UNIVERSIDADE FEDERAL DO PARANÁ

DANILO MICALI FUCCI

PROBING REALITY: EXPLORING REALISM, CLASSICALITY, AND MEASUREMENT

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PROBING REALITY: EXPLORING REALISM, CLASSICALITY, AND MEASUREMENT

Texto apresentado ao Programa de Pós-graduação em Física, do Setor de Ciências Exatas da Universidade Federal do Paraná, como parte dos requisitos necessários para a obtenção do grau de Doutor em Física.

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RESUMO

Este trabalho aborda três questões fundacionais da mecânica quântica: realismo, classicalidade e medição. Utilizando a estrutura de teorias probabilísticas generalizadas (GPT), é proposto um critério de realismo independente de teoria. Fundamentado na premissa de que, em teorias realistas, medições revelam elementos preexistentes da realidade sem perturbar o estado físico, este critério generaliza a noção de realismo de Bilobran-Angelo (BA), ao qual se reduz no contexto da mecânica quântica. Duas medidas de desvio do realismo são introduzidas: a robustez de irrealismo, que quantifica a mistura mínima necessária para que um estado perca suas propriedades de irrealismo, e a divergência de realismo, que mede os desvios entre as probabilidades de um estado e seu correspondente após uma medição não seletiva. Uma análise numérica demonstra a compatibilidade qualitativa entre essas medidas e o quantificador de irrealismo de BA. A partir disso, um critério de classicalidade baseado em realismo também é proposto, fundamentado em teorias simpliciais e contextualizado na mecânica quântica em cenários envolvendo o estado completamente misto. Um modelo simplificado de medições sequenciais não seletivas em pares de observáveis não comutantes é analisado como uma representação de monitoramento ambiental contínuo, demonstrando que um regime operacionalmente clássico pode emergir com apenas dois graus de liberdade no ambiente. Adicionalmente, uma medida de classicalidade baseada em realismo é introduzida, com estudos de caso realizados para qubits e qutrits. Por fim, o problema da medição é explorado dentro de uma estrutura relacional inspirada no cenário do amigo de Wigner. Um arranjo experimental mínimo é proposto, envolvendo um agente, um sistema e um laboratório, juntamente com postulados que enfatizam o papel do agente como um sumidouro de informação. Essa interpretação destaca a estrutura hierárquica dos privilégios informacionais na medição, o descarte de sistemas medidos e as bases conceituais do processo de medição. A segunda lei da termodinâmica é reinterpretada como o aumento de termos de entropia local. Além disso, o paradoxo do amigo de Wigner é reformulado através de um experimento mental modificado, chamado "amigo de Wigner flutuante," que introduz um regime informacional completamente simétrico, onde todas as todos os observadores concordam em suas descrições. O problema da medição é então avaliado sob esta estrutura.

Palavras-chaves: problema da medição. realismo. informação. emaranhamento. amigo de wigner. classicalidade.

ABSTRACT

This work addresses three foundational issues in quantum mechanics: realism, classicality, and measurement. Using the framework of generalized probabilistic theories (GPT), a novel theory-independent criterion for realism is devised. Grounded in the premise that, in realist theories, measurements reveal preexisting elements of reality without disturbing the physical state, this criterion generalizes the Bilobran-Angelo (BA) notion of realism, to which it reduces in the quantum mechanical context. Two measures of deviation from realism are introduced: robustness of irrealism, which quantifies the minimal mixing required for a state to lose its irrealism properties, and the divergence of realism, which measures deviations for the probabilities of a state and its post nonselective measurement counterpart. A numerical analysis demonstrates qualitative compatibility between these measures and BA's irrealism quantifier. Building on this, a realism-based classicality criterion is proposed, rooted in simplicial theories and contextualized in quantum mechanics as scenarios involving the completely mixed state. A toy model of sequential pairwise nonselective measurements on noncommuting observables is analyzed as a simplified representation of continuous environmental monitoring, demonstrating that an operationally classical regime can emerge with only two environmental degrees of freedom. Additionally, a measure of realism-based classicality is introduced, with case studies conducted for qubits and qutrits. Finally, the measurement problem is explored within a relational framework inspired by the Wigner's friend scenario. A minimal experimental setup is proposed, involving an agent, a system, and a laboratory, alongside postulates emphasizing the agent's role as an information sink. This interpretation highlights the hierarchical structure of informational privileges in measurement protocols, the discarding of measured systems, and the conceptual foundation of the measurement process. The second law of thermodynamics is reinterpreted as the increase of local entropy terms. Furthermore, the Wigner's friend paradox is reframed through a modified thought experiment, the floating Wigner's friend, which introduces a completely symmetric informational regimen where all observable descriptions are concordant. The measurement problem is subsequently evaluated under this framework.

Key-words: measurement problem. realism. information. entanglement. wigner's friend. classicality.

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1 INTRODUCTION

Within every body of knowledge lie the seeds for its own sublation. Knowledge arises from questions, and just as a question that interrogates everything is inconceivable, so too is knowledge that resolves everything. Indeed, well-framed questions drive the acquisition of knowledge, and robust knowledge, in turn, fosters new questions. The empirical success of quantum mechanics elevates it to a status of fundamental knowledge. Consequently, the questions it implies are nothing short of fundamental.

Thomas Kuhn's seminal work, *The Structure of Scientific Revolutions* [1], resonates with this idea. According to Kuhn, every scientific discipline relies on the establishment of a paradigm—a set of theories, methods, and assumptions that underpin the normal functioning of science. However, the emergence of a paradigm invariably gives rise to anomalies: questions that the current framework cannot adequately address. During periods of normal science, a field can progress effectively despite the presence of such anomalies. As the field matures, however, these unanswered questions become increasingly pressing. If they are resolved, the paradigm is reinforced. If not, the field enters a period of scientific crisis.

Classical physics, the preceding scientific paradigm, revealed its limitations through anomalies such as the ultraviolet catastrophe [2], the failure to detect the *luminiferous aether* [3], Mercury's anomalous orbital precession [4], and the photoelectric effect [5]. These unresolved problems ushered science into a period of crisis, ultimately culminating in a scientific revolution. The outcomes of this revolution were the development of relativity and quantum mechanics.

Some of the anomalies in quantum mechanics were recognized by the very pioneers of the theory. Einstein initiated the inquiry into the question of realism in his celebrated "EPR paper" [6]. The contrast between quantum mechanics and classical intuition was highlighted by Schrödinger through his famous cat-based thought experiment [7]. The role of measurement and the observer was further explored by Wigner in his thought-provoking paradox [8].

These three foundational issues—realism, classicality, and measurement—are the anomalies that this work seeks to address.

The problem of realism in quantum mechanics questions whether physical properties of a system have well-defined values independently of observation. Einstein, Podolsky, and Rosen (EPR) introduced the concept of "elements of reality" in 1935 [9], arguing that if a physical property can be predicted with certainty without disturbing the system, it must have an element of reality assigned to it. This view, however, was challenged by Bell in 1964 [10], who demonstrated that quantum predictions are incompatible with theories of local hidden variables—a result experimentally verified in numerous loophole-free tests [11–16]. Later developments of the concept of realism, such as those proposed by Fine in Ref. [17], introduced

an approach based on joint probability distributions assigned to measurements of different observables. The probabilistic profile of the joint measurability of non-commuting observables contrasts with this definition, underscoring a fundamental clash between incompatibility and realism.

The problem of classicality, or the quantum-to-classical transition, concerns how classical mechanics emerges as an effective description from the universal framework of quantum mechanics. This transition assumes that quantum mechanics is fundamental and must recover classical physics in certain limits, such as those involving large, isolated systems. Early investigations of this issue include Bohr's emphasis on the role of measurement and complementarity [18], and Ehrenfest's demonstration that the expectation values of quantum observables approximate classical behavior for large objects [19]. More recent advances center on the decoherence program [20–23], where classicality emerges through interactions with an environment, which suppresses quantum interference and encodes system properties redundantly in pointer states. Zurek's quantum Darwinism [24–27] builds on this framework, proposing that objectivity arises when multiple observers independently access environmental fragments and agree on the system's properties.

The measurement problem is one of the most emblematic challenges in the foundations of quantum mechanics. Its symbolic exposition is often attributed to the thought experiment proposed by Wigner in Ref.[8], where conflicting descriptions of physical states arise when measurements are performed within isolated systems. Brukner provides a more structured framing of the problem, dividing it into two key questions [28]: the "small measurement problem", which asks why a specific outcome occurs in a measurement, and the "big measurement problem", which seeks to define what constitutes a measurement. This conundrum has been central to debates on the interpretation of quantum mechanics and has inspired significant contributions from various perspectives, including the Copenhagen interpretation [18], the many-worlds interpretation [29], and the consistent histories approach [30].

These three problems are addressed in sequence within the structure of this work: Chapter 2 lays out the theoretical foundations; Chapter 3 addresses realism; Chapter 4 focuses on classicality; and Chapter 5 examines the measurement problem.

Chapter 2 introduces the foundational formalism of density operator quantum mechanics, classical information theory, and quantum information theory. It is presented in a narrative style to cultivate an operational intuition behind the formalism. Specifically, experimental protocols are framed in terms of three entities: the agent, the system, and the laboratory. These protocols are further structured into three distinct phases: preparation, evolution, and measurement. This operational framework serves as a unifying structure throughout this work and is further developed in detail in Chapter 5.

Chapter 3 draws significant inspiration from the realism criterion introduced by Bilobran and Angelo (BA) [31]. This framework is particularly appealing due to its operational

meaning and its inclusion of a quantifier for the violation of realism. Axiomatized in Ref. [32], the BA criterion has enabled advancements in resource theories [33], the development of novel notions of nonlocality [34–36], and contributions to quantum foundations [37–39], all supported by experimental investigations [40, 41]. A distinguishing feature of this criterion is its capacity to diagnose the presence of elements of reality not solely through the quantum state, but primarily by analyzing how the information encoded in the state is affected by measurements.

Our main focus in this chapter is to establish a theory-independent notion of realism. By generalizing this concept beyond the context of quantum mechanics, we aim to gain deeper insight into the nature of realism itself and explore its relationship with various physical theories. This endeavor is facilitated by the framework of generalized probabilistic theories (GPTs) [42–44], which provide a formalism for characterizing the operational aspects of any physical theory.

There, we propose a theory-independent realism criterion based on the premise that, in realist theories, measurements merely reveal preexisting elements of reality without disturbing the physical state. We demonstrate that this criterion effectively generalizes the BA criterion, recovering it within the framework of quantum mechanics. Additionally, we examine its relationship with Fine's criterion and its implications for the validity of Bayes' rule.

Two measures of deviation from this criterion are introduced. The first leverages the concept of robustness, quantifying the minimal amount of mixing required for a system to lose its irrealism for a physical property. The second utilizes the Kullback-Leibler divergence [45] to measure how the probabilities assigned to measurement outcomes for a given state deviate from those of a state that is realist with respect to a specific physical property. Numerical studies of these quantities are conducted and compared with the BA irrealism quantifier, revealing both qualitative and quantitative similarities.

We then explore the conditions for classicality in a theory-independent context and examine their relationship with our notion of realism. This leads to the proposal of a criterion for realism-based classicality, grounded in the concept of simplicial theories. In the context of quantum mechanics, this criterion corresponds to scenarios where the only available state is the completely mixed state.

Chapter 4 builds from this premise, presenting a mechanism based on sequential nonselective measurements on two noncommuting observables as a simplified model of continuous environmental monitoring. Each measurement represents an entangling interaction with an environmental degree of freedom, followed by the discarding of that component. Repeated cycles of interaction and discard simulate continuous monitoring, and we demonstrate that this process is sufficient for the emergence of a realism-based notion of classicality with only two environmental degrees of freedom.

The results of this chapter are framed within the generalized Bloch sphere formalism [46, 47], which maps density operators onto an Euclidean vector space, similar to the standard treatment of a qubit. This formalism is reviewed, and the BA criterion is reinterpreted through its framework.

We investigate the relationship between irrealism and information, deriving a strict bound for a realism-based notion of classicality. An approximation is developed for sequential measurements involving observables that are only slightly noncommuting, allowing us to analyze the emergence of realism in this context. This framework is applied to states of arbitrary dimension, broadening its applicability.

A measure of realism-based classicality, termed the volume of irrealism, is introduced. This measure facilitates the computation of the average irrealism for any observable given a quantum state, enabling a context-independent quantification of irrealism. Furthermore, a variation of this measure provides a means to assign a classicality measure to specific classes of observables. The results of this chapter are illustrated and validated through analyses of qubits and qutrits.

Chapter 5 focuses on the concept of measurement, the fundamental process underlying all preceding discussions. Given the highly conceptual nature of the measurement problem, this chapter adopts a bottom-to-top approach, beginning with a review of the most fundamental assumptions underlying physics. The Wigner's friend scenario serves as a central theme, guiding the introduction of a relational framework for quantum mechanics

We propose an interpretation in which a minimal experimental setup necessitates the three entities introduced in Chapter 2: the agent, the system, and the laboratory. A series of postulates are presented to support a perspective where all physical descriptions must be made by an agent, who is, in turn, considered a physical system from the perspective of another agent.

A notion is proposed wherein an agent functions as an information sink during the process of information acquisition, leading to a reinterpretation of the second law of thermodynamics. Specifically, this reinterpretation frames the second law as the increase of local entropy within this context.

A distinctive feature of quantum mechanical measurement protocols is postulated: the inevitable discarding of the measured system. This postulate prompts discussions regarding the hierarchical structure of informational privileges among different agents.

The implications of this hierarchy are examined within the traditional Wigner's friend scenario, as well as a modified version proposed by us: the floating Wigner's friend. This new thought experiment introduces a stronger version of the Wigner's friend paradox and offers a resolution by envisioning a scenario of complete symmetry between the agents.

The chapter concludes with an analysis of the measurement problem within the frame-

work developed herein. A third question regarding the measurement problem is introduced, addressing whether it arises intrinsically from the formalism of quantum mechanics or is primarily an interpretational issue.

Before starting, a brief note from the author: I believe that isolating content from form is never fully possible. Even in scientific contexts, the development of new notations has often driven progress. In physics, Feynman diagrams serve as a prime example, while in computer science, many programming languages are isomorphic, yet each has facilitated unique advancements. As Leslie Lamport, the pioneer of LaTeX, aptly stated: “If you’re thinking without writing, you only think you’re thinking.” This work reflects a structured ordering of my thoughts as I explore the subjects at hand. Consequently, each chapter adopts its own tone: Chapter 2 is playful; Chapter 3 is abstract; Chapter 4 is laborious; and Chapter 5 is conceptual. Although this structure may lack the cross-referential capabilities of a more encyclopedic approach, I hope for a trade-off that maximizes the potential for insight. With this spirit in mind, we begin.

2 THEORETICAL FOUNDATIONS

Whenever we learn a physical theory, we usually are exposed to it by one of two different approaches: the historical or the axiomatic. The historical approach has the advantage of reminding us that physics is a human endeavor, made by actual people who lived under particular circumstances, and it sheds light over the contingencies permeating its formalism and conventions. The axiomatic approach is effective at being concise and direct; it also allows us to grasp the most recent development of the theory and to correlate its corpus with different physical theories which can share similarities. In this session, we are going to talk about quantum mechanics, information theory, and quantum information theory yet by another approach. At the expense of rigor, we will try to present those frameworks as they stem from the physical intuition of the experimental setting. Any rigor lost, however, may be retrieved by consulting Refs. [48], [49], and [50], the main sources informing this session.

2.1 QUANTUM MECHANICS

A physical theory is considered successful whenever it can predict correctly, given the uncertainty of the measurement apparatuses, the outcomes of measurements for a particular experiment. Let us break down this process into some of its constituent parts.

There are many ways to conceptualize what an experiment is. One particularly insightful way, we argue, is to understand it as a dynamic process happening between three entities:

1. *The system* is a part of the universe whose properties are being investigated;
2. *The laboratory* is composed by many other systems whose properties are determined *a priori*;
3. *The agent* is that which investigates or determines the properties of physical systems.

2.1.1 Preparation

No experimental process simply occurs *ab nihilo*; it arises from a particular scenario that can be framed as a previous experimental setting.

Consider an experimentalist—here embodying *the agent*—who wants to investigate the behavior of a beam of silver atoms traveling perpendicularly through an inhomogeneous magnetic field. She begins by taking a small piece of pure silver and placing it inside an oven to heat it. The silver is a small metallic sphere, and she knows it is silver because it was labeled in the container from which it was taken. That label reflects the result of a prior experiment that

determined the chemical composition of the object. Thus, the experiment being performed is enabled by the outcomes of previous experiments.

The agent characterizes the system through its preparation—a protocol by which a part of the universe, whose characteristics are already determined to a certain degree, is isolated and then subjected to a particular context in the laboratory. The language used for this characterization is that of a physical theory—in this case, quantum mechanics. The role of the physical theory is to assign the system a mathematical counterpart, encapsulating all the information about the preparation necessary for predicting the relevant outcomes of the experiment.

One way to represent a system is by assigning it a vector in a vector space. This approach is already familiar from classical mechanics, where a system is characterized by a vector in the phase space. In the context of quantum mechanics, however, the vector is not assigned to a phase space but to a Hilbert space—a vector space equipped with an inner product. This vector is typically expressed as a complex column vector, denoted by $|\psi\rangle$.

An alternative formulation of quantum mechanics employs a density operator to represent a system (or, equivalently, to characterize a preparation). Formally, the density operator is a positive semi-definite Hermitian operator with trace one, acting on a Hilbert space. A simple connection between this formulation and the state vector formalism arises by noting that a state described by the state vector $|\psi\rangle$ can be equivalently represented by the density operator $|\psi\rangle\langle\psi|$, usually denoted as ρ, where $\langle\psi|$ is the Hermitian conjugate (transposed complex conjugate) of $|\psi\rangle$.

When an experimental setting is conceptualized as a dynamic process involving the agent—where the agent is responsible for assigning a mathematical representation to the system—it is crucial for our physical theory to provide tools that describe not only the system itself but also the manner in which the agent characterizes it. This is where the density operator formalism surpasses the state vector formalism: it allows the incorporation of the agent's subjective uncertainty regarding the preparation.

Suppose that, in a particular protocol for the preparation of a physical state, one of the steps involves the experimentalist preparing two distinct physical systems, $|\psi\rangle$ and $|\phi\rangle$. Now suppose that, in the next step, she mixes these systems and retrieves only one of them, without knowing which one it is. Using the state vector formalism, the experimentalist must work in parallel with two different scenarios, considering both possibilities: returning $|\psi\rangle$ or $|\phi\rangle$. However, with the density operator formalism, she can incorporate her ignorance about which system was retrieved directly into the description of the state itself, assigning a density operator $\rho = \frac{1}{2}(|\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|)$.

More generally, for a quantum system that could be in states $|\psi_i\rangle$ with probabilities p_i,

the density operator assigned to the set $\{p_i, |\psi_i\rangle\}$ is given by:

$$\rho \coloneqq \sum_i p_i \, |\psi_i\rangle \, \langle\psi_i| \, . \tag{2.1}$$

Whenever the probability distribution is such that any $p_i = 1$, that is, whenever there is no uncertainty about the composition of the system, it is possible to write $\rho = |\Psi\rangle \, \langle\Psi|$, and the state is identified as *pure*. Otherwise, it is *mixed*. For a state of maximum uncertainty, that is, a state composed only of equiprobable states, $\rho = \mathbb{1}/d$, where d is the dimension of the Hilbert space over which ρ acts, and the state is said to be *completely mixed*.

A simple criterion distinguishes pure and mixed states: a state is pure *iff* $\mathrm{Tr}\,(\rho^2) = 1$ and mixed *iff* $\mathrm{Tr}\,(\rho^2) < 1$. Completely mixed states are characterized by $\mathrm{Tr}\,(\rho^2) = 1/d$, where d is the dimension of the Hilbert space.

An apparent trade-off for incorporating the agent's uncertainty into the description of the system is the fact that, for a given mixed state, there are infinitely many different probabilistic distributions of pure states that result in the same density operator ρ. Does this non-uniqueness of the density operator representation imply a loss of information? We shall see that predictions for measurement outcomes depend solely on the density operator. Thus, if two systems yield the same density operator, they are completely indistinguishable by any experimental procedures. Consequently, no information is lost: the density operator fully answers every question it was designed to address.

Now suppose the experimentalist, while following the protocol for preparing the system under investigation, produces two or more distinct quantum systems, $\rho_{\mathcal{A}}, \rho_{\mathcal{B}}, \ldots, \rho_{\mathcal{N}}$. If she wishes to describe these systems as a joint composite system, the procedure inherent to the density operator formalism prescribes that the resulting state is given by $\rho_{\mathcal{AB}\ldots\mathcal{N}} = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \otimes \cdots \otimes \rho_{\mathcal{N}}$, where $\otimes$ denotes the Kronecker product. The resulting density operator acts on the composite Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \cdots \otimes \mathcal{H}_{\mathcal{N}}$.

If, on the other hand, she wishes to discard a subsystem of the joint composite system, or if she is compelled to do so because the subsystem has become inaccessible, the formalism provides her with the appropriate descriptive tool. For a bipartite system composed of two partitions, $\mathcal{A}$ and $\mathcal{B}$, represented as $\rho_{\mathcal{AB}}$, the reduced density operator resulting from discarding partition $\mathcal{B}$ is given by

$$\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho_{\mathcal{AB}}), \tag{2.2}$$

where $\mathrm{Tr}_{\mathcal{B}}$ denotes the partial trace over partition $\mathcal{B}$, defined as

$$\mathrm{Tr}_{\mathcal{B}}(|a_i\rangle \, \langle a_j| \otimes |b_i\rangle \, \langle b_j|) \coloneqq |a_i\rangle \, \langle a_j| \, \langle b_j|b_i\rangle \, , \tag{2.3}$$

with $|a_i\rangle$, $|a_j\rangle$ and $|b_i\rangle$, $|b_j\rangle$ being vectors from partitions $\mathcal{A}$ and $\mathcal{B}$, respectively. The definition is analogous for the case where partition $\mathcal{A}$ is "traced out," and the extension to multipartite systems is straightforward.

In our original scenario, removing a piece of silver from the shelf and placing it inside the oven can be metaphorically understood as "tracing out" the room from the composite system composed of the silver and its environment. More generally, at the end of an experiment, defining a system for the subsequent protocol involves effectively tracing out all other systems, thereby eliminating any correlations that may have been established with the system of interest.

2.1.2 Evolution

With the system prepared and its corresponding density operator assigned, the experimentalist proceeds with the experiment. She activates the magnets and heats the oven, vaporizing the small piece of silver. Through a precisely aligned aperture, the resulting atomic beam is emitted and propagates through the magnetic field.

Initially, her task was to follow a well-defined protocol to isolate a part of the universe, whether by combining, discarding, mixing, or purifying subsystems, until the system of interest was properly prepared. Now, to investigate its properties, she must subject the system to various controlled conditions and observe its responses. After all, the attributes of a system only acquire meaning in relation to something else. The laboratory, with its controlled and well-defined environment, provides the ideal setting for such investigations. Understanding the system ultimately means understanding how it interacts and behaves within the laboratory context.

The way a system evolves over time under specific circumstances—its motion influenced by particular factors—is the focus of what is referred to as mechanics. In classical mechanics, several formalisms describe the motion of physical systems, most notably Newtonian, Lagrangian, and Hamiltonian mechanics. In quantum mechanics, an analogous concept is employed: an operator called the Hamiltonian, denoted by H. Similar to its classical counterpart (in cases where the constraints are time-invariant and the potential energy is independent of velocity), the Hamiltonian represents the total energy of the system:

$$H = T + V. \tag{2.4}$$

The Hamiltonian serves as the generator of time evolution and consists of two components. The term T, representing the kinetic energy, encodes the system's evolution relative to a chosen spatial basis. The term V, representing the potential energy, describes the system's interaction with other systems.

The specific way in which the Hamiltonian generates time evolution depends on a fundamental choice within the formalism of quantum mechanics. We can either describe the state as evolving in time or consider the quantities we measure to evolve in time. Each choice defines a "representation": the former corresponds to Schrödinger's representation, while the latter corresponds to Heisenberg's representation. Both are mathematically equivalent.

In this chapter, we adopt the Schrödinger representation and postulate that a quantum state, represented by a state vector, evolves in time according to the Hamiltonian as described by the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle, \tag{2.5}$$

where $\hbar$ is the reduced Planck constant.

To see how this principle fits under the density operator formalism, let us consider a pure state $\rho = \left|\psi\right\rangle\left\langle\psi\right|$. Taking the time derivative and applying the product rule, we obtain:

$$\frac{\partial\rho}{\partial t} = \frac{\partial\left|\psi\right\rangle}{\partial t}\left\langle\psi\right| + \left|\psi\right\rangle\frac{\partial\left\langle\psi\right|}{\partial t}. \tag{2.6}$$

Using Eq. (2.5) and noting that the time derivative of $\left\langle\psi\right|$, being the complex conjugate, introduces a minus sign, we find:

$$i\hbar\frac{\partial\rho}{\partial t} = [H,\rho]. \tag{2.7}$$

The term $[H,\rho]$ is called the commutator of H and ρ, which is defined as

$$[A,B] \coloneqq AB - BA \tag{2.8}$$

where A and B are arbitrary operators. The identity in Eq. (2.7) is known as the Liouville-von Neumann equation.

Since the commutator is the quantum counterpart for the Poisson brackets in classical mechanics, it is no coincidence that the Liouville-von Neumann equation is so named. In classical mechanics, for a system with coordinates q and conjugate momenta p, the phase space distribution $\rho(p,q)$ represents the density of microstates per unit volume in the phase space. Liouville's theorem asserts that

$$\frac{\partial\rho}{\partial t} = -\{\rho,H\}, \tag{2.9}$$

where $\{\rho,H\}$ is the Poisson bracket with the Hamiltonian H. This theorem ensures that the phase space volume of a classical system is conserved under Hamiltonian evolution.

In quantum mechanics, the density operator serves as the analogue of the classical phase space distribution. Because the density operator encapsulates all probabilities associated with measurements, and these probabilities must remain conserved (summing to one by definition), the evolution of a closed quantum system must preserve the trace and other defining properties of the density operator. This is achieved through unitary evolution, which is a hallmark of quantum mechanics and ensures the conservation of all information about the quantum state.

More formally, an operator U is said to be unitary *iff* $U^\dagger U = UU^\dagger = 1$ where $U^\dagger$ is the transpose complex conjugate of U. For a time-independent Hamiltonian, the solution of Eq. (2.7) is given by

$$\rho(t) = e^{-\frac{i}{\hbar}H^\dagger t}\rho(0)e^{\frac{i}{\hbar}Ht}. \tag{2.10}$$

and $e^{-\frac{i}{\hbar}Ht}$ is referred to as the time evolution operator. Since this evolution must be unitary, the process is expected to be reversible. Rewriting the initial state as

$$\rho(0) = e^{-\frac{i}{\hbar}H'^{\dagger}t}\rho(t)e^{\frac{i}{\hbar}H't} \tag{2.11}$$

and comparing this expression with the forward evolution equation, we find that reversibility requires $H' = -H$. Additionally, the unitarity of $e^{-\frac{i}{\hbar}Ht}$ implies that H must be Hermitian, since Hermiticity ensures the unitarity of the exponential operator.

In the previous discussion, the laboratory was traced out of the system, leaving the system characterized solely by its density operator. However, as the system evolves in time, it interacts with the laboratory, with the interaction governed by the Hamiltonian. Despite this interaction, the system was referred to as "closed", a condition necessary for the unitarity of its evolution. This raises a fundamental question: can the system truly be considered closed if it is influenced by an external entity? To clarify, a system is defined as "closed" if it is not "open". An open system, in turn, is one that interacts with other systems whose properties are not fully known, such that their interactions cannot be accurately incorporated into the system's Hamiltonian.

This discussion naturally leads to the introduction of another entity in the dynamics: *the environment*. The environment consists of many systems, but unlike the laboratory, the properties of these systems are not known to the agent. When the effects of the environment on the system are of such a nature, they are collectively referred to as noise. As a system becomes increasingly subjected to noise, its state transitions from being pure to mixed. This process can be expressed as follows:

$$\rho(0) = |\psi_i\rangle\langle\psi_i| \rightarrow \rho(t) = \sum_i p_i |\psi_i\rangle\langle\psi_i| . \tag{2.12}$$

where $\rho(0)$ represents the initial pure state, and $\rho(t)$ represents the mixed state resulting from the influence of noise. In essence, noise introduces randomness into the system. For the agent, it transforms what is known into what is unknown.

To eliminate this uncertainty, suppose an idealized scenario in which the environment is fully characterized and represented by a pure density operator. By redefining the system of interest to include both the system and the environment, the joint state evolves under a fully determined Hamiltonian controlled entirely by the laboratory settings, resulting in unitary evolution. Mathematically, let $|\psi\rangle$ denote the vector state of the system and $|\epsilon\rangle$ the vector state of the fully determined environment. The joint state is then represented by $\rho = |\Psi\rangle\langle\Psi|$, where $|\Psi\rangle = |\psi\rangle \otimes |\epsilon\rangle$. The evolution of this state is expressed as:

$$\rho(0) = |\Psi\rangle\langle\Psi| \rightarrow \rho(t) = e^{-\frac{i}{\hbar}Ht} |\Psi\rangle\langle\Psi| e^{\frac{i}{\hbar}Ht}. \tag{2.13}$$

If the initial state is pure, $\rho(0) = |\Psi\rangle\langle\Psi|$, with $|\Psi\rangle = |\psi\rangle \otimes |\epsilon\rangle$, the state remains pure throughout the evolution. However, if we trace out the environmental degrees of freedom, $\mathcal{E}$,

from $\rho(t)$, the reduced density matrix of the system is generally mixed:

$$\rho_S(t) = \mathrm{Tr}_{\mathcal{E}}[\rho(t)] = \mathrm{Tr}_{\mathcal{E}}\left[e^{-\frac{i}{\hbar}Ht}\,|\Psi\rangle\,\langle\Psi|\,e^{\frac{i}{\hbar}Ht}\right]. \tag{2.14}$$

This demonstrates that, while the joint system-environment evolution is unitary, simply disregarding (tracing out) part of the composite system results in a mixed state for the subsystem. As will be discussed, this phenomenon of loss of coherence is a direct consequence of the entanglement between the system and the environment.

To contrast this with a classical scenario, consider two deterministic systems evolving jointly under a well-defined classical Hamiltonian. If one of the systems is ignored, the remaining system does not become "mixed" in the quantum sense but retains its deterministic evolution. Lack of ignorance about the joint system implies lack of ignorance about each subsystem.

This key distinction highlights the unique feature of quantum mechanics known as the purification property: any mixed quantum state can always be represented as a reduced state of a pure state in a larger system. This result is formalized by the purification theorem, or Schrödinger–HJW theorem, which ensures that a mixed state ρ_S can always be extended to a pure state $\rho = |\Psi\rangle\,\langle\Psi|$ in a higher-dimensional Hilbert space [51].

By preparing the system, a density operator was assigned to it. Throughout its evolution in the experimental setup, previous knowledge about the laboratory allowed the description of a logically reversible process by means of a Hamiltonian. This last process can be entirely carried out mentally by the agent: the parameters are set and the system evolves accordingly. For the last step, a description cannot be given irrespectively of the agent's action.

This is the one that interests us the most: when information acquisition ensues.

2.1.3 Measurement

The experimentalist cannot have access to magnetic momenta degrees of freedom of each particle directly, but only to infer them indirectly from phenomena she can apprehend with her senses.

Different magnetic momenta prescribe different trajectories along the field, and these trajectories lead each atom to a particular location. Previous knowledge of these causal relations allows the knowledge of the position of each atom to convey information about their magnetic momenta. Because of that, the laboratory is equipped with a series of detectors aligned in parallel with the magnetic field such that they can provide the information of how much each atom was deflected. Each sensor, when it interacts with the atom, produces a signal that is amplified and converted into some kind of signal that will be displayed on a screen. The agent then reads the screen and knows the position of the atoms.

