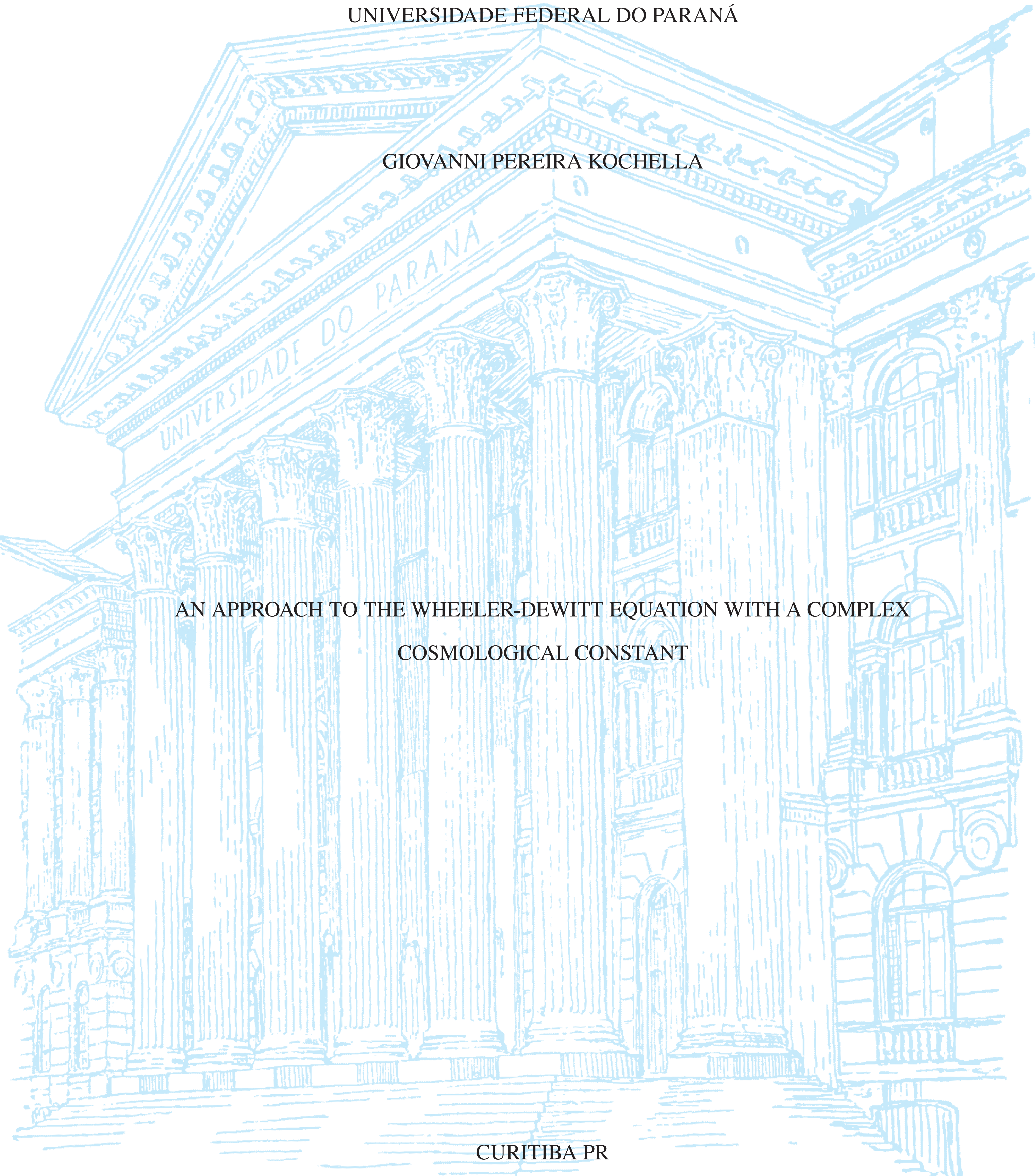

UNIVERSIDADE FEDERAL DO PARANÁ

GIOVANNI PEREIRA KOCHELLA

AN APPROACH TO THE WHEELER-DEWITT EQUATION WITH A COMPLEX COSMOLOGICAL CONSTANT

CURITIBA PR

2025

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Dissertação apresentada como requisito parcial à obtenção do grau de Mestre em Física no Programa de Pós-Graduação em Física, Setor de Ciências Exatas, da Universidade Federal do Paraná.

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RESUMO

Esta dissertação investiga a solução da equação de Wheeler-DeWitt no contexto da gravidade quântica, com ênfase em uma correção quântica derivada da integral de caminhos. A pesquisa começa abordando a evolução histórica, desde Planck até Wheeler e DeWitt, cujas contribuições culminaram na formulação da equação que descreve a gravidade quântica. O primeiro passo no desenvolvimento da dissertação é a dedução da equação de Wheeler-DeWitt a partir da ação de Einstein-Hilbert, utilizando o formalismo de Arnowitt-Deser-Misner. Essa abordagem canônica permite uma descrição detalhada da dinâmica da gravitação, essencial para a quantização do campo gravitacional. No segundo estágio, a dissertação transita para o formalismo de Ashtekar, que reformula a teoria gravitacional através de variáveis auto-duais. Essa reformulação é fundamental para a quantização da gravidade, uma vez que as variáveis de Ashtekar facilitam a integração das teorias clássicas com as abordagens quânticas, permitindo uma melhor compreensão da estrutura matemática que rege a gravitação. A seguir, a dissertação apresenta uma modificação quântica à ação de Einstein-Hilbert, com base na medida funcional proveniente da integral de caminhos. Esta modificação é caracterizada pela introdução de uma constante cosmológica complexa, que emerge como resultado da quantização da gravidade. A constante cosmológica complexa é incorporada na ação modificada, e a equação de Wheeler-DeWitt é resolvida dentro desse novo quadro teórico. Ao considerar os efeitos quânticos da gravitação, a solução obtida oferece uma nova perspectiva sobre a natureza do espaço-tempo e seus componentes fundamentais em escalas quânticas. A pesquisa propõe que a constante cosmológica, derivada dessa medida funcional, desempenha um papel importante na dinâmica quântica do universo, esclarecendo o comportamento das flutuações quânticas do espaço-tempo, através dos efeitos topológicos e quânticos que nossa solução fornece, além de possibilitar novas linhas de pesquisa para a origem da constante cosmológica. Este trabalho contribui para a resolução da equação de Wheeler-DeWitt, ao integrar os formalismos canônicos e da integral de caminhos, propondo uma nova maneira de interpretar a gravidade quântica.

Palavras-chave: Equação de Wheeler-DeWitt. Unificação. Cosmologia Quântica.

ABSTRACT

This dissertation investigates the solution of the Wheeler-DeWitt equation in the context of quantum gravity, with emphasis on a quantum correction derived from the path integral. The research begins by addressing the historical evolution, from Planck to Wheeler and DeWitt, whose contributions culminated in the formulation of the equation that describes quantum gravity. The first step in the development of the dissertation is the derivation of the Wheeler-DeWitt equation from the Einstein-Hilbert action, using the Arnowitt-Deser-Misner formalism. This canonical approach allows a detailed description of the dynamics of gravitation, essential for the quantization of the gravitational field. In the second stage, the dissertation moves to the Ashtekar formalism, which reformulates gravitational theory through self-dual variables. This reformulation is fundamental for the quantization of gravity, since Ashtekar variables facilitate the integration of classical theories with quantum approaches, allowing a better understanding of the mathematical structure that governs gravitation. Next, the dissertation presents a quantum modification to the Einstein-Hilbert action, based on the functional measure derived from the path integral. This modification is characterized by the introduction of a complex cosmological constant, which emerges as a result of the quantization of gravity. The complex cosmological constant is incorporated into the modified action, and the Wheeler-DeWitt equation is solved within this new theoretical framework. The solution obtained, by considering the quantum effects of gravity, the obtained solution offers a new perspective on the nature of spacetime and its fundamental components at quantum scales. The research proposes that the cosmological constant, derived from this functional measure, plays an important role in the quantum dynamics of the universe, shedding light on the behavior of quantum fluctuations of spacetime, through the topological and quantum effects that our solution provides, in addition to enabling new lines of research into the origin of the cosmological constant. This work contributes to the resolution of the Wheeler-DeWitt equation, by integrating the canonical and path integral formalisms, proposing a new way of interpreting quantum gravity.

Keywords: Wheeler-DeWitt equation. Unification. Quantum cosmology.

LIST OF FIGURES

CONTENTS

1 INTRODUCTION

The Planck Era, which began at the end of the 19th century and extended into the early 20th century, marks a crucial period in the evolution of modern physics. This era was characterized by growing dissatisfaction with classical physics, which failed to adequately explain various observed phenomena, especially those related to radiation and the structure of matter at very small scales. Max Planck's work, particularly his introduction of Planck's constant $\hbar$, was instrumental in the development of the new quantum theory. In 1900, Planck proposed that energy is not continuous but quantized, suggesting that energy is emitted or absorbed in discrete units, or "quanta" [1].

Planck's contribution became evident in his efforts to resolve the problem of blackbody radiation, which had perplexed physicists of the time. Classical theories predicted that the radiation emitted by a blackbody would diverge to infinite values at high frequencies, a phenomenon known as the "ultraviolet catastrophe." Planck proposed that the energy of oscillators composing blackbody radiation is quantized, leading to his famous formulation of the Planck law for the spectral distribution of radiation [1]. This idea not only solved the immediate problem of blackbody radiation but also laid out the foundation for the ensuing revolution in physics.

The introduction of Planck's constant represented a paradigm shift, challenging the classical view that energy was a continuous quantity. Quantization not merely provided a new understanding of thermal radiation, but it equally introduced the idea that nature could be described in statistical terms. Quantum mechanics began to emerge as a theory capable of explaining phenomena that classical physics could not, such as the photoelectric effect, later elucidated by Albert Einstein in 1905 [2].

Furthermore, the Planck Era also witnessed the emergence of new ideas about atomic structure. Dalton's atomic theory was revisited, and with the work of Niels Bohr and others, a new understanding of the atom began to take shape. Bohr's model, which introduced quantized energy levels for electrons in hydrogen atoms, demonstrated that quantization was an intrinsic feature of nature at microscopic scales [3]. These discoveries did more than revolutionize chemistry and physics, they also paved the way for a new perspective on the universe, where the deterministic nature of classical physics and the idea of immutable physical laws were no longer universally accepted. The emergence of quantum mechanics introduced inherent uncertainty, exemplified by Heisenberg's uncertainty principle, which highlighted the limitations of predictability at microscopic scales, as we will see later.

Thus, the Planck Era represents a pivotal turning point in the history of physics, where classical ideas began to give way to quantum concepts that defied intuition. However, while quantum mechanics provided new insights into the behavior of subatomic particles, gravity continued to be described by classical Newtonian physics. This disconnect between quantum theory and classical gravity led to an urgent search for a theory that could unify these two domains, a search that would be addressed, in part, by Albert Einstein's contributions [4].

1.1 EINSTEIN'S THEORY OF RELATIVITY

In 1915, Einstein presented his General Theory of Relativity, which completely reshaped the understanding of gravity. The theory not only challenged the Newtonian description of gravity but also demonstrated how mass and energy influence the geometry of spacetime. This groundbreaking framework provided the foundation for a new era in physics, where the unification of quantum mechanics and gravity would become one of the central questions in modern physics research [5].

Einstein's field equation [1], expressed as

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.1}$$

is a fundamental equation in the framework of general relativity and modern physics. Here, $G_{\mu\nu}$ represents the Einstein tensor, while $T_{\mu\nu}$ is the stress-energy tensor, describing the distribution of mass and energy in spacetime. The constant G is the gravitational constant, and c is the speed of light. This equation establishes a profound relationship between the geometry of spacetime and the matter within it [5].

Through this equation, Einstein posited that gravity is not a force in the classical sense but rather a consequence of the curvature of spacetime caused by the presence of mass and energy. Massive objects, such as planets and stars, distort spacetime around them, causing other objects to follow curved trajectories. This idea reshaped our understanding of gravity, with important implications for cosmology and astrophysics..

The General Theory of Relativity profoundly changed our perception of the universe and additionally introduced fundamental new concepts. The notion that space and time are interconnected into a single entity, spacetime, marked a radical departure from previous ideas. General relativity challenged classical intuition, revealing that phenomena such as time dilation and spatial contraction are not mere artifacts but intrinsic features of reality. These insights led to remarkable predictions, including the bending of light as it passes near massive objects, which was experimentally confirmed during a solar eclipse in 1919 [6].

Moreover, Einstein's theory provided the basis for understanding large-scale cosmic phenomena, such as the expansion of the universe, later explored experimentally by scientists like Edwin Hubble [7]. The interplay between gravity, space, and time also spurred the quest for a unified theory that would incorporate gravity alongside the other fundamental principles of physics. This persistent challenge of unification became a driving force in theoretical physics throughout the 20th century and beyond.

With the advent of general relativity, physicists began to question whether a quantum description of gravity could be formulated in alignment with the quantum principles established by Planck and his contemporaries. The idea that the universe could be described by both quantum theories and general relativity led to the search for new frameworks capable of reconciling these two domains, ultimately culminating in the development of a quantum theory of gravity, including the famous Wheeler-DeWitt equation [4].

1.2 EVOLUTION OF QUANTUM MECHANICS

After Einstein's introduction of the theory of relativity, quantum physics began to solidify through the contributions of notable physicists such as Werner Heisenberg, Erwin Schrödinger, and Paul Dirac, who played crucial roles in the formulation of modern quantum mechanics [8].

1.2.1 Werner Heisenberg and Matrix Mechanics

In 1925, Werner Heisenberg proposed matrix mechanics, a new paradigm that represented quantum systems in terms of matrices and operators. This approach was a significant innovation

[1] In this work, Greek indices $\mu, \nu, \rho, \ldots$ run from 0 to 3, representing spacetime coordinates. We adopt the Einstein summation convention, where repeated indices imply summation over their range. Additionally, we use the metric signature $(-, +, +, +)$, which is common in relativistic formulations.

as it shifted the focus of physicists from well-defined trajectories to observables—quantities that can be measured.

The uncertainty principle, introduced by Heisenberg in 1927 [9], is one of the most remarkable features of his theory. This principle states that it is impossible to simultaneously measure the position (x) and momentum (p) of a particle with arbitrary precision, expressed by the relation:

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \tag{1.2}$$

where Δx is the uncertainty in position, Δp is the uncertainty in momentum, and $\hbar$ is the reduced Planck constant. This discovery challenged the classical view of determinism and laid the foundation for a new understanding of the probabilistic nature of quantum mechanics.

1.2.2 Erwin Schrödinger and Wave Mechanics

In contrast to Heisenberg's approach, Erwin Schrödinger introduced wave mechanics in 1926 [10], describing subatomic particles as waves. His famous wave equation, the Schrödinger equation, provides a mathematical description of the temporal evolution of quantum systems:

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Psi(\mathbf{r}, t) + V(\mathbf{r})\Psi(\mathbf{r}, t), \tag{1.3}$$

where $\Psi(\mathbf{r}, t)$ is the wave function of the system, m is the particle's mass, $V(\mathbf{r})$ is the potential, and ∇^2 is the Laplacian operator. The wave function is fundamental as it contains all the information about the quantum system and its evolution.

1.2.3 Paul Dirac and Quantum Field Theory

While Heisenberg and Schrödinger established the groundwork for quantum mechanics, Paul Dirac made a significant contribution by unifying quantum mechanics with special relativity. In 1928, Dirac formulated his equation, the Dirac equation, which describes the behavior of particles like the electron in a way that incorporates both quantum and relativistic principles:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{1.4}$$

where γ^μ are the Dirac matrices, ∂_μ is the derivative operator, and ψ is the electron's wave function.

This equation not only resolved the problem of negative energy but also predicted the possibility of negative mass states, which later contributed to the interpretation of antiparticles and the discovery of the positron in 1932 [11]. This issue arises because the relativistic energy-momentum relation, $E^2 = p^2c^2 + m^2c^4$, allows for both positive and negative energy solutions. In classical mechanics, a particle would continuously lose energy and fall into arbitrarily negative energy states, making the theory unstable. Dirac resolved this by proposing the existence of a filled "sea" of negative energy states, preventing electrons from decaying into them due to the Pauli exclusion principle. A vacancy in this sea would then behave as a positively charged particle, later identified as the positron, providing a natural explanation for antimatter.

1.2.4 The Quest for Unification of Gravity and Quantum Mechanics

Although the contributions of Heisenberg, Schrödinger, and Dirac established quantum mechanics as a fundamental theory, the unification of gravity and quantum mechanics remained an unresolved challenge. Einstein's general relativity, with its geometric description of gravity, posed difficulties when reconciled with the intrinsically probabilistic principles of quantum mechanics.

The interaction between gravity, described as a force arising from the curvature of spacetime, and quantum forces, characterized by superposition and uncertainty, led to profound investigations into the fundamental structure of spacetime [4]. Physicists began exploring new approaches, including theories like loop quantum gravity and supergravity, in the hope of finding a coherent framework that could unify these two pillars of physics.

1.3 THE PATH TO UNIFICATION: WHEELER AND DEWITT

As quantum mechanics solidified in the first half of the 20th century, the need for a deeper understanding of the fundamental interactions, especially between gravity and quantum mechanics, became evident. The challenge of unifying these theories was one of the main motivations for theoretical physics research in the subsequent decades. In this context, prominent figures like John Archibald Wheeler and Bryce DeWitt played crucial roles in formulating the idea of quantum gravity.

1.3.1 John Archibald Wheeler: The Bridge Builder

Wheeler, one of the most influential physicists of the 20th century, began his career collaborating with Einstein and dedicated himself to research in relativity and quantum gravity. He pioneered the idea that gravity should not be treated as an isolated classical force but rather as a manifestation of spacetime geometry. One of his most important concepts was that space and time are dynamic fabrics shaped by the presence of mass and energy [12].

Wheeler also introduced the concept of "geometrodynamics," suggesting that gravity and the geometry of spacetime are interdependent. This idea laid the foundation for quantum gravity research and encouraged the exploration of new ways to understand the relationship between quantum mechanics and general relativity. He popularized the famous phrase, "Physicists should not merely accept the structure of spacetime; they should be able to build that structure" [13].

1.3.2 Bryce DeWitt: The Wheeler-DeWitt Equation

While Wheeler explored the intersection of gravity and quantum mechanics, Bryce DeWitt, one of Wheeler's collaborators, made significant strides in formulating the equation that now bears their names. In 1967, DeWitt introduced the Wheeler-DeWitt equation [14], representing one of the most significant efforts to quantize gravity. This equation can be seen as the quantum version of Einstein's field equations, aiming to describe the evolution of the universe in the context of quantum mechanics.

The general form of the Wheeler-DeWitt equation is:

$$\hat{H}\Psi[g_{ab}, \phi] = 0, \tag{1.5}$$

where $\hat{H}$ is the Hamiltonian operator, g_{ab} is the metric tensor describing spacetime geometry, ϕ represents matter fields, and Ψ is the wavefunction of the universe.

The Wheeler-DeWitt equation presents significant challenges, as it does not include time as an independent variable. Instead, its solution provides a description of the universe's state without a well-defined temporal parameter. This aspect raises fundamental questions about the nature of time and evolution in the context of quantum gravity [15].

1.3.3 The Quest for a Theory of Everything

The research of Wheeler and DeWitt was part of a broader movement to develop a "theory of everything" that unifies all fundamental forces of nature, including gravity, electromagnetism, and nuclear interactions. This quest has become one of the main goals of modern theoretical physics and inspired the development of alternative theories, such as string theory and loop quantum gravity [16].

