UNIVERSIDADE FEDERAL DO PARANÁ

PAULO HENRIQUE DE BRITO SILVA

IS THERE A QUANTUM RESOURCE FOR LOCAL REALISM ERASURE?

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2025

PAULO HENRIQUE DE BRITO SILVA

IS THERE A QUANTUM RESOURCE FOR LOCAL REALISM ERASURE?

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“Science cannot solve the ultimate mystery of nature. And that is because, in the last analysis, we ourselves are a part of the mystery that we are trying to solve."

(Max Planck)

RESUMO

Desde 1935, quando Einstein, Podolsky e Rosen clamaram a incompletude da mecânica quântica recorrendo à hipótese de realismo local, houveram inúmeros debates sobre as verdadeiras implicações dessa conclusão. Neste trabalho, investigamos teoricamente um apagador quântico modificado que é capaz de correlacionar operações em um laboratório com alterações no realismo de observáveis em outro laboratório causalmente desconectado. A suspeita natural sobre o que está por trás desse fenômeno é o emaranhamento. Ao controlar o emaranhamento inicial do estado analisado, nós examinamos o papel desse recurso quântico nesse fenômeno. Nós provamos que, mesmo com um estado inicialmente separável, a correlação entre o apagamento de realidade e as operações remotas se mantém. Além disso, nosso estudo revela que a discórdia quântica também é irrelevante para o fenômeno observado. No entanto, a irrealidade inicial e a não-localidade baseada no realismo são necessárias, embora não suficientes. Além disso, demonstramos que, para os casos analisados neste trabalho, a discórdia quântica da medição de σ_z desempenha um papel fundamental, mostrando que o fenômeno pode ocorrer mesmo para estados inicialmente clássicos-clássicos. Nossos resultados fornecem uma compreensão mais profunda de como as correlações quânticas contribuem para a quebra do realismo local e como as operações remotas influenciam esse processo. Ademais, ao contrário dos cenários do tipo Bell, onde as correlações dizem respeito aos resultados dos observáveis, o experimento considerado correlaciona operações com aspectos ontológicos. Ao empregar um critério de realismo independente de variáveis ocultas realistas locais, nosso trabalho demonstra que a realidade, em geral, não possui um conteúdo local.

Palavras-chaves: Realismo local; Irrealismo; Recursos quânticos; Emaranhamento; Discórdia quântica; Não localidade.

ABSTRACT

Since 1935, when Einstein, Podolsky, and Rosen claimed the incompleteness of quantum mechanics by resorting to the hypothesis of local realism, there have been numerous debates regarding the true implications of this conclusion. In this work, we theoretically investigate a modified quantum eraser that is capable of correlating operations in one laboratory with changes in the realism of observables in another causally disconnected laboratory. The natural suspicion of what lies behind this phenomenon is entanglement. By controlling the initial entanglement of the analyzed state, we examine the role of this quantum resource in the phenomenon. We prove that even with an initially separable state, the correlation remains between the reality erasure and the remote operations. Furthermore, our study reveals that quantum discord is also irrelevant to the observed phenomenon. However, the initial irreality and realism-based nonlocality are necessary, although not sufficient. Additionally, we demonstrated that, for the cases analyzed in this work, the quantum discord of the measurement of σ_z plays a fundamental role, showing that the phenomenon can occur even for initially classical-classical states. Our results provide deeper insights into how quantum correlations contribute to the breakdown of local realism and how remote operations influence this process. Moreover, unlike Bell-type scenarios, where the correlations concern the outcomes of observables, the considered experiment correlates operations with ontological aspects. By employing a realism criterion independent of local realistic hidden variables, our work demonstrates that reality, in general, does not have a local content.

Key-words: Local realism; Irrealism; Quantum resources; Entanglement; Quantum discord; Nonlocality.

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CHAPTER 1

Introduction

The beginning of the twentieth century was profoundly important for the foundations of physics. Trying to explain some experimental data concerning the thermal radiation of bodies, in 1900 Planck proposed the elementary quantum of action $\hbar$, “one of the most significant and momentous contributions ever made in the history of physics", according to Jammer [1], and initiated the development of quantum theory. In the following years, physicists like Planck, Einstein, Bohr, Sommerfeld, de Broglie, Pauli, and others explored the quantum idea, leading to theoretical results with a high degree of agreement with experiments. Quantum theory achieved its peak of development with Heisenberg, Schrödinger, Born, Jordan, and Dirac, and then became known as Quantum Mechanics (QM). Since its inception, QM has dealt with many principles that contradicted those of classical physics. For example, while in classical physics particle-like and wavelike behaviors are mutually exclusive[1], Bohr argued that they are complementary in QM. Beyond that, even the notion of trajectory is lost in QM, due to Heisenberg’s uncertainty principle.

After the establishment of a pragmatic view of QM in the 1930s, many criticisms to the orthodox interpretation of QM emerged, attempting to “fix" the aspects that fall outside the traditional classical interpretation and bring QM closer to Classical Mechanics (CM). Probably the most important paper published in this era was that of Einstein, Podolsky, and Rosen (EPR) in 1935 [2], where the authors claimed the correctness and incompleteness of QM, since it explained the experimental data and was not compatible

[1] For example, in the classical description of light, Newton argued in favor of its particle-like behavior, while Huygens supported its wave-like nature.

with notions such as locality and local realism. According to the modern form of EPR arguments, if two systems A and B are space-like separated, measurements in A cannot immediately influence the system B, and the reality described by each system is completely local, i.e., they can only be affected by events inside their light cones. Moreover, by their realism criterion, they concluded that QM was unable to assign elements of reality to all observables simultaneously. Since QM allows certain kinds of descriptions that are not in agreement with this hypothesis, EPR claimed that, although QM is a correct theory, it is incomplete. Bohr, the most important physicist behind the orthodox interpretation of QM, immediately responded EPR [3], attempting to convince them that the elements of reality observed depend on the experimental setup.

Other criticisms of the orthodox interpretation of QM also emerged in the literature, such as those made by Schrödinger [4, 5], when the term entanglement entered the realm of physics. Entanglement is one of the most important features of QM, and it has been profoundly studied throughout the years [6], marking the beginning of quantum resource theory [7]. Essentially, a quantum resource is a property of quantum systems that enables tasks that are impossible to perform with classical systems. Since the advent of entanglement, many different quantum resources has been explored, such as Bell-nonlocality [8], quantum steering [9], quantum discord [10–12], irrealism [13], and realism-based nonlocality [14, 15].

Regarding the tasks that can be performed using quantum resources, there are the cryptographic processes and quantum communication [8], and quantum teleportation [16]. Additionally, there are also experiments of foundational interest, such as the quantum eraser [17–24], which can correlate path-information of a photon with particle-like or wavelike behavior. By doing a modification in the standard model of a quantum eraser, the authors in [25] proposed an experimental setup that is capable of correlating choices with elements of reality, against the central point in EPR's paper. By using a modern approach of realism [13], the authors showed that the correlation occurs even without any retrodiction hypothesis. Indeed, the experiment was performed in [26], and the same conclusions were reached, confirming the phenomenon. These works provide strong arguments against the notion of local realism introduced by EPR, since the theoretical analysis does not need to consider the distance between the systems, and the experimental verification was conducted in two laboratories space-like separated. Thus, according to this conclusion, the notion of realism is not local.

The phenomenon of correlation of choices and elements of reality was studied for the case of pure states, where quantum resources overlap with each other. Consequently, it was not possible to determine which quantum resource (if any) was responsible for the phenomenon. Although the initial entanglement was controlled and its absence implied the non-occurrence of the phenomenon for that case, it was not possible to conclude that it was responsible for it. Therefore, this is precisely the focus of study in this work.

The present master's thesis focuses on exploring the quantum resource (if any) responsible for the occurrence of the aforementioned phenomenon and to determine whether classical correlations alone are sufficient to establish such a correlation. The approach taken here involved extending the original work with a more general state, including noise in the system, to explore the true role of entanglement in the initial system. Beyond that, exploring the limits of the conclusions for the previous case, general separable systems were also considered to investigate the role of quantum discord and realism-based nonlocality. The calculations were carried out analytically throughout the text, with computational support used when necessary to aid in interpreting the results.

This work is organized as follows: Chapter 2 discusses QM as a whole, since its historical development to the main ideas proposed by EPR. This chapter covers the beginning of the theory in Section 2.1, its mathematical description in modern notation in Section 2.2, and its philosophical aspects, such as interpretations and its criticisms, in Section 2.3. Chapter 3 introduces the basic concepts used throughout the text. Section 3.1 focuses on the study of the entropies used in classical and quantum information theory, such as Shannon and von Neumann entropies. Section 3.2 briefly discusses how to represent a quantum state in the Bloch representation, which will be useful for future chapters. Section 3.3 introduces the idea of quantum resources, along with the main quantum resources used in this work. Finally, Section 3.4 presents the setup of a Mach-Zehnder Interferometer and describes quantum systems within it. Chapter 4 introduces the standard quantum eraser in Section 4.1, and Section 4.2 reviews the papers that introduced the phenomenon studied in this work. The original contributions of this master's thesis are discussed in Chapter 5, where Werner states are applied to the experimental setup in Section 5.1, and Section 5.2 explore general separable states in the modified quantum eraser. Both Sections 5.1 and 5.2 analyze the role of the quantum resources relevant to each case. Section 5.3 presents final remarks on the study, offering new insights into the analysis of the problem, and concluding the evaluation of the resources. Finally, Chapter 6 summarizes the conclusions based on the development presented in this work.

CHAPTER 2

Quantum Mechanics and its Foundations

The goal of this chapter is to introduce the mathematical framework and philosophical aspects underlying the foundations of Quantum Mechanics. The discussion spans from the beginning of Quantum Theory, with Max Planck's introduction of the quantum postulate, to the seminal work of Einstein, Podolsky, and Rosen, as well as modern notions of realism. The mathematical and conceptual development of Quantum Mechanics will be presented, as much as possible, from a historical perspective, based on Max Jammer's books [1, 27], source of all quotes in this chapter (except for those with explicit references). Section 2.1 provides a brief review of the origins of quantum physics and quantum mechanics, although it is not essential to understand the main results of this work. Section 2.2 covers the mathematical framework of quantum mechanics, particularly from the perspective of density operators. Section 2.3 explores the orthodox interpretation of quantum mechanics, known as the Copenhagen Interpretation, alongside its criticisms and philosophical implications concerning realism in quantum theory.

2.1 The Birth of Quantum Mechanics

This section begins by addressing the historical context of the late nineteenth century. By then, what is now known as "Classical Physics" was unable to account for certain experimental results. One such issue was the "ultraviolet catastrophe," which arose in the study of thermal radiation from bodies. Experimental observations provided key results that any theoretical explanation needed to address, such as

Wien's displacement law, which relates the wavelength of maximum radiance to a body's temperature, and Stefan's law, which describes the radiance's dependence on temperature. Additionally, the shape of the radiance curve as a function of wavelength or frequency was well known.

The classical explanation proposed by Lord Rayleigh and Jeans, commonly called the Rayleigh-Jeans blackbody model, agreed with experimental data at low frequencies but predicted diverging radiance at high frequencies, leading to a significant disagreement between theory and experiment.

In this context, at the end of 1900, Max Planck, inspired by Rubens and Kurlbaum's experiments on radiance at low and high frequencies, approached the problem from a thermodynamic perspective. He successfully derived the correct spectral radiance law, which complied with Wien's displacement law and Stefan's law, though initially using interpolations and lacking formal hypotheses. To formalize what he described as a "lucky guess," Planck turned to Boltzmann's probabilistic conception of entropy for an explanation. After meticulous calculations, he deduced a general radiance formula which matched the earlier results, now known as Planck's law of radiation, only if the equipartition theorem was revised, assuming the energy of vibrational modes followed the form $\epsilon = nh\nu$, where n is the vibrational mode, ϵ is the energy of the vibrational mode n, h is Planck's constant and ν is the frequency. This marked the introduction of the quantum of energy.

Planck presented his results on December 14, 1900, at a meeting of the German Physical Society a date often regarded as the birth of Quantum Theory. In a letter to R. W. Wood, Planck admitted that this explanation was "an act of desperation," as "a theoretical explanation *had* to be supplied, at all costs, whatever the price." Indeed, he remained skeptical of his own hypothesis and repeatedly tried alternative ways to incorporate the constant h into his framework.

The discussion of the concept of quantum advanced significantly with Einstein's seminal work [28] "On a Heuristic Viewpoint Concerning the Production and Transformation of Light." In this paper, Einstein concluded that light "behaves as if it consisted of independent energy quanta," with $\epsilon = h\nu$ (in modern notation), aligning with Planck's hypothesis. He proposed that radiation behaves as though it consists of a finite number of localized energy quanta, later termed photons. At that time, the wavelike nature of light was widely accepted due to extensive experimental evidence. However, as Young noted in 1807, although experiments supported wave models, "it is allowed on all sides, that light either consists in the emission of very minute particles from luminous substances." The validity of Einstein's argument was solidified by his explanation of the photoelectric effect, also presented in the same paper, solving a problem that had persisted since Hertz's discovery in 1887.

During this period, the concept of quanta was largely confined to heat theory, even though it had been used to explain the photoelectric effect. Nevertheless, Arthur Erich Haas was the first to link the concept of quanta to the atom. In a 1910 paper, Haas combined Thomson's atomic model with the theory of line spectra. Despite the novelty of his work, it was ridiculed and quickly dismissed after Rutherford introduced his atomic model in 1911. However, Haas's hypothesis gained recognition from Lorentz at the Solvay Conference in 1911 and later by Niels Bohr, who incorporated the quantum concept into his atomic model. Bohr successfully unified the experimental results related to the hydrogen atom into a single model, based on the hypothesis that quanta were indeed fundamental to atomic structure.

Bohr's 1913 atomic model is often regarded as a set of theoretical hypotheses that explained experimental observations without delving into their deeper physical origins. Bohr was likely the first to explore atomic stability from a quantum perspective. As Jammer observed, "In contrast to his predecessors who related Planck's constant h to atomic models with the purpose of finding a mechanical or electromagnetic interpretation of h, Bohr recognized that Planck's constant should be applied to Rutherford's model not in order to elucidate the physical significance of the former, but rather to account for the stability of the latter." Bohr's model, together with its connection to Rydberg's series, shifted the focus of quantum theory toward discrete spectra, yielding significant advances.

In 1915, Sommerfeld, who shared Bohr's view of using Planck's constant to explain atomic stability, generalized Bohr's model. He proposed quantizing all canonical variables of a system, subject to what are now known as the Sommerfeld conditions. This approach introduced the azimuthal quantum number l, associated with orbital angular momentum, and provided the first explanation of the fine structure of the hydrogen atom.

By the time of Sommerfeld's model, Quantum Theory could explain many experimental results available at the time. The next step was to formalize these findings into fundamental principles. Two principles were particularly important to the conceptual development of this era: the Adiabatic Principle and the Correspondence Principle. The Adiabatic Principle[1], as demonstrated by Ehrenfest, formalized quantization conditions, while the Correspondence Principle[2], as explained by Bohr, connected the quantum theory's limits with Maxwell's electromagnetic theory. Both principles were fundamental to the new physics of the twentieth century. Although Bohr, as noted earlier, refused from offering a classical interpretation of Planck's constant, he extensively used the

[1] Adiabatic Principle in this context means that if a quantum system evolves slowly, its quantum numbers remain unchanged [1].

[2] The Correspondence Principle establishes quantum theory must recover classical theory in the limit of of large quantum numbers [1].

Correspondence Principle to bridge Quantum Theory and Classical Electrodynamics. At this stage, Bohr was not yet prepared to fully embrace the quantum nature of reality.

During the early development of quantum theory, before the quantum hypothesis, another significant event occurred: the discovery of the Zeeman Effect. This phenomenon laid the groundwork for the conceptual development of what would later be known as spin. The first to suggest a connection between magnetic effects and light was Michael Faraday in 1845. Unfortunately for Faraday, this connection remained speculative. It was only in 1896 that Pieter Zeeman observed the separation of the spectral lines of sodium under the influence of a magnetic field. According to Zeeman, his experiment was inspired by the idea that, "if Faraday thought of the possibility of the above-mentioned relation (light-magnetism), perhaps it might be worthwhile to try the experiment again with the excellent auxiliaries of spectroscopy of the present time."

Thanks to Lorentz, Larmor, and Zeeman, and aided by the discovery of the electron by J. J. Thomson in 1897, the Zeeman Effect could be explained within the framework of classical electromagnetic theory. However, this explanation only accounted for the triplet splitting of spectral lines. In December 1897, Preston observed, "It is interesting to notice that the two lines of sodium and the blue line 4800 of cadmium do not belong to the class which show as triplets. In fact, the blue cadmium line belongs to the weak-middled quartet class, while one of the D lines shows as a sextet of fine bright lines [...]." This observation, quickly confirmed by Cornu, became known as the Anomalous Zeeman Effect (AZE). Despite Lorentz's attempts in 1898 to extend his theory to explain the AZE, the phenomenon remained an unsolved problem until the end of the "Old Quantum Theory" in 1925.

Following Bohr's groundbreaking contributions to atomic theory, the normal Zeeman Effect was elegantly explained by Sommerfeld (who introduced the azimuthal and magnetic quantum numbers) and Debye using the new concepts of atomic theory. However, the AZE persisted as a problem. As Sommerfeld remarked, "In the present state of quantum treatment, the Zeeman Effect achieves just as much as Lorentz's theory, but not more. It can account for the normal triplet... but hitherto it has not been able to explain the complicated Zeeman types." Even Sommerfeld's relativistic treatment failed in addressing the AZE.

With the advances in spectral analysis available from 1923 onward, studies of spectral line multiplicities increased significantly. This period saw the introduction and establishment of selection rules for n, l, and m. Nevertheless, as is now known, a crucial development in quantum theory was still needed: the concept of spin. Attempting to explain the AZE, Sommerfeld and Landé proposed the "magnetic-core hypothesis." According to this hypothesis, the "atomic core" possesses angular momentum in s units of $\hbar$ and a corresponding magnetic moment. Using this idea, they showed that the valence (optical) electron experienced an internal Zeeman Effect, allowing for the interpretation of the AZE. This explanation appeared consistent with the experimental results obtained by Stern and Gerlach in 1921, which were interpreted as evidence of "space quantization," an idea introduced by Sommerfeld some years before. However, deeper analysis soon revealed mistakes in this explanation.

In the fall of 1924, Wolfgang Pauli identified a discrepancy between Landé's theory and experimental results. From the perspective of relativity, the magnetic-core hypothesis predicted that the Zeeman Effect would depend on the atomic number, contradicting experimental findings. This led Pauli to state, "In particular, the angular momenta of alkali atoms and their energy changes in an external magnetic field have to be considered as due essentially to the exclusive action of the optical electron, which also has to be regarded as the source of the magnetomechanical anomaly. The doublet structure of the alkali spectra, as well as the deviation from Larmor's theorem, are due, according to this view, to a peculiar, classically indescribable two-valuedness in the quantum-theoretic properties of the optical electron." It was at this moment in history that the concept of spin truly began to emerge.

At the same time that Pauli was formulating his arguments against the magnetic-core hypothesis, Edmund C. Stoner presented his theory of electron distribution across the energy levels of the atom. Building on this, and alongside Bohr's advancements in understanding these distributions, Pauli developed a groundbreaking explanation by assuming two hypotheses. The first was the existence of an additional quantum number, now denoted as m_s, to characterize each allowed orbit in the atom. The second was a prohibition principle, which Pauli articulated as: "There never exist two or more equivalent electrons in an atom which, in strong fields, agree in all quantum numbers." This statement is now famously known as Pauli's Exclusion Principle.

Although Pauli's approach succeeded in explaining the observed phenomena, the new quantum number lacked a fundamental explanation. Pauli himself admitted that "no deeper motivation of the rule can be provided." The notion of the electron's spin, initially conceptualized as an "electron spinning" (a view that is known to be outdated), first appeared with Kronig in January 1925, although Compton had briefly mentioned a similar idea in 1921. However, Pauli quickly dismissed Kronig's idea. Later that same year, in November, Uhlenbeck and Goudsmit independently proposed the same concept. While Bohr, one of the era's most influential physicists, accepted these ideas, Pauli initially remained skeptical, primarily due to the theoretical issues they introduced.

By the spring of 1926, when Thomas resolved these apparent problems, Pauli conceded: "Although at first I strongly doubted the correctness of this idea because of its classical mechanical character, I was finally converted to it by Thomas's calculations on the magnitude of doublet splitting. On the other hand, my earlier doubts, as well as the cautious expression ‘classically not describable two-valuedness,’ were partially validated during later developments. Bohr demonstrated, on the basis of wave mechanics, that electron spin cannot be measured by classically describable experiments and must therefore be considered an essentially quantum-mechanical property of the electron."

At this point, quantum theory had achieved significant success in explaining many experimental results. Bohr’s atomic model, combined with the introduction of new ideas, suggested that the theory was converging toward a general description of nature, supported by well-defined principles. While the narrative of spin’s discovery progressed, another enduring discussion, the wave-particle duality, continued to unfold, eventually leading to the emergence of quantum mechanics, as formulated by Born and others.

The wave-particle duality debate intensified after Planck introduced the quantum and Einstein proposed the photon. These ideas challenged the classical understanding of light as purely wavelike, as described by classical electrodynamics. Einstein’s 1905 photon hypothesis successfully explained light-matter interactions, but it was Arthur Compton’s 1921 experiment that firmly established the particle-like behavior of light. By 1923, Compton resolved the failure of Thomson’s theory of x-ray scattering for short wavelengths using the photon concept. He wrote, "The present theory depends essentially upon the assumption that each electron which is effective in the scattering scatters a complete quantum. It involves also the hypothesis that the quanta of radiation are received from definite directions and are scattered in definite directions. The experimental support of the theory indicates very convincingly that a radiation quantum carries with it directed momentum as well as energy."

Though initially met with skepticism, Compton’s theory gained widespread acceptance after extensive debate, solidifying its status by late 1924. Between 1921 and 1924, much attention focused on the particle-like behavior of x-rays. This behavior proved equally useful in explaining other optical phenomena, further reinforcing the photon hypothesis.

In parallel with these developments, Maurice de Broglie played an active role in x-ray studies and spectroscopy. Maurice profoundly influenced his younger brother, Louis de Broglie (often referred to simply as de Broglie). As Louis later recalled: "I had long discussions with my brother on the interpretation of his beautiful experiments on the photoelectric effect and corpuscular spectra... These long conversations with my brother about the properties of x-rays... led me to profound meditations on the need of always associating the aspect of waves with that of particles." Indeed, Louis de Broglie became a central figure in advancing the discussion of wave-particle duality.

At the end of the summer of 1923, de Broglie’s concept of the "phase wave" began to take shape. According to Max Jammer, the paper published at that time can be considered the birth of wave mechanics [29]. Drawing on Hamiltonian mechanics and optical-mechanical analogies, de Broglie concluded that "a stream of electrons passing through a sufficiently narrow hole should also exhibit diffraction phenomena." He proposed that the phase wave guided the particles, thus reconciling the wave-particle duality. Furthermore, de Broglie argued that just as radiation exhibits particle-like behavior, material bodies exhibit wavelike behavior.

On November 29, 1924, de Broglie presented his doctoral thesis, which expanded on his earlier published works. In particular, he explored the optical-mechanical analogy using his phase waves and the concept of electron interference. Regarding this, Einstein remarked, "I shall discuss this interpretation in greater detail because I believe that it involves more than merely an analogy." What was missing to transform this concept from a mere analogy into a true theory was experimental verification of electron diffraction or interference. On this matter, two students of Born, James Franck and Walter Elsasser, had been studying Clinton Joseph Davisson's experiments from 1919 on the scattering of electrons by metals. They noticed that electron diffraction effects were already visible in Davisson's results. Franck and Elsasser calculated the wavelength corresponding to the observed patterns and found excellent agreement with de Broglie's relation $\lambda = h/p$. Subsequent experimental verifications confirmed these findings.

However, as this section aims to present the foundations of what is now called Quantum Mechanics, two other significant achievements of early 20th-century physics must be discussed: matrix mechanics and wave mechanics.

Before 1925, quantum theory largely relied on Bohr's ideas and, most notably, the correspondence principle. Essentially, quantum problems were first solved classically and then translated into the "quantum realm" using quantization hypotheses. However, there was no logical or consistent formalism for these translations. As Max Jammer observed, solving quantum problems was "a matter of skillful guessing and intuition rather than deductive and systematic reasoning."

In 1925, Werner Heisenberg abandoned the classical description in favor of a new framework based on observable magnitudes. Heisenberg later stated that his theory was inspired by Einstein's theory of relativity. The critical point in Heisenberg's theory was the rejection of the classical notion that both position and momentum were directly observable. This hypothesis was motivated by the inability to measure these quantities experimentally and the failure of previous theories that treated them as observables. In summary, Heisenberg replaced kinematic variables with optical quantities.

Unlike Bohr's approach, Heisenberg managed to formally incorporate the correspondence principle into his new theory, providing a more general method for addressing quantum problems. Heisenberg first sent his paper to Pauli in May 1925, who encouraged him to submit it to Born. Born received the paper in July, and it was submitted to the editor of Zeitschrift für Physik [30] on July 29, 1925. This paper introduced the abstract notions now known as "Heisenberg's multiplication rule," which Born recognized as matrix multiplication.

Following this, Born and Jordan initiated a crucial collaboration to mathematically formalize Heisenberg's ideas. Their collaboration led to the first use of the term "Quantum Mechanics" in their paper "On Quantum Mechanics" [31], first published on September 27, 1925. Soon after, Born identified certain weaknesses in their formulation, prompting further development of the mathematical tools used in physics. Born and Norbert Wiener introduced operators into Quantum Mechanics, an idea that Wiener developed and that gained the approval of David Hilbert, one of the most influential mathematicians of the time. Alongside these developments, while Wiener contributed operator-based calculations, Paul Dirac devised an algebraic algorithm equivalent to other descriptions.

Initially, the matrix mechanics developed by Heisenberg, Born, and Jordan faced rejection from experimental physicists but attracted the interest of those with philosophical inclinations. Bohr remarked, "The whole apparatus of quantum mechanics can be regarded as a precise formulation of the tendencies embodied in the correspondence principle," and expressed hope with the statement, "It is to be hoped that a new era of mutual stimulation of mechanics and mathematics has commenced." The initial resistance to matrix mechanics was partly due to its reliance on advanced mathematics unfamiliar to most physicists at the time.

Before Quantum Mechanics, physics was dominated by functions and differential equations, making calculus the primary mathematical tool. However, the algebraic structure of Quantum Mechanics, represented primarily by matrices, was a departure from this tradition and was not commonly taught in physics courses. The turning point for Quantum Mechanics came with Erwin Schrödinger's seminal work on the wave equation, which provided a more accessible and widely accepted framework.

As mentioned earlier, by 1925, de Broglie's ideas about wave-particle duality were no longer mere hypotheses. Stronger verification came in subsequent years, culminating in the 1937 Nobel Prize in Physics for Davisson and George Thomson, awarded for their demonstration of electron interference patterns. Given the fundamental role of waves in nature, it was expected that such waves would obey an equation. Drawing on established principles of physics, this wave theory was anticipated to generalize classical mechanics, reducing to the latter in appropriate limits. Several physicists attempted to develop this wave theory, including E. Madelung, who searched for a "wave theory of atomic levels." However, it was Schrödinger who succeeded in constructing the theory. Schrödinger's background in areas such as the physics of continuous media, eigenvalue problems, and acoustics, as well as his interest in philosophy, uniquely prepared him for this accomplishment.

In 1926, Schrödinger was invited by Debye to give a colloquium at the Technical University of Zurich on de Broglie's ideas. Debye later remarked: "The preparation for that talk really got him started. Only a few months passed between the talk and his publications." Debye's choice was deliberate, as Schrödinger, already interested in statistical mechanics, had been profoundly influenced by Einstein's 1925 paper on the quantum theory of an ideal gas, which underscored the meaning of de Broglie's ideas.

Following insights from de Broglie's 1924 thesis, Schrödinger published his seminal four-part work, "Quantization as a Problem of Proper Values," (chapters 1,2, 5 and 6 of [32]) in 1926. In the first part, published on January 27, he introduced the now-famous time-independent Schrödinger equation, reinterpreting Sommerfeld's quantization rule as a wave-theoretic eigenvalue problem. This paper also featured the first solution of the hydrogen atom using his formalism and introduced the symbol ψ for the wave function. The second part, published on February 23, elaborated on key concepts indispensable to "undulatory mechanics" (as it was called at the time), such as Hamilton's optical-mechanical analogy. Schrödinger applied his formalism to problems like the harmonic oscillator and the rigid rotator, finding full agreement with Heisenberg's theory for Planck's oscillator.

The third part, published on May 10, addressed time-independent perturbation theory and its application to the Stark effect in hydrogen, yielding results in good agreement with experimental data. The fourth part, published on June 21, introduced the time-dependent Schrödinger equation. Initially, Schrödinger required ψ to be real. However, the time-dependent equation seemed a diffusion equation with an imaginary diffusion coefficient, prompting him to accept that ψ must generally be a complex-valued function. Schrödinger concluded this part by discussing the physical significance of ψ, stating "The ψ function is to do no more and no less than provide a survey and mastery of these fluctuations via a single differential equation. It has repeatedly been pointed out that the ψ function itself cannot and may not in general be interpreted directly in terms of three-dimensional space — despite the one-electron problem suggesting otherwise — because it is, in general, a function in configuration space rather than real space."

The importance of Schrödinger's contributions to modern physics is difficult to overstate. As Jammer noted, the 1926 papers were "undoubtedly one of the most influential contributions ever made in the history of science." Unlike Heisenberg's formalism, which relied on less familiar mathematical tools, Schrödinger's wave mechanics was rooted in calculus, facilitating its acceptance among physicists.

In conclusion, the quantum concepts developed between 1900 and 1913 were largely grounded in classical mechanics, relying on analogies with macroscopic phenomena. Bohr's 1913 atomic theory marked a turning point, highlighting key differences between microscopic and macroscopic systems. This philosophical shift culminated in Heisenberg's matrix mechanics, which focused on measurable quantities, and Schrödinger's wave mechanics. Together, these form the mathematical foundations of Quantum Mechanics, which were further formalized in subsequent years.

2.2 Mathematical Foundations and the Postulates of Quantum Mechanics

The mathematical foundations of QM, as discussed earlier, were primarily established by Heisenberg and Schrödinger between 1925 and 1926. However, debates regarding the physical interpretation of the mathematical objects used in the theory persisted. Heisenberg focused on understanding observable quantities through the notion of operators and their evolution (as it is known today), while Schrödinger drew on de Broglie's ideas, developing a theory based on wave functions. Although both theories aligned with experimental results, they were rooted in complementary views of nature.

Before the publication of the third part of "Quantization as a Problem of Proper Values," Schrödinger demonstrated that his formalism led to the same fundamental assumptions as Heisenberg's. This led him to conclude that the two theories were equivalent. However, from a formal perspective, this equivalence was not immediately obvious. Stronger proofs of equivalence were provided in the following months by Wolfgang Pauli and Carl Eckart, and it was only in 1929 that John von Neumann presented a rigorous mathematical proof of their equivalence [33].

Once Quantum Mechanics was established as a mathematical theory, it became necessary to address its physical interpretation, particularly the meaning of the wave function. Schrödinger initially proposed that $\psi\psi^*$ represented a "weight function" for the charge density of an electron. However, this interpretation was quickly shown to be incorrect. Alternatively, while studying electron scattering, Born derived an equation that could only be understood in the particle picture if $\psi\psi^*$ were interpreted as a probability density, as will be discussed further later in this section. This marked the formal introduction of probabilistic concepts into Quantum Mechanics. Although other interpretations of ψ were proposed, Born's approach became the standard, forming a core part of the Copenhagen Interpretation of Quantum Mechanics, which will be discussed in Section 2.3.

In addition to Schrödinger's theory and the interpretations of the wave function, it was noted in the previous section that Born, Jordan, Heisenberg and Dirac [31, 34, 35] were instrumental in developing the algebraic formulation of Quantum Mechanics. They introduced an algebraic algorithm to address quantum problems and developed the transformation theory, which describes transformations in Quantum Mechanics that preserve the invariance of empirically significant formulas. These formulations were crucial for the mathematical foundations of Quantum Mechanics, as they unified all known quantum-mechanical formalisms into a single framework.

Building on the work of Born, Jordan, Heisenberg and Dirac, David Hilbert, together with his assistants L. W. Nordheim and von Neumann, began exploring the mathematical foundations of Quantum Theory. They rederived previously known results using a novel approach grounded in pure mathematics. This work led von Neumann to recognize the possibility of establishing a new, more formal, and rigorous framework for QM. Between 1927 and 1929, he developed a formalism based on Hilbert spaces, which will be discussed in this section. In 1932, von Neumann published his seminal book on the mathematical foundations of QM [33], one of the most significant works in the field.

This section presents the formalism of QM in its most modern form, provided by the references [33, 36–38], encompassing all the previous descriptions. It should be noted that since von Neumann's contributions, the mathematics underlying Quantum Theory has been developed in increasingly sophisticated ways. The purpose here is not to present the formalism in its most abstract or rigorous form, but rather to maintain sufficient formality to preserve the theory's generality and completeness. Additionally, as the Quantum Mechanical formalism constitutes an entire theory, including many aspects that are beyond the scope of this work, only the concepts essential for understanding the topics at hand will be discussed.

The postulates of Quantum Mechanics will now be introduced, with comments provided at appropriate points, including discussions of the underlying mathematics and relevant historical notes.

For Postulate 1, it will be necessary to introduce key concepts such as Hilbert spaces and operators. The goal of these discussions is not to assume that the reader has no prior knowledge but to ensure a clear understanding of the concepts of linear algebra as they are translated into the language of Quantum Mechanics. To begin, let us define the Hilbert space.

- **Hilbert Space:** A complex vector space is called a Hilbert space if it is a Banach space (i.e., a vector space with metric and every Cauchy sequence in the space converges to an element within the space) where the norm is induced from a hermitian inner product (see ref. [39] for mathematical details).

Hilbert spaces are the fundamental mathematical structure required to study Quantum Mechanics. With the definition established, the discussion about their elements can now begin. Using Dirac's notation (introduced by him in 1939), the elements of a Hilbert space are denoted by $|\psi\rangle \in \mathcal{H}$, and their Hermitian conjugates are vectors in the dual space[3] $\mathcal{H}^*$, written as $\langle\psi| \in \mathcal{H}^*$. The inner product mentioned above is defined as

$$\langle\cdot|\cdot\rangle : \mathcal{H}^* \times \mathcal{H} \to \mathbb{C}.$$

With the inner product, it becomes possible to discuss orthogonal and orthonormal vectors. Furthermore, the structure of a vector space allows the exploration of possible bases for such spaces. If $\{|u_i\rangle\}$ is a complete orthonormal basis for $\mathcal{H}$, every element $|\psi\rangle \in \mathcal{H}$ can be expressed as $|\psi\rangle = \sum_i a_i |u_i\rangle$, where a_i are the projections of the original vector onto the basis vectors $|u_i\rangle$. Using the inner product, these projections can be written as $a_i = \langle u_i|\psi\rangle$. Consequently, one can conclude that

$$|\psi\rangle = \sum_i \langle u_i|\psi\rangle |u_i\rangle = \Big(\sum_i |u_i\rangle \langle u_i|\Big) |\psi\rangle \Rightarrow \mathbb{1} = \sum_i |u_i\rangle \langle u_i| \,,$$

where $\mathbb{1}$ is the identity operator. The above equation highlights the structure necessary to understand operators.

A linear operator $A : \mathcal{H} \to \mathcal{H}$ is defined as

$$A|\psi\rangle = |\psi'\rangle \,,$$
$$A\Big(\sum_a a\,|\psi_a\rangle\Big) = \sum_a a\Big(A\,|\psi_a\rangle\Big),$$

and $A^\dagger : \mathcal{H}^* \to \mathcal{H}^*$ is the Hermitian conjugate of the operator A. An operator A is said to be Hermitian if $A = A^\dagger$.

The spectrum[4] of an operator can now be defined. Let A be an operator in a Hilbert space $\mathcal{H}$. The spectrum of A is given by

$$\sigma(A) = \{a \in \mathbb{C} \,|\, \det(A - a \cdot \mathbb{1}) = 0\}.$$

In other words, the spectrum of an operator is the set of its eigenvalues. Using the spectrum, the spectral decomposition of an operator can be written as

$$A = \sum_n a_n P_n,$$

where P_n are projectors (Hermitian and idempotent operators) associated with the eigenvalues a_n.

As a final remark before stating the first postulate, it is important to note that eigenvalues can be degenerate. In such cases, the projectors P_n must account for this

[3] For the case of Hilbert spaces, the dual space is isomorphic to the Hilbert space itself, so that $\mathcal{H} \sim \mathcal{H}^*$.
[4] Formally, this is the pure point spectrum. For more details about the spectral theorem, see ref. [40].

degeneracy. The most general form of the spectral decomposition, including degenerate eigenvalues, is given by

$$A = \sum_n a_n \Big(\sum_{i=1}^{g_n} P_n^{(i)} \Big),$$

where $P_n^{(i)} = \left| u_n^{(i)} \right\rangle \left\langle u_n^{(i)} \right|$ are the orthogonal projectors associated with the eigenvalue a_n, which has degeneracy g_n. These projectors satisfy the conditions $P_n^{(i)} P_m^{(j)} = \delta_{nm}\delta_{ij}$ and $\sum_n \sum_{i=1}^{g_n} P_n^{(i)} = \mathbb{1}$.

