^{\beta}\varphi(x)| < \infty,\ \forall \beta \in \mathbb{N}_0^n\}.$$

The action of a linear partial differential operator

$$p(x,D) = \sum_{|\alpha|\le m} a_{\alpha}(x) D^{\alpha},$$

with coefficients $a_{\alpha} \in \mathcal{B}^{\infty}(\mathbb{R}^n)$, can be expressed as

$$p(x,D)u(x) = \int e^{i\xi x} p(x,\xi)\widehat{u}(\xi) đ\xi, \quad x \in \mathbb{R}^n,\ u \in \mathscr{S}(\mathbb{R}^n), \tag{2.1}$$

where $p(x,\xi)$ is the *symbol* of the operator $p(x,D)$, which is given by

$$p(x,\xi) = \sum_{|\alpha|\le m} a_{\alpha}(x)\xi^{\alpha}, \quad (x,\xi) \in \mathbb{R}^{2n}$$

and $đ\xi = (2\pi)^{-n} d\xi$.

Remark 2.1. *We point that:*

- (2.1) *can be obtained by using Fourier transform and its standard properties.*
- *The symbol $p(x,\xi)$ satisfies: for all $\alpha,\beta \in \mathbb{N}_0^n$, there exists $C_{\alpha,\beta} > 0$ such that*

$$|\partial_{\xi}^{\alpha}\partial_x^{\beta} p(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}, \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{2.2}$$

The right hand side of (2.1) indeed makes sense for any smooth function $p(x,\xi)$ satisfying an estimate like (2.2), for $m \in \mathbb{R}$. Smooth functions with this property are called *symbols*, and operators of the form (2.1) associated with some symbol are called *pseudo-differential operators* (p.d.o.). If a symbol is polynomially bounded like in (2.2), we say that it has *finite order*. When a symbol is not polynomially bounded but it is bounded by an exponential function, the symbol has *infinite order*.

Remark 2.2. *Concerning operators of infinite order, intending to obtain convergence in the integral which defines the right hand side of* (2.1)*, it is required some stronger decay condition for the Fourier transform of u, for this we could take u belonging to some suitable Gelfand-Shilov class.*

Sometimes we will be interested in symbols that have boundedness of polynomial type with respect to variables x and ξ simultaneously, namely: for $m_1, m_2 \in \mathbb{R}$, for each $\alpha, \beta \in \mathbb{N}_0^n$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m_1 - |\alpha|} \langle x \rangle^{m_2 - |\beta|}, \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{2.3}$$

Operators with symbols satisfying (2.3) are called *pseudo-differential operators of* **SG** *type*.

2.1 $\mathbf{S}^m(\mathbb{R}^{2n})$ and $\mathbf{SG}^m(\mathbb{R}^{2n})$ classes

In this section we present the well known *Hörmander classes* $\mathbf{S}^m(\mathbb{R}^{2n})$ and the $\mathbf{SG}^m(\mathbb{R}^{2n})$ classes. In the sequel, we intend to list some results involving these classes of symbols and their principal properties. Proofs and more details can be found in [18], [25], [37], [41], [42] and [45].

Definition 2.1. Let m be a real number. A smooth function p belongs to $\mathbf{S}^m(\mathbb{R}^{2n})$ if for every $\alpha, \beta \in \mathbb{N}_0^n$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m - |\alpha|}, \quad (x,\xi) \in \mathbb{R}^{2n}.$$

The class $\mathbf{S}^m(\mathbb{R}^{2n})$ is a *Fréchet space* equipped with the semi-norms

$$|p|_{\mathbf{S}^m,\ell} := \max_{|\alpha+\beta| \leq \ell} \sup_{(x,\xi)\in\mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \langle \xi \rangle^{-m+|\alpha|}, \quad p \in \mathbf{S}^m(\mathbb{R}^{2n}),\ \ell \in \mathbb{N}_0.$$

If a pseudo-differential operator has its symbol belonging to $\mathbf{S}^m(\mathbb{R}^{2n})$, it is continuous from $\mathscr{S}(\mathbb{R}^n)$ to itself and it can be extended to a continuous map from $\mathscr{S}'(\mathbb{R}^n)$ to itself. Actually, another kind of extension can be established for operators of this nature: an operator with symbol in $\mathbf{S}^m(\mathbb{R}^{2n})$ extends to a bounded map from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$, for all $s \in \mathbb{R}$. The norm of the pseudo-differential operator $p(x,D)\colon H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R})$ is bounded in terms of a finite number of semi-norms of the symbol p.

As usual, when we are working with pseudo-differential operators, it is fundamental to consider the *asymptotic expansion* of symbols. This will make our life easier when we need to compose, to take adjoint and transpose operators.

Definition 2.2. Let $(m_j)_{j\in\mathbb{N}_0}$ be a non-increasing sequence of real numbers which diverges to $-\infty$, and symbols $p \in \mathbf{S}^m(\mathbb{R}^{2n})$, $p_j \in \mathbf{S}^{m_j}(\mathbb{R}^{2n})$, for all $j \in \mathbb{N}_0$, with $m = m_0$. We say that p is asymptotic to $\sum_j p_j$ in $\mathbf{S}^m(\mathbb{R}^{2n})$ if

$$p - \sum_{j<N} p_j \in \mathbf{S}^{m_N}(\mathbb{R}^{2n}), \quad \forall N \in \mathbb{N}.$$

If p is asymptotic to $\sum_j p_j$, it can be used the notation $p \sim \sum_j p_j$ in $\mathbf{S}^m(\mathbb{R}^{2n})$ to indicate it.

The next results concern to the composition, adjoint and transposition of pseudo-differential operators whose symbols are in the Hörmander classes.

Theorem 2.1. *Let $p \in \mathbf{S}^m(\mathbb{R}^{2n})$ and $q \in \mathbf{S}^{m'}(\mathbb{R}^{2n})$. There exists $c \in \mathbf{S}^{m+m'}(\mathbb{R}^{2n})$ such that*

$$c(x,D) = p(x,D) \circ q(x,D)$$

and

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) \text{ in } \mathbf{S}^{m+m'}(\mathbb{R}^{2n}).$$

Theorem 2.2. *Let $p \in \mathbf{S}^m(\mathbb{R}^{2n})$, $p^*(x,D)$ the L^2-adjoint and ${}^tp(x,D)$ the transpose of $p(x,D)$. Then:*

(i) *There exists $a \in \mathbf{S}^m(\mathbb{R}^{2n})$ such that $a(x,D) = p^*(x,D)$ and*

$$a(x,\xi) \sim \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!}\overline{\partial_\xi^\alpha D_x^\alpha p(x,\xi)} \text{ in } \mathbf{S}^m(\mathbb{R}^{2n}).$$

(ii) *There exists $b \in \mathbf{S}^m(\mathbb{R}^{2n})$ such that $b(x,D) = {}^tp(x,D)$ and*

$$b(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(\partial_\xi^\alpha D_x^\alpha p)(x,-\xi) \text{ in } \mathbf{S}^m(\mathbb{R}^{2n}).$$

Now let us introduce a new weight function $\langle x\rangle^m$ bounding our symbols and its derivatives. From this it will raise the so called **SG** classes.

Definition 2.3. Let $m = (m_1, m_2) \in \mathbb{R}^2$. We shall denote by $\mathbf{SG}^m(\mathbb{R}^{2n})$ or by $\mathbf{SG}^{m_1,m_2}(\mathbb{R}^{2n})$ the space of all functions $p \in C^\infty(\mathbb{R}^{2n})$ such that for every $\alpha, \beta \in \mathbb{N}_0^n$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m_1-|\alpha|}\langle x\rangle^{m_2-|\beta|}, \quad (x,\xi) \in \mathbb{R}^{2n}.$$

The class $\mathbf{SG}^m(\mathbb{R}^{2n})$ is a *Fréchet space* equipped with the semi-norms

$$|p|_{\mathbf{SG}^m,\ell} := \max_{|\alpha+\beta|\leq\ell} \sup_{(x,\xi)\in\mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)|\langle\xi\rangle^{-m_1+|\alpha|}\langle x\rangle^{-m_2+|\beta|}, \quad p \in \mathbf{SG}^m(\mathbb{R}^{2n}),\ \ell \in \mathbb{N}_0.$$

If a pseudo-differential operator has its symbol belonging to $\mathbf{SG}^m(\mathbb{R}^{2n})$, it is continuous from $\mathscr{S}(\mathbb{R}^n)$ to itself and it can be extended to a continuous map from $\mathscr{S}'(\mathbb{R}^n)$ to itself. Another kind of extension can be established for operators of this nature: an operator with symbol in $\mathbf{SG}^m(\mathbb{R}^{2n})$ extends to a bounded map from $H^{s_1,s_2}(\mathbb{R}^n)$ to $H^{s_1-m_1,s_2-m_2}(\mathbb{R}^n)$, for all $s = (s_1, s_2) \in \mathbb{R}^2$.

Definition 2.4. An operator $r(x, D)$ is said to be a **SG** smoothing operator if its symbol $r(x, \xi)$ belongs to $\mathbf{SG}^m(\mathbb{R}^{2n})$ for all $m \in \mathbb{R}^2$. In fact, under this condition, $r(x, D)$ maps continuously $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$.

Remark 2.3. *Notice that*

$$\bigcap_{m \in \mathbb{R}^2} \mathbf{SG}^m(\mathbb{R}^{2n}) = \mathscr{S}(\mathbb{R}^{2n}).$$

Definition 2.5. Let $(m_j)_{j \in \mathbb{N}_0}$, $m_j = (m_1^{(j)}, m_2^{(j)})$, be a sequence in $\mathbb{R}^2$ whose coordinates are non-increasing sequences of real numbers which diverge to $-\infty$, and symbols $p \in \mathbf{SG}^m(\mathbb{R}^{2n})$, $p_j \in \mathbf{SG}^{m_j}(\mathbb{R}^{2n})$, for all $j \in \mathbb{N}_0$, with $m = m_0$. We say that p is asymptotic to $\sum_j p_j$ in $\mathbf{SG}^m(\mathbb{R}^{2n})$ if

$$p - \sum_{j<N} p_j \in \mathbf{SG}^{m_N}(\mathbb{R}^{2n}), \quad \forall N \in \mathbb{N}.$$

If p is asymptotic to $\sum_j p_j$, it can be used the notation $p \sim \sum_j p_j$ in $\mathbf{SG}^m(\mathbb{R}^{2n})$ to indicate it.

Theorem 2.3. *Let $p \in \mathbf{SG}^m(\mathbb{R}^{2n})$ and $q \in \mathbf{SG}^{m'}(\mathbb{R}^{2n})$. There exist $c \in \mathbf{SG}^{m+m'}(\mathbb{R}^{2n})$ and a smoothing operator $r(x, D)$ such that*

$$p(x, D) \circ q(x, D) = c(x, D) + r(x, D)$$

and

$$c(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha p(x, \xi) D_x^\alpha q(x, \xi) \text{ in } \mathbf{SG}^{m+m'}(\mathbb{R}^{2n}).$$

Theorem 2.4. *Let $p \in \mathbf{SG}^m(\mathbb{R}^{2n})$, $p^*(x, D)$ the L^2-adjoint and ${}^t p(x, D)$ the transpose of $p(x, D)$. Then:*

(i) *There exist $a \in \mathbf{SG}^m(\mathbb{R}^{2n})$ and a smoothing operator $r_1(x, D)$ such that*

$$p^*(x, D) = a(x, D) + r_1(x, D)$$

and

$$a(x, \xi) \sim \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} \overline{\partial_\xi^\alpha D_x^\alpha p(x, \xi)} \text{ in } \mathbf{SG}^m(\mathbb{R}^{2n}).$$

(ii) *There exist $b \in \mathbf{SG}^m(\mathbb{R}^{2n})$ and a smoothing operator $r_2(x, D)$ such that*

$${}^t p(x, D) = b(x, D) + r_2(x, D)$$

and

$$b(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} (\partial_\xi^\alpha D_x^\alpha p)(x, -\xi) \text{ in } \mathbf{SG}^m(\mathbb{R}^{2n}).$$

2.2 $\mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n})$, $\mathbf{\Gamma}^m_{\mu,\nu}(\mathbb{R}^{2n})$ and $\mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ classes

In the analysis of $p-$evolution equations in Gevrey and Gelfand-Shilov spaces we will need to work with symbols in Hörmander and **SG** classes which are also Gevrey regular and with operators of infinite order. We dedicate this section to introduce these symbols. The theory presented here can be found in [2], Chapter 2, and other recommended works are [15, 19, 27, 44, 46].

Definition 2.6. Let $A > 0$, $\mu, \nu > 1$ and $m \in \mathbb{R}$. We define $\mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$ to be the Banach space of all smooth functions $a(x, \xi)$ satisfying

$$\|a\|_A := \sup_{\alpha,\beta\in\mathbb{N}_0^n} \sup_{x,\xi\in\mathbb{R}^n} A^{-|\alpha+\beta|}\alpha!^{-\mu}\beta!^{-\nu}\langle\xi\rangle^{-m+|\alpha|}|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| < +\infty.$$

We set

$$\boxed{\mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n}) := \bigcup_{A>0} \mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n}; A)}$$

equipped with the inductive limit topology of the Banach spaces $\mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$, and

$$\boxed{\mathbf{\Gamma}^m_{\mu,\nu}(\mathbb{R}^{2n}) := \bigcap_{A>0} \mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n}; A)}$$

equipped with the projective limit topology of the Banach spaces $\mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$.

Definition 2.7. Let $A > 0$, $\mu, \nu > 1$ and $m \in \mathbb{R}$. We define $\tilde{\mathbf{S}}^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$ to be the Banach space of all smooth functions $a(x, \xi)$ satisfying

$$|||a|||_A := \sup_{\alpha,\beta\in\mathbb{N}_0^n} \sup_{x,\xi\in\mathbb{R}^n} A^{-|\alpha+\beta|}\alpha!^{-\mu}\beta!^{-\nu}\langle\xi\rangle^{-m}|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| < +\infty.$$

We set

$$\boxed{\tilde{\mathbf{S}}^m_{\mu,\nu}(\mathbb{R}^{2n}) := \bigcup_{A>0} \tilde{\mathbf{S}}^m_{\mu,\nu}(\mathbb{R}^{2n}; A)}$$

equipped with the inductive limit topology and

$$\boxed{\tilde{\mathbf{\Gamma}}^m_{\mu,\nu}(\mathbb{R}^{2n}) := \bigcap_{A>0} \tilde{\mathbf{S}}^m_{\mu,\nu}(\mathbb{R}^{2n}; A)}$$

equipped with the projective limit topology.

Finally, we define symbols of infinite order.

Definition 2.8. Let consider the constants $\mu, \nu, \theta > 1$ and $A, c > 0$. The space $\mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n}; A, c)$ is defined as the Banach space of all smooth functions $a(x, \xi)$ satisfying

$$\|a\|_{A,c} := \sup_{\alpha,\beta\in\mathbb{N}_0^n} \sup_{x,\xi\in\mathbb{R}^n} A^{-|\alpha+\beta|}\alpha!^{-\mu}\beta!^{-\nu}\langle\xi\rangle^{|\alpha|}e^{-c|\xi|^{\frac{1}{\theta}}}|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| < +\infty.$$

We set

$$\boxed{\mathbf{S}^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}) := \bigcup_{A,c>0} \mathbf{S}^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}; A, c)}$$

endowed with the inductive limit topology of the Banach spaces $\mathbf{S}^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}; A, c)$.

Remark 2.4. *If the constants μ and ν are equal, then the spaces $\mathbf{S}^{m}_{\mu,\mu}(\mathbb{R}^{2n})$, $\tilde{\mathbf{S}}^{m}_{\mu,\mu}(\mathbb{R}^{2n})$ and $\mathbf{S}^{\infty}_{\mu,\mu;\theta}(\mathbb{R}^{2n})$ will be denoted for simplicity by $\mathbf{S}^{m}_{\mu}(\mathbb{R}^{2n})$, $\tilde{\mathbf{S}}^{m}_{\mu}(\mathbb{R}^{2n})$ and $\mathbf{S}^{\infty}_{\mu;\theta}(\mathbb{R}^{2n})$, respectively.*

Remark 2.5. *For all $m \in \mathbb{R}$ and $\theta > 1$, we have $\mathbf{S}^{m}_{\mu}(\mathbb{R}^{2n}) \subset \mathbf{S}^{\infty}_{\mu;\theta}(\mathbb{R}^{2n})$.*

Remark 2.6. *If $\mu_1 < \mu_2$, then for every $A > 0$ we have $\mathbf{S}^{m}_{\mu_1}(\mathbb{R}^{2n}; A) \subset \mathbf{\Gamma}^{m}_{\mu_2}(\mathbb{R}^{2n})$ and $\tilde{\mathbf{S}}^{m}_{\mu_1}(\mathbb{R}^{2n}; A) \subset \tilde{\mathbf{\Gamma}}^{m}_{\mu_2}(\mathbb{R}^{2n})$.*

We dedicate the next subsections to collect some definitions, properties and results concerning the pseudo-differential operators with symbols in $\mathbf{S}^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $\mathbf{S}^{m}_{\mu;\nu}(\mathbb{R}^{2n})$.

2.2.1 Continuity on Gelfand-Shilov spaces

The next result states that any operator of the form (2.1) acts continuously on Gelfand-Shilov spaces and its proof can be checked in [2], Proposition 2.1.

Proposition 2.1. *Let $\mu, \nu, s, \theta > 1$ such that $s > \mu$ and $\theta > \nu$. If $p \in \mathbf{S}^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$, then the operator $p(x, D)$ is continuous from $\Sigma^{\theta}_{s}(\mathbb{R}^{n})$ to $\Sigma^{\theta}_{s}(\mathbb{R}^{n})$ and it can be extended to a linear continuous operator from $(\Sigma^{\theta}_{s})'(\mathbb{R}^{n})$ to $(\Sigma^{\theta}_{s})'(\mathbb{R}^{n})$.*

Remark 2.7. *Using the same argument as in the proof of Proposition 2.1, it can be proved that operators with finite order symbols in $\mathbf{S}^{m}_{\mu,\nu}(\mathbb{R}^{2n})$ are continuous from $\mathcal{S}^{\theta}_{s}(\mathbb{R}^{n})$ into $\mathcal{S}^{\theta}_{s}(\mathbb{R}^{n})$ if $s \geq \mu$ and $\theta \geq \nu$ and it extends continuously to the dual space $(\mathcal{S}^{\theta}_{s})'(\mathbb{R}^{n})$.*

2.2.2 Asymptotic sums

In order to develop a symbolic calculus for the classes of pseudo-differential operators we are dealing with, we will need the following concept of asymptotic sums.

Definition 2.9. We say that:

(i) the formal sum $\sum_{j=0}^{\infty} a_j$ belongs to $\mathbf{FS}^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ if $a_j \in C^{\infty}(\mathbb{R}^{2n})$ and there are constants $H, C, c, B > 0$ such that

$$|\partial^{\alpha}_{\xi}\partial^{\beta}_{x} a_j(x,\xi)| \leq HC^{|\alpha+\beta|+2j}\alpha!^{\mu}\beta!^{\nu}j!^{\mu+\nu-1}\langle\xi\rangle^{-|\alpha|-j}e^{c|\xi|^{\frac{1}{\theta}}},$$

for every $\alpha, \beta \in \mathbb{N}^{n}_{0}$, $x \in \mathbb{R}^{n}$, $j \in \mathbb{N}_0$ and $\langle\xi\rangle \geq B(j) := Bj^{\mu+\nu-1}$.

(ii) the formal sum $\sum_{j=0}^{\infty} a_j$ belongs to $\mathbf{FS}^m_{\mu,\nu}(\mathbb{R}^{2n})$ if $a_j \in C^\infty(\mathbb{R}^{2n})$ and there are constants $H, C, B > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)| \leq HC^{|\alpha+\beta|+2j}\alpha!^\mu \beta!^\nu j!^{\mu+\nu-1}\langle\xi\rangle^{m-|\alpha|-j},$$

for every $\alpha, \beta \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $\langle\xi\rangle \geq B(j) := Bj^{\mu+\nu-1}$.

Remark 2.8. *Notice that $\mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ can be viewed as a subset of $\mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ in the sense that, for $a \in \mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ we set $a_0 := a$ and $a_j := 0$, for $j \geq 1$, then $a = \sum_j a_j \in \mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. On the other hand if $\sum_j b_j \in \mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$, then $b_0 \in \mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Analogous considerations hold for the finite order case $\mathbf{FS}^m_{\mu,\nu}(\mathbb{R}^{2n})$ and $\mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n})$.*

Remark 2.9. *For every $m \in \mathbb{R}$ and $\theta > 1$ the inclusion $\mathbf{FS}^m_{\mu,\nu}(\mathbb{R}^{2n}) \subset \mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ holds.*

Remark 2.10. *Let $a, b \in \mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Define, for $j \geq 0$,*

$$c_j = \sum_{|\alpha|=j} \frac{(-1)^{|\alpha|}}{\alpha!}\overline{\partial_\xi^\alpha D_x^\alpha a}, \quad d_j = \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial_\xi^\alpha a D_x^\alpha b.$$

Then $\sum_j c_j, \sum_j d_j \in \mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. We have analogous considerations for the finite order case $\mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n})$.

Definition 2.10. Let $\sum_j a_j, \sum_j b_j \in \mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. The notation $\sum_j a_j \sim \sum_j b_j$ means that there are constants $H, C, c, B > 0$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta \sum_{j<N}(a_j - b_j)(x,\xi)| \leq HC^{|\alpha+\beta|+2N}\alpha!^\mu \beta!^\nu N!^{\mu+\nu-1}\langle\xi\rangle^{-|\alpha|-N}e^{c|\xi|^{\frac{1}{\theta}}},$$

for every $\alpha, \beta \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$, $N \in \mathbb{N}$ and $\langle\xi\rangle \geq B(N) := BN^{\mu+\nu-1}$. An analogous definition can be established for $\mathbf{FS}^m_{\mu,\nu}(\mathbb{R}^{2n})$.

Proposition 2.2. *If $\sum_j a_j \in \mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$, then there exists a symbol $a \in \mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ such that $a \sim \sum_j a_j$ in $\mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. An analogous result holds for $\mathbf{FS}^m_{\mu,\nu}(\mathbb{R}^{2n})$.*

Proof. We address the reader to Proposition 2.2 in [2]. □

The next result is responsible to provide us a way to define regularizing operators for our classes.

Proposition 2.3. *Let $a, b \in \mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $\sum_j a_j \in \mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. If $a \sim \sum_j a_j \sim b$ in $\mathbf{FS}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $\theta > \mu + \nu - 1$, then there exist constants $H, C, c > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta (a-b)(x,\xi)| \leq HC^{|\alpha+\beta|}\alpha!^\mu \beta!^\nu e^{-c|\xi|^{\frac{1}{r}}}, \quad x, \xi \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n,$$

where $r \geq \mu + \nu - 1$.

Proof. Check Proposition 2.3 of [2]. □

In an analogous way, a similar result can be obtained for the classes of finite order, except that in this case we do not need to ask any hypothesis over the parameter θ.

Proposition 2.4. *Let $a, b \in \mathbf{S}^m_{\mu,\nu}(\mathbb{R}^{2n})$ and $\sum_j a_j \in \mathbf{FS}^m_{\mu,\nu}(\mathbb{R}^{2n})$. If $a \sim \sum_j a_j \sim b$ in $\mathbf{FS}^m_{\mu,\nu}(\mathbb{R}^{2n})$ then there exist constants $H, C, c > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta (a-b)(x,\xi)| \leq HC^{|\alpha+\beta|}\alpha!^\mu \beta!^\nu e^{-c|\xi|^{\frac{1}{r}}}, \quad x, \xi \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n,$$

where $r \geq \mu + \nu - 1$.

Definition 2.11. For any $\tilde{r} > 1$, we define $\mathscr{K}_{\tilde{r}}$ as the space of all symbols $q \in \mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ such that for every $r \geq \tilde{r}$ there exist positive constants C and c such that

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq C^{|\alpha+\beta|+1}\alpha!^r \beta!^r e^{-c|\xi|^{\frac{1}{r}}}, \quad x, \xi \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n.$$

The operator $Q = q(x, D)$ is said to be $\tilde{r}-$regularizing whenever its symbol $q(x,\xi)$ belongs to $\mathscr{K}_{\tilde{r}}$.

Remark 2.11. *Operators with symbols in $\mathscr{K}_{\tilde{r}}$ are regularizing in the sense that they can be extended to linear and continuous maps from $(\gamma_0^\theta)'(\mathbb{R}^n)$ to $\gamma^\theta(\mathbb{R}^n)$ if $\theta > \tilde{r}$. This can be easily proved observing that the Fourier transform of $u \in (\gamma_0^\theta)'(\mathbb{R}^n)$ goes like $\exp(-c|\xi|^{\frac{1}{\theta}})$ for some $c > 0$ and arguing as in [44, Lemma 3.2.12]. See also Theorem 2.8 for other regularizing properties of these operators.*

2.2.3 Adjoint, transpose and composition

This section is dedicated to analyse how to deal with the composition, transpose and adjoint of our pseudo-differential operators. The first result concerns the adjoint and the transpose, and its proof can be checked in [2], Theorem 2.6.

Theorem 2.5. *Let $p \in \mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ with $\mu, \nu > 1$ and $\theta > \mu+\nu-1$. Let moreover $p^*(x, D)$ and ${}^t p(x, D)$ the L^2 adjoint and the transpose of $p(x, D)$ respectively. Then there exist symbols $q_j \in \mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $r_j \in \mathscr{K}_{\mu+\nu-1}$, $j = 1, 2$, such that*

$$p^*(x, D) = q_1(x, D) + r_1(x, D), \quad q_1(x,\xi) \sim \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!}\overline{\partial_\xi^\alpha D_x^\alpha p(x,\xi)} \text{ in } FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$$

and

$${}^t p(x, D) = q_2(x, D) + r_2(x, D), \quad q_2(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(\partial_\xi^\alpha D_x^\alpha p)(x, -\xi) \text{ in } FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n}).$$

By Lemma 2.5 in [2], if $\mu > 1$, $\nu > 1$, θ satisfies $\theta > \mu+\nu-1$, $r \in \mathscr{K}_{\mu+\nu-1}$ and $p \in \mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$, then all the operators ${}^t r(x, D)$, $r^*(x, D)$ and $p(x, D) \circ r(x, D)$ are generated by symbols in $\mathscr{K}_{\mu+\nu-1}$. In other words, for regularizing symbols, the transpose and the adjoint do not lose the characteristic of being given by symbols which still are regularizing. The same holds if we compose an operator of infinite order symbol with an operator with regularizing symbol. The next result concerns about the composition of two pseudo-differential operators whose symbols are in $\mathbf{S}^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. A complete proof for such result can be found in Theorem 2.7 of [2].

Theorem 2.6. *Let us consider the parameters $\mu, \nu > 1$ and $\theta > \mu + \nu - 1$. If p and q are symbols in $\mathbf{S}^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$, then there exist symbols $s \in \mathbf{S}^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $r \in \mathscr{K}_{\mu+\nu-1}$ such that*

$$p(x,D) \circ q(x,D) = s(x,D) + r(x,D), \quad s(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x,\xi) D_x^{\alpha} q(x,\xi) \text{ in } \mathbf{FS}^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}).$$

Remark 2.12. *To conclude this subsection, we point out that analogous results concerning the product, transpose and adjoints of pseudo-differential operators with symbols in $\mathbf{S}^{m}_{\mu,\nu}(\mathbb{R}^{n})$ can be obtained. Besides, in the frame of finite order we do not need to require any hypothesis over the parameter θ.*

Pseudo-differential operators of finite order act in Gevrey-Sobolev spaces according to the next Theorem.

Theorem 2.7. *Let $p \in \mathbf{S}^{m'}_{\mu,\nu}(\mathbb{R}^{2n})$ and consider the parameters $\theta > \mu + \nu - 1$ and $m, \rho \in \mathbb{R}$. Then*

$$p(x,D)\colon H^{m}_{\rho;\theta}(\mathbb{R}^{n}) \to H^{m-m'}_{\rho;\theta}(\mathbb{R}^{n}), \text{ continuously.}$$

Proof. Check Theorem 2.8 in [2]. □

To conclude, the next result tells us how regularizing operators act in Gevrey-Sobolev spaces. The proof can be consulted in [2], Theorem 2.9.

Theorem 2.8. *If $\mu, \nu > 1$, $\theta > \mu + \nu - 1$ and $r \in \mathscr{K}_{\mu+\nu-1}$, then for every $m \in \mathbb{R}$ and $\tilde{\rho} \in \mathbb{R}$*

$$r(x,D)\colon H^{m}_{\tilde{\rho};\theta}(\mathbb{R}^{n}) \to \bigcap_{\rho \in \mathbb{R}} H^{m}_{\rho;\theta}(\mathbb{R}^{n}), \text{ continuously.}$$

2.3 $\mathbf{SG}^{m}_{\mu,\nu}(\mathbb{R}^{2n})$ and $\mathbf{\Gamma G}^{m}_{\mu,\nu}(\mathbb{R}^{2n})$ classes

The symbol classes introduced here satisfy some Gevrey estimates and the boundedness is given in terms of weights with respect to the both variables x and ξ, namely $\langle x \rangle$ and $\langle \xi \rangle$. We start with some definitions.

Definition 2.12. Let $A > 0$, $m := (m_1, m_2) \in \mathbb{R}^2$ and $\mu, \nu \geq 1$. The space $\mathbf{SG}^{m}_{\mu,\nu}(\mathbb{R}^{2n}, A)$ is defined as the Banach space of all smooth functions $p(x,\xi)$ satisfying

$$|p|_A := \sup_{\alpha,\beta \in \mathbb{N}_0^n} \sup x, \xi \in \mathbb{R}^n A^{-|\alpha+\beta|} \alpha!^{-\mu} \beta!^{-\nu} \langle \xi \rangle^{-m_1+|\alpha|} \langle x \rangle^{-m_2+|\alpha|} |\partial_{\xi}^{\alpha} \partial_x^{\beta} p(x,\xi)| < +\infty.$$

We set

$$\boxed{\mathbf{SG}^{m}_{\mu,\nu}(\mathbb{R}^{2n}) := \bigcup_{A>0} \mathbf{SG}^{m}_{\mu,\nu}(\mathbb{R}^{2n}, A)}$$

equipped with the inductive limit topology of the Banach spaces $\mathbf{SG}^{m}_{\mu,\nu}(\mathbb{R}^{2n}, A)$, and

$$\boxed{\mathbf{\Gamma G}^{m}_{\mu,\nu}(\mathbb{R}^{2n}) := \bigcap_{A>0} \mathbf{SG}^{m}_{\mu,\nu}(\mathbb{R}^{2n}, A)}$$

equipped with the projective limit topology of the Banach spaces $\mathbf{SG}^{m}_{\mu,\nu}(\mathbb{R}^{2n}, A)$.

Remark 2.13. *Again, we can use a simpler notation to represent these classes when* $\mu = \nu$*, namely*

$$\mathbf{SG}^m_\mu(\mathbb{R}^{2n}) \quad \text{and} \quad \mathbf{\Gamma G}^m_\mu(\mathbb{R}^{2n}).$$

Remark 2.14. *Notice that, if* $\mu_1 < \mu_2$*, then for every* $A > 0$ *we have* $\mathbf{SG}^m_{\mu_1}(\mathbb{R}^{2n}; A) \subset \mathbf{\Gamma G}^m_{\mu_2}(\mathbb{R}^{2n})$*.*

Besides the continuity properties on weighted Sobolev spaces inherited from $\mathbf{SG}^m(\mathbb{R}^{2n})$ and $\mathbf{\Gamma G}^m(\mathbb{R}^{2n})$ the operators with symbol in $\mathbf{SG}^m_{\mu,\nu}(\mathbb{R}^{2n})$ and $\mathbf{\Gamma G}^m_{\mu,\nu}(\mathbb{R}^{2n})$ are continuous in Gelfand-Shilov-Sobolev in the following sense, cf. [15, Theorem A.18].

Theorem 2.9. *If* $p \in \mathbf{SG}^{m'}_{\mu,\nu}(\mathbb{R}^{2n})$ *for some* $m' \in \mathbb{R}^2$*, then for any* $m, \rho \in \mathbb{R}^2$ *and* s, θ *satisfying* $\min\{s, \theta\} > \mu + \nu - 1$*, the operator* $p(x, D)$ *maps* $H^m_{\rho;s,\theta}(\mathbb{R}^n)$ *into* $H^{m-m'}_{\rho;s,\theta}(\mathbb{R}^n)$ *continuously.*

We conclude this section with a result we shall use intensively in the proof of our main result. This can be regarded as a variant of sharp Gårding inequality for Gevrey regular **SG** symbols. This result follows directly from Theorem 6 in [9].

Theorem 2.10. *Let* $p \in \mathbf{SG}^m_{\mu,\nu}(\mathbb{R}^{2n})$ *such that* $\mathbf{Re}\, p(x, \xi) \geq 0$*. Then there exist* $q \in \mathbf{SG}^m_{\mu,\nu}(\mathbb{R}^{2n})$*,* $r \in \mathbf{SG}^{m-(1,1)}(\mathbb{R}^{2n})$ *and* $r_\infty \in \mathscr{K}_k$*, with* $k > \mu + \nu - 1$ *such that*

$$p(x, D) = q(x, D) + r(x, D) + r_\infty(x, D)$$

and

$$(q(x, D)v, v)_{L^2(\mathbb{R}^n)} \geq 0, \quad \forall v \in \mathscr{S}(\mathbb{R}^n).$$

Chapter 3

CAUCHY PROBLEM FOR p-EVOLUTION OPERATORS WITH DATA IN GEVREY SPACES

3.1 Well-posedness in inductive Gevrey spaces

For the main result of this chapter, we consider an integer number $p \geq 2$ and the class of differential operators of the type

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=1}^{p} a_{p-j}(t,x)D_x^{p-j}, \quad (t,x) \in [0,T] \times \mathbb{R}, \tag{3.1}$$

where $T > 0$, the symbol $a_p(t,\xi) := a_p(t)\xi^p$ is real-valued with order p and $a_{p-j}(t,x,\xi) := a_{p-j}(t,x)\xi^{p-j}$ has order $p-j$ (with respect to ξ), for each $j = 1,...,p$.

Remark 3.1. *It is easy to notice that $a_p(t,\xi) = a_p(t)\xi^p$ is the symbol of the differential operator $a_p(t)D_x^p$ and, for each $j = 1,...,p$, $a_{p-j}(t,x,\xi) = a_{p-j}(t,x)\xi^{p-j}$ is the symbol of the differential operator $a_{p-j}(t,x)D_x^{p-j}$.*

Our goal is to study the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = g(x), & x \in \mathbb{R} \end{cases}, \tag{3.2}$$

with data f and g in Gevrey-Sobolev spaces $H^m_{\rho;\theta}(\mathbb{R})$. In our first main result, we will prove what we call the *well-posedness* in $\mathcal{H}^\infty_\theta(\mathbb{R})$ for the Cauchy problem, which is defined in the following.

Definition 3.1. We say that the Cauchy problem (3.2) is *well-posed* in $\mathcal{H}^\infty_\theta(\mathbb{R})$ when, for any given $\rho > 0$ there exists $\tilde{\rho} > 0$ and a constant $C := C(\rho,T) > 0$ such that, for all $f \in C\left([0,T]; H^m_{\rho;\theta}(\mathbb{R})\right)$ and $g \in H^m_{\rho;\theta}(\mathbb{R})$, there exists a unique solution $u \in C^1\left([0,T]; H^m_{\tilde{\rho};\theta}(\mathbb{R})\right)$ and the following energy estimate holds

$$\|u(t,\cdot)\|_{H^m_{\tilde{\rho};\theta}} \leq C\left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau,\cdot)\|^2_{H^m_{\rho;\theta}} d\tau\right).$$

The first result we intend to prove in this chapter is stated in the following.

Theorem 3.1. *Let $\theta_0 > 1$ and $\sigma \in \left(\frac{p-2}{p-1}, 1\right)$ such that $\theta_0 < \frac{1}{(p-1)(1-\sigma)}$. Let P be an operator of the type (3.1) whose coefficients satisfy the following assumptions:*

(i) $a_p \in C([0,T];\mathbb{R})$ *and there exists* $C_{a_p} > 0$ *such that* $|a_p(t)| \geq C_{a_p}$, *for all* $t \in [0,T]$.

(ii) $|\partial_x^\beta a_{p-j}(t,x)| \leq C_{a_{p-j}}^{\beta+1} \beta!^{\theta_0} \langle x \rangle^{-\frac{p-j}{p-1}\sigma - \beta}$, *for some* $C_{a_{p-j}} > 0$, $j = 1, ..., p-1$ *and for all* $\beta \in \mathbb{N}_0$, $(t,x) \in [0,T] \times \mathbb{R}$.

If $\theta > 1$ *is such that* $\theta_0 \leq \theta < \frac{1}{(p-1)(1-\sigma)}$, *the data* $f \in C\left([0,T]; H^m_{\rho;\theta}(\mathbb{R})\right)$ *and* $g \in H^m_{\rho;\theta}(\mathbb{R})$, *with* $m, \rho \in \mathbb{R}$ *and* $\rho > 0$, *then the Cauchy problem* (3.2) *admits a unique solution* $u \in C^1\left([0,T]; H^m_{\tilde{\rho};\theta}(\mathbb{R})\right)$ *for some* $\tilde{\rho} \in (0,\rho)$, *and the solution satisfies the energy estimate*

$$\|u(t)\|^2_{H^m_{\tilde{\rho};\theta}} \leq C\left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau)\|^2_{H^m_{\rho;\theta}} d\tau\right), \tag{3.3}$$

for all $t \in [0,T]$ *and for some constant* $C > 0$. *In particular, for* $\theta \in \left[\theta_0, \frac{1}{(p-1)(1-\sigma)}\right)$ *the Cauchy problem* (3.2) *is well-posed in* $\mathcal{H}^\infty_\theta(\mathbb{R})$.

