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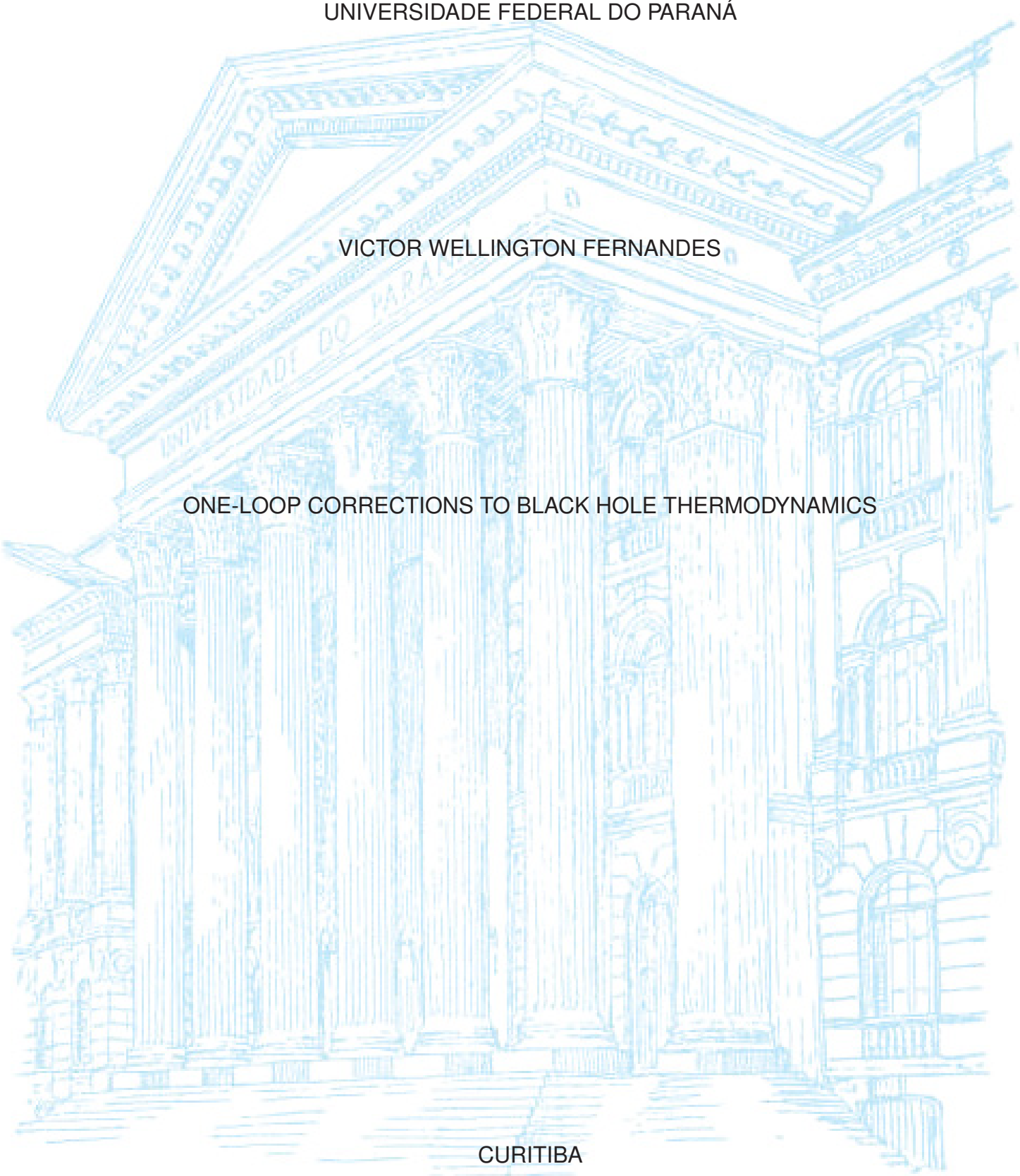


ONE-LOOP CORRECTIONS TO BLACK HOLE THERMODYNAMICS

CURITIBA

2025

VICTOR WELLINGTON FERNANDES

ONE-LOOP CORRECTIONS TO BLACK HOLE THERMODYNAMICS

Dissertação apresentada ao Programa de Pós-Graduação em Física do Setor de Ciências Exatas da Universidade Federal do Paraná como requisito parcial para a obtenção do título de Mestre em Física .

Supervisor: Prof. Iberê Oliveira Kuntz de Souza, PhD.

CURITIBA

2025

DADOS INTERNACIONAIS DE CATALOGAÇÃO NA PUBLICAÇÃO (CIP)
UNIVERSIDADE FEDERAL DO PARANÁ
SISTEMA DE BIBLIOTECAS – BIBLIOTECA DE CIÊNCIA E TECNOLOGIA

Fernandes, Victor Wellington
One-loop corrections to black hole thermodynamics / Victor Wellington Fernandes. – Curitiba, 2025.
1 recurso on-line : PDF.

Dissertação (Mestrado) - Universidade Federal do Paraná, Setor de Ciências Exatas, Programa de Pós-Graduação em Física.

Orientador: Iberê Oliveira Kuntz de Souza

1. Termodinâmica. 2. Buracos negros (Astronomia). 3. Gravitação. 4. Teoria quântica. I. Universidade Federal do Paraná. II. Programa de Pós-Graduação em Física. III. Souza, Iberê Oliveira Kuntz de. IV. Título.

Bibliotecário: Elias Barbosa da Silva CRB-9/1894

TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação FÍSICA da Universidade Federal do Paraná foram convocados para realizar a arguição da Dissertação de Mestrado de **VICTOR WELLINGTON FERNANDES**, intitulada: **"One-Loop Corrections to Black Hole Thermodynamics"**, sob orientação do Prof. Dr. IBERE OLIVEIRA KUNTZ DE SOUZA, que após terem inquirido o aluno e realizada a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.

A outorga do título de mestre está sujeita à homologação pelo colegiado, ao atendimento de todas as indicações e correções solicitadas pela banca e ao pleno atendimento das demandas regimentais do Programa de Pós-Graduação.

CURITIBA, 27 de Fevereiro de 2025.

Assinatura Eletrônica
28/02/2025 08:36:45.0
IBERE OLIVEIRA KUNTZ DE SOUZA
Presidente da Banca Examinadora

Assinatura Eletrônica
28/02/2025 17:15:03.0
JÚLIO CÉSAR FABRIS
Avaliador Externo (UNIVERSIDADE FEDERAL DO ESPÍRITO SANTO)

Assinatura Eletrônica
28/02/2025 16:36:53.0
RENATO MOREIRA ANGELO
Avaliador Interno (UNIVERSIDADE FEDERAL DO PARANÁ)

Acknowledgements

Diante dos desafios enfrentados nos últimos dois anos, gostaria de expressar minha gratidão a todas as pessoas que participaram dessa jornada, oferecendo suporte e tornando essa empreitada mais leve e agradável.

Agradeço aos meus amigos por todo o apoio, tanto nos momentos de alegria e nos cafés após o almoço quanto nos períodos mais difíceis. Um agradecimento especial àqueles que compartilham comigo a paixão pelo conhecimento humano e pela arte, em suas mais diversas formas.

Também quero agradecer às bandas terraplana, Ema Stoned, Adorável Clichê e hateyourmusic, cuja música me acompanhou durante os estudos e o processo de escrita. Boa parte deste trabalho só foi possível graças às amigas Viy e Ofie, e ao amigo Law, por terem produzido o melhor electroqueernoise e gorenoise nas bandas Of The Moon e Nëkrosë, ajudando-me a escapar dos incessantes latidos dos cães na vizinhança.

Sou imensamente grato ao professor Iberê por ter aceitado essa jornada, pela dedicação e orientação, tanto no âmbito acadêmico quanto no pessoal.

Agradeço também aos professores que se dispuseram a avaliar este trabalho, tanto na pré-defesa quanto na defesa.

Por fim, agradeço ao Programa de Pós-Graduação em Física da UFPR pela infraestrutura e à CAPES pelo suporte financeiro.

I am immensely grateful to exist in a time when I can listen to all the music Kristina Esfandiari has created. From The Mountains to Sugar High. Words cannot fully express the depth of my emotional connection with her projects, King Woman and Miserable.

I am especially thankful to the band Ragana for providing emotional support during the lowest points of the past two years, as well as to Ethel Cain for the marvelous album Perverts.

I would also like to thank Locrian, Sunn O))), and Bell Witch for creating some of the heaviest and most breathtaking music in human history. Additionally, I acknowledge Tony Iommi and Dylan Carlson, without whom drone doom metal would not exist.

Hail dro)))ne doom.

"A companhia dos sábios, ó Rei, é para mim o mais caro tesouro! O homem só vale pelo que sabe. Saber é poder. Os sábios educam pelo exemplo e nada há que avassale o espírito humano mais suave e profundamente do que o exemplo. Não deve, porém, o homem cultivar a ciência senão para utilizá-la na prática do bem."

Beremiz Samir, no livro O Homem que Calculava, de Malba Tahan

RESUMO

Nesse trabalho abordamos a termodinâmica de buracos negros utilizando o formalismo da gravitação quântica euclidiana, que se baseia na formulação das integrais de caminho da teoria quântica de campos. Apesar das diversas aplicações em física, essa formulação apresenta alguns problemas referentes a definições matemáticas, especialmente em relação à medida funcional das integrais de caminho. Na formulação usual, as integrais não são invariantes por redefinições de campos, que são similares a mudanças de coordenadas no espaço de configuração dos campos. Explorando uma melhor definição da medida funcional, para garantir a invariância das integrais de caminho, obtemos uma correção de um loop à ação efetiva do sistema considerado. No contexto da termodinâmica de buracos negros, essa correção introduz um termo na ação efetiva, similar à presence de uma constante cosmológica, a qual interpretamos como uma pressão no buraco negro.

Palavras-chaves: Termodinâmica de buracos negros; gravitação quântica; medida funcional; correção de um loop.

ABSTRACT

In this work, we study the thermodynamics of black holes using the formalism of Euclidean quantum gravity. This formalism is based on the path integral formulation of quantum field theory. Despite its wide applications in physics, this formulation presents some issues related to the mathematical definitions, especially regarding the functional measure of the path integrals. In the usual formulation, these integrals are not invariant under field redefinitions, which play the role of coordinate transformations in the configuration space of fields. Exploring a better definition of the functional measure, to render the path integrals invariant, we obtain a one-loop correction to the effective action of the system considered. In the context of black hole thermodynamics, this correction introduces a term in the effective action similar to the presence of a cosmological constant, which is interpreted as a pressure in the black hole.

Key-words: Black hole thermodynamics; quantum gravity; functional measure; one-loop correction.

List of Figures

CONTENTS

CHAPTER 1

Introduction

The theory of general relativity has been remarkably successful in describing the gravitational interaction. The other three fundamental interactions, namely the electromagnetic, strong and weak interactions, are described by the standard model of particle physics. While general relativity is formulated as a classical field theory, the standard model is based on quantum field theory. To address this asymmetry in the theoretical treatment of the fundamental interactions, physicists have been pursuing the development of a quantum theory of gravity.

General relativity successfully explained numerous gravitational phenomena, ranging from the dynamics of planetary motion within solar system to emission of gravitational waves by astronomical objects and the formation of galaxies [1]. This theory radically transformed our understanding about space, time and gravity. In general relativity, space and time are unified into a four-dimensional construct known as *spacetime*, and gravity emerges as a manifestation of the curvature of this structure.

Despite its success, certain issues suggest that general relativity may be incomplete. For instance, the theory predicts regions where the curvature of spacetime becomes infinite, leading to the formation of singularities. Furthermore, the continued collapse of matter is predicted to result in objects of infinite density, known as *black holes*. These infinities are interpreted as breakdowns of general relativity, indicating the need for a more fundamental theory to provide a proper description of gravity [2, 3].

The predictions of the standard model have demonstrated remarkable agreement with the experimental data in particles physics [4]. This model is founded on quantum field theories that describe elementary particles and their interactions. These

quantum field theories are constructed using principles of quantum mechanics and special relativity, excluding gravitational effects. However, according to the equivalence principle in general relativity, gravity influences all forms of matter. Thus, incorporating gravity into the study of elementary particles is essential for achieving a more precise description of the fundamental laws of nature [5]. Consequently, the development of a quantum theory of gravity is necessary to align gravity with the framework of the standard model.

An initial step toward a quantum theory of gravity is the approach known as *quantum field theory in curved spacetimes*. In this framework, gravity is treated as a classical field, while the matter fields are subjected to quantization. Using this treatment, Hawking predicted that black holes emit thermal radiation, a phenomenon not accounted for by general relativity [6]. Within this approach, it is also possible to derive an expression for the black hole entropy, showing it to be proportional to the area of the event horizon. These relationship was first conjectured by Bekenstein in his study of black hole thermodynamics [7]. These predictions suggest a fundamental connection between gravitation, quantum mechanics and, thermodynamics [2].

Quantum field theories are generally formulated using two main approaches. The first relies on operators and the canonical quantization of classical field theories. The second approach is based on path integrals, which provide an immediate connection to statistical mechanics through the imaginary time formalism. This connection allows us to study the thermodynamic properties of quantum fields [8]. The application of this formalism to the study of gravitation forms the basis of what is known as *Euclidean quantum gravity*. Using this approach, Hawking and Gibbons successfully derived the partition function for gravitational systems and reproduced the results previously obtained by Hawking through the operator formalism [9].

Although the path integral formulation is well-stablished and is widely employed across various areas of physics, the path integrals lack a rigorous mathematical definition. One of the primary concerns relates to the functional measure, which is not invariant under general field redefinitions [10, 11]. This is similar to the laws of physics depending on a preferred class of coordinate systems, a highly undesirable feature. To address this issue, the functional measure can be modified to ensure invariance under general field redefinitions. Such modification may introduce one-loop contributions to the effective action of the theory. This approach can be employed to give a better definition to the path integral formalism and open pathways to explore new physics, from extensions of the standard model to quantum gravity [12–14].

With these considerations in mind, the aim of this thesis is to investigate one-loop contributions to the partition function of gravitational systems. These contribution arise from the modification of the functional measure, ensuring its invariance under

general field redefinitions. We are particularly interested in obtaining modifications of the usual thermodynamic quantities of black holes due to quantum effects. To provide a comprehensive introduction to the subject of black hole thermodynamics, this thesis is organized in several chapters, covering the necessary material for a clear understanding of the central topic.

In Chapter 2, we construct the theory of general relativity from basic assumptions. To motivate the mathematical treatment adopted, we begin with a brief review of the evolution of the concepts of space and time, from classical to modern physics. Subsequently, introduce the mathematical foundations of manifolds and its application to obtain the Einstein's field equations.

Chapter 3 is devoted to a general introduction to quantum field theory. We begin with a conceptual discussion of quantum fields, followed by a review of key topics from classical and quantum physics. These reviews aim to provide a foundation for an accessible introduction to the path integral formulation of quantum field theory. The issues related to the functional measure are addressed, and we introduce the topic of statistical field theory.

Chapter 4 focuses on the study of black hole thermodynamics. The first section offers a concise review of black hole physics. subsequently, tools from statistical field theory are employed to study black hole thermodynamics.

In Chapter 5, we extend the calculations from Chapter 4 by including the one-loop contributions arising from modifications to the functional measure.

Finally, in chapter 6, we present our conclusions and outline potential directions for future work.

CHAPTER 2

General Relativity

This chapter is devoted to the presentation of the theory of general relativity. We start by discussing the concepts of space and time in classical physics and how they evolved to give origin to the idea of spacetime. In order to formulate general relativity as a geometrical theory of spacetime, we introduce the notion of manifolds and the tools needed to study their geometry. The last part of the chapter is dedicated to the derivation of the Einstein field equation. All the content presented in this chapter is based on [2] unless otherwise stated.

2.1 Spacetime

In this section, we briefly discuss the concepts of space and time in physics, as well as their evolution, from classical physics to their unification under a new structure called *spacetime* in the twentieth century.

To study a physical problem in classical physics, such as the motion of material bodies in space, we assume that the physical space has the same structure as the Euclidean space $\mathbb{R}^3$. That is, we postulate that to each position in space we can associate a point $x = (x^1, x^2, x^3)$ in $\mathbb{R}^3$, which allows us to define a coordinate system to analyse the physical phenomenon of interest. We also postulate that there exists a real parameter, called *time* and denoted by t, which parametrizes the dynamics of physical systems, i.e., we can write the evolution of physical quantities as functions depending on t. Effectively, what we do is similar to put three rigid rulers, mutually orthogonal and

with coinciding origin, at some position in space, and use a clock to track the dynamics of the system under study. We refer to individuals equipped with a method of measuring distances and a method of measuring time intervals as *observers*. The observer's clock together with the coordinate system will be called *frame of reference*. Note that space and time are assumed to exist on their own right. By saying this we mean that the time parameter is independent of the position in space and evolves in the same way in all points. Also, the structure of the space does not change as time passes. So, if it was possible to put a clock at every position in space, all previously synchronized with each other, they would always agree. Also, with the passing of time, the rulers used to define positions would never stretch, bend, nor have any change.

Since each observer can define its frame of reference at its will, by choosing a particular point to be the origin of the coordinate system and also choosing a particular orientation of coordinate axes, we can ask if all observers will describe physical phenomena in equivalent ways. That is, are the laws of physics the same for all observers? The theory of Galilean relativity is concerned exactly with the way different observers describe the same mechanical phenomena and the relation between their descriptions [15]. In the Newton formulation of classical mechanics, it is assumed that there exist a set of preferred frames of reference, called *inertial frames*. They constitute a special class of frames of reference because Newton's Laws of Motion hold in any inertial frame. This motivates the statement of a key principle in Newtonian mechanics:

> **Restricted Principle of Special Relativity:** All inertial frames are equivalent as far as dynamical experiments are concerned.

The translation from the description of a particular inertial frame to any other is done by means of a class of transformations called *Galilean transformations*. Consider an inertial frame of reference S, which uses coordinates (x^1, x^2, x^3) and time t, and a second inertial frame of reference S', which uses coordinates (x'^1, x'^2, x'^3) and time t'. Let the x^1 and x'^1 axes be parallel and let S' travel with respect to S with velocity $\mathbf{V}$ in the x^1 direction. The Galilean transformations connecting S and S' are given by

$$\begin{cases} x'^1 = x^1 - Vt, \quad (|\mathbf{V}| = V), \\ x'^2 = x^2, \\ x'^3 = x^3, \\ t' = t. \end{cases} \tag{2.1}$$

Differentiating the first expression with respect to t, and noting that $t = t'$, we have that a particle moving with velocity $\mathbf{u}$ with respect to S will move with respect to S' with velocity $\mathbf{u}'$. The relation between these velocities is given by

$$u' = u - V, \quad u = |\mathbf{u}|,\ u' = |\mathbf{u}'|. \tag{2.2}$$

Note that by the Restricted Principle of Special Relativity, the Newton Laws of Motion are invariant under Galilean transformations. There is also a physical quantity invariant under such transformations, namely, the spatial distance between to points. Hence, given two points $x,\ y\ \in \mathbb{R}^3$, their spatial distance D, defined by

$$D^2 = (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2, \tag{2.3}$$

has the same value in all frames of reference. Since this quantity is observer independent, it encodes information about the physical space, i.e., it says that the geometrical structure of space is that of the Euclidean space R^3. Therefore, to summarize, in classical physics we assume that space is immutable and has the structure of $\mathbb{R}^3$, and time is a parameter $t \in \mathbb{R}$. In Newton's formulation of classical mechanics, spatial distances and Newton's Laws of Motion are invariant under Galilean transformations. Next, we discuss the assumptions and principles used to construct the remaining two theories of relativity and what they tell us about the geometrical structure of the physical space.

The departure from the idea that space is immutable and time is absolute has its roots in Maxwell's formulation of classical electromagnetism. In this formulation, four equations give the relation between the electric and magnetic fields, and their sources, namely, the electric charge distribution and the current density. It is possible to manipulate this set of equations and derive the velocity of propagation of electromagnetic waves [15]. Surprisingly, this velocity is precisely that of light and the conclusion drawn from this fact is that light is a electromagnetic wave.

By the time of this discovery, the knowledge was that all wave phenomena required a material medium for waves to propagate. Physicists were then induced to postulate the existence of a material medium, named *luminiferous ether*, that would carry electromagnetic waves. Then, it was expected that the absolute motion of bodies through ether could be detected if experiments using light were performed. Such an experiment was conceived by the physicists Michelson and Morley in 1881 and had the goal to measure the speed with which Earth moves with respect to the ether. The outcome of their experiment was that, if Earth is moving through ether, the value of the speed is experimentally undetectable. To explain this result, two hypothesis were proposed by Lorentz, Fitzgerald and Poincaré in 1985. They postulated that material bodies would contract and clocks would slow down when through ether. The mathematical description of these effects is given by the so-called *Lorentz transformations*. These effects would affect any experimental apparatus conceived to measure the velocity of the Earth relative to the ether so as to cancel the expected results, thus explaining the null outcomes

of the Michaelson and Morley's experiment. It is worth noting that these assumptions had no physical basis, since their effects could not be verified experimentally, and were postulated in order not to abandon the idea of the luminiferous ether.

The natural conclusion drawn from the Michelson-Morley experiment was that light propagates with the same speed for all inertial observers, independently of their relative motion, and that there is no luminiferous ether. This posed a inconsistency between classical mechanics and electromagnetism, since it contradicts the Galilean Relativity (equation (2.2)). The solution to this inconsistency would result in the next theory of relativity, namely, the Special Theory of Relativity.

Starting from two postulates Einstein was able to solve the inconsistency presented previously. The first postulate is a generalization of the Restricted Principle of Special Relativity to encompass other types of physical phenomena other than mechanical ones. By noting that any performed experiment involves not only dynamics but also other branches of physics, Einstein proposed an extension of this principle. The first postulate is then stated as

> **First Postulate (Principle of Special Relativity):** All inertial frames are equivalent.

The second postulate adopted by Einstein is related to the outcome of the Michaelson-Morley experiment, although Einstein was not aware of the experimental results.

> **Second Postulate (Constancy of the Velocity of Light):** The velocity of light is the same in all inertial frames.

Note that, differently from the two hypothesis proposed by Lorentz, Fitzgerald and Poincaré, these postulates are based on physical grounds. One of the implications of the second postulate is that the simultaneity of events is observer-dependent. If two inertial observers are moving with respect to each other, two events that are simultaneous to one observer will not be simultaneous to the other [16]. That is, each observer will measure a different value for the time interval between these two events. Thus, time is not absolute! Another consequence derived from this postulate is that one inertial observer will see the ruler used by the other observer as being contracted due to their relative motion. Therefore, two inertial observers moving with respect to each other will measure different values for the time interval and spatial distance between the same two events. Since these effects are not predicted by the Galilean transformations, we have to find a more fundamental class of transformations connecting the description of two different observers.

By demanding that the speed of light have the same value for any two observers we can derive the desired transformations. These transformations are precisely the Lorentz transformations. The contraction of moving bodies and slowing down of moving clocks come now as predictions, not as ad hoc hypothesis. Consider the same setup used to enunciate the Galilean transformations between the inertial frames of reference S and S'. In this configuration, the Lorentz transformations are given by

$$\begin{cases} t' = \dfrac{t - (V/c^2)x^1}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \quad (|\mathbf{V}| = V), \\ x'^1 = \dfrac{x^1 - Vt}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \\ x'^2 = x^2, \\ x'^3 = x^3, \end{cases} \tag{2.4}$$

where c is the speed of light in vacuum. Just as spatial distances are invariant under Galilean transformations, there is also a quantity that is invariant under Lorentz transformations. From equations (2.4) we see that time intervals and spatial distances are observer-dependent. Nonetheless, there is a certain combination of these quantities that is invariant under these transformations. This fact gives us a hint that time and space are somehow connected and motivates us to merge them together in a new structure called *spacetime*. To do so, we assume that to each point in physical space we can associate a point $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$, with $x^0 = t$ the time coordinate and x^1, x^2 and x^3 the spatial coordinates. Hence, given two points $x,\, y \in \mathbb{R}^4$, we define the *spacetime interval* I between x and y by

$$I^2 = -c^2(x^0 - y^0)^2 + (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2. \tag{2.5}$$

This quantity has the same value for all inertial observers. Again, since this quantity is observer-independent, it gives us information about our physical spacetime. Namely, space and time are not separate quantities, but rather part of a more fundamental structure called the *Minkowski space*. We point out that not only the Galilean transformations need to be replaced but also the Newton laws of motion. This can be motivated by noting that the Lorentz transformations (2.4) become undefined for $V = c$. Also, if $V > c$, the lengths and time instants become imaginary. Then, we may see c as a limiting speed for all bodies. In Newtonian mechanics we do not have such a limit, and thus we must replace Newton's laws.

This new theory about the spacetime is called the *Special Theory of Relativity* and its model of spacetime is called *Minkowski spacetime*. From the Lorentz transfor-

mations (2.4) we see that different inertial observers, moving with respect to each other, will measure different lengths and time intervals for the same two events. Thus, space and time are not absolute, as it was believed in classical physics. The next and last theory of relativity we shall discuss is the so-called *General Theory of Relativity*, which takes gravitational effects into account and describe gravity as a consequence of the geometrical structure of spacetime.

If we are to consider gravitational effects, we must deal with non-inertial frames of reference. Since we cannot shield matter and energy from the influence of gravitation, any physical system that we study is affected by gravity [15]. Thus, consider two frames of reference S and S', with S an inertial one and S' describing a uniformly accelerated motion with respect to S, with acceleration $\mathbf{A}$. Let us examine a particle of mass m, under the influence of an external gravitational field $\mathbf{g}$ and a force $\mathbf{F}$ of non-gravitational origin, as measured by S. We assume that the particle moves in a small region of space, such that the gravitational field is uniform in this region, and that its velocity is small when compared to that of light. In this regime, we can use Newton's laws of motion to study the system. From Newton's second law, with the vertical axis pointing upwards, we find

$$m\mathbf{a} = -m\mathbf{g} + \mathbf{F}, \tag{2.6}$$

where $\mathbf{a}$ is the acceleration of the particle with respect to S. On the other hand, from the Galilean law of composition of velocities we obtain that the acceleration $\mathbf{a}'$ of the test particle with respect to S' is given by

$$\mathbf{a}' = \mathbf{a} - \mathbf{A}, \tag{2.7}$$

which allows us to write (2.6) as

$$m(\mathbf{a}' + \mathbf{A}) = -m\mathbf{g} + \mathbf{F}. \tag{2.8}$$

Note that in writing (2.6) we assumed the equivalence between inertial mass and gravitational mass. In classical mechanics and special relativity, this equivalence comes from experimental data. We shall see below that this equivalence is related to a fundamental principle in general relativity and played a major role in its development. Now, if we consider that S' is freely falling under the influence of the gravitational field, its acceleration with respect to S is given by $\mathbf{A} = -\mathbf{g}$. For this case, (2.8) becomes

$$m\mathbf{a}' = \mathbf{F}. \tag{2.9}$$

That is, for a freely falling frame of reference in a gravitational field, the effects of the gravitational field over the particle disappear. Thus, mechanical experiments cannot distinguish a free fall in a uniform gravitational field from a uniform motion in the absence of gravity. Einstein took a step further and extended this conclusion to the other physical laws. This culminated in the formulation of the following principle [16]

> **Equivalence Principle:** In a sufficiently small container, so that the gravitational field inside it can be taken to be uniform, in free fall in this gravitational field, the laws of physics are the same as in an inertial frame of reference in the absence of a gravitational field.