With these definitions in place, we are now ready to present the first postulate of Quantum Mechanics.

- ***Postulate 1***: The state describing the degrees of freedom of a physical system at a time t_0 is completely described by the density operator ρ acting on a Hilbert space $\mathcal{H}$.

The density operator, written in its spectral decomposition as $\rho = \sum_i p_i P_i$, is a Hermitian and positive semidefinite operator, meaning its eigenvalues p_i are always real and non-negative, representing the population of each pure state $P_i = |\psi_i\rangle \langle\psi_i|$, which are the eigenstates of ρ. Additionally, ρ must have unit trace, i.e., it satisfies $\operatorname{tr}\rho = 1$. These conditions allow for the description of any physical system in Quantum Mechanics. As will be discussed later, the general characterization of a quantum state lies in the properties of ρ. It is also worth noting that, unlike Classical Mechanics, the description of a physical system in Quantum Mechanics is inherently abstract. This abstraction played a significant role in the challenges physicists of the last century faced when trying to understand its meaning.

To correctly understand Postulate 2, it is necessary to define an observable. Consider that $A : \mathcal{H} \to \mathcal{H}$ is a Hermitian operator. Furthermore, assume that the set of eigenvectors of A forms a complete basis of $\mathcal{H}$. If both conditions are satisfied, the operator A is called an observable. Note that the requirement for A to be Hermitian ensures that its eigenvalues are real numbers. With this definition in place, the second postulate can be stated:

- ***Postulate 2***: Every measurable physical quantity $\mathfrak{A}$ is described by an observable A acting on a Hilbert space $\mathcal{H}$, and the only possible outcomes of such measurements are the eigenvalues of A.

This postulate is fundamental to understanding the theory. As mentioned earlier, Quantum Theory began with Planck's concept of the quantum of energy. The term "quantum" itself arose because of quantization; in other words, the continuum notion of energy failed to explain experimental results. The development of the theory reached its maturity through the works of Born, Jordan, Dirac, and von Neumann, culminating in the establishment of Postulate 2. This postulate naturally incorporates the idea of quantization as a consequence of finite-dimensional vector spaces. Although it can be generalized to infinite-dimensional spaces, this work focuses solely on the finite-dimensional case.

Regarding the eigenbasis of an observable A, it is important to discuss orthogonality. Suppose $\{|a_n^i\rangle\}$ is the eigenbasis of A, and $A\left|a_{n,m}^{i,j}\right\rangle = a_{n,m}\left|a_{n,m}^{i,j}\right\rangle$. Then, using the fact that A is an observable, it follows that $(a_n - a_m)\langle a_m^j|a_n^i\rangle = 0$. Thus, if $a_n \neq a_m$, it must hold that $\langle a_n^i|a_m^j\rangle = 0$. In other words, eigenvectors corresponding to distinct eigenvalues are orthogonal.

A clarification regarding Postulate 2 is worth analyzing. Consider that a physical state is described by $\rho = \sum_i p_i P_i$ and that a physical quantity $\mathcal{A}$ is described by $A = \sum_i a_i A_i$, where A_i are the projectors associated with the eigenvectors a_i of the observable A. With this information, how can one describe the final state after measuring A on the state ρ? The answer to this question is provided by the third postulate.

- ***Postulate 3***: If a physical state is described by ρ and a measurement of the quantity $\mathfrak{A}$, represented by an observable A, is performed on ρ yielding an outcome a_n, the state ρ collapses to the final state

$$\rho \longmapsto \rho_{a_n} = \frac{A_n \rho A_n}{\mathrm{tr}(A_n \rho)}, \tag{2.1}$$

where $A_n = \sum_i^{g_n} |a_n^i\rangle\langle a_n^i|$ are the projectors of A.

This postulate will be thoroughly discussed in the next section because it represents a significant departure from Classical Mechanics, as emphasized by Einstein and his collaborators. For now, it is essential to highlight the notion of state preparation and the concept of a complete set of commuting observables.

Suppose a physical system is described by a density operator ρ, which is defined in a multi-partite Hilbert space,

$$\rho : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \to \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N. \tag{2.2}$$

The goal is to obtain a specific eigenstate of ρ to prepare the system for an experiment. The general form of ρ in this scenario is

$$\rho = \sum_{ijk\cdots} p_{ijk\cdots} |a_i, b_j, c_k, \cdots\rangle\langle a_i, b_j, c_k, \cdots|, \tag{2.3}$$

where $|a_i, b_j, c_k, \cdots\rangle = |a_i\rangle \otimes |b_j\rangle \otimes |c_k\rangle \cdots$ is an eigenbasis for the product space. By postulate 3, it is possible to prepare the state via collapse, i.e., by measuring an observable A in the Hilbert space $\mathcal{H}_1$ while leaving the others unchanged[5]. If the measurement outcome is a_1, the resulting state is

$$\rho_{a_1} = \frac{(A_1 \otimes \mathbb{1}_{2,3,\cdots})\rho(A_1 \otimes \mathbb{1}_{2,3,\cdots})}{\mathrm{tr}\Big((A_1 \otimes \mathbb{1}_{2,3,\cdots})\rho\Big)} = |a_1\rangle\langle a_1| \otimes \sum_{jk\cdots} p'_{jk\cdots} |b_j, c_k, \cdots\rangle\langle b_j, c_k, \cdots| \,,$$

where $p'_{jk\cdots} = p_{1jk}/\Big(\sum_{jk\cdots} p_{1jk\cdots}\Big)$. Here, the measurement of A isolates the state $|a_1\rangle\langle a_1|$, but the remaining terms are still in a superposition and not yet factorizable. The process can continue with measurements B, C, and so on, selecting the eigenvalues b_1, c_1, and so on, ultimately leading to

$$\rho \xrightarrow{a_1,b_1,c_1,\cdots} |a_1\rangle\langle a_1| \otimes |b_1\rangle\langle b_1| \otimes |c_1\rangle\langle c_1| \otimes \cdots \,,$$

producing a fully prepared state where each Hilbert space has an eigenstate of the respective measured observable.

Concluding the first part of this discussion concerning the preparation of a state, it is worth mentioning the commutator of two observables. For generic operators A and B, the commutator is defined as $[A, B] = AB - BA$. Essentially, if $[A, B] = 0$, the operators are said to commute. In the previous case, each operator was defined in separated Hilbert spaces. Hence, it becomes trivial noting that $[A \otimes \mathbb{1}, \mathbb{1} \otimes B] = (A\mathbb{1}) \otimes (\mathbb{1}B) - (\mathbb{1}A) \otimes (B\mathbb{1}) = 0$, and it happens to every two observables defined on different Hilbert spaces. Therefore, in order to prepare the previous state, the set of observables $\{A_1 \otimes \mathbb{1}_{2,3,\cdots}, B_2 \otimes \mathbb{1}_{1,3,\cdots}, C_3 \otimes \mathbb{1}_{1,2,4,\cdots}, \cdots\}$ was assumed. If there are no degeneracies, this constitutes a Complete Set of Commuting Observables (CSCO), which is necessary for state preparation.

However, it is yet possible to prepare a state with measurements in a single Hilbert space. Suppose the system is described by $\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$. As previously mentioned it is necessary to measure observables acting on ρ. Thus, suppose that $A = \sum_n a_n \sum_i^{g_n} |a_n^i\rangle\langle a_n^i|$ is measured and the eigenvalue a_1 is obtained, which is g_1 degenerated. In such case, the state collapses to

$$\rho \xrightarrow{a_1} \rho_{a_1} = \frac{A_1 \rho A_1}{\mathrm{tr}(A_1\rho)} = \frac{\sum_n p_n |\psi_n^{g_1}\rangle\langle\psi_n^{g_1}|}{\sum_n p_n ||\psi_n^{g_1}||^2}, \tag{2.4}$$

where $|\psi_n^{g_1}\rangle = \sum_i^{g_1} \langle a_1^i|\psi_n\rangle |a_1^i\rangle$ is the projection of $|\psi_n\rangle$ in the subspace generated by $\{|a_1^i\rangle\}$ and $||\psi_n^{g_1}||^2 = |\langle a_1^i|\psi_n\rangle|^2$. Then, note that the state is not yet an eigenstate of A, once it has g_1 possible states with the same eigenvalue. Consider now that a second observable is measured, say B. Also, suppose that the eigenvalue b_1 was obtained, which is $h_1 \leq g_1$ degenerated. Hence, the final state is

$$\rho_{a_1} \xrightarrow{b_1} \rho_{a_1,b_1} = \frac{\sum_n p_n |\psi_n^{h_1}\rangle\langle\psi_n^{h_1}|}{\sum_n ||\psi_n^{h_1}||^2}, \tag{2.5}$$

5 By "unchanged," I mean that the operators acting on all Hilbert spaces except $\mathcal{H}_1$ are the identity operator $\mathbb{1}$.

where $\left|\psi_n^{h_1}\right\rangle = \sum_i^{h_1} \langle b_1^i|\psi_n^{g_1}\rangle |b_1^i\rangle$ is analogous to the previous state. There is a particularly notable detail to address here. By postulate 3, the state collapses after a measurement of an observable, but now it does not produces an eigenstate of the measured observable. Hence, how is it possible to decrease the degree of degeneracy without losing the information of the first measurement? For example, suppose that $g_1 = h_1 = 1$, so that $\rho_{a_1} = |a_1\rangle \langle a_1|$ and $\rho_{a_1,b_1} = |b_1\rangle \langle b_1|$. In that case, note that if the bases $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ are mutually unbiased bases (MUB)[6], the second measurement destroys the information of the first measurement, in the sense that it is not possible to recover the previous state. However, if the bases are the same, in other words, if both A and B can be expressed by the same eigenbasis (A and B are commuting observables), it is true that $\rho_{a_1} = \rho_{a_1,b_1}$. Thus, the collapse do not destroys the information of the first measurement, and it is possible to prepare a state. To decrease the degree of degeneracy, the idea is to choose a set of observables that further refine the state, continuing until the final state becomes a pure state. When this has been achieved, the set of observables is called a CSCO, as previously discussed.

To conclude this discussion, it is important to clarify the relation between two observables sharing the same eigenbasis and their commutation relation. For this purpose, the following theorem is helpful.

- ***Theorem 2.1***. If A and B are observables in a Hilbert space, and $[A, B] = 0$, it is possible to construct an orthonormal eigenbasis that is common to both observables.

- *Proof*: Firstly, suppose that $[A, B] = 0$ and $A|a_n^i\rangle = a_n |a_n^i\rangle$. So, it is possible to write that $\langle a_m^j| [A, B] |a_n^i\rangle = (a_m - a_n) \langle a_m^j| B |a_n^i\rangle = 0$, and if $a_n \neq a_m$, the matrix representation of B in the eigenbasis of A is diagonal, or $\langle a_m^j| B |a_n^i\rangle = 0$. Hence, considering just one eigenvalue of A, it is possible to diagonalize the observable B in such subspace. If $B|b_n^i\rangle = b_n |b_n^i\rangle$, and since $\{|a_n^i\rangle\}$ is a basis of this subspace, it is possible to write $|b_n^i\rangle = \sum_j^{g_n} c_j |a_n^j\rangle$. However, with this description, it is possible to note that $A|b_n^i\rangle = a_n |b_n^i\rangle$, and $\{|b_n^i\rangle\}$ is an eigenbasis of A and B.

  On the other hand, consider that $AB\left|b_{n,m}^{i,j}\right\rangle = a_n b_m \left|b_{n,m}^{i,j}\right\rangle$, so that $[A, B]\left|b_{n,m}^{i,j}\right\rangle = (a_n b_m - b_m a_n)\left|b_{n,m}^{i,j}\right\rangle$. Hence, as $a_{n,m}$ are real numbers, the previous condition leads to $[A, B] = 0$. □

The preceding postulates pertain to the description of physical states, the possible outcomes of measuring an observable, and the collapse of the state. However, it is important to note that all the states considered so far are fixed in time. The next postulate addresses the time evolution of a state.

6 A pair of bases $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ are said to be MUB iff $|\langle a_i|b_j\rangle|^2 = 1/d$, with $i, j \in \{1, 2, \cdots, d\}$, and d is the dimension of the Hilbert space considered.

- ***Postulate 4***: If a physical state is described by a density operator ρ in a time t_0, and H is the Hamiltonian operator acting on ρ, the evolution of the state to a time $t > t_0$ is described by the Liouville-von Neumann's equation

$$i\hbar\partial_t\rho = [H(t), \rho]. \tag{2.6}$$

The above equation is a generalization of the Schrödinger's equation. Indeed, for a pure state $\rho = |\psi\rangle \langle\psi|$, (2.6) leads to $i\hbar\partial_t |\psi\rangle = H(t) |\psi\rangle$.

This work focuses on discrete cases, however it is worth giving a brief context to introduce the next postulate. In 1926, when Schrödinger introduced his equation, he developed this formalism in terms of wave functions, now written as $\psi(x) = \langle x|\psi\rangle$. This correspondence is uniquely guaranteed by Riesz representation theorem (for more details, see reference [39]), with x being a continuous variable. Schrödinger initially interpreted the wave function as a "weight" function, so that it was possible to describe the charge density of an electron. However, such an interpretation was quickly shown to be erroneous by Heisenberg and Born, with evidence coming from experiments with diffraction of electrons. It was precisely the study of this phenomenon that led to a new interpretation of the wave function proposed by Born. In his own words about Schrödinger's interpretation, Born said, "On this point I could not follow him. This was connected with the fact that my Institute and that of James Franck were housed in the same building of Göttingen University. Every experiment by Franck and his assistants on electron collisions (of the first and second kind) appeared to me as a new proof of the corpuscular nature of the electron".

In July of 1926, while studying electron-atom collisions, Born developed what is known as "Born's approximation". In this work, Born analyzed the hypothesis of a corpuscular interpretation and concluded that for such an interpretation, $|\psi(x)|^2$ should measure the probability of the event [41]. This interpretation of the wave function is summarized in the following postulate.

- ***Postulate 5***: If a measurement of a physical quantity $\mathcal{A}$, described by an observable A, is made on a physical state described by ρ, the probability of obtaining the result a_n is given by the Born's rule

$$p(a_n) = \mathrm{tr}(A_n\rho),$$

with $A_n = \sum_i^{g_n} |a_n^i\rangle \langle a_n^i|$ being the projectors of the observable A.

Further details about this postulate will be discussed in the next section.

2.3 Interpretations and Philosophical Aspects of Quantum Mechanics

This section is structured as follows. Subsection 2.3.1 introduces the Copenhagen Interpretation, providing its historical context and addressing concepts not covered in the previous sections. Subsection 2.3.2 discusses criticisms to the Copenhagen Interpretation, including the EPR arguments, which are of great significance for Subsection 2.3.3, where the concept of realism in quantum theory is explored.

2.3.1 Copenhagen Interpretation

The postulates of Quantum Mechanics presented in the previous section were described using a modern approach, often aligned with the “orthodox" interpretation, known as the Copenhagen Interpretation. This interpretation was mainly developed by Niels Bohr and his collaborators, including Heisenberg and Pauli. The following paragraphs will further explore this interpretation, setting the stage for the scenario of physics in the late 1920s. To achieve this, two fundamental principles of Quantum Mechanics need to be introduced.

In September of 1926, Schrödinger delivered lectures in Copenhagen, at Bohr’s institute, on wave mechanics, and attacked Bohr’s view of the “quantum jumps", which refers to the collapse of the wave function. It is interesting to note that these debates were so intense that Schrödinger once said “If one has to stick to this damned quantum jumping, then I regret having ever been involved in this thing", and Bohr answered “But we others are very grateful to you that you were, since your work did so much to promote this theory". Heisenberg, reflecting on these debates, recognized that the core of this conflict lay in interpreting the quantum formalism, which was in its early stages. Inspired by Einstein’s ideas of questioning the meaning of things in nature, instead of questioning “how nature can be described by a mathematical scheme?", Heisenberg said “Well, is it not so that I find in nature situations which can be described by quantum mechanics?". In a letter to Pauli in October of 1926, Heisenberg already understood some central points of his famous uncertainty principle, writing “[...] meaningless to speak of the place of a particle with definite velocity", and then “But if one does not take it too seriously with the accuracy in using the notions of velocity and position, then it may well make sense". On February 23, 1927, Heisenberg consolidated his thoughts on these problems in a letter to Pauli, who quickly encouraged him to publish them. The final manuscript was read by Pauli and Bohr, and submitted to the Zeitschrift für Physik at the end of March of 1927 [42].

It is essential to acknowledge others who noticed these interpretative nuances before Heisenberg's publication. As mentioned in Section 2.1, Dirac and Jordan were fundamental to the mathematical development of quantum formalism, and both noticed this peculiarity. Dirac, referring to conjugated variables once said “One cannot answer any question on quantum theory, which refers to numerical values for both q and p. One would expect, however, to be able to answer questions in which only q or only p are given numerical values, or, more generally, when any set of constraints of integration ξ that commute with one another are give numerical values". Jordan concluded a similar thing when declared “for a given value of q all values of p are equally possible". With these mentions, it will now be discussed the uncertainty relations by using the modern notation and formalism.

Suppose that $\rho : \mathcal{H} \to \mathcal{H}$ is a density operator and $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H} \to \mathcal{H}$ are observables. In such case, it is possible to define the expectation value of the observables in the state ρ as $\langle A \rangle = \mathrm{tr}(A\rho)$ and $\langle B \rangle = \mathrm{tr}(B\rho)$. By defining $\Delta A = A - \langle A \rangle$, one can write $\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2 \equiv \sigma_A^2$, which is identified as the dispersion of the observable A, and the same applies to the observable B. Now, consider the two states $\alpha = (\Delta A)\rho$ and $\beta = (\Delta B)\rho$. In such case, it is possible to define an hermitian inner product (that use to be the standard hermitian inner product for operators) as

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} &\to \mathbb{C} \\ (\alpha, \beta) &\longmapsto \mathrm{tr}\big(\alpha^\dagger \beta\big). \end{aligned}$$

Since it has the formal structure of an inner product, Cauchy-Schwarz inequality can be used, and then $\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle \geq |\langle \alpha, \beta \rangle|^2$. Using such result and the above definition of inner product, one can obtain that

$$\sigma_A^2 \sigma_B^2 \geq |\langle \Delta A \Delta B \rangle|^2.$$

Analyzing the second term, one can write $\Delta A \Delta B = \frac{1}{2}([A, B] + \{\Delta A, \Delta B\})$, with $[A, B]$ being the commutator between the observables and $\{\Delta A, \Delta B\} = \Delta A \Delta B + \Delta B \Delta A$ being the anti-commutator of ΔA and ΔB. Since the anti-commutator is an hermitian operator and the commutator is a purely imaginary one (due to $[A, B]^\dagger = -[A, B]$), it is possible to state that $|\langle \Delta A \Delta B \rangle|^2 = \frac{1}{2}(|\langle [A, B] \rangle|^2 + |\langle \{\Delta A, \Delta B\} \rangle|^2)$. Therefore, by using Cauchy-Schwarz inequality strengthening the inequality, it is possible to state that

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4} |\langle [A, B] \rangle|^2, \tag{2.7}$$

which is the Uncertainty Principle obtained by Robertson in 1929 [43]. One of the inequalities obtained by Heisenberg in 1927 concerns the operators of position and momentum. Since for these canonical variables the commutation relation is written as $[q, p_q] = i\hbar$, Heisenberg uncertainty principle is written as

$$\sigma_q \sigma_{p_q} \geq \frac{\hbar}{2}, \tag{2.8}$$

where σ_j is the uncertainty of the observable j. The another uncertainty relation obtained by Heisenberg is $\sigma_t \sigma_E \geq \hbar/2$ concerning time and energy.[7]

Due to the importance of this principle to the Copenhagen Interpretation as well as to the Philosophy of Physics, it is worth discussing some of its fundamental aspects. The Eq. (2.7) essentially establishes a lower bound for the product of uncertainties of two observables. Thus, it is not possible to simultaneously obtain well-defined values for two noncommuting observables measurements, such as position and momentum. This conclusion conflicts with Classical Mechanics and the notion of trajectory, or even determinism. In Heisenberg's words, "[...] in the strong formulation of the causal law 'If we know exactly the present, we can predict the future' it is not the conclusion but rather the premise which is false. We *cannot* know, as a matter of principle, the present in all its details". This conclusion had a profound impact on the philosophy of science [1, 27, 44], as pointed by Jammer: "modern philosophy, as Schlick later admitted, was taken by surprise since even the mere possibility of such a solution had never been anticipated in spite of the profusion of discussions on this problem for generations". In his paper's conclusion, Heisenberg says "In view of the intimate connection between the statistical character of quantum theory and the imprecision of all perception it may be suggested that behind the statistical universe of perception there lies hidden a 'real' world ruled by causality. Such speculations seem to us - and this we stress with emphasis - useless and meaningless. For physics has to confine itself to the formal description of the relations among perceptions". It is fair to say that this last statement was in great agreement with Bohr's view and, for sure, was fundamental to the establishment of the Copenhagen Interpretation.

Heisenberg's uncertainty relations were fundamental for Bohr to conclude his own view about the wave-particle duality. Indeed, from July, 1925 to September, 1927, Bohr investigated the implications of wave-particle duality by himself. Heisenberg's publication came as a confirmation of some ideas he had already established on his own, which is also the reason why he was a strong advocate of this view. For him, in contrast with Heisenberg, it was not the formalism that must be the priority to reach an interpretation of the theory, but the logic. In order to unify the characteristics of both waves and particles, which are mutually exclusive, Bohr concluded that a new logical instrument was needed, what is known as *complementarity*. Such an idea leads to his famous Complementarity Principle of Quantum Mechanics [45, 46], which is: *wave-picture and particle-picture are not contradictory behaviors, but complementary. It is necessary to take both behaviors into account for a complete description of the*

[7] There is no direct correspondence between the previous development of the uncertainty relations and this one, because time is not an observable in Quantum Mechanics. Indeed, Heisenberg did not use that kind of procedure. Instead, he analyzed the experiments available at that time and made hypotheses about the observable quantities. For the time-energy uncertainty relation, he analyzed the Stern-Gerlach experiment.

system. The picture arises from the nature of the experiment that is able to bring up only one complementary aspect. According to Bohr, the uncertainty relations were, in Jammer's words, the "price" for violating the rigorous exclusion of notions. For further discussions on this principle and its updating, see [47].

Although Bohr's principle was known to be from 1928, it was discussed in the International Congress of Physics in Como, in September of 1927, and in the 5th Solvay Conference, in October, 1927, on electrons and photons. Historically, it is often seen as the moment when the Copenhagen Interpretation was established. The name reflects Bohr's prominent role in its development, as well as the contributions of others physicists working with him in Copenhagen.

The Copenhagen Interpretation is strongly based on the concepts of probabilities and the collapse, as presented in earlier. To fully grasp its foundations, it is instructive to follow the approach suggested by Primas [48] and reviewed by Omnès [49]. According to them, Copenhagen Interpretation can be summarized as follows. First, the theory deals with *individual objects*, and all that is possible to obtain are probabilities associated with measurements. The probability involved in Quantum Theory is *intrinsic*, that is, it does not reflect the ignorance of the observer, but rather a fundamental aspect of the description of nature. Concerning the description of a system, it is common to admit Heisenberg's view that there is a frontier separating the quantum world to the classical world. However, it is worth noting that, for Bohr, there is no quantum world, and nothing can be explicitly obtained of an atomic system from classical observations. Following these ideas, the observational means must be described in terms of classical physics, and the act of observation is an irreversible event. This irreversibility is due to the collapse occurred when a measurement was performed, and such collapse is not describable by quantum dynamics, but is a *new type of physical law*. In order to make correct assumptions of the system, only the results of a measurement can be taken to be true, i.e., it is meaningless to say what is happening to a system between the time it is prepared and measured. As a measurement reveals one of the complementary aspects of a system, the complementary principle holds for the interpretation. And the last fundamental aspect of the interpretation is that pure quantum states are objective, but not real. Objective means that the states deal with intrinsic properties of the system, and according to Heisenberg, a pure state represents the potentially possible in an objective way. The discussion concerning realism is left to the last part of this section.

Concluding the overview of the Copenhagen Interpretation, Primas [48] pointed out "*there is no uniform opinion of what the Copenhagen Interpretation should be*". The points presented here are those which "survived" over time, that is, that are constantly assumed by the proponents of such interpretation. However, as will be presented in the next part of this section, there are many criticisms to this interpretation.

2.3.2 Criticism to the Copenhagen Interpretation and EPR arguments

The Copenhagen Interpretation has been criticized since its advent, mainly because it understands the nature very differently from the classical perspective. In other words, the materialistic philosophy of classical physics is abandoned, and the notion of the observer is fundamental to the understanding of quantum theory. Naturally, this disruption of the traditional philosophy of the natural sciences led to new attempts of reconciliation between quantum and classical theories. Heisenberg, in his text “Criticism and Counterproposals to the Copenhagen Interpretation of Quantum Theory" (chapter VIII of the reference [50]), categorized these attempts in three groups, which will be discussed below.

The first group of criticisms tries to change the language, that is, the philosophy, of this interpretation to align it more closely with classical physics. As Heisenberg mentioned, “their interpretations cannot be refuted by experiment, since they only repeat the Copenhagen Interpretation in a different language". One of the most relevant counterproposals that Heisenberg puts in this group is that made by David Bohm [51, 52], using the idea of *hidden variables*. Basically, this interpretation considers a new approach to Quantum Mechanics, in terms of the new “hidden content" coded in these hidden variables. With this approach, Bohm could show that it is possible to speak about trajectories and the particles are objectively real, i.e., quantum particles are like the classical particles in Newtonian mechanics. However, Bohm's work leads to some questions since it deals with waves in the configuration space. Following Bohm's ideas, Heisenberg asks “What does it mean to call waves in the configuration space ‘real’?" Moreover, the theory breaks the symmetry between position and momentum, and Heisenberg expresses his dissatisfaction, concluding that Bohm's theory cannot be considered as an improvement of Copenhagen Interpretation. In the mentioned text, the author discusses more attempts at changing the philosophy of the interpretation, but the purpose here is not to deepen so much in the subject, but just to give some examples of these attempts. For more details, it is strongly recommended to read the original text [50].

The second group of criticisms understands that Copenhagen Interpretation is the correct one if the experimental data is always correct, and tries to change the theory itself, in order to arrive at a different philosophical interpretation. To exemplify this group, Heisenberg mentioned Janossy and his work attacking the idea of the wave function collapse [53], where he introduced the damping terms, which make the interference terms disappear after a finite time. However, as pointed out by Heisenberg and Janossy himself, even if this corresponds to reality, there are many alarming consequences of such an interpretation, such as waves propagating faster than the speed of light. Hence, in Heisenberg's words, "we should hardly be ready to sacrifice the simplicity of quantum theory for this kind of view until we are compelled by experiments to do so".

Finally, concerning the third group of criticisms to the Copenhagen Interpretation, which will be more discussed in this work, there are those who have a general dissatisfaction with the philosophical conclusions concerning quantum theory and, as pointed by Heisenberg, do not try to give a better explanation of the facts. In such group, Heisenberg mentioned von Laue, Schrödinger and Einstein, and the latter plays an important role to the the criticisms mentioned previously, mainly because historically this is the first group of criticisms, and the Einstein-Bohr debates concerning the foundations of quantum physics [46] were very famous in the first half of the twentieth century. Heisenberg cited Schrödinger's work "Are there quantum jumps?" [54] as part of the criticisms present in this group. However, as time has shown, probably the most influential article against the Copenhagen Interpretation was published in 1935, by Einstein, Boris Podolsky, and Nathan Rosen (EPR) [2], where the authors argued against the completeness of quantum theory. In order to understand their arguments, it is worth discussing more details of this work.

In 1935, physicists were used to interpret their results using the orthodox interpretation. However, as discussed in the section of the Copenhagen Interpretation, such a set of ideas is completely disruptive from classical ones. It was in this context that EPR published their seminal paper entitled "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" [2]. According to them, by using some reasonable criterion to decide whether a physical theory is correct and complete, they argued in favor of the correctness and incompleteness of quantum theory. The original paper discusses their arguments through an analysis of the position and momentum of two different systems that interacted in the past. However, in order to be more comprehensive and already establish some notation that will be used in the next chapters, this discussion will be made as discussed by David Bohm [55], in 1957, using the concept of spin.

To start the discussion, according to EPR a physical theory is considered correct depending on the degree of agreement between theory and experiment. For such a case, the authors have no concerns with quantum theory, as its agreement with experimental data within its validity regime is great. Thus, they quickly concluded that quantum theory is indeed a correct theory. However, the authors also introduced their criterion for a complete theory, which is: *every element of the physical reality must have a counterpart in the physical theory.* By ***element of physical reality*** they mean: *if, without any way disturbing the system, we can predict with certainty (i. e. with probability equals to unity) the value of a physical quantity, then there is an element of physical* *reality corresponding to this physical quantity.* For such criterion, they found issues in their view with the completeness of quantum theory.

To illustrate their argument, consider the existence of two laboratories that are far from each other, i. e. they are outside of the light cone of each other, say A for Alice's laboratory and B for Bob's one. Both experimentalists, in their respective laboratories, receive a spin-$\frac{1}{2}$ particle that previously interacted with the other particle in a correlation source. This source prepared the global system in a singlet state, that is, a pure state described by

$$|\psi_S\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{2.9}$$

where $|0\rangle$ and $|1\rangle$ are the states for spin up and down, respectively, in z direction in the computational basis. In this context, the EPR arguments are as follows. Consider that Alice chooses to measure the z direction of spin of her particle, establishing an element of reality. In such a case, she could infer what is the result of the same measurement made by Bob, since the global state is well defined and there is a constraint for the total spin. Hence, Alice could obtain the outcome of a measurement in Bob's laboratory for the same direction without perturbing Bob's system, and EPR would say that there is an element of physical reality corresponding to the z direction of spin for Bob's particle.

For instance, if Alice measures $\hbar/2$ for her particle, then $-\hbar/2$ is the measurement outcome in Bob's laboratory. Using the same argument, it is possible to state that Alice does not need to choose measuring the z direction of spin, she could measure any other direction, and, by symmetry properties of the singlet state, the same conclusions would be reached. Therefore, EPR argued that, since it is possible to obtain the elements of reality for any direction of Bob's spin, and measurements made in Alice's site cannot disturb Bob's system since they are spatially separated, Bob's state must be well defined, and the theory is incomplete. The incompleteness came since quantum mechanics does not predict these simultaneous values of spin for any direction because they are incompatible observables. In this point, it is worth noting that EPR's conclusion has its foundational basis on the idea that spin *exists* as a well-defined property of the system.

2.3.3 Realism in Quantum Theory

EPR's arguments had a profound influence on the development of quantum mechanics, particularly in its philosophical aspects. As shown earlier, their arguments are deeply based on a criterion of *realism* in a physical theory. To better understand this concept and its implications to quantum theory, this subsection is dedicated to discuss this topic in detail.

Considering EPR's notion of physical reality, it is noted that CM is completely realistic, since it is governed by deterministic equations. Beyond that, even when relativity is considered, the notion of realism is maintained. This context provides insight into Einstein's perspective and critiques of QM. The development of QM was completely different from that of classical theories. While relativity was primarily developed by Einstein and a few other great physicists and mathematicians, based on physical and mathematical principles and later experimentally verified, QM was developed from a multitude of experimental data that lacked a theoretical explanation. In a way, the origin of quantum theories was not necessarily based on immutable principles of classical physics, as previously discussed in this chapter. To develop a theory that was so consistent with what was being observed experimentally, a departure from well-established concepts of classical physics was required. This abandonment of classical notions led to the emergence of ideas that deviate from common sense when viewed from the perspective of Classical Physics.

In the context of the realism present in a physical theory, Heisenberg mentioned [50] Einstein's concerns and said “When Einstein has criticized quantum theory he has done so from the basis of dogmatic realism. This is a very natural attitude. Every scientist who does research work feels that he is looking for something that is objectively true. His statements are not meant to depend upon the conditions under which they can be verified. Especially in physics the fact that we can explain nature by simple mathematical laws tells us that here we have met some genuine feature of reality, not something that we have - in any meaning of the word - invented ourselves. This is the situation which Einstein had in mind when he took dogmatic realism as the basis for natural science". And he continues “But quantum theory is in itself an example for the possibility of explaining nature by means of simple mathematical laws without this basis. These laws may perhaps not seem quite simple when one compares them with Newtonian mechanics. But, judging from the enormous complexity of the phenomena which are to be explained (for instance, the line spectra of complicated atoms), the mathematical scheme of quantum theory is comparatively simple. Natural science is actually possible without the basis of dogmatic realism".

To explore a peculiarity in EPR's criterion of realism, let some examples be considered. First, consider the case of a pure eigenstate of the observable A, $|\psi\rangle = |a\rangle$. This state is such that $A\,|a\rangle = a\,|a\rangle$, that is, there is an element of reality for the observable A. This is generalized to any state similar to this one: every state that is described by one eigenvector of some observable has an element of reality associated with this observable. Furthermore, in case the state being a linear combination of eigenvectors of an observable, that is, in case the state being in a superposition, there is no element of reality associated with that observable. Both of these cases are contemplated in EPR's criterion, i.e., are natural conclusions in light of their criterion.

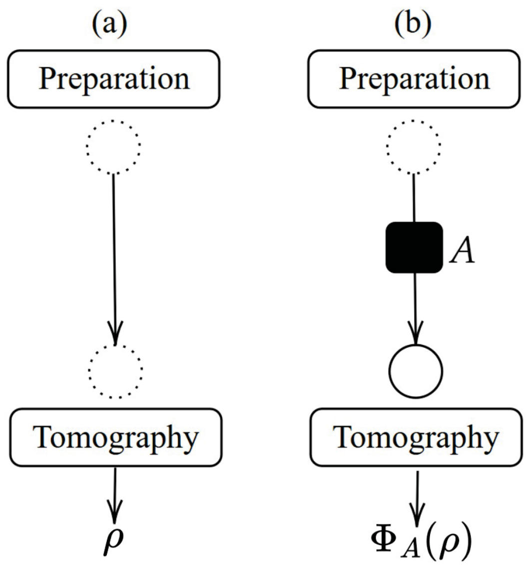

Figure 1 – The figure illustrates the protocol on which the BA-realism criterion is based. The scheme in (a) represents a preparation, understood as an ensemble of identically prepared systems, that is submitted to a tomography process and is uniquely determined to be described by a density operator ρ. The scheme in (b) is similar to (a), but with the addition of an agent measuring an observable A before the tomography process. Since the measurement results remain unknown, the best possible description after tomography is given $\Phi_A(\rho)$.

However, let another case be considered. Suppose now that a state is described by the density operator $\rho = \sum_i p_i |a_i\rangle \langle a_i|$. Such a case is just a mixture of realistic states of the observable A and can be used to describe an ensemble of classical particles. According to EPR's views, this should be an example of a realistic state, since it describes a classical mixture of realistic states for momentum. However, considering just their criterion of realism, such a state is not realistic[8]. Hence, in order to generalize EPR's criterion and establish this concept to those types of physical states, Bilobran and Angelo introduced their criterion of realism in 2015 [13], which will be explained in the following paragraphs.

The Bilobran-Angelo criterion of realism (BA-realism) was formulated based on a protocol using the non-selective measurements map. In order to introduce this criterion, suppose the existence of a physical state of N parts. Then, such a state is submitted to a protocol of tomography to be completely determined, that is, to be described as the state ρ. The protocol of tomography is done by an experimentalist using different measurements in an ensemble of this physical state, so that it is possible to state that the complete description of the system is given by ρ. Now, to make the point of the criterion clear, suppose the existence of an unknown external agent who measures some generic observable $A = \sum_k a_k A_k$ in the same preparation in the first Hilbert space, but before the tomography process. In such a case, as the external agent

[8] Indeed, EPR never used the term “realistic state". However, the terminology here refers to the states that satisty a realism criterion for a given observable.

is performing measurements in the system, by using the collapse of the state, it is possible to state that the final description after each measurement will be described by an eigenstate of A in the correspondent Hilbert space, that is, A_k for some $k \in \mathbb{N}$ in $\mathcal{H}_1$. Hence, the best description to be known after the process of tomography is given by

$$\Phi_A(\rho) = \sum_k (A_k \otimes \mathbb{1}_{2\cdots N})\rho(A_k \otimes \mathbb{1}_{2\cdots N}) = \sum_k p_{a_k} A_k \otimes \rho_{2|a_k}, \tag{2.10}$$

where $p_{a_k} = \mathrm{tr}(A_k \otimes \mathbb{1}_{2\cdots N})\rho(A_k \otimes \mathbb{1}_{2\cdots N})$ are the probabilities of obtaining the eigenvalue a_k for the measurement A in the state ρ, and $\rho_{2|a_k} = \mathrm{tr}_1[(A_k \otimes \mathbb{1}_{2\cdots N})\rho(A_k \otimes \mathbb{1}_{2\cdots N})]/p_{a_k} \in \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ is the collapsed state for the other (multi)partition conditioned to the measurement outcome a_k. Finally, with this information, it is possible to define the BA-realism criterion.