Ultimately, the experimentalist's sensory experience delineates her bound for information acquisition. That which lies beyond her senses can only be accessed via a translation

into elements of her phenomenology. That mediation is done by the measurement apparatus, and its language is that of physical properties, or observables. She knows the position and magnetic momentum of an atom by experiencing a number on the display.

A density operator describes her state of knowledge about a physical system, that is formulated in terms of its physical properties, which are then dependent on what measurement apparatuses can determine. The density operator thus conveys the probabilities assigned to that which the measurement apparatus measures.

To determine these probabilities, we first construct the mathematical representation of observables. In the state vector formalism, observables are represented by self-adjoint operators, where the eigenvalues correspond to the possible outcomes of a measurement, and the associated eigenvectors represent the vector states assigned to each outcome. Mathematically, this is expressed as:

$$A\left|\psi_a\right\rangle = a\left|\psi_a\right\rangle, \tag{2.15}$$

where A represents the physical quantity being measured. According to the Born rule, for an arbitrary vector state $|\psi\rangle$, the probability of obtaining the outcome a in a measurement of A is given by:

$$p(a) = |\left\langle\psi_a|\psi\right\rangle|^2. \tag{2.16}$$

Since probabilities must sum to unity, the eigenvectors satisfy the completeness relation:

$$\sum_a \left|\psi_a\right\rangle\left\langle\psi_a\right| = 1. \tag{2.17}$$

Using this identity, we can multiply both sides of Eq. (2.15) by $\langle\psi_a|$ and sum over all a, yielding an alternative expression for the observable A:

$$A = \sum_a aA_a, \tag{2.18}$$

where $A_a = |\psi_a\rangle\langle\psi_a|$ is the projection operator associated with the outcome a. For measurements where the outcomes are mutually exclusive, the eigenvectors satisfy the orthogonality condition $\langle\psi_a|\psi_{a'}\rangle = \delta_{aa'}$, which characterizes a projective measurement. When considering an ensemble of many identical systems, one can compute the average value of the possible outcomes for a given observable, its expectation value:

$$\left\langle A\right\rangle_\psi = \left\langle\psi\right| A\left|\psi\right\rangle. \tag{2.19}$$

In density operator formalism, Born rule becomes

$$p(a) = \mathrm{Tr}\left(\rho A_a\right), \tag{2.20}$$

and the expectation value is given by

$$\left\langle A\right\rangle_\rho = \mathrm{Tr}\left(\rho A\right). \tag{2.21}$$

.

This allows a way of viewing quantum mechanics as a generalization of probability theory that replaces classical probabilities $\{p_a\}$ with complex amplitudes $\{\psi_a\}$, subject to the normalization condition $\sum_a |\langle\psi_a|\psi\rangle|^2 = \sum_a |\psi_a|^2 = 1$. In classical probability, the normalization $\sum_a p_a = 1$ is preserved by stochastic matrices, which are non-negative matrices with columns summing to 1. In quantum mechanics, analogous transformations are governed by unitary matrices U, which preserve the quadratic norm $\sum_a |\psi_a|^2$, a property intrinsic to the Hilbert space structure of quantum theory. To appreciate this difference, let us proceed with the narrative.

The silver atom contains 47 electrons, with 46 forming a symmetric electron cloud. The magnetic moment of the atom arises primarily from the spin of the 47th electron. Classically, one might expect the magnetic moment to align in any arbitrary direction, leading to the deflection of the atomic beam in such a way that the detection pattern on the detector wall forms a continuous normal distribution. However, experimental results reveal a starkly different behavior: the atoms strike the detector in only two discrete spots, as if the spin of the 47th electron can only orient itself parallel or antiparallel to the applied magnetic field.

This observation, along with considerations of the magnitudes of relevant physical quantities—such as the distance traveled by the atoms, the degree of their deflection, and the strength of the magnetic field—motivates the construction of the operator S_z, representing the spin component in the z-direction (aligned with the magnetic field). Mathematically, it is expressed as:

$$S_z = \frac{\hbar}{2}\left(|S_z;+\rangle\langle S_z;+| - |S_z;-\rangle\langle S_z;-|\right). \tag{2.22}$$

Suppose the experimentalist selects only the atoms that are deflected upward and then subjects them to another magnetic field, this time oriented along the x-axis. One way to conceptualize this scenario is that, by selecting atoms with a specific property, she is effectively preparing a new quantum state described by a density operator, as if a new experiment is beginning. This preparation protocol is made possible because now she knows that every selected atom has a well-defined spin state. It was measured.

When the selected atoms, described by $\rho = |S_z;+\rangle\langle S_z;+|$, pass through a magnetic field oriented along the x-axis, the beam splits again into two distinct components, deflected fully to the left or right. Similar to S_z, the spin operator along the x-axis, S_x, is defined as:

$$S_x = \frac{\hbar}{2}\left(|S_x;+\rangle\langle S_x;+| - |S_x;-\rangle\langle S_x;-|\right). \tag{2.23}$$

The experimentalist then selects only the atoms deflected to the right, corresponding to the state $|S_x;+\rangle$. Intuitively, one might expect that passing these atoms through a second magnetic field aligned along the z-axis would result in a single beam deflected upward, since the atoms were initially prepared in the state $|S_z;+\rangle$. Under the assumption that the magnetic fields alone do not alter the spin orientation, the property of being in $|S_z;+\rangle$ should remain

unchanged. However, when the atoms pass through the z-aligned field, the beam splits into two components, deflected both upward and downward. Why does this occur?

To minimize the effects of environmental noise and ensure reliable results, the experiment requires a well-controlled laboratory setup. Even under ideal conditions, the results persist: measuring S_x appears to erase any information about S_z. Selecting atoms deflected downward in the initial S_z measurement confirms this observation. After selecting atoms based on an S_x outcome, it becomes impossible to infer the prior S_z measurement. Thus, measuring S_x is an irreversible process.

Through systematic investigation, the experimentalist concludes that preparing the state $|S_z;+\rangle$ leads to S_x outcomes consistent with:

$$|S_z;+\rangle = \frac{1}{\sqrt{2}}(|S_x;+\rangle + |S_x;-\rangle). \tag{2.24}$$

This demonstrates that a pure state representing a definite outcome in one basis can be expressed as a sum of eigenstates in another basis. Such a representation is what defines superposition.

By considering that the outcomes of the same observable are mutually exclusive, implying orthogonality, the experimentalist constructs the S_z and S_x operators and relates them by defining a unitary matrix that transforms vectors from one basis to another. This approach provides matrix representations for both observables in a chosen basis, such as S_z. This mathematical framework highlights another fundamental aspect of quantum mechanics, closely tied to superposition: incompatibility.

From a practical perspective, she observes that the order of measurements is crucial. Since measuring S_x erases information about the state with respect to S_z, it becomes clear that to obtain a definite outcome for S_z, this observable must be measured last. Thus, the sequence of measurements directly impacts the results.

In the formalism of quantum mechanics, the product of S_z and S_x differs from its reverse, as expressed using the commutator:

$$[S_z, S_x] \neq 0. \tag{2.25}$$

Two observables A and B are incompatible *iff* their commutator is nonzero. This result applies to any two observables that do not share a common set of eigenvectors. In such cases, once a measurement is performed, the post-measurement state vector in the basis of the incompatible observable is in a superposition state. When a state described by a superposition is measured, the outcome is inherently probabilistic.

This is captured by the Heisenberg uncertainty principle, expressed as:

$$\Delta(A)\Delta(B) \geq \frac{|\langle\psi|[A,B]|\psi\rangle|}{2}. \tag{2.26}$$

This principle states that for an ensemble of identical systems described by $|\psi\rangle$, if measurements of A are performed on some systems and B on others, the standard deviations $\Delta(A)$ and $\Delta(B)$ must satisfy the inequality (2.26).

The mathematical representation of measurement in quantum mechanics is given by the following postulate: let $\{M_m\}$ be a collection of measurement operators, where m labels the possible outcomes, and these operators satisfy the completeness relation $\sum_m M_m^\dagger M_m = \mathbb{1}$. If, immediately before the measurement, the system is described by the state vector $|\psi\rangle$, then for a measurement where the outcome m occurs, the state of the system immediately after the measurement is:

$$\frac{M_m |\psi\rangle}{\sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle}}. \tag{2.27}$$

In the density operator formalism,

Measurement postulate. *Given a set of measurement operators $\{M_m\}$, if ρ describes the state of a quantum system immediately before a measurement, then upon obtaining an outcome m, the state of the system becomes*

$$\frac{M_m \rho M_m^\dagger}{\mathrm{Tr}\,(M_m^\dagger M_m \rho)}. \tag{2.28}$$

Before proceeding, let us briefly revisit the double-slit experiment to illustrate some consequences of the measurement postulate. In this setup, the experimentalist directs a field-emission electron gun at a barrier with two slits and places a screen with detectors behind it. As the electrons are emitted, their initially localized wavefunction spreads due to the uncertainty in their momentum. Upon encountering the barrier, the wavefunction is spatially divided by the slits. At this stage, the state of the electrons is described by:

$$\rho = |\psi\rangle\langle\psi|, \quad \text{where} \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle), \tag{2.29}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ represent the state vectors corresponding to the electron passing through slit 1 and slit 2, respectively.

The probability distribution for the electrons' positions on the detection screen is determined from the density operator via:

$$p(x) = \mathrm{Tr}\,(\rho M_x), \quad M_x = |x\rangle\langle x|, \tag{2.30}$$

where M_x are the measurement operators associated with detecting the particle at position x on the screen. This distribution produces the characteristic interference pattern observed on the screen. The interference arises due to the off-diagonal (coherence) terms in the density operator:

$$\rho = \frac{1}{2}\left(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_1|\right), \tag{2.31}$$

which encode the phase information responsible for the interference fringes.

To further probe the system, the experimentalist places electromagnetic sensors near the slits to detect the passage of each electron. These sensors perform a measurement that determines which slit the electron traverses. The measurement operators corresponding to this process are:

$$M_1 = |\psi_1\rangle \langle\psi_1|, \quad M_2 = |\psi_2\rangle \langle\psi_2|. \tag{2.32}$$

After the measurement, the state of the system is updated according to the measurement postulate. The new density operator becomes:

$$\rho' = \sum_{i=1,2} M_i \rho M_i^\dagger = \frac{1}{2} \left(|\psi_1\rangle \langle\psi_1| + |\psi_2\rangle \langle\psi_2| \right). \tag{2.33}$$

In this post-measurement state, the off-diagonal terms in ρ vanish, indicating the loss of coherence between $|\psi_1\rangle$ and $|\psi_2\rangle$. Consequently, the interference pattern disappears. The resulting probability distribution on the screen is now:

$$p(x) = \mathrm{Tr}\ (\rho' M_x) = \frac{1}{2} \left(|\psi_1(x)|^2 + |\psi_2(x)|^2 \right), \tag{2.34}$$

which corresponds to a classical mixture of probabilities. This reflects the destruction of quantum coherence due to the measurement, a key feature of the quantum-to-classical transition induced by decoherence.

As mentioned, the fundamental aspect of a measurement process is acquisition of information. To explore this concept, we proceed by introducing key principles of information theory.

2.2 CLASSICAL INFORMATION THEORY

The characterization of the agent has received little attention thus far. While addressing this is among our goals, it is prudent to begin with well-defined physical processes, avoiding initial engagement with concepts lacking clear mathematical formulation, such as subjectivity.

In the measurement process, the agent acquires information by observing the outcome displayed on the apparatus. But what do we mean by information? Is it tied to *truth*? Does it exist independently, or does it require a counterpart that it describes? Is information an ontological entity, or is it merely an abstraction to facilitate calculations? The concept of information, like any other, admits various definitions depending on its context. Here, we adopt a pragmatic approach, defining information as that which resolves uncertainty.

2.2.1 Information entropy

This definition enables a quantitative perspective: resolving a highly uncertain situation conveys more information than resolving one with low uncertainty. Conversely, no information is gained from what is already certain. Thus, information is intrinsically linked to

uncertainty, an idea well-formulated within probability theory. Building on this foundation, information theory provides a rigorous framework to quantify uncertainty through *information entropy*.

Claude Shannon laid the groundwork for classical information theory in his seminal 1948 paper, *A Mathematical Theory of Communication* [52] (later renamed *The Mathematical Theory of Communication* in recognition of its generality). In this work, Shannon introduced the concept of information entropy, establishing the basis for the quantitative study of information.

A random variable X is defined by a set $\{x_i, p_i\}$, where x_i are the possible outcomes and p_i are the corresponding probabilities. When measuring X, the outcome x_i occurs with probability p_i. Information entropy quantifies the average amount of information gained when the value of X is observed. Alternatively, it measures the uncertainty of X prior to observation.

The mathematical form of information entropy can be derived by considering a function H of a random variable X that satisfies the following conditions:

1. $H(X)$ depends only on the probabilities $\{p_i\}$, so $H(X) = H(\{p_i\})$;
2. H is a smooth function;
3. For independent events with probabilities p_i and p_j, $H(p_i p_j) = H(p_i) + H(p_j)$.

A function satisfying these conditions is $H(X) = -k \sum_i p_i \log p_i$, where k is a positive constant. This expression defines the information entropy, up to a scaling factor.

Building on the previous developments, we can derive the same function by considering a mathematical formulation motivated by an experimental context. This approach aligns closely with Shannon's original work in [52]. In his seminal paper, Shannon addressed two fundamental questions:

1. How much can a message be compressed without losing information? Equivalently, how can we quantify the redundancy in a message?
2. At what rate can information be transmitted over a noisy channel without loss? In other words, how much redundancy must be added to protect the message's information during communication through a noisy channel?

Here, we focus on the first question, so we need a representation of the concept of a message. A message can be modeled as a string of n letters drawn from an alphabet $\{a_1, a_2, \ldots, a_k\}$. Assuming that the occurrence of each letter is statistically independent of the others, we can assign probabilities $p(a_j)$ to each letter such that:

$$\sum_{j=1}^{k} p(a_j) = 1. \tag{2.35}$$

Consider a message written in a binary alphabet, $\{a_1, a_2\}$, where the probabilities of occurrence are $p(a_1) = p$ and $p(a_2) = 1-p$. For a long message of length $n \gg 1$, the occurrences of a_1 and a_2 are approximately pn and $(1-p)n$, respectively. The number of distinct strings with this composition corresponds to the permutations of these occurrences, given by $\binom{n}{np}$. Using the Stirling approximation, $\log n! \approx n \log n - n$, we compute:

$$\begin{aligned} \log \binom{n}{np} &= \log \frac{n!}{(np)![n(1-p)]!} \\ &\approx n \log n - n - \big[np \log np - np + n(1-p) \log n(1-p) - n(1-p)\big]. \end{aligned} \tag{2.36}$$

Simplifying, we find:

$$\log \binom{n}{np} \approx nH(p), \tag{2.37}$$

where $H(p) \coloneqq -p \log p - (1-p) \log(1-p)$ is the binary entropy function. Taking log to base 2, the number of possible strings is $2^{nH(p)}$, establishing a direct relation between the binary alphabet and bits.

If we compose a code block containing all the typical strings, the code block will have $2^{nH(p)}$ entries. Each entry can be identified by a tag, which is a string of length $nH(p)$. By employing this approach, instead of transmitting the full message of length n, we may reference the message in the code block by its corresponding tag.

Since $0 \leq H(p) \leq 1$, with $H(p) = 1$ only when all letters occur with equal frequency (*i.e.*, $p = 1/2$ for the binary alphabet), the tag will generally be shorter than the original message.

Using this strategy, storing the tag instead of the full message reduces the required physical resources for memory, effectively compressing the message without any loss of information. Shannon demonstrated that any further attempt at compression beyond this limit would result in an irretrievable loss of information.

It is straightforward to generalize the binary alphabet to an alphabet with k symbols. For an alphabet where the probability assigned to a symbol x is $p_x \coloneqq p(x)$, a very long message of length n will have each symbol occurring approximately np_x times. The number of possible strings is given by:

$$\frac{n!}{\prod_x (np_x)!} \approx 2^{nH(X)}, \tag{2.38}$$

where

$$H(X) \coloneqq -\sum_x p_x \log p_x \tag{2.39}$$

is the information entropy of the ensemble $\{x, p_x\}$.

This result reveals that the amount of information required to encode a message from an arbitrary alphabet is $nH(X)$, which provides a lower bound for any compression protocol. In this way, we address the first question posed by Shannon regarding the limits of data compression.

To make this concept explicit, consider an example from [48]. Imagine an information source X emitting symbols α, β, γ, and δ with probabilities assigned to their occurrence given by $1/2$, $1/4$, $1/8$, and $1/8$, respectively. Instead of allocating the same number of bits to represent each symbol, we can assign shorter bit strings to the more frequent symbols. For example:

$$\alpha \mapsto 0, \quad \beta \mapsto 10, \quad \gamma \mapsto 110, \quad \delta \mapsto 111.$$

With this encoding, the average length of a compressed symbol is

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 = \frac{7}{4}. \tag{2.40}$$

Remarkably, this matches the Shannon entropy for the source:

$$\begin{aligned} H(X) &= -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{8}\log\frac{1}{8} - \frac{1}{8}\log\frac{1}{8} \\ &= \frac{7}{4}. \end{aligned} \tag{2.41}$$

Thus, the average length of a message composed of n symbols emitted by this source can be stored in $nH(X)$ bits. Any further attempt to compress the message would result in an irreversible loss of information.

Let us illustrate why an encoding like $\alpha \mapsto 0$, $\beta \mapsto 1$, $\gamma \mapsto 10$, and $\delta \mapsto 11$ would not yield better compression. Consider the average length of a compressed symbol under this scheme. The calculation gives

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{8} \cdot 2 + \frac{1}{8} \cdot 2 = \frac{5}{4}. \tag{2.42}$$

While this average length is smaller than $\frac{7}{4}$, the encoding fails to uniquely identify each symbol in all cases, as it does not satisfy the prefix property (i.e., no code is a prefix of another). For instance, the string "101" could ambiguously decode to either $\beta\alpha\beta$ or $\gamma\beta$. This ambiguity makes the encoding nonviable, highlighting the necessity of prefix-free codes, such as those used in Huffman [53] coding, to achieve optimal compression without loss of information.

To avoid issues of mathematical indefiniteness in cases where $p_i = 0$, we define $0 \log 0 \equiv 0$. From this point onward, we will use the natural logarithm (base e) in the definition of $H(X)$. It is straightforward to verify that $H(X)$ is non-negative and that, for d possible outcomes of the random variable X, the entropy $H(X)$ achieves its maximum value, $\log d$, when all outcomes are equiprobable.

2.2.2 Quantities and properties

Based on Shannon entropy we can define additional entropic measures, analogous to how joint probability, $p(a, b)$, and conditional probability, $p(a|b)$, are defined in probability theory. These measures will be extensively used throughout this work, and further details can be found in [48].

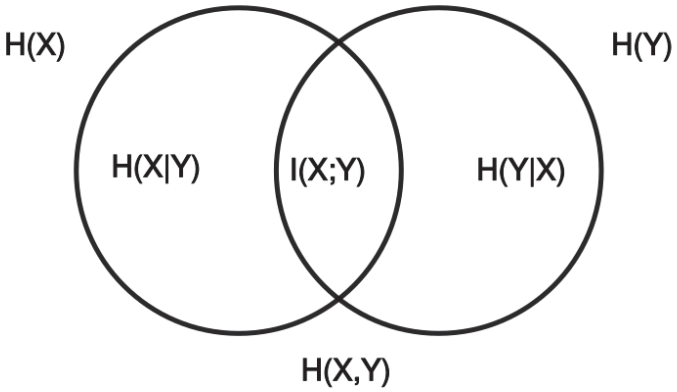

Figure 1 – Measures of entropy and their relationships illustrated in a Venn diagram. Original figure adapted from [54].

- *Joint entropy*: Given two random variables X and Y, the total uncertainty associated with them is defined as:

$$H(X, Y) := -\sum_{x,y} p(x, y) \log p(x, y). \tag{2.43}$$

  The generalization of joint entropy to an arbitrary set of random variables is straightforward.

- *Conditional entropy*: The remaining uncertainty about X after knowing the outcome of Y is given by:

$$H(X|Y) := H(X, Y) - H(Y), \tag{2.44}$$

  which represents the entropy of X conditioned on the knowledge of Y.

- *Mutual information*: The total information shared between X and Y is defined as:

$$I(X; Y) := H(X) + H(Y) - H(X, Y). \tag{2.45}$$

  Since the sum of $H(X)$ and $H(Y)$ includes their mutual information twice, we subtract their joint entropy to obtain $I(X; Y)$. Substituting (2.44) into (2.45), we find:

$$I(X; Y) = H(X) - H(X|Y). \tag{2.46}$$

Fig. 1 provides a visual representation to facilitate a clearer understanding of how these quantities are constructed. This understanding is further enriched by considering their properties, such as those listed and explained below.

1. $H(X, Y)$ and $I(X; Y)$ are symmetric, meaning their values remain unchanged if X and Y are permuted. For the joint entropy, this symmetry reflects the fact that the total uncertainty about two random variables does not depend on their order. For mutual

information, the symmetry indicates that the amount of information gained about X from knowing Y is equal to the amount gained about Y from knowing X.

2. $H(X,Y) \geq H(X)$, with equality holding *iff* $Y = f(X)$, i.e., when Y is a deterministic function of X. This inequality implies that the uncertainty about a pair of variables cannot be smaller than the uncertainty about either variable individually. When Y is fully determined by X, there is no additional uncertainty contributed by Y.

3. $H(Y|X) \geq 0$, which implies $I(X;Y) \leq H(Y)$. Equality holds *iff* $Y = f(X)$. This property indicates that the uncertainty about Y given X cannot be negative, meaning $I(X;Y)$, the mutual information shared between X and Y, is limited by the total uncertainty of Y. When Y is a deterministic function of X, $H(Y|X) = 0$, and the mutual information $I(X;Y)$ reaches its maximum value of $H(Y)$.

4. $H(Y|X) \leq H(Y)$, leading to $I(X;Y) \geq 0$. Equality holds *iff* X and Y are independent. This property shows that knowing X can only decrease or maintain the uncertainty about Y, never increase it. When X and Y are independent, knowing X provides no new information about Y, making the conditional entropy equal to the original entropy.

5. $H(X|Y,Z) \geq H(X|Y)$, meaning that the conditional entropy of X given both Y and Z cannot be smaller than the conditional entropy of X given Y alone. This reflects the intuitive idea that adding more conditioning variables cannot reduce the uncertainty beyond what is conditioned on fewer variables.

6. *Subadditivity*: $H(X,Y) \leq H(X) + H(Y)$. Equality holds *iff* X and Y are independent. This property shows that the total uncertainty about a pair of variables is less than or equal to the sum of their individual uncertainties, with equality occurring when there is no correlation between the variables.

7. *Strong subadditivity*: $H(X,Y,Z) + H(Y) \leq H(X,Y) + H(Y,Z)$. Equality holds *iff* $X \to Y \to Z$ forms a Markov chain. This inequality highlights the interdependence of three random variables, ensuring that the uncertainty relations are consistent with causality in a Markov chain structure.

As mentioned earlier, information theory provides tools for establishing a notion of distance between probability distributions. One such notion is given by the *relative entropy*, also known as the *Kullback–Leibler (KL) divergence* [45], which measures the difference between two probability distributions $p(x)$ and $q(x)$, both defined over the same index set x. It is defined as:

$$\mathcal{D}\left(p\|q\right) \coloneqq \sum_x p(x) \log \frac{p(x)}{q(x)} = -H(X) - \sum_x p(x) \log q(x). \tag{2.47}$$

Here, we adopt the conventions $0 \log 0 \equiv 0$ and $p(x) \log 0 \equiv +\infty$ if $p(x) > 0$.

The KL divergence can be regarded as a measure of "distance" between distributions, as it satisfies non-negativity: $\mathcal{D}(p\|q) \geq 0$, with equality if and only if $p(x) = q(x)$ for all x. This property follows from Gibbs' inequality, which states that for any probability distributions $p(x)$ and $q(x)$:

$$\sum_x p(x) \log \frac{p(x)}{q(x)} \geq 0. \tag{2.48}$$

However, a limitation of KL divergence is that it cannot be considered a true metric, as it is not symmetric; that is, in general, $\mathcal{D}(p\|q) \neq \mathcal{D}(q\|p)$.

This framework enables the experimentalist to systematically account for all the information obtained from the experiment, as recorded by the measurement apparatus. Additionally, it provides tools for addressing protocols that can be described entirely in classical terms, such as the random selection of one quantum system from a collection. However, not all aspects of the experiment can be described classically.

For information theory to be truly comprehensive—such that every element of the experimental setting has an informational counterpart—it is necessary to integrate information theory with quantum mechanics.

2.3 QUANTUM INFORMATION THEORY

The interface of the measurement apparatus, where the measurement outcomes are displayed, is a classical object. Quantum features, such as a superposition of different outcomes on the screen, are typically absent. However, the actual quantum system under investigation exhibits such behavior. How, then, should we quantify the information associated with systems of this nature?

In this section, we will base our discussion on the sources [48], [55], and [56]. Following the same approach used to derive the entropy of information, let us now consider a similar scenario where the source of information is explicitly a quantum system.

2.3.1 Quantum information entropy

Information from a quantum system is obtained exclusively through measurements. We assume that a message of n letters is generated from the outcomes of measurements performed on a quantum system described by a mixed state ρ. In one arrangement, the letters correspond to the outcomes of measurements of an observable $A = \sum_a a\,|a\rangle\langle a|$, with probabilities $p(a) = \mathrm{Tr}\,(|a\rangle\langle a|\,\rho)$.

The entropy of information for the set of letters a arises naturally when ρ is expressed in diagonal form. Representing ρ in an orthonormal basis $\{|a\rangle\}$, we have $\rho = \sum_a \lambda_a\,|a\rangle\langle a|$, where the coefficients λ_a correspond to the probabilities assigned to each letter a. The entropy

of this distribution is:

$$S(\rho) = -\sum_a \lambda_a \log \lambda_a, \tag{2.49}$$

which represents $H(A)$ for the ensemble $A = \{a, \lambda_a\}$.

Eq. (2.49) is equal to $-\mathrm{Tr}\,(\rho \ln \rho)$ because the trace operation is basis-independent. This latter expression provides a general formulation for the quantum entropy of information, known as the *von Neumann entropy*, named after John von Neumann [57]. It is defined as:

$$S(\rho) \coloneqq -\mathrm{Tr}\,(\rho \ln \rho). \tag{2.50}$$

When measuring a particular observable A in a state ρ, where ρ can be diagonalized in the basis of A, the entropy of information is given by $H(A)$. However, complications arise when measuring an observable B for which ρ cannot be diagonalized in the basis $\{|b\rangle\}$. Similarly, challenges occur when ρ represents an ensemble of states $\{\rho_i\}$ that do not commute with each other. In such cases, we can still compute the von Neumann entropy using the expression in Eq. (2.50), but there is no corresponding Shannon entropy expression like in Eq. (2.49). In these scenarios, the von Neumann entropy remains well-defined, but a direct interpretation as a Shannon entropy is no longer valid.

This reasoning suggests that, although the von Neumann entropy was derived as a particular quantum analog of the Shannon entropy, it can be viewed as a generalization of the latter. One might even be tempted to claim that quantum information is inherently more general than classical information. However, making such a claim would require distinguishing between different "types" of information, which would necessitate significantly expanding our definition of information to a more complex framework. To remain grounded, it is more prudent to state that the von Neumann entropy exhibits distinct features compared to the Shannon entropy, reflecting the unique characteristics of quantum systems.

2.3.2 Quantities and properties

But how different are they? The differences between the von Neumann entropy and the Shannon entropy become evident when comparing their properties. To study the properties of the von Neumann entropy, we must first introduce additional quantum entropic measures, just as we did for its classical counterpart. Fortunately, these definitions extend far beyond this discussion and will reappear frequently throughout this work. We now introduce:

- *Quantum joint entropy*: The total uncertainty of a composite system consisting of two partitions, $\mathcal{A}$ and $\mathcal{B}$, is given by:

  $$S(\rho_{\mathcal{AB}}) \coloneqq -\mathrm{Tr}\,(\rho_{\mathcal{AB}} \ln \rho_{\mathcal{AB}}), \tag{2.51}$$

  where $\rho_{\mathcal{AB}}$ is the density matrix of the composite system. This measure extends the concept of joint entropy to quantum systems.

- *Quantum conditional entropy*: The remaining uncertainty about the state of partition $\mathcal{A}$ given knowledge of partition $\mathcal{B}$ is defined as:

$$S(\rho_{\mathcal{A}}|\rho_{\mathcal{B}}) \coloneqq S(\rho_{\mathcal{AB}}) - S(\rho_{\mathcal{B}}). \tag{2.52}$$

  This is the quantum analog of conditional entropy in classical information theory.

- *Quantum mutual information*: The total information shared between partitions $\mathcal{A}$ and $\mathcal{B}$ is given by:

$$\begin{aligned} I_{\mathcal{A}:\mathcal{B}}(\rho) &\coloneqq S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}) - S(\rho_{\mathcal{AB}}) \\ &= S(\rho_{\mathcal{A}}) - S(\rho_{\mathcal{A}}|\rho_{\mathcal{B}}) = S(\rho_{\mathcal{B}}) - S(\rho_{\mathcal{B}}|\rho_{\mathcal{A}}). \end{aligned} \tag{2.53}$$

  This quantity quantifies the correlation between the two partitions, capturing both quantum and classical contributions to the shared information.

With these definitions established, we are now prepared to proceed and list several key properties of the von Neumann entropy:

1. *Non-negativity*: The von Neumann entropy is always greater than or equal to zero, $S(\rho) \geq 0$, with equality holding *iff* ρ is a pure state. A pure state represents maximal knowledge (minimal ignorance) about the quantum system, as it is described by a single state vector.

2. *Maximum value*: The entropy attains its maximum value of $\log d$, where d is the dimension of the Hilbert space on which ρ acts, $S(\rho) \leq \log d$. Equality holds *iff* ρ is a completely mixed state, $\rho = \mathbb{1}/d$. In this case, the state represents minimal knowledge (maximal ignorance) about the system, as all pure states are equally probable.

3. *Invariance under unitary transformations*: The entropy remains invariant under unitary transformations, $S(U\rho U^\dagger) = S(\rho)$, where U is a unitary operator. This invariance reflects the fact that unitary transformations preserve the eigenvalues of ρ, and thus its entropy.

4. *Non-decreasing under projective measurements*: The entropy may increase but never decrease as a result of a projective measurement. Let A_a be a complete set of orthogonal projectors defining the observable $A = \sum_a aA_a$. After a measurement of A, the post-measurement state is

$$\Phi_A(\rho) = \sum_a A_a \rho A_a. \tag{2.54}$$

   The entropy satisfies:

$$S(\Phi_A(\rho)) \geq S(\rho), \tag{2.55}$$

   with equality holding *iff* the measurement does not disturb the state, i.e., $\Phi_A(\rho) = \rho$. This property reflects the irreversibility of information loss during measurement.

5. *Symmetry of entanglement for pure states*: For a pure composite system $\rho_{\mathcal{AB}}$, the entropies of the reduced states are equal:

$$S(\rho_{\mathcal{A}}) = S(\rho_{\mathcal{B}}). \tag{2.56}$$

This reflects the symmetry of entanglement in a pure bipartite system, where the entropy of one subsystem equals the entropy of the other.

6. *Additivity of entropy for product states*: The entropy of a product state, $\rho \otimes \sigma$, is additive:

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma). \tag{2.57}$$

This property holds because the eigenvalues of the composite state are the products of the eigenvalues of the individual states, and the logarithm of a product separates into a sum.

7. *Schumacher's decomposition rule*: For states ρ_i supported on orthogonal subspaces with associated probabilities p_i, the entropy satisfies:

$$S\left(\sum_i p_i \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i). \tag{2.58}$$

Here, $H(p_i)$ is the classical Shannon entropy of the probabilities p_i, representing the uncertainty in choosing the subspace, while $\sum_i p_i S(\rho_i)$ captures the average quantum uncertainty within each subspace.