This principle implies that a body of gravitational mass m, freely falling in a uniform gravitational field, will have inertial mass m, as pointed out previously. Also, in a freely falling frame of reference, the laws of physics must reduce to those of special relativity [3]. This means that locally the spacetime of General Relativity resembles the Minkowski spacetime. Although this gives information about the local structure of spacetime, it does not give any hint on its global structure. So, to choose a mathematical structure to model the spacetime of general relativity, we require this structure to be locally similar to flat space, but do not put any restriction on its global structure. This motivates us to study the spacetime of general relativity as being a differentiable manifold. This is because the vicinity of each point of a manifold can be identified with the Euclidean space, even though the manifold as a whole may exhibit a non-Euclidean geometry. The next section is devoted to present the theory of manifolds and their geometrical properties.

Although our understanding about space and time has evolved and led to the formulation of new theories to describe nature, the emergence of a new theory does not imply that the previous are obsolete. Each theory has a specific range of validity, and the choice of description for modelling a physical phenomenon depends on the level of imprecision we are willing to accept.

2.2 Manifolds

We now introduce the concept of manifold and explore the essential aspects required to describe gravitation as a consequence of spacetime geometry. Many of the tools used to define manifolds in a rigorous manner are unnecessary for our purposes here. So, next we present just a conceptual overview of the topic only.

Roughly speaking, a n-dimensional smooth manifold M is a set of points where the neighbourhood of each point resembles $\mathbb{R}^n$. For example, a continuous

and differentiable curve is a one-dimensional manifold, a sphere is a two-dimensional manifold, etc. But what we mean by saying that the vicinity of each point of M is similar to $\mathbb{R}^n$? To state this precisely, we definite the notion of *smooth* functions.

Definition 2.2.1. Let $U \subset \mathbb{R}^n$ and $k \in \mathbb{N}$. A function $f : U \to \mathbb{R}$ is said to be C^k at $p \in U$ if its partial derivatives

$$\frac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}} \tag{2.10}$$

exist and are continuous at p for all integers $0 \leq j \leq k$. The function f is C^∞ or *smooth* at $p \in U$ if it is C^k for all $k \geq 0$.

Then, given a manifold M, it is possible to find invertible C^∞ maps $\psi_\alpha : O_\alpha \to U_\alpha$ that map any open subset $O_\alpha \subset M$ into open subsets $U_\alpha \in \mathbb{R}^n$. That is, we can identify the neighbourhood of each point of the manifold with a portion of the Euclidean space. This is a desirable feature, since we need a vector space structure to define directional derivatives, line integrals, and other operations that are employed in defining physical quantities. The functions ψ_α are called *charts* or *coordinate systems*. From now on we shall refer to a smooth manifold simply as *manifold*.

We can adopt two different approaches to study manifolds. One of them is the *extrinsic view*, in which we study a n-dimensional manifold as a subset of a higher dimensional structure. An example of this is to consider a two-dimensional sphere, denoted by S_2, as being a subset of $\mathbb{R}^3$. The other approach, called the *intrinsic view*, considers manifolds on their own right and not as being subsets of other structures. In this formulation we do not make any reference to structures other than the manifold itself when performing calculations. As discussed previously, the Equivalence Principle tells us that locally the spacetime looks like the Minkowski space $\mathbb{R}^4$. Thus, we treat spacetime as being a four-dimensional manifold. We shall adopt the intrinsic view when studying spacetime, since spacetime is not embedded in a higher dimensional space, as far as we know. Next, we introduce the tools needed to study manifolds in the intrinsic view.

2.2.1 Vectors

When considering a curved manifold, such as a sphere, we are faced with an apparent difficulty in studying calculus. If we take two points p, $q \in S_2$, the vector sum of their position vectors does not lie in the sphere and then S_2 has not a vector space structure. Then, what is the strategy to study calculus on curved manifolds? Here the notion of *tangent space* comes to help us. In multivariable calculus the tangent space at a point $p \in \mathbb{R}^n$ is defined to be the set of all vectors with origin in p [17].

Consider a line through $p = (p^1, \ldots, p^n)$, with direction $v = (v^1, \ldots, v^n)$ and parametrization

$$c(t) = p + tv, \quad t \in I \subset \mathbb{R}. \tag{2.11}$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function in the neighbourhood of the point $p \in \mathbb{R}^3$ and v a vector tangent to p. The *directional derivative* of f in the direction of v at p, denoted by $D_v f$, is defined to be

$$D_v f = \frac{d}{dt} f(c(t))\bigg|_{t=0}. \tag{2.12}$$

By the chain rule, we obtain

$$\begin{aligned} D_v f &= \sum_{\mu=1}^{n} \frac{dc^\mu}{dt}(0) \frac{\partial f}{\partial x^\mu}(p) \\ &= \sum_{\mu=1}^{n} v^\mu \frac{\partial f}{\partial x^\mu}(p). \end{aligned} \tag{2.13}$$

To each $v \in \mathbb{R}^n$ there corresponds a unique directional derivative D_v, and the converse is also true. Then, we can associate a tangent vector $v \in \mathbb{R}^3$ with the operator D_v, given by

$$D_v = \sum_{\mu=1}^{n} v^\mu \frac{\partial}{\partial x^\mu}\bigg|_p, \tag{2.14}$$

acting on $f : \mathbb{R}^3 \to \mathbb{R}$. This gives the motivation to define the tangent vectors to a manifold M at $p \in M$ as certain operators acting on functions $g : M \to \mathbb{R}$. In doing so, we define the tangent vectors in terms of functions and points on the manifold, without mentioning objects and structures external to the manifold itself. This connects exactly with the idea of the intrinsic view approach. Since derivative operators satisfy the linearity property and Leibniz rule (product rule), we require that the tangent vectors obey these same rules.

Definition 2.2.2. Let M be a manifold and $\mathscr{F}$ the set of C^∞ functions $f : M \to \mathbb{R}$. The *tangent space* at $p \in M$, denoted by T_pM, is the set of all maps $v : \mathscr{F} \to \mathbb{R}$ that satisfy the following rules:

(1) Linearity: $v(af + bg) = av(f) + bv(g), \quad \forall\, f,\, g \in \mathscr{F}$ and $a,\, b \in \mathbb{R}$;

(2) Leibniz rule: $v(fg) = f(p)v(g) + g(p)v(f), \quad f,\, g \in \mathscr{F}$.

The elements of T_pM are called *tangent vectors* or *contravariant vectors*.

Given that M is a smooth manifold of dimension $n \in \mathbb{N}$, T_pM has the structure of a vector space of dimension n. Thus, we are allowed to introduce a basis $\{X_\mu\}_{\mu=1}^n$ in the tangent space and write a tangent vector v as a linear combination of these basis vectors

$$v = \sum_{\mu=1}^{n} v^\mu X_\mu. \tag{2.15}$$

In the case of multivariable calculus, we have expanded the directional derivative operator D_v as a linear combination of partial derivatives $\{\partial/\partial x^\mu\}_{\mu=1}^n$ evaluated at p. This set constitutes the so called *coordinate basis*. Thus, the basis are dependent on the chart ψ (coordinate system) we adopt. Had we chosen another chart ψ', we would have a different basis $\{X'_\nu\}_{\nu=1}^n$. In other words, this amounts to choose a set of coordinates $\{x^\mu\}_{\mu=1}^n$ or another set of coordinates $\{x'^\nu\}_{\nu=1}^n$. Using the chain rule, we can find that the relation between the old basis and this new basis is given by

$$X_\mu = \sum_{\nu=1}^{n} \frac{\partial x'^\nu}{\partial x^\mu} X'_\nu, \tag{2.16}$$

Comparison of equations (2.15) and (2.16) gives the relation between the components of vector v in the two basis

$$v'^\nu = \sum_{\mu=1}^{n} v^\mu \frac{\partial x'^\nu}{\partial x^\mu}. \tag{2.17}$$

Equation (2.17) is known as the *vector transformation law*.

In multivariable calculus we define that a parametric curve $g : I \subset \mathbb{R} \to \mathbb{R}^n$ is smooth or C^∞ when its derivatives of all orders exist and are continuous for all $t \in I$. Since we are going to study calculus on manifolds, we shall define the notion of smoothness for functions defined on a manifold in a similar manner.

Definition 2.2.3. A *smooth curve* on a manifold M is a C^∞ map $\gamma : I \subset \mathbb{R} \to M$.

Given that a smooth curve $\gamma : I \subset \mathbb{R} \to M$ is a one-dimensional manifold, we can construct the tangent space at each point of γ by associating a tangent vector at each of its points as follows.

Definition 2.2.4. Let M be a manifold, $\gamma : I \subset \mathbb{R} \to M$ a smooth curve and $f \in \mathscr{F}$. The tangent vector $T(f)$ to γ at the point $p \in M$ is the vector

$$T(f) = \frac{d}{dt}(f \circ \gamma)\bigg|_p . \tag{2.18}$$

Since we can map all points of a manifold M on $\mathbb{R}^n$ by using its charts, we can map a curve $\gamma : I \subset \mathbb{R} \to M$ on a curve $(x^1(t), x^2(t), \ldots, x^n(t))$ in $\mathbb{R}^n$. Then, we have that the components of the tangent vector to γ are given by

$$T^\mu = \frac{dx^\mu}{dt} \tag{2.19}$$

in any coordinate basis.

We can associate a vector to each point p of a manifold M, giving rise to the notion of vector field. That is, to each point $p \in M$ we can associate a vector $v \in T_pM$. From the definition of tangent vectors, given a smooth function $f \in \mathscr{F}$, $v(f)$ is a number for all $p \in M$. Thus, $v(f)$ is a function on M and we can use this fact to define the notion of smoothness for vector fields.

Definition 2.2.5. Let M be a manifold and v a vector field on M. The vector field v is said to be *smooth* if the function $v(f)$ is smooth for each $f \in \mathscr{F}$. The set of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

Given a basis, we can express a vector v in terms of its components v^μ with respect to that basis. It follows that a vector field v is smooth if and only if its components v^μ are smooth functions.

In order to study the geometry of manifolds, we need another class of mathematical objects called *tensors*. Next, we introduce them and their application in the study of manifolds geometry.

2.2.2 Tensors

In linear algebra, given two vector spaces U and V, a linear transformation T is a function $T : U \to V$ satisfying some axioms. In the case where $V = \mathbb{R}$, we call T a *linear functional* that maps a vector $u \in U$ to a number in $T(u) \in \mathbb{R}$. The concept of tensors is a generalization of linear functionals to take several vectors, possibly from different vector spaces, and map them in a certain number in $\mathbb{R}$.

Definition 2.2.6. Let V be a finite-dimensional vector space and V^* its dual vector space. A *tensor* of type $(k,\, l)$ is a multilinear map

$$T : \underbrace{V^* \times V^* \times \cdots \times V^*}_{k \text{ copies of } V^*} \times \underbrace{V \times V \times \cdots \times V}_{l \text{ copies of } V} \to \mathbb{R}. \tag{2.20}$$

To put it in words, a tensor of type $(k,\, l)$ is function that takes as input k dual vectors and l vectors, and produces a real number. We shall denote the set of all tensors of type (k,l) by $\mathscr{T}(k,l)$. Since tensors are multilinear maps, the sum of two tensors is a tensor. Also, a tensor multiplied by a scalar is a tensor. Thus, we have that $\mathscr{T}(k,l)$ has the structure of a vector space. From the definition, a tensor of type $(1,0)$ it is a map $T : V^* \to \mathbb{R}$, which is exactly a element of V, that is, a vector. Similarly, a tensor of type $(0,1)$ is a map $U : V \to \mathbb{R}$, i.e., a element of V^* or a dual vector.

There are a few ways we can construct tensors out of other tensors. Since tensors are multilinear maps, the sum of two tensors of the same type and the product of a tensor by a scalar gives another tensor. We can also define two new operations to build new tensors. The first operation is the following:

Definition 2.2.7. Let $T \in \mathscr{T}(k,l)$. The *contraction* of T with respect to the ith dual vector and jth vector is a map $C : \mathscr{T}(k,l) \to \mathscr{T}(k-1, l-1)$ defined by

$$C(T) = \sum_{\sigma=1}^{n} T(\ldots,\, v^{*\sigma}, \ldots; \ldots,\, v_\sigma, \ldots), \tag{2.21}$$

where $\{v_\sigma\}_{\sigma=1}^n$ is a basis of V and $\{v^{*\sigma}\}_{\sigma=1}^n$ its dual basis.

The second operation is defined as follows:

Definition 2.2.8. Let $T \in \mathscr{T}(k,l)$ and Let $U \in \mathscr{T}(k',l')$. The *outer product* of T and U, denoted by $T \otimes U$, is a tensor of type $(k+k', l+l')$, defined by

$$\begin{aligned} T \otimes U(v^{*1},\, \ldots, v^{*k+k'};\, v_1,\, \ldots,\, v_{l+l'}) = \\ = T(v^{*1},\, \ldots,\, v^{*k};\, v_1,\, \ldots,\, v_l) U(v^{*k+1},\, \ldots,\, v^{*k+k'};\, v_{l+1},\, \ldots,\, v_{l+l'}). \end{aligned}$$

The outer product of tensors is also referred to as *tensor product*. Thus, we can take outer products of vectors and dual vectors to build tensors. From the definition, the tensor product of two vectors is a tensor of type $(2,0)$, the tensor product of a vector and a dual vector results in a tensor of type $(1,1)$, and so on. Let V be a n-dimensional vector space and consider a basis $\{v_\mu\}_{\mu=1}^n$ of V and $\{v^{*\nu}\}_{\nu=1}^n$ its dual basis. Then, a basis for $\mathscr{T}(k,l)$ is given by the set $\{v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{*\nu_1} \otimes \cdots \otimes v^{*\nu_l}\}$ with $\mu_j,\, \nu_j = 1, \ldots, n$. Thus, we can write every tensor $T \in \mathscr{T}(k,l)$ as a linear combination

$$T = \sum_{\mu_1=1}^{n} \cdots \sum_{\mu_k=1}^{n} \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_l=1}^{n} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, v_{\mu_1} \otimes \cdots \otimes v^{*\nu_l}. \tag{2.22}$$

The coefficients $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$ are called the *components of T* with respect to the basis $\{v_\mu\}_{\mu=1}^n$. Note that the superscripts are associated with the vectors of the basis and the subscripts are associated with the dual vectors of the basis. That is, we have k superscripts μ_j because the elements of the basis are made of a product of k vectors v_{μ_j}. Similarly, we have l subscripts because the elements of the product basis are made of a product of l dual vectors $v^{*\nu_j}$. This notation shows explicitly that the tensor T is of type (k, l). It is worth to bear in mind that, had we chosen a different basis, the values of the components $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$ would be different. Despite that, the tensor itself is basis-independent and is defined by its action on vectors and dual vectors. To avoid this basis-dependence we introduce below a special notation for tensors that also explicitly shows the type of the tensor.

Consider a tensor $T \in \mathscr{T}(k, l)$, with components $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$, and a tensor $U \in \mathscr{T}(k', l')$, with components $U^{\mu_1 \cdots \mu_{k'}}{}_{\nu_1 \cdots \nu_{l'}}$. The contraction of T has components given by

$$(C(T))^{\mu_1 \cdots \mu_{k-1}}{}_{\nu_1 \cdots \nu_{l-1}} = \sum_{\sigma=1}^{n} T^{\mu_1 \cdots \sigma \cdots \mu_{k-1}}{}_{\nu_1 \cdots \sigma \cdots \nu_{l-1}} . \tag{2.23}$$

Now, let $V = T \otimes U$. Then, its components are given by

$$V^{\mu_1 \cdots \mu_{k+k'}}{}_{\nu_1 \cdots \nu_{l+l'}} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} U^{\mu_{k+1} \cdots \mu_{k+k'}}{}_{\nu_{l+1} \cdots \nu_{l+l'}} . \tag{2.24}$$

When introducing tensors we have considered an arbitrary vector space V. Next, we will be concerned with the case where $V = T_pM$. Then, to construct tensors, we need the dual vector space of T_pM.

Definition 2.2.9. Let M be a manifold and T_pM the tangent space at $p \in M$. The *cotangent space* at $p \in M$, denoted by T_p^*M, is the dual vector space of T_pM. The elements of T_p^*M are called *cotangent vectors* or *covariant vectors*.

The coordinate basis of T_pM is denoted by $\{\partial/\partial x^\mu\}_{\mu=1}^n$ and we denote the basis of T_p^* by $\{dx^\mu\}_{\mu=1}^n$. Since $\{dx^\mu\}_{\mu=1}^n$ is the dual basis of the coordinate basis $\{\partial/\partial x^\mu\}_{\mu=1}^n$, we have

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu . \tag{2.25}$$

When changing between coordinate systems $\{x^\mu\}_{\mu=1}^n$ and $\{x'^\nu\}_{\nu=1}^n$, the components v'^ν of a vector in the new basis are related to its components v^μ in the original basis by

$$v'^\nu = \sum_{\mu=1}^{n} v^\mu \frac{\partial x'^\nu}{\partial x^\mu}, \tag{2.26}$$

where

$$\delta^{\mu}_{\nu} = \begin{cases} 0, \ \mu \neq \nu, \\ 1, \ \mu = \nu, \end{cases} \tag{2.27}$$

is the Kronecker delta.

Let ω_μ denote the components of $\omega \in T^*_p M$. Under the same coordinate transformation, the components of ω in the new basis are given by

$$\omega'_\nu = \sum_{\mu=1}^{n} \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu}. \tag{2.28}$$

Using the transformation laws (2.17) and (2.28) in the expansion (2.22) of a tensor $T \in \mathscr{T}(k,l)$, we obtain the transformation law for tensors components

$$T'^{\,\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = \sum_{\mu_1=1}^{n}\cdots\sum_{\mu_k=1}^{n}\sum_{\nu_1=1}^{n}\cdots\sum_{\nu_l=1}^{n} T^{\lambda_1\cdots\lambda_k}{}_{\rho_1\cdots\rho_l} \frac{\partial x'^{\mu_1}}{\partial x^{\lambda_1}}\cdots\frac{\partial x'^{\mu_k}}{\partial x^{\lambda_k}}\frac{\partial x^{\rho_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\rho_l}}{\partial x'^{\nu_l}}. \tag{2.29}$$

This equation is known as *tensor transformation law*. Writing tensorial equations in terms of its components makes the notation cumbersome due to the appearance of many summation symbols. In order to simplify the notation we adopt the *Einstein summation convention*, which allows us to write the equations without the summation symbols. Consider a vector space V of dimension $\dim V = n$ and $v \in V$. Given a basis $\{e_\mu\}_{\mu=1}^n$, we can write the vector v as

$$v = \sum_{\mu=1}^{n} v^\mu e_\mu. \tag{2.30}$$

Note that the summation is performed over the index μ, which appears as an upper index in the components v^μ and as a subscript in the elements of the basis e_μ. Thus, when the same index appears as an upper and lower index, the summation is implied. That is, we adopt the convention in which

$$v^\mu e_\mu = \sum_{\mu=1}^{n} v^\mu e_\mu. \tag{2.31}$$

As another example, consider again the vector v and a matrix M, with elements $M_{\mu\nu}$, $\mu, \nu = 1, \ldots, n$. From the matrix multiplication formula, we have that the vector $u = Mv$ has components

$$u_\mu = \sum_{\nu=1}^{n} M_{\mu\nu} v^\nu, \tag{2.32}$$

or, adopting the summation convention,

$$u_\mu = M_{\mu\nu} v^\nu. \tag{2.33}$$

Note that, when we sum over an index (right-hand side of equation (2.32)), the result (left-hand side) does not depend on that index. To avoid confusions concerning this convention, a repeated index must appear only twice in each term.

We have already introduced the concept of (contravariant) vector fields as the association of a vector $v \in T_pM$ to each point of a manifold M. Similarly, we define the notion of *covariant vector field* as being the association of a covariant vector $\omega \in T_p^*M$ to each point $p \in M$. We can go further and define the notion of *tensor field* by associating a tensor $T \in \mathscr{T}(k,l)$ to each point $p \in M$. Given these definitions, we establish the notion of smoothness for these fields. This can be done noting that $\omega(v)$, $v \in T_pM$, is a function on $\mathbb{R}$, since the covariant vector is map $\omega : T_pM \to \mathbb{R}$.

Definition 2.2.10. Let M be a manifold and ω a covariant vector field on M. The covariant field ω is said to be *smooth* if the function $\omega(v)$ is smooth for each $v \in \mathfrak{X}(M)$. We denote the set of all smooth covariant fields on M by $\mathfrak{X}^*(M)$.

Also, given a tensor $T \in \mathscr{T}(k,l)$, from the definition of tensors we have that the quantity $T(\omega^1, \ldots, \omega^k, v_1, \ldots, , v_l)$, $\omega^i \in T_p^*M \,\forall\, i = 1, \ldots, k$ and $v_j \in T_pM \,\forall\, j = 1, \ldots, l$, is also a function on $\mathbb{R}$.

Definition 2.2.11. Let M be a manifold and T a tensor field on M. The tensor field T is said to be *smooth* if the function $T(\omega^1, \ldots, \omega^k, v_1, \ldots, , v_l)$ is smooth for each $v_j \in \mathfrak{X}(M) \,\forall\, j = 1, \ldots, l$ and $\omega^i \in \mathfrak{X}^*(M) \,\forall\, i = 1, \ldots, k$.

Now we introduce a notation that allows us to write tensorial equations in a basis-independent way. This notation, called *abstract index notation* is similar to the notation used to denote the components of the tensor T in equation (2.22) using Greek superscripts and subscripts. In this notation, given a tensor $T \in \mathscr{T}(k,l)$ we attach to it k contravariant indices and l covariant indices using latin letters, resulting in $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$. We reserve Greek indices to denote the components of tensors. Thus, $T^\mu{}_{\nu\lambda}$ denotes the components of $T^a{}_{bc}$. Since vectors are tensors of type $(1,0)$, we denote a vector v by v^a. Similarly, since dual vectors are tensors of type $(0,1)$, we denote them as ω_a. A tensor T of type $(0,2)$ is written as T_{ab}. If we also consider two vectors v^a and w^a, the number $T(v,w)$ is written as $T_{ab}v^a w^b$. Note that the index a is related to the first position,

occupied by v, and the index b is related to the position occupied by the vector w. Thus, had we written $T_{ab}w^b v^a$, we would still know that this represents the number $T(v,w)$. Had we considered the number $T(w,v)$, in the abstract index notation this would be written as $T_{ab}w^a v^b$. Note that we can write this as $T_{ba}w^b v^a$ or $T_{ba}v^a w^b$, since the first index of the tensor T is the same of the vector w and the second index of T is related to the index of v, thus defining in an unequivocal way the order in which the vectors are taken by the tensor T. It is important to note that, in general, $T(v,w)$ and $T(w,v)$ are different numbers. Or, in index notation, $T_{ab}v^a w^b \neq T_{ba}v^a w^b$. If we have the case where $T(v,w) = T(w,v)$ for any vectors v and w, then $T_{ab}v^a w^b = T_{ba}v^a w^b$ and as a consequence $T_{ab} = T_{ba}$. In this case, we call T_{ab} a *symmetric tensor*. In the situation in which $T_{ab} = -T_{ba}$, we call T_{ab} an *antisymmetric tensor*. We remark that in $T_{ab}v^a$ the summation convention is not implied, since the indices are present only to remind us the type of tensors we are dealing with. The convention only applies when all the indices are Greek.

Given a tensor T_{ab}, we define the notations

$$\begin{aligned} T_{(ab)} &= \frac{1}{2!}\left(T_{ab} + T_{ba}\right), \\ T_{[ab]} &= \frac{1}{2!}\left(T_{ab} - T_{ba}\right). \end{aligned} \tag{2.34}$$

Note that $T_{(ab)}$ gives the symmetric part of the tensor T_{ab} and $T_{[ab]}$ its antisymmetric part. Similarly, given a tensor T_{abc} of type, $(0,3)$, we define

$$\begin{aligned} T_{(abc)} &= \frac{1}{3!}\left(T_{abc} + T_{acb} + T_{bac} + T_{bca} + T_{cab} + T_{cba}\right), \\ T_{[abc]} &= \frac{1}{3!}\left(T_{abc} - T_{acb} - T_{bac} + T_{bca} + T_{cab} - T_{cba}\right). \end{aligned} \tag{2.35}$$

Note that in defining the antisymmetric part of T_{abc}, in the right-hand side there appears all possible permutations of the indices abc, but with a minus sign for the terms of odd permutation. The same happened in defining $T_{[ab]}$, but in this case we only had two possible permutations of ab. The same definitions can be extended to tensors of any type. To *symmetrize* a tensor $T_{a_1\cdots a_l}$ means to take its symmetric part. Analogously, to *antisymmetrize* a tensor means to take its antisymmetric part. In doing so, we may specify in which indices we are symmetrizing or antisymmetrizing. For example, given a tensor T_{abcd}, symmetrizing over the indices a and b gives us $T_{abcd} + T_{bacd}$, and antisymmetrizing over a and c gives $T_{abcd} - T_{cbad}$.

Using the abstract index notation, we can write the operations introduced before in a clearer way. Given a tensor $T^a{}_{bc}$, the contraction of the indices a and b is denoted

by $T^a{}_{ac}$. Given also another tensor $S^a{}_b$, the outer product of $T^a{}_{bc}$ and $S^a{}_b$ is denoted by $T^a{}_{bc}S^d{}_e$ (we omit the $\otimes$ symbol), which corresponds exactly to a tensor of type $(2,3)$ as should be by the definition of outer product of tensors.

As a first example of a tensor used in physics, we introduce the notion of a metric. This object allows us to define a inner product on the tangent space T_pM.

Definition 2.2.12. Let M be a manifold and $p \in M$. A *metric* on M is a linear map $g : T_pM \times T_pM \to \mathbb{R}$ such that it is

1. symmetric: $g(v_1, v_2) = g(v_2, v_1) \quad \forall\, v_1,\, v_2 \in T_pM$, and;
2. nondegenerate: $g(v, v_1) = 0 \quad \forall\, v \in T_pM \iff v_1 = 0$.