- ***BA-Realism:*** An observable $A = \sum_k a_k A_k$, with projectors A_k, acting on the Hilbert space $\mathcal{H}_1$, is said to be real to a given preparation $\rho \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ iff

$$\Phi_A(\rho) = \rho. \tag{2.11}$$

The above criterion is based on the idea that the measurement establishes an element of reality in the state. Then, when there is an agent, the elements of reality are guaranteed. If in the case without an agent we arrive at a tomography identical to that of the case with the agent, then the first case already has an element of reality established. Eq. (2.11) can also be interpreted in another way, without mentioning the protocol with the unknown external agent. A realistic state is one for which the effect of the measurement is merely to reveal an already established element of reality. Thus, if the measurement is non-revealed, the original state of knowledge should be recovered. Classical statistical mechanics meets this criterion perfectly. To exemplify the case of a classical state, consider the previous state that was not in agreement with EPR's criterion, that is, $\rho = \sum_i p_i \left|a_i\right\rangle \left\langle a_i\right|$. In such case, consider that the unknown external agent is measuring the observable A, so that

$$\Phi_A(\rho) = \sum_k A_k \rho A_k = \sum_{ik} p_i \left|a_k\right\rangle \left\langle a_k|a_i\right\rangle \left\langle a_i|a_k\right\rangle \left\langle a_k\right| = \sum_i p_i \left|a_i\right\rangle \left\langle a_i\right| = \rho.$$

Thus, the initial state is indeed realistic with respect to A, contrary to what the EPR criterion established.

One significant advantage of the BA-realism criterion is that it is well-defined and analytical, in the sense that not only works to every physical state, but also allows quantification of the amount of irrealism within a state, i.e, a quantifier of the amount of violation of realism. This quantifier will be further discussed in the next chapter when the concept of *Irreality* is introduced.

Before concluding this chapter, it is worth discussing some important topics related to the subject. First, inspired by EPR's criteria for a complete physical theory and realism, Bell explored these hypotheses for a generic probability theory [56–58] and developed the first form of a Bell inequality, later explored upon by others [59]. Essentially, Bell assumed that the joint probability of outcomes a and b for measurements x and y, respectively, can be expressed as

$$p(a,b|x,y) = \sum_{\lambda} p(a|\lambda,x)p(b|\lambda,y)p(\lambda). \tag{2.12}$$

The discussion concerns the same scenario as that considered to explain the EPR arguments: Alice (who measures x) and Bob (who measures y) are located in space-like separated laboratories. The idea behind Eq. (2.12) is that, even if it is not possible to write $p(a,b|x,y)$ as a product of individual probabilities depending solely on the respective laboratories (locality), the knowledge of λ, the set of hidden variables, enables such a factorization, as shown in the above equation. Therefore, λ encapsulates the knowledge necessary to restore locality. Eq. (2.12) is known as Bell's hypothesis or Local Causality hypothesis, and there is still a debate regarding its underlying assumptions. However, as shown in [60], Eq. (2.12) can be derived from three primitive hypotheses: realistic hidden variables, non-superdeterminism, and locality[9]. Thus, by writing expectation values of the joint outcomes $\langle a_x b_y \rangle$, for $x, y \in \{0,1\}$, as

$$\langle a_x b_y \rangle = \sum_{ab} ab\, p(a,b|x,y),$$

one can calculate the quantity[10]

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle.$$

If Eq. (2.12) holds, then $S \leq 2$ for $a, b \in \{-1, 1\}$. However, for the state in Eq. (2.9), $S = 2\sqrt{2} > 2$, so that QM does not satisfy Eq. (2.12). Through this procedure, Bell demonstrated that QM is incompatible with EPR premises, and it was later experimentally verified as a fact [61–66], culminating in the Nobel Prize in Physics in 2022. These works actively shown and explored the concept of nonlocality in quantum theory, which is, even when the theory is supplemented with hidden variables, it still exhibits the characteristic that the outcome joint probability is not separable, and a measurement in one laboratory affects (in principle) the state in the other laboratory (for a more detailed discussion see reference [60]).

In the following years after the publication of the EPR paper, alongside Bell's works and some misinterpretations of Einstein's ideas, it became widely believed that

[9] for more details on each of these assumptions (which are not unique), the reader is encouraged to consult [60].

[10] The present discussion concerns the Bell inequality proposed in [59], and it is not the original formulation by Bell.

Einstein opposed the probabilistic nature of QM. However, as Bell mentions in [58], Einstein himself admitted that this was not the case. What truly troubled him was, as mentioned by Bell, "what he could not accept was that an intervention at one place could influence, immediately, affairs at the other", that is, the concept of nonlocality. In other words, even for systems outside the light cone of each other, there would still be a disturbance in the elements of reality. What Einstein believed to be the foremost requirement for any physical theory was the hypothesis of local realism: the idea that elements of reality are influenced solely by local events connected through timelike intervals. The locality hypothesis in this context was extensively explored by Bell within the framework of hidden variables [56–59]. The connection between Bell's hypothesis and realism is explored in [60], and [67] establishes an equivalence between the existence of deterministic hidden variable models, well-defined joint probability distributions for noncommunting observables, and the non-violation of Bell's inequality. Regarding more recent and fundamental developments on the topic of realism, the reader is encouraged to explore works such as [68], where an axiomatization of the concept of realism is discussed, and [69], that analyzes the realism from a perspective independent of any specific theoretical framework, thereby treating it as a theory-independent concept.

CHAPTER 3

Concepts of Quantum Information Theory

While the last chapter introduced the quantum mechanical framework and philosophical aspects of quantum mechanics, the goal of this chapter is to present some preliminary concepts of quantum information theory [38], as well as basic notions of optical devices and optical setups. Section 3.1 discusses the Shannon and von Neumann entropies, and establish some important theorems to be used throughout this work. Section 3.2 discusses some useful quantum resources for this work, namely, Irreality, Entanglement, Quantum Discord, and Realism-based Nonlocality. Section 3.3 introduces the important optical devices for the development of this work and some setups, such as the Mach-Zehnder interferometer and quantum erasers. Therefore, to the reader is given the freedom to skip sections as desired to avoid fatigue.

3.1 Shannon Entropy and von Neumann Entropy

The notion of Entropy was introduced in the nineteenth century through a thermodynamical approach, mainly due to the work of Sadi Carnot and Rudolf Clausius, in the context of thermal machines. Later, some of the greatest scientists of all time, say James Clerk Maxwell, Ludwig Boltzmann, and Josiah Willard Gibbs, turned their attention to this concept and established the statistical foundations of Entropy [70]. According to their works, entropy can be understood as a measure of how much disorder, or how much uncertainty, there is in a physical system.

In the first half of the twentieth century, while working at Bell Labs, Claude

Elwood Shannon developed what is usually known as information theory [71]. Basically, Shannon was studying how to mathematically formulate the notion of information of a random variable X. In order to obtain a mathematical formulation, it is expected that the information of a random variable X, say $\mathcal{I}(X)$, is expected to satisfy some basic conditions, such as

1. $\mathcal{I}(X)$ must be a function only of the probability distribution of X, that is, $\mathcal{I}(X) = \mathcal{I}(p(X)) = \mathcal{I}(p)$;
2. As it must be possible to study the change of information, it is reasonable to assume that $\mathcal{I}(p(X))$ is a smooth function of the probability;
3. The information gained when two independent events occur with individual probabilities p_1 and p_2 is the sum of the information gained from each event alone, that is, $\mathcal{I}(p_1 p_2) = \mathcal{I}(p_1) + \mathcal{I}(p_2)$.

From these assumptions, it is possible to take the derivative as follows

$$\frac{\partial}{\partial p_1}\mathcal{I}(p_1 p_2) = p_2 \mathcal{I}'(p_1 p_2) = \mathcal{I}'(p_1),$$
$$\frac{\partial}{\partial p_2}\mathcal{I}(p_1 p_2) = p_1 \mathcal{I}'(p_1 p_2) = \mathcal{I}'(p_2),$$

so that $p_1 \mathcal{I}'(p_1) = p_2 \mathcal{I}'(p_2)$, or even $\mathcal{I}'(p) = c/p$, with $c \in \mathbb{R}$. Hence,

$$\mathcal{I}(p) = k \log p, \tag{3.1}$$

with $k \in \mathbb{R}$. The physical meaning of $\mathcal{I}(p)$ is the amount of information provided by the occurrence of an event with a probability p of happening. Therefore, it represents the information gained when the content of X is learned, or, from a complementary perspective, is the amount of uncertainty about X before knowing its content.

From previous discussion, Shannon introduced his concept of Entropy, nowadays known as Shannon Entropy, as being the amount, in average, of information gained when the value of a random variable is learned, that is, considering that X is a discrete random variable,

$$\langle \mathcal{I}(p) \rangle = k \sum_{x \in \Omega_X} p_x \log p_x,$$

with Ω_X being the set of possible outcomes of X. By analogy with the thermodynamical notion of entropy, this quantity cannot be negative. However, $p_x \leq 1$ for any $x \in \Omega_X$, so that, by definition, $k \equiv -1$, and Shannon Entropy of a random variable X, with probability distribution $p(X)$, $H(X)$, is written as

$$H(X) = -\sum_{x \in \Omega_X} p_x \log p_x. \tag{3.2}$$

It is worth noting that it is very common to have random variables with just two outcomes, and this is the case of this work. In such case, the entropy is said the binary entropy, which is

$$h(p) \equiv -p \log p - (1-p)\log(1-p), \tag{3.3}$$

and $\log = \log_2$, so that the entropy is measured in *bits*, and results in a number between 0 and 1. Another important convention is that, as probability distribution allows $p_x = 0$ for some $x \in \Omega_X$, it is defined that $0 \log 0 \equiv \lim_{x\to 0} x \log x = 0$.

One property of particular interest of entropy is how it behaves when there is a convex combination of probabilities, that is, how $H(\alpha p_1 + (1-\alpha)p_2)$ is related with the individual entropies of p_1 and p_2. For such a case, the theorem below is considered.

- ***Theorem 3.1.*** Shannon entropy is a concave function of probabilities, and for binary entropy it is true that

$$h(\alpha p_1 + (1-\alpha)p_2) \geq \alpha h(p_1) + (1-\alpha)h(p_2). \tag{3.4}$$

- *Proof.* Considering the expression for Shannon Entropy in (3.2), it is possible to state that

$$\frac{\partial^2}{\partial p_i^2} H(p_1, \cdots, p_n) = -\frac{1}{p_i}, \qquad \frac{\partial^2}{\partial p_i \partial p_j} H(p_1, \cdots, p_n) = 0.$$

Hence, the Hessian matrix of $H(p_1, \cdots, p_n)$ has all its eigenvalues negative, since $p_i \geq 0$ for all i, and $H(p_1, \cdots, p_n)$ is a concave function. Considering now the case of the binary entropy, as this is also a Shannon Entropy, it is concave, but it is a single-variable function, so that $h''(p) \geq 0$. As $h(p)$ is analytical, Taylor series is defined for that, and by using the mean-value form of the expansion, it is possible to state that

$$h(p) = h(p_0) + h'(p_0)(p-p_0) + \frac{h''(p^*)}{2}(p-p_0)^2,$$

with $p_0 \leq p^* \leq p$. As $h(p)$ is a concave function,

$$h(p) \leq h(p_0) + h'(p_0)(p-p_0).$$

Assuming $p_0 = \alpha p_1 + (1-\alpha)p_2$,

$$h(p_1) \leq h(\alpha p_1 + (1-\alpha)p_2) + h'(\alpha p_1 + (1-\alpha)p_2)((1-\alpha)(p_1-p_2)),$$
$$h(p_2) \leq h(\alpha p_1 + (1-\alpha)p_2) + h'(\alpha p_1 + (1-\alpha)p_2)(\alpha(p_2-p_1)),$$

and finally

$$\alpha h(p_1) + (1-\alpha)h(p_2) \leq h(\alpha p_1 + (1-\alpha)p_2),$$

as stated by the theorem. □

From the previous theorem, since $H(X)$ is a concave function, it admits a maximum value. Hence, considering the case of $H(X) = H(p_1, p_2, \cdots, p_d)$, with d representing the number of possible outcomes of the random variable X, the maximum value of H is obtained for

$$\frac{\partial}{\partial p_i} H(p_1, p_2, \cdots, p_d) = f(p_i) = 0$$

for all $i \in \Omega_X$ and $f(p_i)$ being a function only of the p_i considered. Since this equation must hold for all p_i, it is straightforward that all the probabilities must be equal. Hence, the probability summation condition leads to $p_i = 1/d$, for any $i \in \Omega_X$. Using this result in the definition of the Shannon Entropy, it is possible to state the following condition of the image of the function

$$0 \leq H(X) \leq \log d. \tag{3.5}$$

Within the context of entropies, it is worth to introduce the concepts of Relative and Conditional Entropy, and Mutual Information. Considering, firstly, the case of Relative Entropy, it measures the closeness of two probability distributions, $p(x)$ and $q(x)$. For these distributions, the Relative Entropy is defined as

$$H(p(x)||q(x)) \equiv \sum_{x\in\Omega_X} p(x) \log \frac{p(x)}{q(x)} = -\sum_{x\in\Omega_X} p(x) \log q(x) - H(p(X)). \tag{3.6}$$

An important result of the relative entropy is that it is non-negative. To achieve such result, consider that $e^{\ln x} \geq 1 + \ln x$, so that $\ln x \leq x - 1$, and $\ln x = \log x \ln 2$. Therefore, it is possible to conclude that

$$\begin{aligned} H(p(x)||q(x)) &= -\sum_{x\in\Omega_X} p(x) \log \frac{q(x)}{p(x)} \\ &\geq \frac{1}{\ln 2} \sum_{x\in\Omega_X} p(x) \Big(1 - \frac{q(x)}{p(x)}\Big) \\ &= \frac{1}{\ln 2} \sum_{x\in\Omega_X} (p(x) - q(x)) = 0 \\ \therefore \quad H(p(x)||q(x)) &\geq 0. \end{aligned} \tag{3.7}$$

From the above development, it is straightforward to see that the equality holds if, and only if, $p(x) = q(x)$.

For the Relative Entropy it is also worth considering the relation between the joint probability and individual probabilities, that is, $H(p(x,y)||p(x)p(y))$. For such a case,

$$H(p(x,y)||p(x)p(y)) = \sum_{\substack{x\in\Omega_X \\ y\in\Omega_Y}} p(x,y) \log \frac{p(x,y)}{p(x)p(y)},$$

and then

$$
\begin{aligned}
H(p(x,y)||p(x)p(y)) &= -H(p(x,y)) - \sum_{\substack{x\in\Omega_X\\ y\in\Omega_Y}} p(x,y)\log\big(p(x)p(y)\big)\\
&= -\sum_{x\in\Omega_X}\Big(\sum_{y\in\Omega_Y} p(x,y)\Big)\log p(x) - \sum_{y\in\Omega_Y}\Big(\sum_{x\in\Omega_X} p(x,y)\Big)\log p(y)\\
&\quad - H(p(x,y))\\
&= -\sum_{x\in\Omega_X} p(x)\log p(x) - \sum_{y\in\Omega_Y} p(y)\log p(y) - H(p(x,y))\\
&= H(p(x)) + H(p(y)) - H(p(x,y)).
\end{aligned}
$$

By using (3.7),

$$H(p(x,y)) \le H(p(x)) + H(p(y)). \tag{3.8}$$

Considering now the case of Conditional Entropy, for two random variables X and Y, the Conditional Entropy $H(X|Y)$ measures how related is the information of X to the information of Y, and is defined as

$$H(X|Y) \equiv H(X,Y) - H(Y). \tag{3.9}$$

It is important that X, indeed, can be a set of random variables, such that the following theorem holds.

- ***Theorem 3.2.*** Let $\{X_1,\cdots,X_n,Y\}$ be a set of random variables. For such a case, the following equation holds

  $$H(X_1,\cdots,X_n|Y) = \sum_{i=1}^{n} H(X_i|Y,\Omega_{i-1}), \tag{3.10}$$

  with Ω_{i-1} is the set with the first $i-1$ variables X_i.

- *Proof.* By using definition (3.9), it is possible to state that

  $$
  \begin{aligned}
  H(X_1,\cdots,X_n|Y) &= H(X_1,\cdots,X_n,Y) - H(Y)\\
  &= H(X_1,\cdots,X_n,Y) - H(X_1,Y) + \Big(H(X_1,Y) - H(Y)\Big)\\
  &= H(X_1,\cdots,X_n,Y) - H(X_1,X_2,Y) + H(X_2|X_1,Y) + H(X_1|Y)
  \end{aligned}
  $$

  and so forth, so that, by induction,

  $$H(X_1,\cdots,X_n|Y) = \sum_{i=1}^{n} H(X_i|Y,\Omega_{i-1}),$$

  as states the theorem. □

Following the definition of Condition Entropy, it is also useful to define the Mutual Information of two random variables X and Y, $\mathcal{I}(X:Y)$, that measures how much information these variables have in common. Its definition is

$$\begin{aligned} \mathcal{I}(X:Y) &\equiv H(X) + H(Y) - H(X,Y) \\ &= H(X) - H(X|Y) \equiv \mathcal{J}(X:Y). \end{aligned} \tag{3.11}$$

A brief comment on this topic clarifies the reasoning supporting the definitions of $\mathcal{I}(X:Y)$ and $\mathcal{J}(X:Y)$. Classically, these terms are equivalent. However, in the case of quantum states, they differ and require greater attention. For that purpose, both quantities are defined here for reference in the subsequent developments of the text.

Since the formalism and properties of Shannon entropy are established, it is time to generalize its results to quantum theory. Shannon entropy deals with uncertainty related to classical probabilistic theory. However, it has been shown in (2.2) that quantum theory has its own rule to calculate probabilities, and this probability theory is different from classical one, since it admits probabilistic amplitudes. The generalization of Shannon entropy to quantum states, say ρ, is the von Neumann entropy, $S(\rho)$, introduced firstly in 1932 [33], and is defined as

$$S(\rho) = -\operatorname{tr}(\rho \log \rho). \tag{3.12}$$

In summary, von Neumann entropy deals with density operators, different from Shannon entropy that deals with probability functions. If $\{\lambda_i\}$ is the set of eigenvalues of the density operator ρ, it is possible to express (3.12) as

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i. \tag{3.13}$$

The above equation is precisely the Shannon entropy, since each λ_i is a probability. In fact, the functional structures are the same. However, quantum mechanics allows different kinds of states, genuinely quantum, so that the entropies represent different quantities. As an example, consider the state described by $\rho = p\,|0\rangle\langle 0| + \frac{1-p}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$. In fact, the states $|0\rangle\langle 0|$ and $\frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$ are different, although not orthogonal. Hence, considering just the Shannon entropy of the distributions p and $1-p$, it would be computed as $H(p, 1-p) = h(p)$, with h being the binary entropy. Considering now the von Neumann entropy of this state, it is necessary to consider the eigenvalues of ρ, that is, $\lambda_\pm = \frac{1}{2}\big(1 \pm \sqrt{1-2p(1-p)}\big)$, and so $S(\rho) = h\Big(\frac{1}{2}\big(1 + \sqrt{1-2p(1-p)}\big)\Big)$. The only cases when both entropies are the same for this state are when $p \in \{0,1\}$. This brief example is useful to visualize a difference between these two entropies, and the fact that quantum states represent a bigger set than the classical one. Indeed, Shannon entropy corresponds to a distribution associated to a specific observable, while von Neumann entropy pertains to the state ρ, which encodes the distributions of all observables.

An important topic related to von Neumann entropy is the image of $S(\rho)$, that as well as $H(X)$,

$$0 \leq S(\rho) \leq \log d, \tag{3.14}$$

with d being the dimension of the Hilbert space on which ρ acts. The bounds 0 and $\log d$ are obtained for pure states and the identity state, respectively. The physical interpretation of these cases is the same as the classical one, that is: for a pure state, it is well defined, so that it has no uncertainty and, consequently, null entropy; for the identity state, that is, $\rho = \mathbb{1}/d$, all probabilities are the same and it is the most random state possible, so that the uncertainty is maximized, as well as the entropy of the state.

Following the procedure made previously for the Shannon entropy, it is possible to define the Quantum Relative Entropy, which is

$$S(\rho||\sigma) \equiv \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma), \tag{3.15}$$

and for this definition, the following theorem holds.

- ***Theorem 3.3.*** The Klein inequality states that

$$S(\rho||\sigma) \geq 0, \tag{3.16}$$

  and the equality holds if, and only if, $\rho = \sigma$.

- *Proof.* Assuming that $\{|i\rangle\}$ is the eigenbasis of ρ, with $\{p_i\}$ being the respective set of eigenvalues, it is possible to write

$$S(\rho||\sigma) = \sum_i p_i \log p_i - \sum_i p_i \langle i| \log \sigma |i\rangle ,$$

  and using that $\{|j\rangle\}$ and $\{q_j\}$ are eigenbasis and the set of eigenvalues of σ, respectively, defining $P_{ij} = \langle i|j\rangle \langle j|i\rangle$, the above equation can be written as

$$S(\rho||\sigma) = \sum_i p_i \Big(\log p_i - \sum_j P_{ij} \log q_j \Big).$$

  As the logarithm is a strictly concave function, it is true that $\sum_j P_{ij} \log q_j \leq \log\Big(\sum_j P_{ij} q_j\Big)$, and the equality holds only if, for each i, there exists some j such that $P_{ij} = 1$. With this result,

$$S(\rho||\sigma) \geq \sum_i p_i \log \Big(\frac{p_i}{\sum_j P_{ij} q_j} \Big), \tag{3.17}$$

  and the above equation is the same of the classical relative entropy, so that (3.7) holds, and then

$$S(\rho||\sigma) \geq 0.$$

  Considering the equality in (3.17), the equality in (3.1) holds only when $q_i = p_i$ for all i, so that ρ and σ share the same eigenvalues, and thus $\rho = \sigma$, as states the theorem. □

The above theorem is useful to prove some important results on von Neumann entropy. Another important result is the consequence of doing projective measurements in a physical state. On this topic, consider the next theorem.

- ***Theorem 3.4.*** Suppose $\{P_i\}$ is a complete set of orthogonal projectors and ρ is a density operator. By defining $\rho' = \sum_i P_i \rho P_i$, the following inequality holds

$$S(\rho') \geq S(\rho). \tag{3.18}$$

- *Proof.* By using the Klein Inequality for ρ and ρ',

$$S(\rho||\rho') = -\operatorname{tr}(\rho \log \rho') - S(\rho) \geq 0.$$

Analyzing the first term on the above equation,

$$-\operatorname{tr}(\rho \log \rho') = -\operatorname{tr}\left(\sum_i P_i \rho \log \rho'\right) = -\operatorname{tr}\left(\sum_i P_i \rho \log \rho' P_i\right),$$

but $\rho' P_i = P_i \rho'$, so that $\log \rho'$ commutes with P_i, and

$$-\operatorname{tr}(\rho \log \rho') = -\operatorname{tr}(P_i \rho P_i \log \rho') = -\operatorname{tr}(\rho' \log \rho') = S(\rho').$$

Hence,

$$S(\rho') \geq S(\rho),$$

as stated by the theorem. □

The above theorem is, indeed, counterintuitive. Physically speaking, measuring an observable with projectors P_i typically results in a more specific state, which would reduce its entropy. However, when a complete measurement is performed, the resulting state is always a mixed state, differing from the pre-measurement state, which could have been a pure state with lower entropy.

Concerning the concavity of von Neumann entropy, it would be straightforward that it is a concave function, since its definition can be made as the Shannon entropy of the eigenvalues of a density state. Given that the eigenvalues establish a probability distribution, Theorem 3.1 holds, and von Neumann entropy is a concave function. However, some caution is necessary, since these entropies are not the same. To establish the concavity of $S(\rho)$, consider the following theorem.

- ***Theorem 3.5.*** Let $\rho = \sum_i p_i \rho_i$ be a density operator describing a system, and $\{p_i\}$ be a probability distribution and ρ_i are density operators. Then

$$S(\rho) \leq \sum_i p_i S(\rho_i) + H(p_i), \tag{3.19}$$

with equality if, and only if, the states ρ_i have support on orthogonal subspaces.

- *Proof.* Suppose that $\rho_i = |\psi_i\rangle\langle\psi_i|$ is a pure state in a Hilbert space $\mathcal{H}_\mathcal{A}$, and $|\rho'\rangle = \sqrt{p_i}\,|\psi_i, i\rangle$ is defined in $\mathcal{H}_\mathcal{A}\otimes\mathcal{H}_\mathcal{B}$, with $\{|i\rangle\}$ being a orthonormal basis for $\mathcal{H}_\mathcal{B}$. Hence, $\rho' = \sum_{ij}\sqrt{p_i p_j}\,|\psi_i\rangle\langle\psi_j|\otimes|i\rangle\langle j|$. Since ρ' define a pure state, it is possible to use the Schmidt's decomposition in order to obtain the eigenvalues of the bipartite state, and then, $S(\rho_\mathcal{A}) = S(\rho_\mathcal{B})$, with $\rho_\mathcal{A} = \mathrm{tr}_\mathcal{B}\,\rho'$ and $\rho_\mathcal{B} = \mathrm{tr}_\mathcal{A}\,\rho'$. However, $\rho_\mathcal{A} = \rho$, so that $S(\rho_\mathcal{B}) = S(\rho)$. If a projective measurement is made on $\mathcal{H}_\mathcal{B}$, so that the final state is written as $\rho_f = \sum_i p_i\,|\psi_i\rangle\langle\psi_i|\otimes|i\rangle\langle i|$, it is possible to state that the description for $\mathcal{H}_\mathcal{B}$ is given by $\mathrm{tr}_\mathcal{A}\,\rho_f = \sum_i p_i\,|i\rangle\langle i|$, and by using the theorem 3.4,

$$S\Big(\sum_i p_i\,|i\rangle\langle i|\Big) = H(p_i) \geq S(\rho_\mathcal{B}) = S(\rho).$$

Since $S(\rho_i) = 0$ for each i, the theorem is proved for this case. Following the above analysis, it is possible to note that the equality holds only if $\{|\psi_i\rangle\}$ are all orthogonal.

Considering the case of a mixed state, let $\rho_i = \sum_j p_j^i\,\big|e_j^i\big\rangle\big\langle e_j^i\big|$ be the orthonormal decomposition for the states ρ_i. Following the above result for pure states, that is, with $p_i \to p_i p_j^i$,

$$S(\rho) \leq H(p_i p_j^i) = -\sum_{ij} p_i p_j^i \log\big(p_i p_j^i\big),$$

and noting that $\sum_{ij} p_i p_j^i \log\big(p_i p_j^i\big) = \sum_i p_i \log p_i + \sum_i p_i \sum_j p_j^i \log p_j^i$, the above equation becomes

$$S(\rho) \leq H(p_i) + \sum_i p_i S(\rho_i).$$

The equality holds from the same conditions for the pure state. □

In order to establish the concavity of $S(\rho)$, suppose that $\rho_{\mathcal{AB}} = \sum_i p_i\rho_i\otimes|i\rangle\langle i|$, with AB denoting the two Hilbert spaces $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$, consider the von Neumann entropy for the joint state $\rho_{\mathcal{AB}}$, so that, by Klein inequality,

$$\begin{aligned}
S(\rho_{\mathcal{AB}}||\rho_\mathcal{A}\otimes\rho_\mathcal{B}) &= -S(\rho_{\mathcal{AB}}) - \mathrm{tr}(\rho_{\mathcal{AB}}\log(\rho_\mathcal{A}\otimes\rho_\mathcal{B})) \\
&= -S(\rho_{\mathcal{AB}}) - \mathrm{tr}_\mathcal{B}(\mathrm{tr}_\mathcal{A}(\rho_{\mathcal{AB}}\log\rho_\mathcal{A})) - \mathrm{tr}_\mathcal{A}(\mathrm{tr}_\mathcal{B}(\rho_{\mathcal{AB}}\log\rho_\mathcal{A})) \\
&= -S(\rho_{\mathcal{AB}}) + S(\rho_\mathcal{B}) + S(\rho_\mathcal{A}) \geq 0
\end{aligned}$$

$$\therefore S(\rho_{\mathcal{AB}}) \leq S(\rho_\mathcal{A}) + S(\rho_\mathcal{B}). \tag{3.20}$$

Hence, for the $\rho_{\mathcal{AB}}$ defined before, the reduced states are $\rho_\mathcal{A} = \sum_i p_i\rho_i$ and $\rho_\mathcal{B} = \sum_i p_i\,|i\rangle\langle i|$, and thus, by applying (3.20) and theorem 3.5,

$$H(p_i) + \sum_i p_i S(\rho_i) \leq S\Big(\sum_i p_i\rho_i\Big) + H(p_i)$$

$$\Rightarrow S\Big(\sum_i p_i\rho_i\Big) \geq \sum_i p_i S(\rho_i), \tag{3.21}$$

and $S(\rho)$ is a concave function.

Up to now, it was possible to define all the analogous of Shannon entropies for the case of von Neumann entropies. There are two more definitions to finish these connections. The first is the Quantum Conditional Entropy, which is more difficult to generalize, once it requires the information of at least one Hilbert space, which is obtained with measurements, and such measurements disturbs the systems. In order to define the quantum conditional entropy, let $\rho_{\mathcal{A}|B_i^{\mathcal{B}}} \equiv B_i^{\mathcal{B}}\rho_{\mathcal{AB}}B_i^{\mathcal{B}}/\operatorname{tr}\big(B_i^{\mathcal{B}}\rho_{\mathcal{AB}}\big)$ be the state defined for $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ given that it was obtained an outcome labeled by i in a measurement defined by the projector $B_i^{\mathcal{B}}$ on $\mathcal{H}_{\mathcal{B}}$, and the probability of obtain $\rho_{\mathcal{A}|B_i^{\mathcal{B}}}$ for $\rho_{\mathcal{AB}}$ is given by $p_i = \operatorname{tr}\big(B_i^{\mathcal{B}}\rho_{\mathcal{AB}}\big)$. Hence, the conditional entropy given the complete measurement defined by the set $\{B_i^{\mathcal{B}}\}$ is defined as

$$S(\rho_{\mathcal{A}|\mathcal{B}}) \equiv \sum_i p_i S(\rho_{\mathcal{A}|B_i^{\mathcal{B}}}). \tag{3.22}$$

Following the definition (3.22), Quantum Mutual Information can be defined analogous as (3.11). Indeed, Quantum Mutual Information represents the amount of information shared bewteen two degrees of freedom described by the Hilbert spaces $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$. In Eq. (3.11), the quantities $\mathcal{I}$ and $\mathcal{J}$ were defined. Both of these quantities are extended to the quantum case, where they are defined as follows:

$$\begin{aligned} \mathcal{I}(\rho_{\mathcal{AB}}) &= S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}) - S(\rho_{\mathcal{AB}}) \\ \mathcal{J}_{[B]}(\rho_{\mathcal{AB}}) &= S(\rho_{\mathcal{A}}) - S(\rho_{\mathcal{A}|\mathcal{B}}). \end{aligned} \tag{3.23}$$

However, an important remark should be made at this point. Different from the classical case, in the quantum case the quantities $\mathcal{I}$ and $\mathcal{J}$ are not necessarily equal. It happens since measurements in quantum mechanics are perturbative. This difference will be further explored in the next section when Quantum Discord comes to light. The interpretations for these entropies are the same as the Shannon entropy, given the correct considerations to quantum cases.

The last result to be shown in this section is the following theorem.

- ***Theorem 3.6.*** Von Neumann Entropy $S(\rho)$ is invariant under unitary transformations.

- *Proof.* Let U be a unitary transformation such that $U^\dagger U = \mathbb{1}$. The transformed state is $\rho' = U\rho U^\dagger$, and then $S(\rho') = -\operatorname{tr}(\rho' \log \rho')$. Suppose that $\{\lambda_i\}$ and $\{\lambda_i'\}$ are the set of eigenvalues of ρ and ρ', respectively. Concerning the set $\{\lambda_i'\}$, each λ_i' is

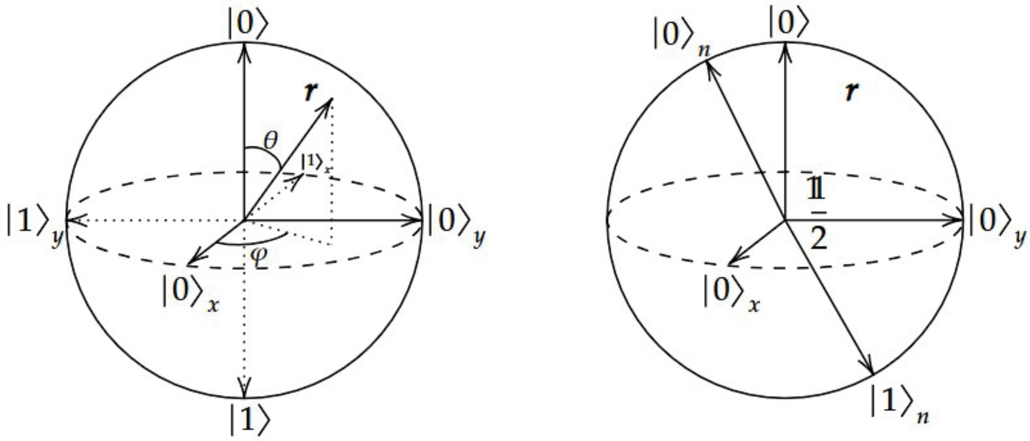

Figure 2 – On the left, a schematic representation of a vector in a Bloch sphere. On the right, an example of the eigenstates of σ_n in a Bloch sphere.

obtained by the relation $\det(\rho' - \lambda_i' \mathbb{1}) = 0$, or even

$$\begin{aligned}\det(\rho' - \lambda_i \mathbb{1}) &= \det\left(U\rho U^\dagger - \lambda_i' \mathbb{1}\right)\\ &= \det\left(U\rho U^\dagger - \lambda_i' U U^\dagger\right)\\ &= \det(U\rho - \lambda_i' U)\det\left(U^\dagger\right)\\ &= \det(U)\det\left(U^\dagger\right)\det(\rho - \lambda_i' \mathbb{1})\\ &= \det\left(UU^\dagger\right)\det(\rho - \lambda_i' \mathbb{1}) = \det(\rho - \lambda_i' \mathbb{1})\end{aligned}$$

so that $\{\lambda_i'\} = \{\lambda_i\}$, and $S(\rho') = S(\rho)$, concluding this proof. □

3.2 Bloch states

This section is dedicated to introduce an important way of describing qubit systems that will be important for the final analysis of this work. In short, when it comes to qubits, they can be geometrically represented in what is called the Bloch sphere (see Figure 2). A great advantage of this approach is that abstract algebraic entities in a Hilbert space can be represented in the euclidean space $\mathbb{R}^3$. Generally speaking, a qubit state can be written as

$$\rho = \frac{\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2}, \tag{3.24}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli vector, with the components being the Pauli matrices, and $\boldsymbol{r}$ is the vector that defines the physical state. For some geometrical intuition, Figure 2 illustrates a vector in a Bloch sphere and a representation of general eigenstates of σ_n.

Concerning the vector $\boldsymbol{r}$, consider the trace of ρ^2, that is,

$$\begin{aligned}\operatorname{tr}\rho^2 &= \operatorname{tr}\Big(\frac{1}{4}\big(\mathbb{1} + 2\boldsymbol{r}\cdot\boldsymbol{\sigma} + (\boldsymbol{r}\cdot\boldsymbol{\sigma})^2\big)\Big)\\ &= \frac{1}{4}\Big(2 + \sum_{ij} r_i r_j \operatorname{tr}\big(\mathbb{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k\big)\Big)\\ &= \frac{1}{2}\big(1 + r^2\big),\end{aligned}$$

where it was used that

$$\sigma_i\sigma_j = \mathbb{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{3.25}$$

with ϵ_{ijk} being the Levi-Civita symbol. The quantity $\operatorname{tr}\rho^2$ is defined as being the purity $\mathcal{P}(\rho)$, so that if $\mathcal{P}(\rho) = 1$, the state is pure, otherwise it is mixed. Then, by the above equality, it is straightforward that $r = 1$ for pure states. In the case of $r = 0$, the state is in the center of the sphere, which is the identity state, as illustrated in Figure 2.

For some visualization and gain of intuition, consider the example of the z-axis in Figure 2, with $\boldsymbol{r} = z\boldsymbol{k}$, so that the state in (3.24) is written as

$$\rho_z = \frac{\mathbb{1} + z\sigma_z}{2} = \frac{(1+z)\left|0\right\rangle\left\langle 0\right| + (1-z)\left|1\right\rangle\left\langle 1\right|}{2}.$$

Hence, for $|z| < 1$, both $\left|0\right\rangle\left\langle 0\right|$ and $\left|1\right\rangle\left\langle 1\right|$ contribute for the description, so that the final state is mixed. On the other hand, if $|z| = 1$, only one term contributes, and

$$\rho_{z=1} = \left|0\right\rangle\left\langle 0\right|, \qquad \rho_{z=-1} = \left|1\right\rangle\left\langle 1\right|,$$

establishing pure states, as previously mentioned. Geometrically, if $\boldsymbol{r}$ is a vector that defines a state and $\mathfrak{s}$ is a line parallel to this vector, the eigenbasis of the state defined by $\boldsymbol{r}$ is given by the states located at the intersection of the Bloch sphere's surface with the line $\mathfrak{s}$. This fact can be easily visualized using the axes of the Bloch sphere, corresponding to the eigenbases of σ_x, σ_y and σ_z, with pure states occurring only when $r = 1$, that is, when the states lie on the sphere's surface.