The work of Wheeler and DeWitt not only laid the groundwork for quantum gravity but also raised profound philosophical questions about the universe's nature and the role of consciousness in quantum observation. The Wheeler-DeWitt equation, in particular, remains an active topic of research and discussion, reflecting the complexity of unifying quantum mechanics and general relativity [15].

1.4 RESEARCH APPROACH

The search for a unified theory combining gravity and quantum interactions remains one of the greatest challenges in modern theoretical physics. Although general relativity and quantum mechanics have been remarkably successful in their respective domains, they have not yet been fully unified. The Wheeler–DeWitt equation, introduced by John Archibald Wheeler and Bryce DeWitt, represents one of the main attempts to describe gravity in a quantum framework. However, the lack of an explicit time variable raises deep conceptual questions about the evolution of the universe in quantum gravity, as we will see throughout the work.

This dissertation investigates a quantum correction to the Wheeler-DeWitt equation that emerges from the functional measure in the path integral formalism, a correction that has been previously established in the literature. This modification introduces a complex component to the cosmological constant, thereby affecting the universe's wave function Ψ. The implications of this modification are far-reaching, potentially influencing the emergence of semiclassical spacetime, quantum fluctuations in the early universe, and the stability of vacuum solutions. These aspects will be explored in detail throughout this work.

Recognizing the limitations posed by the absence of an explicit time variable in the standard Wheeler-DeWitt formulation, we employ the Ashtekar variables formalism as a means to address this issue and develop a more complete approach to the quantum dynamics of spacetime. This choice is motivated by the formalism's ability to recast general relativity in a form that resembles gauge theories, allowing for a more natural quantization procedure. Within this framework, we explore how the introduction of a complex cosmological constant impacts the structure of the Wheeler-DeWitt equation and its solutions.

The main contribution of this dissertation is to extend the analysis of the modified Wheeler-DeWitt equation by investigating the physical consequences of a complex cosmological constant within the Ashtekar formalism. Although previous works have derived this term, we seek to resolve and discuss the impact that this new term has on the Wheeler-DeWitt equation and the Ψ state.

To structure this study, the work is divided into five chapters and two appendices:

- **Chapter 1: Introduction -** This chapter contextualizes the reader about the historical and scientific background of the topic, explaining the motivation for this study and its importance in the context of quantum gravity research.

- **Chapter 2: Wheeler-DeWitt Equation -** This chapter delves into the foundational aspects of quantum mechanics and general relativity, exploring their theoretical frameworks and highlighting the challenges in unifying these theories. It establishes the conceptual groundwork necessary for the subsequent reformulations.

- **Chapter 3: Ashtekar Variables -** The focus shifts to reformulating general relativity using the Ashtekar variables. This chapter provides a detailed explanation of the motivation for introducing these variables, their mathematical structure, and their relevance to quantum gravity. The nuances of this formalism are explored, with a particular emphasis on its implications for the Wheeler-DeWitt equation.

- **Chapter 4: Modified Wheeler-DeWitt Equation -** This chapter introduces a quantum correction to the Wheeler-DeWitt equation, derived from the path integral formulation. It focuses on solving this modified equation using the previously introduced formalism and interpreting the results in the context of quantum gravity.

- **Chapter 5: Conclusions and Future Perspectives -** The final chapter compiles the results obtained throughout the study and discusses their broader implications. Additionally, it provides an outlook on potential future research directions motivated by these findings.

- **Appendix A: Group Theory - Gauss Constraint -** In this appendix, we present a study of the Gauss constraint within the Ashtekar formalism. We begin with fundamental concepts such as gauge symmetries and Lie groups, developing the necessary tools to understand the definition and role of the Gauss constraint in quantum gravity.

- **Appendix B: Exterior Algebra and Differential Forms -** This appendix explores exterior algebra, a fundamental formalism for describing geometric theories such as general relativity. It covers key concepts like differential forms, the wedge product, and the exterior derivative. The appendix also details the calculation of the curvature tensor and its decomposition, highlighting the relevance of this framework in quantum gravity.

Lastly, an important limitation of this work lies in the theoretical and mathematical nature of the analysis, which requires solving hard equations and using approximations, such as assuming the field strength is self-dual. Furthermore, the physical interpretation of the solutions is a conceptual challenge addressed throughout the research.

2 WHEELER-DEWITT EQUATION

2.1 REVIEW OF GENERAL RELATIVITY

General relativity is the cornerstone of modern gravitational theory, providing a geometric description of gravity as the curvature of spacetime caused by mass and energy. In this section, we introduce the fundamental concepts of the metric tensor, curvature, and their relationship in general relativity.

The metric tensor $g_{\mu\nu}$ is a symmetric tensor that defines distances and angles in spacetime. The spacetime interval ds^2 between two events (specific points in spacetime, defined by their spatial and temporal coordinates) is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

where dx^μ are the coordinate differentials, and $g_{\mu\nu}$ encodes the geometric properties of spacetime. The indices $\mu, \nu = 0, 1, 2, 3$ label the temporal ($\mu = 0$) and spatial ($\mu = 1, 2, 3$) components in a four-dimensional spacetime[1] .

The curvature of spacetime is described by the Riemann curvature tensor $R^\rho_{\sigma\mu\nu}$, which measures how much spacetime deviates from being flat or planar. Geometrically, this deviation can be observed through the accumulated change in a vector that is parallel transported around an infinitesimal closed loop[2]. The equation for the Riemann curvature tensor is:

$$R^\rho_{\mu\sigma\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{2.2}$$

where $\Gamma^\sigma_{\mu\nu}$ are the Christoffel symbols, which represent the connections that describe the parallel transport mentioned earlier. The expression for the Christoffel symbol is:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right). \tag{2.3}$$

Other important quantities to define are the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R. The Ricci tensor is obtained by contracting the Riemann tensor over its first and third indices:

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu}. \tag{2.4}$$

This contraction reduces the complexity of the Riemann tensor by focusing on the curvature along specific directions, making the Ricci tensor an essential quantity in Einstein's field equations, which describe how matter and energy influence the curvature of spacetime.

The Ricci scalar R, a further contraction of the Ricci tensor, is given by:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.5}$$

[1]In Einstein's summation convention, whenever an index appears twice in a term (once as a superscript and once as a subscript, or both), it implies a sum over all possible values of that index. For example, the expression $V^i g_{ij}$ implicitly means $\sum_i V^i g_{ij}$, where the index i is summed over all its possible values. This convention helps streamline mathematical expressions, particularly in relativistic and gauge theories, by eliminating the need for explicit summation signs.

[2]An infinitesimal closed loop refers to a small, differential path in spacetime that starts and ends at the same point. The deviation observed after transporting a vector around this loop reveals the presence of curvature.

The Ricci scalar represents the trace of the Ricci tensor and serves as a scalar quantity that encodes the overall curvature of spacetime. It simplifies the complex structure of the Ricci tensor into a single number, which is particularly useful in general relativity, where it appears in the Einstein–Hilbert action, as we will see in the next section.

These quantities describe the curvature of spacetime and are fundamental to Einstein's field equations, which relate the geometry of spacetime to its energy content. Setting the speed of light to unity ($c = 1$) simplifies Einstein's field equations and can then be expressed explicitly as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.6}$$

in which we can verify the importance of the previously mentioned quantities. The term $\Lambda g_{\mu\nu}$ corresponds to the cosmological constant Λ, which was originally introduced by Einstein to allow for a static universe. It is now understood to be related to the energy density of the vacuum and plays a crucial role in modern cosmology, particularly in the context of dark energy and the accelerated expansion of the universe.

This brief overview of general relativity provides the necessary background to introduce the Einstein-Hilbert action and its role in deriving the Einstein field equations, as we will study in the next section.

2.2 EINSTEIN-HILBERT ACTION

The starting point for establishing the connection between quantum mechanics and cosmology begins by exploring and manipulating the Einstein-Hilbert action. The Einstein-Hilbert action, in turn, is simply a way of expressing Einstein's field equations in a variational form, it's given by:

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\,(R - 2\Lambda), \tag{2.7}$$

where:

- G_N is Newton's gravitational constant .
- g is the determinant of the metric $g_{\mu\nu}$, which defines the geometric structure of spacetime [17].
- R is the Ricci scalar, which contains information about the curvature of spacetime [18].
- Λ is the cosmological constant, which describes the energy density of the vacuum [19].

To obtain Einstein's field equations from this action, it is necessary to use the Hamilton principle, which involves calculating the variation of the action with respect to the metric tensor $g_{\mu\nu}$, to find the extremum of our physical system [20].

$$\delta S_{EH} = \frac{1}{16\pi G}\int_{\mathcal{M}} [\delta(R\sqrt{-g}) - 2\Lambda\delta\sqrt{-g}]\, d^4x = 0. \tag{2.8}$$

The variation of the action can be decomposed as:

$$\delta(R\sqrt{-g}) = \sqrt{-g}\delta R + R\delta(\sqrt{-g}), \tag{2.9}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.10}$$

Now only the calculation of the variation of the Ricci scalar R remains.

2.2.0 Variation of the Ricci scalar

The Ricci scalar R is obtained by contracting the Ricci tensor $R_{\mu\nu}$, which describes the curvature of space, with the metric $g^{\mu\nu}$:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.11}$$

When varying R, we apply the product rule, so:

$$\delta R = \delta(g^{\mu\nu} R_{\mu\nu}) = (\delta g^{\mu\nu}) R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}. \tag{2.12}$$

Therefore, to obtain the full variation of$R$, we need to calculate $\delta R_{\mu\nu}$, as a result:

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\lambda\mu}, \tag{2.13}$$

and,

$$\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu}), \tag{2.14}$$

where

- ∇_λ denotes the covariant derivative[3] associated with the metric $g_{\mu\nu}$.
- $\Gamma^\alpha_{\beta\gamma}$ denotes the Christoffel symbol, which is fundamental for understanding the parallel transport in our spacetime over a given curvature [21].

The Christoffel connection is given in terms of the metric tensor $g^{\mu\nu}$, by:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \tag{2.15}$$

The variation of the Christoffel connection $\delta\Gamma^\lambda_{\mu\nu}$ is:

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu}\right). \tag{2.16}$$

Substituting expression (2.16) into (2.14), we obtain the variation of $R_{\mu\nu}$:

$$\delta R_{\mu\nu} = \nabla_\lambda \left(\nabla_\mu \delta g^\lambda_\nu + \nabla_\nu \delta g^\lambda_\mu - \nabla^\lambda \delta g_{\mu\nu}\right). \tag{2.17}$$

Now we can substitute $\delta R_{\mu\nu}$ into the variation of R:

$$\delta R = g^{\mu\nu} \delta R_{\mu\nu} + R_{\mu\nu} \delta g^{\mu\nu}. \tag{2.18}$$

The first term, $g^{\mu\nu}\delta R_{\mu\nu}$, involves covariant derivatives and can be rewritten in terms of a total divergence term, which does not contribute to the variation of the action due to Gauss's theorem (which allows transforming a total divergence term into a boundary integral, which can be ignored if the variation vanishes at the boundaries). Thus, the variation of R can be expressed as:

[3] The covariant derivative is a generalization of the ordinary derivative that accounts for changes in the basis of a curved space. Given a vector field V^μ and a connection $\Gamma^\lambda_{\mu\nu}$, the covariant derivative is defined as:

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda.$$

It ensures tensorial transformation properties under coordinate changes and is essential in differential geometry and general relativity.

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu V^\mu, \tag{2.19}$$

where V^μ is a vector that depends on $\delta g_{\mu\nu}$ and its derivatives. This term $\nabla_\mu V^\mu$ can be ignored in the variation of the action due to Gauss's theorem.

Now, we substitute the variations of R and $\sqrt{-g}$ into the variation of the action (2.8):

$$\delta S_{EH} = \frac{1}{16\pi G}\int_{\mathcal{M}} \left(\sqrt{-g}R_{\mu\nu} - \frac{1}{2}\sqrt{-g}Rg_{\mu\nu} + \Lambda\sqrt{-g}g_{\mu\nu}\right)\delta g^{\mu\nu}\, d^4x, \tag{2.20}$$

alternatively,

$$\delta S_{EH} = \frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\, d^4x. \tag{2.21}$$

For the action to be stationary (either minimum or maximum), this variation must be zero for any $\delta g^{\mu\nu}$, which gives us the condition:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{2.22}$$

These are the Einstein field equations in vacuum [22]. They relate the curvature of spacetime (through the Ricci tensor $R_{\mu\nu}$) to the spacetime metric $g_{\mu\nu}$. To include matter and energy, we introduce the energy-momentum tensor $T_{\mu\nu}$, resulting in the complete field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.23}$$

In equivalent terms:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{2.24}$$

where $G_{\mu\nu}$ is the Einstein tensor, which describes the curvature of spacetime:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{2.25}$$

Thus, the curvature of spacetime is determined by the presence of matter and energy, as described by the tensor $T_{\mu\nu}$ [23].

These observations confirm that the Einstein-Hilbert action is fundamental in general relativity because it encapsulates all the information about the dynamics of the gravitational field. Now that we have seen that this action can precisely express general relativity as described by Einstein's field equations, we need only to quantize gravity and arrive at the Wheeler-DeWitt equation. To do this, we will use the ADM formalism to rewrite the action [24].

2.3 THE ADM FORMALISM

The ADM formalism (Arnowitt-Deser-Misner) was developed to reformulate general relativity in a way that facilitates its quantization. The central idea is to foliate spacetime into three-dimensional spatial slices that evolve over time, so that the four-dimensional spacetime can be described by a three-dimensional metric h_{ab} and auxiliary functions that determine how these slices evolve.

In other words, we take a 3+1 dimensional Riemannian manifold $\mathcal{M}$ with metric g and divide it into spatial slices Σ_t as shown in Figure 2.1.

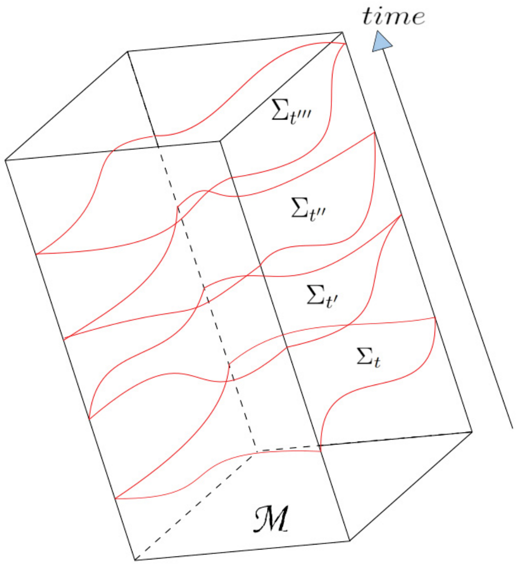

Figure 2.1: A foliation of a manifold $\mathcal{M}$ into spatial hypersurfaces Σ_t, representing different instants of time in the spacetime evolution.

Thus, the spacetime metric in the ADM formalism can be written as[4]:

$$ds^2 = -(N^2 - N_a N^a)\, dt^2 + 2N_a\, dx^a\, dt + h_{ab}\, dx^a\, dx^b, \tag{2.26}$$

where we introduced:

- The lapse function, denoted by N, which controls the time interval between two consecutive spatial slices;
- The shift vector, denoted by N^a, which describes how the spatial coordinates are 'shifted' as time progresses;
- The three-dimensional metric h_{ab} that describes the geometry of each spatial slice.

In this context, a, b represent spatial indices on the slice, while the temporal dimension is denoted separately by t. This separation allows for a clear decomposition of spacetime into spatial and temporal components.

Additionally, we can interpret the lapse function and the shift vector geometrically as shown in Figure 2.2 and Figure 2.3.

[4]The ADM metric is derived naturally by writing $g_{\mu\nu}$ in terms of the geometric quantities associated with the foliation.

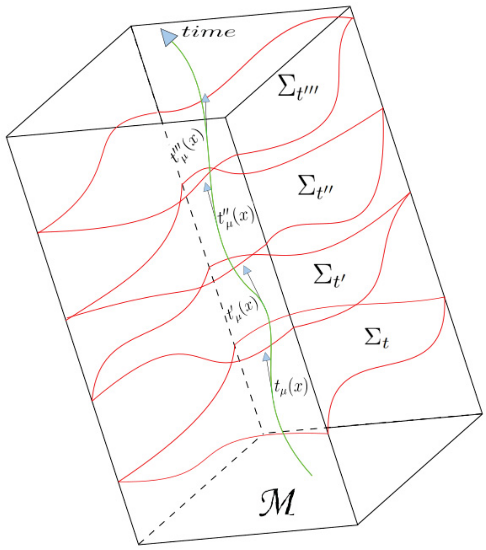

Figure 2.2: An illustration of the tangent vector $t_\mu(x)$, representing the "time function", evolving through a foliated manifold $\mathcal{M}$ at a fixed point x.

The vector $t_\mu(x)$ represents a tangent vector to the manifold M at a point x. It is associated with the "time function", which describes the evolution of a system within a foliated structure of spacetime. In a 3+1 dimensional foliation, the manifold is partitioned into a series of spatial hypersurfaces, and $t_\mu(x)$ encodes the change in the spatial slices as they evolve over time. This vector is essential in the study of dynamics in general relativity, as it defines the direction of time flow within the manifold and is typically used to parametrize the evolution of points across the foliated slices.

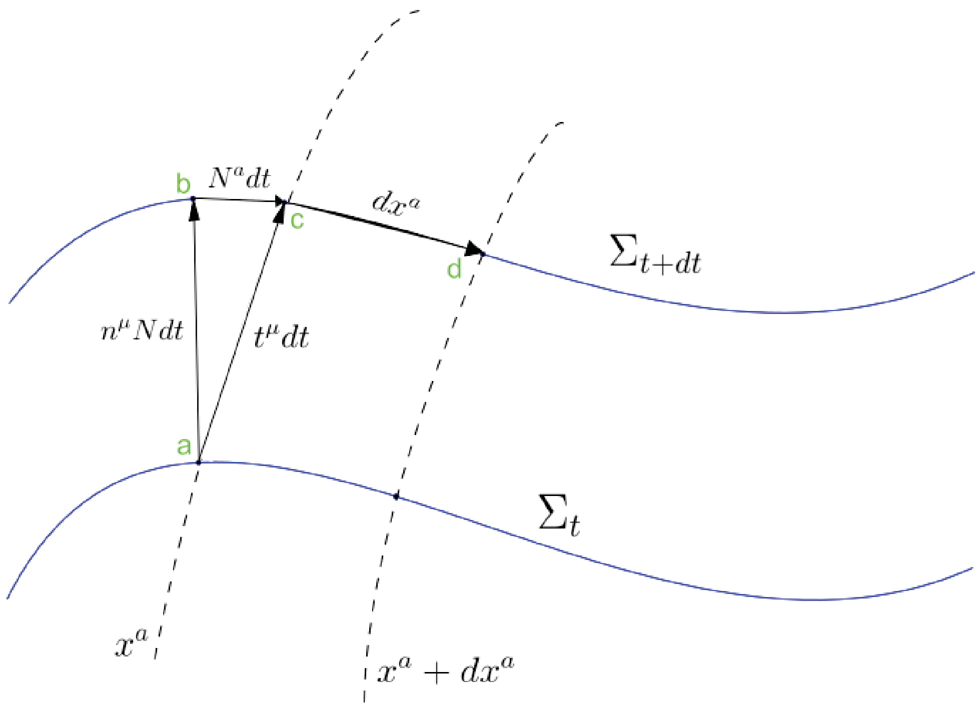

Figure 2.3: An illustration of the lapse function N and the shift vector N^a, which describe the evolution of spatial hypersurfaces in the foliated spacetime.