8. *Joint entropy theorem*: For orthogonal states $|i\rangle$ in partition $\mathcal{A}$, density operators ρ_i in partition $\mathcal{B}$, and probabilities p_i, the joint entropy satisfies:

$$S\left(\sum_i p_i \left|i\right\rangle \left\langle i\right| \otimes \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i). \tag{2.59}$$

This theorem generalizes the previous property to the case of a composite system, where $\mathcal{A}$ and $\mathcal{B}$ jointly contribute to the total entropy.

9. *Concavity of entropy*: The entropy of a mixture of states ρ_i is greater than or equal to the weighted average entropy of the individual states:

$$S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S(\rho_i). \tag{2.60}$$

This property highlights the fact that mixing increases uncertainty. Equality holds *iff* the states ρ_i are identical.

10. *Subadditivity*: For a bipartite quantum state $\rho_{\mathcal{AB}}$, the entropy satisfies:

$$S(\rho_{\mathcal{AB}}) \leq S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}). \tag{2.61}$$

Equality holds *iff* $\rho_{\mathcal{AB}} = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$, meaning there is no correlation (quantum or classical) between the two partitions.

11. *Araki–Lieb inequality*: The entropy of a bipartite quantum system is bounded below by the absolute difference of the entropies of its subsystems:

$$S(\rho_{\mathcal{AB}}) \geq |S(\rho_{\mathcal{A}}) - S(\rho_{\mathcal{B}})|. \tag{2.62}$$

It implies that information about one subsystem places constraints on the entropy of the other, reflecting the interdependence of the subsystems.

12. *Strong subadditivity*: For a tripartite quantum state $\rho_{\mathcal{ABC}}$, the entropy satisfies:

$$S(\rho_{\mathcal{ABC}}) \leq S(\rho_{\mathcal{AB}}) + S(\rho_{\mathcal{BC}}) - S(\rho_{\mathcal{B}}). \tag{2.63}$$

This shows that the total uncertainty cannot exceed the combined contributions of $\mathcal{AB}$ and $\mathcal{BC}$ after accounting for the "shared" contribution of $\mathcal{B}$. The term $S(\rho_{\mathcal{B}})$ prevents double-counting the uncertainty associated with $\mathcal{B}$, ensuring that any correlations spanning $\mathcal{ABC}$ are consistent with the overlapping pairs $\mathcal{AB}$ and $\mathcal{BC}$.

As with the Shannon entropy, we can define an entropy-like measure to quantify the "distance" between two density operators. The quantum analogue of classical relative entropy is the *quantum relative entropy*. For density operators ρ and σ, it is defined as:

$$S(\rho\|\sigma) \coloneqq \mathrm{Tr}\,(\rho \log \rho) - \mathrm{Tr}\,(\rho \log \sigma). \tag{2.64}$$

This quantity is finite as long as the support of ρ lies within the support of σ. If there is a non-trivial intersection between the kernel of σ and the support of ρ, the quantum relative entropy is defined to be $+\infty$.

The quantum relative entropy is non-negative, a result known as *Klein's inequality*:

$$S(\rho\|\sigma) \geq 0, \tag{2.65}$$

with equality holding *iff* $\rho = \sigma$. Note that the quantum relative entropy is not symmetric, meaning $S(\rho\|\sigma) \neq S(\sigma\|\rho)$ in general.

We can now highlight a key difference between the Shannon entropy and the von Neumann entropy. For classical random variables X and Y, the joint entropy always satisfies $H(X, Y) \geq H(X)$. However, this does not generally hold for the von Neumann entropy in quantum systems. In particular, $S(\rho_{\mathcal{AB}})$ is not necessarily greater than or equal to $S(\rho_{\mathcal{A}})$.

To illustrate this, consider a bipartite system with partitions $\mathcal{A}$ and $\mathcal{B}$, described by the pure state:

$$\rho_{\mathcal{AB}} = |\psi\rangle\langle\psi|, \quad \text{where} \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2.66}$$

Since the state is pure, the von Neumann entropy of the composite system is zero: $S(\rho_{\mathcal{AB}}) = 0$, indicating maximal knowledge of the system. However, if we trace out partition $\mathcal{B}$, the reduced state of $\mathcal{A}$ becomes:

$$\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho_{\mathcal{AB}}) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \tag{2.67}$$

which is a mixed state with von Neumann entropy $S(\rho_{\mathcal{A}}) = \log 2$. This discrepancy arises from the entanglement between $\mathcal{A}$ and $\mathcal{B}$ in the state $\rho_{\mathcal{AB}}$. But what exactly is entanglement?

2.3.3 Entanglement

For bipartite pure states, entanglement is present in any system whose state vector cannot be expressed as a product state, $|\psi\rangle \neq |\psi_{\mathcal{A}}\rangle \otimes |\psi_{\mathcal{B}}\rangle$. States that can be written in this form are separable, meaning the partitions are independent. Entanglement represents a form of correlation that arises when two subsystems are connected in such a way that they cannot be treated independently, often resulting from interactions between the subsystems.

To illustrate the concept, we first consider classical correlations. Imagine someone buys a pair of socks, where the only available colors are blue and red. This person gives one sock to Alice and the other to Bob, her friend. Later, Alice travels to Saturn and, at a predetermined time, opens her bag. If she finds a red sock, she instantly knows that Bob's sock must also be red. This is because the pair of socks is correlated: the color of one sock determines the color of the other. Such classical correlations arise from the way the socks were paired and exist regardless of the physical distance between Alice and Bob.

Now, suppose a similar experiment is performed, but instead of socks, Alice and Bob share two spin-1/2 particles prepared in a state with total angular momentum zero. This bipartite state can be represented as:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{2.68}$$

where $|01\rangle$ and $|10\rangle$ denote the spins of Alice's and Bob's particles, respectively. Both particles are in superposition states.

At a predetermined time, Alice measures her particle in the basis $\{|0\rangle, |1\rangle\}$. Upon measurement, the wave function of the entire system collapses: if Alice observes spin 0, the system collapses to $|01\rangle$, and Bob's particle is in the state $|1\rangle$. If Alice observes spin 1, the system collapses to $|10\rangle$, and Bob's particle is in the state $|0\rangle$.

While this resembles the classical correlation described earlier, there is a critical difference: in the quantum case, Bob's particle was also in a superposition state before Alice's measurement. Alice's measurement collapses the entire wave function, altering the state of the system and, consequently, the probabilistic predictions for Bob's particle. This phenomenon arises because the two particles share entanglement, as the state $|\psi\rangle$ cannot be expressed as a product state.

This intriguing feature of quantum mechanics was first brought to prominence in the seminal 1935 paper by Einstein, Podolsky, and Rosen (EPR), titled *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?* [6]. In this work, the authors highlighted what they viewed as the absurdity of entanglement, famously dubbing it "spooky action at

a distance". Their critique stemmed from their tacit assumption of locality throughout their reasoning.

Locality, in this context, refers to the principle that entities in a physical theory can influence only their immediate surroundings, with such influence propagating at most at the speed of light in a vacuum, c. This requirement forbids causal relationships between events that are spacelike separated. The EPR argument proposed that if Alice's measurement cannot causally affect Bob's particle, then quantum mechanics must be incomplete. In their view, there must exist elements of physical reality not accounted for in the theory. These missing elements, often called hidden variables, would allow for a more complete theory that restores locality.

John Bell demonstrated that the EPR proposal of supplementing quantum mechanics with hidden local variables is not viable. More precisely, Bell proved in his seminal 1964 paper, *On the Einstein Podolsky Rosen Paradox* [10], that the predictions of quantum mechanics are fundamentally incompatible with any hidden local variable theory. Bell's theorem has since been supported by numerous experimental tests employing a wide variety of strategies, including [11–15, 58, 59].

In our example, Alice's measurement caused Bob's state to conform to it. As a result, the probabilities assigned to Bob's measurement outcomes may change depending on Alice's measurement. Thus, the probabilities of Alice's outcomes a from measuring A cannot generally be considered independent of Bob's outcomes b from measuring B. Mathematically:

$$p(a, b|A, B) \neq p(a|A)p(b|B), \tag{2.69}$$

since, for independent probabilities, the joint probability would factorize.

Bell hypothesized that by introducing general local hidden variables λ, representing underlying local physical quantities, it might be possible to describe the statistical profiles of Alice's and Bob's measurements independently. This is expressed as:

$$p(a, b|A, B, \lambda) = \int_{\lambda} p(\lambda)p(a|A, \lambda)p(b|B, \lambda)\, d\lambda. \tag{2.70}$$

However, Bell demonstrated that the predictions of quantum mechanics are incompatible with this expression. Specifically, quantum states with entanglement violate this requirement, while product states satisfy it. For product states:

$$p(a, b|A, B) = \mathrm{Tr}\left((A_a \otimes B_b)\, |\Psi\rangle \langle\Psi|\right) = p(a|A)p(b|B), \tag{2.71}$$

where $|\Psi\rangle$ is a product state.

Most empirical validations of Bell's theorem do not follow Bell's original argument directly but instead rely on a result derived from it: the CHSH inequality. This inequality, proposed in 1969 by John Clauser, Michael Horne, Abner Shimony, and Richard Holt [60], states that if the predictions of a theory for a pure state are compatible with Eq. (2.70), then:

$$S = \langle a_0 b_0\rangle + \langle a_0 b_1\rangle + \langle a_1 b_0\rangle - \langle a_1 b_1\rangle \leq 2, \tag{2.72}$$

where $\langle a_i b_j \rangle$ represents the expectation value of the joint outcomes for measurements a_i and b_j on the subsystems.

For a pair of maximally entangled particles, the CHSH parameter reaches $S = 2\sqrt{2}$, violating the inequality. Such states exhibit *Bell nonlocality*, a distinct quantum resource. Various methodologies have been developed to quantify Bell nonlocality for pure and mixed states [61–69]. This remains a complex topic, as different quantification methods often yield varying results. Additionally, puzzling features arise, such as the fact that maximally entangled states do not necessarily exhibit maximal Bell nonlocality [70–73]. Returning to entanglement itself, we should first clarify what is meant by maximally entangled states.

A clear criterion exists for diagnosing entanglement in pure bipartite states, even though determining whether a state can be written as a product state is generally nontrivial. When this task is challenging, various methodologies can quantify the amount of entanglement in a state using entropy-based measures. These measures must satisfy specific criteria, as outlined in [74]. One widely used measure is:

- *Entropy of entanglement*: The entanglement between partitions $\mathcal{A}$ and $\mathcal{B}$ for a pure bipartite state ρ in $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ is given by:

$$E(\rho) = S(\rho_\mathcal{A}), \tag{2.73}$$

  where $\rho_\mathcal{A} = \mathrm{Tr}_\mathcal{B}(\rho)$ and, for pure states, $S(\rho_\mathcal{A}) = S(\rho_\mathcal{B})$.

For a maximally entangled state, tracing out one partition leaves the other in a completely mixed state. In this case, $S(\rho_\mathcal{A})$ (or equivalently $S(\rho_\mathcal{B})$) is maximal, indicating maximal entanglement.

Another particularly useful tool for analyzing the entanglement properties of a state is the Schmidt decomposition. For any bipartite state $|\psi\rangle$, there exist orthonormal states $\{|i_\mathcal{A}\rangle\}$ and $\{|i_\mathcal{B}\rangle\}$ for the partitions $\mathcal{A}$ and $\mathcal{B}$ such that:

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |i_\mathcal{A}\rangle\, |i_\mathcal{B}\rangle\,, \tag{2.74}$$

where $\{\lambda_i\}$ are real, non-negative coefficients satisfying $\sum_i \lambda_i = 1$.

The number of nonzero coefficients λ_i is the Schmidt rank (or Schmidt number) of the state. A state is separable, *i.e.*, unentangled, if and only if the Schmidt rank is 1. In this case, $\lambda_1 = 1$, and the state can be written as a product state. A state is entangled if the Schmidt rank is greater than 1. The state is maximally entangled if all the λ_i are equal.

For example, the following states, with Schmidt rank 2, are maximally entangled:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{2.75}$$

These states, known as Bell states, form a maximally entangled basis for two qubits.

Entanglement research also extends to mixed states, but this case is significantly more complex. A straightforward diagnostic criterion exists, again through negation: a bipartite mixed state is not entangled *iff* it can be written as a *separable state*. Separable states are those that admit a convex sum representation of the form:

$$\rho_{sep} = \sum_i p_i \rho_i^{\mathcal{A}} \otimes \rho_i^{\mathcal{B}}, \tag{2.76}$$

where $\{\rho_i^{\mathcal{A}}\}$ and $\{\rho_i^{\mathcal{B}}\}$ are local states in partitions $\mathcal{A}$ and $\mathcal{B}$, and $\{p_i\}$ are probabilities satisfying $\sum_i p_i = 1$.

A key subtlety arises here: unlike in the pure state case, separable mixed states can still exhibit correlations between measurements in $\mathcal{A}$ and $\mathcal{B}$. These are *classical correlations*, akin to the sock-pair example discussed earlier. For such states, the outcomes of measurements on $\mathcal{A}$ and $\mathcal{B}$ may be correlated such that:

$$\mathrm{Tr}\,(A \otimes B\rho_{sep}) \neq \mathrm{Tr}\,(A \otimes \mathbb{1}_{\mathcal{B}}\rho_{sep})\mathrm{Tr}\,(\mathbb{1}_{\mathcal{A}} \otimes B\rho_{sep}), \tag{2.77}$$

where A and B are observables on $\mathcal{A}$ and $\mathcal{B}$, respectively.

However, even with such correlations, separable states respect the locality condition expressed in Eq. (2.70). If hidden variables are introduced to account for subjective ignorance, these correlations can be explained without invoking entanglement.

This brings us to another subtlety. For the requirement in Eq. (2.70) to be violated, the state cannot be of the form ρ_{sep}. Thus, Bell nonlocal mixed states are necessarily entangled. However, as shown in [75], the converse is not true: there exist entangled states that are Bell local. This implies that the set of entangled states is a strict superset of the set of Bell nonlocal states.

Quantifying entanglement in mixed states is also a challenging task, with current tools often limited to specific classes of states, particularly in low-dimensional systems. For higher-dimensional states, practical quantifiers remain largely unavailable. In simpler cases, such as bipartite two-qubit systems, measures like *concurrence* provide a useful tool. For a bipartite two-qubit state ρ, the concurrence is defined as:

- *Concurrence*:

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{2.78}$$

  where λ_i are the eigenvalues, in decreasing order, of the matrix $\rho\tilde{\rho}$, with:

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{2.79}$$

  ρ^* is the complex conjugate of ρ, and σ_y is the Pauli y operator.

With the ability to diagnose and quantify entanglement for pure and mixed states, it is natural to consider how entanglement may arise or behave in the dynamics of an experimental setting. Suppose our experimentalist prepares a bipartite system with partitions $\mathcal{A}$ and $\mathcal{B}$, initially described by the product state:

$$|\Psi\rangle = |\psi_{\mathcal{A}}\rangle \otimes |\psi_{\mathcal{B}}\rangle\,, \tag{2.80}$$

where $|\psi_{\mathcal{A}}\rangle$ and $|\psi_{\mathcal{B}}\rangle$ are the quantum states of partitions $\mathcal{A}$ and $\mathcal{B}$, respectively. If these partitions are allowed to interact, their evolution, described by Eq. (2.13), may, in general, result in a final state that exhibits entanglement.

Conversely, if the partitions of the bipartite state do not interact—either directly or indirectly—and only local operations are performed on one of the partitions, such as measurements, appending ancillary systems, or applying quantum channels, entanglement cannot be generated. Similarly, if the experimentalist responsible for partition $\mathcal{A}$ transmits information to the experimentalist managing partition $\mathcal{B}$ through a classical medium, such as a text message sent via a classical communication channel, no new entanglement will arise. In summary, entanglement cannot be created through LOCC (local operations and classical communication).

2.3.4 Qubit

Returning to the origins of information theory—quantifying the resources required to store information—we have explored the allocation of bits for messages characterized by specific information entropy. In quantum systems, the carrier of information must also be characterized: this is the *qubit*.

A qubit encodes information in any two-level quantum system, such as a spin-1/2 particle or the orthogonal polarizations of a photon. It is represented by a state vector in a two-dimensional Hilbert space spanned by two orthonormal basis vectors, $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1. \tag{2.81}$$

Unlike a classical bit, which represents information probabilistically as 0 or 1 (*e.g.*, $\rho_{\text{classical}} = \text{diag}(p, 1-p)$ for probabilities p and $1-p$), a qubit can exist in coherent superpositions of these states. This coherence is reflected in the off-diagonal elements of the qubit's density matrix:

$$\rho_{\text{pure}} = |\psi\rangle\langle\psi| = \begin{bmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{bmatrix}. \tag{2.82}$$

A common parametrization of $|\psi\rangle$ employs angles $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$, rewriting it as:

$$|\psi\rangle = \cos\frac{\phi}{2}\,|0\rangle + \sin\frac{\phi}{2}e^{i\theta}\,|1\rangle\,. \tag{2.83}$$

This allows the qubit to be visualized geometrically on the *Bloch sphere* (see Fig. 2).

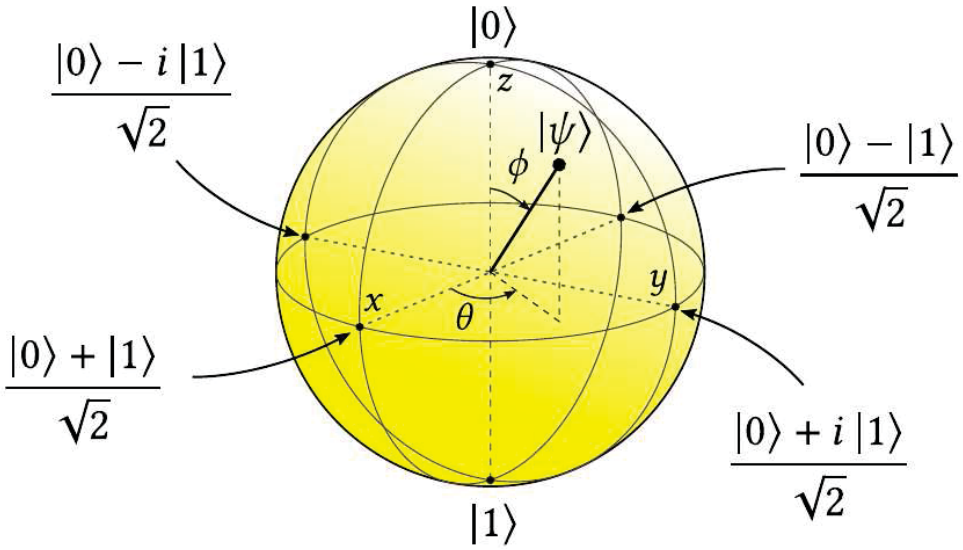

Figure 2 – Bloch sphere. $|\psi\rangle$ represents the qubit according to Eq. (2.83), corresponding to a point in the sphere with azimuthal and polar angles θ and ϕ. Original image at [32]

.

The Bloch sphere representation can also be introduced using the density operator formalism. A qubit's density operator is given by

$$\rho = \frac{1}{2}\left(\mathbb{1} + \vec{r}\cdot\vec{\sigma}\right), \tag{2.84}$$

where $\mathbb{1}$ is the 2×2 identity matrix, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices, and $\vec{r} = (r_x, r_y, r_z)$ is the Bloch vector. The Bloch vector $\vec{r}$ satisfies $\|\vec{r}\| \leq 1$, with $\|\vec{r}\| = 1$ for pure states and $\|\vec{r}\| < 1$ for mixed states.

In this representation, pure states correspond to points on the sphere's surface, with $\vec{r}$ indicating the direction of the state on the sphere. Mixed states are represented by points inside the sphere, with the maximally mixed state, $\rho = \mathbb{1}/2$, located at the origin $\vec{r} = 0$.

Since any d-dimensional quantum system, where $d = 2^n$, can be represented by n qubits, the qubit serves as the fundamental unit of quantum information theory and a cornerstone of quantum mechanics. For instance, a single qubit suffices to describe the state of a two-level system, such as the spin-up and spin-down states of an electron or the horizontal and vertical polarizations of a photon. Similarly, two qubits can describe a four-level system, such as the combined spin states of two entangled electrons: $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$.

Although a three-level system (a qutrit) cannot be represented by a single qubit due to the dimensional mismatch, it can be embedded into the Hilbert space of two qubits. For example, the three basis states $|0\rangle$, $|1\rangle$, and $|2\rangle$ of the qutrit can be mapped to three of the four available two-qubit states: $|00\rangle$, $|01\rangle$, and $|10\rangle$. The fourth two-qubit state, $|11\rangle$, remains unused in this mapping. While this representation is redundant, it allows the qutrit to be modeled within a framework where qubits are the fundamental unit.

Having moved from the general framework of quantum mechanics to its most fundamental symbolic unit, we now turn to deeper questions surrounding the nature of measurement,

classicality, and, more immediately, *realism*.

3 REALISM

Scientific knowledge is typically assumed to be about the nature of the world, going far beyond the mere appearance of things. It tells us about what can be apprehended directly by our senses, but it also tells us about subatomic particles or how light bends in gravitational fields. It tells us about what is directly influencing us, but it also tells us about how nature was way before and beyond human existence. To say that science gives us knowledge not only about what is directly observed but also about what is lying behind the appearances is, in broad terms, what *scientific realism* is about.

If we try to be really cautious about this idea, we will soon be embarking on an old and complicated philosophical enterprise. Trying to specify what we really mean by "world" touches complicated topics in ontology. If we discuss in detail the way by which we may acquire data from the world and how to construct knowledge from those data, we will discuss epistemology. We can, however, evocate a concept that synthesizes some relevant aspects of these questions in the sphere of Philosophy of Science. Alan Chalmers calls it global anti-realism in [76].

Philosophy of Language inherited us a fascinating problem: there is a circular relation of meaning and experience [77]. A description of the world, done through language, can only have meaning if it finds reflection in experience. Conversely, it is not possible to abstract any particular experience of the world independently of the way it is described through language. The perception and description of the world are then inextricably linked. This precludes any direct contact with reality or communication of it independently of the language by which we formulate our physical theories. A radical consideration of this idea is the global anti-realist stance.

Chalmers argues that no serious contemporary philosopher holds the view that we can come face-to-face with reality. However, this fact does not condemn science as a futile endeavor. Neither could we deny the practicality of it. In fact, science is absolutely pragmatic. So should we be when dealing with it, including when concepts as abstract as realism are brought to the discussion.

Investigating a metaphysical concept within the confines of a physical theory demands proposing a definition for it in terms of the theory's structure. Subsequently, the consistency of the definition with the predictions made by the theory is analyzed. We start with the earliest definition of realism for quantum mechanics, introduce BA's criterion, and then we generalize it to a definition that may be incorporated and thus investigated within any physical theory.

3.1 BILOBRAN AND ANGELO'S CRITERION OF REALISM

Physics' seminal definition of realism was published in 1935 by EPR [9], referencing the concept of

- *Element of reality*: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity."

This concept incorporates the assumption that the role of a physical theory is to describe an objective reality.

Physical theories represent physical states through physical properties, conceptualized by measurement protocols. EPR argued that a successful physical theory should correctly predict the statistics described by the measurements (correctness) and find correspondence between every physical property defined by the theory and the elements of reality (completeness).

This correspondence gives the backbone of their criterion of realism, described by a Boolean variable that is either true or false, that discriminates a physical property real or not depending on its correspondence to its element of reality. The truth value of this correspondence is determined by the conjunction of two propositions: that it is possible to predict with certainty the outcome of a measurement of this physical property, and that it is possible to perform a maximally informative measurement of this physical property without disturbing the system.

For example, consider that a spin $1/2$ particle was prepared by a measurement of the observable S_z which yielded the outcome $\hbar/2$. The post-measurement vector state is $|\psi\rangle = |S_z;+\rangle$. Once the theory predicts that any further measurement of S_z would return the same outcome, $\hbar/2$, with certainty and without changing the vector state of the system, EPR's criterion states that there exists an element of reality corresponding to S_z. If, on the other hand, we chose to measure S_x, in this basis the state is written as $|\psi\rangle = \frac{1}{\sqrt{2}}(|S_x;-\rangle + |S_x;+\rangle)$ and we cannot predict the outcome of the measurement with certainty, nor render the state undisturbed after the measurement. Therefore, no element of reality is assigned to S_x for this system.

EPR's criterion was devised to investigate the first hints of nonclassical phenomena, specifically nonlocality, predicted by state vector quantum mechanics. Indeed, for nonlocality to manifest, superposition is mandatory, and the criterion is sensible to that. But once it was defined for state vector quantum mechanics, its direct application on mixed states would lead to divisive accounts.

Consider a statistical mixture between two states: $|S_z;+\rangle$ and $|S_z;-\rangle$. That is, a mixed state $\rho = p_1 |S_z;+\rangle \langle S_z;+| + p_2 |S_z;-\rangle \langle S_z;-|$ with $p_1 \neq 1$, $p_2 \neq 1$ and $p_1 + p_2 = 1$. One could argue that because it is impossible to predict the result of a projective measurement of S_z, there is no corresponding element of reality to it. But one could also say that the corresponding elements of reality are there, they are just obscured by subjective ignorance.

After Bell's work showing the incompatibility of the predictions of quantum mechanics with the assumption of local hidden variables [10], realism can be understood not just as a piece of argument to unravel nonclassical properties of the theory, but as a nonclassical feature of the theory itself. The quantum mechanics interpretability problem derives from its nonclassical aspects and how they relate. A clear definition of realism adds another piece to the puzzle. However, EPR's criterion indefiniteness for mixed states, makes it unfit to explore nonclassical aspects of density operator quantum mechanics, like purification.

Bilobran and Angelo's realism criterion [31] fills this gap, generalizing EPR's criterion, as long as the following premise is accepted: if a projective measurement of a discrete spectrum observable $A = \sum_a aA_a$ is performed over a quantum state, there is an element of reality corresponding to A even if the outcome of the measurement was not revealed.

The implementation of this rationale allows for a refinement on EPR's criterion that assesses the realism of an observable through a positive-valued real number, rooted on an experimental protocol well defined for mixed states.

The experimental protocol defines a procedure in which one investigates whether a preparation has already established realism for an observable A. To this end, a source prepares two large sets of copies of the same physical state. Upon immediate subjecting of the first set to quantum tomography, it is determined that the best description of each copy is given by the density operator ρ. The second set, on the other hand, prior to undergoing tomography, has each of its copies intercepted by an agent that performs a projective measurement of an observable A—which, by hypothesis, establishes its realism—whose outcome they make inaccessible. For those copies, the description is $\Phi_A(\rho)$. This is illustrated in Figure 3.

For generality, we assume that each copy given by the source is bipartite, and the nonselective (*i.e* unreavealed) measurement is performed over one of the partitions. The intercepting action of the agent can be modeled by a completely positive trace-preserving map we call "unrevealed measurement", acting over a state ρ like

$$\Phi_A(\rho) \coloneqq \sum_a (A_a \otimes \mathbb{1}_{\mathcal{B}})\, \rho\, (A_a \otimes \mathbb{1}_{\mathcal{B}}) = \sum_a p_a A_a \otimes \rho_{\mathcal{B}|a}, \tag{3.1}$$

where $p_a = \mathrm{Tr}[(A_a \otimes \mathbb{1}_{\mathcal{B}})\rho]$, $\rho_{\mathcal{B}|a} = \mathrm{Tr}_{\mathcal{A}}[(A_a \otimes \mathbb{1}_{\mathcal{B}})\rho]/p_a$, and $\mathcal{A}$ and $\mathcal{B}$ the first and second partitions of the system. The quantity p_a corresponds to the probability of obtaining the result a after measuring A in partition $\mathcal{A}$. The state $\rho_{\mathcal{B}|a}$ gives the state system's part $\mathcal{B}$ conditioned on the outcome a.

Because the map Φ_A determines a state of realism for A, the criterion determines: *the observable A is real for the state ρ iff*:

$$\rho = \Phi_A(\rho). \tag{3.2}$$

Whenever this condition holds, A was already real for ρ *a priori*, meaning that Φ_A's action leaves the preparation undisturbed.

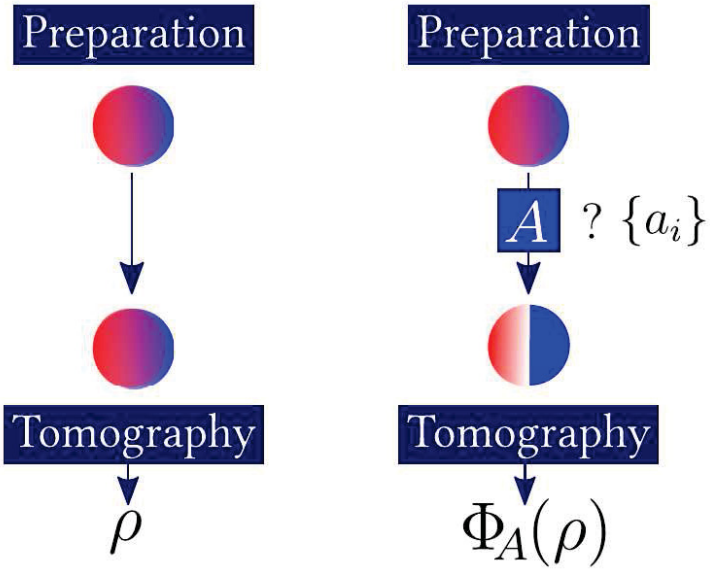

Figure 3 – A source on the left prepares infinitely many copies of a quantum state, which are subsequently sent to a quantum tomography procedure. The tomography assigns the description ρ to the state. On the right, the same source prepares identical copies of the quantum state, but these are intercepted, and the observable A is measured without revealing the measurement outcome. The quantum tomography then assigns the description $\Phi_A(\rho)$ to these states. Original figure in Ref. [32].

As an illustration, consider the subjection of the pure state $\rho = |S_z;+\rangle\langle S_z;+|$ to the unrevealed measurement protocol of S_z. The formalism predicts $\rho = \Phi_{S_z}(\rho)$, accusing realism of S_z and thus agreeing with the EPR's criterion. Now, for the mixed state $\rho = p_1 |S_z;+\rangle\langle S_z;+| + p_2 |S_z;-\rangle\langle S_z;-|$, $\rho = \Phi_{S_z}(\rho)$ follows as well, accusing the realism of S_z. BA's criterion thereby conceives statistical mixtures of states of realism as a state of realism, extending EPR' criterion.

It is also worthwhile to mention that Φ_A is a completely positive trace-preserving unital map [31]. Complete positivity ensures its mapping of positive operators into positive operators whose normalization is guaranteed by the trace-preserving property. Unitality implies that $\Phi_A(\mathbb{1}/d) = \mathbb{1}/d$, so that completely mixed states are mapped onto themselves.

Furthermore, by definition, if $\Phi_A(\rho)$ is a state of A-realism (*i.e.* A is real for ρ), and thus

$$\Phi_A(\Phi_A(\rho)) = \Phi_A(\rho), \tag{3.3}$$

such that Φ_A is idempotent. This property is merely a consequence of the definition of projectors: $A_a A_{a'} = A_a \delta_{aa'}$.

A particularly relevant insight about the quantity $\Phi_A(\rho)$ is given by its relation to the Stinespring dilation theorem [78]. In short, the theorem states that any completely positive trace-preserving map (CPTP) can be realized as a unitary transformation acting on a larger Hilbert space followed by a partial trace. Mathematically,

$$\Phi_A(\rho) = \mathrm{Tr}_{\mathcal{E}}[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger]. \tag{3.4}$$

U is an unitary map acting over $\mathcal{H} \otimes \mathcal{H}_{\mathcal{E}}$, with $\mathcal{H}_{\mathcal{E}}$ being an ancillary space, and $|e_0\rangle \in \mathcal{H}_{\mathcal{E}}$ is a vector in this space. This connection enables the interpretation of a nonselective measurement as a process carried out by an environment $\mathcal{E}$. Represented by the state $|e_0\rangle \langle e_0|$, the environment captures information about ρ during a dynamic interaction and is subsequently discarded.