Another interesting feature of Bloch spheres is the geometrical interpretation of probabilities and expectation values. To visualize the expectation values, consider the expectation value of $\boldsymbol{n}\cdot\boldsymbol{\sigma}$, which is given by $\langle\boldsymbol{n}\cdot\boldsymbol{\sigma}\rangle = \operatorname{tr}\big((\boldsymbol{n}\cdot\boldsymbol{\sigma})\rho\big)$, where ρ is given by Eq. (3.24). Then,

$$\begin{aligned}\langle\boldsymbol{n}\cdot\boldsymbol{\sigma}\rangle &= \operatorname{tr}\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{\sigma} + (\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{r}\cdot\boldsymbol{\sigma})}{2}\Big)\\ &= \operatorname{tr}\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{\sigma} + (\boldsymbol{n}\cdot\boldsymbol{r})\mathbb{1} + i(\boldsymbol{n}\times\boldsymbol{r})\cdot\boldsymbol{\sigma}}{2}\Big)\\ &= \boldsymbol{n}\cdot\boldsymbol{r},\end{aligned} \tag{3.26}$$

where the identity $(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{r}\cdot\boldsymbol{\sigma}) = (\boldsymbol{n}\cdot\boldsymbol{r})\mathbb{1} + i(\boldsymbol{n}\times\boldsymbol{r})\cdot\boldsymbol{\sigma}$ and $\operatorname{tr}\sigma = 0$ have been used. In other words, the vector defining the state also determines the expectation values of each observable through its components. The interpretation of the probabilities follows a similar logic. In this case, consider an eigenstate of σ_n, where the probability of obtaining the eigenvalue 1 when measuring of σ_n is given by

$$\begin{aligned} p(1,\sigma_n) &= \mathrm{tr}\left(\left(\frac{\mathbb{1}+\boldsymbol{n}\cdot\boldsymbol{\sigma}}{2}\right)\left(\frac{\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma}}{2}\right)\right) \\ &= \frac{1}{4}\,\mathrm{tr}\left(\mathbb{1}+\boldsymbol{n}\cdot\boldsymbol{\sigma}+\boldsymbol{r}\cdot\boldsymbol{\sigma}+(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{r}\cdot\boldsymbol{\sigma})\right) \\ &= \frac{1}{4}\,\mathrm{tr}\left(\mathbb{1}+\boldsymbol{n}\cdot\boldsymbol{\sigma}+\boldsymbol{r}\cdot\boldsymbol{\sigma}+\boldsymbol{n}\cdot\boldsymbol{r}\mathbb{1}+i(\boldsymbol{n}\times\boldsymbol{r})\cdot\boldsymbol{\sigma}\right) \\ &= \frac{1+\boldsymbol{n}\cdot\boldsymbol{r}}{2}. \end{aligned} \tag{3.27}$$

Thus, the probability is also determined by the vector $\boldsymbol{r}$.

As an example of the above description, consider the state given by

$$\rho = \eta\,|0\rangle\,\langle 0|_x + (1-\eta)\,|0\rangle\,\langle 0|_y\,, \tag{3.28}$$

where the subscripts x and y denote the bases of σ_x and σ_y, respectively, and $\eta\in\mathbb{R}$ with $\eta\in[0,1]$. In the Bloch representation, this state can be rewritten as

$$\begin{aligned} \rho &= \eta\left(\frac{\mathbb{1}+\boldsymbol{x}\cdot\boldsymbol{\sigma}}{2}\right)+(1-\eta)\left(\frac{\mathbb{1}+\boldsymbol{y}\cdot\boldsymbol{\sigma}}{2}\right) \\ &= \frac{\mathbb{1}+\big(\eta\boldsymbol{x}+(1-\eta)\boldsymbol{y}\big)\cdot\boldsymbol{\sigma}}{2}. \end{aligned} \tag{3.29}$$

To determine whether the state in Eq. (3.28) is pure, we analyze the norm of the vector $\boldsymbol{r}_\eta \equiv \eta\boldsymbol{x}+(1-\eta)\boldsymbol{y}$. Thus,

$$||\boldsymbol{r}_\eta|| = r_\eta = \eta^2+(1-\eta)^2, \tag{3.30}$$

indicates that Eq. (3.28) represents a pure states only if $\eta\in\{0,1\}$. For the cases where $\eta\in(0,1)$, the vector $\boldsymbol{r}_\eta$ is a convex combination of $\boldsymbol{x}$ and $\boldsymbol{y}$, so that it is a vector inside the Bloch sphere.

Additionally, the expectation value of $\boldsymbol{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ can be determined using Eq. (3.26), so that

$$\begin{aligned} \langle\boldsymbol{\sigma}\rangle &= (\boldsymbol{x}\cdot\boldsymbol{r}_\eta, \boldsymbol{y}\cdot\boldsymbol{r}_\eta, \boldsymbol{z}\cdot\boldsymbol{r}_\eta) \\ &= (\eta, 1-\eta, 0). \end{aligned} \tag{3.31}$$

Regarding the probabilities, from Eq. (3.27), one can derive thaat

$$\begin{cases} p(0,\sigma_x) = (1+\eta)/2, \\ p(1,\sigma_x) = (1-\eta)/2, \end{cases} \qquad \begin{cases} p(0,\sigma_y) = 1-\eta/2, \\ p(1,\sigma_y) = \eta/2, \end{cases} \qquad \begin{cases} p(0,\sigma_z) = 1/2, \\ p(1,\sigma_z) = 1/2. \end{cases} \tag{3.32}$$

3.3 Quantum Resources

The goal of this section is to introduce some of the quantum resources used in this work. However, before doing that, it is worth exploring a little bit of the theory of quantum resources.

Broadly speaking, a resource is a feature that is used to accomplish some task. In QM, a quantum resource is a characteristic of a quantum system employed primarily to achieve tasks related to quantum computation. Although quantum computation is usually thought as a better way of doing computation in comparison with classical one, there is yet a debate concerning the real advantage of using quantum theory, instead of classical theory [38]. One of the differences between such approaches is the availability of resources. Quantum systems can be more explored, since they have a huge variety of resources in comparison with classical.

Resource theory is a very well defined mathematical framework, and is formalized for both classical and quantum resources. According to an important review of the subject [7], a quantum resource theory requires mainly two important definitions: *free states*, that are quantum states with no resource, and *free operations*, that are operations made in states that never increase the amount of resource in the state. However, in order to establish the idea of a resource, there has to be a task that cannot be done if this resource is zero, and considering that the quantum resource is zero is to consider, implicitly, the existence of a quantifier.

There are many examples of quantum resources. Probably the most famous one is Entanglement, usually defined as the nonseparability of a state, that will be more explored soon in this section. However, quantum resources theories can be used even to explore the foundations of quantum mechanics, such as the example of Bell nonlocality, that is indeed a quantum resource very used in communication and cryptography, as it was reviewed by Brunner *et al* [8]. Other examples of quantum resources used in quantum foundations and other studies are Contextuality, Incompatibility, Steering, Quantum Correlations and others, all of them reviewed in [7].

Concerning all those mentioned types of quantum resources, it is appropriate to introduce here what is known as the hierarchy of quantum resources. In particular, one of them encompasses some of the resources mentioned earlier. This hierarchy is given by

$$\mathfrak{s}_{\mathcal{E}} \subset \mathfrak{s}_{\mathcal{D}} \subset \mathfrak{s}_{\mathcal{DS}} \subset \mathfrak{s}_{\mathcal{NL}}, \tag{3.33}$$

where $\mathfrak{s}_i$ means the set possessing the quantum resource i, where $\mathcal{E}$ means entanglement; $\mathcal{D}$ means one-way quantum discord; $\mathcal{DS}$ means symmetric quantum discord; $\mathcal{NL}$ means realism-based nonlocality. The purpose of Eq. (3.33) is to show that, although quantum resources are generally employed for different tasks, they maintain a certain ordering regarding the set of states that possess such resources. That is, the presence of one resource does not necessarily exclude the presence of another.

This brief introduction has not the purpose of giving a profound knowledge on the subject. For the interested reader, it is recommended to explore the cited papers to gain a deeper understanding of the role of quantum resources in both technology and quantum theory. In what follows, the quantum resources that are relevant to this work will be explored.

3.3.1 Irreality

In Section 2.3.3 it was introduced the concept of realism in quantum physics. In particular, this section will further explore the BA-realism, and for such discussion, it is important to go deeper in this concept.

First, it is worth to recall that BA-realism encompasses the states of mixture, different from EPR's views. Indeed, as mentioned in section 2.3.3, given a state $\rho = \sum_i p_i A_i$, if A_i are the projectors associated to an observable A, then there is an element of reality for $\mathcal{A}$ in the state ρ according to BA-realism criteria. Additionally, it was previously mentioned that the protocol that the criteria was based is completely dependent on the idea that a measurement always establishes an element of reality. In other words, if a measurement is performed within a state, it will be realistic for the measured observable after the collapse. Following this discussion, it is worth mentioning that the map $\Phi_A(\rho)$ is such that it preserves the pre-existing reality. In other words, if $\Phi_A(\rho) = \rho$, then $\Phi_A(\Phi_A(\rho)) = \Phi_A \circ \Phi_A(\rho) = \rho$.

By using the concept of Φ and noting that this map destroys the off-diagonal elements, the authors Bilobran and Angelo defined the concept of Irreality [13] as being the difference of the entropies of $\Phi_A(\rho)$ and ρ for some observable. That is, the irreality of an observable A is defined as

$$\mathfrak{I}_A(\rho) \equiv S(\Phi_A(\rho)) - S(\rho), \tag{3.34}$$

with S being the von Neumann entropy. According to theorem 3.4 stated in the last section, it is straightforward that $\mathfrak{I}_A(\rho) \geq 0$, and the equality holds if, and only if, the state ρ is a mixture state of the projectors of the observable A, or simply an eigenstate of A. In other words, the irreality is zero if, and only if, the state has an element of reality.

Irreality can also be described in terms of relative entropy, clarifying the fact that irreality is a measure based on an “entropic distance." To achieve such description, Eq. (3.34) is explicitly written as $\mathfrak{I}_A(\rho) = -\operatorname{tr}(\Phi_A(\rho) \log \Phi_A(\rho)) + \operatorname{tr}(\rho \log \rho)$. However, the first term in this equation can be written differently if the properties of projectors are used, i.e, by using that $\sum_a (A_a \otimes \mathbb{1}_B) = \mathbb{1}_A \otimes \mathbb{1}_B$ and $(A_a \otimes \mathbb{1}_B)^2 = A_a \otimes \mathbb{1}_B$, and by noting that $[(A_a \otimes \mathbb{1}_B, \Phi_A(\rho)] = 0$, the first term is written as

$$\begin{aligned}\operatorname{tr}(\Phi_A(\rho)\log\Phi_A(\rho)) &= \operatorname{tr}\left(\sum_a (A_a \otimes \mathbb{1}_B)\rho(A_a \otimes \mathbb{1}_B)\log\Phi_A(\rho)\right)\\ &= \operatorname{tr}\left(\sum_a (A_a \otimes \mathbb{1}_B)\rho\log\Phi_A(\rho)(A_a \otimes \mathbb{1}_B)\right)\\ &= \operatorname{tr}\left(\sum_a (A_a \otimes \mathbb{1}_B)\rho\log\Phi_A(\rho)\right) = \operatorname{tr}\rho\log\Phi_A(\rho),\end{aligned}$$

so that the irreality can be written using the definition of relative entropy

$$\mathfrak{I}_A(\rho) = S(\rho||\Phi_A(\rho)). \tag{3.35}$$

This result reinforces the fact that the irreality is non-negative and is zero if, and only if, $\rho = \Phi_A(\rho)$, which is in agreement with theorem 3.3.

It is worth mentioning that the irreality is strongly state and observable dependent. It means that for a given state ρ, it can be realistic for a given observable A, or $\mathfrak{I}_A(\rho) = 0$, although not for B, or $\mathfrak{I}_B(\rho) > 0$, and vice-versa for another state σ, that is, $\mathfrak{I}_A(\sigma) > 0$ and $\mathfrak{I}_B(\sigma) = 0$.

One last point that is worth noting is the description of $\mathfrak{I}$ in terms of the irreality of the reduced state. For such discussion, consider a physical state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and an observable $A : \mathcal{H}_1 \to \mathcal{H}_1$. Let $\rho_2 \equiv \operatorname{tr}_1 \rho$ be the reduced state in $\mathcal{H}_1$, so that $\Phi_A(\rho_2) = \rho_2$, i.e, A has an element of reality for ρ_2. Hence, in such case, the irreality of A in the state ρ can be written as

$$\begin{aligned}\mathfrak{I}_A(\rho) &= S(\Phi_A(\rho)) - S(\rho)\\ &= S(\Phi_A(\rho)) - S(\rho) + \Big(S(\Phi_A(\rho_1) - S(\Phi_A(\rho_1)\Big) + \Big(S(\rho_1) - S(\rho_1)\Big)\\ &\qquad + \Big(S(\rho_2) - S(\Phi_A(\rho_2))\Big)\\ &= S(\Phi_A(\rho_1)) - S(\rho_1) + \Big(S(\rho_1) + S(\rho_2) - S(\rho)\Big)\\ &\qquad - \Big(S(\Phi_A(\rho_1)) + S(\Phi_A(\rho_2)) - S(\Phi_A(\rho))\Big)\end{aligned}$$

$$\therefore \mathfrak{I}_A(\rho) = \mathfrak{I}_A(\rho_1) + \mathcal{D}_{[A]}(\rho), \tag{3.36}$$

with $\mathcal{D}_{[A]}(\rho)$ being a type of quantum discord (further discussion in the last part of this section) associated with measurements of A. If $\mathcal{D}_1(\rho) = \min_A \mathcal{D}_{[A]}(\rho)$, then

$$\mathfrak{I}_A(\rho) - \mathfrak{I}_A(\rho_1) \geq \mathcal{D}_1(\rho). \tag{3.37}$$

Thus, the above equation demonstrates that quantum correlations prevent irrealism from being purely local, as the irrealism of A for ρ differs from that calculated for ρ_1.

3.3.2 Entanglement

Entanglement is probably the most important quantum resource in quantum mechanics. Actually, quoting Schrödinger on entanglement, “I would not call *one* but rather *the* characteristic trait of quantum mechanics" [4], in the same paper that he introduced the term “entangled states", in 1935. The goal of this brief subsection is to define quantum entanglement and present some of the quantifiers available for the purpose of this work.

Entanglement is a correlation between two parts of the system, that usually arises when these parts interact with each other [6]. An intelligent form of understanding entanglement, in this sense, is to define the set of entangled states as being the set of states that cannot be generated by performing Local Operations and Classical Communication (LOCC) [7]. By local operations it is understood the set of operations locally performed over just one part of a composite system, whereas classical communication is a method of broadcasting information through a classical device.

To establish a mathematical definition of entanglement, let ρ be a density operator in a multipartite Hilbert space $\mathcal{H} = \bigotimes_i^n \mathcal{H}_i$. If, for any bi-partition $\mathcal{A}$ and $\mathcal{B}$, it is possible to describe $\rho = \sum_i p_i \rho_i^{\mathcal{A}} \otimes \rho_i^{\mathcal{B}}$, then the state ρ is said to be separable, otherwise it is entangled. The first important comment on this subject is that $\mathcal{A}$ and $\mathcal{B}$ can be any bi-partition, i.e, it is just required that $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} = \mathcal{H}$. Secondly, even separable states, typically, contain quantum correlations, although not entanglement. Hence, although entanglement is a kind of quantum correlation, it is not unique. Another important point is the case of a pure state. For such case, $p_j = 1$ and $p_i = 0 \, \forall i \neq j$, and $\rho = \rho_j^{\mathcal{A}} \otimes \rho_j^{\mathcal{B}}$. If both $\rho_j^{\mathcal{A}}$ and $\rho_j^{\mathcal{B}}$ represent projectors, say $\rho_j^{\mathcal{A}} = |\psi_j\rangle \langle\psi_j|$ and $\rho_j^{\mathcal{B}} = |\phi\rangle \langle\phi|$, then $\rho = |\psi_j \phi_j\rangle \langle\psi_j \phi_j|$, which is completely described by $|\Psi\rangle = |\psi_j\rangle \otimes |\phi_j\rangle$. Thus, if a pure state is written as a tensor product of two pure states, it is separable, otherwise it is entangled.

Concerning the quantum resource feature of entanglement, it is necessary to establish the free states and the free operations. Essentially, the free states for entanglement are the separable states, as defined above, whereas the free operations are the LOCC. However, as mentioned in the beginning of this section, a quantum resource requires some task of interest and a quantifier. In the case of entanglement, the most celebrate example of task is the quantum teleportation [16], and two important quantifiers are going to be explored now.

Concerning pure states, a quite simple quantifier of entanglement is the von Neumann entropy of the reduced state, i.e, if ρ is a pure state defined in $\mathcal{H}_1 \otimes \mathcal{H}_2$, then the entanglement of ρ is defined as

$$\mathcal{E}(\rho) := S(\rho_1) = S(\rho_2), \tag{3.38}$$

with $\rho_1 = \mathrm{tr}_2\, \rho$, and similarly for ρ_2. As an example, consider the state defined by $|\psi\rangle = |01\rangle$ in the computational basis[1], so that $\rho = |\psi\rangle\langle\psi| = |01\rangle\langle 01| = |0\rangle\langle 0| \otimes |1\rangle\langle 1|$. Hence, $\mathrm{tr}_1\, \rho = \rho_2 = |1\rangle\langle 1|$, and $S(\rho_2) = 0$, concluding the fact that a pure state that is separable is not entangled.

Another important example of pure states is to consider the singlet state in (2.9). For such case, it is worth noting that this is a type of state that cannot be factorized in a simple product state. Hence, there is entanglement is such state. However, it is instructive to calculate the amount of entanglement, and in this case,

$$\rho = \frac{1}{2}(|01\rangle\langle 01| + |10\rangle\langle 10| - |10\rangle\langle 01| - |01\rangle\langle 10|)$$
$$\Rightarrow \rho_2 = \mathrm{tr}_1\, \rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{\mathbb{1}}{2}.$$

Hence, ρ_2 is the most entropic state, so that $\mathcal{E}(\rho) = 1$, and the entanglement is maximized. The last two cases of interest exemplify exactly the point that, by using von Neumann entropy of the reduced state as a quantifier of entanglement for pure states, such function gave values between 0 and 1, and are always well-defined for the studied states.

The above discussion is allowed only for pure states. The discussion concerning mixed states is more complex. Indeed, the discussion of quantifiers for mixed states is huge, with many quantifiers defined in the literature (for more details see reference [6]). However, somehow these quantifiers are dependent on a sophisticated form of extremization which, according to Wootters, “are difficult to handle analytically". Given that, in 1997, Wootters defined the entanglement quantifier for *qubits* (two level systems) called the *Concurrence* [72], based on the idea of entanglement of formation, which can be understood as follows. Given a density operator $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ defined in a bipartite Hilbert space, the entanglement of each pure state $\rho_i = |\psi_i\rangle\langle\psi_i|$ is given by the von Neumann entropy of the subsystems. However, ρ is a mixture of many pure states, so that the entanglement of formation is defined as the average entanglement of the pure states of the decomposition, minimized over all decompositions of ρ:

$$\mathcal{E}(\rho) = \min \sum_i p_i \mathcal{E}(\rho_i). \tag{3.39}$$

Wootters' remarkable contribution was to prove that the above minimization can be expressed as an explicit function of ρ, which is called Concurrence, and is defined as

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{3.40}$$

and λ_i are the eigenvalues, in decreasing order, of the non-Hermitian operator $\rho\tilde{\rho}$, and

$$\tilde{\rho} := (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{3.41}$$

[1] In the computational basis, $|0\rangle$ represents the eigenvector for the $z-$component of spin with eigenvalue $\hbar/2$, and $|1\rangle$ is the eigenvector for the $z-$component of spin with eigenvalue $-\hbar/2$.

with ρ^* being the complex conjugated of ρ and σ_y the Pauli operator. It is worth noting that the complex conjugate is taken in the standard computational basis for two *qubits*. Thus, by the construction of (3.40), it can itself be used as an entanglement quantifier, such that

$$\mathcal{E}(\rho) = C(\rho). \tag{3.42}$$

A very didatic example using the concept of concurrence can be done with the Werner state, defined as

$$\rho_W = \frac{1-\eta}{4}\mathbb{1} + \eta \left|\psi_S\right\rangle \left\langle\psi_S\right|, \tag{3.43}$$

with $|\psi_S\rangle$ being the singlet state defined in (2.9), $\eta \in [0,1]$ is a parameter that control the convex combination between $|\psi_S\rangle$ and the identity state $\mathbb{1}$. It is known that $|\psi_S\rangle$ is a maximally entangled state, while $\mathbb{1}/4$ is the classical state, with zero entanglement. However, it is not clear to state how the mixture of these states affects the entanglement of the final state. In order to understand that, let the concurrence be analyzed. Firstly, the operator $\rho_W\tilde{\rho}_W$ must be written in the computational basis, so that

$$\begin{aligned}\rho_W\tilde{\rho}_W = \Big(\frac{1-\eta}{4}\Big)^2\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) + \Big[\Big(\frac{1-\eta}{4}\Big)^2 + \frac{\eta^2}{4}\Big]\left(|01\rangle\langle 01| + |10\rangle\langle 10|\right) \\ - \eta\Big(\frac{1+\eta}{4}\Big)\left(|01\rangle\langle 10| + |10\rangle\langle 01|\right),\end{aligned} \tag{3.44}$$

whose eigenvalues are $\lambda_1 = \left(\frac{1+3\eta}{4}\right)^2$ and $\lambda_2 = \left(\frac{1-\eta}{4}\right)^2$, with λ_2 being 3 times degenerated. Hence, by the definition of concurrence, the entanglement for the state in (3.43) is written as

$$\mathcal{E}(\rho_W) = C(\rho_W) = \frac{1}{2}\max\{0, 3\eta - 1\}. \tag{3.45}$$

By the above equation, the state ρ_W is entangled if $\eta \in (\frac{1}{3}, 1]$ and not entangled if $\eta \in [0, \frac{1}{3}]$. It means that, depending of the mixture with the identity state (mixture with noisy), even the most entangled state (the singlet state) can result in a separable one.

A noteworthy remark to conclude this subsection is that, as briefly mentioned throughout the text, these methods of calculating entanglement are not unique. In fact, the study of entanglement quantifiers is very extensive, and the reader is encouraged to consult the following references [6, 73].

3.3.3 Quantum Discord

In brief, quantum discord was introduced earlier in this chapter but has not yet been properly discussed. In the study of von Neumann entropy, in section 3.1, it was introduced the concept of quantum mutual information, the quantum counterpart of the classical mutual information. In such discussion, it is worth noting that, considering the classical case, the mutual information could be expressed in two different ways, although has the same values, according to Eq. (3.11). However, as discussed in the end of the mentioned section, quantum mutual information can also be expressed in two different ways, according to Eq. (3.23), but these are not necessarily equal. Indeed, as measurements in quantum mechanics induce perturbations in the state, the idea of a conditional entropy must be seen with more caution, and such feature of quantum theory implies that the quantities $\mathcal{I}$ and $\mathcal{J}$ are not necessarily equal.

Exploring the difference between the two forms of writing the quantum mutual information, Ollivier and Zurek introduced the concept of quantum discord [10], which is understood by the authors as “a measure of the information that cannot be extracted by the reading of the state of the apparatus. Hence, quantum discord is a good indicator of the quantum nature of the correlations". Independently of Ollivier and Zurek, Henderson and Vedral explored the same difference in [11]. The first definition of quantum discord of a measurement of the observable B on $\mathcal{H}_{\mathcal{B}}$ is simply

$$\mathcal{D}_{[B]}(\rho) := \mathcal{I}(\rho) - \mathcal{J}_{[B]}(\rho). \tag{3.46}$$

Such definition is asymmetric and depends on the observables of interest, so that it was also proposed the one-way quantum discord, that is independent of the observable, as being

$$\mathcal{D}_{\mathcal{B}}(\rho) = \min_{B} \mathcal{D}_{[B]}(\rho). \tag{3.47}$$

As an example of calculation of the one-way quantum discord, let the Werner state in (3.43) be considered again. Previously, it was calculated that there is a condition for η such that ρ_W is entangled. Now, let the same state be analyzed by the point of view of the one-way quantum discord. The optimal observable for such state is ${\sigma_z}$[2], so that its eigenbasis already is the computational basis. Thus, by the definitions of $\mathcal{I}$ and $\mathcal{J}$ in (3.23), and using Eq. (3.15), it is possible to calculate for ρ_W the following quantities

$$\begin{aligned} S(\rho_W) &= -\Big(\frac{1+3\eta}{4}\Big)\log\Big(\frac{1+3\eta}{4}\Big) - 3\Big(\frac{1-\eta}{4}\Big)\log\Big(\frac{1-\eta}{4}\Big), \\ S(\operatorname{tr}_{\mathcal{A}}\rho_W) &= 1, \\ S(\rho_{\mathcal{A}|\{\sigma_z\}_B}) &= -\Big(\frac{1+\eta}{2}\Big)\log\Big(\frac{1+\eta}{2}\Big) - \Big(\frac{1-\eta}{2}\Big)\log\Big(\frac{1-\eta}{2}\Big). \end{aligned} \tag{3.48}$$

As the eigenbasis of σ_z is the optimal one, just by the definitions of $\mathcal{I}$ and $\mathcal{J}$, quantum discord is written as $\mathcal{D}_{\mathcal{B}}(\rho_W) = S(\rho_{\mathcal{A}|\{\sigma_z\}_{\mathcal{B}}}) + S(\operatorname{tr}_{\mathcal{A}}\rho_W) - S(\rho_W)$, so that

$$\mathcal{D}_{\mathcal{B}}(\rho_W) = \Big(\frac{1+3\eta}{4}\Big)\log(1+3\eta) + \Big(\frac{1-\eta}{4}\Big)\log(1-\eta) - \Big(\frac{1+\eta}{2}\Big)\log(1+\eta), \tag{3.49}$$

according to [74].

Going even further in this discussion on quantum discord, in a work by Rulli and Sarandy, they explored the asymmetry in the definition of Olivier and Zurek and defined

[2] Indeed, since the singlet state is invariant under rotations, any direction of σ is optimal.

the so called symmetric quantum discord [12]. To achieve such definition, let first the one-way quantum discord be expressed in terms of the relative entropy. In order to do so, Eq. (3.23) must be expressed in terms of relative entropy. $\mathcal{I}(\rho_{\mathcal{AB}})$ is already the own definition of that, so that

$$\mathcal{I}(\rho_{\mathcal{AB}}) = S(\rho_{\mathcal{AB}}||\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}). \tag{3.50}$$

For $\mathcal{J}_{[B]}(\rho_{\mathcal{AB}})$, it is worth noting that

$$\Phi_B(\rho_{\mathcal{AB}}) = \sum_i p_i \rho_{\mathcal{A}|B_i} \otimes B_i,$$
$$\mathrm{tr}_{\mathcal{A}}\, \Phi_B(\rho_{\mathcal{AB}}) = \Phi_B(\rho_{\mathcal{B}}) = \sum_i p_i B_i,$$

and by a direct application of theorem 3.5, the entropies are computated as

$$S(\Phi_B(\rho_{\mathcal{AB}})) = H(p) + \sum_i p_i S(\rho_{\mathcal{A}|B_i^{\mathcal{B}}}),$$
$$S(\Phi_B(\rho_{\mathcal{B}})) = H(p),$$

so that $\mathcal{J}_{[B]}(\rho_{\mathcal{AB}})$ is written as

$$\mathcal{J}_{[B]}(\rho_{\mathcal{AB}}) = S(\Phi_{AB}(\rho_{\mathcal{AB}})||\rho_{\mathcal{A}} \otimes \Phi_B(\rho_{\mathcal{B}})). \tag{3.51}$$

Hence, from (3.50) and (3.51), (3.47) can be written as

$$\mathcal{D}_{\mathcal{B}}(\rho_{\mathcal{AB}}) = \min_{\{B\}} \Big(S(\rho_{\mathcal{AB}}||\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) - S(\Phi_B(\rho_{\mathcal{AB}})||\rho_{\mathcal{A}} \otimes \Phi_B(\rho_{\mathcal{B}}) \Big). \tag{3.52}$$

Finally, by taking measurements in both spaces, the symmetric quantum discord is defined as

$$\mathcal{D}_{\mathcal{AB}}(\rho) = \min_{\{A\}\otimes\{B\}} \big(\mathcal{I}(\rho) - \mathcal{I}(\Phi_{AB}(\rho))\big), \tag{3.53}$$

with $\Phi_{AB}(\rho)$ being the extension of (2.10) for two measurements, i.e.,

$$\Phi_{AB}(\rho) = \sum_{ij} [A_i \otimes B_j]\rho_{\mathcal{AB}}[A_i \otimes B_j]. \tag{3.54}$$

The name symmetric quantum discord is due to the symmetry involved in the measurements in both Hilbert spaces of the state, while in the one-way quantum discord only one of the Hilbert spaces is submitted to measurements. Additionally to this discussion, it is worth mentioning that due to the symmetry of the singlet state, both one-way and symmetric quantum discord are the same for the state in (3.43). Indeed, the optimal observable still is σ_z for both Hilbert spaces, so that

$$\begin{aligned}
\Phi_A(\rho_W) = \Phi_B(\rho_W) = \Phi_{AB}(\rho_W) &= \frac{1-\eta}{4}\mathbb{1} + \frac{\eta}{2}\big(|10\rangle\langle 10| + |01\rangle\langle 01| \big), \\
\mathcal{I}(\rho_W) &= 2 - S(\rho_W), \\
\mathcal{I}(\Phi_{AB}(\rho_W)) &= 2 - S(\Phi_{AB}(\rho_W)),
\end{aligned} \tag{3.55}$$

so that, by using (3.53), the symmetric quantum discord can be written as

$$\begin{aligned}\mathcal{D}_{\mathcal{AB}}(\rho_W) &= \left(\frac{1+3\eta}{4}\right)\log(1+3\eta) + \left(\frac{1-\eta}{4}\right)\log(1-\eta) - \left(\frac{1+\eta}{2}\right)\log(1+\eta) \\ &= \mathcal{D}_{\mathcal{B}}(\rho_W).\end{aligned} \tag{3.56}$$

At this point, it is important to delve into the nuances of entanglement and quantum discord. As obtained in (3.45), the entanglement for the Werner state goes to zero abruptly when η goes to $1/3$. However, it does not occur in the case of the one-way quantum discord. Indeed, according to Eq. (3.49), the one-way quantum discord for Werner state is zero only when η is zero, and the behavior of the function is smooth, differently from entanglement. This brief example illustrates an important point regarding quantum correlations: although entanglement is “the characteristic" of quantum mechanics, according to Schrödinger, it is not the only form of purely quantum correlation. There are other forms of quantum correlations, encompassed by quantum discord, that are considered in this smooth decay. Moreover, such conclusion is the reason why the set of one-way quantum discordant states be bigger than the entangled states, as given by Eq. (3.33). Additional details concerning theoretical and experimental aspects of Quantum Discord can be seen in [75].

3.3.4 Realism-Based Nonlocality

The notion of nonlocality was briefly introduced in Section 2.3.3, that is, if a state cannot be explained by a local model, it is said to be nonlocal. It was also mentioned its relation with Bell inequalities and realism [60], and there are other works that study this concept without mentioning any kind of inequality [76]. For a full review of the subject see reference [8].

The idea of realism-based nonlocality was introduced in [13] and explored in [14, 15, 77]. Suppose $\rho_{\mathcal{AB}}$ describes the global state of Alice and Bob, which are in laboratories space-like separated so that they are outside of the light cone of each other. According to [13], a measure of nonlocality can be written as

$$\mathcal{N}(A,B|\rho_{\mathcal{AB}}) := \mathfrak{I}_A(\rho_{\mathcal{AB}}) - \mathfrak{I}_A(\Phi_B(\rho_{\mathcal{AB}})), \tag{3.57}$$

and it quantifies how much the irreality in the observable A accessible in Alice's site changes due to measurements in B in Bob's site. Since it quantifies how nonlocal were the measurements based on the concept of irreality, (3.57) was defined as being the realism-based nonlocality for the context $\{A, B, \rho_{\mathcal{AB}}\}$. Additionally, the authors in [14] explored this definition to define the context-independent realism-based nonlocality as being

$$\mathcal{N}(\rho_{\mathcal{AB}}) = \max_{A,B} \mathcal{N}(A,B|\rho_{\mathcal{AB}}). \tag{3.58}$$

The above definition is motivated by three main reasons. First, it is expected that the greater the change in irreality, the greater the nonlocality. Second, this definition guarantees that if $\mathcal{N}(\rho_{\mathcal{AB}}) = 0$, there is no context where the irreality can change, and for $\mathcal{N}(\rho_{\mathcal{AB}}) > 0$, there is at least one context $\{A, B, \rho_{\mathcal{AB}}\}$ where the irreality changes. Hence, definition (3.58) provides necessary and sufficient conditions to analyze the realism-based nonlocality. The third reason is that, according to [14], from the above definition, maximally entangled states leads to maximally nonlocal states.

Before finishing this section, it is important to discuss a little about the hierarchy of quantum resources in Eq. (3.33). From this equation, every entangled state has one-way quantum discord, every one-way quantum discordant state has symmetric quantum discord, and every symmetric discordant state has realism-based nonlocality, But the converse is not true. Thus, there exist realism-based nonlocal states that are not symmetric discordant, symmetric discordant states that are not one-way discordant, and one-way discordant states that are not entangled. Understanding this hierarchy is crucial for determining the appropriate quantum resources to be studied in this work.

3.4 Mach-Zehnder Interferometers

The purpose of this section is to introduce some basic concepts on how to operate on quantum states in optical setups. Thus, some important optical devices will be presented, along with how they are addressed within the quantum formalism, as well as two examples that are quite illustrative from both physical and educational perspective. This section serves as a direct introduction to the next chapter, where a slightly more complex setup will be considered. The present analysis is based on reference [78] and references therein.

The first subject of discussion in this section is the Mach-Zehnder Interferometer (MZI) [79, 80]. This optical setup is illustrated in Figure 3 (a). Essentially, the MZI is used to determine the phase shift variations between two collimated beams. Moreover, it is possible to study the behavior of the photon in some situations with a single photon. For the purpose of this work, it is considered that the photon enters in the MZI passing through a Beam-Splitter (BS), that is responsible for controlling the probability of the photon being transmitted or reflected. For both situations, the mirrors (M) are responsible for guiding the photon until the next BS, that again will be the responsible for controlling the probabilities of transmission and reflection. Here, the important point is that the whole analysis is made collecting data from the detectors, so that is possible to study the statistics of the problem. One of the arms of the interferometer has a phase shifter (φ), responsible for changing the optical path from one arm to another.

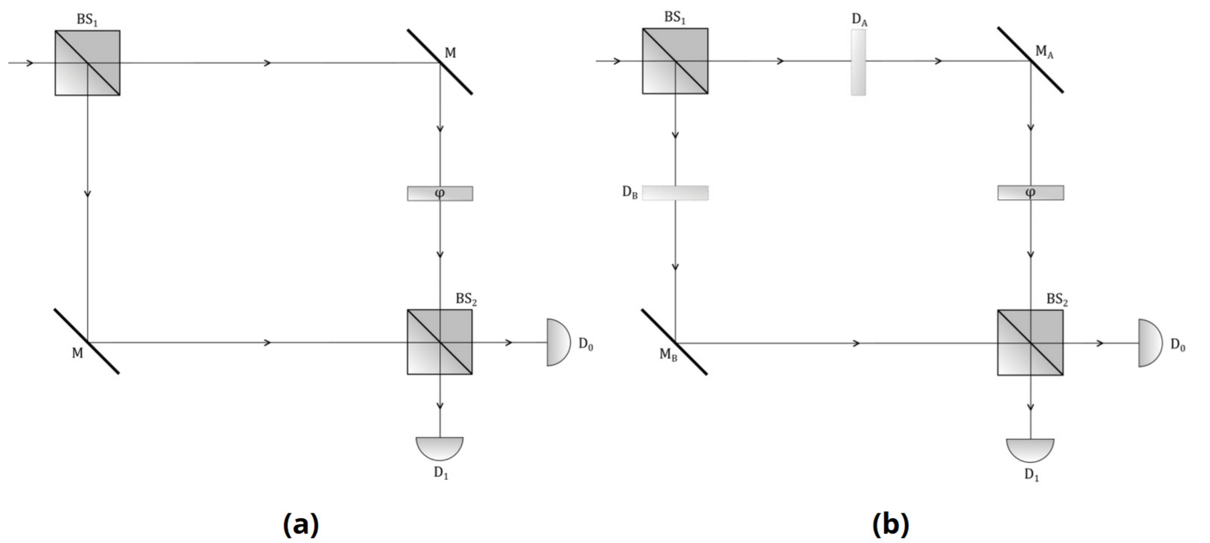

Figure 3 – (a) A schema of the usual Mach-Zehnder interferometer (MZI), and displays its main optical devices, such as Beam-Splitters (BS), mirrors (M), a phase shift (φ) and detectors of the path mode (D's). (b) A schema of a MZI, however with a slightly modification that are the path markers D_A and D_B in the horizontal and vertical path modes, respectively.

