UNIVERSIDADE FEDERAL DO PARANÁ
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GHENT UNIVERSITY

ANDRÉ PEDROSO KOWACS

LINKS BETWEEN REGULARITY AND INEQUALITIES ON COMPACT LIE GROUPS

CURITIBA, PR, BRASIL

2024

ANDRÉ PEDROSO KOWACS

LINKS BETWEEN REGULARITY AND INEQUALITIES ON COMPACT LIE GROUPS

Tese apresentada ao Programa de Pós-Graduação em Matemática, Setor de Ciências Exatas, Universidade Federal do Paraná, como requisito parcial à obtenção do título de Doutor em Matemática.

Orientador: Prof. Dr. Alexandre Kirilov (UFPR-Brasil)

Coorientador: Prof. Dr. Michael Ruzhansky (Ghent University-Bélgica e Queen Mary University of London-Reino Unido)

CURITIBA, PR, BRASIL

19 de Novembro de 2024

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Orientador: Alexandre Kirilov
Coorientador: Michael Ruzhansky

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40001016041P1

ATA Nº052

MINUTES OF PUBLIC DEFENSE SESSION OF DISSERTATION TO OBTAIN THE DEGREE OF PHILOSOPHY DOCTOR IN GRADUATE PROGRAM IN MATHEMATICS

On 18/11/2024 at 11:00, in 300 do Departamento de Matemática da UFPR, remoto via Plataforma Zoom, Centro Politécnico da Universidade Federal do Paraná of the Federal University of Paraná, the activities pertaining to the rite of defense of work entitled: **Links between regularity and inequalities on compact Lie groups**by ANDRÉ PEDROSO KOWACS, were undertaken. The Examining Board, designated by the Faculty of the Graduate Program of the Federal University of Paraná in GRADUATE PROGRAM IN MATHEMATICS was composed of the following Members: BART DE BRUYN (UNIVERSITEIT GENT), WAGNER AUGUSTO ALMEIDA DE MORAES (UNIVERSIDADE FEDERAL DO PARANÁ), MARIAN SLODICKA (UNIVERSITEIT GENT), IGOR AMBO FERRA (UNIVERSIDADE FEDERAL DE SÃO CARLOS), KAREL VAN BOCKSTAL (UNIVERSITEIT GENT), GABRIEL CUEVA CANDIDO SOARES DE ARAÚJO (UNIVERSIDADE DE SÃO PAULO), PAULO LEANDRO DATTORI DA SILVA (UNIVERSIDADE DE SÃO PAULO), . The Board Chair initiated the rites as defined by the Faculty of the Program and, after the remarks of the members of the examining commitee and the respective responses were addressed, the final evaluation of the examining board was read, wich was decided by APPROVAL. This result should be made official by the Faculty of the Program, through the fulfillment of all recommendation and corrections requested by the examining board within the procedural deadlines defined by the program. The granting of the title of philosophy doctor is contigent upon the fulfillment of all the requirements and terms determined int the regulations of the Graduate Program. There being no further business to transact, the Chair adjourned the session, of which, I, BART DE BRUYN, drew up these minutes, which are signed by me and the other members of the Examining Board.

CURITIBA, November 18th, 2024.

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BART DE BRUYN
President of the Examining Board

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APPROVAL MINUTE

The Examining Board is designated by the Faculty of the Graduate Program of the Federal University of Paraná in GRADUATE PROGRAM IN MATHEMATICS where invited to argue the DISSERTATION of PHILOSOPHY DOCTOR by **ANDRÉ PEDROSO KOWACS** , entitled: **Links between regularity and inequalities on compact Lie groups**, which and after assessment of the candidate and the work, the Examining Board decided for the APPROVAL in the present rite.

The granting of the title of philosophy doctor is contingent upon the fulfillment of all the requirements indicated by the Examining Board and terms determined in the regulation of the Graduate Program.

CURITIBA, November 18th, 2024.

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External Member (UNIVERSIDADE DE SÃO PAULO)

ATA Nº052

ATA DE SESSÃO PÚBLICA DE DEFESA DE DOUTORADO PARA A OBTENÇÃO DO GRAU DE DOUTOR EM MATEMÁTICA

No dia dezoito de novembro de dois mil e vinte e quatro às 11:00 horas, na sala 300 do Departamento de Matemática da UFPR, remoto via Plataforma Zoom, Centro Politécnico da Universidade Federal do Paraná, foram instaladas as atividades pertinentes ao rito de defesa de tese do doutorando **ANDRÉ PEDROSO KOWACS**, intitulada**: Links between regularity and inequalities on compact Lie groups**. A Banca Examinadora, designada pelo Colegiado do Programa de Pós-Graduação MATEMÁTICA da Universidade Federal do Paraná, foi constituída pelos seguintes Membros: BART DE BRUYN (UNIVERSITEIT GENT), WAGNER AUGUSTO ALMEIDA DE MORAES (UNIVERSIDADE FEDERAL DO PARANÁ), MARIAN SLODICKA (UNIVERSITEIT GENT), IGOR AMBO FERRA (UNIVERSIDADE FEDERAL DE SÃO CARLOS), KAREL VAN BOCKSTAL (UNIVERSITEIT GENT), GABRIEL CUEVA CANDIDO SOARES DE ARAÚJO (UNIVERSIDADE DE SÃO PAULO), PAULO LEANDRO DATTORI DA SILVA (UNIVERSIDADE DE SÃO PAULO). A presidência iniciou os ritos definidos pelo Colegiado do Programa e, após exarados os pareceres dos membros do comitê examinador e da respectiva contra argumentação, ocorreu a leitura do parecer final da banca examinadora, que decidiu pela APROVAÇÃO. Este resultado deverá ser homologado pelo Colegiado do programa, mediante o atendimento de todas as indicações e correções solicitadas pela banca dentro dos prazos regimentais definidos pelo programa. A outorga de título de doutor está condicionada ao atendimento de todos os requisitos e prazos determinados no regimento do Programa de Pós-Graduação. Nada mais havendo a tratar a presidência deu por encerrada a sessão, da qual eu, BART DE BRUYN, lavrei a presente ata, que vai assinada por mim e pelos demais membros da Comissão Examinadora.

CURITIBA, 18 de Novembro de 2024.

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BART DE BRUYN
Presidente da Banca Examinadora

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TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação MATEMÁTICA da Universidade Federal do Paraná foram convocados para realizar a arguição da tese de Doutorado de **ANDRÉ PEDROSO KOWACS** intitulada: **Links between regularity and inequalities on compact Lie groups**, que após terem inquirido o aluno e realizada a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.

A outorga do título de doutor está sujeita à homologação pelo colegiado, ao atendimento de todas as indicações e correções solicitadas pela banca e ao pleno atendimento das demandas regimentais do Programa de Pós-Graduação.

CURITIBA, 18 de Novembro de 2024.

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Presidente da Banca Examinadora

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First, I would like to express my sincere gratitude to my advisor professor Alexandre Kirilov, who believed in my potential, and helped me countless times with reviewing, writing and counseling. I also learned greatly from him, and his support and comprehension were essential for this and other projects.

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I want to thank professor Wagner Augusto Almeida de Moraes for the great support an collaboration during my thesis.

I would also like to extend my thanks to Ghent University and Universidade Federal do Paraná for granting me the opportunity to realize this double diploma.

A special thanks to all members of the examination board for accepting to review my work. Your comments and suggestions greatly contributed to improve this thesis.

Finally, I would like to thank my entire extended family and friends for the constant support, in many ways, and for believing in me during all these last four years of my PhD.

Preface

This dissertation presents the recent results from my research at the Programa de Pós-Graduação em Matemática (PPGM) at the Universidade Federal do Paraná, as well as at the Ghent Analysis and PDE Center at Ghent University. This work has been supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES). Some results included in this thesis have been submitted for publication and are available in the following references:

- Cardona, D., Kirilov, A., A. A. de Moraes, W. and Pedroso Kowacs, A. (2024). On the Sobolev boundedness of vector fields on compact Riemannian manifolds. *Preprint available at* arXiv:2404.00182

- Kowacs, A. and Ruzhansky, M. (2024). Vector valued Gårding inequality for pseudo -differential operators on compact homogeneous manifolds. *Preprint available at* arXiv:2402.18966.

- Kirilov, A., Pedroso Kowacs, A. and Augusto Almeida de Moraes, W. (2024). Global solvability and hypoellipticity for evolution operators on tori and spheres, Math. Nachr., 1-46, https://doi.org/10.1002/mana.202300506.

RESUMO

Esta tese apresenta uma coleção de resultados em grupos de Lie compactos obtidos pelo autor durante os últimos anos. Esses incluem condições necessárias e suficientes para a hipoeliticidade global e resolubilidade global de uma classe de operadores diferenciais de “evolução” de primeira ordem com coeficientes variáveis complexos, definidos num produto finito qualquer de grupos de Lie compactos. Também apresentamos condições suficientes para a obtenção de uma desigualdade de Gårding “sharp” para operadores pseudo-diferenciais agindo em funções a valores vetoriais definidas em grupos de Lie compactos, e em seções de fibrados vetoriais homogêneos.

Palavras-chave: grupos compactos, hipoeliticidade global, resolubilidade global, grupos de Lie, desigualdade de Gårding, operador pseudo-diferencial.

ABSTRACT

This dissertation presents a collection of results on compact Lie groups obtained by the author over the last few years. These results include necessary and sufficient conditions to have global hypoellipticity and global solvability of a class of first-order “evolution” differential operators with variable complex coefficients, defined on an arbitrary finite product of compact Lie groups. Additionally, we provide sufficient conditions for obtaining a “sharp” Gårding inequality for pseudo-differential operators acting on vector-valued functions, defined on compact Lie groups, and on sections of homogeneous vector bundles.

Keywords: compact groups, global hypoellipticity, global solvability, Lie groups, Gårding inequality, pseudo-differential operator.

NEDERLANDSE SAMENVATTING

Dit proefschrift presenteert een verzameling resultaten over compacte Lie-groepen, verkregen door de auteur in de afgelopen jaren. Deze resultaten omvatten noodzakelijke en voldoende voorwaarden voor globale hypo-ellipticiteit en globale oplosbaarheid van een klasse van eerste-orde "evolutie" differentiaaloperatoren met variabele complexe coëfficiënten, gedefinieerd op een willekeurig eindig product van compacte Lie-groepen. We geven ook voldoende voorwaarden voor het verkrijgen van een "scherpe" Gårding-ongelijkheid voor pseudodifferentiaaloperatoren die werken op vectorgewaardeerde functies, gedefinieerd op compacte Lie-groepen, en op secties van homogene vectorbundels.

trefwoorden: compacte groepen, globale hypo-ellipticiteit, globale oplosbaarheid, Lie-groepen, Gårding-ongelijkheid, pseudodifferentiaaloperator.

Contents

Chapter 1

Introduction

The study of compact Lie groups has garnered significant attention due to their profound mathematical structures and applications in various domains. In this context, the present work aims to contribute to the field by offering a collection of novel results obtained over the past few years. Specifically, this research focuses on two aspects: global hypoellipticity and global solvability of first-order evolution differential operators, and the establishment of sharp Gårding inequalities for pseudo-differential operators.

The investigation begins by exploring the necessary and sufficient conditions for global hypoellipticity and solvability of a class of first-order evolution differential operators with variable complex coefficients. These operators are considered on finite products of compact Lie groups, providing a broad framework for understanding their behavior in more general settings. The techniques used rely on state of the art theories developed for the study of compact Lie groups. Remarkably, the generality of the results obtained presents many challenges and requires notable changes to previously used techniques, as well as new ideas and definitions. The difficulties arise, for instance, in the unknown behavior of the eigenvalues of vector fields on compact Lie groups, in contrast with the well-known simple comportment of such eigenvalues in the case of the torus and 3-sphere. This work can simultaneously be seen a generalization of certain results in [9] and [75]. This is in part because we treat a similar but more general version of the differential operators considered in the first part of the first reference. More specifically the vector fields considered can be any left-invariant vector field on compact Lie groups, and we also consider a zero-order term. At the same time, our results extend certain results from the second reference by considering operators acting on any finite product of compact Lie groups as opposed to only two, and we also investigate the global solvability in the complex case.

In addition, this work delves into the conditions required for obtaining a sharp Gårding inequality for certain classes of pseudo-differential operators. The operators considered act on vector-valued functions defined on compact Lie groups and on sections of homogeneous vector bundles. As a direct consequence, we obtain a sharp Gårding inequality on compact homogeneous manifolds as well. Establishing a sharp Gårding inequality is pivotal for advancing the theory of pseudo-differential operators, as it provides, for instance, a critical tool for proving local solvability and well posedness of certain Cauchy problems. Due to the correspondence between higher order differential equations and differential equations involving vector-valued functions (which can be seen as a system of differential equations), our results allow for the applications of this inequality to this setting also.

Overall, this dissertation presents a comprehensive study of differential operators and inequalities in the context of compact Lie groups, offering valuable insights and advancing the field with significant theoretical contributions.

Outline of the dissertation:

This dissertation is organized as follows:

In Chapter 2, we remind the basic results and notation necessary for studying Fourier analysis on compact Lie groups.

In Chapter 3, we introduce our novel multi-index notation, definitions, and basic results for the study of Fourier analysis on a product of compact Lie groups.

In Chapter 4, we study the global hypoellipticity and global solvability of a class of constant coefficients first-order differential operators defined on any finite product of compact Lie groups.

In Chapter 5, we study a class of first order differential operators with variable real-valued coefficients on finite products of compact Lie groups. We show that we can relate the global hypoellipticity and global solvability of these operators with the corresponding properties of the constant coefficient operators obtained by taking their coefficients' averages.

In Appendix 5.A, we state and prove some auxiliary results used in the previous chapter.

In Chapter 6 we first recall the theory and main results regarding the study of the vector-valued Fourier analysis on compact Lie groups. We also extend the notion of amplitudes to the setting of pseudo-differential operators on vector-valued functions on compact Lie groups. We then present and prove our main results concerning a type of "sharp" Gårding inequality for vector-valued functions on compact Lie groups. As a corollary, we then present a similar result for compact homogeneous vector bundles.

In Chapter 7, we conclude by summarizing our main results and presenting possible applications of these.

Chapter 2

Preliminaries

In this chapter, we introduce the main theory, results, and definitions used in this dissertation. A more detailed consideration of the concepts and demonstrations of the results presented in this chapter can be found in the references [94] (chapters 6 to 10), [45] (chapters 1 and 2) and [99].

2.1 Fourier analysis on compact topological groups

2.1.1 Representation of topological groups

Let G be a topological group and V a vector space. A representation ξ of G is a homomorphism from G into the automorphism group of V. We define its dimension by $\dim \xi = \dim V$. Notice that when V is finite-dimensional, its automorphism group can be seen as a group of matrices under the usual product. If V is also a Hilbert space, and for every $g \in G$, the linear operator $\xi(g)$ is unitary, we say that the representation ξ is *unitary*.

Definition 2.1.1. Let ξ be a representation of G. A subspace $W \subset V$ is said to be ξ-invariant if $\xi(g)W \subset W$, for all $g \in G$. The representation ξ is said to be topologically irreducible if the only closed ξ-invariant subspaces of V are the trivial subspaces $\{0\}$ and V.

Notice that if $W \subset V$ is ξ-invariant, then $\xi : G \to \mathrm{Aut}(W)$ is a representation of G also. Whenever V is a topological vector space, for instance when it is finite dimensional, then we can establish the following definition.

Definition 2.1.2. A representation ξ of G is said to be strongly continuous if for every $v \in V$, the mapping from G to V given by $g \mapsto \xi(g)v$ is continuous.

Definition 2.1.3. An intertwining map between the representations $\xi \in \text{Hom}(G, \text{Aut}(V))$ and $\eta \in \text{Hom}(G, \text{Aut}(W))$ of G, is a linear map $A : V \to W$ such that

$$A\xi(g) = \eta(g)A,$$

for every $g \in G$. If there exists an invertible intertwining map between the representations ξ and η, then they are said to be equivalent, denoted by $\xi \sim \eta$. We also denote their equivalence class by $[\xi]$ or $[\eta]$. Notice that $\xi \sim \eta \implies \dim \xi = \dim \eta$.

If two representations ξ and η are both unitary and topologically irreducible, then it is possible to show that any intertwining map between them is either the zero mapping or an invertible isometry.

2.1.2 The Peter Weyl decomposition

In what follows, we assume that G is a compact group, that is, a topological group which is also a compact topological space. Denote by Rep(G) the set of all strongly continuous unitary irreducible representations of G. We define $\widehat{G}$ to be the set of all equivalence classes of elements in Rep(G). Since G is compact, every element in Rep(G) is finite dimensional. Therefore, for each $[\xi] \in \widehat{G}$, we may choose a matrix-valued representative of $[\xi]$, that is, $\xi \in [\xi]$ is such that $\xi : G \to \text{U}(m)$, for some $m \in \mathbb{N}$, where $\text{U}(m) = \{A \in \mathbb{C}^{m\times m} | A^* = A^{-1}\}$. With this in mind, we will always consider $\xi \in [\xi] \in \widehat{G}$ to be matrix-valued.

For any $[\xi] \in \widehat{G}$, we denote its dimension by $d_{[\xi]} = \dim \xi$, and we may write it as d_ξ for convenience. Therefore, for any $[\xi] \in \widehat{G}$, $1 \leq i, j \leq d_\xi$, and $x \in G$, if we denote by $\xi_{ij}(x) \doteq \xi(x)_{ij}$ the "ij-th coefficient of the matrix $\xi(x)$", we obtain the coefficient functions $\xi_{ij} : G \to \mathbb{C}$, where $1 \leq i, j \leq d_\xi$. These are not only continuous, by definition, but can be shown to be smooth also.

Next, notice that G is locally compact, therefore there exists a unique positive Borel measure μ_G, called the Haar measure of G, characterized by the following properties:

- It is invariant under group translation, in the sense that

$$\int_G f(x) d\mu_G(x) = \int_G f(yx) d\mu_G(x),$$

  for all measurable $f : G \to \mathbb{C}$ and every $y \in G$.

- It is finite and normalized, that is, $\mu_G(G) = 1$.

For convenience, we denote such measure simply by dx and write

$$\int_G f(x)dx \doteq \int_G f(x)d\mu_G(x),$$

for all measurable complex-valued functions f. From the properties above, it can be shown that this measure also satisfies

$$\int_G f(x)dx = \int_G f(xy)dx = \int_G f(x^{-1})dx,$$

for any $y \in G$ and measurable $f : G \to \mathbb{C}$.

We denote by $L^p(G)$ the usual Banach space of p-th integrable (with respect to the Haar measure) complex-valued functions on G, where $1 \leq p < \infty$ and $\|f\|_p = \left(\int_G |f(x)|^p dx\right)^{\frac{1}{p}}$. For $p = \infty$, the space $L^\infty(G)$ will denote the usual space of essentially bounded integrable functions on G, and $\|f\|_\infty = \operatorname{ess\,sup}|f|$.

Note that by Fubini's Theorem one can show that the Haar measure on the product of compact Lie groups is given by the tensor product of the Haar measures of each group.

The following well-known result highlights the importance of the representation theory of G to the study of its function spaces.

Theorem 2.1.4 (Peter-Weyl)**.** *Let G be a compact group. Then the set*

$$\left\{\sqrt{d_\xi}\xi_{ij} \in L^2(G) | \xi = (\xi_{ij})_{i,j=1}^{d_\xi},\ [\xi] \in \widehat{G}\right\}$$

is an orthonormal basis for $L^2(G)$, where from each equivalence class in $\widehat{G}$, we choose only one representative.

2.1.3 Fourier series on compact Lie groups

The results from the previous subsection motivate the definition of Fourier coefficients for functions on compact groups, as follows.

Definition 2.1.5. Let G be a compact group. If $f \in L^2(G)$ and $[\xi] \in \widehat{G}$, where $\xi = (\xi_{ij})_{i,j=1}^{d_\xi} \in$

$\text{Rep}(G)$, define the Fourier coefficient of f at ξ as the matrix

$$\widehat{f}(\xi) \doteq \int_G f(x)\xi^*(x)dx \in \mathbb{C}^{d_\xi \times d_\xi}.$$

That means

$$\widehat{f}(\xi)_{\alpha\beta} = \int_G f(x)\overline{\xi_{\beta\alpha}(x)}dx = \langle f, \xi_{\beta\alpha}\rangle_{L^2(G)},$$

for each $1 \leq \alpha, \beta \leq d_\xi$.

Note that even though the coefficient $\widehat{f}(\xi)$ depends on the choice of representative for $[\xi] \in \widehat{G}$, different representatives yield unitarily similar matrices. Also, the Peter-Weyl Theorem implies we can write any $f \in L^2(G)$ as the sum

$$f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta=1}^{d_\xi} \widehat{f}(\xi)_{\alpha\beta}\xi_{\beta\alpha}(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \text{Tr}\left(\widehat{f}(\xi)\xi(x)\right).$$

This series converges for μ_G-almost every x in G, as well as in $L^2(G)$. The group Fourier transform also preserves the L^2 norm in the following sense.

Proposition 2.1.6 (Plancherel's identity)**.** *Let G be a compact group. Then, for every $f \in L^2(G)$:*

$$||f||^2_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} d_\xi \|\widehat{f}(\xi)\|^2_{HS} = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta=1}^{d_\xi} |\widehat{f}(\xi)_{\alpha\beta}|^2,$$

where $\|A\|^2_{HS} \doteq Tr(AA^*)$ denotes the Hilbert-Schmidt norm.

It is worth noting that the formulas above hold independently of the choice of representative for each equivalence class in $\widehat{G}$, due to the cyclic property of the trace.

Proposition 2.1.7. *Let G be a compact group. Then, for every $f, g \in L^2(G)$:*

$$\langle f, g\rangle_{L^2(G)} = \left\langle \widehat{f}, \widehat{g}\right\rangle_{L^2(\widehat{G})} \doteq \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta=1}^{d_\xi} \widehat{f}(\xi)_{\alpha\beta}\overline{\widehat{g}(\xi)_{\alpha\beta}},$$

or, equivalently,

$$\langle f, g\rangle_G = \left\langle \widehat{f}, \widehat{\overline{g}}\right\rangle_{L^2(\widehat{G})} = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta=1}^{d_\xi} \widehat{f}(\xi)_{\alpha\beta}\widehat{g}(\overline{\xi})_{\alpha\beta}. \tag{2.1}$$

Proof. Indeed, using the orthogonality of the representations $\xi \in [\xi] \in \widehat{G}$, and the fact that the

Fourier series converges in $L^2(G)$, we have that

$$\int_G f(x)\overline{g(x)}dx = \int_G \left(\sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta=1}^{d_\xi} \widehat{f}(\xi)_{\alpha\beta}\xi_{\beta\alpha}(x) \right) \left(\sum_{[\eta]\in\widehat{G}} d_\eta \sum_{\gamma,\kappa=1}^{d_\eta} \overline{\widehat{g}(\xi)_{\gamma\kappa}\eta_{\kappa\gamma}}(x) \right) dx$$
$$= \sum_{[\xi]\in\widehat{G}} \sum_{[\eta]\in\widehat{G}} d_\xi d_\eta \sum_{\alpha,\beta=1}^{d_\xi} \sum_{\gamma,\kappa=1}^{d_\eta} \widehat{f}(\xi)_{\alpha\beta}\overline{\widehat{g}(\xi)_{\gamma\kappa}} \int_G \xi_{\beta\alpha}(x)\overline{\eta_{\kappa\gamma}(x)}dx$$
$$= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta=1}^{d_\xi} \widehat{f}(\xi)_{\alpha\beta}\overline{\widehat{g}(\xi)_{\alpha\beta}}.$$

□

Remark 2.1.8. Note that formula 2.1 makes sense since $[\xi] \in \widehat{G}$ if and only if $[\overline{\xi}] \in \widehat{G}$, where $\overline{\xi}(x) \doteq \overline{\xi(x)}$, for every $x \in G$. Moreover, the equivalence between both expressions above can be seen from the fact that

$$\widehat{\overline{g}}(\xi) = \overline{\widehat{g}(\overline{\xi})},$$

for every $g \in L^2(G)$ and $[\xi] \in \widehat{G}$.

2.2 Fourier analysis on compact Lie groups

We now assume G is a compact Lie group. We will assume the reader is familiar with the theory of smooth manifolds and Lie groups, for which we cite [82, 103]. Let $\mathfrak{g} \equiv \text{Lie}(G)$ denote the space of all left-invariant vector fields on G, which can be identified with the tangent space of G at the identity $e_G \in G$. This real vector space (with dimension equal to $\dim G$) also has a Lie algebra structure (through the Lie bracket for vector fields), said to be the group's Lie algebra. Since G is compact or since they are left-invariant, each $X \in \mathfrak{g}$ is complete. This means that its flow starting at any $x \in G$, given by $(x,t) \mapsto \phi_t^X(x)$, is defined for every $t \in \mathbb{R}$. This way, we may define the exponential map

$$\exp : \mathfrak{g} \to G$$

by $\exp(X) = \phi_1^X(e_G)$. Equivalently, as any compact Lie group is isomorphic to a subgroup of the matrix group $\text{U}(m)$ (see [23], Chapter III, Theorem 4.1), a compact group's Lie algebra may also be identified with a matrix subgroup and the exponential mapping coincides with the usual matrix exponential. That is the mapping given by the power series

$$\exp(X) \doteq \sum_{k=0}^{\infty} \frac{1}{k!} X^k,$$

for any matrix $X \in \mathbb{C}^{n\times n}$. As vector fields, each $X \in \mathfrak{g}$ acts as a first order left-invariant differential operator on G. By definition of the exponential mapping, this action can be expressed by

$$Xf(x) = \frac{d}{dt}\bigg|_{t=0} f(x\exp(tX)),$$

for any $f \in C^\infty(G)$ and $x \in G$. By taking the direct sum of the tensor products of multiple copies of $\mathfrak{g}$, modulo an ideal, we can embed $\mathfrak{g}$ into a unital associative algebra $\mathcal{U}(\mathfrak{g})$, called the universal enveloping algebra of $\mathfrak{g}$. Its elements can thus be viewed as finite order left-invariant differential operators on G. More precisely, define

$$\mathcal{T} \doteq \bigoplus_{m=0}^{\infty} \otimes^m \mathfrak{g}$$

the direct sum of all finite tensor products of $\mathfrak{g}$ with itself, i.e. $\otimes^m \mathfrak{g}$ denotes the m-fold product $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$. Its elements are given by linear combinations of elements of the form

$$\lambda_0 \mathbf{1} + \sum_{j=1}^{J} \sum_{k=1}^{K_j} \lambda_{jk} X_{k_{j1}} \otimes \cdots \otimes X_{k_{jj}},$$

where $\mathbf{1}$ is the formal unit of $\mathcal{T}$, $\lambda_0, \lambda_{jk} \in \mathbb{R}$, $X_{k_{jm}} \in \mathfrak{g}$ and $J, K_j \in \mathbb{N}_0$. The product in $\mathcal{T}$ is defined by the tensor product for the basis elements, that is,

$$(X_1 \otimes \cdots \otimes X_n)(Y_1 \otimes \cdots \otimes Y_m) \doteq X_1 \otimes \cdots \otimes X_n \otimes Y_1 \otimes \cdots \otimes Y_m,$$

for $X_j, Y_k \in \mathfrak{g}$, $1 \le j \le n$, $1 \le k \le m$, and extended linearly to $\mathcal{T}$. Consider the two sided ideal $\mathcal{J} \subset \mathcal{T}$ spanned by the set

$$\{X \otimes Y - Y \otimes X - [X,Y] | X, Y \in \mathfrak{g}\}.$$

The universal enveloping algebra of $\mathfrak{g}$ is then defined as the quotient algebra

$$\mathcal{U}(\mathfrak{g}) \doteq \mathcal{T}/\mathcal{J}.$$

The Killing form of a real Lie algebra $\mathfrak{g}$ is the bilinear mapping $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$B(X, Y) \doteq \text{Tr}(\text{ad}(X)\text{ad}(Y)),$$

where $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$, is the mapping given by $\text{ad}(X)(Y) = [X, Y]$, for any $X, Y \in \mathfrak{g}$.

It can be proven that the Killing form of the Lie algebra of a compact Lie group is negative semi-definite. Using this fact, through an argument similar to the Gram-Schmidt process for constructing orthonormal basis on vector spaces, we can construct a basis $\{X_j\}_{j=1}^n$ of $\mathfrak{g}$ such that

$$B(X_i, X_j) = -\delta_{ij}, \tag{2.2}$$

where δ_{ij} is the Kronecker delta. The Casimir element $\mathcal{L}_G \in \mathcal{U}(\mathfrak{g})$ of G, also known as the positive Laplacian of G, or just the "Laplacian", is defined by

$$\mathcal{L}_G = -\sum_{j=1}^{n} X_j \otimes X_j.$$

The positive Laplacian can be viewed as a second order, partial differential operator on G, which is also positive definite and bi-invariant. It is also independent on the choice of basis $\{X_j\}_{j=1}^n$ satisfying (2.2). One can prove that for every $[\xi] \in \widehat{G}$, its coefficient functions $\xi_{ij} : G \to \mathbb{C}$ are all eigenfunctions of $\mathcal{L}_G$, corresponding to the same eigenvalue. In fact, we have the following result.

Theorem 2.2.1. *Let G be a compact Lie Group and $\mathcal{L}_G$ its positive Laplacian. Then for every $[\xi] \in \widehat{G}$, there exist real numbers $\nu_{[\xi]} \geq 0$ such that*

$$\mathcal{L}_G \xi_{\alpha\beta} = \nu_{[\xi]} \xi_{\alpha\beta},$$

for every $1 \leq \alpha, \beta \leq d_\xi$.

Now let G be a n dimensional compact Lie group, and $\mathcal{B} = \{X_j\}_{j=1}^n$ a basis of its Lie algebra. For any multi-index $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$, we denote by ∂^γ a differential operator given by the composition of exactly $|\gamma| = \gamma_1 + \cdots + \gamma_n$ elements of $\mathcal{B}$, such that X_j appears exactly γ_j times, for $1 \leq j \leq n$. Note here that composing the same vector fields in different orders may yield different differential operators, but this will not be relevant in this work. We will denote the set of all such operators by $\text{Diff}(G, \mathcal{B})$.

Let $C^\infty(G)$ be the set of all smooth complex-valued functions on G. We endow it with the Fréchet space topology given by the countable family of seminorms

$$\{p_{\partial^\gamma} | \partial^\gamma \in \text{Diff}(G, \mathcal{B}), |\gamma| \leq m \in \mathbb{N}_0\},$$

where

$$p_{\partial^\gamma}(f) = \max_{x \in G} |\partial^\gamma f(x)|,$$

for every $f \in C^\infty(G)$.

Definition 2.2.2. For G a compact Lie group, let $\mathcal{D}'(G)$ be the space of distributions on G, that is, the space of all continuous linear functionals on $C^\infty(G)$, with the usual notion of convergence, that is, $u_j \to u$ in $\mathcal{D}'(G)$ if and only if $u_j \to u$ point-wise.

Throughout this work we will denote the duality between $\mathcal{D}'(G)$ and $C^\infty(G)$ by

$$\langle u, f \rangle_G \doteq u(f),$$

for $u \in \mathcal{D}'(G)$ and f in $C^\infty(G)$. By definition, for each $u \in \mathcal{D}'(G)$ there exist $C, N > 0$ such that

$$|\langle u, f \rangle_G| \leq C \max_{x \in G} \sum_{|\gamma| \leq N} |\partial^\gamma f(x)|,$$

for every $f \in C^\infty(G)$.

Notice that as G is compact, $C^\infty(G) \subset L^2(G)$. We can also extend the definitions and results from the previous subsection to the space $\mathcal{D}'(G)$ as follows.

Definition 2.2.3. Let G be a compact Lie group. For $u \in \mathcal{D}'(G)$, and $[\xi] \in \widehat{G}$, define the Fourier coefficient of u at ξ as the matrix

$$\widehat{u}(\xi) \doteq \langle u, \xi^* \rangle \in \mathbb{C}^{d_\xi \times d_\xi}.$$

More precisely, $\widehat{u}(\xi)$ is the $d_\xi \times d_\xi$-matrix with coefficients

$$\widehat{u}(\xi)_{\alpha\beta} \doteq \langle u, \overline{\xi_{\beta\alpha}} \rangle_G \in \mathbb{C},$$

where $1 \leq \alpha, \beta \leq d_\xi$.

Similar to the Peter-Weyl theorem, one can prove that for every $u \in \mathcal{D}'(G)$ the series

$$\sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta=1}^{d_\xi} \widehat{u}(\xi)_{\alpha\beta}\xi_{\beta\alpha}$$

converges to $u \in \mathcal{D}'(G)$ in the sense of distributions. It follows then that for any $u \in \mathcal{D}'(G)$ and $f \in C^\infty(G)$:

$$\langle u, f\rangle_G = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta=1}^{d_\xi} \widehat{u}(\xi)_{\alpha\beta}\widehat{f}(\overline{\xi})_{\alpha\beta},$$

similarly to formula (2.1).

If $u \in L^p(G)$, $1 \leq p \leq \infty$, there is a natural way to identify u with a distribution (still denoted by u) through the formula

$$\langle u, f\rangle_G = \int_G u(x)f(x)dx, \tag{2.3}$$

for every $f \in C^\infty(G)$. Also, for any differential operator $\partial^\gamma \in \text{Diff}(G, \mathcal{B})$, we define $\partial^\gamma u \in \mathcal{D}'(G)$ by

$$\langle \partial^\gamma u, f\rangle_G \doteq (-1)^{|\gamma|}\langle u, \partial^\gamma f\rangle_G,$$

for every $f \in C^\infty(G)$. This extends a similar identity which holds for smooth functions and coincides with the notion of weak derivatives for L^p functions via the duality (2.3).

Definition 2.2.4. Let G be a compact Lie Group and $\mathcal{L}_G$ its positive Laplacian. Denote by

$$\langle\xi\rangle \doteq (1 + \nu_{[\xi]})^{1/2}$$

the common eigenvalue of the linear operator $(\text{Id}+\mathcal{L}_G)^{1/2}$ corresponding to the eigenfunctions $\xi_{\alpha\beta}$ with $1 \leq \alpha, \beta \leq d_\xi$, $[\xi] \in \widehat{G}$.

From the discussion above, we know these eigenvalues are strictly positive. They also satisfy the following inequality:

Proposition 2.2.5. *Let G be a compact Lie group. There exists $C > 0$ such that*

$$\nu_{[\xi]} \leq \langle\xi\rangle^2 \leq C\nu_{[\xi]},$$

for all non trivial $[\xi] \in \widehat{G}$.

Proof. Indeed, let λ be the smallest non-zero eigenvalue of $\mathcal{L}_G$. Then if $[\xi]$ is not trivial

$$1+\nu_{[\xi]} \leq \left(\frac{1}{\lambda}+1\right)\nu_{[\xi]},$$

therefore $\nu_{[\xi]} \leq \langle\xi\rangle^2 \leq \left(\frac{1}{\lambda}+1\right)\nu_{[\xi]}$, for all non-trivial $[\xi] \in \widehat{G}$. □

Also, by Weyl's eigenvalue counting formula for the Laplacian (see [104]), one can also prove the following.

Proposition 2.2.6. *Let G be a compact Lie group. There exists $K > 0$ such that*

$$d_\xi \leq K\langle\xi\rangle^{\frac{\dim G}{2}},$$

for all $[\xi] \in \widehat{G}$.

Remark 2.2.7. Weyl's eigenvalue counting formula also implies that given $R > 0$, there exist only finitely many $[\xi] \in \widehat{G}$ such that $\langle\xi\rangle < R$. This fact will be relevant later in this work.

As in the Euclidean setting, it is well known that the asymptotic behaviour of the Fourier coefficients characterizes smoothness as shown in the following theorem.

Theorem 2.2.8. *Let G be a compact Lie Group. The following are equivalent:*

i) $f \in C^\infty(G)$;

ii) For every $N > 0$, there exists $M_N > 0$ such that

$$|\widehat{f}(\xi)_{\alpha\beta}| \leq M_N\langle\xi\rangle^{-N},$$

for all $[\xi] \in \widehat{G}$ and $1 \leq \alpha, \beta \leq d_\xi$.

The following statements are equivalent:

i) $u \in \mathcal{D}'(G)$;

ii) There exist $N > 0, M > 0$ such that

$$|\widehat{u}(\xi)_{\alpha\beta}| \leq M\langle\xi\rangle^{N},$$

for all $[\xi] \in \widehat{G}$ and $1 \leq \alpha, \beta \leq d_\xi$.

In fact, the Sobolev space $H^s(G)$, where $s \in \mathbb{R}$, is defined as the Banach space given by the set of all $u \in \mathcal{D}'(G)$ such that

$$\|u\|^2_{H^s(G)} \doteq \sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{2s} \|\widehat{u}(\xi)\|^2_{HS} = \sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{2s} \sum_{\alpha,\beta=1}^{d_\xi} |\widehat{u}(\xi)_{\alpha\beta}|^2 < \infty,$$

with the Sobolev norm $\|\cdot\|_{H^s(G)}$.

Proposition 2.2.9. *Let G be a compact Lie group. Then*

$$\sum_{[\xi]\in\widehat{G}} d_\xi^2 \langle\xi\rangle^{-2t} < \infty \iff t > \frac{\dim G}{2}.$$

Proof. Indeed, notice that for the delta distribution centered at the group indentity δ, we have $\widehat{\delta}(\xi) = \mathrm{Id}_{d_\xi}$ is the $d_\xi \times d_\xi$ identity matrix, for every $[\xi] \in \widehat{G}$. Therefore

$$\sum_{[\xi]\in\widehat{G}} d_\xi^2 \langle\xi\rangle^{-2t} = \sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{-2t} \|\widehat{\delta}(\xi)\|^2_{HS} = \|\delta\|^2_{H^{-t}(G)}.$$

Through a localization argument, this norm is finite if and only if $t > \dim G/2$. □

Theorem 2.2.8 then implies that

$$\bigcap_{s\in\mathbb{R}} H^s(G) = C^\infty(G), \qquad \bigcup_{s\in\mathbb{R}} H^s(G) = \mathcal{D}'(G).$$

We now present the concept of global symbol for pseudo-differential operators on compact Lie groups, introduced by Ruzhansky and Turunen in [94], as follows.

Definition 2.2.10. Let $A : C^\infty(G) \to C^\infty(G)$ be a continuous linear operator. Its symbol is defined as the mapping given by

$$\sigma_A(x,\xi) \doteq \xi(x)^*(A\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi},$$

for every $x \in G$, $[\xi] \in \widehat{G}$, where $(A\xi)(x)_{\alpha\beta} \doteq (A\xi_{\alpha\beta})(x)$, for all $1 \le \alpha, \beta \le d_\xi$.

We can also obtain the symbol of a continuous linear operator as follows. In the following, we denote by $C^\infty(G)\,\widehat{\otimes}\,\mathcal{D}'(G)$ the complete locally convex tensor product of the nuclear spaces $C^\infty(G)$ and $\mathcal{D}'(G)$.

Theorem 2.2.11 (Schwartz kernel)**.** *Let $A : C^\infty(G) \to C^\infty(G)$ be a continuous linear operator. Then there exists a unique mapping $K \in C^\infty(G)\,\widehat{\otimes}\mathcal{D}'(G)$ such that*

$$\langle f, A\phi\rangle = \langle K, f \otimes \phi\rangle.$$

Proposition 2.2.12. *Let $A : C^\infty(G) \to C^\infty(G)$ be a continuous linear operator. Denote by K_A denote its Schwartz kernel, and $R_A \doteq K(x, y^{-1}x)$ its right convolution kernel. Then*

$$\sigma_A(x,\xi) = \widehat{R_A(x,\cdot)}(\xi),$$

for every $x \in G$, $[\xi] \in \widehat{G}$.