The deviation of this realism criterion is quantified by an entropic difference, computing the *irrealism* of A for ρ:

$$\mathfrak{I}_A(\rho) \coloneqq S(\Phi_A(\rho)) - S(\rho). \tag{3.5}$$

That is, BA's criterion quantifies the irrealism (or realism, conversely), in terms of a real number.

An equivalent mathematical formulation is facilitated by the quantum relative entropy (Eq. (2.64)):

$$\mathfrak{I}_A(\rho) = S(\rho||\Phi_A(\rho)), \tag{3.6}$$

such that, together with Klein's inequality (Eq. (2.65)), irrealism's nonnegativity is guaranteed:

$$\mathfrak{I}_A(\rho) \geq 0, \tag{3.7}$$

with equality holding *iff* $\rho = \Phi_A(\rho)$. This excludes the possibility of negative irrealism. Moreover, the conjunction of (3.6) and the monotonicity of the relative entropy implies that $\mathfrak{I}_A(\rho)$ does not increase under CPTP maps. Thus, subjecting a state ρ to quantum channels cannot increase the irrealism of an observable A. This is expressed by

$$\mathfrak{I}_A(\rho) = S(\rho||\Phi_A(\rho)) \geq S(C(\rho)||\Phi_A(C(\rho))) = \mathfrak{I}_A(C(\rho)), \tag{3.8}$$

where C is a CPTP map.

It is noteworthy that, while for single-partite systems irreality reduces to a measure of coherence, for bipartite systems the following decomposition holds:

$$\mathfrak{I}_A(\rho) = \mathfrak{I}_A(\rho_{\mathcal{A}}) + D_A(\rho), \tag{3.9}$$

with $D_A(\rho) \coloneqq I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\Phi_A(\rho))$ being called the nonminimized one-way quantum discord. We should thus deem the degree of realism of A sensible to both the coherence of the reduced state $\rho_{\mathcal{A}}$, understood as local coherence, and the quantum correlations attributed to measurements of A.

Realism's responsiveness to quantum correlations suggests a connection with nonlocality that is encapsulated by a quantifier called realism-based nonlocality, firstly introduced in [34]. BA considered changes in the realism status of an observable A acting on $\mathcal{H}_{\mathcal{A}}$ upon a measurement of an observable B acting on $\mathcal{H}_{\mathcal{B}}$ conducted in a remote location given a bipartite state ρ. For the context $\{A, B, \rho\}$, they defined a contextual realism-based nonlocality as

$$\eta_{AB}(\rho) \coloneqq \mathfrak{I}_A(\rho) - \mathfrak{I}_A(\Phi_B(\rho)). \tag{3.10}$$

This nonnegative quantity is symmetrical upon permutations $A \rightleftarrows B$, vanishes only for states of realism for A or B and uncorrelated states $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$. The context dependency is disposed by defining

$$N_{\text{rb}}(\rho) := \max_{A,B} \eta_{AB}(\rho), \tag{3.11}$$

realism-based nonlocality. Also nonnegative, its zero conditions carry from its contextual analogue, vanishing for uncorrelated states, and it is a nonanomalous measure of nonlocality, discriminating maximally entangled states as maximally nonlocal states. Although reducing to entanglement and Bell nonlocality for pure states, when considering mixed states this quantity determines a quantumness measure that is responsive to features others are not, delineating a strict superset of entangled and Bell nonlocal states.

Further relations between irrealism and quantumness are highlighted in [79] with the authors investigating incompatibility. They define context incompatibility as a resource for safety testing tasks of quantum channel communications. Given a context $\mathbb{C} := \{\rho, A, B\}$, context incompatibility is quantified by

$$\mathcal{I}_{\mathbb{C}} := I(\Phi_A(\rho)) - I(\Phi_{BA}(\rho)) = \mathfrak{I}_B(\Phi_A(\rho)), \tag{3.12}$$

such that $I(\rho) := \ln d - S(\rho)$ is the information of a state ρ acting on a d-dimensional Hilbert space, and $\Phi_{BA}(\rho) = \Phi_B(\Phi_A(\rho))$. Vanishing for maximally mixed states, it is otherwise responsive to the commutation relation between A and B, vanishing if $[A, B] = 0$ $(\forall\rho)$. Considering the case where A and B act on the same partition, the last term of (3.12) can be understood as an assessment of B coherence in a A-incoherent state $\Phi_A(\rho)$.

We believe that this exposition highlights the pervasiveness of the concept of realism in quantum mechanics by showcasing its interconnectedness with the distinctive features of the theory. To further expand our understanding of realism *per se*, we elevate it to a theory-independent status. So, we briefly introduce a framework that facilitates this task, exposing the minimal amount necessary of it for the follow-up discussion.

3.2 GENERALIZED PROBABILISTIC THEORIES

Physical theories are subject to the particularities of historical moments in physics. These include previous knowledge accumulation, the purpose the field is serving, limitations in technological development, what is deemed as an acceptable way of thinking, among others. Those constraints impose at least some commonalities across every theory of such a period. One research program dedicated to abstracting those commonalities into a general mathematical framework is called generalized probabilistic theories (GPT).

GPT was a term crafted by Barret in 2007 [42], but its earliest ideas date back to 1947, with Segal's work [43]. The framework, which enables the characterization of the operational aspects of any physical theory, currently has several different formulations.

One possible construction of GPT employs category theory [80, 81] or, more specifically, symmetric monoidal categories to develop GPT from process theories [82, 83]. Process theories allow for a rigorous treatment of physical processes in terms of pictorial descriptions. This could be seen as a "top-to-bottom" approach. A "bottom-to-top" construction is made by directly defining the building blocks of arbitrary theories. This approach was taken by Plávala in [44], the work on which this session is largely based.

The main so-called building blocks for this construction of GPT are given by three objects: *state spaces*, *effect algebras* and *channels*. State spaces generalize the set of all possible density operators in quantum mechanics. Effect algebras generalized the positive operator valued measurements (POVM). Channels generalize quantum channels.

It is always possible to specify the state of a physical system in a physical theory by means of a preparation procedure — a set of instructions to be followed at the beginning of an experiment. Another way of characterizing a physical state, other than specifying how it was prepared, is through the specification of its properties. The properties of a physical state may be known by getting answers to questions which could be made about the system in the experiment. And each question can be structured as a series of "yes/no" questions. If every question about two states has equal answers, the preparation procedures leading to each state are equivalent, and the states are the same.

The set of equivalent classes of preparation procedures is called state space. Axiomatically, its description is given in terms of a set of points $\mathscr{K}$ forming a subset of a real and finite-dimensional vector space with Euclidian topology. In addition, this subset is convex, bounded, and closed.

Each point in this vector space represents set of every possible instructions leading to the preparation of an identical physical state. To incorporate states characterized by subjective ignorance about the preparation, we require convexity. To clarify, suppose we are unsure if a particular state was prepared according to a particular protocol, leading to the state x or to another protocol leading to y. The probabilities of having x or y are then given by p and $1 - p$, respectively. For the state $px + (1 - p)y$ to be part of the state space, regardless of the value of p, the space must be convex.

Just as many preparations can lead to the same state, thereby defining equivalent preparations, many sets of questions can characterize states equivalently. We call this set of equivalence classes of all possible "yes/no" questions effect algebra. This construction of state space and effect algebra highlights their duality: a state can be characterized by its preparation or its properties.

Formally, an effect algebra over a state space $\mathscr{K}$, $\mathscr{F}(\mathscr{K})$, is simply the set of all affine functions $f : \mathscr{K} \to [0, 1]$. Each function f can be understood as a question about a physical state in K whose answer goes from "yes" (1) to "no" (0) but could also be of a probabilistic nature.

For example, if we ask whether a coin is showing heads or tails, we may receive the answer $1/2$ when there is lack of knowledge, indicating a probability of 50%. The affine requirement is clarified if we consider a state of the kind $px + (1-p)y$. Consistency with the conservation of probability demands $f(px + (1-p)y) = pf(x) + (1-p)f(y)$. "Mixed" questions are also permitted. For example: "is the coin state heads or tails with 50% of chance each?".

The coin example can be illustrated by representing it in a bi-dimensional real Euclidian vector space. The nature of this system, as conceived through classical physics, determines that each possible state is mutually exclusive. We implement this by assigning orthogonal vectors to each pure state. One possible construction is to assign H (heads) with the vector $(1, 0)$ and T (tails) with $(0, 1)$. The state space is defined by the convex hull of these points: it determines a 1-simplex, a line segment whose extremities are $(1, 0)$ and $(0, 1)$.

The effect algebra can be represented in the same space. We can assign to $(1, 0)$ the equivalent questions to "is it H?" and proceed similarly to the corresponding equivalent questions for T. We also implement the "null" and "identity" questions: the former always returns "no" and the latter always returns "yes". Respectively, they can be put in $(0, 0)$ and $(1, 1)$. We end up with a square representing the effect space.

It is also possible to append a list of instructions at the end of a preparation or, equivalently, to prepend it at the beginning of a "yes/no" question. By doing this, we are able to transform the state already inside the experimental setting. This is the rationale for channel. A channel $\mathscr{C}$ is simply an affine map such that, for two states spaces $\mathscr{K}_1$ and $\mathscr{K}_2$ we have $\mathscr{C} : \mathscr{K}_1 \to \mathscr{K}_2$. For the coin, an example of a channel is the operation of flipping the coin. It maps the states $(1, 0) \to (0, 1)$ and $(0, 1) \to (1, 0)$.

This formalism allows for the conception of realism in an arbitrary theory context.

3.3 THEORY-INDEPENDENT REALISM

In a theory where a physical property is real — possessing a definite value independent of observation —, a measurement simply serves to disclose the pre-existing element of reality, without affecting it. If the result of a measurement is lost or remains inaccessible, the state of knowledge remains unchanged from before the measurement. Therefore, if the outcome of that measurement is inaccessible, the measurement is innocuous: it neither disturbs the system nor conveys information about it. This principle forms the foundation of the criterion we are about to introduce.

To apply this principle, we consider a generic state ϵ within a state space $\mathscr{K}$ of dimension d. Physical properties $\mathscr{X} = \{x_i\}_{i=1}^{d}$ and $\mathscr{Y} = \{y_j\}_{j=1}^{d}$ are defined by their set of possible outcomes. In terms of effects, each possible outcome is understood by the relation $f_{x_i}(\epsilon) = p_\epsilon(x_i)$.

We require, as the criterion of realism, that the probabilities assigned to any outcome

x_i resulting from the measurement of any property $\mathcal{X}$ on a state ϵ must remain unchanged when an unrevealed measurement of $\mathcal{Y}$ was conducted beforehand. Explicitly, we formulate the core of this chapter by:

Criterion of realism. *A physical property $\mathcal{Y}$ is real for a state ϵ if and only if*

$$p_\epsilon(x_i) = p_{\Phi_{\mathcal{Y}}(\epsilon)}(x_i) \qquad \left(\forall\, \mathcal{X} = \{x_i\}_{i=1}^{d}\right). \tag{3.13}$$

The subscript $\Phi_{\mathcal{Y}}(\epsilon)$ appearing in the right-hand term of Eq. (3.13) denotes the state ϵ after undergoing an unrevealed measurement of $\mathcal{Y}$. This term is represented in terms of conditional probabilities as follows:

$$p_{\Phi_{\mathcal{Y}}(\epsilon)}(x_i) \coloneqq \sum_j p_\epsilon(x_i|y_j)\, p_\epsilon(y_j). \tag{3.14}$$

It is important to note that, depending on the particularities of a theory, knowing y_j may change the state ϵ. The conditional probability notation is meant to indicate that ϵ represents the original state prior to the information obtained through a precise measurement of $\mathcal{Y}$.

Criterion (3.13) does not require every physical state ϵ to satisfy it. When all states in a given physical theory comply with this criterion, the theory is classified as realist. If only a subset of states ϵ satisfies the criterion, those states are referred to as $\mathcal{Y}$-realist.

By presupposing that, when considering $\mathcal{Y}$-realist states, joint probability distributions $p(x_i, y_j)$ exist for all $\mathcal{X}$, our criterion is seen through a perspective that resonates with Fine's approach to determinism [17]. This assumption implies that $p(x_i, y_j)$ is well-defined irrespective of the order in which $\mathcal{X}$ and $\mathcal{Y}$ are measured. If the standard definition of conditional probability is also valid, we can express the joint probability as

$$p(x_i, y_j) = p_\epsilon(x_i|y_j)\, p_\epsilon(y_j) = p_\epsilon(y_j|x_i)\, p_\epsilon(x_i), \tag{3.15}$$

Bayes' rule. Given that the conditional probabilities satisfy $\sum_j p_\epsilon(y_j|x_i) = 1$, we plug (3.15) into definition (3.14) and retrieve the realism criterion:

$$\begin{aligned} p_{\Phi_{\mathcal{Y}}(\epsilon)}(x_i) &\coloneqq \sum_j p_\epsilon(x_i|y_j)\, p_\epsilon(y_j) \\ &= \sum_j p_\epsilon(y_j|x_i)\, p_\epsilon(x_i) \\ &= p_\epsilon(x_i). \end{aligned} \tag{3.16}$$

Let $\mathscr{C}$ denote the state space containing generic states ϵ, and let $\mathscr{C}_{\mathcal{Y}}$ represent the subset of $\mathcal{Y}$-realist states. While $\mathscr{C}$ is convex by construction, it remains to be shown that $\mathscr{C}_{\mathcal{Y}}$ also possesses this property. Convexity of $\mathscr{C}_{\mathcal{Y}}$ can be established by proving that any convex combination of two $\mathcal{Y}$-realist states also belongs to $\mathscr{C}_{\mathcal{Y}}$.

Consider a one-parameter state $\epsilon'_\lambda = (1-\lambda)\epsilon'_1 + \lambda\epsilon'_2$, where $\lambda \in [0,1]$ and $\epsilon'_1, \epsilon'_2 \in \mathscr{C}_\mathcal{Y}$. To show that $\epsilon'_\lambda \in \mathscr{C}_\mathcal{Y}$, suppose a binary random variable Λ with probabilities $(1-\lambda)$ and λ. Depending on the outcome of Λ, a rule is applied such that the observable $\mathcal{X}$ is measured either in ϵ'_1 or ϵ'_2. The resulting probabilistic profile for this scenario is expressed as

$$p_{\epsilon'_\lambda}(x_i) = p_{(1-\lambda)\epsilon'_1+\lambda\epsilon'_2}(x_i). \tag{3.17}$$

Since ϵ'_1 and ϵ'_2 are $\mathcal{Y}$-realist states, we can write:

$$p_{(1-\lambda)\epsilon'_1+\lambda\epsilon'_2}(x_i) = p_{(1-\lambda)\Phi_\mathcal{Y}(\epsilon'_1)+\lambda\Phi_\mathcal{Y}(\epsilon'_2)}(x_i). \tag{3.18}$$

The right-hand side of this equation corresponds to the preparation and mixing of two ensembles, one of ϵ'_1 and the other of ϵ'_2, with relative populations $(1-\lambda)$ and λ, respectively. In this process, $\mathcal{Y}$ is measured first, and $\mathcal{X}$ is measured second on a randomly selected state from the mixed ensemble ϵ'_λ. Thus, we have:

$$p_{(1-\lambda)\Phi_\mathcal{Y}(\epsilon'_1)+\lambda\Phi_\mathcal{Y}(\epsilon'_2)}(x_i) = p_{\Phi_\mathcal{Y}(\epsilon'_\lambda)}(x_i). \tag{3.19}$$

This final equality reflects the fact that Φ is an affine map, ensuring that the mapping preserves convex combinations. Connecting this result with the previous equations completes the proof that $\epsilon'_\lambda \in \mathscr{C}_\mathcal{Y}$, establishing the convexity of $\mathscr{C}_\mathcal{Y}$.

Classical mechanics provides the most straightforward example of a realist theory. In this case, the state space $\mathscr{C}$ corresponds to a $2n$-dimensional phase space, where n represents the number of dimensions for position and momentum. A state ϵ within this space is a single point, fully specifying the configuration of a physical system. Realism in this framework is evident, as the set of $\mathcal{Y}$-realist states coincides with the entire phase space, meaning $\mathscr{C} = \mathscr{C}_\mathcal{Y}$ for any observable $\mathcal{Y}$.

This conclusion remains unchanged in classical statistical mechanics, where classical mechanics is extended to incorporate subjective uncertainties and governed by Liouville's equation. For any generalized coordinate q_i and its conjugate momentum π_j, it is possible to define a joint probability distribution $\mathcal{P}(q_i, \pi_j)\, dq_i d\pi_j$. As outlined earlier, this joint probability satisfies the realism criterion given in Eq. (3.13), further reinforcing the compatibility of classical statistical mechanics with realist principles.

In the framework of quantum mechanics, the expressions for the left-hand and right-hand sides of the realism criterion in Eq. (3.13) translate into:

$$p_\epsilon(x_i) \xrightarrow{\text{QM}} \text{Tr}\,[X_i\rho], \tag{3.20}$$

$$p_{\Phi_\mathcal{Y}(\epsilon)}(x_i) \xrightarrow{\text{QM}} \text{Tr}\,[X_i\Phi_\mathcal{Y}(\rho)], \tag{3.21}$$

where ρ is the density operator representing the state ϵ, and $X = \sum_i x_i X_i$ is the observable associated with the physical property $\mathcal{X}$. Here, X_i are the projectors corresponding to the measurement outcomes x_i, and $\Phi_\mathcal{Y}$ is defined as in Eq. (3.1).

Substituting these expressions into Eq. (3.13), the criterion becomes:

$$\mathrm{Tr}\left[X_i\rho\right] = \mathrm{Tr}\left[X_i\Phi_{\mathcal{Y}}(\rho)\right]. \tag{3.22}$$

Using the linearity of the trace, this equation can be rewritten as

$$\begin{aligned}\mathrm{Tr}\left[X_i\Phi_{\mathcal{Y}}(\rho)\right] - \mathrm{Tr}\left[X_i\rho\right] &= 0\\ \mathrm{Tr}\left[X_i(\Phi_Y(\rho) - \rho)\right] &= 0.\end{aligned} \tag{3.23}$$

It is straightforward to see that ρ being a state of Y-realism, according to BA criterion (Eq. (3.2)), provides a sufficient condition for the above equation to hold:

$$\begin{aligned}\mathrm{Tr}\left[X_i(\Phi_Y(\rho) - \rho)\right] &= \mathrm{Tr}\left[X_i(\rho - \rho)\right]\\ &= 0.\end{aligned} \tag{3.24}$$

That this condition is also necessary is a consequence of the duality between state spaces and effect algebra. That is, as established in [44], given a state space $\mathscr{K}$, one can construct the corresponding effect algebra $E(\mathscr{K})$, and conversely, the effect algebra can be used to reconstruct the state space. The proof provided below illustrates a specific instance of this duality, relying solely on the formalism of quantum mechanics [1].

We start by representing Eq. (3.23) in terms of the generalized Bloch sphere formalism:

$$\mathrm{Tr}\ \left[X_i\left(\Phi_{\mathcal{Y}}(\rho_{\vec{r}}) - \rho_{\vec{r}}\right)\right] = 0. \tag{3.25}$$

Using Eqs. (4.1) for $\rho_{\vec{r}}$, (4.15) for X_i, and (4.24) for $\Phi_{\mathcal{Y}}(\rho_{\vec{r}})$, with $\vec{u} \coloneqq P_X\vec{r}$, we have:

$$\mathrm{Tr}\left[\frac{1}{d}\left(\mathbb{1} + C_d\vec{x}_i\cdot\vec{\Lambda}\right)\left(\frac{1}{d}(\mathbb{1} + C_d\vec{u}\cdot\vec{\Lambda}) - \frac{1}{d}(\mathbb{1} + C_d\vec{r}\cdot\vec{\Lambda})\right)\right] = 0. \tag{3.26}$$

Simplifying the term inside the second parentheses:

$$\begin{aligned}\Phi_{\mathcal{Y}}(\rho_{\vec{r}}) - \rho_{\vec{r}} &= \frac{1}{d}\left(\mathbb{1} + C_d\vec{u}\cdot\vec{\Lambda}\right) - \frac{1}{d}\left(\mathbb{1} + C_d\vec{r}\cdot\vec{\Lambda}\right),\\ &= \frac{1}{d}C_d\left(\vec{u} - \vec{r}\right)\cdot\vec{\Lambda}.\end{aligned} \tag{3.27}$$

Substituting this back, the trace becomes:

$$\mathrm{Tr}\left[\frac{1}{d}\left(\mathbb{1} + C_d\vec{x}_i\cdot\vec{\Lambda}\right)\frac{1}{d}C_d\left(\vec{u} - \vec{r}\right)\cdot\vec{\Lambda}\right] = 0. \tag{3.28}$$

Expanding the product inside the trace:

$$\mathrm{Tr}\left[\frac{1}{d^2}\left(\mathbb{1}\cdot C_d\left(\vec{u} - \vec{r}\right)\cdot\vec{\Lambda}\right) + \frac{1}{d^2}C_d^2\left(\left(\vec{x}_i\cdot\vec{\Lambda}\right)\cdot\left(\vec{u} - \vec{r}\right)\cdot\vec{\Lambda}\right)\right] = 0. \tag{3.29}$$

[1] For this proof, we employ the generalized Bloch sphere formalism, which will be introduced in the next chapter. If the reader is not familiar with it, they can read the review we provide before proceeding or skip this proof to come back later if so they wish.

The first term vanishes because $\mathrm{Tr}\left[\mathbb{1}\cdot\vec{\Lambda}\right]=0$. Therefore, we are left with:

$$\frac{1}{d^2}C_d^2\mathrm{Tr}\left[(\vec{x}_i\cdot\vec{\Lambda})\cdot(\vec{u}-\vec{r})\cdot\vec{\Lambda}\right]=0. \tag{3.30}$$

Now let $\vec{v}\coloneqq\vec{r}-\vec{u}$, so the equation becomes:

$$\mathrm{Tr}\left[(\vec{x}_i\cdot\vec{\Lambda})(\vec{v}\cdot\vec{\Lambda})\right]=0. \tag{3.31}$$

Given Eq. (4.21), multiplying this equation by x_i and summing over i gives:

$$\mathrm{Tr}\left[(\vec{x}\cdot\vec{\Lambda})(\vec{v}\cdot\vec{\Lambda})\right]=0. \tag{3.32}$$

Lastly, using the identity $\mathrm{Tr}\left[(\vec{r}_1\cdot\vec{\Lambda})(\vec{r}_2\cdot\vec{\Lambda})\right]=2(\vec{r}_1\cdot\vec{r}_2)$, we obtain:

$$\vec{x}\cdot\vec{v}=0. \tag{3.33}$$

This condition must be true for any observable X, represented by its associated vector $\vec{x}$. The only way this holds universally is if $\vec{v}=\vec{0}$, which implies $\vec{r}=\vec{u}$. Consequently, this leads to Eq. (3.2), completing the proof.

This proof demonstrates the equivalence between our realism criterion and the one proposed by BA within the framework of quantum mechanics, thereby validating our formulation as a legitimate generalization of their approach.

We are ready for the next step, by which we quantify irrealism.

3.4 QUANTIFYING IRREALISM

In physical theories, when two states fall outside $\mathscr{C}_{\mathcal{Y}}$, indicating a violation of the realism criterion in Eq. (3.13) for a given property $\mathcal{Y}$, it becomes important to assess the extent of this violation. This motivates the need for a framework to quantify irrealism, which we define as the complement of realism. To this end, we propose two distinct methodologies for measuring irrealism, providing a systematic way to compare the degrees of violation exhibited by different states.

The irrealism measure introduced by BA, Eq. (3.5), is fundamentally based on von Neumann entropy, a quantity that is inherently tied to the formalism of quantum mechanics. While this makes it a powerful tool within quantum theory, it also limits its applicability to broader contexts.

To bridge this gap, we develop irrealism measures that extend beyond the quantum domain while remaining consistent with BA's formulation in the quantum mechanical limit. A comparison between our definitions and BA's will elucidate their similarities, differences, and the potential for broader applicability, reinforcing the validity of our generalized approach.

3.4.1 Robustness of irrealism

For our first methodology, we adopt the standard approach of robustness under state perturbation. Robustness measures how much a state must be perturbed, or "mixed" with another state, to satisfy a given criterion.

The *robustness of irrealism* is based on a simple physical intuition: suppose that you have a state ϵ and complete access to any other state ϵ'. The robustness $\mathcal{R}_{\mathcal{Y}}(\epsilon)$ tells you the smallest fraction η of ϵ' that must be mixed with ϵ to form the perturbed state

$$\epsilon_\eta = (1-\eta)\epsilon + \eta\epsilon', \tag{3.34}$$

such that $\epsilon_\eta \in \mathscr{C}_{\mathcal{Y}}$, making it a realist state for the property $\mathcal{Y}$. In essence, robustness quantifies the minimum amount of mixing required to eliminate the irrealism of a given state. Mathematically, it is expressed as:

$$\mathcal{R}_{\mathcal{Y}}(\epsilon) \coloneqq \min_{\epsilon'} \left\{ \eta \in [0,1] \;\middle|\; (1-\eta)\epsilon + \eta\epsilon' \in \mathscr{C}_{\mathcal{Y}} \right\}. \tag{3.35}$$

This measure leverages the convex structure of the state spaces $\mathscr{C}$ and $\mathscr{C}_{\mathcal{Y}}$, which is particularly useful because it avoids reliance on a specific metric for these spaces. Instead, it provides a generalizable way to quantify irrealism, satisfying desirable properties for such a measure.

Notably, the robustness of irrealism satisfies the condition $\mathcal{R}_{\mathcal{Y}}(\epsilon) \geq 0$, with equality holding if and only if $\mathcal{Y}$ is an element of reality for the state, *i.e.*, $\epsilon \in \mathscr{C}_{\mathcal{Y}}$. This ensures that the measure is zero only for states that already satisfy the realism criterion.

It is important to note that the state ϵ' used in the perturbation must belong to the broader state space $\mathscr{C}$ rather than the subset $\mathscr{C}_{\mathcal{Y}}$. If ϵ' were restricted to $\mathscr{C}_{\mathcal{Y}}$, and $\dim(\mathscr{C}_{\mathcal{Y}}) < \dim(\mathscr{C})$, the measure $\mathcal{R}_{\mathcal{Y}}(\epsilon)$ would always be zero. This would render the robustness measure meaningless for distinguishing states based on their irrealism, as it would fail to capture any relevant information about how far a state lies from satisfying the realism criterion.

Despite its generality, a geometrical intuition is close at hand. Let $\bar{\eta}$ correspond to the minimized quantity defining, for a state ϵ, $\mathcal{R}_{\mathcal{Y}}(\epsilon) = \bar{\eta}$ and the corresponding $\mathcal{Y}$-realist state $\epsilon_{\bar{\eta}} = (1-\bar{\eta})\epsilon + \bar{\eta}\epsilon'$. Together with an appropriate metric, we can express $\bar{\eta}$ as

$$\bar{\eta} = \frac{||\epsilon_{\bar{\eta}} - \epsilon||}{||\epsilon' - \epsilon||}. \tag{3.36}$$

By minimizing η while sweeping through every ϵ', one finds the smallest distance $||\epsilon_{\bar{\eta}} - \epsilon||$ relative to $||\epsilon' - \epsilon||$, subject to the constraint $\epsilon_\eta \in \mathscr{C}_{\mathcal{Y}}$.

This process is illustrated by a case study, where we calculated the robustness of irrealism of the spin observable in the $\hat{z}$ direction, $S_{\hat{z}}$, for a single qubit, given by:

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{\rho}\cdot\vec{\sigma}), \tag{3.37}$$

with $\vec{\rho} = (x, y, z) \in \mathbb{R}^3$.

The concept of irrealism as defined in Eq. (3.5) was shown to be equivalent to coherence for single qubit systems [31]. This equivalence also extends to the robustness of irrealism. Notably, our results for this scenario align with those presented in [84].

Computations such as those involved in this case study are mediated by semidefinite programming (SDP) algorithms, designed to handle optimizations involving linear operators subject to the constraint of positive semidefiniteness. However, the symmetries involved in the qubit case allow simpler methods to be employed. By recognizing that the set $\mathscr{C}_{S_{\hat{z}}}$ describes a vertical line in the Bloch sphere, going from $(0, 0, -1)$ to $(0, 0, 1)$, we can drastically reduce the number of states ρ' in the computation, such that $\rho_\eta = (1 - \eta)\rho + \eta\rho' \in \mathscr{C}_{S_{\hat{z}}}$ for some η.

More specifically, given a vector $\vec{\rho} = (x, y, z)$, the set of vectors $\vec{\rho}' = (x', y', z') \in \mathbb{R}^3$ corresponding to the operators ρ' that are eligible describe a 2-dimensional shape within the Bloch sphere. To construct this set, we take the point (x, y, z) and consider all rays that originate from it and intersect $\mathscr{C}_{S_{\hat{z}}}$, which lies along the vertical axis. The portion of each ray before it reaches $\mathscr{C}_{S_{\hat{z}}}$ is discarded, leaving only the section beyond the intersection. The resulting shape is a semicircle with $\mathscr{C}_{S_{\hat{z}}}$ as its diameter, and the arc lies on the surface of the Bloch sphere, oriented in the $-\hat{x}, -\hat{y}$ direction.

Further computational analysis reveals that η is inversely proportional to the distance between the coordinates (x', y', z') and the set $\mathscr{C}_{S_{\hat{z}}}$. In other words, for any given $\vec{\rho} = (x, y, z)$, the optimal $\vec{\rho}'$ always lies along the "equator" of the Bloch sphere, oriented in the $-\hat{x}, -\hat{y}$ direction. Precisely, given the state ρ with $\vec{\rho} = (x, y, z)$, the state ρ' with $\vec{\rho}' = (x', y', z')$ that yields $\mathcal{R}_{S_{\hat{z}}}(\rho) = \bar{\eta}$ is given by the coordinates

$$x' = -\frac{x}{\sqrt{x^2 + y^2}} =: \bar{x}, \qquad y' = -\frac{y}{\sqrt{x^2 + y^2}} =: \bar{y}, \qquad z' = 0. \tag{3.38}$$

Once the ray that goes from (x, y, z) to $(\bar{x}, \bar{y}, 0)$ is intercepted by $\mathscr{C}_{S_{\hat{z}}}$ at a single point, this result, illustrated in Figure 4, fully characterizes $\mathcal{R}_{S_{\hat{z}}}(\rho)$.

As exposed in Eq. (3.36), the parameter $\bar{\eta}$ can be interpreted geometrically as the ratio between $||\vec{\rho}_{\bar{\eta}} - \vec{\rho}'||$, where $\vec{\rho}_{\bar{\eta}}$ represents the vector corresponding to $\rho_{\bar{\eta}}$, and $||\vec{\rho} - \vec{\rho}'||$. Using this geometric relationship, the expression for the robustness of irrealism is straightforwardly derived as:

$$\mathcal{R}_{S_{\hat{z}}}(\rho) = \bar{\eta} = \frac{||\hat{z} \times \vec{\rho}||}{1 + ||\hat{z} \times \vec{\rho}||}, \tag{3.39}$$

where $||\hat{z} \times \vec{\rho}|| = r \sin\theta$, with r being the radius and θ the polar angle in the spherical coordinate representation of the vector $\vec{\rho}$.

Using Eq. (3.39), we can see that, indeed, every $S_{\hat{z}}$-realist state shows null robustness of irrealism, once $||\hat{z} \times \vec{\rho}|| = 0$ for those cases. Eigenstates of observables maximally incompatible with $S_{\hat{z}}$ — represented along the equator of the Bloch sphere —, in turn, account for $\mathcal{R}_{S_{\hat{z}}}(\rho) = 1/2$, because $||\hat{z} \times \vec{\rho}|| = 1$. Given a single qubit system, Eq. (3.39) determines $0 \leq \mathcal{R}_{S_{\hat{z}}}(\rho) \leq 1/2$.