We may expand g in the dual coordinate basis $\{dx^\mu\}_{\mu=1}^n$ as

$$g = \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} g_{\mu\nu}\, dx^\mu \otimes dx^\nu. \tag{2.36}$$

Considering the expansion of tangent vectors in the coordinate basis

$$\begin{aligned} u &= \sum_{\mu=1}^{n} u^\mu \frac{\partial}{\partial x^\mu}, \\ v &= \sum_{\nu=1}^{n} v^\nu \frac{\partial}{\partial x^\nu}, \end{aligned} \tag{2.37}$$

we have

$$g(u,v) = \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} g_{\mu\nu} dx^\mu \left(\sum_{\sigma=1}^{n} u^\sigma \frac{\partial}{\partial x^\sigma}\right) \otimes dx^\nu \left(\sum_{\lambda=1}^{n} v^\lambda \frac{\partial}{\partial x^\lambda}\right). \tag{2.38}$$

Using the fact that the maps dx^μ are linear and the relation (2.25), we obtain

$$g(u,v) = \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} g_{\mu\nu} u^\mu v^\nu, \tag{2.39}$$

which provides an inner product between the vectors u, $v \in T_pM$. From this formula, we define the *infinitesimal line element* ds by

$$ds^2 = \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} g_{\mu\nu}\, dr^\mu dr^\nu. \tag{2.40}$$

where dr^a is an infinitesimal displacement vector.

Given a metric g, in terms of an orthonormal basis of T_pM, we can represent g by the diagonal matrix

$$g = \text{diag}(-1, \ldots, -1, +1, \ldots, +1, 0, \ldots, 0). \tag{2.41}$$

The *signature* of g is the number of negative and positive eigenvalues of the metric [3]. If all eigenvalues are positive, the metric is called *Euclidean*. If we have only one negative eigenvalue, and all the others are positive, we call g a *Lorentzian* or *pseudo-Riemannian* metric.

The Lorentzian metrics provide a way to classify smooth curves according to their tangent vector field.

Definition 2.2.13. Let M be a manifold, g a Lorentzian metric on M, and $\gamma : I \subset \mathbb{R} \to M$ a smooth curve, with tangent field T. The curve γ is said to be

1. *timelike*, if $g(T, T) < 0$;
2. *null*, if $g(T, T) = 0$;
3. *spacelike*, if $g(T, T) > 0$.

Since a metric g is a tensor of type $(0, 2)$, we denote it by g_{ab}. If we apply g_{ab} to a vector v^b, we get $g_{ab}v^b$, which is a dual vector. We associate $g_{ab}v^b$ with the dual vector v_a. Thus, applying the metric to a vector, it "lowers" its index. We can also show that there exists the inverse of the metric g_{ab}, which we denote by g^{ab}. Thus, $g_{ac}g^{cb} = \delta^a{}_b$. The inverse metric constitutes a tensor of type $(2, 0)$ and applying it to a dual vector ω_b results in a vector $g^{ab}\omega_b$. We identify $g^{ab}\omega_b$ with ω^a. We can also raise and lower the indices of tensor, thus we have the following rules to manipulate indices

$$\begin{aligned} v^a &= g^{ab}v_b, \\ v_a &= g_{ab}v^b, \\ T^a{}_b &= g^{ac}T_{cb} = g_{bc}T^{ac}. \end{aligned} \tag{2.42}$$

The contraction of the two indices of $R^a{}_b$ results in $R^a{}_a$ and we shall denote it simply by R, since the two indices "cancel out" and the result should not depend on them.

Having introduced the basics on manifolds and tensors, we are now able to study the geometry of manifolds.

2.3 Curvature

Intuitively, we are familiar to the concept of curvature and we have used it informally up to this point in the text. In a qualitative way, we know that a two-dimensional plane is flat, whereas a sphere is curved. To study the geometry of manifolds, however, we need a quantitative definition of curvature. This section focuses on presenting the mathematical concepts necessary to define curvature. One of the characterizations of curvature is based on the concept of *parallel transport* of vectors and how this process alters the original vector. Since this process involves analysing changes in the transported vector, we begin with the definition of a derivative operator on manifold.

Definition 2.3.1. A *derivative operator*, denoted by ∇_a, on a manifold M is a map

$$\nabla_a : \mathscr{T}(k,l) \to \mathscr{T}(k,l+1) \tag{2.43}$$

satisfying the following properties:

1. Linearity: $\forall\, A,\, B \in \mathscr{T}(k,l)and\alpha,\, \beta \in \mathbb{R}$,

$$\nabla_c \left(\alpha A^{a_1\cdots a_k}{}_{b_1\cdots b_l} + \beta B^{a_1\cdots a_k}{}_{b_1\cdots b_l}\right) = \alpha \nabla_c A^{a_1\cdots a_k}{}_{b_1\cdots b_l} + \beta \nabla_c B^{a_1\cdots a_k}{}_{b_1\cdots b_l}. \tag{2.44}$$

2. Leibniz rule: $\forall\, A \in \mathscr{T}(k,l)andB \in \mathscr{T}(k',l')$,

$$\begin{aligned}\nabla_e \left(A^{a_1\cdots a_k}{}_{b_1\cdots b_l} B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{l'}}\right) = &\left(\nabla_e A^{a_1\cdots a_k}{}_{b_1\cdots b_l}\right) B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{l'}} \\ &+ A^{a_1\cdots a_k}{}_{b_1\cdots b_l} \left(\nabla_e B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{l'}}\right).\end{aligned} \tag{2.45}$$

3. Commutativity with contraction: $\forall\, A \in \mathscr{T}(k,l)$,

$$\nabla_d \left(A^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_l}\right) = \nabla_d A^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_l}. \tag{2.46}$$

4. Consistency with the notion of tangent vectors as directional derivatives on scalar fields: $\forall\, A \in \mathscr{T}(k,l)andT^a \in T_pM$,

$$T(f) = T^a \nabla_a f. \tag{2.47}$$

5. Torsion free: $\forall\, f \in \mathscr{F}$,

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f. \tag{2.48}$$

Although we attach a subscript to the derivative operator ∇, this operator is not a dual vector. We only adopt this practice to emphasize that the action of ∇ in a tensor of type (k,l) results in a tensor of type $(k,l+1)$, that is, the resulting tensor has one more subscript than the original tensor.

We may ask, is always possible to define a derivative operator on a manifold M? The answer is yes and we proceed as follows. Consider a manifold M and let $T^{a_1\cdots a_k}{}_{b_1\cdots b_l} \in \mathscr{T}(k,l)$ be a smooth tensor field on M. Let ψ be a coordinate system, with associated bases $\{v^\mu\}_{\mu=1}^n$ and $\{v_\mu\}_{\mu=1}^n$, in which the tensor $T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ has components $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$. We define the derivative operator ∂_a, called *ordinary derivative*, as the derivative operator such that $\partial_c T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ has components $\partial(T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l})/\partial x^\sigma$. Note that the definition of an ordinary derivative depends on the coordinate system chosen, since the components of $\partial_c T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ are defined to be partial derivatives with respect to the coordinates defined by ψ. Had we chosen another coordinate system, with coordinates $\{x'^\mu\}_{\mu=1}^n$, the partial derivatives would be different from the previous case.

We may also ask, if there is a relation between different derivative operators. To investigate this question, consider two derivative operators ∇ and $\widehat{\nabla}$. From condition (4) of the derivative operator definition, we have that the tangent vector t to $f \in \mathscr{F}$ can be written in terms of ∇_a and also $\widehat{\nabla}_a$. From this we obtain

$$\begin{aligned} T^a \nabla_a f &= T^a \widehat{\nabla}_a f \\ \Rightarrow \nabla_a f &= \widehat{\nabla}_a f. \end{aligned} \tag{2.49}$$

That is, the action of any two derivative operators on a scalar field f must always agree. Now we investigate if ∇ and $\widehat{\nabla}$ agree or not when acting on tensor quantities. Considering a scalar field $f \in \mathscr{F}$ and a dual vector field ω_b, we have that

$$\widehat{\nabla}_a(f\omega_b) - \nabla_a(f\omega_b) = (\widehat{\nabla}_a)\omega_b + f\widehat{\nabla}_a\omega_b - (\nabla_a f)\omega_b - f\nabla_a\omega_b, \tag{2.50}$$

and using (2.49), we obtain

$$\widehat{\nabla}_a(f\omega_b) - \nabla_a(f\omega_b) = f(\widehat{\nabla}_a\omega_b - \nabla_a\omega_b). \tag{2.51}$$

The quantities $\nabla_a\omega_b$ and $\widehat{\nabla}_a\omega_b$ depend on the value of ω_b at a point p as well as how ω_b varies in the neighbourhood of p, since derivatives are related to rates of change. However, the difference $\widehat{\nabla}_a\omega_b - \nabla_a\omega_b$ depends only on the value of ω_b at p. Consider another dual vector field ω'_b, such that $\omega'_b(p) = \omega_b(p)$. We can write the difference between these two dual vector fields as

$$\omega'_b - \omega_b = \sum_{j=1}^{n} g_j \, \xi^j_b, \tag{2.52}$$

where g_j are smooth functions, such that $g_j(p) = 0$, and ξ^j_b are a smooth dual vector fields. Applying the operator $(\nabla_a - \widehat{\nabla}_a)$ to equation (2.52) gives

$$\begin{aligned}(\widehat{\nabla}_a - \nabla_a)(\omega'_b - \omega_b) &= \sum_{j=1}^{n}(\widehat{\nabla}_a - \nabla_a)g_j\,\xi^j_b \\ &= \sum_{j=1}^{n}\left[g_j(\widehat{\nabla}\xi^j_b - \nabla\xi^j_b)\right],\end{aligned} \tag{2.53}$$

where relation (2.51) was used in the second equality. Evaluating this expression at p and using the fact that $g_j(p) = 0$, $\forall\, j = 1, \ldots, n$, we find that the right-hand side of (2.53) vanishes. The result is then

$$(\widehat{\nabla}_a - \nabla_a)\omega'_b\Big|_p = (\widehat{\nabla}_a - \nabla_a)\omega_b\Big|_p. \tag{2.54}$$

Thus, given any two smooth dual vector fields ω_b and ω'_b, which agree on their value at $p \in M$, the action of the operator $(\widehat{\nabla}_a - \nabla_a)$ on them give the same result. From this we conclude that $(\widehat{\nabla}_a - \nabla_a)\omega_b$ depends only on the value of ω_b at $p \in M$. Thus, the operator $(\widehat{\nabla}_a - \nabla_a)$ takes a dual vector ω_b at p and maps it to the tensor $(\widehat{\nabla}_a - \nabla_a)\omega_b$ of type $(0, 2)$ at p. This property characterizes a tensor field $C^c{}_{ab}$ of type $(1, 2)$, since a tensor of this type maps the dual vector ω_c into the tensor $C^c{}_{ab}\omega_c$ of type $(0, 2)$. Hence,

$$\begin{aligned}(\widehat{\nabla}_a - \nabla_a)\omega_b &= C^c{}_{ab}\omega_c \\ \Rightarrow \nabla_a\omega_b &= \widehat{\nabla}_a\omega_b - C^c{}_{ab}\omega_c.\end{aligned} \tag{2.55}$$

This gives the general relation between two derivative operators acting on dual vector fields. Consider now the case where $\omega_b = \nabla_b f$, $f \in \mathscr{F}$. From the relation (2.49) we have $\omega_b = \widehat{\nabla}_b f$. Substitution of these two relations on (2.55) yields

$$\nabla_a\nabla_b f = \widehat{\nabla}_a\widehat{\nabla}_b f - C^c{}_{ab}\nabla_c f. \tag{2.56}$$

Rewriting this equation with indices a and b exchanged, and using the property (5) of the definition of derivative operators, gives

$$\nabla_a\nabla_b f = \widehat{\nabla}_a\widehat{\nabla}_b f - C^c{}_{ba}\nabla_c f. \tag{2.57}$$

Subtracting one of this from equations from the other leads to the symmetry property of $C^c{}_{ab}$ under exchange of its lower indices

$$C^{c}{}_{ab} = C^{c}{}_{ba}. \tag{2.58}$$

To see the difference in the action of the derivative operators ∇ and $\widehat{\nabla}$ in vector fields, consider a dual vector field ω_b and a vector field v^b. Since $f = \omega_b v^b$ is a scalar, from the relation (2.49) we have

$$\begin{aligned} 0 &= (\widehat{\nabla}_a - \nabla_a)(\omega_b v^b) \\ &= [(\widehat{\nabla}_a - \nabla_a)\omega_b]v^b + \omega_b(\widehat{\nabla}_a - \nabla_a)v^b, \end{aligned} \tag{2.59}$$

and using (2.55), we obtain

$$(C^{c}{}_{ab}\omega_c)v^b + \omega_b(\widehat{\nabla}_a - \nabla_a)v^b = 0. \tag{2.60}$$

Note that in the first term in the right-hand side both indices b and c are contracted. So, we can substitute them by any other index we want. By exchanging $b \leftrightarrow c$, we obtain

$$\begin{aligned} 0 &= (C^{b}{}_{ac}\omega_b)v^c + \omega_b(\widehat{\nabla}_a - \nabla_a)v^b \\ &= \left[(\widehat{\nabla}_a - \nabla_a)v^b + C^{b}{}_{ac}v^c\right]\omega_b. \end{aligned} \tag{2.61}$$

This must be valid for any dual vector field ω_b and therefore

$$\nabla_a v^b = \widehat{\nabla}_a v^b + C^{b}{}_{ac}v^c. \tag{2.62}$$

Thus, the difference between the action of two derivative operators on vector fields is also characterized by the tensor field $C^{a}{}_{bc}$. Following the same reasoning employed to obtain (2.55) and (2.62) we can show that, for a smooth tensor field $T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$, we have the relation

$$\nabla_a T^{b_1\cdots b_k}{}_{c_1\cdots c_l} = \widehat{\nabla}_a T^{b_1\cdots b_k}{}_{c_1\cdots c_l} + \sum_{i=1}^{k} C^{b_i}{}_{ad}T^{b_1\cdots d\cdots b_k}{}_{c_1\cdots c_l} - \sum_{i=1}^{l} C^{d}{}_{ac_i}T^{b_1\cdots b_k}{}_{c_1\cdots d\cdots c_l}. \tag{2.63}$$

The choice of the tensor fields $C^{a}{}_{bc}$ is made based on the physical system under study. As an example, when considering electromagnetic interactions, the field $C^{c}{}_{ab}$ is related to the electromagnetic four-potential $A_a(x)$.

If we take the operator $\widehat{\nabla}_a$ to be the ordinary derivative ∂_a, it is customary to denote $C^{c}{}_{ab}$ by $\Gamma^{c}{}_{ab}$ and call it a *Christoffel symbol*. In this case, equation (2.62) become

$$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ac} v^c. \tag{2.64}$$

As defined here, Christoffel symbols are tensor fields associated with the ordinary derivative operator ∂_a. They are tensors in the sense that they are linear maps on vectors. However, if we perform a coordinate transformation, we also change the derivative operator ∂_a to another derivative ∂'_a. Then, the fields $\Gamma^c{}_{ab}$ associated with the derivative operator ∂_a are different from the fields $\Gamma'^c{}_{ab}$ associated with ∂'_a. This implies that the components of $\Gamma^c{}_{ab}$ will not be related to the components of $\Gamma'^c{}_{ab}$ by the transformation law (2.29). We remark that in the literature it is common to introduce tensors as certain mathematical objects that satisfy the transformation law (2.29). Then, Christoffel symbols are usually said not to be tensors.

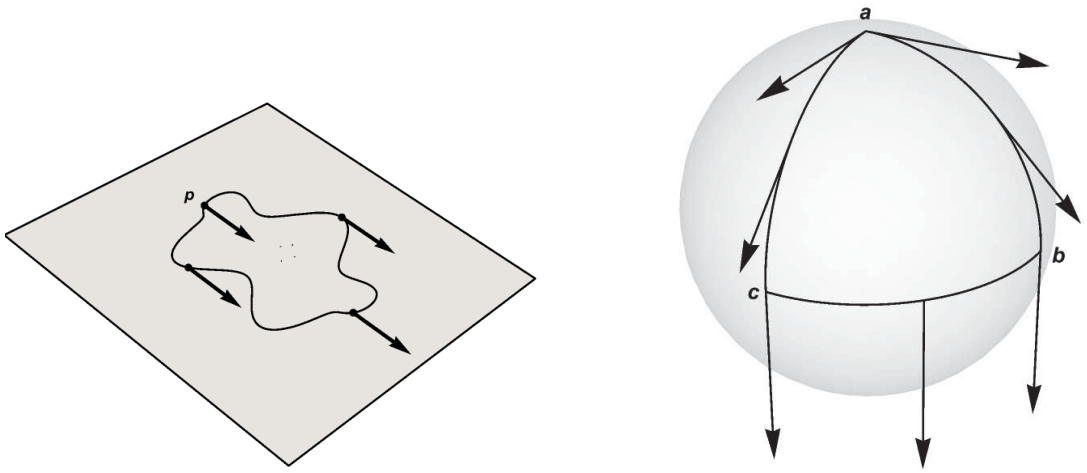
We can use the derivative operator ∇_a, defined on a manifold M, to study how quantities change as we move through M. A particular case of interest arises when we examine how a vector field varies along a curve on M. To illustrate this, we introduce the concept of *parallel transport*. Imagine moving a vector on a surface in any direction, but without rotating it about any axis. First, we consider the case where the surface in question is a plane (Figure 1(a)). We begin by placing a vector at a point p and moving it through the plane, along a given trajectory, returning it to its starting point. In this case, at the end of this process, the vector is parallel to its original configuration, regardless of the trajectory followed on the plane. However, the result is different for a curved surface. Now, consider a sphere and let us start with a vector tangent to its north pole a (Figure 1(b)). We move the vector along a trajectory, ensuring that it always remain tangent to the sphere's surface. The trajectory begins at the point a and follows a meridian until it reaches the equator at the point b. After that, we move the vector along the equator to point c. We then move the vector along another meridian, returning to the north pole a. At the end of this process, we observe that the vector's orientation differs from its initial direction.

In both cases, the vectors we transported parallel to their respective surfaces, however, a difference in the results arises due to the curvature of the sphere. Thus, we can use parallel transport to characterize the curvature of a surface. First, we provide the mathematical definition of parallel transport.

Definition 2.3.2. Let M be a manifold, $\gamma : I \subset \mathbb{R} \to M$ a smooth curve on M, with tangent field T^a, $a = 1, \ldots, 4$. A vector field v^a, defined along γ, is said to be *parallelly transported* along the curve γ if

$$T^a \nabla_a v^b \Big|_{\gamma(t)} = 0, \quad \forall\, t \in I. \tag{2.65}$$

That is, a vector field is said to be parallely transported along a curve if its directional derivative, in the direction of T^a, is zero along the curve. This does not imply that all the



(a) Parallel transport on a plane. (b) Parallel transport on a sphere.

Figure 1 – (a)The parallel transport of a vector along a closed curve on a plane does not alter the vector direction. (b) In the case of a curved surface, the parallel transport changes the vector's direction.

vectors constituting v^a are parallel. For example, consider the vector field v^a to be the one constructed by parallel transport along a meridian as in figure 1(b). Since we have drawn the sphere and the vectors as lying in $\mathbb{R}^3$, thus adopting the extrinsic view, we observe that the vectors along the meridians are not parallel. However, we must keep in mind that we are working from the intrinsic point of view. Thus, an individual living on the surface of the sphere and standing at point a would observe that the vector field is tangent to the surface. As this individual walks along the meridian, it would note that the vector field remains tangent to the surface, and have not changed with respect to the tangent vectors to the meridian. In this sense, parallelly transported vector fields are the ones that most resemble a constant vector field when we are dealing with curved manifolds. We can express equation (2.65) in terms of its components by choosing a coordinate system and using equation (2.64). Thus, we obtain

$$\begin{aligned} 0 &= T^a \nabla_a v^b \\ &= T^a \left(\partial_a v^b + \Gamma^b{}_{ac} v^c \right) \end{aligned} \tag{2.66}$$

or, in terms of components,

$$0 = T^\mu \left(\frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu{}_{\mu\lambda} v^\lambda \right). \tag{2.67}$$

Choosing a coordinate system amounts to map the curve $\gamma(t)$ to a curve with components $x^\mu(t)$ in $\mathbb{R}^4$, and thus the components of the tangent vector are given by $T^\mu = dx^\mu/dt$. Then, we write equation (2.67) as

$$0 = \frac{dx^\mu}{dt}\frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu{}_{\mu\lambda}\frac{dx^\mu}{dt}v^\lambda. \tag{2.68}$$

By the chain rule, observing that v^b is defined along γ, that is, $v^\nu = v^\nu(x^\mu(t))$, we have

$$\sum_{\mu=1}^{4}\frac{dx^\mu}{dt}\frac{\partial v^\nu}{\partial x^\mu} = \frac{dv^\nu}{dt}, \tag{2.69}$$

which allows us to write equation (2.68) as

$$\frac{dv^\nu}{dt} + \Gamma^\nu{}_{\mu\lambda}\frac{dx^\mu}{dt}v^\lambda = 0. \tag{2.70}$$

We extend the definition of parallel transport to include the case of transporting tensors along curves on a manifold.

Definition 2.3.3. Let M be a manifold, $\gamma : I \subset \mathbb{R} \to M$ a smooth curve on M, with tangent field t^a, $a = 1, \ldots, 4$. A tensor field $T^{b_1\cdots b_k}{}_{c_1\cdots c_l}$, defined along γ, is said to be *parallelly transported* along the curve γ if

$$t^a\nabla_a T^{b_1\cdots b_k}{}_{c_1\cdots c_l}\Big|_{\gamma(s)} = 0, \quad \forall\, s \in I. \tag{2.71}$$

We have seen that it is possible to define several distinct derivative operators on a manifold M, and any two of them are related by the equation (2.63). We can narrow down the possible choices of derivative operators on M by introducing a metric on our manifold and by considering the parallel transport of vectors. If two vector fields, u^a and v^a, are both parallelly transported along a curve γ, the value of the inner product $g_{ab}u^a v^b$ remains constant along γ. Thus, we require

$$T^a\nabla_a(g_{bc}u^b v^c) = 0, \tag{2.72}$$

with u^a and v^a satisfying equation (2.65). Using the Leibniz rule, we have

$$T^a u^b v^c \nabla_a g_{bc} = 0. \tag{2.73}$$

This equality hold for any curve γ and parallelly transported vector fields u^a and v^a if and only if

$$\nabla_a g_{bc} = 0. \tag{2.74}$$

This condition is known as *metric compatibility* and a derivative operator satisfying this property is called a *Levi-Civita connection*. From now on, we impose this additional

condition on our derivative operators. In fact, for a given metric g_{ab}, there exists a unique derivative operator ∇_a that satisfies this condition. Considering an arbitrary derivative operator $\widehat{\nabla}_a$ and using equation (2.63), we can write (2.74) as

$$\begin{aligned} 0 &= \nabla_a g_{bc} \\ &= \widehat{\nabla}_a g_{bc} - C^d{}_{ab} g_{dc} - C^d{}_{ac} g_{bd} \\ \Rightarrow \widehat{\nabla}_a g_{bc} &= C^d{}_{ab} g_{dc} + C^d{}_{ac} g_{bd}. \end{aligned}$$

Lowering indices using the metric, we get $C^d{}_{ab} g_{dc} = C_{cab}$ and $C^d{}_{ac} g_{bd} = C_{bac}$. Thus, we can express this equation as

$$C_{cab} + C_{bac} = \widehat{\nabla}_a g_{bc}. \tag{2.75}$$

Rewriting this same equation, but with indices interchanged, gives

$$C_{cba} + C_{abc} = \widehat{\nabla}_b g_{ac}. \tag{2.76}$$

$$C_{bca} + C_{acb} = \widehat{\nabla}_c g_{ab}. \tag{2.77}$$

Adding equations (2.75) and (2.76), subtracting equation (2.77), and using the property (2.58), we obtain the solution to $C^c{}_{ab}$

$$C^c{}_{ab} = \frac{1}{2} g^{cd} \left[\widehat{\nabla}_a g_{bd} + \widehat{\nabla}_b g_{ad} - \widehat{\nabla}_d g_{ab} \right]. \tag{2.78}$$

This choice of $C^c{}_{ab}$ solves the equation (2.74). Since $\widehat{\nabla}_a$ and $C^c{}_{ab}$ uniquely determines ∇_a, we have that ∇_a is unique.

As discussed before, in the case where we take the derivative operator $\widehat{\nabla}_a$ to be the ordinary derivative ∂_a, the tensor fields $C^a{}_{bc}$ are the Christoffel symbols $\Gamma^a{}_{bc}$. Using equation (2.78) we have

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \left[\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right], \tag{2.79}$$

or, in terms of components,

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} \sum_{\sigma=0}^{3} g^{\lambda\sigma} \left[\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right]. \tag{2.80}$$

This formula provides a straightforward way to calculate the Christoffel symbols for a given metric g_{ab}.

As discussed above, the parallel transport of a vector field along a closed curve, in general, generates a vector field different from the original one. This failure of the vector field to return to its initial configuration is related to the curvature of the space. The curvature also causes the consecutive action of derivative operators ∇_a and ∇_b to depend on the order in which the derivatives act on tensor quantities. In other words, the actions of $\nabla_a\nabla_b$ and $\nabla_b\nabla_a$ on a tensor field yield different results. Similar to the expression (2.55), we will show that this difference is characterized by a tensor field, which we will use to describe the curvature of space. To do so, let M be a manifold, ∇_a a derivative operator on M. Consider the dual vector field $\omega_a \in \mathfrak{X}^*(M)$ and $f \in \mathscr{F}$. The consecutive application of ∇_a on $f\omega_a$ gives

$$\begin{aligned}\nabla_a\nabla_b(f\omega_c) &= \nabla_a\left[(\nabla_b f)\omega_c + f\nabla_b\omega_c\right] \\ &= (\nabla_a\nabla_b f)\omega_c + \nabla_b f\nabla_a\omega_c + \nabla_a f\nabla_b\omega_c + f\nabla_a\nabla_b\omega_c.\end{aligned} \tag{2.81}$$

Evaluating $\nabla_b\nabla_a(f\omega_c)$ and subtracting the result from the above equation yields

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)(f\omega_c) = f(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c. \tag{2.82}$$

We have seen that, given two derivative operators ∇_a and $\widehat{\nabla}_a$, and a covariant vector field ω_b, all defined on the manifold M, the difference $(\widehat{\nabla}_a - \nabla_a)\omega_b$ depends only on the value of ω_b at the point $p \in M$. Following the same reasoning, we can show that the difference $(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c$ at the point $p \in M$ depends only on the value of ω_c at $p \in M$. Therefore, $(\nabla_a\nabla_b - \nabla_b\nabla_a)$ maps covariant vectors ω_c at p to tensors of type $(0,3)$ at $p \in M$. This map thus characterizes a tensor of type $(1,3)$. We shall denote this tensor field by ${R_{abc}}^d$ and we have

$$\nabla_a\nabla_b\omega_c - \nabla_b\nabla_a\omega_c = {R_{abc}}^d\omega_d. \tag{2.83}$$

The tensor ${R_{abc}}^d$ is called the *Riemann curvature tensor*. This expression shows how the action of two covariant derivatives does no commute, due to the curvature of the manifold M. Thus, the Riemann curvature tensor carries information about the curvature of M. If we had $M = \mathbb{R}^n$, the covariant derivatives would be the ordinary derivative operators, which commute. In this case, from the above expression we would obtain ${R_{abc}}^d = 0$ (the zero tensor) indicating that the manifold has no curvature, or in other words, the manifold is flat.