Theorem 2.2.13. *Let $A : C^\infty(G) \to C^\infty(G)$ be a continuous linear operator. Then*

$$Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathit{Tr}[\xi(x)\sigma_A(x,\xi)\widehat{f}(\xi)] = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta,\gamma=1}^{d_\xi} \xi_{\beta\gamma}(x)\sigma_A(x,\xi)_{\gamma\alpha}\widehat{f}(\xi)_{\alpha\beta} \tag{2.4}$$

for every $f \in C^\infty(G)$ and $x \in G$.

Ruzhansky, Turunen and Wirth would also define the global symbol classes $\mathscr{S}^m_{\rho,\delta}(G)$ later in [99]. These classes extend the usual Hörmander symbol classes $S^m_{\rho,\delta}(G)$ in the sense that they define the same class of pseudo-differential operators in the range of ρ and δ on which $S^m_{\rho,\delta}(G)$ are well defined, that is, $0 \le \delta < \rho \le 1$, and $\rho \ge 1-\delta$. We shall adopt the alternative notation $S^m_{\rho,\delta}(G \times \widehat{G}) \equiv \mathscr{S}^m_{\rho,\delta}(G)$. More precisely, these symbol classes are defined as follows.

Definition 2.2.14. Let $q \in C^\infty(G)$. We define the difference operator Δ_q by $\Delta_q\widehat{f}(\xi) = \widehat{qf}(\xi)$, for every $f \in \mathcal{D}'(G)$. We say Δ_q is a difference operator of order $k \in \mathbb{N}$ if q has a zero of order k at the group identity e_G. A family of order 1 difference operators is said to be admissible if the gradients at e_G of the corresponding functions q span the tangent space of G at e_G. Furthermore, such collection is said to be strongly admissible if e_G is the only common zero of the functions q as well.

Consider a fixed ordered family of order 1 difference operators $\{\Delta_{q_1}, \ldots, \Delta_{q_k}\}$. For every multi-index $\alpha \in \mathbb{N}_0^k$ we denote by Δ^α the difference operator $\Delta_{q_1}^{\alpha_1} \ldots \Delta_{q_k}^{\alpha_k}$ of order $|\alpha|$. We also denote $q_\alpha = q_1^{\alpha_1} \ldots q_k^{\alpha_k}$. This way, $\Delta^\alpha = \Delta_{q_\alpha}$ as well.

The following result provides an analog of the "Leibniz's rule" for difference operators, its proof can be found in [96].

Proposition 2.2.15 (Leibniz's formula for Difference Operators)**.** *For any multi-index α there exist constants $C_{\lambda,\mu} \geq 0$ such that*

$$\Delta_\xi^\alpha[\widehat{f}(\xi)\widehat{g}(\xi)] = \sum_{|\mu|,|\lambda|\leq|\alpha|\leq|\lambda+\mu|} C_{\lambda,\mu}(\Delta_\xi^\lambda \widehat{f}(\xi))(\Delta_\xi^\mu \widehat{g}(\xi)),$$

for any $f, g \in \mathcal{D}'(G)$, and all $[\xi] \in \widehat{G}$.

Definition 2.2.16 (Symbol classes $S^m_{\rho,\delta}(G \times \widehat{G})$)**.** Let $\{\Delta_{q_1}, \ldots \Delta_{q_k}\}$ be a strongly admissible ordered family of difference operators, and $\{X_1, \ldots, X_n\}$ an ordered basis for the Lie algebra of G. We say that $\sigma : G\times\widehat{G} \to \bigcup_{[\xi]\in\widehat{G}} \mathbb{C}^{d_\xi\times d_\xi}$ is in $S^m_{\rho,\delta}(G\times\widehat{G})$, where $m \in \mathbb{R}$, $0 \leq \delta, \rho \leq 1$, if $\sigma(x,\xi) = \widehat{\kappa(x,\cdot)}(\xi)$ for some $\kappa \in C^\infty(G)\,\widehat{\otimes}\mathcal{D}'(G)$ and for every multi-index $\alpha \in \mathbb{N}_0^k$, $\beta \in \mathbb{N}_0^n$, there exists a constant $C_{\alpha\beta} > 0$ such that

$$\|\Delta^\alpha \partial^\beta \sigma(x,\xi)\|_{op} \leq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|},$$

for every $(x, [\xi]) \in G \times \widehat{G}$.

Remark 2.2.17. It can be shown that the previous definitions does not depend on the choice of strongly admissible ordered family of difference operators, nor on the choice of ordered basis for the Lie algebra of G.

Definition 2.2.18. For $\sigma_A \in S^m_{\rho,\delta}(G \times \widehat{G})$ we define by $A = \mathrm{Op}(\sigma_A)$ the pseudo-differential operator given by formula (2.4). We denote the set of all such operators by $\Psi^m_{\rho,\delta}(G \times \widehat{G})$.

We also denote the set of smoothing symbols and smoothing operators by

$$S^{-\infty}(G \times \widehat{G}) = \bigcap_{m\in\mathbb{R}} S^m_{1,0}(G \times \widehat{G}), \qquad \Psi^{-\infty}(G \times \widehat{G}) = \bigcap_{m\in\mathbb{R}} \Psi^m_{1,0}(G \times \widehat{G}),$$

respectively.

Remark 2.2.19. It can be shown that a pseudo-differential operator of order m is a bounded operator from $H^s(G)$ into $H^{s-m}(G)$. Consequently, for a pseudo-differential $A \in \Psi^{-\infty}(G \times \widehat{G})$, and any $u \in \mathcal{D}'(G)$, $Au \in C^\infty(G)$, hence the name "smoothing".

Remark 2.2.20. If $A : C^\infty(G) \to C^\infty(G)$ is a continuous left-invariant linear operator, that is,

$$(A \circ \pi_L(y)f)(x) = (\pi_L(y) \circ A)f(x),$$

where $\pi_L(y)f(x) = f(y^{-1}x)$, for every $f \in C^\infty(G)$ and $x, y \in G$, then σ_A is independent of $x \in G$. Theorem 2.2.13 then implies that

$$\widehat{Af}(\xi) = \sigma_A(\xi)\widehat{f}(\xi),$$

for every $\xi \in \text{Rep}(G)$ and $f \in C^\infty(G)$. By duality, this remains true for $f \in \mathcal{D}'(G)$.

Viewed as a left-invariant first order partial differential operator, any left invariant vector field X on a compact Lie group G is pseudo differential operator of order 1. In fact, its symbol σ_X belongs to the symbol class $S^1_{1,0}(G \times \widehat{G})$. Therefore there exists a constant $C > 0$ such that

$$\|\sigma_X(\xi)\|_{op} \leq C\langle\xi\rangle, \tag{2.5}$$

for every $[\xi] \in \widehat{G}$.

In this thesis, we shall make use of the following theorem, whose proof can be found in [44].

Theorem 2.2.21 (Taylor Expansion in Compact Lie Groups). *Let G be a compact Lie group, $d = \dim G$. Let also $\{\Delta_{q_1}, \ldots, \Delta_{q_d}\}$ be a strongly admissible collection of difference operators. There exists a basis of left-invariant vector fields $\{X_1, \ldots, X_d\}$ such that*

$$X_j q_k(x^{-1})|_{x=e_G} = \delta_{jk},$$

for all $1 \leq j, k \leq d$. Moreover, any $f \in C^\infty(G)$, can be written as

$$f(xy) = f(x) + \sum_{1 \leq |\alpha| < N} \frac{1}{\alpha!}\partial^\alpha f(x) q_\alpha(y^{-1}) + R^f_{x,N}(y),$$

for any $x, y \in G$, $N \in \mathbb{N}$, where the last term is referred to as the Taylor remainder of order N and satisfies

$$|R^f_{x,N}(y)| \lesssim |y|^N \max_{|\alpha| \leq N} \|\partial^\alpha f\|_{L^\infty(G)}.$$

Here $|y|$ denotes the distance from the group identity e_G to y, given by any Riemmanian metric defined on G.

Part I

Global hypoellipticity and solvability of differential operators

Chapter 3

Fourier analysis on the product of compact Lie groups

3.1 Introduction

In this chapter, we propose to study the regularity of solution and solvability of a class of first-order differential operators on a product of compact $n \in \mathbb{N}$ compact Lie groups. More precisely, denoting by $\mathcal{D}'(G)$ the space of distributions on $G = G_1 \times \cdots \times G_n$, where each G_j is a compact Lie group, and by $L : \mathcal{D}'(G) \to \mathcal{D}'(G)$ a first-order differential operator, we are interested in establishing conditions that ensure that u is smooth whenever Lu is smooth, a property known as global hypoellipticity. Regarding the global solvability, we want to identify under what conditions it is possible to guarantee that the equation $Lu = f \in C^\infty(G)$ admits a smooth solution.

The study of compact Lie groups has significant relevance in the context of mathematics and physics. For instance, the compact Lie group $SO(3)$ can be identified with the group of symmetries of the 2-dimensional sphere, and therefore to rotations on $\mathbb{R}^3$. Similarly, there is a connection between the compact Lie group $\mathrm{Sp}(3)$ and the quaternions which are also related to rotations in $\mathbb{R}^3$. One possible application of compact Lie groups is to the study of the Landau-Lifshitz equation, which describes the magnetization of a ferromagnetic materials. In [101] the author introduces an alternate formulation of the Landau-Lifshitz equation on the Lie group $SO(3)$, and develops a numerical method to solve these equations. The advantage of basing the numerical method on the $SO(3)$ interpretation is, that it allows us to preserve quadratic first integrals of the differential equation (which define the geometry of this problem).

Extensive research has been dedicated to exploring these global properties for operators defined on the torus, as well as on other compact Lie groups, as in [1, 41], and also on arbitrary compact manifolds, as in [2, 18, 34, 37, 38].

The specific scenario where the operator L is a vector field defined on a torus deserves special attention. As conjectured by S. Greenfield and N. Wallach, see [46], if a smooth closed manifold M admits a globally hypoelliptic real vector field L, then M is diffeomorphic to a torus, and L can be conjugated to a constant vector field that satisfies a Diophantine condition. Consequently, the investigation of global hypoellipticity for vector fields on closed manifolds primarily focuses on the tori. Some references in this subject include [29, 43, 55, 57, 65, 89]. It is worth noting that the literature contains numerous other references on this topic beyond the ones mentioned here.

Most of the studies that deal with the question of global hypoellipticity and global solvability on compact Lie groups make use of Fourier analysis, which allows one to reduce partial differential equations into ordinary differential equations, or even algebraic equations. This is possible through the Fourier analysis and quantization of pseudo-differential operators on compact Lie groups introduced by Ruzhansky and Turunen in [94]. This technique is used, for example, in [1], [28], [74], [75], [76], [77], [78], [79] and [98]. In the particular case where the compact Lie group is a torus, Ruzhasnky and Turunen's theory coincides with the usual Fourier analysis of complex-valued periodic functions, and so the same technique dates much further back, and can be found in a large selection of papers, see for instance [5, 8, 9, 10, 11, 12, 34, 36, 50, 55, 56, 57, 64, 89]. In this thesis, we adapt these techniques, using state of the art theories, to the study of global hypoellipticity and global solvability of first-order differential operators on finite products of compact Lie groups.

The main difficulties in studying this case, compared to the case of the torus, is that in general we have no information about the rate of growth of the eigenvalues of an arbitrary left-invariant vector field on a compact Lie group. There is also the problem that there is a lack of "dilation" properties corresponding to the eigenvalues of such vector fields. In the n-torus on the other hand, for every $\xi_j \in \mathbb{Z}$, for every $k \in \mathbb{Z}$ the complex numbers $ik\xi_j$ correspond to eigenvalues of the vector field ∂_{x_j}. There is also the fact that the Fourier coefficients are matrix-valued, and therefore required a multi-index notation to address specific entries of these matrices.

In this section, we introduce the theory and notation, developed by the author, used for considering the Fourier analysis on the finite product of compact Lie groups. This theory and notation will then be used in the following sections for the study of global properties of first-order differential operators.

Let $n \in \mathbb{N}$ and consider $G_1, \dots, G_n$ compact Lie groups. Their product $G = G_1 \times \cdots \times G_n$ is also a compact Lie group, so we can apply definitions and results of Chapter 2 to it. Moreover, it can be shown that given any $[\xi] \in \widehat{G}$, there exist $[\xi^1] \in \widehat{G}_1, \dots, [\xi^n] \in \widehat{G}_n$ such that

$$[\xi] = [\xi^1 \otimes \cdots \otimes \xi^n],$$

where $\otimes$ denotes the external tensor product of representations. As seen in [23, 74], the external tensor product of strongly continuous irreducible unitary representations is also a strongly continuous irreducible unitary representation, whose dimension is equal to the product of the dimensions of each of its factors. Not only that, any two such representations are equivalent if and only if each of its corresponding factors are pairwise equivalent. Therefore, we may identify $\widehat{G}$ with $\widehat{G}_1 \times \cdots \times \widehat{G}_n$. Inspired by the works of Kirilov, Almeida and Ruzhansky in [74, 75], we define the partial Fourier transforms on the product group G as follows.

Definition 3.1.1. Let $n \in \mathbb{N}$, $n \geq 2$. Consider $G = G_1 \times \cdots \times G_n$ a product of n compact Lie groups. Given $f \in L^2(G)$, define its partial Fourier transform at $\xi^n_{\alpha_n\beta_n}$ by

$$\widehat{f}(x_1, \dots, x_{n-1}, \xi^n)_{\alpha_n\beta_n} = \int_{G_n} f(x_1, x_2, \dots, x_n)\overline{\xi^n_{\beta_n\alpha_n}}(x_n)dx_n,$$

where $[\xi^n] \in \widehat{G}_n$ and $1 \leq \alpha_n, \beta_n \leq d_{\xi^n}$. Similarly, define its partial Fourier transform at $\xi^{n-1}_{\alpha_{n-1}\beta_{n-1}} \otimes \xi^n_{\alpha_n\beta_n}$ by

$$\begin{aligned}&\widehat{f}(x_1, \dots, x_{n-2}, \xi^{n-1}, \xi^n)_{\alpha\beta} = \\ &= \int_{G_{n-1}} \int_{G_n} f(x_1, x_2, \dots, x_n)\overline{\xi^{n-1}_{\beta_{n-1}\alpha_{n-1}}}(x_{n-1})\overline{\xi^n_{\beta_n\alpha_n}}(x_n)dx_n d_{x_{n-1}},\end{aligned}$$

where $[\xi^j] \in \widehat{G_j}$, $\alpha = (\alpha_{n-1}, \alpha_n)$, $\beta = (\beta_{n-1}, \beta_n)$ and $1 \leq \alpha_j, \beta_j \leq d_{\xi^j}$, for $j = n-1, n$. Proceeding in a similar manner, we define up to the "total" partial Fourier Transform of f at

$\xi^1_{\alpha_1\beta_1} \otimes \cdots \otimes \xi^n_{\alpha_n\beta_n}$ by

$$\widehat{f}(\xi^1, \ldots, \xi^n)_{\alpha\beta} = \int_{G_1} \cdots \int_{G_n} f(x_1, \ldots, x_n)\overline{\xi^1_{\beta_1\alpha_1}}(x_1) \ldots \overline{\xi^n_{\beta_n\alpha_n}}(x_n)\, dx_n \ldots dx_1,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $[\xi^j] \in \widehat{G_j}$, $1 \leq \alpha_j, \beta_j \leq d_{\xi^j}$, for $j = 1, \ldots, n$.

Notice that $\widehat{f}(\xi^1, \ldots, \xi^n)$ may be viewed as a $2n$-dimensional array with complex entries given by $\widehat{f}(\xi^1, \ldots, \xi^n)_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_n\beta_n} = \widehat{f}(\xi^1, \ldots, \xi^n)_{\alpha\beta}$.

If $f \in L^2(G)$, it then follows from the Peter-Weyl Theorem and the previous definitions that

$$f(x_1, \ldots, x_n) =$$
$$= \sum_{[\xi^1]\in\widehat{G_1}} \cdots \sum_{[\xi^n]\in\widehat{G_n}} d_{\xi^1} \ldots d_{\xi^n} \sum_{\alpha_1,\beta_1=1}^{d_{\xi 1}} \cdots \sum_{\alpha_n,\beta_n=1}^{d_{\xi n}} \widehat{f}(\xi_1, \ldots, \xi^n)_{\alpha\beta}\xi^1_{\beta_1\alpha_1}(x_1) \ldots \xi^n_{\beta_n\alpha_n}(x_n), \tag{3.1}$$

where convergence holds almost everywhere as well as in $L^2(G)$.

To avoid cumbersome notation such as the one above, we establish the following definitions.

Definition 3.1.2. Let $G = G_1 \times \cdots \times G_n$ be a product of compact Lie groups. For $[\xi] = [\xi^1 \otimes \cdots \otimes \xi^n] \in \widehat{G}$, we define:

$$J_\xi = \{\gamma \in \mathbb{N}^n | 1 \leq \gamma_j \leq d_{\xi^j},\ 1 \leq j \leq n\}.$$

Also, for $\alpha, \beta \in J_\xi$, let

$$\xi_{\alpha\beta}(x) \doteq \xi^1_{\alpha_1\beta_1}(x_1) \ldots \xi^n_{\alpha_n\beta_n}(x_n),$$

for all $x = (x_1, \ldots, x_n) \in G$.

Since $d_\xi = d_{\xi^1} \ldots d_{\xi^n}$, for every $[\xi] \in \widehat{G}$, equality (3.1) may be rewritten as:

$$f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \widehat{f}(\xi)_{\alpha\beta}\, \xi_{\beta\alpha}(x),$$

for $x=(x_1,\ldots,x_n)$ and $\xi=\xi^1\otimes\cdots\otimes\xi^n$. Similarly, for any $1\leq j<n$ the formula

$$f(x_1,\ldots,x_n)=$$
$$=\sum_{[\xi^j]\in\widehat{G_j}}\cdots\sum_{[\xi^n]\in\widehat{G_n}} d_{\xi^j}\ldots d_{\xi^n}\sum_{\alpha_j,\beta_j=1}^{d_{\xi^j}}\cdots\sum_{\alpha_n,\beta_n=1}^{d_{\xi^n}}\widehat{f}(x_1,\ldots,x_{j-1},\xi^j,\ldots,\xi^n)_{\alpha\beta}$$
$$\times\,\xi^j_{\beta_j\alpha_j}(x_j)\ldots\xi^n_{\beta_n\alpha_n}(x_n)$$

can be rewritten as

$$f(x)=\sum_{[\xi]\in(\widehat{G_j\times\cdots\times G_n})} d_\xi\sum_{\alpha,\beta\in J_\xi}\widehat{f}(x_1,\ldots,x_{j-1},\xi)_{\alpha\beta}\xi_{\beta\alpha}(x_j,\ldots,x_n),$$

where $\alpha=(\alpha_j,\ldots\alpha_n)$ and $\beta=(\beta_j,\ldots,\beta_n)$.

We can extend these definitions to distributions on G as follows.

Definition 3.1.3. For $u\in\mathcal{D}'(G)$, where $G=G_1\times\cdots\times G_n$ is the product of $n\geq 2$ compact Lie groups, we define its partial Fourier transform at $[\xi^n_{\alpha_n\beta_n}]\in\widehat{G}_n$ as the distribution acting on $C^\infty(G_1\times\cdots\times G_{n-1})$ given by

$$\langle\widehat{u}(\cdot,\ldots,\cdot,\xi^n)_{\alpha_n\beta_n},\psi\rangle_{G_1\times\cdots\times G_{n-1}}=\left\langle u,\psi\otimes\overline{\xi^n_{\beta_n\alpha_n}}\right\rangle_{G_1\times\cdots\times G_n},$$

for every $\psi\in C^\infty(G_1\times\cdots\times G_{n-1})$. Similarly, we define its partial Fourier transform at $\xi^{n-1}_{\alpha_{n-1}\beta_{n-1}}\otimes\xi^n_{\alpha_n\beta_n}$ as the distribution acting on $C^\infty(G_1\times\cdots\times G_{n-2})$ given by

$$\left\langle\widehat{u}(\cdot,\ldots,\xi^{n-1},\xi^n)_{\alpha\beta},\psi\right\rangle_{G_1\times\cdots\times G_{n-2}}=\left\langle u,\psi\otimes\overline{\xi^{n-1}_{\beta_{n-1}\alpha_{n-1}}\otimes\xi^n_{\beta_n\alpha_n}}\right\rangle_{G_1\times\cdots\times G_n},$$

for every $\psi\in C^\infty(G_1\times\cdots\times G_{n-2})$, where $\alpha=(\alpha_{n-1},\alpha_n)$, $\beta=(\beta_{n-1},\beta_n)$. Proceeding like this, we define up to the "total" partial Fourier transform of u at $\xi^1\otimes\cdots\otimes\xi^n$, which corresponds to the complex number

$$\widehat{u}(\xi^1,\ldots,\xi^n)_{\alpha\beta}=\left\langle u,\overline{\xi^1_{\beta_1\alpha_1}}\otimes\cdots\otimes\overline{\xi^n_{\beta_n\alpha_n}}\right\rangle_{G_1\times\cdots\times G_n}\in\mathbb{C}.$$

As before, we can also write

$$\begin{aligned} u &= \\ &= \sum_{[\xi^1]\in\widehat{G_1}} \cdots \sum_{[\xi^n]\in\widehat{G_n}} d_{\xi^1}\dots d_{\xi^n} \sum_{\alpha_j,\beta_j=1}^{d_{\xi^j}} \cdots \sum_{\alpha_n,\beta_n=1}^{d_{\xi^n}} \widehat{u}(\xi^1,\dots,\xi^n)_{\alpha\beta}\xi^1_{\beta_1\alpha_1}\dots\xi^n_{\beta_n\alpha_n} \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \widehat{u}(\xi)_{\alpha\beta}\xi_{\beta\alpha}. \end{aligned}$$

Similarly, for $1 \leq j < n$:

$$u = \sum_{[\xi]\in G_j \widehat{\times\cdots\times} G_n} d_\xi \sum_{\alpha,\beta\in J_\xi} \widehat{u}(\cdot,\dots,\cdot,\xi)_{\alpha\beta}\xi_{\beta\alpha},$$

where the convergence holds in the sense of distributions.

As the tangent space of a product of manifolds can be identified with the direct sum of the tangent spaces of each of its factors, we have that $\mathcal{L}_{G_1\times\cdots\times G_n} = \mathcal{L}_{G_1} + \cdots + \mathcal{L}_{G_n}$, and so $\nu_{[\xi^1\otimes\cdots\otimes\xi^n]} = \nu_{[\xi^1]} + \cdots + \nu_{[\xi^n]}$, for every $[\xi^1\otimes\cdots\otimes\xi^n] \in \widehat{G}$. Therefore

$$\begin{aligned} \langle \xi^1\otimes\cdots\otimes\xi^n\rangle^2 &= \big(1+\nu_{[\xi^1\otimes\cdots\otimes\xi^n]}\big) \\ &= \big(1+\nu_{[\xi^1]}+\cdots+\nu_{[\xi^n]}\big) \\ &\leq \big(1+\nu_{[\xi^1]}\big)+\cdots+\big(1+\nu_{[\xi^n]}\big) \\ &= \langle\xi^1\rangle^2+\cdots+\langle\xi^n\rangle^2 \leq (\langle\xi^1\rangle+\cdots+\langle\xi^n\rangle)^2. \end{aligned}$$

Also

$$\begin{aligned} \langle\xi^1\rangle+\cdots+\langle\xi^n\rangle &\leq \langle\xi^1\otimes\cdots\otimes\xi^n\rangle+\cdots+\langle\xi^1\otimes\cdots\otimes\xi^n\rangle \\ &= n\langle\xi^1\otimes\cdots\otimes\xi^n\rangle. \end{aligned}$$

From which we conclude that

$$\frac{1}{n}(\langle\xi^1\rangle+\cdots+\langle\xi^n\rangle) \leq \langle\xi^1\otimes\cdots\otimes\xi^n\rangle \leq (\langle\xi^1\rangle+\cdots+\langle\xi^n\rangle), \tag{3.2}$$

for every $[\xi^j] \in \widehat{G_j}$, $1 \leq j \leq n$. With this inequality in mind, we extend Theorem 2.2.8 to the product setting, in the following lemma.

Lemma 3.1.4. *Let $G = G_1 \times \cdots \times G_n$ be a product of compact Lie groups. Then the following are equivalent:*

1. $f \in C^\infty(G)$;

2. *For every $N > 0$, there exists $M_N > 0$ such that*

$$|\widehat{f}(\xi)_{\alpha\beta}| \leq M_N(\langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle)^{-N},$$

for all $[\xi] \in \widehat{G}$, $\alpha, \beta \in J_\xi$.

Furthermore, the following are also equivalent:

1. $u \in \mathcal{D}'(G)$;

2. *There exist $N, M > 0$, such that*

$$|\widehat{u}(\xi)_{\alpha\beta}| \leq M(\langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle)^{N},$$

for all $[\xi] \in \widehat{G}$, $\alpha, \beta \in J_\xi$.

Proof. Notice that the product of Lie groups is a Lie group as well, therefore this follows from Theorem 2.2.8 and inequalities (3.2). This proof for the case $n = 2$ can also be found in [74].

□

Lemma 3.1.5. *Let $G = G_0 \times G_1 \times \cdots \times G_n$ be a product of compact Lie groups, $\dim G_0 = n_0$. Then the following are equivalent:*

1. $f \in C^\infty(G)$;

2. *For every $\gamma \in \mathbb{N}_0^{n_0}$, for every $N > 0$, $\exists M_{\gamma N} > 0$ such that*

$$|\partial_t^\gamma \widehat{f}(t, \xi^1, \ldots, \xi^n)_{\alpha\beta}| \leq M_{\gamma N} \left(\langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle\right)^{-N},$$

for all $t \in G_0$, $[\xi^j] \in \widehat{G_j}$, $1 \leq \alpha_j, \beta_j \leq d_{\xi^j}$, $j = 1, \ldots, n$.

Furthermore, the following are also equivalent:

1. $u \in \mathcal{D}'(G)$;

2. *There exists* $N \in \mathbb{N}$, $M > 0$, *such that:*

$$|\langle \widehat{u}(\cdot, \xi^1, \dots, \xi^n)_{\alpha\beta}, \psi \rangle_{G_0}| \leq M p_N(\psi)(\langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle)^N,$$

for all $[\xi^j] \in \widehat{G_j}$, $1 \leq \alpha_j, \beta_j \leq d_{\xi^j}$, $j = 1, \dots, n$ *and* $\psi \in C^\infty(G_0)$.

where $p_N(\psi) \doteq \sum_{|\gamma| \leq N} \max_{t \in G_0} |\partial^\gamma \psi(t)|$.

Proof. We will prove the case $n = 1$, since we can always write $G = G_0 \times G'$, where $G' = G_1 \times \cdots \times G_n$ and then the result follows from inequalities (3.2). This result and proof were inspired by [74].

First suppose $f \in C^\infty(G)$. Since f is smooth, given $N > 0$, $\gamma \in \mathbb{N}_0^{n_0}$, $(\text{Id} + \mathcal{L}_{G_1})\partial^\gamma f$ is also smooth and so

$$\begin{aligned} |\langle \xi^1 \rangle^N \partial_t^\gamma \widehat{f}(t, \xi^1)_{\alpha_1\beta_1}| &= |(\text{Id} + \widehat{\mathcal{L}_{G_1})^{\frac{N}{2}} \partial_t^\gamma f}(t, \xi^1)_{\alpha_1\beta_1}| \\ &\leq \int_{G_1} |(\text{Id} + \mathcal{L}_{G_1})^{\frac{N}{2}} \partial_t^\gamma f(t, x_1)| |\overline{\xi^1_{\beta_1\alpha_1}}(x_1)| dx_1 \\ &\leq M, \end{aligned}$$

for every $t \in G_0$, $1 \leq \alpha_1, \beta_1 \leq d_{\xi^1}$,where $M = \max_{(t,x_1) \in G_0 \times G_1} |(\text{Id} + \mathcal{L}_{G_1})\partial^\gamma f(t, x_1)|$ is finite since G is compact and also since

$$\begin{aligned} 1 = \xi^1_{\alpha_1\alpha_1}(e_{G_1}) &= \sum_{\beta_1=1}^{d_{\xi^1}} \xi^1_{\alpha_1\beta_1}(x_1) \xi^1_{\beta_1\alpha_1}(x_1^{-1}) \\ &= \sum_{\beta_1=1}^{d_{\xi^1}} \xi^1_{\alpha_1\beta_1}(x_1) \xi^{1*}_{\beta_1\alpha_1}(x_1) \\ &= \sum_{\beta_1=1}^{d_{\xi^1}} \xi^1_{\alpha_1\beta_1}(x_1) \overline{\xi^1_{\alpha_1\beta_1}(x_1)} = \sum_{\beta_1=1}^{d_{\xi^1}} |\xi^1_{\alpha_1\beta_1}(x_1)|^2, \end{aligned} \tag{3.3}$$

so $|\xi^1_{\alpha_1\beta_1}(x_1)| \leq 1$ for every $x_1 \in G_1$.

Now suppose the converse holds. Given $N' > 0$, take $N \in \mathbb{N}$ such that $N \geq N'$, notice that

$$\begin{aligned}
\nu_{[\xi^0]}^N |\widehat{f}(\xi^0, \xi^1)_{\alpha\beta}| &= |\widehat{\mathcal{L}_{G_0}^N f}(\xi^0, \xi^1)_{\alpha\beta}| \\
&\leq \int_{G_0} |\mathcal{L}_{G_0}^N \widehat{f}(t, \xi^1)||\overline{\xi^0_{\beta_0\alpha_0}}(t)|dt \\
&\leq \sum_{|\gamma|=2N} \max_{t \in G_0} |\partial_t^\gamma \widehat{f}(t, \xi^1)| \\
&\leq M,
\end{aligned}$$

from the definition of $\mathcal{L}_{G_0}$ and the fact that $|\xi^0(t)_{\beta_0\alpha_0}| \leq 1$ by (3.3). From Proposition 2.2.5 and our hypothesis we then conclude that

$$\begin{aligned}
|\widehat{f}(\xi^0, \xi^1)_{\alpha\beta}| \leq C_N \langle \xi^0 \rangle^{-N} \langle \xi^1 \rangle^{-N} &\leq C_N 2^N (\langle \xi^0 \rangle + \langle \xi^1 \rangle)^{-N} \\
&\leq C_N 2^N (\langle \xi^0 \rangle + \langle \xi^1 \rangle)^{-N'},
\end{aligned}$$

for every $[\xi] = [\xi^0 \otimes \xi^1] \in \widehat{G}$, $\alpha, \beta \in J_\xi$. The fact that $f \in C^\infty(G)$ then follows from Lemma 3.1.4. Now suppose $u \in \mathcal{D}'(G)$. Then notice that

$$\langle \widehat{u}(\cdot, \xi^1)_{\alpha_1\beta_1}, \psi \rangle_{G_0} = \langle u, \psi \otimes \overline{\xi^1_{\beta_1\alpha_1}} \rangle_{G_0 \times G_1}, \tag{3.4}$$

by definition. On the other hand, since $u \in \mathcal{D}'(G)$, there exist $N_0, N_1 \in \mathbb{N}_0$, $C > 0$ such that

$$\begin{aligned}
|\langle u, \psi \otimes \overline{\xi^1_{\beta_1\alpha_1}} \rangle| &\leq C \max_{(t,x_1) \in G_0 \times G_1} \sum_{|\gamma| \leq N_0} \sum_{|\lambda| \leq N_1} |\partial_t^\gamma \psi(t) \overline{\partial_{x_1}^\lambda \xi^1_{\beta_1\alpha_1}}(x_1)| \\
&= C \max_{t \in G_0} \sum_{|\gamma| \leq N_0} |\partial_t^\gamma \psi(t)| \max_{x_1 \in G_1} \sum_{|\lambda| \leq N_1} |\partial_{x_1}^\lambda \xi^1_{\beta_1\alpha_1}(x_1)|. \tag{3.5}
\end{aligned}$$

Since the symbol of the differential operator $\partial_{x_1}^\lambda$ satisfies $\sigma_{\partial_{x_1}^\lambda} \in \Psi_{1,0}^{|\lambda|}(G_1 \times \widehat{G_1})$, and since

$\sigma_{\partial^\lambda_{x_1}}(\xi^1) = \xi^1(x_1)^* \partial^\lambda_{x_1}\xi^1(x_1) \implies \partial^\lambda_{x_1}\xi(x_1) = \xi^1(\xi)\sigma_{\partial^\lambda_{x_1}}(\xi^1)$, we have that

$$\begin{aligned}
|\partial^\lambda_{x_1}\xi^1_{\beta_1\alpha_1}(x_1)| &\leq \sum_{\gamma=1}^{d_{\xi^1}} |\xi^1_{\alpha_1\gamma}(x_1)||\sigma_{\partial^\lambda_{x_1}}(\xi^1)_{\gamma\beta_1}| \\
&\leq \left(\sum_{\gamma=1}^{d_{\xi^1}} |\xi^1_{\alpha_1\gamma}(x_1)|^2\right)^{1/2} \left(\sum_{\gamma=1}^{d_{\xi^1}} |\sigma_{\partial^\lambda_{x_1}}(\xi^1)_{\gamma\beta_1}|^2\right)^{1/2} \\
&\leq \|\sigma_{\partial^\lambda_{x_1}}(\xi^1)\|_{HS} \leq \sqrt{d_{\xi^1}}\|\sigma_{\partial^\lambda_{x_1}}(\xi^1)\|_{op} \leq C\langle\xi^1\rangle^{\frac{\dim G_1}{4}}\langle\xi^1\rangle^{|\lambda|},
\end{aligned}$$

for some $C > 0$, by Proposition 2.2.6 and (3.3). Therefore

$$\sum_{|\lambda|\leq N_1} |\partial^\lambda_{x_1}\xi^1_{\beta_1\alpha_1}(x_1)| \leq C'\langle\xi^1\rangle^{N_1+\frac{\dim G_1}{4}}, \tag{3.6}$$

for some $C' > 0$. Hence, from (3.4) and (3.5) it follows that

$$|\langle u(\cdot,\xi^1)_{\alpha_1\beta_1}, \psi\rangle_{G_0}| \leq C'' p_{N'}(\psi)\langle\xi^1\rangle^{N'},$$

for some $C'' > 0$, and where $N' = \max\{N_0, N_1 + \lceil\frac{\dim G_1}{4}\rceil\}$, for every $[\xi^1] \in \widehat{G_1}$, $1 \leq \alpha_1, \beta_1 \leq d_{\xi^1}$ as claimed. For the converse, applying the definition of $p_N(\cdot)$ and inequality (3.6) with ξ^0 instead of ξ^1, we obtain that there exist $N \in \mathbb{N}$ such that

$$\begin{aligned}
|\widehat{u}(\xi^0,\xi^1)_{\alpha\beta}| &= \left|\left\langle \widehat{u}(\cdot,\xi^1)_{\alpha_1\beta_1}, \overline{\xi^0_{\beta_0\alpha_0}}\right\rangle_{G_0}\right| \\
&\leq M p_N(\xi^0_{\beta_0\alpha_0})\langle\xi^1\rangle^N \\
&\leq C\langle\xi^0\rangle^{N+\frac{\dim G_0}{4}}\langle\xi^1\rangle^N \\
&\leq C'(\langle\xi^0\rangle + \langle\xi^1\rangle)^{N+\frac{\dim G_0}{4}},
\end{aligned}$$

for some constant $C' > 0$ and all $[\xi] = [\xi^0 \otimes \xi^1] \in \widehat{G}$. Therefore, Lemma 3.1.4 implies that $u \in \mathcal{D}'(G)$, as claimed. □

Next we extend Proposition 2.1.7 to the product of compact Lie groups, as in the following propositions.

Proposition 3.1.6. *Let $G = G_1 \times \cdots \times G_n$ be a product of compact Lie groups, and $f, g \in$*

$L^2(G)$. *Then*

$$\langle f, g\rangle_{L^2(G)} = \int_G f(x)\overline{g(x)}dx = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \widehat{f}(\xi)_{\alpha\beta}\widehat{\overline{g}}(\overline{\xi})_{\alpha\beta}.$$

Also, if $f \in \mathcal{D}'(G)$, $g \in C^\infty(G)$

$$\langle f, g\rangle_G = \int_G f(x)g(x)dx = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \widehat{f}(\xi)_{\alpha\beta}\widehat{g}(\overline{\xi})_{\alpha\beta}.$$

Proof. Indeed

$$\begin{aligned}\int_G f(x)\overline{g(x)}dx &= \int_G \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \widehat{f}(\xi)_{\alpha\beta}\xi_{\beta\alpha}(x)\overline{g(x)}dx \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \widehat{f}(\xi)_{\alpha\beta} \int_G \overline{g(x)}\xi_{\beta\alpha}(x)dx \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \widehat{f}(\xi)_{\alpha\beta}\widehat{\overline{g}}(\overline{\xi})_{\alpha\beta}.\end{aligned}$$

Note that the interchange of order of summation and integration is justified due to absolute convergence of the integral and series in the first case, and by the convergence in the sense of distributions of the Fourier series in the second case. This finishes the proof. □

Proposition 3.1.7. *Let* $G = G_1 \times \cdots \times G_n$ *be a product of compact Lie groups,* G_0 *a compact Lie group. Then for* $f, g \in L^2(G_0 \times G)$*:*

$$\langle f, g\rangle_{L^2(G_0\times G)} = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \int_{G_0} \widehat{f}(t,\xi)_{\alpha\beta}\widehat{\overline{g}}(t,\overline{\xi})_{\alpha\beta}dt.$$

In particular

$$\langle f, g\rangle_{G_0\times G} \doteq \int_{G_0\times G} f(x)g(x)dx = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \int_{G_0} \widehat{f}(t,\xi)_{\alpha\beta}\widehat{g}(t,\overline{\xi})_{\alpha\beta}dt.$$

Similarly, if $f \in \mathcal{D}'(G_0 \times G)$ *and* $g \in C^\infty(G_0 \times G)$*:*

$$\langle f, g\rangle_{G_0\times G} = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \left\langle \widehat{f}(\cdot,\xi)_{\alpha\beta}, \widehat{g}(\cdot,\overline{\xi})_{\alpha\beta}\right\rangle_{G_0}.$$

Proof. Indeed, by applying Proposition 3.1.6 twice, we get that

$$\begin{aligned}\langle f,g\rangle_{L^2(G_0\times G)} &= \sum_{[\xi_0]\in\widehat{G_0}}\sum_{[\xi]\in\widehat{G}} d_{\xi_0}d_\xi \sum_{\alpha_0,\beta_0\in J_{\xi_0}}\sum_{\alpha,\beta\in J_\xi}\widehat{f}(\xi^0,\xi)_{\alpha'\beta'}\widehat{\overline{g}}(\overline{\xi^0},\overline{\xi})_{\alpha'\beta'}\\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi}\sum_{[\xi_0]\in\widehat{G_0}} d_{\xi_0}\sum_{\alpha_0,\beta_0\in J_{\xi_0}}\widehat{f}(\xi^0,\xi)_{\alpha'\beta'}\widehat{\overline{g}}(\overline{\xi^0},\overline{\xi})_{\alpha'\beta'}\\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi}\int_{G_0}\widehat{f}(t,\xi)_{\alpha\beta}\widehat{\overline{g}}(t,\overline{\xi})_{\alpha\beta}dt,\end{aligned}$$

where $\alpha' = (\alpha_0,\alpha_1,\ldots,\alpha_n), \beta' = (\beta_0,\beta_1,\ldots,\beta_n)$. Since $\langle f,g\rangle_{G_0\times G} = \langle f,\overline{g}\rangle_{L^2(G_0\times G)}$, this implies

$$\langle f,g\rangle_{G_0\times G} = \langle f,\overline{g}\rangle_{L^2(G_0\times G)} = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi}\int_{G_0}\widehat{f}(t,\xi)_{\alpha\beta}\widehat{g}(t,\overline{\xi})_{\alpha\beta}dt,$$

which proves the second formula. Since $C^\infty(G_0\times G)\subset L^2(G_0\times G)$, the third claim follows similarly. □

Let $G = G_1\times\cdots\times G_n$ be a product of compact Lie groups. Fix $j\in\{1,2,\ldots,n\}$ and X_j a left-invariant vector field on G_j. Notice that viewed as a linear differential operator, iX_j acts as a symmetric operator on $L^2(G_j)$, that is,

$$\begin{aligned}\langle iX_jf,g\rangle_{L^2(G)} &= \int_G i\left.\frac{d}{dt}\right|_{t=0} f(x\exp(tX_j))\overline{g(x)}dx\\ &= \int_G f(x)\overline{(-i)\left.\frac{d}{dt}\right|_{t=0} g(x\exp(-tX_j))}dx\\ &= \langle f,iX_jg\rangle_{L^2(G)},\end{aligned}$$

where we have used the right invariance of the Haar measure. In particular, for a fixed $[\xi^j]\in\widehat{G_j}$, we may consider its action on the vector subspace $\mathcal{H}_{\xi^j,\alpha_j\cdot}\doteq \text{span}\{\xi^j_{\alpha_j\beta_j}|\beta_j=1,\ldots,d_{\xi_j}\}$, which is invariant by iX_j as seen below. It follows from the Fourier inversion formula, and Remark 2.2.20, that

$$iX_j\xi^j_{\alpha_j\beta_j}(x) = \sum_{\gamma_j=1}\xi_{\alpha_j\gamma_j}(x)\sigma_{iX_j}(\xi^j)_{\gamma_j\beta_j},$$

where $\sigma_{iX_j}(\xi^j)\in\mathbb{C}^{d_{\xi^j}\times d_{\xi^j}}$ is the symbol of the differential operator iX_j evaluated at $[\xi^j]\in\widehat{G_j}$, as defined in Definition 2.2.10. Thus, the symmetric linear operator iX_j is given, with respect to the basis $\{\xi^j_{\alpha_j\beta_j}|\beta_j=1,\ldots,d_{\xi_j}\}$, by the (symmetric) matrix $\sigma_{iX_j}(\xi^j)$. But then there exists a unitary matrix A such that $A^*\sigma_{iX_j}(\xi^j)A$ is diagonal with real coefficients.