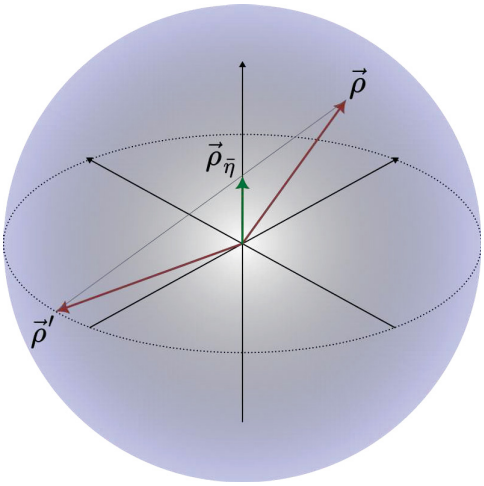

Figure 4 – Bloch sphere representation of the robustness of irrealism for $S_{\hat{z}}$ in a qubit system. The extended vertical axis illustrates the set of states $\mathscr{C}_{S_{\hat{z}}}$. The three colored arrows represent the vectors parametrizing the states in Eq. (3.35) after minimizing η with respect to ρ'. For $\rho = \frac{1}{2}(\mathbb{1} + \vec{\rho} \cdot \vec{\sigma})$, where $\vec{\rho} = (x, y, z) \in \mathbb{R}^3$, the robustness $\mathcal{R}_{S_{\hat{z}}}(\rho) = \bar{\eta}$ [see Eq. (3.39)] is achieved with $\rho' = \frac{1}{2}(\mathbb{1} + \vec{\rho}' \cdot \vec{\sigma})$, where $\vec{\rho}' = (\bar{x}, \bar{y}, 0) \in \mathbb{R}^3$.

For a generic spin observable $S_{\hat{n}}$, the robustness of irrealism can still be expressed using Eq. (3.39) after a basis rotation that aligns $\hat{z}$ with $\hat{n}$.

To prove Eq. (3.39), we notice that the Bloch vector corresponding to $\rho_{\bar{\eta}}$ is:

$$\vec{\rho}_{\bar{\eta}} = (1 - \bar{\eta})\vec{\rho} + \bar{\eta}\vec{\rho}', \tag{3.40}$$

and that for $\rho_{\bar{\eta}} \in \mathscr{C}_{S_{\hat{z}}}$, it must lie on the vertical axis of the Bloch sphere, *i.e.*, $(\vec{\rho}_{\bar{\eta}})_x = 0$ and $(\vec{\rho}_{\bar{\eta}})_y = 0$. Substituting $\vec{\rho}' = (\bar{x}, \bar{y}, 0)$ into $\vec{\rho}_{\bar{\eta}}$, we find

$$(\vec{\rho}_{\bar{\eta}})_x = (1 - \bar{\eta})x - \bar{\eta}\frac{x}{\sqrt{x^2 + y^2}} = 0, \tag{3.41}$$

$$(\vec{\rho}_{\bar{\eta}})_y = (1 - \bar{\eta})y - \bar{\eta}\frac{y}{\sqrt{x^2 + y^2}} = 0. \tag{3.42}$$

Factoring x and y, this gives

$$1 - \bar{\eta} = \bar{\eta}\frac{1}{\sqrt{x^2 + y^2}}, \tag{3.43}$$

which simplifies to:

$$\bar{\eta} = \frac{\sqrt{x^2 + y^2}}{1 + \sqrt{x^2 + y^2}}. \tag{3.44}$$

By recognizing that $||\hat{z} \times \vec{\rho}|| = \sqrt{x^2 + y^2}$ represents the distance from $\vec{\rho}$ to the vertical axis $\mathscr{C}_{S_{\hat{z}}}$, the proof is complete.

A comparison between the BA's irreality defined in Eq. (3.5) and the robustness of irrealism was conducted numerically. Figure 5 shows the irrealism of $S_{\hat{z}}$ for a pure qubit state as the polar angle θ varies from 0 to π in the Bloch sphere. The robustness of irrealism was normalized to match the absolute maximum value of the irreality for a direct comparison.

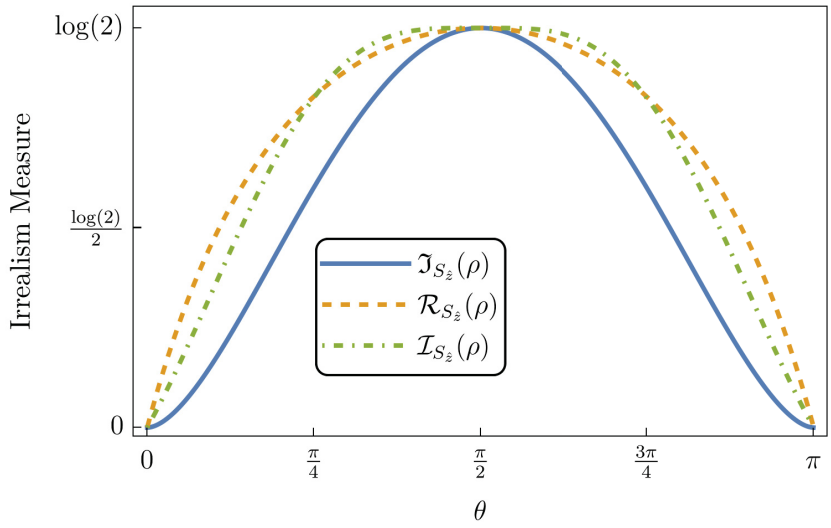

Figure 5 – Graph of $\mathfrak{I}_{S_{\hat{z}}}(\rho)$, $\mathcal{R}_{S_{\hat{z}}}(\rho)$, and $\mathcal{I}_{S_{\hat{z}}}(\rho)$ for a pure qubit state as the polar angle $\theta \in [0, \pi]$.

Both measures reach zero and attain their maximum at the same points. Additionally, the parametric curve in Fig. 6 confirms a monotonic relationship between the two, highlighting their conceptual alignment.

3.4.2 Divergence of realism

The second method for defining a theory-independent irrealism quantifier emerges naturally from the structure of the definition in Eq. (3.5). This method involves quantifying the divergence between the left- and right-hand terms in the BA irrealism criterion. To formalize this, we adopt the Kullback-Leibler (KL) divergence, Eq. (2.47), a widely recognized and well-established measure of statistical distance between probability distributions.

The realism criterion applies to any physical property $\mathcal{X}$, allowing us to determine the degree of realism violation by identifying the property that maximizes the divergence between the associated probability distributions. This leads to the following quantifier for the irreality of $\mathcal{Y}$ in a given physical state ϵ:

$$\mathcal{I}_{\mathcal{Y}}(\epsilon) = \max_{\mathcal{X}} \mathcal{D}\left(P_\epsilon^{\mathcal{X}} \middle\| P_{\Phi_{\mathcal{Y}}(\epsilon)}^{\mathcal{X}}\right), \tag{3.45}$$

where $\mathcal{I}_{\mathcal{Y}}(\epsilon)$ is referred to as the *divergence of realism* of $\mathcal{Y}$ for the state ϵ.

In the notation of this chapter, the KL divergence is defined for two discrete probability distributions $P_\epsilon^{\mathcal{X}} = \{p_\epsilon(x_i)\}_{i=1}^d$ and $Q_\epsilon^{\mathcal{X}} = \{q_\epsilon(x_i)\}_{i=1}^d$ as:

$$\mathcal{D}\left(P_\epsilon^{\mathcal{X}} \middle\| Q_\epsilon^{\mathcal{X}}\right) := \sum_{i=1}^{d} p_\epsilon(x_i) \log\left[\frac{p_\epsilon(x_i)}{q_\epsilon(x_i)}\right]. \tag{3.46}$$

This divergence is non-negative by definition, reflecting the Gibbs inequality, and equals zero if and only if $P_\epsilon^{\mathcal{X}}$ and $Q_\epsilon^{\mathcal{X}}$ are identical. In the present context, $\mathcal{I}_{\mathcal{Y}}(\epsilon)$ inherits these

properties, ensuring it is non-negative and vanishes only when the realism criterion in Eq. (3.13) is satisfied. Specifically, $\mathcal{I}_{\mathcal{Y}}(\epsilon) = 0$ *iff* $\epsilon \in \mathscr{C}_{\mathcal{Y}}$.

While the robustness of irrealism relies on the convex geometric structure of state spaces and physical intuition, the divergence of realism is grounded exclusively in classical probability theory. In the context of quantum mechanics, Eq. (3.45) can be reformulated as:

$$\mathcal{I}_Y(\rho) = \max_X S\big(\Phi_X(\rho)||\Phi_{X,Y}(\rho)\big), \tag{3.47}$$

where $\Phi_{X,Y}(\rho) = \Phi_X\big(\Phi_Y(\rho)\big)$ denotes the successive application of the nonselective measurement state mappings associated with the observables X and Y.

To derive Eq. (3.47), consider that for any Hermitian operator A with orthonormal eigenbasis $|a\rangle$ and eigenvalues a, the action of a generic function f is given by:

$$f(A)\,|a\rangle = f(a)\,|a\rangle\,. \tag{3.48}$$

Additionally, notice that the post nonselective states can be expressed as:

$$\Phi_X(\rho) = \sum_i p_\rho(x_i) X_i, \quad \Phi_{X,Y}(\rho) = \sum_i p_{\Phi_Y(\rho)}(x_i) X_i, \tag{3.49}$$

highlighting the fact that such states commute, by sharing the same set of eigenvectors. Using this commutativity property, and $X_i = |x_i\rangle\langle x_i|$, we work through the quantum relative entropy as follows:

$$\begin{aligned}
S\big(\Phi_X(\rho)||\Phi_{X,Y}(\rho)\big) &= \mathrm{Tr}\left[\Phi_X(\rho)\log\Phi_X(\rho)\right] - \mathrm{Tr}\left[\Phi_X(\rho)\log\Phi_{X,Y}(\rho)\right] \\
&= \mathrm{Tr}\left[\left(\sum_i p_\rho(x_i)\,|x_i\rangle\langle x_i|\right)\log\left(\sum_j p_\rho(x_j)\,|x_j\rangle\langle x_j|\right)\right] \\
&\quad - \mathrm{Tr}\left[\left(\sum_i p_\rho(x_i)\,|x_i\rangle\langle x_i|\right)\log\left(\sum_j p_{\Phi_Y(\rho)}(x_j)\,|x_j\rangle\langle x_j|\right)\right] \\
&= \sum_i p_\rho(x_i)\,\langle x_i|\log\left(\sum_j p_\rho(x_j)\,|x_j\rangle\langle x_j|\right)|x_i\rangle \\
&\quad - \sum_i p_\rho(x_i)\,\langle x_i|\log\left(\sum_j p_{\Phi_Y(\rho)}(x_j)\,|x_j\rangle\langle x_j|\right)|x_i\rangle \\
&= \sum_i p_\rho(x_i)\log p_\rho(x_i) - \sum_i p_\rho(x_i)\log p_{\Phi_Y(\rho)}(x_i) \\
&= \sum_i p_\rho(x_i)\log\left[\frac{p_\rho(x_i)}{p_{\Phi_Y(\rho)}(x_i)}\right] \\
&= \mathcal{D}\big(P^X_\rho\big\|P^X_{\Phi_Y(\rho)}\big).
\end{aligned} \tag{3.50}$$

Like this, the equality of the arguments inside the maximization functions ensures the equivalence between Eqs. (3.45) and (3.47). This derivation follows the proof originally developed in Ref. [85].

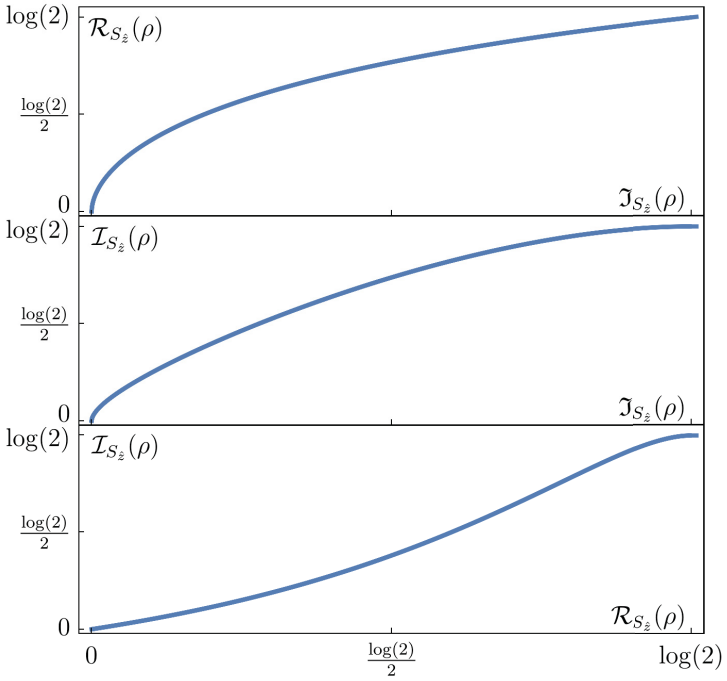

Figure 6 – Three parametric graphs are presented for a pure qubit state, with the polar angle $\theta \in [0, \pi]$. The top graph displays $\mathfrak{I}_{S_{\hat{z}}}(\rho)$ against $\mathcal{R}_{S_{\hat{z}}}(\rho)$, the middle $\mathfrak{I}_{S_{\hat{z}}}(\rho)$ against $\mathcal{I}_{S_{\hat{z}}}(\rho)$, and the bottom $\mathcal{R}_{S_{\hat{z}}}(\rho)$ against $\mathcal{I}_{S_{\hat{z}}}(\rho)$.

Eq. (3.47) was used for a case study of qubits, following the same parameters as before. Its results are illustrated in Figs. 5 and 6, exhibiting a behavior similar to that of $\mathcal{R}_{S_{\hat{z}}}(\rho)$, and $\mathcal{I}_{S_{\hat{z}}}(\rho)$. Specifically, its peak and vanishing points are coincident. Notably, the apparent plateau observed in $\mathcal{I}_{S_{\hat{z}}}(\rho)$ in Fig. 5 is not a true plateau. Instead, it represents a region where the absolute value of the slope gradually approaches zero, with the slope reaching zero only at the peak.

Furthermore, the monotonic relationships observed in Fig. 6 among the irrealism quantifiers $\mathfrak{I}_{S_{\hat{z}}}(\rho)$, $\mathcal{R}_{S_{\hat{z}}}(\rho)$, and $\mathcal{I}_{S_{\hat{z}}}(\rho)$ suggest that these measures may be qualitatively equivalent, highlighting their consistency in capturing the degree of realism violations.

3.5 REALISM AND CLASSICALITY

Generalized probabilistic theories (GPT) readily accommodate not just full-fledged physical theories but also their sub-theories. A sub-theory may comprise, for instance, a single system with only a subset of possible states and operations. The coin example illustrates this idea: it can be regarded as a sub-theory of classical mechanics containing a one-dimensional state space (the line segment between $(1, 0)$ and $(0, 1)$) and a single flipping operation.

The coin sub-theory is trivially realist because there is only one physical property available, $\mathcal{X} = \{H, T\}$. More generally, any sub-theory of classical mechanics is also determined to be a realist theory. This follows naturally, as classical sub-theories inherit their realism from classical mechanics, which is the paradigmatic realist theory.

Classical theories also possess another hallmark: classicality. Among the many definitions of classicality suitable for quantum contexts, a widely accepted theory-agnostic criterion is that a theory is classical if its state space is a simplex. Concretely, a set of states is simplicial if every mixed state ω has a unique convex decomposition into pure states. Formally, if

$$\omega = \sum_i p_i \omega_i \quad \text{with} \quad \sum_i p_i = 1,\ p_i \geq 0, \tag{3.51}$$

then $\{\omega_i\}$ and $\{p_i\}$ must be unique. Quantum mechanics stands in contrast: a given mixed state on the Bloch sphere admits infinitely many such decompositions.

Due to the duality between state spaces and effect algebras in GPT, classical (simplicial) state spaces are paired with hypercubic effect algebras. For instance, the one-dimensional simplicial state space $[0, 1]$ for a coin corresponds to a square $[0, 1]^2$ for the effect algebra (after including the null and identity effects). In such a hypercubic effect algebra, there always exists a single effect that fully characterizes the state — meaning one yes/no question suffices to pinpoint the system's condition. Quantum-mechanical states, however, generally demand tomography via multiple measurements on many copies of the system, since any single measurement disturbs the state and yields incomplete information.

In Ref. [86], Schmidt shows that any simplicial theory implies macrorrealism [87], while D'Ariano demonstrates in Ref. [88] that respecting "information without disturbance" also forces a simplicial state space. Consequently, every simplicial theory — hence every classical theory — is necessarily a realist theory.

One might then ask: is there a classical sub-theory embedded in quantum mechanics? The answer is yes, and an especially relevant example is the trivial sub-theory containing only a single — completely mixed — state. Its corresponding effect space reduces to a line. In other words, no measurements are available to distinguish any further structure, and no richer geometry emerges. Operationally, this amounts to complete ignorance about the system — no accessible measurement can alter or reveal the state. Under these circumstances, one obtains a sub-theory that is both trivially classical (the state space is a single point) and trivially realist (no nontrivial observables exist to disturb or reveal otherwise).

In the chapter to follow, we use this kind of construction to propose a general notion of *realism-based classicality* in the quantum context, showing how a classical description may effectively emerge under appropriate operational constraints.

4 CLASSICALITY

Quantum mechanics is not rendered non-classical solely due to its counterintuitive aspects. Classical theories also contain elements that challenge our explanatory intuition, such as the principle of least action in Hamiltonian mechanics or the nature of entropy in classical statistical mechanics.

From an operational perspective, quantum mechanics describes physical systems that can perform tasks strictly forbidden within a classical framework. When quantum features that distinguish these systems from classical ones are suppressed, the resulting behavior effectively restores an operationally classical regime.

This idea motivates the quantum-to-classical paradigms which involve decoherence [20–23]—without complex amplitudes, a system's probabilistic profile evolves classically.

When discussing the BA irrealism notion in Eq. (3.5), we noted that for single-partite systems, it reduces to a measure of coherence. Indeed, coherence can be understood in terms of irrealism: realism dictates decoherence.

Furthermore, irrealism is fundamentally essential for the manifestation of quantum behavior, such as coherence, as formalized by other quantumness markers. In Ref. [79], it is demonstrated that, without irrealism, a notion of incompatibility cannot emerge (see Eq. (3.12)). Nonlocal features—such as symmetrical discord [89], discord [90], entanglement [91], steering [92], and Bell nonlocality [75]—also vanish without irrealism [34], as realism-based nonlocality (Eq. (3.11)) reduces to zero.

We now introduce and analyze a realism-based classicality notion. Its mathematical structure is developed, and specific cases are investigated. To avoid fragmenting the conceptual discussion across walls of equations, the results are interpreted in the final section of this chapter.

4.1 GENERALIZED BLOCH SPHERE

In statistical quantum mechanics, preparations are represented by density operators, which are inherently abstract entities. These mathematical objects are algebraically well-behaved due to their properties: they are Hermitian, positive semidefinite, and have unit trace. However, envisioning them geometrically to gain intuition is far from straightforward.

In contrast, other physical theories, such as classical mechanics, naturally conceive states as real-valued vectors in a Euclidean vector space, making them more intuitive to visualize. Statistical quantum mechanics can adopt a similar geometric framework by establishing an isomorphism between the space of density operators and real-valued vectors. For a two-

dimensional system, this correspondence gives rise to the Bloch sphere.

The generalized Bloch sphere representation is comprehensively introduced in Refs. [46, 47]. Here, we focus on the essential aspects needed to understand the results presented in this chapter.

4.1.1 Density operators

A density operator ρ of dimension d has d real numbers in its diagonal and $d(d-1)$ complex numbers off the diagonal. Because $\rho^\dagger = \rho$, the off-diagonal terms are given by $d(d-1)$ real numbers and, since $\mathrm{Tr}\,[\rho] = 1$, the diagonal requires $d-1$ real numbers. This suggests representing ρ by a $d(d-1) + d - 1 = d^2 - 1$ dimensional real-valued vector $\vec{r}$.

A possible formulation is as follows:

$$\rho_{\vec{r}} = \frac{1}{d}\left(\mathbb{1} + C_d \vec{r}\cdot\vec{\Lambda}\right), \tag{4.1}$$

which generalizes Eq. (2.84). We define an orthonormal basis $\{\hat{e}_i\}_{i=1}^{d^2-1}$ in $\mathbb{R}^{d^2-1}$ such that $\vec{r} = \sum_{i=1}^{d^2-1} r_i \hat{e}_i$ and $\vec{\Lambda} = \sum_{i=1}^{d^2-1} \Lambda_i \hat{e}_i$ is a matrix-valued vector. By taking Λ_i Hermitian, the same property is secured for ρ. Trace 1 is secured once we require Λ_i to be traceless. Furthermore, generalizing the construction of the Pauli matrices, we use the Hilbert-Schmidt norm and impose the normalization condition $\mathrm{Tr}\,(\Lambda_i\Lambda_j) = 2\delta_{ij}$. With that, we calculate:

$$\begin{aligned}
\mathrm{Tr}\left[\left(\vec{r}_1\cdot\vec{\Lambda}\right)\left(\vec{r}_2\cdot\vec{\Lambda}\right)\right] &= \mathrm{Tr}\left[\sum_{i,j} r_{1,i} r_{2,j}\Lambda_i\Lambda_j\right] \\
&= \sum_{i,j} r_{1,i} r_{2,j}\,\mathrm{Tr}[\Lambda_i\Lambda_j] \\
&= \sum_i r_{1,i} r_{2,i}\cdot 2 \\
&= 2(\vec{r}_1\cdot\vec{r}_2).
\end{aligned} \tag{4.2}$$

Now, by demanding that pure states have $||\vec{r}|| = 1$, and using the equation above, we have

$$\begin{aligned}
\mathrm{Tr}\,[\rho_{\vec{r}}^2] &= \frac{1}{d^2}\mathrm{Tr}\,\left[\mathbb{1} + 2C_d\vec{r}\cdot\vec{\Lambda} + C_d^2(\vec{r}\cdot\vec{\Lambda})^2\right] \\
&= \frac{1}{d^2}\left(\mathrm{Tr}\,[\mathbb{1}] + C_d^2\mathrm{Tr}\,\left[(\vec{r}\cdot\vec{\Lambda})^2\right]\right) \\
&= \frac{1}{d} + \frac{2C_d^2}{d^2} = 1.
\end{aligned} \tag{4.3}$$

Solving for C_d, we get

$$C_d = \sqrt{\frac{d(d-1)}{2}}. \tag{4.4}$$

The relation between the purity of the system and the norm $||\vec{r}||$ is close at hand. Using the equation above:

$$\begin{aligned}\mathrm{Tr}\left[\rho_{\vec{r}}^2\right] &= \frac{1}{d^2}\left(\mathrm{Tr}\ [\mathbb{1}] + \frac{d(d-1)}{2}\mathrm{Tr}\ \left[(\vec{r}\cdot\vec{\Lambda})^2\right]\right) \\ &= \frac{1}{d} + \frac{d-1}{d}||\vec{r}||, \end{aligned} \tag{4.5}$$

which ascertains completely mixed states, $\mathrm{Tr}\left[\rho^2\right] = 1/d$, giving $||\vec{r}|| = 0$.

Eq. (4.1) shows how we can represent a linear operator acting on a d-dimensional state space on the basis $\{\mathbb{1}, \Lambda_1, \ldots, \Lambda_{d^2-1}\}$. This set gives the generators of the special unitary group of degree d, SU(d), the Lie group of unitary $d \times d$ matrices of determinant 1. Once their commutators, $[\Lambda_i, \Lambda_j]$, and anticommutators, $\{\Lambda_i, \Lambda_j\}$ are also self-adjoint operators, they can be expressed by the same basis. Generalizing the case for the Pauli matrices, one possible formulation is like

$$\left[\Lambda_i, \Lambda_j\right] = 2i\sum_{k=1}^{d^2-1} f_{ijk}\Lambda_k, \qquad \{\Lambda_i, \Lambda_j\} = \frac{4}{d}\delta_{ij}\mathbb{I} + 2\sum_{k=1}^{d^2-1} g_{ijk}\Lambda_k, \tag{4.6}$$

with the first term for the anticommutator chosen to satisfy $\mathrm{Tr}\left[\Lambda_i^2\right] = 2$. To determine the structure constants f_{ijk} and d_{ijk}, we multiply both sides of the equations above by Λ_k and take the trace. Using the Hilbert-Schmidt norm $\mathrm{Tr}\left[\Lambda_i, \Lambda_j\right] = 2\delta_{i,j}$, the commutator gives:

$$\begin{aligned}\mathrm{Tr}\ \left[\left[\Lambda_i, \Lambda_j\right]\Lambda_k\right] &= \mathrm{Tr}\ \left[2i\sum_{l=1}^{d^2-1} f_{ijl}\Lambda_l\Lambda_k\right] \\ &= 2i\sum_{l=1}^{d^2-1} f_{ijl}\mathrm{Tr}\ \left[\Lambda_l\Lambda_k\right] \\ &= 2i\sum_{l=1}^{d^2-1} f_{ijl}2\delta_{l,k} \\ &= 4if_{ijk}, \end{aligned} \tag{4.7}$$

and the anticommutator:

$$\begin{aligned}\mathrm{Tr}\ \left[\{\Lambda_i, \Lambda_j\}\Lambda_k\right] &= \mathrm{Tr}\ \left[\frac{4}{d}\delta_{ij}\Lambda_k + 2\sum_{l=1}^{d^2-1} g_{ijl}\Lambda_l\Lambda_k\right] \\ &= \frac{4}{d}\delta_{ij}\mathrm{Tr}\left[\Lambda_k\right] + 2\sum_{l=1}^{d^2-1} g_{ijl}\mathrm{Tr}\left[\Lambda_l\Lambda_k\right] \\ &= 2\sum_{l=1}^{d^2-1} g_{ijl}2\delta_{lk} \\ &= 4g_{ijk}. \end{aligned} \tag{4.8}$$

We obtain, then:

$$f_{ijk} = \frac{1}{4i}\mathrm{Tr}\left[\left[\Lambda_i, \Lambda_j\right]\Lambda_k\right], \qquad g_{ijk} = \frac{1}{4}\mathrm{Tr}\left[\{\Lambda_i, \Lambda_j\}\Lambda_k\right]. \tag{4.9}$$

Given a (d^2-1)-dimensional vector $\vec{r}$ corresponding to a d-dimensional density operator, Eq. (4.1) gives its full characterization upon a choice of the generators $\{\mathbb{1}, \Lambda_1, \ldots, \Lambda_{d^2-1}\}$. To do the converse, obtaining $\vec{r}$ given ρ, we start by calculating

$$\begin{aligned}\rho_{\vec{r}}\Lambda_j &= \frac{1}{d}\left[\Lambda_j + C_d(\vec{r}\cdot\Lambda)\Lambda_j\right] \\ &= \frac{1}{d}\left(\Lambda_j + C_d\sum_i r_i\Lambda_i\Lambda_j\right),\end{aligned} \tag{4.10}$$

notice that

$$\Lambda_j\rho_{\vec{r}} = \frac{1}{d}\left(\Lambda_j + C_d\sum_i r_i\Lambda_j\Lambda_i\right), \tag{4.11}$$

and obtain the anticommutator function:

$$\begin{aligned}\frac{1}{2}\{\rho_{\vec{r}}, \Lambda_j\} &= \frac{1}{2}(\rho_{\vec{r}}\Lambda_j + \Lambda_j\rho_{\vec{r}}) \\ &= \frac{1}{2d}\left(\Lambda_j + C_d\sum_i r_i\Lambda_i\Lambda_j + \Lambda_j + C_d\sum_i r_i\Lambda_j\Lambda_i\right) \\ &= \frac{1}{d}\left[\Lambda_j + \frac{C_d}{2}\sum_i r_i(\Lambda_i\Lambda_j + \Lambda_j\Lambda_i)\right] \\ &= \frac{1}{d}\left[\Lambda_j + \frac{C_d}{2}\sum_i r_i\{\Lambda_i, \Lambda_j\}\right] \\ &= \frac{1}{d}\left(\Lambda_j + \frac{2C_d}{d}\sum_i r_i\delta_{i,j}\mathbb{1} + C_d\sum_{i,k} r_i g_{ijk}\Lambda_k\right) \\ &= \frac{1}{d}\left(\Lambda_j + \frac{2C_d}{d}r_j\mathbb{1} + C_d\sum_{i,k} r_i g_{ijk}\Lambda_k\right).\end{aligned} \tag{4.12}$$

Now, by taking the trace of the equation above, we obtain:

$$\begin{aligned}\mathrm{Tr}\left[\frac{1}{2}\{\rho_{\vec{r}}, \Lambda_j\}\right] &= \frac{1}{d}\left(\mathrm{Tr}\left[\Lambda_j\right] + \frac{2C_d}{d}r_j\mathrm{Tr}\left[\mathbb{1}\right] + C_d\sum_{i,k} r_i g_{ijk}\mathrm{Tr}\left[\Lambda_j\right]\right) \\ &= \frac{1}{2}\left(\frac{2C_d}{d}r_j d\right) \\ &= \frac{2C_d}{d}r_j.\end{aligned} \tag{4.13}$$

By substituting $j \longrightarrow i$, we solve for r_i and, using Eq. (4.4), one gets

$$\begin{aligned} r_i &= \frac{d}{2C_d}\mathrm{Tr}\left[\frac{1}{2}\{\rho_{\vec{r}}, \Lambda_i\}\right] \\ &= \frac{d}{2C_d}\frac{1}{2}\left(\mathrm{Tr}\left[\rho_{\vec{r}}\Lambda_i\right] + \mathrm{Tr}\left[\Lambda_i\rho_{\vec{r}}\right]\right) \\ &= \frac{d}{2C_d}\frac{1}{2}\left(\mathrm{Tr}\left[\rho_{\vec{r}}\Lambda_i\right] + \mathrm{Tr}\left[\rho_{\vec{r}}\Lambda_i\right]\right) \\ &= \frac{d}{2C_d}\mathrm{Tr}\left[\rho_{\vec{r}}\Lambda_i\right] \\ &= \frac{d}{2}\sqrt{\frac{2}{d(d-1)}}\mathrm{Tr}\left[\rho_{\vec{r}}\Lambda_i\right] \\ &= \sqrt{\frac{d}{2(d-1)}}\mathrm{Tr}\left[\rho_{\vec{r}}\Lambda_i\right]. \end{aligned} \tag{4.14}$$

Such equation determines the $d^2 - 1$ components of the vector $\vec{r}$ upon the statistics given by the measurements of a set of informationally complete observables, like Λ_i, with $i = 1, 2, ..., d^2 - 1$. This gives the means by which one can encode a density operator as a real vector $\vec{r}$ in a real ball $B(\mathbb{R}^{d^2-1})$ of radius 1.