We can find a similar formula for the action of $(\nabla_a\nabla_b - \nabla_b\nabla_a)$ on a vector fields v^a. Since $v^a\omega_a$ is a scalar, by equation (2.48) we have

$$\begin{aligned} 0 &= (\nabla_a\nabla_b - \nabla_b\nabla_a)(v^c\omega_c) \\ &= \nabla_a\left[(\nabla_b v^c)\omega_c + v^c\nabla_b\omega_c\right] - \nabla_b\left[(\nabla_a v^c)\omega_c + v^c\nabla_a\omega_c\right] \\ &= \left[(\nabla_a\nabla_b - \nabla_b\nabla_a)v^c\right]\omega_c + v^c(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c, \end{aligned} \tag{2.84}$$

and using equation (2.83), we obtain

$$\begin{aligned} 0 &= (\nabla_a\nabla_b - \nabla_b\nabla_a)v^c\omega_c + v^c R_{abc}{}^d\omega^d \\ &= \left[(\nabla_a\nabla_b - \nabla_b\nabla_a)v^d + R_{abc}{}^d v^c\right]\omega_d. \end{aligned} \tag{2.85}$$

This expression must be valid for any smooth dual vector field ω_{cc} and then we conclude that

$$\nabla_a\nabla_b v^c - \nabla_b\nabla_a v^c = -R_{abd}{}^c v^d. \tag{2.86}$$

For the general case of a tensor field $T^{c_1\cdots c_k}{}_{d_1\cdots d_l}$ of type (k,l), we have

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)T^{c_1\cdots c_k}{}_{d_1\cdots d_l} = -\sum_{i=1}^{k} R_{abe}{}^{c_i}T^{c_1\cdots e\cdots c_k}{}_{d_1\cdots d_l} + \sum_{i=1}^{k} R_{abd_i}{}^{e}T^{c_1\cdots c_k}{}_{d_1\cdots e\cdots d_l}\,. \tag{2.87}$$

We can derive a formula to evaluate the components of the Riemann tensor in a coordinate basis. From equation (2.55), by using $\widehat{\nabla}_a = \partial_a$ and $C^a{}_{bc} = \Gamma^a{}_{bc}$, we find

$$\nabla_b\omega_c = \partial_b\omega_c - \Gamma^d{}_{bc}\omega_d. \tag{2.88}$$

Applying equation (2.63), we then have

$$\nabla_a\nabla_b\omega_c = \partial_a\left(\partial_b\omega_c - \Gamma^d{}_{bc}\omega_d\right) - \Gamma^e{}_{ab}\left(\partial_e\omega_c - \Gamma^d{}_{ec}\omega_d\right) - \Gamma^e{}_{ac}\left(\partial_b\omega_e - \Gamma^d{}_{be}\omega_d\right). \tag{2.89}$$

This equation allows us to express equation (2.83) as

$$R_{abc}{}^d\omega_d = \left[-(\partial_a\Gamma^d{}_{bc}) + (\partial_b\Gamma^d{}_{ac}) + \Gamma^e{}_{ac}\Gamma^d{}_{be} - \Gamma^e{}_{bc}\Gamma^d{}_{ae}\right]\omega_d, \tag{2.90}$$

where we have used the fact that the partial derivatives commute and the symmetry of the Christoffel symbols in their lower indices. Since the dual field ω_d is arbitrary, we obtain

$$R_{abc}{}^{d} = -(\partial_a \Gamma^{d}{}_{bc}) + (\partial_b \Gamma^{d}{}_{ac}) + \Gamma^{e}{}_{ac}\Gamma^{d}{}_{be} - \Gamma^{e}{}_{bc}\Gamma^{d}{}_{ae}. \tag{2.91}$$

By choosing a coordinate basis, the components of the Riemann tensor are given by

$$R_{\mu\nu\rho}{}^{\sigma} = -\frac{\partial}{\partial x^{\mu}}\Gamma^{\sigma}{}_{\nu\rho} + \frac{\partial}{\partial x^{\nu}}\Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\sigma}{}_{\nu\lambda} - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\sigma}{}_{\mu\lambda}. \tag{2.92}$$

Next, we obtain some properties of the Riemann curvature tensor. Consider equation (2.86) and exchange the indices a and b on both sides. This leads to the symmetry property

$$R_{abc}{}^{d} = -R_{bac}{}^{d}. \tag{2.93}$$

To obtain a second property, consider a dual vector field ω_a and a derivative operator ∇_a. For any ∇_a and any ω_a we have

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c - (\nabla_a\nabla_c - \nabla_c\nabla_a)\omega_b + (\nabla_b\nabla_c - \nabla_c\nabla_b)\omega_a = 0. \tag{2.94}$$

This can be shown using equation (2.63), with $\widehat{\nabla}_a = \partial_a$ and $C^{a}{}_{bc} = \Gamma^{a}{}_{bc}$. Using equation (2.87), the above equation yields

$$R_{abc}{}^{d} - R_{acb}{}^{d} + R_{bca}{}^{d} = 0. \tag{2.95}$$

For a derivative operator ∇_a and a metric g_{ab}, such that $\nabla_a g_{bc} = 0$, we have

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)g_{cd} = 0. \tag{2.96}$$

On the other hand, using (2.87), we can write this equation as

$$\begin{aligned} &R_{abc}{}^{e}g_{ed} + R_{abd}{}^{e}g_{ce} = 0 \\ &\Rightarrow R_{abcd} = -R_{abdc}, \end{aligned} \tag{2.97}$$

where in the last line we have lowered the last index of $R_{abc}{}^{d}$ using the metric g_{ab}. Combining equations (2.93), (2.95) and (2.97), we find

$$R_{abcd} = R_{cdab}. \tag{2.98}$$

Consider now the quantity $(\nabla_a\nabla_b - \nabla_b\nabla_a)\nabla_c\omega_d$. Using equation (2.87), this can be written as

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\nabla_c\omega_d = R_{abc}{}^e\nabla_e\omega_d + R_{abd}{}^e\nabla_c\omega_e. \tag{2.99}$$

Using the same equation, the quantity $\nabla_a(\nabla_b\nabla_c\omega_d - \nabla_c\nabla_d\omega_d)$ can be written as

$$\begin{aligned}\nabla_a(\nabla_b\nabla_c\omega_d - \nabla_c\nabla_d\omega_d) &= \nabla_a R_{bcd}{}^e\omega_e \\ &= (\nabla_a R_{bcd}{}^e)\omega_e + R_{bcd}{}^e\nabla_a\omega_e.\end{aligned} \tag{2.100}$$

Antisymmetrization over indices a, b and c in equations (2.99) and (2.100) render the left-hand side of both equations to be equal. Then, equality of the right-hand sides gives

$$(\nabla_a R_{bcd}{}^e - \nabla_a R_{cbd}{}^e - \nabla_b R_{acd}{}^e + \nabla_b R_{cad}{}^e - \nabla_c R_{bad}{}^e + \nabla_c R_{abd}{}^e)\omega_e = 0. \tag{2.101}$$

Since this must hold for any ω_a, we conclude that

$$\nabla_a R_{bcd}{}^e - \nabla_a R_{cbd}{}^e - \nabla_b R_{acd}{}^e + \nabla_b R_{cad}{}^e - \nabla_c R_{bad}{}^e + \nabla_c R_{abd}{}^e = 0. \tag{2.102}$$

This identity is called the *Bianchi identity*.

Another useful tensor, also related to the curvature of space, can be obtained from the Riemann tensor $R_{abc}{}^d$. Contracting indices b and d, yields the *Ricci tensor*

$$R_{ac} = R_{abc}{}^b. \tag{2.103}$$

From equation (2.98), we observe that R_{ab} is a symmetric tensor. The *scalar curvature* R is define to be the contraction of the two indices of the Ricci tensor. That is,

$$R = R_a{}^a = g^{ab}R_{ab}. \tag{2.104}$$

Contraction of indices a and e in the Bianchi identity (2.102), using properties (2.93) and (2.98), leads to

$$\nabla_a R_{bcd}{}^a + \nabla_b R_{cd} - \nabla_c R_{bc} = 0. \tag{2.105}$$

Now, using the metric g^{ed} to raise the index d in the above equation, and contracting the indices e and b, results in

$$\nabla_a R_c{}^a + \nabla_b R_c{}^b - \nabla_c R = 0. \tag{2.106}$$

Note that the first two terms are equal, since the index of the derivative is contracted with the upper index of $R_c{}^a$. Thus, defining the *Einstein tensor* as

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}, \tag{2.107}$$

we can express equation (2.106) as

$$\nabla^a G_{ab} = 0, \tag{2.108}$$

where $\nabla^a = g^{ab}\nabla_b$. The Einstein tensor is a key component in the Einstein field equation, which relates the geometry of spacetime with its matter and energy distribution.

2.3.1 Geodesics

When dealing with curved manifolds, we may not able to draw a straight line connecting two points of the manifold in such a way that this line lies entirely inside the manifold. Consider again the two-dimensional sphere S_2. Given two points p, $q \in S_2$, there are infinitely many curves on S_2 connecting these points. We may ask, which of these curves is the straightest? Such a curve is called a *geodesics*. Note that the curves that most look like a straight line are those whose tangent vector field most closely resembles a constant vector field. This idea connects to the concept of parallel transport of vector fields and motivates the following definition:

Definition 2.3.4. Let M be a manifold, $\gamma : I \subset \mathbb{R} \to M$ a smooth curve on M, with tangent field T^a, $a = 1, \ldots, 4$. A *geodesics* is a curve γ such that

$$T^a \nabla_a T^b \Big|_{\gamma(t)} = 0, \quad \forall\, t \in I. \tag{2.109}$$

We may write equation (2.109) in terms of its components. From equation (2.70), with $v^a = T^a$, we have

$$\frac{dT^\mu}{dt} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{dt}t^\lambda = 0. \tag{2.110}$$

Also, from equation (2.19), the tangent vector field has components

$$T^\mu(t) = \frac{dx^\mu(t)}{dt}. \tag{2.111}$$

Thus, the geodesic equation, in a given coordinate basis, is written as

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = 0. \tag{2.112}$$

Note that equation (2.112) represents a coupled system of second-order ordinary differential equations for the functions $x^\mu(t)$. From the theory of ordinary differential equations, there always exists a unique solution to this system for any given initial value of x^μ and dx^μ/dt. Hence, for a given point $p \in M$ and any tangent vector field $T^a \in \mathfrak{X}(M)$, there always exists a unique geodesics through p with tangent vector field T^a. The existence and uniqueness of geodesics allow us to use them to construct coordinate systems on the manifold.

Consider two distinct geodesics, with tangent vector fields initially parallel (Figure 2). Due to the curvature of the manifold, the geodesics may bend towards or away from one another.

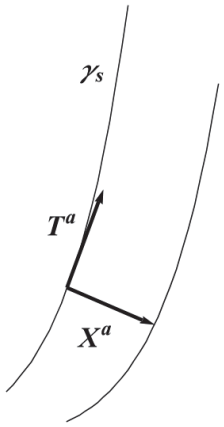


Figure 2 – Two geodesics, with tangent vector field T^a and deviation vector X^a.

This provides another way to characterize the curvature of the manifold. Let $\gamma_s(t)$ denote a family of geodesics, where for each $s \in \mathbb{R}$, the curve γ_s is a geodesic. Since the parameter $s \in \mathbb{R}$ relates two nearby geodesics, the vector field $X^a = (\partial/\partial s)^a$ represents the displacement toward a infinitely close geodesic and is called the *deviation vector*. We can construct the vector fields T^a and X^a to be orthogonal everywhere. Since T^a and X^a commute, we have

$$T^b \nabla_b X^a = X^b \nabla_b T^a. \tag{2.113}$$

Consider the quantity

$$v^a = T^b \nabla_b X^a, \tag{2.114}$$

which gives the rate of change of X^a as we move along the geodesic. Thus, v^a is interpreted as the relative velocity between two nearby geodesics. In a similar manner, we interpret

$$a^a = T^c \nabla_c v^a \tag{2.115}$$

as the relative acceleration of two nearby geodesics. Using equation (2.114), we find

$$a^a = T^c \nabla_c (T^b \nabla_b X^a), \tag{2.116}$$

and with equation (2.113), this becomes

$$\begin{aligned} a^a &= T^c \nabla_c (X^b \nabla_b T^a) \\ &= (T^c \nabla_c X^b)(\nabla_b T^a) + X^b T^c \nabla_c \nabla_b T^a. \end{aligned} \tag{2.117}$$

Using the relation (2.87) to invert the order of the derivatives in the last term gives

$$\begin{aligned} a^a &= (T^c \nabla_c X^b)(\nabla_b T^a) + X^b T^c \nabla_b \nabla_c T^a - R_{cbd}{}^a X^b T^c T^d \\ &= (T^c \nabla_c X^b)(\nabla_b T^a) + X^c T^b \nabla_c \nabla_b T^a - R_{cbd}{}^a X^b T^c T^d, \end{aligned} \tag{2.118}$$

where we have made the substitution of indices $b \leftrightarrow c$ in the last term. We can rewrite this equation as

$$\begin{aligned} a^a &= (T^c \nabla_c X^b)(\nabla_b T^a) + X^c T^b \nabla_c \nabla_b T^a - R_{cbd}{}^a X^b T^c T^d \\ &= X^c \nabla_c (T^b \nabla_b T^a) - R_{cbd}{}^a X^b T^c T^d, \end{aligned} \tag{2.119}$$

and using equation (2.109), we find

$$a^a = -R_{cbd}{}^a T^c X^b T^d. \tag{2.120}$$

This equation (2.120) is known as the *geodesic deviation equation*. In the next section we shall use this equation to obtain the Einstein Field Equation.

2.4 Einstein Field Equation

The Einstein field equation is a non-linear partial differential equation for the metric tensor g_{ab} of spacetime. This equation relates the geometry of spacetime to the matter and energy distribution. In the previous section, we have developed the mathematical framework used to describe the geometrical part. Now, we will explore how to describe the matter and energy content. From now on we follow a unit system, called the *geometrized units* system, in which the gravitational G constant and the speed of light c are dimensionless and set to $G = c = 1$.

2.4.1 Special Relativity

As previously discussed, in special Relativity there exists a special class of observers, called *inertial observers*. These observers can define a coordinate system using coordinates $(x^0, x^1, x^2, x^3) \in \mathbb{R}^4$. Note that in the context of special and general relativity, the components of tensor quantities are enumerated starting from 0. In any inertial coordinate system, the spacetime interval I, defined by

$$I^2 = -(x^0 - \bar{x}^0)^2 + (x^1 - \bar{x}^1)^2 + (x^2 - \bar{x}^2)^2 + (x^3 - \bar{x}^3)^2, \tag{2.121}$$

has the same value. If we define the quantity $\eta_{\mu\nu}$ by

$$\eta_{\mu\nu} = \begin{cases} -1, & \mu = \nu = 0, \\ 0, & \mu \neq \nu, \\ +1, & \mu = \nu = 1, 2, 3, \end{cases} \tag{2.122}$$

we can write

$$I^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu}(x^\mu - \bar{x^\mu})(x^\nu - \bar{x^\nu}). \tag{2.123}$$

This expression resembles equation (2.39) for the inner product, with $u^\mu = v^\mu = (x^\mu - \bar{x}^\mu)$. We can represent the metric η_{ab} by a square matrix, given by

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.124}$$

Then, η_{ab} is a metric of Lorentzian signature. Since the components of this metric are constant, using the ordinary partial derivative operator ∂_a, we have

$$\partial_a \eta_{bc} = 0, \tag{2.125}$$

and ∂_a is the derivative operator naturally associated with η_{ab}. From equations (2.125) and (2.80), the Christoffel symbols vanish. Consequently, from equation (2.92), the Riemann curvature tensor also vanishes. In this case, we say that η_{ab} is *flat* metric. By the geodesic equation (2.112), we conclude that the geodesics of η_{ab} are straight lines. Since the trajectories of inertial observers are straight lines, we conclude that their trajectories are timelike geodesics. This provides another way of stating that nothing can

travel faster than the speed of light in vacuum. Therefore, in the framework of special relativity, spacetime has the manifold structure of $\mathbb{R}^4$, with a flat metric η_{ab} of Lorentz signature defined on it.

We now turn our attention to the motion of material particles. Since material particles have mass $m > 0$, they travel along timelike curves w^a. These curves can be parametrized by the *proper time* τ, which is defined as

$$\tau = \int \sqrt{-\eta_{ab} T^a T^b} ds, \tag{2.126}$$

where $s \in \mathbb{R}$ is an arbitrary parametrization of the curve, and $T^a = dw^a/ds$ is the tangent vector to the curve, when parametrized by s. The parametrization of w^a can be changed by using $s = s(\tau)$, leading to a reparametrized curve $w^a = w^a(\tau)$. With this parametrization, the tangent vector u^a to the timelike curve w^a is called the *4-velocity* of the curve. The 4-velocity has unit lenght, since

$$\begin{aligned} u^a &= \frac{dw^a}{d\tau} \\ &= \frac{dw^a}{ds}\frac{ds}{d\tau} \\ &= T^a \frac{ds}{d\tau}. \end{aligned} \tag{2.127}$$

From the definition (2.126), we obtain

$$\begin{aligned} u^a &= T^a \frac{1}{\sqrt{-\eta_{bc} T^b T^c}} \\ \Rightarrow u^a u_a &= -T^a T_a \frac{1}{T^b T_b} \\ &= -1, \end{aligned} \tag{2.128}$$

where we have lowered the index of T^c using the metric η_{bc}. Since free particles move along geodesics, their 4-velocity satisfy the geogesic equation

$$u^a \partial_a u^b = 0, \tag{2.129}$$

in agreement with our previous discussion. Note that in the rest frame of a particle, with trajectory w^a, we have

$$\frac{dw^1}{d\tau} = \frac{dw^2}{d\tau} = \frac{dw^3}{d\tau} = 0. \tag{2.130}$$

Using this together with $u^a u_a = -1$ we conclude that in the rest frame of the particle, the 4-velocity u^a has components $(1, 0, 0, 0)$ (in units where $c = 1$). We define the *4-momentum* p^a of a massive particle by

$$p^a = mu^a. \tag{2.131}$$

The *energy* of the particle, with 4-momentum $p^a = mu^a$, as measured by an observer with 4-velocity v^a, is defined to be

$$E = -p_a v^a. \tag{2.132}$$

If the observer is at rest with respect to the particle, we have $v^a = u^a$ and then

$$E = -mu_a u^a = m, \tag{2.133}$$

which is the formula $E = mc^2$ for the rest energy of a massive particle. As defined in equation (2.132), the energy is measured by an observer at the position of the particle. Recall that the parallel transport of a vector in flat spacetime does not alter its orientation or magnitude. Thus, we define the energy of a particle, as measured by an observer not at the particle's position, to be the same as measured by an observer at the site of the particle, with both observers having parallel 4-velocities.

To describe continuous matter distributions in special relativity, we introduce the *stress-energy momentum tensor* T_{ab}, a symmetric tensor that encodes information about the physical properties of matter, such as energy, pressure, and momentum. For example, for an observer with 4-velocity v^a, the quantity $T_{ab}v^a v^b$ represents the mass-energy density of matter. A *perfect fluid* is defined as a continuous matter distribution that can be fully characterized by its energy density ρ and pressure P [3]. The energy-momentum tensor T_{ab} for a perfect fluid is given by

$$T_{ab} = \rho u_a u_b + P(\eta_{ab} + u_a u_b), \tag{2.134}$$

where u^a is the field representing the 4-velocity of the fluid. The equation of motion for a perfect fluid, in the absence of external forces, is given by

$$\partial_a T^{ab} = 0. \tag{2.135}$$

Substituting equation (2.134) in equation (2.135), and contracting with u_b, gives

$$\begin{aligned} 0 &= u_b \partial_a T^{ab} \\ &= -(\partial_a \rho + \partial_a P) u^a + (\rho + P)\left[-(\partial_a u^a) + u^a u^b (\partial_a u^b)\right] + (\partial_a P) u^a. \end{aligned} \tag{2.136}$$

From equation (2.128) we have

$$\begin{aligned} \partial_a (u^b u_b) &= 0 \\ \Rightarrow u_b \partial_a u^b &= 0. \end{aligned} \tag{2.137}$$

Then, the equation of motion (2.136) becomes

$$u^a \partial_a \rho + (\rho + P) \partial_a u^a = 0. \tag{2.138}$$

In the non-relativistic limit, the particles composing the fluid are expected to move with speed $|\mathbf{v}| \ll 1$. The pressure is associated with the random motion of the fluid particles. In this limit, we expect that $P \ll \rho$ [3]. Since u^a has components $(1,\, v_x,\, v_y,\, v_z)$, equation (2.138) can be expressed in terms of components as

$$\begin{aligned} 0 &= u^\mu \partial_\mu \rho + (\rho + P) \partial_\mu u^\mu \\ &\approx u^\mu \partial_\mu \rho + \rho \partial_\mu u^\mu \\ &= \frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} v^j \frac{\partial \rho}{\partial x^j} + \rho \sum_{j=1}^{3} \frac{\partial v^j}{\partial x^j}. \end{aligned} \tag{2.139}$$

Using the nabla operator ∇ from vector calculus, we can write this equation as

$$\begin{aligned} 0 &= \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} \\ &= \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}), \end{aligned} \tag{2.140}$$

which is the continuity equation for ρ, where $\mathbf{v}$ is the usual velocity from mechanics. In the next section, we present the description of physical quantities in the scope of General Relativity.

2.4.2 General Relativity

In the discussion about the structure of spacetime, under the light of general relativity, we pointed out that no body can be shielded from the gravitational influence.

Thus, any observer measuring the properties of a test particle would move in the same way as the test particle, since gravity would affect both in the same manner. As a result, the observer cannot detect the action of gravity on the test particle.

To construct the theory of relativity that describes gravity, we assume that the spacetime metric is not flat, differently from special relativity. In addition, we assume that the trajectory of freely falling bodies in a gravitational field are the geodesics of the spacetime metric. Thus, the motion of bodies under the influence of gravity alone will be determined by the geometry of spacetime. In this way, we interpret the gravitational influence as a consequence of the curvature of spacetime, rather than a force, as in Newton's theory of gravity.

We assume two guiding principles to formulate the laws of physics in curved spacetime:

> **Principle of General Covariance:** The laws of physics are invariant under coordinate transformations and must be expressible in tensorial form.
>
> **Reduction to Special Relativity:** The equations of General Relativity must reduce to those of Special Relativity in the case where the metric g_{ab} is flat.

Tensorial equations hold in any coordinate system we choose. Also, all tensors satisfy the transformation law (2.29), so every term in a tensorial equation transforms in the same way under a coordinate transformation. This is the essence of covariance and ensures that the equations retain the same form under coordinate transformations. Therefore, it is desirable express physical laws in tensorial form, since coordinate systems are a human construct and do not exist in nature. Adopting the Principle of General Covariance guarantees that our equations will be valid, no matter the coordinate system we choose.

The second principle we invoked is motivated by the Equivalence Principle, which asserts that locally, the spacetime takes the same form as the Minkowski spacetime, and the laws of Special Relativity hold. Together, these two principles suggest that we replace the metric η_{ab} in the equations of Special Relativity by g_{ab} and the derivative operator ∂_a by ∇_a, where ∇_a the derivative operator associated with g_{ab}, satisfying $\nabla_a g_{ab} = 0$. This procedure helps us to generalize the laws of physics from flat spacetime to curved spacetimes.

Since we are considering the possibility that spacetime M is curved, the spacetime metric g_{ab} may not be flat, and the structure of spacetime M may differ from that of $\mathbb{R}^4$. However, since we are only considering these two modifications, we may continue to describe physical quantities using tensors. Thus, the motion of particles will still be

described by a timelike curve, and perfect fluids will be described in terms of a 4-velocity u^a, a density ρ and pressure P. Hence, similarly to special relativity, we define the 4-velocity u^a of a particle as the unit vector tangent to its trajectory. A particle under the influence of gravity alone satisfies the geodesic equation

$$u^a \nabla_a u^b = 0. \tag{2.141}$$

The 4-momentum p^a of a particle is defined by

$$p^a = m u^a. \tag{2.142}$$

The energy of a particle with 4-velocity u^a, measured by and observer with 4-velocity v^a at the site of the particle, is given by

$$E = -p_a v^a. \tag{2.143}$$

Here, we encounter an important difference from special relativity. Since spacetime is curved and parallel transport is path-dependent, we lose the notion of parallel vectors at different points in spacetime. Thus, we cannot define the energy of a particle as measure by a distant observer. As before, we describe a perfect fluid using the energy-momentum tensor T_{ab} given by

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b), \tag{2.144}$$

satisfying the equation of motion

$$\nabla^a T_{ab} = 0. \tag{2.145}$$

We now seek the equation that describes the relation between spacetime geometry and matter distribution. To gain insight on the form of this equation, we compare the description of tidal forces by Newtonian theory of gravity and general relativity. In the Newtonian formulation, the gravitational field generated by a mass M can be described in terms of a potential Φ, with the acceleration (force per unit mass) of a particle of mass m immersed in this gravitational field given by the Newton's Law of Motion

$$\frac{d^2 \mathbf{r}(t)}{dt^2} = -\nabla \Phi, \tag{2.146}$$

where $\mathbf{r}(t)$ represent the position of the particle at time t. Now, consider two test particles falling toward a spherical mass distribution M (figure 3). The separation between the particles is described by a vector $\mathbf{n}$.

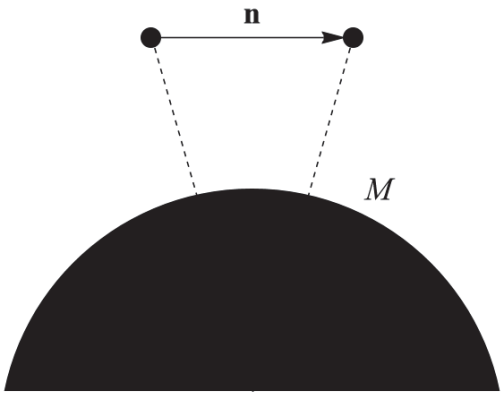


Figure 3 – Two particles in the vicinity of a body of mass M, with the separation of the particles given by the vector $\mathbf{n}$.