Therefore, setting $\eta^j(x) \doteq A^*\xi^j(x)A$, for every $x \in G$, we have that $\eta^j \in [\xi^j]$ and $\sigma_{iX_j}(\eta^j) = A^*\sigma_{iX_j}(\xi^j)A$ is diagonal. In conclusion, we may always choose a representative of $[\xi^j]$ such that $\sigma_{X_j}(\xi^j)$ is diagonal with imaginary coefficients.

Definition 3.1.8. Given a left-invariant vector field X_j, for every $1 \leq j \leq n$, we define the real numbers $\mu_{\alpha_j}(\xi^j)$ such that

$$\sigma_{X_j}(\xi^j)_{\alpha_j\beta_j} = i\mu_{\alpha_j}(\xi^j)\delta_{\alpha_j\beta_j}, \text{ for } 1 \leq \alpha_j, \beta_j \leq d_{\xi^j}, \tag{3.7}$$

where δ_{mn} is the Kronecker delta.

Remark 3.1.9. With inequality (2.5) in mind, this implies that if X_j is a left-invariant vector field on a compact Lie group X_j, then with the notation above, the inequality

$$|\mu_{\alpha_j}(\xi^j)| \leq C\langle\xi^j\rangle, \tag{3.8}$$

holds for some $C > 0$ and every $[\xi^j] \in \widehat{G_j}$, $1 \leq \alpha_j \leq d_{\xi_j}$.

Chapter 4

Constant coefficients operators

This chapter is structured as follows:

-In Section 4.1, we present our new results regarding the global solvability and hypoellipticity of constant coefficient first-order differential operators on the product of n compact Lie groups.

-In Section 4.2, we show how our results recover results already known in the literature on the products of tori and spheres, as well as new results in these setting.

First, we recall the definition of the global properties of differential operators which we are concerned.

Definition 4.0.1. Let G be a compact Lie group. We say that a differential operator $L : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is globally hypoelliptic if $Lu = f \in C^\infty(G) \implies u \in C^\infty(G)$.

Definition 4.0.2. Let G be a compact Lie group. Let $L : \mathcal{D}'(G) \to \mathcal{D}'(G)$ be a differential operator. We define

$$(\ker {}^tL)^0 \doteq \{u \in \mathcal{D}'(G) \,|\, \langle u, f\rangle_G = 0,\ \forall f \in \ker {}^tL\}$$

where ${}^tL : C^\infty(G) \to C^\infty(G)$ is the called the transpose of L, and is the differential operator given by $\langle Lu, f\rangle = \langle u, {}^tL\, f\rangle$, for every $u \in \mathcal{D}'(G),\ f \in C^\infty(G)$. We say L is globally solvable if for every $f \in (\ker {}^tL)^0 \cap C^\infty(G)$, $\exists u \in C^\infty(G)$ such that $Lu = f$.

Remark 4.0.3. It is worth mentioning a classical argument using functional analysis (as proved in a more general context in [102], Proposition 35.4, and later more explicitly in [31], Theorem 1.4 and Corollary 1.5) shows that a differential operator $L : C^\infty(M) \to C^\infty(M)$ defined on a compact manifold M is globally solvable as in the definition above if and only if L has closed range. In fact, this equivalent condition is frequently used as the definition for global solvability in several papers. Also, it is worth mentioning that this fact could possibly simplify some of the proofs in this thesis, however we chose not to rely on this fact and prove our results based only on the definition above.

4.1 Global regularity on compact Lie groups

Let $G = G_1 \times \cdots \times G_n$ be a product of compact Lie groups. Consider the constant-coefficients left-invariant first order differential operator $L : \mathcal{D}'(G) \to \mathcal{D}'(G)$ given by

$$L \doteq c_1 X_1 + \cdots + c_n X_n + q, \tag{4.1}$$

where $c_j \in \mathbb{C}$, X_j is a left-invariant vector field on G_j, for $j = 1, \dots, n$ and $q \in \mathbb{C}$. Since $G_1 \times \{e_{G_2}\} \cong G_1$, we will assume every G_j to be non-trivial. Also, since we allow the c_j's to be equal to zero, we will also assume that every X_j is non-zero.

For each $u \in \mathcal{D}'(G)$, we may write the partial Fourier transforms of Lu, one variable at a time, as follows:

$$\begin{aligned}\widehat{Lu}(x_1, \dots, x_{n-1}, \xi^n)_{\alpha_n\beta_n} = {} & (c_1X_1 + \cdots + c_{n-1}X_{n-1})\widehat{u}(x_1, \dots, x_{n-1}, \xi^n)_{\alpha_n\beta_n} + \\ & + ic_n\mu_{\alpha_n}(\xi^n)\widehat{u}(x_1, \dots, x_{n-1}, \xi^n)_{\alpha_n\beta_n} + q\widehat{u}(x_1, \dots, x_{n-1}, \xi^n)_{\alpha_n\beta_n},\end{aligned}$$

$$\begin{aligned}\widehat{Lu}(x_1, \dots, x_{n-2}, \xi^{n-1}, \xi^n)_{\alpha\beta} = {} & (c_1X_1 + \cdots + c_{n-2}X_{n-2})\widehat{u}(x_1, \dots, x_{n-2}, \xi^{n-1}, \xi^n)_{\alpha\beta} + \\ & + ic_{n-1}\mu_{\alpha_{n-1}}(\xi^{n-1})\widehat{u}(x_1, \dots, x_{n-2}, \xi^{n-1}, \xi^n)_{\alpha\beta} + \\ & + ic_n\mu_{\alpha_n}(\xi^n)\widehat{u}(x_1, \dots, x_{n-2}, \xi^{n-1}, \xi^n)_{\alpha\beta} + \\ & + q\widehat{u}(x_1, \dots, x_{n-2}, \xi^{n-1}, \xi^n)_{\alpha\beta},\end{aligned}$$

for $\alpha = (\alpha_{n-1}, \alpha_n)$ and $\beta = (\beta_{n-1}, \beta_n)$, and so on, until we have taken the total partial Fourier transform of Lu, given by

$$\widehat{Lu}(\xi^1, \dots, \xi^n)_{\alpha\beta} = i\left(c_1\mu_{\alpha_1}(\xi^1) + \cdots + c_n\mu_{\alpha_n}(\xi^n) - iq\right)\widehat{u}(\xi^1, \dots, \xi^n)_{\alpha\beta}. \tag{4.2}$$

For convenience, let $c = (c_1, \dots, c_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $[\xi] = [\xi^1 \otimes \cdots \otimes \xi^n] \in$

$\widehat{G}$, and define

$$\langle c, \mu_\alpha(\xi)\rangle \doteq c_1\mu_{\alpha_1}(\xi^1) + \cdots + c_n\mu_{\alpha_n}(\xi^n). \tag{4.3}$$

This way, equation (4.2) can be rewritten as

$$\widehat{Lu}(\xi)_{\alpha\beta} = i\left(\langle c, \mu_\alpha(\xi)\rangle - iq\right)\widehat{u}(\xi)_{\alpha\beta}. \tag{4.4}$$

Defining

$$\sigma_L(\xi)_{\alpha\alpha} \doteq i\left(\langle c, \mu_\alpha(\xi)\rangle - iq\right),$$

we can view L as a Fourier multiplier with symbol σ_L.

With Lemma 3.1.4 in mind, we can study the global properties mentioned above by studying the behaviour of the coefficients $(\langle c, \mu_\alpha(\xi)\rangle - iq)$ as $\langle\xi\rangle \to \infty$.

In order to prove the necessary and sufficient conditions for the global solvability of the operator L, we first obtain an equivalent characterization of global solvability for a differential operator.

Definition 4.1.1. Let $P : \mathcal{D}'(G) \to \mathcal{D}(G)$ be a continuous linear operator whose symbol $\sigma_P(\xi)$ is diagonal, so that $\widehat{Pf}(\xi)_{\alpha\beta} = \sigma_P(\xi)_{\alpha\alpha}\widehat{f}(\xi)_{\alpha\beta}$ for each $\alpha, \beta \in J_\xi,\ [\xi] \in \widehat{G},\ f \in \mathcal{D}'(G)$. We define $\mathcal{K}_P \subset \mathcal{D}'(G)$ by

$$\mathcal{K}_P \doteq \{u \in \mathcal{D}'(G) |\text{ such that } \widehat{u}(\xi)_{\alpha\beta} = 0,\text{ for all } \beta \in J_\xi,\text{ if } \sigma_P(\xi)_{\alpha\alpha} = 0\}.$$

Proposition 4.1.2. *Let P be as in the previous definition. Suppose ${}^tP$ is also diagonal, that is, $\widehat{{}^tPu}(\xi)_{\alpha\beta} = \sigma_{{}^tP}(\xi)_{\alpha\alpha}\widehat{u}(\xi)_{\alpha\beta}$, for every $\alpha, \beta \in J_\xi$ and $u \in \mathcal{D}'(G)$. If it satisfies $\sigma_P(\xi) = \lambda\sigma_{{}^tP}(\overline{\xi})$, for some $\lambda \in \mathbb{C}\backslash\{0\}$, and every $[\xi] \in \widehat{G}$, then*

$$\mathcal{K}_P = (\ker {}^tP)^0.$$

Proof. Suppose $f \in \mathcal{K}_P$ and let $v \in C^\infty(G)$ be such that $v \in \ker {}^tP$. Then

$$\langle f, v\rangle_G = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \widehat{f}(\xi)_{\alpha\beta}\widehat{v}(\overline{\xi})_{\alpha\beta}.$$

If $\sigma_P(\xi)_{\alpha\alpha} = 0$, then $\widehat{f}(\xi)_{\alpha\beta} = 0$. On the other hand, if $\sigma_P(\xi)_{\alpha\alpha} \neq 0$, then

$$0 = \widehat{{}^tP\,v}(\overline{\xi})_{\alpha\beta} = \sigma_{{}^tP}(\overline{\xi})_{\alpha\alpha}\widehat{v}(\overline{\xi})_{\alpha\beta} = \frac{1}{\lambda}\sigma_P(\xi)_{\alpha\alpha}\widehat{v}(\overline{\xi})_{\alpha\beta}$$

implies that $\widehat{v}(\overline{\xi})_{\alpha\beta} = 0$, so every term in the sum above is zero. Therefore $\mathcal{K}_P \subset (\ker {}^tP)^0$. Now let $f \in (\ker {}^tP)^0$. Suppose first that $\sigma_P(\xi)_{\alpha\alpha} = 0$ for some $[\xi] \in \widehat{G}$, $\alpha \in J_\xi$. For each $\beta \in J_\xi$, let $v_{\xi,\alpha,\beta} \in C^\infty(G)$ be given by

$$\widehat{v_{\xi,\alpha,\beta}}(\eta)_{\gamma\kappa} = \begin{cases} 1 \text{ if } \eta = \overline{\xi},\ \gamma = \alpha,\ \kappa = \beta \\ 0, \text{ otherwise.} \end{cases}$$

Then $\widehat{{}^tP\,v_{\xi,\alpha,\beta}}(\overline{\xi})_{\alpha\beta} = \sigma_{{}^tP}(\overline{\xi})_{\alpha\alpha}\widehat{v_{\xi,\alpha,\beta}}(\overline{\xi})_{\alpha\beta} = \frac{1}{\lambda}\sigma_P(\xi)_{\alpha\alpha} = 0$, so $v \in \ker {}^tP$. Therefore

$$0 = \langle f, v\rangle_G = d_\xi \widehat{f}(\xi)_{\alpha\beta},$$

since this holds for every $1 \leq \beta \leq d_\xi$, we conclude that $f \in \mathcal{K}_P$. □

Proposition 4.1.3. *The operator L defined in* (4.1) *is globally solvable if and only if*

$$\mathcal{K}_L \cap C^\infty(G) = L(C^\infty(G)).$$

Proof. As in equation (4.4), under our choice of representations

$$\sigma_L(\xi)_{\alpha\alpha} = i(\langle c, \mu_\alpha(\xi)\rangle - iq)$$

and

$$\sigma_{{}^tL}(\xi)_{\alpha\alpha} = i(-\langle c, \mu_\alpha(\xi)\rangle - iq),$$

for every $[\xi] \in \widehat{G}$, $\alpha \in J_\xi$, since ${}^tL = -(c_1X_1 + \cdots + c_nX_n) + q$. Next, we claim that X_j satisfies $\mu_{\alpha_j}(\overline{\xi^j}) = -\mu_{\alpha_j}(\xi^j)$, so that $\sigma_L(\xi) = \sigma_{{}^tL}(\overline{\xi})$, for every $[\xi] \in \widehat{G}$, and the claim follows from Proposition 4.1.2. Indeed, note that

$$\begin{aligned} i\mu_{\alpha_j}(\overline{\xi^j}) &= \sigma_{X_j}(\overline{\xi^j})_{\alpha_j\alpha_j} = X_j\overline{\xi^j}(e_{G_j}) = \overline{X_j\xi^j_{\alpha_j\alpha_j}(e_{G_j})} \\ &= \overline{i\mu_{\alpha_j}(\xi^j)} = -i\mu_{\alpha_j}(\xi^j), \end{aligned}$$

as claimed. □

Next, we present necessary and sufficient conditions for global solvability and hypoellipticity of the operator L.

Proposition 4.1.4. *The operator L is globally solvable if and only if there exist $M, N > 0$ such that*

$$|\sigma_L(\xi)_{\alpha\alpha}| = |c_1\mu_{\alpha_1}(\xi^1) + \cdots + c_n\mu_{\alpha_n}(\xi^n) - iq| \geq M\left(\langle\xi^1\rangle + \cdots + \langle\xi^n\rangle\right)^{-N}, \tag{4.5}$$

for every $[\xi] \in \widehat{G}$, $\alpha \in J_\xi$, such that the left hand side of the inequality is not zero. It is globally hypoelliptic if and only if the equation

$$c_1\mu_{\alpha_1}(\xi^1) + \cdots + c_n\mu_{\alpha_n}(\xi^n) - iq = 0$$

has only finitely many solutions for $[\xi] \in \widehat{G}$, $\alpha \in J_\xi$ and inequality (4.5) also holds whenever its left-hand side is not zero.

Proof. Suppose the first condition is not satisfied, that is, for every $m \in \mathbb{N}$, there exist distinct $\xi_m = \xi_m^1 \otimes \cdots \otimes \xi_m^n$, where $[\xi_m^j] \in \widehat{G_j}$, $j = 1, \ldots, n$, and $\alpha(m) \in \mathbb{N}^n$ such that

$$0 < |\langle c, \mu_{\alpha(m)}(\xi_m)\rangle - iq| \leq (\langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle)^{-m}. \tag{4.6}$$

Indeed, if only finitely many ξ satisfied the inequality above, by taking N large enough and M small enough, we could obtain that inequality (4.5) holds for all $[\xi] \in \widehat{G}$ and $\alpha \in J_\xi$. Define the Fourier coefficients

$$\widehat{f}(\xi_m)_{\alpha(m)\bar{1}} = i(\langle c, \mu_{\alpha(m)}(\xi_m)\rangle - iq),\ m \in \mathbb{N},$$

and

$$\widehat{f}(\xi)_{\alpha\beta} = 0, \text{ otherwise.}$$

Here $\bar{1} = (1, \ldots, 1) \in \mathbb{N}^n$ denotes the n-dimensional vector of ones. By Lemma 3.1.4 and inequality (4.6), these coefficients define $f \in C^\infty(G)$. Moreover, by Proposition 4.1.2 we have that $f \in (\ker {}^tL)^0$ since $\widehat{f}(\xi)_{\alpha\beta} = 0$ whenever $\sigma(\xi)_{\alpha\alpha} = i(\langle c, \mu_\alpha(\xi)\rangle - iq) = 0$. Now suppose there exists $u \in C^\infty(G)$ such that $Lu = f$. Then its partial Fourier coefficients would satisfy

$$\widehat{Lu}(\xi)_{\alpha\beta} = i\left(\langle c, \mu_\alpha(\xi)\rangle - iq\right)\widehat{u}(\xi)_{\alpha\beta} = \widehat{f}(\xi)_{\alpha\beta}, \tag{4.7}$$

and so $\widehat{u}(\xi_m)_{\alpha(m)\,\overline{1}} = 1$, for every $m \in \mathbb{N}$. But then $u \notin C^\infty(G)$ by Lemma 3.1.4, which is a contradiction. We conclude no such u can exist and therefore L is not globally solvable. Considering the same Fourier coefficients for $\widehat{f}$ and $\widehat{u}$, if we define all other coefficients of $\widehat{u}$ to be 0, then by Lemma 3.1.4 these coefficients define $u \in \mathcal{D}'(G) \setminus C^\infty(G)$. Moreover, by comparing Fourier coefficients, it clearly satisfies $Lu = f$, from which we conclude that L is not globally hypoelliptic as well. Suppose now that inequality (4.5) holds in the correct domain and let $f \in (\ker {}^tL)^0 \cap C^\infty(G)$. Then, by Proposition 4.1.2, $\widehat{f}(\xi)_{\alpha\beta} = 0$ whenever $\langle c, \mu_\alpha(\xi)\rangle - iq = 0$. Define

$$\widehat{u}(\xi)_{\alpha\beta} = \frac{\widehat{f}(\xi)_{\alpha\beta}}{i\left(\langle c, \mu_\alpha(\xi)\rangle - iq\right)}, \tag{4.8}$$

whenever $\langle c, \mu_\alpha(\xi)\rangle - iq \neq 0$ and $\widehat{u}(\xi)_{\alpha\beta} = 0$, otherwise. Notice that

$$|\widehat{u}(\xi)_{\alpha\beta}| \leq |\widehat{f}(\xi)_{\alpha\beta}| \frac{1}{M}\left(\langle \xi^1\rangle + \cdots + \langle \xi^n\rangle\right)^N.$$

Therefore, by Lemma 3.1.4, for every $N' > 0$ there exists $M' > 0$ such that

$$|\widehat{u}(\xi)_{\alpha\beta}| \leq \frac{M'}{M}\left(\langle \xi^1\rangle + \cdots + \langle \xi^n\rangle\right)^{-N'+N}$$

By Lemma 3.1.4 we conclude these Fourier coefficients define $u \in C^\infty(G)$, which proves that L is globally solvable. Note further that if $u \in \mathcal{D}'(G)$ is such that $Lu = f \in C^\infty(G)$, then by comparing partial Fourier transform as before, $\widehat{u}(\xi)_{\alpha\beta}$ is uniquely determined by equality (4.8) whenever the denominator is not zero. Hence, if the set of $[\xi]$, α for which the denominator is zero is finite, the same argument as before implies that the coefficients of u must decay faster than any power of $(\langle \xi^1\rangle + \cdots + \langle \xi^n\rangle)$. Lemma 3.1.4 then implies that $u \in C^\infty(G)$ and so L is globally hypoelliptic. □

Remark 4.1.5. We can apply a very similar argument to prove the inequality (4.5) is also a necessary and sufficient condition for L to be globally solvable in the sense of distributions, that is: for all $f \in (\ker {}^tL)^0$, there exists $u \in \mathcal{D}'(G)$ such that $Lu = f$ if and only if inequality (4.5) holds whenever its left hand side is not zero.

Corollary 4.1.6. *If L defined in (4.1) is globally hypoelliptic, then*

$$\ker(L) \subset \textit{span}\left\{\xi_{\alpha\beta} \,|\, [\xi] \in \widehat{G},\ \alpha, \beta \in J_\xi\right\},$$

and in fact $\dim \ker(L) < +\infty$.

Proof. Indeed, notice that if $u \in \ker(L)$, then by comparing Fourier coefficients as in equality (4.7), we have that

$$u \in \text{span}\left\{\xi_{\alpha\beta} \mid [\xi] \in \widehat{G},\ \langle c, \mu_\alpha(\xi)\rangle - iq = 0\right\}.$$

Therefore,

$$\dim \ker(L) \leq \sum_{\substack{[\xi]\in\widehat{G} \\ \langle c,\mu_\alpha(\xi)\rangle - iq = 0}} d_\xi^2.$$

Since L is globally hypoelliptic, the sum above is finite by Proposition 4.1.4, proving the statement. □

4.2 Global regularity on tori and spheres

Our goal in this section is to study the global regularity of the operator L in the particular cases where the groups G_j are equal to different copies of the one dimensional torus or the three dimensional sphere. Not only we recover results already proven in the literature using the techniques developed in the last section, but also we present some new results.

4.2.1 Tori

First, let $G = \mathbb{T}^1 \times \cdots \times \mathbb{T}^1 = \mathbb{T}^n$, be the n dimensional torus and consider $X_j = \partial_{x_j}$ the usual partial differentiation with respect to the j-th variable, for $j = 1, \ldots, n$. Then, $\dim \xi = 1$ for every $[\xi] = [t \mapsto e^{ikt}] \in \widehat{\mathbb{T}^1}$ and we may identify $\xi \sim \mu_1(\xi) \sim k \in \mathbb{Z}$. In this case the operator L described in the last section is given by

$$L = c_1\partial_{x_1} + \cdots + c_n\partial_{x_n} + q, \tag{4.9}$$

and its symbol is

$$\sigma_L(k) = i(\langle c, \mu_{\bar{1}}(k)\rangle - iq) = i(c_1k_1 + \cdots + c_nk_n - iq),$$

where $k \in \mathbb{Z}^n$, $c \in \mathbb{C}^n$ and $q \in \mathbb{C}$. Since $\langle k_j\rangle = \sqrt{1+k_j^2}$ has the same asymptotic behaviour as $|k_j|$, for $k_j \to \pm\infty$, Proposition 4.1.4 implies that L is globally solvable if and only if there exist $M, N > 0$ such that

$$|c_1k_1 + \cdots + c_nk_n - iq| \geq M(|k_1| + \cdots + |k_n|)^{-N}, \tag{4.10}$$

for every $k \in \mathbb{Z}^n \setminus \{0\}$ whenever the left hand side of (4.10) is not zero.

With this in mind we obtain the following corollaries.

Corollary 4.2.1. *Suppose that either*

(i) $c \in (\mathbb{Q} + i\mathbb{Q})^n$*;*

(ii) There exist $1 \leq j_1, j_2 \leq n$ *such that* $\alpha = \frac{a_{j_1}}{a_{j_2}}$ *is an irrational non-Liouville number,* $\frac{a_j}{a_{j_2}} \in \mathbb{Q}$*, for every* $j_1 \neq j = 1, \ldots, n$*, and* $\frac{\operatorname{Im}(q)}{a_{j_2}} \in \mathbb{Q}$*;*

(iii) There exist $1 \leq j_1, j_2 \leq n$ *such that* $\lambda = \frac{b_{j_1}}{b_{j_2}}$ *is an irrational non-Liouville number,* $\frac{b_j}{b_{j_2}} \in \mathbb{Q}$*, for every* $j_1 \neq j = 1, \ldots, n$*, and* $\frac{\operatorname{Re}(q)}{b_{j_2}} \in \mathbb{Q}$*;*

where $c_j = a_j + ib_j$*, for* $j = 1, \ldots n$*. Then* L *defined in* (4.9) *is globally solvable.*

Remark 4.2.2. The claim above still holds if we replace the quotients in its statement by their respective numerators. The proof is very similar.

Proof. First assume (i). We claim that there exists $\varepsilon = \varepsilon(c, q) > 0$ such that

$$|c_1k_1 + \cdots + c_nk_n - iq| \geq \varepsilon,$$

whenever the left hand side of the inequality is not zero. This proves that inequality (4.10) holds trivially, and so L is globally solvable by Proposition 4.1.4. Indeed, note that if $\operatorname{Re}(q) \in \mathbb{Q}$, then there are, $r_j, r \in \mathbb{Z}$ and $s_j, s \in \mathbb{N}$, $j = 1, \ldots, n$, such that

$$\begin{aligned} |c_1k_1 + \cdots + c_nk_n - iq| &\geq \left|\frac{r_1}{s_1}k_1 + \cdots + \frac{r_n}{s_n}k_n - \frac{r}{s}\right| \\ &= |r_1k_1s_2\ldots s_ns + \cdots + r_nk_ns_1\ldots s_{n-1}s + rs_1\ldots s_n|\,\frac{1}{s_1\ldots s_ns} \end{aligned}$$

which is greater than or equal to $\varepsilon = \frac{1}{s_1 \dots s_n s}$ whenever it is not zero, since the first term on the last line is an integer. A similar argument can be applied if $\mathrm{Im}(q) \in \mathbb{Q}$, now using the imaginary part of the symbol. If both the real and imaginary part of q are irrational, then with the same notation as before, $\mathrm{Re}(q)s_1 \dots s_n \notin \mathbb{Z}$, therefore

$$
\begin{aligned}
|c_1k_1 + \cdots + c_nk_n - iq| &\geq \left| \frac{r_1}{s_1}k_1 + \cdots + \frac{r_n}{s_n}k_n - \mathrm{Re}(q) \right| \\
&= |r_1k_1s_2 \dots s_n s + \cdots + r_nk_ns_1 \dots s_{n-1}s + \mathrm{Re}(q)s_1 \dots s_n| \frac{1}{s_1 \dots s_n} \\
&\geq \frac{\| \mathrm{Re}(q)s_1 \dots s_n\|_{\mathbb{R}/\mathbb{Z}}}{s_1 \dots s_n},
\end{aligned}
$$

where $\|x\|_{\mathbb{R}/\mathbb{Z}}$ denotes the distance of the real number x to the nearest integer, so the inequality also holds as claimed. Now assume (ii). Suppose, without loss of generality, that $j_1 = 1$, $j_2 = 2$ and $\frac{a_j}{a_2} = \frac{r_j}{s_j}$, $\frac{\mathrm{Im}(q)}{a_2} = \frac{r}{s}$ where $r, r_j \in \mathbb{Z}$ and $s, s_j \in \mathbb{N}$, for $j = 2, \dots, n$. Then

$$
\begin{aligned}
&|c_1k_1 + \cdots + c_nk_n - iq| \\
&\qquad \geq |a_2| \left| \alpha k_1 + \frac{k_2s_3 \dots s_n s + k_3r_3s_4 \dots s_n s + \cdots + k_ns_3 \dots s_{n-1}s + rs_3 \dots s_n}{s_3 \dots s_n s} \right|.
\end{aligned}
$$

Set $\omega(k_2, \dots, k_n) \doteq k_2s_3 \dots s_n s + k_3r_3s_4 \dots s_n s + \cdots + k_ns_3 \dots s_{n-1}s + rs_3 \dots s_n \in \mathbb{Z}$. Then the last inequality can be rewritten as

$$
|c_1k_1 + \cdots + c_nk_n - iq| \geq |a_2||k_1| \left| \alpha - \frac{-\omega(k_2, \dots, k_n)}{|k_1|s_3 \dots s_n s} \right|,
$$

for $k_1 \neq 0$. But then, as $|k_1| > 0$ and α is an irrational non-Liouville number, there exist $M' > 0$, $N' > 1$ such that

$$
\begin{aligned}
|c_1k_1 + \cdots + c_nk_n - iq| &\geq |a_2||k_1|M'(|k_1|s_3 \dots s_n s)^{-N'} \\
&\geq |a_2|M'(s_3 \dots s_n s)^{-N'}|k_1|^{-N'} \\
&\geq M''(|k_1| + \cdots + |k_n|)^{-N'},
\end{aligned}
$$

for $M'' = |a_2|M(s_3 \dots s_n s)^{-N'} > 0$. When $k_1 = 0$, we can apply the same argument from part (i). Together, these inequalities imply (4.10) holds, therefore by Proposition 4.1.4 the operator L is globally solvable. The proof of (iii) is very similar and is left to the reader. □

Corollary 4.2.3. *If $c \in \mathbb{R}^n$ and there exist $1 \leq j_1, j_2 \leq n$ such that $\lambda = \frac{c_{j_1}}{c_{j_2}}$ is a Liouville number and $\frac{q}{c_{j_2}} = i\ell \in i\,\mathbb{Z}$, then L defined in (4.9) is not globally solvable.*

Proof. First, without loss of generality, we may suppose $j_1 = 1$, $j_2 = 2$. Note that as λ is a Liouville number there exist sequences $(p_m)_{m\in\mathbb{N}} \subset \mathbb{Z}$, $(r_m)_{m\in\mathbb{N}} \subset \mathbb{N}$, with $r_m \to +\infty$ and such that

$$\left|\lambda - \frac{p_n}{r_m}\right| < \frac{1}{(r_m)^m} \iff |\lambda r_m - p_m| < (r_m)^{-m+1},\ \forall m \in \mathbb{N}\,.$$

By taking $k(m) = (r_m, -(p_m + \ell), 0, \ldots, 0) \in \mathbb{Z}^n$, for each $m \in \mathbb{N}$, we have that

$$|c_1 k(m)_1 + \cdots + c_n k(m)_n - iq| = |c_2|\,|\lambda r_m - p_m| < |c_2|(r_m)^{-m+1}. \tag{4.11}$$

It follows the inequality (4.10) cannot hold, since if it did, there would exist $M, N > 0$ such that

$$|c_1 k(m)_1 + \cdots + c_n k(m)_n - iq| \geq M(|k(m)_1| + \cdots + |k(m)_n|)^{-N}$$

for every $m \in \mathbb{N}$. But then, by (4.11) that would imply

$$M(r_m + |p_m + \ell|)^{-N} < |c_2|(r_m)^{-m+1},$$

which we can rewrite as

$$\begin{aligned} |c_2| > M(r_m + |p_m + \ell|)^{-N}(r_m)^{m-1} &= M\left(\frac{r_m^{\frac{m-1}{N}}}{r_m + |p_m + \ell|}\right)^N \\ &= M\left(\frac{r_m^{\frac{m-1}{N}-1}}{1 + \frac{|p_m+\ell|}{r_m}}\right)^N \end{aligned}$$

for every $m \in \mathbb{N}$. By the definition of the sequences $(r_m)_{m\in\mathbb{N}}$ and $(p_m)_{m\in\mathbb{N}}$, $|p_m| \leq |\lambda r_m| + 1$, so $\frac{|p_m+\ell|}{r_m}$ is bounded. Therefore, the right hand side of the inequality above tends to $+\infty$ as $m \to +\infty$, so that we obtain a contradiction. Hence (4.10) does not hold and so by Proposition 4.1.4 the operator L is not globally solvable. □

As for global hypoellipticity, in this particular case Proposition 4.1.4 implies that L defined in (4.9) is globally hypoelliptic if and only if it is globally solvable and the set

$$\mathcal{N}_L = \{k \in \mathbb{Z}^n \,|c_1 k_1 + \cdots + c_n k_n - iq = 0\}$$

is finite. Note that if $q = 0$, then surely $k = (0, \ldots, 0) \in \mathcal{N}_L$, and if there exists any other $k \in \mathcal{N}_L$, then $mk = (mk_1, \ldots, mk_n) \in \mathcal{N}_L$ for every $m \in \mathbb{Z}$. Therefore the only way $\mathcal{N}_L$ can be finite is if $\mathcal{N}_L = \{0\}$. This is equivalent to there being no non-trivial integer solutions to the system of Diophantine equations

$$\begin{cases} a_1k_1 + \cdots + a_nk_n = 0 \\ b_1k_1 + \cdots + b_nk_n = 0. \end{cases}$$

Also, if $q \notin c_1\,\mathbb{Z} + \cdots + c_n\,\mathbb{Z}$, then clearly $\mathcal{N}_L$ is empty and therefore finite, and so L is globally hypoelliptic if and only if it is globally solvable. This means that, for instance, Corollary 4.2.1 allows us to conclude that if $c \in \mathbb{Q}^n$ and $q \notin i\mathbb{Q}$, then L is globally hypoelliptic.

An interesting case occurs when $n \geq 2$, $b_1 = b_2 = \cdots = b_n = 0$, $a_1, \ldots, a_n \in \mathbb{Z}$ and $q = -im \in i\,\mathbb{Z}$. Then

$$\mathcal{N}_L = \{k \in \mathbb{Z}^n \,|a_1k_1 + \cdots + a_nk_n = m\},$$

so in this case L is not globally hypoelliptic if and only if the Diophantine equation above has infinitely many solutions $k \in \mathbb{Z}$. This is true if and only if $\gcd(a_1, \ldots, a_n)$ divides m. Indeed, if such equation has infinitely many solutions, choose one solution $k \in \mathbb{Z}^n$. Then since

$$a_1k_1 + \cdots + a_nk_n = m,$$

we see that $\gcd(a_1, \ldots, a_n)$ divides the left-hand side, and so it must divide m also. The converse follows by induction on $n \in \mathbb{N}$. If $n = 2$, then by Bézout's identity, there exists $k \in \mathbb{Z}^2$ such that $a_1k_1 + a_2k_2 = \gcd(a_1, a_2) = d$. Then since $m = dl$, for some $l \in \mathbb{Z}$, we have that $k' = (k_1l, k_2l)$ solves the equation. Consequently, for every $w \in \mathbb{Z}$, every

$$k_w = \left(k_{1w} = k_1' + w\frac{a_2}{\gcd(a_1, a_2)}, \, k_{2w} = k_2' - w\frac{a_1}{\gcd(a_1, a_2)}\right)$$

solves the equation, and so the claim follows. Now suppose the claim holds for the case with $n - 1$ variables. If the Diophantine equation

$$a_1k_1 + \cdots + a_nk_n = m$$

admits solution then the equation

$$\gcd(a_1,\dots,a_{n-1})z + a_n k_n = m$$

admits solution. By the reasoning above, as $\gcd(\gcd(a_1,\dots a_{n-1}), a_n) = \gcd(a_1,\dots a_{n-1}, a_n)$ divides m, this last equation has infinitely many solutions z', while each

$$a_1k_1 + \dots + a_{n-1}k_{n-1} = \gcd(a_1,\dots,a_{n-1})z'$$

also has infinitely many solutions by our inductive hypothesis, proving the case of n variables. In sum, we have proved:

Corollary 4.2.4. *Let L be the first-order differential operator acting on the torus $\mathbb{T}^n$, $n \geq 2$ given by*

$$L = a_1\partial_{x_1} + \dots + a_n\partial_{x_n} + q,$$

where $a_1,\dots,a_n \in \mathbb{Z}$, $q \in \mathbb{C}$. Then L is globally hypoelliptic if and only if $q \notin i\,\mathbb{Z}$ or $\gcd(a_1,\dots,a_n)$ does not divide $iq \in \mathbb{Z}$.

4.2.2 Spheres

Let $G_1,\dots,G_n = \mathbb{S}^3$, and consider $X_j = i\partial_{0,j}$ the smooth vector field in $\mathbb{S}^3$, where $\partial_{0,j}$ is the neutral operator ∂_0 on the j-th copy of $\mathbb{S}^3$. We recall there exists a natural identification $\widehat{\mathbb{S}^3} \sim \frac{1}{2}\mathbb{N}_0$, where to each $\ell \in \frac{1}{2}\mathbb{N}_0$ there corresponds a $2\ell+1$ dimensional representation which we also denote by ℓ. With this in mind, we shall denote $[\ell] \in \widehat{\mathbb{S}^3 \times \dots \times \mathbb{S}^3}$ with $\ell \in \frac{1}{2}\mathbb{N}_0^s$. Following the notation used so far, as seen in [94], the symbol of each X_j is given by

$$\sigma_{i\partial_0}(\ell_j)_{\alpha_j\beta_j} = i(\alpha_j - \ell_j - 1)\delta_{\alpha_j\beta_j},$$

for $1 \leq \alpha_j, \beta_j \leq 2\ell_j + 1$, $j = 1,\dots,n$, where $\delta_{\alpha_j\beta_j}$ is the Kronecker's delta. Note that this means

$$-\ell_j \leq \mu_{\alpha_j}(\ell_j) \leq \ell_j,\ \ell_j - \mu_{\alpha_j}(\ell_j) \in \mathbb{N}_0,$$

for every $1 \leq \alpha_j \leq 2\ell_j + 1$, $j = 1,\dots,n$. This means in case $\ell_j \in \frac{1}{2}\mathbb{N}_0 \setminus \mathbb{N}_0$, then $\mu_{\alpha_j}(\ell_j)$ assumes values in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ between $-\ell$ and ℓ, and for the case $\ell_j \in \mathbb{N}_0$ we have that $\mu_{\alpha_j}(\ell_j)$ assumes values in $\mathbb{Z}$. Also, note that $\langle \ell_j \rangle = \sqrt{1 + \ell_j(\ell_j + 1)}$ has the same asymptotic behaviour as ℓ_j for $\ell_j \to +\infty$. Therefore the condition for global solvability can be rewritten as follows.

Corollary 4.2.5. *The operator L given by*

$$L = c_1 i\partial_{0,1} + \cdots + c_n i\partial_{0,n} + q$$

is globally solvable if and only if there exist $M, N > 0$ such that

$$|\langle c, \mu_\alpha(\ell)\rangle - iq| \geq M(\ell_1 + \cdots + \ell_n)^{-N},$$

for every $-\ell \leq \mu_\alpha(\ell) \leq \ell$, $\ell - \mu_\alpha(\ell) \in \mathbb{N}_0^s$, $\ell \in \frac{1}{2}\mathbb{N}_0^s \backslash \{0\}$ such that the left hand side of the inequality is not zero.