4.1.2 Observables

Observables, given by projective operators, also find a fit in this framework. We recall that an observable A respects $\sum_i A_i = \mathbb{1}$ and $\mathrm{Tr}\left(A_i A_j\right) = \delta_{ij}$, where A_i are the projectors. Once projectors have the same structure as the eigenstates of the observable, we start by writing

$$A_i = \frac{1}{d}\left(\mathbb{1} + C_d\hat{a}_i \cdot \vec{\Lambda}\right). \tag{4.15}$$

The closure condition gives

$$\begin{aligned} \sum_i A_i &= \frac{1}{d}\sum_i\left[\mathbb{1} + C_d\hat{a}_i \cdot \vec{\Lambda}\right] \\ &= \mathbb{1} + \frac{C_d}{d}\sum_i\left[\hat{a}_i \cdot \vec{\Lambda}\right] = \mathbb{1}, \end{aligned} \tag{4.16}$$

which, due to the orthogonality of the Λ_i, implies

$$\sum_i \hat{a}_i = \vec{0}. \tag{4.17}$$

Now, for the orthogonality of the projectors, we have, taking $i \neq j$:

$$\begin{aligned} \mathrm{Tr}\left(A_i A_j\right) &= \frac{1}{d^2}\mathrm{Tr}\left[\mathbb{1} + C_d\hat{a}_i \cdot \vec{\Lambda} + C_d\hat{a}_j \cdot \vec{\Lambda} + C_d^2(\hat{a}_i \cdot \vec{\Lambda})(\hat{a}_j \cdot \vec{\Lambda})\right] \\ &= \frac{1}{d} + \frac{d-1}{d}(\hat{a}_i \cdot \hat{a}_j) = 0, \end{aligned} \tag{4.18}$$

giving

$$\hat{a}_i \cdot \hat{a}_j = \frac{\delta_{ij} d - 1}{d - 1}. \tag{4.19}$$

Let a_i be the eigenvalues of A. Since A is traceless, $\sum_i a_i = 0$, and we express

$$\begin{aligned} A &= \sum_i a_i A_i \\ &= \sum_i a_i \left(\frac{1}{d} \left(\mathbb{1} + C_d \hat{a}_i \cdot \vec{\Lambda} \right) \right) \\ &= \frac{1}{d} \left(\sum_i a_i \right) \mathbb{1} + \frac{C_d}{d} \sum_i a_i \, \hat{a}_i \cdot \vec{\Lambda} \\ &= \vec{a} \cdot \vec{\Lambda}, \end{aligned} \tag{4.20}$$

using the definition:

$$\vec{a} \coloneqq \frac{C_d}{d} \sum_i a_i \hat{a}_i. \tag{4.21}$$

The geometrical structure defined by Eq. (4.19) represents a simplex embedded within the Bloch hypersphere. Recall that a simplex is the generalization of a triangle or tetrahedron, serving as the simplest polytope in a given dimension. Each of the vectors $\hat{a}_i$ touches the hypersphere's surface, and by taking the convex hull of these points, we obtain a regular simplex whose center coincides with the hypersphere's center, corresponding to the completely mixed state. We denote the $(d-1)$-dimensional simplex defined by the observable A as $\triangle_A$.

We now have enough tools to express the probability assigned to obtaining an outcome a_i in a measurement, given the preparation of a state $\rho_{\vec{r}}$. This formalism's elegance is displayed by accomplishing this through an expression of d and the inner product $\hat{a}_i \cdot \vec{r}$:

$$\begin{aligned} p_{a_i} &= \mathrm{Tr}[A_i \rho_{\vec{r}}] \\ &= \mathrm{Tr}\left[\frac{1}{d} \left(\mathbb{1} + C_d \hat{a}_i \cdot \vec{\Lambda} \right) \cdot \frac{1}{d} \left(\mathbb{1} + C_d \vec{r} \cdot \vec{\Lambda} \right) \right] \\ &= \frac{1}{d^2} \mathrm{Tr}\left[\mathbb{1} + C_d \hat{a}_i \cdot \vec{\Lambda} + C_d \vec{r} \cdot \vec{\Lambda} + C_d^2 (\hat{a}_i \cdot \vec{\Lambda})(\vec{r} \cdot \vec{\Lambda}) \right] \\ &= \frac{1}{d^2} \left(\mathrm{Tr}[\mathbb{1}] + C_d^2 \, \mathrm{Tr}[(\hat{a}_i \cdot \vec{\Lambda})(\vec{r} \cdot \vec{\Lambda})] \right) \\ &= \frac{1}{d^2} \left(d + 2C_d^2 (\hat{a}_i \cdot \vec{r}) \right) \\ &= \frac{1}{d} \left(1 + \frac{2C_d^2}{d} (\hat{a}_i \cdot \vec{r}) \right) \\ &= \frac{1}{d} \left[1 + (d-1)(\hat{a}_i \cdot \vec{r}) \right]. \end{aligned} \tag{4.22}$$

A remarkable fact about this formalism is that only in the case $d = 2$ is the Bloch ball fully populated by vectors corresponding to valid density operators. To understand why this occurs, consider a unit vector $\hat{r}_\phi$ located between $\hat{a}_i$ and $\hat{a}_j$, rotated relative to $\hat{a}_i$ by an angle

ϕ. This vector can be expressed in terms of $\hat{a}_i$ and $\hat{a}_j$ using the spherical linear interpolation function, yielding:

$$\hat{r}_\phi = \frac{\sin(\theta - \phi)\hat{a}_i + \sin\phi\hat{a}_j}{\sin\theta}, \tag{4.23}$$

where $\theta = \arccos(\hat{a}_i \cdot \hat{a}_j)$, as defined by Eq. (4.19). For cases where $d > 2$, a numerical evaluation of p_{a_j}, performed by inserting $\hat{r}_\phi$ into Eq. (4.22), returns negative probabilities for $\phi \in (0, \theta)$. Indeed, using Eq. (4.1), one can see that the matrices defined by such vectors are not positive semidefinite and, therefore, do not correspond to valid physical states.

This limitation arises because the matrices Λ_i for $d > 2$ are not rotationally invariant. As a result, a simplex cannot simply be rotated within its own hyperplane without some vertices corresponding to invalid physical states, as illustrated in the example above. It is physically nonsensical to define an observable whose eigenstates are not valid quantum states.

Completely characterizing the region described by valid Bloch vectors within the hypersphere is a highly nontrivial task. The characterization depends on the choice of the generators of SU(d), and the resulting shape is generally not spherical. Nevertheless, since any mixture of two physical states must also correspond to a valid physical state, we can at least confirm that the shape is convex.

A result presented in Ref. [93] shows that, within the unit-radius Bloch ball $B(\mathbb{R}^{d^2-1})$, there exists a smaller ball $B(\mathbb{R}^{d^2-1})$, centered at the origin, that is fully populated by valid Bloch vectors. This inscribed ball has a radius of $1/(d-1)$ and corresponds to the ball inscribed in every simplex representative of a basis. For $d = 2$, this radius becomes 1, reflecting the familiar geometry of the Bloch sphere for qubits.

We conclude this review by expressing the map Φ_A for singlepartite states in terms of real vectors. Using equations (4.15), (4.17), and (4.15), we obtain

$$\begin{aligned}
\Phi_A(\rho_{\vec{r}}) &= \sum_i p_{a_i} A_i \\
&= \sum_i \frac{1}{d^2}\left[1 + (d-1)(\hat{a}_i \cdot \vec{r})\right]\left(\mathbb{1} + C_d \hat{a}_i \cdot \vec{\Lambda}\right) \\
&= \sum_i \frac{1}{d^2}\left[\mathbb{1} + (d-1)(\hat{a}_i \cdot \vec{r})\mathbb{1} + C_d(\hat{a}_i \cdot \vec{\Lambda}) + (d-1)C_d(\hat{a}_i \cdot \vec{r})(\hat{a}_i \cdot \vec{\Lambda})\right] \\
&= \frac{1}{d^2}\left[\sum_i \mathbb{1} + \sum_i (d-1)(\hat{a}_i \cdot \vec{r})\mathbb{1} + \sum_i C_d(\hat{a}_i \cdot \vec{\Lambda}) + \sum_i (d-1)C_d(\hat{a}_i \cdot \vec{r})(\hat{a}_i \cdot \vec{\Lambda})\right] \\
&= \frac{1}{d^2}\left[d\mathbb{1} + (d-1)C_d(\hat{a}_i \cdot \vec{r})(\hat{a}_i \cdot \vec{\Lambda})\right] \\
&= \frac{1}{d}\mathbb{1} + \frac{(d-1)C_d}{d^2}(\hat{a}_i \cdot \vec{r})(\hat{a}_i \cdot \vec{\Lambda}) \\
&= \frac{1}{d}(\mathbb{1} + C_d P_A \vec{r} \cdot \vec{\Lambda}).
\end{aligned} \tag{4.24}$$

where we define

$$P_A \bullet := \frac{d-1}{d} \sum_i (\hat{a}_i \cdot \bullet) \hat{a}_i. \tag{4.25}$$

This result was first obtained in [79].

4.2 GEOMETRIC BA CRITERION

Once this has not been done in the literature before, let us inspect the behavior of the map P_A in detail. We verify that its action on a vector corresponding to A_i is innocuous. Using Eqs. (4.17) and (4.19),

$$\begin{aligned} P_A \hat{a}_i &= \frac{d-1}{d} \sum_j (\hat{a}_j \cdot \hat{a}_i) \hat{a}_j \\ &= \frac{d-1}{d} \sum_j \left(\frac{\delta_{ij} d - 1}{d-1} \right) \hat{a}_j \\ &= \frac{d-1}{d} \left(\hat{a}_i - \frac{1}{d-1} \sum_{i \neq j} \hat{a}_j \right) \\ &= \frac{d-1}{d} \left(\hat{a}_i + \frac{\hat{a}_i}{d-1} \right) \\ &= \hat{a}_i. \end{aligned} \tag{4.26}$$

This result allows the study of the case for a vector $\vec{r}_A \in \triangle_A$, that is, $\vec{r}_A = \sum_i r_i \hat{a}_i$:

$$\begin{aligned} P_A \vec{r}_A &= P_A \sum_i r_i \hat{a}_i \\ &= \sum_i r_i P_A \hat{a}_i \\ &= \sum_i r_i \hat{a}_i \\ &= \vec{r}_A, \end{aligned} \tag{4.27}$$

revealing the same behavior and generalizing the previous result. It is straightforward to see the idempotence for vectors within $\triangle_A$:

$$\begin{aligned} P_A^2 \vec{r}_A &\equiv P_A P_A \vec{r}_A \\ &= P_A \vec{r}_A \\ &= \vec{r}_A. \end{aligned} \tag{4.28}$$

Now, for a generic rector $\vec{r}$, we obtain

$$\begin{aligned} P_A P_A \vec{r} &= P_A \frac{d-1}{d} \sum_i (\hat{a}_i \cdot \vec{r}) \hat{a}_i \\ &= \frac{d-1}{d} \sum_i (\hat{a}_i \cdot \vec{r}) P_A \hat{a}_i \\ &= \frac{d-1}{d} \sum_i (\hat{a}_i \cdot \vec{r}) \hat{a}_i \\ &= P_A \vec{r}, \end{aligned} \tag{4.29}$$

revealing P_A as an idempotent projector.

A result that will be particularly important for this chapter is that:

$$||\vec{r}|| \geq ||P_A \vec{r}||. \tag{4.30}$$

To prove this, we start by considering that once the $\triangle_A$ is contained in a $d-1$ dimensional hypersurface while the Bloch hypersphere's dimension is d^2-1, there are always infinitely many vectors within the hypersphere that are orthogonal to $\triangle_A$. In other words, there are infinitely many vectors $\vec{r}_\perp \notin \triangle_A$ such that $\vec{r}_\perp \cdot \hat{a}_i = 0\ (\forall i)$. Now, we consider a vector $\vec{r}$ with $||\vec{r}|| \leq 1$ and write it like $\vec{r} = \vec{r}_{||} + \vec{r}_\perp$ such that $\vec{r}_{||} \cdot \vec{r}_\perp = 0$. An explicit construction of $\vec{r}_{||}$ involves using the Gram-Schmidt process to construct an orthonormal basis $\{\hat{u}_i\}_{i=1}^{d-1}$ from the basis $\{\hat{a}_i\}_{i=1}^{d}$ such that

$$\vec{r}_{||} = \sum_{i=1}^{d-1} (\hat{u}_i \cdot \vec{r}) \hat{u}_i. \tag{4.31}$$

Continuing, we manipulate $\vec{r}$ like

$$\vec{r} = \alpha \hat{r}_{||} + \beta \hat{r}_\perp \tag{4.32}$$

with $\{\alpha, \beta\} \in \mathbb{R}$ so that we can write the norm

$$||\vec{r}|| = \sqrt{\alpha^2 + \beta^2}. \tag{4.33}$$

Then, using the linearity of P_A, we evaluate

$$\begin{aligned} P_A \vec{r} &= \alpha P_A \hat{r}_{||} + \beta P_A \hat{r}_\perp \\ &= \alpha \hat{r}_{||}. \end{aligned} \tag{4.34}$$

It is possible that $\hat{r}_{||} \notin \triangle_A$ by falling in the simplex hyperplane but beyond the simplex limits. In cases like that, $\vec{r}_{||} = k\vec{r}_A$, with $k \in \mathbb{R}$, so that Eq. (4.27) guarantees the result above, and we have:

$$||P_A \vec{r}|| = \sqrt{\alpha}. \tag{4.35}$$

Given that

$$\sqrt{\alpha^2 + \beta^2} \geq \sqrt{\alpha}, \tag{4.36}$$

the inequality (4.30) is proven, showing that the action of P_A never increases the norm of $\vec{r}$.

To make this more complete, we determine $||P_A\vec{r}||$. Taking its squared form,

$$\begin{aligned}||P_A\vec{r}||^2 &= P_A\vec{r}\cdot P_A\vec{r}\\ &= \left(\frac{d-1}{d}\sum_i(\hat{a}_i\cdot\vec{r})\hat{a}_i\right)\cdot\left(\frac{d-1}{d}\sum_j(\hat{a}_j\cdot\vec{r})\hat{a}_j\right)\\ &= \left(\frac{d-1}{d}\right)^2\sum_i\sum_j(\hat{a}_i\cdot\vec{r})(\hat{a}_j\cdot\vec{r})(\hat{a}_i\cdot\hat{a}_j).\end{aligned} \tag{4.37}$$

Considering Eq. (4.19), we split the summations for $i=j$ and $i\neq j$ arriving at

$$||P_A\vec{r}||^2 = \left(\frac{d-1}{d}\right)^2\left(\sum_{i=1}^{d}(\hat{a}_i\cdot\vec{r})^2 - \frac{1}{d-1}\sum_{i\neq j}(\hat{a}_i\cdot\vec{r})(\hat{a}_j\cdot\vec{r})\right). \tag{4.38}$$

Now, we take the square root of it:

$$||P_A\vec{r}|| = \sqrt{\left(\frac{d-1}{d}\right)^2\left(\sum_{i=1}^{d}(\hat{a}_i\cdot\vec{r})^2 - \frac{1}{d-1}\sum_{i\neq j}(\hat{a}_i\cdot\vec{r})(\hat{a}_j\cdot\vec{r})\right)}. \tag{4.39}$$

Just as worthy of scrutiny is whether P_A is an orthogonal map. The conditions to be satisfied are $P_A^2 = P_A = P_A^\dagger$. Since idempotence has already been proven, we must show that P_A is self-adjoint. Let $\vec{s}$ be a vector with the same dimension of $\vec{r}$ and $||\vec{s}|| \leq 1$, if P_A is self-adjoint, it is symmetric with respect to the inner product: $P_A\vec{r}\cdot\vec{s} = \vec{r}\cdot P_A\vec{s}$. We check that:

$$\begin{aligned}P_A\vec{r}\cdot\vec{s} &= \left(\frac{d-1}{d}\sum_{i=1}^{d}(\hat{a}_i\cdot\vec{r})\hat{a}_i\right)\cdot\vec{s}\\ &= \frac{d-1}{d}\sum_{i=1}^{d}(\hat{a}_i\cdot\vec{r})(\hat{a}_i\cdot\vec{s})\\ &= r\cdot\left(\frac{d-1}{d}\sum_{i=1}^{d}(\hat{a}_i\cdot\vec{s})\hat{a}_i\right)\\ &= \vec{r}\cdot P_A\vec{s},\end{aligned} \tag{4.40}$$

confirming the orthogonality. This implies that P_A projects a generic vector in the closest point possible of the plane of $\triangle_A$, enabling the equation

$$(P_A\vec{r})\cdot(\vec{r} - P_A\vec{r}) = 0. \tag{4.41}$$

Once a nonselective measurement protocol should always retrieve a physical state, this result gives a different way of testing if a given vector $\vec{r}$ corresponds to a valid density operator: if $P_A\vec{r}\notin\triangle_A$ then $\vec{r}$ does not correspond to a valid density operator.

We conclude this section by restating BA's criterion in geometrical terms:

Geometric BA criterion for realism. *A is real for $\rho_{\vec{r}}$ iff*

$$\vec{r} = P_A\vec{r} \quad\Longleftrightarrow\quad \vec{r}\in\triangle_A. \tag{4.42}$$

4.3 EMERGENCE OF REALISM

A prototypical scenario illustrating the emergence of realism through measurements of noncommuting observables involves sequential pairwise measurements. In this experimental protocol, two agents alternately perform nonselective measurements of noncommuting observables A and B. For example, Eva measures A without recording the outcome, followed by Eve measuring B, also without preserving the result. This procedure constitutes a protocol of pairwise measurements. By repeating these steps, we generate a sequence of pairwise measurements. With that in mind, our main result in this session is stated:

Theorem. *For any state $\rho_{\vec{r}}$, there always exists a sufficiently large number of sequential pairwise measurements of noncommuting observables A and B that can reduce the irrealism $\mathfrak{I}_X$ of any observable X to an arbitrarily small value.*

Proof.—For the first part of this proof, we start by examining the bound for the difference of von Neumann entropies between two quantum states ρ and σ of dimension d defined by Fanne's inequality [94]:

$$|S(\rho) - S(\sigma)| \leq T \ln (d-1) + H_{\text{bin}}(T), \tag{4.43}$$

with $d \in \mathbb{N}_{\geq 2}$. Above, $H_{\text{bin}}(T) = -T \ln T - (1-T) \ln (1-T)$ is Shannon's binary entropy, and the parameter T is defined as

$$T = T(\rho, \sigma) = \frac{1}{2}||\rho - \sigma||_1, \tag{4.44}$$

where $||\rho - \sigma||_1$ is the trace distance between ρ and σ, or the Schatten 1-norm.

Such norm is expressed by

$$||O||_1 = \text{Tr}\sqrt{O^\dagger O}, \tag{4.45}$$

with O a generic operator. To clarify its meaning, taking O Hermitian, we have:

$$\begin{aligned} ||O||_1 &= \text{Tr}\sqrt{O^\dagger O} \\ &= \text{Tr}\sqrt{O^2} \\ &= \text{Tr}\,|O| \\ &= \sum_i |o_i|, \end{aligned} \tag{4.46}$$

which is just the sum of the module of the eigenvalues o_i of O. In the case where O is also positive semidefinite, we have $|o_i| = o_i$ $(\forall i)$. For our context, this implies

$$||\rho - \sigma||_1 = \sum_i |\lambda_i|, \tag{4.47}$$

such that λ_i are the eigenvalues of $\rho - \sigma$, allowing for the interpretation of the trace norm as a measure of distinguishability between ρ and σ. In fact, due to the unity trace of density

operators, the equation above is bounded by $0 \leq ||\rho - \sigma||_1 \leq 2$, with the first equality holding *iff* $\rho = \sigma$ and the second holding *iff* ρ and σ are orthogonal. This fact, together with Eq. (4.44), allows the conclusion that $T \in [0, 1]$.

Resorting to computational methods, it can be graphically verified that $H_{\text{bin}}(T) \leq \sqrt{2T}$. Furthermore, since $T \leq \sqrt{T}$, we derive

$$\begin{aligned} T \ln(d-1) + H_{\text{bin}}(T) &\leq \sqrt{T} \ln(d-1) + \sqrt{2T} \\ &= \left(\sqrt{2} + \ln(d-1)\right) \sqrt{T}, \end{aligned} \tag{4.48}$$

which allows for a new bound:

$$|S(\rho) - S(\sigma)| \leq \left(\sqrt{2} + \ln(d-1)\right) \sqrt{T}. \tag{4.49}$$

To proceed, we introduce the Schatten p-norm,

$$||O||_p = \left(\text{Tr}\left[(O^\dagger O)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}, \tag{4.50}$$

so that we can bring Hölder's inequality:

$$||OU||_1 \leq ||O||_p ||U||_q, \tag{4.51}$$

where U is a generic operator with the same dimension of O, $p, q \in [1, \infty]$, and $\frac{1}{p} + \frac{1}{q} = 1$. A convenient result for this proof is given by setting $p = q = 2$ and $U = \mathbb{1}/d$. Explicitly, Schatten's 2-norm gives $||O||_2 = \sqrt{\text{Tr}\left[O^\dagger O\right]}$, and we evaluate

$$\begin{aligned} ||U||_2 &= \sqrt{\text{Tr}\left[U^\dagger U\right]} \\ &= \sqrt{\frac{1}{d^2}\text{Tr}\left[\mathbb{1}\right]} \\ &= \frac{1}{\sqrt{d}}. \end{aligned} \tag{4.52}$$

This result, together with the specified parameters, gives the inequality

$$||O||_1 \leq \sqrt{d}||O||_2, \tag{4.53}$$

enabling a final bound. From Eqs. (4.44) and (4.49) we rewrite

$$\begin{aligned} |S(\rho) - S(\sigma)| &\leq \left(\sqrt{2} + \ln(d-1)\right) \sqrt{\frac{1}{2}||\rho - \sigma||_1} \\ &= \left(1 + \frac{\ln(d-1)}{2}\right) \sqrt{||\rho - \sigma||_1}, \end{aligned} \tag{4.54}$$

such that with Eq. (4.53) we derive:

$$|S(\rho) - S(\sigma)| \leq d^{1/4} \left(1 + \frac{\ln(d-1)}{\sqrt{2}}\right) \sqrt{||\rho - \sigma||_2}. \tag{4.55}$$

At this point, we bring the generalized Bloch sphere formalism to rewrite the last bound obtained. First, we adjust the notation by taking $\rho \to \rho_{\vec{r}_1}$ and $\sigma \to \rho_{\vec{r}_2}$. Now, calculating

$$\begin{aligned}\rho_{\vec{r}_1} - \rho_{\vec{r}_2} &= \frac{1}{d}\left(\mathbb{1} + C_d\vec{r}_1 \cdot \vec{\Lambda}\right) - \frac{1}{d}\left(\mathbb{1} + C_d\vec{r}_2 \cdot \vec{\Lambda}\right) \\ &= \frac{1}{d}\left[\mathbb{1} + C_d\vec{r}_1 \cdot \vec{\Lambda} - \mathbb{1} - C_d\vec{r}_2 \cdot \vec{\Lambda}\right] \\ &= \frac{1}{d}\left[C_d(\vec{r}_1 - \vec{r}_2) \cdot \vec{\Lambda}\right] \\ &= \frac{C_d}{d}(\vec{r}_1 - \vec{r}_2) \cdot \vec{\Lambda}, \end{aligned} \tag{4.56}$$

allows to evaluate, together with Eqs. (4.2) and (4.4), the term:

$$\begin{aligned}||\rho_{\vec{r}_1} - \rho_{\vec{r}_2}||_2 &= \sqrt{\mathrm{Tr}[(\rho_{\vec{r}_1} - \rho_{\vec{r}_2})^\dagger(\rho_{\vec{r}_1} - \rho_{\vec{r}_2})]} \\ &= \sqrt{\mathrm{Tr}\left[\left(\frac{C_d}{d}(\vec{r}_1 - \vec{r}_2) \cdot \vec{\Lambda}\right)^2\right]} \\ &= \frac{C_d}{d}\sqrt{\mathrm{Tr}\left[(\vec{r}_1 - \vec{r}_2) \cdot \vec{\Lambda}\,(\vec{r}_1 - \vec{r}_2) \cdot \vec{\Lambda}\right]} \\ &= \frac{C_d}{d}\sqrt{2\,(\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2)} \\ &= \sqrt{\frac{d-1}{d}}\,||\vec{r}_1 - \vec{r}_2||. \end{aligned} \tag{4.57}$$

Now, we rewrite the bound,

$$\begin{aligned}|S(\rho) - S(\sigma)| &\leq d^{1/4}\left(1 + \frac{\ln(d-1)}{\sqrt{2}}\right)\sqrt{||\rho - \sigma||_2} \\ &= d^{1/4}\left(1 + \frac{\ln(d-1)}{\sqrt{2}}\right)\sqrt{\sqrt{\frac{d-1}{d}}\,||\vec{r}_1 - \vec{r}_2||} \\ &= (d-1)^{1/4}\left(1 + \frac{\ln(d-1)}{\sqrt{2}}\right)\sqrt{||\vec{r}_1 - \vec{r}_2||}, \end{aligned} \tag{4.58}$$

resulting in

$$|S(\rho) - S(\sigma)| \leq g(d)\sqrt{||\vec{r}_1 - \vec{r}_2||}, \tag{4.59}$$

with the definition of the monotonically increasing function

$$g(d) \coloneqq (d-1)^{1/4}\left(1 + \frac{\ln(d-1)}{\sqrt{2}}\right). \tag{4.60}$$

For the second part of this proof, we consider a scenario of sequential pairwise measurements of the observables $\{A, B\}$. Upon a nonselective measurement of $A = \vec{a} \cdot \vec{\Lambda}$ over the state $\rho_{\vec{r}}$, the state $\Phi_A(\rho_{\vec{r}})$ is obtained. Then, for the nonselective measurement of $B = \vec{b} \cdot \vec{\Lambda}$, a similar mathematical structure holds, resulting in the state

$$\Phi_B\Phi_A(\rho_{\vec{r}}) = \frac{1}{d}\big(\mathbb{1} + C_d P_B P_A \vec{r} \cdot \vec{\Lambda}\big). \tag{4.61}$$

Upon inspection, the transition implied by this step in the sequence of measurements is given in the Bloch formalism by $\vec{r} \longrightarrow \vec{r}_1 \equiv P_B P_A \vec{r}$.

It follows from Eq. (4.30) that $||\vec{r}|| \leq ||P_A\vec{r}|| \leq ||P_A P_B \vec{r}||$. With that in mind, we write

$$||\vec{r}|| = \epsilon_1 ||\vec{r}|| \tag{4.62}$$

with $\epsilon_1 \in \mathbb{R}_{[0,1]}$. Still from Eq. (4.30), one can verify that $||\vec{r}|| = ||\vec{r}_1||$ when $||\vec{r}|| = 0$, the case for maximally mixed states. Besides that, given the context $\{\rho, A, B\}$, the same equality happens when A and B share the same simplex. In this circumstance, it holds that

$$[A, B] = [A, \rho] = [B, \rho] = 0 \implies \vec{r} = \vec{r}_1, \tag{4.63}$$

and for states such that $||\vec{r}|| \neq 0$, we have $\epsilon_1 = 1$.

Working with the general cases, after n sequential pairwise measurements, the state is given by

$$\vec{r}_n \equiv (P_B P_A)^n \vec{r}, \tag{4.64}$$

such that

$$||\vec{r}_n|| = \left(\Pi_{k=1}^{n} \epsilon_k\right) ||\vec{r}|| \quad (\forall\, n \in \mathbb{N}_{>0})\,. \tag{4.65}$$

It is important to notice that once $(P_B P_A)^n \rho \leq P_B P_A \rho$, the operator $P_B P_A$ is not idempotent and, thus, nonorthogonal. Once P_A is applied, the resulting state's corresponding Bloch vector is of the form $\vec{r}_A = \sum_i r_i \hat{a}_i$, and, after P_B's action, of the form $\vec{r}_B = \sum_i r_i \hat{b}_i$. With the decreasing action on the norm and the iterative projection over different simplexes, it also holds that $\epsilon_k \geq \epsilon_{k+1}$ whenever the conditions for $||\vec{r}|| = ||\vec{r}_1||$ are not met.

Turning to the definition of irrealism (Eq. (3.5)), we now consider the irrealism of a generic observable $X = \vec{x} \cdot \vec{\Lambda}$:

$$\mathfrak{I}_X(\rho_{\vec{r}_n}) = S\big(\Phi_X(\rho_{\vec{r}_n})\big) - S(\rho_{\vec{r}_n}), \tag{4.66}$$

such that, relying on the bound (4.58), we can write

$$\begin{aligned}
\mathfrak{I}_X(\rho_{\vec{r}_n}) &\leq g(d)\sqrt{||\vec{r}_n - P_X\vec{r}_n||} \\
&= g(d)\sqrt{||(1 - P_X)\vec{r}_n||} \\
&= g(d)\sqrt{||(1 - P_X)(P_B P_A)^n \vec{r}||} \\
&= g(d)\sqrt{\frac{||(1 - P_X)(P_B P_A)^n \vec{r}||}{||(P_B P_A)^n \vec{r}||}}\sqrt{||(P_B P_A)^n \vec{r}||} \\
&= g(d)\sqrt{||(1 - P_X)\hat{r}_B||}\,\sqrt{||(P_B P_A)^n \vec{r}||},
\end{aligned} \tag{4.67}$$

where we defined

$$\hat{r}_B \coloneqq \hat{r}_n = \frac{\vec{r}_n}{||\vec{r}_n||}. \tag{4.68}$$

Recall that the subindex B is used once after $P_A P_B$'s action, the resulting vector falls on $\triangle_B$.

Bound (4.67) makes explicit that whenever $\vec{r} = 0$, irrealism for any X vanishes, regardless of the choices on A, B, and X, and regardless of the magnitude of n, justifying why fully mixed states $\rho = \mathbb{1}/d$ are termed classical states.

For cases where $\vec{r} \cdot \hat{a}_i = 0$ $(\forall i)$, rendering $P_A\vec{r} = 0$, we find $\mathfrak{I}_X(\rho_{\vec{r}_n}) = 0$ even for $n = 1$, showing that a single measurement is sufficient for establishing classical realism.

Another instance worth mentioning is when $[X, B] = 0$. Once, P_B projects a vector into $\triangle_B$, B is always real for $\rho_{\vec{r}_n}$, and for every X commuting with B we obtain $\mathfrak{I}_X(\rho_{\vec{r}_n}) = 0$.

We finish this proof under the assumption that $[A, B] \neq 0$. Turning to the notation $\epsilon_k = \mathrm{O}(\epsilon)$, with $\epsilon \in \mathbb{R}_{[0,1)}$, we argue that

$$\begin{aligned} \frac{||\vec{r}_n||}{||\vec{r}||} &= \Pi_{k=1}^{n} \epsilon_k \\ &\leq \mathrm{O}(\epsilon^n) \\ &= [\mathrm{O}(\epsilon)]^n. \end{aligned} \tag{4.69}$$

Now, we consider $\mathfrak{I}_X(\rho_{\vec{r}_n}) \leq \delta$ when $n \geq [n_{\min}]$, where $[\bullet]$ stands integer part of $\bullet$. To find n_{min} we plug Eq. (4.69) into Eq. (4.67):

$$\begin{aligned} \delta &= g(d)\sqrt{||(1-P_X)\hat{r}_B||}\,\sqrt{[\mathrm{O}(\epsilon)]^{n_{min}}||\vec{r}||} \\ &= g(d)\sqrt{||\vec{r}||\,||(1-P_X)\hat{r}_B||}\,[\mathrm{O}(\epsilon)]^{\frac{n_{min}}{2}}, \end{aligned} \tag{4.70}$$

rearrange it like

$$\frac{\delta}{g(d)\sqrt{||\vec{r}||\,||(1-P_X)\hat{r}_B||}} = [\mathrm{O}(\epsilon)]^{\frac{n_{min}}{2}} \tag{4.71}$$

and take $\log \bullet$ on both sides,

$$\begin{aligned} \log\left(\frac{\delta}{g(d)\sqrt{||\vec{r}||\,||(1-P_X)\hat{r}_B||}}\right) &= \log\left([\mathrm{O}(\epsilon)]^{\frac{n_{min}}{2}}\right) \\ &= \frac{n_{\min}}{2}\log[\mathrm{O}(\epsilon)]. \end{aligned} \tag{4.72}$$

We solve for $n_{\min}$ to arrive at our final expression:

$$n_{\min} = 2\frac{\log\left(\frac{\delta}{g(d)\sqrt{||\vec{r}||\,||(1-P_X)\hat{r}_B||}}\right)}{\log[\mathrm{O}(\epsilon)]}. \tag{4.73}$$

The equation above guarantees that, given δ vanishingly small, with $n_{\min}$ pairwise sequential measurements of A and B, such that $[A, B] \neq 0$, the irrealism of X, $\mathfrak{I}_X$, becomes smaller than δ, as we wanted to show. ∎

4.4 INFORMATION-IRREALISM BOUND

The proof above relied solely on bounds defined in terms of geometric entities, as functions of vectors, projectors, and the function $g(d)$, without requiring entropy calculations.

However, this approach comes at the cost of the bound not being strict. For those interested in a more precise understanding of how $\mathfrak{I}_X$ behaves when $||\vec{r}_n||$ does not approach zero, alternative bounds become useful.