This decomposition allows us to rewrite the determinant of the metric $g_{\mu\nu}$ and the Ricci scalar R in terms of these variables. This facilitates the analysis of the temporal evolution of the geometry of space in terms of variables that can be quantized.

2.3.1 Decomposition of the Ricci Scalar

The decomposition of the Ricci scalar R in the ADM formalism can be derived from the general expression for the four-dimensional Ricci scalar and utilizing the foliation of spacetime into three-dimensional spatial slices. The four-dimensional Ricci scalar is given by:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.27}$$

Using the metric (2.26) and the definition of the Ricci tensor (2.13), we can rewrite the Ricci scalar as:

$$R = {}^{(3)}R + K_{ab}K^{ab} - K^2, \tag{2.28}$$

where:

- ${}^{(3)}R$ is the Ricci scalar associated with the three-dimensional metric h_{ab}, which describes the intrinsic curvature of each spatial slice.
- K_{ab} is the extrinsic curvature tensor, which describes how each spatial slice is "curved"[5] within the four-dimensional spacetime. It can also be thought of as the velocity field of $h_{ab}(x)$ that 'slides' over the foliation.
- K is the trace of the extrinsic curvature tensor, defined as $K = h^{ab}K_{ab}$.

This decomposition is useful because it separates the contributions from the intrinsic curvature of the spatial slices (${}^{(3)}R$) and the extrinsic curvature ($K_{ab}K^{ab}$ and K^2) in the study of gravity in the ADM formalism.

From (2.26), we also see that:

$$\sqrt{-g} = N\sqrt{h}, \tag{2.29}$$

where h is the determinant of the three-dimensional metric h_{ij}. Substituting (2.28) and (2.29) into the Einstein-Hilbert action (2.7), we obtain:

$$S_{ADM} = \frac{1}{16\pi G} \int dt\, d^3x\, N\sqrt{h} \left({}^{(3)}R + K_{ab}K^{ab} - K^2 - 2\Lambda \right). \tag{2.30}$$

This formulation forms the basis for writing and interpreting the Wheeler-DeWitt equation concisely [25].

2.3.2 The Hamiltonian of General Relativity

To derive the Wheeler-DeWitt equation, we will define the DeWitt metric G^{abcd} as follows:

$$G^{abcd} := \frac{\sqrt{h}}{2}(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd}). \tag{2.31}$$

[5] This curvature arises as a consequence of the separation between the spatial slices in the foliation of spacetime. Unlike intrinsic curvature, extrinsic curvature not only bends but also 'stretches' spacetime, encoding how the hypersurface is embedded in the higher-dimensional manifold.

This metric is used to define the geometry the spatial configuration, which allows us to quantify differences between metric configurations. It also provides the foundation for determining the geometric structure of the space of spatial metrics and incorporating the necessary dynamical terms into the equation.

By substituting (2.31) into (2.30), we can explicitly observe the separation of the kinetic and potential terms:

$$S_{ADM} = \frac{1}{16\pi G_N} \int dt\, d^3x\, N \left(\underbrace{G^{abcd} K_{ab} K_{cd}}_{kinetic} + \underbrace{\sqrt{h} \left({}^{(3)}R - 2\Lambda \right)}_{potential} \right). \tag{2.32}$$

From this point onward, and within the framework of the ADM formalism, where spacetime is foliated into spatial slices, we will use the expression for

$$K_{ab} = \frac{1}{2N} \left(\dot{h}_{ab} - D_a N_b - D_b N_a \right), \tag{2.33}$$

where,

- D_a represents the covariant derivative associated with the three-dimensional spatial metric h_{ab} [6] , and it acts within the spatial slices of the ADM formalism. It projects the shift vector N_a onto these slices, capturing variations entirely within the spatial geometry;
- $\dot{h}_{ab}$ represents the ordinary time derivative of the induced metric h_{ab}.

In this way, we can apply (2.33) to (2.32), resulting in:

$$S_{ADM} = \frac{1}{16\pi G} \int dt\, d^3x \left(p^{ab} \dot{h}_{ab} - \underbrace{N \mathcal{H}^g_\perp - N^a \mathcal{H}^g_a}_{\mathcal{H}_{constraints}} \right), \tag{2.34}$$

where,

- p^{ab} is the conjugate momentum with respect to the metric h_{ab}, definined by:

$$p^{ab} = \frac{\sqrt{h}}{16\pi G_N} \left(K^{ab} - h^{ab} K \right); \tag{2.35}$$

- $\mathcal{H}^g_\perp$ is the Hamiltonian density responsible for the Hamiltonian constraint (as we will discuss shortly), defined by:

$$\mathcal{H}^g_\perp = 16\pi G_N G_{abcd} p^{ab} p^{cd} - \frac{\sqrt{h}}{16\pi G_N} \left({}^{(3)}R - 2\Lambda \right); \tag{2.36}$$

[6] D_a is the covariant derivative compatible with the three-dimensional spatial metric h_{ab}, meaning that it satisfies $D_a h_{bc} = 0$. It acts within the spatial slices of the ADM formalism and ensures that spatially projected tensors transform consistently under coordinate changes within each slice. Unlike the four-dimensional covariant derivative ∇_μ, which includes the full spacetime connection $\Gamma^\lambda_{\mu\nu}$, D_a only involves the Christoffel symbols Γ^c_{ab} constructed from h_{ab}.

- $\mathcal{H}_a^g$ is the Hamiltonian density responsible for the momentum constraint, defined by:

$$\mathcal{H}_a^g = -2D_b p_a^b. \tag{2.37}$$

Similarly to what we did in (2.8), we will vary our action with respect to the shift vector N^a and the lapse function N (since the metric now depends on these new parameters) and extremize in such a way that:

$$\frac{\delta S}{\delta N} = 0, \tag{2.38}$$

$$\frac{\delta S}{\delta N^a} = 0. \tag{2.39}$$

We can now begin to notice that the first variation will only affect the term $N\mathcal{H}_\perp^g$, so:

$$\frac{\delta S}{\delta N} = \frac{\delta(N\mathcal{H}_\perp^g)}{\delta N} = \mathcal{H}_\perp^g + N\frac{\delta\mathcal{H}_\perp^g}{\delta N} = 0. \tag{2.40}$$

It is important to note that the term $\frac{\delta H_g^\perp}{\delta N}$ implicitly depends on N, as expanding the conjugate momentum p_{ab} will include extrinsic curvature terms K_{ab} that depend on N. Under certain approximations, such as considering small variations of the 3-metric or a stationary background, we assume that K_{ab} is of order $1/N$, i.e., $K_{ab} \sim \frac{1}{N}$, leading to the following expression:

$$\frac{\delta\mathcal{H}_\perp^g}{\delta N} \to 0. \tag{2.41}$$

This approximation is valid under the assumption that the extrinsic curvature K_{ab} becomes approximately constant in time. This assumption is reasonable because, in the regime we are considering, the variations in K_{ab} over time are small enough to be neglected. Specifically, the shift vector N_a and lapse function N do not exhibit significant time dependence, which implies that the extrinsic curvature terms are approximately constant. Consequently, the time variation of the extrinsic curvature can be treated as negligible, simplifying the calculation. This approximation is particularly useful when the system is in a stable configuration, where the effects of time-dependent changes in the geometry are minimal.

Now that we have this term tending to zero due to the approximation made, we can return to (2.40) and find our first constraint (the Hamiltonian constraint):

$$\mathcal{H}_\perp^g \approx 0 \qquad \text{(Hamiltonian Constraint)}. \tag{2.42}$$

Performing the variation $\frac{\delta S}{\delta N^a}$. This variation will only affect the term $N^a\mathcal{H}_a^g$, from which we obtain:

$$\frac{\delta S}{\delta N^a} = \frac{\delta(N^a\mathcal{H}_a^g)}{\delta N^a} = 0. \tag{2.43}$$

If we observe the dependence of N^a in the extrinsic curvature tensor K_{ab} , we find that, similarly to what was done above, we can derive the momentum constraint, given by:

$$\mathcal{H}_a^g = -2D_b p_a^b \approx 0 \qquad \text{(Momentum Constraint)}. \tag{2.44}$$

Now that we have derived the Hamiltonian constraints responsible for describing the dynamics of the gravitational field, the next step will be to quantize these equations [26].

2.3.3 Quantization and the Wheeler-DeWitt Equation

To obtain the quantum version of general relativity, we perform the usual quantization procedure by transforming the momenta p^{ab} and h_{ab} into differential operators that act on the wave functional (also known as the wave function of the universe) $\Psi[h_{ab}(\mathbf{x})]$:

$$p^{ab} \rightarrow \hat{p}^{ab},$$

$$h_{ab} \rightarrow \hat{h}_{ab},$$

and act on $\Psi[h_{ab}(\mathbf{x})]$ as follows:

$$\hat{p}^{ab}\Psi[h_{ab}(\mathbf{x})] = -i\hbar\frac{\delta}{\delta h_{ab}}\Psi[h_{ab}(\mathbf{x})], \tag{2.45}$$

$$\hat{h}_{ab}\Psi[h_{ab}(\mathbf{x})] = h_{ab}\Psi[h_{ab}(\mathbf{x})], \tag{2.46}$$

and the Poisson bracket becomes:

$$[\hat{h}_{ab}(\mathbf{x}), \hat{p}^{cd}(\mathbf{y})] = i\hbar\delta_a^c\delta_b^d\delta(\mathbf{x}, \mathbf{y}). \tag{2.47}$$

Now that we know how the Hamiltonian elements act on the wave functional, we can substitute and apply the Hamiltonian $\hat{\mathcal{H}_\perp}^g$ to $\Psi[h_{ab}(\mathbf{x})]$, as follows:

$$\hat{\mathcal{H}}_\perp^g\Psi[h_{ab}(\mathbf{x})] = \left(-16\pi G_N\hbar^2 G_{abcd}\frac{\delta^2}{\delta h_{ab}\delta h_{cd}} - \frac{\sqrt{h}}{16\pi G_N}({}^{(3)}R - 2\Lambda)\right)\Psi[h_{ab}(\mathbf{x})] = 0. \tag{2.48}$$

This is the Wheeler-DeWitt equation, which describes the quantum dynamics of the gravitational field. It is a wave equation for the universe, and the wave function of the universe $\Psi[h_{ab}(\mathbf{x})]$ contains all the information about the possible quantum states of the universe at a given point $\mathbf{x}$.

Additionally, we can replicate the analysis for the second constraint, the momentum constraint $\hat{\mathcal{H}}_a$, which results in:

$$\hat{\mathcal{H}}_a\Psi[h_{ab}(\mathbf{x})] = \left(-2D_b H_{ac}\frac{\hbar}{i}\frac{\delta}{\delta h_{bc}}\right)\Psi[h_{ab}(\mathbf{x})] = 0. \tag{2.49}$$

This equation is known as the quantum diffeomorphism equation and, together with the Wheeler-DeWitt equation, enforces invariance under spatial coordinate transformations. This equation is related to the conservation of momentum and also ensures that the wave functional of the universe respects such symmetries.

2.4 PROBLEMS AND DETAILS OF THE WHEELER-DEWITT EQUATION

Now that we have defined the Wheeler-DeWitt equations, we will seek to analyze them to better understand the nuances and results that we can obtain from them.

2.4.1 Problem of time

One of the most notable features of the Wheeler-DeWitt equation is the absence of an explicit time term. This characteristic gives rise to one of the major problems associated with the

Wheeler-DeWitt equation, known as the "Problem of Time" [27, 28]. This problem arises from the conceptual incompatibility of time between general relativity and quantum mechanics. While general relativity treats time as a geometric variable dependent on the structure of spacetime, in quantum mechanics, time is an external, absolute variable that serves as a parameter for the evolution of the system [29]. This mismatch between the two theories creates a profound dilemma in the quest for a quantum theory of gravity.

The problem of time is intrinsically related to the Wheeler-DeWitt equation because, when quantized, it results in a static equation, meaning there is no explicit dependence on time. This suggests that, according to this equation, the state of the universe is timeless, which contradicts our intuition based on standard quantum mechanics, where systems evolve over time.

In general relativity, time is part of the structure of spacetime and is not an independent entity. Time is tied to the spacetime metric, and the absence of temporal evolution in the Wheeler-DeWitt equation implies that there is no global notion of time, as exists in conventional quantum mechanics. Furthermore, the lack of time dependence of the spatial metric h_{ab} can be understood by considering that in the ADM formalism, h_{ab} represents the intrinsic geometry of the three-dimensional spatial hypersurfaces. These hypersurfaces are "frozen" in the Wheeler-DeWitt equation, which reflects the fact that the evolution of the spatial geometry is encoded in the Hamiltonian constraints, and not explicitly in terms of time. Hence, the absence of time dependence in h_{ab} arises due to the formulation of the theory, where the evolution of the system is encoded in the constraints, rather than through a direct time parameter.

2.4.1.1 Proposed Solutions

Several approaches have been proposed to resolve or reinterpret the problem of time. Among them, the following stand out:

- **Emergent Time:** One of the most widely accepted proposals is that time, as we understand it, is an emergent entity [30] that arises approximately in certain regimes, such as in the semiclassical limit of gravity. In this scenario, time may emerge when gravity is coupled with other quantum field theories, recovering temporal evolution.

- **Deparametrization:** Another approach is deparametrization, where the aim is to identify an internal variable of the system that can act as a substitute for time [31]. This means that time can be viewed as a function of other parameters of the system, allowing the evolution to be described in terms of internal variables.

- **Many-Worlds Interpretations:** In the context of the many-worlds interpretation of quantum mechanics, the problem of time can be reinterpreted as a reflection of the branching of the universe's states [32, 33]. In this framework, the absence of a global time would be an inherent feature of the many-worlds formalism, where observers in different "branches" perceive distinct temporal evolutions.

2.4.1.2 Implications and Difficulties

The difficulty in defining a clear notion of time in a quantum gravity theory has profound implications. The absence of a classical time variable complicates the interpretation of the equations of quantum gravity in terms of evolving observables [28]. Additionally, the emergence of a notion of time and causality may be deeply connected to resolving fundamental issues such as the origin of the universe and the singularity of the Big Bang. Thus, the problem of time remains one of the most important and challenging issues in the quest for a unified theory of quantum gravity.

Despite the problem of time, the Wheeler-DeWitt equation is capable of providing significant results in various contexts. For example, it plays a fundamental role in quantum cosmology, specifically in the minissuperspace model, where the universe's metric is simplified and the equation describes the evolution of the universe in its early moments [34]. In this scenario, the Wheeler-DeWitt equation can predict the emergence of a universe from the quantum vacuum, offering a theoretical description for the origin of the Big Bang and the creation of spacetime [35]. Additionally, in semiclassical regimes, when quantum gravity is coupled with quantum matter, the equation recovers the classical temporal evolution of the universe, providing insights into the transition between the quantum and classical regimes. Thus, while the problem of time remains a fundamental challenge, the equation still offers a powerful framework for studying the quantum aspects of gravity and cosmology.

2.4.2 Group Theory

The Wheeler-DeWitt equation emerges as the quantum expression of gravity in the ADM formalism, imposing that the state of the universe be independent of any explicit temporal evolution. An essential part of this formulation is the invariance under the diffeomorphism group, which governs coordinate transformations in spacetime. This subsection explores the role of the diffeomorphism group in the Wheeler-DeWitt equation and the concept of superspace, which captures the geometric structure of the problem.

2.4.2.1 The Role of the Diffeomorphism Group in the Wheeler-DeWitt Equation

In the context of general relativity, the central symmetry group describing coordinate transformations in spacetime is the diffeomorphism group of a manifold $\mathcal{M}$ [36], denoted as $\mathit{Diff}(\mathcal{M})$. This group consists of all smooth and invertible transformations that can be applied to the coordinates of$\mathcal{M}$. In simple terms, any coordinate transformation that preserves the differentiability of the manifold belongs to the group $\mathit{Diff}(\mathcal{M})$.

The group $\mathit{Diff}(\mathcal{M})$ represents the transformations that keep the physics of gravity invariant, regardless of the choice of coordinates. This reflects general covariance, one of the fundamental principles of general relativity. General relativity, and by extension, the Wheeler-DeWitt equation, are formulated such that the laws of physics remain invariant under any coordinate change. This means that there is no privileged coordinate system. Thus, the group $\mathit{Diff}(\mathcal{M})$ encodes this fundamental symmetry.

Therefore, the group $\mathrm{Diff}(\mathcal{M})$ is an infinitely dimensional group of geometric transformations. This reflects the fact that diffeomorphisms can be thought of as smooth, invertible mappings between different configurations of the manifold. Since there are infinitely many ways to deform the spacetime geometry in a smooth manner, the group becomes infinitely dimensional. It encompasses transformations that allow for local variations in the manifold without introducing any singularities, capturing the full range of smooth deformations of the spacetime.

These deformations refer specifically to changes in the spatial metric h_{ab}, which describes the intrinsic geometry of the three-dimensional spatial hypersurfaces in the ADM formalism. Diffeomorphisms act on this metric by smoothly shifting its components, altering the geometry in a continuous way without changing its fundamental smooth structure. This highlights the flexibility of the ADM formalism in describing different spatial configurations through smooth, continuous transformations.

2.4.2.2 Superspace

The concept of "superspace" is central to the canonical analysis of general relativity and the quantization of gravity. Superspace is the space of all possible three-dimensional spatial geometries, meaning it is the space of metrics h_{ab} on a three-dimensional hypersurface, identified up to diffeomorphisms. This abstract space is crucial for the formulation of the Wheeler-DeWitt equation.

More precisely, superspace is the quotient space defined as:

$$\text{Superspace} = \frac{Riem(\Sigma)}{Diff(\Sigma)}, \tag{2.50}$$

where

- $Riem(\Sigma)$ is the set of all Riemannian metrics h_{ab} on a spatial hypersurface Σ;
- $Diff(\Sigma)$ is the group of diffeomorphisms on Σ.

Thus, superspace can be described as the space of all possible spatial geometries, with two metrics being equivalent if they can be related by a diffeomorphism.

The Wheeler-DeWitt equation, when interpreted in superspace, describes the quantum evolution of spatial geometries independent of an external time. Since time does not appear explicitly in the Wheeler-DeWitt equation, this formulation reflects the problem of time mentioned earlier in quantum gravity, where there is no global time parameter.

2.4.2.3 Comparison Between $Diff(\mathcal{M})$ and $SU(2)$

In terms of group theory, there is a significant distinction between the diffeomorphism group $Diff(\mathcal{M})$ and the internal symmetry group SU(2), which appears in Ashtekar variables and will be discussed in the next chapter. While $Diff(\mathcal{M})$ describes geometric symmetries related to spacetime itself, SU(2) describes internal symmetries associated with gauge connections.

- **Diff**$(\mathcal{M})$: It is an infinite-dimensional group that includes all smooth transformations of spacetime. It reflects the freedom to choose different coordinates in spacetime without altering the underlying physical laws. Its role in the Wheeler-DeWitt equation is to ensure the invariance of quantum gravity under coordinate changes.
- **SU**(2): It is a finite-dimensional Lie group with 3 generators. In the context of Ashtekar variables, SU(2) is the internal symmetry group associated with the gauge connection, used to reformulate quantum gravity in terms of a gauge field theory, facilitating the quantization of gravity.