The global solvability in this case is determined by conditions very similar to the ones in the previous subsection. On the other hand, in terms of conditions for global hypoellipticity, there is a significant difference. Note that now if

$$c_1\mu_{\alpha_1}(\ell_1) + \cdots + c_n\mu_{\alpha_n}(\ell_n) - iq = 0 \tag{4.12}$$

for some $\ell \in \frac{1}{2}\mathbb{N}_0^s$, $\alpha \in J_\ell$, then as $\mu_{\alpha_j}(\ell_j) = \mu_{\alpha_j+m}(\ell_j + m)$ for every $m \in \mathbb{N}_0$, we have that the set of all ℓ and α such that $c_1\mu_{\alpha_1}(\ell_1) + \cdots + c_n\mu_{\alpha_n}(\ell_n) - iq = 0$ is either empty or infinite. In particular, if $q = 0$, then L is not globally hypoelliptic, as in this case clearly $\ell = 0$ and $\alpha = \overline{1}$ is a solution to the previous equation.

Hence, we can rewrite the condition condition for global hypoellipticity as follows.

Corollary 4.2.6. *The operator L given by*

$$L = c_1 i\partial_{0,1} + \cdots + c_n i\partial_{0,n} + q$$

is globally hypoelliptic if and only if

$$c_1 z_1 + \cdots + c_n z_n - iq \neq 0$$

for all $z_1, \ldots, z_n \in \frac{1}{2}\mathbb{Z}^n$ and there exist $M, N > 0$ such that

$$|\langle c, \mu_\alpha(\ell)\rangle - iq| \geq M(\ell_1 + \cdots + \ell_n)^{-N},$$

for every $-\ell \leq \mu_\alpha(\ell) \leq \ell$, $\ell - \mu_\alpha(\ell) \in \mathbb{N}_0^s$, $\ell \in \frac{1}{2}\mathbb{N}_0^s \setminus \{0\}$.

Chapter 5

Variable coefficients operators

5.1 Real coefficients

Let G_0 be a compact Lie group and $G = G_1 \times \cdots \times G_n$ be a product of compact Lie groups. In this chapter we will prove necessary and sufficient conditions for the global solvability and global hypoellipticity for operators of the form

$$L = X_0 + a_1(x_0)X_1 + \cdots + a_n(x_0)X_n + q(x_0, x_1, \ldots, x_n), \tag{5.1}$$

where each a_j is a smooth real-valued function on G_0, q is a smooth complex-valued function on $G_0 \times G$ and each X_j is a non-zero left-invariant vector field on G_j, for $j = 0, 1, \ldots, n$.

We will show that under suitable conditions, the global solvability and global hypoellipticity of the operator L is completely determined by whether or not these properties hold for the constant coefficients operator L_0 obtained by taking the averages of the coefficients of the operator L. First, however, we will need the following lemma.

Lemma 5.1.1. *For a compact Lie group G', let $L_1, L_2 : \mathcal{D}'(G') \to \mathcal{D}'(G')$ be differential operators on G', such that $\exists \Psi : \mathcal{D}'(G') \to \mathcal{D}'(G')$ smooth automorphism which satisfies $\Psi(C^\infty(G')) = C^\infty(G')$ and $L_1 \circ \Psi = \Psi \circ L_2$. Then L_1 is globally hypoelliptic if and only if L_2 is globally hypoelliptic.*

Proof. Suppose L_1 is globally hypoelliptic and $L_2 u = f \in C^\infty(G')$. Then $L_1 \circ \Psi u = \Psi \circ L_2 u = \Psi f \in C^\infty(G')$. Since L_1 is globally hypoelliptic, $\Psi u \in C^\infty(G')$ and so $u \in C^\infty(G')$, from which we conclude L_2 is globally hypoelliptic also. In case L_2 is globally hypoelliptic, then note that Ψ^{-1} satisfies $L_2 \circ \Psi^{-1} = \Psi^{-1} \circ L_1$, so the result follows by symmetry. □

Proposition 5.1.2. *Suppose that there exist real functions $A_j \in C^\infty(G_0)$ such that $X_0 A_j(x_0) = a_j(x_0) - a_{j0}$, for some $a_{j0} \in \mathbb{R}$, $j = 1, \ldots, n$, and $Q \in C^\infty(G_0 \times G)$ such that $(L - q)Q = q - q_0$, for some $q_0 \in \mathbb{C}$. Then L defined in (5.1) is globally hypoelliptic (resp. solvable) if and only if L_0 is globally hypoelliptic (resp. solvable), where L_0 is the operator given by*

$$L_0 \doteq X_0 + a_{10}X_1 + \cdots + a_{n0}X_n + q_0.$$

Proof. By Lemma 5.1.1, it suffices to exhibit an automorphism Ψ' on $\mathcal{D}'(G_0 \times G)$ that preserves $C^\infty(G_0 \times G)$ and conjugates L and L_0, to prove L is globally hypoelliptic if and only if L_0 is globally hypoelliptic. Consider the mapping $\Psi' : \mathcal{D}'(G_0 \times G) \to \mathcal{D}'(G_0 \times G)$ given by

$$\Psi' \doteq \Psi_q \circ \Psi,$$

where

$$\Psi u(x_0, x_1, \ldots, x_n) \doteq \sum_{[\xi] \in \widehat{G}} d_\xi \sum_{\alpha, \beta \in J_\xi} e^{-i\left(\sum_{j=1}^n \mu_{\alpha_j}(\xi^j) A_j(x_0)\right)} \widehat{u}(x_0, \xi)_{\alpha\beta} \xi_{\beta\alpha}(x), \tag{5.2}$$

and

$$\Psi_q u(x_0, x) \doteq u(x_0, x) e^{-Q(x_0, x)},$$

in the sense of distributions, for every $u \in \mathcal{D}'(G_0 \times G)$. A simple computation verifies that $L \circ \Psi_q \circ \Psi = \Psi_q \circ \Psi \circ L_0$. We claim that Ψ' is an automorphism on $\mathcal{D}'(G_0 \times G)$ which preserves $C^\infty(G_0 \times G)$. Indeed, first we will show that if $u \in C^\infty(G_0 \times G)$, then also $\Psi' u \in C^\infty(G_0 \times G)$. Indeed, let $\gamma \in \mathbb{N}_0^{\dim G_0}$. Then by (5.2) we have

$$\begin{aligned}
|\partial^\gamma \widehat{\Psi u}(x_0, \xi)_{\alpha\beta}| &= \left| \partial^\gamma \left[e^{-i\left(\sum_{j=1}^n \mu_{\alpha_j}(\xi^j) A_j(x_0)\right)} \widehat{u}(x_0, \xi)_{\alpha\beta} \right] \right| \\
&\leq \sum_{\gamma' + \gamma'' = \gamma} \left| \partial^{\gamma'} e^{-i\left(\sum_{j=1}^n \mu_{\alpha_j}(\xi^j) A_j(x_0)\right)} \right| \left| \partial^{\gamma''} \widehat{u}(x_0, \xi)_{\alpha\beta} \right| \\
&\leq \sum_{\gamma' + \gamma'' = \gamma} \sum_{j=1}^n C_{\gamma' j} |\mu_{\alpha_j}(\xi^j)|^{|\gamma'|} \left| \partial^{\gamma''} \widehat{u}(x_0, \xi)_{\alpha\beta} \right|,
\end{aligned}$$

for some $C_{\gamma' j} > 0$. Therefore, given $N > 0$, as $u \in C^\infty(G_0 \times G)$, by Lemma 3.1.5 and

Remark 3.1.9, there exists $C'_{\gamma''N} > 0$ such that

$$|\partial^\gamma \widehat{\Psi u}(x_0,\xi)_{\alpha\beta}| \leq \sum_{\gamma'+\gamma''=\gamma} \sum_{j=1}^{n} C_{\gamma' j} \langle \xi^j \rangle^{|\gamma'|} C'_{\gamma''N} \left(\langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle\right)^{-(N+|\gamma|)}$$
$$\leq C''_{\gamma N} \left(\langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle\right)^{-N}$$

for some $C''_{\gamma N} > 0$ and all $x_0 \in G_0$, $[\xi] \in \widehat{G}$, $\alpha, \beta \in J_\xi$. By Lemma 3.1.5 we conclude that $\Psi u \in C^\infty(G_0 \times G)$, and so clearly $\Psi' u \in C^\infty(G_0 \times G)$. The proof that $u \in \mathcal{D}'(G_0 \times G) \implies \Psi' u \in \mathcal{D}'(G_0 \times G)$ is similar. Notice that Ψ'^{-1} is given by $\Psi^{-1} \circ \Psi_q^{-1}$, where

$$\Psi^{-1} u(x_0, x_1, \ldots, x_n) \doteq \sum_{[\xi] \in \widehat{G}} d_\xi \sum_{\alpha,\beta \in J_\xi} e^{i\left(\sum_{j=1}^n \mu_{\alpha_j}(\xi^j) A_j(x_0)\right)} \widehat{u}(x_0,\xi)_{\alpha\beta} \xi_{\beta\alpha}(x),$$

and

$$\Psi_q^{-1} u(x_0, x) \doteq u(x_0, x) e^{Q(x_0,x)},$$

in the sense of distributions, for every $u \in \mathcal{D}'(G_0 \times G)$, so it is clear that Ψ' is an automorphism. This proves the claim on global hypoellipticity. It remains to prove that L is globally solvable if and only if L_0 is globally solvable. First assume that L is globally solvable. Note that L and L_0 satisfy ${}^tL = -L + 2q$ and ${}^tL_0 = -L_0 + 2q_0$. Also, $(-L + 2q) \circ \Psi_q^{-1} \circ \Psi = \Psi_q^{-1} \circ \Psi \circ (-L_0 + 2q_0)$. Indeed, this follows from the following facts that can be verified by a simple calculation:

- $-(X_0 + \sum_{j=1}^n a_j(x_0) X_j) \circ \Psi_q^{-1} u = \Psi_q^{-1} \circ (-(X_0 + \sum_{j=1}^n a_j(x_0) X_j)) u + (-q + q_0) \Psi_q u$
- $(-X_0) \circ \Psi u = \Psi \circ (\sum_{j=1}^n \mu_{\alpha_j}(\xi^j)(a_j - a_{j0})) u + \Psi \circ (-X_0) u$
- $\Psi \circ (-\sum_{j=1}^n \mu_{\alpha_j}(\xi^j)(a_{j0})) u = \Psi \circ (-\sum_{j=1}^n a_{j0}(x_0) X_j) u$
- $(-\sum_{j=1}^n a_j(x_0) X_j) \circ \Psi u = -\sum_{j=1}^n \mu_{\alpha_j}(\xi^j) a_j(x_0) \Psi u.$

To see why these last assertions are true, note that from Definition 2.2.10, we have that

$$\sigma_{X_j}(\xi^j) = \xi^j(x_j)^* X_j \xi^j(x_j) = X_j \xi^j(e_{G_j}),$$

so that

$$\begin{aligned} X_j\xi^j(x_j) &= \xi^j(x_j)X_j\xi^j(e_{G_j}) \\ &= \xi^j(x_j)\sigma_{X_j}(\xi^j) \\ &= \xi^j(x_j)\text{diag}(\mu_{\alpha_j}(\xi^j)). \end{aligned}$$

Therefore, $X_j\xi^j(x_j)_{\beta_j\alpha_j} = \mu_{\alpha_j}(\xi^j)\xi^j(x_j)_{\beta_j\alpha_j}$.

Using this fact, we may prove that $\Psi_q^{-1} \circ \Psi(\ker {}^tL_0) = \ker {}^tL$. Indeed if $u \in \ker {}^tL_0$, then

$$\begin{aligned} 0 &= \Psi_q^{-1} \circ \Psi \circ {}^tL_0\, u \\ &= \Psi_q^{-1} \circ \Psi \circ (-L_0 + 2q_0)u \\ &= (-L + 2q) \circ \Psi_q^{-1} \circ \Psi u \\ &= {}^tL \circ \Psi_q^{-1} \circ \Psi u, \end{aligned}$$

therefore $\Psi_q^{-1} \circ \Psi u \in \ker({}^tL)$ and $\Psi_q^{-1} \circ \Psi(\ker {}^tL_0) \subset \ker {}^tL$. Conversely, if $v \in \ker {}^tL$ then

$$\begin{aligned} 0 &= \Psi^{-1} \circ \Psi_q \circ {}^tL\, v \\ &= \Psi^{-1} \circ \Psi_q \circ (-L + 2q)v \\ &= (-L_0 + 2q_0) \circ \Psi^{-1} \circ \Psi_q v \\ &= {}^tL_0 \circ \Psi^{-1} \circ \Psi_q v, \end{aligned}$$

so $\Psi^{-1} \circ \Psi_q(\ker {}^tL) \subset \ker {}^tL_0$ which implies $\ker {}^tL \subset \Psi_q^{-1} \circ \Psi(\ker {}^tL_0)$, which finishes the proof that $\Psi_q^{-1} \circ \Psi(\ker {}^tL_0) = \ker {}^tL$.

Therefore every $v \in \ker {}^tL$ may be written as $v = (\Psi_q^{-1} \circ \Psi)u$, where $u \in \ker {}^tL_0$. Now if $f \in (\ker {}^tL_0)^0 \cap C^\infty(G_0 \times G)$, then $(\Psi_q \circ \Psi)f \in (\ker {}^tL)^0 \cap C^\infty(G_0 \times G)$. Indeed, for any $v = \Psi_q^{-1} \circ \Psi u$, $u \in \ker {}^tL_0$, Proposition 3.1.7 implies that

$$\begin{aligned} &\int_{G_0\times G} \Psi_q \circ \Psi f(x_0, x)\Psi_q^{-1} \circ \Psi u(x_0, x)dx_0dx \\ &= \int_{G_0\times G} \Psi f(x_0, x)e^{-Q(x_0,x)}\Psi u(x_0, x)e^{Q(x_0,x)}dx_0dx \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \int_{G_0} \widehat{f}(x_0, \xi)_{\alpha\beta}e^{-i\left(\sum_{j=1}^n \mu_{\alpha_j}(\xi^j)A_j(x_0)\right)}\widehat{u}(x_0, \overline{\xi})_{\alpha\beta}e^{-i\left(\sum_{j=1}^n \mu_{\alpha_j}(\overline{\xi^j})A_j(x_0)\right)}\, dx_0\,. \end{aligned}$$

Since $\mu_{\alpha_j}(\overline{\xi^j}) = -\mu_{\alpha_j}(\xi^j)$, for every $[\xi^j] \in \widehat{G_j}$, the exponentials cancel out. Therefore

$$\int_{G_0\times G} (\Psi_q \circ \Psi) f(x_0, x)(\Psi_q^{-1} \circ \Psi) u(x_0, x) dx_0 dx = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \int_{G_0} \widehat{f}(x_0, \xi)_{\alpha\beta} \widehat{u}(x_0, \overline{\xi})_{\alpha\beta}\, dx_0$$
$$= \int_{G_0\times G} f(x_0, x) u(x_0, x)\, dx_0\, dx = 0,$$

where we used Proposition 3.1.7 once again, and the fact that $f \in (\ker {}^tL_0)^0$, $u \in \ker {}^tL_0$. This proves the claim that $(\Psi_q \circ \Psi) f \in (\ker {}^tL)^0 \cap C^\infty(G_0 \times G)$, so by the definition of global solvability there exists $u \in C^\infty(G_0 \times G)$ such that $Lu = (\Psi_q \circ \Psi)f$. But then $f = (\Psi^{-1} \circ \Psi_q^{-1} \circ L)u = (L_0 \circ \Psi^{-1} \circ \Psi_q^{-1})u$, so that L_0 is globally solvable. The converse follows by symmetry, which finishes the proof. □

Example 5.1.3. The following example has been extracted from [40], but with slightly different notation. Let $G_0 = G_1 = \mathbb{S}^3$ and consider the operator $L = X_0 + a_1(x_0)X_1$, where X_j corresponds to the vector field $D_{3,j} = \partial_{\psi_j} = i\partial_{0,j}$ on the x_j variable, and $a_1 : \mathbb{S}^3 \to \mathbb{R}$ is expressed in Euler's angles by

$$a_1(x_0(\phi_0, \theta_0, \psi_0)) = -\cos\left(\frac{\theta_0}{2}\right) \sin\left(\frac{\phi_0 + \psi_0}{2}\right) + \sqrt{2},$$

which satisfies

$$X_0 \operatorname{Tr}(x_0) = a_1(x_0) - \sqrt{2}.$$

Since $a_{10} = \sqrt{2}$, by propositions 4.1.4 and 5.1.2 we have that L is not globally hypoelliptic, but is globally solvable.

5.1.1 The case of the torus

Assume now $G_0 = \mathbb{T}^1$, $X_0 = \partial_t$. In other words, the operator L can be seen as an "evolution operator" on a periodic time variable. In this case, the conditions for Proposition 5.1.2 are satisfied for any $a_1, \ldots, a_n \in C^\infty(\mathbb{T}^1)$ and $q \in C^\infty(\mathbb{T}^1)$. Indeed, taking

$$A_j(t) = \int_0^t a_j(\tau) d\tau - a_{j0} t,$$

for every $t \in [0, 2\pi]$, where $a_{j0} = \frac{1}{2\pi}\int_0^{2\pi} a_j(\tau)d\tau$, $j = 1, \dots, n$, and

$$Q(t) = \int_0^t q(\tau)d\tau - q_0 t,$$

for every $t \in [0, 2\pi]$, where $q_0 = \frac{1}{2\pi}\int_0^{2\pi} q(\tau)d\tau$, these satisfy

$$X_0 A_j(t) = \partial_t A_j(t) = a_j(t) - a_{j0}, \qquad j = 1, \dots, n,$$

and

$$(L - q)Q(t) = \partial_t Q(t) = q(t) - q_0,$$

for every $t \in [0, 2\pi]$. Therefore Proposition 5.1.2 implies the following corollary:

Corollary 5.1.4. *Let X_j be a left-invariant vector field on a compact Lie group G_j, for each $j = 1, \dots, n$, and consider the partial differential operator on $\mathbb{T}^1 \times G_1 \times \cdots \times G_n$ given by*

$$L = \partial_t + a_1(t)X_1 + \cdots + a_n(t)X_n + q,$$

where $a_1, \dots, a_n$ are smooth real-valued functions on $\mathbb{T}^1$, $q \in C^\infty(\mathbb{T}^1)$. The operator L is globally solvable if and only if there exist $M, N > 0$ such that

$$|k + a_{10}\mu_{\alpha_1}(\xi^1) + \cdots + a_{n0}\mu_{\alpha_n}(\xi^n) - iq_0| \geq M\left(|k| + \langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle\right)^{-N} \tag{5.3}$$

for every $(k, [\xi]) \in \mathbb{Z} \times \widehat{G}$, $\alpha \in J_\xi$, such that the left hand side of the inequality is not zero, where $a_{j0} = \frac{1}{2\pi}\int_0^{2\pi} a_j(t)dt$, $j = 1, \dots, n$, $q_0 = \frac{1}{2\pi}\int_0^{2\pi} q(t)dt$. It is globally hypoelliptic if and only if the equation

$$k + a_{10}\mu_{\alpha_1}(\xi^1) + \cdots + a_{n0}\mu_{\alpha_n}(\xi^n) - iq_0 = 0$$

has only finitely many solutions for $(k, [\xi]) \in \mathbb{Z} \times \widehat{G}$, $\alpha \in J_\xi$, and inequality (5.3) also holds whenever its left-hand side is not zero.

Example 5.1.5. Consider L as given in Corollary 5.1.4. Suppose also that $\mathrm{Re}(q_0) \neq 0$. Then L is both globally solvable and globally hypoelliptic, as

$$|k + a_{10}\mu_{\alpha_1}(\xi^1) + \cdots + a_{n0}\mu_{\alpha_n}(\xi^n) - iq_0| \geq |\,\mathrm{Re}(q_0)| > 0,$$

for every $(k, [\xi]) \in \mathbb{Z} \times \widehat{G}$, $\alpha \in J_\xi$.

Example 5.1.6. Let L be as in Corollary 5.1.4. Assume also that $G_1, \ldots, G_n = \mathbb{T}^1$. If

i) a_{l0} is an irrational non-Liouville number, but $a_{j0}, \operatorname{Im}(q) \in \mathbb{Q}$, for $1 \leq j \neq l \leq n$, or;

ii) $a_{10}, \ldots, a_{n0}, \operatorname{Im}(q_0) \in \mathbb{Q}$.

Then L is globally solvable and, in the second case, L is not globally hypoelliptic.

Indeed, both cases (i) and (ii) follow from Corollaries 4.2.1 and 5.1.4 and their respective comments.

Example 5.1.7. For $G_1, \ldots, G_n = \mathbb{T}^1$, if $q_0 \in i\,\mathbb{Z}$ and there exists $1 \leq j \leq n$ such that a_{j0} is a Liouville number, then L is not globally solvable (nor hypoelliptic). Indeed, this follows from Corollaries 4.2.3 and 5.1.4.

5.2 Complex coefficients

The goal of this section is to study the case where the operator L has complex-valued coefficients. The standard approach is to apply the partial Fourier transform to the "space" variables $x_1, \ldots, x_n$, in order to obtain a uncoupled system of differential equations in only the "time" variable t. Since this technique requires solving a differential equation on the compact Lie group G_0, we will only consider the case $G_0 = \mathbb{T}^1$, and $X_0 = \partial_t$. This way we may apply the standard theory for solving ordinary differential equations on the torus, which boils down to solving these equations on the real line, but with periodic boundary conditions. With this in mind, in this chapter we study the operator

$$L \doteq \partial_t + \left[\sum_{j=1}^{n} c_j(t) X_j\right] + q(t), \tag{5.4}$$

where $c_j(t) = a_j(t) + ib_j(t)$ are smooth complex-valued functions on $\mathbb{T}^1$, X_j are non-zero left-invariant vector fields on the compact Lie groups G_j, $j = 1, \ldots, n$, and $q \in C^\infty(\mathbb{T}^1)$. The same arguments used in Proposition 5.1.2 can be used to prove that the global hypoellipticity and solvability of L is equivalent to those same properties of the operator given by

$$\partial_t + \left[\sum_{j=1}^{n} (a_{j0} + ib_j(t)) X_j\right] + q_0,$$

where $\mathbb{R} \ni a_{j0} = \frac{1}{2\pi}\int_0^{2\pi} a_j(t)dt$, $j = 1, \ldots, n$, and $\mathbb{C} \ni q_0 = \frac{1}{2\pi}\int_0^{2\pi} q(t)dt$, so we will assume these functions are all constant. In other words, we will consider $a_j = a_{j0} \in \mathbb{R}$, $j = 1, \ldots, n$, and $q = q_0 \in \mathbb{C}$. We also define the constant coefficients operator

$$L_0 \doteq \partial_t + \left[\sum_{j=1}^{n} c_{j0}(t) X_j\right] + q, \tag{5.5}$$

where $c_{j0} = \frac{1}{2\pi}\int_0^{2\pi} c_j(t)dt$.

First, we present some notation and prove some auxiliary lemmas.

Notice that if $u, f \in C^\infty(\mathbb{T}^1 \times G)$ satisfy $Lu = f$, comparing partial Fourier coefficients on $\mathbb{T}^1 \times G$ yields the following uncoupled system of ordinary differential equations on the torus:

$$\partial_t \widehat{u}(t,\xi)_{\alpha\beta} + i(\langle c(t), \mu_\alpha(\xi)\rangle - iq)\, \widehat{u}(t,\xi)_{\alpha\beta} = \widehat{f}(t,\xi)_{\alpha\beta}, \tag{5.6}$$

for $t \in \mathbb{T}^1$, $[\xi] \in \widehat{G}$, and $\alpha, \beta \in J_\xi$, where $\langle c(t), \mu_\alpha(\xi)\rangle \doteq \sum_{j=1}^{n} c_j(t)\mu_{\alpha_j}(\xi^j)$.

Lemma 5.2.1. *Let L be as defined in (5.4). Suppose that the sequence of smooth functions $\widehat{u}(\cdot,\xi)_{\alpha\beta} \in C^\infty(\mathbb{T}^1)$ satisfies*

$$[\partial_t + i(\langle c(t), \mu_\alpha(\xi)\rangle - iq)]\, \widehat{u}(t,\xi)_{\alpha\beta} = \widehat{f}(t,\xi)_{\alpha\beta}, \tag{5.7}$$

for all $t \in \mathbb{T}^1$, $[\xi] \in \widehat{G}$, and $\alpha, \beta \in J_\xi$, where $f \in C^\infty(\mathbb{T}^1 \times G)$. If there exist $K, N > 0$, such that

$$|\widehat{u}(t,\xi)_{\alpha\beta}| \leq K\left(\langle \xi^1\rangle + \cdots + \langle \xi^n\rangle\right)^N \|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty,$$

for all $t \in \mathbb{T}^1$, $[\xi] \in \widehat{G}$, and $\alpha, \beta \in J_\xi$, then these Fourier coefficients define a smooth function $u \in C^\infty(\mathbb{T}^1 \times G)$, which also satisfies $Lu = f$.

Proof. To prove these coefficients define a smooth function, by Lemma 3.1.5 we must prove that $\forall m \in \mathbb{N}_0, \forall N' > 0$, $\exists K'_{mN'} > 0$ such that

$$|\partial_t^m \widehat{u}(t,\xi)_{\alpha\beta}| \leq K'_{mN'}\left(\langle \xi^1\rangle + \cdots + \langle \xi^n\rangle\right)^{-N},$$

for all $t \in \mathbb{T}^1$, $[\xi] \in \widehat{G}$, and $\alpha, \beta \in J_\xi$. We prove this by strong induction on m. The case $m = 0$, follows from the hypothesis and Lemma 3.1.5. Suppose the claim holds for every case

$m \leq k \in \mathbb{N}_0$. Then, by (5.7), we have that

$$\begin{aligned}
\partial_t^{k+1}\widehat{u}(t,\xi)_{\alpha\beta} &= \partial_t^k(\partial_t\widehat{u}(t,\xi)_{\alpha\beta}) = \\
&= \partial_t^k\left(-i(\langle c(t),\mu_\alpha(\xi)\rangle - iq)\widehat{u}(t,\xi)_{\alpha\beta} + \widehat{f}(t,\xi)_{\alpha\beta}\right) \\
&= -i\sum_{j=0}^{k}\binom{k}{j}(\langle\partial_t^j c(t),\mu_\alpha(\xi)\rangle\partial_t^{k-j}\widehat{u}(t,\xi)_{\alpha\beta}) + q\partial_t^k\widehat{u}(t,\xi)_{\alpha\beta} + \partial_t^k\widehat{f}(t,\xi)_{\alpha\beta},
\end{aligned}$$

and so

$$\begin{aligned}
|\partial_t^{k+1}\widehat{u}(t,\xi)_{\alpha\beta}| \leq 2^k \max_{l=1,\ldots,k}\max_{j=1,\ldots,n}\|\partial_t^j c_l\|_\infty(\langle\xi^1\rangle+\cdots+\langle\xi^n\rangle) \\
\times \max_{j=1,\ldots,k}|\partial_t^j\widehat{u}(t,\xi)_{\alpha\beta}| + |q||\partial_t^k\widehat{u}(t,\xi)_{\alpha\beta}| + \|\partial_t^k\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty,
\end{aligned} \tag{5.8}$$

for every $t \in \mathbb{T}^1$, $[\xi] \in \widehat{G}$ and $\alpha,\beta \in J_\xi$. The case $m = k+1$ then follows from the previous cases and the inequality above. Indeed, given $N > 0$, by the inductive hypothesis there exists $M > 0$ be such that

$$\max_{j=1,\ldots,k}|\partial_t^j\widehat{u}(t,\xi)_{\alpha\beta}| \leq M(\langle\xi^1\rangle+\cdots+\langle\xi^n\rangle)^{-(N+1)},$$

for every $t \in \mathbb{T}^1$, $[\xi] \in \widehat{G}$ and $\alpha,\beta \in J_\xi$. Also, since f is smooth, by Lemma 3.1.5 there exists $M' > 0$ such that

$$|\partial_t^k\widehat{f}(t,\xi)_{\alpha\beta}| \leq M'(\langle\xi^1\rangle+\cdots+\langle\xi^n\rangle)^{-N},$$

for every $t \in \mathbb{T}^1$, $[\xi] \in \widehat{G}$ and $\alpha,\beta \in J_\xi$. Applying this inequalities to (5.8) we obtain

$$|\partial_t^{k+1}\widehat{u}(t,\xi)_{\alpha\beta}| \leq ((2^k \max_{l=1,\ldots,k}\max_{j=1,\ldots,n}\|\partial_t^j c_l\|_\infty + |q|)M + M')(\langle\xi^1\rangle+\cdots+\langle\xi^n\rangle)^{-N},$$

for every $t \in \mathbb{T}^1$, $[\xi] \in \widehat{G}$ and $\alpha,\beta \in J_\xi$, which finishes the proof by induction. The fact that $Lu = f$ can be seen by comparing partial Fourier coefficients. □

Finally, for $c_0 = (c_{10},\ldots,c_{n0}) \in \mathbb{C}^n$, define the following sets

$$\mathcal{Z}_L = \left\{([\xi],\alpha) \in \widehat{G}\times\mathbb{N}^n \mid \alpha \in J_\xi,\ \langle c_0,\mu_\alpha(\xi)\rangle - iq \in \mathbb{Z}\right\} \tag{5.9}$$

$$\mathcal{Z}_L^c = \left\{([\xi],\alpha) \in \widehat{G}\times\mathbb{N}^n \mid \alpha \in J_\xi,\ \langle c_0,\mu_\alpha(\xi)\rangle - iq \notin \mathbb{Z}\right\}, \tag{5.10}$$

where $\langle c_0, \mu_\alpha(\xi)\rangle - iq$ is as in (4.3).

Sufficient Conditions

We are now ready to prove sufficient conditions for the operator L to be globally solvable and/or globally hypoelliptic. The first condition we present establishes a connection between the sublevel sets of the coefficients of L the Fourier coefficients of solutions to the equation $Lu = f$. This is similar to the conditions found in [9].

Proposition 5.2.2. *Suppose equation* (5.6) *admits solution, for some* $([\xi], \alpha) \in \mathcal{Z}_L$*,* $\beta \in J_\xi$*. Suppose also that for every* $r \in \mathbb{R}$ *each sublevel set*

$$\Omega_r^{\xi,\alpha} = \left\{ t \in \mathbb{T}^1 \middle| \int_0^t \langle b(\tau), \mu_\alpha(\xi)\rangle - \mathrm{Re}(q) d\tau < r \right\}$$

is connected. Then there exists $\widehat{u}(t,\xi)_{\alpha\beta}$*, solution to* (5.6)*, which satisfies*

$$|\,\widehat{u}(t,\xi)_{\alpha\beta}\,| \leq 2\pi \|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty,$$

for every $t \in \mathbb{T}^1$.

Proof. Let $([\xi], \alpha) \in \mathcal{Z}_L$, $\beta \in J_\xi$ be as stated. By compactness of $\mathbb{T}^1$, there exists $t_{\xi\alpha} \in \mathbb{T}^1$ such that

$$-\int_0^{t_{\xi\alpha}} \langle b(\tau), \mu_\alpha(\xi)\rangle - \mathrm{Re}(q) d\tau = \inf_{t\in\mathbb{T}^1} \left\{ -\int_0^t \langle b(\tau), \mu_\alpha(\xi)\rangle - \mathrm{Re}(q) d\tau \right\}.$$

According to the hypothesis, the differential equation given by $\partial_t\, \widehat{u}(t,\xi)_{\alpha\beta} + i(\langle c(t), \mu_\alpha(\xi)\rangle - iq)\, \widehat{u}(t,\xi)_{\alpha\beta} = \widehat{f}(t,\xi)_{\alpha\beta}$ admits solution, so by Lemma 5.A.1 the mapping

$$t \mapsto \widehat{u}(t,\xi)_{\alpha\beta} = \int_{t_{\xi\alpha}}^t \exp\left\{ -i \int_s^t \left(\langle c(\tau), \mu_\alpha(\xi)\rangle - iq\right) d\tau \right\} \widehat{f}(s,\xi)_{\alpha\beta}\, ds \tag{5.11}$$

defines one such solution. Let us prove it satisfies the inequality as stated. For $t \in \mathbb{T}^1$, define

$$r_t \doteq -\int_0^t \langle b(\tau), \mu_\alpha(\xi)\rangle - \mathrm{Re}(q) d\tau.$$

Note that t and $t_{\xi\alpha} \in \tilde{\Omega}^{\xi,\alpha}_{r_t}$, where

$$\tilde{\Omega}^{\xi,\alpha}_r = \left\{ t \in \mathbb{T}^1 \middle| -\int_0^t \langle b(\tau), \mu_\alpha(\xi)\rangle - \mathrm{Re}(q) d\tau \leq r \right\},$$

which is connected by Lemma 5.A.5 and our assumptions. Therefore there exists $\gamma_t \subset \tilde{\Omega}^{\xi,\alpha}_{r_t}$ circumference arc connecting $t_{\xi\alpha}$ to t. Since γ_t is contained in $\tilde{\Omega}^{\xi,\alpha}_{r_t}$, we have that

$$-\int_0^s \langle b(\tau), \mu_\alpha(\xi)\rangle - \mathrm{Re}(q) d\tau \leq r_t = -\int_0^t \langle b(\tau), \mu_\alpha(\xi)\rangle - \mathrm{Re}(q) d\tau, \tag{5.12}$$

for all $s \in \gamma_t$, by definition. Note that for any $t \in \mathbb{T}^1$, there are precisely two arcs connecting $t_{\xi\alpha}$ to t, whose union is $\mathbb{T}^1$. As equation (5.6) admits solution, Lemma 5.A.1 implies that

$$\begin{aligned} 0 &= \int_0^{2\pi} \exp\left\{ i \int_0^s \left(\langle c(\tau), \mu_\alpha(\xi)\rangle - iq \right) d\tau \right\} \widehat{f}(s,\xi)_{\alpha\beta}\, ds \\ &= \int_0^{2\pi} \exp\left\{ -i \int_s^0 \left(\langle c(\tau), \mu_\alpha(\xi)\rangle - iq \right) d\tau \right\} \widehat{f}(s,\xi)_{\alpha\beta}\, ds \,. \end{aligned}$$

Hence

$$\int_0^{2\pi} \exp\left\{ -i \int_s^t \left(\langle c(\tau), \mu_\alpha(\xi)\rangle - iq \right) d\tau \right\} \widehat{f}(s,\xi)_{\alpha\beta} ds = 0,$$

and so since the integration on the torus is oriented, the integral from $t_{\xi\alpha}$ to t in the definition of $\widehat{u}$ is independent on the choice of path connecting these two points. Integrating over γ_t, we can rewrite formula (5.11) as

$$\begin{aligned} \widehat{u}(t,\xi)_{\alpha\beta} &= \int_{\gamma_t} \exp\left\{ -i \int_s^t \left(\langle c(\tau), \mu_\alpha(\xi)\rangle - iq \right) d\tau \right\} \widehat{f}(s,\xi)_{\alpha\beta}\, ds \\ &= \int_{\gamma_t} \exp\left\{ \int_0^t \langle b(\tau), \mu_\alpha(\xi)\rangle - \mathrm{Re}(q) d\tau - \int_0^s \langle b(\tau), \mu_\alpha(\xi)\rangle - \mathrm{Re}(q) d\tau \right\} \\ &\times \exp\left\{ i(s-t)(\langle a_0, \mu_\alpha(\xi)\rangle + \mathrm{Im}(q)) \right\} \widehat{f}(s,\xi)_{\alpha\beta}\, ds \,. \end{aligned}$$

Therefore, by (5.12) we can estimate

$$\begin{aligned} |\, \widehat{u}(t,\xi)_{\alpha\beta} \,| &\leq \int_{\gamma_t} |\widehat{f}(s,\xi)_{\alpha\beta}| ds \\ &\leq 2\pi \|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty . \end{aligned}$$

Since $t \in \mathbb{T}^1$ was arbitrarily chosen, the inequality holds for every $t \in \mathbb{T}^1$. □

Corollary 5.2.3. *If $\mathcal{Z}_L^c$ is finite and for all sufficiently large $\langle\xi\rangle$ every sublevel $\Omega_r^{\xi,\alpha}$ is connected, then L defined in (5.4) is globally solvable.*

Proof. Let $f \in (\ker {}^tL)^0 \cap C^\infty(\mathbb{T}^1 \times G)$. Then by Lemma 5.A.3, the equation

$$\widehat{Lu}(t,\xi)_{\alpha\beta} = \widehat{f}(t,\xi)_{\alpha\beta} \iff [\partial_t + i(\langle c(t), \mu_\alpha(\xi)\rangle - iq)]\,\widehat{u}(t,\xi)_{\alpha\beta} = \widehat{f}(t,\xi)_{\alpha\beta} \tag{5.13}$$

admits solution $\widehat{u}(\cdot,\xi)_{\alpha\beta} \in C^\infty(\mathbb{T}^1)$ for every $[\xi] \in \widehat{G}$, $\alpha, \beta \in J_\xi$. We claim that the inverse Fourier transform of this sequence of functions defines a smooth function on $\mathbb{T}^1 \times G$. Let $R > 0$, be such that every sublevel $\Omega_r^{\xi,\alpha}$ is connected, for $\langle\xi\rangle > R$. Since $\mathcal{Z}_L^c$ is finite, we may take $R' \geq R$ such that $\langle\xi\rangle \leq R'$, for every $([\xi],\alpha) \in \mathcal{Z}_L^c$. Let

$$K = \max_{t\in\mathbb{T}^1} \max_{\substack{\langle\xi\rangle\leq R', \alpha\,\beta\in J_\xi \\ \widehat{f}(\cdot,\xi)_{\alpha\beta}\neq 0}} |\,\widehat{u}(t,\xi)_{\alpha\beta}\,|(\|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty)^{-1} \geq 0.$$

Then

$$|\,\widehat{u}(t,\xi)_{\alpha\beta}\,| \leq K\|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty,$$

for every $t \in \mathbb{T}^1$, $\langle\xi\rangle \leq R'$ and $\alpha, \beta \in J_\xi$. If $\langle\xi\rangle > R'$, then $([\xi],\alpha) \in \mathcal{Z}_L$, and each sublevel $\Omega_r^{\xi,\alpha}$ is connected. Proposition 5.2.2 then implies that for every $\beta \in J_\xi$ there exists a solution to equation (5.13) which satisfies

$$|\,\widehat{u}(t,\xi)_{\alpha\beta}\,| \leq 2\pi\|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty,$$

for every $t \in \mathbb{T}^1$. Taking $\tilde{K} = \max\{K, 2\pi\}$, then $|\,\widehat{u}(t,\xi)_{\alpha\beta}\,| \leq \tilde{K}\|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty$ for every $[\xi] \in \widehat{G}$, and $\alpha, \beta \in J_\xi$. Finally, Lemma 5.2.1 implies $u \in C^\infty(\mathbb{T}^1 \times G)$ and satisfies $Lu = f$, which proves L is globally solvable. □

Remark 5.2.4. It is worth noting that if G is non-trivial, then $\widehat{G}$ is infinite. Therefore, if $\mathcal{Z}_L^c$ is finite then $\mathcal{Z}_L$ must be infinite. Also, as we shall prove in Proposition 5.2.12, every vector field X_j admits a sequence $[\xi_m^j] \in \widehat{G_j}$ such that $\mu_1(\xi_m^j) \to \infty$ as $m \to \infty$. Therefore the only way for $\mathcal{Z}_L^c$ to be finite (and so $\mathcal{Z}_L$ infinite) is if $\mathrm{Re}(q) = 0$ and $b_{j0} = 0$ for every $j = 1, \ldots, n$. We will prove this by contradiction. Indeed, suppose first that $\mathrm{Re}(q) \neq 0$. Then for $[\xi] \in \widehat{G}$, $\alpha \in J_\xi$

$$\mathrm{Im}(\langle c_0, \mu_\alpha(\xi)\rangle - iq) = \langle b_0, \mu_\alpha(\xi)\rangle - \mathrm{Re}(q).$$

Therefore $\langle c_0, \mu_\alpha(\xi)\rangle - iq \in \mathbb{Z}$ only if $\langle b_0, \mu_\alpha(\xi)\rangle = \mathrm{Re}(q)$. If all b_0 are zero, then

$([\xi],\alpha) \in \mathcal{Z}_L^c$ for all $[\xi] \in \widehat{G}$, $\alpha \in J_\xi$. On the other hand, if some $b_{j0} \neq 0$, then let $[\xi_m^j] \in \widehat{G_j}$ be a sequence such that $\mu_1(\xi_m^j) \to \infty$ as $m \to \infty$. Let $\xi_m = \mathbb{1}^1 \otimes \cdots \otimes \mathbb{1}^{j-1} \otimes \xi_m^j \otimes \mathbb{1}^{j+1} \otimes \cdots \otimes \mathbb{1}^n$, where $\mathbb{1}^j \in U(1)$ is the trivial representation on G_j, that is, $\mathbb{1}^j(x_j) = 1 \in U(1)$, $\forall x_j \in G_j$, $j = 1, \ldots, n$. Notice that, in particular, $\mathbb{1}^j$ satisfies $\mu_1(\mathbb{1}^j) = 0$, for all $j = 1, \ldots, n$. Therefore, we have that $([\xi_m], \overline{1}) \in \mathcal{Z}_L$ for only finitely many m, since in this case $\langle b_0, \mu_\alpha(\xi_m) \rangle = b_{j0}\mu_1(\xi_m^j) \to \pm\infty$. Therefore $([\xi_m], \overline{1}) \in \mathcal{Z}_L^c$ for infinitely many m and $\mathcal{Z}_L^c$ must be infinite. The case where some $b_{j0} \neq 0$ and $\operatorname{Re}(q) = 0$ is analogous.