We now proceed to derive a strict bound and analyze the sequential pairwise measurement scenario under its constraints.

To express the von Neumann entropy of $\rho_{\vec{r}}$ in terms of $\vec{r}$, we recall that $S(\rho) = H(\lambda_i)$, where λ_i are the eigenvalues of ρ, and $H(\lambda_i) = -\sum_i \lambda_i \log \lambda_i$ is the Shannon entropy. Since ρ is Hermitian, it can always be represented in a purely diagonal form, and the bases that diagonalize ρ correspond to observables for which ρ exhibits no coherence. Geometrically, these observables, denoted by R, define simplexes within which $\vec{r}$ lies. With the eigenvalues λ_i corresponding to the probabilities p_i of obtaining the outcomes of an observable R given $\rho_{\vec{r}}$, we use Eq. (4.22) to obtain

$$S(\rho_{\vec{r}}) = H(p_i) = H\left(\frac{1}{d}\left[1 + (d-1)\hat{r}_i \cdot \vec{r}\right]\right), \tag{4.74}$$

with vectors $\hat{r}_i$ representing the projectors of R in the Bloch sphere. Fully mixed states, with $\vec{r} = 0$, cause $\{p_i\} = 1/d$, maximizing the entropy:

$$S\left(\rho_{\vec{0}}\right) = H\left(\frac{1}{d}\right) = \log d. \tag{4.75}$$

Pure states, $\vec{r} = \hat{r}_j$, minimize it. Using Eq. (4.19):

$$\begin{aligned} S\left(\rho_{\hat{r}_i}\right) &= H\left(\frac{1}{d}\left[1 + (d-1)\hat{r}_i \cdot \hat{r}_i\right]\right) \\ &= \log 1 + \sum_{i \neq j}\left(\frac{1}{d}\left[1 + (d-1)\hat{r}_i \cdot \hat{r}_j\right]\right)\log\left(\frac{1}{d}\left[1 + (d-1)\hat{r}_i \cdot \hat{r}_j\right]\right) \\ &= \sum_{i \neq j}\left(\frac{1}{d}\left[1 - \frac{d-1}{d-1}\right]\right)\log\left(\frac{1}{d}\left[1 - \frac{d-1}{d-1}\right]\right) \\ &= 0 \log 0 \\ &:= 0. \end{aligned} \tag{4.76}$$

Given a physical state, to assess the maximal irrealism assigned to any observable, one maximizes Eq. (4.66) over every X:

$$\mathfrak{I}_X(\rho) \leq \max_{\{X\}} \mathfrak{I}_X(\rho) = \log d - S(\rho) = I(\rho). \tag{4.77}$$

This result demonstrates that the maximal irrealism associated with ρ is bounded by the information content of ρ, providing a strict limit on irrealism. In terms of real vectors, this becomes

$$\mathfrak{I}_X(\rho_{\vec{r}}) \leq \log d - H\left(\frac{1}{d}\left[1 + (d-1)\hat{r}_i \cdot \vec{r}\right]\right), \tag{4.78}$$

which, together with the Eq. (4.75), corroborates to the idea that the state $\mathbb{1}/d$ determines full realism, once $\mathfrak{I}_X\left(\rho_{\vec{0}}\right) = 0\ (\forall X)$.

Using Eqs. (4.24), (4.66) and (4.74), we display the exact value for $\mathfrak{I}_X\left(\rho_{\vec{r}}\right)$:

$$\mathfrak{I}_X(\rho_{\vec{r}}) = H\left(\frac{1}{d}\left[1 + (d-1)\hat{x}_i \cdot P_X\vec{r}\right]\right) - H\left(\frac{1}{d}\left[1 + (d-1)\hat{r}_i \cdot \vec{r}\right]\right), \tag{4.79}$$

with $\hat{x}_i$ corresponding to the projectors X_i. The exact value for $\mathfrak{I}_X\left(\rho_{\vec{r}_n}\right)$ is enabled by this equation simply by replacing $\vec{r} \to \vec{r}_n$, as long as $\vec{r}_n$ is characterized.

Determining $(P_B P_A)^n \vec{r} = \vec{r}_n$ given a context $\{\rho_{\vec{r}}, A, B\}$ involves the study of a pattern expressed by the following iterative process:

$$\begin{aligned}
P_A\vec{r} = &\quad \frac{d-1}{d}\sum_i (\hat{a}_i \cdot \vec{r})\hat{a}_i \\
P_B P_A\vec{r} = \vec{r}_1 &\quad \left(\frac{d-1}{d}\right)^2 \sum_j \sum_i (\hat{a}_i \cdot \vec{r})(\hat{a}_i \cdot \hat{b}_j)\hat{b}_j \\
P_A\vec{r}_1 = &\quad \left(\frac{d-1}{d}\right)^3 \sum_k \sum_j \sum_i (\hat{a}_i \cdot \vec{r})(\hat{a}_i \cdot \hat{b}_j)(\hat{b}_j \cdot \hat{a}_k)\hat{a}_k \\
P_B P_A\vec{r}_1 = \vec{r}_2 = &\quad \left(\frac{d-1}{d}\right)^4 \sum_l \sum_k \sum_j \sum_i (\hat{a}_i \cdot \vec{r})(\hat{a}_i \cdot \hat{b}_j)(\hat{b}_j \cdot \hat{a}_k)(\hat{a}_k \cdot \hat{b}_l)\hat{b}_l,
\end{aligned} \tag{4.80}$$

and so on. One can verify that the analytic formula capturing this pattern is:

$$\vec{r}_n = \left(\frac{d-1}{d}\right)^{2k} \sum_{i_1, i_2, \ldots, i_{2k+1}} (\vec{r} \cdot \hat{a}_{i_1}) \left(\prod_{j=1}^{k} (\hat{a}_{i_{2j-1}} \cdot \hat{b}_{i_{2j}})\right) \left(\prod_{j=1}^{k-1} (\hat{a}_{i_{2j+1}} \cdot \hat{b}_{i_{2j}})\right) \hat{b}_{i_{2k+1}}. \tag{4.81}$$

Using Eq. (4.81), and the index swap $i_{2k+1} \to l$, one writes

$$\vec{r}_n = \sum_l r_{nl}\hat{b}_l, \tag{4.82}$$

with

$$r_{nl} \coloneqq \left(\frac{d-1}{d}\right)^{2k} \sum_{i_1, i_2, \ldots, i_{2k+1}} (\vec{r} \cdot \hat{a}_{i_1}) \left(\prod_{j=1}^{k} (\hat{a}_{i_{2j-1}} \cdot \hat{b}_{i_{2j}})\right) \left(\prod_{j=1}^{k-1} (\hat{a}_{i_{2j+1}} \cdot \hat{b}_{i_{2j}})\right). \tag{4.83}$$

It is convenient to put Eq. (4.79) in the form:

$$\begin{aligned}
\mathfrak{I}_X(\rho_{\vec{r}_n}) &= H\left(\frac{1}{d}\left[1 + (d-1)\hat{x}_i \cdot P_X\vec{r}_n\right]\right) - H\left(\frac{1}{d}\left[1 + (d-1)\hat{r}_i \cdot \vec{r}_n\right]\right) \\
&= H\left(\frac{1}{d}\left[1 + (d-1)\hat{x}_i \cdot P_X \sum_l r_{nl}\hat{b}_l\right]\right) - H\left(\frac{1}{d}\left[1 + (d-1)\hat{b}_i \cdot \sum_l r_{nl}\hat{b}_l\right]\right) \\
&= H\left(\frac{1}{d} + \left(\frac{d-1}{d}\right)^2 \sum_{l,m} r_{nl}(\hat{b}_l \cdot \hat{x}_m)(\hat{x}_i \cdot \hat{x}_m)\right) - H\left(\frac{1}{d} + \frac{d-1}{d}\sum_l r_{nl}(\hat{b}_i \cdot \hat{b}_l)\right),
\end{aligned} \tag{4.84}$$

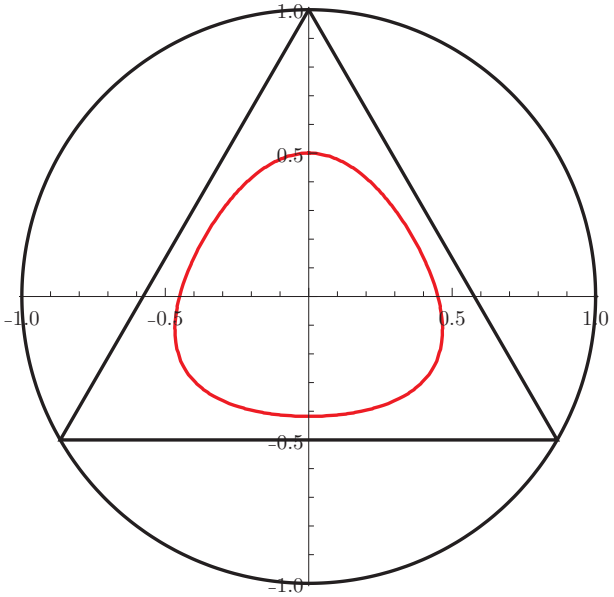

Figure 7 – 2-section of the Bloch sphere for a $d = 3$ system made parallel to the triangle simplex determined by an observable. The contour in red corresponds to the set of states with $I(\rho_{\vec{r}}) = 0.231049$.

and its upper bound in the form:

$$\Im_X(\rho_{\vec{r}_n}) \leq I(\rho_{\vec{r}_n}) = \log d - H\left(\frac{1}{d} + \frac{d-1}{d}\sum_l r_{nl}(\hat{b}_i \cdot \hat{b}_l)\right). \tag{4.85}$$

A possibly misguided intuition inherited from the qubit scenario is that information is simply a measure of purity. When $d = 2$, the information is a function of Shannon's binary entropy. The probabilities p_1 and p_2 are fully determined by the purity $\mathrm{Tr}\,[\rho^2]$, mapped to the norm $||\vec{r}||$, and the condition $\mathrm{Tr}\,[\rho] = 1$, which gives $p_2 = (1 - p_1)$. For $d \geq 3$, such constraints cannot fully determine the set of probabilities $\{p_i\}_{i=1}^d$.

Figure 7 displays a 2-section of the qutrit 8-d hypersphere. The triangle corresponds to the simplex determined by an observable, $\triangle_R$. We consider a state $\rho_{\vec{r}}$ with $\vec{r} = \frac{\hat{r}_1}{2}$, corresponding in the image to the point $(0, 0.5)$ and evaluate its information: $I(\rho_{\vec{r}}) = 0.231049$. Then, within the simplex we search for every other state with the same information value, finding the contour displayed in red. Notice that if the information were completely determined by $||\vec{r}||$, the shape in red would be described by a circle centered at the origin. Instead, we get a rounded triangle shape, revealing that the information depends on $||\vec{r}||$ together with the placement of $\vec{r}$ within the simplex. In the direction of the midpoints of the triangle's edge, such as $-\hat{r}_1$, we obtain the minimal values for $||\vec{r}||$, here approximately 0.42. For sufficiently large $||\vec{r}||$ values, one cannot achieve a closed contour. The extreme case of pure states determines the triplet $\{\hat{r}_i\}_{i=1}^d$ corresponding to $I(\rho_{\vec{r}}) = \log 3$. A closed contour can be obtained only for states such that $||\vec{r}|| \leq 1/(d-1)$ in the directions of the midpoints of $\triangle_R$.

4.5 SLIGHTLY INCOMPATIBLE OBSERVABLES

It is true that, once the entities $\{\rho_{\vec{r}}, A, B, X\}$ are well-defined, so is $\Im_X(\rho_{\vec{r}_n})$. Nevertheless, the plethora of indices, summations, products, inner products, Shannon entropies, and similar constructs serve as a testament to the complexity and boldness required for explicit calculations. To simplify the landscape, we proceed by investigating relevant scenarios that allow for useful approximations, revealing insightful results.

One scenario of this kind is given by $\vec{r} = ||\vec{r}||\hat{b}_1$ and $\hat{a} \cdot \hat{b} = 1 - \epsilon$, such that $\epsilon << 1$. That is, we are considering a situation where our physical state is characterized by a density operator with $\vec{r} \in \triangle_B$, and the pairwise sequential measurements performed with observables A and B that are just slightly incompatible, $[A, B] << 1$. Without loss of generality, this situation allows the description of $\hat{a}_1$ rotated with respect to $\hat{b}_1$ by $\theta << 1$.

Our first step is to investigate $P_A\vec{r}$ in a scenario like this. Avoiding a shortage of variables, we recycle α and β, and define

$$\vec{r} \cdot \hat{a}_1 \coloneqq \alpha \qquad \vec{r} \cdot \hat{a}_i \coloneqq \beta_i, \ (i \neq 1). \tag{4.86}$$

Given the structure $\sum_{i=1}^{d} \hat{a}_i = 0$ (Eq. (4.17)), we can do

$$\begin{aligned} 0 &= \vec{r} \cdot \sum_{i=1}^{d} \hat{a}_i \\ &= \sum_{i=1}^{d} (\vec{r} \cdot \hat{a}_i) \\ &= \alpha + \sum_{i=2}^{d} (\vec{r} \cdot \hat{a}_i) \\ &= \alpha + \sum_{i=2}^{d} \beta_i, \end{aligned} \tag{4.87}$$

obtaining the relation

$$\alpha = -\sum_{i=2}^{d} \beta_i. \tag{4.88}$$

$||P_A\vec{r}||^2$, as obtained in Eq. (4.38), is:

$$||P_A\vec{r}||^2 = \left(\frac{d-1}{d}\right)^2 \left(\sum_{i=1}^{d} (\hat{a}_i \cdot \vec{r})^2 - \frac{1}{d-1} \sum_{i \neq j} (\hat{a}_i \cdot \vec{r})(\hat{a}_j \cdot \vec{r})\right). \tag{4.89}$$

Informed by Eq. (4.88), we investigate both summations. The first one gives:

$$\begin{aligned} \sum_{i=1}^{d} (\hat{a}_i \cdot \vec{r})^2 &= (\hat{a}_1 \cdot \vec{r})^2 + \sum_{i=2}^{d} (\hat{a}_i \cdot \vec{r})^2 \\ &= \alpha^2 + \sum_{i=2}^{d} \beta_i^2. \end{aligned} \tag{4.90}$$

As for the second one, we start by manipulating it like

$$\sum_{i\neq j}(\hat{a}_i\cdot\vec{r})(\hat{a}_j\cdot\vec{r}) = \left(\sum_{i=1}^{d}(\hat{a}_i\cdot\vec{r})\right)^2 - \sum_{i=1}^{d}(\hat{a}_i\cdot\vec{r})^2. \tag{4.91}$$

Because the first term on the right-hand side is zero, we have:

$$\sum_{i\neq j}(\hat{a}_i\cdot\vec{r})(\hat{a}_j\cdot\vec{r}) = -\alpha^2 - \sum_{i=2}^{d}\beta_i^2. \tag{4.92}$$

Plugging Eqs. (4.90) and (4.92) into Eq. (4.89) gives

$$\begin{aligned}||P_A\vec{r}||^2 &= \left(\frac{d-1}{d}\right)^2\left[\alpha^2 + \sum_{i=2}^{d}\beta_i^2 + \frac{1}{d-1}\left(\alpha^2 + \sum_{i=2}^{d}\beta_i^2\right)\right]\\ &= \left(\frac{d-1}{d}\right)^2\left[\left(1+\frac{1}{d-1}\right)\left(\alpha^2 + \sum_{i=2}^{d}\beta_i^2\right)\right]\\ &= \frac{d-1}{d}\left(\alpha^2 + \sum_{i=2}^{d}\beta_i^2\right).\end{aligned} \tag{4.93}$$

As a sanity check, one may consider $\alpha = 1$ and $\beta_i = -1/(d-1)$ to retrieve $||P_A\vec{r}||^2 = 1$.

The relation we defined between $\hat{a}_1$ and $\hat{b}_1$ is such that we obtain

$$\begin{aligned}\vec{r}\cdot\hat{a}_1 &= ||\vec{r}||\hat{b}_1\cdot\hat{a}_1\\ &= ||\vec{r}||\cos\theta.\end{aligned} \tag{4.94}$$

Since $\theta << 0$, a Taylor expansion gives the approximation up to the second order:

$$\vec{r}\cdot\hat{a}_1 \approx ||\vec{r}||\left(1 - \frac{\theta^2}{2}\right). \tag{4.95}$$

To determine the correction terms for $\vec{r}\cdot\hat{a}_i$ with $i \neq 1$, we recycle the variable δ, set

$$\vec{r}\cdot\hat{a}_i = ||\vec{r}||\left(-\frac{1}{d-1} + \delta_i\right), \qquad (i \neq 1) \tag{4.96}$$

where we used Eq. (4.19), and, using Eq. (4.17), calculate:

$$\begin{aligned}
0 &= \vec{r}\cdot\hat{a}_1 + \sum_{i=2}^{d}\vec{r}\cdot\hat{a}_i \\
&\approx ||\vec{r}||\left(1-\frac{\theta^2}{2}\right) + \sum_{i=2}^{d}||\vec{r}||\left(-\frac{1}{d-1}+\delta_i\right) \\
&= ||\vec{r}||\left[1-\frac{\theta^2}{2}+\sum_{i=2}^{d}\left(-\frac{1}{d-1}+\delta_i\right)\right] \\
&= ||\vec{r}||\left(1-\frac{\theta^2}{2}-\frac{d-1}{d-1}+\sum_{i=2}^{d}\delta_i\right) \\
&= ||\vec{r}||\left(-\frac{\theta^2}{2}+\sum_{i=2}^{d}\delta_i\right) \\
&\approx ||\vec{r}||\left(-\frac{\theta^2}{2}+(d-1)\delta\right),
\end{aligned} \tag{4.97}$$

where we assumed in the last step equal corrections for every $i \neq 1$. Solving for δ, we get

$$\delta = \frac{\theta^2}{2(d-1)} \tag{4.98}$$

wielding, for $i \neq 1$:

$$\begin{aligned}
\vec{r}\cdot\hat{a}_i &\approx ||\vec{r}||\left[-\frac{1}{d-1}+\frac{\theta^2}{2(d-1)}\right] \\
&= ||\vec{r}||\left[\frac{\theta^2-2}{2(d-1)}\right].
\end{aligned} \tag{4.99}$$

Given definitions (4.86) and the approximations we obtained in Eqs. (4.95) and (4.99), we are in a good position to determine $||P_A\vec{r}||$. We need to calculate α^2 and β^2 for that, then,

$$\begin{aligned}
\alpha^2 &= (\vec{r}\cdot\hat{a}_1)^2 \\
&= ||\vec{r}||^2\left(\frac{2-\theta^2}{2}\right)^2 \\
&= ||\vec{r}||^2\frac{(2-\theta^2)^2}{4}
\end{aligned} \tag{4.100}$$

and

$$\begin{aligned}
\beta^2 &= (\vec{r}\cdot\hat{a}_i)^2 \\
&= ||\vec{r}||^2\left[\frac{\theta^2-2}{2(d-1)}\right]^2 \\
&= ||\vec{r}||^2\frac{(2-\theta^2)^2}{4(d-1)^2}.
\end{aligned} \tag{4.101}$$

Substituting these terms in equation (4.93), gives

$$\begin{aligned}
||P_A\vec{r}||^2 &= \frac{d-1}{d}\left(\alpha^2 + \sum_{i=2}^{d}\beta_i^2\right) \\
&= \frac{d-1}{d}\left(||\vec{r}||^2\frac{(2-\theta^2)^2}{4} + \sum_{i=2}^{d}||\vec{r}||^2\frac{(2-\theta^2)^2}{4(d-1)^2}\right) \\
&= ||\vec{r}||^2\frac{d-1}{4d}\left((2-\theta^2)^2 + \frac{(2-\theta^2)^2}{d-1}\right) \\
&= ||\vec{r}||^2\frac{(2-\theta^2)^2}{4}, \qquad (4.102)
\end{aligned}$$

such that

$$||P_A\vec{r}|| = ||\vec{r}||\frac{2-\theta^2}{2}. \qquad (4.103)$$

This result mirrors the fact that the mixing caused by the map Φ_A does not depend on the dimension of the system but on the commutation relations between A and the observables R such that $\vec{r} \in \triangle_R$.

To determine the direction where $P_A\vec{r}$ falls within the hypersphere, we can calculate directly, using the approximations we obtained, that

$$\begin{aligned}
P_A\vec{r} &= \left(\frac{d-1}{d}\right)\sum_{i=1}^{d}(\hat{a}_i\cdot\vec{r})\hat{a}_i \\
&= \left(\frac{d-1}{d}\right)\left[(\hat{a}_1\cdot\vec{r})\hat{a}_1 + \sum_{i=2}^{d}(\hat{a}_i\cdot\vec{r})\hat{a}_i\right] \\
&= ||\vec{r}||\left(\frac{d-1}{d}\right)\left(\frac{2-\theta^2}{2}\hat{a}_1 + \sum_{i=2}^{d}\left[\frac{\theta^2-2}{2(d-1)}\right]\hat{a}_i\right) \\
&= ||\vec{r}||\left(\frac{d-1}{d}\right)\left(\frac{2-\theta^2}{2}\hat{a}_1 + \left[\frac{2-\theta^2}{2(d-1)}\right]\hat{a}_1\right) \\
&= ||\vec{r}||\left(\frac{d-1}{d}\right)\left(\frac{d}{d-1}\right)\left(\frac{2-\theta^2}{2}\right)\hat{a}_1 \\
&= ||\vec{r}||\left(\frac{2-\theta^2}{2}\right)\hat{a}_1. \\
&= ||P_A\vec{r}||\hat{a}_1. \qquad (4.104)
\end{aligned}$$

This result confirms the norm we obtained before and ascertains that the symmetry relations of the simplex cause the projected vector to fall along $\hat{a}_1$.

Calculating $\vec{r}_n$ in this situation becomes easy. The sequential pairwise measurements

give:

$$\begin{aligned}
P_A\vec{r} &= && ||\vec{r}||\frac{2-\theta^2}{2}\hat{a}_1 \\
P_BP_A\vec{r} &= \vec{r}_1 && ||\vec{r}||\left(\frac{2-\theta^2}{2}\right)^2\hat{b}_1 \\
P_AP_B\vec{r}_1 &= && ||\vec{r}||\left(\frac{2-\theta^2}{2}\right)^3\hat{a}_1 \\
P_BP_A\vec{r}_1 &= \vec{r}_2 = && ||\vec{r}||\left(\frac{2-\theta^2}{2}\right)^4\hat{b}_1,
\end{aligned} \tag{4.105}$$

such that

$$\vec{r}_n = ||\vec{r}||\left(\frac{2-\theta^2}{2}\right)^{2n}\hat{b}_1, \tag{4.106}$$

and we find

$$||\vec{r}_n|| = ||\vec{r}||\left(\frac{2-\theta^2}{2}\right)^{2n}. \tag{4.107}$$

With that, using Eq. (4.74), we obtain the bound (4.78):

$$\begin{aligned}
\mathfrak{I}_X(\rho_{\vec{r}_n}) &\leq \log d - H\left(\frac{1}{d}\left[1+(d-1)\hat{b}_i\cdot\vec{r}_n\right]\right) \\
&= \log d - H\left(\frac{1}{d}\left[1+||\vec{r}_n||(d-1)\hat{b}_i\cdot\hat{b}_1\right]\right) \\
&= \log d + \left(\frac{1}{d}+||\vec{r}_n||\frac{d-1}{d}\right)\log\left(\frac{1}{d}+||\vec{r}_n||\frac{d-1}{d}\right) \\
&\quad + \sum_{i=2}^{d}\left(\frac{1}{d}\left[1+||\vec{r}_n||(d-1)\hat{b}_i\cdot\hat{b}_1\right]\right)\log\left(\frac{1}{d}\left[1+||\vec{r}_n||(d-1)\hat{b}_i\cdot\hat{b}_1\right]\right) \\
&= \log d + \left(\frac{1}{d}+||\vec{r}_n||\frac{d-1}{d}\right)\log\left(\frac{1}{d}+||\vec{r}_n||\frac{d-1}{d}\right) \\
&\quad + \frac{d-1}{d}\left(1-||\vec{r}_n||\right)\log\left[\frac{1}{d}\left(1-||\vec{r}_n||\right)\right].
\end{aligned} \tag{4.108}$$

Extreme cases are worth considering to check the consistency of our calculations in this scenario. We start by taking $\vec{r} = \hat{b}_1$ and $n = 0$. That is, the case where we consider the information of a pure state of dimension d. It suffices to introduce $||\vec{r}_n|| = 1$ in the equation above:

$$\begin{aligned}
\mathfrak{I}_X(\rho_{\hat{b}_1}) &\leq \log d + \left(\frac{1}{d}+\frac{d-1}{d}\right)\log\left(\frac{1}{d}+\frac{d-1}{d}\right) \\
&\quad + \frac{d-1}{d}(1-1)\log\left[\frac{1}{d}(1-1)\right] \\
&= \log d + \log 1 + 0\log 0 \\
&= \log d,
\end{aligned} \tag{4.109}$$

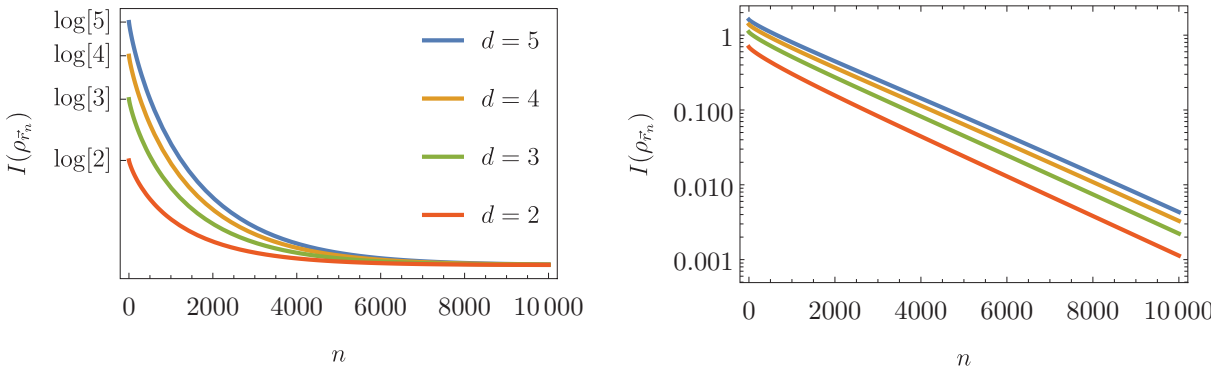

Figure 8 – Plots of the irrealism bound $I(\rho_{\vec{r}_n})$ as a function of the number n of sequential pairwise measurements for $\vec{r} = \hat{b}_1$, $\vec{a} \cdot \vec{b} = \cos\theta$ with $\theta = 1°$, and $d \in \mathbb{N}_{\leq 5}$. Displayed on the left is a linear plot and on the right is a logarithmic plot, following the same color legends.

as we expected. The other extreme case concerns $n \to \infty$, causing $||\vec{r}_n|| \to 0$. We obtain:

$$\begin{aligned}
\mathfrak{I}_X(\rho_{\vec{0}}) &\leq \log d + \frac{1}{d}\log\left(\frac{1}{d}\right) + \frac{d-1}{d}\log\left(\frac{1}{d}\right) \\
&= \log d + \left(\frac{1}{d} + \frac{d-1}{d}\right)\log\frac{1}{d} \\
&= \log d - \log d \\
&= 0,
\end{aligned} \tag{4.110}$$

also, as we expected.

We argue that the decrease in $||\vec{r}||$ caused by P_A does not depend on d. As indicated by Eq. (4.107), the same holds for the projector $(P_B P_A)^n$. Information, in contrast, depends on the dimension d. As a consequence, in a sequential pairwise measurement scenario, the maximal irrealism accessible to any observable X will depend both on d and θ, which corresponds to the degree of noncommutability between A and B.

To investigate this in more detail, we performed a numerical analysis where the information bound was evaluated for pure systems, $\vec{r} = \hat{b}_1$, of dimension $d \in \mathbb{N}_{\leq 5}$ and $n \in [0, 10^4]$. Using the approximations obtained for only slightly incompatible observables, we take $\vec{a} \cdot \vec{b} = \cos\theta$ with $\theta = 1°$. The results shown in Figure 8 suggest that higher-dimensional systems display greater irrealism and irrealism resilience, once the curves do not intercept each other. This fact can be visualized in the logarithmic plot, with the relative spreading of the curves as n increases.

In addition to the information bound of a higher-dimensional system being strictly larger than that of a lower-dimensional one, we can also observe from the graph that the shapes of their curves differ. Quantifying this difference involves taking Eq. (4.108), dividing it by $\log d$, and then evaluating its limit as $d \to \infty$. Analyzing term by term, the first term approaches 1,

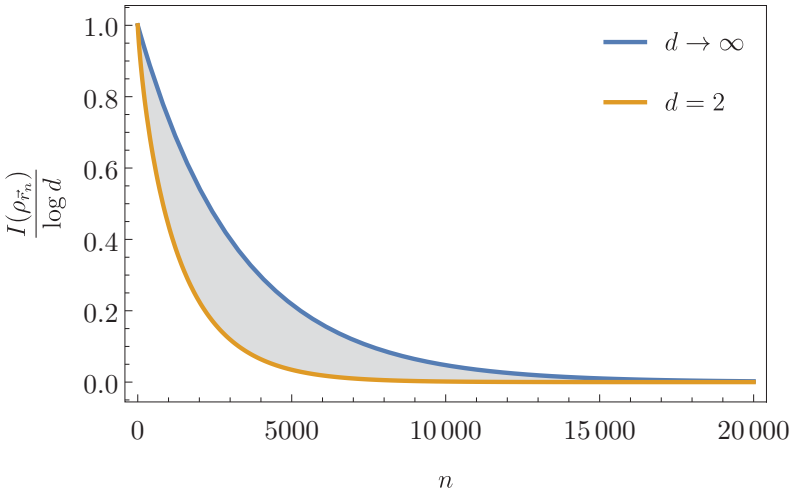

Figure 9 – Plot of $I(\rho_{\vec{r}_n})/\log d$ as a function of the number n of sequential pairwise measurements for $\vec{r} = \hat{b}_1$ and $\vec{a} \cdot \vec{b} = \cos\theta$ with $\theta = 1°$. Pictured in orange and blue systems of $d = 2$ and $d \to \infty$ respectively, the gray area represents the region covered by the curves for systems of every other dimensionality.

while the second term tends to zero,

$$\lim_{d\to\infty} \left(\frac{1}{d} + ||\vec{r}_n|| \frac{d-1}{d}\right) \frac{\log\left(\frac{1}{d} + ||\vec{r}_n|| \frac{d-1}{d}\right)}{\log d} = \lim_{d\to\infty} ||\vec{r}_n|| \frac{\log ||\vec{r}_n||}{\log d} = 0, \tag{4.111}$$

and the third:

$$\begin{aligned} \lim_{d\to\infty} \frac{d-1}{d} (1 - ||\vec{r}_n||) \frac{\log\left[\frac{1}{d}(1 - ||\vec{r}_n||)\right]}{\log d} &= \lim_{d\to\infty} (1 - ||\vec{r}_n||) \frac{\log\left[\frac{1-||\vec{r}_n||}{d}\right]}{\log d} \\ &= \lim_{d\to\infty} (1 - ||\vec{r}_n||) \frac{-\frac{1}{d}}{\frac{1}{d}} \\ &= ||\vec{r}_n|| - 1, \end{aligned} \tag{4.112}$$

where we used L'Hôpital's rule. Putting everything together, we obtain

$$\lim_{d\to\infty} \frac{\mathfrak{I}_X(\rho_{\vec{r}_n})}{\log d} \leq \lim_{d\to\infty} \frac{I(\rho_{\vec{r}_n})}{\log d} = ||\vec{r}_n||. \tag{4.113}$$

Figure 9 illustrates the results of this analysis. Regardless of the magnitude of d, we always obtain a monotonically decreasing function. We also verify that this function is convex, but the degree of its convexity attenuates as d increases, converging to the shape described by the upper limit of the shaded region, which corresponds to the filling given by every $d \geq 2$.