In [6], Theorem 1, the authors have proved the above theorem for $p = 3$. They noticed that the conclusion of the result cannot be achieved in a straight way, however, by performing a suitable change of variable in order to obtain an equivalent Cauchy problem, the theorem can be proved after several steps. Another similar result (by changing some hypothesis and/or functional classes) can be found in other works developed by the same authors, namely [4, 5]. We recommend the P.h.D. thesis [2], Section 3.2, for a very well detailed description about the obstacles found to obtain the energy estimate in the Cauchy problem for 3-evolution equations. Similar results had been proved in [33] for $p = 2$. The novelty in this thesis is that we will prove Gevrey well-posedness for a general p. This extension leads to non-trivial technical difficulties as we will see in the sequel.

Remark 3.2. *Comparing the assumptions of Theorem 3.1 with the known literature for 3-evolution equations, e.g., [6, Theorem 1.2], we notice that our decay at infinity condition (ii) is given on the whole coefficients of the lower-order terms (without distinguishing the behaviour of the real and the imaginary parts) and prescribes for the derivatives of the coefficients a decay that increases with the order of the derivatives. This extra decay is used in Subsection 3.5 to absorb a growth in* x *of some terms appearing, for instance, in* (3.32) *and* (3.33). *Without this assumption, the estimate of these terms is possible but more involved. Moreover, under the condition (ii), the symbols* $a_{p-j}(t,x)\xi^{p-j}$ *are symbols in* **SG** *classes, and this allows to apply Theorem 2.10 in Section 3.6. As far as we know, a version of this theorem for Gevrey regular Hörmander symbols with a precise estimate of the regularity of the remainders is still missing in the literature, although it is a somewhat expected result. In conclusion, we believe that we could have distinguished conditions on real and imaginary parts of the coefficients as in [13, Theorem 1.1] or [6, Theorem 1.2], and we could have assumed (ii) only for a finite number of derivatives or avoided the extra decay for the* x*-derivatives, but we preferred to skip these refinements in order to not add further technicality to the proof and to work in the frame of the* **SG** *calculus.*

3.2 The description of the change of variable

As explained in the introduction, in order to prove Theorem 3.1, it is necessary to make a change of variable and to transform the Cauchy problem (3.2) into an equivalent problem which will be well-posed in Sobolev spaces and then obtain Gevrey well-posedness for (3.2) by using the inverse change of variable. To perform this change of variable, we shall consider an invertible pseudo-differential operator of infinite order given by

$$Q_{\Lambda,K,\rho'}(t,x,D) = e^{\Lambda_{K,\rho'}}(t,D) \circ e^{\Lambda}(x,D), \tag{3.4}$$

where

$$\Lambda(x,\xi) = \sum_{k=1}^{p-1} \lambda_{p-k}(x,\xi) \in \mathbf{SG}^{0,1-\sigma}_{\mu}(\mathbb{R}^2) \cap \mathbf{S}^{(p-1)(1-\sigma)}_{\mu}(\mathbb{R}^2)$$

for some $\mu > 1$ and functions λ_{p-k} which will be defined in (3.6), and

$$\Lambda_{K,\rho'}(t,\xi) = K(T-t)\langle\xi\rangle_h^{(p-1)(1-\sigma)} + \rho'\langle\xi\rangle_h^{1/\theta}$$

with $0 < \rho' < \rho$, $K > 0$ and $\langle\xi\rangle_h := \sqrt{h^2+\xi^2}$ for $h >> 1$ to be chosen later. The inverse operator $Q_{\Lambda,K,\rho'}(t,x,D)^{-1}$ will recover the solution $u = Q_{\Lambda,K,\rho'}(t,x,D)^{-1}v$ of (3.2), where v is the solution of the auxiliary problem. The properties of the operator $Q_{\Lambda,K,\rho'}(t,x,D)$ and its inverse are responsible to determine the space where the Cauchy problem (3.2) is well-posed. Now, let us describe each part of the operator $Q_{\Lambda,K,\rho'}(t,x,D)$.

- The role of each factor $e^{\lambda_{p-j}}$ of the symbol $e^{\Lambda} = e^{\lambda_{p-1}} \cdot \ldots \cdot e^{\lambda_{p-1}}$ in the conjugation with $e^{\Lambda}(x,D)$ is to turn $\mathbf{Im}\, a_{p-j}(t,x,D)$, $j = 1,\ldots,p-1$, into the sum of a positive operator plus a term of lower order, without changing the parts of order $p, p-1, ..., p-j+1$ of the operator. Summing up, the conjugation with $e^{\Lambda}(x,D)$ turns the operator $P(t,x,D_t,D_x)$ into a sum of positive operators plus a remainder of order $(p-1)(1-\sigma)$.
- The operator $e^{K(T-t)\langle D\rangle_h^{(p-1)(1-\sigma)}}$ does not change terms of order $1, ..., p$, but it corrects the error of order $(p-1)(1-\sigma)$ coming from the previous transformation by changing it into the sum of a positive operator plus a remainder of order zero. This is obtained by choosing K sufficiently large.
- The term $e^{\rho'\langle D\rangle_h^{1/\theta}}$ is the leading term of the transformation $Q_{\Lambda,K,\rho'}(t,x,D)$, since we are assuming $(p-1)(1-\sigma) < 1/\theta$ and $\rho' > 0$: it changes the setting of the Cauchy problem from Gevrey-Sobolev-type spaces to the standard Sobolev spaces. Moreover, since $\rho' > 0$, the inverse operator $(Q_{\Lambda,K,\rho'}(t,x,D))^{-1}$ has regularizing properties with respect to the spaces $H^m_{\rho;\theta}(\mathbb{R})$, i.e., it maps $H^m(\mathbb{R})$ into a Gevrey-Sobolev space $H^m_{\rho'-\delta;\theta}(\mathbb{R})$ for every positive δ.

In next sections we will give more details about this change of variable. By denoting

$$P_{\Lambda,K,\rho'}(t,x,D_t,D_x) := Q_{\Lambda,K,\rho'}(t,x,D_x) \circ (iP)(t,x,D_t,D_x) \circ (Q_{\Lambda,K,\rho'}(t,x,D_x))^{-1},$$

just notice that the Cauchy problem (3.2) is equivalent to the auxiliary Cauchy problem

$$\begin{cases} P_{\Lambda,K,\rho'}(t,x,D_t,D_x)v(t,x) = Q_{\Lambda,K,\rho'}(t,x,D_x)f(t,x), & (t,x)\in[0,T]\times\mathbb{R} \\ v(0,x) = Q_{\Lambda,K,\rho'}(0,x,D_x)g(x), & x\in\mathbb{R} \end{cases}, \tag{3.5}$$

in the sense that if u solves (3.2), then $v = Q_{\Lambda,K,\rho'}(t,x,D_x)u$ solves (3.5) and, if v solves (3.5), then $u = (Q_{\Lambda,K,\rho'}(t,x,D_x))^{-1}v$ solves (3.2).

3.3 The functions $\lambda_{p-k}(x,\xi)$, $k = 1, ..., p-1$

As we described in the previous section, one of the parts used in the change of variable is the operator $e^{\Lambda}(x,D)$, where $\Lambda = \lambda_{p-1} + \cdots + \lambda_1$. Let us define each one of the functions λ_{p-k}. For each $k = 1, ..., p-1$, let $M_{p-k} > 0$ to be chosen later on and define

$$\lambda_{p-k}(x,\xi) := M_{p-k}\omega\left(\frac{\xi}{h}\right)\langle\xi\rangle_h^{1-k}\int_0^x \langle y\rangle^{-\frac{p-k}{p-1}\sigma}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy, \tag{3.6}$$

where ω and ψ are C^∞ functions such that

$$\omega(\xi) = \begin{cases} 0, & |\xi|\le 1 \\ -\mathrm{sgn}(a_p(t)), & |\xi| > R_{a_p} \end{cases}, \qquad \psi(y) = \begin{cases} 1, & |y|\le\frac{1}{2} \\ 0, & |y|\ge 1 \end{cases}$$

and these functions also satisfy

$$|\partial_\xi^\alpha\omega(\xi)| \le C_\omega^{\alpha+1}\alpha!^\mu \qquad \text{and} \qquad |\partial_y^\beta\psi(y)| \le C_\psi^{\beta+1}\beta!^\mu.$$

Notice that, by assumption (i) in Theorem 3.1, ω is constant for $|\xi|\ge R_{a_p}$, hence, if $\alpha\ne 0$ we have that $\omega^{(\alpha)}\left(\frac{\xi}{h}\right)$ is supported for $|\xi|/h \le R_{a_p}$, which gives us

$$h^{-\alpha} \le \langle\xi\rangle_h^{-\alpha}\langle R_{a_p}\rangle^\alpha.$$

Remark 3.3. *The $h-$bracket $\langle\cdot\rangle_h$ satisfies*

$$|\partial_\xi^\alpha\langle\xi\rangle_h^m| \le C_m^\alpha\alpha!\langle\xi\rangle_h^{m-\alpha}, \quad \xi\in\mathbb{R},\ \alpha\in\mathbb{N}_0,$$

where $C_m > 0$ is independent of h. We can replace $\langle\xi\rangle$ by $\langle\xi\rangle_h$ because this does not change the symbol classes and it is very useful in the proofs of our results (for instance, to obtain the invertibility $Q_{\Lambda,K,\rho'}(t,x,D)$). The new class obtained by replacing $\langle\xi\rangle$ for $\langle\xi\rangle_h$ is $\mathbf{S}^m_{h,\mu,\nu}(\mathbb{R}^2)$, and a smooth function p belongs to $\mathbf{S}^m_{h,\mu,\nu}(\mathbb{R}^2)$ if there exists $C > 0$ such that

$$|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)| \le C^{\alpha+\beta+1}\alpha!^\mu\beta!^\nu\langle\xi\rangle_h^{m-\alpha}, \quad \alpha,\beta\in\mathbb{N}_0,\ x,\xi\in\mathbb{R}.$$

Since $h >> 1$, it follows that

$$\langle\xi\rangle \le \langle\xi\rangle_h \le h\langle\xi\rangle, \quad \xi\in\mathbb{R},$$

hence $\mathbf{S}^m_{\mu,\nu}(\mathbb{R}^2) = \mathbf{S}^m_{h,\mu,\nu}(\mathbb{R}^2)$ *with equivalent topologies. Besides,* $\langle\xi\rangle$ *and* $\langle\xi\rangle_h$ *are asymptotically equivalent, which means that*

$$\lim_{|\xi|\to\infty} \frac{\langle\xi\rangle_h}{\langle\xi\rangle} = 1.$$

The above limit implies that for any $\epsilon > 0$ *there exists* $R_{h,\epsilon} > 0$ *such that*

$$\langle\xi\rangle_h \leq (1+\epsilon)\langle\xi\rangle, \quad |\xi| \geq R_{h,\epsilon}.$$

Finally, the exchange of $\langle\xi\rangle$ *for* $\langle\xi\rangle_h$ *does not change the dependence of the constants appearing in this work, which will be independent of* h*.*

The following result summarizes some properties about the functions λ_{p-k}.

Lemma 3.1. *For each* $k = 1, ..., p-1$*, the following statements are true:*

(i) $|\lambda_{p-k}(x,\xi)| \leq \frac{M_{p-k}}{1-\frac{p-k}{p-1}\sigma}\langle\xi\rangle_h^{(p-k)(1-\sigma)}$.

(ii) $|\partial_\xi^\alpha \lambda_{p-k}(x,\xi)| \leq C^{\alpha+1}\alpha!^\mu \langle\xi\rangle_h^{(p-k)(1-\sigma)-\alpha}$, *for all* $\alpha \geq 1$.

(iii) $|\partial_\xi^\alpha \lambda_{p-k}(x,\xi)| \leq C^{\alpha+1}\alpha!^\mu \langle\xi\rangle_h^{1-k-\alpha}\langle x\rangle^{1-\frac{p-k}{p-1}\sigma}$, *for all* $\alpha \geq 0$.

(iv) $|\partial_\xi^\alpha \lambda_{p-k}(x,\xi)| \leq C^{\alpha+1}\alpha!^\mu \langle\xi\rangle_h^{-\alpha}\langle x\rangle^{\frac{p-k}{p-1}(1-\sigma)}$, *for all* $\alpha \geq 0$.

(v) $|\partial_\xi^\alpha \partial_x^\beta \lambda_{p-k}(x,\xi)| \leq C^{\alpha+\beta+1}(\alpha!\beta!)^\mu \langle\xi\rangle_h^{1-k-\alpha}\langle x\rangle^{-\frac{p-k}{p-1}\sigma-(\beta-1)}$, *for all* $\alpha \geq 0$ *and* $\beta \geq 1$.

Proof. Let us denote by $\chi_\xi(x)$ the characteristic function of the set $\{x \in \mathbb{R}^n; \langle x\rangle \leq \langle\xi\rangle_h^{p-1}\}$. Then, it follows that

$$\begin{aligned}
|\lambda_{p-k}(x,\xi)| &= M_{p-k}\left|\omega\left(\frac{\xi}{h}\right)\right|\langle\xi\rangle_h^{-k+1}\left|\int_0^x \langle y\rangle^{-\frac{p-k}{p-1}\sigma}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy\right| \\
&\leq M_{p-k}\langle\xi\rangle_h^{-k+1}\int_0^{|x|}\langle y\rangle^{-\frac{p-k}{p-1}\sigma}\chi_\xi(y)dy \\
&\leq M_{p-k}\langle\xi\rangle_h^{-k+1}\int_0^{\min\{|x|,\langle\xi\rangle_h^{p-1}\}}\langle y\rangle^{-\frac{p-k}{p-1}\sigma}dy \\
&\leq M_{p-k}\langle\xi\rangle_h^{-k+1}\int_0^{\min\{\langle x\rangle,\langle\xi\rangle_h^{p-1}\}} y^{-\frac{p-k}{p-1}\sigma}dy \\
&= \frac{M_{p-k}}{1-\frac{p-k}{p-1}\sigma}\langle\xi\rangle_h^{-k+1}\left[y^{1-\frac{p-k}{p-1}\sigma}\right]_0^{\min\{\langle x\rangle,\langle\xi\rangle_h^{p-1}\}} \\
&= \frac{M_{p-k}}{1-\frac{p-k}{p-1}\sigma}\langle\xi\rangle_h^{-k+1}(\min\{\langle x\rangle,\langle\xi\rangle_h^{p-1}\})^{1-\frac{p-k}{p-1}\sigma} \\
&\leq \frac{M_{p-k}}{1-\frac{p-k}{p-1}\sigma}\langle\xi\rangle_h^{-k+1}(\langle\xi\rangle_h^{p-1})^{1-\frac{p-k}{p-1}\sigma} \\
&= \frac{M_{p-k}}{1-\frac{p-k}{p-1}\sigma}\langle\xi\rangle_h^{-k+1}\langle\xi\rangle_h^{p-1-(p-k)\sigma} \\
&= \frac{M_{p-k}}{1-\frac{p-k}{p-1}\sigma}\langle\xi\rangle_h^{(p-k)(1-\sigma)}
\end{aligned}$$

which gives us (i). Similarly, we have

$$|\lambda_{p-k}(x,\xi)| \leq \frac{M_{p-k}}{1-\frac{p-k}{p-1}\sigma}\langle\xi\rangle_h^{1-k}\langle x\rangle^{1-\frac{p-k}{p-1}\sigma}.$$

Now, we want to obtain the estimate (ii) for $|\partial_\xi^\alpha \lambda_{p-k}(x,\xi)|$. Note that, by Leibniz rule we obtain

$$\begin{aligned}
\partial_\xi^\alpha \lambda_{p-k}(x,\xi) &= \partial_\xi^\alpha \left\{ M_{p-k}\omega\left(\frac{\xi}{h}\right)\langle\xi\rangle_h^{1-k}\int_0^x \langle y\rangle^{-\frac{p-k}{p-1}\sigma}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy\right\} \\
&= M_{p-k}\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\left[\partial_\xi^{\alpha_1}\omega\left(\frac{\xi}{h}\right)\right]\left[\partial_\xi^{\alpha_2}\langle\xi\rangle_h^{-1+k}\right] \\
&\times \int_0^x \langle y\rangle^{-\frac{p-k}{p-1}\sigma}\left[\partial_\xi^{\alpha_3}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right]dy \qquad (3.7)
\end{aligned}$$

Now we need to find a way to deal with the derivatives which appears in (3.7). The first one is very simple; the second one we can estimate by $\partial_\xi\langle\xi\rangle_h^m \leq C_m^\alpha \alpha!\langle\xi\rangle_h^{m-\alpha}$, $\xi\in\mathbb{R}$, $\alpha\in\mathbb{N}$, where C_m is a positive constant independent of h; the third derivative needs to be computed by using Faà di Bruno's formula in the following way

$$\partial_\xi^{\alpha_3}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right) = \sum_{j=1}^{\alpha_3}\frac{1}{j!}\psi^{(j)}\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)\sum_{\gamma_1+\cdots+\gamma_j=\alpha_3}\frac{\alpha_3!}{\gamma_1!\cdots\gamma_j!}\prod_{\ell=1}^{j}\partial_\xi^{\gamma_\ell}\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right).$$

Hence, we can estimate in (3.7) by taking the modulo

$$\begin{aligned}
|\partial_\xi^\alpha \lambda_{p-k}(x,\xi)| &\leq M_{p-k}\sum_{\alpha+\alpha_2+\alpha_3=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\left|\omega^{(\alpha_1)}\left(\frac{\xi}{h}\right)\right| h^{-\alpha_1}|\partial_\xi^{\alpha_2}\langle\xi\rangle_h^{-k+1}| \\
&\times \int_0^{|x|}\chi_\xi(y)\langle y\rangle^{-\frac{p-k}{p-1}\sigma}\sum_{j=1}^{\alpha_3}\frac{1}{j!}\left|\psi^{(j)}\left(\frac{\langle y\rangle}{\langle\xi\rangle^{p-1}}\right)\right| \\
&\times \sum_{\gamma_1+\cdots+\gamma_j=\alpha_3}\frac{\alpha_3!}{\gamma_1!\cdots\gamma_j!}\prod_{\ell=1}^{j}\partial_\xi^{\gamma_\ell}\langle\xi\rangle_h^{1-p}\langle y\rangle dy \\
&\leq M_{p-k}\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}C_\omega^{\alpha_1+1}\alpha_1!^\mu\langle\xi\rangle_h^{-\alpha_1}\langle R_{a_p}\rangle^{\alpha_1}C^{\alpha_2}\alpha_2!\langle\xi\rangle_h^{1-k-\alpha_2} \\
&\times \int_0^{|x|}\chi_\xi(y)\langle y\rangle^{-\frac{p-k}{p-1}\sigma}\sum_{j=1}^{\alpha_3}\frac{1}{j!}C_\psi^{j+1} \\
&\times \sum_{\gamma_1+\cdots+\gamma_j=\alpha_3}\frac{\alpha_3!}{\gamma_1!\cdots\gamma_j!}\langle y\rangle\prod_{\ell=1}^{j}C^{\gamma_\ell}\gamma_\ell!\langle\xi\rangle_h^{1-p-\gamma_\ell}dy \\
&\leq M_{p-k}\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}C_\omega^{\alpha_1+1}\alpha_1!^\mu\langle\xi\rangle_h^{-\alpha_1}\langle R_{a_p}\rangle^{\alpha_1}C^{\alpha_2}\alpha_2!\langle\xi\rangle_h^{1-k}\langle\xi\rangle_h^{-\alpha_2} \\
&\times \int_0^{|x|}\langle y\rangle^{-\frac{p-k}{p-1}\sigma}\sum_{j=1}^{\alpha_3}\frac{1}{j!}C_\psi^{\alpha_3+1}\alpha_3!^\mu\sum_{\gamma_1+\cdots+\gamma_j=\alpha_3}\frac{\alpha_3}{\gamma_1!\cdots\gamma_j!}C^{\alpha_3}\gamma_1!\cdots\gamma_j!\langle\xi\rangle_h^{-\alpha_3}.
\end{aligned}$$

We obtain

$$
\begin{aligned}
|\partial_\xi^\alpha \lambda_{p-k}(x,\xi)| &\leq M_{p-k} C_{\omega,\psi,R_{a_p}}^{\alpha+1} \alpha!^\mu \langle \xi \rangle_h^{-\alpha} \langle \xi \rangle_h^{1-k} \int_0^{|x|} \chi_\xi(y) \langle y \rangle^{-\frac{p-k}{p-1}\sigma} dy \\
&\leq \frac{M_{p-k}}{1-\frac{p-k}{p-1}\sigma} C_{\omega,\psi,R_{a_p}}^{\alpha+1} \langle \xi \rangle_h^{-\alpha} \langle \xi \rangle_h^{(p-k)(1-\sigma)} \\
&= \frac{M_{p-k}}{1-\frac{p-k}{p-1}\sigma} C_{\omega,\psi,R_{a_p}}^{\alpha+1} \langle \xi \rangle_h^{(p-k)(1-\sigma)-\alpha},
\end{aligned}
$$

and we get (ii). To obtain (iii), we just need to observe that

$$\int_0^{|x|} \chi_\xi(y) \langle y \rangle^{-\frac{p-k}{p-1}\sigma} dy \leq \langle x \rangle^{1-\frac{p-k}{p-1}\sigma}.$$

To obtain (iv), we use the fact that, on the support of $\psi(\langle y \rangle / \langle \xi \rangle_h^{p-1})$, we have $\langle \xi \rangle_h^{1-k} \leq \langle x \rangle^{\frac{1-k}{p-1}}$, which implies

$$\langle \xi \rangle_h^{1-k-\alpha} \langle x \rangle^{1-\frac{p-k}{p-1}\sigma} \leq \langle \xi \rangle_h^{-\alpha} \langle x \rangle^{\frac{p-k}{p-1}(1-\sigma)}.$$

Now, by considering $\beta \geq 1$, we have

$$\partial_x^\beta \lambda_{p-k}(x,\xi) = M_{p-k} \omega\left(\frac{\xi}{h}\right) \langle \xi \rangle_h^{-k+1} \partial_x^\beta \int_0^x \langle y \rangle^{-\frac{p-k}{p-1}\sigma} \psi\left(\frac{\langle y \rangle}{\langle \xi \rangle_h^{p-1}}\right) dy. \tag{3.8}$$

Note that, we need to compute the derivative of order β of the integral. If $\beta = 1$, we have

$$\partial_x \int_0^x \langle y \rangle^{-\frac{p-k}{p-1}\sigma} \psi\left(\frac{\langle y \rangle}{\langle \xi \rangle_h^{p-1}}\right) dy = \langle x \rangle^{-\frac{p-k}{p-1}\sigma} \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}}\right),$$

and then we can compute by using Leibniz formula

$$
\begin{aligned}
\partial_x^\beta \int_0^x \langle y \rangle^{-\frac{p-k}{p-1}\sigma} \psi\left(\frac{\langle y \rangle}{\langle \xi \rangle_h^{p-1}}\right) dy &= \partial_x^{\beta-1}\left[\langle x \rangle^{-\frac{p-k}{p-1}\sigma} \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}}\right)\right] \\
&= \sum_{\beta_1+\beta_2=\beta-1} \frac{(\beta-1)!}{\beta_1!\beta_2!} \partial_x^{\beta_1} \langle x \rangle^{-\frac{p-k}{p-1}\sigma} \partial_x^{\beta_2} \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}}\right).
\end{aligned}
$$

Returning to (3.8), we can write

$$
\begin{aligned}
\partial_x^\beta \lambda_{p-k}(x,\xi) &= M_{p-k} \omega\left(\frac{\xi}{h}\right) \langle \xi \rangle_h^{-k+1} \sum_{\beta_1+\beta_2=\beta-1} \frac{(\beta-1)!}{\beta_1!\beta_2!} \partial_x^{\beta_1} \langle x \rangle^{-\frac{p-k}{p-1}\sigma} \partial_x^{\beta_2} \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}}\right) \\
&= M_{p-k} \sum_{\beta_1+\beta_2=\beta-1} \frac{(\beta-1)!}{\beta_1!\beta_2!} \partial_x^{\beta_1} \langle x \rangle^{-\frac{p-k}{p-1}\sigma} \omega\left(\frac{\xi}{h}\right) \langle \xi \rangle_h^{-k+1} \partial_x^{\beta_2} \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}}\right)
\end{aligned}
$$

and from this, it follows that

$$\begin{aligned}
\partial_\xi^\alpha \partial_x^\beta \lambda_{p-k}(x,\xi) &= M_{p-k} \sum_{\beta_1+\beta_2=\beta-1} \frac{(\beta-1)!}{\beta_1!\beta_2!} \partial_x^{\beta_1} \langle x\rangle^{-\frac{p-k}{p-1}\sigma} \partial_\xi^\alpha \left[\omega\left(\frac{\xi}{h}\right) \langle\xi\rangle_h^{-k+1} \partial_x^{\beta_2} \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right] \\
&= M_{p-k} \sum_{\beta_1+\beta_2=\beta-1} \frac{(\beta-1)!}{\beta_1!\beta_2!} \partial_x^{\beta_1} \langle x\rangle^{-\frac{p-k}{p-1}\sigma} \\
&\times \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} \partial_\xi^{\alpha_1} \omega\left(\frac{\xi}{h}\right) \partial_\xi^{\alpha_2} \langle\xi\rangle_h^{-k+1} \partial_\xi^{\alpha_3} \partial_x^{\beta_2} \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right) \\
&= M_{p-k} \sum_{\beta_1+\beta_2=\beta-1} \frac{(\beta-1)!}{\beta_1!\beta_2!} \partial_x^{\beta_1} \langle x\rangle^{-\frac{p-k}{p-1}\sigma} \\
&\times \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} \omega^{(\alpha_1)}\left(\frac{\xi}{h}\right) h^{-\alpha_1} \partial_\xi^{\alpha_2} \langle\xi\rangle_h^{-k+1} \\
&\times \sum_{j=1}^{\alpha_3+\beta_2} \frac{\psi^{(j)}\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)}{j!} \sum_{\gamma_1+\cdots+\gamma_j=\alpha_3} \sum_{\delta_1+\cdots+\delta_j=\beta_2} \frac{\alpha_3!\beta_2!}{\gamma_1!\delta_1!\cdots\gamma_j!\delta_j!} \prod_{\ell=1}^{j} \partial_x^{\delta_\ell} \langle x\rangle \partial_\xi^{\gamma_\ell} \langle\xi\rangle_h^{1-p}.
\end{aligned}$$

Now, we can estimate

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta \lambda_{p-k}(x,\xi)| &\leq M_{p-k} \sum_{\beta_1+\beta_2=\beta-1} \frac{(\beta-1)!}{\beta_1!\beta_2!} |\partial_x^{\beta_1} \langle x\rangle^{-\frac{p-k}{p-1}\sigma}| \\
&\times \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} \left|\omega^{(\alpha_1)}\left(\frac{\xi}{h}\right)\right| h^{-\alpha_1} |\partial_\xi^{\alpha_2} \langle\xi\rangle_h^{-k+1}| \\
&\times \chi_\xi(x) \sum_{j=1}^{\alpha_3+\beta_2} \frac{\left|\psi^{(j)}\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right|}{j!} \\
&\times \sum_{\gamma_1+\cdots+\gamma_j=\alpha_3} \sum_{\delta_1+\cdots+\delta_j=\beta_2} \frac{\alpha_3!\beta_2!}{\gamma_1!\delta_1!\cdots\gamma_j!\delta_j!} \prod_{\ell=1}^{j} |\partial_x^{\delta_\ell} \langle x\rangle| |\partial_\xi^{\gamma_\ell} \langle\xi\rangle_h^{1-p}| \\
&\leq M_{p-k} \sum_{\beta_1+\beta_2=\beta-1} \frac{(\beta-1)!}{\beta_1!\beta_2!} C^{\beta_1} \beta_1! \langle x\rangle^{-\frac{p-k}{p-1}\sigma-\beta_1} \\
&\times \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} C_\omega^{\alpha_1+1} \alpha_1!^\mu \langle\xi\rangle_h^{-\alpha_1} \langle R_{a_p}\rangle^{\alpha_1} C^{\alpha_1} \alpha_1! \langle\xi\rangle_h^{-k+1-\alpha_1} \\
&\times \sum_{j=1}^{\alpha_3+\beta_2} \frac{1}{j!} C_\psi^{j+1} j!^\mu \sum_{\gamma_1+\cdots+\gamma_j=\alpha_3} \sum_{\delta_1+\cdots+\delta_j=\beta_2} \frac{\alpha_3!\beta_2!}{\gamma_1!\delta_1!\cdots\gamma_j!\delta_j!} \\
&\times \prod_{\ell=1}^{j} C^{\delta_\ell} \delta_\ell! \langle x\rangle^{1-\delta_\ell} C^{\gamma_\ell} \gamma_\ell! \langle\xi\rangle_h^{1-p-\gamma_\ell} \\
&\leq M_{p-k} C_{\psi,\omega,R_{a_p}}^{\alpha+\beta+1} \alpha!^\mu (\beta-1)!^\mu \langle\xi\rangle_h^{1-k-\alpha} \langle x\rangle^{-\frac{p-k}{p-1}\sigma-(\beta-1)}.
\end{aligned}$$

□

Remark 3.4. *We can conclude that* $\lambda_{p-k} \in \mathbf{SG}_\mu^{0,\frac{p-k}{p-1}(1-\sigma)}(\mathbb{R}^2)$ *and* $\lambda_{p-k} \in \mathbf{S}_\mu^{(p-k)(1-\sigma)}(\mathbb{R}^2)$*, for each* $k = 1, ..., p-1$*. Hence*

$$\Lambda = \sum_{k=1}^{p-1} \lambda_{p-k} \in \mathbf{SG}_\mu^{0,1-\sigma}(\mathbb{R}^2) \cap \mathbf{S}_\mu^{(p-1)(1-\sigma)}(\mathbb{R}^2). \tag{3.9}$$

3.4 Invertibility of the operator $e^{\Lambda}(x,D)$

In this section we construct the inverse operator of $e^{\Lambda}(x,D)$, for $\Lambda = \lambda_{p-1} + \cdots + \lambda_1$. In order to do this, we need to introduce the notion of *reverse* operator. Let $\lambda \in \mathbf{S}_{\mu}^{\frac{1}{\kappa}}(\mathbb{R}^2)$, $1 < \mu \leq \kappa$, be a real-valued symbol satisfying

$$|\partial_\xi^\alpha \partial_x^\beta \lambda(x,\xi)| \leq \rho_0 A^{\alpha+\beta}(\alpha!\beta!)^\kappa \langle \xi \rangle^{\frac{1}{\kappa}-\alpha}. \tag{3.10}$$

Note that $e^{\pm\lambda} \in \mathbf{S}^\infty_{\kappa;\kappa}(\mathbb{R}^2)$. Let

$$e^{\pm\lambda}(x,D)u(x) = \mathbf{op}(e^{\pm\lambda(x,\xi)})u(x) = \int_{\mathbb{R}} e^{ix\xi \pm \lambda(x,\xi)}\widehat{u}(\xi)đ\xi.$$

Let us consider the *reverse operator* ${}^R(e^{\pm\lambda}(x,D))$ which is defined as the transpose of $e^{\pm\lambda}(x,-D)$. Namely, this operator is given by the oscillatory integral

$$\begin{aligned} {}^R(e^{\pm\lambda}(x,D))u(x) &= Os - \iint e^{i\xi(x-y)\pm\lambda(y,\xi)}u(y)dyđ\xi \\ &= \lim_{\epsilon\to 0}\iint e^{i\xi(x-y)\pm\lambda(y,\xi)}\chi(\epsilon y,\epsilon\xi)u(y)dyđ\xi \end{aligned}$$

for some $\chi \in \mathcal{S}_\kappa(\mathbb{R}^2)$ such that $\chi(0,0) = 1$. The reader can find more details about the reverse operator in [34] and [35, Proposition 2.13].

The operators $e^{\lambda}(x,D)$ and ${}^R(e^{\lambda}(x,D))$, for a symbol λ satisfying (3.10), have continuity properties given by the next result.

Proposition 3.1. *Let us consider a symbol λ satisfying (3.10), $\rho, m \in \mathbb{R}$ and $1 < \theta \leq \kappa$. Then:*

(i) *If $\kappa > \theta$, the operators $e^{\lambda}(x,D)$ and ${}^R(e^{\lambda}(x,D))$ map continuously $H^m_{\rho;\theta}(\mathbb{R})$ into $H^m_{\rho-\delta;\theta}(\mathbb{R})$ for every $\delta > 0$.*

(ii) *If $\kappa = \theta$, there exists $\tilde{\delta} > 0$ such that $e^{\lambda}(x,D)\colon H^m_{\rho;\theta}(\mathbb{R}) \to H^m_{\rho-\delta;\theta}(\mathbb{R})$ is continuous for every $\delta > 0$ satisfying*

$$|\delta - \rho| < \tilde{\delta}A^{-1/\theta} \quad \textit{and} \quad \delta > C(\lambda) := \sup_{(x,\xi)\in\mathbb{R}^2} \frac{\lambda(x,\xi)}{\langle\xi\rangle^{1/\theta}}.$$

Moreover, the reverse ${}^R(e^{\lambda}(x,D))\colon H^m_{\rho;\theta}(\mathbb{R}) \to H^m_{\rho-\delta;\theta}(\mathbb{R})$ is continuous for every $|\delta| < \tilde{\delta}A^{-1/\theta}$ and $\delta > C(\lambda)$.

Proof. Check [34, Proposition 6.7]. □

In the following we shall consider the operator ${}^R(e^{-\Lambda}(x,D))$ with Λ defined by (3.9). Notice that

$$|\partial_\xi^\alpha \Lambda(x,\xi)| \leq \rho_0 A^{|\alpha|}\alpha!^\mu \langle\xi\rangle_h^{(p-1)(1-\sigma)} \tag{3.11}$$

and

$$|\partial_\xi^\alpha \partial_x^\beta \Lambda(x,\xi)| \leq \rho_0 A^{\alpha+\beta}(\alpha!\beta!)^\rho \langle\xi\rangle_h^{-\alpha}. \tag{3.12}$$

whenever $\beta \geq 1$. This estimate means that for $\beta \neq 0$, $\partial_x^\beta \Lambda$ behaves like a symbol of order 0. Taking into account conditions (3.11) and (3.12) the following result holds, cf. Lemma 4 in [6].

Lemma 3.2. *Let $\mu > 1$. Then, for $h > 0$ large enough, the operator $e^\Lambda(x,D)$ is invertible and its inverse is given by*

$$(e^\Lambda(x,D))^{-1} = {}^R(e^{-\Lambda}(x,D)) \circ (I + r(x,D))^{-1} = {}^R(e^{-\Lambda}(x,D)) \circ \sum_{j\geq 0}(-r(x,D))^j,$$

where $r = \tilde{r} + \bar{r}$ for some $\tilde{r} \in \mathbf{SG}_\mu^{-1,-\sigma}(\mathbb{R}^2)$ and $\bar{r} \in \mathscr{K}_\kappa$, that is, satisfying

$$|\partial_\xi^\alpha \partial_x^\beta \bar{r}(x,\xi)| \leq C^{\alpha+\beta+1}(\alpha!\beta!)^\kappa e^{-c\{\langle x\rangle^{\frac{1}{\kappa}} + \langle \xi\rangle^{\frac{1}{\kappa}}\}}, \tag{3.13}$$

with $\kappa > 2\mu - 1$ and for some $C, c > 0$. Moreover, for every $N \in \mathbb{N}$, we have

$$\tilde{r} - \sum_{1\leq\gamma\leq N} \frac{1}{\gamma!}\partial_\xi^\gamma \left(e^\Lambda D_x^\gamma e^{-\Lambda}\right) \in \mathbf{SG}_\mu^{-(N+1),-\sigma(N+1)}(\mathbb{R}^2), \tag{3.14}$$

and the symbol of the operator $\sum(-r(x,D))^j$ is of the form $q + q_\infty$, where $q \in \mathbf{SG}_\mu^{0,0}(\mathbb{R}^2)$ and q_∞ satisfies (3.13) for all $C, c > 0$ and $\kappa > 2\mu - 1$.

Now we can prove some results about the symbol $r(x,\xi)$ of the operator $r(x,D)$ and on the corresponding Neumann series.

Lemma 3.3. *The symbol $r(x,\xi)$ which appears in the above lemma can be expressed as*

$$r = -\partial_\xi D_x \Lambda + b_{-2} + \cdots + b_{-(p-2)} + b_{-(p-1)} + b_{-p} \tag{3.15}$$

where $b_{-m} \in \mathbf{SG}_\mu^{-m,-\frac{p-m+1}{p-1}\sigma}(\mathbb{R}^2)$ depends only on $\lambda_{p-1}, ..., \lambda_{p-(m-1)}$, for $m = 2, ..., p-1$. Moreover, b_{-p} is the sum of a symbol in $\mathbf{SG}_\mu^{-p,-\frac{1}{p-1}\sigma}(\mathbb{R}^2)$ and a symbol in $\mathscr{K}_\kappa$ for $\kappa > 2\mu - 1$.