Concerning the mathematical description, each optical device in the MZI is described by an unitary operator, once there is no loss of information when the photon passes through each of these devices. In order to construct the unitary operators, it is important to understand the consequence of each device. First, suppose that the initial state is given by $|\psi_0\rangle = |0\rangle$, that describes the horizontal path mode of the photon, as illustrated in Figure 3 (a). After the BS, considering that the coefficients of transmission and reflection are, respectively, $T = \cos\theta$ and $R = \sin\theta$ for $\theta \in [0, \frac{\pi}{2}]$, the state is a superposition of the path modes mediated by the probabilities associated to the BS. Hence, the final state is $|\psi_{BS}\rangle = \cos\theta\,|0\rangle + i\sin\theta\,|1\rangle$. The associated phase i to the reflected term is due to the phases of the electric field of the photon [81]. Otherwise, if the initial state was $|1\rangle$, the final one would be $\cos\theta\,|1\rangle + i\sin\theta\,|0\rangle$. Hence, by choosing the computational basis in the order $|0\rangle\,,|1\rangle$, the matrix representation of the BS is written as

$$U_{BS}(\theta) \doteq \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}. \tag{3.59}$$

Following the same rationale, the matrix representations for the M's and φ in the spatial mode 1 are

$$U_M \doteq \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad U_\varphi(1) \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}. \tag{3.60}$$

Since the mathematical description of the devices are done, let the state be developed through the MZI. Considering that both BS are equal, the evolution of the state is as follows

$$\begin{aligned}
|\psi_0\rangle &= |0\rangle\,, \\
|\psi_0\rangle \xrightarrow{\text{BS}_1} |\psi_{BS}\rangle &= \cos\theta\,|0\rangle + i\sin\theta\,|1\rangle\,, \\
|\psi_{BS}\rangle \xrightarrow{\text{M's}} |\psi_M\rangle &= -\sin\theta\,|0\rangle + i\cos\theta\,|1\rangle\,, \\
|\psi_M\rangle \xrightarrow{\varphi} |\psi_\varphi\rangle &= -\sin\theta\,|0\rangle + ie^{i\varphi}\cos\theta\,|1\rangle\,, \\
|\psi_\varphi\rangle \xrightarrow{\text{BS}_2} |\psi_f\rangle &= -(1+e^{i\varphi})\sin\theta\cos\theta\,|0\rangle + i(e^{i\varphi}\cos^2\theta - \sin^2\theta)\,|1\rangle\,.
\end{aligned} \tag{3.61}$$

The final state $|\psi_f\rangle$ is going to be analyzed in the detectors D_0 and D_1, so that the probabilities distributions obtained at the end are given by

$$\begin{aligned}
p(\text{D}_0) &= |\langle 0|\psi_f\rangle|^2 = 2\sin^2\theta\cos^2\theta(1+2\cos\varphi), \\
p(\text{D}_1) &= |\langle 1|\psi_f\rangle|^2 = \sin^4\theta + \cos^4\theta - 4\sin^2\theta\cos^2\theta\cos\varphi.
\end{aligned} \tag{3.62}$$

The above result is interesting to study the behavior of the photon. Even if the initial state is a well-defined path state, the probability distributions are sensitive to the relative phase φ. Hence, it is fair interpreting the behavior of the photon as being wavelike. Indeed, after the first BS, the best description to be given to the state is that of a wave. However, it cannot be dismissed that this interpretation involves a form of retrodiction, since the final interpretation concerns using the statistical results obtained after the MZI to make assertions about the photon's behavior within the MZI.

Now, consider a slight modification to the previously discussed MZI, as shown in Figure 3 (b). In this case, in addition to the relative phase being introduced in one of the interferometer arms, two nondestructive detectors (i.e., path markers) are placed in the system, one in each arm of the interferometer. The purpose of these markers are, in a way, to reveal the photon's path, which will be analyzed through its entanglement with these degrees of freedom. Since the markers are quantum systems that interact with the photon, each one needs a state in a Hilbert space to be described. For such description, let the initial state be $|\psi_0\rangle = |0\rangle_\gamma |0\rangle_A |0\rangle_B = |000\rangle_{\gamma AB}$, with γ representing the path mode of the photon, and A and B being the corresponding states of the path markers D_A and D_B, respectively. The previous discussion on the mathematical description of the system is the same, but now each optical device must be accounted in the corresponding Hilbert space. In order to analyze the final state of this setup, consider that the path markers 'click' if the photon's path coincides with the arm where the marker is placed, that is,

$$\begin{aligned}
U_{D_A}|000\rangle_{\gamma AB} &= |010\rangle_{\gamma AB}\,, \quad U_{D_A}|100\rangle_{\gamma AB} = |100\rangle_{\gamma AB}\,, \\
U_{D_B}|000\rangle_{\gamma AB} &= |000\rangle_{\gamma AB}\,, \quad U_{D_B}|100\rangle_{\gamma AB} = |101\rangle_{\gamma AB}\,.
\end{aligned}$$

With these definitions, also considering that both BS are equal, the evolution of the state is as follows

$$
\begin{aligned}
|\psi_0\rangle &= |000\rangle_{\gamma AB} \\
|\psi_0\rangle \xrightarrow{\mathsf{BS}_1} |\psi_{BS}\rangle &= (\cos\theta\,|0\rangle + i\sin\theta\,|1\rangle)_\gamma\,|00\rangle_{AB} \\
|\psi_{BS}\rangle \xrightarrow{\mathsf{D}'s} |\psi_D\rangle &= \cos\theta\,|010\rangle_{\gamma AB} + i\sin\theta\,|101\rangle_{\gamma AB} \\
|\psi_D\rangle \xrightarrow{\mathsf{M}'s} |\psi_M\rangle &= i\cos\theta\,|110\rangle_{\gamma AB} - \sin\theta\,|001\rangle_{\gamma AB} \\
|\psi_M\rangle \xrightarrow{\varphi} |\psi_\varphi\rangle &= ie^{i\varphi}\cos\theta\,|110\rangle_{\gamma AB} - \sin\theta\,|001\rangle_{\gamma AB} \\
|\psi_\varphi\rangle \xrightarrow{\mathsf{BS}_2} |\psi_f\rangle &= i\,|1\rangle_\gamma\left(e^{i\varphi}\cos^2\theta\,|10\rangle_{AB} - \sin^2\theta\,|01\rangle_{AB}\right) \\
&\quad - \sin\theta\cos\theta\,|0\rangle_\gamma\left(|01\rangle_{AB} + e^{i\varphi}\,|10\rangle_{AB}\right),
\end{aligned}
\tag{3.63}
$$

so that the probability distributions obtained after the markers D_1 and D_2 being analyzed are

$$
\begin{aligned}
p(\mathsf{D}_0) &= \mathrm{tr}_{AB\gamma}\left(\mathbb{1}_{AB}\otimes|0\rangle\langle 0|_\gamma\,|\psi_f\rangle\langle\psi_f|\right) = 2\sin^2\theta\cos^2\theta, \\
p(\mathsf{D}_1) &= \mathrm{tr}_{AB\gamma}\left(\mathbb{1}_{AB}\otimes|1\rangle\langle 1|_\gamma\,|\psi_f\rangle\langle\psi_f|\right) = \sin^4\theta + \cos^4\theta.
\end{aligned}
\tag{3.64}
$$

Analyzing Eq. (3.64), although the initial state in the path mode is the same as it was when (3.62) was obtained, it is clear that, when the path markers are considered, the relative phase φ has no influence in the final description. This situation occurs here due to the entanglement of the 3 degrees of freedom, generated in $|\psi_D\rangle$ in (3.63). Analogously to what was interpreted in the previous case, this situation can be interpreted as if the photon had a particle-like behavior, since the relative phase is inconsequential to the final probabilistic description.

As the reader may think, the two situations presented so far are very similar to the double-slit experiment. The first case, illustrated in Figure 3 (a) represents the standard experiment, with no detection of the electron in any slit. The conclusions of the experiment are analogous to the one presented in that case, i.e, wavelike behavior of the photon (or electron in the double-slit). When a marker is placed at the slit to determine which one the electron passed through, the situation is analogous as when the path markers were placed in the MZI to entangle the path of the photon, so that the final conclusion is the same: particle-like behavior for the photon (or electron). This brief example using MZI is very illustrative, since its calculations are quite easy and the physics behind them is deep.

CHAPTER 4

Correlation between Choices and Ontological Descriptions

This chapter is dedicated to introduce the quantum eraser experiment using the same approach applied in the previous chapter to introduce the MZI. Additionally, this chapter aims to discuss two fundamental papers for this work, which introduce the main ideas addressed in the present study. The purpose of this brief chapter, therefore, is to establish the foundations and the key questions to be analyzed in the next chapter, which constitutes the original contribution of this study. This chapter is organized as follows: Section 4.1 introduces the quantum eraser experiment, explaining its key developments, conclusions, and physics. Section 4.2 reviews two important papers on a modified version of the quantum eraser, presenting its main ideas, the theoretical and experimental development as well as the establishment of the phenomenon of local realism correlation.

4.1 Quantum Eraser

This section is dedicated to the study of a system that will be of fundamental importance for the next chapter, the quantum eraser [17, 78, 82, 83]. A very simple setup that works as a quantum eraser is illustrated in Figure 4. In this section, the well-known experimentalists, Alice and Bob, return to each of their designated laboratories. The setup of the quantum eraser is as follows. A pair of entangled photons is generated by a nonlinear crystal of beta barium borate (BBO) and each photon is sent to a different, spatially separated, laboratory, Alice's and Bob's one. The regions must be outside of the lightcone of each other, ensuring that there is no causal communication via light

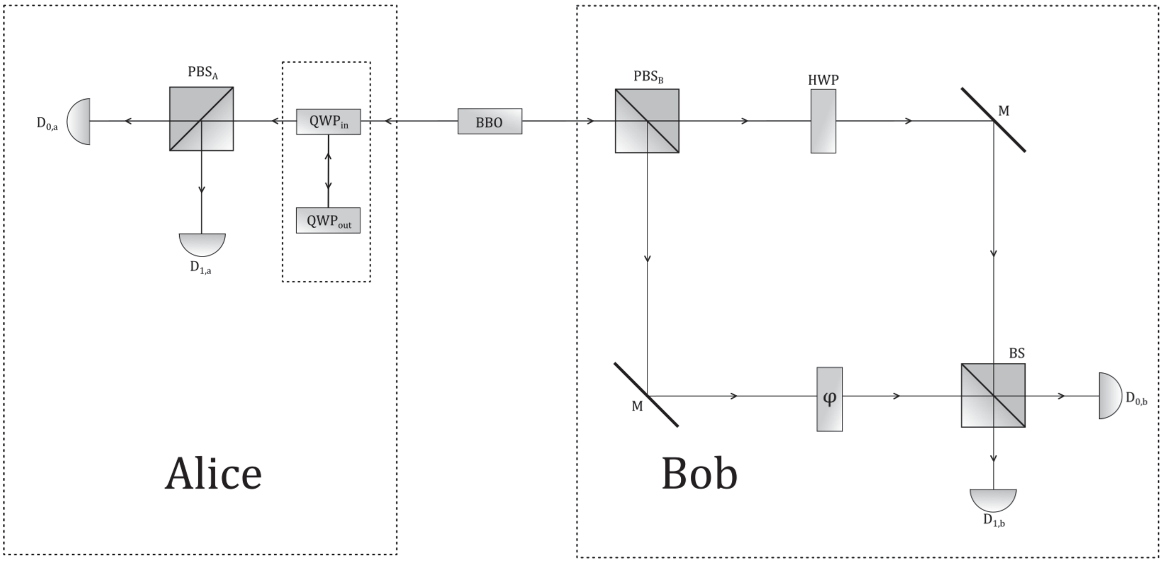

Figure 4 – A schema of an experimental optical setup used as a quantum eraser. The BBO represents the crystal of beta barium borate that generates two photons that travel to different laboratories, represented by Alice and Bob. In Alice's laboratory, she can choose whether or not to include quarter-wave plate (QWP), represented by $\mathrm{QWP_{in}}$ and $\mathrm{QWP_{out}}$, respectively. The photon then passes through a polarized beam-splitter ($\mathrm{PBS_A}$) before being detected on the detectors $\mathrm{D}_{i,\mathrm{a}}$, where $i \in \{0,1\}$. In Bob's laboratory, there is a modified Mach-Zehnder Interferometer (MZI). $\mathrm{PBS_B}$ represents the polarized beam-splitter, after which the photon passes through a half-wave plate (HWP), is reflected by mirrors (M), passes through a phase shift (φ), and then a beam-splitter (BS), until it is detected by detectors $\mathrm{D}_{i,\mathrm{b}}$, where $i \in \{0,1\}$.

signals between the laboratories. Bob's laboratory has a modified version of the MZI presented before. The main differences is that the first BS is changed to a Polarized Beam Splitter (PBS), that allows the transmission of the photon with polarization $|0\rangle$ and reflects the $|1\rangle$ polarization, and the inclusion of a Half Wave Plate (HWP), that changes the linear polarization of the photon. On the other hand, Alice's laboratory has at maximum two components, which are a Quarter Wave Plate (QWP), responsible for changing the linear polarization to circular polarization, that can be choosen whether it is considered in the experiment, and a PBS as well as Bob's laboratory. The whole point of the quantum eraser is that the path information of Bob's photon is erased depending on Alice's choice of putting the QWP.

In order to make the mathematical description of the quantum eraser, the initial state must be defined. Then, as the BBO produces a pair of entangled photons, let the initial state be written as

$$|\psi_0\rangle = \left(\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\right)_{AB} \otimes |00\rangle_{ab}\,. \tag{4.1}$$

The convention for the polarization and path modes established here will be used throughout this work. Uppercase letters are used for polarization Hilbert spaces, so that A and B represent the polarizations of Alice's and Bob's photons, respectively;

lowercase letters are used to specify the path modes of each photon, so that a and b represent the path modes of Alice's and Bob's photon, respectively.

Applying the unitary operators corresponding to Bob's laboratory, the evolution of the state in (4.1) is as follows

$$\begin{aligned} |\psi_0\rangle &\xrightarrow{\mathsf{PBS_B}} |\psi_{PBS_B}\rangle = \frac{1}{\sqrt{2}}(i\,|0101\rangle + |1000\rangle)_{ABab}, \\ |\psi_{PBS_B}\rangle &\xrightarrow{\mathsf{HWP}} |\psi_{HWP}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)_{Ab} \otimes |00\rangle_{Ba}\,, \\ |\psi_{HWP}\rangle &\xrightarrow{\mathsf{M's},\varphi} |\psi_{\varphi}\rangle = \frac{1}{\sqrt{2}}(i\,|11\rangle - e^{i\varphi}\,|00\rangle)_{Ab} \otimes |00\rangle_{Ba}\,, \\ |\psi_{\varphi}\rangle &\xrightarrow{\mathsf{BS}} |\psi_{Bob}\rangle = \frac{1}{2}\big[e^{i\varphi}\,|0\rangle \otimes (|0\rangle + i\,|1\rangle) + |1\rangle \otimes (|0\rangle - i\,|1\rangle)\big]_{Ab} \otimes |00\rangle_{Ba}\,. \end{aligned} \tag{4.2}$$

Concerning Alice's laboratory, she has two options to be performed, i.e, to include or not the QWP in the experiment. Firstly, suppose that she decides to not include the QWP, so that the final state that describes the situation is given by

$$|\psi_{Bob}\rangle \xrightarrow{\mathsf{PBS_A}} |\psi_{out}\rangle = \frac{1}{2}\big[e^{i\varphi}\,|000\rangle + ie^{i\varphi}\,|001\rangle + i\,|110\rangle + |111\rangle\,\big]_{Aab} \otimes |0\rangle_B\,. \tag{4.3}$$

On the other hand, if Alice chooses to include the QWP, with

$$\begin{aligned} U_{QWP}\,|0\rangle_A &= \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)_A, \\ U_{QWP}\,|1\rangle_A &= \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)_A, \end{aligned} \tag{4.4}$$

the final state would be

$$\begin{aligned} |\psi_{Bob}\rangle \xrightarrow{\mathsf{QWP,\,PBS_A}} |\psi_{in}\rangle = \frac{1}{\sqrt{2}}\Big[\cos\frac{\varphi}{2}\,|0000\rangle - i\sin\frac{\varphi}{2}\,|1100\rangle - \sin\frac{\varphi}{2}\,|0001\rangle \\ - i\cos\frac{\varphi}{2}\,|1101\rangle\,\Big]_{AaBb}. \end{aligned} \tag{4.5}$$

The central point of the quantum eraser is that the behavior of Bob's photon varies depending on Alice's choice to consider or not the QWP alongside post-selection. To illustrate this feature, consider the case without post-selection, i.e, let the probability distribution of Bob's laboratory be obtained without considering anything on Alice's laboratory. For such situation, both states $|\psi_{in}\rangle$ and $|\psi_{out}\rangle$ leads to

$$\begin{aligned} p(\mathsf{D_{0,b}}) &= \mathrm{tr}_{ABab}\big(\mathbb{1}_{ABa} \otimes |0\rangle\langle 0|_b\,|\psi_{in}\rangle\langle\psi_{in}|\big) = \mathrm{tr}_{ABab}\big(\mathbb{1}_{ABa} \otimes |0\rangle\langle 0|_b\,|\psi_{out}\rangle\langle\psi_{out}|\big) = \frac{1}{2}, \\ p(\mathsf{D_{1,b}}) &= \mathrm{tr}_{ABab}\big(\mathbb{1}_{ABa} \otimes |1\rangle\langle 1|_b\,|\psi_{in}\rangle\langle\psi_{in}|\big) = \mathrm{tr}_{ABab}\big(\mathbb{1}_{ABa} \otimes |1\rangle\langle 1|_b\,|\psi_{out}\rangle\langle\psi_{out}|\big) = \frac{1}{2}. \end{aligned} \tag{4.6}$$

Thus, there is no difference if post-selection is not considered.

Suppose now that post-selection is considered, i.e, the probability distribution obtained by Bob is conditioned to the results obtained by Alice. It means that the relevant probability distribution to the problem is written as $p(\mathsf{D}_{i,\mathsf{b}}|\mathsf{D}_{j,\mathsf{a}})$, and it reads as the probability of the detector $\mathsf{D}_{i,\mathsf{b}}$ click at Bob's laboratory given that the detector $\mathsf{D}_{j,\mathsf{a}}$ clicked at Alice's laboratory. Then, when Alice's detector clicks, the state collapses as given by (2.1),

$$\begin{aligned}
\big(\,|0\rangle\,\langle 0|\,\big)_a\,|\psi_{out}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)_b \otimes |000\rangle_{AaB}\,,\\
\big(\,|1\rangle\,\langle 1|\,\big)_a\,|\psi_{out}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)_b \otimes |110\rangle_{AaB}\,,\\
\big(\,|0\rangle\,\langle 0|\,\big)_a\,|\psi_{in}\rangle &= \Big(\cos\frac{\varphi}{2}\,|0\rangle - \sin\frac{\varphi}{2}\,|1\rangle\,\Big)_b \otimes |000\rangle_{AaB}\,,\\
\big(\,|1\rangle\,\langle 1|\,\big)_a\,|\psi_{in}\rangle &= \Big(\sin\frac{\varphi}{2}\,|0\rangle - \cos\frac{\varphi}{2}\,|1\rangle\,\Big)_b \otimes |110\rangle_{AaB}\,,
\end{aligned}$$

and the probability distribution obtained by Bob is given by (2.2), so that when the QWP is not considered, the state (4.3) is used, and

$$\begin{cases} p(\mathsf{D}_{0,\mathsf{b}}|\mathsf{D}_{0,\mathsf{a}}) = p(\mathsf{D}_{0,\mathsf{b}}|\mathsf{D}_{1,\mathsf{a}}) = 1/2,\\ p(\mathsf{D}_{1,\mathsf{b}}|\mathsf{D}_{0,\mathsf{a}}) = p(\mathsf{D}_{1,\mathsf{b}}|\mathsf{D}_{1,\mathsf{a}}) = 1/2, \end{cases} \tag{4.7}$$

and when QWP is considered, the state (4.5) is used, and

$$\begin{cases} p(\mathsf{D}_{0,\mathsf{b}}|\mathsf{D}_{0,\mathsf{a}}) = \cos^2\frac{\varphi}{2}, \quad p(\mathsf{D}_{0,\mathsf{b}}|\mathsf{D}_{1,\mathsf{a}}) = \sin^2\frac{\varphi}{2},\\ p(\mathsf{D}_{1,\mathsf{b}}|\mathsf{D}_{0,\mathsf{a}}) = \sin^2\frac{\varphi}{2}, \quad p(\mathsf{D}_{1,\mathsf{b}}|\mathsf{D}_{1,\mathsf{a}}) = \cos^2\frac{\varphi}{2}. \end{cases} \tag{4.8}$$

Thus, analyzing the above results, it is straightforward that when QWP is included alongside post-selection, the probability distribution obtained by Bob is that of a wavelike behavior, since the relative phase φ is relevant to the final description; on the other hand, when QWP is not considered, the probability distribution is that of a particle-like behavior, since the relative phase φ is inconsequential to the final description. The idea of the quantum eraser, therefore, is that the presence of the QWP, when properly analyzed, erases the path information of Bob's photon.

4.2 Modified Quantum Eraser and the Ontological Correlation

The setup for the quantum eraser was presented in the previous section, and it was discussed that depending on Alice's choice of putting the QWP in the experiment alongside post-selection, the path-information of Bob's photon could be erased. Exploring the idea of this phenomenon and going further on EPR's concept of local realism, the authors in [25] proposed a modified version of the quantum eraser, which is adapted and illustrated in Figure 5.

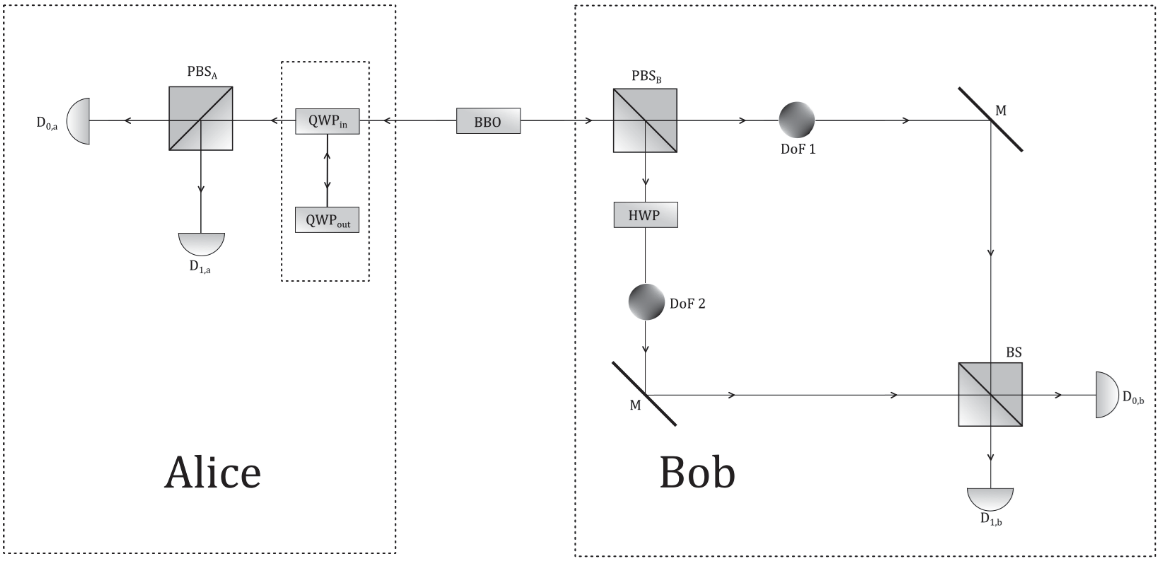

Figure 5 – A modified version of the quantum eraser experiment in Figure 4. As before, the BBO represents the crystal of beta barium borate that generates two photons that travel to laboratories of Alice and Bob. In Alice's laboratory, she can choose whether or not to include quarter-wave plate (QWP), represented by QWP$_{in}$ and QWP$_{out}$, respectively. The photon then passes through a polarized beam-splitter (PBS$_A$) before being detected on the detectors $D_{i,a}$, where $i \in \{0,1\}$. In Bob's laboratory, there is a modified version of the Mach-Zehnder Interferometer (MZI). PBS$_B$ represents the polarized beam-splitter, after which the photon passes through a half-wave plate (HWP). The crucial difference from the previous quantum eraser is the introduction of two degrees of freedom in Bob's laboratory, named DoF 1 and DoF 2, each one in a different arm of the interferometer. The photon is then reflected by mirrors (M) and then passes through a beam-splitter (BS), until it is detected by detectors $D_{i,b}$, where $i \in \{0,1\}$.

Comparing with the quantum eraser introduced in Figure 4, the difference remains in the presence of the degrees of freedom[1] DoF 1 and DoF 2. The idea of the present setup is the following. The nonlinear crystal of beta barium borate produces a pair of entangled photons that are sent to different laboratories, Alice and Bob. In Alice's laboratory, the procedure is the same as presented in the previous section, i.e., Alice can decide whether the QWP is considered in the experiment. On the other hand, different from the previous case that Bob analyzes the path information of the photon, now he is going to analyze the states of DoF 1 and DoF 2. The main difference in such approach, although adding complexity to the experimental setup, is that the final analysis is excluding the hypothesis of retro-inference, i.e., there will be no hypotheses on the past behavior of the photon by measuring the probability distributions at a later moment, as pointed out in [25]. However, the key point here is to utilize the entanglement between the degrees of freedom to certify the photon's wavelike behavior within the

[1] In the original paper, these are two atoms, A_1 and A_2. However, it is enough to consider instead two general degrees of freedom, mainly because of the practical implementation experimentally. Although it is possible to use stimulated emission as a non-destructive detector, as shown in [84], the practical use of general markers is preferable.

MZI.

In a very similar way done in the analysis of the usual quantum eraser, it is necessary to describe the Hilbert spaces for the photons' polarizations, named $\mathcal{H}_A$ and $\mathcal{H}_B$; their path modes, named $\mathcal{H}_a$ and $\mathcal{H}_b$; but it is also necessary to describe the degrees of freedom DoF 1 and DoF 2, and the associated Hilbert spaces are named $\mathcal{H}_{d_1}$ and $\mathcal{H}_{d_2}$, respectively. In order to achieve a higher degree of generality, the initial state is written as

$$|\psi_0\rangle = \Big(\cos\frac{\theta}{2}\,|01\rangle + \sin\frac{\theta}{2}\,|10\rangle\Big)_{AB} \otimes |0011\rangle_{abd_1d_2}\,, \tag{4.9}$$

with $\theta \in [0, \frac{\pi}{2}]$ controlling the initial entanglement of the polarizations. As well as mentioned in the case of the quantum eraser, the laboratories of Alice and Bob are said to be spatially separated, thus ensuring that they cannot causally influence each other during Alice's procedures.

Since the background is established, the evolution of the state (4.9) throughout Bob's laboratory is as follows.

$$\begin{aligned} |\psi_0\rangle &\xrightarrow{\text{PBS}_\text{B}} |\psi_{PBS_B}\rangle = \Big(i\cos\frac{\theta}{2}\,|011\rangle + \sin\frac{\theta}{2}\,|100\rangle\Big)_{ABb} |011\rangle_{ad_1d_2} \\ |\psi_{PBS_B}\rangle &\xrightarrow{\text{HWP}} |\psi_{HWP}\rangle = \Big(i\cos\frac{\theta}{2}\,|01\rangle + \sin\frac{\theta}{2}\,|10\rangle\Big)_{Ab} |0011\rangle_{Bad_1d_2}\,. \end{aligned} \tag{4.10}$$

Now, when the photons interact with the degrees of freedom, they mark the path, and then

$$\begin{aligned} |\psi_{HWP}\rangle &\xrightarrow{\text{DoF 1,2}} |\psi_d\rangle = \Big(i\cos\frac{\theta}{2}\,|0110\rangle + \sin\frac{\theta}{2}\,|1001\rangle\Big)_{Abd_1d_2} |00\rangle_{Ba} \\ |\psi_d\rangle &\xrightarrow{\text{M's, BS}} |\psi_{Bob}\rangle = \Big(\cos\frac{\theta}{2}\,|010\omega_+\rangle + \sin\frac{\theta}{2}\,|101\omega_-\rangle\Big)_{Ad_1d_2b} |00\rangle_{Ba}\,, \end{aligned} \tag{4.11}$$

with $|\omega_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i\,|1\rangle)$.

Concerning Alice's laboratory, firstly let the analysis be done in the case when the QWP is out of the experiment. In such case,

$$|\psi_{Bob}\rangle \xrightarrow{\text{PBS}_\text{A}} |\psi_{out}\rangle = \cos\frac{\theta}{2}\,|000\omega_+10\rangle_{AaBbd_1d_2} + i\sin\frac{\theta}{2}\,|110\omega_-01\rangle_{AaBbd_1d_2}\,. \tag{4.12}$$

Measurements of a and b post-selected on the detectors D_0 on the above state result in the state described by Bob as being $|\psi_{PSout}\rangle = |0010\rangle_{Bbd_1d_2}$. In this case, it is straightforward that the final description has an element of reality for the degrees of freedom, since their states are neither entangled nor in superposition. Hence, if $\Omega_{out}^{Bob} := |\psi_{PSout}\rangle\langle\psi_{PSout}|$, by BA-realism criteria in (2.11), it is true that $\Phi_{d_i}(\Omega_{out}^{Bob}) = \Omega_{out}^{Bob}$ for any $i \in \{1, 2\}$, ensuring a realistic description of the degrees of freedom, so that

$$\mathfrak{I}_{d_i}(\Omega_{out}^{Bob}) = 0 \tag{4.13}$$

for any $i \in \{1, 2\}$.

Analyzing now the case when QWP is considered in the experiment, defining

$$|\zeta_{\pm}\rangle_{d_1 d_2} = \sin\frac{\theta}{2}\,|01\rangle_{d_1 d_2} \pm \cos\frac{\theta}{2}\,|10\rangle_{d_1 d_2}\,, \tag{4.14}$$

the evolution in Alice's laboratory is such that

$$\begin{aligned} |\psi_{Bob}\rangle &\xrightarrow{\text{QWP}_\text{A}} |\psi_{QWP}\rangle = \frac{1}{2}\Big(\,|0000\zeta_+\rangle - i\,|0001\zeta_-\rangle - i\,|1000\zeta_-\rangle - |1001\zeta_+\rangle\,\Big)_{AaBbd_1d_2} \\ |\psi_{QWP}\rangle &\xrightarrow{\text{PBS}_\text{A}} |\psi_{in}\rangle = \frac{1}{2}\Big(\,|0000\zeta_+\rangle - i\,|0001\zeta_-\rangle + i\,|1100\zeta_-\rangle - i\,|1101\zeta_+\rangle\,\Big)_{AaBbd_1d_2}. \end{aligned} \tag{4.15}$$

As the previous case, measurements of a and b post-selected on the detectors D_0 in the state $|\psi_{in}\rangle$ lead to the final description obtained by Bob, given by $|\psi_{PSin}\rangle = |00\rangle_{Bb} \otimes |\zeta_+\rangle_{d_1 d_2}$. Hence, since $|\zeta_+\rangle$ is defined in (4.14), the state that describe the degrees of freedom are entangled in general, so that there is no element of reality for them. Indeed, defining $\Omega_{in}^{Bob} = |\psi_{PSin}\rangle\langle\psi_{PSin}|$ and applying the definition of irreality in (3.34),

$$\mathfrak{I}_{d_i}(\Omega_{in}^{Bob}) = -\cos^2\frac{\theta}{2}\log\Big(\cos^2\frac{\theta}{2}\Big) - \sin^2\frac{\theta}{2}\log\Big(\sin^2\frac{\theta}{2}\Big), \tag{4.16}$$

for any $i \in \{1, 2\}$, which is numerically *equal to the amount of entanglement of the initial state*[2], when analyzed by the von Neumann entropy of the reduced state.

Before finishing the theoretical discussion, it is worth noting that the authors in [25] proved that there is no difference in the results if Alice's unitary operations are made before the interaction with DoF 1 and DoF 2, and the time of Alice's actions plays no relevant role in the final conclusions. Furthermore, they were able to simulate these results in a quantum computer, corroborating their analytical results. Then, as shown by the calculations of this section, Alice's actions were sufficient to change the realism in the degrees of freedom of Bob's laboratory, and since their setups are outside of the light cone of each other, it is fair to understand these results as being against the notion of local realism, as introduced by EPR.

Following the theoretical discussion and motivated by the physics behind the correlation established by the previous analysis, the authors in [26] were able to perform an experiment to test the previous discussion. The experimental setup is very similar to that in Figure 5. In such implementation, the degrees of freedom are managed by two beam displacers, introducing two qubit systems into the description as needed. Different from the first simulation of the problem, this experiment gives a stronger argument against locality, since the laboratories of Alice and Bob are guaranteed space-like separated.

[2] Indeed, this observation led to the hypothesis that entanglement is the underlying cause of the phenomenon.

A slight modification of the experiment leads to a difference on the entanglement observed in the system. In order to set a real experiment, the analysis was made so that the final entanglement that (in certain way) destroys the elements of reality is tripartite, different from the theoretical analysis given above that was a bipartite entanglement. Since the irreality is of major importance in such analysis, and it is the same as observed before, the foundations of the conclusions remain the same. The conclusions of the experiment corroborate the theoretical analysis, so that the correlation between Alice's choices and the ontological description in Bob's laboratory was indeed observed. This result is of great importance for the analysis of the present work, and this paper provides "strong evidence against local realism," quoting the authors.

The next step in the theoretical analysis, from the perspective of quantum information theory and quantum resources theory, is to investigate whether a quantum resource underlies this phenomenon. Naturally, since the final irrealism that establishes the occurrence of the phenomenon is equal to the initial entanglement, the direct hypothesis is that entanglement is the quantum resource responsible for it. Therefore, this work is devoted to investigating this possibility and its implications, which are presented in detail in the next chapter.

CHAPTER 5

Investigation of the Quantum Resource for Local Realism Erasure

This chapter represents the original contribution and the core of this study. In this chapter, states with a higher degree of generality in the modified quantum eraser experiment will be analyzed to investigate the quantum resources essential for the phenomenon described in the previous chapter. Section 5.1 examines the occurrence of the phenomenon in the context of post-selection choices, introducing a quantity that aims to properly encapsulate the effect of post-selection. In Section 5.2, the Werner state will be examined to introduce a fully symmetric state, enhancing the generality of the analysis. This section will also delve into the dependencies of irreality on entanglement and quantum discord. Section 5.3 focuses on states parameterized by the Bloch sphere, aiming to uncover the true roles of quantum discord, and realism-based nonlocality in the erasure of local realism. Finally, Section 5.4 presents final remarks related to the problem, offering further insights into the findings.

5.1 The occurrence of the phenomenon

As presented in Chapter 4, the phenomenon analyzed in this work is the influence of the QWP on the final descriptions of irreality for the degrees of freedom DoF 1 and 2 in Bob's laboratory. This section aims to define a criterion for identifying the occurrence of this phenomenon. The motivation for this discussion lies in the impact of post-selection on the final description.

The approach developed in [25], reviewed in Section 4.2, along with the discussion of the quantum eraser in Section 4.1, highlights the fundamental role of post-selection in shaping the final description of the problem. In particular, when Eqs. (4.13) and (4.16) were derived, a specific post-selection was considered. Although in that case the results were independent of the chosen post-selection, it is not trivial that this holds for any class of states. Therefore, the goal of this discussion is to establish a criterion that accounts for all possible post-selection choices, ensuring that the analysis does not rely on a particular one.

To establish an analysis that avoids interpretative loopholes based on the choice of post-selection, it is necessary to define a quantity that is both sensitive to the presence of the phenomenon and non-selective, meaning it does not favor any particular post-selection. Since Bob's description depends on the post-selections determined by his and Alice's data, it is natural for this quantity to incorporate both Alice's and Bob's choices, weighted by the probabilities associated with each possible post-selection.