As the particles fall towards the center of M, the norm of $\mathbf{n}$ decreases, and the particles accelerate toward one another. We consider the case where the two particles are infinitesimally close, so the potential at the position of the second particle is $\Phi(\mathbf{r}+\mathbf{n})$. We can approximate it at first order by the Taylor expansion around $\mathbf{r}$ as

$$\Phi(\mathbf{r}+\mathbf{n}) = \Phi(\mathbf{r}) + \sum_{j=1}^{3} \left.\frac{\partial \Phi}{\partial x^j}\right|_{\mathbf{r}} n^j. \tag{2.147}$$

Then, the acceleration of the second particle is given by

$$\begin{aligned} &\frac{d^2}{dt^2}(\mathbf{r}(t)+\mathbf{n}(t)) = -\nabla\Phi(\mathbf{r}) - \sum_{j=1}^{3} \left.\frac{\partial(\nabla\Phi)}{\partial x^j}\right|_{\mathbf{r}} n^j \\ &\Rightarrow \left(\frac{d^2\mathbf{r}(t)}{dt^2} + \nabla\Phi(\mathbf{r})\right) = -\left(\frac{d^2\mathbf{n}(t)}{dt^2} + \sum_{j=1}^{3} \left.\frac{\partial(\nabla\Phi)}{\partial x^j}\right|_{\mathbf{r}} n^j\right). \end{aligned} \tag{2.148}$$

From equation (2.146), the left-hand side of this equation vanishes and we obtain

$$\begin{aligned} \frac{d^2\mathbf{n}(t)}{dt^2} &= -\sum_{j=1}^{3} \left.\frac{\partial(\nabla\Phi)}{\partial x^j}\right|_{\mathbf{r}} n^j \\ &= (-\mathbf{n}\cdot\nabla)\nabla\Phi(\mathbf{r}). \end{aligned} \tag{2.149}$$

Thus, the acceleration of the particles toward one another is given by $(-\mathbf{n}\cdot\nabla)\nabla\Phi$ and is called the *tidal acceleration*. In terms of components, the tidal acceleration $\mathbf{a}$ is given by

$$a_i = -n^j \frac{\partial^2 \Phi}{\partial x^j \partial x^i}. \tag{2.150}$$

Writing in terms of tensors, we have

$$a^b = -n^c \partial_c \partial^b \Phi. \tag{2.151}$$

On the other hand, if we view gravity as a consequence of the curvature of spacetime, the trajectory of two infinitesimally close particles will be geodesics, and the relative acceleration between them will be described by the geodesic deviation equation (2.120). Then, we have

$$a^a = -R_{cbd}{}^a v^c x^b v^d, \tag{2.152}$$

where v^a is the 4-velocity of the particles and x^a their separation vector. These two results for the relative acceleration suggest the correspondence

$$R_{cbd}{}^a v^c v^d \longleftrightarrow \partial_b \partial^a \Phi. \tag{2.153}$$

Nonetheless, from Poisson's equation, we have

$$\nabla^2 \Phi = 4\pi\rho, \tag{2.154}$$

where ρ is the mass density of matter. As discussed before, in general relativity the mass and energy properties of matter are described by the stress-energy tensor T_{ab} and we have the correspondence

$$T_{ab} v^a v^b \longleftrightarrow \rho, \tag{2.155}$$

where v^a is the 4-velocity of the observer. Therefore, considering equation (2.154) and the two correspondences (2.153) and (2.155), we are led to write

$$\begin{aligned} R_{cad}{}^a v^c v^d &= 4\pi T_{cd} v^c v^d \\ \Rightarrow R_{cd} &= 4\pi T_{cd}. \end{aligned} \tag{2.156}$$

On the other hand, we have defined the Einstein tensor in equation (2.107) which, together with (2.156) becomes

$$G_{ab} = 4\pi \left(T_{ab} - \frac{1}{2} T g_{ab} \right). \tag{2.157}$$

From (2.108) and (2.145), with ∇_a the derivative operator associated with g_{ab}, we obtain

$$\begin{aligned} 0 &= \nabla^a G_{ab} \\ \Rightarrow 0 &= \nabla^a T_{ab} - \frac{1}{2}\nabla^a T g_{ab} \\ &= -\frac{1}{2}\nabla^a T g_{ab} \\ &= -\frac{1}{2}\nabla_b T. \end{aligned} \tag{2.158}$$

This result suggests that $\nabla_a T = 0$, from which we can conclude that T is constant throughout the universe. This is unphysical, since $T = 0$ in vacuum and $T \neq 0$ in the presence of matter. Equation (2.156) was postulated by Einstein, but due to this unphysical conclusion, it was rejected. If, instead of equation (2.156), we consider

$$G_{ab} = 8\pi T_{ab}, \tag{2.159}$$

or, writing G_{ab} explicitly,

$$R_{ab} - \frac{1}{2}R g_{ab} = 8\pi G T_{ab} \tag{2.160}$$

the consistency between the identity (2.108) and the local conservation of energy given by (2.145) is guaranteed. This equation is called the *Einstein field equation* and was first presented by Einstein in 1915. Therefore, in the framework of general relativity, the spacetime is a manifold M with a metric g_{ab} of Lorentzian signature defined on it. The relation between the curvature of g_{ab} and the matter distribution in spacetime is given by the Einstein field equation (2.160). We have restored the constant G to avoid possible confusions on the next chapters.

We have shown that the components $\Gamma^{\lambda}{}_{\mu\nu}$ of the Christoffel symbols depend on the derivatives of $g_{\mu\nu}$ through equation (2.80). From equation (2.92), the components of the Riemann tensor depend on derivatives of $\Gamma^{\lambda}{}_{\mu\nu}$. Then, the components $R_{\mu\nu\rho}{}^{\sigma}$ depend on second-order derivatives of $g_{\mu\nu}$ and we find that the components of Einstein field equation are second-order partial differential equations for $g_{\mu\nu}$. If this set of equations is solvable, we can determine the metric g_{ab} and describe the geometry of spacetime for a given T_{ab}. As we shall see in the next chapter, even for the simplest cases, solving equation (2.160) is not straightforward.

CHAPTER 3

Quantum Field Theory

This chapter is dedicated to the presentation of quantum field theory. We begin by introducing the notion of quantum fields in a conceptual way. To stablish the foundations for the Lagrangian and Hamiltonian formulations of classical field theory, we first review the fundamental principles of classical mechanics of point particles. Following this, we introduce the procedure of canonical quantization and derive a path integral representation for transition amplitudes in quantum mechanics. After reviewing some topics of classical field theory, we apply the canonical quantization to systems characterized by fields. Finally, imaginary time formalism is introduced and used to derive the partition function for field theories.

3.1 What are Quantum Fields?

Light played a fundamental role in the revolution that physics underwent at the beginning of the 20th century. Up to that time, classical physics thrived in explaining the natural world. However, certain phenomena remained unexplained with the existing knowledge. As discussed in the previous chapter, the constancy of the speed of light led to the development of special relativity. This new theory transformed our understanding of the macroscopic world, from the structure of space and time to the dynamics of particles and bodies. Almost at the same time, a revolution occurred in the microscopic domain, culminating in the development of quantum mechanics. The starting point was the inconsistency between theory and experiment involving hot bodies and the emission of electromagnetic radiation.

The aspect of the electromagnetic radiation emitted by a hot body depends on its temperature. As a piece of metal is heated, it first acquires a red coloration. If the temperature continues to rise, the metal may turn yellow and eventually white. This indicates that the color of predominant emission shifts to higher frequencies as the temperature increases [16]. To study thermal radiation, a small hole can be made in the wall of a heated oven or cavity at thermal equilibrium, allowing the radiation escaping through the hole to be analysed. The walls of the cavity are composed by atoms, which themselves consists of charged particles. These atoms can be modelled as charged oscillators capable of absorbing and emitting electromagnetic radiation. Consequently, the radiation inside the cavity attains thermal equilibrium through energy exchange with the wall's atoms.

Classically, the charged oscillators are assumed to oscillate at all frequencies, emitting and absorbing radiation across the entire electromagnetic spectrum. For a given temperature, the spectral distribution of the intensity of the radiation has a specific shape. The classical physics prediction agrees with the experimental data at low frequencies. However, at high frequencies, there is a significant discrepancy. The classical theory predicts that the intensity increases as a power law with frequency, resulting in divergent values for high frequencies. For this reason, this effect became known as the *ultraviolet catastrophe*.

This inconsistency led Max Planck to postulate that an oscillator can only absorb and emit energy in discrete amounts, corresponding to integer multiples of a "quantum" of energy

$$E = h\nu, \tag{3.1}$$

where ν is the frequency of the oscillator and h is the *Planck constant*. Planck's postulate enabled him to derive the correct expression for the intensity, successfully resolving the ultraviolet catastrophe. Later, Einstein expanded on Planck's idea, interpreting the quantum of energy as evidence that light is composed of particles, called *photons*. These ideas, along with subsequent developments led to the establishment of the quantum theory.

In classical physics, when studying the dynamics of particles and rigid bodies, we can characterize the system with a finite set of numbers, such as its coordinates and velocity components [18]. In this case, the system is said to have a finite number of degrees of freedom. The situation is rather different for systems composed of fields. Fields are represented by continuous functions $f(\mathbf{x},\, t)$, which assigns a certain value to each point of space $\mathbf{x}$ at every instant of time t. The nature of f depends on the type of field being described. For example, in classical electromagnetism, the electric $\mathbf{E}(\mathbf{x},\, t)$

and magnetic $\mathbf{B}(\mathbf{x}, t)$ fields associate a vector with each point in space at any given time. To fully characterize a system composed of fields, it is necessary to specify the values of the fields at all points of space. Consequently, such systems are said to have an infinite number of degrees of freedom.

Although the most common examples of fields are related to interactions, such as the gravitational and electromagnetic ones, fields can also be used to describe material systems. Consider a system of N equal masses m, connected by massless stretchable strings with fixed endpoints. In the mechanical equilibrium, the tension T in the strings is constant, and the masses are equally spaced at a distance Δx [19]. Assuming only small oscillations of the string, in the limit $m \to 0$ and $\Delta x \to 0$, provided that the ratio $m/\Delta x$ remains constant, we obtain a continuous string. In this case, while the transversal displacement of each mass m is described by a coordinate $q_i(t)$, $i = 1, \ldots, N$, the transversal displacement of the continuous string at any point $\mathbf{x}$ is represented by a function $u(x, t)$. The study of systems characterized by a field $\phi(\mathbf{x}, t)$, within the framework of classical physics is known as *classical field theory*. To develop a quantum description of fields, it is necessary to review some concepts of quantum mechanics.

In classical physics, systems are characterized by coordinates and momenta. On the other hand, in quantum mechanics, the state of the system is characterized by a vector $|\psi\rangle$, called a *ket*, belonging to a Hilbert space $\mathscr{E}$ [20]. As a specific example, the state of the quantum harmonic oscillator can be characterized by a set of kets $\{|n\rangle\}_{n\in\mathbb{N}}$, where each state corresponds to a specific energy value E_n. The value E_n correspond to $(n + 1/2)$ quanta of energy $h\nu$. The dynamics of the oscillator can be studied using two operators, a and $a^\dagger$, which satisfy certain commutation relations. The application of operator $a^\dagger$ to the state $|n\rangle$ promotes the system to the state $|n + 1\rangle$ and increases its energy by $h\nu$. Similarly, the application of a to the state $|n\rangle$ demotes the system to the state $|n - 1\rangle$ and the energy is decreased by a quantum. We remark that the value of energy is bounded from below, and $a\,|0\rangle = 0$. The operators $a^\dagger$ and a can be interpreted as creating or annihilating a quantum of energy, and thus raising or lowering the energy of the systems. Hence, $a^\dagger$ and a are called the *ladder operators* or *raising* and *lowering operators* of the harmonic oscillator.

By studying the spontaneous emission of electromagnetic radiation by atoms, Dirac obtained the quantum description of the electromagnetic interaction [21], thus founding the *quantum electrodynamics*, the first quantum field theory. In his work, Dirac expressed the vector potential $\mathbf{A}(\mathbf{x}, t)$ as a Fourier series and demonstrated that the coefficients $a_k(t)$ in the expansion satisfy the same commutation relation as the ladder operators of harmonic oscillator. Consequently, the state of free electromagnetic radiation can be characterized by integer numbers $n_1, n_2, \ldots$, with n_k representing the

number of quanta in the normal mode k [22]. The operators $a_k^\dagger$ and a_k are interpreted as creating or annihilating a quantum, or a photon, in the mode k. These operators thus provide quantum mechanical description of electromagnetic interaction.

This technique can also be applied to describe matter [23]. For bosons, the procedure follows the same steps as for the electromagnetic field. However, for fermions, the procedure must be adapted. In this case, each mode k can only accommodate 0 or 1 fermion, due to the Pauli exclusion principle. This can be achieved imposing anticommutation relations on the creation and annihilation operators. This approach is know as *second quantization*, because the fields being quantized are the wave functions of particles from usual quantum mechanics, meaning the system was already quantized once before.

Historically, this method of quantizing fields was the first to be developed, with other methods emerging later. In section 3.5, we introduce the method of quantization via path-integrals, which is more suitable for our purposes.

3.2 Classical Mechanics

Classical mechanics can be formulated in several different ways, with the Newtonian, Lagrangian, and Hamiltonian formulations being some of the most well-known. While all these formulations must lead to the same conclusions, since the laws of nature are independent of the specific theoretical framework, each formalism is better suited for different purposes. In this section, we shall review some concepts of Lagrangian and Hamiltonian mechanics, as they are essential for understanding quantum mechanics and field theories. Both Lagrangian and Hamiltonian mechanics involve the use of functionals, which, roughly speaking, are functions of functions. Before studying classical mechanics specifically, we first review some concepts of calculus of variations

3.2.1 Calculus of Variations

The calculus of variations is a branch of mathematics concerned with finding the extrema of functionals, which are typically defined by integrals, such as the action (3.29) [19]. In calculus, finding the extrema of real-valued functions involves evaluating derivatives to identify the points at which the function reaches a maximum or minimum. In contrast, when dealing with functional, the tasks is not to find a point, but rather a function that extremizes the functional.

Definition 3.2.1. Let F be the set of all integrable C^2 functions $f : \mathbb{R}^3 \to \mathbb{R}$ and

$y : \mathbb{R} \to \mathbb{R}$ be a C^2 function. An *integral functional* is a map $J : F \to \mathbb{R}$, defined by

$$J[y] = \int_{x^1}^{x^2} f(y(x),\, y'(x),\, x)dx, \tag{3.2}$$

where $y'(x) = dy(x)/dx$.

Then, given a functional $J[y]$, the problem of calculus of variations consists in finding the function y, passing through the fixed points (x^1, y^1) and (x^2, y^2), that extremizes $J[y]$.

We can reduce the problem of extremizing functionals to the problem of finding the extrema of real functions. Let $y(x)$ be the function that extremizes the functional $J[y]$ and consider the curve

$$\bar{y}(x) = y(x) + \epsilon\eta(x), \tag{3.3}$$

with $\epsilon \in \mathbb{R}$ and $\eta(x)$ a C^1 function such that $\eta(x^1) = \eta(x^2) = 0$. Given the functional $J[y]$, defined as in equation (3.2), we define the function

$$\begin{aligned} \Phi(\epsilon) &= J[\bar{y}] \\ &= \int_{x^1}^{x^2} f(\bar{y}(x),\, \bar{y}'(x),\, x)dx. \end{aligned} \tag{3.4}$$

By hypothesis, y gives a extremum of J and, by construction, $\bar{y} = y$ for $\epsilon = 0$. Thus, $\Phi(\epsilon)$ have an extremum for $\epsilon = 0$. Hence, the condition of extremization of J by $y(x)$ is

$$\left(\frac{d\Phi}{d\epsilon}\right)_{\epsilon=0} = 0. \tag{3.5}$$

Thus, from the definitions (3.2) and (3.4), we have

$$\begin{aligned} 0 &= \left(\frac{d\Phi}{d\epsilon}\right)_{\epsilon=0} \\ &= \int_{x^1}^{x^2} \left(\frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \epsilon} + \frac{\partial f}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \epsilon}\right)_{\epsilon=0} dx. \end{aligned} \tag{3.6}$$

From the definition (3.3) of $\bar{y}$, we have

$$\begin{aligned} \frac{\partial \bar{y}}{\partial \epsilon} &= \eta(x), \\ \frac{\partial \bar{y}'}{\partial \epsilon} &= \eta'(x). \end{aligned} \tag{3.7}$$

Substituting this in the previous integral, we obtain

$$\int_{x^1}^{x^2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' \right) dx = 0, \tag{3.8}$$

where we have set $\epsilon = 0$, and consequently, $\bar{y} = y$. We can integrate the second term by parts, to give

$$\int_{x^1}^{x^2} \frac{\partial f}{\partial y'}\eta' dx = \left(\frac{\partial f}{\partial y'}\eta \right) \Bigg|_{x^1}^{x^2} - \int_{x^1}^{x^2} \eta \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) dx. \tag{3.9}$$

The first term on the right-hand side vanishes, since by definition, $\eta(x^1) = \eta(x^2) = 0$. With this result, equation (3.8) becomes

$$\int_{x^1}^{x^2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \eta \, dx = 0. \tag{3.10}$$

Apart from the fact that the function η vanishes at the endpoints, η is an arbitrary function. So, the previous equality will hold for every η if and only if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0. \tag{3.11}$$

This equation is known as the *Euler equation* and provides a second-order differential equation that can be solved to obtain the function $y(x)$ that extremizes the functional $J[y]$.

Now, let us consider the general case where the functional J takes several C^2 functions y_i, $i = 1, \ldots, n$, each of which depend on multiple variables, i.e., $y_i = y_i(x^1, \ldots, x^m)$. We now consider a functional of the form

$$J[y_1, \ldots, y_n] = \int_\Omega f\left(y_1, \ldots, y_n, \partial_1 y_1, \ldots, \partial_m y_1, \ldots, \partial_1 y_n, \ldots, \partial_m y_n, x^1, \ldots x^m \right) d\Omega, \tag{3.12}$$

where Ω is a volume in $\mathbb{R}^m$ and we have employed the notation

$$\partial_i y_j = \frac{\partial y_j}{\partial x^i}, \quad i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, m. \tag{3.13}$$

Proceeding as before, suppose that the n functions y_i extremize J and define auxiliary functions

$$\bar{y}_i(x) = y_i(x) + \epsilon\eta_i(x), \quad i = 1, \ldots, n, \tag{3.14}$$

where all the functions η_i are C^1 and vanish at the boundary of the volume Ω. Then, for

$$\Phi(\epsilon) = J[\bar{y}_1, \ldots, \bar{y}_n], \tag{3.15}$$

the condition for extremization of J is

$$\left(\frac{d\Phi}{d\epsilon}\right)_{\epsilon=0} = 0. \tag{3.16}$$

Evaluating the derivative for the functional defined as in (3.12), we obtain:

$$\begin{aligned} \frac{d\Phi}{d\epsilon} &= \int_\Omega \frac{df}{d\epsilon}\, d^m x \\ &= \int_\Omega \left[\sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i}\eta_i + \sum_{j=1}^{m}\frac{\partial f}{\partial(\partial_j y_i)}\partial_j\eta_i\right)\right] d\Omega. \end{aligned} \tag{3.17}$$

For fixed values of i and j, we integrate the second term in the brackets by parts and obtain

$$\int_\Omega \left[\frac{\partial f}{\partial(\partial_j y_i)}\partial_j\eta_i\right] d\Omega = -\int_\Omega \frac{d}{dx^j}\left(\frac{\partial f}{\partial(\partial_j y_i)}\right)\eta_i\, d\Omega \tag{3.18}$$

where we already evaluated the first term of the integral on the boundary. Then we obtain,

$$\frac{d\Phi}{d\epsilon} = \int_\Omega \left[\sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i} - \sum_{j=1}^{m}\frac{d}{dx^j}\left(\frac{\partial f}{\partial(\partial_j y_i)}\right)\right)\right]\eta_i\, d\Omega. \tag{3.19}$$

Following the same reasoning as before, considering that each function η_i are independent and arbitrary, we obtain the Euler equation for the general case

$$\frac{\partial f}{\partial y_i} - \sum_{j=1}^{m}\frac{d}{dx^j}\left(\frac{\partial f}{\partial(\partial_j y_i)}\right) = 0 \quad i = 1, \ldots, m. \tag{3.20}$$

In the following section we are going to use equations (3.11) and (3.20) together with Hamilton's principle to obtain equations of motion for physical systems.

In physics, it is customary to use an alternative notation when dealing with calculus of variations. The variation δy is defined by $\delta y = \epsilon\eta$, so that equation (3.3) is expressed as

$$\bar{y} = y + \delta y. \tag{3.21}$$

The variation δJ of the functional J is defined from equation (3.8) by

$$\begin{aligned} \delta J &= \epsilon \Phi'(0) \\ &= \int_{x^1}^{x^2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx. \end{aligned} \tag{3.22}$$

Note that,

$$\delta y' = \epsilon \eta' = (\epsilon \eta)' = (\delta y)'. \tag{3.23}$$

Substituting this into the integral and performing an integration by parts leads to

$$\delta J = \int_{x^1}^{x^2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \delta y \, dx. \tag{3.24}$$

The variation δy is arbitrary, except for the condition that it vanishes at the endpoints. Therefore, the Euler equation follows from the condition $\delta J = 0$.

3.2.2 Lagrangian Mechanics

In the Lagrangian formulation of classical mechanics, systems are characterized by a set of generalized coordinates $\{q_i(t)\}_{i=1}^n$ and a set of generalized velocities $\{\dot{q}_i(t)\}_{i=1}^n$, where $\dot{q} = dq/dt$ [19]. Here, n is the minimum number of independent coordinates needed to define the system's configuration. These coordinates are referred to as *generalized* because they may not correspond to the usual Euclidean coordinates $(x^1, x^2, x^3) \in \mathbb{R}^3$. The space of all points $(q_1, \ldots, q_n)$ is called the *configuration space* of the system. By knowing the trajectory of the system in the configuration space, we can determine its trajectory in the usual space $\mathbb{R}^3$. We shall categorize the systems in two groups, based on the constraints between the coordinates used to describe the system. If the relationship between the generalized coordinates can be written in the form

$$f(q_1, \ldots, q_n, t) = 0, \tag{3.25}$$

we say that the system is a *holonomic system*. Otherwise, the system is refered to as a *non-holonomic system*. Consider the motion of a simple pendulum of mass m, with a

rod of size l. The relation between the Cartesian coordinates $x(t)$, $y(t)$ of m obey the relation

$$x^2(t) + y^2(t) - l^2 = 0 \tag{3.26}$$

and, hence, the system is holonomic. For the case of a particle confined to move inside a sphere of radius R, the coordinates $x(t)$, $y(t)$, $z(t)$ of the particle satisfy

$$x^2(t) + y^2(t) + z^2(t) - R^2 < 0, \tag{3.27}$$

and, thus, the system is non-holonomic.

The main object in this formulation is the *Lagrangian function* $L(q_1, \ldots, q_n, \dot{q}_1, \dot{q}_n, t)$. To simplify the notation, we write it as $L(q, \dot{q}, t)$. The Lagrangian is defined as

$$L(q, \dot{q}, t) = T(\dot{q}) - V(q), \tag{3.28}$$

where T is the kinetic energy of the system and V the potential energy associated with the forces acting on the system. The equations of motion follow from the following principle:

> **Hamilton's Principle:** Let a holonomic mechanical system be described by the Lagrangian $L(q, \dot{q}, t)$. The motion of the system from instant t_1 to instant t_2 is such that the action, defined as
>
> $$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt, \tag{3.29}$$
>
> is stationary for fixed points $q(t_1)$ and $q(t_2)$.

That is, to find the trajectory of the system we must find the extremum of the action.

The action S has the same form as the functional J defined in (3.2) for the case of a system characterized by a single generalized coordinate $q(t)$. Therefore, the results obtained for the condition of extremization of J apply to S with the replacements

$$\begin{aligned} f &\rightarrow L, \\ y &\rightarrow q, \\ x &\rightarrow t. \end{aligned} \tag{3.30}$$

Performing these substitutions in equation (3.11), we find

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0, \quad i = 1, \ldots, n. \tag{3.31}$$

This equation is called the *Euler-Lagrange equation*, and provides a differential equation whose solution yields the coordinate $q(t)$ that extremizes the action $S[q]$. According to Hamilton's principle, this function corresponds to the motion of the system.

An example of a system characterized by a single generalized coordinate q is a simple pendulum with a rod of length l and mass m. The gravitational potential energy V depends only on the y coordinate of the pendulum. We define the zero of potential energy such that $V(\theta = 0)$ = 0.

The kinetic and the potential energies of the system are given by

$$\begin{cases} T &= \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2, \\ V &= mgl(1 - \cos\theta). \end{cases} \tag{3.32}$$

Given T and V, we obtain the Lagrangian of the system, which depends on the single generalized coordinate θ. We then find that the Euler-Lagrange equation of the system is

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0, \tag{3.33}$$

which is the differential equation of the simple harmonic oscillator.

In more general cases, the Lagrangian L may depend on several generalized coordinates q_i, $i = 1, \ldots, n$. The action functional then takes the form

$$S[q_1, \ldots, q_n] = \int_{t_1}^{t_2} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)dt. \tag{3.34}$$

Now, the action takes the same form of the functional defined in (3.12), with $m = 1$. Then, we obtain a Euler-Lagrange equation for each coordinate, each of which similar to equations in (3.20), and is given by

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{3.35}$$

The application of these equations follows the same steps as in the case of the simple pendulum, once the form of L is known.

In the treatment presented here, the time t is the only independent variable. However, as we will discuss later, the situation changes when we study relativistic field theories, where space and time variables are treated on equal footing.

3.2.3 Hamiltonian Mechanics

Hamilton's formulation of mechanics is related to that of Lagrange and is obtained by introducing the canonically conjugate momentum p_i as an independent variable, which is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{3.36}$$

where L is the Lagrangian of the system and q_i is a generalized coordinate. In this formulation, the conjugate momenta replace the generalized velocities $\dot{q}_i$. Thus, the systems are characterized by a set of generalized coordinates $\{q_i\}_{i=1}^n$ and a set of generalized momenta $\{p_i\}_{i=1}^n$.