Proof. We have that $r \sim \sum_{\gamma\geq 1} \frac{1}{\gamma!}\partial_\xi^\gamma \left(e^\Lambda D_x^\gamma e^{-\Lambda}\right)$, hence

$$r = -\partial_\xi D_x \Lambda + \sum_{\gamma=2}^{p-1} \frac{1}{\gamma!}\partial_\xi^\gamma \left(e^\Lambda D_x^\gamma e^{-\Lambda}\right) + q_{-p}. \tag{3.16}$$

For $\gamma \geq 2$, by Faà di Bruno's formula we get

$$e^\Lambda D_x^\gamma e^{-\Lambda} = \sum_{j=1}^{\gamma} \frac{(-1)^j}{j!} \sum_{\substack{\gamma_1+\cdots+\gamma_j=\gamma \\ \gamma_\ell\geq 1}} \frac{\gamma!}{\gamma_1!\cdots\gamma_j!} \prod_{\ell=1}^{j} D_x^{\gamma_\ell}\Lambda.$$

Now, let us analyse $\partial_\xi^\gamma \left(\prod_{\ell=1}^j D_x^{\gamma_\ell}\Lambda\right)$. Since $\gamma \geq 2$ in the formula (3.16), it is sufficient to prove that

$$\prod_{\ell=1}^{j} D_x^{\gamma_\ell}\Lambda = \tilde{b}_{-2} + \cdots + \tilde{b}_{-(p-1)} + \tilde{b}_{-p},$$

for some $\tilde{b}_{-m} \in \mathbf{SG}_{\mu}^{-m+2,-\frac{p-m+1}{p-1}\sigma}(\mathbb{R}^2)$, $m = 2, ..., p$, depending only on $\lambda_{p-1}, ..., \lambda_{p-(m-1)}$. Since $\Lambda = \lambda_{p-1} + \cdots + \lambda_1$, we have that $D_x^{\gamma_\ell}\Lambda = D_x^{\gamma_\ell}(\lambda_{p-1} + \cdots + \lambda_1)$, $\ell = 1, ..., j$, which implies that

$$\prod_{\ell=1}^{j} D_x^{\gamma_\ell}\Lambda = (D_x^{\gamma_1}\lambda_{p-1} + \cdots + D_x^{\gamma_1}\lambda_1)\cdots(D_x^{\gamma_j}\lambda_{p-1} + \cdots + D_x^{\gamma_j}\lambda_1).$$

By (v) of Lemma 3.3, the only term in the above product with order exactly 0 with respect to ξ is

$$D_x^{\gamma_1}\lambda_{p-1}\cdots D_x^{\gamma_j}\lambda_{p-1} \in \mathbf{SG}_{\mu}^{0,-\sigma}(\mathbb{R}^2).$$

Similarly we notice that the only term of order exactly -1 with respect to ξ is the sum of products of the form

$$D_x^{\gamma_s}\lambda_{p-2}\prod_{\substack{\ell\le j\\ \ell\ne s}} D_x^{\gamma_\ell}\lambda_{p-1} \in \mathbf{SG}_{\mu}^{-1,-\frac{p-2}{p-1}\sigma}(\mathbb{R}^2),$$

which do not depend on $\lambda_{p-3}, ..., \lambda_1$. In general, we note that products containing at least one factor of the type $D_x^{\gamma_\ell}\lambda_{p-k}$ with $k \ge m$ have order at most $-m+1$, so the terms of order $-m+2$ with respect to ξ cannot depend on λ_{p-k} for $k \ge m$. Among these terms, the ones of highest order in x are obviously those depending on λ_{p-m+1} that is products of the form

$$D_x^{\gamma_s}\lambda_{p-m+1}\prod_{\substack{\ell\le j\\ \ell\ne s}} D_x^{\gamma_\ell}\lambda_{p-1} \in \mathbf{SG}_{\mu}^{-m+2,-\frac{p-m+1}{p-1}\sigma}(\mathbb{R}^2).$$

This gives the assertion. □

By using Lemma 3.3 it can be proved the next result.

Lemma 3.4. *For each $j \in \mathbb{N}$, $j \ge 2$, the operator $(-r(x, D))^j$ has order $-j$. Moreover, for $2 \le j < p$, its symbol d_j is of the form*

$$d_j = b_{-j}^{[j]} + b_{-(j+1)}^{[j]} + \cdots + b_{-(p-1)}^{[j]} + b_{-p}^{[j]}, \tag{3.17}$$

for some $b_{-m}^{[j]} \in \mathbf{SG}_{\mu}^{-m,-\frac{jp-m}{p-1}\sigma}(\mathbb{R}^2)$ depending only on $\lambda_{p-1}, ..., \lambda_{p-m+1}$, for each $m = j, ..., p-1$.

Proof. We can argue by induction on j. For $j = 2$, we have

$$\begin{aligned}
(-r(x, D))^2 \quad &= \sum_{k=1}^{p-1}\sum_{\ell=1}^{p-1}(\partial_\xi D_x\lambda_{p-k})(x, D)\circ(\partial_\xi D_x\lambda_{p-\ell})(x, D) \\
&- \left(\sum_{k=1}^{p-k}\partial_\xi D_x\lambda_{p-1}\right)(x, D)\circ\left(\sum_{\ell=2}^{p} b_{-\ell}\right)(x, D) \\
&- \left(\sum_{\ell=2}^{p} b_{-\ell}\right)(x, D)\circ\left(\sum_{k=1}^{p-1}\partial_\xi D_x\lambda_{p-k}\right)(x, D) \\
&+ \sum_{s=2}^{p}\sum_{\ell=2}^{p} b_{-s}(x, D)\circ b_{-\ell}(x, D) \\
&=: \mathtt{F} + \mathtt{G} + \mathtt{H} + \mathtt{L}.
\end{aligned}$$

Consider

$$\mathtt{F} = \sum_{k=1}^{p-1}\sum_{\ell=1}^{p-1}(\partial_\xi D_x \lambda_{p-k})(x,D)\circ(\partial_\xi D_x\lambda_{p-\ell})(x,D)$$

We immediately notice that, since the order of each term of the sum F with respect to ξ is $-s-\ell$, then the terms with order exactly $-m$ w.r.t. ξ cannot depend on λ_{p-s} for $s \geq m$ or $\lambda_{p-\ell}$ for $\ell \geq m$, then these terms are of the form $(\partial_\xi D_x\lambda_{p-s})(x,D)\circ(\partial_\xi D_x \lambda_{p-\ell})(x,D)$ for $s+\ell = m$, hence they symbols belong to $\mathbf{SG}_\mu^{-m,-\frac{2p-m}{p-1}\sigma}(\mathbb{R}^2)$. Concerning

$$\mathtt{G} = \left(\sum_{k=1}^{p-k}\partial_\xi D_x\lambda_{p-1}\right)(x,D)\circ\left(\sum_{\ell=2}^{p} b_{-\ell}\right)(x,D),$$

we notice that for $k \geq m$ the compositions of $(\partial_\xi D_x\lambda_{p-k})(x,D)$ with $\sum_{l=2}^{p} b_{-\ell}(x,D)$ give operators with order less than or equal to $-m-2$ w.r.t. ξ. Hence, the terms of order exactly $-m$ in this sum do not depend on λ_{p-2} for $s \geq m$. Moreover, these terms are of the form $\lambda_{p-s}(x,D)\circ b_{-\ell}(x,D)$ with $\ell = m-s$, hence they belong to $\mathbf{SG}_\mu^{-m,-\frac{2p-m+1}{p-1}}(\mathbb{R}^2)$. We can argue in a very similar way about $\mathtt{H}$. Finally, let's treat the term

$$\mathtt{L} = \sum_{s=2}^{p}\sum_{\ell=2}^{p} b_{-s}(x,D)\circ b_{-\ell}(x,D).$$

The terms depending on λ_{p-k} with $k \geq m$ appear only in compositions of the form $b_{-s}(x,D)\circ b_{-\ell}(x,D)$ with $\ell > m$ and/or $s > m$, which have order $-s-\ell < -m-2$. Moreover, every term of order exactly $-m$ is obtained as a product of the form $b_{-s}(x,D)\circ b_{-\ell}(x,D)$ for $s+\ell = m$ and it decays like $\langle x\rangle^{-\frac{2p-m+2}{p-1}}\sigma$. In conclusion we obtain the assertion for $j=2$. The inductive step follows from similar considerations. Assume now that the statement is true for $j \leq N$ and consider

$$\begin{aligned}(-r(x,D))^{N+1} &= (-r(x,D))\circ(-r(x,D))^N \\ &= \mathbf{op}(-\partial_\xi D_x\lambda_{p-1} - \cdots \partial_\xi D_x \lambda_1 + b_{-2} + \cdots + b_{-p})\circ \mathbf{op}(b_{-N}^{[N]} + \cdots + b_{-p}^{[N]}).\end{aligned}$$

Notice that the only term in the composition with order exactly $-N-1$ w.r.t. ξ is $(-\partial_\xi D_x\lambda_{p-1})(x,D)\circ b_{-N}^{[N]}(x,D)$ whose symbol depends only on λ_{p-1} and belongs to $\mathbf{SG}_\mu^{-N-1,-\frac{(N+1)p-N-1}{p-1}\sigma}(\mathbb{R}^2)$. The only term with order exactly $-N-2$ w.r.t. ξ is given by

$$(-\partial_\xi D_x\lambda_{p-1})(x,D)\circ b_{-N-1}^{[N]}(x,D) + (-\partial_\xi D_x \lambda_{p-2})(x,D)\circ b_{-N}^{[N]}(x,D)$$

whose symbol depends only on λ_{p-1}, λ_{p-2} and belongs to $\mathbf{SG}_\mu^{-N-2,-\frac{(N+1)p-N-2}{p-1}}(\mathbb{R}^2)$, and so on. □

From Lemmas 3.3, 3.4 and Lemma 3 in [6], it follows the next result.

Proposition 3.2. *Let $\mu > 1$. For h large enough, say $h > h_0$, the operator $e^\Lambda(x,D)$ is invertible and its inverse is given by*

$$(e^\Lambda(x,D))^{-1} = {}^R(e^{-\Lambda}(x,D))\circ\mathbf{op}(1 - i\partial_\xi\partial_x\Lambda + q_{-2} + \cdots + q_{-(p-1)} + q_{-p}), \tag{3.18}$$

where $q_{-m} \in \mathbf{SG}_\mu^{-m,-\frac{2p-m}{p-1}\sigma}(\mathbb{R}^2), m = 2,...,p-1,$ depend only on $\lambda_{p-1},...,\lambda_{p-(m-1)}$ and q_{-p} is the sum of a symbol in $\mathbf{SG}_\mu^{-p,-\frac{p}{p-1}\sigma}(\mathbb{R}^2)$ and a symbol satisfying (3.13) for all $C,c>0$ and $\kappa > 2\mu-1$.

3.5 Conjugation of the operator iP

In this section the goal is to perform the conjugation of the operator iP by

$$Q_{\Lambda,K,\rho'}(t,x,D) = e^{\Lambda_{K,\rho'}}(t,D) \circ e^{\Lambda}(x,D),$$

where

$$\Lambda(x,\xi) = \sum_{k=1}^{p-1} \lambda_{p-k}(x,\xi) \qquad \text{and} \qquad \Lambda_{K,\rho'}(t,\xi) = K(T-t)\langle\xi\rangle_h^{(p-1)(1-\sigma)} + \rho'\langle\xi\rangle_h^{1/\theta}.$$

Since the inverse of $e^{\Lambda}(x,D)$ is ${}^R(e^{-\Lambda}(x,D)) \circ \sum_{j\geq 0}(-r(x,D))^j$, it is necessary to work with compositions of the form $e^{\Lambda}(x,D) \circ p(x,D) \circ {}^R(e^{-\Lambda}(x,D))$, with p an operator whose symbol has finite order. The next result (Theorem 2 of [6]) will be used to perform this computation. For more details of the proof the reader also can check Section 3.5.1 of [2].

Theorem 3.2. *Let p be a symbol satisfying*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_A A^{\alpha+\beta}(\alpha!\beta!)^\kappa \langle\xi\rangle_h^{m-\alpha},$$

and let λ satisfying (3.11) and (3.12) for $\beta \neq 0$. Then there exist $\tilde{\delta} > 0$ and $h_0 = h_0(A) \geq 1$ such that if $\rho_0 \leq \tilde{\delta}A^{-1/\mu}$ and $h \geq h_0$, then

$$\begin{aligned} e^{\lambda}(x,D) \circ p(x,D) \circ {}^R(e^{-\lambda}(x,D)) &= p(x,D) + \mathbf{op}\left(\sum_{1\leq \alpha+\beta<N} \frac{1}{\alpha!\beta!} \partial_\xi^\alpha \left(\partial_\xi^\beta e^{\lambda(x,\xi)} D_x^\beta p(x,\xi) D_x^\alpha e^{-\lambda(x,\xi)} \right) \right) \\ &+ r_N(x,D) + r_\infty(x,D) \end{aligned}$$

where

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta r_N(x,\xi)| &\leq C_{\rho_0,A,\kappa}(C_\kappa A)^{\alpha+\beta+2N}(\alpha!\beta!)^\kappa N!^{2\kappa-1}\langle\xi\rangle_h^{m-(1-\frac{1}{\kappa})N-\alpha}, \\ |\partial_\xi^\alpha \partial_x^\beta r_\infty(x,\xi)| &\leq C_{\rho_0,A,\kappa}(C_\kappa A)^{\alpha+\beta+2N}(\alpha!\beta!)^\kappa N!^{2\kappa-1} e^{-c_\kappa A^{-1/\kappa}\langle\xi\rangle_h^{1/\kappa}}. \end{aligned} \tag{3.19}$$

In particular, $r_\infty \in \mathscr{K}_\kappa$.

3.5.1 Conjugation of iP by e^{Λ}

Before we start making the conjugation, note that by Lemma 3.3 the function Λ satisfies

$$|\partial_\xi^\alpha \partial_x^\beta \Lambda(x,\xi)| \leq C_\Lambda^{\alpha+\beta+1}(\alpha!\beta!)^\mu \langle\xi\rangle_h^{(p-1)(1-\sigma)-\alpha}$$

and

$$|\partial_\xi^\alpha \partial_x^\beta \Lambda(x,\xi)| \leq C_\Lambda^{\alpha+\beta+1}(\alpha!\beta!)^\mu \langle\xi\rangle_h^{-\alpha}, \quad \beta \geq 1$$

where C_Λ is a constant depending on $M_{p-1}, ..., M_1, C_\omega, C_\psi, \mu, \sigma$. By the assumption $(p-1)(1-\sigma) < \frac{1}{\theta}$ it follows that

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta \Lambda(x,\xi)| &\leq C_\Lambda^{\alpha+\beta+1}(\alpha!\beta!)^\mu \langle\xi\rangle_h^{(p-1)(1-\sigma)-\alpha} \\ &= C_\Lambda^{\alpha+\beta+1}(\alpha!\beta!)^\mu \langle\xi\rangle_h^{\frac{1}{\theta}-\alpha} \langle\xi\rangle_h^{(p-1)(1-\sigma)-\frac{1}{\theta}} \\ &\leq C_\Lambda h^{(p-1)(1-\sigma)-\frac{1}{\theta}} C_\Lambda^{\alpha+\beta}(\alpha!\beta!)^\mu \langle\xi\rangle_h^{\frac{1}{\theta}-\alpha}, \end{aligned}$$

therefore

$$\rho(\Lambda) := h^{(p-1)(1-\sigma)-\frac{1}{\theta}} C_\Lambda$$

can be taken as small as we want, provided that h can be taken suitably large, and now we are able to compute $e^\Lambda(x,D) \circ (iP) \circ (e^\Lambda(x,D))^{-1}$ by using Theorem 3.2.

We conjugate each term of the operator iP, which is given by

$$iP(t,x,D_t,D_x) = \partial_t + ia_p(t)D_x^p + \sum_{j=1}^{p} ia_{p-j}(t,x)D_x^{p-j}. \tag{3.20}$$

The next items are dedicated to this purpose.

- **The conjugation of ∂_t.** Since Λ does not depend on t, the conjugation of ∂_t is trivial, namely

$$e^\Lambda(x,D) \circ \partial_t \circ (e^\Lambda(x,D))^{-1} = \partial_t.$$

- **The conjugation of $ia_p(t)D_x^p$.** First of all, we will treat only the conjugation of iD_x^p by $e^\Lambda(x,D)$ (since a_p does not depend on x). By Theorem 3.2 it follows that

$$e^\Lambda(x,D) \circ iD_x^p \circ {}^R(e^{-\Lambda}(x,D)) = iD_x^p + \mathbf{op}\left(i \sum_{1 \le \alpha \le p-1} \frac{1}{\alpha!} \partial_\xi^\alpha \left(e^\Lambda \xi^p D_x^\alpha e^{-\Lambda} \right) \right) + r_0(x,D) + r_\infty(x,D). \tag{3.21}$$

Note that we can write

$$\sum_{1 \le \alpha \le p-1} \frac{1}{\alpha!} \partial_\xi^\alpha \left(e^\Lambda \xi^p D_x^\alpha e^{-\Lambda} \right) = \partial_\xi \left(\xi^p D_x(-\Lambda) \right) + c_{-2} + c_{-3} + \cdots + c_{-(p-1)} + c_{-p}, \tag{3.22}$$

where $c_{-m} \in \mathbf{SG}_\mu^{p-m, -\frac{p-m+1}{p-1}\sigma}(\mathbb{R}^2)$ and it depends only on $\lambda_{p-1}, ..., \lambda_{p-m+1}$, for each $m = 2, ..., p$. Now we can substitute (3.22) in (3.21) to obtain

$$\begin{aligned} e^\Lambda(x,D) \circ iD_x^p \circ {}^R\{e^{-\Lambda}(x,D)\} &= iD_x^p + i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) + ic_{-2}(x,D) + \cdots + ic_{-p}(x,D) \\ &+ r_0(x,D) + r_\infty(x,D). \end{aligned}$$

From Proposition 3.2, we have that the Neumann series is given by

$$\sum_{j \ge 0} (-r(x,D))^j = \mathbf{op}(1 - i\partial_\xi \partial_x \Lambda + q_{-2} + \cdots + q_{-(p-1)} + q_{-p}). \tag{3.23}$$

Since we have to perform a composition between the two operators given in (3.22) and (3.23), let us analyse what happens with

$$[iD_x^p + i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) + i(c_{-2}(x,D) + \cdots + ic_{-p}(x,D))] \circ \mathbf{op}(1 - i\partial_\xi \partial_x \Lambda + q_{-2} + \cdots + q_{-p}).$$

We can rewrite it as

$$\begin{aligned}
& iD_x^p + i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) + i(c_{-2}(x,D) + \cdots + c_{-p}(x,D)) \\
+ \quad & iD_x^p \circ \mathbf{op}(\partial_\xi D_x\Lambda) + i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) \circ \mathbf{op}(\partial_\xi D_x\Lambda) \\
+ \quad & i(c_{-2}(x,D) + \cdots + c_{-p}(x,D)) \circ (\partial_\xi D_x\Lambda)(x,D) \\
+ \quad & iD_x^p \circ (q_{-2}(x,D) + \cdots + q_{-p}(x,D)) + i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) \circ (q_{-2}(x,D) + \cdots + q_{-p}(x,D)) \\
+ \quad & i(c_{-2}(x,D) + \cdots + c_{-p}(x,D)) \circ (q_{-2}(x,D) + \cdots + q_{-p}(x,D)) \\
= \quad & iD_x^p + i\mathbf{op}(p\xi^{p-1}D_x(-\Lambda) + \xi^p\partial_\xi D_x(-\Lambda)) + i(c_{-2}(x,D) + \cdots + c_{-p}(x,D)) \\
+ \quad & i\mathbf{op}(\xi^p\partial_\xi D_x\Lambda) + i\mathbf{op}\left(\sum_{\gamma=1}^{p}\frac{1}{\gamma!}\partial_\xi^\gamma\xi^p\partial_\xi D_x^{\gamma+1}\Lambda\right) + i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) \circ \mathbf{op}(\partial_\xi D_x\Lambda) \\
+ \quad & i(c_{-2}(x,D) + \cdots + c_{-p}(x,D)) \circ (\partial_\xi D_x\Lambda)(x,D) \\
+ \quad & iD_x^p \circ (q_{-2}(x,D) + \cdots + q_{-p}(x,D)) + i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) \circ (q_{-2}(x,D) + \cdots + q_{-p}(x,D)) \\
+ \quad & i(c_{-2}(x,D) + \cdots + c_{-p}(x,D)) \circ (q_{-2}(x,D) + \cdots + q_{-p}(x,D)).
\end{aligned}$$

After some simplifications in the above expression, we get

$$\begin{aligned}
& [iD_x^p + i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) + i(c_{-2}(x,D) + \cdots + c_{-p}(x,D))] \\
\circ \quad & [I + \mathbf{op}(\partial_\xi D_x\Lambda) + q_{-2}(x,D) + \cdots + q_{-p}(x,D)] \\
= \quad & iD_x^p + i\mathbf{op}(p\xi^{p-1}D_x(-\Lambda)) + i(c_{-2}(x,D) + \cdots + c_{-p}(x,D)) \\
+ \quad & i\mathbf{op}\left(\sum_{\gamma=1}^{p}\frac{1}{\gamma!}\partial_\xi^\gamma\xi^p\partial_\xi D_x^{\gamma+1}\Lambda\right) + i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) \circ \mathbf{op}(\partial_\xi D_x\Lambda) \\
+ \quad & i(c_{-2}(x,D) + \cdots + c_{-p}(x,D)) \circ (\partial_\xi D_x\Lambda)(x,D) + iD^p \circ (q_{-2}(x,D) + \cdots + q_{-p}(x,D)) \\
+ \quad & i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\}) \circ (q_{-2}(x,D) + \cdots + q_{-p}(x,D)) \\
+ \quad & i(c_{-2}(x,D) + \cdots + c_{-p}(x,D)) \circ (q_{-2}(x,D) + \cdots + q_{-p}(x,D)). \qquad (3.24)
\end{aligned}$$

Next, let us make some remarks about the order and dependence on the constants M_{p-k} of some terms.

Let us first look at the term

$$i\mathbf{op}\left(\sum_{\gamma=1}^{p}\frac{1}{\gamma!}\partial_\xi^\gamma\xi^p\partial_\xi D_x^{\gamma+1}\Lambda\right).$$

From the definition of Λ, we have for each $\gamma \geq 1$

$$\partial_\xi^\gamma\xi^p D_x^{\gamma+1}\partial_\xi\Lambda = \partial_\xi^\gamma\xi^p D_x^{\gamma+1}\partial_\xi\lambda_{p-1} + \cdots + \partial_\xi^\gamma\xi^p D_x^{\gamma+1}\partial_\xi\lambda_1.$$

For each $k = 1, ..., p-1$, we obtain that $\partial_\xi^\gamma\xi^p D_x^{\gamma+1}\partial_\xi\lambda_{p-k}$ has order

$$p - k - \gamma \leq p - k - 1 \text{ w.r.t. } \xi$$

and

$$-\frac{p-k}{p-1}\sigma-\gamma\leq-\frac{p-k}{p-1}\sigma-1 \text{ w.r.t. } x.$$

Notice that the highest order with respect to ξ in this case is $p-2$.

The term $i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\})\circ\mathbf{op}(\partial_\xi D_x\Lambda)$ can be written by using the definition of Λ as

$$i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\})\circ\mathbf{op}(\partial_\xi D_x\Lambda)=-i\sum_{\ell=1}^{p-1}\sum_{s=1}^{p-1}\mathbf{op}(\partial_\xi\{\xi^p D_x\lambda_{p-\ell}\})\circ\mathbf{op}(\partial_\xi D_x\lambda_{p-s}).$$

Note that, for $\ell=1,...,p-1$, the symbol $\partial_\xi\{\xi^p D_x\lambda_{p-\ell}\}$ has order $p-\ell$ w.r.t. ξ and $-\frac{p-\ell}{p-1}\sigma$ w.r.t. x and, for $s=1,...,p-1$, the symbol $\partial_\xi D_x\lambda_{p-s}$ has order $-s$ w.r.t. ξ and $-\frac{p-s}{p-1}\sigma$ w.r.t. x. Hence, the operator $\mathbf{op}(\partial_\xi\{\xi^p D_x\lambda_{p-\ell}\})\circ\mathbf{op}(\partial_\xi D_x\lambda_{p-s})$ has order

$$p-(\ell+s) \text{ w.r.t. } \xi$$

and

$$-\frac{2p-(\ell+s)}{p-1}\sigma \text{ w.r.t. } x,$$

for each $\ell=1,...,p-1$ and $s=1,...,p-1$. If we consider terms of order $p-j$ with respect to ξ, they cannot depend on terms λ_{p-k} with $k\geq j$, because otherwise the order would be strictly less than $p-j$. With respect to x, the order is

$$-\frac{2p-j}{p-1}\sigma\leq-\frac{p-j+1}{p-1}\sigma,$$

because $2p-j\geq p-j+1$, for $p>1$. Again, the highest order with respect to ξ is $p-2$, since ℓ and s runs through the set $\{1,...,p-1\}$.

Concerning the term

$$i\mathbf{op}(\partial_\xi\{\xi^p D_x(-\Lambda)\})\circ[q_{-2}(x,D)+\cdots+q_{-p}],$$

we see that the first term in this composition can be rewritten as

$$-i\mathbf{op}(\partial_\xi\{\xi^p D_x(\lambda_{p-1}+\cdots+\lambda_1)\}=-i\mathbf{op}(\partial_\xi\{\xi^p D_x\lambda_{p-1}\}+\cdots+\partial_\xi\{\xi^p D_x\lambda_1\}).$$

Hence the composition is a sum of terms whose symbols are of the form $\partial_\xi\{\xi^p D_x\lambda_{p-k}\}\cdot q_{-\ell}$, for $k=1,...,p-1$ and $\ell=2,...,p$, whose order is $p-(k+\ell)$ depending only on $\lambda_{p-1},...,\lambda_{p-k-\ell+1}$. Therefore, the highest order here is $p-3$, which is more than we need.

The other terms are all of negative order, then there is no need to make the same analysis as the listed above. Summarizing this discussion: The conjugation $e^\Lambda(x,D)\circ(iD_x^p)\circ(e^\Lambda(x,D))^{-1}$ can be expressed as

$$e^\Lambda(x,D)\circ(iD_x^p)\circ(e^\Lambda(x,D))^{-1}=iD_x^p-\mathbf{op}\left(\partial_\xi\xi^p\partial_x\Lambda+\sum_{m=2}^{p-1}\mathsf{c}_{p-m}\right)+r_0(x,D)+r_\infty(x,D),$$

with r_0 of order zero, r_∞ a regularizing term, $\mathtt{c}_{p-m}$ has order $p-m$ with respect to ξ and $-\frac{p-m+1}{p-1}\sigma$ with respect to x and depends only on $\lambda_{p-1}, ..., \lambda_{p-m+1}$, for each $m = 2, ..., p-1$. Summarizing, the conjugation of the leading term is

$$\begin{aligned} e^{\Lambda}(x,D) \circ (ia_p(t)D_x^p) \circ (e^{\Lambda}(x,D))^{-1} &= ia_p(t)D_x^p - \mathbf{op}\left(\partial_\xi a_p(t)\xi^p \partial_x \Lambda + a_p(t) \sum_{m=2}^{p-1} \mathtt{c}_{p-m}\right) \\ &+ r_0(x,D) + r_\infty(x,D), \end{aligned} \tag{3.25}$$

where r_0 and r_∞ new order zero and regularizing terms, respectively.

- **The conjugation of $ia_{p-j}(t,x)D_x^{p-j}$, for $j = 1, ..., p-1$.** For some $N \in \mathbb{N}$ to be chosen later, by Theorem 3.2 it follows that there exist an order zero term r_0 and a regularizing term r_∞ such that

$$\begin{aligned} & e^{\Lambda}(x,D) \circ a_{p-j}(t,x)D_x^{p-j} \circ {}^R(e^{-\Lambda}(x,D)) \\ = \; & a_{p-j}(t,x)D_x^{p-j} + \mathbf{op}\left(\sum_{1 \le \alpha+\beta \le N} \frac{1}{\alpha!\beta!} \partial_\xi^\alpha \left(\partial_\xi^\beta e^{\Lambda} \cdot D_x^\beta a_{p-j}(t,x)\xi^{p-j} \cdot D_x^\alpha e^{-\Lambda} \right) \right) \\ + \; & r_0(x,D) + r_\infty(x,D). \end{aligned} \tag{3.26}$$

For the next computations, we shall omit (x,ξ) and (x,D) looking for a simple way to write. Our goal here is to analyse what happens with the terms in

$$\sum_{1 \le \alpha+\beta \le N} \frac{1}{\alpha!\beta!} \partial_\xi^\alpha \left(\partial_\xi^\beta e^{\Lambda} \cdot D_x^\beta a_{p-j}\xi^{p-j} \cdot D_x^\alpha e^{-\Lambda} \right). \tag{3.27}$$

By Faà di Bruno formula, the terms $\partial_\xi^\beta e^{\Lambda}$ and $D_x^\alpha e^{-\Lambda}$ become

$$\partial_\xi^\beta e^{\Lambda} = \sum_{\mathtt{j}=1}^{\beta} \frac{1}{\mathtt{j}!} e^{\Lambda} \sum_{\substack{\beta_1+\dots+\beta_{\mathtt{j}}=\beta \\ \beta_{\mathtt{k}} \ge 1}} \frac{\beta!}{\beta_1! \cdots \beta_{\mathtt{j}}!} \prod_{\mathtt{k}=1}^{\mathtt{j}} \partial_\xi^{\beta_{\mathtt{k}}} \Lambda$$

and

$$D_x^\alpha e^{-\Lambda} = \sum_{\ell=1}^{\alpha} \frac{(-1)^\ell}{\ell!} e^{-\Lambda} \sum_{\substack{\alpha_1+\dots+\alpha_\ell=\alpha \\ \alpha_{\mathtt{m}} \ge 1}} \frac{\alpha!}{\alpha_1! \cdots \alpha_\ell!} \prod_{\mathtt{m}=1}^{\ell} D_x^{\alpha_{\mathtt{m}}} \Lambda.$$

By fixing $\alpha+\beta$, the general term in (3.27) will be a sum of terms of the form

$$I_{(\alpha,\beta)}^{[j]} := \partial_\xi^\alpha \left(D_x^\beta a_{p-j}\xi^{p-j} \cdot \partial_\xi^{\beta_1}\lambda_{p-k_1} \cdots \partial_\xi^{\beta_{\mathtt{r}}}\lambda_{p-k_{\mathtt{r}}} \cdot D_x^{\alpha_1}\lambda_{p-\tilde{k}_1} \cdots D_x^{\alpha_{\mathtt{s}}}\lambda_{p-\tilde{k}_{\mathtt{s}}} \right), \tag{3.28}$$

where

$$k_1, ..., k_{\mathtt{r}}, \tilde{k}_1, ..., \tilde{k}_{\mathtt{s}} \in \{1, ..., p-1\} \qquad \text{and} \qquad 1 \le \mathtt{r} \le \beta, \quad 1 \le \mathtt{s} \le \alpha.$$

Now, we want to show that *if λ_{p-k} appears in $I_{(\alpha,\beta)}^{[j]}$ with $k \ge M$, then $I_{(\alpha,\beta)}^{[j]}$ is of order at most $p-M-1$ with respect to ξ.*

Case 1: Let us suppose that there is a term of the type $D_x^{\alpha_1}\lambda_{p-\tilde{k}_1}$ with $\tilde{k}_1 \ge m$, with $m \ge 2$ in (3.28).

By using (v) of Lemma 3.3 and the fact that $\tilde{k}_1 \geq m$ and each $\tilde{k}_{\mathtt{a}} \leq p-1$, for all $1 \leq \mathtt{a} \leq \mathtt{s}$, it follows that

$$\left|\left(D_x^{\alpha_1}\lambda_{p-\tilde{k}_1}\cdots D_x^{\alpha_{\mathtt{s}}}\lambda_{p-\tilde{k}_{\mathtt{s}}}\right)(x,\xi)\right| \leq C^{\alpha+1}(\alpha!)^{\mu}\langle\xi\rangle^{-M+1}\langle x\rangle^{-\alpha+\mathtt{s}-\frac{\mathtt{s}\sigma}{p-1}}. \tag{3.29}$$

Now, from (iv) of Lemma 3.3 and the fact that $k_{\mathtt{a}} \geq 1$, for all $1 \leq \mathtt{a} \leq \mathtt{r}$, we obtain the estimate

$$\left|\left(\partial_\xi^{\beta_1}\lambda_{p-k_1}\cdots\partial_\xi^{\beta_{\mathtt{r}}}\lambda_{p-k_{\mathtt{r}}}\right)(x,\xi)\right| \leq C^{\beta+1}(\beta!)^{\mu}\langle\xi\rangle^{-\beta}\langle x\rangle^{\mathtt{r}(1-\sigma)}. \tag{3.30}$$

By hypothesis, we have that

$$\left|D_x^{\beta}a_{p-j}(t,x)\xi^{p-j}\right| \leq C^{\beta}(\beta!)^{s_0}\langle\xi\rangle^{p-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma-\beta} \leq C^{\beta}(\beta!)^{s_0}\langle\xi\rangle^{p-j}\langle x\rangle^{-\beta-\sigma}. \tag{3.31}$$

From (3.29), (3.30) and (3.31), we get

$$|I^{[j]}_{(\alpha,\beta)}(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-m+1-\beta+p-j-\alpha}\langle x\rangle^{-\alpha+\mathtt{s}-\frac{\mathtt{s}\sigma}{p-1}+\mathtt{r}(1-\sigma)-\frac{p-j}{p-1}\sigma-\beta}. \tag{3.32}$$

By a simple computation, the exponent in $\langle\xi\rangle$ is $p-m-\alpha-\beta$. About the exponent in $\langle x\rangle$, note that

$$-\alpha+\mathtt{s}\underbrace{-\frac{\mathtt{s}\sigma}{p-1}}_{<0}+\underbrace{\mathtt{r}(1-\sigma)}_{<\mathtt{r}}-\frac{p-j}{p-1}\sigma-\beta \leq -\sigma+\underbrace{(\mathtt{s}-\alpha)}_{\mathtt{s}\leq\alpha}+\underbrace{(\mathtt{r}-\beta)}_{\mathtt{r}\leq\beta} \leq -\frac{p-j}{p-1}\sigma.$$

Hence

$$|I^{[j]}_{(\alpha,\beta)}(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{p-m-j+1-\alpha-\beta}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}, \tag{3.33}$$

which implies that $I^{[j]}_{(\alpha,\beta)} \in \mathbf{SG}_{\mu}^{p-m-j+1-\alpha-\beta,-\frac{p-j}{p-1}\sigma}(\mathbb{R}^2)$.

Case 2: Assume that $\partial_\xi^{\beta_1}\lambda_{p-k_1}$, with $k_1 \geq m$, appears in $I^{[j]}_{(\alpha,\beta)}$.

Here we need to present more details in the computations, because this case is a little bit trickier than the previous one. By using estimates (iii) and (iv) of Lemma 3.3 and $k_1 \geq m$,

$$\begin{aligned}
&\left|\left(\partial_\xi^{\beta_1}\lambda_{p-k_1}\cdots\partial_\xi^{\beta_{\mathtt{r}}}\lambda_{p-k_{\mathtt{r}}}\right)(x,\xi)\right| \\
\leq\ & C^{\beta+1}(\beta!)^{\mu}\underbrace{\langle\xi\rangle^{1-k_1-\beta_1}\langle x\rangle^{1-\frac{p-k_1}{p-1}\sigma}}_{\text{from (iii)}}\underbrace{\langle\xi\rangle^{-\beta_2}\langle x\rangle^{\frac{p-k_2}{p-1}(1-\sigma)}\cdots\langle\xi\rangle^{-\beta_{\mathtt{r}}}\langle x\rangle^{\frac{p-k_{\mathtt{r}}}{p-1}(1-\sigma)}}_{\text{from (iv)}} \\
\leq\ & C^{\beta+1}(\beta!)^{\mu}\langle\xi\rangle^{1-m-\beta}\langle x\rangle^{1-\frac{p-k_1}{p-1}\sigma+\frac{p-k_2}{p-1}(1-\sigma)+\cdots+\frac{p-k_{\mathtt{r}}}{p-1}(1-\sigma)}.
\end{aligned} \tag{3.34}$$

By hypothesis and the fact that $j \geq 1$,

$$\left|D_x^{\beta}a_{p-j}(t,x)\xi^{p-j}\right| \leq C^{\beta}(\beta!)^{s_0}\langle\xi\rangle^{p-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma-\beta}. \tag{3.35}$$

By using (v) of Lemma 3.3 and $1 \leq \tilde{k}_{\mathtt{a}} \leq p-1$, for each $1 \leq \mathtt{a} \leq \mathtt{s}$, we obtain

$$\begin{aligned}
& \left| \left(D_x^{\alpha_1} \lambda_{p-\tilde{k}_1} \cdots D_x^{\alpha_{\mathtt{s}}} \lambda_{p-\tilde{k}_{\mathtt{s}}} \right)(x,\xi) \right| \\
\leq\ & C^{\alpha+1}(\alpha!)^{\mu} \langle \xi \rangle^{1-\tilde{k}_1+1-\tilde{k}_2+\cdots+1-\tilde{k}_{\mathtt{s}}} \langle x \rangle^{-\frac{p-\tilde{k}_1}{p-1}\sigma-\alpha_1+1-\cdots-\frac{p-\tilde{k}_{\mathtt{s}}}{p-1}\sigma-\alpha_{\mathtt{s}}+1} \\
=\ & C^{\alpha+1}(\alpha!)^{\mu} \langle \xi \rangle^{\mathtt{s} \overbrace{-\tilde{k}_1-\cdots-\tilde{k}_{\mathtt{s}}}^{\text{all } \tilde{k}_{\mathtt{a}} \geq 1}} \langle x \rangle^{-\alpha+\mathtt{s}-\frac{\mathtt{s}p \overbrace{-\tilde{k}_1-\cdots-\tilde{k}_{\mathtt{s}}}^{\text{all } \tilde{k}_{\mathtt{a}} \leq p-1}}{p-1}\sigma} \\
\leq\ & C^{\alpha+1}(\alpha!)^{\mu} \langle \xi \rangle^{\mathtt{s}-\mathtt{s}} \langle x \rangle^{-\alpha+\mathtt{s}-\overbrace{\frac{\mathtt{s}p-\mathtt{s}(p-1)}{p-1}\sigma}^{\leq 0}} \\
\leq\ & C^{\alpha+1}(\alpha!)^{\mu} \langle x \rangle^{-\alpha+\mathtt{s}}. \qquad (3.36)
\end{aligned}$$

It follows from (3.34), (3.35) and (3.36) that

$$|I^{[j]}_{(\alpha,\beta)}(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{1-m-\beta+p-j-\alpha} \langle x \rangle^{1-\frac{p-k_1}{p-1}\sigma+\frac{p-k_2}{p-1}(1-\sigma)+\cdots+\frac{p-k_{\mathtt{r}}}{p-1}(1-\sigma)-\beta-\frac{p-j}{p-1}\sigma-\alpha+\mathtt{s}}.$$

Now we perform some computations on the exponents. Since $k_{\mathtt{a}} \geq 1$, for each $2 \leq \mathtt{a} \leq \mathtt{r}$, we obtain

$$\begin{aligned}
& 1-\frac{p-k_1}{p-1}\sigma+\frac{p-k_2}{p-1}(1-\sigma)+\cdots+\frac{p-k_{\mathtt{r}}}{p-1}(1-\sigma)-\beta-\frac{p-j}{p-1}\sigma-\alpha+\mathtt{s} \\
\leq\ & 1-\frac{p-k_1}{p-1}\sigma+\frac{p(\mathtt{r}-1)-(\mathtt{r}-1)}{p-1}(1-\sigma)-\alpha-\beta-\frac{p-j}{p-1}\sigma+\mathtt{s} \\
=\ & 1-\frac{p-k_1}{p-1}\sigma+(\mathtt{r}-1)(1-\sigma)-\beta-\frac{p-j}{p-1}\sigma \underbrace{+\mathtt{s}-\alpha}_{\leq 0} \\
\leq\ & 1-\frac{p-k_1}{p-1}\sigma+\underbrace{(\mathtt{r}-1)(1-\sigma)}_{\leq \mathtt{r}-1}-\beta-\frac{p-j}{p-1}\sigma \\
\leq\ & \underbrace{1-\frac{p-k_1}{p-1}\sigma}_{\leq 1}+\mathtt{r}-1-\beta-\frac{p-j}{p-1}\sigma \\
\leq\ & -\frac{p-j}{p-1}\sigma+1-1\underbrace{-\beta+\mathtt{r}}_{\leq 0} \leq -\frac{p-j}{p-1}\sigma.
\end{aligned}$$

Hence

$$|I^{[j]}_{(\alpha,\beta)}(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{p-m-j+1-\alpha-\beta} \langle x \rangle^{-\frac{p-j}{p-1}\sigma}.$$

From (3.26) and the above discussion it follows that

$$e^{\Lambda} \circ (ia_{p-j}(t,x)D_x^{p-j}) \circ {}^{R}\{e^{-\Lambda}\} = ia_{p-j}(t,x)D_x^{p-j} + \mathbf{op}\left(\sum_{m=2}^{p-j} c^{[j]}_{p-m}\right) + r_0(t,x,D) + r_\infty(t,x,D),$$

where r_0 has order zero, r_∞ is regularizing and $c^{[j]}_{p-m}$ has order $p-j-m+1$ with respect to ξ, $-\frac{p-j}{p-1}\sigma$ with respect to x and depends only on $\lambda_{p-1}, ..., \lambda_{p-m+1}$. By composing with the Neumann series and using similar arguments as in the conjugation of the leading term, the previous structure of the composition does not change, which means that

$$e^{\Lambda}(x,D)\circ(ia_{p-j}(t,x)D_x^{p-j})\circ(e^{\Lambda}(x,D))^{-1} = ia_{p-j}(t,x)D_x^{p-j}+\mathbf{op}\left(\sum_{m=2}^{p-1} d_{p-m}^{[j]}\right)+r_0(t,x,D)+r_\infty(t,x,D), \quad (3.37)$$

with $d_{p-m}^{[j]}$ of order $p-j-m+1$ with respect to ξ and $-\frac{p-j}{p-1}\sigma$ with respect to x, depending only on $\lambda_{p-1}, ..., \lambda_{p-m+1}$, r_0 and r_∞ new terms with the same properties as before.