In summary, $Diff(\mathcal{M})$ is an infinitely dimensional group of geometric transformations, while SU(2) is a three-dimensional group of internal symmetries. These topics will be revisited in the next chapter.

2.4.2.4 Conclusions

This chapter aimed to establish the connection between quantum mechanics and general relativity, presenting all the crucial calculations derived from the Einstein-Hilbert action. Additionally, we outlined new topics for the upcoming chapters, such as Ashtekar variables and their significance with the SU(2) group, which we will explore in greater depth to understand the reasons for this choice.

3 ASHTEKAR VARIABLES

In this chapter, we will seek to reformulate our Wheeler-DeWitt equation in order to address the problem of time and also reduce its mathematical complexity. We will write our spacetime using Ashtekar variables [37], based on the tetrad formalism, replacing the metric $g_{\mu\nu}$ with tetrads e^a_μ to obtain a simpler and more efficient local description.

Through this approach, we will be able to not only reduce the mathematical difficulty of describing curvature but also express gravity in terms of a gauge theory. This will allow for a connection with algebraic structures, such as those in group theory, simplifying the manipulation of the equations. Moreover, by using Ashtekar variables, we will be able to address the problem of time more effectively, paving the way for a possible quantization of gravity.

In summary, the tetrad formalism will provide us with a powerful tool to deal with the geometric and topological aspects of spacetime, while the Ashtekar variables will facilitate the reformulation of Einstein's equations in a quantum context.

3.1 TETRAD FORMALISM

The tetrad formalism replaces the metric $g_{\mu\nu}$ of spacetime with a set of tetrads $e^a{}_\mu$, which are 1-forms that act as a local basis at each point in spacetime. These tetrads provide a coordinate-independent way of describing the geometry of spacetime, making them particularly useful in general relativity and quantum gravity. Instead of working directly with the metric, the use of tetrads allows for a more flexible description of the geometry, as they directly connect the geometric properties of spacetime with those of quantum fields. This is particularly important in contexts of quantum gravity quantization, where the classical metric poses difficulties due to the complexity of quantum fluctuations.

The tetrads $e^a{}_\mu$ can be viewed as a set of vectors that map the curved spacetime to a locally flat tangent space. The spacetime metric is expressed in terms of the tetrads as follows:

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \tag{3.1}$$

where η_{ab} represents the Minkowski metric[1] in the local tangent space. The tetrads $e^a{}_\mu$ thus serve to bridge the curved spacetime and the locally flat Minkowski space, allowing the Minkowski metric to be applied locally at each point in spacetime. This relationship introduces the effects of spacetime curvature through the connections between the fields and the local geometry of spacetime.

By making this transition, we can see that the tetrad formalism facilitates the transition to quantum theories of gravity, such as loop quantum gravity, where gravity is treated as a gauge theory. The tetrad formalism provides a simpler and more natural algebraic structure for incorporating quantum fields, allowing gravity to be quantized in a manner similar to the other fundamental forces described in terms of gauge fields [39, 40].

Below is an illustration that exemplifies the relationship between the tetrad e^a_μ and the vectors of the local tangent space and the curved spacetime:

[1]The Minkowski metric describes the geometry of flat space-time in special relativity and is given by $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, where ds^2 is the line element, dt represents the time interval, and dx, dy, and dz represent the spatial intervals. The negative sign in front of the time term reflects the difference between space and time in the four-dimensional spacetime framework.

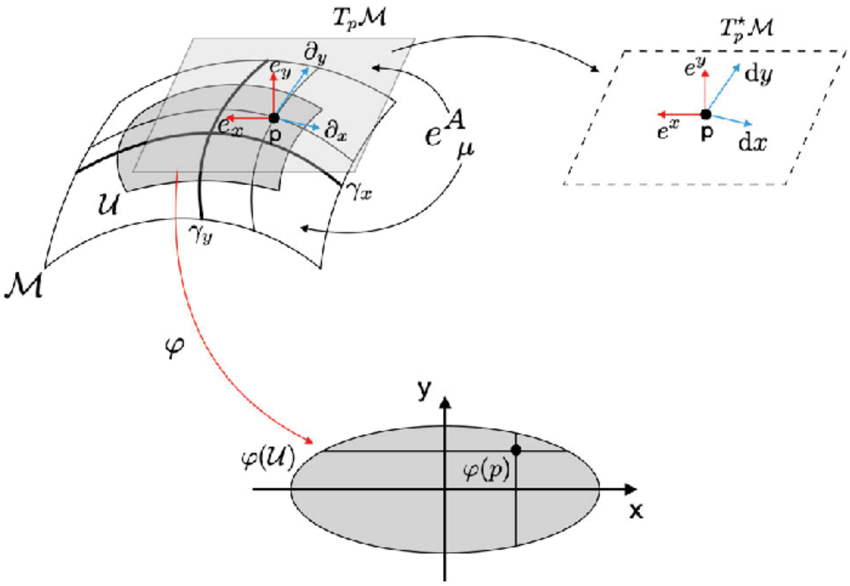

Figure 3.1: A two-dimensional illustration to clarify the tetrad formalism. The tetrads e^A_μ serve to relate the coordinate chart (U, φ) on the manifold M to the orthonormal basis $\{e_x, e_y\}$ in the tangent bundle T_pM. These tetrads also represent the coefficients in the natural (holonomic) basis $\{\partial_x, \partial_y\}$. The coordinate map φ assigns to each point $p \in U \subseteq M$ the coordinates $\varphi(p) = (x, y) \in \varphi(U) \subseteq \mathbb{R}^4$. By transitioning from the tangent space T_pM to the cotangent bundle T^*_pM through the spacetime metric $g_{\mu\nu}$ and the Minkowski metric η_{AB}, the natural basis $\{dx, dy\}$ is transformed into the orthonormal basis $\{e_x, e_y\}$ using the tetrads e^A_μ. [38]

$$\begin{array}{c}\text{Curved spacetime}\\ \text{with metric } g_{\mu\nu}\end{array} \xrightarrow{e^a_\mu} \begin{array}{c}\text{Local tangent space}\\ \text{with metric } \eta_{ab}\end{array}$$

In this illustration, the tetrads act as the vectors that transform the local tangent space of Minkowski, with the flat metric η_{ab}, into the curved spacetime, with the metric $g_{\mu\nu}$. This simplifies the description of curvature and makes the formalism more suitable for the quantization of gravity.

3.2 CURVATURE

One of the advantages of using the tetrad formalism is that it facilitates the treatment of the curvature of spacetime, primarily through the use of differential forms. Differential forms provide a more natural and flexible framework for describing geometric objects like curvature, and we will define them in more detail in Appendix B.

In the metric formalism, curvature is defined through the Levi-Civita connection. However, in the tetrad formalism, we express the connection in terms of tetrads, which simplifies the treatment of curvature and provides a more intuitive geometric interpretation.

The 2-form of curvature R_{ab}, which represents the curvature of spacetime in this formalism, is given by:

$$R_{ab} = d\omega_{ab} + \omega_a^{\ c} \wedge \omega_c^{\ b}, \tag{3.2}$$

where ω_{ab} is the spin connection. The spin connection ω_{ab} is a 1-form that encodes how the tetrads change from point to point, and it is crucial for defining how vectors are parallel transported in curved spacetime. The term $d\omega_{ab}$ represents the exterior derivative of the spin connection, which can be seen as the "rate of change" of the spin connection itself.

The wedge product $\wedge$ is the operation used in differential forms to combine two 1-forms into a 2-form, which is the correct mathematical object for representing curvature. The first term, $d\omega_{ab}$, describes how the spin connection itself changes, while the second term, $\omega_a^{\ c} \wedge \omega_c^{\ b}$, describes how the spin connections at different points in spacetime interact.

This reformulation of curvature is more suitable in geometric and topological contexts because it enables a more efficient manipulation of the equations. It also simplifies the expressions for curvature, allowing for more straightforward calculations. Moreover, the use of differential forms facilitates the exploration of the topological properties of spacetime, which are crucial in quantum gravity theories, where the underlying geometry may have quantum fluctuations.

In the context of the metric formalism, the Ricci tensor $R_{\mu\nu}$ is obtained by contracting the Riemann curvature tensor $R^{\rho}_{\ \mu\nu\rho}$. The Ricci tensor provides information about the curvature of spacetime, specifically the trace of the curvature in a given direction. In the tetrad formalism, the curvature can be expressed similarly, but the use of differential forms and the spin connection provides a more flexible and geometrically insightful description.

In situations where the metric $g_{\mu\nu}$ becomes singular, such as in the case of black holes or the singularity of the Big Bang, the metric formalism faces significant challenges. The tetrad formalism offers a way to bypass these singularities [41, 42]. While the metric may become undefined or singular, the tetrads can continue to provide a well-defined description of the local structure of spacetime, even in regions of high curvature. This makes it a useful tool in the formulation of quantum gravity theories, which seek to describe these regions where the classical theory of gravity is not applicable.

3.3 ACTION IN THE TETRAD FORMALISM

Now that we know how to express our metric tensors $g_{\mu\nu}$ and curvature R^{ab} in these new variables, we will seek to rewrite the Einstein-Hilbert action in this new formalism. We know that the Einstein-Hilbert action is given by:

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\,(R - 2\Lambda).$$

To express $\sqrt{-g}\, d^4x$ in terms of this new formalism, we will need to rewrite our metric g as follows:

$$g = \det(g_{\mu\nu}) = \det(e^a_\mu e^b_\nu \eta^{ab}). \tag{3.3}$$

We know from Binet's theorem that:

$$det(A.B) = det(A).det(B). \tag{3.4}$$

Using (3.4) in (3.3), we have:

$$g = \det(e^a_\mu)\det(e^b_\nu)\underbrace{\det(\eta)}_{=-1}. \tag{3.5}$$

therefore,

$$\sqrt{-g} = \sqrt{\det(e^a_\mu)\det(e^b_\nu)} = \sqrt{\det(e^a_\mu)^2} = det(e). \tag{3.6}$$

We also know that the volume element d^4x can be written as:

$$d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{3.7}$$

or alternatively, using the tetrads e^a, where $e^a = e^a_\mu \, dx^\mu$, we can rewrite dx^μ as:

$$dx^\mu = e^\mu_a e^a, \tag{3.8}$$

and expanding the implicit sum of a, we get:

$$dx^\mu = e^\mu_a e^a = e^\mu_0 \, dx^0 + e^\mu_1 \, dx^1 + e^\mu_2 \, dx^2 + e^\mu_3 \, dx^3. \tag{3.9}$$

Using (3.9) in (3.7) results in:

$$\begin{aligned} d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = & \underbrace{\left(e^0_0 \, dx^0 + e^0_1 \, dx^1 + e^0_2 \, dx^2 + e^0_3 \, dx^3\right)}_{=e^0} \wedge \\ & \wedge \underbrace{\left(e^1_0 \, dx^0 + e^1_1 \, dx^1 + e^1_2 \, dx^2 + e^1_3 \, dx^3\right))}_{=e^1} \wedge \\ & \wedge \underbrace{\left(e^2_0 \, dx^0 + e^2_1 \, dx^1 + e^2_2 \, dx^2 + e^2_3 \, dx^3\right)}_{=e^2} \wedge \\ & \wedge \underbrace{\left(e^3_0 \, dx^0 + e^3_1 \, dx^1 + e^3_2 \, dx^2 + e^3_3 \, dx^3\right)}_{=e^3}. \end{aligned} \tag{3.10}$$

Thus, the volumetric element d^4x, can be represented through the tetrads e^a, as:

$$d^4x = e^0 \wedge e^1 \wedge e^2 \wedge e^3. \tag{3.11}$$

Using (3.6), (3.11), and introducing the Levi-Civita symbol ϵ_{abcd} to correctly represent the volume in spacetime, we have:

$$\epsilon_{abcd} = \begin{cases} +1, & \text{if } (a, b, c, d) \text{ is an even permutation of } (0, 1, 2, 3), \\ -1, & \text{if } (a, b, c, d) \text{ is an odd permutation of } (0, 1, 2, 3), \\ 0, & \text{if any index is repeated.} \end{cases}$$

With this definition, we can now express the volume differential in spacetime in terms of the tetrads e_a. Therefore, we have:

$$\sqrt{-g} \, d^4x = \frac{1}{4!} \epsilon^{abcd} e_a \wedge e_b \wedge e_c \wedge e_d. \tag{3.12}$$

The factor $\frac{1}{4!}$ arises in the expression for the differential volume d^4x in terms of the Levi-Civita symbol due to the normalization of the outer product of tetrads. The Levi-Civita symbol ϵ_{abcd} is used in this context to represent the volume of spacetime in a compact and oriented way, and when writing the volume d^4x as a combination of the wedge product of tetrads, the constant $1/4!$ appears to correct for the number of possible permutations of the indices. This term is a consequence of the number of ways to arrange four distinct indices, since the Levi-Civita symbol has a total of 4! permutations, and by dividing by the factor 4! we ensure that the expression is normalized correctly. For more details on the use of the Levi-Civita symbol and its application in differential geometry and theoretical physics, the reader is referred to references such as Misner et al. [43] and Wald [18].

In addition, we need to rewrite the scalar curvature R with these new variables. We know that:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{3.13}$$

Using (3.1), we have:

$$R = e^{\mu}_{a} e^{\nu}_{b} \eta^{ab} R_{\mu\nu}, \tag{3.14}$$

and we can also write it as:

$$R = \frac{1}{2} R^{ab} \wedge e_a \wedge e_b, \tag{3.15}$$

or alternatively,

$$R = \frac{1}{2} \epsilon^{abcd} R_{ab} \wedge e_c \wedge e_d. \tag{3.16}$$

The equations (3.15) and (3.16) represent transformations of the curvature tensor $R_{\mu\nu}$ into the differential form R^{ab}, using the Levi-Civita symbol. The calculation leading to expression (3.15) and its alternative (3.16) is extensive and complicated, with the main goal of rewriting the curvature tensor in terms of differential forms and connecting the components of the tensor with the Levi-Civita symbol to represent the wedge product of 2-forms. Readers interested in more details about these steps can refer to the books by Wald (1984) *General Relativity* [18] and Misner, Thorne, and Wheeler (1973) *Gravitation* [25], which extensively cover the tetrad formalism and curvature in terms of differential forms.

Once we have obtained (3.16) and (3.12), we can write the action as:

$$S = \frac{1}{16\pi G} \int \epsilon_{abcd} \left(\frac{1}{2} e^a \wedge e^b \wedge R^{cd} - \frac{2\Lambda}{4!} e^a \wedge e^b \wedge e^c \wedge e^d \right),$$

$$S = \frac{1}{32\pi G} \int \epsilon_{abcd} \left(e^a \wedge e^b \wedge R^{cd} - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d \right). \tag{3.17}$$

Furthermore, we can separate the first term into real and imaginary parts, see Appendix B, in order to obtain:

$$\epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} \longrightarrow \star(e^a \wedge e^b) \wedge R_{ab} + i e^a \wedge e^b \wedge R_{ab}, \tag{3.18}$$

where its mathematical meanings are:

- The **real part**, that is, the term $\star(e^a \wedge e^b)$, is the dualized curvature, and the $\star$ denotes the Hodge dual, applied to the 2-forms. This gives us a classical description of curvature.
- The **imaginary part** is the term without dualization, $e^a \wedge e^b \wedge R^{ab}$, with a factor of i, indicating the complexification.

Thus, we can rewrite the Einstein-Hilbert action as:

$$S = \frac{1}{32\pi G} \int_{M^4} \left(\star(e^a \wedge e^b) \wedge R_{ab} + i e^a \wedge e^b \wedge R_{ab} - \frac{\Lambda}{6} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \right). \tag{3.19}$$

This expression has three main terms:

1. $\star(e^a \wedge e^b) \wedge R_{ab}$: Represents the real part of curvature, which describes classical gravity. It corresponds to the Hilbert-Palatini action [18];

2. $ie^a \wedge e^b \wedge R_{ab}$: The imaginary part of curvature, which is the Holst term, proportional to the first Bianchi identity in the absence of torsion [25].

3. $-\frac{\Lambda}{6}\epsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d$: The cosmological constant term, which provides the vacuum energy [18].

And as expected, although there are differences in the way we interpret the geometry of space-time, we arrive at the same physical results derived from the Einstein-Hilbert action, since we have only modified the way we express this geometry using the tetrad e^μ_a. Now that we have obtained this transformation, we will seek to derive its corresponding Hamiltonian, with the goal of rewriting and reinterpreting the Wheeler-DeWitt equation in this new formalism.

3.4 HAMILTONIAN IN ASHTEKAR FORMALISM

Now, we will seek to write a Hamiltonian so we can perform the canonical quantization of general relativity in this new formalism. We know that a total Hamiltonian can be written as a linear combination of its constraints, see reference [8], such that:

$$H_{Total} = \frac{1}{8\pi G}\int d^4x \sum_{i=1}^{N} \lambda_i C_i, \tag{3.20}$$

where λ_i is the Lagrange multiplier related to the constraint C_i, which can be interpreted as auxiliary parameters that do not necessarily have physical meanings but ensure the validity of the restrictions imposed by the theory.

If we observe that the work done so far was to change the way we interpret spacetime and describe it through other variables, it is reasonable to assume that the physics must be preserved. Therefore, we can assume that the total Hamiltonian is similar to the one calculated in the previous chapter (2.34), of the form:

$$H_{Total} = \frac{1}{8\pi G}\int d^4x \left(N\mathcal{H} + N^a\mathcal{V}_a + \sum_{i=3}^{N} \lambda_i C_i\right) \tag{3.21}$$

In this equation, we open the sum up to 2 and keep the remainder contained in the sum, where the first term, $N\mathcal{H}$, refers to the Hamiltonian constraint or scalar constraint, together with its Lagrange multiplier N (lapse function). This constraint is responsible for guaranteeing the temporal invariance of the theory, being the expression of the dynamical content of gravity in the canonical context. Physically, N controls the passage of time between the spatial slices in the ADM decomposition.

The second term, $N^a\mathcal{V}_a$, refers to the diffeomorphism constraint or vector constraint, with its Lagrange multiplier N^a (displacement vector). This term guarantees invariance under spatial diffeomorphisms, allowing the theory to be invariant with respect to changes in spatial coordinates. The multiplier N^a adjusts how the system evolves toward three-dimensional space, allowing "shifts" between spatial slices.

Finally, the term $\sum_{i=3}^{N} \lambda^i C_i$ encompasses all other constraints, such as the Gaussian constraint, which appears in gauge theories (such as in the Ashtekar formalism, see Appendix

A.3), with λ^j representing the associated Lagrange multipliers. The Gaussian constraint in this context enforces the conservation of the internal charge $SU(2)$ and guarantees the internal gauge invariance of the theory. For example, in our formalism, it will guarantee that the gauge symmetry $SU(2)$ of the connection A^i_a and the densified triad E^a_i is preserved under local rotations in the internal space [44].