Thus, considering the aforementioned points and selecting post-selections of the type $\{\mathsf{D}_{k,\mathsf{a}}, \mathsf{D}_{l,\mathsf{b}}\}$, the reality description made by Bob becomes conditioned on each post-selection, which is induced by the following state:

$$\begin{aligned}\Omega^{Bob|kl}_{in/out} &= \mathrm{tr}_{Aa}\big(\rho_{in/out|kl}\big)\\ &= \mathrm{tr}_{Aa}\left(\frac{\big(M^k_a \otimes M^l_b \otimes \mathbb{1}_{ABd_1d_2}\big)\rho_{in/out}\big(M^k_a \otimes M^l_b \otimes \mathbb{1}_{ABd_1d_2}\big)}{p^{in/out}_{kl}}\right),\end{aligned} \tag{5.1}$$

where $p^{in/out}_{kl} = \mathrm{tr}\left[(M^k_a \otimes M^l_b \otimes \mathbb{1}_{ABd_1d_2})\rho_{in/out}\right]$, and $\rho_{in/out}$ are the final general descriptions considering and not considering the QWP in the system, respectively. Therefore, Eq. (5.1) refers to the Bob state conditioned on the post-selection $\{\mathsf{D}_{k,\mathsf{a}}, \mathsf{D}_{l,\mathsf{b}}\}$. The occurrence of the phenomenon is established when the presence of the QWP in the system changes the final irreality for the degrees of freedom d_1 and d_2 for Bob's description. Thus, we propose the following quantity

$$\delta\mathfrak{I} := |\mathfrak{I}_{in} - \mathfrak{I}_{out}|, \tag{5.2}$$

where $\mathfrak{I}_{in/out}$ represents the irreality for the cases $\mathrm{QWP}_{\mathrm{in/out}}$, and to take into account all post-selections in each of the possible configurations, these terms are defined as

$$\mathfrak{I}_{in/out} := \sum_{kl} p^{in/out}_{kl}\mathfrak{I}_{d_j}(\Omega^{Bob|kl}_{in/out}). \tag{5.3}$$

This definition relates the quantities $\mathfrak{I}_{in/out}$ to irrealities conditioned on the possible post-selections, or even to the average value of the final irreality considering all the possible post-selections. Thus, Eq. (5.2) can be written as

$$\delta\mathfrak{I} := \Big|\sum_{kl}\Big(p^{in}_{kl}\mathfrak{I}_{d_j}\big(\Omega^{Bob|kl}_{in}\big) - p^{out}_{kl}\mathfrak{I}_{d_j}\big(\Omega^{Bob|kl}_{out}\big)\Big)\Big|. \tag{5.4}$$

Eq. (5.4) can be understood as the absolute value of the difference between the average values of irreality with and without the QWP, based on the post-selections at $\{\mathsf{D}_{k,\mathsf{a}}, \mathsf{D}_{l,\mathsf{b}}\}$ for $k, l \in \{0,1\}$. Therefore, $\delta\mathfrak{I}$ quantifies how different the averages of such descriptions are. Based on this, and assuming that averaging inherently removes the hypothesis of a privileged post-selection, a nonzero difference in the averages serves as a strong indication that the presence of the QWP indeed influenced the final description, thereby establishing the occurrence of the phenomenon.

A noteworthy remark regarding the motivation that underlies the definition in Eq. (5.3) is its connection to the non-signaling principle[1]. A crucial point in the description of Eq. (5.3) is that it takes into account revealed post-selections, i.e., the post-selected state is determined by the chosen detectors, meaning that both choices must be known. In the case of non-revealed post-selections, the problem reduces to the use of the maps Φ, so that Eq. (5.1) would return only the description of an average ensemble of the form $\sum_{kl} p_{kl}^{in/out} \Omega_{in/out}^{Bob|kl} = \mathrm{tr}_{Aa}(\rho_{in/out})$. Since the trace over Alice's spaces is unable to alter Bob's description, due to the non-signaling principle, these states cannot exhibit the phenomenon. Therefore, the approach taken in this work is to include all possible post-selections in an unbiased manner, as proposed in Eq. (5.3).

The following sections explore different classes of states within the context of the modified quantum eraser, and the analysis of the occurrence of the phenomenon will be made by using the concept of $\delta\mathfrak{I}$.

5.2 Werner State in the Modified Quantum Eraser

This section generalizes the results of the previous chapter by using a Werner state for the initial state of the photon's polarizations. In such approach, the initial state is given by

$$\rho_0 = \left(\frac{1-\eta}{4}\mathbb{1} + \eta\,|\psi_\theta\rangle\,\langle\psi_\theta|\right)_{AB} \otimes |0011\rangle\,\langle 0011|_{abd_1d_2}\,, \tag{5.5}$$

with $|\psi_\theta\rangle = \cos\frac{\theta}{2}|01\rangle + \sin\frac{\theta}{2}|10\rangle$. The only difference compared to (4.9) lies in the polarization Hilbert spaces. The parameter η controls the mixing of the previously considered pure state with the identity state. Specifically, when $\eta = 1$, we recover the analysis presented in [25], while for $\eta = 0$ the initial state of polarization is the identity itself. The above state is going to be submitted to the previous setup, i.e., that introduced in Figure 5. The role of each component was also discussed in the previous chapters, so that the evolution of the state in (5.5) throughout Bob's laboratory can be now analyzed.

[1] This principle states that information cannot be transmitted instantaneously between spatially separated systems, i.e., systems that are spacelike separated.

Firstly, after the PBS, the state changes to

$$\rho_{PBS_B} = \Big[\frac{1-\eta}{4}\Big(\mathbb{1}_A \otimes |00\rangle\langle 00|_{Bb} + \mathbb{1}_B \otimes |11\rangle\langle 11|_{Ab}\Big) + \eta \left|\psi_\theta^{PBS}\right\rangle\left\langle\psi_\theta^{PBS}\right|_{ABb}\Big] \otimes |011\rangle\langle 011|_{ad_1d_2}, \tag{5.6}$$

with $\left|\psi_\theta^{PBS}\right\rangle_{ABb} = i\cos\frac{\theta}{2}|011\rangle_{ABb} + \sin\frac{\theta}{2}|100\rangle_{ABb}$. The HWP is responsible to factorize the Hilbert space related to Bob's polarization, so that

$$\rho_{HWP_B} = \Big(\frac{1-\eta}{4}\mathbb{1} + \eta\left|\psi_\theta^{HWP}\right\rangle\left\langle\psi_\theta^{HWP}\right|\Big)_{Ab} \otimes |0011\rangle\langle 0011|_{Bad_1d_2}, \tag{5.7}$$

with $\left|\psi_\theta^{HWP}\right\rangle_{Ab} = i\cos\frac{\theta}{2}|01\rangle_{Ab} + \sin\frac{\theta}{2}|10\rangle_{Ab}$. The above state then interacts with the degrees of freedom DoF 1 and 2, and the state changes to

$$\rho_d = \Big[\frac{1-\eta}{4}\mathbb{1} \otimes \big(|001\rangle\langle 001| + |110\rangle\langle 110|\big) + \eta\left|\psi_\theta^{d}\right\rangle\left\langle\psi_\theta^{d}\right|\Big]_{Abd_1d_2} \otimes |00\rangle\langle 00|_{Ba}, \tag{5.8}$$

with $\left|\psi_\theta^{d}\right\rangle_{Abd_1d_2} = i\cos\frac{\theta}{2}|0110\rangle_{Abd_1d_2} + \sin\frac{\theta}{2}|1001\rangle_{Abd_1d_2}$. Such interaction leads to an entanglement between the polarization of Alice's photon, the path mode of Bob's photon and the degrees of freedom that mark the path. In principle, the state $\left|\psi_\theta^{d}\right\rangle$ has a genuinely multipartite entanglement, although the global state in (5.8) is not properly a Werner state, since the mixture is not made with the identity state. Finishing the developent of the state in Bob's laboratory, the final state after the mirrors and the last BS is given by

$$\rho_{Bob} = \Big[\frac{1-\eta}{4}\mathbb{1} \otimes \big(|\omega_-01\rangle\langle\omega_-01| + |\omega_+10\rangle\langle\omega_+10|\big) + \eta\left|\psi_\theta^{Bob}\right\rangle\left\langle\psi_\theta^{Bob}\right|\Big]_{Abd_1d_2} \otimes |00\rangle\langle 00|_{Ba}, \tag{5.9}$$

with $|\omega_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$ and $\left|\psi_\theta^{Bob}\right\rangle_{Abd_1d_2} = \cos\frac{\theta}{2}|0\omega_+10\rangle_{Abd_1d_2} + \sin\frac{\theta}{2}|1\omega_-01\rangle_{Abd_1d_2}$.

Concerning Alice's choice of whether the QWP is considered or not in the experiment, her choices are going to be analyzed as it was made in the previous chapter. Considering first the case when the QWP is not considered, then the state in (5.9) is submitted to a PBS in Alice's site, so that the final state is given by

$$\rho_{out} = \Big[\frac{1-\eta}{4}\big(|00\omega_-01\rangle\langle 00\omega_-01| + |00\omega_+10\rangle\langle 00\omega_+10|\big) + \eta\left|\psi_\theta^{out}\right\rangle\left\langle\psi_\theta^{out}\right|\Big]_{Aabd_1d_2} \otimes |0\rangle\langle 0|_B, \tag{5.10}$$

with $|\psi_\theta^{out}\rangle_{Aabd_1d_2} = \cos\frac{\theta}{2}|00\omega_+10\rangle_{Aabd_1d_2} + i\sin\frac{\theta}{2}|11\omega_-01\rangle_{Aabd_1d_2}$. After post-selecting the above state in the detectors corresponding to $|k\rangle_a$ and $|l\rangle_b$, the state is

$$\begin{aligned}\rho_{out}(D_{k,a}, D_{l,b}) &= \frac{\big(M_a^k \otimes M_b^l \otimes \mathbb{1}_{ABd_1d_2}\big)\rho_{out}\big(M_a^k \otimes M_b^l \otimes \mathbb{1}_{ABd_1d_2}\big)}{\mathrm{tr}\big(\big(M_a^k \otimes M_b^l \otimes \mathbb{1}_{ABd_1d_2}\big)\rho_{out}\big)} \\ &= \left(\frac{\frac{1-\eta}{4}\big(|01\rangle\langle 01| + |10\rangle\langle 10|\big) + \eta\cos^2\frac{\theta}{2}|10\rangle\langle 10|}{\frac{1-\eta}{2} + \eta\cos^2\frac{\theta}{2}}\right)_{d_1d_2} \otimes |0k0l\rangle\langle 0k0l|_{AaBb},\end{aligned}$$

for any $k,l\in\{0,1\}$, so that the final description of Bob, $\Omega_{out}^{Bob|kl}\equiv\Omega_{out}^{Bob}=\mathrm{tr}_{Aa}\big(\rho_{out}(D_{k,a},D_{l,b})\big)$, is given by

$$\Omega_{out}^{Bob}=|0l\rangle\langle 0l|_{Bb}\otimes\left(\frac{\frac{1-\eta}{4}\big(|01\rangle\langle 01|+|10\rangle\langle 10|\big)+\eta\cos^2\frac{\theta}{2}|10\rangle\langle 10|}{\frac{1-\eta}{2}+\eta\cos^2\frac{\theta}{2}}\right)_{d_1d_2}, \tag{5.11}$$

that is, it does not depend on post-selection. The above equation is clearly a generalization of the state $|\psi_{PSout}\rangle$ mentioned in the previous chapter. However, it is worth noting that (5.11) is a diagonal state in the computational basis, and therefore a realistic state for any d by BA-realism criterion. Thus, formally speaking,

$$\Phi_{d_i}(\Omega_{out}^{Bob})=\Omega_{out}^{Bob}\Rightarrow\mathfrak{I}_{d_i}(\Omega_{out}^{Bob})=0, \tag{5.12}$$

for any $i\in\{1,2\}$, and it is the same conclusion obtained previously.

Considering now the case in which the QWP is placed in the experiment, the state in (5.9) is submitted to the QWP, so that the state changes to

$$\begin{aligned}\rho_{QWP}=\frac{1}{4}\Big[&(1-\eta)\big(|0\omega_-01\rangle\langle 0\omega_-01|+|0\omega_+10\rangle\langle 0\omega_+10|+|1\omega_-01\rangle\langle 0\omega_-01|+\\ &|1\omega_+10\rangle\langle 1\omega_+10|\big)+\eta|\psi_\theta^\zeta\rangle\langle\psi_\theta^\zeta|\Big]_{Abd_1d_2}\otimes|00\rangle\langle 00|_{aB},\end{aligned} \tag{5.13}$$

with $|\psi_\theta^\zeta\rangle_{Abd_1d_2}=|00\rangle_{Ab}|\zeta_+\rangle_{d_1d_2}-i\,|01\rangle_{Ab}|\zeta_-\rangle_{d_1d_2}-i\,|10\rangle_{Ab}|\zeta_-\rangle_{d_1d_2}-|11\rangle_{Ab}|\zeta_+\rangle_{d_1d_2}$. Then, after the PBS in Alice's site, the final state is given by

$$\begin{aligned}\rho_{in}=\frac{1}{4}\Big[&(1-\eta)\big(|00\omega_-01\rangle\langle 00\omega_-01|+|00\omega_+10\rangle\langle 00\omega_+10|+|11\omega_-01\rangle\langle 11\omega_-01|+\\ &|11\omega_+10\rangle\langle 11\omega_+10|\big)+\eta\,|\psi_\theta^{in}\rangle\langle\psi_\theta^{in}|\Big]_{Aabd_1d_2}\otimes|0\rangle\langle 0|_B,\end{aligned} \tag{5.14}$$

with $|\psi_\theta^{in}\rangle_{Aabd_1d_2}=|000\rangle_{Aab}|\zeta_+\rangle_{d_1d_2}-i\,|110\rangle_{Aab}|\zeta_-\rangle_{d_1d_2}+|110\rangle_{Aab}|\zeta_-\rangle_{d_1d_2}-i\,|111\rangle_{Aab}|\zeta_+\rangle_{d_1d_2}$. By following the same procedure as before, post-selecting on the detectors corresponding to $|k\rangle_a$ and $|l\rangle_b$, the state becomes

$$\begin{aligned}\rho_{in}(D_{k,a},D_{l,b})&=\frac{\big(M_a^k\otimes M_b^l\otimes\mathbb{1}_{ABd_1d_2}\big)\rho_{in}\big(M_a^k\otimes M_b^l\otimes\mathbb{1}_{ABd_1d_2}\big)}{\mathrm{tr}\big(\big(M_a^k\otimes M_b^l\otimes\mathbb{1}_{ABd_1d_2}\big)\rho_{out}\big)}\\ &=\frac{1}{2}\Big[(1-\eta)\big(|01\rangle\langle 01|+|10\rangle\langle 10|\big)+2\eta\,|\zeta_+\rangle\langle\zeta_-|\Big]_{d_1d_2}\otimes|0k0l\rangle\langle 0k0l|_{AaBb},\end{aligned}$$

for any $k,l\in\{0,1\}$, and the final description of Bob, $\Omega_{in}^{Bob|kl}\equiv\Omega_{in}^{Bob}=\mathrm{tr}_{Aa}\big(\rho_{in}(D_{k,a},D_{l,b})\big)$, is

$$\Omega_{in}^{Bob}=|00\rangle\langle 00|_{Bb}\otimes\Big(\frac{1-\eta}{2}\big(|01\rangle\langle 01|+|10\rangle\langle 10|\big)+\eta\,|\zeta_+\rangle\langle\zeta_+|\Big)_{d_1d_2}. \tag{5.15}$$

Exactly as before, the above state is the generalization of the state $|\psi_{PSin}\rangle$ mentioned in the previous chapter. In what follows, it is worth noting that $|\zeta_+\rangle\langle\zeta_+|$ is not a diagonal state in the computational basis, so there are off-diagonal elements, or coherence. For such case, as BA-realism is based on the complete positive trace preserving (CPTP) map Φ, that destroys the off-diagonal elements, it is straightforward that Ω_{in}^{Bob} is not realistic for the observables associated to the degrees of freedom DoF 1 and 2. Thus, $\Phi_{d_i}(\Omega_{in}^{Bob}) \neq \Omega_{in}^{Bob}$, and $\mathfrak{I}_{d_i}(\Omega_{in}^{Bob}) \geq 0$ for any $i \in \{1, 2\}$.

On the irreality of the state (5.15), recovering the concept of the binary entropy in (3.3), $\mathfrak{I}_{d_i}(\Omega_{in}^{Bob})$ is written as

$$\mathfrak{I}_{d_i}(\Omega_{in}^{Bob}) = h\left(\frac{1-\eta}{2} + \eta \sin^2\frac{\theta}{2}\right) - h\Big(\frac{1-\eta}{2}\Big). \tag{5.16}$$

If the previous limit is considered, i.e., when $\eta = 1$ (Figure 5, the above equation reduces to the same result given in (4.16), as expected for a generalization. Another important comment on Eq. (5.16) is that, as discussed in Chapter 3, the binary entropy is a concave function, and its functional form is very well-known. By these premises, it is possible to state that the $\mathfrak{I}_{d_i}(\Omega_{in}^{Bob}) = 0$ only when $\eta \sin^2(\theta/2) = 0$, which means that the irreality is zero if $\eta = 0$, $\theta = 0$, or both[2], corroborating the discussion of [25].

Before finishing this discussion, it is worth noting the strength of the argument of the influence of post-selection. In the above discussion, concerning the Werner states, it is clear that the results in Eqs. (5.12) and (5.16) are independent of the post-selection chosen. Therefore, from the perspective of the quantity introduced in Eq. (5.4), the presence of the phenomenon is indicated by

$$\begin{aligned} \delta\mathfrak{I} &= \Big|\sum_{kl}\Big(p_{kl}^{in}\mathfrak{I}_{d_i}\big(\Omega_{in}^{Bob|kl}\big) - p_{kl}^{out}\mathfrak{I}_{d_i}\big(\Omega_{out}^{Bob|kl}\big)\Big)\Big| \\ &= \Big|\sum_{kl}\Big(p_{kl}^{in}\mathfrak{I}_{d_i}\big(\Omega_{in}^{Bob}\big) - p_{kl}^{out}\mathfrak{I}_{d_i}\big(\Omega_{out}^{Bob}\big)\Big)\Big| \\ &= \Big|\Big(\mathfrak{I}_{d_i}\big(\Omega_{in}^{Bob}\big) - \mathfrak{I}_{d_i}\big(\Omega_{out}^{Bob}\big)\Big)\sum_{kl}p_{kl}^{in}\Big| \\ &= \Big|\mathfrak{I}_{d_i}\big(\Omega_{in}^{Bob}\big) - \mathfrak{I}_{d_i}\big(\Omega_{out}^{Bob}\big)\Big|. \end{aligned} \tag{5.17}$$

This result would immediately represent the trivial way of quantifying the phenomenon. However, from the construction proposed by Eqs. (5.2) and (5.3), it is clear that this quantity is only consistent with the unbiased description of the problem when the results do not effectively depend on post-selection. Moreover, from Eq. (5.12),

$$\delta\mathfrak{I} = \mathfrak{I}_{d_i}\big(\Omega_{in}^{Bob}\big). \tag{5.18}$$

Therefore, indeed, the phenomenon occurs only when the final irreality for the QWP_{in} configuration is greater than zero. In what follows, the discussion on the role of entanglement and quantum discord for the occurrence of the phenomenon of local irreality erasure will be presented.

[2] Note that none of these cases are entangled. When $\eta = 0$, the polarization state becomes the identity state, and when $\theta = 0$, the identity state is mixed with a single ket rather than a superposition.

5.2.1 The role of Entanglement

The goal of this subsection is to investigate the entanglement of the initial state, and then to study whether it is necessary or not for the phenomenon occurring. In subsection 3.2.2 it was discussed an example of calculating the entanglement for the Werner state in (3.43). In this section, the concept of concurrence is recovered so that the entanglement of (5.5) can be calculated as a function of η and θ, generalizing the previous results.

Starting from the state (5.5), it is straightforward that the interesting correlation is within the state of polarizations, since the state $|0011\rangle_{abd_1d_2}$ is separable. Thus, let the polarization state be considered by its own right as being

$$\rho_{AB} = \frac{1-\eta}{4}\mathbb{1} + \eta\,|\psi_\theta\rangle\,\langle\psi_\theta|\,, \tag{5.19}$$

with $\mathbb{1}$ being the identity operator in the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $|\psi_\theta\rangle = \cos\frac{\theta}{2}\,|01\rangle + \sin\frac{\theta}{2}\,|10\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ the state previously defined. Using the Eq. (3.41), the state $\tilde{\rho}_{AB}$ is written as

$$\tilde{\rho}_{AB} = \frac{1-\eta}{4}\mathbb{1} + \eta|\tilde{\psi_\theta}\rangle\langle\tilde{\psi_\theta}|, \tag{5.20}$$

with $|\tilde{\psi_\theta}\rangle = \sin\frac{\theta}{2}\,|01\rangle + \cos\frac{\theta}{2}\,|10\rangle$. Therefore, the state $\rho_{AB}\tilde{\rho}_{AB}$ is explicitly written in the bipartite computational basis as

$$\begin{aligned}\rho_{AB}\tilde{\rho}_{AB} =&\Big(\frac{1-\eta}{4}\Big)^2\big(\,|00\rangle\,\langle 00| + |11\rangle\,\langle 11|\,\big)\\ &+\left[\Big(\frac{1-\eta}{4}\Big)^2 + \eta\Big(\frac{1-\eta}{4}\Big) + 2\eta^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right]\big(\,|01\rangle\,\langle 01| + |10\rangle\,\langle 10|\,\big)\\ &+\eta\sin\theta\left[\Big(\frac{1-\eta}{4} + \eta\sin^2\frac{\theta}{2}\Big)\,|10\rangle\,\langle 01| + \Big(\frac{1-\eta}{4} + \eta\cos^2\frac{\theta}{2}\Big)\,|01\rangle\,\langle 10|\,\right].\end{aligned} \tag{5.21}$$

For the eigenvalues of the state in (5.21), it is worth noting that the diagonalization is reduced to a 2×2 matrix, since $|00\rangle$ and $|11\rangle$ are both eigenvectors with the same eigenvalue $\lambda_0 = \left(\frac{1-\eta}{4}\right)^2$. Thus, it suffices to diagonalize the matrix representation of the subspace spanned by the kets $|01\rangle$ and $|10\rangle$. Then, by defining the following functions

$$\begin{aligned} f(\eta,\theta) &= \Big(\frac{1-\eta}{4}\Big)^2 + \eta\Big(\frac{1-\eta}{4}\Big) + 2g^2(\eta,\theta),\\ g(\eta,\theta) &= \frac{1}{2}\eta\sin\theta,\end{aligned} \tag{5.22}$$

the eigenvalues for the subspace mentioned are written as

$$\lambda_\pm(\eta,\theta) = f(\eta,\theta) \pm 2|g(\eta,\theta)|\sqrt{f(\eta,\theta) - g^2(\eta,\theta)}. \tag{5.23}$$

In order to construct the concurrence function, it is necessary to organize the square roots of the eigenvalues in decreasing order. Starting from Eq. (5.23), it is straightforward to check that $\lambda_+ \geq \lambda_-$, so that $\sqrt{\lambda_+} \geq \sqrt{\lambda_-}$, since both are nonnegative. Beyond that, as λ_+ can be written as λ_0 increased by a non-negative term, it is true that $\lambda_+ \geq \lambda_0$, so that $\sqrt{\lambda_+} \geq \sqrt{\lambda_0}$, and then the entanglement can be written as

$$\begin{aligned}\mathcal{E}(\rho_{AB}) &= \max\{0, \sqrt{\lambda_+} - \sqrt{\lambda_-} - 2\sqrt{\lambda_0}\} \\ &= \max\Big\{0, \sqrt{f(\eta,\theta) + 2|g(\eta,\theta)|\sqrt{f(\eta,\theta) - g^2(\eta,\theta)}} \\ &\qquad - \sqrt{f(\eta,\theta) - 2|g(\eta,\theta)|\sqrt{f(\eta,\theta) - g^2(\eta,\theta)}} - \frac{1-\eta}{2}\Big\}.\end{aligned} \tag{5.24}$$

The above result is the generalization of the Eq. (3.45). Indeed, some important cases to analyze the validity of (5.24) are the following.

- Pure state: $\eta = 1 \Rightarrow \mathcal{E}(\rho_{AB}) = |\sin\theta|$.
- Singlet state: $\eta = 1, \sin\frac{\theta}{2} = \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \mathcal{E}(\rho_{AB}) = 1$.
- Werner singlet state: $\sin\frac{\theta}{2} = \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \mathcal{E}(\rho_{AB}) = \frac{1}{2}\max\{0, 3\eta - 1\}$.

As shown in the above result and Eq. (3.45), even when the the most entangled state is mixed with the identity state there exist an interval of values of η such that the mixed state is not entangled anymore. However, in order to make the point for the discussion, it is important to study the real dependence of this interval with the parameters η and θ. For such discussion, it suffices to analyze the function in (5.24), and

$$\sqrt{\lambda_+} - \sqrt{\lambda_-} \geq 2\sqrt{\lambda_0} \Rightarrow \mathcal{E}(\rho_{AB}) \geq 0,$$

and by substituting (5.22) and (5.23) and manipulating the inequality, it is possible to write that

$$4g^2(\eta,\theta) \geq \left(\frac{1-\eta}{2}\right)^2 \Rightarrow (4\sin^2\theta - 1)\eta^2 + 2\eta - 1 \geq 0 \Rightarrow \mathcal{E}(\rho_{AB}) \geq 0. \tag{5.25}$$

Solving for η, the solutions are given by $\eta_\pm = \frac{\pm 2|\sin\theta| - 1}{4\sin^2\theta - 1}$. However, in order to satisfy the inequality, and by using that $\theta \in [0, \frac{\pi}{2}]$, the only solution with physical meaning is η_+, so that

$$\eta(\theta) \geq \frac{2\sin\theta - 1}{4\sin^2\theta - 1} \Rightarrow \mathcal{E}(\rho_{AB}) \geq 0. \tag{5.26}$$

Figure 6 provides an illustration of both irreality, given by (5.16), and entanglement, given by (5.24). The Figure 6 clearly shows that $\mathfrak{I}_{d_i}(\Omega_{in}^{Bob})$ is zero only over the axes, while $\mathcal{E}(\rho_{AB})$ has an area over the $\eta \times \theta$ plane delimitated by the equality in (5.26) where the entanglement is identically zero. This analysis leads to the conclusion that, for a state given by (5.19), even if the entanglement is zero, the irreality is not necessarily zero. In other words, this generalization shows that the entanglement is not a necessary quantum resource to establish the phenomenon.

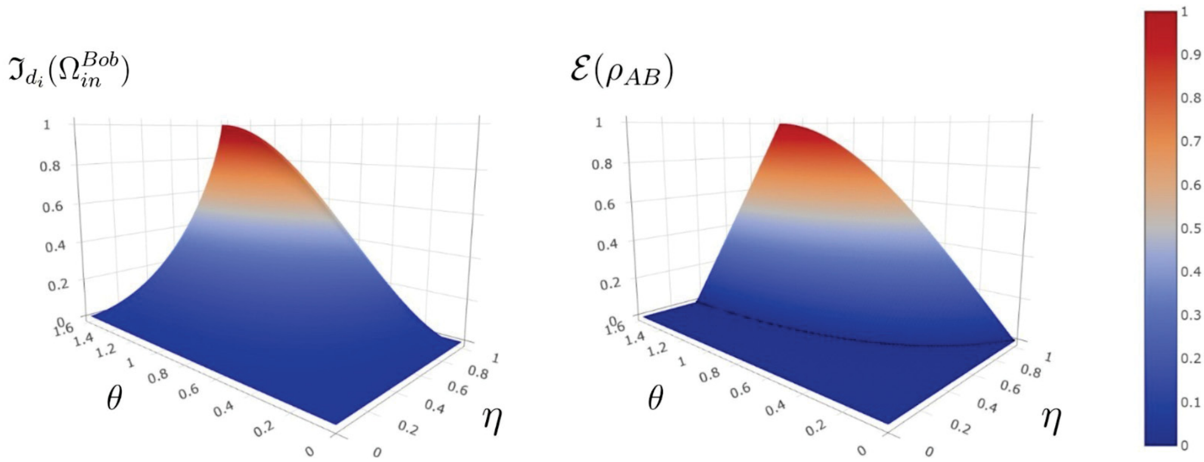

Figure 6 – On the left, the figure shows the plot of $\mathfrak{I}_{d_i}(\Omega^{in}_{Bob})$ as a function of η and θ. On the right, the figure shows the plot of $\mathcal{E}(\rho_{AB})$ as a function of η and θ. The black curve on the right represents the values of η as a function of θ for which the entanglement $\mathcal{E}(\rho_{AB})$ is zero, as determined by the equality in (5.26). Both figures use the same scale.

5.2.2 The role of Quantum Discord

Turning our attention to the role of quantum discord in establishing the phenomenon of ontological correlation, it is important to present some initial results. In section 3.2.3 it was shown that one-way and symmetric quantum discord are the same for the Werner state when mixed with a singlet state. However, this is also the case for the state (5.19). Let the one-way quantum discord be first calculated and then the comparison with the symmetrical quantum discord will be made.

As provided by Eq. (3.47), it is necessary to calculate $S(\rho_{AB}), S(\rho_B)$ and $S(\rho_{A|\{\sigma_n\}_B})$. The first two are simple, and are given by

$$\begin{aligned} S(\rho_{AB}) &= -3\Big(\frac{1-\eta}{4}\Big)\log\Big(\frac{1-\eta}{4}\Big) - \Big(\frac{1+3\eta}{4}\Big)\log\Big(\frac{1+3\eta}{4}\Big), \\ S(\rho_B) &= h\Big(\frac{1-\eta}{2} + \eta\sin^2\frac{\theta}{2}\Big). \end{aligned} \tag{5.27}$$

For the conditional entropy, it is necessary to take the average on each measurement, so that the final description for a general observable σ_n, with eigenstates given by

$$\begin{aligned} |0\rangle_n &= \cos\frac{\phi}{2}|0\rangle + e^{i\varphi}\sin\frac{\phi}{2}|1\rangle, \\ |1\rangle_n &= \sin\frac{\phi}{2}|0\rangle - e^{i\varphi}\cos\frac{\phi}{2}|1\rangle, \end{aligned}$$

is the following

$$\begin{aligned}
S(\rho_{A|\{\sigma_n\}_B}) = &-\Big(\frac{1-\eta}{4}+\eta\sin^2\frac{\phi}{2}\cos^2\frac{\theta}{2}\Big)\log\Big(\frac{1-\eta}{4}+\eta\sin^2\frac{\phi}{2}\cos^2\frac{\theta}{2}\Big)\\
&-\Big(\frac{1-\eta}{4}+\eta\cos^2\frac{\phi}{2}\sin^2\frac{\theta}{2}\Big)\log\Big(\frac{1-\eta}{4}+\eta\cos^2\frac{\phi}{2}\sin^2\frac{\theta}{2}\Big)\\
&-\Big(\frac{1-\eta}{4}+\eta\cos^2\frac{\phi}{2}\cos^2\frac{\theta}{2}\Big)\log\Big(\frac{1-\eta}{4}+\eta\cos^2\frac{\phi}{2}\cos^2\frac{\theta}{2}\Big)\\
&-\Big(\frac{1-\eta}{4}+\eta\sin^2\frac{\phi}{2}\sin^2\frac{\theta}{2}\Big)\log\Big(\frac{1-\eta}{4}+\eta\sin^2\frac{\phi}{2}\sin^2\frac{\theta}{2}\Big)\\
&-h\Big[\frac{1-\eta}{2}+\eta\Big(\sin^2\frac{\phi}{2}\cos^2\frac{\theta}{2}+\cos^2\frac{\phi}{2}\sin^2\frac{\theta}{2}\Big)\Big].
\end{aligned} \tag{5.28}$$

Since $\mathcal{D}_B(\rho_{AB}) = \min_{\sigma_n}\big(S(\rho_{A|\{\sigma_n\}_B}) + S(\rho_B) - S(\rho_{AB})\big)$, the minimization occurs when $S(\rho_{A|\{\sigma_n\}_B})$ is minimum. Taking the derivative of (5.28) with respect to ϕ,

$$\begin{aligned}
\frac{\mathrm{d}S(\rho_{A|\{\sigma_n\}_B})}{\mathrm{d}\phi} = \frac{\eta\sin\phi}{2}\Bigg(&\cos^2\frac{\theta}{2}\log\Big(\frac{\frac{1-\eta}{4}+\eta\sin^2\frac{\phi}{2}\cos^2\frac{\theta}{2}}{\frac{1-\eta}{4}+\eta\cos^2\frac{\phi}{2}\cos^2\frac{\theta}{2}}\Big)+\\
&\sin^2\frac{\theta}{2}\log\Big(\frac{\frac{1-\eta}{4}+\eta\cos^2\frac{\phi}{2}\sin^2\frac{\theta}{2}}{\frac{1-\eta}{4}+\eta\sin^2\frac{\phi}{2}\sin^2\frac{\theta}{2}}\Big)\Bigg).
\end{aligned} \tag{5.29}$$

Optimizing the above equation, the solutions for ϕ are both $\phi = 0$ (minimum point)[3] and $\phi = \pi/2$ (maximum point). Indeed, when $\phi = 0$, then $\sigma_n = \sigma_z$, so that

$$\begin{aligned}
S(\rho_{A|\{\sigma_z\}_B}) = &-\Big(\frac{1-\eta}{4}+\eta\sin^2\frac{\theta}{2}\Big)\log\Big(\frac{1-\eta}{4}+\eta\sin^2\frac{\theta}{2}\Big)\\
&-\Big(\frac{1-\eta}{4}+\eta\cos^2\frac{\theta}{2}\Big)\log\Big(\frac{1-\eta}{4}+\eta\cos^2\frac{\theta}{2}\Big)\\
&-2\Big(\frac{1-\eta}{4}\Big)\log\Big(\frac{1-\eta}{4}\Big)-h\Big(\frac{1-\eta}{2}+\eta\sin^2\frac{\theta}{2}\Big)
\end{aligned} \tag{5.30}$$

is the minimized conditional entropy. Thus, from (3.47), (5.27) and (5.30), the one-way quantum discord is given by

$$\begin{aligned}
\mathcal{D}_B(\rho_{AB}) = &\Big(\frac{1-\eta}{4}\Big)\log\Big(\frac{1-\eta}{4}\Big)+\Big(\frac{1+3\eta}{4}\Big)\log\Big(\frac{1+3\eta}{4}\Big)\\
&-\Big(\frac{1-\eta}{4}+\eta\sin^2\frac{\theta}{2}\Big)\log\Big(\frac{1-\eta}{4}+\eta\sin^2\frac{\theta}{2}\Big)\\
&-\Big(\frac{1-\eta}{4}+\eta\cos^2\frac{\theta}{2}\Big)\log\Big(\frac{1-\eta}{4}+\eta\cos^2\frac{\theta}{2}\Big).
\end{aligned} \tag{5.31}$$

Now, following the definition of the symmetric quantum discord in (3.53), the optimal observable is the same for the previous case by using the exact same argument. Moreover, due to the form of the state in (5.19), it is true that $\Phi_{\sigma_z^A}(\rho_{AB}) = \Phi_{\sigma_z^B}(\rho_{AB}) =$

[3] Indeed, this result was statistically verified.

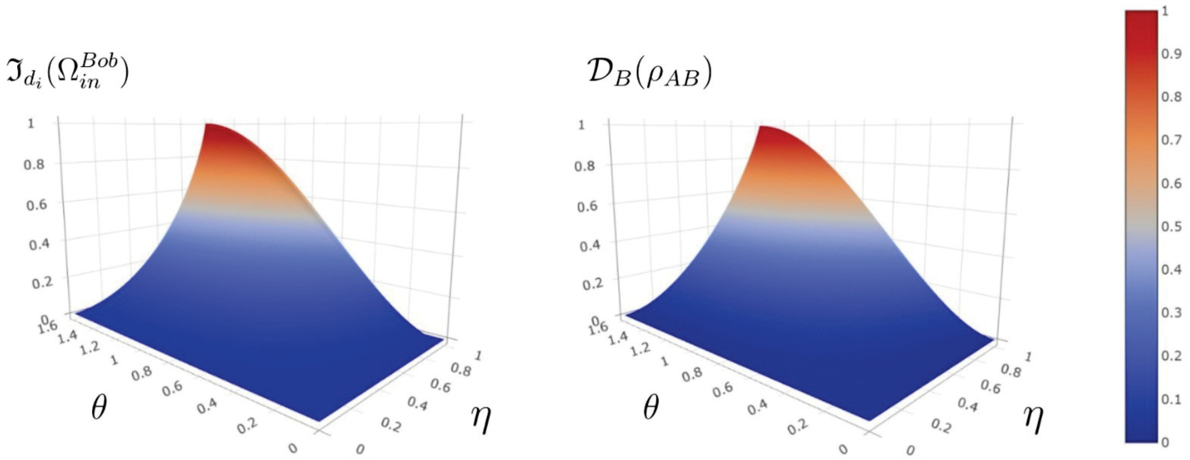

Figure 7 – On the left, the figure shows the plot of $\mathfrak{I}_{d_i}(\Omega^{in}_{Bob})$ as a function of η and θ. On the right, the figure shows the plot of $\mathcal{D}_B(\rho_{AB})$ as a function of η and θ. Both figures use the same scale.