4.6 VOLUME OF IRREALISM

While the information bound (Eq. (4.77) provides the maximal amount of information an observable wields, it does not *a priori* specify how irrealism is distributed among all possible observables. To address this, we introduce a new measure, which we call the *volume of irrealism*.

Considering a quantum state ρ, our aim is to determine the average irrealism associated with all possible observables. To this end, we define the volume of irrealism as:

$$\overline{\mathfrak{I}}(\rho) \coloneqq \frac{1}{N}\int_X \mathfrak{I}_X(\rho)dX. \tag{4.114}$$

Here, the measure dX is the Haar measure over the SU(d) group, ensuring a uniform sampling of observables in the space of all possible d-dimensional quantum observables, represented by the integration domain. The normalization factor $N = \int_X dX$ corresponds to the total volume of the SU(d) group under the Haar measure. Consequently, $\overline{\mathfrak{I}}(\rho)$ quantifies the average irrealism of the state ρ across the entire space of observables, ensuring that the measure is unbiased and mathematically well defined.

It is a direct consequence of Klein's inequality (equation (2.65)) that $\mathfrak{I}_X(\rho) \geq 0$, with equality holding *iff* $\rho = \Phi_X(\rho)$. To guarantee $\overline{\mathfrak{I}}(\rho) = 0$, it is necessary that $\mathfrak{I}_X(\rho) = 0$ $(\forall X)$. If the domain includes at least two observables that do not commute, this condition is satisfied only by $\rho = \mathbb{1}/d$. Thus, completely mixed states have zero volume of irrealism. Moreover, because irrealism is always limited by information, we have the non-strict upper bound $\overline{\mathfrak{I}}(\rho) \leq I(\rho)$.

In practice, $\overline{\mathfrak{I}}(\rho)$ can be approximated using Monte Carlo sampling. By randomly generating N observables X_i, the integral can be approximated as:

$$\overline{\mathfrak{I}}(\rho) \approx \frac{1}{N}\sum_{i=1}^{N} \mathfrak{I}_{X_i}(\rho). \tag{4.115}$$

This approach is computationally efficient and provides a reasonable estimate of the integral for a sufficiently large N.

Expressing this quantity in the generalized Bloch sphere formalism, leads to the equation

$$\overline{\mathfrak{I}}(\rho_{\vec{r}}) = \frac{1}{S}\int_S \mathfrak{I}_X(\rho_{\vec{r}})d\Omega(\vec{x}), \tag{4.116}$$

where the integration domain S corresponds to the region at the surface of the hypersphere populated by the vectors $\vec{x}$. The normalization factor $S = \int_S d\Omega(\vec{x})$ gives the area of this region, and $d\Omega(\vec{x})$ gives one of its surface elements.

Using generalized spherical coordinates, we can define $\phi_1 \in [0, 2\pi]$ and $\phi_i \in [0, \pi]$, with $i \in \{2, 3, .., d-1\}$, such that

$$d\Omega(\vec{x}) = \prod_{i=1}^{d-1} \sin^{d-i-1}(\phi_i)d\phi_i. \tag{4.117}$$

With this, we obtain

$$\overline{\mathfrak{I}}(\rho_{\vec{r}}) = \frac{1}{S}\int_S \mathfrak{I}_X(\rho_{\vec{r}}) \prod_{i=1}^{d-1} \sin^{d-i-1}(\phi_i)d\phi_i. \tag{4.118}$$

One practical problem we face with this definition is the necessity of characterization of the region expressed by S. Similarly to the problem of determining the region of the Bloch sphere

populated by vectors corresponding to valid density operators, each vector $\vec{x}$ should correspond to a valid observable. Once the definition of irrealism is based upon projective measurements, the eigenstates of the observables should also correspond to valid post-measurement states. That is, the vector $\vec{x}$ needs to correlate with a simplex $\triangle_X$ whose vertices correlate to valid physical states given the generators Λ_i.

We also consider the situation where the domain of integration does not correspond to all possible observables, but only to a specific class of possible observables $\{X'\}$. Expressing

$$\overline{\mathfrak{I}}_{\{X'\}}(\rho) \coloneqq \frac{1}{N'} \int_{X'} \mathfrak{I}_X(\rho) dX, \tag{4.119}$$

with the domain of integration $X' \subset X$, and $N' = \int_{X'} dX$, we define the *volume of irrealism for observables of the class* $\{X'\}$ *given* ρ. Similarly to the situation presented above, if the set $\{X'\}$ contains incompatible observables, we obtain $\overline{\mathfrak{I}}_{\{X'\}}(\rho) = 0$ *iff* $\rho = \mathbb{1}/d$. The maximum value for $\overline{\mathfrak{I}}_{\{X'\}}(\rho)$ occurs when ρ is a pure and a state of reality for some observable that is maximally incompatible with all observables in the set $\{X'\}$, returning $\overline{\mathfrak{I}}_{\{X'\}}(\rho) = I(\rho) = \log d$.

4.7 CASE STUDIES

We proceed to investigate specific scenarios by fixing the dimension of the system. Specifically, we set $d = 2$ and $d = 3$. The qubit, being the fundamental entity in quantum information, illustrates the formalism and tools we developed so far in the scenario of the usual Bloch sphere representation. Its simplicity allows for a didactic illustration. The qutrit situation already showcases the particularities of a higher-dimensional system, while being still simple enough to be computationally permissible for most of the analysis we are interested in.

4.7.1 Qubit analysis

Qubit systems are represented as $\rho_{\vec{r}} = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma})$, where the generators $\vec{\Lambda}$ reduce to the Pauli matrices $\vec{\sigma}$, as previously exposed. Observables, written as $A = \hat{a} \cdot \vec{\sigma}$, determine one-dimensional simplexes: straight lines diametrically crossing the Bloch sphere. This is clearly envisioned by recalling that the observables of the qubit system have two projectors that determine vectors such that $\hat{a}_1 \cdot \hat{a}_2 = -1$, according to the relation (4.19). Taking the convex set generated by all the possible mixtures of those projectors, we end with the one-dimensional simplex $\triangle_A$.

Using the fact that $\hat{a}_2 = -\hat{a}_1$, and setting the eigenvalues of the observables as $a_1 = 1$ and $a_2 = -1$, we use equation (4.21)

$$\begin{aligned} \vec{a} &= \frac{C_d}{d}(a_1\hat{a}_1 + a_2\hat{a}_2) \\ &= \frac{1}{2}[\hat{a}_1 - (-1)\hat{a}_1] \\ &= \hat{a}_1 \end{aligned} \tag{4.120}$$

Using this result, the post unrevealed measurement state becomes, according to equation (4.25):

$$\begin{aligned} P_A\vec{r} &= \frac{d-1}{d}\sum_i(\hat{a}_i\cdot\vec{r})\hat{a}_i \\ &= \frac{1}{2}\left[(\hat{a}_1\cdot\vec{r})\hat{a}_1+(\hat{a}_2\cdot\vec{r})\hat{a}_2\right] \\ &= \frac{1}{2}\left[(\hat{a}_1\cdot\vec{r})\hat{a}_1-(-\hat{a}_1\cdot\vec{r})\hat{a}_1\right] \\ &= (\hat{a}_1\cdot\vec{r})\hat{a}_1 \\ &= (\hat{a}\cdot\vec{r})\hat{a}, \end{aligned} \tag{4.121}$$

using the notation $\vec{a}=\hat{a}$. With that, we obtain the sequence

$$\begin{aligned} &P_A\vec{r} = && (\hat{a}\cdot\vec{r})\hat{a} \\ &P_BP_A\vec{r}=\vec{r}_1 && (\hat{a}\cdot\hat{b})(\hat{a}\cdot\vec{r})\hat{b} \\ &P_A\vec{r}_1 = && (\hat{a}\cdot\hat{b})^2(\hat{a}\cdot\vec{r})\hat{a} \\ &P_BP_A\vec{r}_1=\vec{r}_2= && (\hat{a}\cdot\hat{b})^3(\hat{a}\cdot\vec{r})\hat{b}, \end{aligned} \tag{4.122}$$

such that

$$\vec{r}_n=(\hat{a}\cdot\hat{b})^{2n-1}(\hat{a}\cdot\vec{r})\hat{b}. \tag{4.123}$$

With this construction, P_X's action on $\vec{r}_n$ takes the form of

$$P_X\vec{r}_n=(\hat{a}\cdot\hat{b})^{2n-1}(\hat{a}\cdot\vec{r})(\hat{b}\cdot\hat{x})\hat{x}. \tag{4.124}$$

We now use the computationally verifiable inequality

$$H_{\text{bin}}\left(\frac{1+\mu\lambda}{2}\right)-H_{\text{bin}}\left(\frac{1+\lambda}{2}\right)\leq\lambda^2(1-\mu^4)\,\log 2, \tag{4.125}$$

where $H_{\text{bin}}(p)=-p\ln p-(1-p)\ln(1-p)$ is the Shannon binary entropy, $\lambda\in\mathbb{R}_{[0,1]}$, and $\mu\in\mathbb{R}_{[-1,1]}$. For $\mu=0$ and $\lambda=1$, the upper bound becomes tight. Substituting $\mu=0$ and $\lambda=1$ into the left-hand side:

$$\begin{aligned} H_{\text{bin}}\left(\frac{1+\mu\lambda}{2}\right)-H_{\text{bin}}\left(\frac{1+\lambda}{2}\right) &= H_{\text{bin}}\left(\frac{1+0\cdot1}{2}\right)-H_{\text{bin}}\left(\frac{1+1}{2}\right) \\ &= H_{\text{bin}}\left(\frac{1}{2}\right)-H_{\text{bin}}(1). \end{aligned}$$

Since $H_{\text{bin}}\left(\frac{1}{2}\right)=-\frac{1}{2}\ln\frac{1}{2}-\frac{1}{2}\ln\frac{1}{2}=\ln 2$ and $H_{\text{bin}}(1)=0$, we confirm:

$$H_{\text{bin}}\left(\frac{1+\mu\lambda}{2}\right)-H_{\text{bin}}\left(\frac{1+\lambda}{2}\right)=\log 2. \tag{4.126}$$

Using the above inequality, we calculate $\mathfrak{I}_X(\rho_{\vec{r}_n})$ (Eq. (4.79)) explicitly. Recall that

$$\mathfrak{I}_X(\rho_{\vec{r}_n})=H_{\text{bin}}\left(\frac{1+||P_X\vec{r}_n||}{2}\right)-H_{\text{bin}}\left(\frac{1+||\vec{r}_n||}{2}\right). \tag{4.127}$$

Using the inequality, we substitute $\lambda = ||\vec{r}_n||$ and $\mu = \frac{||P_X\vec{r}_n||}{||\vec{r}_n||}$:

$$\mathfrak{I}_X(\rho_{\vec{r}_n}) \leq ||\vec{r}_n||^2 \left(1 - \left(\frac{||P_X\vec{r}_n||}{||\vec{r}_n||}\right)^4\right) \log 2. \tag{4.128}$$

Substituting $P_X\vec{r}_n = (\hat{a}\cdot\hat{b})^{2n-1}(\hat{a}\cdot\vec{r})(\hat{b}\cdot\hat{x})\hat{x}$ and $||P_X\vec{r}_n|| = |(\hat{b}\cdot\hat{x})|(\hat{a}\cdot\hat{b})^{2n-1}|\hat{a}\cdot\vec{r}|$, we find:

$$||\vec{r}_n|| = (\hat{a}\cdot\hat{b})^{2n-1}|\hat{a}\cdot\vec{r}|, \tag{4.129}$$

$$\frac{||P_X\vec{r}_n||}{||\vec{r}_n||} = \hat{b}\cdot\hat{x}. \tag{4.130}$$

Thus,

$$\mathfrak{I}_X(\rho_{\vec{r}_n}) \leq \left(\hat{a}\cdot\vec{r}\right)^2\left(\hat{a}\cdot\hat{b}\right)^{2(2n-1)}\left[1 - \left(\hat{x}\cdot\hat{b}\right)^4\right]\log 2. \tag{4.131}$$

Since $|\hat{a}\cdot\hat{b}| < 1$ whenever $[A, B] \neq 0$, this formula supports our main result: for any observable X, we can always choose n large enough such that irrealism is upper bounded by arbitrarily small values.

For a system of dimension $d = 2$, the volume of irrealism (Eq. (4.114)) takes the form

$$\overline{\mathfrak{I}}(\rho_{\vec{r}}) = \frac{1}{4\pi}\int_S \mathfrak{I}_X(\rho_{\vec{r}})\, d\Omega(\vec{x}), \tag{4.132}$$

where S is the surface area of the unit sphere in three dimensions, and $d\Omega(\hat{x})$ represents one of its surface elements.

To work through this integral, we use Eq. (4.127) and set $||P_X\vec{r}|| = |\hat{x}\cdot\vec{r}| = ||\vec{r}||\cos\theta$. The surface element in spherical coordinates is $d\Omega(\hat{x}) = \sin\theta\, d\theta\, d\phi$ and, due to the integrand independence on the azimuthal angle ϕ, the volume of irrealism simplifies to:

$$\overline{\mathfrak{I}}(\rho_{\vec{r}}) = \frac{1}{2}\int_0^{\pi} \mathfrak{I}_X(\rho_{\vec{r}})\sin\theta\, d\theta. \tag{4.133}$$

The parity of the dot product allows us to further simplify the integral. We restrict the polar angle θ to the range $[0, \pi/2]$ and multiply the result by 2:

$$\overline{\mathfrak{I}}(\rho_{\vec{r}}) = \int_0^{\pi/2} \mathfrak{I}_X(\rho_{\vec{r}})\sin\theta\, d\theta. \tag{4.134}$$

Introducing the substitution $\alpha = \cos\theta$, with $d\alpha = -\sin\theta\, d\theta$, the limits of integration become $\alpha \in [1, 0]$. Finally, the integral becomes:

$$\overline{\mathfrak{I}}(\rho_{\vec{r}}) = \int_0^1 H_{\text{bin}}\left(\frac{1 + ||\vec{r}||\alpha}{2}\right) d\alpha - H_{\text{bin}}\left(\frac{1 + ||\vec{r}||}{2}\right). \tag{4.135}$$

The volume of irrealism for a qubit is thus a function of the norm of $\vec{r}$, establishing a direct relationship between this quantity and the information of the qubit:

$$I(\rho_{\vec{r}}) = \log 2 - H_{\text{bin}}\left(\frac{1 + ||\vec{r}||}{2}\right) \geq \mathfrak{I}_X(\rho_{\vec{r}}). \tag{4.136}$$

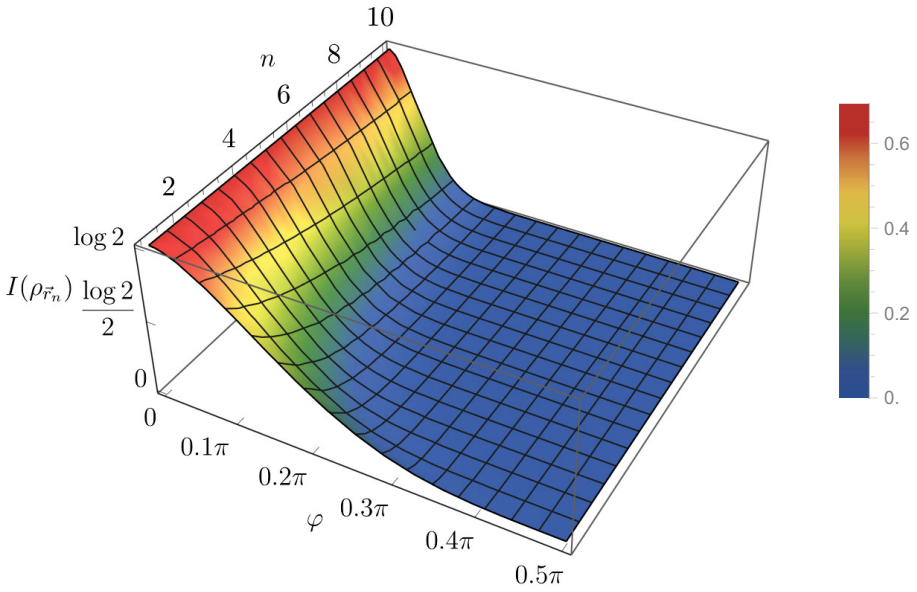

Figure 10 – Numerical results illustrating the information-irrealism boun $I(\rho_{\vec{r}_n})$ a function of n and φ in a sequential pairwise measurements scenario. It was considered an initial state S_z with eigenvalue 1, $A = S_{\vec{z}} \cos\varphi + S_{\vec{x}} \sin\varphi$ and $B = S_{\vec{z}}$, with $\hat{a} \cdot \hat{b} = \cos\varphi$.

A parametric plot of $\overline{\Im}(\rho_{\vec{r}})$ against $I(\rho_{\vec{r}})$ shows a straight line. From its slope we determine the equation:

$$I(\rho_{\vec{r}}) = (2\log 2)\overline{\Im}(\rho_{\vec{r}}), \tag{4.137}$$

such that $0 \leq \overline{\Im}(\rho_{\vec{r}}) \leq 1/2$. These bounds correspond to the completely mixed state and a pure state, respectively.

A numerical analysis was performed to determine the information-irrealism bound of $\rho_{\vec{r}_n}$, using $\rho_{\vec{r}}$ as the qubit eigenstate of S_z with eigenvalue 1. The analysis considered $n \in \mathbb{N}_{\leq 10}$ and $\varphi \in [0, \pi/2]$ and the observables were taken as $A = S_{\vec{z}} \cos\varphi + S_{\vec{x}} \sin\varphi$ and $B = S_{\vec{z}}$, yielding $\hat{a} \cdot \hat{b} = \cos\varphi$.

The results, illustrated in Fig. 10, show that the information bound approaches zero for small values of n when $\varphi > \pi/4$, but it persists when $\varphi \ll 1$. Specifically, numerical analysis reveals that achieving $I(\rho_{\vec{r}_n}) < 0.001$ for $\varphi = 0.1^\circ$ requires n on the order of 10^6. However, as n increases, the bound decreases more steeply as a function of φ. This behavior aligns with the result demonstrated in Eq. (4.131), which shows that for sufficiently large n, the bound approaches zero even when $[A, B] \approx 0$, corresponding here to small values of φ.

4.7.2 Qutrit analysis

When analyzing qutrits, we encounter distinctive features unique to systems with $d > 2$. As previously mentioned, the surface of the generalized Bloch sphere is not entirely populated by vectors corresponding to valid physical states. For qutrits, the Bloch hypersphere

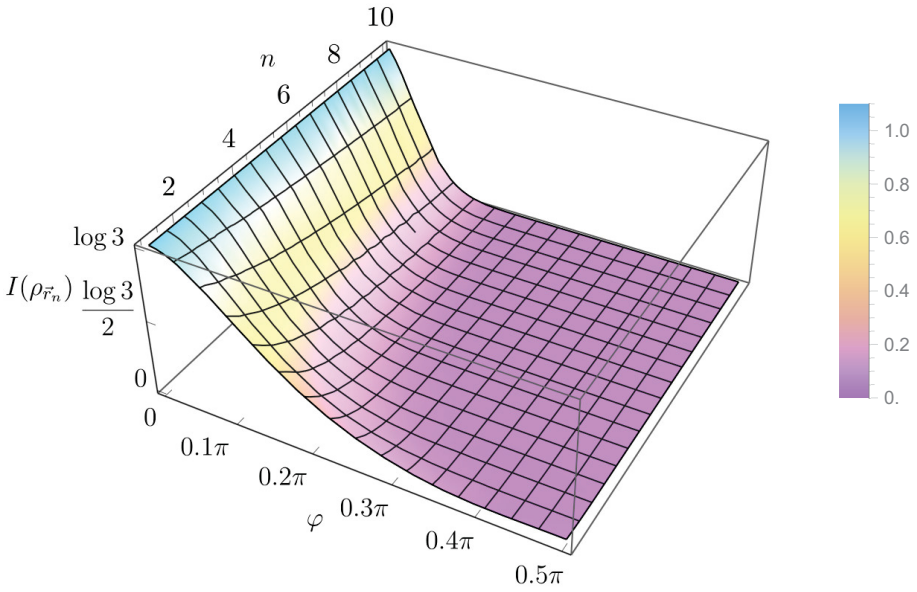

Figure 11 – Numerical results showcasing the behavior of the information-irrealism boun $I(\rho_{\vec{r}_n})$ in a sequential pairwise measurement scenario for qutrits. The analysis considers an initial state given by the eigenstate of S_z with eigenvalue -1. The observables are parameterized as $A = S_{\vec{z}} \cos\varphi + S_{\vec{x}} \sin\varphi$ and $B = S_{\vec{z}}$, with their alignment described by $\hat{a} \cdot \hat{b} = \cos\varphi$.

resides in 8 dimensions and is fully populated only when $r \leq 1/2$. Observables in this context are characterized by triangles (simplexes) rather than straight lines, as in the qubit case.

Moreover, the condition $\vec{a} \cdot \vec{b} = 0$ is not sufficient to ensure that two observables A and B are maximally incompatible. For instance, consider the $d = 3$ spin observables $S_{\hat{z}}$ and $S_{\hat{x}}$ in the $\hat{z}$ and $\hat{x}$ directions, respectively. Let $\hat{z}_i$ and $\hat{x}_i$ represent the vectors corresponding to their projectors. In this case, we find that $\hat{z}_i \cdot \hat{x}_j \neq 0$ for $i, j \in \{1, 2, 3\}$.

We examined how the information-irrealism bound of $\rho_{\vec{r}_n}$ evolves under sequential measurements numerically. The state $\rho_{\vec{r}}$ was chosen as the eigenstate of S_z with eigenvalue -1, while the observables were parameterized as $A = S_{\vec{z}} \cos\varphi + S_{\vec{x}} \sin\varphi$ and $B = S_{\vec{z}}$. For these observables, the alignment parameter satisfies $\hat{a} \cdot \hat{b} = \cos\varphi$. The ranges explored where, as before, $n \leq 10$ and $\varphi \in [0, \pi/2]$.

The results, shown in Fig. 11, indicate a behavior consistent with the qubit case: the bound decreases as φ and n increase, and for sufficiently large n, it approaches zero even when $[A, B] \approx 0$, corresponding to small values of φ. However, unlike qubits, $\varphi = \pi/2$ does not correspond to a mutually unbiased basis (MUB) for qutrits. For instance, when $n = 1$, the bound remains finite, with $I(\rho_{\vec{r}_n}) \approx 0.015$.

Due to the lack of rotational invariance of the Gell-Mann matrices, the 3-d analogue of Pauli matrices, explicit calculations of the volume of irrealism in its integral form, Eq. (4.114), are inherently complex. In contrast, spin operators are rotationally invariant, allowing for a more tractable case study of the volume of irrealism for spin observables given a qutrit state,

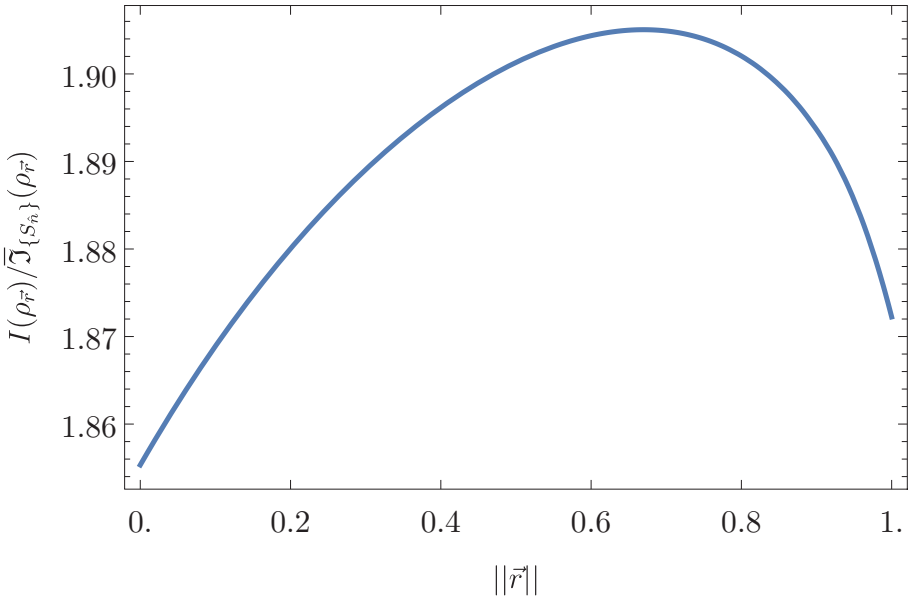

Figure 12 – Ratio of the information $I(\rho_{\vec{r}})$ to the volume of irrealism for spin observables $\overline{\mathfrak{I}}_{\{S_{\hat{n}}\}}(\rho_{\vec{r}})$ as a function of the Bloch vector norm $||\vec{r}||$ in qutrit states.

denoted as $\overline{\mathfrak{I}}_{\{S_{\hat{n}}\}}(\rho_{\vec{r}})$, in accordance with Eq. (4.119).

Following a similar approach to the qubit case, we exploit the symmetries of the spin observable scenario and derive an expression of the form:

$$\overline{\mathfrak{I}}_{\{S_{\hat{n}}\}}(\rho_{\vec{r}}) = \int_0^{\pi/2} \mathfrak{I}_X(\rho_{\vec{r}}) \sin\theta \, d\theta. \tag{4.138}$$

Originally, the integration is performed over a 2-dimensional projection of the 7-dimensional hypersurface of the Bloch sphere. This reduction, expressed in the equation above, is made possible by the symmetry of the spin observables, which enables the computation to be carried out along a single curve within this projection.

Our analysis revealed an approximately proportional relationship between the information and the volume of irrealism for spin observables:

$$I(\rho_{\vec{r}}) \approx (\sqrt{3}\log 3)\overline{\mathfrak{I}}_{\{S_{\hat{n}}\}}(\rho_{\vec{r}}). \tag{4.139}$$

We evaluated the ratio $I(\rho_{\vec{r}})/\overline{\mathfrak{I}}_{\{S_{\hat{n}}\}}(\rho_{\vec{r}})$ as a function of $||\vec{r}||$, finding a concave dependence, as illustrated in Fig. 12. Notably, the average availability of irrealism relative to the state's information is minimal when $||\vec{r}|| = 2/3$. This value corresponds to the maximal norm for a Bloch vector such that states with the same information trace a closed contour inside the simplex $\triangle_R$ (see Fig. 7).

4.8 REALISM-BASED CLASSICALITY

The main result of this chapter, formalized as a theorem, is built on a protocol of sequential pairwise measurements. This scenario can be understood as a simplified model for a

more general process. By considering Eq. (3.4), nonselective measurements can be reinterpreted as a coupling between the system and an environmental degree of freedom, followed by the tracing out (discard) of that environment.

Within this framework, we demonstrated that classical behavior can emerge without requiring a dynamical interaction between the system and large environments. Each measurement step reduces the modulus of the Bloch vector, leading to an asymptotic state $\rho_{\vec{r}_n} \to \mathbb{1}/d$ and vanishing irrealism $\mathfrak{I}_X(\rho_{\vec{r}_n}) \to 0$ as $n \to \infty$ (Eq. (4.73)). This result underscores that a continuous coupling with an environment comprising just two degrees of freedom suffices for the emergence of realism-based classicality.

For the special case of mutually unbiased observables (A and B forming an MUB), a single interaction ($n = 1$) is sufficient to achieve classical realism. In this regime, the Bloch vector norm vanishes after one step, $||\vec{r}_1|| = ||P_B P_A \vec{r}|| = 0$, and the irrealism of any observable also disappears, $\mathfrak{I}_X(\rho_{\vec{r}_1}) = 0$.

This efficiency contrasts with quantum Darwinism [24–27], which ties classicality to objectivity by requiring redundant encoding of information across large environments. In our case, when retrieving environmental fragments involved in the monitoring process, the incompatibility between A and B implies that the encoded information may not be concordant.

The establishment of classicality through information leakage into the environment is formalized using the strict information-irrealism bound, Eq. (4.78).

By applying this bound and considering the monitoring of slightly incompatible observables, as described in Eq. (4.108), one can model a scenario where the monitoring is performed by a slightly faulty measurement apparatus. In this case, each measurement projects the system onto states corresponding to observables that are only approximately the same. The relationship between the emergence of classicality and the system's dimension is illustrated in Fig. 9. As d increases, the rate of classicality emergence slows but eventually stabilizes at a lower bound, corresponding to the upper edge of the shaded region in the figure.

The mathematical formulation of the volume of irrealism, Eq. (4.114), can be directly interpreted as the average entropic distance between a quantum state and its post-measurement state after a nonselective measurement of a totally randomized observable. By averaging over all possible observables, this measure provides a context-independent notion of irrealism for a given state.

This interpretation also motivates a broader perspective: the volume of irrealism serves as a direct quantifier of realism-based classicality. Pure states, which maximize the volume of irrealism, exhibit the greatest capability for performing inherently quantum tasks. In contrast, completely mixed states, with zero volume of irrealism, behave classically in any experimental context.

When restricted to a specific class of observables, the volume of irrealism quantifies

nonclassical behavior in experimental setups involving only a limited class of degrees of freedom. This measure allows for a tailored assessment of quantum behavior in scenarios where access to the full space of observables is constrained.

As demonstrated in the qutrit case study, this measure reveals a nonlinear relationship between the classicality of certain observables and the total information available about the state for $d > 2$ dimension states. This analysis provides insights into the optimal level of mixing required for tasks where nonclassical behavior plays a critical role, aiding in the design of experimental protocols to harness quantum effects effectively.

The broader perspective outlined in this section enables a quantitative assessment of the interplay between realism, classicality, and information. Pairwise sequential measurements were employed as a model to conceptualize system-environment interactions, offering a clear mathematical framework under the BA realism formalism. However, while this protocol provides a rigorous description, the measurement process *per se* remains conceptually challenging to define.

The foundational questions surrounding the nature of measurement are addressed in the next chapter.

5 MEASUREMENT

Quantum mechanics poses a persistent conceptual challenge that resists definitive description. Fundamental features such as wave-particle duality, superposition, entanglement, indeterminacy, and uncertainty represent distinct aspects of this challenge, each depending on the interpretative lens applied to the theory.

In the study of quantum foundations, three primary questions encapsulate this difficulty: the ontological versus epistemological status of the wave function [6], the nature of locality [10], and the measurement problem [18]. Different approaches to these questions result in a variety of interpretations of the theory.

Interpretations of a theory are not intrinsic components of the theory itself. Instead, they serve as heuristic tools that help users develop an intuitive understanding of the framework. For an interpretation to be meaningful, it must remain consistent with the empirical predictions of the theory.

Any set of observations can support an infinite number of physical theories by introducing sufficient hidden variables or complex auxiliary assumptions. This inherent flexibility means that identifying a single "true" theory is fundamentally unattainable. At best, philosophical considerations can rule out certain theories, but none of such theories can be definitively disproven [95]. The same reasoning applies to the various interpretations of a theory.

Developing and evaluating an interpretation involves proposing a set of metaphysical assumptions, such as locality, realism, or determinism. These assumptions must be clearly expressed in the language of the theory, checked for consistency with the framework, and evaluated for consistency with the empirical data. By formulating and testing these assumptions, we can delineate a set of interpretations that remain consistent with the theory. Over time, new insights or empirical evidence may further constrain this set, eliminating interpretations that fail to align with the framework or observations.

As a result, numerous theories can explain the same set of observations, and each theory can support multiple interpretations.

In classical contexts, such as the motion and dynamics of macroscopic objects, we apply classical mechanics. From the perspective adopted in this work, classical mechanics is best understood not as a specific physical theory but as a mathematical framework that supports the construction of various physical theories. For example, Newtonian mechanics, Lagrangian mechanics, and Hamiltonian mechanics are distinct theories derived from this framework, each with its own interpretations suited to specific applications.

Similarly, as argued by Nielsen and Chuang in Ref. [48], quantum mechanics is not