- **The conjugation of $ia_0(t,x)$.** For this term, the conjugation is given by

$$e^{\Lambda}\circ(ia_0(t,x))\circ(e^{\Lambda})^{-1} = r_0(t,x,D)+r_\infty(t,x,D), \quad (3.38)$$

  where r_0 has order zero and r_∞ is a regularizing term.

Finally, let us gather all the previous computations. It follows that

$$\begin{aligned} e^{\Lambda}\circ(iP)\circ(e^{\Lambda})^{-1} &= \partial_t + e^{\Lambda}\circ(ia_p(t)D_x^p)\circ(e^{\Lambda})^{-1} + \sum_{j=1}^{p-1} e^{\Lambda}\circ(ia_{p-j}(t,x)D_x^{p-j})\circ(e^{\Lambda})^{-1} \\ &+ e^{\Lambda}\circ(ia_0(t,x))\circ(e^{\Lambda})^{-1} \\ &= \partial_t + ia_p(t)D_x^p - \mathbf{op}\left(\partial_\xi a_p(t)\xi^p\partial_x\Lambda\right) + \mathbf{op}\left(a_p(t)\sum_{m=2}^{p-1}\mathtt{c}_{p-m}\right) \\ &+ \sum_{j=1}^{p-1} ia_{p-j}(t,x)D_x^{p-j} + \mathbf{op}\left(\sum_{j=1}^{p-1}\sum_{m=2}^{p-j} d_{p-m}^{[j]}\right) + (r_0+r_\infty)(t,x,D). \end{aligned} \quad (3.39)$$

Now we define

$$d_{p-j}(t,x,\xi) := a_p(t)\mathtt{c}_{p-j}(x,\xi) + \sum_{\mathtt{h}+m=j+1} d_{p-m}^{[\mathtt{h}]}(x,\xi), \qquad j=2,...,p-1,$$

which has ξ-order $p-j$, x-order $-\frac{p-j}{p-1}\sigma$ and depends only on $\lambda_{p-1}, ..., \lambda_{p-j+1}$. Hence we can rewrite (3.39) as

$$\begin{aligned} e^{\Lambda}(x,D)\circ(iP)\circ(e^{\Lambda}(x,D))^{-1} &= \partial_t + ia_p(t)D_x^p + \sum_{j=1}^{p-1} ia_{p-j}(t,x)D_x^{p-j} \\ &- \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1}) - \cdots - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_1) \\ &+ \mathbf{op}\left(\sum_{j=2}^{p-1} d_{p-j}\right) + (r_0+r_\infty)(t,x,D), \end{aligned} \quad (3.40)$$

where r_0 is a zero order term, r_∞ is regularizing, and d_{p-j} has ξ-order $p-j$, x-order $-\frac{p-j}{p-1}\sigma$ and depends only on $\lambda_{p-1}, ..., \lambda_{p-j+1}$.

In order to perform the next conjugation, let us improve (3.40). Essentially we must put the terms level by level according to the ξ-order. Namely

$$\begin{aligned}
e^{\Lambda}(x,D)\circ(iP)\circ(e^{\Lambda}(x,D))^{-1} &= \partial_t + a_p(t)D_x^p\\
&+ ia_{p-1}(t,x)D_x^{p-1} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1})\\
&+ ia_{p-2}(t,x)D_x^{p-2} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-2}) + \mathbf{op}(d_{p-2})\\
&+ \cdots\\
&+ ia_2(t,x)D_x^2 - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_2) + \mathbf{op}(d_2)\\
&+ ia_1(t,x)D_x - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_1) + \mathbf{op}(d_1)\\
&+ (r_0 + r_\infty)(t,x,D), \qquad (3.41)
\end{aligned}$$

where d_{p-j}, $j = 2,...,p-1$, r_0 and r_∞ are as described previously.

3.5.2 Conjugation of $e^{\Lambda}(iP)(e^{\Lambda})^{-1}$ by $e^{\Lambda_{K,\rho'}}$

For some $K > 0$ and $\rho' < \rho$, we consider the operator $e^{\Lambda_{K,\rho'}}(t,D)$, where

$$\Lambda_{K,\rho'}(t,\xi) := K(T-t)\langle\xi\rangle_h^{(p-1)(1-\sigma)} + \rho'\langle\xi\rangle_h^{1/\theta}. \qquad (3.42)$$

is a symbol of order $1/\theta$, since $(p-1)(1-\sigma) < 1/\theta$ by assumption.

- **Conjugation of ∂_t.** For this term we get

$$e^{\Lambda_{K,\rho'}}(t,D)\circ\partial_t\circ e^{-\Lambda_{K,\rho'}}(t,D) = \partial_t + K\langle D_x\rangle_h^{(p-1)(1-\sigma)}.$$

- **Conjugation of $ia_p(t)D_x^p$.** Since $a_p(t)$ does not depend on x, the conjugation is simply given by

$$e^{\Lambda_{K,\rho'}}(t,D)\circ(ia_p(t)D_x^p)\circ e^{-\Lambda_{K,\rho'}}(t,D) = ia_p(t)D_x^p.$$

- **Conjugation of $ia_{p-1}(t,x)D_x^{p-1} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1})$.** We have that

$$\begin{aligned}
&e^{\Lambda_{K,\rho'}}(t,D)\circ\big(ia_{p-1}(t,x)D_x^{p-1} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1})\big)\circ e^{-\Lambda_{K,\rho'}}(t,D)\\
&= ia_{p-1}(t,x)D_x^{p-1} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1}) + A_{K,\rho'}^{[p-1]}(t,x,D),
\end{aligned}$$

  with $A_{K,\rho'}^{[p-1]}(t,x,D)$ a remainder term whose symbol satisfies

$$\left|\partial_\xi^\alpha\partial_x^\beta A_{K,\rho'}^{[p-1]}(t,x,\xi)\right| \leq C_{T,K,\rho',M_{p-1}}\langle\xi\rangle_h^{p-2+\frac{1}{\theta}}\langle x\rangle^{-\sigma-1}, \qquad (3.43)$$

  for all $\alpha,\beta\in\mathbb{N}_0$, $t\in[0,T]$ and $x,\xi\in\mathbb{R}$.

- **Conjugation of $ia_{p-j}(t,x)D_x^{p-j} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-j}) + \mathbf{op}(d_{p-j})$, $j = 2,...,p-1$.**

$$\begin{aligned}
&e^{\Lambda_{K,\rho'}}(t,D)\circ\big(ia_{p-j}(t,x)D_x^{p-j} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-j}) + \mathbf{op}(d_{p-j})\big)\circ e^{-\Lambda_{K,\rho'}}(t,D)\\
&= ia_{p-j}(t,x)D_x^{p-j} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-j}) + \mathbf{op}(d_{p-j}) + A_{K,\rho'}^{[p-j]}(t,x,D),
\end{aligned}$$

with $A^{[p-j]}_{K,\rho'}(t,x,D)$ a remainder term whose symbol satisfies

$$\left|\partial_\xi^\alpha\partial_x^\beta A^{[p-j]}_{K,\rho'}(t,x,\xi)\right| \le C_{T,K,\rho',M_{p-1},\dots,M_{p-j}}\langle\xi\rangle_h^{p-j-1+\frac{1}{\theta}}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}, \tag{3.44}$$

for all $\alpha,\beta\in\mathbb{N}_0$, $t\in[0,T]$ and $x,\xi\in\mathbb{R}$.

Gathering all these computations, we get

$$\begin{aligned}
& Q_{\Lambda,K,\rho'}(t,x,D)\circ(iP)\circ(Q_{\Lambda,K,\rho'}(t,x,D))^{-1}\\
= \;& e^{\Lambda_{K,\rho'}}(t,D)\circ e^{\Lambda}(x,D)\circ(iP)\circ(e^{\Lambda}(x,D))^{-1}\circ e^{-\Lambda_{K,\rho'}}(t,D) = \partial_t + ia_p(t)D_x^p\\
+ \;& ia_{p-1}(t,x)D_x^{p-1} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1}) + A^{[p-1]}_{K,\rho'}(t,x,D)\\
+ \;& ia_{p-2}(t,x)D_x^{p-2} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-2}) + \mathbf{op}(d_{p-2}) + A^{[p-2]}_{K,\rho'}(t,x,D)\\
+ \;& \cdots\\
+ \;& ia_2(t,x)D_x^2 - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_2) + \mathbf{op}(d_2) + A^{[2]}_{K,\rho'}(t,x,D)\\
+ \;& ia_1(t,x)D_x - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_1) + \mathbf{op}(d_1) + A^{[1]}_{K,\rho'}(t,x,D)\\
+ \;& K\langle D_x\rangle_h^{(p-1)(1-\sigma)} + (r_0+r_\infty)(t,x,D),
\end{aligned} \tag{3.45}$$

where $A^{[p-j]}_{K,\rho'}$ is a remainder term depending on $M_{p-1},\dots,M_{p-j}$ satisfying (3.43) and (3.44), for $j=2,\dots,p-1$.

3.6 Estimates from below for the real parts

This section is devoted to prove some estimates from below for the real parts of the lower order terms of

$$P_{\Lambda,K,\rho'} := Q_{\Lambda,K,\rho'}\circ iP\circ(Q_{\Lambda,K,\rho'})^{-1},$$

intending to apply to these terms the sharp Gårding Theorem 2.10 and finally to achieve a well-posedness result for the Cauchy problem (3.5). For each $j=1,\dots,p-1$, by the definition of λ_{p-j} it follows that

$$-\partial_\xi\xi^p\partial_x\lambda_{p-j}(x,\xi) = -p\xi^{p-1}M_{p-j}\omega\left(\frac{\xi}{h}\right)\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right).$$

By the definition of ω and provided that $|\xi|>2h$ we can rewrite the above expression as

$$\begin{aligned}
-\partial_\xi\xi^p\partial_x\lambda_{p-j}(x,\xi) &= p|\xi|^{p-1}M_{p-j}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\\
&= p|\xi|^{p-1}M_{p-j}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\\
&- p|\xi|^{p-1}M_{p-j}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right].
\end{aligned}$$

The first term $p|\xi|^{p-1}M_{p-j}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}$ has order $p-j$ in ξ and $-\frac{p-j}{p-1}\sigma$ in x. For the term $p|\xi|^{p-1}M_{p-j}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right]$, let us make some considerations: first of all, since

$$\psi(y)=\begin{cases}1, & |y|\leq\frac{1}{2}\\ 0, & |y|\geq 1\end{cases}$$

it follows that $1-\psi\left(\langle x\rangle/\langle\xi\rangle_h^{p-1}\right)$ is supported for $2\langle x\rangle\geq\langle\xi\rangle_h^{p-1}$. From this, for each $j=1,...,p-1$, we have

$$\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\leq 2\langle\xi\rangle_h^{-(p-j)\sigma},$$

hence

$$\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\leq 2\langle\xi\rangle_h^{1-j-(p-j)\sigma}.$$

Therefore, $p|\xi|^{p-1}M_{p-j}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right]$ has order less than or equal to $(p-1)(1-\sigma)$ in ξ and order zero in x, for each $j=1,...,p-1$, and then the sum of these terms still have order less than or equal to $(p-1)(1-\sigma)$ in ξ and zero in x. It follows that

$$\sum_{j=1}^{p-1}\mathbf{op}(-\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-j}(x,\xi))=\sum_{j=1}^{p-1}pM_{p-j}a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}-B^{[(p-1)(1-\sigma)]}(t,x,D),$$

where

$$B^{[(p-1)(1-\sigma)]}(t,x,\xi):=\sum_{j=1}^{p-1}pa_p(t)|\xi|^{p-1}M_{p-j}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right],\tag{3.46}$$

that is, $B^{[(p-1)(1-\sigma)]}$ has order less than or equal to $(p-1)(1-\sigma)$ in ξ, order zero in x and it depends on $M_{p-1},...,M_1$. Finally, we can write (3.45) as

$$\begin{aligned}
&Q_{\Lambda,K,\rho'}(t,x,D)\circ(iP)\circ(Q_{\Lambda,K,\rho'}(t,x,D))^{-1}=\partial_t+ia_p(t)D_x^p\\
+\;&ia_{p-1}(t,x)D_x^{p-1}+\mathbf{op}\left(pM_{p-1}a_p(t)|\xi|^{p-1}\langle x\rangle^{-\sigma}\right)+A_{K,\rho'}^{[p-1]}(t,x,D)\\
+\;&ia_{p-2}(t,x)D_x^{p-2}+\mathbf{op}\left(pM_{p-2}a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{p-2}{p-1}\sigma}\right)+\mathbf{op}(d_{p-2})+A_{K,\rho'}^{[p-2]}(t,x,D)\\
+\;&\cdots\\
+\;&ia_2(t,x)D_x^2+\mathbf{op}\left(pM_2a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{-p+3}\langle x\rangle^{-\frac{2}{p-1}\sigma}\right)+\mathbf{op}(d_2)+A_{K,\rho'}^{[2]}(t,x,D)\\
+\;&ia_1(t,x)D_x+\mathbf{op}\left(pM_1a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{-p+2}\langle x\rangle^{-\frac{1}{p-1}\sigma}\right)+\mathbf{op}(d_1)+A_{K,\rho'}^{[1]}(t,x,D)\\
+\;&K\langle D_x\rangle_h^{(p-1)(1-\sigma)}-B^{[(p-1)(1-\sigma)]}(t,x,D)+(r_0+r_\infty)(t,x,D),
\end{aligned}\tag{3.47}$$

with d_{p-m}, $m=2,...,p-1$, $A_{K,\rho'}^{[p-j]}$, $j=1,...,p-1$, r_0, r_∞ and $B^{[(p-1)(1-\sigma)]}$ as described previously.

- **Estimates for level** $(p-1)$**.** Let

$$\tilde{a}_{p-1}(t,x,D_x):=ia_{p-1}(t,x)D_x^{p-1}+\mathbf{op}\left(pM_{p-1}a_p(t)|\xi|^{p-1}\langle x\rangle^{-\sigma}\right)+A_{K,\rho'}^{[p-1]}(t,x,D).$$

We have that

$$\mathbf{Re}\ \tilde{a}_{p-1}(t,x,\xi) = -\mathbf{Im}\ a_{p-1}(t,x)\xi^{p-1} + pM_{p-1}a_p(t)|\xi|^{p-1}\langle x\rangle^{-\sigma} + \mathbf{Re}\ A^{[p-1]}_{K,\rho'}(t,x,\xi). \quad (3.48)$$

By the assumption (ii) in Theorem 3.1 we obtain

$$|\mathbf{Im}\ a_{p-1}(t,x)\xi^{p-1}| \leq C_{a_{p-1}}\langle x\rangle^{-\sigma}|\xi|^{p-1},$$

and from (3.43),

$$\begin{aligned} |\mathbf{Re}\ A^{[p-1]}_{K,\rho'}(t,x,\xi)| &\leq C_{T,K,\rho',M_{p-1}}\langle\xi\rangle_h^{p-1-1+\frac{1}{\theta}}\langle x\rangle^{-\sigma} \\ &\leq C_{T,K,\rho',M_{p-1}}\langle\xi\rangle_h^{p-1}h^{-1+\frac{1}{\theta}}\langle x\rangle^{-\sigma}. \end{aligned}$$

Moreover, $|\xi| \geq 2h$ implies that there exists $C > 0$ independent of h such that $\langle\xi\rangle_h^{p-1} \geq C|\xi|^{p-1}$. Therefore we can estimate in (3.48)

$$\mathbf{Re}\ \tilde{a}_{p-1}(t,x,\xi) \geq \left(-C_{a_{p-1}} + pM_{p-1}C_{a_p} - C_{T,K,\rho',M_{p-1}}h^{-1+\frac{1}{\theta}}\right)\langle x\rangle^{-\sigma}|\xi|^{p-1}. \quad (3.49)$$

Now we can choose the real number M_{p-1} such that $-C_{a_{p-1}} + pM_{p-1}C_{a_p} \geq 1$, which is equivalent to

$$M_{p-1} \geq \frac{1 + C_{a_{p-1}}}{pC_{a_p}},$$

and then we choose h large enough such that

$$C_{T,K,\rho',M_{p-1}}h^{-1+\frac{1}{\theta}} \leq \frac{1}{2}.$$

By these choices of M_{p-1} and h depending only on M_{p-1}, it follows that

$$\mathbf{Re}\ \tilde{a}_{p-1}(t,x,\xi) \geq \frac{1}{2}\langle x\rangle^{-\sigma}|\xi|^{p-1}, \quad |\xi| \geq 2h,$$

By Theorem 2.10, the operator $\tilde{a}_{p-1}(t,x,D)$ can be decomposed as

$$\tilde{a}_{p-1}(t,x,D) = Q_+^{[p-1]}(t,x,D) + \tilde{r}^{[p-2]}(t,x,D), \quad (3.50)$$

where $\mathbf{Re}\left(Q_+^{[p-1]}(t,x,D)u,u\right)_{L^2} \geq 0$, $u \in L^2(\mathbb{R})$, and $\tilde{r}^{[p-2]}(t,x,\xi) \in \mathbf{SG}^{p-2,-1-\sigma}_{\mu'}(\mathbb{R}^2)$, $t \in [0,T]$, depends only on M_{p-1} and h. We point that, the term $\tilde{r}^{[p-2]}(t,x,D)$ will be put together with the terms of order $p-2$ in (3.47).

- **Estimates for level** $(p-2)$**.** Let

$$\begin{aligned} \tilde{a}_{p-2}(t,x,D) &:= ia_{p-2}(t,x)D_x^{p-2} + \mathbf{op}\left(pM_{p-2}a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{p-2}{p-1}\sigma}\right) \\ &+ A^{[p-2]}_{K,\rho'}(t,x,D) + d_{p-2}(t,x,D) + \tilde{r}^{[p-2]}(t,x,D), \end{aligned}$$

hence

$$\begin{aligned}\mathbf{Re}\ \tilde{a}_{p-2}(t,x,\xi) &= -\mathbf{Im}\ a_{p-2}(t,x)\xi^{p-2} + pM_{p-2}a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{p-2}{p-1}\sigma} \\ &+ \mathbf{Re}\ A_{K,\rho'}^{[p-2]}(t,x,\xi) + \mathbf{Re}\ d_{p-2}(t,x,\xi) + \tilde{r}^{[p-2]}(t,x,\xi). \end{aligned} \tag{3.51}$$

Since d_{p-2} and $\tilde{r}^{[p-2]}$ depends only on M_{p-1} and satisfy

$$|\mathbf{Re}\ d_{p-2}(t,x,\xi) + \mathbf{Re}\ \tilde{r}^{[p-2]}(t,x,\xi)| \leq C_{M_{p-1}}\langle\xi\rangle_h^{p-2}\langle x\rangle^{-\frac{p-2}{p-1}\sigma},$$

by similar arguments and estimates that we have used in level $(p-1)$, we estimate (3.51) as

$$\begin{aligned}&\mathbf{Re}\ \tilde{a}_{p-2}(t,x,\xi) \\ \geq\ &\left(-C_{a_{p-2}} + pM_{p-2}C_{a_p} - C_{M_{p-1}} - C_{T,K,\rho',M_{p-1},M_{p-2}}h^{-1+\frac{1}{\theta}}\right)|\xi|^{p-2}\langle x\rangle^{-\frac{p-2}{p-1}\sigma}. \end{aligned} \tag{3.52}$$

Now, picking M_{p-2} such that $-C_{a_{p-2}} + pM_{p-2}C_{a_p} - C_{M_{p-1}} \geq 1$, we obtain

$$M_{p-2} \geq \frac{1 + C_{a_{p-2}} + C_{M_{p-1}}}{pC_{a_p}}$$

depending only on M_{p-1}. Also, we can choose h large enough such that

$$C_{T,K,\rho',M_{p-1},M_{p-2}}h^{-1+\frac{1}{\theta}} \leq \frac{1}{2}$$

depending only on M_{p-1} and M_{p-2}. Therefore

$$\mathbf{Re}\ \tilde{a}_{p-2}(t,x,\xi) \geq \frac{1}{2}\langle x\rangle^{-\frac{p-2}{p-1}\sigma}|\xi|^{p-2}, \quad |\xi| \geq 2h.$$

Again by Theorem 2.10, it follows that

$$\tilde{a}_{p-2} = Q_+^{[p-2]}(t,x,D) + \tilde{r}^{[p-3]}(t,x,D), \tag{3.53}$$

with $\mathbf{Re}\left(Q_+^{[p-2]}(t,x,D)u,u\right)_{L^2} \geq 0$, $u \in L^2(\mathbb{R})$, and $\tilde{r}^{[p-3]}(t,x,\xi) \in \mathbf{SG}_{\mu'}^{p-3,-\frac{p-2}{p-1}\sigma-1}(\mathbb{R}^2)$, $t \in [0,T]$, depends only on M_{p-1}, M_{p-2}, and h.

- **Estimates for a generic level** $(p-j)$**,** $j \geq 2$**.** We call

$$\begin{aligned}\tilde{a}_{p-j}(t,x,D) &:= ia_{p-j}(t,x)D_x^{p-j} + \mathbf{op}\left(pM_{p-j}a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\right) \\ &+ A_{K,\rho'}^{[p-j]}(t,x,D) + d_{p-j}(t,x,D) + \tilde{r}^{[p-j]}(t,x,D). \end{aligned}$$

By similar arguments as in the previous steps, we get

$$\begin{aligned}&\mathbf{Re}\ \tilde{a}_{p-j}(t,x,\xi) \\ \geq\ &\left(-C_{a_{p-j}} + pM_{p-j}C_{a_p} - C_{M_{p-1},\ldots,M_{p-j+1}} - C_{T,K,\rho',M_{p-1},\ldots,M_{p-j}}h^{-1+\frac{1}{\theta}}\right) \\ \times\ &|\xi|^{p-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}. \end{aligned} \tag{3.54}$$

By choosing M_{p-j} such that $-C_{a_{p-j}} + pM_{p-j}C_{a_p} - C_{M_{p-1},...,M_{p-j+1}} \geq 1$, we obtain

$$M_{p-j} \geq \frac{1 + C_{a_{p-j}} + C_{M_{p-1},...,M_{p-j+1}}}{pC_{a_p}}$$

depending on $M_{p-1}, ..., M_{p-j+1}$. At this point, we choose h large enough such that

$$C_{T,K,\rho',M_{p-1},...,M_{p-j}} h^{-1+\frac{1}{\theta}} \leq \frac{1}{2}$$

depending on $M_{p-1}, ..., M_{p-j}$. Hence

$$\mathbf{Re}\ \tilde{a}_{p-j}(t,x,\xi) \geq \frac{1}{2}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}|\xi|^{p-j}, \quad |\xi| \geq 2h.$$

By Theorem 2.10

$$\tilde{a}_{p-j}(t,x,D) = Q_+^{[p-j]}(t,x,D) + \tilde{r}^{[p-j-1]}(t,x,D), \tag{3.55}$$

where

$$\mathbf{Re}\left(Q_+^{[p-j]}(t,x,D)u,u\right)_{L^2} \geq 0, \quad u \in L^2(\mathbb{R}),$$

and $\tilde{r}^{[p-j-1]}(t,x,\xi) \in \mathbf{SG}_{\mu'}^{p-j-1,-1-\frac{p-j}{p-1}\sigma}(\mathbb{R}^2)$ depends on $M_{p-1}, ..., M_{p-j+1}, h$.

At this point, after we choose all the parameters $M_{p-1}, ..., M_1$ and h, we have

$$\begin{aligned} & Q_{\Lambda,K,\rho'}(t,x,D) \circ (iP) \circ (Q_{\Lambda,K,\rho'}(t,x,D))^{-1} = \partial_t + ia_p(t)D_x^p \\ + \ & Q_+^{[p-1]}(t,x,D) + Q_+^{[p-2]}(t,x,D) + \cdots + Q_+^{[2]}(t,x,D) + Q_+^{[1]}(t,x,D) \\ + \ & K\langle D_x\rangle_h^{(p-1)(1-\sigma)} - B^{[(p-1)(1-\sigma)]}(t,x,D) + (r_0 + r_\infty)(t,x,D). \end{aligned}$$

Now we can choose $K \geq K_0(M_{p-1}, ..., M_1)$ large enough such that

$$\mathbf{Re}\left(K\langle\xi\rangle_h^{(p-1)(1-\sigma)} - B^{[(p-1)(1-\sigma)]}(t,x,\xi)\right) \geq 0, \tag{3.56}$$

and by applying Theorem 2.10 once more we obtain

$$\begin{aligned} P_{\Lambda,K,\rho'} \ &= \ Q_{\Lambda,K,\rho'}(t,x,D) \circ (iP) \circ (Q_{\Lambda,K,\rho'}(t,x,D))^{-1} = \partial_t + ia_p(t)D_x^p \\ &+ \ \sum_{j=1}^{p-1} Q_+^{[p-j]}(t,x,D) + Q_+^{[(p-1)(1-\sigma)]}(t,x,D) + (r_0 + r_\infty)(t,x,D). \end{aligned} \tag{3.57}$$

Summing up, for ρ' small enough, choosing, in order, $M_{p-1}, ..., M_1, K$ and h large enough, precisely

$$C_{T,K,\rho',M_{p-1},...,M_1} h^{-1+\frac{1}{\theta}} \leq \frac{1}{2} \quad \text{and} \quad h > h_0$$

enlarging the parameter h_0 given in Proposition 3.2 if necessary, we obtain that formula (3.57) holds.

By these estimates, we can assert the next proposition.

Proposition 3.3. *Suppose $\rho' > 0$ sufficiently small. There exist $M_{p-1}, ..., M_1 > 0$, $K > 0$ and $h_0 = h_0(K, M_{p-1}, ..., M_1, T, \rho') > 0$ such that for every $h > h_0$ the Cauchy problem associated to the conjugated operator $P_{\Lambda,K,\rho'}$ is well-posed in $H^m(\mathbb{R})$. More precisely, for any Cauchy data $\tilde{f} \in C([0,T]; H^m(\mathbb{R}))$ and $\tilde{g} \in H^m(\mathbb{R})$, there exists a unique solution $v \in C([0,T]; H^m(\mathbb{R})) \cap C^1([0,T]; H^{m-p}(\mathbb{R}))$ such that the following energy estimates holds*

$$\|v(t)\|^2_{H^m} \leq C \left(\|\tilde{g}\|^2_{H^m} + \int_0^t \|\tilde{f}(\tau)\|^2_{H^m} d\tau \right), \quad t \in [0,T].$$

Proof. First of all, let us consider the choices of the constants which give us the decompositions in (3.50), (3.55) and that turns the inequality (3.56) true. By Leibniz formula and the identity $z + \overline{z} = 2\mathbf{Re}\, z$, we obtain

$$\frac{d}{dt}\|v(t)\|^2_{L^2} = 2\,\mathbf{Re}\,(\partial_t v(t), v(t))_{L^2}.$$

From this and (3.57), it follows that

$$\begin{aligned}
\frac{d}{dt}\|v(t)\|^2_{L^2} &= 2\mathbf{Re}\ (P_{\Lambda,K,\rho'}(t,x,D)v(t), v(t))_{L^2} - 2\mathbf{Re}\ (ia_p(t)D^p_x v(t), v(t))_{L^2} \\
&- 2\mathbf{Re}\ \left(\sum_{j=1}^{p-1} Q_+^{[p-j]}(t,x,D)v(t), v(t) \right)_{L^2} - 2\mathbf{Re}\left(Q_+^{[(p-1)(1-\sigma)]}(t,x,D)v(t), v(t) \right)_{L^2} \\
&- 2\mathbf{Re}\ (r_0(t,x,D)v(t), v(t))_{L^2} \\
&\leq 2\|P_{\Lambda,K,\rho'}v(t)\|_{L^2}\|v(t)\|_{L^2} + 2\|r_0 v(t)\|_{L^2}\|v(t)\|_{L^2},
\end{aligned} \tag{3.58}$$

where, in order to obtain this inequality, we have used the Cauchy-Schwarz inequality in the first and last terms and that

$$\mathbf{Re}\,(ia_p(t)D^p_x v(t), v(t))_{L^2} = 0, \quad \mathbf{Re}\left(Q_+^{[p-j]}v(t), v(t)\right)_{L^2} \geq 0 \quad \text{and} \quad \mathbf{Re}\left(Q_+^{[(p-1)(1-\sigma)]}v(t), v(t)\right)_{L^2} \geq 0.$$

By the inequality $2ab \leq a^2 + b^2$, for $a, b \in \mathbb{R}$, and since r_0 has order zero, we still can estimate in (3.58)

$$\frac{d}{dt}\|v(t)\|^2_{L^2} \leq C'\left(\|v(t)\|^2_{L^2} + \|P_{\Lambda,K,\rho'}v(t)\|^2_{L^2}\right).$$

Finally, by using Gronwall inequality and taking to account that $e^{C't} \leq e^{C'T} =: C$, for $t \in [0,T]$, we get the energy estimate

$$\|v(t)\|^2_{L^2} \leq C\left(\|v(0)\|^2_{L^2} + \int_0^t \|P_{\Lambda,K,\rho'}v(\tau)\|^2_{L^2} d\tau\right),$$

which implies the well-posedness in $H^m(\mathbb{R})$, by the standard energy method. □

3.7 Proof of Theorem 3.1

Given $\theta > 1$ satisfying the hypothesis of Theorem 3.1, $m \in \mathbb{R}$, $\rho > 0$, take the initial data

$$f \in C\left([0,T]; H^m_{\rho;\theta}(\mathbb{R})\right) \quad \text{and} \quad g \in H^m_{\rho;\theta}(\mathbb{R}).$$

Set the positive constants $M_{p-1}, ..., M_1, K, h_0$ for which Proposition 3.3 holds and $\rho' \in (0, \rho)$. We know that both symbols Λ and $K(T-t)\langle\cdot\rangle_h^{(p-1)(1-\sigma)}$ have order $(p-1)(1-\sigma) < 1/\theta$, hence by Proposition 3.1 it follows that

$$\begin{aligned} f_{\Lambda,K,\rho'} &:= Q_{\Lambda,K,\rho'}(t,x,D)f \quad \in \quad C([0,T]; H^m(\mathbb{R})) \\ g_{\Lambda,K,\rho'} &:= Q_{\Lambda,K,\rho'}(t,x,D)g \quad \in \quad H^m(\mathbb{R}), \end{aligned}$$

for $\rho' < \rho$. By Proposition 3.3, there exists a unique solution $v \in C([0,T]; H^m(\mathbb{R}))$ to the Cauchy problem

$$\begin{cases} P_{\Lambda,K,\rho'}v(t,x) = f_{\Lambda,K,\rho'}(t,x) \\ v(0,x) = g_{\Lambda,K,\rho'}(x) \end{cases}, \quad (t,x) \in [0,T] \times \mathbb{R},$$

satisfying the energy estimate

$$\|v(t)\|^2_{H^m} \leq C\left(\|g_{\Lambda,K,\rho'}\|^2_{H^m} + \int_0^t \|f_{\Lambda,K,\rho'}(\tau)\|^2_{H^m} d\tau\right), \quad t \in [0,T]. \tag{3.59}$$

By setting $u := (Q_{\Lambda,K,\rho'}(t,x,D))^{-1} v$, we obtain a solution for the original Cauchy problem (3.2), namely

$$\begin{cases} Pu(t,x) = f(t,x) \\ u(0,x) = g(x) \end{cases}, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

The next step is to figure out which space the solution u belongs to. Notice that

$$Q_{\Lambda,K,\rho'}(t,x,D) = e^{\Lambda_{K,\rho'}}(t,D) \circ e^{\Lambda}(x,D) \quad \Rightarrow \quad (Q_{\Lambda,K,\rho'}(t,x,D))^{-1} = \left(e^{\Lambda}(x,D)\right)^{-1} \circ \left(e^{\Lambda_{K,\rho'}}(t,D)\right)^{-1},$$

and then by the definitions of $\Lambda_{K,\rho'}$ and Λ, and by Lemma 3.2 we may write

$$u(t,x) = {}^R\left(e^{-\Lambda}(x,D)\right)\sum_j(-r(x,D))^j e^{-K(T-t)\langle D\rangle_h^{(p-1)(1-\sigma)}} e^{-\rho'\langle D\rangle_h^{\frac{1}{\theta}}} v(t,x), \quad v \in H^m(\mathbb{R}),$$

but $v \in H^m(\mathbb{R})$ implies that $e^{-\rho'\langle D\rangle_h^{\frac{1}{\theta}}} v =: u_1 \in H^m_{\rho';\theta}(\mathbb{R})$, hence

$$u(t,x) = {}^R\left(e^{-\Lambda}(x,D)\right)\sum_j(-r(x,D))^j e^{-K(T-t)\langle D\rangle_h^{(p-1)(1-\sigma)}} u_1, \quad u_1 \in H^m_{\rho';\theta}(\mathbb{R}).$$

Notice that, for every $\delta_1 > 0$, we have

$$e^{-K(T-t)\langle D\rangle_h^{(p-1)(1-\sigma)}} u_1 = \underbrace{e^{-K(T-t)\langle D\rangle_h^{(p-1)(1-\sigma)}} e^{-\delta_1\langle D\rangle_h^{\frac{1}{\theta}}}}_{\text{order zero}} e^{\delta_1\langle D\rangle_h^{\frac{1}{\theta}}} u_1 =: u_2 \in H^m_{\rho'-\delta_1;\theta}(\mathbb{R}),$$

and since $\sum_j(-r(x,D))^j$ has order zero, $u_3 := \sum_j(-r(x,D))^j u_2 \in H^m_{\rho'-\delta_1;\theta}(\mathbb{R})$, which allows us to write

$$u(t,x) = {}^R\left(e^{-\Lambda}(x,D)\right)\sum_j(-r(x,D))^j u_2 = {}^R\left(e^{-\Lambda}(x,D)\right) u_3, \quad u_3 \in H^m_{\rho'-\delta_1;\theta}(\mathbb{R}).$$

By Proposition 3.1, we have that ${}^R\left(e^{-\Lambda}(x,D)\right)$ maps $H^m_{\rho;\theta}(\mathbb{R})$ into $H^m_{\rho-\delta_2;\theta}(\mathbb{R})$, for any $\delta_2 > 0$, and we can assert that $u(t,\cdot) \in H^m_{\rho'-\delta;\theta}(\mathbb{R})$ for all $\delta > 0$, $t \in [0,T]$. Notice that the solution is less regular than the Cauchy data, in the sense that it exhibits a loss in the coefficient of the exponential weight. If we set $\tilde{\rho} := \rho' - \delta$, it follows from (3.59) that

$$\begin{aligned} \|u(t)\|^2_{H^m_{\tilde{\rho};\theta}} &= \| \left(Q_{\Lambda,K,\rho'}(t,\cdot,D)\right)^{-1} v(t)\|^2_{H^m_{\tilde{\rho};\theta}} \leq C_1 \|v(t)\|^2_{H^m} \\ &\leq C_2 \left(\|g_{\Lambda,K,\rho'}\|^2_{H^m} + \int_0^t \|f_{\Lambda,K,\rho'}(\tau)\|^2_{H^m} d\tau \right) \\ &\leq C_3 \left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau)\|^2_{H^m_{\rho;\theta}} d\tau \right), \quad t \in [0,T]. \end{aligned} \tag{3.60}$$

The conclusion here is: if we take the data $f \in C\left([0,T]; H^m_{\rho;\theta}(\mathbb{R})\right)$ and $g \in H^m_{\rho;\theta}(\mathbb{R})$, for some $m \in \mathbb{R}$ and $\rho > 0$, then we find a solution u for the Cauchy problem associated with the operator P with initial data f and g which satisfies

$$u \in C\left([0,T]; H^m_{\tilde{\rho};\theta}(\mathbb{R})\right), \quad \tilde{\rho} < \rho.$$

To prove the uniqueness of the solution, let consider $u_1, u_2 \in C\left([0,T]; H^m_{\tilde{\rho};\theta}(\mathbb{R})\right)$ such that

$$\begin{cases} Pu_j = f \\ u_j(0) = g \end{cases}, \quad j = 1, 2.$$

By shrinking ρ' if necessary, we can find new parameters $M'_{p-1}, ..., M'_1 > 0$, $K' > 0$ and $h'_0 > 0$ in order to apply Proposition 3.3 again and obtain that the Cauchy problem associated with the conjugated operator

$$P_{\Lambda',K',\rho'} := Q_{\Lambda',K',\rho'} \circ (iP) \circ Q^{-1}_{\Lambda',K',\rho'}$$

is well-posed in $H^m(\mathbb{R})$, where $Q_{\Lambda',K',\rho'}$ comes from the new choice of parameters. Proposition 3.3 allows us to conclude that $Q_{\Lambda',K',\rho'}f,\ Q_{\Lambda',K',\rho'}g,\ Q_{\Lambda',K',\rho'}u_j \in H^m(\mathbb{R})$ and satisfy

$$\begin{cases} P_{\Lambda',K',\rho'} Q_{\Lambda',K',\rho'} u_j = Q_{\Lambda',K',\rho'} f \\ Q_{\Lambda',K',\rho'} u_j(0) = Q_{\Lambda',K',\rho'} g \end{cases}, \quad j = 1, 2,$$

hence $Q_{\Lambda',K',\rho'}u_1 = Q_{\Lambda',K',\rho'}u_2$, which implies that $u_1 = u_2$. This concludes the proof.