$\Phi_{\sigma_z^A\sigma_z^B}(\rho_{AB})$, and

$$\begin{aligned}
\operatorname{tr}_A \rho_{AB} = \rho_B &= \frac{1-\eta}{2}\mathbb{1}_B + \eta\big(\cos^2\frac{\theta}{2}\,|1\rangle\langle 1| + \sin^2\frac{\theta}{2}\,|0\rangle\langle 0|\big)_B \\
&= \operatorname{tr}_A \Phi_{\sigma_z^B}(\rho_{AB}) \equiv \Phi_{\sigma_z^B}(\rho_B), \\
\operatorname{tr}_B \rho_{AB} = \rho_A &= \frac{1-\eta}{2}\mathbb{1}_A + \eta\big(\cos^2\frac{\theta}{2}\,|0\rangle\langle 0| + \sin^2\frac{\theta}{2}\,|1\rangle\langle 1|\big)_A \\
&= \operatorname{tr}_B \Phi_{\sigma_z^A}(\rho_{AB}) \equiv \Phi_{\sigma_z^A}(\rho_A),
\end{aligned}$$

so that equal states have the same von Neumann entropy, and then the symmetric quantum discord is simply written as

$$\begin{aligned}
\mathcal{D}_{AB}(\rho_{AB}) = S(\Phi_{AB}(\rho_{AB})) - S(\rho_{AB}) &+ \underbrace{\big(S(\rho_A) - S(\Phi_{\sigma_z^A}(\rho_A))\big)}_{0} + \underbrace{\big(S(\rho_B) - S(\Phi_{\sigma_z^B}(\rho_B))\big)}_{0} \\
= &\left(\frac{1-\eta}{4}\right)\log\left(\frac{1-\eta}{4}\right) + \left(\frac{1+3\eta}{4}\right)\log\left(\frac{1+3\eta}{4}\right) \\
&- \left(\frac{1-\eta}{4} + \eta\sin^2\frac{\theta}{2}\right)\log\left(\frac{1-\eta}{4} + \eta\sin^2\frac{\theta}{2}\right) \\
&- \left(\frac{1-\eta}{4} + \eta\cos^2\frac{\theta}{2}\right)\log\left(\frac{1-\eta}{4} + \eta\cos^2\frac{\theta}{2}\right) \\
= &\ \mathcal{D}_{\mathcal{B}}(\rho_{AB}).
\end{aligned} \tag{5.32}$$

In order to verify the connection between irreality in (5.16) and the quantum discord in (5.31) or (5.32), consider the cases where $\delta\mathfrak{I} = \mathfrak{I}_{d_i}(\Omega_{in}^{Bob}) = 0$. This equality holds only if $\eta\sin^2\frac{\theta}{2} = 0$, or, considering the domain of each variable, if $\eta = 0$ or if $\theta = 0$. Mathematically, $\eta = 0$ or $\theta = 0 \;\Rightarrow\; \mathfrak{I}_{d_i}(\Omega_{in}^{Bob}) = 0$, and the phenomenon does not occur. Now, considering the cases where quantum discord is null, that is, when (5.31) is zero, the analysis is completely analogous to the previous case of

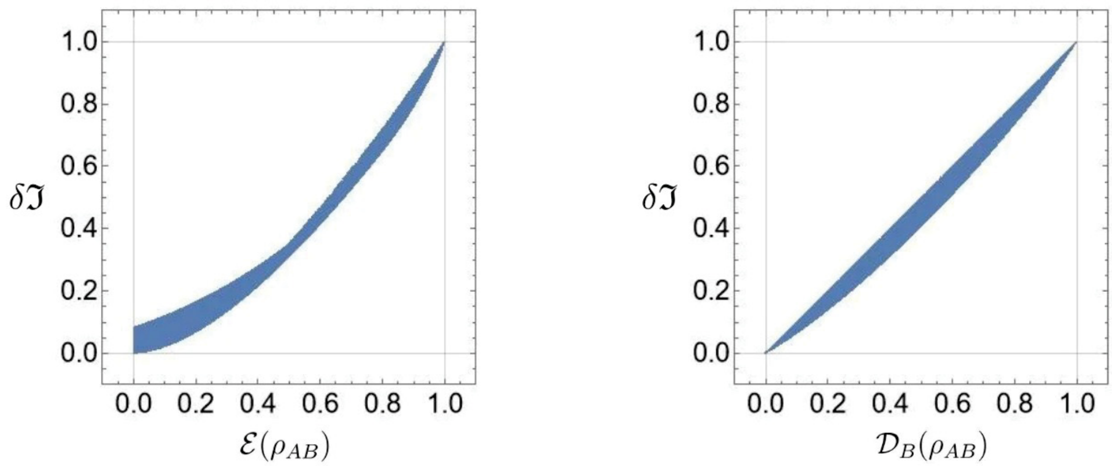

Figure 8 – On the left, the figure shows the parametric plot of $\delta\mathfrak{I}$ as a function of $\mathcal{E}(\rho_{AB})$. On the right, the figure shows the parametric plot of $\delta\mathfrak{I}$ as a function of $\mathcal{D}_B(\rho_{AB})$. Each graph consists of 10^6 points, each corresponding to a randomly generated set $\{\eta, \theta\}$ via the method Mersenne Twister of Mathematica.

irreality, since $\frac{1+3\eta}{4} = \frac{1-\eta}{4} + \eta$, and considering only the solutions in agreement within the domain of each variable, (5.31) is zero when $\eta = 0$ or $\theta = 0$, leading to the same conclusions as those of irreality. Thus, $\eta = 0$ or $\theta = 0 \Rightarrow \mathcal{D}_B(\rho_{AB}) = 0$. Therefore, there is a perfect agreement between the non-zero quantum discord and the presence of the phenomenon. Figure 7 illustrates both graphs of $\mathfrak{I}_{d_i}(\Omega_{in}^{Bob})$ and $\mathcal{D}_B(\rho_{AB})$ as functions of η and θ, clearly showing that the agreement is complete, unlike the case with entanglement. Furthermore, Figure 8 shows a statistical analysis between $\delta\mathfrak{I}$ and both $\mathcal{E}(\rho_{AB})$ and $\mathcal{D}_B(\rho_{AB})$. In the case of entanglement, Figure 8 clearly shows that even when entanglement is zero, $\delta\mathfrak{I}$ can still be greater than zero. Moreover, the analysis of the second graph establishes the following equivalence

$$\delta\mathfrak{I} > 0 \Leftrightarrow \mathcal{D}_B(\rho_{AB}) > 0, \tag{5.33}$$

and there is an equivalence of the form $\delta\mathfrak{I} \Leftrightarrow \mathcal{D}_B(\rho_{AB})$.

This analysis suggests that discord is the quantum resource responsible for the phenomenon. Therefore, it is necessary to study particular cases of discord by considering a classical-quantum state and verifying whether eliminating discord is sufficient to eliminate the correlation.

5.3 Bloch states in the Modified Quantum Eraser

This section delves deeper into the investigation initiated in the previous section. Here, cases where the state is certainly not entangled and discord exists in only one of the polarization spaces will be considered to examine the role of one-way quantum discord in the occurrence of the phenomenon. Thus, entanglement plays no role for the considered cases in this section. Additionally, realism-based nonlocality is analyzed for the same types of states. Section 5.3.1 considers one-way quantum discord solely in Alice's polarization, while Section 5.3.2 provides an analogous analysis for Bob.

5.3.1 Bloch state at Alice's site

To begin with quantum discord on Alice's Hilbert space of polarization, consider the initial state given by

$$\rho_0 = \Big[\eta\Big(\frac{\mathbb{1}+\boldsymbol{r_0}\cdot\boldsymbol{\sigma}}{2}\Big)\otimes|0\rangle\langle 0| + (1-\eta)\Big(\frac{\mathbb{1}+\boldsymbol{r_1}\cdot\boldsymbol{\sigma}}{2}\Big)\otimes|1\rangle\langle 1|\Big]_{AB}\otimes|0011\rangle\langle 0011|_{abd_1d_2}\,. \tag{5.34}$$

The above state is, then, consisted by a generic qubit state for Alice (not entangled) and projectors for Bob, considering only the polarizations of the photons, and it is a quantum-classical state. Thus, as made before for Werner state in (5.5), the state in (5.34) must be evolved throughout the modified quantum eraser. However, as each component in the experiment is conducted in the computational basis, considering that $\boldsymbol{r_i} = (x_i, y_i, z_i)$, the above state can be rewritten explicitly in terms of the eigenbasis of σ_z, so that it becomes

$$\begin{aligned}\rho_0 =&\eta\Big[\Big(\frac{1+z_0}{2}\Big)|000011\rangle\langle 000011| + \Big(\frac{1-z_0}{2}\Big)|100011\rangle\langle 100011|\\ &+\Big(\frac{x_0+iy_0}{2}\Big)|000011\rangle\langle 100011| + \Big(\frac{x_0-iy_0}{2}\Big)|100011\rangle\langle 000011|\Big]_{AaBbd_1d_2}\\ &+(1-\eta)\Big[\Big(\frac{1+z_1}{2}\Big)|001011\rangle\langle 001011| + \Big(\frac{1-z_1}{2}\Big)|101011\rangle\langle 101011|\\ &+\Big(\frac{x_1+iy_1}{2}\Big)|001011\rangle\langle 101011| + \Big(\frac{x_1-iy_1}{2}\Big)|101011\rangle\langle 001011|\Big]_{AaBbd_1d_2}\end{aligned} \tag{5.35}$$

The evolution of ρ_0 in Bob's laboratory is as follows. First, after the PBS, the state becomes

$$\begin{aligned}\rho_{PBS} =&\eta\Big[\Big(\frac{1+z_0}{2}\Big)|000011\rangle\langle 000011| + \Big(\frac{1-z_0}{2}\Big)|100011\rangle\langle 100011|\\ &+\Big(\frac{x_0+iy_0}{2}\Big)|000011\rangle\langle 100011| + \Big(\frac{x_0-iy_0}{2}\Big)|100011\rangle\langle 000011|\Big]_{AaBbd_1d_2}\\ &+(1-\eta)\Big[\Big(\frac{1+z_1}{2}\Big)|001111\rangle\langle 001111| + \Big(\frac{1-z_1}{2}\Big)|101111\rangle\langle 101111|\\ &+\Big(\frac{x_1+iy_1}{2}\Big)|001111\rangle\langle 101111| + \Big(\frac{x_1-iy_1}{2}\Big)|101111\rangle\langle 001111|\Big]_{AaBbd_1d_2}.\end{aligned} \tag{5.36}$$

Then, the HWP changes the polarization of the photon with $|1\rangle_b$, so that

$$\begin{aligned}\rho_{HWP} =&\eta\Big[\Big(\frac{1+z_0}{2}\Big)|000011\rangle\langle000011| + \Big(\frac{1-z_0}{2}\Big)|100011\rangle\langle100011|\\ &+\Big(\frac{x_0+iy_0}{2}\Big)|000011\rangle\langle100011| + \Big(\frac{x_0-iy_0}{2}\Big)|100011\rangle\langle000011|\Big]_{AaBbd_1d_2}\\ &+(1-\eta)\Big[\Big(\frac{1+z_1}{2}\Big)|000111\rangle\langle000111| + \Big(\frac{1-z_1}{2}\Big)|100111\rangle\langle100111|\\ &+\Big(\frac{x_1+iy_1}{2}\Big)|000111\rangle\langle100111| + \Big(\frac{x_1-iy_1}{2}\Big)|100111\rangle\langle000111|\Big]_{AaBbd_1d_2}.\end{aligned} \tag{5.37}$$

The above state is then submitted to the interaction with the degrees of freedom DoF 1 and 2, and

$$\begin{aligned}\rho_{d} =&\eta\Big[\Big(\frac{1+z_0}{2}\Big)|000001\rangle\langle000001| + \Big(\frac{1-z_0}{2}\Big)|100001\rangle\langle100001|\\ &+\Big(\frac{x_0+iy_0}{2}\Big)|000001\rangle\langle100001| + \Big(\frac{x_0-iy_0}{2}\Big)|100001\rangle\langle000001|\Big]_{AaBbd_1d_2}\\ &+(1-\eta)\Big[\Big(\frac{1+z_1}{2}\Big)|000110\rangle\langle000110| + \Big(\frac{1-z_1}{2}\Big)|100110\rangle\langle100110|\\ &+\Big(\frac{x_1+iy_1}{2}\Big)|000110\rangle\langle100110| + \Big(\frac{x_1-iy_1}{2}\Big)|100110\rangle\langle000110|\Big]_{AaBbd_1d_2}.\end{aligned} \tag{5.38}$$

Finally, the above state passes through the mirrors and the BS, and then becomes

$$\begin{aligned}\rho_{Bob} =&\eta\Big[\Big(\frac{1+z_0}{2}\Big)|000\omega_{-}01\rangle\langle000\omega_{-}01| + \Big(\frac{1-z_0}{2}\Big)|100\omega_{-}01\rangle\langle100\omega_{-}01|\\ &+\Big(\frac{x_0+iy_0}{2}\Big)|000\omega_{-}01\rangle\langle100\omega_{-}01| + \Big(\frac{x_0-iy_0}{2}\Big)|100\omega_{-}01\rangle\langle000\omega_{-}01|\Big]_{AaBbd_1d_2}\\ &+(1-\eta)\Big[\Big(\frac{1+z_1}{2}\Big)|000\omega_{+}10\rangle\langle000\omega_{+}10| + \Big(\frac{1-z_1}{2}\Big)|100\omega_{+}10\rangle\langle100\omega_{+}10|\\ &+\Big(\frac{x_1+iy_1}{2}\Big)|000\omega_{+}10\rangle\langle100\omega_{+}10| + \Big(\frac{x_1-iy_1}{2}\Big)|100\omega_{+}10\rangle\langle000\omega_{+}10|\Big]_{AaBbd_1d_2}.\end{aligned} \tag{5.39}$$

Considering Alice's choices, first suppose that she decided to now include the QWP in the experiment, so that

$$\begin{aligned}\rho_{out} =&\eta\Big[\Big(\frac{1+z_0}{2}\Big)|000\omega_{-}01\rangle\langle000\omega_{-}01| + \Big(\frac{1-z_0}{2}\Big)|110\omega_{-}01\rangle\langle110\omega_{-}01|\\ &-i\Big(\frac{x_0+iy_0}{2}\Big)|000\omega_{-}01\rangle\langle110\omega_{-}01| + i\Big(\frac{x_0-iy_0}{2}\Big)|110\omega_{-}01\rangle\langle000\omega_{-}01|\Big]_{AaBbd_1d_2}\\ &+(1-\eta)\Big[\Big(\frac{1+z_1}{2}\Big)|000\omega_{+}10\rangle\langle000\omega_{+}10| + \Big(\frac{1-z_1}{2}\Big)|110\omega_{+}10\rangle\langle110\omega_{+}10|\\ &-i\Big(\frac{x_1+iy_1}{2}\Big)|000\omega_{+}10\rangle\langle110\omega_{+}10| + i\Big(\frac{x_1-iy_1}{2}\Big)|110\omega_{+}10\rangle\langle000\omega_{+}10|\Big]_{AaBbd_1d_2}.\end{aligned} \tag{5.40}$$

The above state is symmetric regarding the choice of post-selection and is nonzero only for post-selections of the type $\mathrm{D}_{k,\mathrm{a}}$ and $\mathrm{D}_{k,\mathrm{b}}$. Thus, by analyzing Bob's description, the final state is written as

$$\begin{aligned}\Omega_{out}^{Bob} &= \Omega_{out}^{Bob|kk} = \mathrm{tr}_{Aa}\left(\rho_{out}(\mathsf{D}_{k,\mathsf{a}},\mathsf{D}_{k,\mathsf{b}})\right)\\ &= |0k\rangle\langle 0k|_{Bb}\otimes\frac{\eta\left(\frac{1+z_0}{2}\right)|01\rangle\langle 01|_{d_1d_2}+(1-\eta)\left(\frac{1+z_1}{2}\right)|10\rangle\langle 10|_{d_1d_2}}{\eta\left(\frac{1+z_0}{2}\right)+(1-\eta)\left(\frac{1+z_1}{2}\right)}.\end{aligned} \tag{5.41}$$

The above state is clearly diagonal, i.e., the description correspondent to the degrees of freedom DoF 1 and 2 is merely a statistical mixture, and therefore realistic. Mathematically,

$$\Phi_{d_i}(\Omega_{out}^{Bob}) = \Omega_{out}^{Bob} \Rightarrow \mathfrak{I}_{d_i}(\Omega_{out}^{Bob}) = 0. \tag{5.42}$$

Considering now the case where Alice decides to include the QWP in the experiment, the state after the interaction with that becomes

$$\begin{aligned}\rho_{QWP} =&\eta\Big[\left(\frac{1+z_0}{2}\right)|\omega_+00\omega_-01\rangle\langle\omega_+00\omega_-01| + \left(\frac{1-z_0}{2}\right)|\omega_-00\omega_-01\rangle\langle\omega_-00\omega_-01|\\ &- i\left(\frac{x_0+iy_0}{2}\right)|\omega_+00\omega_-01\rangle\langle\omega_-00\omega_-01| + i\left(\frac{x_0-iy_0}{2}\right)|\omega_-00\omega_-01\rangle\langle\omega_+00\omega_-01|\Big]_{AaBbd_1d_2}\\ &+(1-\eta)\Big[\left(\frac{1+z_1}{2}\right)|\omega_+00\omega_+10\rangle\langle\omega_+00\omega_+10| + \left(\frac{1-z_1}{2}\right)|\omega_-00\omega_+10\rangle\langle\omega_-00\omega_+10|\\ &- i\left(\frac{x_1+iy_1}{2}\right)|\omega_+00\omega_+10\rangle\langle\omega_-00\omega_+10| + \left(\frac{x_1-iy_1}{2}\right)|\omega_-00\omega_+10\rangle\langle\omega_+00\omega_+10|\Big]_{AaBbd_1d_2}.\end{aligned} \tag{5.43}$$

Then, after the last PBS on Alice's laboratory, the final description of the system is given by

$$\begin{aligned}\rho_{in} =& |0\rangle\langle 0|_B\otimes\\ &\Bigg[|00\rangle\langle 00|\otimes\Big[\eta\left(\frac{1+y_0}{2}\right)|\omega_-01\rangle\langle\omega_-01| + (1-\eta)\left(\frac{1+y_1}{2}\right)|\omega_+10\rangle\langle\omega_+10|\Big]\\ &+|11\rangle\langle 11|\otimes\Big[\eta\left(\frac{1-y_0}{2}\right)|\omega_-01\rangle\langle\omega_-01| + (1-\eta)\left(\frac{1-y_1}{2}\right)|\omega_+10\rangle\langle\omega_+10|\Big]\\ &+|10\rangle\langle 10|\otimes\Big[\eta\left(\frac{-z_0+ix_0}{2}\right)|\omega_-01\rangle\langle\omega_-01| + (1-\eta)\left(\frac{-z_1+ix_1}{2}\right)|\omega_+10\rangle\langle\omega_+10|\Big]\\ &-|01\rangle\langle 01|\otimes\Big[\eta\left(\frac{z_0+ix_0}{2}\right)|\omega_-01\rangle\langle\omega_-01| + (1-\eta)\left(\frac{z_1+ix_1}{2}\right)|\omega_+10\rangle\langle\omega_+10|\Big]\Bigg]_{Aabd_1d_2}.\end{aligned} \tag{5.44}$$

Choosing the post-selection on $\mathsf{D}_{k,\mathsf{a}}$ and $\mathsf{D}_{l,\mathsf{b}}$, Bob's description is

$$\begin{aligned}\Omega_{in}^{Bob} &= \Omega_{in}^{Bob|kl} = \mathrm{tr}_{Aa}\left(\rho_{in}(\mathsf{D}_{k,\mathsf{a}},\mathsf{D}_{l,\mathsf{b}})\right)\\ &= |0l\rangle\langle 0l|_{Bb}\otimes\frac{\eta\left(\frac{1+y_0}{2}\right)|01\rangle\langle 01|_{d_1d_2}+(1-\eta)\left(\frac{1+y_1}{2}\right)|10\rangle\langle 10|_{d_1d_2}}{\eta\left(\frac{1+y_0}{2}\right)+(1-\eta)\left(\frac{1+y_1}{2}\right)},\end{aligned} \tag{5.45}$$

that is very similar to (5.41). Moreover, the state in (5.45) is realistic, since it is a statistical mixture, so that the same conclusion as obtained in (5.42) are reached, that is

$$\Phi_{d_i}(\Omega_{in}^{Bob}) = \Omega_{in}^{Bob} \Rightarrow \mathfrak{I}_{d_i}(\Omega_{in}^{Bob}) = 0. \tag{5.46}$$

It is important to address one final point before concluding this discussion. Both results in (5.42) and (5.46) are independent of the post-selection chosen due to the form of the states (5.40) and (5.44). Beyond that, it is interesting that both results are independent of the initial vectors $\boldsymbol{r}_i$, in the sense that every vector leads to a statistical mixture.

From the perspective of $\delta\mathfrak{I}$, since the conclusions for both configurations are independent of post-selection, Eq. (5.4) gives

$$\delta\mathfrak{I} = 0, \tag{5.47}$$

and the phenomenon never occurs. Since the initial state in (5.34) may have one-way quantum discord for Alice's polarization in general, and even symmetric quantum discord, it is possible to conclude here that neither is sufficient to establish the phenomenon in study. Furthermore, since the result is independent of the vectors defining the initial state, allowing one-way quantum discord on Alice's site while Bob has classical states will never lead to the phenomenon, independently of the quantum resources present in the initial state.

5.3.2 Bloch state at Bob's site

The study here is analogous to the one made in the previous subsection, however with the following initial state.

$$\rho_0 = \left[\eta \left|0\right\rangle \left\langle 0\right| \otimes \left(\frac{\mathbb{1} + \boldsymbol{r_0}\cdot\boldsymbol{\sigma}}{2}\right) + (1-\eta)\left|1\right\rangle\left\langle 1\right| \otimes \left(\frac{\mathbb{1} + \boldsymbol{r_1}\cdot\boldsymbol{\sigma}}{2}\right)\right]_{AB} \otimes \left|0011\right\rangle \left\langle 0011\right|_{abd_1d_2}. \tag{5.48}$$

The above state throughout the experimental setup in Figure 5 is evolved in a completely analogous manner as made for the state (5.34). Then, in terms of the computational basis, the evolution is as follows. First, in Bob's laboratory, after the PBS the state becomes

$$\begin{aligned}\rho_{PBS} =&\eta\Bigg[\Big(\frac{1+z_0}{2}\Big)\left|000011\right\rangle\left\langle 000011\right| + \Big(\frac{1-z_0}{2}\Big)\left|001111\right\rangle\left\langle 001111\right| \\ &- i\Big(\frac{x_0+iy_0}{2}\Big)\left|000011\right\rangle\left\langle 001111\right| + i\Big(\frac{x_0-iy_0}{2}\Big)\left|001111\right\rangle\left\langle 000011\right|\Bigg]_{AaBbd_1d_2} \\ &+ (1-\eta)\Bigg[\Big(\frac{1+z_1}{2}\Big)\left|100011\right\rangle\left\langle 100011\right| + \Big(\frac{1-z_1}{2}\Big)\left|101111\right\rangle\left\langle 101111\right| \\ &- i\Big(\frac{x_1+iy_1}{2}\Big)\left|100011\right\rangle\left\langle 101111\right| + i\Big(\frac{x_1-iy_1}{2}\Big)\left|101111\right\rangle\left\langle 100011\right|\Bigg]_{AaBbd_1d_2}.\end{aligned} \tag{5.49}$$

After the HWP,

$$\begin{aligned}\rho_{HWP} =&\eta\Bigg[\Big(\frac{1+z_0}{2}\Big)\left|000011\right\rangle\left\langle 000011\right| + \Big(\frac{1-z_0}{2}\Big)\left|000111\right\rangle\left\langle 000111\right| \\ &- i\Big(\frac{x_0+iy_0}{2}\Big)\left|000011\right\rangle\left\langle 000111\right| + i\Big(\frac{x_0-iy_0}{2}\Big)\left|000111\right\rangle\left\langle 000011\right|\Bigg]_{AaBbd_1d_2} \\ &+ (1-\eta)\Bigg[\Big(\frac{1+z_1}{2}\Big)\left|100011\right\rangle\left\langle 100011\right| + \Big(\frac{1-z_1}{2}\Big)\left|100111\right\rangle\left\langle 100111\right| \\ &- i\Big(\frac{x_1+iy_1}{2}\Big)\left|100011\right\rangle\left\langle 100111\right| + i\Big(\frac{x_1-iy_1}{2}\Big)\left|100111\right\rangle\left\langle 100011\right|\Bigg]_{AaBbd_1d_2}.\end{aligned} \tag{5.50}$$

When the degrees of freedom DoF 1 and 2 are considered, the state becomes

$$\begin{aligned}\rho_{d} =&\eta\Bigg[\Big(\frac{1+z_0}{2}\Big)\left|000001\right\rangle\left\langle 000001\right| + \Big(\frac{1-z_0}{2}\Big)\left|000110\right\rangle\left\langle 000110\right| \\ &- i\Big(\frac{x_0+iy_0}{2}\Big)\left|000001\right\rangle\left\langle 000110\right| + i\Big(\frac{x_0-iy_0}{2}\Big)\left|000110\right\rangle\left\langle 000001\right|\Bigg]_{AaBbd_1d_2} \\ &+ (1-\eta)\Bigg[\Big(\frac{1+z_1}{2}\Big)\left|100001\right\rangle\left\langle 100001\right| + \Big(\frac{1-z_1}{2}\Big)\left|100110\right\rangle\left\langle 100110\right| \\ &- i\Big(\frac{x_1+iy_1}{2}\Big)\left|100001\right\rangle\left\langle 100110\right| + i\Big(\frac{x_1-iy_1}{2}\Big)\left|100110\right\rangle\left\langle 100001\right|\Bigg]_{AaBbd_1d_2}.\end{aligned} \tag{5.51}$$

Then, the above state is submitted to the unitary correspondent to the mirrors and the last BS, so that after Bob's laboratory the state becomes

$$\begin{aligned}\rho_{Bob} =&\eta\Bigg[\Big(\frac{1+z_0}{2}\Big)\left|000\omega_{-}01\right\rangle\left\langle 000\omega_{-}01\right| + \Big(\frac{1-z_0}{2}\Big)\left|000\omega_{+}10\right\rangle\left\langle 000\omega_{+}10\right| \\ &- i\Big(\frac{x_0+iy_0}{2}\Big)\left|000\omega_{-}01\right\rangle\left\langle 000\omega_{+}10\right| + i\Big(\frac{x_0-iy_0}{2}\Big)\left|000\omega_{+}10\right\rangle\left\langle 000\omega_{-}01\right|\Bigg]_{AaBbd_1d_2} \\ &+ (1-\eta)\Bigg[\Big(\frac{1+z_1}{2}\Big)\left|100\omega_{-}01\right\rangle\left\langle 100\omega_{-}01\right| + \Big(\frac{1-z_1}{2}\Big)\left|100\omega_{+}10\right\rangle\left\langle 100\omega_{+}10\right| \\ &- i\Big(\frac{x_1+iy_1}{2}\Big)\left|100\omega_{-}01\right\rangle\left\langle 100\omega_{+}10\right| + i\Big(\frac{x_1-iy_1}{2}\Big)\left|100\omega_{+}10\right\rangle\left\langle 100\omega_{-}01\right|\Bigg]_{AaBbd_1d_2}.\end{aligned} \tag{5.52}$$

Analyzing the case when Alice decides to not include the QWP in the experiment, the final description, after the PBS in Alice's laboratory, is given by

$$\begin{aligned}\rho_{out} =&\eta\left|000\right\rangle\left\langle 000\right|_{AaB}\otimes\Bigg[\Big(\frac{1+z_0}{2}\Big)\left|\omega_{-}01\right\rangle\left\langle \omega_{-}01\right| + \Big(\frac{1-z_0}{2}\Big)\left|\omega_{+}10\right\rangle\left\langle \omega_{+}10\right| \\ &+ \Big(\frac{x_0+iy_0}{2}\Big)\left|\omega_{-}01\right\rangle\left\langle \omega_{+}10\right| + \Big(\frac{x_0-iy_0}{2}\Big)\left|\omega_{+}10\right\rangle\left\langle \omega_{-}01\right|\Bigg]_{bd_1d_2} \\ &+ (1-\eta)\left|110\right\rangle\left\langle 110\right|_{AaB}\otimes\Bigg[\Big(\frac{1+z_1}{2}\Big)\left|\omega_{-}01\right\rangle\left\langle \omega_{-}01\right| + \Big(\frac{1-z_1}{2}\Big)\left|\omega_{+}10\right\rangle\left\langle \omega_{+}10\right| \\ &+ \Big(\frac{x_1+iy_1}{2}\Big)\left|\omega_{-}01\right\rangle\left\langle \omega_{+}10\right| + \Big(\frac{x_1-iy_1}{2}\Big)\left|\omega_{+}10\right\rangle\left\langle \omega_{-}01\right|\Bigg]_{bd_1d_2}\end{aligned} \tag{5.53}$$

The state in (5.53) must be considered with some caution, since the choice of post-selection is relevant for the final description. From (5.53), the post-selection chosen by Bob is irrelevant. However, if Alice decides to post-select in $\mathsf{D}_{0,\mathsf{a}}$, all the physics of the problem lies on the properties of the state defined by $\boldsymbol{r}_0$, and it is analogous if she chooses $\mathsf{D}_{1,\mathsf{a}}$ with $\boldsymbol{r}_1$. Moreover, the final form of the state is the same, so that if the post-selection is taken into $\mathsf{D}_{k,\mathsf{a}}$ by Alice and $\mathsf{D}_{l,\mathsf{b}}$ by Bob, his final description is given by

$$\begin{aligned}\Omega_{out}^{Bob|k} &= \mathrm{tr}_{Aa}\left(\rho_{out}(\mathsf{D}_{k,\mathsf{a}},\mathsf{D}_{l,\mathsf{b}})\right) \\ &= \left|0l\right\rangle\left\langle 0l\right|_{Bb}\otimes\Big[\Big(\frac{1+z_k}{2}\Big)\left|01\right\rangle\left\langle 01\right| + \Big(\frac{1-z_k}{2}\Big)\left|10\right\rangle\left\langle 10\right| \\ &\quad+ \Big(\frac{x_k+iy_k}{2}\Big)\left|01\right\rangle\left\langle 10\right| + \Big(\frac{x_k-iy_k}{2}\Big)\left|10\right\rangle\left\langle 01\right|\Big]_{d_1d_2},\end{aligned} \tag{5.54}$$

for $\eta \in (0,1)$ and $k,l \in \{0,1\}$. Therefore, there is another important point to discuss. As the eigenbasis of the observables of the degrees of freedom DoF 1 and 2 are those of σ_z, the state in (5.54) is not realistic in general, since it is described by a generic $\boldsymbol{r}_k$. Then, the irreality of d_j for $\Omega_{out}^{Bob|k}$ is calculated as being

$$\mathfrak{I}_{d_j}(\Omega_{out}^{Bob|k}) = h\Big(\frac{1+z_k}{2}\Big) - h\Big(\frac{1+r_k}{2}\Big), \tag{5.55}$$

and clearly depends on Alice's post-selection.

It is worth calculating the cases when the irreality is zero. Due to the form of the binary entropy, and as $\boldsymbol{r}_k = (x_k, y_k, z_k)$, the following conclusion is reached:

$$\mathfrak{I}_{d_j}(\Omega_{out}^{Bob|k}) \begin{cases} = 0, \text{ if } \boldsymbol{r}_k = z_k \boldsymbol{k}, \\ > 0, \text{ if } r_k > |z_k|. \end{cases} \tag{5.56}$$

Considering now the case when Alice decides to include the QWP, (5.52) is submitted to the unitary of the QWP, then becoming

$$\begin{aligned} \rho_{QWP} =& \eta \left|\omega_+ 00\right\rangle \left\langle\omega_+ 00\right|_{AaB} \otimes \Bigg[\Big(\frac{1+z_0}{2}\Big) \left|\omega_- 01\right\rangle \left\langle\omega_- 01\right| + \Big(\frac{1-z_0}{2}\Big) \left|\omega_+ 10\right\rangle \left\langle\omega_+ 10\right| \\ &+ \Big(\frac{x_0+iy_0}{2}\Big) \left|\omega_- 01\right\rangle \left\langle\omega_+ 10\right| + \Big(\frac{x_0-iy_0}{2}\Big) \left|\omega_+ 10\right\rangle \left\langle\omega_- 01\right| \Bigg]_{bd_1d_2} \\ &+ (1-\eta) \left|\omega_- 00\right\rangle \left\langle\omega_- 00\right|_{AaB} \otimes \Bigg[\Big(\frac{1+z_1}{2}\Big) \left|\omega_- 01\right\rangle \left\langle\omega_- 01\right| + \Big(\frac{1-z_1}{2}\Big) \left|\omega_+ 10\right\rangle \left\langle\omega_+ 10\right| \\ &+ \Big(\frac{x_1+iy_1}{2}\Big) \left|\omega_- 01\right\rangle \left\langle\omega_+ 10\right| + \Big(\frac{x_1-iy_1}{2}\Big) \left|\omega_+ 10\right\rangle \left\langle\omega_- 01\right| \Bigg]_{bd_1d_2}, \end{aligned} \tag{5.57}$$

and after the PBS in Alice's laboratory, the final description is

$$\begin{aligned} \rho_{in} =& \frac{\eta}{2}\big(\left|00\right\rangle \left\langle 00\right| + \left|11\right\rangle \left\langle 11\right| - \left|10\right\rangle \left\langle 10\right| - \left|01\right\rangle \left\langle 01\right| \big)_{Aa} \otimes \left|0\right\rangle \left\langle 0\right|_B \otimes \\ &\Bigg[\Big(\frac{1+z_0}{2}\Big) \left|\omega_- 01\right\rangle \left\langle\omega_- 01\right| + \Big(\frac{1-z_0}{2}\Big) \left|\omega_+ 10\right\rangle \left\langle\omega_+ 10\right| \\ &+ \Big(\frac{x_0+iy_0}{2}\Big) \left|\omega_- 01\right\rangle \left\langle\omega_+ 10\right| + \Big(\frac{x_0-iy_0}{2}\Big) \left|\omega_+ 10\right\rangle \left\langle\omega_- 01\right| \Bigg]_{bd_1d_2} \\ &+ \frac{1-\eta}{2}\big(\left|00\right\rangle \left\langle 00\right| + \left|11\right\rangle \left\langle 11\right| + \left|10\right\rangle \left\langle 10\right| + \left|01\right\rangle \left\langle 01\right| \big)_{Aa} \otimes \left|0\right\rangle \left\langle 0\right|_B \otimes \\ &\Bigg[\Big(\frac{1+z_1}{2}\Big) \left|\omega_- 01\right\rangle \left\langle\omega_- 01\right| + \Big(\frac{1-z_1}{2}\Big) \left|\omega_+ 10\right\rangle \left\langle\omega_+ 10\right| \\ &+ \Big(\frac{x_1+iy_1}{2}\Big) \left|\omega_- 01\right\rangle \left\langle\omega_+ 10\right| + \Big(\frac{x_1-iy_1}{2}\Big) \left|\omega_+ 10\right\rangle \left\langle\omega_- 01\right| \Bigg]_{bd_1d_2}. \end{aligned} \tag{5.58}$$

Interestingly, different from what occurred in (5.53), the post-selection chosen is not relevant for the final description of the state (5.58). Thus, choosing the post-selection in

$\mathsf{D}_{k,\mathsf{a}}$ and $\mathsf{D}_{l,\mathsf{b}}$, Bob's final description is given by

$$\begin{aligned}\Omega_{in}^{Bob} &= \Omega_{in}^{Bob|kl} = \mathrm{tr}_{Aa}\left(\rho_{in}(\mathsf{D}_{k,\mathsf{a}},\mathsf{D}_{l,\mathsf{b}})\right)\\ &= |00\rangle\langle 0l|_{Bb}\otimes\left\{\left[\eta\Big(\frac{1+z_0}{2}\Big)+(1-\eta)\Big(\frac{1+z_1}{2}\Big)\right]|01\rangle\langle 01|\right.\\ &+\left[\eta\Big(\frac{1-z_0}{2}\Big)+(1-\eta)\Big(\frac{1-z_1}{2}\Big)\right]|10\rangle\langle 10|\\ &+\left[\eta\Big(\frac{x_0+iy_0}{2}\Big)+(1-\eta)\Big(\frac{x_1+iy_1}{2}\Big)\right]|01\rangle\langle 10|\\ &\left.+\left[\eta\Big(\frac{x_0-iy_0}{2}\Big)+(1-\eta)\Big(\frac{x_1-iy_1}{2}\Big)\right]|10\rangle\langle 01|\right\}_{d_1d_2}.\end{aligned} \tag{5.59}$$

The Eq. (5.59) has a very similar form as (5.54), the difference is that, in the first, the vectors defining the initial state are combined convexly, whereas in the second, the vectors are defined based on post-selection. Calculating the irreality of the state (5.59), it is obtained that

$$\mathfrak{I}_{d_j}(\Omega_{in}^{Bob}) = h\Big(\frac{1+|\eta z_0+(1-\eta)z_1|}{2}\Big) - h\Big(\frac{1+||\eta\boldsymbol{r_0}+(1-\eta)\boldsymbol{r_1}||}{2}\Big). \tag{5.60}$$

By the same analysis as that made before for $\mathfrak{I}(\Omega_{out}^{Bob|k})$, it is possible to investigate the cases when $\mathfrak{I}_{d_j}(\Omega_{in}^{Bob})$, and due to the form of the binary entropy alongside the vectors,

$$\mathfrak{I}_{d_j}(\Omega_{in}^{Bob})\begin{cases}=0, & \text{if } \boldsymbol{r_k}=z_k\boldsymbol{k},\ \forall k,\\ >0, & \text{if } \exists k\,|\,\boldsymbol{r}_k\neq z_k\boldsymbol{k}.\end{cases} \tag{5.61}$$

One additional discussion to the irrealities is to indeed understand when the phenomenon occurs. Regarding the present work, it is assumed that if there exists a single case, for the same post-selection, where the irreality without the QWP and with the QWP differ, i.e., the presence of the QWP influences the final ontological description, the phenomenon is occurring. Analyzing its occurrence using $\delta\mathfrak{I}$, from Eq. (5.4), it is necessary to obtain the probability distribution associated to each post-selection choice. In the case of $\mathrm{QWP_{out}}$ configuration, Eq. (5.53) provides

$$p_{kl}^{out}\equiv p_k^{out}=\begin{cases}\eta, & \text{if } k=0,\\ 1-\eta, & \text{if } k=1.\end{cases}$$

Thus,

$$\sum_{kl} p_{kl}^{out}\mathfrak{I}_{d_j}(\Omega_{out}^{Bob|kl}) = \sum_k p_k^{out}\mathfrak{I}_{d_j}(\Omega_{out}^{Bob|k}) = \eta\mathfrak{I}_{d_j}(\Omega_{out}^{Bob|0}) + (1-\eta)\mathfrak{I}_{d_j}(\Omega_{out}^{Bob|1}), \tag{5.62}$$

where Eq. (5.54) was also used. Moreover, since it was shown that for the $\mathrm{QWP_{in}}$ configuration the results are independent of post-selection, one can write

$$\sum_{kl} p_{kl}^{in}\mathfrak{I}_{d_j}(\Omega_{in}^{Bob|kl}) = \mathfrak{I}_{d_j}(\Omega_{in}^{Bob})\sum_{kl} p_{kl}^{in} = \mathfrak{I}_{d_j}(\Omega_{in}^{Bob}). \tag{5.63}$$

Therefore, from Eqs. (5.62) and (5.63),

$$\delta\mathfrak{I} = |\mathfrak{I}_{d_j}(\Omega_{in}^{Bob}) - \eta\mathfrak{I}_{d_j}(\Omega_{out}^{Bob|0}) - (1-\eta)\mathfrak{I}_{d_j}(\Omega_{out}^{Bob|1})|. \tag{5.64}$$

Eq. (5.64) can be analyzed by defining $\Delta\mathfrak{I}_i := |\mathfrak{I}_{d_j}(\Omega_{in}^{Bob}) - \mathfrak{I}_{d_j}(\Omega_{out}^{Bob|i})|$, that is an indicator of the phenomenon for a given post-selection[4]. Then,

$$\delta\mathfrak{I} \leq \eta\Delta\mathfrak{I}_0 + (1-\eta)\Delta\mathfrak{I}_1, \tag{5.65}$$

so that when the phenomenon does not occur for any post-selection, then $\delta\mathfrak{I} = 0$. The case when the phenomenon never occurs for that situation is when both vectors defining the initial state are parallel to $\boldsymbol{k}$.