Finally, note that $\mathcal{Z}_L^c$ finite also implies that L_0 is globally solvable but not globally hypoelliptic, by Proposition 4.1.4.

Lemma 5.2.5. *The following are equivalent:*

1. *There exist $M, N > 0$ such that*

$$|k + \langle c_0, \mu_\alpha(\xi) \rangle - iq| \geq M(|k| + \langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle)^{-N},$$

for every $(k, [\xi], \alpha) \in \mathbb{Z} \times \widehat{G} \times \mathbb{N}^n$, $\alpha \in J_\xi$ such that $k + \langle c_0, \mu_\alpha(\xi) \rangle - iq \neq 0$.

2. *There exist $M, N > 0$ such that*

$$|1 - e^{\pm 2\pi i(\langle c_0, \mu_\alpha(\xi) \rangle - iq)}| \geq M\left(\langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle\right)^{-N},$$

for every $([\xi], \alpha) \in \mathcal{Z}_L^c$.

Proof. Suppose 2. does not hold. This means that for every $m \in \mathbb{N}$, there exists $[\xi_m] \in \widehat{G}$, $\alpha(m) \in J_{\xi_m}$, such that

$$0 < \left|1 - e^{\pm 2\pi i(\langle c_0, \mu_{\alpha(m)}(\xi_m) \rangle - iq)}\right| < \frac{1}{m}(\langle \xi_m^1 \rangle + \cdots + \langle \xi_m^n \rangle)^{-m}. \tag{5.14}$$

Note that this implies $|\operatorname{Re}(q) - \langle b_0, \mu_{\alpha(m)}(\xi_m) \rangle| \to 0$ and that there exists a sequence of integers $(k_m)_{m \in \mathbb{N}}$ such that

$$|k_m + \langle a_0, \mu_{\alpha(m)}(\xi_m) \rangle + \operatorname{Im}(q)| \to 0.$$

It is easy to verify that for any real numbers a and b:

$$|1 - e^{a+bi}| \geq |1 - e^a|.$$

Therefore, for all sufficiently large $m \in \mathbb{N}$, the mean value theorem implies that

$$\begin{aligned} \left|1 - e^{\pm 2\pi i(\langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq)}\right| &\geq \left|1 - e^{\pm 2\pi(\operatorname{Re}(q) - \langle b_0, \mu_{\alpha(m)}(\xi_m)\rangle)}\right| \\ &\geq e^{-1} 2\pi |\operatorname{Re}(q) - \langle b_0, \mu_{\alpha(m)}(\xi_m)\rangle|. \end{aligned} \tag{5.15}$$

Also, since $\frac{\sin(2\pi x)}{2\pi x} \to 1$ as $x \to 0$, we have that

$$|\sin(2\pi(k_m + \langle a_0 \mu_{\alpha(m)}(\xi_m)\rangle + \operatorname{Im}(q))| \geq \pi |k_m + \langle a_0, \mu_{\alpha(m)}(\xi_m)\rangle + \operatorname{Im}(q)|, \tag{5.16}$$

for all sufficiently large $m \in \mathbb{N}$. For sufficiently large m, we also have that $e^{2\pi(\operatorname{Re}(q) - \langle b_0, \mu_{\alpha(m)}\rangle}$ $\geq \frac{1}{2}$, as this sequence has limit equal to 1. Therefore, for all such m, we have that

$$\begin{aligned} \pi|k_m + \langle a_0, \mu_{\alpha(m)}(\xi_m)\rangle + \operatorname{Im}(q)| &\leq |\sin(2\pi(k_m + \langle a_0, \mu_{\alpha(m)}(\xi_m)\rangle + \operatorname{Im}(q))| \\ &\leq 2e^{2\pi(\operatorname{Re}(q) - \langle b_0, \mu_{\alpha(m)}(\xi_m)\rangle)} |\sin(2\pi(k_m + \langle a_0, \mu_{\alpha(m)}(\xi_m)\rangle + \operatorname{Im}(q))| \\ &= 2\left|\operatorname{Im}(1 - e^{\pm 2\pi i(\langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq)})\right| \\ &\leq 2\left|1 - e^{\pm 2\pi i(\langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq)}\right|. \end{aligned} \tag{5.17}$$

Together, inequalities (5.14), (5.15), (5.16) and (5.17) imply that for $C = \frac{4+e}{2\pi} > 0$ and all sufficiently large m

$$\begin{aligned} 0 &< |k_m + \langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq| \\ &\leq |k_m + \langle a_0, \mu_{\alpha(m)}(\xi_m)\rangle + \operatorname{Im}(q)| + |\operatorname{Re}(q) - \langle b_0, \mu_{\alpha(m)}(\xi_m)\rangle| \\ &\leq \frac{4+e}{2\pi} \left|1 - e^{\pm 2\pi i(\langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq)}\right| \leq \frac{C}{m} \left(\langle \xi_m^1 \rangle + \cdots + \langle \xi_m^n \rangle\right)^{-m}. \end{aligned}$$

We claim that this implies 1. cannot hold. Indeed, suppose that it holds. Then there exist $M, N > 0$, such that for every $m \in \mathbb{N}$ we have that

$$\begin{aligned} M\left(|k_m| + \langle \xi_m^1 \rangle + \cdots + \langle \xi_m^n \rangle\right)^{-N} &\leq |k_m + \langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq| \\ &< \frac{C}{m} \left(\langle \xi_m^1 \rangle + \cdots + \langle \xi_m^n \rangle\right)^{-m}. \end{aligned}$$

But then

$$0 < \frac{M}{C} < \frac{1}{m}\left(\langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle\right)^{-m}\left(|k_m| + \langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle\right)^N$$
$$= \frac{1}{m}\left(\langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle\right)^{-m+N}\left(\frac{|k_m|}{\langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle} + 1\right)^N.$$

Since $|k_m + \langle a_0, \mu_{\alpha(m)}(\xi_m)\rangle + \mathrm{Im}(q)| < 1$ for all sufficiently large $m \in \mathbb{N}$, this implies that for all such m we have that

$$|k_m| \leq 1 + K(\langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle) + |\,\mathrm{Im}(q)|,$$

where

$$0 \leq K = \max\{C_1|a_{10}|, \ldots, C_n|a_{n0}|\}.$$

Here, $C_j > 0$ satisfies $|\mu_{\alpha_j}(\xi^j)| \leq C_j\langle\xi^j\rangle$, for every $\alpha_j \in J_{\xi^j}$, $[\xi^j] \in \widehat{G_j}$ and $j = 1, \ldots, n$. Therefore

$$0 < \frac{M}{C} < \frac{1}{m}\left(\langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle\right)^{-m+N}\left(\frac{1 + |\,\mathrm{Im}(q)|}{\langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle} + K + 1\right)^N,$$

for every sufficiently large $m \in \mathbb{N}$. But this cannot be true since the right hand side of the inequality above tends to 0 as $m \to +\infty$. This proves the claim and so $1.$ cannot hold if $2.$ does not hold.

Next, suppose $1.$ does not hold. Then, for every $m \in \mathbb{N}$, there exist $k_m \in \mathbb{Z}$, $[\xi_m] \in \widehat{G}$, $\alpha(m) \in J_{\xi_m}$ such that

$$0 < |k_m + \langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq| < \frac{1}{m}(|k_m| + \langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle)^{-m}.$$

In particular, every $([\xi_m], \alpha(m))$ is in $\mathcal{Z}_L^c$ and $k_m + \langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq$ tends to 0 as $m \to +\infty$. Let us define for every $m \in \mathbb{N}$, the real numbers

$$\gamma_m \doteq \mathrm{Re}(k_m + \langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq),$$
$$\nu_m \doteq \mathrm{Im}(k_m + \langle c_0, \mu_{\alpha(m)}(\xi_m)\rangle - iq).$$

Note that

$$\left|1-e^{\mp 2\pi i(\langle c_0,\mu_{\alpha(m)}(\xi_m)\rangle - iq)}\right| \leq |1-e^{\pm 2\pi\nu_m}\cos(2\pi\gamma_m)| + e^{\pm 2\pi\nu_m}|\sin(2\pi\gamma_m)|, \tag{5.18}$$

for every $m\in\mathbb{N}$. Also, for any real numbers x and y we have the inequality

$$|1-e^x\cos(y)| \leq |1-\cos(y)| + |\cos(y)||1-e^x| \leq |1-\cos(y)| + |1-e^x|.$$

Therefore inequality (5.18) implies that for each m sufficiently large

$$\begin{aligned}\left|1-e^{\pm 2\pi i(\langle c_0,\mu_{\alpha(m)}(\xi_m)\rangle - iq)}\right| &\leq |1-\cos(2\pi\gamma_m)| + |1-e^{\mp 2\pi\nu_m}| + e^{\mp 2\pi\nu_m}|\sin(2\pi\gamma_m)| \\ &\leq 2\pi|\gamma_m| + 2\pi|\nu_m|e + 2\pi|\nu_m|e \\ &\leq C(|\gamma_m|+|\nu_m|),\end{aligned}$$

for $C = 6\pi e$, where for the second inequality we have applied the mean value theorem on each factor of the sum in the first line, and also the fact that both $\nu_m,\gamma_m\to 0$. Therefore

$$\begin{aligned}\left|1-e^{\pm 2\pi i(\langle c_0,\mu_{\alpha(m)}(\xi_m)\rangle - iq)}\right| &\leq 2C|\gamma_n + i\nu_m| \\ &< 2\frac{C}{m}(|k_m| + \langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle)^{-m} \\ &\leq 2\frac{C}{m}(\langle\xi_m^1\rangle + \cdots + \langle\xi_m^n\rangle)^{-m},\end{aligned}$$

for all sufficiently large m, so that 2. cannot hold. □

Proposition 5.2.6. *Suppose that there exists $R>0$ such that for all $\langle\xi\rangle > R$, we have that*

- $\mathbb{T}^1\ni t\mapsto\langle b(t),\mu_\alpha(\xi)\rangle \doteq \sum_{j=1}^n b_j(t)\mu_{\alpha_j}(\xi^j)$ *does not change sign if* $(\xi,\alpha)\in\mathcal{Z}_L^c$*, and*
- *the sublevel* $\Omega_r^{\xi,\alpha}$ *is connected for every* $(\xi,\alpha)\in\mathcal{Z}_L$*,* $r\in\mathbb{R}$*.*

Suppose also L_0 defined in (5.5) is globally solvable. Then L defined in (5.4) is globally solvable, and if $\mathcal{Z}_L$ is finite, then L is globally hypoelliptic.

Proof. Let $f \in (\ker{}^tL)^0\cap C^\infty(\mathbb{T}^1\times G)$ and $R>0$ be as stated. We shall exhibit $u\in C^\infty(\mathbb{T}^1\times G)$ such that $Lu=f$, by choosing the appropriate partial Fourier coefficients for u. By Lemma 5.A.3 we know that for every $[\xi]\in\widehat{G}$, $\alpha,\beta\in J_\xi$, the differential equation

$$\widehat{Lu}(t,\xi)_{\alpha\beta} = [\partial_t + i(\langle c(t),\mu_\alpha(\xi)\rangle - iq)]\,\widehat{u}(t,\xi)_{\alpha\beta} = \widehat{f}(t,\xi)_{\alpha\beta} \tag{5.19}$$

admits smooth solution. For each $\langle\xi\rangle \leq R$, and $\alpha, \beta \in J_\xi$, define $\widehat{u}(\cdot,\xi)_{\alpha\beta}$ to be one such solution. Since there are only finitely many such coefficients (see Remark 2.2.7), we can find, as in the proof of Corollary 5.2.3, $K_1 > 0$ such that

$$|\,\widehat{u}(t,\xi)_{\alpha\beta}\,| \leq K_1 \|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty, \tag{5.20}$$

for every $t \in \mathbb{T}^1$, $\langle\xi\rangle \leq R$, and $\alpha, \beta \in J_\xi$. Now suppose $\langle\xi\rangle > R$ and $([\xi], \alpha) \in \mathcal{Z}_L$. In this case, every sublevel $\Omega_r^{\xi,\alpha}$ is connected. Hence by Lemma 5.2.2, for every $\beta \in J_\xi$, we can define $\widehat{u}(\cdot,\xi)_{\alpha\beta}$ to be the smooth solution to (5.19) which satisfies

$$|\,\widehat{u}(t,\xi)_{\alpha\beta}\,| \leq 2\pi \|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty. \tag{5.21}$$

for every $t \in \mathbb{T}^1$. On the other hand, if $\langle\xi\rangle > R$, $([\xi], \alpha) \in \mathcal{Z}_L^c$, and $\beta \in J_\xi$, according to Lemma 5.A.1 we can choose $\widehat{u}(\cdot,\xi)_{\alpha\beta}$ to be the unique solution to the differential equation (5.19) given by the equivalent expressions

$$\widehat{u}(t,\xi)_{\alpha\beta} = (\tilde{c}_{\xi,\alpha})^{-1} \int_0^{2\pi} e^{-q\tau} e^{-i\tau\langle a_0,\mu_\alpha(\xi)\rangle} e^{\int_{t-\tau}^{t}\langle b(w),\mu_\alpha(\xi)\rangle dw} \widehat{f}(t-\tau,\xi)_{\alpha\beta}\, d\tau, \tag{5.22}$$

and

$$\widehat{u}(t,\xi)_{\alpha\beta} = (\tilde{d}_{\xi,\alpha})^{-1} \int_0^{2\pi} e^{q\tau} e^{i\tau\langle a_0,\mu_\alpha(\xi)\rangle} e^{-\int_{t}^{t+\tau}\langle b(w),\mu_\alpha(\xi)\rangle dw} \widehat{f}(t+\tau,\xi)_{\alpha\beta}\, d\tau, \tag{5.23}$$

where $\tilde{c}_{\xi,\alpha} = 1 - e^{-2\pi i(\langle c_0,\mu_\alpha(\xi)\rangle - iq)}$, $\tilde{d}_{\xi,\alpha} = e^{2\pi i(\langle c_0,\mu_\alpha(\xi)\rangle - iq)} - 1$. Note also that since L_0 is globally solvable, and as $|k|$ has the same asymptotic behaviour as $\langle k\rangle = \sqrt{1+k^2}$ for $k \in \widehat{\mathbb{T}^1} \sim \mathbb{Z}$, by Proposition 4.1.4 and Lemma 5.2.5 there exist $M, N > 0$ such that

$$|(\tilde{c}_{\xi,\alpha})|^{-1} \leq M \left(\langle\xi^1\rangle + \cdots + \langle\xi^n\rangle\right)^N,$$

$$|(\tilde{d}_{\xi,\alpha})|^{-1} \leq M \left(\langle\xi^1\rangle + \cdots + \langle\xi^n\rangle\right)^N.$$

As $t \mapsto \langle b(t), \mu_\alpha(\xi)\rangle$ does not change sign, first assume it is non-positive. Then

$$e^{\int_{t-\tau}^{t}\langle b(w),\mu_\alpha(\xi)\rangle dw} \leq 1,$$

for every $0 \leq t, \tau \leq 2\pi$. Therefore, for each $\beta \in J_\xi$, formula (5.22) implies that

$$|\,\widehat{u}(t,\xi)_{\alpha\beta}\,| \leq 2\pi M e^{2\pi|Re(q)|} \left(\langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle\right)^N \|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty,$$

for every $t \in \mathbb{T}^1$. If we now assume $\langle b(t), \mu_\alpha(\xi) \rangle \geq 0$ for all $t \in \mathbb{T}^1$, then

$$e^{-\int_t^{t+\tau} \langle b(w), \mu_\alpha(\xi) \rangle dw} \leq 1,$$

so by formula (5.23), for every $\beta \in J_\xi$, the inequality

$$|\,\widehat{u}(t,\xi)_{\alpha\beta}\,| \leq 2\pi M e^{2\pi|Re(q)|} \left(\langle \xi^1 \rangle + \cdots + \langle \xi^n \rangle\right)^N \|\widehat{f}(\cdot,\xi)_{\alpha\beta}\|_\infty$$

is also satisfied for every $t \in \mathbb{T}^1$. Taking $K = \max\{2\pi, 2\pi M e^{2\pi|Re(q)|}, K_1\}$, it follows from Lemmas 3.1.5 and 5.2.1 that these Fourier coefficients indeed define $u \in C^\infty(\mathbb{T}^1 \times G)$, which clearly satisfies $Lu = f$, therefore L is globally solvable. Finally, note that if $\mathcal{Z}_L$ is finite and if $Lu = f \in C^\infty(\mathbb{T}^1 \times G)$, then by the unicity of the solutions of (5.19) for all $([\xi], \alpha) \in \mathcal{Z}_L^c$, by Lemma 5.A.1, and from the fact that these solutions satisfy the previous inequalities, $u \in C^\infty(\mathbb{T}^1 \times G)$ by Lemma 3.1.5. This proves that L is globally hypoelliptic as well. □

5.2.1 Necessary Conditions

Next we state and prove necessary conditions for the operator L to be globally solvable and/or globally hypoelliptic. Some of these will require estimates from below to the symbols of vector fields, which shall be explained.

The first necessary condition we present states that the operator L can be globally solvable, or globally hypoelliptic, only if the operator L_0 is globally solvable.

Proposition 5.2.7. *If L defined in (5.4) is globally solvable or globally hypoelliptic, then L_0 defined in (5.5) is globally solvable.*

Proof. Suppose that L_0 is not globally solvable. Then by Proposition 4.1.4 and the fact that $\sqrt{1+k^2}$ has the same asymptotic behavior as $|k|$ for $k \in \mathbb{Z}$, $k \to \pm\infty$, there exist sequences $([\xi_m], \alpha(m))_{m\in\mathbb{N}} \subset \mathcal{Z}_L^c$, $(k_m)_{m\in\mathbb{N}} \subset \mathbb{Z}$, such that

$$0 < |k_m + i(\langle c_0, \mu_{\alpha(m)}(\xi_m) \rangle - iq)| \leq (\langle \xi_m^1 \rangle + \cdots + \langle \xi_m^n \rangle + |k_m|)^{-m},$$

for every $m \in \mathbb{N}$. Up to a change of representative we may assume, without loss of generality, that $\alpha(m) = \bar{1} \doteq (1, \dots, 1)$ for all $m \in \mathbb{N}$. Then, as in the proof of Lemma 5.2.5, there exists $K' > 0$ such that

$$|d_m| \leq K'(\langle \xi_m^1 \rangle + \cdots + \langle \xi_m^n \rangle)^{-m},$$

where

$$d_m \doteq 1 - e^{-2\pi i(\langle c_0, \mu_{\bar{1}}(\xi_m) \rangle - iq)},$$

for every $m \in \mathbb{N}$. Also, for all such m, let $t_m \in [0, 2\pi]$ be such that

$$\int_0^{t_m} \langle b(s), \mu_{\bar{1}}(\xi_m) \rangle - \mathrm{Re}(q) ds = \max_{t \in [0,2\pi]} \int_0^t \langle b(s), \mu_{\bar{1}}(\xi_m) \rangle - \mathrm{Re}(q) ds.$$

Due to the $[0, 2\pi]$ being compact, by taking a subsequence we can assume $t_m \to t_0 \in [0, 2\pi]$. After a translation, we may further suppose $t_0 \in (0, 2\pi)$. Take $\phi \in C^\infty(\mathbb{T}^1)$, such that $\mathrm{supp}(\phi) \subset I$, $0 \leq \phi(t) \leq 1$, $\int_{t_0}^{2\pi} \phi(s) ds > \frac{1}{2}$ and $\int_0^{2\pi} \phi(s) ds = 1$, where $I \subset (0, 2\pi)$ is a closed interval not containing t_0. For every $m \in \mathbb{N}$, define $\widehat{f}(\cdot, \xi_m)_{\bar{1}\bar{1}} \in C^\infty(\mathbb{T}^1)$ to be the 2π-periodic extension of the function

$$[0, 2\pi] \ni t \mapsto d_m \exp\left\{-i \int_{t_m}^t (\langle c(w), \mu_{\bar{1}}(\xi_m) \rangle - iq) dw \right\} \phi(t).$$

Otherwise, define $\widehat{f}(\cdot, \xi)_{\alpha\beta} \equiv 0$. We claim that these partial Fourier coefficients define a smooth function on $\mathbb{T}^1 \times G$. Indeed, notice that for every $\gamma \in \mathbb{N}_0$, and every $t \in [0, 2\pi]$:

$$|\partial_t^\gamma \widehat{f}(t, \xi_m)_{\bar{1}\bar{1}}| = |d_m| \left| \sum_{j=0}^{\gamma} \binom{\gamma}{j} \partial_t^j \exp\left\{-i \int_{t_m}^t (\langle c(w), \mu_{\bar{1}}(\xi_m) \rangle - iq) dw \right\} \partial_t^{\gamma - j} \phi(t) \right|.$$

By Faa-di-Bruno's formula

$$\partial_t^j \exp\left\{-i \int_{t_m}^t (\langle c(w), \mu_{\bar{1}}(\xi_m) \rangle - iq) dw \right\} = \sum_{r \in \Delta(j)} \frac{j!}{r!} \exp\left\{-i \int_{t_m}^t (\langle c(w), \mu_{\bar{1}}(\xi_m) \rangle - iq) dw \right\} \times \prod_{l=1}^{j} \left(\frac{-i\partial_t^l \int_{t_m}^t (\langle c(w), \mu_{\bar{1}}(\xi_m) \rangle - iq) dw}{l!} \right)^{r_l},$$

where $\Delta(j) = \{r \in \mathbb{N}_0^j | \sum_{l=1}^j l r_l = j\}$. Notice that $|\mu_1(\xi_m^k)| \leq C_j \langle \xi_m^k \rangle$, for some $C_j > 0$ and for every $m \in \mathbb{N}$, $k = 1, \ldots, n$, and

$$\operatorname{Re}\left(-i\int_{t_m}^{t}(\langle c(w), \mu_{\bar{1}}(\xi_m)\rangle - iq)dw\right) = \int_{t_m}^{t}\langle b(w), \mu_{\bar{1}}(\xi_m)\rangle - \operatorname{Re}(q)dw \leq 0,$$

for all $t \in [0, 2\pi]$, by the definition of t_m. Therefore, we obtain that

$$\left|\partial_t^j \exp\left\{-i\int_{t_m}^{t}(\langle c(w), \mu_{\bar{1}}(\xi_m)\rangle - iq)dw\right\}\right| \leq D_j(\langle \xi_m^1\rangle + \cdots + \langle \xi_m^n\rangle)^j,$$

for some $D_j > 0$ and all $t \in [0, 2\pi]$, $j \in \{0, 1, \ldots, \gamma\}$. Finally, we can estimate

$$|\partial_t^\gamma \widehat{f}(t, \xi_m)_{\bar{1}\bar{1}}| \leq \tilde{K}_{\gamma m}(\langle \xi_m^1\rangle + \cdots + \langle \xi_m^n\rangle)^{\gamma - m},$$

for all $t \in \mathbb{T}^1$, $m \in \mathbb{N}$. Therefore, it follows from Lemma 3.1.5 that $f \in C^\infty(\mathbb{T}^1 \times G)$. Furthermore, Lemma 5.A.2 implies that $f \in (\ker {}^tL)^0$, since $\widehat{f}(\cdot, \xi)_{\alpha\beta} \equiv 0$ for all $([\xi], \alpha) \in \mathcal{Z}_L$. We now show that there is no $u \in C^\infty(\mathbb{T}^1 \times G)$ which satisfies $Lu = f$, and so L is not globally solvable. Indeed, suppose that such a u exists. Then its Fourier coefficients satisfy

$$\widehat{Lu}(t, \xi_m)_{\bar{1}\bar{1}} = [\partial_t + i(\langle c(t), \mu_{\bar{1}}(\xi_m)\rangle - iq)]\widehat{u}(t, \xi_m)_{\bar{1}\bar{1}} = \widehat{f}(t, \xi_m)_{\bar{1}\bar{1}},$$

for every $t \in \mathbb{T}^1$, $m \in \mathbb{N}$. From Lemma 5.A.1, $\widehat{u}(\cdot, \xi_m)_{\bar{1}\bar{1}}$ must satisfy

$$\widehat{u}(t, \xi_m)_{\bar{1}\bar{1}} = (d_m)^{-1}\int_0^{2\pi} \exp\left\{-i\int_{t-\tau}^{t}(\langle c(w), \mu_{\bar{1}}(\xi_m)\rangle - iq)dw\right\}\widehat{f}(t-\tau, \xi_m)_{\bar{1}\bar{1}}d\tau, \quad (5.24)$$

for every $t \in [0, 2\pi]$, $m \in \mathbb{N}$. In order to properly calculate the value of $\widehat{u}(t, \xi_m)_{\bar{1}\bar{1}}$, at each $t \in \mathbb{T}^1$, we first split the integral above into two integrals: one where τ ranges from 0 to t and another where τ ranges from t to 2π. Hence, for any $t \in [0, 2\pi]$ this yields

$$\begin{aligned}
&\widehat{u}(t,\xi_m)_{\overline{1}\,\overline{1}} = \\
&\int_0^t \exp\left\{-i\int_{t-\tau}^{t}(\langle c(w),\mu_{\overline{1}}(\xi_m)\rangle - iq)dw - i\int_{t_m}^{t-\tau}(\langle c(w),\mu_{\overline{1}}(\xi_m)\rangle - iq)dw\right\}\phi(t-\tau)d\tau \\
&+\int_t^{2\pi} \exp\left\{-i\int_{t-\tau}^{t}(\langle c(w),\mu_{\overline{1}}(\xi_m)\rangle - iq)dw - i\int_{t_m}^{t-\tau+2\pi}(\langle c(w),\mu_{\overline{1}}(\xi_m)\rangle - iq)dw\right\} \\
&\times \phi(t-\tau+2\pi)d\tau \\
&= \exp\left\{-i\int_{t_m}^{t}(\langle c(w),\mu_{\overline{1}}(\xi_m)\rangle - iq)dw\right\}\int_0^t \phi(t-\tau)d\tau \\
&+\exp\left\{-i\int_{t_m}^{t+2\pi}(\langle c(w),\mu_{\overline{1}}(\xi_m)\rangle - iq)dw\right\}\int_t^{2\pi} \phi(t-\tau+2\pi)d\tau.
\end{aligned}$$

After a change of variables, and due to the definition of c_0, we have that

$$\begin{aligned}
\widehat{u}(t,\xi_m)_{\overline{1}\,\overline{1}} &= \exp\left\{-i\int_{t_m}^{t}(\langle c(w),\mu_{\overline{1}}(\xi_m)\rangle - iq)dw\right\}\int_0^t \phi(\tau')d\tau' \\
&+\exp\left\{-i\int_{t_m}^{t+2\pi}(\langle c(w),\mu_{\overline{1}}(\xi_m)\rangle - iq)dw\right\}\int_t^{2\pi} \phi(\tau')d\tau \\
&= \exp\left\{-i\int_{t_m}^{t}(\langle c(w),\mu_{\overline{1}}(\xi_m)\rangle - iq)dw\right\}\Bigg(\int_0^t \phi(\tau')d\tau' \\
&+\exp\left\{-2\pi i(\langle c_0,\mu_{\overline{1}}(\xi_m)\rangle - iq)\right\}\int_t^{2\pi} \phi(\tau')d\tau'\Bigg).
\end{aligned}$$

For $t = t_m$, we obtain

$$\widehat{u}(t_m,\xi_m)_{\overline{1}\,\overline{1}} = \int_0^{t_m} \phi(\tau')d\tau' + \exp\left\{-2\pi i(\langle c_0,\mu_{\overline{1}}(\xi_m)\rangle - iq)\right\}\int_{t_m}^{2\pi} \phi(\tau')d\tau'.$$

By continuity, both $|e^{-2\pi i(\langle c_0,\mu_{\overline{1}}(\xi_m)\rangle - iq)}|$ and $\int_{t_m}^{2\pi}\phi(\tau')d\tau'$ are greater than $\frac{1}{2}$ for m big enough, and so

$$|\widehat{u}(t_m,\xi_m)_{\overline{1}\,\overline{1}}| \geq \frac{1}{4},$$

for all such $m \in \mathbb{N}$. Lemma 3.1.5 then implies $u \notin C^\infty(\mathbb{T}^1 \times G)$, which is a contradiction, and therefore no such function u can exist. As we have exhibited $f \in (\ker {}^tL)^0 \cap C^\infty(\mathbb{T}^1 \times G)$ such that there is no $u \in C^\infty(\mathbb{T}^1 \times G)$ which satisfies $Lu = f$, we conclude that L is not globally solvable. To prove L is also not globally hypoelliptic, we exhibit $u \in \mathcal{D}'(\mathbb{T}^1 \times G)\backslash C^\infty(\mathbb{T}^1 \times G)$ which satisfies $Lu = f$. For that, define the Fourier coefficients $\widehat{u}(\cdot,\xi)_{\alpha\beta}$ as identically zero for $[\xi] \neq [\xi_m]$ or $\alpha, \beta \neq \overline{1}$, and $\widehat{u}(\cdot,\xi_m)_{\overline{1}\,\overline{1}}$ by formula (5.24), for every $m \in \mathbb{N}$. Then as we have seen, these coefficients cannot define $u \in C^\infty(\mathbb{T}^1 \times G)$ by Lemma 3.1.5. On the other hand, note that for all $t \in [0, 2\pi]$ and $m \in \mathbb{N}$ sufficiently big,

$$\begin{aligned} |\widehat{u}(t,\xi_m)_{\bar{1}\bar{1}}| &\leq \exp\left\{\int_{t_m}^{t} (\langle b(w), \mu_{\bar{1}}(\xi_m)\rangle - \mathrm{Re}(q)dw\right\} \\ &\quad + \exp\left\{2\pi(\langle b_0, \mu_{\bar{1}}(\xi_m)\rangle - \mathrm{Re}(q))\right\} \\ &\leq 1 + \exp\left\{2\pi(\langle b_0, \mu_{\bar{1}}(\xi_m)\rangle - \mathrm{Re}(q))\right\} \leq 3, \end{aligned}$$

since $e^{2\pi(\langle b_0, \mu_{\bar{1}}(\xi_m)\rangle - \mathrm{Re}(q))} \to 1$ as $m \to +\infty$. This implies that for any $\psi \in C^\infty(\mathbb{T}^1)$, $[\xi] \in \widehat{G}$ and $\alpha, \beta \in J_\xi$,

$$|\langle \widehat{u}(\cdot,\xi)_{\alpha\beta}, \psi\rangle_{\mathbb{T}^1}| \leq 6\pi\|\psi\|_\infty.$$

Therefore, by Lemma 3.1.5 these coefficients define $u \in \mathcal{D}'(\mathbb{T}^1 \times G)$. By comparing Fourier coefficients, we see that $Lu = f$, from which we conclude that L is not globally hypoelliptic.

□

Corollary 5.2.8. *If $t \mapsto \langle b(t), \mu_\alpha(\xi)\rangle$ does not change sign for all but finitely many $[\xi] \in \widehat{G}$, and $\alpha \in J_\xi$, then L defined in (5.4) is globally hypoelliptic if and only if L_0 defined in (5.5) is globally hypoelliptic. If also, $\mathrm{Re}(q) = 0$, then L is globally solvable if and only if L_0 is globally solvable.*

Proof. Notice that for all $[\xi] \in \widehat{G}$, $\alpha \in J_\xi$ such that $t \mapsto \langle b(t), \mu_\alpha(\xi)\rangle$ does not change sign, the sublevel $\Omega_r^{\xi,\alpha}$ is connected for every $r \in \mathbb{R}$, as the integral in its definition is monotone when $\mathrm{Re}(q) = 0$. From Proposition 5.2.6 we then conclude that L is globally solvable if L_0 is also globally solvable. Conversely, if L_0 is not globally solvable, then by Proposition 5.2.7, L is not globally solvable. Now, for any $\mathrm{Re}(q)$, from Proposition 5.2.6 it also follows that if L_0 is globally hypoelliptic, then so is L. If however L_0 is not globally hypoelliptic, by Proposition 4.1.4 either L_0 is not globally solvable, or $\mathcal{Z}_L$ is infinite. In the first case, Proposition 5.2.7 implies that L is not globally hypoelliptic. There only remains to prove that if $\mathcal{Z}_L$ is infinite, then L is not globally hypoelliptic. Suppose $\mathcal{Z}_L$ is infinite and let $([\xi_m], \alpha(m))_{m\in\mathbb{N}} \subset \mathcal{Z}_L$ be a sequence of distinct terms. For each $m \in \mathbb{N}$, let $d_m \in \mathbb{R}$, $t_m \in [0, 2\pi]$ be such that

$$d_m \doteq \max_{0\leq t\leq 2\pi} \int_0^t (\langle b(s), \mu_{\alpha(m)}(\xi_m)\rangle - \mathrm{Re}(q))ds = \int_0^{t_m} (\langle b(s), \mu_{\alpha(m)}(\xi_m)\rangle - \mathrm{Re}(q))ds.$$

For every $m \in \mathbb{N}$, define

$$\widehat{u}(t,\xi_m)_{\alpha(m)\,\bar{1}} = e^{-d_m} e^{-i\int_0^t (\langle c(w), \mu_{\alpha(m)}(\xi_m)\rangle - iq)dw},$$

for every $t \in \mathbb{T}^1$. Also define $\widehat{u}(\cdot,\xi)_{\alpha\beta} \equiv 0$ for every other partial Fourier coefficient. Notice that these partial Fourier coefficients are all well defined since $(\xi_m, \alpha(m)) \in \mathcal{Z}_L$, for every $m \in \mathbb{N}$. Also,

$$|\widehat{u}(t,\xi_m)_{\alpha(m)\,\bar{1}}| = e^{-d_m} e^{\int_0^t (\langle b(w), \mu_{\alpha(m)}(\xi_m)\rangle - \mathrm{Re}(q))dw} \leq 1$$

and

$$|\widehat{u}(t_m,\xi_m)_{\alpha(m)\,\bar{1}}| = 1,$$

for all $m \in \mathbb{N}$, so these coefficients define $u \in \mathcal{D}'(\mathbb{T}^1 \times G) \backslash C^\infty(\mathbb{T}^1 \times G)$, by Lemma 3.1.5. On the other hand, $\widehat{Lu}(\cdot,\xi_m)_{\alpha(m)\,\bar{1}} \equiv 0$ for all m, since

$$\begin{aligned}\widehat{Lu}(t,\xi_m)_{\alpha(m)\,\bar{1}} &= \partial_t \widehat{u}(t,\xi_m)_{\alpha(m)\,\bar{1}} + i(\langle c(t), \mu_{\alpha(m)}(\xi_m)\rangle - iq)\widehat{u}(t,\xi_m)_{\alpha(m)\,\bar{1}} \\ &= 0,\end{aligned}$$

for every $t \in \mathbb{T}^1$. We conclude that $Lu = 0 \in C^\infty(\mathbb{T}^1 \times G)$, and therefore L is not globally hypoelliptic. □

Remark 5.2.9. Notice that in the corollary above we can replace "$t \mapsto \langle b(t), \mu_\alpha(\xi)\rangle$ does not change sign for all but finitely many $[\xi] \in \widehat{G}$" by "All b_i are linearly dependent and do not change sign". Indeed, if $b_1, \ldots b_n$ are linearly dependent and do not change sign, then there exists a smooth real-valued function $b_0 : \mathbb{T}^1 \to \mathbb{R}$, which does not change sign, and a vector $v = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such

$$(b_1, \ldots, b_n) = b_0(\lambda_1, \ldots, \lambda_n).$$

Then clearly $t \mapsto \langle b(t), \mu_\alpha(\xi)\rangle = \sum_{j=1}^n b_j(t)\mu_{\alpha_j}(\xi^j) = \left(\sum_{j=1}^n \lambda_j \mu_{\alpha_j}(\xi^j)\right) b_0(t)$ does not change sign, for every $[\xi] \in \widehat{G}$.

In order to obtain further necessary conditions we will need the following definition.

Definition 5.2.10. We say that a vector field X on a compact Lie group G admits an Archimedean sequence if there exists a sequence of distinct elements $[\xi_m] \in \widehat{G}$, and integers

$1 \leq r_m \leq d_{\xi_m}$, for $m \in \mathbb{N}$, and $K, \varepsilon > 0$ such that

$$|\mu_{r_m}(\xi_m)| \geq K \langle \xi_m \rangle^{\varepsilon}, \ \forall m \in \mathbb{N}. \tag{5.25}$$

Here $\sigma_X(\xi) = \operatorname{diag}(i\mu_1(\xi), \ldots, i\mu_{d_\xi}(\xi))$. Note we can always assume $0 < K, \varepsilon \leq 1$. By composing ξ_m with a change of variables, we may further assume $r_m = 1$, for all $m \in \mathbb{N}$. Also, as $\langle \overline{\xi} \rangle = \langle \xi \rangle$ and $\mu_r\left(\overline{\xi}\right) = -\mu_r(\xi)$ for $[\xi] \in \widehat{G}$, $r \in \{1, \ldots, d_\xi\}$, we can also assume that

$$\mu_1\left(\xi_m\right) \geq K \langle \xi_m \rangle, \ \forall m \in \mathbb{N}. \tag{5.26}$$

By taking a subsequence, we will also suppose that the sequence of real numbers $\mu_1(\xi_m)$ is strictly increasing. Furthermore, for each $m \in \mathbb{N}$, we define

$$\xi_{-m} = \overline{\xi_m},$$

and $\xi_0 = \mathbb{1}$ is the trivial representation $\mathbb{1}(x) = 1$, $\forall x \in G$. In particular, $\lim_{m \to \pm\infty} \mu_1(\xi_m) = \pm\infty$.