□

Remark 3.5. *In Theorem 3.1 we assumed that a_p depends only on t. In the $H^\infty-$setting it is possible to allow also a dependence on x in a_p assuming a suitable decay for $|x| \to \infty$ as for the other coefficients, cf. [12, Section 4]. In the Gevrey setting, this is difficult since in the conjugation $e^{K(T-t)\langle D\rangle_h^{1/\theta}} i a_p(t,x) D_x^p e^{-K(T-t)\langle D\rangle_h^{1/\theta}}$ it gives*

$$i a_p(t,x) D_x^p + \mathbf{op}\left(K(T-t)\partial_\xi \langle \xi\rangle_h^{\frac{1}{\theta}} \partial_x a_p(t,x)\xi^p\right) + l.o.t.$$

and the second term behaves like $\langle\xi\rangle_h^{p-1+\frac{1}{\theta}}$, that is of order strictly greater than $p-1$ and it cannot be compared with other terms as we did in Section 3.6.

Remark 3.6. *In this final remark, we observe that we could prove a slightly more general version of Theorem 3.1, which is a kind of sufficiency counterpart for the main theorem in [7]. More to the point, we may consider the following hypotheses on the decay of the coefficients $a_{p-j}(t,x), j = 1, \dots, p-1$:*

$$|\partial_x^\beta a_{p-j}(t,x)| \leq C_{p-j}^{\beta+1} \beta!^{\theta_0} \langle x \rangle^{-\sigma_{p_j} - \beta}, \tag{3.61}$$

for some $1 < \theta_0$ and $\sigma_{p-j} \in (0,1)$. Then the condition $\theta_0 \leq \theta < \frac{1}{(p-1)(1-\sigma)}$ for $\mathcal{H}_\theta^\infty(\mathbb{R})$ well-posedness of (3.2) would be:

$$\Xi < \frac{1}{\theta}, \tag{3.62}$$

where Ξ is given by (6). In the next lines, we shall explain how to obtain this improvement using the ideas developed in this thesis.

To prove the sufficiency of (3.62) under (3.61), we just need to consider a slightly different change of variables (cf. Section 3.3):

$$\lambda_{p-j}(x,\xi) = M_{p-j} w(\xi h^{-1}) \langle \xi \rangle_h^{-(j-1)} \int_0^x \langle y \rangle^{-\sigma_{p-j}} \psi\left(\frac{\langle y \rangle}{\langle \xi \rangle_h^{p-1}} \right) dy, \quad j = 1, \dots, p-1,$$

and

$$\Lambda_{K,\rho'}(t,\xi) = K(T-t)\langle \xi \rangle_h^{\Xi} + \rho' \langle \xi \rangle_h^{\frac{1}{\theta}}.$$

Arguing as in Lemma 3.1, one gets that

$$\Lambda = \sum_{j=1}^{p-1} \lambda_{p-j} \in \boldsymbol{S}_\mu^{\Xi}(\mathbb{R}^2) \cap \boldsymbol{SG}_\mu^{0,\Theta}(\mathbb{R}^2),$$

where

$$\Theta := \max_{1 \leq j \leq p-1} \left\{ (1 - \sigma_{p-j}) - \frac{j-1}{p-1} \right\}.$$

Thus, we still have that $e^{\rho' \langle \xi \rangle_h^{\frac{1}{\theta}}}$ is the leading part of the operator

$$Q_{\Lambda,K,\rho'}(t,x,D) = e^{\Lambda_{K,\rho'}}(t,D) \circ e^{\Lambda}(x,D).$$

So, due to the hypotheses $\Xi < \frac{1}{\theta}$, $1 < \theta_0 \leq \theta$ and (3.61), one can still run the change of variable argument and conclude $\mathcal{H}_\theta^\infty(\mathbb{R})$ well-posedness for (3.2). Of course the proof of the result described in this remark is even more technical than the proof of Theorem 3.1. For this reason, we decided to present a simpler result and leave the details of this more general version to the interested reader.

Remark 3.7. *Theorem 1 in [7] shows that if $\theta > \frac{1}{(p-1)(1-\sigma)}$, the Cauchy problem (3.2) is not well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$. In critical case $\theta = \frac{1}{(p-1)(1-\sigma)}$ by modifying the terms of the change of variable it should be possible to show local in time well-posedness. We do not treat this case in detail for our Cauchy problem but just refer to similar situations, cf. for instance [17, Remark 8].*

3.8 Well-posedness in projective Gevrey spaces

In this section, we study the same Cauchy problem (3.2), but now we aim to obtain a well-posedness result in projective Gevrey spaces $H^{\infty}_{\theta}(\mathbb{R}^n)$. Also we point out that, for this case, it is necessary a slightly change on the hypotheses on the coefficients of the operator $P(t,x,D_t,D_x)$. The second main result of this work is given in the following.

Theorem 3.3. *Let $\theta_0 > 1$ and $\sigma \in \left(\frac{p-2}{p-1}, 1\right)$ such that $\theta_0 < \frac{1}{(p-1)(1-\sigma)}$. Let P be an operator of the type (3.1) whose coefficients satisfy the following assumptions:*

(i) *$a_p \in C([0,T];\mathbb{R})$ and there exists $C_{a_p} > 0$ such that $|a_p(t)| \geq C_{a_p}$, for all $t \in [0,T]$.*

(ii) *For every $A > 0$ there exists $C_{A,a_{p-j}} > 0$ such that $|\partial_x^{\beta} a_{p-j}(t,x)| \leq C_{A,a_{p-j}} A^{\beta} \beta!^{\theta_0} \langle x \rangle^{-\frac{p-j}{p-1}\sigma - \beta}$, for all $j = 1, ..., p-1$ and for $\beta \in \mathbb{N}_0$, $(t,x) \in [0,T] \times \mathbb{R}$.*

If $\theta > 1$ is such that $\theta_0 \leq \theta < \frac{1}{(p-1)(1-\sigma)}$, the data $f \in C\left([0,T]; H^m_{\rho;\theta}(\mathbb{R})\right)$ and the data $g \in H^m_{\rho;\theta}(\mathbb{R})$, with $m \in \mathbb{R}$ and $\rho > 0$, then the Cauchy problem (3.2) admits a unique solution $u \in C^1\left([0,T]; H^m_{\tilde{\rho};\theta}(\mathbb{R})\right)$ for every $\tilde{\rho} \in (0,\rho)$ and the solution satisfies the energy estimate

$$\|u(t)\|^2_{H^m_{\tilde{\rho};\theta}} \leq C\left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau)\|^2_{H^m_{\rho;\theta}} d\tau\right), \tag{3.63}$$

for all $t \in [0,T]$ and for some constant $C > 0$. In particular, for $\theta \in \left[s_0, \frac{1}{(p-1)(1-\sigma)}\right)$ the Cauchy problem (3.2) is well-posed in $H^{\infty}_{\theta}(\mathbb{R})$.

To prove Theorem 3.3 we will proceed in a very similar way as we have done in Theorem 3.1 with the necessary adjustments. Let us consider the functions $\lambda_{p-k}(x,\xi)$ given by (3.6) and $\Lambda_{K,\rho'}(t,\xi)$ given by (3.42) which will be employed in the change of variable, where the parameters $K > 0$, $\rho' \in (0,\rho)$ will be chosen later in a suitable way.

We observe that under the condition (ii) of Theorem 3.3, the symbol $a_{p-j}(t,x)\xi^{p-j}$ belongs to $C\left([0,T]; \mathbf{\Gamma G}^{p-j,-\frac{p-j}{p-1}\sigma}_{\theta_0}(\mathbb{R}^{2n})\right)$ which is contained in $C\left([0,T]; \mathbf{\Gamma G}^{p-j}_{\theta_0}(\mathbb{R}^{2n})\right)$. Then we can use the following results to deal with the related operators.

Proposition 3.4. *If p is a symbol of $\tilde{\mathbf{\Gamma}}^m_{\theta}(\mathbb{R}^{2n})$, then the operator $p(x,D)$ maps continuously $H^{m'}_{\rho;\theta}(\mathbb{R}^n)$ into $H^{m'-m}_{\rho;\theta}(\mathbb{R}^n)$ for every $m', \rho \in \mathbb{R}$.*

If $p \in \mathbf{\Gamma}^m_{\theta}(\mathbb{R}^{2n})$ and $q \in \mathbf{\Gamma}^{m'}_{\theta}(\mathbb{R}^{2n})$, by Proposition 6.4 of [34] the operator $p(x,D)q(x,D)$ is a pseudo-differential operator with symbol s given by

$$s(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p(x,\xi) D_x^{\alpha} q(x,\xi) + r_N(x,\xi),$$

for all $N \geq 1$, where $r_N \in \mathbf{\Gamma}^{m+m'-N}_{\theta}(\mathbb{R}^{2n})$.

And now we have the following conjugation theorem, cf. Theorem 2.9 [5], which is a reformulation of Theorem 3.2 for symbols in $\mathbf{\Gamma}_\theta^m(\mathbb{R}^{2n})$.

Theorem 3.4. *Let p be a symbol in $\mathbf{\Gamma}_\theta^m(\mathbb{R}^{2n})$ and Φ satisfying (3.11) and (3.12), and $\mu < \theta < \kappa$. Then there exists $h_0 = h_0(C_\Phi) > 1$ such that if $h \geq h_0$, then*

$$\begin{aligned} e^{\Phi}(x,D)p(x,D)\,{}^R(e^{-\Phi}(x,D)) &= p(x,D) + \mathbf{op}\left(\sum_{1\leq|\alpha+\beta|<N} \frac{1}{\alpha!\beta!}\partial_\xi^\alpha\{\partial_\xi^\beta e^{\Phi(x,\xi)} D_x^\beta p(x,\xi) D_x^\alpha e^{-\Phi(x,\xi)}\}\right) \\ &+ r_N(x,D) + r_\infty(x,D), \end{aligned} \tag{3.64}$$

where r_N and r_∞ satisfy, respectively, the following conditions: there exists $c' = c'(\Phi) > 0$ and for every $A > 0$ there exists $C_A > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta r_N(x,\xi)| \leq C_A A^{|\alpha+\beta|+2N} (\alpha!\beta!)^\theta N!^{2\theta-1} \langle\xi\rangle_h^{m-(1-\frac{1}{\kappa})N-|\alpha|}, \tag{3.65}$$

and

$$|\partial_\xi^\alpha \partial_x^\beta r_\infty(x,\xi)| \leq C_A A^{|\alpha+\beta|+2N} (\alpha!\beta!)^\theta N!^{2\theta-1} e^{-c'\langle\xi\rangle_h^{\frac{1}{\kappa}}}. \tag{3.66}$$

Proof. Check [34, Theorem 6.9]. □

Remark 3.8. *Concerning the remainder terms r_N and r_∞ appearing in the latter result, we can point that:*

- *By choosing N large enough depending on κ, we notice that $m - (1 - \frac{1}{\kappa})N$ shrinks, hence r_N can be considered as a symbol of order zero.*
- *The operator corresponding to r_∞ has regularizing properties on Gevrey classes. In other words, it maps $(G_0^\theta)'(\mathbb{R}^n)$ into $G^\theta(\mathbb{R}^n)$. However, for our goals we shall consider r_∞ as a symbol of $\tilde{\mathbf{\Gamma}}_\theta^0(\mathbb{R}^{2n})$.*
- *In the computations of Subsection 3.8.1, if N is sufficiently large, the remainder $r_N + r_\infty$ can be considered as a symbol of $\tilde{\mathbf{\Gamma}}_\theta^0(\mathbb{R}^{2n})$.*
- *Since we are dealing now with projective Gevrey regular symbols p, it is possible to conclude that the remainders r_N and r_∞ satisfy estimates like (3.65) and (3.66).*

We shall apply Theorem 3.4 to $\Phi = \Lambda$, where Λ is defined by (3.9). In fact, it is immediate to verify that Λ satisfies the assumptions of Theorem 3.4.

The next result (we are considering the particular case of symbols in $\mathbb{R}^2$) is used to perform conjugations with operators $e^{\Lambda_{K,\rho'}}(x,D)$, where $\Lambda_{K,\rho'}$ is given by (3.42) for some $K > 0$ and $\rho' \in (0,\rho)$ (where $\rho > 0$ is the same as in the statement of Theorem 3.3). Compared to the latter result we have a small difference in the sense that the conjugation now can be performed for every $\rho' > 0$, since the symbol of the operator satisfies projective Gevrey estimates.

Proposition 3.5. *If $p \in \tilde{\mathbf{\Gamma}}^m_\theta(\mathbb{R}^2)$, then we can write*

$$e^{\Lambda_{K,\rho'}}(t,D)p(x,D)e^{-\Lambda_{K,\rho'}}(t,D) = \mathbf{op}\left(\sum_{\alpha<N} \frac{1}{\alpha!}\partial_\xi^\alpha e^{\Lambda_{K,\rho'}(t,\xi)} D_x^\alpha p(x,\xi) e^{-\Lambda_{K,\rho'}(x,\xi)}\right) + r_N(t,x,D),$$

where r_N satisfies the following condition: for every $A > 0$ there exists $C_{K,\rho',A,N} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta r_N(t,x,\xi)| \leq |||p|||_A C_{K,\rho',A,N} A^{\alpha+\beta} \langle \xi \rangle_h^{m-N(1-\frac{1}{\theta})}.$$

3.8.1 Conjugation of the operator

Now we are ready to perform the conjugation for the projective case, that is, let us obtain a suitable representation for $P_{\Lambda,K,\rho'} := Q_{\Lambda,K,\rho'}(iP)Q^{-1}_{\Lambda,K,\rho'}$. The reader must have in mind that some details in the computations will be omitted since they follow the same ideas as in the inductive case, so it can be checked in Section 3.5.1.

Conjugation of iP by e^Λ

The goal here is to perform the conjugation of iP by the operator $e^\Lambda(x,D)$ for $\Lambda = \sum_{k=1}^{p-1} \lambda_{p-k}$. Since $iP(t,x,D_t,D_x) = \partial_t + ia_p(t)D_x^p + ia_{p-1}(t,x)D_x^{p-1} + \cdots + ia_1(t,x)D_x + ia_0(t,x)$, let us split the computations term by term.

- **The conjugation of ∂_t.** It is trivially given by

$$e^\Lambda(x,D) \circ \partial_t \circ (e^\Lambda(x,D))^{-1} = \partial_t.$$

- **The conjugation of $ia_p(t)D_x^p$.** We notice that $ia_p(t)D_x^p$ does not depend on x, hence we can use Theorem 3.4 to obtain

$$\begin{aligned} e^\Lambda(x,D) \circ (ia_p(t)D_x^p) \circ {}^R(e^{-\Lambda}(x,D)) &= ia_p(t)D_x^p + i\mathbf{op}\left(\sum_{1\leq\alpha<p} \frac{1}{\alpha!}\partial_\xi^\alpha \left(e^\Lambda a_p(t)\xi^p D_x^\alpha e^{-\Lambda}\right)\right) \\ &+ r_p(t,x,D) + r_\infty(t,x,D), \end{aligned} \tag{3.67}$$

where $r_p \in \Gamma^0_\theta(\mathbb{R}^2)$ and r_∞ satisfies an estimate like (3.66). As in (3.21), (3.67) becomes

$$\begin{aligned} e^\Lambda(x,D) \circ (ia_p(t)D_x^p) \circ {}^R(e^{-\Lambda}(x,D)) &= ia_p(t)D_x^p + i\mathbf{op}\left(\partial_\xi\{a_p(t)\xi^p D_x(-\Lambda)\}\right) \\ &+ ia_p(t)c_{-2}(x,D) + \cdots + ia_p(t)c_{-p}(x,D) \\ &+ r_\infty(t,x,D), \end{aligned}$$

where $c_{-m} \in \mathbf{SG}_\mu^{p-m,-\frac{p-m+1}{p-1}\sigma}(\mathbb{R}^2)$ and depends only on $\lambda_{p-1},...,\lambda_{p-m+1}$, for each $m = 2,...,p$. Now, composing with the Neumann series, it follows that

$$\begin{aligned} e^\Lambda(x,D) \circ (ia_p(t)D_x^p) \circ (e^\Lambda(x,D))^{-1} &= ia_p(t)D_x^p - \mathbf{op}\left(\partial_\xi a_p(t)\xi^p \partial_x\Lambda + a_p(t)\sum_{m=2}^{p-1} \mathsf{c}_{p-m}\right) \\ &+ \tilde{r}_0(t,x,D), \end{aligned} \tag{3.68}$$

with $\tilde{r}_0 \in C\left([0,T];\tilde{\mathbf{\Gamma}}^0_\theta(\mathbb{R}^2)\right)$ and c_{p-m} of order $p-m$ with respect to ξ and $-\frac{p-m+1}{p-1}\sigma$ with respect to x depending only on $\lambda_{p-1},...,\lambda_{p-(m-1)}$, for each $m=2,...,p-1$.

From now on, symbols of $C\left([0,T],\tilde{\mathbf{\Gamma}}^0_\theta(\mathbb{R}^2)\right)$ are denoted by $\tilde{r}_0$.

- **Conjugation of** $ia_{p-j}(t,x)D_x^{p-j}$**,** $j=1,...,p-1$**.** For $N\in\mathbb{N}$ sufficiently large to be chosen later, by Theorem 3.4 we get

$$\begin{aligned} & e^\Lambda(x,D)\circ(ia_{p-j}(t,x)D_x^{p-j})\circ\ {}^R(e^{-\Lambda}(x,D)) \\ = \;& ia_{p-j}(t,x)D_x^{p-j}+\mathbf{op}\left(\sum_{1\le\alpha+\beta<N}\frac{1}{\alpha!\beta!}\partial_\xi^\alpha\left(\partial_\xi^\beta e^\Lambda\cdot iD_x^\beta a_{p-j}(t,x)\xi^{p-j}D_x^\alpha e^{-\Lambda}\right)\right) \\ + \;& \tilde{r}_0(t,x,D). \end{aligned} \tag{3.69}$$

By analogous arguments as used to the conjugation of the inductive case, (3.69) turns into

$$e^\Lambda\circ(ia_{p-j}(t,x)D_x^{p-j})\circ\ {}^R(e^{-\Lambda}) = ia_{p-j}(t,x)D_x^{p-j}+\mathbf{op}\left(\sum_{m=2}^{p-j}c^{[j]}_{p-m}\right)+\tilde{r}_0(t,x,D),$$

where $c^{[j]}_{p-m}\in\mathbf{\Gamma G}_\theta^{p-j-(m-1),-\frac{p-j}{p-1}\sigma}(\mathbb{R}^2)$ and depends only on $\lambda_{p-1},...,\lambda_{p-(m-1)}$. And now composing with the Neumann series, we keep the structure of the above expression in the following way

$$\begin{aligned} & e^\Lambda(x,D)\circ(ia_{p-j}D_x^{p-j})\circ(e^\Lambda(x,D))^{-1} \\ = \;& ia_{p-j}(t,x)D_x^p+\mathbf{op}\left(\sum_{m=2}^{p-1}d^{[j]}_{p-m}\right)+\tilde{r}_0(t,x,D), \end{aligned} \tag{3.70}$$

with $d^{[j]}_{p-m}\in\mathbf{\Gamma G}_\theta^{p-j-(m-1),-\frac{p-j}{p-1}\sigma}(\mathbb{R}^2)$ depending only on $\lambda_{p-1},...,\lambda_{p-(m-1)}$.

- **Conjugation of** $ia_0(t,x)$**.** We have that

$$e^\Lambda(x,D)\circ(ia_0(t,x))\circ(e^\Lambda(x,D)) = \tilde{r}_0(t,x,D). \tag{3.71}$$

Now, putting all these computations together and arguing like in the inductive case, we have

$$\begin{aligned} e^\Lambda(x,D)\circ(iP)\circ(e^\Lambda(x,D))^{-1} \;=\;& \partial_t+a_p(t)D_x^p \\ +\;& ia_{p-1}(t,x)D_x^{p-1}-\mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1}) \\ +\;& ia_{p-2}(t,x)D_x^{p-2}-\mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-2})+\mathbf{op}(d_{p-2}) \\ +\;& \cdots \\ +\;& ia_2(t,x)D_x^2-\mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_2)+\mathbf{op}(d_2) \\ +\;& ia_1(t,x)D_x-\mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_1)+\mathbf{op}(d_1) \\ +\;& \tilde{r}_0(t,x,D), \end{aligned} \tag{3.72}$$

with d_{p-j} of order $p-j$ with respect to ξ and $-\frac{p-j}{p-1}\sigma$ with respect to x, depending only on $\lambda_{p-1}, ..., \lambda_{p-(j-1)}$, for every $j = 2, ..., p-1$.

Conjugation of $e^{\Lambda}(iP)(e^{\Lambda})^{-1}$ by $e^{\Lambda_{K,\rho'}}$

Before we start with the conjugation, just recall that $e^{\Lambda_{K,\rho'}}(t, D)$ is such that

$$\Lambda_{K,\rho'}(t, x, \xi) = K(T-t)\langle\xi\rangle_h^{(p-1)(1-\sigma)} + \rho'\langle\xi\rangle_h^{1/\theta},$$

for some $K > 0$ and $\rho' < \rho$, and it is a symbol of order $1/\theta$, since $(p-1)(1-\sigma) < 1/\theta$ by assumption. In the projective case, we shall conjugate operators with symbols of Gevrey regularity $\theta > 1$ with Gevrey constant $A > 0$ which can be taken arbitrarily small. We notice that, to perform this conjugation in the inductive case, we had to consider ρ' small enough to apply Theorem 3.2. However, in the projective setting, no assumptions appear on ρ' in Theorem 3.4 neither in Proposition 3.5, since A can be chosen as small as we want. Summing up, we can choose $\rho' \in (0, \rho)$ arbitrarily and then perform the conjugation.

- **Conjugation of ∂_t.** For this term we get

$$e^{\Lambda_{K,\rho'}}(t, D) \circ \partial_t \circ e^{-\Lambda_{K,\rho'}}(t, D) = \partial_t + K\langle D_x\rangle_h^{(p-1)(1-\sigma)}.$$

- **Conjugation of $ia_p(t)D_x^p$.** Since $a_p(t)$ does not depend on x, the conjugation is simply given by

$$e^{\Lambda_{K,\rho'}}(t, D) \circ (ia_p(t)D_x^p) \circ e^{-\Lambda_{K,\rho'}}(t, D) = ia_p(t)D_x^p.$$

- **Conjugation of $ia_{p-1}(t, x)D_x^{p-1} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1})$.** We have that

$$\begin{aligned} & e^{\Lambda_{K,\rho'}}(t, D) \circ \left(ia_{p-1}(t, x)D_x^{p-1} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1})\right) \circ e^{-\Lambda_{K,\rho'}}(t, D) \\ = \quad & ia_{p-1}(t, x)D_x^{p-1} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1}) + A_{K,\rho'}^{[p-1]}(t, x, D), \end{aligned}$$

  with $A_{K,\rho'}^{[p-1]}(t, x, D)$ a remainder term whose symbol satisfies: for any $A > 0$, there exists

$$C_{A,T,K,\rho',M_{p-1}} > 0$$

  such that

$$\left|\partial_\xi^\alpha\partial_x^\beta A_{K,\rho'}^{[p-1]}(t, x, \xi)\right| \leq C_{A,T,K,\rho',M_{p-1}} A^{\alpha+\beta}(\alpha!\beta!)^\theta\langle\xi\rangle_h^{p-2+\frac{1}{\theta}}\langle x\rangle^{-\sigma-1}, \tag{3.73}$$

  for all $\alpha, \beta \in \mathbb{N}_0$, $t \in [0, T]$ and $x, \xi \in \mathbb{R}$.

- **Conjugation of $ia_{p-j}(t, x)D_x^{p-j} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-j}) + \mathbf{op}(d_{p-j})$, $j = 2, ..., p-1$.** For each j, we have

$$\begin{aligned} & e^{\Lambda_{K,\rho'}}(t, D) \circ \left(ia_{p-j}(t, x)D_x^{p-j} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-j}) + \mathbf{op}(d_{p-j})\right) \circ e^{-\Lambda_{K,\rho'}}(t, D) \\ = \quad & ia_{p-j}(t, x)D_x^{p-j} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-j}) + \mathbf{op}(d_{p-j}) + A_{K,\rho'}^{[p-j]}(t, x, D), \end{aligned}$$

with $A^{[p-j]}_{K,\rho'}(t,x,D)$ a remainder term whose symbol satisfies: for any $A>0$, there exists

$$C_{A,T,K,\rho',M_{p-1},...,M_{p-j}} > 0$$

such that

$$\left|\partial_\xi^\alpha \partial_x^\beta A^{[p-j]}_{K,\rho'}(t,x,\xi)\right| \leq C_{A,T,K,\rho',M_{p-1},...,M_{p-j}} A^{\alpha+\beta}(\alpha!\beta!)^\theta \langle\xi\rangle_h^{p-j-1+\frac{1}{\theta}} \langle x\rangle^{-\frac{p-j}{p-1}\sigma}, \tag{3.74}$$

for all $\alpha,\beta \in \mathbb{N}_0$, $t\in[0,T]$ and $x,\xi\in\mathbb{R}$.

Gathering all these computations, we get

$$\begin{aligned}
& Q_{\Lambda,K,\rho'}(t,x,D)\circ(iP)\circ(Q_{\Lambda,K,\rho'}(t,x,D))^{-1} \\
= \;& e^{\Lambda_{K,\rho'}}(t,D)\circ e^{\Lambda}(x,D)\circ(iP)\circ(e^{\Lambda}(x,D))^{-1}\circ e^{-\Lambda_{K,\rho'}}(t,D) = \partial_t + ia_p(t)D_x^p \\
+ \;& ia_{p-1}(t,x)D_x^{p-1} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-1}) + A^{[p-1]}_{K,\rho'}(t,x,D) \\
+ \;& ia_{p-2}(t,x)D_x^{p-2} - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_{p-2}) + \mathbf{op}(d_{p-2}) + A^{[p-2]}_{K,\rho'}(t,x,D) \\
+ \;& \cdots \\
+ \;& ia_2(t,x)D_x^2 - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_2) + \mathbf{op}(d_2) + A^{[2]}_{K,\rho'}(t,x,D) \\
+ \;& ia_1(t,x)D_x - \mathbf{op}(\partial_\xi a_p(t)\xi^p\partial_x\lambda_1) + \mathbf{op}(d_1) + A^{[1]}_{K,\rho'}(t,x,D) \\
+ \;& K\langle D_x\rangle_h^{(p-1)(1-\sigma)} + \tilde{r}_0(t,x,D),
\end{aligned} \tag{3.75}$$

where $A^{[p-j]}_{K,\rho'}$ is a remainder term depending on $M_{p-1},...,M_{p-j}$ satisfying (3.73) for $j=1$ and (3.74) for $j=2,...,p-1$, and $\tilde{r}_0$ is a projective symbol of order zero.

3.8.2 Lower bound estimates for the real parts

At this moment, we do not intend to repeat all the computations to obtain the desired estimates. For this reason, we recommend that the reader checks Section 3.6 for more details. The operator given in (3.75) can be rewritten as

$$\begin{aligned}
& P_{\Lambda,K,\rho'}(t,x,D) := Q_{\Lambda,K,\rho'}(t,x,D)\circ(iP)\circ(Q_{\Lambda,K,\rho'}(t,x,D))^{-1} \\
= \;& \partial_t + ia_p(t)D_x^p \\
+ \;& ia_{p-1}(t,x)D_x^{p-1} + \mathbf{op}\left(pM_{p-1}a_p(t)|\xi|^{p-1}\langle x\rangle^{-\sigma}\right) + A^{[p-1]}_{K,\rho'}(t,x,D) \\
+ \;& ia_{p-2}(t,x)D_x^{p-2} + \mathbf{op}\left(pM_{p-2}a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{p-2}{p-1}\sigma}\right) + \mathbf{op}(d_{p-2}) + A^{[p-2]}_{K,\rho'}(t,x,D) \\
+ \;& \cdots \\
+ \;& ia_2(t,x)D_x^2 + \mathbf{op}\left(pM_2a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{-p+3}\langle x\rangle^{-\frac{2}{p-1}\sigma}\right) + \mathbf{op}(d_2) + A^{[2]}_{K,\rho'}(t,x,D) \\
+ \;& ia_1(t,x)D_x + \mathbf{op}\left(pM_1a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{-p+2}\langle x\rangle^{-\frac{1}{p-1}\sigma}\right) + \mathbf{op}(d_1) + A^{[1]}_{K,\rho'}(t,x,D) \\
+ \;& K\langle D_x\rangle_h^{(p-1)(1-\sigma)} - B^{[(p-1)(1-\sigma)]}(t,x,D) + \tilde{r}_0(t,x,D),
\end{aligned} \tag{3.76}$$

with d_{p-m}, $m=2,...,p-1$, $A^{[p-j]}_{K,\rho'}$, $j=1,...,p-1$, as described previously, $\tilde r_0 \in C\Big([0,T], \tilde{\mathbf{\Gamma}}^0_\theta(\mathbb{R}^2)\Big)$ and

$$B^{[(p-1)(1-\sigma)]}(t,x,\xi) := \sum_{j=1}^{p-1} pa_p(t)|\xi|^{p-1} M_{p-j} \langle \xi \rangle_h^{1-j} \langle x \rangle^{-\frac{p-j}{p-1}\sigma} \left[1 - \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \right]$$

has order less than or equal to $(p-1)(1-\sigma)$ in ξ, order zero in x and it depends on $M_{p-1},...,M_1$. Next, we shall fix $A=1$ in the estimates and omit the dependence on A in the constants. And now we are ready to obtain the desired estimates from below. We summarize the estimates for each level in the following:

- **Estimates for level** $(p-1)$**.** Set

$$\tilde a_{p-1}(t,x,D_x) := ia_{p-1}(t,x)D_x^{p-1} + \mathbf{op}\left(pM_{p-1}a_p(t)|\xi|^{p-1}\langle x\rangle^{-\sigma}\right) + A^{[p-1]}_{K,\rho'}(t,x,D).$$

  Then

$$\begin{aligned}
&\mathbf{Re}\ \tilde a_{p-1}(t,x,\xi) \geq \tfrac{1}{2}^{-\sigma}|\xi|^{p-1}, \quad |\xi| \geq 2h,\\
&\tilde a_{p-1}(t,x,D) = Q_+^{[p-1]}(t,x,D) + \tilde r^{[p-2]}(t,x,D),\\
&\mathbf{Re}\left(Q_+^{[p-1]}(t,x,D)u,u\right)_{L^2} \geq 0, \quad u \in L^2(\mathbb{R}), \text{ and}\\
&\tilde r^{[p-2]}(t,x,\xi) \in \mathbf{SG}^{p-2,-1-\sigma}_{\mu'}(\mathbb{R}^2),\ t\in[0,T] \text{ dep. on } M_{p-1}, h.
\end{aligned}$$

- **Estimates for a generic level** $(p-j)$**,** $j \geq 2$**.** We set

$$\begin{aligned}
\tilde a_{p-j}(t,x,D) &:= ia_{p-j}(t,x)D_x^{p-j} + \mathbf{op}\left(pM_{p-j}a_p(t)|\xi|^{p-1}\langle\xi\rangle_h^{1-j}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}\right)\\
&+ A^{[p-j]}_{K,\rho'}(t,x,D) + d_{p-j}(t,x,D) + \tilde r^{[p-j]}(t,x,D).
\end{aligned}$$

  Then

$$\begin{aligned}
&\mathbf{Re}\ \tilde a_{p-j}(t,x,\xi) \geq \tfrac{1}{2}\langle x\rangle^{-\frac{p-j}{p-1}\sigma}|\xi|^{p-j}, \quad |\xi| \geq R,\\
&\tilde a_{p-j}(t,x,D) = Q_+^{[p-j]}(t,x,D) + \tilde r^{[p-j-1]}(t,x,D),\\
&\mathbf{Re}\left(Q_+^{[p-j]}(t,x,D)u,u\right)_{L^2} \geq 0, \quad u \in L^2(\mathbb{R}), \text{ and}\\
&\tilde r^{[p-j-1]}(t,x,\xi) \in \mathbf{SG}^{p-j-1,-1-\frac{p-j}{p-1}\sigma}_{\mu'}(\mathbb{R}^2) \text{ dep. on } M_{p-1},...,M_{p-(j-1)}, h.
\end{aligned}$$

At this point, after the choice of all the parameters $M_{p-1},...,M_1$ and h, our conjugation can be expressed as

$$\begin{aligned}
P_{\Lambda,K,\rho'} &= \partial_t + ia_p(t)D_x^p\\
&+ Q_+^{[p-1]}(t,x,D) + Q_+^{[p-2]}(t,x,D) + \cdots + Q_+^{[2]}(t,x,D) + Q_+^{[1]}(t,x,D)\\
&+ K\langle D_x\rangle_h^{(p-1)(1-\sigma)} - B^{[(p-1)(1-\sigma)]}(t,x,D) + \tilde r_0(t,x,D).
\end{aligned}$$

Now, recalling the definition of $B^{[(p-1)(1-\sigma)]}$ in (3.46) we can choose $K \geq K_0(M_{p-1},...,M_1)$ large enough such that

$$\mathbf{Re}\left(K\langle\xi\rangle_h^{(p-1)(1-\sigma)} - B^{[(p-1)(1-\sigma)]}(t,x,\xi)\right) \geq 0,$$

and by applying Theorem 2.10 once more we obtain

$$\begin{aligned} P_{\Lambda,K,\rho'} &= \partial_t + ia_p(t)D_x^p + \sum_{j=1}^{p-1} Q_+^{[p-j]}(t,x,D) \\ &+ Q_+^{[(p-1)(1-\sigma)]}(t,x,D) + \tilde{r}_0(t,x,D). \end{aligned} \tag{3.77}$$

Summing up, for ρ' small enough, choosing, in order, $M_{p-1}, ..., M_1, K$ and h large enough, precisely

$$C_{T,K,\rho',M_{p-1},...,M_1} h^{-1+\frac{1}{\theta}} \leq \frac{1}{2} \quad \text{and} \quad h > h_0$$

enlarging the parameter h_0 given in Proposition 3.2 if necessary, we obtain that formula (3.77) holds. Now we are ready to prove Theorem 3.3.