The central object in Hamiltonian mechanics is the Hamiltonian function $H(q,\, p,\, t)$, where q denotes all coordinates and p all momenta. The function H is obtained from the Lagrangian L by the Legendre transformation

$$H(q,\, p) = \sum_{i=1}^{n} p_i \dot{q}_i(p) - L(q,\, \dot{q}(p)). \tag{3.37}$$

The equations of motion of the system are obtained from the *Hamilton equations*, defined as

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}. \end{aligned} \tag{3.38}$$

We now recall the definition of *Poisson brackets*, which allows us to write equations in a more compact form. Given two functions $F(q,\, p,\, t)$ and $G(q,\, p,\, t)$, the Poisson bracket of F and G is defined as [19]

$$\{F,\, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}. \tag{3.39}$$

This allows us to write the total time derivative of a function $A(q,\, p,\, t)$ as

$$\frac{dA}{dt} = \{A,\, H\} + \frac{\partial A}{\partial t}. \tag{3.40}$$

From the definition of Poisson brackets, we have the following relations

$$\begin{aligned} \{q_i,\, q_j\} &= 0, \\ \{p_i,\, p_j\} &= 0, \\ \{q_i,\, p_j\} &= \delta_{ij}, \end{aligned} \tag{3.41}$$

for all $i,\, j = 1,\, \ldots,\, n$, where δ_{ij} is the Kronecker delta.

3.3 Quantum Mechanics

In this section, we review key concepts of quantum mechanics essential for the study of quantum fields and derive a path integral representation of transition amplitudes. For simplicity, we restrict the discussion for one-dimensional systems.

3.3.1 Canonical Quantization

The Hamiltonian formalism provides a recipe to obtain the quantum description of a system from its classical counterpart, when one exists [20]. The process, known as *canonical quantization*, involves replacing

$$\begin{aligned} x &\to X, \\ p &\to P, \\ \{\cdot\,, \cdot\} &\to \frac{1}{i\hbar}[\,\cdot\,,\, \cdot\,], \end{aligned} \tag{3.42}$$

where X and P are the position and momentum operators, respectively, and $\hbar$ is the *reduced Planck constant*. The commutator $[\,\cdot\,,\, \cdot\,]$ is defined by

$$[A, B] = AB - BA, \tag{3.43}$$

for any operators A and B. With these substitutions, the classical Hamiltonian becomes a Hermitian operator $H(X,\, P,\, t)$, called the *Hamiltonian operator*.

In quantum mechanics, the relations (3.41) become

$$\begin{aligned} [X_i, X_j] &= 0, \\ [P_i, P_j] &= 0, \\ [X_i, P_j] &= i\hbar\delta_{ij}. \end{aligned} \tag{3.44}$$

Consider a one-dimensional system, with generalized coordinate x and momentum p. In the context of classical mechanics, the products xp and px are equivalent. On the other hand, from the last relation above we see that in quantum mechanics $XP \neq PX$. Then, if the classical Hamiltonian contains terms involving products of x and p, we must adopt a symmetrization procedure to obtain the corresponding Hamiltonian operator. In the example considered, the term xp must be replaced by $(XP + PX)/2$ when performing the canonical quantization.

As mentioned before, in quantum mechanics, the state of the system is characterized by a vector $|\psi\rangle$, called a *ket*, belonging to a Hilbert space $\mathscr{E}$. Then, the operators X, P and H are operators in the Hilbert space of the system. The eigenvectors or eigenstates of these operators constitute bases for the Hilbert space [20].

To study the dynamics of quantum systems, we can adopt two different approaches. The first approach, known as the *Schrödinger picture*, treats the state kets $|\psi(t)\rangle$ as time-dependent, while the operators do not carry any time dependence. We add a label S to the kets in the Schrödinger picture to explicitly indicate the picture we are using. In this approach, the time evolution of the system is governed by the *Schrödinger equation*

$$H\,|\psi(t)\rangle_S = i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle_S\,. \tag{3.45}$$

We restrict our discussion to time-independent Hamiltonians. In this situation, the solution of Schrödinger equation is given by

$$|\psi(t)\rangle_S = e^{-\frac{i}{\hbar}H(t-t_0)}\,|\psi(t_0)\rangle_S\,, \tag{3.46}$$

where $|\psi(t_0)\rangle_S$ is the state ket of the system at a instant of time $t_0 < t$. Thus, the connection between the state ket $|\psi(t_0)\rangle_S$ and $|\psi(t)\rangle_S$ is given by the operator

$$U(t,\,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}, \tag{3.47}$$

called the *time evolution operator*. Multiplying equation (3.46) on both sides by the adjoint of U leads to

$$\begin{aligned} |\psi(t_0)\rangle_S &= U^\dagger(t,\,t_0)\,|\psi(t)\rangle_S \\ &= e^{\frac{i}{\hbar}H(t-t_0)}\,|\psi(t)\rangle_S\,, \end{aligned} \tag{3.48}$$

and thus we interpret $U^\dagger$ as the operator that evolves the system backwards in time.

We label the eigenkets X and P by their corresponding eigenvalues, that is,

$$\begin{aligned} X\left|x\right\rangle_S &= x\left|x\right\rangle_S, \\ P\left|p\right\rangle_S &= p\left|p\right\rangle_S, \end{aligned} \tag{3.49}$$

and both sets of eigenstates $\{\left|x\right\rangle_S\}_{x\in\mathbb{R}}$ and $\{\left|p\right\rangle_S\}_{p\in\mathbb{R}}$ constitute an orthonormal basis for the Hilbert space of the system. We remark that in the Schrödinger picture, the basis kets do not carry time dependence.

The other approach we can adopt to study the dynamics of quantum systems is known as the *Heisenberg picture*. In this approach, the state kets are time-independent and the operators carry the time dependence. In this picture we will label the kets and operators with a subscript H. Thus, if we know the system is in the state $\left|\psi\right\rangle_H$ at some time t_0, it will remain in this same state at all subsequent times. Note that this state corresponds to $\left|\psi(t_0)\right\rangle_S$ in the Schrödinger picture. From equation (3.48), we then have

$$\left|\psi\right\rangle_H = e^{\frac{i}{\hbar}H(t-t_0)}\left|\psi(t)\right\rangle_S. \tag{3.50}$$

An operator A_H in the Heisenberg picture is related to A_S in the Schrödinger picture by

$$\begin{aligned} A_H(t) &= e^{\frac{i}{\hbar}H(t-t_0)}A_S e^{-\frac{i}{\hbar}H(t-t_0)}\left|\psi(t)\right\rangle_S \\ &= U^\dagger(t,\,t_0)A_S U(t,\,t_0). \end{aligned} \tag{3.51}$$

Using this relation, assuming that A_S does not depend explicitly on time, we can show that the operator $A_H(t)$ satisfies

$$\frac{d}{dt}A_H(t) = [A_H(t), H]. \tag{3.52}$$

This equation is known as the *Heisenberg equation of motion*. Particularly, the position operator in the two pictures are related by

$$X_H(t) = U^\dagger(t,\,t_0)X_S U(t,\,t_0) \tag{3.53}$$

and its eigenstates satisfy

$$X_H(t)\left|x,\,t\right\rangle_H = x\left|x,\,t\right\rangle_H. \tag{3.54}$$

From this, we see that the eigenstates in the two pictures are related by

$$\left|x,\,t\right\rangle_H = U^\dagger(t,\,t_0)\left|x\right\rangle_S. \tag{3.55}$$

Since the sets of eigenstates $\{|x\rangle_S\}_{x\in\mathbf{R}}$, $\{|p\rangle_S\}_{p\in\mathbf{R}}$ and $\{|x,\,t\rangle_H\}_{x\in\mathbf{R}}$ constitute orthonormal bases of the Hilbert space of the system, they satisfy

$$\begin{aligned} {}_S\langle x|x'\rangle_S &= \delta(x-x'),\\ {}_H\langle x,\,t|x',\,t\rangle_H &= \delta(x-x'),\\ {}_S\langle p|p'\rangle_S &= \delta(p-p'), \end{aligned} \tag{3.56}$$

and also

$$\begin{aligned} \int dx\,|x\rangle_S\,{}_S\langle x| &= \mathbb{1},\\ \int dx\,|x,\,t\rangle_H\,{}_H\langle x,\,t| &= \mathbb{1},\\ \int dp\,|p\rangle_S\,{}_S\langle p| &= \mathbb{1}, \end{aligned} \tag{3.57}$$

which we call the *completeness relations*. The inner product between momentum eigenkets and position eigenkets reads

$${}_S\langle p|x\rangle_S = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}px}. \tag{3.58}$$

The Schrödinger and Heisenberg pictures are equivalent, and the dynamics of a quantum system can be studied in either of these approaches. However, there exists another formalism for studying the dynamics of the system, based on the classical Hamiltonian, that does not rely on non-commuting operators. This formulation involves path integrals, and we now turn our attention to this approach.

3.3.2 Path Integrals on Quantum Mechanics

In quantum mechanics, we are usually interested in evaluating probability amplitudes of the form $\langle f|i\rangle$, where $|i\rangle$ represents the initial state of the system and $|f\rangle$ the final state. If we know $|i\rangle$ and the appropriate operators, we can solve the equation of motion in either Schrödinger or Heisenberg pictures to determine the state of the system at any given time t. However, we will use an alternative method, which expresses transition amplitudes as path integrals.

We define the transition amplitude

$$\begin{aligned} F(x_f,\,t_f;\,x_i,\,t_i) &= {}_H\langle x_f,\,t_f|x_i,\,t_i\rangle_H\\ &= \langle x_f|\,e^{-\frac{i}{\hbar}H(t_f-t_i)}\,|x_i\rangle\,, \end{aligned} \tag{3.59}$$

where we have dropped out the labels of basis kets in the Schrödinger picture. Here, we assume that the Hamiltonian is given by

$$H(X,\, P) = \frac{P^2}{2m} + V(X). \tag{3.60}$$

Let us divide the time interval $t_f - t_i$ in N equal segments of length δt. In the end we will be interested in taking the continuum limit $\delta t \to 0$ and $N \to \infty$. Then, we have

$$\delta t = \frac{t_f - t_i}{N}, \tag{3.61}$$

so any intermediate time t_n can be expressed as

$$t_n = t_i + n\delta t, \quad t_i < t_n < t_f. \tag{3.62}$$

We can introduce a completeness relation in the expression for $F(x_f,\, t_f;\, x_i,\, t_i)$ for each intermediate time t_n and we obtain

$$\begin{aligned} F(x_f,\, t_f;\, x_i,\, t_i) = \int dx_{N-1} \cdots \int dx_1 {}_H\langle x_f,\, t_f | x_{N-1},\, t_{N-1}\rangle_H \\ {}_H\langle x_{N-1},\, t_{N-1} | x_{N-2},\, t_{N-2}\rangle_H \cdots {}_H\langle x_1,\, t_1 | x_i,\, t_i\rangle_H \\ = \prod_{j=1}^{N-1} \int dx_j \prod_{k=1}^{N} {}_H\langle x_k,\, t_k | x_{k-1},\, t_{k-1}\rangle_H, \end{aligned} \tag{3.63}$$

where $tf > t_{N-1} > t_{N-2} > \cdots > t_1$ and we have identified $t_i = t_0$ and $t_f = t_N$. From relation (3.55) and the definition (3.47) of the time evolution operator, any intermediate term has the form

$$\begin{aligned} {}_H\langle x_n,\, t_n | x_{n-1},\, t_{n-1}\rangle_H &= \langle x_n |\, e^{-\frac{i}{\hbar}H(X,P)(t_n - t_0)} e^{\frac{i}{\hbar}H(t_{n-1} - t_0)}\, |x_{n-1}\rangle \\ &= \langle x_n |\, e^{-\frac{i}{\hbar}H(X,P)(t_n - t_{n-1})}\, |x_{n-1}\rangle \\ &= \langle x_n |\, e^{-\frac{i}{\hbar}H(X,P)\delta t}\, |x_{n-1}\rangle\,. \end{aligned} \tag{3.64}$$

Inserting a completeness relation written in terms of momentum basis we obtain

$$
\begin{aligned}
{}_H\langle x_n,\, t_n | x_{n-1},\, t_{n-1}\rangle_H &= \int dp\, \langle x_n|\, e^{-\frac{i}{\hbar}H(X,P)\delta t}\, |p\rangle\, \langle p|x_{n-1}\rangle \\
&= \int dp\, \langle x_n|p\rangle\, \langle p|x_{n-1}\rangle\, e^{-\frac{i}{\hbar}H(x_n,p)\delta t} \\
&= \int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}p(x_n-x_{n-1})} e^{-\frac{i}{\hbar}H(x_n,p)\delta t}.
\end{aligned}
\tag{3.65}
$$

Using the explicit form of the Hamiltonian, we find

$$
\begin{aligned}
{}_H\langle x_n,\, t_n | x_{n-1},\, t_{n-1}\rangle_H &= \int \frac{dp}{2\pi\hbar} \exp\left\{-\frac{i}{\hbar}\left[\left(\frac{p^2}{2m}+V(x_n)\right)\delta t - p(x_n-x_{n-1})\right]\right\} \\
&= \exp\left[-\frac{i}{\hbar}V(x_n)\delta t\right] \int \frac{dp}{2\pi\hbar} \exp\left[-\frac{i}{\hbar}\left(\frac{\delta t}{2m}p^2 - p(x_n-x_{n-1})\right)\right].
\end{aligned}
\tag{3.66}
$$

The integral can be evaluated using the Gaussian integral formula [4]

$$
\int_{-\infty}^{\infty} dp \exp\left(-\frac{1}{2}ap^2 + Jp\right) = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{J^2}{2a}\right). \tag{3.67}
$$

Then, we obtain

$$
{}_H\langle x_n,\, t_n | x_{n-1},\, t_{n-1}\rangle_H = \left(-\frac{im}{2\pi\hbar\delta t}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\delta t\left[\frac{(x_n-x_{n-1})^2}{\delta t^2} - V(x_n)\right]\right\}. \tag{3.68}
$$

Substitution of this result in equation (3.63) leads to

$$
\begin{aligned}
F(x_f,\, t_f;\, x_i,\, t_i) &= \prod_{j=1}^{N-1} \int dx_j \prod_{k=1}^{N} \left(-\frac{im}{2\pi\hbar\delta t}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\delta t\left[\frac{m}{2}\frac{(x_n-x_{n-1})^2}{\delta t^2} - V(x_n)\right]\right\} \\
&= \left(-\frac{im}{2\pi\hbar\delta t}\right)^{N/2} \prod_{j=1}^{N-1} \int dx_j \exp\left\{\frac{i}{\hbar}\delta t \sum_{k=1}^{N}\left[\frac{m}{2}\frac{(x_k-x_{k-1})^2}{\delta t^2} - V(x_k)\right]\right\}.
\end{aligned}
\tag{3.69}
$$

From usual calculus, in the limits $\delta t \to 0$ and $N \to \infty$, such that $N\delta t = t_f - t_i$ is fixed, we have

$$
\frac{x_k - x_{k-1}}{\delta t} \longrightarrow \dot{x}(t_n) \quad \text{and} \quad \delta t \sum_{k=1}^{N} f(t_k) \longrightarrow \int_{t_i}^{t^f} dt f(t). \tag{3.70}
$$

In this limit, the argument of the exponential becomes

$$\delta t \sum_{k=1}^{N} \left[\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\delta t^2} - V(x_k) \right] \longrightarrow \int_{t_i}^{t_f} dt \left(\frac{1}{2} m \dot{x}^2(t) - V(t) \right), \tag{3.71}$$

which we recognize as the action functional S of the system. Thus, we express the transition amplitude (3.69) as

$$F(x_f,\, t_f;\, x_i,\, t_i) = \int \mathscr{D}x\, e^{\frac{i}{\hbar} S[x]}, \tag{3.72}$$

where we have defined the notation

$$\mathscr{D}x = \lim_{\substack{\delta t \to 0 \\ N \to \infty}} \left(-\frac{im}{2\pi\hbar\delta t} \right)^{N/2} \prod_{j=1}^{N-1} dx_j \tag{3.73}$$

and we call $\mathscr{D}x$ the *functional measure*. The integrations must be taken over all paths $x(t)$ satisfying the boundary conditions

$$x(t_i) = x_i \quad \text{and} \quad x(t_f) = x_f. \tag{3.74}$$

We point out that the factor

$$\lim_{\substack{\delta \to 0 \\ N \to \infty}} \left(-\frac{im}{2\pi\hbar\delta t} \right)^{N/2} \tag{3.75}$$

is divergent, which can bring some issues. Therefore, the expression (3.72) gives the desire result, allowing us to obtain the transition amplitude from the classical action.

In classical mechanics, from the Hamilton's principle, the action determines the path that the system follows. On the other hand, in quantum mechanics all paths contribute to the transition amplitude. We remark that the expression for $F(x_f,\, t_f;\, x_i,\, t_i)$ presents some issues. We note that the integrand is purely a phase factor, which has the same modulus for all trajectories and we may have problems of convergence [24]. Also, in most cases we are not even able to perform the integration.

3.4 Classical Field Theory

In the previous sections, we studied systems characterized by a discrete set of generalized coordinates. Now we turn to the study of systems characterized by fields. In

this section, we present the Lagrangian and Hamiltonian formulations of classical field theory, introducing the basic concepts that will be needed when studying quantum fields later.

As briefly discussed in the beginning of the chapter, fields are a generalization of the concept of coordinates, associating a certain value to each point $\mathbf{x}$ of space at a given time t. Thus, fields are characterized by a continuous function, denoted by $\phi(\mathbf{x},\, t)$, and play the role of generalized coordinates. In the discrete case, the generalized coordinates $q_i(t)$ and generalized velocities $\dot{q}_i(t)$ are a function of time only, and the Lagrangian depends on q and its time derivative $\dot{q}$. By analogy, since that our generalized coordinates are functions of both time and space, we expect the Lagrangian to depend on ϕ and its time and spacial derivatives $\dot{\phi}$ and $\nabla\phi$ [25]. In relativistic theories, the time is treated as just another coordinate in spacetime. So, from now on we write $\phi = \phi(x)$, where it is understood that x is a point in spacetime. The Lagrangian function of fields must have a dependence of the form

$$L = L\left[\phi,\, \dot{\phi},\, \frac{\partial\phi}{\partial x},\, \frac{\partial\phi}{\partial y},\, \frac{\partial\phi}{\partial z}\right]. \tag{3.76}$$

Recalling the tensor notation used in the previous chapter, we have

$$(x^0, x^1,\, x^2,\, x^3) = (t,\, x,\, y,\, z) \tag{3.77}$$

so that

$$\partial_\mu\phi = \frac{\partial\phi}{\partial x^\mu}, \quad \mu = 0,\, 1,\, 2,\, 3. \tag{3.78}$$

Then, we express the dependence of the Lagrangian as $L = L[\phi,\, \partial_\mu\phi]$. It is customary to write the Lagrangian as

$$L[\phi,\, \partial_\mu\phi] = \int d^3x \mathscr{L}[\phi,\, \partial_\mu\phi], \tag{3.79}$$

where $\mathscr{L}$ is the *Lagrangian density*, but we shall refer to it simply as the *Lagrangian*. The action functional is then given by

$$\begin{aligned} S[\phi] &= \int_{t_1}^{t_2} dt \int d^3x \mathscr{L}[\phi,\, \partial_\mu\phi] \\ &= \int d^4x \mathscr{L}[\phi,\, \partial_\mu\phi], \end{aligned} \tag{3.80}$$

where d^4x denotes the volume element in the Minkowski spacetime. The time interval of for integration in the time variable is implicitly understood.

The equations of motion for fields also follow from the Hamilton's principle, which states that the field assume the configuration that extremizes the action. We see that the action (3.80) has the same form as that of the function (3.12) with $n = 1$ and $m = 4$. Then, from equation (3.20), the equation of motion for fields is given by

$$\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{3.81}$$

If our Lagrangian depend on several different fields ϕ_j, $j = 1, \ldots, n$, then we would have a Euler-Lagrange equation for each ϕ_j, just as in the discrete case.

The idea of a Hamiltonian formulation of field theory is completely analogous to the case of discrete systems. Our goal is to obtain the Hamiltonian functional H from the Lagrangian L. We begin by defining the Hamiltonian density $\mathscr{H}$ by

$$H = \int d^3x \mathscr{H}. \tag{3.82}$$

In Hamiltonian mechanics, we defined the canonically conjugate momenta p to the variable q, which replaced the generalized velocity as an independent variable. To apply the Hamiltonian formulation to field theories, we need to define a "momentum" that is canonically conjugate to our field variable $\phi(\mathbf{x}, t)$ [4]. In analogy with the definition (3.36) p, we define the *canonically conjugate momentum* (density) π by

$$\pi(\mathbf{x}, t) = \frac{\partial \mathscr{L}}{\partial \dot{\phi}(\mathbf{x}, t)}. \tag{3.83}$$

The Hamiltonian density is obtained from the Lagrangian density by

$$\mathscr{H}[\phi, \pi] = \pi \dot{\phi}[\phi, \pi] - \mathscr{L}[\phi, \dot{\phi}[\phi, \pi]], \tag{3.84}$$

where $\dot{\phi}$ denotes the time derivative of ϕ. Although the Lagrangian may depend on all derivatives $\partial_\mu \pi$, $\mu = 0, \ldots, 3$, the conjugate momentum is defined as the derivative of $\mathscr{L}$ with respect to $\dot{\phi}$. In this way, the time coordinate is treated differently from the spatial coordinates. Thus, the Hamiltonian formulation is not manifestly covariant, while the Lagrangian formulation is. Therefore, field theories are usually presented in terms of Lagrangians, so that the equations are expressed in a covariant form that is valid in any inertial frame of reference.

To develop quantum field theory from classical field theory, in a similar manner to the canonical quantization of quantum mechanics, we introduce the Poisson brackets for fields. The analogous of the relations (3.41) for fields are given by [24]

$$\begin{aligned} \{\phi(\mathbf{x},t),\, \phi(\mathbf{x}',t)\} &= 0, \\ \{\pi(\mathbf{x},t),\, \pi(\mathbf{x}',t)\} &= 0, \\ \{\phi(\mathbf{x},t),\, \pi(\mathbf{x}',t)\} &= i\hbar\delta(\mathbf{x}-\mathbf{x}'), \end{aligned} \tag{3.85}$$

where $\delta(\mathbf{x}-\mathbf{x})$ is the Dirac delta.

3.4.1 General Relativity as a Classical Field Theory

In the previous chapter we have derived the Einstein's field equations making a connection between the geodesic deviation and tidal acceleration of nearby particles in a gravitational field. An alternative derivation can be done by formulating general relativity as a classical field theory. In this formulation, the metric tensor $g_{ab}(x)$ is treated as the field variable and the Einstein's field equations are the corresponding equations of motion.

Since general relativity relates the geometry of spacetime to the matter distribution, the action for general relativity must contain a contribution $S_G[g]$ from the gravitational field $g_{ab}(x)$ and a contribution $S_M[\phi,\, g]$, related to the matter fields, denoted by ϕ. The gravitational action is given by

$$S_G[g] = S_{EH}[g] + S_B[g] - S_0, \tag{3.86}$$

where S_{EH} is the *Einstein-Hilbert action*, S_B is a boundary term and S_0 is a term that does not alters the equations of motion. These terms are defined as

$$\begin{aligned} S_{EH}[g] &= \frac{1}{16\pi G}\int_\Omega d^4x\,\sqrt{-g}R, \\ S_B[g] &= \frac{1}{8\pi G}\oint_{\partial\Omega} d^3y\sqrt{h}\varepsilon K, \\ S_0[h] &= \frac{1}{8\pi G}\oint_{\partial\Omega} d^3y\sqrt{h}\varepsilon K_0, \end{aligned} \tag{3.87}$$

where, R is the Ricci scalar in the volume Ω, g is the determinant of the metric g_{ab} defined in Ω, and h is the determinant of the metric defined on the boundary $\partial\Omega$. The quantities K and K_0 are associated with the curvature of the boundary $\partial\Omega$ and will be

discussed below. The parameter ε is equal to +1 where the surface $\partial\Omega$ is timelike and -1 where $\partial\Omega$ is spacelike. We assume that $\partial\Omega$ is nowhere null.

If a matter field ϕ is present, the matter action has the form

$$S_M[\phi,\, g] = \int_\Omega d^4x\, \sqrt{-g}\mathscr{L}[\phi,\, \partial_\mu\phi,\, g_{\mu\nu}], \tag{3.88}$$

where $\mathscr{L}$ is the Lagrangian density describing ϕ. Note that the form of S_M is similar to the action of field theories, given in equation (3.80). The difference is that here we have a factor $\sqrt{-g}$ in the integrand, which is present to ensure invariance under general coordinate transformations.

The complete action functional is given by

$$S[g,\, \phi] = \int_\Omega dx^4\sqrt{-g}\left(\frac{R}{16\pi G} + \mathscr{L}\right) + \frac{1}{8\pi G}\oint_{\partial\Omega} dy^3\sqrt{h}\varepsilon(K - K_0). \tag{3.89}$$

The Einstein's field equations (4.1) can be obtained from the variation of S with respect to $g_{\mu\nu}$, subjected to the condition $\delta g_{\mu\nu} = 0$ on the boundary $\partial\Omega$ [3, 26]. In the presence of a cosmological constant Λ, the complete action becomes

$$S[g,\, \phi] = \int_\Omega dx^4\sqrt{-g}\left(\frac{R}{16\pi G} + \mathscr{L} - 2\Lambda\right) + \frac{1}{8\pi G}\oint_{\partial\Omega} dy^3\sqrt{h}\varepsilon(K - K_0). \tag{3.90}$$

Since we are interested in vacuum solutions ($R = \mathscr{L} = 0$), without a cosmological constant, the first integral in equation (3.89) vanishes and we are left with

$$S[g] = \frac{1}{8\pi G}\oint_{\partial\Omega} dy^3\sqrt{h}\varepsilon(K - K_0). \tag{3.91}$$

The quantity K is the trace of the quantity K_{ab}, called the *extrinsic curvature*, and is obtained from

$$K = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, n^\mu\right), \tag{3.92}$$

where n_μ denote the components of the unit normal vector to the surface $\partial\Omega$. The components n_μ is given by

$$n_\mu = \varepsilon\frac{\partial_\mu\Phi}{|g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi|}, \tag{3.93}$$

where Φ is the function that characterizes the surface. This is similar to the case of multivariable calculus, where a surface is characterized by a function $f(x^1,\, \ldots,\, x^n)$ and a vector normal to this surface is given by the gradient of f.

We recall that to characterize the curvature of spacetime, we used the Riemann tensor, which was obtained by considering the intrinsic approach in the study of manifolds. The situation is different for the surface $\partial\Omega$, since it is embedded in spacetime. Thus, we adopt the extrinsic approach and the extrinsic curvature of $\partial\Omega$ is characterized by K_{ab}.