Example 5.2.11. Suppose $G = \mathbb{T}^n$ is the n-dimensional torus, $n \in \mathbb{N}$. Then every $[\xi] \in \widehat{\mathbb{T}^n}$ is one dimensional, and we can identify $[\xi] \sim k \in \mathbb{Z}^n$. Also, the vector fields ∂_{t_j}, $j = 1, \ldots, n$ have symbol $\sigma_{\partial_{t_j}}(k) = ik_j$, and $\langle k \rangle = \sqrt{1 + |k|^2}$. This means the sequence $(k(m))_{m \in \mathbb{N}}$ given by $k(m) = (0, \ldots, m, \ldots, 0)$ is an Archimedean sequence for the vector field X_j, where m is in the j-th coordinate, as

$$|\mu_1(k(m))| = |m| \geq \frac{1}{\sqrt{2}} \sqrt{1 + m^2} = \frac{1}{\sqrt{2}} \langle k(m) \rangle, \ \forall m \in \mathbb{N}\,.$$

Similarly, for $G = \mathbb{S}^3$, we can identify $[\xi] \in \widehat{\mathbb{S}^3}$ with $\ell \in \frac{1}{2}\mathbb{N}_0$. Also, the vector field $i\partial_0 = D_3$ on $\mathbb{S}^3$ corresponding with the partial derivative $\frac{\partial}{\partial \psi}$ in the local chart given by the Euler angles, has symbol

$$\sigma_{D_3}(\ell) = (i(\alpha - \ell - 1)\delta_{\alpha\beta})_{\alpha,\beta=1}^{2\ell+1}.$$

Also, here $\langle \ell \rangle = \sqrt{1 + \ell + \ell^2}$. This means the sequence $(\ell_m)_{m \in \mathbb{N}}$ given by $\ell_m = m$ is an Archimedean sequence for the vector field D_3, as

$$|\mu_1(\ell_m)| = m \geq \frac{1}{\sqrt{3}} \sqrt{1 + m + m^2} = \frac{1}{\sqrt{3}} \langle \ell_m \rangle, \ \forall m \in \mathbb{N}\,.$$

On the other hand, if G is any compact Lie group and X is the 0 vector field, that is, $Xf \equiv 0$, for any $f \in C^\infty(G)$, then clearly X does not admit Archimedean sequence, as $\sigma_X(\xi) = 0 \in \mathbb{C}^{d_\xi \times d_\xi}$ for every $\xi \in \widehat{G}$.

Proposition 5.2.12. *Every non-zero left-invariant vector field on a compact Lie group admits an Archimedean sequence.*

Proof. Let $X \neq 0$ be a left-invariant vector field on a compact Lie group G, $0 < \varepsilon < 1$. Viewed as a linear operator, X is not bounded from the Sobolev space $H^s(G)$ to $H^{s-\varepsilon}(G)$, for any $s \in \mathbb{R}$, by Theorem 1.1 in [26], a result by Cardona, Kirilov, Moraes and the author. We claim that this implies the existence of an Archimedean sequence with exponent ε. Indeed, if not, then for $\sigma_X(\xi) = \text{diag}(\mu_1(\xi), \dots, \mu_{d_\xi}(\xi))$, there would exist $C > 0$ such that

$$|\mu_r(\xi)| \leq C\langle\xi\rangle^\varepsilon,$$

for all $[\xi] \in \widehat{G}$, $1 \leq r \leq d_\xi$. But then, for any $v \in H^s(G)$:

$$\begin{aligned}
\|Xv\|^2_{H^{s-\varepsilon}(G)} &= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{r,m=1}^{d_\xi} \langle\xi\rangle^{2(s-\varepsilon)} |\mu_r(\xi)|^2 |\widehat{v}(\xi)_{rm}|^2 \\
&\leq C \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{r,m=1}^{d\xi} \langle\xi\rangle^{2s} |\widehat{v}(\xi)_{rm}|^2 \\
&= C\|v\|^2_{H^s(G)}.
\end{aligned}$$

This implies $X : H^s(G) \to H^{s-\varepsilon}(G)$ is bounded which is a contradiction. Therefore, we conclude such an Archimedean sequence must exist. □

Remark 5.2.13. From the previous examples, we see that in some cases it is also possible to obtain Archimedean sequences with exponent $\varepsilon = 1$. In fact, this seems to be the case in all known examples, and we conjecture Proposition 5.2.12 is also true for $\varepsilon = 1$. Also, note that necessarily we have that $0 < \varepsilon \leq 1$, due to inequality (3.8).

Proposition 5.2.14. *Suppose that for some $1 \leq j \leq n$, the function b_j changes sign. If one of the following also holds:*

1. *$\mathcal{Z}_L$ is finite;*

2. *$\text{Re}(q) \neq 0$;*

3. $b_{j0} \neq 0$ *or*

4. $b_{r0} \neq 0$*, for some* $j \neq r \in \{1, \ldots, n\}$*,*

then L *as defined in* (5.4) *is not globally solvable nor globally hypoelliptic.*

Proof. We will prove the claim by way of contradiction. First, suppose L is globally solvable. Then, by Proposition 5.2.7, this implies that L_0 defined in (5.5) is also globally solvable. Without loss of generality, we may assume $j = 1$. Let $([\xi_m^1])_{m\in\mathbb{Z}} \subset \widehat{G}_1$ be an Archimedean sequence for X_1. Assuming case 1, 2 or 3 hold, we define the sequence

$$([\xi_m])_{m\in\mathbb{N}} \subset \widehat{G},$$

by

$$[\xi_m] \doteq [\xi_m^1 \otimes \mathbb{1}^2 \otimes \cdots \otimes \mathbb{1}^n],$$

for every $m \in \mathbb{N}$, where $\mathbb{1}^j \in U(1)$ is the trivial representation on G_j, that is, $\mathbb{1}^j(x_j) = 1 \in U(1)$, $\forall x_j \in G_j$, $j = 1, \ldots, n$. Notice that, in particular, $\mathbb{1}^j$ satisfies $\mu_1(\mathbb{1}^j) = 0$, for all $j = 1, \ldots, n$. By taking a subsequence we may also assume $([\xi_m], \overline{1}) \in \mathcal{Z}_L^c$ for all $m \in \mathbb{N}$. Indeed, this is possible due to our assumptions and Remark 5.2.4. That is, since Archimedean sequences tend to infinity, as explained in that remark, the sequence $[\xi_m]$ can only have finitely many elements contained in $\mathcal{Z}_L$.
Suppose first $b_{10} = 0$ or $b_{10} > 0$. In either case, set

$$G(t,\tau) \doteq \int_t^{t+\tau} a_{10} + ib_1(w)dw,$$

and let $B \in \mathbb{R}$, $t_0, \tau_0 \in [0, 2\pi]$ be such that

$$B = \min_{0\leq t,\tau\leq 2\pi} \operatorname{Im}(G(t,\tau)) = \int_{t_0}^{t_0+\tau_0} b_1(w)dw.$$

Notice that, since b_1 changes sign, $B < 0$. Also, by composing b_1 with an appropriate translation if necessary, we may assume $t_0, \tau_0 \in (0, 2\pi)$ and that $b_1(0) \neq 0$, so that $b_1(t_0 + \tau_0) = 0$ and $t_0 + \tau_0 \in (0, 2\pi)$. Take $\phi \in C^\infty(\mathbb{T}^1)$ and $\delta > 0$ such that

$$\operatorname{supp}(\phi) \subset [t_0 + \tau_0 - \delta, t_0 + \tau_0 + \delta] \subset (0, 2\pi),$$

$$\phi \equiv 1 \text{ in } [t_0 + \tau_0 - \delta/2, t_0 + \tau_0 + \delta/2],$$

and also $0 \leq \phi(t) \leq 1$ for all $t \in \mathbb{T}^1$. Define $\widehat{f}(\cdot, \xi_m)_{\bar{1}\bar{1}}$ to be the 2π-periodic extension of the mapping

$$[0, 2\pi] \ni t \mapsto d_m e^{B\mu_1(\xi_m^1)} e^{-i\mu_1(\xi_m^1)a_{10}(t-t_0)} \phi(t) e^{-q(t-t_0)},$$

where $d_m \doteq e^{2\pi i(c_{10}\mu_1(\xi_m)-iq)} - 1$, for every $m \in \mathbb{N}$. For every other partial Fourier coefficient, define it by $\widehat{f}(\cdot, \xi)_{\alpha\beta} \equiv 0$. We claim that these partial Fourier coefficients define $f \in C^\infty(\mathbb{T}^1 \times G)$.

Indeed, first notice that since $b_{10} \geq 0$ and $\mu_1(\xi_m^1) \to +\infty$,

$$|d_m| \leq e^{2\pi(-b_{10}\mu_1(\xi_m^1)+\mathrm{Re}(q))} + 1 \leq e^{2\pi(\mathrm{Re}(q))} + 1,$$

for all $m \in \mathbb{N}$ sufficiently large. It follows that there exists $K > 0$ such that $|d_m| \leq K$, for all $m \in \mathbb{N}$. Next, let $\psi \in C^\infty([0, 2\pi])$ be given by: $[0, 2\pi] \ni t \mapsto \phi(t)e^{-q(t-t_0)}$. Then

$$|\partial_t^\gamma \widehat{f}(t, \xi_m)_{\bar{1}\bar{1}}| \leq K e^{B\mu_1(\xi_m^1)} \sum_{j=0}^{\gamma} \binom{\gamma}{j} |\partial_t^j e^{-i\mu_1(\xi_m^1)a_{10}(t-t_0)}||\partial_t^{\gamma-j}\psi(t)|, \tag{5.27}$$

for every $t \in [0, 2\pi]$, $m \in \mathbb{N}$. Note that

$$\begin{aligned} |\partial_t^j e^{-i\mu_1(\xi_m^1)a_{10}(t-t_0)}| &= |(-i\mu_1(\xi_m^1)a_{10})^j e^{-i\mu_1(\xi_m^1)a_{10}(t-t_0)}| \\ &\leq (1 + |a_{10}|)^j \langle \xi_m^1 \rangle^j \\ &\leq (1 + |a_{10}|)^\gamma \langle \xi_m^1 \rangle^\gamma, \end{aligned} \tag{5.28}$$

for every $t \in [0, 2\pi]$, $m \in \mathbb{N}$. Also, since $B < 0$ and $([\xi_m^1])_m$ is an Archimedean sequence, we have that

$$e^{B\mu_1(\xi_m^1)} \leq e^{K'B\langle \xi_m^1 \rangle^\varepsilon} \tag{5.29}$$

for some $K', \varepsilon > 0$ and every $m \in \mathbb{N}$. Therefore, from (5.27), (5.28) and (5.29) we have that

$$|\partial_t^\gamma \widehat{f}(t, \xi_m)_{\bar{1}\bar{1}}| \leq K(1 + |a_{10}|)^\gamma Q_\gamma e^{K'B\langle \xi_m^1 \rangle^\varepsilon} \langle \xi_m^1 \rangle^\gamma,$$

for every $t \in \mathbb{T}^1$, $m \in \mathbb{N}$, where $Q_\gamma \doteq 2^\gamma \max_{1 \leq j \leq \gamma} \max_{t \in [0, 2\pi]} |\partial_t^j \psi(t)|$. Therefore, for any

$N > 0$ there exist $Q_{\gamma N}, Q'_{\gamma N} > 0$ such that

$$\begin{aligned}|\partial_t^\gamma \widehat{f}(t,\xi_m)_{\bar{1}\bar{1}}| &\le Q_{\gamma N}\langle \xi_m^1\rangle^{-N}\\ &\le Q'_{\gamma N}(\langle \xi_m^1\rangle + \langle \mathbb{1}^2\rangle + \cdots + \langle \mathbb{1}^n\rangle)^{-N},\end{aligned}$$

for every $t \in \mathbb{T}^1$, $m \in \mathbb{N}$, so by Lemma 3.1.5 these coefficients define $f \in C^\infty(\mathbb{T}^1 \times G)$ by their inverse partial Fourier transform, as claimed. Next, notice that since $\{([\xi_m], \bar{1})\}_{m\in\mathbb{N}}$ is contained in $\mathcal{Z}_L^c$, this implies that $\widehat{f}(\cdot,\xi)_{\alpha\beta} \equiv 0$ for all $([\xi],\alpha) \in \mathcal{Z}_L$. And so we conclude, from Lemma 5.A.2, that $f \in (\ker {}^tL)^0 \cap C^\infty(\mathbb{T}^1 \times G)$.

Now suppose $Lu = f$, for some $u \in C^\infty(\mathbb{T}^1 \times G)$. Since $\{([\xi_m], \bar{1})\}_{m\in\mathbb{N}}$ is contained in $\mathcal{Z}_L^c$, by Lemma 5.A.1, $\widehat{u}(\cdot,\xi_m)_{\bar{1}\bar{1}}$ satisfies

$$\begin{aligned}\widehat{u}(t,\xi_m)_{\bar{1}\bar{1}} &= (d_m)^{-1}\int_0^{2\pi} \exp\left\{\int_t^{t+\tau} (\langle c(w), \mu_{\bar{1}}(\xi_m)\rangle - iq)dw\right\}\widehat{f}(t+\tau,\xi_m)_{\bar{1}\bar{1}}d\tau\\ &= e^{-i\mu_1(\xi_m^1)a_{10}(t-t_0)}e^{-q(t-t_0)}\int_0^{2\pi} e^{\mu_1(\xi_m^1)(B-\mathrm{Im}(G(t,\tau)))}\phi(t+\tau)d\tau,\end{aligned}$$

for every $t \in [0,2\pi]$, and every $m \in \mathbb{N}$. Let $\theta(\tau) \doteq \mathrm{Im}(G(t_0,\tau)) - B$, $\tau \in \mathbb{T}^1$. Then

$$\begin{aligned}|\widehat{u}(t_0,\xi_m)_{\bar{1}\bar{1}}| &= \left|\int_0^{2\pi} e^{-\mu_1(\xi_m^1)\theta(\tau)}\phi(t_0+\tau)d\tau\right|\\ &\ge \int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^{-\mu_1(\xi_m^1)\theta(\tau)}d\tau\\ &\ge \int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^{-K''\langle\xi_m^1\rangle\theta(\tau)}d\tau,\end{aligned}$$

for every $m \in \mathbb{N}$, since $\theta(\tau) \ge 0$ for all $\tau \in [0,2\pi]$, and $0 < \mu_1(\xi_m^1) \le K''\langle\xi_m^1\rangle$ for some $K'' > 0$ and for all $m \in \mathbb{N}$. It follows from Lemma 5.A.4 that for all m sufficiently big, there exists $M > 0$ such that

$$|\widehat{u}(t_0,\xi_m)_{\bar{1}\bar{1}}| \ge M(\langle\xi_m^1\rangle)^{-\frac{1}{2}},$$

so

$$|\widehat{u}(t_0,\xi_m)_{\bar{1}\bar{1}}| \ge M'(\langle\xi_m^1\rangle + \langle\mathbb{1}^2\rangle + \cdots + \langle\mathbb{1}^n\rangle)^{-\frac{1}{2}}$$

for all $m \in \mathbb{N}$ and some $M' > 0$. We conclude then that $u \notin C^\infty(\mathbb{T}^1 \times G)$, by Lemma

3.1.5. But that is a contradiction, which means no such an u can exist and so L is not globally solvable. To prove that L is not globally hypoelliptic, consider the same function f and partial Fourier coefficients $\widehat{u}(\cdot, \xi_m)_{\bar{1}\bar{1}}$, but now define every other partial Fourier coefficient $\widehat{u}(\cdot, \xi)_{\alpha\beta}$ to be identically 0. By the previous argument, these coefficients do not define $u \in C^\infty(\mathbb{T}^1 \times G)$. However, they do define $u \in \mathcal{D}'(\mathbb{T}^1 \times G)$, since for every $\zeta \in C^\infty(\mathbb{T}^1)$

$$
\begin{aligned}
&|\langle \widehat{u}(\cdot, \xi_m^1)_{\bar{1}\bar{1}}, \zeta \rangle_{\mathbb{T}^1}| = \\
&= \left| \int_0^{2\pi} e^{-i(\mu_1(\xi_m^1)a_{10} + \mu_1(\eta^2)a_{20})(t-t_0)} e^{-q(t-t_0)} \int_0^{2\pi} e^{\mu_1(\xi_m^1)(B - \operatorname{Im}(G(t,\tau))} \phi(t+\tau)\zeta(t) d\tau dt \right| \\
&\leq \int_0^{2\pi} \int_0^{2\pi} e^{2\pi |\operatorname{Re}(q)|} e^{\mu_1(\xi_m^1)(B - \operatorname{Im}(G(t,\tau))} |\phi(t+\tau)||\zeta(t)| d\tau dt \\
&\leq (2\pi)^2 \|\phi\|_\infty \|\zeta\|_\infty e^{2\pi |\operatorname{Re}(q)|} \\
&\leq Q' p_1(\zeta) \langle \xi_m^1 \rangle \\
&\leq Q'' p_1(\zeta) (\langle \xi_m^1 \rangle + \langle \mathbb{1}^2 \rangle + \cdots + \langle \mathbb{1}^n \rangle)^1,
\end{aligned}
$$

for some $Q', Q'' > 0$ and every $m \in \mathbb{N}$. By comparing partial Fourier coefficients, it is clear that $Lu = f$, and so we conclude that L is not globally hypoelliptic. In the case $b_{10} < 0$, the proof is similar, though now define

$$
A \doteq \max_{0 \leq t, \tau \leq 2\pi} \operatorname{Im}(H(t,\tau)) = \int_{t_1 - \tau_1}^{t_1} b(w) dw > 0,
$$

$$
d'_m \doteq 1 - e^{-2\pi i (c_0 \mu_1(\xi_m^1) - iq)},
$$

and let $\widehat{f}(\cdot, \xi_m)_{\bar{1}\bar{1}}$ be the 2π-periodic extension of

$$
[0, 2\pi] \ni t \mapsto d'_m e^{A\mu_1(\xi_m^1)} e^{-i\mu_1(\xi_m^1)a_{10}(t-t_0)} \tilde{\phi}(t) e^{-q(t-t_1)},
$$

where $\tilde{\phi}$ has properties similar to those of ϕ, and then make use of the other equivalent formula for the solutions of the differential equations $[\partial_t + i(\langle c_j(t), \mu_\alpha(\xi) \rangle - iq)]\widehat{u}(t, \xi_m)_{\bar{1}\bar{1}} = \widehat{f}(t, \xi)_{\bar{1}\bar{1}}$. Finally, we prove case 4. In this case, we may assume, without loss of generality, that $r = 2$. Choose $\eta^2 \in \widehat{G}_2$ such that $\mu_1(\eta^2) \neq 0$, which is possible as X_2 is non-zero. Indeed, if not, then since $\widehat{X_2 f}(\eta^2)_{\alpha_2\beta_2} = \mu_{\alpha_2}(\eta^2) \widehat{f}(\eta^2)_{\alpha_2\beta_2}$ for every $f \in C^\infty(G_2)$, $1 \leq \alpha_2, \beta_2 \leq d_{\eta^2}$, this would imply $X_2 f \equiv 0$, for every $f \in C^\infty(G_2)$, which would imply $X_2 = 0$, a contradiction with our assumptions. Define

$$[\tilde{\xi_m}] \doteq [\xi_m^1 \otimes \eta^2 \otimes \mathbb{1}^3 \cdots \otimes \mathbb{1}^n],$$

for each $m \in \mathbb{N}$. We may assume $b_{10} = 0$ and $\mathrm{Re}(q) = 0$, as we have already proved the result otherwise (in the previous cases). In this setting we may also suppose (again) that $([\tilde{\xi_m}], \overline{1}) \in \mathcal{Z}_L^c$ for all $m \in \mathbb{N}$. Notice that we may further assume that b_2 does not change sign, since, if it did, we could reorder it to be b_1 and apply the proof of case 3 once again. Next, we split the proof in two subcases: $b_{20} > 0$ or $b_{20} < 0$. Assume first that $b_{20} > 0$. In this subcase, substituting η^2 by $\overline{\eta^2}$, we may further assume $\mu_1(\eta^2) > 0$. Using the same notation as in the first part of this proof, define $\widehat{f}(\cdot, \tilde{\xi_m})_{\overline{1}\,\overline{1}}$ to be the 2π-periodic extension of

$$[0, 2\pi] \ni t \mapsto \tilde{d_m} e^{B\mu_1(\xi_m^1)} e^{-i(\mu_1(\xi_m^1)a_{10} + \mu_1(\eta^2)a_{20})(t-t_0)} \phi(t) e^{-q(t-t_0)},$$

where now $\tilde{d_m} \doteq e^{2\pi i(c_{10}\mu_1(\xi_m^1) + c_{20}\mu_1(\eta^2) - iq)} - 1$. Define also $\widehat{f}(\cdot, \xi)_{\alpha\beta} \equiv 0$ for every other $[\xi] \in \widehat{G}$, $\alpha, \beta \in J_\xi$. Again we claim that these coefficients define a smooth function f. Indeed, since $b_{10} = 0 = \mathrm{Re}(q)$,

$$|\tilde{d_m}| \leq K_1,$$

where $K_1 = e^{2\pi(-\mu_1(\eta^2)b_{20})} + 1$. For each $\gamma \in \mathbb{N}_0$, we estimate as before

$$\left|\partial_t^\gamma \widehat{f}(t, \tilde{\xi_m})_{\overline{1}\,\overline{1}}\right| \leq K e^{B\mu_1(\xi_m^1)} \sum_{j=0}^{\gamma} \binom{\gamma}{j} \left|\partial_t^j e^{-i(\mu_1(\xi_m^1)a_{10} + \mu_1(\eta^2)a_{20})(t-t_0)}\right| \left|\partial_t^{\gamma-j}\psi(t)\right|,$$

for every $t \in [0, 2\pi]$. Notice that

$$\begin{aligned}
|\partial_t^j e^{-i(\mu_1(\xi_m^1)a_{10} + \mu_1(\eta^2)a_{20})(t-t_0)}| &= \left|(-i(\mu_1(\xi_m^1)a_{10} + \mu_1(\eta^2)a_{20})^j e^{-i(\mu_1(\xi_m^1)a_{10} + \mu_1(\eta^2)a_{20})(t-t_0)}\right| \\
&\leq (1 + |a_{10}| + |a_{20}|)^j \left(\langle \xi_m^1 \rangle + \langle \eta^2 \rangle\right)^j \\
&\leq (1 + |a_{10}| + |a_{20}|)^\gamma \left(\langle \xi_m^1 \rangle + \langle \eta^2 \rangle\right)^\gamma,
\end{aligned}$$

for every $t \in [0, 2\pi]$ and $m \in \mathbb{N}$. Also, since $B < 0$ and $([\xi_m^1])_m$ is an Archimedean sequence,

$$e^{B\mu_1(\xi_m^1)} \leq e^{BK_2\langle \xi_m^1 \rangle^\varepsilon}$$

for some $K_2, \varepsilon > 0$. Therefore

$$|\partial_t^\gamma \widehat{f}(t, \tilde{\xi}_m)_{\bar{1}\bar{1}}| \leq K_1(1 + |a_{10}| + |a_{20}|)^\gamma Q_\gamma e^{BK_2\langle \xi_m^1 \rangle^\varepsilon} \left(\langle \xi_m^1 \rangle + \langle \eta^2 \rangle \right)^\gamma,$$

for every $t \in \mathbb{T}^1$, where $Q_\gamma = 2^\gamma \max_{1 \leq j \leq \gamma} \max_{t \in [0, 2\pi]} |\partial_t^j \psi(t)|$. Moreover, as $B < 0$, for any $N > 0$ there exists a constant $Q_{\gamma N} > 0$ such that

$$|\partial_t^\gamma \widehat{f}(t, \tilde{\xi}_m)_{\bar{1}\bar{1}}| \leq Q_{\gamma N} \langle \xi_m^1 \rangle^{-(N+\gamma)} \left(\langle \xi_m^1 \rangle + \langle \eta^2 \rangle \right)^\gamma,$$

for every $t \in \mathbb{T}^1$, $m \in \mathbb{N}$. Therefore

$$|\partial_t^\gamma \widehat{f}(t, \tilde{\xi}_m)_{\bar{1}\bar{1}}| \leq Q'_{\gamma N} \left(\langle \xi_m^1 \rangle + \langle \eta^2 \rangle + \langle \mathbb{1}^3 \rangle + \cdots + \langle \mathbb{1}^n \rangle \right)^{-N},$$

for some constant $Q'_{\gamma N} > 0$, and every $t \in \mathbb{T}^1$, $m \in \mathbb{N}$, $\gamma \in \mathbb{N}_0$. By Lemma 3.1.5, we conclude that these coefficients define $f \in C^\infty(\mathbb{T}^1 \times G)$. Once again, notice that $\{([\tilde{\xi}_m], \bar{1}) \mid m \in \mathbb{N}\}$ is contained in $\mathcal{Z}_L^c$, which implies that $\widehat{f}(\cdot, \xi)_{\alpha\beta} \equiv 0$ for all $([\xi], \alpha) \in \mathcal{Z}_L$. By Lemma 5.A.2 we conclude that $f \in (\ker {}^t L)^0 \cap C^\infty(\mathbb{T}^1 \times G)$. Now suppose there exists $u \in C^\infty(\mathbb{T}^1 \times G)$ such that $Lu = f$. Since $([\tilde{\xi}_m], \bar{1}) \in \mathcal{Z}_L^c$ for every $m \in \mathbb{N}$, because $\mathrm{Re}(q) = 0$ and $b_{20} \neq 0 \neq \mu_1(\eta^2)$, Lemma 5.A.1 implies that the partial Fourier coefficients $\widehat{u}(\cdot, \tilde{\xi}_m)_{\bar{1}\bar{1}}$ must satisfy

$$\begin{aligned} \widehat{u}(t, \tilde{\xi}_m)_{\bar{1}\bar{1}} &= (\tilde{d_m})^{-1} \int_0^{2\pi} \exp\left\{ \int_t^{t+\tau} (\langle c(w), \mu_{\bar{1}}(\tilde{\xi}_m) \rangle - iq) dw \right\} \widehat{f}(t+\tau, \tilde{\xi}_m)_{\bar{1}\bar{1}} d\tau \\ &= e^{-i(\mu_1(\xi_m^1)a_{10} + (\mu_1(\eta^2)a_{20})(t-t_0)} e^{-q(t-t_0)} \int_0^{2\pi} e^{\mu_1(\xi_m^1)(B - \mathrm{Im}(G(t,\tau))} e^{-\mu_1(\eta^2) \int_t^{t+\tau} b_2(w) dw} \phi(t+\tau) d\tau, \end{aligned} \tag{5.30}$$

for every $t \in \mathbb{T}^1$, $m \in \mathbb{N}$. As we are assuming $b_{20} < 0$, this implies b_2 is non-positive on $\mathbb{T}^1$. Therefore

$$\int_t^{t+\tau} b_2(w) dw \leq 0$$

for all $t, \tau \in [0, 2\pi]$. Let $\theta(\tau) \doteq \operatorname{Im}(G(t_0, \tau)) - B$, for each $\tau \in [0, 2\pi]$. Then

$$\begin{aligned}
|\widehat{u}(t_0, \tilde{\xi}_m)_{\bar{1}\bar{1}}| &= \left| \int_0^{2\pi} e^{-\mu_1(\xi^1_m)\theta(\tau)} e^{-\mu_1(\eta^2)\int_{t_0}^{t_0+\tau} b_2(w)dw} \phi(t_0+\tau)d\tau \right| \\
&\geq \int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^{-\mu_1(\xi^1_m)\theta(\tau)} e^{-\mu_1(\eta^2)\int_{t_0}^{t_0+\tau} b_2(w)dw} d\tau \\
&\geq \int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^{-K''\langle \xi^1_m \rangle \theta(\tau)} d\tau,
\end{aligned}$$

where we used the fact that $\theta(\tau) \geq 0$, and $0 < \mu_1(\xi^1_m) \leq K''\langle \xi^1_m \rangle$ for some $K'' > 0$ and for every $m \in \mathbb{N}$, as well as the fact that $e^{-\mu_1(\eta^2)\int_{t_0}^{t_0+\tau} b_2(w)dw} \geq 1$ for all $\tau \in [0, 2\pi]$. Applying Lemma 5.A.4 once again, there exists $\tilde{M} > 0$ such that, for all sufficiently large m

$$|\widehat{u}(t_0, \tilde{\xi}_m)_{\bar{1}\bar{1}}| \geq \tilde{M} \frac{1}{\langle \xi^1_m \rangle^{\frac{1}{2}}}. \tag{5.31}$$

Therefore

$$|\widehat{u}(t_0, \tilde{\xi}_m)_{\bar{1}\bar{1}}| \geq \tilde{M}' \left(\langle \xi^1_m \rangle + \langle \eta^2 \rangle + \langle \mathbb{1}^3 \rangle + \cdots + \langle \mathbb{1}^n \rangle \right)^{-\frac{1}{2}}$$

for some $\tilde{M}' > 0$, and all $m \in \mathbb{N}$. By Lemma 3.1.5 $u \notin C^\infty(\mathbb{T}^1 \times G)$, which is a contradiction. Therefore no such function u exists, which means that L is not globally solvable. Next we prove that L is not globally hypoelliptic. Indeed, similar to the previous case, we choose the partial Fourier coefficients of u by formula (5.30) for $\widehat{u}(t, \tilde{\xi}_m)_{\bar{1}\bar{1}}$ and let $\widehat{u}(\cdot, \xi)_{\alpha\beta} \equiv 0$ for every other $[\xi] \in \widehat{G}$, and $\alpha, \beta \in J_\xi$. Then the inequalities (5.31) imply that these partial Fourier coefficients do not define a smooth function. On the other hand, for every $\zeta \in C^\infty(\mathbb{T}^1)$ and

$m \in \mathbb{N}$, we have that

$$
\begin{aligned}
&|\langle \widehat{u}(\cdot, \tilde{\xi}_m)_{\overline{1}\,\overline{1}}, \zeta \rangle_{\mathbb{T}^1}| = \\
&= \left| \int_0^{2\pi} e^{-i(\mu_1(\xi_m^1)a_{10} + \mu_1(\eta^2)a_{20})(t-t_0)} e^{-q(t-t_0)} \right. \\
&\times \left. \int_0^{2\pi} e^{\mu_1(\xi_m^1)(B - \mathrm{Im}(G(t,\tau))} \phi(t+\tau)\zeta(t) e^{-\mu_1(\eta^2)\int_t^{t+\tau} b_2(w)dw} d\tau dt \right| \\
&\leq \int_0^{2\pi}\int_0^{2\pi} e^{2\pi|\mathrm{Re}(q)|} e^{\mu_1(\xi_m^1)(B - \mathrm{Im}(G(t,\tau))} |\phi(t+\tau)||\zeta(t)| e^{-\mu_1(\eta^2)\int_t^{t+\tau} b_2(w)dw} d\tau dt \\
&\leq (2\pi)^2 \|\phi\|_\infty \|\zeta\|_\infty \max_{t,\tau\in[0,2\pi]} e^{-\mu_1(\eta^2)\int_t^{t+\tau} b_2(w)dw} e^{2\pi|\mathrm{Re}(q)|} \\
&\leq \tilde{Q}' p_1(\zeta) \langle \xi_m^1 \rangle \\
&\leq \tilde{Q}'' p_1(\zeta) (\langle \xi_m^1 \rangle + \langle \eta^2 \rangle + \langle \mathbb{1}^3 \rangle + \cdots + \langle \mathbb{1}^n \rangle)^1
\end{aligned}
$$

for some constants $\tilde{Q}', \tilde{Q}'' > 0$. Thus, by Lemma 3.1.5 these partial Fourier coefficients define $u \in \mathcal{D}'(\mathbb{T}^1 \times G)$. By comparing partial Fourier coefficients, it is clear that $Lu = f$, therefore we conclude that L is not globally hypoelliptic. The proof for the sub-case $b_{20} > 0$ is very similar, so we omit it. But the main idea is to substitute η^2 by $\overline{\eta^2}$ (since $\mu_1(\overline{\eta^2}) < 0$). □

Next, exhibit a condition which allows us to recover certain results analogous to the case of the Torus. This condition guarantees that the eigenvalues in an Archimedean sequence are not too "far apart".

Definition 5.2.15. We say a vector field X on a compact Lie group admits a non-sparse Archimedean sequence if there exists an Archimedean sequence $([\xi_k])_{k\in\mathbb{Z}}$ for X and $M > 0$ such that

$$|\mu_1(\xi_{k+1}) - \mu_1(\xi_k)| \leq M,$$

for all $k \in \mathbb{Z}$.

Remark 5.2.16. Note that as a direct consequence of the definition above, for every $r \in \mathbb{R}$, there exists $k_r \in \mathbb{Z}$ such that

$$r < \mu_1(\xi_{k_r}) < r + M. \tag{5.32}$$

Indeed, suppose that is not true. Then, as $\mu_1(\xi_k) \to +\infty$ as $k \to +\infty$, and $\mu(\xi_k)$ is strictly increasing, there exists $r \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $\mu_1(\xi_k) < r < r + M < \mu_1(\xi_{k+1})$. Then $\mu_1(\xi_{k+1}) - \mu_1(\xi_k) > M$, a contradiction.

With non-sparse Archimedean sequences, we can relate linear independence of the coefficients with the change of sign of their linear combinations, as follows.

Proposition 5.2.17. *Let $X_1, \dots, X_n$ be left invariant vector fields on the compact Lie groups G_j, $j = 1, \dots, n$, $n \in \mathbb{N}$. Suppose every X_j admits a non-sparse Archimedean sequence. Then $t \mapsto \langle b(t), \mu_{\bar{1}}(\xi) \rangle = \sum_{j=1}^{n} b_j(t)\mu_1(\xi^j)$ changes sign for infinitely many $[\xi] \in \widehat{G}$, if and only if some b_j changes sign or $\dim$ span$\{b_1, \dots, b_n\} \geq 2$.*

Proof. First suppose some b_j changes sign. Without loss of generality we can assume $j = 1$. Let $(\xi^1_m)_m$ be an Archimedean sequence for X_1. For every $m \in \mathbb{N}$, define

$$\xi_m = \xi^1_m \otimes \mathbb{1}^2 \otimes \cdots \otimes \mathbb{1}^n.$$

It is clear then that the function $t \mapsto \langle b(t), \mu_{\bar{1}}(\xi_m) \rangle = \sum_{j=1}^{n} b_j(t)\mu_1(\xi^j_m) = b_1(t)\mu_1(\xi^1_m)$ changes sign for infinitely many $[\xi] \in \widehat{G}$, as there are infinitely many $m \in \mathbb{N}$ such that $\mu_1(\xi^j_m) \neq 0$. Suppose now $g \doteq b_{j_1}$, $h \doteq b_{j_2}$ are $\mathbb{R}$-linearly independent, for some $j_1, j_2 \in \{1, \dots, n\}$. Without loss of generality we may assume $j_1 = 1$ and $j_2 = 2$. Let $(\xi^1_k)_k$ and $(\xi^2_k)_k$ be non-sparse Archimedean sequences for X_1 and X_2, respectively. We shall exhibit sequence $(\xi_m)_{m \in \mathbb{N}}$, of distinct terms, such that $t \mapsto \langle b(t), \mu_{\bar{1}}(\xi_m) \rangle$ changes sign, for each $m \in \mathbb{N}$. From the previous case, we may assume both g and h do not change sign. Since $g \neq \lambda h$, for some non-zero constant λ, the mapping $t \mapsto \frac{h(t)}{g(t)}$ is either identically zero or non constant on $A \doteq \{t \in \mathbb{T}^1 \,|g(t) \neq 0\}$. Take $t_0, t_1 \in \mathbb{T}^1$ such that $g(t_0) \neq 0 \neq h(t_1)$. If the quotient $\frac{h(t)}{g(t)}$ is identically zero on A and $\operatorname{sign}(g(t_0)) = \operatorname{sign}(h(t_1))$, then

$$t \mapsto \mu_1(\xi^1_{-m})g(t) + \mu_1(\xi^2_1)h(t)$$

changes sign since $h(t) = 0$ whenever $g(t) \neq 0$ and $\mu_1(\xi^1_{-m}) < 0$ while $\mu_1(\xi^2_1) > 0$. Therefore we can take $\xi_m = \xi^1_{-m} \otimes \xi^2_1 \otimes \mathbb{1}^3 \otimes \cdots \otimes \mathbb{1}^n$, for each $m \in \mathbb{N}$. If $\operatorname{sign}(g(t_0)) \neq \operatorname{sign}(h(t_1))$, we can apply the same idea and take the sequence $\xi_m = \xi^1_{+m} \otimes \xi^2_1 \otimes \mathbb{1}^3 \otimes \cdots \otimes \mathbb{1}^n$. Now assume that the mapping $t \mapsto \frac{h(t)}{g(t)}$ is non constant in A. Suppose first that $h(t) \geq 0$, then choose $t_0, t_1 \in [0, 2\pi]$, such that

$$0 \neq a = \frac{h(t_0)}{g(t_0)} < \frac{h(t_1)}{g(t_1)} = b \neq 0.$$

These exist since the function $t \mapsto \frac{h(t)}{g(t)}$ is continuous and non-constant by our assumptions.

We claim there exists $k_{1m}, k_{2m} \in \mathbb{Z}$ such that

$$0 < \mu_1(\xi^1_{k_{1m}})g(t_0) + \mu_1(\xi^2_{k_{2m}})h(t_0),$$
$$0 > \mu_1(\xi^1_{k_{1m}})g(t_1) + \mu_1(\xi^2_{k_{2m}})h(t_1),$$

for each $m \in \mathbb{N}$. Indeed, as $h(t_0), h(t_1) > 0$, the inequalities above are satisfied if

$$-\mu_1(\xi^1_{k_{1m}})\frac{g(t_0)}{h(t_0)} < \mu_1(\xi^2_{k_{2m}}) < -\mu_1(\xi^1_{k_{1m}})\frac{g(t_1)}{h(t_1)}. \tag{5.33}$$

As $\frac{g(t_1)}{h(t_1)} < \frac{g(t_0)}{h(t_0)}$, there exists $\varepsilon > 0$, such that $\frac{g(t_1)}{h(t_1)} = \frac{g(t_0)}{h(t_0)} - \varepsilon$. Then, if k_{1m} is sufficiently big, $\mu_1(\xi^1_{k_{1m}})\varepsilon \geq M$, where $M > 0$ is such that for every $r \in \mathbb{R}$, there exists $k_r \in \mathbb{Z}$ such that

$$r < \mu_1(\xi^2_{k_r}) < r + M$$

(notice that M exists because the sequence $(\xi^2_k)_k$ is non-sparse). Therefore, for each $r_m = -\mu_1(\xi^1_{k_{1m}})\frac{g(t_0)}{h(t_0)}$, there exists $k_{2m} \in \mathbb{Z}$ such that

$$r_m < \mu_1(\xi^2_{k_{2m}}) < r_m + M \leq r_m + \mu_1(\xi_{k_{1m}})\varepsilon.$$

Choosing $k_{1m} = k_{10}+m$, where k_{10} satisfies $\mu_1(\xi^1_{k_{10}}) \geq M/\varepsilon$ and k_{2m} satisfying the inequality above, we have $(k_{1m}, k_{2m}) \in \mathbb{Z}^2$ for each $m \in \mathbb{N}$, such that (5.33) holds, therefore $t \mapsto \langle b(t), \mu_{\bar{1}}(\xi_m)\rangle$ changes sign for every $\xi_m = \xi^1_{k_{1m}} \otimes \xi^2_{k_{2m}} \otimes \mathbb{1}^3 \otimes \cdots \otimes \mathbb{1}^n$, where $m \in \mathbb{N}$. The case $h(t) \leq 0$ is analogous. Finally, the reverse implication follows from Remark 5.2.9. $\square$

To conclude this section, we collect some of the results of this section in two final corollaries, as follows.