3.8.3 Proof of Theorem 3.3

Given $\theta > 1$ satisfying the hypothesis of Theorem 3.1, $m \in \mathbb{R}$, $\rho > 0$, take the initial data

$$f \in C\left([0,T]; H^m_{\rho;\theta}(\mathbb{R})\right) \quad \text{and} \quad g \in H^m_{\rho;\theta}(\mathbb{R}).$$

Set the positive constants $M_{p-1}, ..., M_1, K, h_0$ for which Proposition 3.3 holds and $\rho' \in (0, \rho)$. We know that both symbols Λ and $K(T-t)\langle\cdot\rangle_h^{(p-1)(1-\sigma)}$ have order $(p-1)(1-\sigma) < 1/\theta$, hence by Proposition 3.1 it follows that

$$\begin{aligned} f_{\Lambda,K,\rho'} &:= Q_{\Lambda,K,\rho'}(t,x,D)f \quad \in \quad C\left([0,T]; H^m(\mathbb{R})\right) \\ g_{\Lambda,K,\rho'} &:= Q_{\Lambda,K,\rho'}(t,x,D)g \quad \in \quad H^m(\mathbb{R}), \end{aligned}$$

for $\rho' < \rho$. By Proposition 3.3, we have well-posedness in Sobolev spaces $H^m(\mathbb{R})$ for the Cauchy problem associated with the operator $P_{\Lambda,K,\rho'} = Q_{\Lambda,K,\rho'} \circ (iP) \circ Q^{-1}_{\Lambda,K,\rho'}$ given by (3.57), that is, there exists a unique solution $v \in C\left([0,T]; H^m(\mathbb{R})\right)$ to the Cauchy problem

$$\begin{cases} P_{\Lambda,K,\rho'} v(t,x) = f_{\Lambda,K,\rho'}(t,x) \\ v(0,x) = g_{\Lambda,K,\rho'}(x) \end{cases}, \quad (t,x) \in [0,T] \times \mathbb{R},$$

satisfying the energy estimate

$$\|v(t)\|^2_{H^m} \leq C\left(\|g_{\Lambda,K,\rho'}\|^2_{H^m} + \int_0^t \|f_{\Lambda,K,\rho'}(\tau)\|^2_{H^m} d\tau\right), \quad t \in [0,T]. \tag{3.78}$$

By setting $u := \left(Q_{\Lambda,K,\rho'}(t,x,D)\right)^{-1} v$, we obtain a solution for the original Cauchy problem (3.2), namely

$$\begin{cases} Pu(t,x) = f(t,x) \\ u(0,x) = g(x) \end{cases}, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

As before,

$$u(t,x) = {}^R\left(e^{-\Lambda}(x,D)\right)\sum_j(-r(x,D))^j e^{-K(T-t)\langle D\rangle_h^{(p-1)(1-\sigma)}} e^{-\rho'\langle D\rangle_h^{\frac{1}{\theta}}} v(t,x), \quad v\in H^m(\mathbb{R}),$$

but $v\in H^m(\mathbb{R})$ implies that $e^{-\rho'\langle D\rangle_h^{\frac{1}{\theta}}} v =: u_1 \in H^m_{\rho';\theta}(\mathbb{R})$, hence

$$u(t,x) = {}^R\left(e^{-\Lambda}(x,D)\right)\sum_j(-r(x,D))^j e^{-K(T-t)\langle D\rangle_h^{(p-1)(1-\sigma)}} u_1, \quad u_1\in H^m_{\rho';\theta}(\mathbb{R}).$$

Notice that, for every $\delta_1>0$, we have

$$e^{-K(T-t)\langle D\rangle_h^{(p-1)(1-\sigma)}} u_1 = \underbrace{e^{-K(T-t)\langle D\rangle_h^{(p-1)(1-\sigma)}} e^{-\delta_1\langle D\rangle_h^{\frac{1}{\theta}}}}_{\text{order zero}} e^{\delta_1\langle D\rangle_h^{\frac{1}{\theta}}} u_1 =: u_2 \in H^m_{\rho'-\delta_1;\theta}(\mathbb{R}),$$

and since $\sum_j(-r(x,D))^j$ has order zero, $u_3 := \sum_j(-r(x,D))^j u_2 \in H^m_{\rho'-\delta_1;\theta}(\mathbb{R})$, which allows us to write

$$u(t,x) = {}^R\left(e^{-\Lambda}(x,D)\right)\sum_j(-r(x,D))^j u_2 = {}^R\left(e^{-\Lambda}(x,D)\right) u_3, \quad u_3\in H^m_{\rho'-\delta_1;\theta}(\mathbb{R}).$$

By Proposition 3.1, we have that ${}^R\left(e^{-\Lambda}(x,D)\right)$ maps $H^m_{\rho;\theta}(\mathbb{R})$ into $H^m_{\rho-\delta_2;\theta}(\mathbb{R})$, for every $\delta_2>0$, and we can assert that $u(t,\cdot)\in H^m_{\rho'-\delta;\theta}(\mathbb{R})$ for all $\delta>0$, $t\in[0,T]$. If we set $\tilde\rho := \rho'-\delta$, it follows from (3.78) that

$$\begin{aligned}
\|u(t)\|^2_{H^m_{\tilde\rho;\theta}} &= \|\left(Q_{\Lambda,K,\rho'}(t,\cdot,D)\right)^{-1} v(t)\|^2_{H^m_{\tilde\rho;\theta}} \le C_1\|v(t)\|^2_{H^m} \\
&\le C_2\left(\|g_{\Lambda,K,\rho'}\|^2_{H^m} + \int_0^t \|f_{\Lambda,K,\rho'}(\tau)\|^2_{H^m} d\tau\right) \\
&\le C_3\left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau)\|^2_{H^m_{\rho;\theta}} d\tau\right), \quad t\in[0,T].
\end{aligned} \tag{3.79}$$

To conclude, we notice that if the data f and g are valued in $H^m_{\rho;\theta}(\mathbb{R})$ for all $\rho>0$, then the solution belongs to $H^m_{\rho'';\theta}(\mathbb{R})$, for every $\rho''\in(0,\rho)$, which means that

$$u\in C\left([0,T];H^\infty_\theta(\mathbb{R})\right).$$

The uniqueness follows as in the proof of Theorem 3.1.

□

Chapter 4

CAUCHY PROBLEM FOR p-EVOLUTION OPERATORS WITH DATA IN GELFAND-SHILOV SPACES

4.1 Gelfand-Shilov well-posedness and main result

In this chapter, we shall consider the same p-evolution operator given by (3.1) as in Chapter 3, for $p \geq 2$, that is,

$$P(t, x, D_t, D_x) = D_t + a_p(t)D_x^p + \sum_{j=1}^{p} a_{p-j}(t, x)D_x^{p-j}, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Our goal is to study the same Cauchy problem

$$\begin{cases} P(t, x, D_t, D_x)u(t, x) = f(t, x), & (t, x) \in [0, T] \times \mathbb{R} \\ u(0, x) = g(x), & x \in \mathbb{R} \end{cases}, \tag{4.1}$$

except that now the data f and g will be considered in Gelfand-Shilov-Sobolev spaces, that is, $f \in C\left([0, T]; H^0_{(\rho_1,\rho_2);s,\theta}(\mathbb{R})\right)$ and $g \in H^0_{(\rho_1,\rho_2);s,\theta}(\mathbb{R})$, where

$$H^0_{(\rho_1,\rho_2);s,\theta}(\mathbb{R}) = \{u \in \mathscr{S}'(\mathbb{R}) \ : \ e^{\rho_2\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}} u \in L^2(\mathbb{R})\}.$$

In Gelfand-Shilov setting, we also have the definition of well-posedness.

Definition 4.1. We say that the Cauchy problem (4.1) is *well-posed* in $\mathcal{S}^\theta_s(\mathbb{R})$ when, for any given $m \in \mathbb{R}^2$, $\rho = (\rho_1, \rho_2)$, $\rho_1\rho_2 > 0$, there exists $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)$, $\tilde{\rho}_1, \tilde{\rho}_2 > 0$ and a constant $C := C(\rho, T) > 0$ such that, for all $f \in C\left([0, T]; H^m_{\rho;s,\theta}(\mathbb{R})\right)$ and $g \in H^m_{\rho;s,\theta}(\mathbb{R})$, there exists a unique solution $u \in C^1\left([0, T]; H^m_{\tilde{\rho};s,\theta}(\mathbb{R})\right)$ and the following energy estimate holds

$$\|u(t, \cdot)\|_{H^m_{\tilde{\rho};s,\theta}} \leq C\left(\|g\|^2_{H^m_{\rho;s,\theta}} + \int_0^t \|f(\tau, \cdot)\|^2_{H^m_{\rho;s,\theta}} d\tau\right).$$

It is easy to verify, eventually conjugating our operator by $\langle x\rangle^{m_2}\langle D\rangle^{m_1}$, that we can replace $m \in \mathbb{R}^2$ by $(0, 0)$ in Definition 4.1. The first main result that will be proved in this chapter, is stated in the following.

Theorem 4.1. *Let $\theta_0 > 1$ and $\sigma \in \left(\frac{p-2}{p-1}, 1\right)$ such that $\theta_0 < \frac{1}{(p-1)(1-\sigma)}$. Let P be an operator of the type (3.1) whose coefficients satisfy the following assumptions:*

(i) $a_p \in C([0,T];\mathbb{R})$ *and there exists* $C_{a_p} > 0$ *such that* $|a_p(t)| \geq C_{a_p}$, *for all* $t \in [0,T]$.

(ii) $|\partial_x^\beta a_{p-j}(t,x)| \leq C_{a_{p-j}}^{\beta+1} \beta!^{\theta_0} \langle x \rangle^{-\frac{p-j}{p-1}\sigma - \beta}$, *for some* $C_{a_{p-j}} > 0$, $j = 1, ..., p-1$ *and for all* $\beta \in \mathbb{N}_0$, $(t,x) \in [0,T] \times \mathbb{R}$.

Let $s, \theta > 1$ *such that* $(p-1)\theta < \min\left\{\frac{1}{1-\sigma}, s\right\}$ *and* $\theta \geq \theta_0$, *and let* $f \in C\left([0,T]; H^0_{\rho;s,\theta}(\mathbb{R})\right)$ *and* $g \in H^0_{\rho;s,\theta}(\mathbb{R})$, *for some* $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ *with* $\rho_1, \rho_2 > 0$. *Then there exists a unique solution* $u \in C^1\left([0,T]; H^0_{(\tilde{\rho}_1,\delta);s,\theta}(\mathbb{R})\right)$ *of* (4.1) *for some* $\tilde{\rho}_1 \in (0, \rho_1)$, $\delta \in (0, \rho_2)$ *and it satisfies the energy estimate*

$$\|u(t)\|^2_{H^0_{(\tilde{\rho}_1,\delta);s,\theta}} \leq C\left(\|g\|^2_{H^0_{\rho;s,\theta}} + \int_0^t \|f(\tau)\|^2_{H^0_{\rho;s,\theta}} d\tau\right), \tag{4.2}$$

for all $t \in [0,T]$ *and for some constant* $C > 0$. *In particular, the Cauchy problem* (4.1) *is well-posed in* $\mathcal{S}^\theta_s(\mathbb{R})$.

By similar reasons as we have discussed in Chapter 3, the proof of Theorem 4.1 cannot be achieved in a straight way. Then we need to work in a strategy to conjugate the operator P, which means that a change of variable must be done in the original Cauchy problem. Essentially, it is sufficient to add a new term to the change of variable used before.

As a matter of fact, the Cauchy problem (4.1) has the data f ang g in Gelfand-Shilov-Sobolev spaces, namely

$$f(t),\ g \in H^0_{(\rho_1,\rho_2);s,\theta}(\mathbb{R}),\ t \in [0,T].$$

The idea is to multiply f and g and to conjugate P by the term $e^{\delta\langle x\rangle^{1/s}}$, with $\delta \in (0,\rho_2)$ to be chosen later. Notice that this change of variable pulls back the data f and g to some Gevrey-Sobolev space (of course, the coefficients of the conjugated operator $P_\delta := e^{\delta\langle x\rangle^{1/s}} P e^{-\delta\langle x\rangle^{1/s}}$ should preserve the properties of the coefficients a_{p-j} of P) and this will give us an auxiliary Cauchy problem whose data will be in some Gevrey-Sobolev classes, that is,

$$e^{\delta\langle x\rangle^{1/s}} f(t),\ e^{\delta\langle x\rangle^{1/s}} g \in H^0_{\rho_1;\theta}(\mathbb{R}).$$

Indeed, the auxiliary Cauchy problem becomes

$$\begin{cases} P_\delta(t,x,D_t,D_x)v(t,x) = e^{\delta\langle x\rangle^{1/s}} f(t,x) \\ v(0,x) = e^{\delta\langle x\rangle^{1/s}} g(x) \end{cases}, \quad (t,x) \in [0,T] \times \mathbb{R}. \tag{4.3}$$

Once we conclude that this conjecture works, the next step is simply apply Theorem 3.1 to the Cauchy problem (4.3), provided that $\theta \in \left[\theta_0, \min\left\{\frac{1}{(p-1)(1-\sigma)}, s\right\}\right)$, and then the proof of Theorem 4.1 is a consequence of it.

4.2 Conjugation of P by $e^{\delta\langle x\rangle^{1/s}}$

After all the previous, let us perform the conjugation of the operator $P(t,x,D_t,D_x)$ by $e^{\delta\langle x\rangle^{1/s}}$ and its inverse, where $\delta\in(0,\rho_2)$. As before, the conjugation will be made term-by-term, and the next steps are dedicated to this purpose.

- **The conjugation of D_t.** Since $e^{\delta\langle x\rangle^{1/s}}$ does not depend on t, the conjugation is trivially given by
$$e^{\delta\langle x\rangle^{1/s}}\circ D_t\circ e^{-\delta\langle x\rangle^{1/s}}=D_t.$$

- **The conjugation of $a_p(t)D_x^p$.** First of all, the computation is made just with D_x^p (and then we can put the factor $a_p(t)$ later). By using Leibniz formula, it can be computed
$$\begin{aligned}e^{\delta\langle x\rangle^{1/s}}D_x^pe^{-\delta\langle x\rangle^{1/s}} &= D_x^p+\mathbf{op}\left(\sum_{k=1}^p(k!)^{-1}\partial_\xi^k\xi^pe^{\delta\langle x\rangle^{1/s}}D_x^ke^{-\delta\langle x\rangle^{1/s}}\right)\\ &= D_x^p+\mathbf{op}\left(\sum_{k=1}^p\frac{1}{k!}\cdot\frac{p!}{(p-k)!}\xi^{p-k}e^{\delta\langle x\rangle^{1/s}}D_x^ke^{-\delta\langle x\rangle^{1/s}}\right)\\ &= D_x^p+\sum_{k=1}^pb_{p-k}^{(\delta)}(x)D_x^{p-k},\end{aligned}$$
where
$$b_{p-k}^{(\delta)}(x):=\binom{p}{k}e^{\delta\langle x\rangle^{1/s}}D_x^ke^{-\delta\langle x\rangle^{1/s}}.$$
Now, let us check what happens with the terms $b_{p-k}^{(\delta)}$. By Faà di Bruno formula, it follows that
$$e^{\delta\langle x\rangle^{1/s}}D_x^ke^{-\delta\langle x\rangle^{1/s}}=\sum_{\ell=1}^k\frac{1}{\ell!}\sum_{\substack{k_1+\cdots+k_\ell=k\\k_\nu\geq1}}\frac{k!}{k_1!\cdots k_\ell!}\prod_{\nu=1}^\ell D_x^{k_\nu}\left(-\delta\langle x\rangle^{1/s}\right).$$
Notice that, since $|\partial_x^\beta\langle x\rangle^m|\leq C^\beta\beta!\langle x\rangle^{m-\beta}$, we obtain the estimate
$$\begin{aligned}\left|\prod_{\nu=1}^\ell D_x^{k_\nu}\left(-\delta\langle x\rangle^{1/s}\right)\right| &= \delta^\ell\prod_{\nu=1}^\ell\left|D_x^{k_\nu}\langle x\rangle^{1/s}\right|\leq\delta^\ell\prod_{\nu=1}^\ell C^{k_\nu}k_\nu!\langle x\rangle^{\frac{1}{s}-k_\nu}\\ &= \delta^\ell C^kk_1!\cdots k_\ell!\langle x\rangle^{\frac{\ell}{s}-k}\\ &\leq C_\delta^{k+1}k_1!\cdots k_\ell!\langle x\rangle^{k\left(\frac{1}{s}-1\right)}.\end{aligned}$$
Hence
$$\begin{aligned}|b_{p-k}^{(\delta)}(x)| &\leq \binom{p}{k}\sum_{\ell=1}^k\frac{1}{\ell!}\sum_{\substack{k_1+\cdots+k_\ell=k\\k_\nu\geq1}}\frac{k!}{k_1!\cdots k_\ell!}C_\delta^{k+1}k_1!\cdots k_\ell!\langle x\rangle^{k\left(\frac{1}{s}-1\right)}\\ &= \frac{p!}{k!(p-k)!}\sum_{\ell=1}^k\frac{1}{\ell!}\sum_{\substack{k_1+\cdots+k_\ell=k\\k_\nu\geq1}}k!C_\delta^{k+1}\langle x\rangle^{k\left(\frac{1}{s}-1\right)},\end{aligned}$$

which gives us that $b_{p-k}^{(\delta)} \sim \langle x \rangle^{k\left(\frac{1}{s}-1\right)}$. Similarly we obtain that

$$|D_x^\beta b_{p-k}^{(\delta)}(x)| \leq \tilde{C}^{\beta+k+1}\beta!\langle x \rangle^{k\left(\frac{1}{s}-1\right)-\beta}, \tag{4.4}$$

for all $\beta \in \mathbb{N}_0$. Therefore, the conjugation of $a_p(t)D_x^p$ is given by

$$e^{\delta\langle x \rangle^{1/s}}(a_p(t)D_x^p)e^{-\delta\langle x \rangle^{1/s}} = a_p(t)D_x^p + \sum_{k=1}^{p} \tilde{a}_{p-k}^{(\delta)}(t,x)D_x^{p-k},$$

where $\tilde{a}_{p-k}^{(\delta)}(t,x) = a_p(t)b_{p-k}^{(\delta)}(x)$ satisfy, for every $k = 1, ..., p$, the following estimate

$$\sup_{t\in[0,T]} |D_x^\beta \tilde{a}_{p-k}^{(\delta)}(x)| \leq C_1^{\beta+1}\beta!\langle x \rangle^{k\left(\frac{1}{s}-1\right)-\beta}. \tag{4.5}$$

- **The conjugation of $a_{p-j}(t,x)D_x^{p-j}$, $j = 1, ..., p-1$.** At this moment, we need to be concerned what happens with the lower order terms. For each $j = 1, ..., p-1$, by using again the Leibniz formula

$$\begin{aligned}
e^{\delta\langle x \rangle^{1/s}}(a_{p-j}(t,x)D_x^{p-j})e^{-\delta\langle x \rangle^{1/s}} &= a_{p-j}(t,x)D_x^{p-j} \\
&+ \mathbf{op}\left(\sum_{k=1}^{p-j} \frac{1}{k!}a_{p-j}(t,x)\partial_\xi^k \xi^{p-j} e^{\delta\langle x \rangle^{1/s}} D_x^k e^{-\delta\langle x \rangle^{1/s}}\right) \\
&= a_{p-j}(t,x)D_x^{p-j} \\
&+ \mathbf{op}\left(\sum_{k=1}^{p-j} \binom{p-j}{k} a_{p-j}(t,x)\xi^{p-j-k} e^{\delta\langle x \rangle^{1/s}} D_x^k e^{-\delta\langle x \rangle^{1/s}}\right) \\
&= a_{p-j}(t,x)D_x^{p-j} + \sum_{k=1}^{p-j} a_{p-j}(t,x)c_{p-j-k}^{(\delta)}(x)D_x^{p-j-k}
\end{aligned}$$

where

$$c_{p-j-k}^{(\delta)}(x) = \binom{p-j}{k} e^{\delta\langle x \rangle^{1/s}} D_x^k e^{-\delta\langle x \rangle^{1/s}}.$$

By the same argument used before, it follows that $c_{p-j-k}^{(\delta)}$ satisfy (4.4), for every $j = 1, ..., p-1$ and $k = 1, ..., p-j$.

- **The conjugation of $a_0(t,x)$.** It is simply given by

$$e^{\delta\langle x \rangle^{1/s}} a_0(t,x) e^{-\delta\langle x \rangle^{1/s}} = a_0(t,x).$$

By the previous analysis, we can assert that the conjugated form of the operator $P(t,x,D_t,D_x)$ by $e^{\delta\langle x \rangle^{1/s}}$ and its inverse can be written as

$$\begin{aligned}
e^{\delta\langle x \rangle^{1/s}} \circ P \circ e^{-\delta\langle x \rangle^{1/s}} &= D_t + a_p(t)D_x^p + \sum_{j=1}^{p-1} a_{p-j}(t,x)D_x^{p-j} + a_0(t,x) \\
&+ \sum_{k=1}^{p-1} \tilde{a}_{p-k}^{(\delta)}(t,x)D_x^{p-k} + \tilde{a}_0^{(\delta)}(t,x) + \sum_{j=1}^{p-1}\sum_{\ell=1}^{p-j} a_{p-j}(t,x)c_{p-j-\ell}^{(\delta)}(x)D_x^{p-j-\ell}.
\end{aligned}$$

Notice that the double sum in the last term can be expressed as (omitting the variables t and x for simplicity)

$$\sum_{j=1}^{p-1} a_{p-j} \sum_{\ell=1}^{p-j} c_{p-j-\ell}^{(\delta)} D_x^{p-j-\ell} = \sum_{k=2}^{p} d_{p-k}^{(\delta)} D_x^{p-k},$$

where

$$d_{p-k}^{(\delta)} := c_{p-k}^{(\delta)} \sum_{\ell=k-1}^{p} a_{p-\ell}, \quad k = 2, ..., p.$$

In particular, we have that $d_{p-k}^{(\delta)}$ satisfy (4.5). Hence

$$\begin{aligned}
e^{\delta\langle x\rangle^{1/s}} \circ P \circ e^{-\delta\langle x\rangle^{1/s}} &= D_t + a_p(t)D_x^p + \sum_{j=1}^{p-1} a_{p-j}(t,x)D_x^{p-j} + a_0(t,x) \\
&+ \sum_{k=1}^{p-1} \tilde{a}_{p-k}^{(\delta)}(t,x)D_x^{p-k} + \tilde{a}_0^{(\delta)}(t,x) \\
&+ \sum_{k=2}^{p} d_{p-k}^{(\delta)}(t,x)D_x^{p-k} \\
&= D_t + a_p(t)D_x^p + a_{p-1}(t,x)D_x^{p-1} + \sum_{j=2}^{p-1} a_{p-j}(t,x)D_x^{p-j} + a_0(t,x) \\
&+ \tilde{a}_{p-1}^{(\delta)}(t,x)D_x^{p-1} + \sum_{k=2}^{p-1} \tilde{a}_{p-k}^{(\delta)}(t,x)D_x^{p-k} + \tilde{a}_0^{(\delta)}(t,x) \\
&+ \sum_{k=2}^{p-1} d_{p-k}^{(\delta)}(t,x)D_x^{p-k} + d_0^{(\delta)}(t,x),
\end{aligned}$$

and finally

$$\begin{aligned}
e^{\delta\langle x\rangle^{1/s}} \circ P \circ e^{-\delta\langle x\rangle^{1/s}} &= D_t + a_p(t)D_x^p + \left\{a_{p-1}(t,x) + \tilde{a}_{p-1}^{(\delta)}(t,x)\right\} D_x^{p-1} \\
&+ \sum_{j=2}^{p} \left\{a_{p-j}(t,x) + \tilde{a}_{p-j}^{(\delta)}(t,x) + d_{p-j}^{(\delta)}(t,x)\right\} D_x^{p-j} \\
&= D_t + ia_p(t)D_x^p + \sum_{j=1}^{p} a_{p-j}^{(\delta)}(t,x)D_x^{p-j} \qquad (4.6)
\end{aligned}$$

where

$$\begin{aligned}
a_{p-1}^{(\delta)}(t,x) &:= a_{p-1}(t,x) + \tilde{a}_{p-1}^{(\delta)}(t,x), \\
a_{p-j}^{(\delta)}(t,x) &:= a_{p-j}(t,x) + \tilde{a}_{p-j}^{(\delta)}(t,x) + d_{p-j}^{(\delta)}(t,x), \quad 2 \leq j \leq p,
\end{aligned}$$

From the previous estimates we obtain that $a_{p-j}^{(\delta)}$ satisfy the following estimate:

$$\sup_{t\in[0,T]} \left(|D_x^\beta \tilde{a}_{p-j}^{(\delta)}(t,x)| + |D_x^\beta d_{p-j}^{(\delta)}(t,x)|\right) \leq C^{\beta+1}\beta!^{\theta_0}\langle x\rangle^{-j\left(1-\frac{1}{s}\right)-\beta}. \qquad (4.7)$$

4.3 Proof of Theorem 4.1

To prove Theorem 4.1 we want to apply Theorem 3.1 to the operator P_δ. So we need to show that the operator P_δ satisfies the assumptions of Theorem 3.1. With this purpose we distinguish two cases.

The case $s \geq \frac{1}{1-\sigma}$. In this case, the assumption of Theorem 4.1 reads as

$$\theta < \frac{1}{(p-1)(1-\sigma)} \leq \frac{s}{p-1}.$$

In this case we observe that the coefficients of P_δ satisfy the assumptions of Theorem 3.1, that is,

$$j\left(1-\frac{1}{s}\right) \geq \frac{p-j}{p-1}\sigma, \quad \forall j = 1, ..., p.$$

In fact, we notice that as j increases, the left-hand side of the above inequality increases and the right-hand side decreases, so this inequality holds for all $j = 1, ..., p$ if, and only if, it holds for $j = 1$, that is,

$$\left(1-\frac{1}{s}\right) \geq \frac{p-1}{p-1}\sigma \quad \Leftrightarrow \quad 1-\frac{1}{s} \geq \sigma \quad \Leftrightarrow \quad s \geq \frac{1}{1-\sigma},$$

which is true. Hence we have

$$\sup_{t\in[0,T]} |D_x^\beta a_{p-j}^{(\delta)}(t,x)| \leq C^{\beta+1}\beta!^{\theta_0}\langle x\rangle^{-\frac{p-j}{p-1}\sigma-\beta}. \tag{4.8}$$

Let us then consider the Cauchy problem (4.1) with data

$$f \in C\left([0,T]; H^0_{\rho;s,\theta}(\mathbb{R})\right) \quad \text{and} \quad g \in H^0_{\rho;s,\theta}(\mathbb{R}),$$

for $\rho = (\rho_1, \rho_2)$, with $\rho_1, \rho_2 > 0$. If $\delta \in (0, \rho_2)$, then

$$f_\delta := e^{\delta\langle x\rangle^{1/s}} f \in C\left([0,T]; H^0_{(\rho_1,\rho_2-\delta);s,\theta}(\mathbb{R})\right) \quad \text{and} \quad g_\delta := e^{\delta\langle x\rangle^{1/s}} g \in H^0_{(\rho_1,\rho_2-\delta);s,\theta}(\mathbb{R}),$$

because, if $\phi \in H^0_{\rho;s,\theta}(\mathbb{R})$, then

$$\phi_{(\rho_1,\rho_2)}(x) := e^{\rho_2\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}}\phi \in L^2(\mathbb{R}), \tag{4.9}$$

and

$$\begin{aligned} e^{(\rho_2-\delta)\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}} \underbrace{e^{\delta\langle x\rangle^{1/s}}\phi}_{:=\phi_\delta} &= e^{(\rho_2-\delta)\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}} e^{\delta\langle x\rangle^{1/s}} \left(e^{\rho_2\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}}\right)^{-1} \underbrace{e^{\rho_2\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}}\phi}_{=\phi_{(\rho_1,\rho_2)}} \\ &= \underbrace{e^{(\rho_2-\delta)\langle x\rangle^{1/s}} e^{\rho_1\langle D\rangle^{1/\theta}} e^{\delta\langle x\rangle^{1/s}} e^{-\rho_1\langle D\rangle^{1/\theta}} e^{-\rho_2\langle x\rangle^{1/s}}}_{:=A_{(\rho_1,\rho_2,\delta)}} \phi_{(\rho_1,\rho_2)} \in L^2(\mathbb{R}), \end{aligned}$$

since $A_{(\rho_1,\rho_2,\delta)}$ has order zero and $\phi_{(\rho_1,\rho_2)} \in L^2(\mathbb{R})$, and this means that $\phi_\delta \in H^0_{(\rho_1,\rho_2-\delta);s,\theta}(\mathbb{R})$.

Since $\rho_2 - \delta > 0$, it follows that

$$f_\delta \in C\left([0,T]; H^0_{\rho_1;\theta}(\mathbb{R})\right) \quad \text{and} \quad g_\delta \in H^0_{\rho_1;\theta}(\mathbb{R}).$$

Once we are considering $\theta \in \left[\theta_0, \min\left\{\frac{1}{(p-1)(1-\sigma)}, s\right\}\right)$, we obtain that the auxiliary Cauchy problem given by (4.3) is well-posed in $\mathcal{H}^\infty_\theta(\mathbb{R})$, that is, there exists a unique solution $v \in C^1\left([0,T]; H^0_{\tilde{\rho}_1;\theta}(\mathbb{R})\right)$ of (4.3), with initial data f_δ and g_δ, satisfying

$$\|v(t,\cdot)\|_{H^0_{\tilde{\rho}_1;\theta}} \leq C(\rho_1,T)\left(\|g_\delta\|^2_{H^0_{\rho_1;\theta}} + \int_0^t \|f_\delta(\tau,\cdot)\|^2_{H^0_{\rho_1;\theta}} d\tau\right), \tag{4.10}$$

for some $\tilde{\rho}_1 \in (0,\rho_1)$. Now we set $u(t,x) := e^{-\delta\langle x\rangle^{1/s}}v(t,x)$. Notice that $u \in C^1\left([0,T]; H_{(\tilde{\rho}_1,\delta);s,\theta}(\mathbb{R})\right)$ and it solves the Cauchy problem (4.1), because

$$u(0,x) = e^{-\delta\langle x\rangle^{1/s}}v(0,x) = e^{-\delta\langle x\rangle^{1/s}}g_\delta(x) = e^{-\delta\langle x\rangle^{1/s}}e^{\delta\langle x\rangle^{1/s}}g(x) = g(x),$$

and $P_\delta(t,x,D_t,D_x)v(t,x) = f_\delta(t,x)$ implies that

$$e^{\delta\langle x\rangle^{1/s}}P(t,x,D_t,D_x)\underbrace{e^{-\delta\langle x\rangle^{1/s}}v(t,x)}_{=u(t,x)} = e^{\delta\langle x\rangle^{1/s}}f(t,x) \quad \Rightarrow \quad P(t,x,D_t,D_x)u(t,x) = f(t,x).$$

Finally, we can use estimate (4.10) to obtain

$$\begin{aligned}
\|u(t,\cdot)\|_{H^0_{(\tilde{\rho}_1,\delta);s,\theta}} &\leq C(\delta)\|v(t,\cdot)\|_{H^0_{\tilde{\rho}_1;\theta}} \\
&\leq C(\rho_1,\delta,T)\left(\|g_\delta\|^2_{H^0_{\rho_1;\theta}} + \int_0^t \|f_\delta(\tau,\cdot)\|^2_{H^0_{\rho_1;\theta}} d\tau\right) \\
&\leq C(\rho_1,\delta,T)\left(\|g\|^2_{H^0_{(\rho_1,\delta);s,\theta}} + \int_0^t \|f(\tau,\cdot)\|^2_{H^0_{(\rho_1,\delta);s,\theta}} d\tau\right) \\
&\leq C(\rho_1,\delta,T)\left(\|g\|^2_{H^0_{\rho;s,\theta}} + \int_0^t \|f(\tau,\cdot)\|^2_{H^0_{\rho;s,\theta}} d\tau\right),
\end{aligned}$$

where $C(\rho_1,\delta,T)$ is a positive constant depending on ρ_1, δ and T.

To prove the uniqueness of the solution, let us consider two solutions

$$u_j \in C^1\left([0,T]; H^0_{(\tilde{\rho}_1,\delta);s,\theta}(\mathbb{R})\right), \quad j = 1,2,$$

with $\delta < \rho_2$, for the Cauchy problem (4.1). By taking any $\tilde{\delta} \in (0,\delta)$, notice that

$$v_j := e^{\tilde{\delta}\langle x\rangle^{1/s}}u_j, \quad j = 1,2,$$

are solutions of (4.3), for δ replaced by $\tilde{\delta}$, and also $v_j \in C^1\left([0,T]; H^0_{\tilde{\rho}_1;\theta}(\mathbb{R})\right)$. The well-posedness in $\mathcal{H}^\infty_\theta(\mathbb{R})$ of (4.3) gives us $v_1 = v_2$, hence $e^{\tilde{\delta}\langle x\rangle^{1/s}}u_1 = e^{\tilde{\delta}\langle x\rangle^{1/s}}u_2$, which implies that $u_1 = u_2$.

The case $s < \frac{1}{1-\sigma}$. Now the assumption of Theorem 4.1 becomes

$$\theta_0 \leq \theta < \frac{s}{p-1} < \frac{1}{(p-1)(1-\sigma)}.$$

In this case we observe that $\langle x\rangle^{-\frac{p-j}{p-1}\sigma} \leq \langle x\rangle^{-\frac{p-j}{p-1}\left(1-\frac{1}{s}\right)}$ and $\langle x\rangle^{-j\left(1-\frac{1}{s}\right)} \leq \langle x\rangle^{-\frac{p-j}{p-1}\left(1-\frac{1}{s}\right)}$, so we get

$$\sup_{t\in[0,T]} |D_x^\beta a^{(\delta)}_{p-j}(t,x)| \leq C^{\beta+1}\beta!^{\theta_0}\langle x\rangle^{-\frac{p-j}{p-1}(1-\frac{1}{s})-\beta}.$$

Then we can repeat the same argument used in the first case with σ replaced by $1-\frac{1}{s}$. Notice that $1-\frac{1}{s}\in\left(\frac{p-2}{p-1},1\right)$ since

$$1-\frac{1}{s}>\frac{p-2}{p-1}\quad\Leftrightarrow\quad\frac{1}{s}<\frac{1}{p-1}\quad\Leftrightarrow\quad s>p-1$$

which holds true since $\frac{s}{p-1}>\theta>1$. Then in this case the Cauchy problem is well-posed in $\mathcal{S}_s^\theta(\mathbb{R})$ for $\theta_0\leq\theta<\frac{1}{(p-1)\left(1-1+\frac{1}{s}\right)}=\frac{s}{p-1}=\min\left\{\frac{1}{(p-1)(1-\sigma)},\frac{s}{p-1}\right\}$. This concludes the proof.

□

Remark 4.1. *Under the assumptions of Theorem 3.3 for $P(t,x,D_t,D_x)$, arguing as in the proof of Theorem 4.1 we may prove a similar well-posedness result for the Cauchy problem with data in $\Sigma_s^\theta(\mathbb{R})$ with minor modifications. We leave the details for the reader.*

4.4 Ill-posedness results for model operators

In the previous section we have proved Theorem 4.1 under the assumption

$$(p-1)\theta<\min\left\{\frac{1}{1-\sigma},s\right\}.$$

The aim of this section is to prove that if

$$(p-1)\theta>\min\left\{\frac{1}{1-\sigma},s\right\},\tag{4.11}$$

then the Cauchy problem (4.1) is not well-posed in general for an operator P satisfying the conditions (i) and (ii) of Theorem 4.1. With this purpose we distinguish two cases.

The case $s\leq\frac{1}{1-\sigma}$. In this case, (4.11) turns into the condition $s<(p-1)\theta$. Then we have the following result.

Proposition 4.1. *Let $s,\theta>1$ such that $s<(p-1)\theta$. Then there exists $g\in\mathcal{S}_s^\theta(\mathbb{R})$ such that the Cauchy problem*

$$\begin{cases}(D_t+D_x^p)u=0\\ u(0,x)=g(x)\end{cases},\quad(t,x)\in[0,T]\times\mathbb{R},\tag{4.12}$$

is not well-posed in $\mathcal{S}_s^\theta(\mathbb{R})$.

Proof. The proof relies in the application of Theorem 1.2 in [10] which states that for every polynomial $q(\xi)$ of degree p with real coefficients and for every $s,\theta>1$ such that $1<s<(p-1)\theta$ there exists $\varphi\in\mathcal{S}_\theta^s(\mathbb{R})$ such that

$$e^{iq(\xi)}\varphi(\xi)\notin\mathcal{S}_\theta^s(\mathbb{R}).\tag{4.13}$$

Taking $q(t,\xi) := t\xi^p$ for fixed $t > 0$ and $\varphi \in \mathcal{S}_\theta^s(\mathbb{R})$ such that (4.13) holds and choosing $g = \mathscr{F}^{-1}(\varphi)$ as initial datum of (4.1), by elementary arguments we have that the solution of (4.1) is

$$u_g(t,x) = \mathscr{F}^{-1}(e^{i\xi^p t}\widehat{g}(\xi)) = \mathscr{F}^{-1}(e^{i\xi^p t}\varphi(\xi)) \notin \mathcal{S}_s^\theta(\mathbb{R}).$$

□

The operator $D_t + D_x^p$ obviously satisfies the assumptions (i), (ii) for every $\sigma \in \left(\frac{p-2}{p-1}, 1\right)$. Hence we are in the situation $s \leq \frac{1}{1-\sigma}$. Then we have proved that if $s < (p-1)\theta$, the Cauchy problem (4.1) is not well-posed in general.

The case $s > \frac{1}{1-\sigma}$. Now (4.11) turns into $\frac{1}{1-\sigma} < (p-1)\theta$. Then we can prove the following Theorem by considering the operator

$$M = D_t + D_x^p + i\langle x\rangle^{-\sigma} D_x^{p-1}. \tag{4.14}$$

Theorem 4.2. *Let M be the operator in (4.14), with $\sigma \in \left(\frac{p-2}{p-1}, 1\right)$. If the Cauchy problem*

$$\begin{cases} Mu = 0 \\ u(0,x) = g(x) \end{cases}, \quad (t,x) \in [0,T] \times \mathbb{R} \tag{4.15}$$

is well-posed in $\mathcal{S}_s^\theta(\mathbb{R})$, then

$$\max\left\{\frac{1}{(p-1)\theta}, \frac{1}{s}\right\} \geq 1 - \sigma.$$

Now, if $s > \frac{1}{1-\sigma}$ and $(p-1)\theta > \frac{1}{1-\sigma}$, this yields $1 - \sigma > \max\left\{\frac{1}{(p-1)\theta}, \frac{1}{s}\right\}$. Then Theorem 4.2 implies that the Cauchy problem (4.15) is not well-posed. Since M is of the form (3.1) and satisfies conditions (i) and (ii) of Theorem 4.1 then we conclude that if (4.11) holds, then in general the Cauchy problem (4.1) is ill-posed.