The quantity K_0 depends only on the metric h_{ab} and its variation with respect to g_{ab} vanishes. Thus, this term does not affect the equations of motion and its presence only alter the numerical value of the gravitational action. This term corresponds to the extrinsic curvature of the boundary $\partial\Omega$ embedded in flat spacetime. Its purpose is to cancel the divergence arising from K when the boundary is pushed to infinity. This extrinsic curvature is defined as

$$K_0 = \frac{1}{\sqrt{-\eta}}\partial_\mu\left(\sqrt{-\eta}\, n^\mu\right), \tag{3.94}$$

where η is the determinant of the metric η_{ab} of flat spacetime.

3.5 Quantum Field Theory

In this section we present the basics of quantum field theory, which arises from the combination of classical field theory, quantum mechanics, and special relativity. Our main goal is to derive the analogue of expression (3.72) for systems described by fields. This formulation, based on path integrals, facilitates the connection with statistical mechanics and quantum field theory, since the partition function of a system can be expressed in terms of a path integral.

3.5.1 Path Integral in Quantum Field Theory

The process of quantizing field theories is analogous to the canonical quantization of non-relativistic quantum mechanics. Since the field ϕ plays the role of generalized coordinates, the field quantization involves the promotion of ϕ to the Schrödinger picture field operator Φ [4]. The counterparts of the eigenstates $|x\rangle$ are the eigenstates $|\phi\rangle$ of the operator Φ, which satisfy

$$\Phi(x)\,|\phi\rangle = \phi(x)\,|\phi\rangle\,. \tag{3.95}$$

To avoid confusions between the field $\pi(x)$ and the constant π, we always write the field's dependence on x. The analogue of momentum eigenstates $|p\rangle$ are the eigenstates of the conjugate momentum field operator Π, which satisfy

$$\Pi\,|\pi\rangle = \pi(x)\,|\pi\rangle\,. \tag{3.96}$$

The inner product of $|\phi\rangle$ and $|\pi\rangle$ is given by

$$\langle\pi|\phi\rangle = \exp\left(-\frac{i}{\hbar}\int d^3x\,\pi(x)\phi(x)\right), \tag{3.97}$$

which is the analogous of $\langle p|x\rangle$ from quantum mechanics. The equivalent of expressions (3.57) for the completeness relations for fields are

$$\begin{aligned}\int \mathscr{D}\phi\ |\phi\rangle\,\langle\phi| &= \mathbb{1},\\ \int \mathscr{D}\pi\ |\pi\rangle\,\langle\pi| &= \mathbb{1},\end{aligned} \tag{3.98}$$

where we have tacitly introduced the symbols $\mathscr{D}\phi$ and $\mathscr{D}\pi$. We return to the meaning of these in the future and for now, it is understood that the integration is performed over all possible field configurations.

In quantum mechanics, an object of great interest is the *vacuum to vacuum transition amplitude* $Z = \langle 0,\, t_f|0,\, t_i\rangle$. From this quantity we can obtain the S-matrix of the theory and then study particle scattering, thus making a connection with experimental data [18]. This object can also be interpreted as a version of the partition function from statistical mechanics, enabling us to study thermodynamic property of systems [8]. The vacuum state is understood as the ground state of the theory, in which no particles are present. Note that the two states that compose this amplitude are defined at different times $t_i \neq t_f$. Interaction disturb the vacuum and create excitations in the field $|0,\, t_i\rangle$. By evaluating Z, we can obtain the probability that these particles have been annihilated, leaving a vacuum state at the time t_f. Next, we will derive a path integral representation of this object.

With the promotion of fields to operators, the Hamiltonian (density) of the theory becomes an operator, which we assume to have the form

$$H = \frac{1}{2}\Pi^2 + \mathscr{V}(\Phi). \tag{3.99}$$

Then, we have all the ingredients to obtain the path integral representation of Z repeating the steps performed to obtain the transition amplitude (3.72). By comparison, the vacuum to vacuum transition amplitude is given by

$$Z = \int \mathscr{D}\phi(x)\, e^{iS[\phi(x)]}, \tag{3.100}$$

where $S[\phi]$ is the classical action functional for the field ϕ. The functional measure in this case is given by

$$\phi(x) = \lim_{\substack{\delta t \to 0 \\ N \to \infty}} \left(-\frac{im}{2\pi\hbar\delta t} \right)^{N/2} \prod_{j=1}^{N-1} d\phi(x_j). \tag{3.101}$$

The integration is performed over all possible intermediate field configurations that satisfy the boundary conditions. We can visualize this process by imagining our field as being represented by a two-dimensional sheet (Figure 4). At each instant of time, the field has a different configuration, and the transition from one configuration to another occurs smoothly. Naturally, there are infinitely many intermediate configurations that match the boundary conditions, and the integration takes all of them into account.

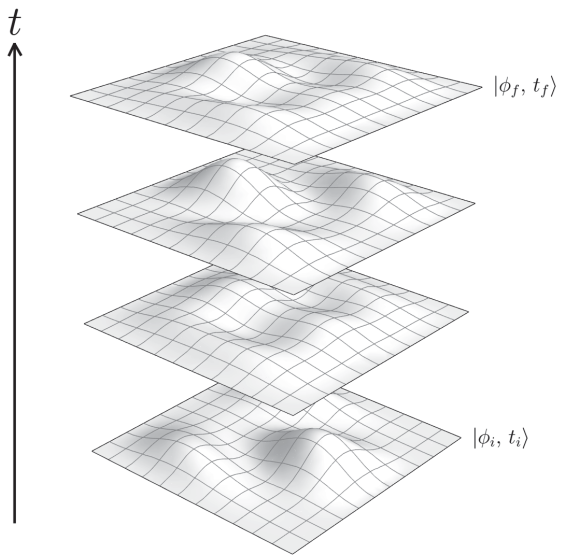


Figure 4 – Field configurations for a field with initial configuration described by $|\phi_i,\, t_i\rangle$ and final configuration described by $|\phi_i,\, t_i\rangle$.

3.5.2 Quantum Field Theory at Finite Temperature

Now we make the connection of quantum field theory with statistical mechanics. This allows us to study the thermodynamic properties of systems composed by fields. This framework is known as *statistical field theory*.

When studying statistical mechanics, one of the main objects is the partition function Z, defined by

$$Z(\beta) = \operatorname{tr} \exp(-\beta H), \tag{3.102}$$

where H is the Hamiltonian operator of the system and $\beta = 1/(k_B T)$, with k_B the Boltzmann constant [18, 25]. We may evaluate the trace using a complete set of eigenstates $\{|x\rangle\}_{x\in\mathbb{R}}$ of the position operator X and obtain

$$Z(\beta) = \int dx \, \langle x| \, e^{-\beta H} \, |x\rangle \, . \tag{3.103}$$

Once we have evaluated the partition function, we can obtain several thermodynamic properties of the system, as the internal energy U and the entropy S through the relations

$$\begin{aligned} U &= -\frac{\partial \ln Z}{\partial \beta}, \\ S &= k_B \left(\ln Z + \beta U\right). \end{aligned} \tag{3.104}$$

To make the connection of statistical mechanics with quantum field theories, we introduce the imaginary time formalism. We perform an analytical continuation of the real time t to the imaginary time via

$$t \to -i\tau, \tag{3.105}$$

where $\tau \in \mathbb{R}$. With this transformation, called *Wick rotation*, we see that the spacetime interval I becomes

$$I = \tau^2 + x^2 + y^2 + z^2 = g_{\mu\nu} x^\mu x^\nu \tag{3.106}$$

and, as a consequence, all the eigenvalues of the metric tensor become positive, which results in a Euclidean metric. For this reason, τ is often referred to as the *Euclidean time*. With the introduction of the imaginary time, the transition amplitude $F(x_f,\, t_f;\, x_i,\, t_i)$, defined in (3.59), becomes

$$F(x_f,\, -i\tau_f;\, x_i,\, -i\tau_i) = \langle x_f| \, e^{-H(\tau_f - \tau_i)} \, |x_i\rangle \, . \tag{3.107}$$

From equation (3.72), the path integral representation of this amplitude is

$$F(x_f,\, -i\tau_f;\, x_i,\, -i\tau_i) = \int \mathscr{D}x \, e^{-S_E[x]}, \tag{3.108}$$

where S_E is the *Euclidean action*, defined by $S_E[x] = -iS[x]$. Using the expression (3.107) we can express the partition function given in (3.103) as

$$\begin{aligned} Z(\beta) &= \int dx\, F(x,\, -i\beta;\, x,\, 0) \\ &= \int dx\, \langle x|\, e^{-\beta H}\, |x\rangle\,, \end{aligned} \tag{3.109}$$

so that we interpret the operator $e^{-\beta H}$ as a time evolution operator in imaginary time [8]. Therefore, we obtain a path integral representation of the partition function, given by

$$Z(\beta) = \int \mathscr{D}x\, e^{-S_E[x]} \tag{3.110}$$

where the Euclidean action is given by

$$S_E[x] = \int_0^\beta d\tau\, L[x,\, \dot{x}]. \tag{3.111}$$

In the path integral in equation (3.110) we are assuming an integration over endpoints, which is equivalent to taking the trace. Since the endpoints are labelled by the same position x, the boundary conditions are now

$$x(\tau + \beta) = x(\tau) \tag{3.112}$$

so the integration is performed over paths with period β.

The generalization of expression (3.110) for fields is given by

$$Z(\beta) = \int \mathscr{D}\phi\, e^{-S_E[\phi]}, \tag{3.113}$$

with the fields $\phi(t)$ satisfying the boundary conditions

$$\phi(\mathbf{x},\, \tau + \beta) = \phi(\mathbf{x},\, \tau). \tag{3.114}$$

The introduction of imaginary time is not limited to establishing a connection with statistical mechanics. It can be employed to facilitate certain calculations in quantum field theory [18, 25]. After completing the necessary calculations, we can perform the inverse rotation to return to Minkowski spacetime. However, in the case of statistical field theory, we are not interested in performing the inverse transformation, since the object of interest is the partition function itself. In the next chapter we will employ the methods presented here to obtain the partition function for a black hole and study its thermodynamic properties.

3.5.3 Effective Action

As discussed before, the path integral formulation presents some issues and one additional problem arises when studying field theories on curved spacetimes. The functional measure, as defined in (3.101), is not invariant under general coordinate transformations on the configuration space of fields. This is an undesirable feature, because it renders the observables to be dependent on coordinate choices in the space of fields. This motivates a modification in the definition of the functional measure, so that it is invariant under such transformations.

In multivariable calculus, when performing a change of coordinates, the presence of the Jacobian ensures that the integration over some volume in space is independent of the coordinate system chosen. In a similar manner, we shall introduce a factor in the path integral that plays the role of a Jacobian and renders the path integral to be invariant under coordinate changes in the configuration space of fields. Then, we redefine the functional measure to be [10, 13]

$$d\mu[\phi] = \mathscr{D}\phi\sqrt{\mathrm{Det}G_{ij}}, \tag{3.115}$$

where $\mathscr{D}\phi$ is the functional measure defined in (3.101) and Det G_{ij} denotes the functional determinant of a metric G_{ij} defined in the configuration-space. The definition of the functional determinant will not be need here and we discuss this later. Here, the factor $\sqrt{\mathrm{Det}G_{ij}}$ plays the role of the Jacobian for path integrals and ensures the invariance of the integral under coordinate transformations in field space.

There is no general procedure to obtain G_{ij}, so we must seek for metrics that leave the path integral invariant. The general form of a metric G_{ij} is given by

$$G_{ij} = G_{IJ}(\phi(x))\delta(x - x'), \tag{3.116}$$

where G_{IJ} is functional of the fields ϕ describing the system under study and $\delta^{(4)}(x - x')$ denotes the Dirac delta in four dimensions. To explain the notation employed here, we consider a specific example. For systems involving only the gravitational fields $g_{\mu\nu}$, the simplest metric $G_{IJ}(g)$ is the *DeWitt metric*, defined as [27]

$$\begin{aligned} G_{IJ}(g) &= G_{\mu\nu\rho\sigma} \\ &= \frac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - a g_{\mu\nu}g_{\rho\sigma}\right), \end{aligned} \tag{3.117}$$

where $a \in \mathbb{R}$ is a free parameter. For the value $a = -1/2$ (the case of 4 spacetime dimensions), this metric is degenerate, so we must avoid it to guarantee that the metric

tensor possesses an inverse. The index I of G_{IJ} labels both the spacetime indices $\mu\nu$ and J labels $\rho\sigma$. For other types of fields, such as scalar, fermionic, gauge, etc, the indices I and J label other types of indices [14]. The inverse of $G_{IJ} = G_{\mu\nu\rho\sigma}$ is denoted by $G^{IJ} = G^{\mu\nu\rho\sigma}$, and have the form

$$G^{\mu\nu\rho\sigma} = \frac{1}{2}\left(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - ag^{\mu\nu}g^{\rho\sigma}\right). \tag{3.118}$$

Using the identity $g^{\mu\sigma}g_{\sigma\nu} = \delta^{\mu}_{\nu}$, where δ^{μ}_{ν} denotes the Kronecker delta, it can be easily checked that

$$G^{\mu\nu\rho\sigma}G_{\rho\sigma\alpha\beta} = \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right). \tag{3.119}$$

The identity in the space of fields, denoted $\mathbb{1}^{\mu\nu}_{\alpha\beta}$, is given by [27]

$$\mathbb{1}^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right). \tag{3.120}$$

In the expression (3.116), the lower case index i labels the discrete index I of G_{IJ} together with the continuous label x of the Dirac delta $\delta(x - x')$. To assert this, we shall write $i = (I,\, x)$. Similarly, the index j labels J and x and we write $j = (J,\, x')$.

For the case where $G_{ij} = \mathbb{1}$, we recover the functional measure (3.101) and call G_{ij} a *trivial metric*. We refer to a metric G_{ij}, that is not the identity, as a *non-trivial metric*.

We can write

$$\sqrt{\mathsf{Det} G_{ij}} = \exp\left(\frac{1}{2}\ln\ \mathsf{Det}\, G_{ij}\right), \tag{3.121}$$

so that the functional measure $d\mu[\phi]$ defined in equation (3.115) can be written as

$$d\mu[\phi] = \mathscr{D}\phi \exp\left(\frac{1}{2}\ln\ \mathsf{Det}\, G_{ij}\right). \tag{3.122}$$

With this redefinition of the functional measure, the partition function (3.113) becomes

$$\begin{aligned} Z(\beta) &= \int d\mu[\phi] \exp\left(-\frac{1}{\hbar}S_E[\phi]\right) \\ &= \int \mathscr{D}\phi\ \exp\left(-\frac{1}{\hbar}S_E[\phi] - \frac{1}{2}\ln \mathsf{Det} G_{ij}\right) \\ &= \int \mathscr{D}\phi\ \exp\left[-\frac{1}{\hbar}\left(S_E[\phi] - \frac{\hbar}{2}\ln \mathsf{Det} G_{ij}\right)\right], \end{aligned} \tag{3.123}$$

where we have recovered the constant $\hbar$. We see that the Euclidean action S_E describing the system receives a correction due to the redefinition of the functional measure. This is only one of the corrections that the action receives. Below we briefly discuss another correction that arises from a perturbative treatment of gravity.

Going back to the evaluation of the term $\sqrt{\mathrm{Det}G_{ij}}$, consider an operator O_{ij} defined as

$$O_{ij} = O_{IJ}(x,\, x'). \tag{3.124}$$

The operator O_{IJ} has a dependence of the fields ϕ of the theory, but we omit the dependence to simplify the notation. Using the relation

$$\ln \mathrm{Det}O_{ij} = \mathrm{Tr} \ln O_{ij}, \tag{3.125}$$

where the functional trace Tr of O_{ij} is defined by

$$\mathrm{Tr}O_{ij} = \int d^4x \sqrt{-g}\, \mathrm{tr}\, O_{IJ}(x,\, x), \tag{3.126}$$

where g is the determinant of the spacetime metric g_{ab} and $\mathrm{tr}\, O_{IJ}$ denotes the usual trace, involving a summation over the elements of O_{IJ} with $I = J$. Also, we have an integration over x after setting $x = x'$. We note that from the definition (3.116) of G_{ij} contains a Dirac delta, so that

$$\mathrm{Tr} \ln G_{ij} = \delta^{(4)}(0) \int d^4x \sqrt{-g}\, \mathrm{tr} \ln G_{IJ}. \tag{3.127}$$

Due to the presence of $\delta^{(4)}(0)$, a divergence shows up. To deal with this divergence, a regularization procedure must be adopted to render this expression finite. This involves the insertion of a cut-off λ, associated with the energy scale, and the adoption of a Gaussian regularization, where [14, 28]

$$\delta^{(4)}(x) = \frac{\lambda^4}{(2\pi)^2} \exp\left(-\frac{\lambda^2}{2}x^2\right). \tag{3.128}$$

Using this form of $\delta^{(4)}(x)$, equation (3.127) becomes

$$\mathrm{Tr} \ln G_{ij} = \zeta \int d^4x \sqrt{-g}\, \mathrm{tr} \ln G_{IJ}, \tag{3.129}$$

where

$$\zeta = \frac{\lambda^4}{(2\pi)^2}. \tag{3.130}$$

This allows us to write the relation (3.125) as

$$\begin{aligned} \ln \mathsf{Det} G_{ij} &= \zeta \int d^4x \sqrt{-g}\, \mathrm{tr} \ln G_{IJ} \\ &= \zeta \int d^4x \sqrt{-g} \ln \det G_{IJ}, \end{aligned} \tag{3.131}$$

where we have used the identity $\ln \det M = \mathrm{tr} \ln M$, for some operator M, and $\det$ denotes the usual matrix determinant [3].

In many branches of physics it is common to employ perturbative methods. Such an approach allows to simplify the calculations and obtain approximate solutions to a problem, when the exact solution cannot be obtained. Usually, we start by modelling a simplified version of the physical system, to which we are able to obtain an exact solution. Then, to obtain an approximate solution of the full problem, we consider some kind of perturbation to the simplified model, that describes the effects that we have neglected in first place [29].

The perturbative solution A to a problem is usually expressed as a power series in some parameter ε, as

$$A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \tag{3.132}$$

where A_0 is the exact solution to the simplified version of the system, and the terms $\epsilon^n A_n$, $n = 1, 2, \ldots$, are called the n*th order* terms and are related to the perturbations considered. The parameter ϵ must be such that this power series converge and a given term is small when compared to the previous one.

As we have seen in the previous chapter, in general relativity, our main goal is to solve Einstein's field equations and obtain the metric tensor $g_{ab}(x)$. The implementation of perturbative methods in this case involves splitting the metric tensor as

$$g_{ab}(x) = \bar{g}_{ab}(x) + h_{ab}(x), \tag{3.133}$$

where, $\bar{g}_{ab}$ is a know solution to Einstein's field equations and h_{ab} is a perturbation whose components satisfy $|h_{\mu\nu}| << |\bar{g}_{\mu\nu}|$, for $\mu, \nu = 0, 1, 2, 3$ [2, 3]. A specific example of this treatment is given by the study of gravitational waves. In this case, $\bar{g}_{ab}$ is seen as a given background and h_{ab} is the disturbance in this background, related to the propagation of gravitational waves through spacetime [1, 30].

Perturbative techniques are commonly employed in quantum field theory as well and one of the main objects in this treatment is the *effective action* S_{eff} [18, 25]. The effective action can be expanded in a power series, similar to the expansion (3.132) as

$$S_{eff} = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots . \tag{3.134}$$

Here, the reduced Planck's constant $\hbar$ plays the role of the parameter ε and S_0 is the classical action. The expansion (3.134) is referred to as a *loop expansion* and the nth term, called the $n-$loop term, is proportional to $\hbar^n$, $n \in \mathbb{N}$. Here we will be concerned with one-loop corrections, thus we shall deal only with actions of the form

$$S_{eff} = S_0 + \hbar S_1. \tag{3.135}$$

Looking back to the expression (3.123) we see that the correction to the action, with origin in the functional measure, corresponds to a one-loop correction.

We are interested in studying quantum effects in gravitation. So, we may view these effects as a deviation from the classical regime. In this way, we are motivated to split the metric tensor g_{ab} as in expression (3.133), with $\bar{g}_{ab}$ representing a solution to the Einstein's field equations and h_{ab} a perturbation due to the quantum effects.

From the definitions (3.87), the Einstein-Hilbert action is of the form

$$S_{EH}[g] = \frac{1}{16\pi G}\int_{\Omega} d^4x\sqrt{-g}R, \tag{3.136}$$

where R is the Ricci scalar, which is obtained from the Riemann tensor $R_{abc}{}^{d}$. We recall that the components of $R_{abc}{}^{d}$ are given by

$$R_{\mu\nu\rho}{}^{\sigma} = -\frac{\partial}{\partial x^{\mu}}\Gamma^{\sigma}{}_{\nu\rho} + \frac{\partial}{\partial x^{\nu}}\Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\sigma}{}_{\nu\lambda} - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\sigma}{}_{\mu\lambda}, \tag{3.137}$$

where

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left[\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right]. \tag{3.138}$$

Then, the Ricci scalar involves products of the metric tensor components $g_{\mu\nu}$ with itself and also with its derivatives. Considering a splitting of $g_{\mu\nu}$ as in (3.133), we can expand the R in a series, with each term proportional to some power of $h_{\mu\nu}$. It can be shown that, to first order in $h_{\mu\nu}$, the action S_{EH} is of the form

$$S_{EH}[g] = S_{EH}[\bar{g}] + \frac{\hbar}{2}\ln \mathrm{Det} H_{ij}, \tag{3.139}$$

where

$$H_{ij} = H_{IJ}(\phi)\delta^{(4)}(x - x') \tag{3.140}$$

is an operator in the configuration space of fields ϕ [27, 28]. For the gravitational case, H_{IJ} is of the form

$$H_{IJ} = K_{IJ}\Box + U_{IJ}, \tag{3.141}$$

or in terms of spacetime indices,

$$\begin{aligned} H_{IJ}(g) &= H_{\mu\nu\rho\sigma} \\ &= K_{\mu\nu\rho\sigma}\Box + U_{\mu\nu\rho\sigma}, \end{aligned} \tag{3.142}$$

where $\Box$ denotes de D'Alambertian operator and

$$K_{\mu\nu\rho\sigma} = \frac{1}{4}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}\right). \tag{3.143}$$

The form of $U_{\mu\nu\rho\sigma}$ will not concern us, since it will appear only as a second order contribution and we may discard it.

With the result (3.139), the gravitational action S_G, defined in (3.86), receives a one-loop correction and the result is the action $S[g]$, given by

$$S[g] = S_G[\bar{g}] + \frac{\hbar}{2}\ln \mathsf{Det} H_{ij}. \tag{3.144}$$

Performing a Wick rotation (see equations (3.105) and (3.108)), this action becomes the Euclidean gravitational action $S_E[g]$, given by

$$S_E[g] = S_G^E[\bar{g}] + \frac{\hbar}{2}\ln \mathsf{Det} H_{ij}, \tag{3.145}$$

where $S_G^E[\bar{g}]$ is obtained from (3.86) after the Wick rotation. With $S_E[g]$, the partition function (3.123), for the gravitational field, becomes

$$Z(\beta) = \int \mathscr{D}g \exp\left[-\frac{\hbar}{2}\left(S_G^E[\bar{g}] + \frac{\hbar}{2}\ln \mathsf{Det} H_{ij} - \frac{\hbar}{2}\ln \mathsf{Det} G_{ij}\right)\right]. \tag{3.146}$$

Now that we have obtained the two corrections for the classical action, up to one-loop, we see that the gravitational effective action S_{eff} is given by

$$S_{eff}[g] = S_G^E[\bar{g}] + \frac{\hbar}{2}\ln \mathsf{Det} H_{ij} - \frac{\hbar}{2}\ln \mathsf{Det} G_{ij}. \tag{3.147}$$

Previously, we have reinstated the constant $\hbar$, so we could see explicitly that the corrections correspond to a one-loop term. From now on, we adopt again a system of units where $\hbar = c = 1$.