Corollary 5.2.18 (Global solvability)**.** *Let L and L_0 be the operators defined in* (5.4) *and* (5.5)*, respectively.*

1. *If $b \equiv 0$ then L is globally solvable if and only if L_0 is globally solvable.*

2. *If some $b_j \not\equiv 0$, $\mathcal{Z}^c_L$ is finite (and so $\mathrm{Re}(q) = 0$ and all $b_{j0} = 0$ by Remark 5.2.4), and the sublevels $\Omega^{\xi,\alpha}_r$ are connected for all $\langle\xi\rangle$ large enough, then L is globally solvable.*

3. *If some $b_j \not\equiv 0$, $\mathrm{Re}(q) = 0$, $\dim \mathit{span}\{b_1, \ldots, b_n\} = 1$ and no b_j changes sign then L is globally solvable if and only if L_0 is globally solvable.*

4. *If* $b \not\equiv 0$*,* $t \mapsto \langle b(t), \mu_\alpha(\xi)\rangle$ *does not change sign for all but finitely many* $[\xi] \in \widehat{G}$*, and* $\alpha \in J_\xi$*, and also* $\mathrm{Re}(q) = 0$*, then* L *is globally solvable if and only if* L_0 *is globally solvable.*

Proof. First notice that case 1 follows from Corollary 5.1.4. Case 2, is given by Corollary 5.2.3. Remark 5.2.9 implies that Case 3 follows from Corollary 5.2.8. Finally, case 4 corresponds to Corollary 5.2.8. □

Corollary 5.2.19 (Global hypoellipticity)**.** *Let* L *and* L_0 *be the operators defined in* (5.4) *and* (5.5)*, respectively.*

1. *If* $b \equiv 0$*, then* L *is globally hypoelliptic if and only if* L_0 *is globally hypoelliptic.*

2. *If* $b \not\equiv 0$ *and* $\dim \mathit{span}\{b_1, \ldots, b_n\} = 1$*, then* L *is globally hypoelliptic if and only if, no* b_j *changes sign,* $\mathcal{Z}_L$ *is finite and* L_0 *is globally solvable (and therefore* L_0 *is also globally hypoelliptic).*

3. *If* $t \mapsto \langle b(t), \mu_\alpha(\xi)\rangle$ *does not change sign for all but finitely many* $[\xi] \in \widehat{G}$*, and* $\alpha \in J_\xi$*, then* L *is globally hypoelliptic if and only if* L_0 *is globally hypoelliptic.*

Proof. Case 1 is a direct consequence of Corollary 5.1.4. Now, from Remark 5.2.9 we have that if all the conditions of case 2 hold, then L is globally hypoelliptic by Corollary 5.2.8. On the other hand, if L_0 is not globally solvable or if $\mathcal{Z}_L$ is infinite, then L_0 is not globally hypoelliptic and L is not globally hypoelliptic also by Corollary 5.2.8. If we suppose that $\mathcal{Z}_L$ is finite, but now that some b_j changes sign, then L is not globally hypoelliptic by Proposition 5.2.14. Finally, case 3 corresponds to Corollary 5.2.8. □

Remark 5.2.20. It is worth mentioning that by Theorem 3.5 in [2] we have that in this context every globally hypoelliptic operator is globally solvable. This could be used to simplify some of the results and proofs in this section. We chose not to rely on this fact as we do not include its proof, and also to present original proofs using different techniques.

5.2.2 Examples

Example 5.2.21. Consider the operator L defined on $\mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{S}^3$ given by

$$L = \partial_t + ((1+\sin(t)) + i(4+\cos(4t)\partial_x + ((2+\sin(t)) + 2i(4+\cos(4t)))i\partial_0 + (4+\sin(3t))$$

then

$$L_0 = \partial_t + (1+4i)\partial_x + (2+8i)i\partial_0 + 4.$$

Since $b_1 = 2b_2$, $b_1, b_2 \geq 0$ and

$$|k + (1+4i)\xi + (2+8i)\alpha - 4i| = \sqrt{(k+\xi+2\alpha)^2 + (4\xi+8\alpha)^2},$$

which is either 0 or ≥ 1 for $k, \xi \in \mathbb{Z}$ and $\alpha \in \frac{1}{2}\mathbb{Z}$. Therefore L_0 is globally solvable by Corollary 5.2.18, item 3. Notice that it is not globally hypoelliptic by Corollary 5.2.19 because $\mathcal{Z}_L$ is infinite since its symbol is zero for $k = 0$, $\xi \in \mathbb{Z}$ and $\alpha = \frac{-k+\xi}{2}$.

Example 5.2.22. Consider the operator L defined on $\mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{S}^3$ given by

$$L = \partial_t + ((1+\cos(t)) + i\sin(t))\partial_x + ((2+\sin(t)) + 2i\cos(t))i\partial_0 + (4i + \sin(3t)).$$

Then

$$L_0 = \partial_t + 1\partial_x + 2i\partial_0 + 4i, \tag{5.34}$$

and so

$$\mathcal{Z}_L^c = \left\{(k,\xi,\alpha) \in \mathbb{Z}^2 \times \frac{1}{2}\mathbb{Z} \mid k + \xi + 2\alpha + 4 \notin \mathbb{Z}\right\} = \emptyset.$$

Hence $\mathcal{Z}_L^c$ is finite. Moreover, one can show that the sublevels

$$\begin{aligned}\Omega_r^{\xi,\alpha} &= \left\{t \in \mathbb{T}^1 \mid \int_0^t \xi \cdot \sin(\tau) + 2\alpha \cdot \cos(\tau) d\tau < r\right\} \\ &= \left\{t \in \mathbb{T}^1 \mid -\xi \cdot \cos(t) + 2\alpha \cdot \sin(t) < r\right\},\end{aligned}$$

are connected for all $\xi \in \mathbb{Z}$, $\alpha \in \frac{1}{2}\mathbb{Z}$ and $r \in \mathbb{R}$. Therefore L is globally solvable by Corollary 5.2.18 item 2.

Example 5.2.23. Consider the operator L defined on $\mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{S}^3$ given by

$$L = \partial_t + (\kappa + 10i\sin^3(t))\partial_x + (\sqrt{2} + 20i\cos^3(t))i\partial_0 + (4i + \sin(3t)),$$

where $\nu = \sum_{k=1}^{\infty} \frac{1}{10^{k!}}$ is a Liouville number. Then

$$L_0 = \partial_t + \nu\partial_x + \sqrt{2}i\partial_0 + 4i,$$

so $a_0 = (\nu, \sqrt{2})$, $b_0 = (0,0)$ and $q_0 = 4i$. Since ν is a Liouville number and $q_0 \in i\mathbb{Z}$, by an argument similar to the one used in Corollary 4.2.3 we have that L_0 is not globally solvable. Therefore by Proposition 5.2.7 we have that L is not globally solvable.

Indeed, by the reasoning of Corollary 4.2.3 the sequence $(-(\sum_{k=1}^{m} 10^{m!-k!}+4), 10^{m!}, 0) \in \mathbb{Z}^2 \times \frac{1}{2}\mathbb{Z}$, $m \in \mathbb{N}$, it satisfies

$$\left| -\left(\sum_{k=1}^{m} 10^{m!-k!}+4\right) + \nu \cdot 10^{m!} + \sqrt{2}\cdot 0 + 4 \right| < \frac{1}{10^{m!m}} \\ \leq C\left(|10^{m!}| + |-\left(\sum_{k=1}^{m} 10^{m!-k!}+4| + |0|\right)\right)^{m},$$

for some $C > 0$. Therefore, consider the distribution given by

$$\widehat{u}(k,\xi,\ell)_{\alpha\beta} = \begin{cases} 1, & \text{if } (k,\xi,\ell,\alpha,\beta) = \left(-\left(\sum_{k=1}^{m} 10^{m!-k!}+4\right), 10^{m!}, 0, 1, 1\right),\ m \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

It clearly is not smooth, however $Lu = f \in C^{\infty}(\mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{S}^3)$.

5.3 Application to the product of tori and 3-spheres

Not only can we apply the results in the last section to the setting where $G_1 \times \cdots \times G_n$ is the product of tori $\mathbb{T}^1$ and spheres $\mathbb{S}^3$, but also, in this case, obtain better results concerning the global properties of the operator L. In fact, we are able to obtain necessary and sufficient conditions for both the global solvability and hypoellipticity of this operator. A more detailed exposition of the global properties of L in this case can be found in [79].

Henceforth, we shall consider first-order evolution differential operators on $\mathbb{T}^{r+1} \times (\mathbb{S}^3)^s$, where r and s are non-negative integers not both zero. By composing it with a change of variables, we may assume the operator L can be written as

$$L = \partial_t + \sum_{j=1}^{r} c_j(t)\partial_{x_j} + \sum_{l=1}^{s} d_l(t) i\partial_{0,l} + q(t), \tag{5.35}$$

where c_j, d_l, q are smooth functions on $\mathbb{T}^1$, $j = 1,\dots,r$, $l = 1,\dots,s$, and $\partial_{0,l} = \frac{1}{i}D_{3,l}$ is the neutral operator on a different copy of $\mathbb{S}^3$, for each l. Again we may assume that the real part of c_j and d_l, as well as q are all constant. This means that for $u \in \mathcal{D}'(\mathbb{T}^{r+1} \times (\mathbb{S}^3)^s)$ by taking the partial Fourier transform of Lu of the last $r+s$ variables, with the same notation as in Chapter 2, Section 4.2, we obtain

$$\widehat{Lu}(t,k,\ell)_{\alpha\beta} = \partial_t \widehat{u}(t,k,\ell)_{\alpha\beta} + i\left(\sum_{j=1}^{r} c_j(t)k_j + \sum_{l=1}^{s} d_l(t)(\alpha_l - \ell_l - 1) - iq\right)\widehat{u}(t,k,\ell)_{\alpha\beta},$$

for every $(k,\ell) \in \mathbb{Z}^r \times \mathbb{N}_0^s$, $\alpha, \beta \in J_\ell$. We can simplify this expression by allowing our matrix indexes to assume values in $\frac{1}{2}\mathbb{Z}$, and performing the change of variables

$$\alpha_l' = \alpha_l - \ell_l - 1,$$
$$\beta_l' = \beta_l - \ell_l - 1.$$

This way, writing

$$\sum_{j=1}^{r} c_j(t)k_j \doteq \langle c(t), k\rangle,\ \sum_{l=1}^{s} d_l(t)\alpha_l' \doteq \langle d(t), \alpha'\rangle,$$

we can rewrite the equality above as

$$\widehat{Lu}(t,k,\ell)_{\alpha'\beta'} = \partial_t \widehat{u}(t,k,\ell)_{\alpha'\beta'} + i(\langle c(t),k\rangle + \langle d(t),\alpha'\rangle - iq)\widehat{u}(t,k,\ell)_{\alpha'\beta'}.$$

From now on we will omit the prime notation on the indexes α and β and assume they are in the appropriate range.

The first step in adapting the results in the previous section to this setting concerns the cardinality of $\mathcal{Z}_L$ and $\mathcal{Z}_L^c$, as defined in (5.9) and (5.10), respectively. Notice that, setting

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} c(t)dt,\ d_0 = \frac{1}{2\pi}\int_0^{2\pi} d(t)dt,$$

we have that

$$\mathcal{Z}_L = \left\{((k,\ell),\alpha) \in \mathbb{Z}^r \times \frac{1}{2}\mathbb{N}_0^s \times \frac{1}{2}\mathbb{Z}^s \mid -\ell \leq \alpha \leq \ell,\ \langle c_0, k\rangle + \langle d_0, \alpha\rangle - iq \in \mathbb{Z}\right\},$$

$$\mathcal{Z}_L^c = \left\{((k,\ell),\alpha) \in \mathbb{Z}^r \times \frac{1}{2}\mathbb{N}_0^s \times \frac{1}{2}\mathbb{Z}^s \mid -\ell \leq \alpha \leq \ell,\ \langle c_0, k\rangle + \langle d_0, \alpha\rangle - iq \notin \mathbb{Z}\right\}.$$

It follows immediately that if $s \neq 0$ these are either empty or infinite. Indeed, note that if $((k,\ell),\alpha) \in \mathcal{Z}_L^c$ or $((k,\ell),\alpha) \in \mathcal{Z}_L$ for some $(k,\ell) \in \mathbb{Z}^r \times \frac{1}{2}\mathbb{N}_0^s$, then $((k,\ell+m),\alpha) \in \mathcal{Z}_L^c$ or $((k,\ell+m),\alpha) \in \mathcal{Z}_L$, for every $m \in \mathbb{Z}$, respectively. Moreover, notice that either $\mathcal{Z}_L$ or $\mathcal{Z}_L^c$ is infinite (and thus not empty), but not both.

Another distinguishing characteristic of this particular case is that given a representation corresponding to a triple $((k,\ell),\alpha)$, there exist representations corresponding to the triples $((mk,m\ell),m\alpha)$, for every $m\in\mathbb{Z}_+$. This fact allows the construction of singular solutions and is responsible for obtaining more results than in the general case. For instance, we are able to obtain a converse of Corollary 5.2.3 as follows.

Proposition 5.3.1. *If $\mathcal{Z}_L^c$ is empty, or equivalently,*

$$c_0\in\mathbb{Z}^r,\ d_0\in 2\,\mathbb{Z}^s,\ q\in i\,\mathbb{Z},$$

then L, as defined in (5.35), is globally solvable if and only if the sublevel set

$$\Omega_m^{k,\alpha}=\left\{t\in\mathbb{T}^1 \mid \int_0^t\Big(\sum_{j=1}^r \mathrm{Im}(c_j)(\tau)k_j+\sum_{l=1}^s \mathrm{Im}(d_l)(\tau)\alpha_l\Big)d\tau<m\right\}$$

is connected, for every $m\in\mathbb{R}$, $k\in\mathbb{Z}^r$, $\alpha\in\frac{1}{2}\mathbb{Z}^s$.

The "if" part follows from Corollary 5.2.3. So we only need to prove the reverse implication. The proof was inspired by [9] and relies essentially on the following lemma adapted from a result by Hörmander in [70].

Lemma 5.3.2. *Let L be a globally solvable differential operator on a compact Lie group G. Then, there exist $m\in\mathbb{N}$ and $C>0$ such that for every $f\in(\ker{}^tL)^0$ and $v\in C^\infty(G)$, the following inequality holds:*

$$\left|\int_G f(x)v(x)\,dx\right|\leq C\left(\sum_{|\alpha|\leq m}\sup_G|\partial^\alpha f|\right)\left(\sum_{|\alpha|\leq m}\sup_G|\partial^\alpha({}^tLv)|\right). \tag{5.36}$$

The sum $\sum_{|\alpha|\leq m}$ is taken over all left-invariant differential operators on G of order at most m, or equivalently, it is taken over all linear combinations of at most m compositions of elements of a basis for the Lie algebra of G.

Proof of Proposition 5.3.1. Denote $b_j\doteq\mathrm{Im}(c_j)$, $e_l\doteq\mathrm{Im}(d_l)$, for $1\leq j\leq r$, $1\leq l\leq s$. Suppose that L is globally solvable and that, for some $\tilde{k}\in\mathbb{Z}^r$, $\tilde{\alpha}\in\frac{1}{2}\mathbb{Z}^s$, and $m_0'\in\mathbb{R}$, the sublevel set

$$\Omega_{m_0'}^{\tilde{k},\tilde{\alpha}}=\left\{t\in\mathbb{T}^1;\int_0^t\big(\langle b(\tau),\tilde{k}\rangle+\langle f(\tau),\tilde{\alpha}\rangle\big)d\tau<m_0'\right\}\text{ is disconnected.}$$

It is worth noting that this assumption implies that either $\tilde{k} \neq \overline{0}$ or $\tilde{\alpha} \neq \overline{0}$. By applying Lemma 5.A.6, we can find $m_0 \in \mathbb{R}$ as well as functions g_0 and v_0 in $C^\infty(\mathbb{T}^1)$ that satisfy the following conditions:

$$\int_0^{2\pi} g_0(t)dt = 0, \quad \text{supp } g_0 \cap \Omega_{m_0}^{\tilde{k},\tilde{\alpha}} = \varnothing, \text{ and}$$
$$\int_0^{2\pi} g_0(t)v_0(t)\, dt = \ell_0 > 0, \ \ \text{supp } v_0' \subset \Omega_{m_0}^{\tilde{k},\tilde{\alpha}},$$

For each $n \in \mathbb{N}$, we define $n\tilde{\ell} = (n|\tilde{\alpha}_1|, \ldots, n|\tilde{\alpha}_s|)$ and $g_n, v_n \in C^\infty(\mathbb{T}^{r+1} \times (\mathbb{S}^3)^s)$ as follows:

$$g_n(t,x,y) = \sqrt{d_{n\tilde{\ell}}}\exp\left(ni\int_0^t \langle c(\tau), \tilde{k}\rangle + \langle d(\tau), \tilde{\alpha}\rangle d\tau - qt\right) g_0(t)e^{-ni\langle x,\tilde{k}\rangle}\mathfrak{t}^{n\tilde{\ell}}_{(-n\tilde{\ell})(-n\tilde{\ell})}(y)$$
$$v_n(t,x,y) = \sqrt{d_{n\tilde{\ell}}}\exp\left(-ni\int_0^t \langle c(\tau), \tilde{k}\rangle + \langle d(\tau), \tilde{\alpha}\rangle d\tau + qt\right) v_0(t)e^{ni\langle x,\tilde{k}\rangle}\mathfrak{t}^{n\tilde{\ell}}_{(n\tilde{\ell})(n\tilde{\ell})}(y)$$

for every $(t, x, y) \in \mathbb{T}^{r+1} \times (\mathbb{S}^3)^s$. It is important to note that these functions are well-defined based on the given hypotheses for c_0, d_0, and q.

We claim that $g_n \in (\ker {}^tL)^0$. Indeed, if $u \in \ker {}^tL$ then

$$\widehat{-{}^tLu}(t,k,\ell)_{\alpha\beta} = 0 \implies [\partial_t + i(\langle c(t), k\rangle + \langle d(t), \alpha\rangle + iq)]\widehat{u}(t,k,\ell)_{\alpha\beta} = 0. \tag{5.37}$$

Next, we will use the following lemma:

Lemma 5.3.3. *The following equalities hold:*

i. If $f, g \in L^2(\mathbb{S}^3)$, then

$$\int_{\mathbb{S}^3} f(x)g(x)\, dx = \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) \sum_{m,n=-\ell}^{\ell} \widehat{f}(\ell)_{mn}\widehat{g}(\ell)_{(-m)(-n)}(-1)^{n-m}, \tag{5.38}$$

ii. if $f, g \in L^2((\mathbb{S}^3)^s)$, then

$$\int_{(\mathbb{S}^3)^s} f(x)g(x)\, dx = \sum_{\ell \in \frac{1}{2}\mathbb{N}_0^s} d_\ell \sum_{-\ell \leq \alpha,\beta \leq \ell} \widehat{f}(\ell)_{\alpha\beta}\widehat{g}(\ell)_{(-\alpha)(-\beta)}(-1)^{\Sigma(\alpha_j - \beta_j)}, \tag{5.39}$$

iii. if $f, g \in L^2((\mathbb{S}^3)^s)$, then

$$\int_{\mathbb{T}^{r+1}\times(\mathbb{S}^3)^s} fg = (2\pi)^r \sum_{\xi\in\mathbb{Z}^r} \sum_{\ell\in\frac{1}{2}\mathbb{N}_0^s} d_\ell \tag{5.40}$$

$$\times \sum_{-\ell\leq\alpha,\beta\leq\ell} \int_0^{2\pi} \widehat{f}(t,\xi,\ell)_{\alpha\beta}\widehat{g}(t,-\xi,\ell)_{(-\alpha)(-\beta)}(-1)^{\Sigma(\alpha_j-\beta_j)}\, dt\,. \tag{5.41}$$

Proof. These equalities follow immediately from Propositions 2.1.7 and 3.1.7, by considering the property $\overline{\mathfrak{t}^\ell_{mn}} = \mathfrak{t}^\ell_{(-m)(-n)}(-1)^{n-m}$. This last fact can be deduced from the definition itself (see [94]), as evaluating $\mathfrak{t}^\ell_{mn}$ with Euler angles yields, as follows:

$$\begin{aligned}
\overline{\mathfrak{t}^\ell_{mn}(\omega(\phi,\theta,\psi))} &= e^{i(m\phi+n\psi)}\overline{P^\ell_{mn}(\cos(\theta))} \\
&= e^{-i((-m)\phi+(-n)\psi)}(-1)^{n-m}P^\ell_{mn}(\cos(\theta)) \\
&= (-1)^{n-m}e^{-i((-m)\phi+(-n)\psi)}P^\ell_{(-m)(-n)}(\cos(\theta)) \\
&= (-1)^{n-m}\mathfrak{t}^\ell_{(-m)(-n)}(\omega(\phi,\theta,\psi))
\end{aligned}$$

where

$$P^\ell_{mn}(x) = c^\ell_{mn}\frac{(1-x)^{(n-m)/2}}{(1+x)^{(m+n)/2}}\left(\frac{d}{dx}\right)^{\ell-m}[(1-x)^{\ell-n}(1+x)^{\ell+n}]$$

with

$$c^\ell_{mn} = 2^{-\ell}\frac{(-1)^{\ell-n}i^{n-m}}{\sqrt{(\ell-n)!(\ell+n)!}}\sqrt{\frac{(\ell+m)!}{(\ell-m)!}}.$$

□

Now, using the formula from Lemma 5.3.3, we have

$$\begin{aligned}
\langle u, g_n\rangle &= (2\pi)^r \sum_{k\in\mathbb{Z}^r}\sum_{\ell\in\frac{1}{2}\mathbb{N}_0} d_\ell \sum_{-\ell\leq\alpha,\beta\leq\ell}\int_0^{2\pi} \widehat{u}(t,k,\ell)_{\alpha\beta}\widehat{g_n}(t,-k,\ell)_{(-\alpha)(-\beta)}\,dt(-1)^{\Sigma(\beta_j-\alpha_j)} \\
&= \sqrt{d_{n\tilde{\ell}}}(2\pi)^r\int_0^{2\pi}\widehat{u}(t,n\tilde{k},n\tilde{\ell})_{(n\tilde{\ell})(n\tilde{\ell})}\exp\left(ni\int_0^t\langle c(\tau),\tilde{k}\rangle+\langle d(\tau),\tilde{\alpha}\rangle d\tau - qt\right)g_0(t)\,dt \\
&= \sqrt{d_{n\tilde{\ell}}}(2\pi)^r\int_0^{2\pi}\widehat{u}(t,n\tilde{k},n\tilde{\ell})_{(n\tilde{\ell})(n\tilde{\ell})}\exp(niw(t)-qt)g_0(t)\,dt,
\end{aligned}$$

where $w(t) = \int_0^t \langle c(\tau),\tilde{k}\rangle + \langle d(\tau),\tilde{\alpha}\rangle d\tau$.

Since

$$\partial_t \left(\exp(niw(t) - qt)\widehat{u}(t, n\tilde{k}, n\tilde{\ell})_{(n\tilde{\alpha})(n\tilde{\alpha})} \right) = \exp(niw(t) - qt) \times$$
$$\times \left[\partial_t \widehat{u}(t, n\tilde{k}, n\tilde{\ell})_{(n\tilde{\alpha})(n\tilde{\alpha})} + \left(ni[\langle c(t), \tilde{k}\rangle + \langle d(t), \tilde{\alpha}\rangle] - q \right) \widehat{u}(t, n\tilde{k}, n\tilde{\ell})_{(n\tilde{\alpha})(n\tilde{\alpha})} \right] = 0,$$

then for every $n \in \mathbb{N}$ we have

$$\exp(niw(t) - qt)\widehat{u}(t, n\tilde{k}, n\tilde{\ell})_{(n\tilde{\alpha})(n\tilde{\alpha})} \equiv k_n \in \mathbb{C}.$$

Consequently

$$\langle u, g_n \rangle = \sqrt{d_{n\tilde{\ell}}}\, (2\pi)^r k_n \int_0^{2\pi} g_0(t)\, dt = 0,$$

which implies that $g_n \in (\ker {}^tL)^0$ for all $n \in \mathbb{N}$, as claimed.

Therefore, by Lemma 5.3.2 there exists $C > 0$, $\lambda \in \mathbb{N}_0$ such that

$$\left| \int_{\mathbb{T}^{r+1} \times (\mathbb{S}^3)^s} g_n v_n \right| \leq C \left(\sum_{|\mu| \leq \lambda} \sup_{(t,x,y) \in \mathbb{T}^{r+1} \times (\mathbb{S}^3)^s} |\partial^\mu g_n(t, x, y)| \right)$$
$$\times \left(\sum_{|\mu| \leq \lambda} \sup_{(t,x,y) \in \mathbb{T}^{r+1} \times (\mathbb{S}^3)^s} |\partial^\mu ({}^tLv_n)(t, x, y)| \right). \tag{5.42}$$

However, note that by the definitions of g_n and v_n, we have

$$\int_{\mathbb{T}^{r+1} \times (\mathbb{S}^3)^s} g_n v_n = (2\pi)^r \frac{d_{n\tilde{\ell}}}{d_{n\tilde{\ell}}} \int_0^{2\pi} f_0(t) v_0(t) dt = (2\pi)^r l_0 \geq \ell_0 > 0, \ n \in \mathbb{N}. \tag{5.43}$$

On the other hand, since $\operatorname{supp} f_0 \cap \Omega_{m_0}^{\tilde{k},\tilde{\alpha}} = \varnothing$, by the Leibniz formula, there exist positive constants M_1, M_μ, and M_1' such that

$$
\begin{aligned}
&\sum_{|\mu|\leq\lambda} \sup_{(t,x,y)\in\mathbb{T}^{r+1}\times(\mathbb{S}^3)^s} |\partial^\mu g_n(t,x,y)| \\
&\leq \sqrt{d_{n\tilde{\ell}}} \sum_{|\mu|\leq\lambda} \sup_{(t,x,y)\in\mathbb{T}^{r+1}\times(\mathbb{S}^3)^s} \left| \sum_{\gamma_1+\gamma_2+\gamma_3+\gamma_4=\mu} C_{\mu\gamma}\partial^{\gamma_1}[\exp(niw(t)-qt)]\partial^{\gamma_2}[e^{-in\langle x,\tilde{k}\rangle}] \right| \\
&\quad \times \left| \partial^{\gamma_3}[g_0(t)]\partial^{\gamma_4}\mathfrak{t}^{n\tilde{\ell}}_{(-n\tilde{\alpha})(-n\tilde{\alpha})}(y) \right| \\
&\leq \sqrt{d_{n\tilde{\ell}}} \sum_{|\mu|\leq\lambda} M_1 M_\mu n^{|\mu|} \sup_{t\in\mathbb{T}^1\setminus\Omega^{\tilde{k},\tilde{\alpha}}_{m_0}} |\exp(niw(t)-qt)| \sum_{\gamma_4\leq\mu}\sup_{y\in(\mathbb{S}^3)^s} |\partial^{\gamma_4}\mathfrak{t}^{n\tilde{\ell}}_{(-n\tilde{\alpha})(-n\tilde{\alpha})}(y)| \\
&\leq \sqrt{d_{n\tilde{\ell}}} M_1' n^\lambda \sup_{t\in\mathbb{T}^1\setminus\Omega^{\tilde{k},\tilde{\alpha}}_{m_0}} \left[\exp\left(-n\int_0^t \langle b(\tau),\tilde{k}\rangle + \langle f(\tau),\tilde{\alpha}\rangle d\tau \right) \right] \\
&\times \sup_{\gamma_4\leq\mu}\sup_{y\in(\mathbb{S}^3)^s} |\partial^{\gamma_4}\mathfrak{t}^{n\tilde{\ell}}_{(-n\tilde{\alpha})(-n\tilde{\alpha})}(y)|,
\end{aligned}
$$

where $C_{\mu\gamma} = \binom{\mu}{\gamma_1,\gamma_2,\gamma_3,\gamma_4}$. For $t \notin \Omega^{\tilde{k},\tilde{\alpha}}_{m_0}$ we have $\int_0^t \big(\langle b(\tau),\tilde{k}\rangle + \langle f(\tau),\tilde{\alpha}\rangle\big)d\tau > m_0$, hence

$$
\sup_{t\in\mathbb{T}^1\setminus\Omega^{\tilde{k},\tilde{\alpha}}_{m_0}} \left[\exp\left(-n\int_0^t \langle b(\tau),\tilde{k}\rangle + \langle f(\tau),\tilde{\alpha}\rangle d\tau \right) \right] \leq e^{-nm_0}, n\in\mathbb{N}.
$$

Furthermore, for any $-\ell_j \leq \alpha_j \leq \ell_j$ and $y_j \in \mathbb{S}^3$ the unitarity identity holds:

$$
\begin{aligned}
1 = &\mathfrak{t}^{\ell_j}_{\alpha_j\alpha_j}(\mathsf{e}) = \sum_{\beta_j=-\ell_j}^{\ell_j} \mathfrak{t}^{\ell_j}_{\alpha_j\beta_j}(y_j)\mathfrak{t}^{\ell_j}_{\beta_j\alpha_j}(y_j^{-1}) \\
&= \sum_{\beta_j=-\ell_j}^{\ell_j} \mathfrak{t}^{\ell_j}_{\alpha_j\beta_j}(y_j)\overline{\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j}}(y_j) = \sum_{\beta_j=-\ell_j}^{\ell_j} \left|\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j}(y_j)\right|^2,
\end{aligned}
$$

where $\mathsf{e}\in\mathbb{S}^3$ is the neutral element of the Lie group $\mathbb{S}^3$.

This implies that $\left|\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j}(y_j)\right| \leq 1$ for all α_j, β_j, and y_j. Now, if ∂^{γ_4} is a left-invariant differential operator of order less than or equal to λ, then it can be expressed as a linear combination of at most λ (including 0) compositions of the vector fields in $\{D_{1,j}, D_{2,j}, D_{3,j}\}_{j=1}^s$, where each $D_{1,j}$ and $D_{2,j}$ represent the vector fields on $\mathbb{S}^3$ given by $\partial/\partial\phi_j$ and $\partial/\partial\theta_j$, respectively, in local coordinates. Here, (ϕ_j,θ_j,ψ_j) are Euler angle coordinates on $\mathbb{S}^3$, and so their associated coordinate vector fields form a basis for the Lie algebra of the group.

It is clear that $D_{k,j}\mathfrak{t}^{\ell_{j'}}_{\alpha_{j'}\beta_{j'}} = 0$ if $j \neq j'$. Furthermore, according to [94] Chapter 11, we have the following result:

$$
\begin{aligned}
D_{3,j}\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j} &= i\beta_j\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j};\\
D_{2,j}\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j} &= \frac{\sqrt{(\ell_j-\beta_j)(\ell_j+\beta_j+1)}}{2}\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j+1} - \frac{\sqrt{(\ell_j+\beta_j)(\ell_j-\beta_j+1)}}{2}\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j-1};\\
D_{1,j}\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j} &= \frac{\sqrt{(\ell_j-\beta_j)(\ell_j+\beta_j+1)}}{-2i}\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j+1} + \frac{\sqrt{(\ell_j+\beta_j)(\ell_j-\beta_j+1)}}{-2i}\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j-1}.
\end{aligned}
$$

Therefore, for all $\ell_j, \alpha_j, \beta_j$, and k, we have

$$
|D_{k,j}\mathfrak{t}^{\ell_j}_{\alpha_j\beta_j}(y_j)| \leq 2\ell_j.
$$

By induction, we obtain

$$
|\partial^{\gamma_4}\mathfrak{t}^{n|\tilde{\alpha}_j|}_{(-n\tilde{\alpha}_j)(-n\tilde{\alpha}_j)}| \leq 2^{|\gamma_4|}n^{|\gamma_4|}|\tilde{\alpha}_j|^{|\gamma_4|}.
$$

Since

$$
d_{n\tilde{\ell}} = \prod_{j=1}^{s}(2n|\tilde{\alpha}_j|+1) \leq \prod_{j=1}^{s}4(\|\tilde{\alpha}\|_\infty+1)n = 4^s(\|\tilde{\alpha}\|_\infty+1)^s n^s,
$$

substituting this inequality into the previous inequality, we obtain:

$$
\begin{aligned}
\sum_{|\mu|\leq\lambda}\sup_{(t,x,y)\in\mathbb{T}^{r+1}\times(\mathbb{S}^3)^s}|\partial^\mu g_n(t,x,y)| &\leq M_1' n^{\lambda+s/2}e^{-nr_0}2^{\lambda+s}n^\lambda\|\tilde{\alpha}\|_\infty^\lambda(\|\tilde{\alpha}\|_\infty+1)^{s/2}\\
&= M_1'' n^{2\lambda+s/2}e^{-nm_0}.
\end{aligned}
$$

Also, observe that ${}^tL\left(\exp(-inw(t)+qt)e^{in\langle x,\tilde{k}\rangle}\mathfrak{t}^{n\tilde{\ell}}_{(n\tilde{\alpha})(n\tilde{\alpha})}(y)\right) = 0$. Therefore, we have

$$
{}^tLv_n(t,x,y) = \sqrt{d_{n\tilde{\ell}}}({}^tL+\partial_t)\left[v_0'(t)\exp(-inw(t)+qt)e^{in\langle x,\tilde{k}\rangle}\mathfrak{t}^{n\tilde{\ell}}_{(n\tilde{\alpha})(n\tilde{\alpha})}(y)\right].
$$

Hence,

$$
\begin{aligned}
&\sum_{|\mu|\leq\lambda} \sup_{(t,x,y)\in\mathbb{T}^{r+1}\times(\mathbb{S}^3)^s} |\partial^\mu({}^tLv_n)(t,x,y)| \\
&\leq \sqrt{d_{n\tilde{\ell}}} \sum_{|\mu|\leq\lambda+1} \sup_{(t,x,y)\in\mathbb{T}^{r+1}\times(\mathbb{S}^3)^s} \left|\partial^\mu \left[v_0'(t)\exp(-niw(t)+qt)e^{in\langle x,\tilde{k}\rangle}\mathfrak{t}^{n\tilde{\ell}}_{(n\tilde{\alpha})(n\tilde{\alpha})}(y)\right]\right| \\
&\leq \sqrt{d_{n\tilde{\ell}}} M_2 n^{(\lambda+1)} \sup_{t\in\operatorname{supp} v_0'} \left[\exp\left(n\int_0^t \langle b(\tau),\tilde{k}\rangle + \langle f(\tau),\tilde{\alpha}\rangle d\tau\right) \sup_{|\mu|\leq\lambda+1}\left|\partial^\mu \mathfrak{t}^{n\tilde{\ell}}_{(n\tilde{\alpha})(n\tilde{\alpha})}(y)\right|\right] \\
&\leq 2^s M_2 n^{2(\lambda+1)+s/2}\|\tilde{\alpha}\|_\infty^{\lambda+1} e^{n\int_0^{t_1}\langle b(\tau),\tilde{k}\rangle+\langle f(\tau),\tilde{\alpha}\rangle d\tau}(\|\tilde{\alpha}\|_\infty+1)^{s/2} \\
&= M_2' n^{2(\lambda+1)+s/2} e^{n\int_0^{t_1}\langle b(\tau),\tilde{k}\rangle+\langle f(\tau),\tilde{\alpha}\rangle d\tau},
\end{aligned}
$$

where $t_1 \in \operatorname{supp}(v_0') \subset \Omega^{\tilde{k},\tilde{\alpha}}_{m_0}$ is a point where the restriction of the exponential function achieves its maximum value over $\operatorname{supp}(v_0')$.

We define

$$
\omega = \int_0^{t_1} \langle b(\tau),\tilde{k}\rangle + \langle f(\tau),\tilde{\alpha}\rangle d\tau - m_0.
$$

Notice that $\omega < 0$, since $t_1 \in \Omega^{\tilde{k},\tilde{\alpha}}_{m_0}$. By using the previous inequalities, we can establish the following:

$$
\left(\sum_{|\mu|\leq\lambda} \sup_{(t,x,y)\in\mathbb{T}^{r+1}\times(\mathbb{S}^3)^s} |\partial^\mu g_n(t,x,y)|\right)\left(\sum_{|\mu|\leq\lambda} \sup_{(t,x,y)\in\mathbb{T}^{r+1}\times(\mathbb{S}^3)^s} |\partial^\mu({}^tLv_n)(t,x,y)|\right) \leq M_1'' M_2' n^{4\lambda+3+s} e^{n\omega}.
$$

Consequently, by (5.42) and (5.43), we have:

$$
0 = \lim_{n\to\infty}\left|\int_{\mathbb{T}^{r+1}\times(\mathbb{S}^3)^s} g_n v_n\right| \geq \ell_0 > 0.
$$

This leads to a contradiction, proving that L cannot be globally solvable. □

It is also evident that in this context every vector field in L admits non-sparse Archimedean sequences, which allow us to relate the linear independence of the coefficient functions with the change of sign of their linear combinations by Proposition 5.2.17. Not only that, notice that if $t \mapsto \langle \operatorname{Im}(c)(t), k\rangle + \langle \operatorname{Im}(d)(t),\alpha\rangle$ changes sign for some $(k,\alpha) \in \mathbb{Z}^r \times \frac{1}{2}\mathbb{Z}^s$, then the same is true for the "dilations" $(m\cdot k, m\cdot\alpha) \in \mathbb{Z}^r \times \frac{1}{2}\mathbb{Z}^s$, for every

$m \in \mathbb{Z}$. Using this, we can obtain another version of Proposition 5.2.14:

Proposition 5.3.4. *If there exist vectors $\tilde{k} \in \mathbb{Z}^r$ and $\tilde{\alpha} \in \frac{1}{2}\mathbb{Z}^s$ not both zero such that*

$$\langle c_0, \tilde{k}\rangle + \langle d_0, \tilde{\alpha}\rangle - iq \notin \mathbb{Z},$$

and the real-valued smooth function

$$\theta(t) \doteq \langle b(t), \tilde{k}\rangle + \langle f(t), \tilde{\alpha}\rangle, t \in \mathbb{T}^1$$

changes sign, then the operator L defined in (5.35) is neither globally hypoelliptic nor globally solvable.

And also the following proposition:

Proposition 5.3.5. *Suppose that at least one of the functions b_j or f_k is non-null. Then the operator L, as defined in (5.35), is neither globally solvable nor globally hypoelliptic if:*

(A) $(b_0, f_0) = \overline{0}$ *and* $(a_0, e_0, q) \notin \mathbb{Z}^r \times 2\mathbb{Z}^s \times i\mathbb{Z}$ *or*

(B) $(b_0, f_0) \neq \overline{0}$ *and either* $\dim$ *span* $\{b_1, \ldots, b_r, f_1, \ldots, f_s\} \geq 2$ *or some of the functions b_j or f_k changes sign.*

Putting all these results together and using the properties mentioned above, we are able to obtain both necessary and sufficient conditions for global solvability and hypoellipticity in this case, that is:

Theorem 5.3.6 (Global solvability)**.** *The operator L is globally solvable if and only if one of the following conditions holds:*

i. $\dim$ *span* $\{b_1, \ldots, b_r, f_1, \ldots, f_s\} \leq 1$, *none of the functions b_j and f_l change sign, and L_0 is globally solvable.*

ii. $(b, f) \not\equiv 0$, $(b_0, f_0) = 0$, $(a_0, e_0) \in \mathbb{Z}^r \times 2\mathbb{Z}^s$, $q \in i\mathbb{Z}$ *and every sublevel set*

$$\Omega_m^{k,\alpha} = \left\{ t \in \mathbb{T}^1; \int_0^t \Big(\sum_{j=1}^r b_j(t) k_j + \sum_{l=1}^s f_l(t)\alpha_l \Big) dt < m \right\}$$

is connected, for every $m \in \mathbb{R}$, $k \in \mathbb{Z}^r$, and $\alpha \in \frac{1}{2}\mathbb{Z}^s$.