In view of the the considerations above, we devote the rest of this section to prove Proposition 4.2.

4.4.1 Proof of Theorem 4.2

Let us consider the Cauchy problem associated to the model operator M given by (4.14), that is,

$$\begin{cases} (D_t + D_x^p + i\langle x\rangle^{-\sigma} D_x^{p-1})u(t,x) = 0 \\ u(0,x) = g(x) \end{cases}, \quad (t,x) \in [0,T] \times \mathbb{R}, \tag{4.16}$$

where $\sigma > 0$. Our goal is to prove that, if the Cauchy problem (4.15) is well-posed in $\mathcal{S}_s^\theta(\mathbb{R})$, then $1 - \sigma > \max\left\{\frac{1}{(p-1)\theta}, \frac{1}{s}\right\}$ leads to a contradiction.

To keep the development of our work, we need to establish some notations, concepts and results which were not mentioned before in the background chapters.

Definition 4.2. For any $m \in \mathbb{R}$, the set $S^m_{0,0}(\mathbb{R})$ is defined as the space of all functions $p \in C^\infty(\mathbb{R}^2)$ satisfying: for all $\alpha, \beta \in \mathbb{N}_0$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^m.$$

The topology in $S^m_{0,0}(\mathbb{R})$ é induced by the family of semi-norms

$$|p|_\ell^{(m)} := \max_{\alpha \leq \ell, \beta \leq \ell} \sup_{x,\xi \in \mathbb{R}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \langle \xi \rangle^{-m}, \quad p \in S^m_{0,0}(\mathbb{R}),\ \ell \in \mathbb{N}_0.$$

Theorem 4.3 (Calderón-Vaillancourt)**.** *If $p \in S^m_{0,0}(\mathbb{R})$ then, for all $s \in \mathbb{R}$, there exist $\ell := \ell(s,m) \in \mathbb{N}_0$ and $C := C_{s,m} > 0$ such that*

$$\|p(x,D)u\|_{H^s(\mathbb{R})} \leq C |p|_\ell^{(m)} \|u\|_{H^{s+m}(\mathbb{R})}, \quad \forall u \in H^{s+m}(\mathbb{R}).$$

Besides, when $m = s = 0$, $|p|_\ell^{(m)}$ can be replaced by

$$\max_{\alpha,\beta \leq 2} \sup_{x,\xi \in \mathbb{R}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)|.$$

Proof. Check Theorem 1.6 of [37]. □

Now we consider the algebra properties of $S^m_{0,0}(\mathbb{R})$ with respect to the composition of operators. Let $p_j \in S^{m_j}_{0,0}(\mathbb{R})$, $j = 1, 2$, and define

$$\begin{aligned} q(x,\xi) &= Os - \iint e^{-iy\eta} p_1(x, \xi+\eta) p_2(x+y, \xi) dy đ\eta \\ &= \lim_{\epsilon \to 0} \iint e^{-iy\eta} p_1(x,\xi+\eta) p_2(x+y,\xi) e^{-\epsilon^2 y^2} e^{-\epsilon^2 \eta^2} dy đ\eta. \end{aligned} \tag{4.17}$$

We often write $q(x,\xi) = p_1(x,\xi) \circ p_2(x,\xi)$, but we need to have in mind that this means the expression given in the right-hand side of (4.17). Therefore, we have the following result concerning the composition.

Theorem 4.4. *Let $p_j \in S^{m_j}_{0,0}(\mathbb{R})$, $j = 1, 2$, and q defined by (4.17). Then $q \in S^{m_1+m_2}_{0,0}(\mathbb{R})$ and $q(x,D) = p_1(x,D) p_2(x,D)$. Moreover,*

$$q(x,\xi) = \sum_{\alpha < N} \frac{1}{\alpha!} \partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi) + r_N(x,\xi), \tag{4.18}$$

where

$$r_N(x,\xi) = N \int_0^1 \frac{(1-\vartheta)^{N-1}}{N!} Os - \iint e^{-iy\eta} \partial_\xi^N p_1(x, \xi + \vartheta\eta) D_x^N p_2(x+y,\xi) dy đ\eta d\vartheta,$$

and the semi-norms of r_N can be estimated in the following way: for any $\ell_0 \in \mathbb{N}_0$, there exists $\ell_1 = \ell_1(\ell_0)$ such that

$$|r_N|_{\ell_0}^{(m_1+m_2)} \leq C_{\ell_0} |\partial_\xi^N p_1|_{\ell_1}^{(m_1)} |\partial_x^N p_2|_{\ell_1}^{(m_2)}.$$

Proof. For a proof of this result, we recommend to the reader [37], Lemma 2.4 and Theorem 1.4. □

In order to obtain the desired contradiction mentioned in the first lines of this section, let us define some essential elements. Let $\phi \in G^{\theta}(\mathbb{R})$ satisfying

$$\widehat{\phi}(\xi) = e^{-\rho_0 \langle \xi \rangle^{1/\theta}}, \tag{4.19}$$

for some $\rho_0 > 0$. Notice that $\phi \in \mathcal{S}_1^{\theta}(\mathbb{R})$. In order to reach this conclusion, we need to prove that $\phi(x) = \int e^{i\xi x}\widehat{\phi}(\xi) đ\xi$ satisfies $|\phi(x)| \leq Ce^{-c|x|}$, for some positive constants C and c, due to Proposition 6.1.7 of [41]. Integration by parts leads us to

$$x^{\beta}\phi(x) = (-1)^{\beta} \int e^{i\xi x} D_{\xi}^{\beta} e^{-\rho_0 \langle x \rangle^{1/\theta}} đ\xi,$$

and Faà di Bruno formula together with some factorial inequalities imply the following estimate

$$\begin{aligned} |x^{\beta}\phi(x)| &\leq \int \left| e^{i\xi x} D_{\xi}^{\beta} e^{-\rho_0 \langle x \rangle^{1/\theta}} \right| đ\xi \\ &= \int \sum_{j=1}^{\beta} \frac{1}{j!} e^{-\rho_0 \langle \xi \rangle^{1/\theta}} \sum_{\substack{\beta_1 + \cdots + \beta_j = \beta \\ \beta_\ell \geq 1}} \frac{\beta!}{\beta_1! \cdots \beta_j!} \prod_{\ell=1}^{j} \left| \partial_{\xi}^{\beta_\ell} \left(-\rho_0 \langle \xi \rangle^{1/\theta} \right) \right| đ\xi \\ &\leq C_{\rho_0}^{\beta} \beta!, \quad \forall x \in \mathbb{R},\ \beta \in \mathbb{N}_0, \end{aligned} \tag{4.20}$$

hence $\phi \in \mathcal{S}_1^{\theta}(\mathbb{R}) \subset \mathcal{S}_s^{\theta}(\mathbb{R})$, for all $s \geq 1$. Then, there exists $\rho = (\rho_1, \rho_2)$, with $\rho_1, \rho_2 > 0$, such that $\phi \in H^0_{\rho;s,\theta}(\mathbb{R})$, for all $s \geq 1$.

Now let us consider a sequence $(\sigma_k)_{k \in \mathbb{N}_0}$ of positive real numbers such that $\sigma_k \to \infty$, as $k \to \infty$. Then we define the sequence of functions $(\phi_k)_{k \in \mathbb{N}_0}$ by setting

$$\phi_k(x) := e^{-\rho_2 4^{1/s} \sigma_k^{\frac{p-1}{s}}} \phi\left(x - 4\sigma_k^{p-1}\right), \quad k \in \mathbb{N}_0.$$

For each $k \in \mathbb{N}_0$, $\phi_k \in H^0_{\rho;s,\theta}(\mathbb{R})$ and satisfies

$$\begin{aligned} \|\phi_k\|^2_{H^0_{\rho;s,\theta}} &= \int \left| e^{\rho_2 \langle x \rangle^{1/s}} e^{\rho_1 \langle D \rangle^{1/\theta}} \phi_k(x) \right|^2 dx \\ &= \int e^{2\rho_2 \langle x \rangle^{1/s}} \left| e^{\rho_1 \langle D \rangle^{1/\theta}} \phi_k(x) \right|^2 dx \\ &= \int e^{2\rho_2 \langle x \rangle^{1/s}} \left| e^{\rho_1 \langle D \rangle^{1/\theta}} e^{-\rho_2 4^{1/s} \sigma_k^{\frac{p-1}{s}}} \phi\left(x - 4\sigma_k^{p-1}\right) \right|^2 dx \\ &= \int e^{2\rho_2 \left\langle x + 4\sigma_k^{p-1} \right\rangle^{1/s}} e^{-2\rho_2 4^{1/s} \sigma_k^{\frac{p-1}{s}}} \left| e^{\rho_1 \langle D \rangle^{1/\theta}} \phi(x) \right|^2 dx \\ &\leq \int e^{2\rho_2 \langle x \rangle^{1/s}} \left| e^{\rho_1 \langle D \rangle^{1/\theta}} \phi(x) \right|^2 dx \\ &= \|\phi\|^2_{H^0_{\rho;s,\theta}} = \texttt{constant}, \end{aligned}$$

which means that the sequence $\left(\|\phi_k\|_{H^0_{\rho;s,\theta}} \right)_{k \in \mathbb{N}_0}$ is uniformly bounded in k.

By assuming that the Cauchy problem (4.15) is well-posed in $\mathcal{S}_s^\theta(\mathbb{R})$, we can consider, for every $k \in \mathbb{N}_0$,

$$u_k \in C^1\left([0,T]; H^0_{\tilde{\rho};s,\theta}(\mathbb{R})\right),$$

where $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)$, with $\tilde{\rho}_1, \tilde{\rho}_2 > 0$, the solution of (4.15) with initial datum ϕ_k, that is,

$$\begin{cases} M u_k(t,x) = 0 \\ u_k(0,x) = \phi_k(x) \end{cases}.$$

The energy inequality gives us the estimate

$$\|u_k(t,\cdot)\|_{L^2} \leq \|u_k(t,\cdot)\|_{H^0_{\tilde{\rho};s,\theta}} \leq C_{T,\rho}\|\phi_k\|_{H^0_{\rho;s,\theta}} \leq C_{T,\rho}\|\phi\|_{H^0_{\rho;s,\theta}}. \tag{4.21}$$

From the above inequality, it can be concluded that the sequence $(\|u_k(t)\|_{L^2})_{k\in\mathbb{N}_0}$ is uniformly bounded with respect to $k \in \mathbb{N}_0$ and $t \in [0,T]$.

Now we consider a Gevrey cut-off function $h \in G_0^{\theta_h}(\mathbb{R})$, for $\theta_h > 1$, such that $h(x) = 1$ for $|x| \leq \frac{1}{2}$ and $h(x) = 0$ for $|x| \geq 1$; besides, we assume that its Fourier transform satisfies $\widehat{h}(0) > 0$ and $\widehat{h}(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. By using the sequence $(\sigma_k)_{k\in\mathbb{N}_0}$, let us define the sequence of symbols

$$w_k(x,\xi) = h\left(\frac{x - 4\sigma_k^{p-1}}{\sigma_k^{p-1}}\right) h\left(\frac{\xi - \sigma_k}{\frac{1}{4}\sigma_k}\right). \tag{4.22}$$

Remark 4.2. *For each k, notice that $w_k(x,\xi)$ is a symbol localized around the bicharacteristic curve of ξ^p passing through the point $(0,\sigma_k)$ at some fixed time t. Indeed, it is also known as the Hamilton flow generated by the operator $D_t + D_x^p$ passing by a point $(x_0,\xi_0) \in \mathbb{R}^2$, is the solution of*

$$\begin{cases} x'(t) = p\xi(t)^{p-1}, \ x(0) = x_0 \\ \xi'(t) = 0, \ \xi(0) = \xi_0 \end{cases}, \tag{4.23}$$

that is, $(x(t),\xi(t)) = \left(x_0 + pt\xi_0^{p-1}, \xi_0\right)$.

Remark 4.3. *On the support of w_k, x is comparable with σ_k^{p-1} and ξ is comparable σ_k, because $x \in$ supp $w_k(\cdot,\xi)$ implies that*

$$\left|\frac{x - 4\sigma_k^{p-1}}{\sigma_k^{p-1}}\right| \leq 1 \quad \Longleftrightarrow \quad 3\sigma_k^{p-1} \leq x \leq 5\sigma_k^{p-1},$$

and $\xi \in$ supp $w_k(x,\cdot)$ implies that

$$\left|\frac{\xi - \sigma_k}{\frac{1}{4}\sigma_k}\right| \leq 1 \quad \Longleftrightarrow \quad \frac{3}{4}\sigma_k \leq \xi \leq \frac{5}{4}\sigma_k,$$

for each $k \in \mathbb{N}_0$.

For some $\lambda \in (0,1)$ to be chosen later, let $\theta_1 > 1$ such that $\theta_h \leq \theta_1$ and, for each $k \in \mathbb{N}_0$, let

$$N_k := \left\lfloor \sigma_k^{\lambda/\theta_1} \right\rfloor = \max\left\{\alpha \in \mathbb{N}_0 : \ \alpha \leq \sigma_k^{\lambda/\theta_1}\right\}. \tag{4.24}$$

For each k, we consider the *energy* $E_k(t)$ given by

$$E_k(t) = \sum_{\alpha\leq N_k,\beta\leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \left\| w_k^{(\alpha\beta)}(x,D)u_k(t,x) \right\|_{L^2} = \sum_{\alpha\leq N_k,\beta\leq N_k} E_{k,\alpha,\beta}(t), \tag{4.25}$$

where

$$w_k^{(\alpha\beta)}(x,\xi) = h^{(\alpha)}\left(\frac{x-4\sigma_k^{p-1}}{\sigma_k^{p-1}}\right) h^{(\beta)}\left(\frac{\xi-\sigma_k}{\frac{1}{4}\sigma_k}\right).$$

From Remark 4.3 it can be easily established the next lemma which gives us an estimate for the norms of w_k.

Lemma 4.1. *Let $\alpha,\beta,\gamma,\delta,\ell \in \mathbb{N}_0$. Then $w_k^{(\alpha\beta)} \in S^0_{0,0}(\mathbb{R}^2)$ and satisfies the estimate*

$$|\partial_x^\delta \partial_\xi^\gamma w_k^{(\alpha\beta)}(x,\xi)|_\ell^{(0)} \leq C^{\alpha+\beta+\gamma+\delta+\ell+1}(\alpha!\beta!\gamma!\delta!\ell!^2)^{\theta_h}\sigma_k^{-\gamma}\sigma_k^{-\delta(p-1)},$$

for some constant $C>0$ which does not depend on k,α,β,γ and δ.

We also need the two following propositions to prove the main result of this section. Proposition 4.2 has a simple proof, which will be done in the sequel. However, Proposition 4.3 requires a lot of steps to be proved, so we will dedicate a particular subsection to prove it. In the sequel of this section we shall give it as true and use it to prove Theorem 4.2.

Proposition 4.2. *Suppose the Cauchy problem (4.15) is well-posed in $\mathcal{S}_s^\theta(\mathbb{R})$. Then there exists $C>0$ such that, for all $t\in[0,T]$ and $k\in\mathbb{N}_0$:*

$$E_{k,\alpha,\beta}(t) \leq C^{\alpha+\beta+1}(\alpha!\beta!)^{\theta_h-\theta_1} \tag{4.26}$$

and

$$E_k(t) \leq C. \tag{4.27}$$

Proof. Since $w_k^{(\alpha\beta)} \in S^0_{0,0}(\mathbb{R}^2)$, Calderón-Vaillancourt Theorem together with (4.21) and Lemma 4.1 implies that

$$\begin{aligned} \left\| w_k^{(\alpha\beta)}(x,D)u_k(t) \right\|_{L^2} &\leq C\|u_k(t)\|_{L^2} \max_{\alpha,\beta\leq 2} \sup_{(x,\xi)\in\mathbb{R}^2} |\partial_\xi^\alpha \partial_x^\beta w_k(x,\xi)| \\ &\leq C^{\alpha+\beta+1}(\alpha!\beta!)^{\theta_h}, \end{aligned}$$

hence by the definition of $E_{k,\alpha,\beta}(t)$, we have

$$E_{k,\alpha,\beta}(t) = \frac{1}{(\alpha!\beta!)^{\theta_1}} \left\| w_k^{(\alpha\beta)}(x,D)u_k(t) \right\|_{L^2} \leq C^{\alpha+\beta+1}(\alpha!\beta!)^{\theta_h-\theta_1}.$$

Since $\theta_1 > \theta_h$, we finally obtain (4.27), that is,

$$E_k(t) = \sum_{\alpha\leq N_k,\beta\leq N_k} E_{k,\alpha,\beta}(t) \leq C \sum_{\alpha,\beta\in\mathbb{N}_0} C^{\alpha+\beta}(\alpha!\beta!)^{\theta_h-\theta_1} = \texttt{constant}.$$

□

Proposition 4.3. *Let us suppose that the Cauchy problem* (4.15) *is well-posed in* $\mathcal{S}^\theta_s(\mathbb{R})$*. Then there exist positive constants* C*,* c*,* c_1 *such that, for all* $t \in [0,T]$ *and all* k *sufficiently large, the inequality*

$$\partial_t E_k(t) \geq \left(c_1 \sigma_k^{(p-1)(1-\sigma)} - C \sum_{\ell=1}^{p} \frac{\sigma_k^{\ell\lambda}}{\sigma_k^{p(\ell-1)}} \right) E_k(t) - C^{N_k+1} \sigma_k^{C-cN_k},$$

holds.

Now we are ready to go on with the proof of Theorem 4.2 assuming that Proposition 4.3 holds true.

Proof of Theorem 4.2. First of all, we set

$$A_k := c_1 \sigma_k^{(p-1)(1-\sigma)} - C \sum_{\ell=1}^{p} \frac{\sigma_k^{\ell\lambda}}{\sigma_k^{p(\ell-1)}}, \quad R_k := C^{N_k+1} \sigma_k^{C-cN_k}.$$

By Proposition 4.3, we have that

$$\partial_t E_k(t) \geq A_k E_k(t) - R_k. \tag{4.28}$$

Picking $\lambda < \min\{(p-1)(1-\sigma), 1\}$, we notice that

$$(\ell-1) \underbrace{(\lambda - p)}_{<0} + \lambda < \lambda < (p-1)(1-\sigma) \quad \Rightarrow \quad \ell\lambda - p(\ell-1) < (p-1)(1-\sigma),$$

so the leading term in A_k is the first one, hence for k sufficiently large we have

$$A_k \geq \frac{c_1}{2} \sigma_k^{(p-1)(1-\sigma)}. \tag{4.29}$$

From now on, let us consider k sufficiently large. Then, by using Gronwall's inequality, if follows from (4.28) that

$$E_k(t) \geq e^{A_k t} \left(E_k(0) - R_k \int_0^t e^{-A_k \tau} d\tau \right).$$

Then, for any $T^* \in [0,T]$, by using (4.29), the above inequality turns into

$$E_k(T^*) \geq e^{T^* \frac{c_1}{2} \sigma_k^{(p-1)(1-\sigma)}} \left(E_k(0) - T^* R_k \right). \tag{4.30}$$

The next step is to estimate R_k and $E_k(0)$. Since $N_k = \left\lfloor \sigma_k^{\lambda/\theta_1} \right\rfloor$, R_k can be estimated as

$$R_k \leq C e^{-c\sigma_k^{\lambda/\theta_1}}. \tag{4.31}$$

To obtain an estimate from below for $E_k(0)$, we notice that, by definition of $E_k(t)$, $w_k(x,\xi)$ and ϕ_k, if

$\mathscr{F}$ denotes the Fourier transform, we get

$$\begin{aligned}
E_k(0) &\geq \|w_k(x,D)\phi_k\|_{L^2(\mathbb{R}_x)} \\
&= \left\| h\left(\frac{x-4\sigma_k^{p-1}}{\sigma_k^{p-1}}\right) h\left(\frac{D-4\sigma_k}{\frac{1}{4}\sigma_k}\right)\phi_k \right\|_{L^2(\mathbb{R}_x)} \\
&= e^{-\rho_2 4^{1/s}\sigma_k^{(p-1)/s}} \left\| h\left(\frac{x-4\sigma_k^{p-1}}{\sigma_k^{p-1}}\right) h\left(\frac{D-4\sigma_k}{\frac{1}{4}\sigma_k}\right)\phi(x-4\sigma_k^{p-1}) \right\|_{L^2(\mathbb{R}_x)} \\
&= e^{-\rho_2 4^{1/s}\sigma_k^{(p-1)/s}} \left\| \mathscr{F}\left[h\left(\frac{x-4\sigma_k^{p-1}}{\sigma_k^{p-1}}\right)\right](\xi) * h\left(\frac{\xi-\sigma_k}{\frac{1}{4}\sigma_k}\right) e^{-4i\sigma_k^{p-1}\xi}\widehat{\phi}(\xi) \right\|_{L^2(\mathbb{R}_\xi)} \\
&= e^{-\rho_2 4^{1/s}\sigma_k^{(p-1)/s}} \sigma_k^{p-1} \left\| e^{-4i\sigma_k^{p-1}\xi}\widehat{h}\left(\sigma_k^{p-1}\xi\right) * h\left(\frac{\xi-\sigma_k}{\frac{1}{4}\sigma_k}\right) e^{-4i\sigma_k^{p-1}\xi}\widehat{\phi}(\xi) \right\|_{L^2(\mathbb{R}_\xi)},
\end{aligned}$$

hence

$$E_k^2(0) \geq e^{-2\rho_2 4^{1/s}\sigma_k^{(p-1)/s}} \sigma_k^{2(p-1)} \int_{\mathbb{R}_\xi} \left| \int_{\mathbb{R}_\eta} \widehat{h}\left(\sigma_k^{p-1}(\xi-\eta)\right) h\left(\frac{\eta-\sigma_k}{\frac{1}{4}\sigma_k}\right)\widehat{\phi}(\eta)d\eta \right|^2 d\xi.$$

Since $\widehat{h}(0) > 0$ and $\widehat{h}(\xi) \geq 0$, for all $\xi \in \mathbb{R}$, it will be possible to obtain estimates from below to $E_k^2(0)$ performing a restriction in the integration domain. Set

$$G_{1,k} := \left[\frac{7}{8}\sigma_k, \frac{7}{8}\sigma_k + \sigma_k^{-p}\right] \quad \text{and} \quad G_{2,k} := \left[\frac{7}{8}\sigma_k - \sigma_k^{-p}, \frac{7}{8}\sigma_k + \sigma_k^{-p}\right].$$

Notice that, if $\eta \in G_{1,k}$ then $|\eta - \sigma_k| \leq \frac{\sigma_k}{8}$, because

$$\eta \in G_{1,k} \ \Rightarrow\ \frac{7}{8}\sigma_k \leq \eta \leq \frac{7}{8}\sigma_k + \sigma_k^{-p} \ \Rightarrow\ -\frac{\sigma_k}{8} \leq \eta - \sigma_k \leq -\frac{\sigma_k}{8} + \sigma_k^{-p}.$$

Also, if $\eta \in G_{1,k}$ and $\xi \in G_{2,k}$, we notice that

$$\sigma_k^{p-1}|\xi - \eta| \leq 2\sigma_k^{-1},$$

because $\eta \in G_{1,k}$ implies that

$$-\sigma_k^{-p} - \frac{7}{8}\sigma_k \leq -\eta \leq -\frac{7}{8}\sigma_k, \tag{4.32}$$

$\xi \in G_{2,k}$ implies that

$$\frac{7}{8}\sigma_k - \sigma_k^{-p} \leq \xi \leq \frac{7}{8}\sigma_k + \sigma_k^{-p}, \tag{4.33}$$

and from (4.32) and (4.33) we get

$$-2\sigma_k^{-p} \leq \xi - \eta \leq \sigma_k^{-p} \leq 2\sigma_k^{-p} \ \Rightarrow\ |\xi - \eta| \leq 2\sigma_k^{-p} \ \Rightarrow\ \sigma_k^{p-1}|\xi - \eta| \leq 2\sigma_k^{-1}.$$

If we pick $(\xi, \eta) \in G_{2,k} \times G_{1,k}$, then $\sigma_k^{p-1}(\xi - \eta)$ is close to zero for k large enough, hence by the choices of $\widehat{h}(0) > 0$ and $\widehat{h}(\xi) \geq 0$, there exists a constant $C > 0$ such that

$$\widehat{h}\left(\sigma_k^{p-1}(\xi - \eta)\right) > C.$$

Furthermore, if $\eta \in G_{1,k}$ then $h\left(\frac{\eta-\sigma_k}{\frac{1}{4}\sigma_k}\right) = 1$. Finally, since η is comparable with σ_k in $G_{1,k}$, it follows from $\widehat{\phi}(\eta) = e^{-\rho_0\langle\eta\rangle^{1/\theta}}$ that

$$\widehat{\phi}(\eta) \geq e^{-c_{\rho_0}\sigma_k^{1/\theta}},$$

for a positive constant c_{ρ_0} depending on ρ_0. Finally, we obtain

$$\begin{aligned} E_k^2(0) &\geq e^{-2\rho_2 4^{1/s}\sigma_k^{(p-1)/s}}\sigma_k^{2(p-1)}\int_{G_{2,k}}\left|\int_{G_{1,k}} Ce^{-c_{\rho_0}\sigma_k^{1/\theta}}d\eta\right|^2 d\xi \\ &= Ce^{-2\rho_2 4^{1/s}\sigma_k^{(p-1)/s}}\sigma_k^{-(p+2)}e^{-2c_{\rho_0}\sigma_k^{1/\theta}} \end{aligned}$$

which implies that

$$E_k(0) \geq C\sigma_k^{-(p+2)/2}\exp\left(-\rho_2 4^{1/s}\sigma_k^{(p-1)/s}\right)\exp\left(-c_{\rho_0}\sigma_k^{1/\theta}\right) \geq C\exp\left(-\tilde{c}_{\rho_0,\rho_2}\sigma_k^{\max\left\{\frac{p-1}{s},\frac{1}{\theta}\right\}}\right). \tag{4.34}$$

From (4.30), (4.31) and (4.34) we get

$$\begin{aligned} E_k(T^*) &\geq C\exp\left(T^*\frac{c_1}{2}\sigma_k^{(p-1)(1-\sigma)}\right)\left[\sigma_k^{-\frac{p+2}{2}}\exp\left(-\rho_2 4^{\frac{1}{s}}\sigma_k^{\frac{p-1}{s}} - c_{\rho_0}\sigma_k^{\frac{1}{\theta}}\right) - T^*\exp\left(-c\sigma_k^{\frac{\lambda}{\theta_1}}\right)\right] \\ &\geq C_1\exp\left(\frac{c_1}{2}T^*\sigma_k^{(p-1)(1-\sigma)}\right)\left[\exp\left(-\tilde{c}_{\rho_0,\rho_2}\sigma_k^{\max\left\{\frac{p-1}{s},\frac{1}{\theta}\right\}}\right) - T^*\exp\left(-c\sigma_k^{\frac{\lambda}{\theta_1}}\right)\right]. \end{aligned} \tag{4.35}$$

for all $T^* \in (0,T]$, once $\lambda < (p-1)(1-\sigma)$ and k is sufficiently large. Assume by contradiction that $\max\left\{\frac{1}{(p-1)\theta}, \frac{1}{s}\right\} < 1-\sigma$, which is equivalent to $\max\left\{\frac{1}{\theta}, \frac{p-1}{s}\right\} < (p-1)(1-\sigma)$, and take $\lambda < (p-1)(1-\sigma)$. Then, we can pick θ_1 very close to 1 such that

$$\frac{\lambda}{\theta_1} > \max\left\{\frac{1}{\theta}, \frac{p-1}{s}\right\}.$$

It follows from (4.35) that

$$E_k(T^*) \geq C_2\exp\left(\frac{\tilde{c}T^*}{2}\sigma_k^{(p-1)(1-\sigma)}\right)\exp\left(-c''\sigma_k^{\max\left\{\frac{1}{\theta},\frac{p-1}{s}\right\}}\right) \longrightarrow \infty, \text{ as } k \to \infty,$$

since $(p-1)(1-\sigma) > \max\left\{\frac{1}{\theta}, \frac{p-1}{s}\right\}$, which gives us a contradiction, due to Proposition 4.2.

□

4.4.2 Proof of Proposition 4.3

As we mentioned before, the proof of Proposition 4.3 is very long, so we will dedicate this section to prove it.

Let us set

$$v_k^{(\alpha\beta)}(t,x) := w_k^{(\alpha\beta)}(x,D)u_k(t,x).$$

By denoting $[H,K] = HK - KH$, for H, K operators, we can write

$$\begin{aligned} Mv_k^{(\alpha\beta)} &= Mw_k^{(\alpha\beta)}u_k \\ &= w_k^{(\alpha\beta)}\underbrace{Mu_k}_{=0} + [M, w_k^{(\alpha\beta)}]u_k \\ &= [M, w_k^{(\alpha\beta)}]u_k =: f_k^{(\alpha\beta)}. \end{aligned}$$

In order to obtain an estimate from below for $\partial_t E_k$, let us compute

$$\begin{aligned}
\|v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})}\partial_t\|v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})} &= \left(\int_{\mathbb{R}}|v_k^{(\alpha\beta)}(t,x)|^2dx\right)^{\frac{1}{2}}\partial_t\left(\int_{\mathbb{R}}|v_k^{(\alpha\beta)}(t,x)|^2dx\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}}|v_k^{(\alpha\beta)}(t,x)|^2dx\right)^{\frac{1}{2}}\cdot\frac{1}{2}\left(\int_{\mathbb{R}}|v_k^{(\alpha\beta)}(t,x)|^2dx\right)^{-\frac{1}{2}} \\
&\times\ \partial_t\int_{\mathbb{R}}|v_k^{(\alpha\beta)}(t,x)|^2dx \\
&= \frac{1}{2}\partial_t\left(\|v_k^{(\alpha\beta)}(t)\|^2_{L^2(\mathbb{R})}\right) \\
&= \mathbf{Re}\left(\partial_t v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})}. \qquad (4.36)
\end{aligned}$$

The definition of M implies that $\partial_t = iM - iD_x^p + \langle x\rangle^{-\sigma}D_x^{p-1}$, hence we can rewrite (4.36) and estimate from below

$$\begin{aligned}
&\|v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})}\partial_t\|v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})} \\
=\ & \mathbf{Re}\left(iMv_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})} - \underbrace{\mathbf{Re}\left(iD_x^p v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})}}_{\text{purely imaginary}} \\
+\ & \mathbf{Re}\left(\langle x\rangle^{-\sigma}D_x^{p-1}v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})} \\
\geq\ & -\|f_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})}\|v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})} + \mathbf{Re}\left(\langle x\rangle^{-\sigma}D_x^{p-1}v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})}, \qquad (4.37)
\end{aligned}$$

and now we need to estimate:

1. $\mathbf{Re}\left(\langle x\rangle^{-\sigma}D_x^{p-1}v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})}$ from below.
2. $\|f_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})}$ from above.

Estimates from below to Re $\left(\langle x\rangle^{-\sigma}D_x^{p-1}v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})}$

Let χ_k and ψ_k cut-off functions defined by

$$\chi_k(\xi) = h\left(\frac{\xi-\sigma_k}{\frac{3}{4}\sigma_k}\right) \quad \text{and} \quad \psi_k(x) = h\left(\frac{x-4\sigma_k^{p-1}}{3\sigma_k^{p-1}}\right), \qquad (4.38)$$

respectively. Recalling the definition of h, we notice that in the support of $\psi_k(x)\chi_k(\xi)$ we have

$$|\xi-\sigma_k| \leq \frac{3}{4}\sigma_k \quad \Leftrightarrow \quad \frac{\sigma_k}{4} \leq \xi \leq \frac{7\sigma_k}{4}$$

and

$$|x-4\sigma_k^{p-1}| \leq 3\sigma_k^{p-1} \quad \Leftrightarrow \quad \sigma_k^{p-1} \leq x \leq 7\sigma_k^{p-1},$$

for all $k \in \mathbb{N}_0$. From these inequalities, it follows that if $(x,\xi) \in \mathrm{supp}\psi_k(x)\chi_k(\xi)$ then

$$\xi^{p-1} \geq \frac{\sigma_k^{p-1}}{4^{p-1}} \quad \text{and} \quad \langle x\rangle^{-\sigma} \geq 7^{-\sigma}\langle\sigma_k^{p-1}\rangle^{-\sigma},$$

hence

$$\xi^{p-1}\langle x\rangle^{-\sigma} \geq \frac{7^{-\sigma}}{4^{p-1}}\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1}.$$

By setting $\Theta := \frac{7^{-\sigma}}{4^{p-1}}$, let us consider a decomposition of the symbol of $\langle x\rangle^{-\sigma}D_x^{p-1}$ in the following way

$$\begin{aligned}
\langle x\rangle^{-\sigma}\xi^{p-1} &= \Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1} + \langle x\rangle^{-\sigma}\xi^{p-1} - \Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1} \\
&= \underbrace{\Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1}}_{=:\mathtt{I}_{1,k}} + \underbrace{\left(\langle x\rangle^{-\sigma}\xi^{p-1} - \Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1}\right)\psi_k(x)\chi_k(\xi)}_{=:\mathtt{I}_{2,k}(x,\xi)} \\
&+ \underbrace{\left(\langle x\rangle^{-\sigma}\xi^{p-1} - \Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1}\right)(1-\psi_k(x)\chi_k(\xi))}_{=:\mathtt{I}_{3,k}(x,\xi)} \\
&= \mathtt{I}_{1,k} + \mathtt{I}_{2,k}(x,\xi) + \mathtt{I}_{3,k}(x,\xi).
\end{aligned}$$

In the following, we shall estimate each $\mathbf{Re}\left(\mathtt{I}_{\ell,k}(x,D)v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})}$, $\ell \in \{1,2,3\}$.

- **Estimate of Re** $\left(I_{1,k}v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta}(t)\right)_{L^2(\mathbb{R})}$**.** Since $\mathtt{I}_{1,k} = \Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1}$, we get

$$\begin{aligned}
\mathbf{Re}\left(\mathtt{I}_{1,k}v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})} &= \mathbf{Re}\left(\Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1}v_k^{(\alpha\beta)}(t), v_k^{\alpha\beta}(t)\right)_{L^2(\mathbb{R})} \\
&= \Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1}\|v_k^{(\alpha\beta)}(t)\|^2_{L^2(\mathbb{R})} \\
&\geq 2^{-\frac{\sigma}{2}}\Theta\sigma_k^{(p-1)(1-\sigma)}\|v_k^{(\alpha\beta)}(t)\|^2_{L^2(\mathbb{R})}. \qquad (4.39)
\end{aligned}$$

- **Estimate of Re** $\left(\mathtt{I}_{2,k}(x,D)v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})}$**.** Just recall that

$$\mathtt{I}_{2,k}(x,\xi) = \left(\langle x\rangle^{-\sigma}\xi^{p-1} - \Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1}\right)\psi_k(x)\chi_k(\xi),$$

and this symbol belongs to $\mathbf{SG}^{p-1,-\sigma}(\mathbb{R}^2)$ with uniform estimates with respect to k. Indeed, we have that

$$\begin{aligned}
|\partial_\xi^\gamma\partial_x^\nu\mathtt{I}_{2,k}(x,\xi)| &= \left|\partial_\xi^\gamma\partial_x^\nu\left[\left(\langle x\rangle^{-\sigma}\xi^{p-1} - \Theta\langle\sigma_k^{p-1}\rangle^{-\sigma}\sigma_k^{p-1}\right)\psi_k(x)\chi_k(\xi)\right]\right| \\
&\leq \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \nu_1+\nu_2=\nu}}\frac{\gamma!\nu!}{\gamma_1!\gamma_2!\nu_1!\nu_2!}\left|\partial_\xi^{\gamma_1}\partial_x^{\nu_1}\left(\langle x\rangle^{-\sigma}\xi^{p-1} - \Theta\langle\sigma_k^{p-1}\rangle\sigma_k^{p-1}\right)\right| \\
&\times |\partial_x^{\nu_2}\psi_k(x)||\partial_\xi^{\gamma_2}\chi_k(\xi)|.
\end{aligned}$$

Since $h \in G_0^{\theta_h}(\mathbb{R})$, ψ_k and χ_k satisfy, respectively,

$$|\partial_x^{\nu_2}\psi_k(x)| = \left|\partial_x^{\nu_2}h\left(\frac{x-4\sigma_k^{p-1}}{3\sigma_k^{p-1}}\right)\right| \leq C^{\nu_2+1}\nu_2!^{\theta_h}\sigma_k^{-(p-1)\nu_2}$$

and

$$|\partial_\xi^{\gamma_2}\chi_k(\xi)| = \left|\partial_\xi^{\gamma_2}h\left(\frac{\xi-\sigma_k}{\frac{3}{4}\sigma_k}\right)\right| \leq C^{\gamma_2+1}\gamma_2!^{\theta_h}\sigma_k^{-\gamma_2}.$$

Now, by using the above inequalities and the fact that x is comparable with σ_k^{p-1} and ξ is comparable with σ_k on the support of $\psi_k(x)\chi_k(\xi)$, we can compute and estimate

$$\begin{aligned} & |\partial_\xi^\gamma \partial_x^\nu \mathtt{I}_{2,k}(x,\xi)| \\ \leq & \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \nu_1+\nu_2=\nu}} \frac{\gamma!\nu!}{\gamma_1!\gamma_2!\nu_1!\nu_2!} C^{\gamma_1+\nu_1+1}\gamma_1!\nu_1!\langle x\rangle^{-\sigma-\nu_1}\langle\xi\rangle^{p-1-\gamma_1} C^{\gamma_1+\nu_2+1}\gamma_2!^{\theta_h}\nu_2!^{\theta_h}\sigma_k^{-\gamma_2}\sigma_k^{-\nu_2(p-1)} \\ \leq & C^{\gamma+\nu+1}(\gamma!\nu!)^{\theta_h}\langle\xi\rangle^{p-1-\gamma}\langle x\rangle^{-\sigma-\nu}. \end{aligned}$$

Moreover, by the choice of Θ, we have that $\mathtt{I}_{2,k}(x,\xi) \geq 0$, for all $x, \xi \in \mathbb{R}$, then it follows by Theorem 2.10 that

$$\mathtt{I}_{2,k}(x,D) = \mathtt{p}_{2,k}(x,D) + \mathtt{r}_{2,k}(x,D),$$

where $\mathtt{p}_{2,k}(x,D)$ is a positive operator and $\mathtt{r}_{2,k} \in \mathbf{SG}^{p-2,-\sigma-1}(\mathbb{R}^2)$. Once the semi-norms of $\mathtt{I}_{2,k}$ are uniformly bounded with respect to k, the same holds to the semi-norms of $\mathtt{r}_{2,k}$. In this way, we have that

$$\begin{aligned} \mathbf{Re}\left(\mathtt{I}_{2,k}(x,D)v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})} & \geq \mathbf{Re}\left(\mathtt{r}_{2,k}(x,D)v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})} \\ & = \mathbf{Re}(\underbrace{\mathtt{r}_{2,k}\langle x\rangle^{-\sigma+1}}_{\text{order } p-2}\langle x\rangle^{-\sigma-1}v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t))_{L^2(\mathbb{R})} \\ & \geq -C\|\langle x\rangle^{-\sigma-1}v_k^{(\alpha\beta)}(t)\|_{H^{p-2}(\mathbb{R})}\|v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})} \\ & \geq -C\|v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})}\sum_{\ell=0}^{p-2}\left\|D_x^\ell\left(\langle x\rangle^{-\sigma-1}v_k^{(\alpha\beta)}(t)\right)\right\|_{L^2(\mathbb{R})}. \end{aligned}$$

To deal with the terms $\left\|D_x^\ell\left(\langle x\rangle^{-\sigma-1}v_k^{(\alpha\beta)}(t)\right)\right\|_{L^2(\mathbb{R})}$ we need to use the Leibniz formula and write

$$D_x^\ell\left(\langle x\rangle^{-\sigma-1}v_k^{(\alpha\beta)}(t)\right) = \sum_{\ell'=0}^{\ell}\binom{\ell}{\ell'}D_x^{\ell'}\langle x\rangle^{-\sigma-1}D_x^{\ell-\ell'}v_k^{(\alpha\beta)}(t).$$

On the support of $D_x^{\ell-\ell'}v_k^{(\alpha\beta)}(t)$, x is comparable with σ_k^{p-1}, hence

$$\left\|D_x^\ell\left(\langle x\rangle^{-\sigma-1}v_k^{(\alpha\beta)}(t)\right)\right\|_{L^2(\mathbb{R})} \leq C^{\ell+1}\sigma_k^{-(p-1)(\sigma+1)}\sum_{\ell'=0}^{\ell}\frac{\ell!}{(\ell-\ell')!}\|D_x^{\ell-\ell'}v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})}.$$

Now, we need the following lemma to conclude the desired estimate. The proof follows readily the same argument of the proof of Lemma 2 in [7], the only difference is that here we require well-posedness in $\mathcal{S}_s^\theta(\mathbb{R})$.