Our task now is to evaluate the one-loop term in (3.147), which we denote by S_1, so that

$$S_1 = \frac{1}{2}\left(\ln \mathsf{Det} H_{ij} - \mathsf{Det} G_{ij}\right). \tag{3.148}$$

Using the property

$$(\mathsf{Det} G_{ij})^{-1} = \mathsf{Det}(G_{ij})^{-1} = \mathsf{Det} G^{ij} \tag{3.149}$$

we can rewrite S_1 as

$$S_1 = \frac{1}{2}\ln \mathsf{Det}\left(G^{il} H_{lj}\right). \tag{3.150}$$

Here G^{ij} denotes the inverse of the metric G_{ij} and we have written the indices in the product $G^{il}H_{lj}$ to be consistent with the formula for the product of operators. This product, is proportional to $G^{IL}H_{LJ}$ and to evaluate it, we rewrite H_{IJ} as in expression (3.141), obtaining

$$G^{IL}H_{LJ} = G^{IL}(K_{LJ}\Box + U_{LJ}), \tag{3.151}$$

or using the definitions (3.118) and (3.142),

$$\begin{aligned} G^{IL}H_{LJ} &= G^{\mu\nu\rho\sigma} H_{\rho\sigma\alpha\beta} \\ &= G^{\mu\nu\rho\sigma}\left(K_{\rho\sigma\alpha\beta}\Box + U_{\rho\sigma\alpha\beta}\right). \end{aligned} \tag{3.152}$$

It will be convenient to factor $K_{\rho\sigma\alpha\beta}$ in this expression. To do so, note that we can write

$$K_{\rho\sigma\alpha\beta} = K_{\rho\sigma\lambda\gamma}\mathbb{1}^{\lambda\gamma}_{\alpha\beta} \tag{3.153}$$

where $\mathbb{1}^{\lambda\gamma}_{\alpha\beta}$ is defined in (3.120). Then, we can express $G^{IL}H_{LJ}$ as

$$\begin{aligned} G^{IL}H_{LJ} &= G^{\mu\nu\rho\sigma}\left(K_{\rho\sigma\lambda\gamma}\mathbb{1}^{\lambda\gamma}_{\alpha\beta}\Box + U_{\rho\sigma\alpha\beta}\right) \\ &= G^{\mu\nu\rho\sigma}K_{\rho\sigma\lambda\gamma}\left(\mathbb{1}^{\lambda\gamma}_{\alpha\beta}\Box + K^{\lambda\gamma\tau\xi}U_{\tau\xi\alpha\beta}\right), \end{aligned} \tag{3.154}$$

where $K^{\lambda\gamma\tau\xi}$ denotes the inverse of $K_{\lambda\gamma\tau\xi}$. Using this result, together with the definitions (3.116) and (3.140), we obtain

$$\begin{aligned} G^{il}H_{il} &= G^{IL}H_{LJ}\delta^4(x-x')\delta^4(x-x') \\ &= G^{\mu\nu\rho\sigma}K_{\rho\sigma\lambda\gamma}\left(\mathbb{1}^{\lambda\gamma}_{\alpha\beta}\Box + K^{\lambda\gamma\tau\xi}U_{\tau\xi\alpha\beta}\right)\delta^4(x-x')\delta^4(x-x') \end{aligned} \tag{3.155}$$

so the action (3.150) is given by

$$S_1 = \frac{1}{2}\ln \mathsf{Det}\left[G^{\mu\nu\rho\sigma}K_{\rho\sigma\lambda\gamma}\left(\mathbb{1}^{\lambda\gamma}_{\alpha\beta}\Box + K^{\lambda\gamma\tau\xi}U_{\tau\xi\alpha\beta}\right)\delta^4(x-x')\delta^4(x-x')\right]. \tag{3.156}$$

Using the property $\mathsf{Det}(AB) = \mathsf{Det}(A)\mathsf{Det}(B)$, this can be written as

$$\begin{aligned} S_1 &= \frac{1}{2}\ln\left\{\mathsf{Det}\left[G^{\mu\nu\rho\sigma}K_{\rho\sigma\lambda\gamma}\delta^{(4)}(x-x')\right]\mathsf{Det}\left[\left(\mathbb{1}^{\lambda\gamma}_{\alpha\beta}\Box + K^{\lambda\gamma\tau\xi}U_{\tau\xi\alpha\beta}\right)\delta^{(4)}(x-x')\right]\right\} \\ &= \frac{1}{2}\ln \mathsf{Det}\left[G^{\mu\nu\rho\sigma}K_{\rho\sigma\lambda\gamma}\delta^{(4)}(x-x')\right] + \frac{1}{2}\ln \mathsf{Det}\left[\left(\mathbb{1}^{\lambda\gamma}_{\alpha\beta}\Box + K^{\lambda\gamma\tau\xi}U_{\tau\xi\alpha\beta}\right)\delta^{(4)}(x-x')\right]. \end{aligned} \tag{3.157}$$

The second term in this expression provides subdominant contributions to S_1, when compared to the first term. Then, we will discard this term and our one-loop correction is then given by

$$\begin{aligned} S_1 &= \frac{1}{2}\ln \mathsf{Det}\left[G^{\mu\nu\rho\sigma}K_{\rho\sigma\lambda\gamma}\delta^4(x-x')\right] \\ &= \frac{\zeta}{2}\int d^4x\sqrt{-g}\ln\det G_{IJ}, \end{aligned} \tag{3.158}$$

where we have used the result (3.131) in the last equality. With the definitions (3.118) and (3.143), we obtain

$$\begin{aligned} S_1 &= \frac{\zeta}{2}\int d^4x\sqrt{-g}\ln\det\left[\frac{1}{8}\left(g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-ag^{\mu\nu}g^{\rho\sigma}\right)\left(g_{\rho\lambda}g_{\sigma\gamma}+g_{\rho\gamma}g_{\sigma\lambda}-g_{\rho\sigma}g_{\lambda\gamma}\right)\right] \\ &= \frac{\zeta}{2}\int d^4x\sqrt{-g}\ln\det\left[\frac{1}{8}\left(2\delta^\mu_\lambda\delta^\nu_\gamma+2\delta^\mu_\gamma\delta^\nu_\lambda-2\delta^\mu_\sigma g^{\nu\sigma}g_{\lambda\gamma}-a\delta^\rho_\gamma g^{\mu\nu}g_{\rho\lambda}-a\delta^\rho_\lambda g^{\mu\nu}g_{\rho\gamma}+g^{\mu\nu}g_{\lambda\gamma}(g^{\rho\sigma}g_{\rho\sigma})\right)\right] \end{aligned} \tag{3.159}$$

which, with $\delta^\mu_\sigma g^{\nu\sigma} = g^{\mu\nu}$ and $g^{\rho\sigma}g_{\rho\sigma} = 4$, reads

$$S_1 = \frac{\zeta}{2}\int d^4x\sqrt{-g}\ln\det\left[\frac{1}{4}\left(\delta^\mu_\lambda\delta^\nu_\gamma+\delta^\mu_\gamma\delta^\nu_\lambda+(a-1)g^{\mu\nu}g_{\lambda\gamma}\right)\right]. \tag{3.160}$$

Note that the first two terms together are proportional to the identity $\mathbb{1}^{\mu\nu}_{\lambda\gamma}$. Then, using the matrix determinant lemma

$$\det\left[\mathbb{1}^{\mu\nu}_{\lambda\gamma} + (a-1)g^{\mu\nu}g_{\lambda\gamma} = 1 + 4(a-1)\right] \tag{3.161}$$

together with the property $\det(aM) = a^n \det M$, for $a \in \mathbb{R}$ and M a $n \times n$ matrix, we obtain

$$S_1 = \frac{\zeta}{2}\int d^4x\sqrt{-g}\ln\left[\frac{1+4(a-1)}{256}\right]. \tag{3.162}$$

Finally, the effect action (3.147), at one-loop level, is given by

$$S_{eff}[g] = S^E_G[\bar{g}] + \int d^4x\sqrt{-g}\Lambda_C, \tag{3.163}$$

where we have defined

$$\Lambda_C = \frac{\zeta}{2}\ln\left[\frac{1+4(a-1)}{256}\right]. \tag{3.164}$$

We remark that Λ_C originated from the redefinition of the functional measure of the path integral and by considering a perturbative expansion of the gravitational action. The latter involved splitting the metric tensor g_{ab} in two parts, one being a classical background $\bar{g}_{ab}$ and the other a perturbation h_{ab}, describing quantum effects. We also have used the partition function, derived using the formalism of quantum field theory. Thus, Λ_C is a purely quantum effect, not predicted by the classical theory of general relativity.

Note that we have tacitly assumed that, for the gravitational field, the partition function has the same form as that obtained from the formalism of quantum field theory. We observe that for gravitation, we are not able to obtain a path integral as we did in the quantum mechanical case, so we assume that the formalism can be generalized to any type of field [9, 27]. We shall discuss this issue in the future.

The missing part to obtain the effective action is the Euclidean gravitational action $S^E_G[\bar{g}]$. In the next chapter we will obtain $S^E_G[\bar{g}]$ for a specific metric $\bar{g}_{ab}$ and present the basics of black hole thermodynamics.

CHAPTER 4

Black Hole Thermodynamics

4.1 Schwarzschild Solution

As we have seen in chapter 2, according to the theory of general relativity, gravitation is an effect of the curvature of spacetime. This curvature is related to the matter distribution by the equation, given by

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab}. \tag{4.1}$$

Our goal is to derive the g_{ab} that solves this equation for a given energy-momentum tensor T_{ab}. The spacetime has four dimensions, so the metric tensor has sixteen components, but only ten are independent, since g_{ab} is a symmetric tensor.

Taking the trace of equation (4.1), we obtain

$$R = -8\pi G T. \tag{4.2}$$

Plugging this result in equation (4.1), we find

$$R_{ab} = 8\pi G \left(T_{ab} - \frac{1}{2} T g_{ab} \right). \tag{4.3}$$

We are interested in the solution outside a body. In this region, we assume that there is no matter distribution, so $T_{ab} = 0$. From this we conclude that the vacuum solutions follow from

$$R_{ab} = 0. \tag{4.4}$$

Recall that the Ricci tensor R_{ab} is defined by $R_{ab} = R_{acb}{}^{c}$, where $R_{abc}{}^{d}$ is the Riemann tensor. We will be concerned with a solution of (4.4) describing the gravitational field for a static and spherically symmetric black hole. The solution to this system is given by the Schwarzschild metric, whose corresponding line element is given by [3]

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \phi^2, \tag{4.5}$$

in a spherical coordinate system $(t,\, r,\, \theta,\, \phi)$. We note that the this line element is ill defined for $r = 0$ and $r = 2GM$. The former represents a physical singularity, while the latter is only an apparent singularity and can be removed by a coordinate change. The Kruskal coordinate system is given by changing the coordinate r to a coordinate X and the time t to a coordinate T, defined by

$$\begin{aligned} X &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right), \\ T &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right), \end{aligned} \tag{4.6}$$

for $0 < r < 2GM$, and

$$\begin{aligned} X &= \left(1 - \frac{r}{2GM}\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right), \\ T &= \left(1 - \frac{r}{2GM}\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right), \end{aligned} \tag{4.7}$$

for $r > 2GM$. With these definitions, we have

$$\frac{X + T}{X - T} = \exp\left(\frac{t}{2GM}\right). \tag{4.8}$$

The quantity $r_S = 2GM$ is called the *Schwarzschild radius*, and corresponds to the radius of the event horizon of the black hole.

4.2 Black Hole Thermodynamics

We are now interested in studying the thermodynamics of a Schwarzschild black hole. To do so, we must compute the corresponding partition function and apply

the usual relations from thermodynamics. In this discussion, we adopt the geometrized system of units, in which $G = c = \hbar = 1$.

The line element in Schwarzschild spacetime is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4.9}$$

which, with the Wick rotation $t = -i\tau$, becomes

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{4.10}$$

Taking $r \to \infty$, the metric becomes that of flat spacetime and have a Euclidean signature. Then, for all the region $r > 2M$, the spacetime is Euclidean. However, in the region $r < 2M$, the first two terms of ds^2 become negative and, consequently, the metric components g_{00} and g_{11} acquire a negative sign, so the spacetime is not Euclidean. Thus, the two regions correspond to different manifolds. In this way, since the partition function is defined on a Euclidean manifold, the region $r < 2GM$ is excluded from the spacetime of interest.

In the imaginary time formalism, the Euclidean time is cyclic and has period β. To determine the value of β for the Euclidean Schwarzschild spacetime, we observe that from the expression (4.8), it follows that τ has period $\beta = 8\pi M$.

To evaluate the action (3.91), we choose the boundary $\partial\Omega$ to be the timelike hypersurface characterized by $r = r_0$. Thus, our hypersurface Φ corresponds to the coordinate restriction $\Phi(r) = 0$ where

$$\Phi(r) = r - r_0. \tag{4.11}$$

Let us evaluate the extrinsic curvature K for the hypersurface embedded in the Euclidean Schwarzschild spacetime. First, using the formula (3.93) for the unit normal vector to $\partial\Omega$, the only component that does not vanish is n_1, which is given by

$$n_1 = \sqrt{1 - \frac{2M}{r_0}}. \tag{4.12}$$

From (4.10) we can read off the components of the metric g_{ab} and obtain that its determinant is equal to $g = r^4 \sin^2\theta$. Then, using (3.92), the extrinsic curvature is given by

$$\begin{aligned} K &= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r} \left(r^2 \sin\theta \sqrt{1 - \frac{2M}{r}} \right) \Bigg|_{r=r_0} \\ &= \frac{2}{r_0} \sqrt{1 - \frac{2M}{r_0}} + \frac{2M}{2r_0^2} \left(\sqrt{1 - \frac{2M}{r_0}} \right)^{-1}. \end{aligned} \tag{4.13}$$

The evaluation of K_0 is performed considering that $\partial\Omega$ is embedded in the Minkowski spacetime. The line element for the Minkowski space, in Euclidean signature, for spherical coordinates, is given by [3]

$$ds^2 = d\tau^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{4.14}$$

From this we can read the components of the metric tensor η_{ab} for Minkowski spacetime. Thus, the determinant of this metric is given by $\eta = r^4 \sin^2\theta$. Using formula (3.93), we see that the only component of the unit normal vector that does not vanish is $n_1 = 1$. With this, from equation (3.94), we find

$$\begin{aligned} K_0 &= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r} \left(r^2 \sin\theta \right) \Big|_{r=r_0} \\ &= \frac{2}{r_0} \end{aligned} \tag{4.15}$$

The restriction of the line element (4.10) to this hypersurface is given by

$$ds^2 = \left(1 - \frac{2M}{r} \right) d\tau^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4.16}$$

from which we can read off the components of the metric h_{ab} induced in this hypersurface and obtain that the determinant of h_{ab} is given by

$$h = r^4 \sin^2\theta \left(1 - \frac{2M}{r} \right). \tag{4.17}$$

Then, we have all the ingredients needed to evaluate the action (3.91). Substituting all of these and evaluating the integral, we obtain

$$I_E[g] = \beta \left[r_0 \left(1 - \sqrt{1 - \frac{2M}{r_0}} \right) - \frac{M}{2} \left(\sqrt{1 - \frac{2M}{r_0}} \right)^{-1} \right]. \tag{4.18}$$

We are interested in taking the limit $r_0 \to \infty$, and then we consider the expansion

$$\sqrt{1-\frac{2M}{r_0}} = 1 - \frac{M}{r_0} + O(r_0^{-2}), \tag{4.19}$$

where $O(r_0^{-2})$ denotes the terms of order 2 in $1/r_0$ or higher. With this expansion, the Euclidean action is given by

$$I_E[g] = \beta \left[M - \frac{M}{2} \left(\sqrt{1-\frac{2M}{r_0}} \right)^{-1} + O(r_0^{-1}) \right]. \tag{4.20}$$

Finally, taking the limit $r_0 \to \infty$, we obtain

$$I_E[g] = \frac{\beta^2}{16\pi}, \tag{4.21}$$

where we have used $\beta = 8\pi M$. Now that we possess the Euclidean action for the Schwarzschild spacetime, we turn our attention to obtain the corresponding partition function.

By comparison with the partition function (3.113), we introduce the partition function for the gravitational case, given by

$$Z(\beta) = \int \mathscr{D}g e^{-I_E}, \tag{4.22}$$

where $\mathscr{D}g$ is the functional measure for the metric tensor. Differently from quantum field theory, we are not able to derive this path integral and then we postulate that the integral has this form. Furthermore, we may not be able to compute the path integral explicitly, so we resort to approximations. We expect that the dominant contributions to the path integral to come from the metrics g_{ab} which solve the classical field equations [9]. Thus, we expand the metric as

$$g_{ab} = \tilde{g}_{ab} + \delta g_{ab}, \tag{4.23}$$

where $\tilde{g}_{ab}$ is the solution to the classical field equations and δg_{ab} is a perturbation. We also expand the action as

$$I[g] = I[\tilde{g}] + I_2[\delta g], \tag{4.24}$$

where $I[\tilde{g}]$ is the classical action and $I_2[\delta g]$ is quadratic in the perturbation δg_{ab}. With this considerations, the partition function satisfies

$$\ln Z[g] = -I[\tilde{g}] + \ln \int \mathscr{D}(\delta g) e^{-I_2[\delta g]} + \ldots, \tag{4.25}$$

where the dots denote the higher order terms.

We have evaluated the action (4.21) considering the Schwarzschild solution to Einstein's field equations, which is a classical solution. Then, the leading contribution to the partition function is given by

$$-\ln Z(\beta) = \frac{\beta^2}{16\pi}. \tag{4.26}$$

Using the relations (3.104), we obtain that for the Schwarzschild black hole, the internal energy U and the entropy S are given by

$$\begin{aligned} U &= \frac{\beta}{8\pi} = M, \\ S &= \frac{\beta^2}{16\pi} = 4\pi r_S^2, \end{aligned} \tag{4.27}$$

where we have used $\beta = 8\pi M$ and $r_S = 2\pi M$. Since r_S is the radius of the event horizon, we can express S in terms of the area of the event horizon as

$$S = \frac{A}{4}, \tag{4.28}$$

which is the Bekenstein-Hawking formula for the entropy of a black hole. The relation between entropy and the area of the event horizon of black holes was first conjectured by Bekenstein, although he did not obtained a mathematical formula for it [7]. Later, Hawking was able to derive this formula, studying quantum field theory in curved spacetimes [6]. In SI units, the expressions given in (4.27) are given by

$$\begin{aligned} U &= c^2 M, \\ S &= \frac{k_B c^3}{4G\hbar} A, \end{aligned} \tag{4.29}$$

where c is the speed of light, k_B the Boltzmann's constant, G the gravitational constant and $\hbar$ the reduced Planck's constant.

We remark that the calculations performed here considered the usual definition of path integrals, without considering the correction coming from the definition of functional measure. The consequences of this corrections to the thermodynamics of black holes will be explored in the next chapter.

CHAPTER 5

One-loop Corrections to Black Hole Thermodynamics

In this chapter we study a one-loop correction to the thermodynamics quantities of a Schwarzschild black hole, with origin in the redefinition of the functional measure of the path integral formulation. We discuss how this correction can be viewed as giving rise a pressure in the system. This lead to the interpretation of the black hole mass as the enthalpy of the system, rather than with the internal energy the system.

5.1 One-loop Correction

In chapter 3 we have presented the general formalism of quantum field theory, using the path integral formalism. As we have discussed, the functional measure of the path integral needs to be redefined, so the path integral is invariant under coordinate transformations in the configuration space of fields. This redefinition introduces a one-loop correction to the action of the system. Also, by considering a perturbative treatment of the gravitational field, another contribution arises in the one-loop level. Combined together, these contributions yields an Euclidean effective action I_{eff} of the system, given by

$$I_{eff}[g] = I_G^E[\bar{g}] + \frac{\hbar}{2}\ln \mathrm{Det} H_{ij} - \frac{\hbar}{2}\ln \mathrm{Det} G_{ij}. \tag{5.1}$$

where $I_G^E[\bar{g}]$ is the classical action for some background field $\bar{g}_{ab}$. For the gravitational case, considering the DeWitt metric

$$G_{IJ} = \frac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - a g_{\mu\nu}g_{\rho\sigma}\right), \tag{5.2}$$

the effective action becomes

$$I_{eff}[g] = I_G^E[\bar{g}] + \int d^4x\sqrt{\bar{g}}\Lambda_C, \tag{5.3}$$

where

$$\Lambda_C = \frac{\zeta}{2}\ln\left[\frac{1+4(a-1)}{256}\right]. \tag{5.4}$$

As we have seen, the Euclidean gravitational action $I_G^E[\bar{g}]$ is given by

$$I_G^E[\bar{g}] = -\frac{1}{16\pi G}\int_\Omega d^4x\sqrt{\bar{g}}R + \frac{1}{8\pi G}\oint_\Omega d^3y\sqrt{h}(K-K_0), \tag{5.5}$$

where it is understood that the integration is taken considering an Euclidean signature. Substituting this into (5.3), we obtain the effective action

$$\begin{aligned} I_{eff}[g] &= -\frac{1}{16\pi G}\int_\Omega d^4x\sqrt{\bar{g}}R + \int d^4x\sqrt{\bar{g}}\Lambda_C + \frac{1}{8\pi G}\oint_\Omega d^3y\sqrt{h}(K-K_0) \\ &= -\int_\Omega d^4x\sqrt{\bar{g}}\left(\frac{R}{16\pi G} - \Lambda_C\right) + \frac{1}{8\pi G}\oint_\Omega d^3y\sqrt{h}(K-K_0). \end{aligned} \tag{5.6}$$

Thus, the one-loop correction introduces the constant Λ_C in the effective action, just as a cosmological constant would appear in the classical action (see equation (3.90)), apart from a factor of 2 [3]. We shall then investigate the thermodynamics of a black hole in the presence of a cosmological constant.

In the usual study of black hole thermodynamics, the cosmological constant Λ is treated as a fixed parameter, typically taken to be zero. However, some treatments take Λ to be a thermodynamic variable [31]. In this case, Λ is associated with a pressure P in the system by the relation

$$P = -\frac{\Lambda}{8\pi G}. \tag{5.7}$$

The consideration of a pressure proportional to Λ alters some identifications we have made in the last chapter, as we shall see below.

When a cosmological constant Λ is taken into account, it introduces the notion of a *vacuum energy*. In the absence of matter, the only contribution to the energy momentum tensor comes from Λ, giving origin to an energy density ρ,

$$\rho = \frac{\Lambda}{8\pi G}. \tag{5.8}$$

Then, ρ is a constant energy density throughout the spacetime. As we have seen, the internal energy U of a black hole is associated with its mass M by $U = M$. Considering that the black hole occupies a volume V, the total energy E in this volume is given by

$$\begin{aligned} E &= U + \rho V \\ &= M - PV, \end{aligned} \tag{5.9}$$

where we have used the definitions (5.7) and (5.8). From this, we obtain

$$M = E + PV, \tag{5.10}$$

which is precisely the definition of the enthalpy function $H(S, P)$. This suggests that we identify the mass M with an enthalpy, rather than with the internal energy. The definition of the volume V is not know at first sight, but we will obtain an expression for it based on thermodynamic relations.

In the presence of the cosmological constant, the line element ds is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2 d\phi^2, \tag{5.11}$$

where $r_h = 8GM$ is the Schwarzschild radius and

$$f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2. \tag{5.12}$$

As in the purely Schwarzschild case, the event horizon radius r_h is defined by $f(r_h) = 0$,

$$\frac{\Lambda}{3}r_h^3 - r_h + 2GM = 0. \tag{5.13}$$

For the case $\Lambda > 0$, this equation has two solutions, corresponding to two distinct event horizons each one with a distinct temperature and the system is not in thermal equilibrium [31]. On the other hand, for $\Lambda < 0$, there is only one event horizon and the system is in thermal equilibrium. Hence, we assume $\Lambda < 0$ in the following.

From the relation (5.13), we can determine the value of M in terms of r_h, and this gives

$$M = \frac{r_h}{2G}\left(1 - \frac{\Lambda}{3}r_h^2\right). \tag{5.14}$$

The enthalpy H is a function of entropy S and pressure P, so in associating the black hole mass M with H, we view M as a function

$$M = H(S, P). \tag{5.15}$$

So, we must rewrite the expression (5.14) in terms of S and P. Recalling that the entropy S is given by

$$\begin{aligned} S &= \frac{\pi A}{4\hbar G} \\ &= \frac{\pi r_h^2}{\hbar G}, \end{aligned} \tag{5.16}$$

we invert this relation and obtain

$$\frac{r_h}{2G} = \left(\frac{\hbar}{4\pi G} S\right)^{1/2}. \tag{5.17}$$

Also, using the definition (5.7), we can write

$$1 - \frac{\Lambda}{3} r_h^2 = 1 + \frac{8\hbar G^2}{3} SP. \tag{5.18}$$

With relations (5.17) and (5.18) in equation (5.14), gives

$$\begin{aligned} M &= H(S, P) \\ &= \left(\frac{\hbar}{4\pi G}\right)^{1/2} \left(S^{1/2} + \frac{8\hbar G^2}{3} S^{3/2} P\right). \end{aligned} \tag{5.19}$$

Now that we have obtained M as a function of entropy S and pressure P, we can employ the usual relations from thermodynamics to evaluate some thermodynamic quantities for this system. The temperature T can be obtained from the enthalpy function by

$$T = \left(\frac{\partial H}{\partial S}\right)_P, \tag{5.20}$$

and the volume V is given by

$$V = \left(\frac{\partial H}{\partial P}\right)_S. \tag{5.21}$$

Then, from equation (5.19), we find

$$
\begin{aligned}
T &= \left(\frac{\hbar}{16\pi G}\right)^{1/2} \left(S^{-1/2} + 8\hbar G^2 S^{1/2} P\right), \\
V &= \left(\frac{\hbar}{16\pi G}\right)^{1/2} \frac{8\hbar G^2}{3} S^{3/2}.
\end{aligned}
\tag{5.22}
$$

Using (5.16) and (5.7), these are given by

$$
\begin{aligned}
T &= \frac{\hbar}{4\pi r_h}\left(1 - \Lambda r_h^2\right), \\
V &= \frac{4\pi r_h^3}{3}.
\end{aligned}
\tag{5.23}
$$

Then we see that the volume V occupied by the system is equal to the volume of a sphere of radius r_h.

Since we have considered the case where $\Lambda < 0$, the pressure P is positive and, thus, exerted by the black hole on its surroundings. We also recall that the constant Λ_C depends on the cut-off λ, which is related to the energy scale adopted (see equation (3.130)). Then, the value of P varies with the energy in the processes that the system undergo and we may view it as a function $P = P(\zeta)$.

We remark that the cosmological constant Λ_C has a purely quantum origin, related to the redefinition of the functional measure of the path integral. This was motivated by seeking a better definition of the path integral formulation of quantum field theory, to cope with some of the issues that arise in this formalism. This being a quantum effect, it can shed a new light in the pursue of a quantum theory of gravity. Despite the fact that we have considered the particular study of thermodynamics of a Schwarzschild black hole, the formalism is general and can be applied to any system.

The derivation of the constant Λ_C was made considering a particular metric G_{ij} for the configuration space of fields, namely, the DeWitt metric. Since we do not have a procedure to obtain the metric for the field space, there can be more possible metrics that also render the path integral invariant. The adoption of other metrics could lead to different results and we seek to investigate this in future works.

As a first speculation, we turn our attention to the question of singularity formations in spacetime. The theory of general relativity predicts that the formation of black holes by the gravitational collapse of matter ultimately leads to a singularity, with all the matter being concentrated in a single point. If we are allowed to view the pressure $P(\zeta)$ as exerted by the matter composing the black hole, this pressure would counteract the gravitational collapse of matter. Since ζ depends on the energy scale, it would vary as the matter collapses, generating greater values of pressure as the matter is infalling. If ζ

can attain higher enough values, this effect could prevent that all the matter to collapse into a single point. In this case, this would prevent the formation of a singularity. At this point, this remains only as a conjecture of the effects that $P(\zeta)$ may cause and the formal investigation of such phenomena is yet to be done.

CHAPTER 6

Conclusion

In this work we have presented the study of black hole thermodynamics using the Euclidean quantum gravity approach. Such an approach is based on the path integral formulation of quantum field theory. This approach provides a direct connection with statistical mechanics and allows us to study the thermodynamic property of quantum systems.

The path integral formulation of quantum field theory presents some mathematical issues, such as the definition of functional measure. Since this measure is not invariant under coordinate changes in the configuration space of fields, we explore a redefinition of this object and investigate some of its consequences. This redefinition involves the introduction of a metric in this field space, to render the path integral invariant. In doing so, a term appears in the effective action of the system and can be viewed as a one-loop correction.

There is no procedure to define a specific metric in the field space and, in principle, any metric that renders the path integral invariant can be considered. For purely gravitational systems, the simplest metric is given by the DeWitt metric. With this metric, the effective action of the system receives a constant correction Λ_C, which plays the role of a cosmological constant. Although this corresponds to a cosmological constant, it has a purely quantum origin and presents a new effect, not accounted by general relativity.

For the special case of a Schwarzschild black hole, the constant Λ_C alters the internal energy by introducing a work-like term, leading to the interpretation of Λ_C as giving rise to a pressure in the system. Following the literature, this suggests that we

identify the black hole mass with the enthalpy, rather than with the internal energy, as it is usually done. At last, we speculate how this pressure could affect the gravitational collapse of matter and the black hole formation, possibly counteracting the formation of a singularity.

For future works, we aim to explore the implications of the one-loop correction to other gravitational systems, such as more general black holes, and to the gravitational collapse of matter. We also seek to study the question of black hole stability in the presence of Λ_C . As we have considered a specific metric in the configuration space, we also aim to explore different metrics for the field space and their implications as well as the effects of higher order corrections to the effective action.

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