As mentioned in the last chapter, by reformulating our theory with Ashtekar variables, we reduce our theory to the group $SU(2)$, as the densitized triad E^a_i and the self-dual connection A^i_a are defined in terms of this internal gauge group. The connection A^i_a is an $SU(2)$ connection, representing the degrees of freedom of gravity, while E^a_i acts as its conjugate momentum, reflecting the metric properties of space. This reduction arises naturally because the internal symmetry of the tetrad formalism is $SO(3,1)$ (special orthogonal group in 3+1 dimensions, is the group of linear transformations that preserve the Minkowski interval), and when restricted to self-dual components, it simplifies to $SU(2)$, preserving the internal gauge invariance under local rotations in the internal space [44]. Consequently, we can express the total Hamiltonian in terms of only three constraints, which constitute the main symmetries of the theory: invariance under temporal evolutions, spatial diffeomorphisms, and gauge transformations. Thus, the total sum of the constraints, multiplied by their respective Lagrange multipliers, results in the total canonical Hamiltonian of gravity given by:

$$H_{Total} = \frac{1}{8\pi G}\int d^4x\,(N\mathcal{H} + N^a\mathcal{V}_a + \lambda_i\mathcal{G}_i)\,, \tag{3.22}$$

where this last term represents the Gauss constraint, which imposes the internal gauge freedom and ensures that the evolution of the system respects the internal symmetries of the gauge connection.

Once we have expression (3.22), we only need to explicitly determine the constraints $\mathcal{H}$, $\mathcal{V}_a$, and $\mathcal{G}_i$. At this point, it is important to emphasize that we are looking to work with a system similar to that performed in the ADM formalism. Therefore, to formulate the theory in canonical terms with the variables N and N^a, it is convenient to choose a gauge with $e^0_\mu = 0$, fixing the temporal component of the tetrad.

In the context of the tetrads e^a_μ, where μ represents the spacetime coordinate indices and a represents the local basis indices, e^0_μ is the temporal component, while e^1_μ, e^2_μ, and e^3_μ correspond to the spatial ones. By adopting $e^0_\mu = 0$, we align the temporal vector of the tetrad with the normal to the spatial surfaces, a natural condition in the ADM formalism, which separates temporal and spatial directions in spacetime and acts as if it were the foliation in this new formalism.

To facilitate the canonical quantization of gravity, we rewrite the geometric variables in conjugate pairs, analogous to the "$p\dot{q}$" that occur in classical systems. In this context, the densitized triad E^a_i and the Ashtekar variable, or self-dual connection A^i_a, are introduced to represent, respectively, a spatial geometric density and a conjugate connection. This pair will simplify our description of gravity and be fundamental for the quantization of gravity.

The densitized triad E^a_i is defined as:

$$E^a_i = \epsilon_{ijk}\epsilon^{abc}e^j_b e^k_c, \tag{3.23}$$

where e^j_b represents the components of the spatial triad, derived from the tetrad e^a_μ. This density preserves the three-dimensional geometric structure of space, and E^a_i behaves as a momentum in the canonical theory, as we will observe throughout the chapter.

On the other hand, the Ashtekar variable A^i_a, conjugate to E^a_i, is expressed as:

$$A_i^a = \Gamma_i^a + iK_i^a,$$

$$A_a^i(x) \equiv -\frac{1}{2}\epsilon_k^{ij}\omega_{aj}^k - i\omega_{a0}^i, \tag{3.24}$$

where $\Gamma_a^i \equiv -\frac{1}{2}\epsilon_k^{ij}\omega_{aj}^k$ is the spin connection[2], ω_{aj}^k is the Lorentz connection, which describes how the tetrad field is transported under local Lorentz transformations in the space-time. Specifically, the Lorentz connection ω_{aj}^k governs the behavior of vectors and spinors under infinitesimal local rotations and boosts in the tangent space of the space-time. It is associated with the curvature of the internal gauge group $SO(3,1)$ and controls how the local frame (tetrad) changes as we move through the manifold. $K_i^a = -\omega_{a0}^i$ is the extrinsic curvature, and ω_{a0}^i is the temporal component of the Lorentz connection. Since the term ω_{a0}^i is associated with the variation of the tetrad field e_μ^0, it acts as the equivalent of extrinsic curvature, which controls the rate of temporal variation of the spatial structure, and therefore will be associated with the "temporal displacement" in the ADM decomposition. With these definitions, we can recover the physical meanings of the Lagrange multipliers.

Moreover, we can easily verify that, although the variables used to describe the geometry of spacetime are complex, they still ensure that we obtain corresponding physical quantities that are real, and thus obey the realism condition:

$$A_a^i + A_a^{i*} = 2\Gamma_a^i[E], \tag{3.25}$$

where the symbol $*$ denotes the complex conjugate. This choice simplifies the canonical structure of the theory, as it reduces the number of complex variables without losing consistency with the reality of spacetime.

Thus, E and A form the core of the Ashtekar formulation for gravity, with a Poisson structure analogous to the conjugate pairs in quantum mechanics, as shown in (3.26), which allows us to treat canonical gravity appropriately for quantization:

$$\{A_a^i(x), E_j^b(y)\} = i8\pi G\, \delta_a^b \delta_j^i \delta(x-y). \tag{3.26}$$

Now that we have an improved structure to describe our problem in terms of the Ashtekar variable and its conjugate, respecting the ADM formalism and the physics obtained in the previous chapter, we can write our Hamiltonian.

To do this, we introduce two fields that be fundamental to solving our problem: the magnetic field B^{ai} and the gauge field strength F_{ab}^k. Both of these fields are electromagnetic objects, where B^{ai} represents the magnetic field and F_{ab}^k is the electromagnetic field strength tensor. The gauge field strength F_{ab}^k is obtained from the Ashtekar connection A_a^i through the following expression:

$$F_{ab}^k = \partial_a A_b^k - \partial_b A_a^k + (8\pi G)\epsilon_{ij}^k A_a^i A_b^j, \tag{3.27}$$

where ∂_a denotes the partial derivative with respect to the spatial coordinates. This structure allows the gauge field strength F_{ab}^k to act as a curvature tensor, reflecting the variation of the connection in space as a function of the gauge connection. Additionally, the magnetic field B^{ai} is defined as:

[2] We changed the notation from ω to Γ for consistency throughout the rest of the text, and to avoid confusion with the Lorentz connection ω_{aj}^k.

$$B^{ai} \equiv \frac{1}{2}\epsilon^{abc}F^{i}_{bc}. \tag{3.28}$$

These definitions allow us to interpret gravitational dynamics in a context analogous to electromagnetism, where the magnetic field is related to variations in the gauge field strength. Just as in electrodynamics, where the magnetic field facilitates the description of interactions in terms of geometric flows, in gravity, the fields B and F translate the curvature of spacetime into three-dimensional variables, aiding in the understanding of the interactions between the geometry of spacetime and the gravitational fields. Furthermore, this analogy provides a basis for constructing a Hamiltonian that incorporates these new fields, leading to a richer formulation of quantum gravitational theories.

When we combine all our definitions, we find the resulting Hamiltonian:

$$\begin{aligned} H = \int_{M^3} \Big[& N'\epsilon_{ijk}E^{ai}E^{bj}\left(F^{k}_{ab} + \frac{\Lambda}{3}\epsilon_{abc}E^{ci}\right) + \\ & + N^{a\prime}E^{b}_{i}F^{i}_{ab} + \lambda'_{i}\left(\partial_a E^{a}_{i} + (8\pi G)\epsilon_{ijk}A^{j}_{a}E^{ak}\right)\Big]\, d^3x. \end{aligned} \tag{3.29}$$

This expression captures the interaction between the magnetic and gauge fields, allowing for a more complete description of gravitational dynamics in a quantum context. Thus, the introduction of the fields B^{ai} and F^{k}_{ab} not only enriches our understanding of gravity but also establishes a promising path for the investigation of quantum gravity theories, where the geometry and dynamics of spacetime are interconnected through these new variables.

Moreover, we can write our action as:

$$\begin{aligned} S = & \frac{1}{32\pi G}\int dt \int_{M^3}\Big[\dot{A}^{a}_{i}E^{i}_{a} + N'\epsilon_{ijk}E^{ai}E^{bj}\left(F^{k}_{ab} + \frac{\Lambda}{3}\epsilon_{abc}E^{ci}\right) \\ & + N'^{a}E^{b}_{i}F^{i}_{ab} + \lambda'_{i}\left(\partial_a E^{a}_{i} + (8\pi G)\epsilon_{ijk}A^{j}_{a}E^{ak}\right)\Big]\, d^3x. \end{aligned} \tag{3.30}$$

It is similar to the case presented in the previous chapter; if we use Hamilton's principle, we obtain the following constraints:

$$\frac{\delta\mathcal{H}}{\delta N'} = 0 \to \mathcal{H} = \epsilon_{ijk}E^{ai}E^{bj}\left(F^{k}_{ab} + \frac{\Lambda}{3}\epsilon_{abc}E^{ci}\right) = 0, \tag{3.31}$$

$$\frac{\delta\mathcal{H}}{\delta N'^{a}} = 0 \to E^{b}_{i}F^{i}_{ab} = 0, \tag{3.32}$$

$$\frac{\delta\mathcal{H}}{\delta \lambda'_{i}} = 0 \to \partial_a E^{a}_{i} + (8\pi G)\epsilon_{ijk}A^{j}_{a}E^{ak} = 0. \tag{3.33}$$

Here we again obtain the three constraints mentioned earlier, which are:

- $\frac{\delta\mathcal{H}}{\delta N'} = 0$ is the Hamiltonian constraint, responsible for generating our Wheeler-DeWitt equation, which is fundamental for quantizing general relativity in our system.
- $\frac{\delta\mathcal{H}}{\delta N'^{a}}$ is the diffeomorphism constraint, responsible for showing the invariance of our system under diffeomorphisms.

- $\frac{\delta \mathcal{H}}{\delta \lambda'_i}$ is the Gauss constraint, responsible for ensuring that the system respects the internal gauge symmetry.

Now that we have found the Hamiltonian of general relativity in this new formalism, we will defer the discussion of its solution to the next chapter and interpret the results obtained so far in a bit more detail.

3.4 CONNECTION WITH GAUGE THEORIES

As we have seen, the tetrad formalism also allows us to treat gravity as a gauge theory, similar to Yang-Mills theories [45]. In gauge theories, the connection is identified with the gauge field, and the curvature $F^a_{\mu\nu}$ represents the curvature of the associated fiber bundle. The tetrads function as a gauge field, connecting different points in spacetime. Yang-Mills theories describe fundamental interactions between particles through gauge fields, where the connection plays a crucial role in describing the properties of local symmetry. For example, the Yang-Mills action can be written as:

$$S_{YM} = \frac{1}{4} \int d^4x \, F^a_{\mu\nu} F^{\mu\nu a}, \tag{3.34}$$

where $F^a_{\mu\nu}$ is the Yang-Mills field tensor, given by:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{3.35}$$

with A^a_μ representing the gauge field associated with the symmetry group, g the coupling constant, and f^{abc} the structure constants of the group.

The connection between the tetrad formalism and gauge theories is evident in how gravity can be understood in terms of symmetries and connections. The Lorentz connection ω, which appears in the description of curvature, can be related to gauge transformations, where gravity manifests as a field that influences the geometry of spacetime. The tetrad connection is expressed as:

$$\omega^{ab}_\mu = e^a_\lambda e^b_\nu \nabla_\mu e^{\lambda\nu}, \tag{3.36}$$

where e^a_μ are the tetrads and ∇_μ represents the covariant derivative. The curvature is then given by the relation:

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^b{}_c, \tag{3.37}$$

where R^{ab} is the curvature tensor and d is the exterior derivative.

Furthermore, the concept of invariance under local transformations is fundamental for both gravity and Yang-Mills theories. In Yang-Mills theory, local invariance ensures that the equations of motion do not change under gauge transformations, resulting in constraint conditions [45], such as the Bianchi identities:

$$\nabla^\mu F^a_{\mu\nu} = 0, \tag{3.38}$$

where $F^a_{\mu\nu}$ is the field tensor. Similarly, in gravity, the conditions for invariance under diffeomorphisms impose restrictions on the variables that describe the geometry of spacetime.

The tetrads, which are used to represent the metric of spacetime, can be viewed as a set of gauge fields that connect the local geometry to the global features of spacetime. When considering the Ashtekar formalism, gravity is reinterpreted in terms of variables that possess gauge symmetry. The Ashtekar formalism provides a foundation for the quantization of gravity by establishing an analogy with Yang-Mills theories, where the connection and curvature play central roles.

In the Ashtekar formalism, the Hamiltonian is expressed in terms of energy density and the conjugate momentum. The Hamiltonian in terms of gauge fields is given by:

$$H = \int_{M^3} d^3x \left(N' \epsilon_{ijk} \left(E^{ai} E^{bj} F^k_{ab} + \frac{\Lambda}{3} \epsilon_{abc} E^{ci} \right) + N^a E^b_i F^i_{ab} + \right. \\ \left. + \lambda_i \left(\partial_a E^a_i + (8\pi G) \epsilon_{ijk} A^j_a E^{ak} \right) \right), \tag{3.39}$$

where E^{ai} is the conjugate momentum to the gauge field A^i_a.

Finally, the relationship between the tetrad formalism and Yang-Mills theories is not only conceptual but also technical. Both formalisms benefit from a common mathematical language, where gauge theory serves as a powerful framework for describing fundamental interactions. The ability to reinterpret gravity as a gauge theory allows for new approaches to be explored in the search for a quantum theory of gravity, where gravitational interactions can be described similarly to particle interactions mediated by gauge fields. The Einstein-Hilbert action without the cosmological constant in the tetrad formalism can be written in terms of the curvature form R^{ab} and the tetrads e^a:

$$S \propto \int \left[\epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} \right].$$

This action has a structure similar to that of Yang-Mills theories, with the difference that here the gauge symmetry is associated with the Lorentz group, which governs the symmetries of spacetime in terms of rotations and boosts. This is distinct from diffeomorphisms, which are more general transformations preserving the smooth structure of the manifold. This provides a theoretical foundation for using techniques developed in gauge theories for the study of gravity, including quantization.

3.5 INTERPRETATIONS OF THIS NEW FORMALISM

Now we will seek to provide some interpretations of how the Ashtekar formalism transforms our understanding of gravity, especially through its canonical variables, which directly connect the geometry and topology of spacetime. This approach not only simplifies the equations governing the dynamics of gravity but also offers new insights into the fundamental properties of spacetime.

3.5.1 Geometric Structure of Gravity

In the Ashtekar approach, the connection A^i_a and the densitized triad E^a_i have important geometric interpretations. The connection A^i_a generalizes the notion of covariant derivatives from differential geometry and plays a central role in defining how local geometric structures (such as the curvature of spacetime) change from point to point. It determines how the tetrads or local frames rotate and stretch across spacetime.

On the other hand, the densitized triad E^a_i, which can be understood as a momentum conjugate to the connection, is closely related to the three-dimensional geometry of space. More precisely, the components of E^a_i are associated with the *area* of a two-dimensional surface in three-dimensional space. This is because, in the Ashtekar formalism, E^a_i represents the density of the triad field, which encodes the geometric properties of space. Specifically, it is proportional to the area element of a surface in the three-dimensional manifold, giving a measure of the "amount of space" associated with each point.

In classical differential geometry, it is the *metric* that determines how distances and angles are measured. The connection A_a^i, while determining how vectors are transported across spacetime, does not directly measure distances. Instead, the metric, which can be derived from the tetrads in the Ashtekar formalism, defines the actual geometry of space, including distances, areas, and volumes.

The relationship between E_i^a and curvature is expressed in the Einstein field equations. In particular, the energy density ρ and pressure p can be related to the components of curvature of spacetime. Thus, gravitational dynamics is intricately linked to the geometry of spacetime, where the Ashtekar connection provides a clear mapping between physical properties (such as energy density) and geometric structure (such as curvature).

3.5.2 Topology of Phase Space

The phase space in the Ashtekar formalism is composed of pairs (A_a^i, E_i^a). The structure of this space is highly topological, with topology associated with the possible configurations of the system. In particular, Ashtekar's variables allow us to consider the configuration space in terms of gauge groups, such as $SU(2)$.

Gauge transformations correspond to local changes in the connection, reflecting how different configurations can represent the same physical state. This gauge invariance suggests that the topology of the phase space has a rich structure, where homotopy classes can be defined, and different Wilson loops can be used to represent fundamental interactions.

The presence of Wilson loops $W[C] = \mathcal{P}\exp\left(\int_C A\right)$ is crucial for understanding the interaction between particles and fields in spacetime. The loop C represents a curve in space, while $\mathcal{P}$ is the path product operator. This topological interpretation shows that gravity, when described with Ashtekar's variables, is not just a matter of curvature and geometry but is also deeply rooted in the topological properties of spacetime.

3.5.3 Physical Interpretations of the Hamiltonian Constraints

The Hamiltonian constraints play a fundamental role in defining the dynamics of the system and in its physical and geometric interpretation.

1. **Gauss Constraint**:

$$\mathcal{G}_i = D_a E_i^a = 0.$$

This equation ensures the conservation of charge associated with local gauge symmetries. Physically, this implies that gravity should be viewed as an interaction that preserves certain quantities, while geometrically it ensures that the curvature of spacetime does not contain monopoles, reflecting a cohesive topological structure.

2. **Diffeomorphism Constraint**:

$$\mathcal{V}_a = E_i^b F_{ab}^i = 0.$$

This term reflects invariance under diffeomorphic transformations, which is essential for the consistency of the theory. The physical interpretation is that the dynamics of the system is independent of the parameterization of points in spacetime, ensuring that the configuration space maintains its topological structure regardless of how coordinates are chosen.

3. **Hamiltonian Constraint**:

$$\mathcal{H} = \epsilon_{ijk} E^{ai} E^{bj} F^k_{ab} + \Lambda \epsilon_{abc} E^{ci} = 0.$$

This equation links the temporal evolution of the system to its geometric properties. Geometrically, the Hamiltonian constraint can be seen as a restriction on the possible configurations of spacetime, reflecting the interconnection between geometric structure and the presence of matter and energy, while physically it implies that gravity not only curves space but also defines its evolution.

3.5.4 Resolution of the Problem of Time

The Ashtekar approach provides a new understanding of the problem of time in quantum gravity. Instead of treating time as an external entity, Ashtekar's variables allow time to be considered an integral part of the equations themselves. As we will see in the next chapter, the imposition of the Wheeler-DeWitt equation:

$$\hat{\mathcal{H}} |\Psi\rangle = 0,$$

suggests that the state of the universe does not evolve in the classical sense but instead coexists in a space of superposition of states. This implies that "time" in quantum gravity may not be linear and absolute, but rather an emergent property of the dynamics of spacetime.

3.5.4 Conclusion

The introduction of Ashtekar variables in the description of gravity not only simplifies the equations of general relativity but also provides a new perspective on the interconnection between geometry, topology, and the dynamics of spacetime. By treating gravity as a gauge theory, the Ashtekar variables open new avenues for understanding quantum gravity, including the resolution of the problem of time and the interrelationship between curvature and the topological structure of the universe.

4 MODIFIED WHEELER-DEWITT EQUATION

In this chapter, we explore how the quantization of gravity, via the path integral formalism, introduces quantum corrections to the classical gravitational action. These corrections arise from the inclusion of a non-trivial functional measure in the configuration space, which adds a term dependent on the functional determinant of the metric $g_{\mu\nu}$. This term manifests as a purely imaginary contribution to the effective action, being proportional to $\hbar$ and representing loop corrections.

We identify that this contribution can be interpreted as a modification of the cosmological constant, leading to the introduction of a complex cosmological constant $\Lambda_{\mathbb{C}}$. This is composed of the classical cosmological constant Λ and an additional term Λ_i, which encapsulates the quantum effects of the functional measure. We rewrite the Einstein-Hilbert action to incorporate this new constant, demonstrating how quantization alters the fundamental structure of the gravitational theory.