Lemma 4.2. *If the Cauchy problem (4.15) is well-posed in $\mathcal{S}_s^\theta(\mathbb{R})$, then for all $N \in \mathbb{N}$, the following estimate holds:*

$$\|D_x^r v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})} \leq C\sigma_k^r\|v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})} + C^{\alpha+\beta+N+1}(\alpha!\beta!)^{\theta_h}N!^{2\theta_h-1}\sigma_k^{r-N},$$

for some positive constant C which does not depend on k.

Applying Lemma 4.2 for $N = N_k$, the following estimate can be achieved

$$\begin{aligned}\left\| D_x^\ell \left(\langle x \rangle^{-\sigma-1} v_k^{(\alpha\beta)}(t) \right) \right\|_{L^2(\mathbb{R})} &\leq C\sigma_k^{-(p-1)(\sigma+1)+\ell} \| v_k^{(\alpha\beta)} \|_{L^2(\mathbb{R})} \\ &+ C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} \sigma_k^{-(p-1)(\sigma+1)} \sigma_k^{\ell-N_k}.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\mathbf{Re} \left(\mathtt{I}_{2,k}(x,D) v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t) \right)_{L^2(\mathbb{R})} &\geq -C\sigma_k^{-(p-1)\sigma-1} \| v_k^{(\alpha\beta)}(t) \|^2_{L^2(\mathbb{R})} \\ &- C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} \sigma_k^{-(p-1)\sigma-N_k-1} \| v_k^{(\alpha\beta)}(t) \|_{L^2(\mathbb{R})}.\end{aligned} \tag{4.40}$$

- **Estimate of Re** $\left(\mathtt{I}_{3,k}(x,D) v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t) \right)_{L^2(\mathbb{R})}$. Recalling that

$$\mathtt{I}_{3,k}(x,\xi) = \left(\langle x \rangle^{-\sigma} \xi^{p-1} - \Theta \langle \sigma_k^{p-1} \rangle^{-\sigma} \sigma_k^{p-1} \right) (1 - \psi_k(x)\chi_k(\xi)),$$

due to the fact that the supports of $w_k^{(\alpha\beta)}(x,\xi)$ and $1 - \psi_k(x)\chi_k(\xi)$ are disjoint, we may write

$$\mathtt{I}_{3,k}(x,D) \circ w_k^{(\alpha\beta)}(x,D) = R_k^{(\alpha\beta)}(x,D),$$

where

$$R_k^{(\alpha\beta)}(x,\xi) = N_k \int_0^1 \frac{(1-\vartheta)^{N_k-1}}{N_k!} Os - \iint e^{-iy\eta} \partial_\xi^{N_k} \mathtt{I}_{3,k}(x, \xi + \vartheta\eta) D_x^{N_k} w_k^{(\alpha\beta)}(x+y, \xi) dy đ\eta d\vartheta.$$

The semi-norms of $R_k^{(\alpha\beta)}$ can be estimated as: for each $\ell_0 \in \mathbb{N}_0$, there exists $\ell_1 = \ell_1(\ell_0)$ such that

$$|R_k|_{\ell_0}^{(0)} \leq C(\ell_0) \frac{N_k}{N_k!} |\partial_\xi^{N_k} \mathtt{I}_{3,k}|_{\ell_1}^{(0)} |\partial_x^{N_k} w_k^{(\alpha\beta)}|_{\ell_1}^{(0)}.$$

From Lemma 4.1, we have that

$$|\partial_x^{N_k} w_k^{(\alpha\beta)}|_{\ell_1}^{(0)} \leq C^{\ell_1+\alpha+\beta+N_k+1} \ell_1!^{2\theta_h} (\alpha!\beta!N_k!)^{\theta_h} \sigma_k^{-N_k(p-1)}.$$

Since $\sigma_k \to +\infty$, we can assume that $N_k \geq p$, then

$$\partial_\xi^{N_k} \mathtt{I}_{3,k}(x,\xi) = - \sum_{\substack{N_1+N_2=N_k \\ N_1 \leq p-1}} \frac{N_k!}{N_1!N_2!} \partial_\xi^{N_1} \left(\langle x \rangle^{-\sigma} \xi^{p-1} - \Theta \langle \sigma_k^{p-1} \rangle^{-\sigma} \sigma_k^{p-1} \right) \psi_k(x) \partial_\xi^{N_2} \chi_k(\xi),$$

hence

$$|\partial_\xi^{N_k} \mathtt{I}_{3,k}(x,\xi)|_{\ell_1}^{(0)} \leq C^{\ell_1+N_k+1} \ell_1!^{2\theta_h} N_k!^{\theta_h} \sigma_k^{p-1-N_k}.$$

By gathering the previous estimates, we obtain

$$|R_k^{(\alpha\beta)}|_{\ell_0}^{(0)} \leq C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} \sigma_k^{p(1-N_k)-1},$$

which provides that

$$\| I_{3,k}(x,D) v_k^{(\alpha\beta)}(t) \|_{L^2(\mathbb{R})} \leq C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} \sigma_k^{p(1-N_k)-1}.$$

Finally, we get

$$\mathbf{Re}\left(\mathtt{I}_{3,k}(x,D)v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})} \geq -C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1}\sigma_k^{p(1-N_k)-1}\|v_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})}. \tag{4.41}$$

From (4.39), (4.40) and (4.41), we obtain the following Lemma.

Lemma 4.3. *If the Cauchy problem* (4.15) *is well-posed in* $\mathcal{S}_s^\theta(\mathbb{R})$, *then for all* k *sufficiently large the estimate*

$$\begin{aligned}\mathbf{Re}\left(\langle x\rangle^{-\sigma} D_x^{p-1} v_k^{(\alpha\beta)}(t), v_k^{(\alpha\beta)}(t)\right)_{L^2(\mathbb{R})} \;&\geq\; c_1\sigma_k^{(p-1)(1-\sigma)}\|v_k^{(\alpha\beta)}(t)\|^2_{L^2(\mathbb{R})} \\ &-\; C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1}\sigma_k^{p(1-N_k)-1}\|v_k^{\alpha\beta}(t)\|_{L^2(\mathbb{R})}\end{aligned}$$

holds for some $c_1 > 0$ *independent of* k, α, β *and* N_k.

Estimates from above of $\|f_k^{(\alpha\beta)}(t)\|_{L^2(\mathbb{R})}$

We know that

$$f_k^{(\alpha\beta)} = [M, w_k^{(\alpha\beta)}]u_k = [D_t + D_x^p, w_k^{(\alpha\beta)}]u_k + [i\langle x\rangle^{-\sigma} D_x^{p-1}, w_k^{(\alpha\beta)}]u_k,$$

so our goal is to estimate the above commutators.

- **Estimate of** $[D_t + D_x^p, w_k^{(\alpha\beta)}]u_k$**.** Since $w_k^{(\alpha\beta)}$ does not depend on t, we have that

$$[D_t + D_x^p, w_k^{(\alpha\beta)}]u_k = \sum_{\gamma=1}^{p}\binom{p}{\gamma} D_x^\gamma w_k^{(\alpha\beta)} D_x^{p-\gamma} u_k, \tag{4.42}$$

because

$$\begin{aligned}[D_t + D_x^p, w_k^{(\alpha\beta)}]u_k &= [D_x^p, w_k^{(\alpha\beta)}]u_k \\ &= D_x^p(w_k^{(\alpha\beta)}u_k) - w_k^{(\alpha\beta)}D_x^p u_k \\ &= \sum_{\gamma=0}^{p}\binom{p}{\gamma} D_x^\gamma w_k^{(\alpha\beta)} D_x^{p-\gamma}u_k - w_k^{(\alpha\beta)}D_x^p u_k \\ &= \left\{\sum_{\gamma=1}^{p}\binom{p}{\gamma} D_x^\gamma w_k^{(\alpha\beta)} D_x^{p-\gamma} + w_k^{(\alpha\beta)}D_x^p - w_k^{(\alpha\beta)}D_x^p\right\}u_k \\ &= \left\{\sum_{\gamma=1}^{p}\binom{p}{\gamma} D_x^\gamma w_k^{(\alpha\beta)} D_x^{p-\gamma}\right\}u_k.\end{aligned}$$

To deal with the right-hand side of (4.42), we will need the identities:

$$f(x)D_x^\ell g(x) = \sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j} D_x^{\ell-j}(g(x)D^j f(x)), \tag{4.43}$$

for f and g smooth functions,

$$\sum_{\gamma=1}^{p}\sum_{j=0}^{p-\gamma} c_{\gamma,j}a_{\gamma+j} = \sum_{\ell=1}^{p}\left(\sum_{\mathtt{q}=1}^{\ell} c_{\mathtt{q},\ell-\mathtt{q}}\right)a_\ell$$

and

$$\frac{(-1)^{\ell+1}}{\ell!} = \sum_{\mathtt{q}=1}^{\ell}\frac{(-1)^{\ell-\mathtt{q}}}{\mathtt{q}!(\ell-\mathtt{q})!}.$$

Then (4.42) becomes

$$\begin{aligned}
& [D_t + D_x^p, w_k^{(\alpha\beta)}] \\
= & \sum_{\gamma=1}^{p}\sum_{j=0}^{p-\gamma}\frac{(-1)^j p!}{\gamma! j!(p-\gamma-j)!}\left(\frac{1}{i\sigma_k^{p-1}}\right)^{\gamma+j} D_x^{p-\gamma-j}\circ w_k^{((\alpha+\ell+j)\beta)}(x,D) \\
= & \sum_{\ell=1}^{p}\left(\sum_{\mathtt{q}=1}^{\ell}\frac{(-1)^{\ell-\mathtt{q}}}{\mathtt{q}!(\ell-\mathtt{q})!}\right)\frac{p!}{(p-\ell)!}\left(\frac{1}{i\sigma_k^{p-1}}\right)^{\ell} D_x^{p-\ell}\circ w_k^{((\alpha+\ell)\beta)}(x,D) \\
= & \sum_{\ell=1}^{p}(-1)^{\ell+1}\binom{p}{\ell}\left(\frac{1}{i\sigma_k^{p-1}}\right)^{\ell} D_x^{p-\ell}\circ w_k^{((\alpha+\ell)\beta)}(x,D).
\end{aligned}$$

By Lemma 4.2, we get

$$\begin{aligned}
& \left\|[D_t + D_x^p, w_k^{(\alpha\beta)}]u_k\right\|_{L^2(\mathbb{R})} \\
\leq & C\sum_{\ell=1}^{p}\binom{p}{\ell}\frac{1}{\sigma_k^{\ell(p-1)}}\|D_x^{p-\ell}v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})} \\
\leq & C\sum_{\ell=1}^{p}\frac{1}{\sigma_k^{p(\ell-1)}}\|v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})} + C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h}N_k!^{2\theta_h-1}\sigma_k^{p-1-N_k}. \qquad (4.44)
\end{aligned}$$

- **Estimate of** $[i\langle x\rangle^{-\sigma}D_x^{p-1}, w_k^{(\alpha\beta)}]u_k$**.** Notice that the commutator can be written in the following way:

$$\begin{aligned}
[i\langle x\rangle^{-\sigma}D_x^{p-1}, w_k^{(\alpha\beta)}] &= i\langle x\rangle^{-\sigma}\sum_{\gamma=1}^{p-1}\binom{p-1}{\gamma}D_x^{\gamma}w_k^{(\alpha\beta)}(x,D)D_x^{p-1-\gamma} \\
&- i\sum_{\gamma=1}^{N_k-1}\frac{1}{\gamma!}D_x^{\gamma}\langle x\rangle^{-\sigma}\partial_\xi^{\gamma}w_k^{(\alpha\beta)}(x,D)D_x^{p-1} + r_k^{(\alpha\beta)}(x,D), \qquad (4.45)
\end{aligned}$$

where the symbol of $r_k^{(\alpha\beta)}(x,D)$ is

$$r_k^{(\alpha\beta)}(x,\xi) = -iN_k\int_0^1\frac{(1-\vartheta)^{N_k-1}}{N_k!}Os-\iint e^{-iy\eta}\partial_\xi^{N_k}w_k^{(\alpha\beta)}(x,\xi+\vartheta\eta)D_x^{N_k}\langle x+y\rangle^{-\sigma}\xi^{p-1}dy\,đ\eta\, d\vartheta.$$

In order to estimate $r_k^{(\alpha\beta)}$, we use the support properties of $w_k^{(\alpha\beta)}$ and write

$$\xi^{p-1} = (\xi+\vartheta\eta-\vartheta\eta)^{p-1} = \sum_{\ell=0}^{p-1}\binom{p-1}{\ell}(\xi+\vartheta\eta)^{\ell}(-\vartheta\eta)^{p-1-\ell}.$$

Now we deal with the oscillatory integral by using the above identity and integration by parts, which lead us to

$$
\begin{aligned}
& Os-\iint e^{-iy\eta}\partial_\xi^{N_k} w_k^{(\alpha\beta)}(x,\xi+\vartheta\eta)D_x^{N_k}\langle x+y\rangle^{-\sigma}\xi^{p-1}dy đ\eta \\
= & \sum_{\ell=0}^{p-1}\binom{p-1}{\ell}\vartheta^{p-1-\ell} \\
\times & Os-\iint D_y^{p-1-\ell}e^{-iy\eta}\partial_\xi^{N_k} w_k^{(\alpha\beta)}(x,\xi+\vartheta\eta)(\xi+\vartheta\eta)^\ell D_x^{N_k}\langle x+y\rangle^{-\sigma}dy đ\eta \\
= & \sum_{\ell=0}^{p-1}\binom{p-1}{\ell}(-\vartheta)^{p-1-\ell} \\
\times & Os-\iint e^{-iy\eta}\partial_\xi^{N_k} w_k^{(\alpha\beta)}(x,\xi+\vartheta\eta)(\xi+\vartheta\eta)^\ell D_x^{N_k+p-1-\ell}\langle x+y\rangle^{-\sigma}dy đ\eta.
\end{aligned}
$$

Hence we can estimate the semi-norms of $r_k^{(\alpha\beta)}$ in the following way:

$$
\begin{aligned}
|r_k^{(\alpha\beta)}|_{\ell_0}^{(0)} & \leq \frac{C^{N_k+p-1}}{N_k!}\sum_{\ell=0}^{p-1}|\xi^\ell\partial_\xi^{N_k}w_k^{(\alpha\beta)}|_{\ell_1}^{(0)}|D_x^{N_k+p-1-\ell}\langle x\rangle^{-\sigma}|_{\ell_1}^{(0)} \\
& \leq C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h}N_k!^{2\theta_h-1}\sigma_k^{p-1-N_k}. \qquad (4.46)
\end{aligned}
$$

By using Calderón-Vaillancourt Theorem and the well-posedness in $S_s^\theta(\mathbb{R})$, it can be concluded that

$$
\|r_k^{(\alpha\beta)}(x,D)u_k\|_{L^2(\mathbb{R})}\leq C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h}N_k!^{2\theta_h-1}\sigma_k^{p-1-N_k}. \qquad (4.47)
$$

In $\sum_{\gamma=1}^{N_k-1}\frac{1}{\gamma!}D_x^\gamma\langle x\rangle^{-\sigma}\partial_\xi^\gamma w_k^{(\alpha\beta)}(x,D)D_x^{p-1}$, the formula (4.43) can be applied, that is,

$$
\begin{aligned}
& \sum_{\gamma=1}^{N_k-1}\frac{1}{\gamma!}D_x^\gamma\langle x\rangle^{-\sigma}\partial_\xi^\gamma w_k^{(\alpha\beta)}(x,D)D_x^{p-1} \\
= & \sum_{\gamma=1}^{N_k-1}\sum_{\ell=0}^{p-1}\frac{1}{\gamma!}\binom{p-1}{\ell}\left(\frac{4}{\sigma_k}\right)^\gamma\left(\frac{1}{\sigma_k^{p-1}}\right)^\ell D_x^\gamma\langle x\rangle^{-\sigma}D_x^{p-1-\ell}\circ w_k^{((\alpha+\ell)(\beta+\gamma))}(x,D). \qquad (4.48)
\end{aligned}
$$

Using the support of $D_x^{p-1-\ell}v_k^{((\alpha+\ell)(\beta+\gamma))}$ and the fact that $|D_x^\gamma\langle x\rangle^{-\sigma}|\leq C^{\gamma+1}\gamma!\langle x\rangle^{-\sigma-\gamma}$, it follows that

$$
\|D_x^\gamma\langle x\rangle^{-\sigma}D_x^{p-1-\ell}v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})}\leq C^{\gamma+1}\gamma!\sigma_k^{-(p-1)(\sigma+\gamma)}\|D_x^{p-1-\ell}v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})}.
$$

By Lemma 4.2, with $N=N_k-\gamma$, we get

$$
\begin{aligned}
& \|D_x^{p-1-\ell}v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})}\leq C\sigma_k^{p-1-\ell}\|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})} \\
+ & C^{\alpha+\ell+\beta+N_k+1}(\alpha!\beta!\ell!\gamma!)^{\theta_h}(N_k-\gamma)^{2\theta_h-1}\sigma_k^{p-1-\ell-N_k+\gamma},
\end{aligned}
$$

hence

$$
\begin{aligned}
&\|D_x^\gamma\langle x\rangle^{-\sigma}D_x^{p-1-\ell}v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})}\leq C^{\gamma+1}\gamma!\sigma_k^{-(\sigma+\gamma)}\sigma_k^{p-1-\ell}\|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})}\\
+\;& C^{\gamma+1}\gamma!\sigma_k^{-(p-1)(\sigma+\gamma)}C^{\alpha+\ell+\beta+N_k+1}(\alpha!\beta!\ell!\gamma!)^{\theta_h}(N_k-\gamma)!^{2\theta_h-1}\sigma_k^{p-1-\ell-(N_k-\gamma)}\\
=\;& C^{\gamma+1}\gamma!\sigma_k^{-(p-1)(\sigma+\gamma)+p-1-\ell}\|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})}\\
+\;& C^{\gamma+1}\gamma!\sigma_k^{-(p-1)(\sigma+\gamma)}C^{\alpha+\ell+\beta+N_k}(\alpha!\beta!\ell!\gamma!)^{\theta_h}(N_k-\gamma)!^{2\theta_h-1}\sigma_k^{p-1-\ell-(N_k-\gamma)}.
\end{aligned}
$$

Now we can estimate

$$
\begin{aligned}
&\left\|\sum_{\gamma=1}^{N_k-1}\frac{1}{\gamma!}D_x^\gamma\langle x\rangle^{-\sigma}\partial_\xi^\gamma w_k^{(\alpha\beta)}(x,D)D_x^{p-1}u_k\right\|_{L^2(\mathbb{R})}\\
\leq\;& \sum_{\gamma=1}^{N_k-1}\sum_{\ell=0}^{p-1}\frac{1}{\gamma!}\binom{p-1}{\ell}\left(\frac{4}{\sigma_k}\right)^\gamma\left(\frac{1}{\sigma_k^{p-1}}\right)^\ell\|D_x^\gamma\langle x\rangle^{-\sigma}D_x^{p-1-\ell}\circ\underbrace{w_k^{((\alpha+\ell)(\beta+\gamma))}(x,D)u_k}_{=v_k^{((\alpha+\ell)(\beta+\gamma))}}\|_{L^2(\mathbb{R})}\\
\leq\;& \sum_{\gamma=1}^{N_k-1}\sum_{\ell=0}^{p-1}\frac{1}{\gamma!}\binom{p-1}{\ell}4^\gamma\sigma_k^{-\gamma}\sigma_k^{-\ell(p-1)}\Big\{C^{\gamma+1}\gamma!\sigma_k^{-(p-1)(\sigma+\gamma)+p-1-\ell}\|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})}\\
+\;& C^{\gamma+1}\gamma!\sigma_k^{-(p-1)(\sigma+\gamma)}C^{\alpha+\ell+\beta+N_k}(\alpha!\beta!\ell!\gamma!)^{\theta_h}(N_k-\gamma)!^{2\theta_h-1}\sigma_k^{p-1-\ell-(N_k-\gamma)}\Big\}\\
\leq\;& C\sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1}C^\gamma\sigma_k^{p-1-(p-1)\sigma-p(\ell+\gamma)}\|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})}\\
+\;& C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h}N_k!^{2\theta_h-1}\sigma_k^{1-N_k-(p-1)\sigma}. \qquad (4.49)
\end{aligned}
$$

To estimate the first term of (4.45), we can write

$$
\begin{aligned}
&\sum_{\gamma=1}^{p-1}\binom{p-1}{\gamma}D_x^\gamma w_k^{(\alpha\beta)}(x,D)D_x^{p-1-\gamma}\\
=\;& \sum_{\ell=1}^{p-1}(-1)^{\ell+1}\binom{p-1}{\ell}\frac{1}{(i\sigma_k^{p-1})^\ell}D_x^{p-1-\ell}\circ w_k^{((\alpha+\ell)\beta)}(x,D),
\end{aligned}
$$

and then, by using Lemma 4.2 once again, it follows that

$$
\begin{aligned}
&\left\|i\langle x\rangle^{-\sigma}\sum_{\gamma=1}^{p-1}\binom{p-1}{\gamma}D_x^\gamma w_k^{(\alpha\beta)}(x,D)D_x^{p-1-\gamma}u_k\right\|_{L^2(\mathbb{R})}\\
=\;& \left\|i\langle x\rangle^{-\sigma}\sum_{\ell=1}^{p-1}(-1)^{\ell+1}\binom{p-1}{\ell}\frac{1}{(i\sigma_k^{p-1})^\ell}D_x^{p-1-\ell}\circ w_k^{((\alpha+\ell)\beta)}(x,D)u_k\right\|_{L^2(\mathbb{R})}\\
\leq\;& \tilde{C}\sum_{\ell=1}^{p-1}\sigma_k^{-\sigma-\ell(p-1)}\Big\{C\sigma_k^{p-1-\ell}\|v_k^{((\alpha+\ell)\beta))}\|_{L^2(\mathbb{R})}+C^{\alpha+\ell+\beta+N_k+1}(\alpha!\beta!)^{\theta_h}N_k!^{2\theta_h-1}\sigma_k^{p-1-\ell-N_k}\Big\}\\
\leq\;& C\sum_{\ell=1}^{p-1}\sigma_k^{-(\sigma+p(\ell-1)+1)}\|v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})}+C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h}N_k!^{2\theta_h-1}\sigma_k^{p-1-\sigma-N_k}. \qquad (4.50)
\end{aligned}
$$

From (4.47), (4.49) and (4.50), we get

$$
\begin{aligned}
\left\|[i\langle x\rangle^{-\sigma}D_x^{p-1}, w_k^{(\alpha\beta)}]u_k\right\|_{L^2(\mathbb{R})} &\leq \left\|i\langle x\rangle^{-\sigma}\sum_{\gamma=1}^{p-1}\binom{p-1}{\gamma}D_x^{\gamma}w_k^{(\alpha\beta)}(x,D)D_x^{p-1-\gamma}u_k\right\|_{L^2(\mathbb{R})} \\
&+ \left\|i\sum_{\gamma=1}^{N_k-1}\frac{1}{\gamma!}D_x^{\gamma}\langle x\rangle^{-\sigma}\partial_\xi^{\gamma}w_k^{(\alpha\beta)}(x,D)D_x^{p-1}u_k\right\|_{L^2(\mathbb{R})} \\
&+ \|r_k^{(\alpha\beta)}(x,D)u_k\|_{L^2(\mathbb{R})} \\
&\leq C\sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1}C^{\gamma}\sigma_k^{p-1-(p-1)\sigma-p(\gamma+\ell)}\|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})} \\
&+ C\sum_{\ell=1}^{p-1}\sigma_k^{p(1-\ell)-(\sigma+1)}\|v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})} \\
&+ C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h}N_k!^{2\theta_h-1}\sigma_k^{p-1-N_k}. \qquad (4.51)
\end{aligned}
$$

After all the previous computations, the inequalities (4.44) and (4.51) lead us to the next result, which provides the desired estimate from above to $f_k^{(\alpha\beta)}$.

Lemma 4.4. *Let suppose that the Cauchy problem* (4.15) *is well-posed in* $\mathcal{S}_s^{\theta}(\mathbb{R})$*. Then*

$$
\begin{aligned}
\|f_k^{(\alpha\beta)}\|_{L^2(\mathbb{R})} &\leq C\sum_{\ell=1}^{p}\frac{1}{\sigma_k^{p(\ell-1)}}\|v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})} + C\sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1}C^{\gamma}\sigma_k^{p-1-(p-1)\sigma-p(\ell+\gamma)}\|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})} \\
&+ C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h}N_k!^{2\theta_h-1}\sigma_k^{p-1-N_k},
\end{aligned}
$$

for some positive constant C which does not depend on k, α, β and N_k.

Finally, we are able to conclude the proof of Proposition 4.3.

Proof of Proposition 4.3. Inequality (4.37) can be combined with Lemmas 4.3 and 4.4 in order to obtain that

$$
\begin{aligned}
\partial_t\|v_k^{(\alpha\beta)}\|_{L^2(\mathbb{R})} &\geq -\|f_k^{(\alpha\beta)}\|_{L^2(\mathbb{R})} + \mathbf{Re}\left(\langle x\rangle^{-\sigma}D_x^{p-1}v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right)_{L^2(\mathbb{R})}\|v_k^{(\alpha\beta)}\|_{L^2(\mathbb{R})}^{-1} \\
&\geq c_1\sigma_k^{(p-1)(1-\sigma)}\|v_k^{(\alpha\beta)}\|_{L^2(\mathbb{R})} - C\sum_{\ell=1}^{p}\frac{1}{\sigma_k^{p(\ell-1)}}\|v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})} \\
&- C\sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1}C^{\gamma}\sigma_k^{p-1-(p-1)\sigma-p(\ell+\gamma)}\|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})} \\
&- C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h}N_k!^{2\theta_h-1}\sigma_k^{p-1-N_k}.
\end{aligned}
$$

Then, by definition of $E_k(t)$, it follows that

$$\begin{aligned}
\partial_t E_k(t) &= \sum_{\alpha\le N_k,\beta\le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \partial_t \|v_k^{(\alpha\beta)}(t,\cdot)\|_{L^2(\mathbb{R})} \\
&\ge \sum_{\alpha\le N_k,\beta\le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} c_1 \sigma_k^{(p-1)(1-\sigma)} \|v_k^{(\alpha\beta)}\|_{L^2(\mathbb{R})} \\
&- C\sum_{\ell=1}^{p} \frac{1}{\sigma_k^{p(\ell-1)}} \sum_{\alpha\le N_k,\beta\le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \|v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})} \\
&- C\sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1} C^{\gamma}\sigma_k^{p-1-(p-1)\sigma-p(\ell+\gamma)} \sum_{\alpha\le N_k,\beta\le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})} \\
&- \sum_{\alpha\le N_k,\beta\le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1}\sigma_k^{p-1-N_k}.
\end{aligned}$$

Now our task is to treat all the terms in the right-hand side of the above inequality. Starting with the first one, we simply have

$$\sum_{\alpha\le N_k,\beta\le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} c_1\sigma_k^{(p-1)(1-\sigma)} \|v_k^{(\alpha\beta)}\|_{L^2(\mathbb{R})} = c_1\sigma_k^{(p-1)(1-\sigma)} E_k(t). \tag{4.52}$$

To deal with the second term, just recall that $E_{k,\alpha+\ell,\beta}(t) = (\alpha+\ell)!^{-\theta_1}\beta!^{-\theta_1}\|v_k^{((\alpha+\ell)\beta)}(t)\|_{L^2(\mathbb{R})}$, hence

$$\begin{aligned}
\sum_{\ell=1}^{p} \frac{C}{\sigma_k^{p(\ell-1)}} \sum_{\alpha\le N_k,\beta\le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \|v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})} &= \sum_{\ell=1}^{p} \frac{C}{\sigma_k^{p(\ell-1)}} \sum_{\alpha\le N_k,\beta\le N_k} \frac{(\alpha+\ell)!^{\theta_1}}{\alpha!^{\theta_1}} E_{k,\alpha+\ell,\beta} \\
&\le \sum_{\ell=1}^{p} \frac{CN_k^{\ell\theta_1}}{\sigma_k^{p(\ell-1)}} \left(E_k + \sum_{\alpha=N_k-\ell+1}^{N_k} E_{k,\alpha+\ell,\beta} \right).
\end{aligned}$$

From (4.26) in Proposition 4.2 it holds that

$$E_{k,\alpha+\ell,\beta} \le C^{\alpha+\beta+\ell+1}((\alpha+\ell)!\beta!)^{\theta_h-\theta_1},$$

and, since $\alpha+\ell \ge N_k$ and $\theta_h-\theta_1 \le 0$, we obtain that

$$\sum_{\ell=1}^{p} \frac{C}{\sigma_k^{p(\ell-1)}} \sum_{\alpha\le N_k,\beta\le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \|v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})} \le \sum_{\ell=1}^{p} \frac{CN_k^{\ell\theta_1}}{\sigma_k^{p(\ell-1)}} \left(E_k + C^{N_k+1}N_k!^{\theta_h-\theta_1} \right).$$

The definition $N_k = \lfloor \sigma_k^{\frac{\lambda}{\theta_1}} \rfloor$ and the inequality $N_k^{N_k} \le e^{N_k}N_k!$ allow us to estimate

$$\begin{aligned}
&\sum_{\ell=1}^{p} \frac{C}{\sigma_k^{p(\ell-1)}} \sum_{\alpha\le N_k,\beta\le N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \|v_k^{((\alpha+\ell)\beta)}\|_{L^2(\mathbb{R})} \\
&\le C\sum_{\ell=1}^{p} \frac{(\sigma_k^{\frac{\lambda}{\theta_1}})^{\ell\theta_1}}{\sigma_k^{p(\ell-1)}} \left(E_k + C^{N_k+1}e^{N_k(\theta_h-\theta_1)}\sigma_k^{\frac{\lambda}{\theta_1}(\theta_h-\theta_1)N_k} \right) \\
&\le C\sum_{\ell=1}^{p} \frac{\sigma_k^{\lambda\ell}}{\sigma_k^{p(\ell-1)}} E_k + C^{N_k+1}\sigma_k^{C-cN_k},
\end{aligned} \tag{4.53}$$

where, from now on, c is a positive constant independent from k.

In the third term, we employ the identity $\|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})} = ((\alpha+\ell)!(\beta+\gamma)!)^{\theta_1} E_{k,\alpha+\ell,\beta+\gamma}$, the estimate

$$\frac{(\beta+\gamma)!}{\beta!} \leq (\beta+\gamma)^\gamma \leq (rN_k)^\gamma \leq r^\gamma(\sigma_k^{\frac{\lambda}{\theta_1}})^\gamma, \quad \text{provided that } \beta+\gamma \leq rN_k,\ r\in\mathbb{N},$$

and the fact that, if $\lambda \in (0,1)$, then for k sufficiently large it holds $C\sigma_k^{\lambda-1} < 1$. Notice that

$$\begin{aligned}
&\sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1} C^\gamma \sigma_k^{p-1-(p-1)\sigma-p(\ell+\gamma)} \sum_{\alpha\leq N_k,\beta\leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_h}} \|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})} \\
= &\sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1} C^\gamma \sigma_k^{p-1-(p-1)\sigma-p(\ell+\gamma)} \sum_{\alpha\leq N_k,\beta\leq N_k} \frac{((\alpha+\ell)!(\beta+\gamma)!)^{\theta_1}}{(\alpha!\beta!)^{\theta_1}} E_{k,\alpha+\ell,\beta+\gamma} \\
\leq &\sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1} \sum_{\alpha\leq N_k,\beta\leq N_k} \left\{ \sum_{\substack{\alpha\leq N_k-\ell \\ \beta\leq N_k-\gamma}} + \sum_{\substack{\alpha\leq N_k,\beta\leq N_k \\ \alpha+\ell>N_k \text{ or } \beta+\gamma>N_k}} \right\} (C\sigma_k^{\lambda-1})^\gamma \sigma_k^{p-1-(p-1)\sigma-p(\ell+\gamma)} E_{k,\alpha+\ell,\beta+\gamma} \\
\leq &\ E_k + \sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1} \sum_{\substack{\alpha\leq N_k,\beta\leq N_k \\ \alpha+\ell>N_k \text{ or } \beta+\gamma>N_k}} C^{\alpha+\ell+\beta+\gamma+1}((\alpha+\ell)!(\beta+\gamma)!)^{\theta_h-\theta_1} \\
\leq &\ E_k + C^{N_k+1} N_k!^{\theta_h-\theta_1}.
\end{aligned}$$

Since $N_k = \lfloor \sigma_k^{\frac{\lambda}{\theta_1}} \rfloor$, the above inequality turns into

$$\sum_{\ell=0}^{p-1}\sum_{\gamma=1}^{N_k-1} C^\gamma \sigma_k^{p-1-(p-1)\sigma-p(\ell+\gamma)} \sum_{\alpha\leq N_k,\beta\leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_h}} \|v_k^{((\alpha+\ell)(\beta+\gamma))}\|_{L^2(\mathbb{R})} \leq E_k + C^{N_k+1}\sigma_k^{C-cN_k}, \tag{4.54}$$

for all k sufficiently large, where C and c are positive constants which do not depend on k.

For the fourth and last term, by using definition of $N_k = \lfloor \sigma_k^{\frac{\lambda}{\theta_1}} \rfloor$ and recalling that $\theta_h < \theta_1$, it can be concluded that

$$\sum_{\alpha\leq N_k,\beta\leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} C^{\alpha+\beta+N_k+1}(\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1}\sigma_k^{p-1-N_k} \leq C^{N_k+1}\sigma_k^{C-cN_k}. \tag{4.55}$$

Therefore, gathering (4.52), (4.53), (4.54) and (4.55) the proof is concluded. □

4.5 Open problems and concluding remarks

In this last section we outline some possible improvements which could be done and some open problems related to $p-$evolutions equations in Gevrey and Gelfand-Shilov spaces.

1. First of all, one important extension is the study of the Cauchy problem for a $p-$evolution operator of the form (2) in arbitrary space dimension, that is for $x \in \mathbb{R}^n$, $n > 1$. Essentially, the main difficulty in this case is the definition of the functions λ_{p-k} appearing in the change of variable. The only known results are for $p = 2$, see [24, 32]. In these papers, the functions λ_2, λ_1 are chosen such that they satisfy suitable differential inequalities. The case $p > 2$ is completely unexplored.

2. The class of $p-$evolution equations defined in [39] is wider than the one given by operators of the form (2). Namely, it includes equations of the form

$$D_t^m u + \sum_{j=0}^{pm} \sum_{|\alpha|=j} a_{j\alpha}(t,x) D_x^\alpha u = f(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^n,$$

   for fixed $m \geq 1$. In the thesis we treated the case $m = 1$. For the case $m > 1$, the only known results in H^∞ are for $p = 2$, cf. [11] and the Gevrey case is particularly involved since the techniques used in [11] seem not to work in this setting.

3. Finally, it would be interesting to apply the results obtained here to the study of non-linear $p-$evolution equations such as the ones mentioned in the introduction. This problem is highly non-trivial and probably requires strong decay conditions on the coefficients of the lower order terms in order to avoid the loss of regularity in the energy estimates and to make possible the application of fixed point arguments.

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