In what follows, let the initial one-way quantum discord be analyzed for the two possible configurations in order to investigate its role for the presence of the phenomenon.

The one-way quantum discord is going to be calculated from Eq. (3.52). However, the observable for that case will be free, that is, it is written as $\sigma_n = \boldsymbol{n}\cdot\boldsymbol{\sigma}$. Considering the state (5.48), it is necessary to calculate $S(\rho_0), S(\mathrm{tr}_A\,\rho_0), S(\Phi_{\sigma_n^B}(\rho_0))$ and $S(\Phi_{\sigma_n^B}(\mathrm{tr}_A\,\rho_0))$. Before the von Neumann entropies, it is necessary calculating the term $\Phi_{\sigma_n^B}(\rho_0)$. For such, considering the Bloch representation for the projectors of σ_n,

$$\begin{aligned}
\Phi_{\sigma_n}\Big[\frac{1}{2}(\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma})\Big] &= \frac{1}{8}\Big[(\mathbb{1}+\boldsymbol{n}\cdot\boldsymbol{\sigma})(\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma})(\mathbb{1}+\boldsymbol{n}\cdot\boldsymbol{\sigma}) \\
&\qquad + (\mathbb{1}-\boldsymbol{n}\cdot\boldsymbol{\sigma})(\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma})(\mathbb{1}-\boldsymbol{n}\cdot\boldsymbol{\sigma})\Big] \\
&= \frac{1}{4}\Big[\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma}+(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2+(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{r}\cdot\boldsymbol{\sigma})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\Big] \\
&= \frac{1}{4}\Big[2\cdot\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma}+(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{n}\cdot\boldsymbol{r})+i\big((\boldsymbol{n}\times\boldsymbol{r})\cdot\boldsymbol{\sigma}\big)(\boldsymbol{n}\cdot\boldsymbol{\sigma})\Big] \\
&= \frac{1}{4}\Big[2\cdot\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma}+(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{n}\cdot\boldsymbol{r})-\big((\boldsymbol{n}\times\boldsymbol{r})\times\boldsymbol{n}\big)\cdot\boldsymbol{\sigma}\Big] \\
&= \frac{1}{2}\Big[\mathbb{1}+\big((\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{n}\big)\cdot\boldsymbol{\sigma}\Big].
\end{aligned} \tag{5.66}$$

This result shows that the measurement of σ_n projects the vector $\boldsymbol{r}$ in the direction of $\boldsymbol{n}$. Thus, it follows that

$$\Phi_{\sigma_n^B}(\rho_0) = \left[\eta\,|0\rangle\langle 0|\otimes\Big(\frac{\mathbb{1}+\big((\boldsymbol{n}\cdot\boldsymbol{r}_0)\boldsymbol{n}\big)\cdot\boldsymbol{\sigma}}{2}\Big)+(1-\eta)\,|1\rangle\langle 1|\otimes\Big(\frac{\mathbb{1}+\big((\boldsymbol{n}\cdot\boldsymbol{r}_1)\boldsymbol{n}\big)\cdot\boldsymbol{\sigma}}{2}\Big)\right]_{AB} \otimes |0011\rangle\langle 0011|_{abd_1d_2}. \tag{5.67}$$

Therefore, from the above state, the necessary entropies are

$$\begin{aligned}
S(\Phi_{\sigma_n^B}(\rho_0)) &= h(\eta)+\eta h\Big(\frac{1+\boldsymbol{r}_0\cdot\boldsymbol{n}}{2}\Big)+(1-\eta)h\Big(\frac{1+\boldsymbol{r}_1\cdot\boldsymbol{n}}{2}\Big), \\
S(\mathrm{tr}_A\,\Phi_{\sigma_n^B}(\rho_0)) &= h\Big(\frac{1+(\eta\boldsymbol{r}_0+(1-\eta)\boldsymbol{r}_1)\cdot\boldsymbol{n}}{2}\Big),
\end{aligned} \tag{5.68}$$

[4] Because of the structure of the state in (5.53), this quantity is well-defined only when $\eta\in(0,1)$.

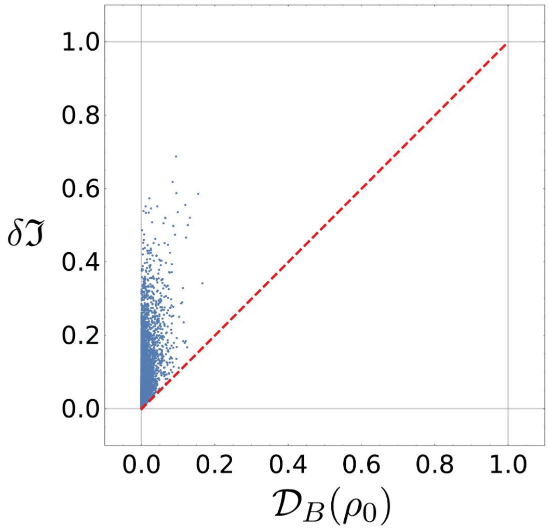

Figure 9 – The Figure illustrates the parametric plot of $\delta\mathfrak{I}$ as a function of $\mathcal{D}_B(\rho_0)$. The graph consists of 10^4 points, each corresponding to a randomly generated set $\{\eta, x_0, y_0, z_0, x_1, y_1, z_1, \theta_n, \varphi_n\}$ via the method Mersenne Twister of Mathematica.

and from the initial state (5.48),

$$\begin{aligned} S(\rho_0) &= h(\eta) + \eta h\Big(\frac{1+r_0}{2}\Big) + (1-\eta)h\Big(\frac{1+r_1}{2}\Big), \\ S(\operatorname{tr}_A \rho_0) &= h\Big(\frac{1+||\eta\boldsymbol{r_0} + (1-\eta)\boldsymbol{r_1}||}{2}\Big). \end{aligned} \tag{5.69}$$

Then, from (5.68) and (5.69), one-way quantum discord can be written as

$$\begin{aligned} \mathcal{D}_B(\rho_0) = \min_{\boldsymbol{n}} \Bigg[& h\Big(\frac{1+|\eta\boldsymbol{r_0} + (1-\eta)\boldsymbol{r_1}|}{2}\Big) - h\Big(\frac{1+(\eta\boldsymbol{r}_0 + (1-\eta)\boldsymbol{r_1})\cdot\boldsymbol{n}}{2}\Big) \\ & + \eta\Bigg(h\Big(\frac{1+\boldsymbol{r}_0\cdot\boldsymbol{n}}{2}\Big) - h\Big(\frac{1+r_0}{2}\Big)\Bigg) + (1-\eta)\Bigg(h\Big(\frac{1+\boldsymbol{r}_1\cdot\boldsymbol{n}}{2}\Big) - h\Big(\frac{1+r_1}{2}\Big)\Bigg)\Bigg]. \end{aligned} \tag{5.70}$$

As previously discussed, it was concluded that the correlation between Alice's choices and Bob's ontological description does not occur for (5.48) only when $\boldsymbol{r}_0 = z_0\boldsymbol{k}$ and $\boldsymbol{r}_1 = z_1\boldsymbol{k}$. Hence, in this situation the state is $\rho_{0,z}$, and (5.70) becomes

$$\begin{aligned} \mathcal{D}_{\mathcal{B}}(\rho_{0,z}) = \min_{\boldsymbol{n}} \Bigg[& h\Big(\frac{1+\eta z_0 + (1-\eta)z_1}{2}\Big) - h\Big(\frac{1+(\eta z_0 + (1-\eta)z_1)\boldsymbol{k}\cdot\boldsymbol{n}}{2}\Big) \\ & + \eta\Bigg(h\Big(\frac{1+z_0\boldsymbol{k}\cdot\boldsymbol{n}}{2}\Big) - h\Big(\frac{1+z_0}{2}\Big)\Bigg) + (1-\eta)\Bigg(h\Big(\frac{1+z_1\boldsymbol{k}\cdot\boldsymbol{n}}{2}\Big) - h\Big(\frac{1+z_1}{2}\Big)\Bigg)\Bigg], \end{aligned}$$

and the minimization is reached for $\boldsymbol{n} = \boldsymbol{k}$, so that

$$\mathcal{D}_B(\rho_{0,z}) = 0. \tag{5.71}$$

On the other hand, as $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ are generic vectors, (5.70) is non-zero in general, since there is no choice of $\boldsymbol{n}$ so that there is no perturbation in the initial state. However, there are other possibilities of having zero discord according to (5.70), which occur when the vectors $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ are parallel. In this case, the minimization takes place with the choice of an observable defined by $\boldsymbol{n} = \boldsymbol{r}_i$. Nevertheless, for such cases the phenomenon occurs, so that the one-way quantum discord is zero and even though the phenomenon is observed. Moreover, since the absence of discrepancy between the final irrealities depends solely on the z components of the vectors, there exist states with zero symmetric quantum discord that still exhibit the phenomenon.

Resorting to computational methods, the parametric plot in Figure 9 shows the dependence of $\delta\Im$, as written in (5.64), with $\mathcal{D}_B(\rho_0)$. From the plot, it is clear that even when quantum discord is zero, there is still a non-zero value for $\delta\Im$, and the phenomenon occurs. On the other hand, the non-occurrence of the phenomenon guarantees that the one-way quantum discord is zero. In summary, one-way quantum discord, in the sense of strictly quantum correlations, cannot be fundamental for this phenomenon.

5.3.3 The role of Realism-Based Nonlocality

As previously discussed in section 3.3.4, if entanglement and quantum discord are zero, realism-based nonlocality is not necessarily zero. Then, the investigation of this resource is of great importance for sake of completeness. Firstly, it is worth mentioning that it was shown that both entanglement and one-way quantum discord are not necessary for establishing the observed phenomenon discussed so far. This is already a significant result, demonstrating that the set of initial states capable of correlating elements of reality is vast.

Concerning realism-based nonlocality, it is necessary to investigate only one case, which is that with Bloch states in Bob's site, given by Eq. (5.48). The reason for analyzing only this state is that it is separable, and therefore not entangled, while the cases in which its discord is null are well-known. Thus, as the previous case demonstrated that discord is not necessary for observing the phenomenon, it remains to be determined what can be concluded regarding the realism-based nonlocality of this state. Another important point to be mentioned is that, as discussed in subsection 3.3.4, if one single case occurs so that $\mathcal{N}(\sigma_n^A, \sigma_m^B|\rho) > 0$, then $\mathcal{N}(\rho) > 0$ and there is realism-based nonlocality.

Considering a context given by $\{\sigma_n^A, \sigma_m^B, \rho_{AB}\}$, with

$$\rho_{AB} = \eta \left|0\right\rangle \left\langle 0\right|_A \otimes \left(\frac{\mathbb{1} + \boldsymbol{r}_0 \cdot \boldsymbol{\sigma}}{2}\right)_B + (1-\eta) \left|1\right\rangle \left\langle 1\right|_A \otimes \left(\frac{\mathbb{1} + \boldsymbol{r}_1 \cdot \boldsymbol{\sigma}}{2}\right)_B, \tag{5.72}$$

we calculate

$$\Phi_{\sigma_n^A}(\rho_{AB}) = \eta\Big(\frac{\mathbb{1} + (\boldsymbol{n}\cdot\boldsymbol{k})\sigma_n}{2}\Big)_A \otimes \Big(\frac{\mathbb{1} + \boldsymbol{r}_0\cdot\boldsymbol{\sigma}}{2}\Big)_B + (1-\eta)\Big(\frac{\mathbb{1} - (\boldsymbol{n}\cdot\boldsymbol{k})\sigma_n}{2}\Big)_A \otimes \Big(\frac{\mathbb{1} + \boldsymbol{r}_1\cdot\boldsymbol{\sigma}}{2}\Big)_B, \tag{5.73}$$

Then, choosing the eigenbasis of σ_n for $\mathcal{H}_A$, the above state becomes

$$\begin{aligned}\Phi_{\sigma_m^A}(\rho_{AB}) = \frac{1}{4}\Bigg[&\mathbb{1} + \big(\eta(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1\big)\cdot\boldsymbol{\sigma} + (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\mathbb{1}\Bigg]_B \\ &\otimes |0\rangle\langle 0|_A^n + \frac{1}{4}\Bigg[\mathbb{1} + \big(\eta(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1\big)\cdot\boldsymbol{\sigma} \\ &- (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\mathbb{1}\Bigg]_B \otimes |1\rangle\langle 1|_A^n,\end{aligned} \tag{5.74}$$

where

$$|0\rangle\langle 0|^n = \frac{\mathbb{1}+\sigma_n}{2}, \qquad |1\rangle\langle 1|^n = \frac{\mathbb{1}-\sigma_n}{2}. \tag{5.75}$$

Therefore, the above state is diagonal if one considers the subspaces spanned by $|0\rangle_n$ and $|1\rangle_n$ and the eigenvectors of the Bloch representation of each bracket. In other words, the idea is that the identity state is diagonal in any basis, so that choosing the eigenbasis of the states specified by the vectors $\eta(1\pm(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1\mp(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1$, say $|\boldsymbol{r}_{0\eta}^{\pm}\rangle$ for the first bracket and $|\boldsymbol{r}_{1\eta}^{\pm}\rangle$ for the second, the above state is diagonal in the basis $\{|0^n\boldsymbol{r}_{0\eta}^{+}\rangle, |0^n\boldsymbol{r}_{0\eta}^{-}\rangle, |1^n\boldsymbol{r}_{1\eta}^{+}\rangle, |1^n\boldsymbol{r}_{1\eta}^{-}\rangle\}_{AB}$, so that its eigenvalues are the own terms as written in the Bloch representation, and thus

$$\begin{aligned}S\big(\Phi_{\sigma_n^A}(\rho_{AB})\big) = &-\frac{1}{4}\Big[1 + ||\eta(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1|| + (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big] \\ &\cdot\log\left(\frac{1}{4}\Big[1 + ||\eta(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1|| + (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\right) \\ &-\frac{1}{4}\Big[1 - ||\eta(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1|| + (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big] \\ &\cdot\log\left(\frac{1}{4}\Big[1 - ||\eta(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1|| + (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\right) \\ &-\frac{1}{4}\Big[1 + ||\eta(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1|| - (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big] \\ &\cdot\log\left(\frac{1}{4}\Big[1 + ||\eta(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1|| - (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\right) \\ &-\frac{1}{4}\Big[1 - ||\eta(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1|| - (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big] \\ &\cdot\log\left(\frac{1}{4}\Big[1 - ||\eta(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0 + (1-\eta)(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1|| - (2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\right).\end{aligned} \tag{5.76}$$

For the calculation of $\Phi_{\sigma_m^B}\Big(\Phi_{\sigma_n^A}(\rho_{AB})\Big)$ the procedure is similar to the previous one. The difference is that when $\Phi_{\sigma_m^B}$ is applied, the state (5.73) becomes

$$\Phi_{\sigma_m^B}\Big(\Phi_{\sigma_n^A}(\rho_{AB})\Big) = \eta\Big(\frac{\mathbb{1}+(\boldsymbol{n}\cdot\boldsymbol{k})\sigma_n}{2}\Big)_A \otimes \Big(\frac{\mathbb{1}+(\boldsymbol{m}\cdot\boldsymbol{r}_0)\sigma_m}{2}\Big)_B + (1-\eta)\Big(\frac{\mathbb{1}-(\boldsymbol{n}\cdot\boldsymbol{k})\sigma_n}{2}\Big)_A \otimes \Big(\frac{\mathbb{1}+(\boldsymbol{m}\cdot\boldsymbol{r}_1)\sigma_m}{2}\Big)_B, \tag{5.77}$$

and the eigenvalues are obtained in a similar for of those in (5.74), but now the state is diagonal in the basis $\{|0^n0^m\rangle, |0^n1^m\rangle, |1^n0^m\rangle, |1^n1^m\rangle\}_{AB}$, so that the von Neumann entropy of the above state, $S\Big(\Phi_{\sigma_m^B}\big(\Phi_{\sigma_n^A}(\rho_{AB})\big)\Big) = S\Big(\Phi_{\sigma_m^B\sigma_n^A}(\rho_{AB})\Big)$, is given by

$$\begin{aligned}
S\Big(\Phi_{\sigma_m^B\sigma_n^A}(\rho_{AB})\Big) = &-\frac{1}{4}\Big[1+|(\eta(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0+(1-\eta)(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1)\cdot\boldsymbol{m}|+(2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\\
&\cdot\log\left(\frac{1}{4}\Big[1+|(\eta(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0+(1-\eta)(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1)\cdot\boldsymbol{m}|+(2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\right)\\
&-\frac{1}{4}\Big[1-|(\eta(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0+(1-\eta)(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1)\cdot\boldsymbol{m}|+(2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\\
&\cdot\log\left(\frac{1}{4}\Big[1-|(\eta(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0+(1-\eta)(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1)\cdot\boldsymbol{m}|+(2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\right)\\
&-\frac{1}{4}\Big[1+|(\eta(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0+(1-\eta)(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1)\cdot\boldsymbol{m}|-(2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\\
&\cdot\log\left(\frac{1}{4}\Big[1+|(\eta(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0+(1-\eta)(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1)\cdot\boldsymbol{m}|-(2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\right)\\
&-\frac{1}{4}\Big[1-|(\eta(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0+(1-\eta)(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1)\cdot\boldsymbol{m}|-(2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\\
&\cdot\log\left(\frac{1}{4}\Big[1-|(\eta(1-(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_0+(1-\eta)(1+(\boldsymbol{n}\cdot\boldsymbol{k}))\boldsymbol{r}_1)\cdot\boldsymbol{m}|-(2\eta-1)(\boldsymbol{n}\cdot\boldsymbol{k})\Big]\right).
\end{aligned} \tag{5.78}$$

From (5.76) and (5.78), the irreality for $\Phi_{\sigma_m^B}\Big(\Phi^A_{\sigma_n}(\rho_{AB})\Big)$ is obtained.

For the calculation of $\mathfrak{I}_{\sigma_m^B}(\rho_{AB})$, from (5.66), $\Phi_{\sigma_m^B}(\rho_{AB})$ is written as

$$\Phi_{\sigma_m^B}(\rho_{AB}) = \eta\,|0\rangle\langle 0|_A \otimes \Big(\frac{\mathbb{1}+(\boldsymbol{m}\cdot\boldsymbol{r}_0)\sigma_m)}{2}\Big)_B + (1-\eta)\,|1\rangle\langle 1|_A \otimes \Big(\frac{\mathbb{1}+(\boldsymbol{m}\cdot\boldsymbol{r}_1)\sigma_m}{2}\Big)_B, \tag{5.79}$$

so that

$$S(\Phi_{\sigma_m^B}(\rho_{AB})) = h(\eta) + \eta h\Big(\frac{1+\boldsymbol{m}\cdot\boldsymbol{r_0}}{2}\Big) + (1-\eta)h\Big(\frac{\mathbb{1}+\boldsymbol{m}\cdot\boldsymbol{r_1}}{2}\Big). \tag{5.80}$$

From Eq. (5.69) and (5.80), the irreality for σ_m^B is

$$\mathfrak{I}_{\sigma_m^B}(\rho_{AB}) = \eta\left[h\Big(\frac{1+\boldsymbol{m}\cdot\boldsymbol{r_0}}{2}\Big) - h\Big(\frac{1+r_0}{2}\Big)\right] + (1-\eta)\left[h\Big(\frac{1+\boldsymbol{m}\cdot\boldsymbol{r_1}}{2}\Big) - h\Big(\frac{1+r_1}{2}\Big)\right]. \tag{5.81}$$

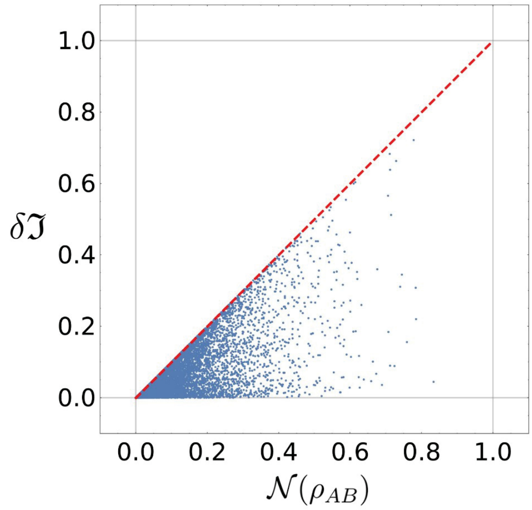

Figure 10 – The Figure illustrates the parametric plot of $\delta\mathfrak{I}$ as a function of $\mathcal{N}(\rho_{AB})$. The graph consists of 10^4 points, each corresponding to a randomly generated set $\{\eta, x_0, y_0, z_0, x_1, y_1, z_1, \theta_n, \varphi_n, \theta_m, \varphi_m\}$ via the method Mersenne Twister of Mathematica.

Note that if $\boldsymbol{m} = \boldsymbol{k}$, the above equation becomes $\mathfrak{I}_{\sigma_z^B}(\rho_{AB}) = \eta\mathfrak{I}_d(\Omega_{out}^{Bob|0}) + (1 - \eta)\mathfrak{I}_d(\Omega_{out}^{Bob|1})$, with $\mathfrak{I}_d(\Omega_{out}^{Bob|k})$ given in (5.55). Thus, from (3.58), (5.76), (5.78) and (5.81), the realism-based nonlocality is given by

$$\begin{aligned}\mathcal{N}(\rho_{AB}) = \max_{\sigma_n^A, \sigma_m^B} \Bigg[\eta \Big[h\Big(\frac{1 + \boldsymbol{m} \cdot \boldsymbol{r_0}}{2}\Big) - h\Big(\frac{1 + r_0}{2}\Big)\Big] + (1 - \eta)\Big[h\Big(\frac{1 + \boldsymbol{m} \cdot \boldsymbol{r_1}}{2}\Big) \\ - h\Big(\frac{1 + r_1}{2}\Big)\Big] - S\big(\Phi_{\sigma_m^B}(\Phi_{\sigma_n^A}(\rho_{AB}))\big) + S\big(\Phi_{\sigma_n^A}(\rho_{AB})\big)\Bigg].\end{aligned} \tag{5.82}$$

The first point to be mention (as a sanity check for the calculations) is that, indeed, if both vectors are in the $\boldsymbol{k}$ direction, $\mathcal{N}(\rho_{AB}) = 0$, as expected. However, in order to truly analyze the role of realism-based nonlocality it is necessary to resort to computational methods. Figure 10 illustrates how $\delta\mathfrak{I}$ depends on the presence of $\mathcal{N}(\rho_{AB})$. Thus, it is possible to note that $\mathcal{N} = 0$ indeed guarantees the absence of the phenomenon. However, the converse is not true, that is, the absence of the phenomenon does not guarantees a local state in the sense of $\mathcal{N} = 0$.

5.4 Final discussion

There are some important points to mention regarding the last results. As we have shown, the conclusion of the previous analysis was that none of the presented resources are fundamental for establishing the correlation. The first hypothesis to understand this result is that some of the measurements within the process may remove the dependence of the final state on the initial state. Moreover, since post-selection is essential for the occurrence of the phenomenon, it could also contribute to eliminating this dependence.

The case where both initial vectors $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ are parallel to $\boldsymbol{k}$ establishes a classical-classical state in the eigenbasis of the observable chosen to measure the degrees of freedom, i.e., the initial state is diagonal in this basis. It was shown that for these types of states, the phenomenon does not occur. However, if the observable chosen to measure the DoF is σ_n, the same conclusions would apply to vectors in the $\boldsymbol{n}$ direction. Therefore, the off-diagonal elements in the space of Bob's photon's polarization appear to be fundamental here. To deepen this discussion, the off-diagonal elements of the initial state can be assessed by its irreality, given by

$$\mathfrak{I}_{\sigma_z^B}(\rho_{AB}) = \eta \mathfrak{I}_d(\Omega_{out}^{Bob|0}) + (1-\eta)\mathfrak{I}_d(\Omega_{out}^{Bob|1}). \tag{5.83}$$

Regarding the description in terms of $\delta\mathfrak{I}$, for the case of Bloch states at Bob's site, Eq. (5.64) contains the same expression as Eq. (5.83). Moreover, due to the form of the initial state and the final descriptions of irreality, given by Eqs. (5.55) and (5.60), it is also possible to state that, for $\rho_B = \mathrm{tr}_A\,\rho_{AB}$, where ρ_{AB} is defined in Eq. (5.72),

$$\begin{aligned}\mathfrak{I}_{\sigma_z^B}(\rho_B) &= h\Big(\frac{1+|\eta z_0 + (1-\eta)z_1|}{2}\Big) - h\Big(\frac{1+||\eta \boldsymbol{r_0} + (1-\eta)\boldsymbol{r_1}||}{2}\Big) \\ &= \mathfrak{I}_{d_j}(\Omega_{in}^{Bob}).\end{aligned} \tag{5.84}$$

Thus, since $\delta\mathfrak{I}$ is written as the difference of the results in Eqs. (5.83) and (5.84), Eq. (5.64) can be rewritten as

$$\delta\mathfrak{I} = |\mathfrak{I}_{d_j}(\rho_B) - \mathfrak{I}_{\sigma_z^B}(\rho_{AB})|,$$

and from Eq. (3.36) we show that

$$\delta\mathfrak{I} = \mathcal{D}_{\sigma_z^B}(\rho_{AB}). \tag{5.85}$$

Thus, the quantity that indicates the occurrence of the phenomenon is the discord of the σ_z^B measurement on Bob's photon polarization, which is precisely the observable chosen to measure the DoF.

It is important to highlight the role of classical-classical states. Revisiting the earlier discussion, since the phenomenon's occurrence is tied to the presence of discord for the σ_z^B measurement, any initial state with off-diagonal elements in the σ_z^B basis leads to $\delta\mathfrak{I} > 0$, establishing a deep connection with measurement incompatibility. In other words, some classical-classical states result in $\delta\mathfrak{I} > 0$, indicating that the phenomenon is fundamentally linked to discord-type correlations, in addition to the irreality of σ_z^B.

Figure 11 illustrates the behavior of $\delta\mathfrak{I}$ when plotted as a function of $\mathfrak{I}_{\sigma_z^B}(\rho_{AB})$. Thus, statistically speaking it is possible to conclude from the data in the mentioned

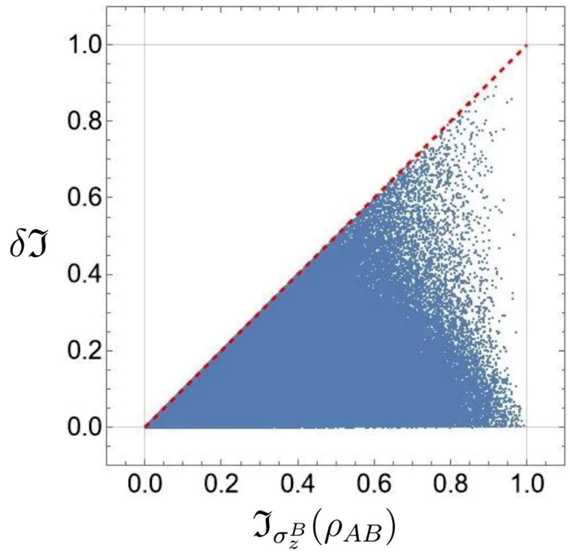

Figure 11 – The Figure illustrates the parametric plot of $\delta\mathfrak{I}$ as a function of $\mathfrak{I}_{\sigma_z^B}(\rho_{AB})$. The graph consists of 10^6 points, each corresponding to a randomly generated set $\{\eta, x_0, y_0, z_0, x_1, y_1, z_1\}$ via the method Mersenne Twister of Mathematica.

Figure that $\mathfrak{I}_{\sigma_z^B}(\rho_{AB}) \geq \delta\mathfrak{I}$, indeed establishing a condition of necessity of the initial irreality of σ_z^B for the occurrence of the phenomenon. Moreover, from the data in Figures 9 and 10, the following chain of inequalities is shown to hold[5]:

$$\mathcal{N}(\rho_{AB}) \geq \delta\mathfrak{I} \geq \mathcal{D}_B(\rho_{AB}). \tag{5.86}$$

Thus, although it is not possible to establish an equivalence between these resources, the above equation shows that $\delta\mathfrak{I}$ is bounded by the realism-based nonlocality and the minimized discord. Consequently, the phenomenon never occurs when $\mathcal{N}(\rho_{AB}) = 0$. Furthermore, Eq. (5.86) establishes that $\mathcal{D}_B(\rho_0) > 0 \Rightarrow \delta\mathfrak{I} > 0$, although the converse does not hold.

The previous discussion is easier understood when analyzed by the perspective of the result in Eq. (5.85). Indeed, as $\mathcal{D}_B(\rho_{AB}) = \min_n \mathcal{D}_{\sigma_n^B}(\rho_{AB})$, it is clear that $\mathcal{D}_{\sigma_z^B}(\rho_{AB}) \geq \mathcal{D}_B(\rho_{AB})$. Additionally, the relation between the discord of the measurement and realism-based nonlocality is considered in the hierarchy of quantum resources, with the latter representing a bigger set[6]. Then, the phenomenon occurs only when the state ρ_{AB} does not have element of reality for σ_z^B and the map $\Phi_{\sigma_z^B}$ reduces correlations. Indeed, these correlations are not genuinely quantum, since the discord of the measurement is greater than the minimized discord in general, with the latter representing the genuinely quantum correlations.

Recovering now the results of the analysis made with the Werner state, it was shown by Eq. (5.33) that the minimized discord was fundamental. However, it does not

[5] since $\mathcal{D}_B(\rho_0) = \mathcal{D}_B(\rho_{AB})$.
[6] That is, the set of states pursuing realism-based nonlocality is bigger than the discordant one.

go against our conclusions on the previous case, since for the Werner state in (5.19), Eq. (5.31) is precisely the discord of the measurement of σ_z^B. This is a particular case where the discord of the measurement encodes the genuinely quantum correlations.

The case of local realism erasure also warrants discussion. Regarding the discussion of Werner states, from Eq. (5.16), the only cases where there is no realism erasure are when $\mathfrak{I}_{d_i}(\Omega_{in}^{Bob}) = 0$, i.e., when $\eta = 0$ or $\theta = 0$. These cases align with the last point mentioned in the previous discussion: if there are no off-diagonal elements in the space of Bob's photon's polarization, there is no realism erasure. Regarding the Bloch states, Section 5.3.1 demonstrated that a classical state of polarization in Bob's photon, as described in (5.34), is incapable of generating the phenomenon, and thus, there is no realism erasure. On the other hand, for a general state of Bob's polarization, as given by (5.48), the only case where all the final irrealities are zero is when both initial vectors are aligned with $\boldsymbol{k}$, thereby establishing a classical state as before.

CHAPTER 6

Conclusions

This work deepens the analysis of the experiment proposed in [25, 26], where a modified version of a quantum eraser experiment was introduced. In this setup, depending on Alice's choice to insert a QWP into her optical arrangement, the elements of reality associated with two general degrees of freedom in Bob's laboratory are altered. Thus, the aim of this work was to investigate the quantum resource responsible for this phenomenon.

The approach considered in this work was to define an unbiased quantity from the perspective of post-selection choices, named $\delta\mathfrak{I}$. Such quantity proved to be reasonable in the detection of the phenomenon. Thus, the analysis of each quantum resource for each class of states was done by comparing the presence of each of them with the values of $\delta\mathfrak{I}$, so that it was possible to conclude the relation of necessity and sufficiency of each of the analyzed quantum resources.

Firstly, we generalized the original results by introducing noise into the system through a Werner state. This study not only recovered the original findings, but also provided new insights into the role of entanglement. While the original description focused on pure states, where quantum resources overlap, our initial approach demonstrated that entanglement is not essential for the occurrence of the phenomenon. However, this approach established a one-to-one relationship between the phenomenon and the presence of quantum discord, a measure of genuinely quantum correlations. To further investigate the role of quantum discord in this context, the next step was to consider separable states expressed in the Bloch representation.

For a class of states, which we call Bloch local states, surely not entangled, it has been shown that when Bob's system is described by an eigenstate of σ_z, regardless of Alice's state, the phenomenon never occurs. On the other hand, if Bob's system is described by a Bloch state, the phenomenon generaly occurs. Moreover, even when Bob state is nondiscordant, in the sense of possessing exclusively quantum correlations, the phenomenon still happens, demonstrating that neither entanglement nor quantum discord are essential for its occurrence.

Since entanglement and quantum discord are inconsequential for observing distinct final irrealities, realism-based nonlocality was the next quantum resource analyzed. Our approach, that considered the quantity $\delta\mathfrak{I}$, demonstrated that the presence of the phenomenon is upper bounded by realism-based nonlocality. Furthermore, a similar behavior of $\delta\mathfrak{I}$ with both $\mathcal{N}$ and $\mathfrak{I}_{\sigma_z^B}$ was observed, so that the initial irreality for σ_z^B also serves as an upper bound for $\delta\mathfrak{I}$. Consequently, the phenomenon cannot occur if $\mathcal{N}$ and $\mathfrak{I}_{\sigma_z^B}$ are zero, establishing that both these resources are necessary, although not sufficient, for the occurrence of the phenomenon.

However, while the occurrence of the phenomenon is possible for nondiscordant states in the sense discussed earlier (based on the analysis of Bloch states), it remains deeply connected to the discord of the measurement of σ_z^B, establishing equality in certain cases. In the case of the Werner state, the optimal measurement was precisely σ_z^B, leading to an equivalence between the phenomenon and quantum discord. On the other hand, for Bloch states, σ_z^B is not the optimal observable, resulting in a quantity that accounts for both classical and quantum correlations. Thus, the phenomenon is constrained to states exhibiting correlations (classical and/or quantum) and lacking elements of reality for σ_z^B. Therefore, the studies conducted in this work suggest that classical correlations and coherence in the σ_z-basis establish the phenomenon of local realism erasure.

A promising direction for future work is to extend this analysis to general two-qubit states. In this study, we examined two classes of states and derived general conclusions for them. A natural next step is to consider an arbitrary two-qubit initial state and investigate whether our conclusions hold in a broader context, potentially identifying new cases beyond those covered in this work. Additionally, future research could explore the conditions under which $\delta\mathfrak{I}$ is not just equivalent to but exactly equal to the discord of the measurement.

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