The introduction of the complex cosmological constant not only modifies the traditional understanding of gravity but also suggests implications for physical phenomena, such as instabilities in spacetime or topological effects, as we will explore throughout the chapter. Finally, we will show that this analysis provides a solid theoretical foundation for future investigations into the impacts of quantization on gravitational models and their potential connections to high-energy physics.

4.1 CORRECTION FROM THE FUNCTIONAL MEASURE

4.1.1 Functional measurement review

At this point, we will use an equation capable of encapsulating all the information about a physical system, while providing a complete description of its quantum and gravitational behavior and possible interactions. The introduction of the generating functional, as done in the work *"Massive Graviton from Diffeomorphism Invariance"* [46], appears as a natural solution to this problem. In this mentioned work, the authors describe a mechanism through which the graviton acquires a mass through the functional measure, without violating the symmetry of the diffeomorphism or introducing Stückelberg fields. In addition, their work deals in more detail with the derivation of the complex cosmological constant, which we will use in the rest of the research.

The generating functional contains all the information about the quantum system, and its importance goes beyond simply calculating observables. By performing functional differentiations with respect to the external source J, we can obtain the correlation functions, which describe how the fields interact with each other and with external sources. Additionally, these functions are fundamental for obtaining scattering amplitudes, which directly connect to experimental results through the LSZ (Lehmann–Symanzik–Zimmermann) formula. Although we will not work with this formula in our research, it can serve as inspiration for future work.

Thus, the generating functional $Z[J]$ is defined as:

$$Z[J] = \int d\mu[\varphi]\, e^{i\left(S[\varphi^i]+J_i\varphi^i\right)}, \tag{4.1}$$

where $d\mu[\varphi]$ represents the functional measure with φ representing the set of fields that describe the physical system, $S[\varphi^i]$ is the classical action of the system, which describes the interactions and dynamics of the fields, and $J_i\varphi^i$ represents the interaction with an external source, which will not play a significant role in our work.

From this definition, we can obtain, for example, correlation functions, which are essential for characterizing the interactions and quantum behaviors of the fields. These functions are obtained by performing a functional differentiation of $Z[J]$ with respect to J, which allows us to study how the field $\varphi(x)$ responds to external perturbations. However, for this work, we will focus only on the term related to the functional measure $d\mu[\varphi]$.

Despite the central role of the generating functional in quantum field theory, the rigorous mathematical definition of the functional measure $d\mu[\varphi]$ remains a challenge. Operationally, it can be expressed as:

$$d\mu[\varphi] = \mathcal{D}\varphi^i \sqrt{\mathrm{Det}G_{ij}}, \tag{4.2}$$

where $D\varphi^i = \prod_i d\varphi$ represents the infinitesimal volume element in the configuration space, and G_{ij} is the configuration-space metric, defined for each type of field. The factor $\sqrt{\mathrm{Det}G_{ij}}$ is necessary to account for the non-trivial geometry of the configuration space [1], as exemplified in models like the non-linear sigma model. In such models, the fields φ^i live on a curved target space rather than a flat Euclidean space, leading to a configuration space with a metric G_{ij} that depends on the geometry of the target space. The determinant of G_{ij} encodes this curvature, ensuring that the functional measure correctly reflects the underlying geometry. For instance, in a non-linear sigma model, the kinetic term of the fields naturally involves G_{ij}, and neglecting $\sqrt{detG_{ij}}$ would result in inconsistencies in the path integral quantization, such as a failure to properly account for the degrees of freedom constrained by the geometry. Therefore, this factor plays a crucial role in preserving the consistency and invariance of the theory under transformations related to the symmetry of the target space.

In the context of gravitational systems, the determinant of G_{ij} introduces corrections to the classical action $S[\varphi^i]$. Specifically, we write:

$$\mathrm{Det}G_{ij} = e^{\delta^{(4)}(0)\int d^4x\sqrt{-g}\,\mathrm{tr}\log G_{ij}}, \tag{4.3}$$

where $\delta^{(n)}(0)$ represents a divergent term. In dimensional regularization, this is often set to zero, simplifying the determinant to unity. However, such an approach can obscure important physical phenomena, such as anomalies. To avoid this, we adopt a more controlled regularization scheme, as proposed in Wilsonian effective field theory, where a natural cutoff is implemented [47]. For instance, using a Gaussian regularization, we define:

$$\delta^{(4)}(x) = \frac{\Lambda^4}{(2\pi)^2} e^{\frac{-x^2\Lambda^2}{2}}. \tag{4.4}$$

When evaluated at the origin, we thus find:

$$\delta^{(4)}(0) = \frac{\Lambda^4}{(2\pi)^2}, \tag{4.5}$$

where Λ is a soft cutoff scale. This approach regularizes the divergent term and introduces a correction to the action, resulting in an effective action:

[1] A non-linear sigma model is a type of quantum field theory where the fields take values in a curved manifold (often called the target space). The action typically includes a kinetic term for the fields that involves the metric of the target space. This model is called "non-linear" because the fields do not transform linearly under the symmetries of the target space. These models are important in various areas of theoretical physics, such as string theory and the study of spontaneous symmetry breaking.

$$S_{\text{eff}} = \int d^4x\sqrt{-g}\left(\mathcal{L} - i\zeta \operatorname{tr}\log G_{ij}\right), \tag{4.6}$$

here, ζ is a coefficient dependent on the cutoff Λ and the regularization scheme. This additional term encapsulates quantum corrections arising from the non-trivial geometry of the configuration space and is fundamental in theories involving gravity, where it can manifest as modifications to the cosmological constant or other non-perturbative effects.

It is important to recognize that the configuration space metric G_{ij} must be considered as an intrinsic part of the definition of the theory. Although it is commonly associated with the bilinear form in the kinetic term, this connection lacks a direct physical justification. However, the requirement that the functional measure be invariant under the underlying symmetries allows the determination of G_{ij} in a manner similar to that in effective field theories. In our specific case, this approach yields the DeWitt metric. In prime order, the most general form of the configuration space metric for arbitrary fields takes the form:

$$S_{\text{eff}} = \int d^4x\sqrt{-g}\left(L - i\gamma \operatorname{tr}\log|g_{\mu\nu}|\right), \tag{4.7}$$

where γ is related to ζ through a finite renormalization [48]. The finite renormalization refers to the process by which the parameters of the model are adjusted in a controlled manner to handle divergences that arise during the calculation of integrals in field theories. Instead of simply removing the divergences through infinite renormalization, finite renormalization introduces a correction factor that addresses these divergences while ensuring the theory remains physically meaningful and consistent. This correction factor is often linked to a cutoff parameter, which allows divergent contributions to be regularized without discarding the underlying physics. This approach guarantees that the theory retains its internal consistency, with the corrections being finite and controlled, preserving the essential properties of the system.

Despite the correction term, it is crucial to emphasize that this equation does not break diffeomorphism invariance. The apparent violation arises because $\sqrt{\operatorname{Det} G_{ij}}$ transforms as a (functional) scalar density, and therefore, so does the final term in the action. However, the measure $D\varphi_i$ also behaves as a scalar density in such a way that the full product $D\varphi_i\sqrt{\operatorname{Det} G_{ij}}$ remains invariant. As a result, any variations in the apparent symmetry-breaking term under spacetime diffeomorphisms are canceled by the Jacobian that emerges from $D\varphi_i$, preserving the invariance of the quantum theory and all its observables.

Since both the functional measure and the classical action respect diffeomorphism invariance, the background-field effective action $\Gamma[g]$ naturally reflects this symmetry. Specifically, at the one-loop level [2], it is expressed as:

$$\begin{aligned}\Gamma[\varphi^i] &= S[\varphi^i] - \frac{i}{2}\log\operatorname{Det} G_{ij} + \frac{i}{2}\log\operatorname{Det}\mathcal{H}_{ij} \\ &= S[\varphi^i] + \frac{i}{2}\log\operatorname{Det}\mathcal{H}^i_j,\end{aligned} \tag{4.8}$$

where the configuration-space metric G_{ij} contributes to the usual correction $\log\operatorname{Det}\mathcal{H}_{ij}$, which is derived from the second derivative of the action $S[\varphi^i]$ with respect to the fields φ^i. The term

[2] The "one-loop level" refers to the approximation where only the lowest-order quantum fluctuations are considered in the perturbative expansion of the effective action. In this context, it involves the contribution from fluctuations around a classical background, accounting for one loop in the Feynman diagram expansion. This is often used to calculate quantum corrections to the classical action, such as the effective potential or propagators.

$\mathcal{H}_{ij}$ represents the Hessian of the action, describing the field fluctuations around the classical configuration. While its determinant is basis-dependent, we express it as a linear operator, ensuring that the determinant of $\mathcal{H}_{ij}$ remains invariant. This invariance is crucial for maintaining symmetries in the quantum formulation of the theory, particularly when considering the effective action and quantum corrections.

Finally, the results presented above were derived within the framework of the Lorentzian path integral. The appearance of the imaginary factor in Eq. (2.6) (and consequently in Eq. (2.8)) arises when the factor $i = \sqrt{-1}$ is extracted from the argument of the exponential in the Lorentzian path integral[3], allowing the measure to be written as a correction to the classical action. By adopting the Euclidean formalism, the functional measure is defined as:

$$Z_E[J] = \int d\mu[\varphi]\, e^{-(S^E_{\text{eff}}[\varphi^i] + J_i \varphi^i)}, \tag{4.9}$$

which leads to a real one-loop correction:

$$S^E_{\text{eff}} = \int d^4x \sqrt{-g}\, \left(L - \gamma \,\text{tr} \log |g_{\mu\nu}|\right). \tag{4.10}$$

This raises the issue of whether the path-integral measure should be defined in Euclidean space (followed by a rotation back to real time) or directly in Lorentzian space. The former choice is typically preferred for a more rigorous mathematical construction of the path integral, although it remains largely formal. In contrast, since the physical processes we observe are described by Lorentzian spacetime, we define the measure within this framework. As a result, different approaches may lead to distinct mathematical treatments, each with its own implications for the theory.

Now that we have made this brief review of functional measure, we will use this knowledge to analyze the correction that arises from the functional measure of Einstein Hilbert's action.

4.1.2 Complex Cosmological Constant

We know that the Einstein-Hilbert action is given by:

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\,(R - 2\Lambda), \tag{4.11}$$

The Hessian associated with this action can be expressed as:

$$H_{\mu\nu\rho\sigma} = K_{\mu\nu\rho\sigma}\Box + U_{\mu\nu\rho\sigma} \tag{4.12}$$

where $K_{\mu\nu\rho\sigma}$ is given by:

$$K_{\mu\nu\rho\sigma} = \frac{1}{4}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}\right), \tag{4.13}$$

[3]The *Lorentzian path integral* is an approach in quantum theory that uses the Lorentzian metric $(-,+,+,+)$ of spacetime to calculate transition amplitudes, preserving the relativistic structure and the nature of real time.

$\Box$ is the D'alembert operator[4] and $U_{\mu\nu\rho\sigma}$ represents a tensor that depends on the spacetime curvature. The exact form of $U_{\mu\nu\rho\sigma}$ is not critical for our discussion, though it can be found in references such as [49].

A standard choice for the configuration-space metric in gravity is the DeWitt metric:

$$G_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - a g_{\mu\nu}g_{\rho\sigma}\right), \tag{4.14}$$

where a is a dimensionless parameter that adjusts the deformation of the configuration-space metric. This parameter modifies the standard DeWitt metric, which is used to describe the geometry of the space of field configurations. The value of a controls the relative contributions of different components in the metric and can affect the behavior of the theory in various contexts, such as in the formulation of quantum gravity or asymptotic safety. By combining Eqs. (4.12) and (4.14), the expression from Eq. (4.8) leads to the following effective action:

$$\begin{aligned}\Gamma[g] = \int d^4x\,\sqrt{-g}\Bigg(&\overbrace{\frac{(R-2\Lambda)}{16\pi G}}^{Einstein-Hilbert} + \frac{i\zeta}{2}\log\det\left(\frac{1}{2}\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} + (a-1)g_{\mu\nu}g^{\rho\sigma}\right)\right) + \\ &+\frac{i}{2}\log\det\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\Box + (K^{-1})^{\mu\nu\rho\sigma}U_{\rho\sigma\alpha\beta}\right).\end{aligned} \tag{4.15}$$

In this expression, the determinant is the finite-dimensional determinant, denoted by det, and the indices are arranged properly, with an equal number of covariant and contravariant indices. Under diffeomorphisms, the determinant yields equal factors of the Jacobian and its inverse, which cancel each other, ensuring the invariance of the effective action, as expected from the correct transformation of the functional measure.

At this point, we focus on simplifying the expression. The last term in Eq. (4.15) can be evaluated using asymptotic expansions in either the curvature or spacetime derivatives. At low energies, the terms involving these derivatives are less significant compared to the second term, which contains no curvature or derivative factors and arises solely from the contribution of the functional measure. Thus, we concentrate on the first line of Eq. (4.15).

Using the matrix determinant lemma [5] , we obtain:

$$\det\left(\delta_{IJ} + (a-1)g_I g_J\right) = 1 + 4(a-1), \tag{4.16}$$

which simplifies Eq. (4.15) to:

$$\Gamma[g] = \int d^4x\,\sqrt{-g}\left(\frac{(R-2\Lambda)}{16\pi G} + i\Lambda'_i\right), \tag{4.17}$$

[4]The operator $\Box$, called the d'Alembert operator or wave operator, is defined as $\Box = \nabla_\mu\nabla^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$, where ∇_μ is the covariant derivative in a curved spacetime, and $g_{\mu\nu}$ is the metric tensor. In flat Minkowski spacetime, this simplifies to $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, where $\eta^{\mu\nu}$ is the Minkowski metric.

[5]The Matrix Determinant Lemma states that for a square invertible matrix $\mathbf{A}$ and vectors $\mathbf{u}$ and $\mathbf{v}$, the determinant of the rank-one updated matrix $\mathbf{A} + \mathbf{u}\mathbf{v}^T$ can be computed as:

$$\det(\mathbf{A} + \mathbf{u}\mathbf{v}^T) = \det(\mathbf{A}) \cdot (1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}).$$

This lemma is useful for simplifying determinant calculations, especially in the context of perturbations in linear algebra and statistical computations.

where Λ_i is defined as:

$$\Lambda_i' = \frac{\zeta}{2} \log\left(1 + 4(a-1)\right). \tag{4.18}$$

This result shows that, for the DeWitt metric, the functional measure contributes a complex term to the cosmological constant, Λ_i, which can influence the interpretation of the cosmological constant in quantum gravity.

4.1.3 Hamiltonian with correction

Once we have seen how we can introduce this complex cosmological constant, we can replicate the development carried out in the previous chapter, so that:

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\left(R - 2\Lambda_{\mathbb{C}}\right),$$

where $\Lambda_{\mathbb{C}} = \Lambda + i\Lambda_i$, with $\Lambda_i = -8\pi G \Lambda_i'$.

Rewriting in the exterior algebra formalism, we have an analogous expression to (3.17):

$$S = \frac{1}{32\pi G} \int \epsilon_{abcd}\left(e^a \wedge e^b \wedge R^{cd} - \frac{\Lambda_{\mathbb{C}}}{6} e^a \wedge e^b \wedge e^c \wedge e^d\right). \tag{4.19}$$

With this, we can separate the first term as in equation (3.18) and obtain:

$$S = \frac{1}{32\pi G} \int_{M^4} \left(\star(e^a \wedge e^b) \wedge R_{ab} + i e^a \wedge e^b \wedge R_{ab} - \frac{\Lambda_{\mathbb{C}}}{6} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d\right). \tag{4.20}$$

Having (4.20), we can write our Hamiltonian and action as (according to equations (3.29) and (3.30)):

$$\begin{aligned} H = \int_{M^3} \Big[& N' \epsilon_{ijk} E^{ai} E^{bj} \left(F^k_{ab} + \frac{\Lambda_{\mathbb{C}}}{3} \epsilon_{abc} E^{ci}\right) \\ & + N^{a\prime} E^b_i F^i_{ab} + \lambda_i' \left(\partial_a E^a_i + (8\pi G)\epsilon_{ijk} A^j_a E^{ak}\right)\Big] d^3x, \end{aligned} \tag{4.21}$$

$$\begin{aligned} S = & \frac{1}{32\pi G} \int dt \int_{M^3} \Big[\dot{A}^a_i E^i_a + N' \epsilon_{ijk} E^{ai} E^{bj} \left(F^k_{ab} + \frac{\Lambda_{\mathbb{C}}}{3} \epsilon_{abc} E^{ci}\right) \\ & + N'^a E^b_i F^i_{ab} + \lambda_i' \left(\partial_a E^a_i + (8\pi G)\epsilon_{ijk} A^j_a E^{ak}\right)\Big] d^3x. \end{aligned} \tag{4.22}$$

Finally, using the principle of least action of Hamilton, we find our new Wheeler-DeWitt Hamiltonian given by:

$$\mathcal{H}_{\text{WDW}} = \epsilon^{ijk} E^{ai} E^{bj} \left(F^k_{ab} + \frac{\Lambda_{\mathbb{C}}}{3} \epsilon_{abc} E^{ck}\right). \tag{4.23}$$

4.2 SOLUTION OF THE WHEELER-DEWITT EQUATION

Now, we know that the Wheeler-DeWitt Hamiltonian, responsible for the evolution of the wave function in the gravitational quantization formalism, can be expressed as:

$$\mathcal{H}_{\text{WDW}} = \epsilon^{ijk} E^{ai} E^{bj} \left(F^k_{ab} + \frac{\Lambda + i\Lambda_i}{3} \epsilon_{abc} E^{ck} \right), \tag{4.24}$$

where $\Lambda_{\mathbb{C}} = \Lambda + i\Lambda_i$ represents the complex cosmological constant, introducing an imaginary component to explore possible effects of complex gravity in our quantum theory.

To quantize the theory, we promote the field E^{ai} to a functional derivative operator. Thus, we have:

$$E^{ai} \longrightarrow -\kappa\hbar \frac{\delta}{\delta A_{ai}}, \tag{4.25}$$

where κ is a normalization factor that preserves the correct dimensionality of the action and facilitates the transition to the quantum regime. We then substitute E^{ai} with the operator above and apply the Hamiltonian $\mathcal{H}_{\text{WDW}}$ to the field's wave function, $\Psi[A]$, to obtain the quantum Wheeler-DeWitt equation.

By applying the Hamiltonian $\mathcal{H}_{\text{WDW}}$ on $\Psi[A]$, we perform the substitution of E^{ai} with its quantized expression. Thus, the Hamiltonian becomes:

$$\hat{\mathcal{H}}_{WDW} \Psi[A] = (\kappa\hbar)^2 \epsilon_{ijk} \frac{\delta}{\delta A_{ai}} \frac{\delta}{\delta A_{bj}} \left(F^k_{ab} - \frac{\kappa\hbar(\Lambda + i\Lambda_i)}{3} \epsilon_{abc} \frac{\delta}{\delta A_{ck}} \right) \Psi[A] = 0. \tag{4.26}$$

The first term $\epsilon^{ijk} \frac{\delta^2}{\delta A_{ai} \delta A_{bj}} F^k_{ab}$ represents the "quantum curvature" and describes the interaction of the Yang-Mills field with the conjugate momentum operator to the connection potential A_{ai}. The presence of $\frac{\delta^2}{\delta A_{ai} \delta A_{bj}}$ indicates a second-order functional dependence on the connection potential, highlighting the complexity of the phase space of quantum gravity. The second term introduces the complex cosmological constant $\Lambda_{\mathbb{C}} = \Lambda + i\Lambda_i$, which here generates a "complex evolution" in the wave function $\Psi[A]$. This term suggests that $\Psi[A]$ is not invariant under a temporal evolution defined by the Hamiltonian $\mathcal{H}_{\text{WDW}}$, which associates complex variations to the functional momentum operator.

Additionally, the expression (4.26) is a functional wave equation for $\Psi[A]$, where each term reflects topological and gravitational contributions. The term with $\Lambda_{\mathbb{C}}$ implies that the wave function may exhibit wave behavior with complex oscillations or decay, depending on the sign and value of the real and imaginary parts of Λ.

The equation is geometrically interpreted as a constraint on the curvature induced by the connection field A_{ai}, and topologically, it suggests that the ground state may be associated with vacuum states with non-trivial structure, such as the Chern-Simons vacuum states, which have direct interpretations in terms of topological invariants. Thus, quantization using Ashtekar variables enables a description that unites curvature and topology in a quantum gravity scenario.

4.2.0 Self-duality of the field strong

To simplify the equation, we assume that $F^k_{ab} = -\frac{\Lambda_{\mathbb{C}}}{3} \epsilon_{abc} E^{ck}$, which leads to the self-duality of the field strength, an important condition in quantum gravity theories due to symmetry and topological properties. With this assumption, we rewrite F^k_{ab} in terms of E^{ai}, reducing the dependence on the wave function and allowing identification with an exponential function of the Chern-Simons functional.

Furthermore, this simplification will serve as an approximation for low energies, as when the cosmological constant $\Lambda_{\mathbb{C}}$ takes specific values and spacetime is close to a vacuum state with a particular symmetry, this approximation becomes a natural simplification, allowing the reduction of the complex quantum formalism to a system where long-range behavior (gravitational effects on large scales) is dominant.

Consequently, we can verify the simplification by using (4.25) in

$$F^k_{ab} = -\frac{\Lambda_{\mathbb{C}}}{3}\epsilon_{abc}E^{ck}, \tag{4.27}$$

which gives:

$$F^k_{ab} \longrightarrow -\frac{\Lambda + i\Lambda_i}{3}\epsilon_{abc}\left(-\kappa\hbar\frac{\delta}{\delta A_{ck}}\right) = \frac{\kappa\hbar(\Lambda + i\Lambda_i)}{3}\epsilon_{abc}\frac{\delta}{\delta A_{ck}}. \tag{4.28}$$

Applying (4.28) to (4.26) we find:

$$(\kappa\hbar)^2\epsilon_{ijk}\frac{\delta}{\delta A_{ai}}\frac{\delta}{\delta A_{bj}}\left(\frac{\kappa\hbar(\Lambda + i\Lambda_i)}{3}\epsilon_{abc}\frac{\delta}{\delta A_{ck}} - \frac{\kappa\hbar(\Lambda + i\Lambda_i)}{3}\epsilon_{abc}\frac{\delta}{\delta A_{ck}}\right)\Psi[A] = 0. \tag{4.29}$$

Once we consider a non-trivial solution, i.e., $\frac{\delta}{\delta A_{ai}}\frac{\delta}{\delta A_{bj}}\Psi[A] \neq 0$, we can observe that our simplification is sufficient to satisfy the equation, and thus, we only need to solve:

$$\epsilon^{abc}\frac{\delta\Psi}{\delta A^k_c} = \frac{3}{\kappa\hbar(\Lambda + i\Lambda_i)}F^k_{ab}\Psi[A]. \tag{4.30}$$

Here we want to explore how the functional variation of Ψ with respect to the gauge potential A relates to the field strength F. To simplify this expression, we contract both sides with ϵ_{dab}, so as to eliminate the indices a and b on the right-hand side. Contracting both sides with ϵ_{dab}, we obtain:

$$\epsilon_{dab}\epsilon^{abc}\frac{\delta\Psi}{\delta A^k_c} = \frac{3}{\kappa\hbar(\Lambda + i\Lambda_i)}\epsilon_{dab}F^k_{ab}\Psi[A]. \tag{4.31}$$

The contraction with ϵ_{dab} is a common technique in field theory to simplify expressions involving antisymmetric tensors, such as the structure tensor ϵ^{abc}, as it helps reduce the number of indices and thus simplify the resulting equation. Applying this to the left-hand side we can use the identity $\epsilon_{dab}\epsilon^{abc} = 2\delta^c_d$, so we obtain:

$$2\delta^c_d\frac{\delta\Psi}{\delta A^k_c} = \frac{3}{\kappa\hbar(\Lambda + i\Lambda_i)}\epsilon_{dab}F^k_{ab}\Psi[A]. \tag{4.32}$$

Isolating the variation of Ψ with respect to A and renaming the indices $d \longrightarrow a$ and $k \longrightarrow i$ to better represent the final expression:

$$\frac{\delta\Psi}{\delta A^i_a} = \frac{3}{2\kappa\hbar(\Lambda + i\Lambda_i)}\epsilon^{abc}F^i_{bc}\Psi[A]. \tag{4.33}$$

To solve the functional equation for $\Psi[A]$, we will multiply (4.33) by $\frac{\delta A^i_a}{\Psi[A]}$ and integrate over its entire domain, thus we have:

$$\int\frac{\delta\Psi}{\Psi[A]} = \int\frac{3}{2\kappa\hbar(\Lambda + i\Lambda_i)}\epsilon^{abc}F^i_{bc}\delta A^i_a = \frac{3}{2\kappa\hbar}\left(\frac{\Lambda - i\Lambda_i}{\Lambda^2 + \Lambda_i^2}\right)\int\epsilon^{abc}F^i_{bc}\delta A^i_a. \tag{4.34}$$

Being in a geometric context, and having defined δA^i_a as a differential 1-form in the three-dimensional space, we can rewrite this term as (Appendix B):

$$\delta A^i_a = A^i_a dx^a = A^i_a d^3x, \tag{4.35}$$

where dx^a is the cotangent basis and is equal to our differential volume element d^3x constituted from the 3-form $dx^1 \wedge dx^2 \wedge dx^3$. By using (4.35) in (4.34), we obtain:

$$\int \frac{\delta\Psi}{\Psi[A]} = \frac{3}{2\kappa\hbar}\left(\frac{\Lambda - i\Lambda_i}{\Lambda^2 + \Lambda_i^2}\right)\int \epsilon^{abc} F^i_{bc} A^i_a d^3x. \tag{4.36}$$

Integrating the left-hand side, we obtain:

$$\ln(\Psi[A]) = \frac{3}{2\kappa\hbar}\left(\frac{\Lambda - i\Lambda_i}{\Lambda^2 + \Lambda_i^2}\right)\int \epsilon^{abc} F^i_{bc} A^i_a d^3x. \tag{4.37}$$

Exponentiating both sides, we get:

$$\Psi[A] = \exp\left(\frac{3}{2\kappa\hbar}\left(\frac{\Lambda - i\Lambda_i}{\Lambda^2 + \Lambda_i^2}\right)\int \epsilon^{abc} F^i_{bc} A^i_a d^3x\right). \tag{4.38}$$

Expanding the term F^i_{bc} given by (3.27), we have:

$$\Psi[A] = \exp\left(\frac{3}{2\kappa\hbar}\left(\frac{\Lambda - i\Lambda_i}{\Lambda^2 + \Lambda_i^2}\right)\int \epsilon^{abc}\left(\partial_a A^k_b A^i_a - \partial_b A^k_a A^i_a + (8\pi G)\epsilon^k_{ij} A^i_a A^j_b A^i_a\right) d^3x\right). \tag{4.39}$$

The integrand in the exponential of (4.38) is known as the Chern-Simons Functional. To highlight this, we simply write this expression in its differential form, thus we have:

$$\Psi[A] = \exp\left(\frac{3}{2\kappa\hbar}\left(\frac{\Lambda - i\Lambda_i}{\Lambda^2 + \Lambda_i^2}\right)\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right), \tag{4.40}$$

alternatively, we can compact our solution as:

$$\Psi[A] = \exp\left(-\frac{3}{2\kappa\hbar}\left(\frac{\Lambda - i\Lambda_i}{\Lambda^2 + \Lambda_i^2}\right)\int Y_{\mathrm{CS}}[A]\right), \tag{4.41}$$

where $Y_{\mathrm{CS}}[A]$ is the Chern-Simons functional, defined by:

$$Y_{\mathrm{CS}}[A] = \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = -\frac{1}{2}\left(A_i dA_i + \frac{1}{3}\epsilon_{ijk} A_i A_j A_k\right) \tag{4.42}$$

Furthermore, we can calculate $|\Psi[A]|^2$ for the obtained state. Looking at (3.24), we can easily see that the Ashtekar variable is complex, and thus the Chern-Simmons functional (4.42) is also complex. If it is complex, we can represent it as:

$$Y_{\mathrm{CS}}[A] = Re(Y_{\mathrm{CS}}[A]) + iIm(Y_{\mathrm{CS}}[A]), \tag{4.43}$$

so the wave function becomes:

$$\Psi[A] = \exp\left(-\frac{3}{2\kappa\hbar}\left(\frac{\Lambda - i\Lambda_i}{\Lambda^2 + \Lambda_i^2}\right)\left(\int Re(Y_{\mathrm{CS}}[A]) + i\int Im(Y_{\mathrm{CS}}[A])\right)\right), \tag{4.44}$$

or, we can separate the real and imaginary parts, obtaining:

$$\begin{aligned}\Psi[A] = &\underbrace{\exp\left[-\frac{3}{2\kappa\hbar}\left(\frac{1}{\Lambda^2 + \Lambda_i^2}\right)\left(\Lambda\int Re(Y_{\mathrm{CS}}[A]) + \Lambda_i\int Im(Y_{\mathrm{CS}}[A])\right)\right]}_{Real}\times \\ &\times\underbrace{\exp\left[-\frac{3}{2\kappa\hbar}\left(\frac{i}{\Lambda^2 + \Lambda_i^2}\right)\left(\Lambda\int Im(Y_{\mathrm{CS}}[A]) - \Lambda_i\int Re(Y_{\mathrm{CS}}[A])\right)\right]}_{Imaginary}.\end{aligned} \tag{4.45}$$

This way, it is possible to verify that your $\Psi^*[A]$ is:

$$\begin{aligned}\Psi^*[A] = &\exp\left[-\frac{3}{2\kappa\hbar}\left(\frac{1}{\Lambda^2 + \Lambda_i^2}\right)\left(\Lambda\int Re(Y_{\mathrm{CS}}[A]) + \Lambda_i\int Im(Y_{\mathrm{CS}}[A])\right)\right]\times \\ &\times\exp\left[\frac{3}{2\kappa\hbar}\left(\frac{i}{\Lambda^2 + \Lambda_i^2}\right)\left(\Lambda\int Im(Y_{\mathrm{CS}}[A]) - \Lambda_i\int Re(Y_{\mathrm{CS}}[A])\right)\right].\end{aligned} \tag{4.46}$$

Finally, the modulus squared of the wave function is given by:

$$|\Psi[A]|^2 = \Psi[A]\Psi^*[A], \tag{4.47}$$

$$|\Psi[A]|^2 = \exp\left[-\frac{3}{\kappa\hbar}\left(\frac{1}{\Lambda^2 + \Lambda_i^2}\right)\left(\Lambda\int Re(Y_{\mathrm{CS}}[A]) + \Lambda_i\int Im(Y_{\mathrm{CS}}[A])\right)\right]. \tag{4.48}$$

Once we obtain the solution (4.41) and (4.48), we now need to interpret it physically, seeking nuances and new interpretive results.

4.3 INTERPRETATIONS

In this section, we explore the mathematical and physical elements of this solution, highlighting its interpretations and implications.

4.3.1 Mathematical Interpretations of the state

The expression of our state exhibits a complex exponential dependence on a topological functional, the Chern-Simons functional:

$$Y_{\mathrm{CS}}[A] = \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right). \tag{4.49}$$

This functional is constructed from the self-dual connection A_a^i, and describes topological properties of the three-dimensional space. The dependence of $\Psi[A]$ on $Y_{\text{CS}}[A]$ indicates that the state is sensitive to the topological structure of spacetime but not to the specific metric.

The statement that the state is sensitive to the topological structure of spacetime, but not to the specific metric, arises from the mathematical properties of the Chern-Simons functional, $Y_{\text{CS}}[A]$, and how it is constructed.

The reason for this independence is primarily tied to the fact that it is formulated exclusively in terms of the connection A_a^i and its exterior derivative, without directly involving metric components, such as g_{ab} or the determinant of the metric $\sqrt{g}$, which are necessary to define metric properties like distances or angles. Moreover, the functional is invariant under diffeomorphisms, i.e., transformations that preserve the smoothness of space but not the metric. This invariance reflects its topological nature, as $Y_{\text{CS}}[A]$ measures global properties associated with the topological class of the connection A_a^i, such as the winding number. These global properties are independent of the choice of local metric, reinforcing the idea that the functional is related only to the connection structure and the topology of space [50].

In the context of the Ashtekar formalism, the three-dimensional metric h_{ab} can be reconstructed from the triadic density variables E_i^a, but the obtained state depends only on the Chern-Simons functional, which is directly constructed with A_a^i. Thus, the metric emerges only as a secondary entity, while the state remains solely linked to the topological structure of space. Therefore, the obtained state, being based on $Y_{\text{CS}}[A]$, reflects sensitivity to the topological properties of spacetime, disregarding specific metrics.

4.3.2 Physical Interpretations of the state

The obtained state $\Psi[A]$ is an exact solution of the Wheeler-DeWitt equation in the Ashtekar formalism, with a non-zero cosmological constant. As we saw earlier, its expression contains topological terms provided by the Chern-Simons functional, and together with the cosmological constant, it becomes capable of connecting topological aspects with quantum aspects of gravity.

The first interpretation we can have is that the obtained state describes a universe in which the connection variables A_a^i capture the curvature of spacetime in a self-similar [6] way. Specifically, A_a^i are related to the spacetime curvature through the field strength F_{ab}^i, which encodes how the geometry of spacetime is influenced by gravitational fields. In de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$) spacetimes, these connection variables reflect the constant curvature of the space, while in the quantum context, it can be interpreted as a gravitational vacuum state.

Furthermore, $Y_{\text{CS}}[A]$ measures global topological properties of the space, such as loops or windings associated with the connection A_a^i. This makes our state sensitive to the topology of spacetime, but independent of local metric properties such as distances and angles. This independence reflects the mathematical construction of $Y_{\text{CS}}[A]$, which is invariant under diffeomorphisms and does not depend on the metric g_{ab}.

From a quantum perspective, this state can be understood as a representation of the "vacuum" in quantum gravity, where the cosmological constant $\Lambda_{\mathbb{C}} = \Lambda + i\Lambda_i$ plays a central role. The real part, Λ, corresponds to the classical curvature scale of spacetime, which is directly related to the geometry of the universe. The imaginary part, Λ_i, arises from the functional measure in path integrals and reflects quantum corrections that introduce fluctuations in the gravitational field at small scales. These quantum corrections can affect the overall properties of spacetime, potentially influencing phenomena such as the origin of the cosmological constant.

[6] The term "self-similar" refers to the fact that the connection preserves the symmetries of these constant curvature spacetimes.

Furthermore, the presence of Λ_i links to issues in black hole thermodynamics, where imaginary components in quantum field theory have been suggested as playing a role in understanding black hole entropy and the emergence of topological phase transitions [51, 52].

In summary, the state offers a quantum description of a spacetime dominated by a cosmological constant, highlighting its dependence on topology and providing insights into the interaction between quantum gravity, cosmology, and quantum properties of spacetime.

The state has several physical implications, including:

- **Topology and Quantum Gravity:** The dependence on the Chern-Simons functional connects the solution to global properties of spacetime, such as topological invariants. This suggests that the state can be interpreted as a quantum description of spacetimes with a positive cosmological constant ($\Lambda > 0$).
- **Quantum Cosmology:** In de Sitter universes, the state is interpreted as the quantum vacuum. The imaginary component of the cosmological constant (Λ_i) may be associated with quantum effects or topological phases.
- **Quantum Phase State:** The presence of a complex exponential implies that $\Psi[A]$ encodes information about the quantum phase of the system, which may be relevant for topological transitions or non-trivial vacuum states.

Furthermore, we can compare this with traditional quantum mechanics, where we have the expression:

$$\langle x|\Psi\rangle = \Psi(x), \tag{4.50}$$

which represents the probability amplitude of finding the system in state Ψ at position x. When we take the modulus squared, we obtain the probability density of finding the particle at position x, given by the expression $|\langle x|\Psi\rangle|^2$.

In our context, the configuration A can be understood as a variable that describes the geometry or topology of spacetime, and in this case, the expression becomes:

$$\langle A|\Psi\rangle = \Psi[A], \tag{4.51}$$

which represents the probability amplitude of finding the universe in the state Ψ with the configuration A. The modulus squared of this amplitude, $|\langle A|\Psi\rangle|^2$, gives us the probability that the system is associated with this configuration A, i.e., the probability that the universe is in a specific cosmic era that can be described by configuration A.

For example, if the configuration A is associated with a specific cosmic era (such as the inflationary era, radiation era, or matter era), the expression $|\langle A|\Psi\rangle|^2$ gives us the probability that the universe is in that stage of evolution, given the quantum state Ψ.

In summary, the analogy with traditional quantum mechanics helps us understand that, just as in quantum systems where we can measure the probabilities of a particle being at positions x, in quantum gravity the configuration A can represent the topology or geometry of spacetime, and $|\langle A|\Psi\rangle|^2$ gives us the probability that the universe is in the cosmological era associated with this configuration of A. This is because, in principle, if we can express a cosmic era through the Lorentz connection, i.e., the curvature the universe has, as in (3.24), we can calculate the probability that our state is in this configuration.

Furthermore, the equation (4.48) represents the probability density associated with the modified Kodama state, incorporating a complex cosmological constant $\Lambda + i\Lambda_i$. Unlike the original Kodama state [53], which involved only the real Chern-Simons functional, this expression includes separate real and imaginary components, each weighted by the corresponding part of the cosmological constant. This structure indicates that the wave function of the universe in this formalism depends not only on the curvature determined by the connection field A but also on additional terms that modify the probability distribution.

The exponential dependence on the Chern-Simons functional suggests that field configurations maximizing this functional have a higher probability weight. In particular, when $\Lambda_i \neq 0$, the wave function includes an oscillatory component associated with the imaginary term. This behavior contrasts with the original Kodama state, where the probability was a purely real exponential function. The presence of an imaginary part alters the weight given to different configurations, modifying the classical limit of the theory.

Another relevant aspect is that the introduction of a nonzero Λ_i affects the normalizability of the state. The original Kodama state was known to be non-normalizable in certain contexts, raising concerns about its physical validity. The presence of an imaginary term in the exponent may modify the asymptotic behavior of the wave function, potentially influencing the convergence of the probability measure over the space of connections. This affects the interpretation of the modified state and its applicability to quantum gravity models.

The structure of this probability density also indicates that the preferred configurations of the gauge field are determined by a balance between the real and imaginary parts of the Chern-Simons functional.[7] This suggests that the evolution of the system is not governed solely by real curvature effects but also by additional contributions from the imaginary sector of the theory. The interplay between these terms could lead to different predictions compared to those obtained using the original Kodama state.

Finally, the modification introduced by the complex cosmological constant changes the weighting of different geometries in the quantum description. This affects how classical and semiclassical limits are recovered, as well as how fluctuations in the cosmological constant contribute to the probability distribution. The dependence on both the real and imaginary parts of $Y_{\text{CS}}[A]$ implies that the probability measure over connections is influenced by additional parameters, which may lead to different physical implications when considering quantum corrections.

4.3.3 Limitations and Challenges

Although our state is mathematically elegant, there are important issues to be considered, similar to the Kodama state, such as:

- **Normalizability:** The integral over the configuration spaces of A_a^i may diverge depending on the type of configuration integrated, raising questions about the quantum consistency of the state.
- **Physical Interpretation of Λ_i:** Despite its theoretical applications, the imaginary cosmological constant (Λ_i) still lacks a clear experimental or observational interpretation.

[7] The real part of the Chern-Simons functional, $\text{Re}(Y_{\text{CS}}[A])$, is associated with classical solutions of the self-dual connection equations, which are relevant in Ashtekar's formalism for gravity. The imaginary part, $\text{Im}(Y_{\text{CS}}[A])$, arises when considering extensions to a complexified phase space, where additional terms contribute to the weight of different configurations. The preferred configurations refers to field configurations of the connection A that maximize the probability density $|\Psi[A]|^2$. These are the configurations that contribute most significantly to the path integral formulation of quantum gravity, as they dominate the probability distribution. In the classical limit, they correspond to gauge fields that satisfy self-duality conditions, while in the quantum regime, they are influenced by both the real and imaginary components of the Chern-Simons functional, leading to modifications in how classical solutions emerge from the quantum state.

4.3.3 Conclusion

The obtained state provides a formally rich solution to the Wheeler-DeWitt equation in the Ashtekar formalism, relating quantum gravity to topological properties of spacetime through the Chern-Simons functional. Despite its potential as a vacuum state in quantum cosmology, challenges related to normalizability and physical interpretation remain open, making it a topic of great interest in modern theoretical physics.

The resulting wave function can be interpreted according to the values of the real and imaginary parts of $\Lambda_{\mathbb{C}}$:

- **If** $\Lambda_{\mathbf{i}} = 0$: The wave function $\Psi[A]$ reduces to the pure Kodama state, associated with a Chern-Simons vacuum state.
- **If** $\Lambda_{\mathbf{i}} \neq 0$: The wave function is partially oscillatory and partially exponential, representing a vacuum state with moderate decay or growth, depending on the sign and magnitudes of Λ and Λ_{i}.

5 CONCLUSIONS

In this dissertation, we investigated the solution of the Wheeler-DeWitt equation in the context of quantum gravity, with an emphasis on the introduction of a complex cosmological constant. This final chapter aims to synthesize the main contributions of the work, discuss the implications of the obtained results, highlight limitations, and suggest avenues for future research.

Throughout this chapter, we will address how the advancements proposed in this work connect with current challenges in quantum gravity and the potential to impact future theories and observations.

5.1 CONTRIBUTIONS OF THE WORK

The main contribution of this work was solving the Wheeler-DeWitt equation using the Ashtekar formalism with a complex cosmological constant, emerging from the correction of the functional measure in the path integral. The key advancements include:

- Incorporation of a complex cosmological constant, offering new perspectives on the quantum dynamics of spacetime, connecting classical gravity concepts with emerging quantum properties.

- Exploration of the physical interpretations of the modified solution, including implications for the origin of the cosmological constant and quantum effects in high-curvature regions. This study points to the possibility of new phase transitions in spacetime.

These contributions not only expand the understanding of the Wheeler-DeWitt equation but also open pathways for explorations that may unify the topological structures of spacetime with the fundamental equations of theoretical physics. The adopted approach also provides a foundation for studying the implications of quantum effects in astrophysical systems, such as black holes and the primordial universe.

5.2 INTERPRETATIONS OF THE RESULTS

The results obtained provide insights into quantum gravity and the nature of spacetime. The introduction of the complex cosmological constant suggests a deep connection between the topological aspects of spacetime and the quantum description of gravity. Specifically:

- The dependence of the solution on the Chern-Simons functional indicates the relevance of topological structures for quantum gravity. This approach enables the exploration of equivalence classes of spacetime at quantum scales.

- Quantum fluctuations of spacetime, associated with the complex cosmological constant, may offer an interpretation of the universe's origin and black hole dynamics. Such phenomena may open pathways to new observables in quantum gravity.

- The role of the imaginary part of the cosmological constant, proportional to $\hbar$, suggests a pathway to understanding the transition between classical and quantum regimes, directly connecting gravity with quantum mechanics at a fundamental scale.

One of the conceptual contributions is the perception that spacetime is not merely a passive container for events but also possesses an emergent dynamics intrinsically connected with quantum degrees of freedom. This view promotes greater integration between classical and quantum perspectives in the description of the universe.

5.3 STUDY LIMITATIONS

Although this work has achieved interesting results, some limitations must be highlighted:

- Dependence on mathematical approximations, such as the assumption of self-duality of the force field, limits the generality of the results and requires further studies for validation in broader contexts.
- The physical interpretation of the complex cosmological constant still lacks more experimental or phenomenological support, especially in the direct detection of its effects.
- The proposed solution is restricted to specific cases and does not fully address the problem of time in quantum gravity. This remains one of the central challenges in the quest for a unified theory.

Additionally, future reformulations should include more general scenarios, with greater interactions between matter, energy, and spacetime. This will allow validating the predictions made here in experimental contexts.

5.4 FUTURE PERSPECTIVES

This work opens several possibilities for future research:

- Exploring the physical implications of the complex cosmological constant in more realistic cosmological models, including scenarios with matter and radiation. Models with anisotropies may reveal new quantum dynamics.
- Investigating the role of additional quantum corrections to resolve the problem of time and obtain more general dynamic solutions. This may include the use of string theory and emergent gravity theories.
- Developing clearer connections between the topological aspects of the solution and astrophysical or cosmological observations. This could involve investigations of primordial black holes and gravitational waves.
- Expanding the study to include effects of quantum field theories in curved spacetimes, with greater emphasis on computational applications.

5.5 FINAL CONSIDERATIONS

The resolution of the Wheeler-DeWitt equation presented in this dissertation offers an interesting vantage point to the understanding of quantum gravity. Despite the limitations, the results obtained highlight the potential of Ashtekar variables and the complex cosmological constant as tools to unify classical and quantum gravity theories.

The theoretical impact may extend to other areas of physics, as the methods used in quantum gravity could offer new approaches to problems in fields like high-energy physics, quantum field theory, and cosmology. By providing a framework for linking quantum mechanics and general relativity, it could help explore aspects that are not fully understood within these areas.

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APPENDIX A – GROUP THEORY - GAUSS CONSTRAINT

In this appendix, we present a detailed study of the Gauss constraint within the Ashtekar formalism. We start from fundamental concepts such as gauge symmetries and Lie groups, and develop the tools necessary to understand the definition and role of the Gauss constraint in the context of quantum gravity.

A.1 INTRODUCTION TO GAUGE SYMMETRIES AND LIE GROUPS

Gravity in the Ashtekar formalism is formulated as a gauge theory with symmetries associated with the group SU(2). To understand the Gauss constraint, it is first necessary to review the basic concepts:

A.1.1 Lie Groups and their Algebras

A Lie group is a continuous group that is also a differentiable manifold. The group SU(2) is defined as the set of 2×2 unitary matrices with unit determinant:

$$\mathrm{SU}(2) = \{U \in \mathbb{C}^{2\times 2} \mid U^\dagger U = I,\ \det(U) = 1\}. \tag{A.1}$$

Its Lie algebra $\mathfrak{su}(2)$ is the space of anti-Hermitian matrices with zero trace:

$$\mathfrak{su}(2) = \{X \in \mathbb{C}^{2\times 2} \mid X^\dagger = -X,\ \mathrm{tr}(X) = 0\}. \tag{A.2}$$

The generators of the algebra $\mathfrak{su}(2)$ are denoted by T^i (with $i = 1, 2, 3$) and satisfy the commutation relations:

$$[T^i, T^j] = \epsilon^{ij}{}_k T^k, \tag{A.3}$$

where $\epsilon^{ij}{}_k$ is the Levi-Civita symbol.

A.1.2 Gauge Connections

A gauge connection A^i_a in SU(2) is a mathematical object that allows the definition of parallel transport and curvature in a gauge theory. It is defined as a 1-form valued in the Lie algebra $\mathfrak{su}(2)$, encoding how the internal degrees of freedom of the system transform under local gauge transformations.

In three-dimensional space, the gauge connection is expressed as:

$$A_a = A^i_a T^i, \tag{A.4}$$

where A^i_a are the components of the connection, a is the spatial index (running over the spatial coordinates), and i is the internal index associated with the generators T^i of the Lie algebra $\mathfrak{su}(2)$.

Physically, the gauge connection plays a role analogous to the electromagnetic potential A_μ in electrodynamics, where it defines how charged fields interact with the gauge field. However, in the case of gravity formulated within the Ashtekar formalism, the gauge group is SU(2) rather than U(1), reflecting the internal rotational symmetry of the triad formulation. This reinterpretation of general relativity as a gauge theory, extensively discussed in [44, 26], provides a natural setting for canonical quantization and loop quantum gravity.

The field strength F^i_{ab} associated with the gauge connection, also known as the curvature of the connection, is given by:

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b. \quad \text{(A.5)}$$

This quantity measures the non-triviality of the gauge field configuration. The first two terms represent the ordinary curl of the connection, while the last term encodes the non-Abelian nature of the SU(2) gauge group. Unlike the electromagnetic field strength, which is simply the antisymmetric derivative of the potential, the presence of the commutator term in the expression for F^i_{ab} results in self-interaction of the gauge field, characteristic of non-Abelian gauge theories.

In the context of general relativity reformulated in the Ashtekar variables, the connection A^i_a is related to the spin connection that describes how local frames rotate with respect to each other in curved space. This allows for a parallel with Yang-Mills gauge theories, where curvature and parallel transport play fundamental roles, as detailed in [44, 26].

A.1.3 Densitized Triads

The densitized triad E^a_i is a fundamental quantity in the Ashtekar formalism, encoding the spatial geometry of a given hypersurface. It is related to the usual triad e^a_i by the determinant density of the spatial metric h:

$$E^a_i = \sqrt{h} e^a_i. \quad \text{(A.6)}$$

Here, e^a_i are the components of the triad field, which provide a local orthonormal basis for the spatial slice. The factor $\sqrt{h}$ ensures that E^a_i transforms appropriately under coordinate transformations and has density weight one.

The densitized triad satisfies the key relation:

$$E^a_i E^b_j \delta^{ij} = h h^{ab}, \quad \text{(A.7)}$$

where h^{ab} is the inverse metric on the three-dimensional hypersurface. This equation demonstrates how the densitized triad acts as the bridge between the gauge-theoretic description and the conventional metric formulation of general relativity.

From a physical perspective, the triad e^a_i determines distances and angles in the spatial manifold, similar to the metric tensor h_{ab}. However, using E^a_i as a fundamental variable provides a more natural formulation in terms of canonical variables, particularly when transitioning to the quantum theory.

One of the crucial aspects of using densitized triads is their role in defining the Poisson bracket structure of general relativity in the Ashtekar formalism. The fundamental phase space variables are given by the pair (A^i_a, E^a_i), satisfying the canonical relation:

$$\{A^i_a(x), E^b_j(y)\} = \delta^b_a \delta^i_j \delta^3(x - y). \quad \text{(A.8)}$$

This structure closely resembles that of a Yang-Mills gauge theory, reinforcing the interpretation of general relativity as a constrained gauge system. The role of densitized triads in loop quantum gravity, particularly in their connection with spin networks and discrete quantum geometry, is extensively discussed in [54, 55].

A.2 THE GAUSS CONSTRAINT

The Gauss constraint is one of the first-class constraints in the Ashtekar formalism. It is defined by:

$$G^i = \partial_a E_i^a + \epsilon^i{}_{jk} A_a^j E^{ak}. \tag{A.9}$$

A.2.1 Physical Significance

The Gauss constraint ensures the invariance of the theory under internal gauge transformations of the group SU(2). For any infinitesimal parameter α^i, a gauge transformation is given by:

$$\delta A_a^i = D_a \alpha^i, \quad \delta E_i^a = \epsilon_{ijk} \alpha^j E^{ak}, \tag{A.10}$$

where $D_a \alpha^i = \partial_a \alpha^i + \epsilon^i{}_{jk} A_a^j \alpha^k$ is the covariant derivative.

The constraint $G^i \approx 0$ ensures that physical observables are invariant under these transformations.

A.2.2 Hamiltonian Form

In the Hamiltonian formalism, the Gauss constraint appears as a condition on the allowed states in the phase space:

$$\int_\Sigma \lambda_i G^i \, dv = 0, \tag{A.11}$$

where λ_i are the associated Lagrange multipliers and Σ is the three-dimensional hypersurface.

A.3 THE GAUSS CONSTRAINT IN THE QUANTUM CONTEXT

In the quantum formulation, the operators corresponding to the triads $\hat{E}_i^a$ and connections $\hat{A}_a^i$ satisfy the commutation relations:

$$[\hat{A}_a^i(x), \hat{E}_j^b(y)] = i\hbar \delta_a^b \delta_j^i \delta^3(x - y). \tag{A.12}$$

The Gauss constraint is implemented as a condition on the quantum states $\Psi[A]$:

$$\hat{G}^i \Psi[A] = 0. \tag{A.13}$$

This implies that $\Psi[A]$ must be invariant under gauge transformations of the group SU(2).

A.3.0 Final Considerations

The Gauss constraint in the Ashtekar formalism is essential to ensure the consistency of the gauge theory associated with gravity. It guarantees that physical configurations are invariant under internal gauge transformations, both in the classical and quantum formulations. In the context of the Kodama state, this constraint is automatically satisfied, reinforcing its consistency as a physical solution.

APPENDIX B – EXTERIOR ALGEBRA AND DIFFERENTIAL FORMS

This appendix presents exterior algebra, a central formalism for describing geometric theories such as general relativity. We cover fundamental concepts like differential forms [56], the wedge product, and the exterior derivative, the calculation of the curvature tensor within this framework, and the decomposition

$$\epsilon_{abcde} e^a \wedge e^b \wedge R^{cd} \longrightarrow \star(e^a \wedge e^b) \wedge R^{ab} + i e^a \wedge e^b \wedge R^{ab}.$$

B.1 FOUNDATIONS OF EXTERIOR ALGEBRA

B.1.1 Differential Forms: Definitions

Differential forms are mathematical objects used to describe physical fields in a covariant manner [56]. Formally, a k-form on a manifold M is a completely antisymmetric multilinear function:

$$\omega : T_pM \times \cdots \times T_pM \to \mathbb{R}, \tag{B.1}$$

where T_pM is the tangent space at a point p on M. The function ω maps k vector fields $X_1, \ldots, X_k$ to real numbers:

$$\omega(X_1, \ldots, X_k) \in \mathbb{R}. \tag{B.2}$$

B.1.1.1 Examples of k-Forms

- A 0-form is a scalar function $f(x)$.
- A 1-form is a linear combination of differentials:

$$\omega = \omega_\mu \, dx^\mu, \tag{B.3}$$

  where ω_μ are components and dx^μ are coordinate differentials, representing infinitesimal displacements in the spacetime coordinates..

- A 2-form is given by:

$$\omega = \frac{1}{2}\omega_{\mu\nu} \, dx^\mu \wedge dx^\nu, \tag{B.4}$$

  with $\omega_{\mu\nu} = -\omega_{\nu\mu}$, reflecting antisymmetry.

B.1.1.2 Properties of Differential Forms

- **Antisymmetry:** For any vectors X, Y, we have:

$$\omega(X, Y) = -\omega(Y, X). \tag{B.5}$$

- **Linearity:** For vectors X, Y and scalars a, b:

$$\omega(aX + bY, Z) = a\omega(X, Z) + b\omega(Y, Z).$$

B.1.2 Exterior Product

The exterior product is a bilinear and anticommutative operation, combining k- and l-forms into a $(k+l)$-form. For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$:

$$\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega. \tag{B.6}$$

Example: If $\omega = dx^1$ and $\eta = dx^2$, then:

$$dx^1 \wedge dx^2 = -dx^2 \wedge dx^1. \tag{B.7}$$

B.1.3 Exterior Derivative

The exterior derivative is an operation mapping a k-form ω to a $(k+1)$-form $d\omega$. It is defined as:

$$d\omega(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \omega(X_0, \ldots, \hat{X}_i, \ldots, X_k), \tag{B.8}$$

where $\hat{X}_i$ indicates that X_i is omitted.

Properties:

- $d^2 = 0$ (the exterior derivative applied twice is zero).
- **Product Rule:** $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$, where ω is a k-form.

B.1.4 Volume Form and the Levi-Civita Symbol

The volume form on an n-dimensional manifold is a non-degenerate n-form. In local coordinates, it is given by:

$$\text{vol} = \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \tag{B.9}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric.

The Levi-Civita symbol $\epsilon_{\mu_1\mu_2\ldots\mu_n}$ is related to the volume form by:

$$\text{vol} = \epsilon_{\mu_1\mu_2\ldots\mu_n}\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}. \tag{B.10}$$

B.2 CURVATURE TENSOR IN EXTERIOR ALGEBRA

In the connection formalism, curvature is represented as a 2-form associated with the connection operator ∇. The curvature tensor is defined as:

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{B.11}$$

where $\omega^a{}_b$ are 1-form connections.

B.2.1 Properties of the Curvature Tensor

- **Antisymmetry:** $R^a{}_b = -R_b{}^a$.
- **Bianchi Identity:** $dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0$.

B.3 DECOMPOSITION OF $\epsilon_{abcde} e^a \wedge e^b \wedge R^{cd}$

The term $\epsilon_{abcde} e^a \wedge e^b \wedge R^{cd}$ frequently appears in the description of gravity as a gauge theory. The mentioned decomposition involves splitting this term into real and imaginary parts [25]:

$$\epsilon_{abcde} e^a \wedge e^b \wedge R^{cd} = \star(e^a \wedge e^b) \wedge R^{ab} + i e^a \wedge e^b \wedge R^{ab}. \tag{B.12}$$

B.3.1 Hodge Star

The Hodge star $\star$ is an operator mapping k-forms to $(n-k)$-forms, defined by the manifold's metric. In an orthonormal basis:

$$\star(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}) = \frac{1}{(n-k)!} \epsilon^{\mu_1 \cdots \mu_k}{}_{\mu_{k+1} \cdots \mu_n} dx^{\mu_{k+1}} \wedge \cdots \wedge dx^{\mu_n}. \tag{B.13}$$

B.3.2 Physical Interpretation

The decomposition reflects the division between the "electric" and "magnetic" parts of curvature in terms of differential forms:

- $\star(e^a \wedge e^b) \wedge R^{ab}$: Related to oriented volumes.
- $i e^a \wedge e^b \wedge R^{ab}$: Related to topological complexity.

B.3.2 Final Considerations

Exterior algebra enables a compact and coordinate-independent description of differential geometry, facilitating the analysis of gauge and gravitational theories. The mentioned decomposition reflects fundamental properties of curvature in the context of modified gravity theories. It also highlights the relevance of differential forms in the context of quantum gravity, where geometric and topological aspects play a crucial role in understanding the structure of spacetime. The separation of terms involving real and imaginary components of the curvature operator suggests an interplay between the classical and quantum aspects of the theory, offering insights into how gravity could be unified with other fundamental forces. Furthermore, the use of the Hodge star operator emphasizes the importance of duality and symmetry in understanding the fundamental nature of spacetime and curvature.