Theorem 5.3.7 (Global hypoellipticity). *L defined in (5.35) is globally hypoelliptic if and only if* $\dim$ *span* $\{b_1, \dots, b_r, f_1, \dots, f_s\} \leq 1$*, none of the functions b_j and f_k changes sign, and L_0 is globally hypoelliptic.*

Example 5.3.8. Consider the first order differential operator

$$L = \partial_t + (a + e^{it})\partial_x + (b + e^{it})i\partial_0 + iq(t)$$

acting on $\mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{S}^3$, where $a, b \in \mathbb{R}$ and $q \in C^\infty(\mathbb{T}^1)$ is real valued. Then since $\int_0^{2\pi} \cos(t)dt = 0$, by Theorem 5.3.6 L is globally solvable if and only if $(a, b) \in \mathbb{Z} \times 2\,\mathbb{Z}$ and $\frac{1}{2\pi}\int_0^{2\pi} q(t)dt \in \mathbb{Z}$. Also, by Theorem 5.3.7 it is not globally hypoelliptic since $\cos(t)$ changes sign on $[0, 2\pi]$. On the other hand, let L be given by

$$L = \partial_t + a(\sin(t) + 1)\partial_x + b(\sin(t) + 1)i\partial_0 + 1,$$

where $a, b \in \mathbb{R}$. Then L is globally hypoelliptic. Indeed, taking $M = N = 1$ it is true that

$$|k_1 + ak_2 + b\alpha - i| \geq 1 \geq M(|k_1| + |k_2| + \ell)^{-N},$$

for every $k_1, k_2 \in \mathbb{Z}$, $\ell \in \frac{1}{2}\mathbb{N}_0$, not all zero, and $-\ell \leq \alpha \leq \ell$, $\ell - \alpha \in \mathbb{N}_0$. Also, notice that

$$k_1 + ak_2 + b\alpha - i \neq 0$$

for every $k_1, k_2 \in \mathbb{Z}$, $\alpha \in \frac{1}{2}\mathbb{Z}$. Therefore the claim follows from Theorem 5.3.7 and Proposition 4.1.4.

Example 5.3.9. Consider the operator L defined on $\mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{S}^3$ given by

$$L = \partial_t + ((1 + \cos(t)) + i\sin^3(t))\partial_x + ((2 + \sin(t)) + 2i\cos^3(2t))i\partial_0 + (4i + \sin(3t)).$$

Then

$$L_0 = \partial_t + 1\partial_x + 2i\partial_0 + 4i,$$

and so $(b_0, f_0) = 0$, $(a_0, e_0) \in \mathbb{Z} \times 2\,\mathbb{Z}$ and $q \in i\,\mathbb{Z}$. However, for $\xi = 1$, $\alpha = -\frac{3}{2}$ and $r = 1.1$ the sublevel

$$\Omega_r^{\xi,\alpha} = \left\{ t \in \mathbb{T}^1 \mid \int_0^t \xi \cdot \sin^3(\tau) + 2\alpha \cdot \cos^3(2\tau)d\tau < r \right\}$$

is not connected. Therefore L is not globally solvable by Theorem 5.3.6.

Appendices

5.A Technical lemmas

Lemma 5.A.1. *Let $f, \theta \in C^\infty(\mathbb{T}^1)$, and set $\theta_0 = \frac{1}{2\pi}\int_0^{2\pi} \theta(t)dt$. If $\theta_0 \notin i\mathbb{Z}$, then the differential equation*

$$\partial_t u(t) + \theta(t)u(t) = f(t), \qquad t \in \mathbb{T}^1, \tag{5.44}$$

admits unique solution in $C^\infty(\mathbb{T}^1)$ given by

$$u(t) = \frac{1}{1-e^{-2\pi\theta_0}} \int_0^{2\pi} f(t-s)e^{-\int_{t-s}^{t} \theta(\tau)d\tau} ds, \tag{5.45}$$

or equivalently by

$$u(t) = \frac{1}{e^{2\pi\theta_0}-1} \int_0^{2\pi} f(t+s)e^{\int_{t}^{t+s} \theta(\tau)d\tau} ds. \tag{5.46}$$

If $\theta_0 \in i\mathbb{Z}$, then equation (5.44) admits infinitely many solutions given by

$$u_\lambda(t) = \lambda e^{-\int_0^t \theta(\tau)d\tau} + \int_0^t f(s)e^{-\int_s^t \theta(\tau)d\tau} ds, \tag{5.47}$$

for every $\lambda \in \mathbb{R}$, if and only if

$$\int_0^{2\pi} f(t)e^{\int_0^t \theta(\tau)d\tau} dt = 0.$$

Proof. This can easily be verified by treating functions on the torus as periodic functions on $\mathbb{R}$. □

Lemma 5.A.2. *Let $G = G_1 \times \cdots \times G_n$ be a product of compact Lie groups and*

$$L = \partial_t + \sum_{j=1}^{n} c_j(t)X_j + q$$

where $c_j(t) = a_j + ib_j(t)$ are smooth functions on the one-dimensional torus $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$,

$X_j \in \mathfrak{g}_j$ $j = 1, \ldots, n$, are left-invariant vector fields on G_j, $q \in \mathbb{C}$. Define

$$\mathcal{N}_L = \Big\{([\xi], \alpha) \in \widehat{G} \times \mathbb{N}^n \,|\alpha \in J_\xi,\ \langle c_0, \mu_\alpha(\xi)\rangle - iq \in \mathbb{Z}\Big\}$$

where $c_0 = \frac{1}{2\pi}\int_0^{2\pi} c(t)dt \in \mathbb{C}^n$, $\mu_\alpha(\xi) = (\mu_{\alpha_1}(\xi^1), \ldots, \mu_{\alpha_n}(\xi^n))$ as defined in Definition 3.1.8. If $f \in C^\infty(\mathbb{T}^1 \times G)$ is such that $\widehat{f}(\cdot, \xi)_{\alpha\beta} \equiv 0$ whenever $([\xi], \alpha) \in \mathcal{N}_L$, then $f \in (\ker {}^tL)^0 \cap C^\infty(\mathbb{T}^1 \times G)$.

Proof. Let $f \in C^\infty(\mathbb{T}^1 \times G)$ be as claimed. Suppose $v \in \ker {}^tL$. Then $-{}^tL\,v = 0$, therefore, taking the partial Fourier transform on the last n variables yields:

$$\widehat{-{}^tL\,v}(t, \overline{\xi})_{\alpha\beta} = \partial_t\widehat{v}(t, \overline{\xi})_{\alpha\beta} + i(\langle c(t), \mu_\alpha(\overline{\xi})\rangle + iq)\widehat{v}(t, \overline{\xi})_{\alpha\beta} = 0 \tag{5.48}$$

for every $t \in \mathbb{T}^1$, $[\xi] \in \widehat{G}$, $\alpha, \beta \in J_\xi$. Therefore, for every $([\xi], \alpha) \notin \mathcal{N}_L$, as $\langle c_0, \mu_\alpha(\xi)\rangle - iq \notin \mathbb{Z}$ then also

$$\begin{aligned}\langle c_0, \mu_\alpha(\xi)\rangle - iq &= -\langle c_0, \mu_\alpha(\overline{\xi})\rangle - iq \\ &= -(\langle c_0, \mu_\alpha(\overline{\xi})\rangle + iq) \notin \mathbb{Z}\end{aligned}$$

so $\langle c_0\mu_\alpha(\overline{\xi})\rangle + iq \notin \mathbb{Z}$. It follows from Lemma 5.A.1 that equation 5.48 implies $\widehat{v}(\cdot, \overline{\xi})_{\alpha\beta} \equiv 0$. From Proposition 3.1.7 we conclude that

$$\begin{aligned}\langle v, f\rangle_{\mathbb{T}^1 \times G} &= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{\alpha,\beta\in J_\xi} \int_0^{2\pi} \widehat{f}(t, \xi)_{\alpha\beta}\widehat{v}(t, \overline{\xi})_{\alpha\beta}dt \\ &= 0\end{aligned}$$

as every term in the sum above is zero. Since $v \in \ker {}^tL$ was arbitrary, we conclude $f \in (\ker {}^tL)^0$. □

Lemma 5.A.3. *Let $G = G_1 \times \cdots \times G_n$ be a product of compact lie groups and*

$$L = \partial_t + \sum_{j=1}^{n} c_j(t)X_j + q$$

where $c_j(t) = a_{j0} + ib_j(t)$, $a_{j0} \in \mathbb{R}$, b_j are smooth real functions on $\mathbb{T}^1$, X_j are left-invariant vector fields on G_j, $j = 1, \ldots, n$, $q \in \mathbb{C}$. Let $c_{j0} = \frac{1}{2\pi}\int_0^{2\pi} c_j(t)dt$, $j = 1, \ldots, n$. If $f \in$

$(\ker {}^tL)^0 \cap C^\infty(G)$*, then the ordinary differential equations on* $\mathbb{T}^1$ *given by*

$$\widehat{Lu}(t,\xi)_{\alpha\beta} = [\partial_t + i(\langle c_j(t), \mu_\alpha(\xi)\rangle - iq)]\,\widehat{u}(t,\xi)_{\alpha\beta} = \widehat{f}(t,\xi)_{\alpha\beta} \tag{5.49}$$

admits solution for every $[\xi] \in \widehat{G}$*,* $\alpha, \beta \in J_\xi$*.*

Proof. Let $f \in (\ker {}^tL)^0 \cap C^\infty(\mathbb{T}^1 \times G)$ and consider equation (5.49). By Lemma 5.A.1 it admits (unique) solution whenever $([\xi],\alpha) \in \mathcal{N}_L^c$, $\beta \in J_\xi$. Now suppose $([\xi],\alpha) \in \mathcal{N}_L$ and let $\beta \in J_\xi$. Define

$$\widehat{v}(t,\overline{\xi})_{\alpha\beta} = \exp\left\{\int_0^t i(\langle c(\tau), \mu_\alpha(\xi)\rangle - iq)d\tau\right\}$$

and $\widehat{v}(t,\xi')_{\alpha'\beta'} \equiv 0$ for all other $[\xi'] \in \widehat{G}$, $\alpha', \beta' \in J_{\xi'}$. Note that these are well defined since $([\xi],\alpha) \in \mathcal{N}_L$. By taking the partial inverse Fourier transform of these coefficients, we obtain that $v \in (\ker {}^tL)$, since:

$$\begin{aligned}
\widehat{-\,{}^tL\,v}(t,\overline{\xi})_{\alpha\beta} &= \partial_t\widehat{v}(t,\overline{\xi})_{\alpha\beta} + i(\langle c(t), \mu_\alpha(\overline{\xi})\rangle + iq)\widehat{v}(t,\overline{\xi})_{\alpha\beta} \\
&= i(\langle c(t), \mu_\alpha(\xi)\rangle - iq)\widehat{v}(t,\overline{\xi})_{\alpha\beta} + i(-\langle c(t), \mu_\alpha(\xi)\rangle + iq)\widehat{v}(t,\overline{\xi})_{\alpha\beta} \\
&= 0.
\end{aligned}$$

Since $f \in (\ker {}^tL)^0$, by Proposition 3.1.7:

$$\begin{aligned}
0 = \int_{\mathbb{T}^1}\int_G f(t,x)v(t,x)dxdt &= \int_0^{2\pi} \widehat{f}(t,\xi)_{\alpha\beta}\,\widehat{v}(t,\overline{\xi})_{\alpha\beta}dt \\
&= \int_0^{2\pi} \widehat{f}(t,\xi)_{\alpha\beta}\exp\left\{\int_0^t i(\langle c(\tau), \mu_\alpha(\xi)\rangle - iq)d\tau\right\}dt
\end{aligned}$$

so by Lemma 5.A.1 equation (5.49) admits (infinitely many) solutions. □

Lemma 5.A.4. *Let* $\psi \in C^\infty(\mathbb{T}^1)$ *be a smooth non-negative real-valued function with a zero of order greater than one at* $s_0 \in \mathbb{T}^1$*, that is,* $\psi(s_0) = \psi'(s_0) = 0$*. Then, there exists* $M > 0$ *such that*

$$\int_{s_0-\delta}^{s_0+\delta} e^{-\lambda\psi(s)}ds \geq \sqrt{\pi}\,\mathrm{erf}(\delta)\lambda^{-1/2}M^{-1/2},$$

for all $\lambda > 0$ *sufficiently big and* $\delta > 0$*.*

Proof. By Taylor's theorem, for each $s \in (s_0-\delta, s_0+\delta)$, there exists $s' \in (s_0-\delta, s_0+\delta)$, such that

$$\psi(s) = \frac{\psi''(s')}{2}(s - s_0)^2.$$

Let $\tilde{M} = \sup_{s \in [s_0-\delta, s_0+\delta]} \left| \frac{\psi''(s)}{2} \right| \geq 0$. If $\tilde{M} = 0$, then $\psi \equiv 0$ on $[s_0 - \delta, s_0 + \delta]$ and the inequality is trivial with $M = 1$, as $\sqrt{\pi}\,\mathrm{erf}(\delta) \leq 2\delta$. Otherwise, let $M = \tilde{M}$ and then for $\lambda M > 1$ we have:

$$\int_{s_0-\delta}^{s_0+\delta} e^{-\lambda\psi(s)} ds \geq \int_{s_0-\delta}^{s_0+\delta} e^{-(\sqrt{\lambda M}(s-s_0))^2} ds \geq \frac{1}{\sqrt{\lambda M}} \left(\int_{-\delta\sqrt{\lambda M}}^{\delta\sqrt{\lambda M}} e^{-s^2} ds \right)$$
$$\geq \frac{1}{\sqrt{\lambda M}} \left(\int_{-\delta}^{\delta} e^{-s^2} ds \right) = \sqrt{\pi}\,\mathrm{erf}(\delta)\lambda^{-1/2} M^{-1/2}.$$

□

Lemma 5.A.5. *Let $\phi \in C^\infty(\mathbb{T}^1)$ be such that $\int_0^{2\pi} \phi(t)\, dt = 0$ and for every $r \in \mathbb{R}$, the set $\Omega_r = \left\{ t \in \mathbb{T}^1 \mid \int_0^t \phi(\tau) d\tau < r \right\}$ is connected. Then*

$$\tilde{\Omega}_r = \left\{ t \in \mathbb{T}^1 \mid \int_0^t \phi(\tau) d\tau \geq r \right\} = \left\{ t \in \mathbb{T}^1 \mid -\int_0^t \phi(\tau) d\tau \leq -r \right\}$$

is also connected.

Proof. This follows from $\tilde{\Omega}_r = \mathbb{T}^1 \setminus \Omega_r$ and the general fact that any $A \subset \mathbb{T}^1$ is connected if and only if $\mathbb{T}^1 \setminus A$ is connected. To see this, let A be connected. Note that the claim $\mathbb{T}^{1^1} \setminus A$ is connected is trivially true if $A = \mathbb{T}^{1^1}$. Otherwise, then $\mathbb{T}^1 \setminus A$ contains at least one point, which, without loss of generality we may assume it is $0 = 0 + 2\pi\,\mathbb{Z}$. If we consider

$$f : (0, 2\pi) \to \mathbb{T}^1 \setminus \{0 + 2\pi\,\mathbb{Z}\}$$
$$x \mapsto x + 2\pi\,\mathbb{Z}$$

then f is an homeomorphism. Since A is connected, $I = f^{-1}(A)$ also is. But then I is an interval, so $I^c = (0, 2\pi) \setminus I$ is either an interval containing $(0, \epsilon)$ or $(2\pi - \epsilon, 2\pi)$ for some $\epsilon > 0$ or the disjoint union of two intervals containing $(0, \epsilon) \cup (2\pi - \epsilon, 2\pi)$, for some $\epsilon > 0$. In both cases, since $\mathbb{T}^1 \setminus A = f(I^c) \cup \{0 + 2\pi\,\mathbb{Z}\}$ it is clearly connected. Switching the roles of A and $\mathbb{T}^1 \setminus A$, the converse also follows. □

Lemma 5.A.6. *Let $\phi \in C^\infty(\mathbb{T}^1)$ be a non-null function, and let Φ be a function such that*

$\Phi' = \phi$. Suppose there exists $m \in \mathbb{R}$ such that the sublevel set

$$\Omega_m = \{t \in \mathbb{T}^1; \Phi(t) < m\}$$

is not connected. Then, there exists $m_0 < m$ such that Ω_{m_0} has two connected components with disjoint closures. Consequently, we can define functions $g_0, v_0 \in C^\infty(\mathbb{T}^1)$ such that:

$$\int_0^{2\pi} g_0(t)\,dt = 0,\ \textit{supp}\ (g_0) \cap \Omega_{m_0} = \varnothing,\ \textit{supp}\ (v_0') \subset \Omega_{m_0}\ \textit{and}\ \int_0^{2\pi} g_0(t)v_0(t)\,dt > 0.$$

Proof. Let $C_1 \subset \mathbb{T}^1$ be a connected component of Ω_m. Notice that C_1 is homeomorphic to an open interval and has two distinct boundary points: $\partial C_1 = \{t_1, t_2\}$. Choose $t_3 \in C_1$ such that $\Phi(t) < m$. Since Ω_m is not connected, there exists another connected component C_2 of Ω_m such that $C_1 \cap C_2 = \emptyset$. Similar to C_1, the component C_2 is also homeomorphic to an open interval and its boundary is given by two distinct points: $\partial C_2 = \{t_4, t_5\}$. Choose $t_6 \in C_2$ such that $\Phi(t_6) < m$.

Now, choose $\epsilon > 0$ such that $m_0 \doteq \max\{\Phi(t_3), \Phi(t_6)\} + \epsilon < r$. Since $\Phi(t_1) = m$, by the continuity of Φ, there exists an open set $U_1 \subset \mathbb{T}^1$ containing t_1 such that $\Phi(t) > m_0$ for each $t \in U_1$. Similarly, we can find an open set U_2 containing t_2 with the same property.

Let I and J be the connected components of Ω_{m_0} that contain t_3 and t_6, respectively. It is important to note that U_1 and U_2 are contained in $\mathbb{T}^1 \backslash (I \cup J)$. Moreover, $I \subset C_1$ and $J \subset C_2$ are "separated" by U_1 and U_2, which implies that their closures do not intersect. In other words, if $x \in \overline{I} \cap \overline{J}$, then there exist sequences $(x_n)_n \subset I$ and $(y_n)_n \subset J$ such that $x_n \to x$ and $y_n \to x$. However, since $x_n \in I \subset C_1$, it follows that $\Phi(x_n) < m_0$ for all n, which implies $\Phi(x) \leq m_0 < m$. Therefore, we have $x \in C_1$. The same logic applies to y_n, J, and C_2, which leads to $x \in C_1 \cap C_2$, which is a contradiction.

Let us consider the previously defined set as contained in the interval $K = [t_1, t_1 + 2\pi] \subset \mathbb{R}$. Without loss of generality, we can assume that

$$t_1 < t_3 < t_2 \leq t_4 < t_6 < t_5 \leq t_1 + 2\pi$$

$$t_3 \in I \subset C_1 = (t_1, t_2), \quad t_6 \in J \subset C_2 = (t_4, t_5)$$

$$U_1 = [t_1, t_1 + \epsilon') \cup (t_1 + 2\pi - \epsilon', t_1 + 2\pi],$$

where $0 < \epsilon'$, and $t_1 + \epsilon' < t_3$, and

$$U_2 = (t_2 - \epsilon'', t_2 + \epsilon''),$$

where $0 < \epsilon''$ and $t_3 < t_2 - \epsilon'' < t_2 + \epsilon'' < t_6$.

Now, for $j = 1, 2$, let $g_j \in C_c^\infty(U_j)$ be a bump function such that $\int_0^{2\pi} g_j(t)\,dt = 1$. Set $g_0 = g_2 - g_1$, so that supp $(g_0) \subset U_1 \cup U_2$ and so supp $(g_0) \cap \Omega_{m_0} = \emptyset$. Also,

$$\int_0^{2\pi} g_0 = \int_{U_2} g_2 - \int_{U_1} g_1 = 1 - 1 = 0.$$

Finally, let $\delta > 0$ be such that $t_3 + \delta \in I$ and $t_6 - \delta \in J$. Choose $v_0 \in C_c^\infty((t_3, t_6))$ such that $v_0 \equiv 1$ in $[t_3 + \delta, t_6 - \delta]$. In this case,

$$\int_0^{2\pi} g_0 v_0 = \int_{U_2} g_2 = 1 > 0$$

and supp $(v_0') \subset I \cup J \subset \Omega_{m_0}$. $\square$

Part II

Sharp Gårding inequality

Chapter 6

Vector-valued sharp Gårding inequality on compact Lie groups

6.1 Introduction

The so called Gårding inequality was first proved by Gårding in his paper [47] and can be stated as follows:

Let P be an elliptic self-adjoint pseudo-differential operator of order $m \in \mathbb{R}$ on an open set $U \subset \mathbb{R}^n$. Then, for any compact set $Q \subset U$ and $\gamma < m/2$, there exist constants $c_{\gamma,Q}, C_{\gamma,Q} > 0$ such that

$$\langle Pu, u\rangle_{L^2} \geq c_{\gamma,Q}\|u\|^2_{H^{m/2}} - C_{\gamma,Q}\|u\|^2_{H^\gamma},$$

for every $u \in C_0^\infty(Q)$. Here, $\langle\cdot,\cdot\rangle_{L^2}$ denotes the usual L^2 inner product and $\|\cdot\|_{H^r}$, denotes the usual norm on the Sobolev space of order $r \in \mathbb{R}$. In order to apply it to different problems, Hörmander [71] and Lax and Nirenberg [81] then adapted this result and proved the so called sharp Gårding inequality for pseudo-differential operators for symbols in the Kohn-Nirenberg class $S^m(\mathbb{R}^{2n})$. Their result states that if a symbol $p \in S^m(\mathbb{R}^{2n})$, for $m \in \mathbb{R}$ satisfies $\operatorname{Re} p(x,\xi) \geq 0$, then the pseudo-differential operator $p(x, D)$ associated to this symbol satisfies the estimate

$$\operatorname{Re}\langle p(x,D)u, u\rangle_{L^2(\mathbb{R}^n)} \geq -C\|u\|^2_{H^{\frac{m-1}{2}}},$$

for some $C > 0$ and every $u \in H^{\frac{m-1}{2}}(\mathbb{R}^n)$, where the norm on the right hand side of the inequality denotes the usual norm in the Sobolev space $H^{\frac{m-1}{2}}(\mathbb{R}^n)$. Since then, this result has been extended, by many authors, to different settings and symbol classes. In particular, this type of inequality has been extended to the global symbol classes on compact Lie groups for scalar valued functions. Inspired by the works on Gårding inequality on compact Lie group [100], on sharp Gårding inequality on compact Lie groups [96], on sharp Gårding inequality for subelliptic operators on compact Lie groups [25] and on Gårding inequality on graded Lie groups [24], we prove a type of sharp Gårding inequality for pseudo-differential operators acting on vector-valued functions on compact Lie groups. In order to do so, we also extend the notion of pseudo-differential operators defined by amplitudes to the vector-valued compact Lie group setting. As a consequence, we obtain a sharp Gårding inequality for pseudo-differential operators on compact homogeneous vector bundles and compact homogeneous manifolds. Finally, in the end we present an application of this result proving existence and uniqueness of solution to a class of systems of vector-valued Cauchy problems of pseudo-differential equations. These results could have many applications in analysis, such as in local solvability and well-posedness of certain Cauchy problems. We remark that our results use state of the art ideas, such as the vector valued Fourier analysis on compact Lie groups developed in [27]. It is also relevant to mention that the sharp Gårding inequality is the strongest lower bound estimate known to hold for systems on $\mathbb{R}^n$ (vector-valued functions). The aim of this chapter is to extend this property for the global quantization of operators on compact Lie groups.

6.2 Preliminaries

6.2.1 Vector-valued Fourier analysis

In this section we recall some of the work by Cardona, Kumar and Ruzhansky in [27] about the vector-valued analog of the Fourier analysis on compact Lie groups presented in Chapter 2. The theory here presented differs slightly from the one found in that paper, as they considered subelliptic Hörmander classes, and here we have adapted their results to consider the usual elliptic Hörmander classes.

Let E_0 be a n-dimensional $\mathbb{C}$-vector space, where $n \in \mathbb{N}$. Consider $B_0 = \{e_1, \dots, e_n\}$ an orthonormal basis on E_0. We may identify $E_0 \cong \mathbb{C}^n \cong \mathbb{C}^{n\times 1}$ as vector spaces, by identifying each element in B_0 with the canonical basis in $\mathbb{C}^n$. Hence, given a mapping $f : G \to E_0$, we may write it as

$$f(x) = (f_1(x), \dots, f_n(x))^t \in \mathbb{C}^{n\times 1},$$

where $f_j(x) = \langle f(x), e_j \rangle_{E_0} \in \mathbb{C}$, for each $x \in G$, where $\langle \cdot, \cdot \rangle_{E_0}$ denotes the canonical inner product on E_0 which it inherits from $\mathbb{C}^n$. Here also the superscript "t" denotes the transpose of the corresponding vector.

Definition 6.2.1. For $f \in L^1(G, E_0)$, define the Fourier coefficients of f at $[\xi] \in \widehat{G}$ by

$$\widehat{f}(\xi) = (\widehat{f}(1,\xi), \ldots, \widehat{f}(n,\xi))^t \doteq (\widehat{f}_1(\xi), \ldots, \widehat{f}_n(\xi))^t,$$

where $\widehat{f}_i(\xi)$ is the group Fourier transform defined in Chapter 2.

As before, this definition can be extended to the set of distributions $\mathcal{D}'(G, E_0)$ acting on $C^\infty(G, E_0)$. For a scalar function $g : G \to \mathbb{C}$, we fix the notation $g(x) \otimes e_i$ to denote the vector-valued function defined which has $x \mapsto g(x)$ as its i-th component, and zero for its other components, i.e.: $(g \otimes e_i)_j(x) = g(x)\delta_{ij}$.
From the Peter-Weyl Theorem, the inversion formula can be written as

$$f(x) = \sum_{i=1}^{n} \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}\left(\xi(x)\widehat{f}(i,\xi)\right) \otimes e_i,$$

for every $f \in L^2(G, E_0)$.

Definition 6.2.2. Let $A : C^\infty(G, E_0) \to C^\infty(G, E_0)$ be a continuous linear operator. Its matrix-valued symbol is given by

$$\sigma_A(i, r, x, \xi) = \xi(x)^* \langle A(\xi \otimes e_i)(x), e_r \rangle_{E_0}, \tag{6.1}$$

for $1 \leq i, r \leq n$, $x \in G$, $[\xi] \in \widehat{G}$.

A similar quantization formula holds in the vector-valued case. More precisely, the following result was proved in [27]:

Proposition 6.2.3. *Let $A : C^\infty(G, E_0) \to C^\infty(G, E_0)$ be a continuous linear operator. Then*

$$Af(x) = \sum_{i,r=1}^{n} \sum_{[\xi] \in \widehat{G}} d_\xi Tr[\xi(x)\sigma_A(i, r, x, [\xi])\widehat{f}_i(\xi)] \otimes e_r, \tag{6.2}$$

for every $x \in G$, $f \in C^\infty(G, E_0)$.

Equivalently, formula (6.2) can also be written as

$$Af(x) = \left(\sum_{i=1}^{n} \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma_A(i,1,x,[\xi])\widehat{f}_i(\xi)], \ldots, \sum_{i=1}^{n} \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma_A(i,n,x,[\xi])\widehat{f}_i(\xi)] \right)^t,$$

or

$$Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)^{\otimes n}\sigma_A(x,[\xi])\widehat{f}(\xi)].$$

Here the matrix $\sigma_A(x,[\xi]) \in (\mathbb{C}^{d_\xi\times d_\xi})^{n\times n}$ is given by:

$$\sigma_A(x,[\xi])_{ir} = \sigma_A(i,r,x,[\xi]),$$

for $1 \le i, r \le n$, $(x,[\xi]) \in G \times \widehat{G}$, and $\xi(x)^{\otimes n}$ is the block diagonal $(\mathbb{C}^{d_\xi\times d_\xi})^{n\times n}$-matrix given by

$$\xi(x)^{\otimes n} = \mathrm{diag}(\xi(x), \ldots, \xi(x)), \tag{6.3}$$

for all $(x,[\xi]) \in G \times \widehat{G}$. Also, this last trace should be understood component-wise, i.e.: $(\mathrm{Tr}(v))_j = \mathrm{Tr}(v_j)$, for $1 \le j \le n$.

As before, we introduce the symbol classes with respect to these quantization formulas.

Definition 6.2.4. Let $J_n \doteq \{1, \ldots, n\}$, $0 \le \rho, \delta \le 1$. A symbol $\sigma \in S^m_{\rho,\delta}((G \times \widehat{G}) \otimes \mathrm{End}(E_0))$ is a mapping from $J_n \times J_n \times G \times \mathrm{Rep}(G)$, smooth in x, such that, $\sigma(i,r,x,\xi) \in \mathbb{C}^{d_\xi\times d_\xi}$, for any $\xi \in \mathrm{Rep}(G)$, $i, r \in J_n$, and it satisfies

$$\|\Delta_\xi^\alpha \partial_x^\beta \sigma(i,r,x,\xi)\|_{op} \le C_{\alpha\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

for some $C_{\alpha,\beta} > 0$, for all multi-indices α, β, and all $(i,r,x,\xi) \in J_n \times J_n \times G \times \mathrm{Rep}(G)$. As before, due to the trace invariance under cyclic permutations, we will abuse the notation and identify $\mathrm{Rep}(G)$ with $\widehat{G}$.

For a symbol σ, its associated pseudo-differential operator $\mathrm{Op}(\sigma) : C^\infty(G, E_0) \to \mathcal{D}'(G, E_0)$ is defined by the expression

$$(\mathrm{Op}(\sigma)u)_r(x) \doteq \sum_{[\eta]\in\widehat{G}} d_\eta \mathrm{Tr}\left[\eta(x) \sum_{i=1}^{n} \sigma(i,r,x,\eta)\widehat{u_i}(\eta) \right], \tag{6.4}$$

for every $1 \leq r \leq n$. We then say Op(σ) is a vector-valued pseudo-differential operator of order m, and write Op$(\sigma) \in \Psi^m_{\rho,\delta}((G \times \widehat{G}) \otimes \text{End}(E_0))$.

We also denote the set of smoothing symbols and smoothing operators by

$$S^{-\infty}((G \times \widehat{G}) \otimes \text{End}(E_0)) = \bigcap_{m \in \mathbb{R}} S^m_{1,0}((G \times \widehat{G}) \otimes \text{End}(E_0)), \tag{6.5}$$

$$\Psi^{-\infty}((G \times \widehat{G}) \otimes \text{End}(E_0)) = \bigcap_{m \in \mathbb{R}} \Psi^m_{1,0}((G \times \widehat{G}) \otimes \text{End}(E_0)), \tag{6.6}$$

As in Chapter 2, for every $s \in \mathbb{R}$, the vector-valued Sobolev space $H^s(G, E_0)$, is defined as the set of all distributions in $\mathcal{D}'(G, E_0)$ such that

$$\|u\|^2_{H^s(G,E_0)} = \sum_{[\xi] \in \widehat{G}} d_\xi \langle \xi \rangle^{2s} \sum_{i=1}^n \|\widehat{u}(i, \xi)\|^2_{HS} < \infty.$$

In this case, the equalities

$$\bigcap_{s \in \mathbb{R}} H^s(G, E_0) = C^\infty(G, E_0), \qquad \bigcup_{s \in \mathbb{R}} H^s(G, E_0) = \mathcal{D}'(G, E_0). \tag{6.7}$$

also hold.

The following theorem justifies the term "smoothing" for the symbols in (6.5), because of equalities (6.7). Its proof can be found in [27, Theorem 3.17].

Theorem 6.2.5. *Let $A : C^\infty(G, E_0) \to C^\infty(G, E_0)$ be a continuous linear operator with symbol $a \in S^m_{\rho,\delta}((G \times \widehat{G}) \otimes \text{End}(E_0))$, $0 \leq \delta < \rho \leq 1$. Then $A : H^s(G, E_0) \to H^{s-m}(G, E_0)$ extends to a bounded operator for all $s \in \mathbb{R}$.*

Remark 6.2.6. As in the scalar case, if $A : C^\infty(G, E_0) \to C^\infty(G, E_0)$ is a continuous left-invariant linear operator, that is,

$$(A \circ \pi_L(y) f)(x) = (\pi_L(y) \circ A) f(x),$$

where $(\pi_L(y) f)(x) = f(y^{-1}x)$, for every $f \in C^\infty(G, E_0)$ and $x, y \in G$, then σ_A is independent of $x \in G$. Proposition 6.2.3 then implies that

$$\widehat{(Af)}(r, \xi) = \sum_{i=1}^n \sigma_A(i, r, \xi) \widehat{f}(i, \xi),$$

or equivalently

$$\widehat{(Af)}_r(\xi) = \sum_{i=1}^{n} \sigma_A(i, r, \xi) \widehat{f}_i(\xi),$$

for every $1 \le r \le n$, $[\xi] \in \widehat{G}$ and $f \in C^\infty(G, E_0)$. By duality, this remains true for $f \in \mathcal{D}'(G, E_0)$. This implies A can be seen as a "matrix Fourier multiplier".

Definition 6.2.7. An amplitude $a \in \mathcal{A}^m_{\rho,\delta}((G \times \widehat{G}) \otimes \text{End}(E_0))$, where $m \in \mathbb{R}$, $0 \le \rho, \delta \le 1$, is a mapping from $J_n \times J_n \times G \times G \times \text{Rep}(G)$, smooth in x and y, such that, for any $\xi \in \text{Rep}(G)$, $a(i, r, x, y, \xi) \in \mathbb{C}^{d_\xi \times d_\xi}$, and for any admissible collection of difference operators Δ_ξ^α, it satisfies

$$\|\Delta_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(i, r, x, y, \xi)\|_{op} \le C_{\alpha\beta\gamma} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta+\gamma|}, \tag{6.8}$$

for some $C_{\alpha\beta\gamma} > 0$ and for all multi-indices α, β, γ, $(i, r, x, y, [\xi]) \in J_n \times J_n \times G \times G \times \widehat{G}$.

For an amplitude a, the amplitude operator $\text{Op}(a) : C^\infty(G, E_0) \to \mathcal{D}'(G, E_0)$ is defined by

$$(\text{Op(a)}u)_r(x) \doteq \sum_{[\eta] \in \widehat{G}} d_\eta \, \text{Tr} \left[\eta(x) \int_G \sum_{i=1}^{n} a(i, r, x, y, \eta) u_i(y) \eta^*(y) \, dy \right] \tag{6.9}$$

for every $1 \le r \le n$, $u \in C^\infty(G, E_0)$. Notice that if A is a pseudo-differential operator $A : C^\infty(G, E_0) \to C^\infty(G, E_0)$ defined as before, and if $a(i, r, x, y, \eta) = \sigma_A(i, r, x, \eta)$, then $\text{Op}(a) = A$.

Proposition 6.2.8. *Let $0 \le \delta < 1$ and $0 \le \rho \le 1$, and $a \in \mathcal{A}^m_{\rho,\delta}((G \times \widehat{G}) \otimes End(E_0))$. Then $Op(a)$ is a continuous linear operator from $C^\infty(G, E_0)$ to $C^\infty(G, E_0)$.*

Proof. Recall that by definition, $(\text{Id} + \mathcal{L}_G)\eta = \langle \eta \rangle^2 \eta$. Consequently, we may rewrite (6.9) as

$$\sum_{[\eta] \in \widehat{G}} d_\eta \, \text{Tr} \left[\eta(x) \int_G \sum_{i=1}^{n} a(i, r, x, y, \eta) u_i(y) \langle \eta \rangle^{-2r} (\text{Id} + \mathcal{L}_G)^r \eta^*(y) \, dy \right]. \tag{6.10}$$

Since $(\text{Id} + \mathcal{L}_G)$ is self-adjoint, the equation above can be rewritten as

$$\sum_{[\eta] \in \widehat{G}} d_\eta \, \text{Tr} \left[\eta(x) \int_G \sum_{i=1}^{n} (\text{Id} + \mathcal{L}_G)^s [a(i, r, x, y, \eta) u_i(y)] \langle \eta \rangle^{-2r} \eta^*(y) \, dy \right], \tag{6.11}$$

for any $s \in \mathbb{N}$. By (6.9) we have that

$$\|(\text{Id} + \mathcal{L}_G)^s a(i, r, x, y, \eta)\|_{op} \langle \eta \rangle^{-2s} \le C \langle \eta \rangle^{m - 2s(1-\delta)}.$$

Since $0 \leq \delta < 1$, we can take $s \in \mathbb{N}$ sufficiently big so that the series above converges absolutely. By a similar argument, the same is true for all derivatives in x of the above expression. It follows that $\text{Op}(a)u \in C^\infty(G, E_0)$ provided $u \in C^\infty(G, E_0)$. The continuity of $\text{Op}(a)$ follows similarly. $\square$

Next we show that any operator defined by an amplitude may also be defined by a symbol, and these two objects are related through an asympotic expansion as follows.

Proposition 6.2.9. *Let $m \in \mathbb{R}$, $0 \leq \delta < \rho \leq 1$ and $a \in \mathcal{A}^m_{\rho,\delta}((G \times \widehat{G}) \otimes End(E_0))$. Then $A = Op(a)$ is a pseudo-differential operator with matrix symbol $\sigma_A \in S^m_{\rho,\delta}((G \times \widehat{G}) \otimes End(E_0))$. Moreover, σ_A has the asymptotic expansion*

$$\sigma_A(i, r, x, \xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_y^\alpha \Delta_\xi^\alpha a(i, r, x, y, \xi)|_{y=x},$$

for each $1 \leq i, r, \leq n$, in the sense that

$$\left(\sigma_A(i, r, x, \xi) - \sum_{0 \leq |\alpha| < N} \frac{1}{\alpha!} \partial_y^\alpha \Delta_\xi^\alpha a(i, r, x, y, \xi)|_{y=x} \right) \in S^{m-(\rho-\delta)N}_{\rho,\delta}((G \times \widehat{G}) \otimes End(E_0)),$$

for every $N \in \mathbb{N}$ sufficiently big.

Proof. Indeed, by Proposition 6.2.8, A is a continuous linear operator acting on $C^\infty(G, E_0)$ and therefore admits symbol σ_A, the matrix symbol of A, given by:

$$\sigma_A(i, r, x, \xi) = \xi(x)^* \langle A(\xi \otimes e_i)(x), e_r \rangle_{E_0}.$$

Therefore

$$\begin{aligned}
\sigma_A(i, r, x, \xi)_{mn} &= \sum_{l=1}^{d_\xi} \xi(x^{-1})_{ml} (A(\xi_{ln} \otimes e_i))_r(x) \\
&= \sum_{l=1}^{d_\xi} \xi(x^{-1})_{ml} \int_G \sum_{[\eta] \in \widehat{G}} d_\eta \, \text{Tr}[\eta(x) a(i, r, x, y, \eta) \xi(y)_{ln} \eta(y)^*] dy \\
&= \int_G \sum_{l=1}^{d_\xi} \xi(x^{-1})_{ml} \xi(y)_{ln} \sum_{[\eta] \in \widehat{G}} d_\eta \, \text{Tr}[\eta(y^{-1}) \eta(x) a(i, r, x, y, \eta)] dy \\
&= \int_G \xi(x^{-1} y)_{mn} \sum_{[\eta] \in \widehat{G}} d_\eta \, \text{Tr}[\eta(y^{-1} x) a(i, r, x, y, \eta)] dy.
\end{aligned}$$

Performing the change of variables $z = y^{-1}x$ (justified by the invariance of the Haar measure), we obtain that

$$\begin{aligned}
\sigma_A(i,r,x,\xi)_{mn} &= \int_G \xi(z^{-1})_{mn} \sum_{[\eta]\in\widehat{G}} d_\eta \sum_{j,k=1}^{d_\eta} \eta(z)_{jk}\, a(i,r,x,xz^{-1},\eta)_{kj} dz \\
&= \sum_{0\le|\alpha|<N} \frac{1}{\alpha!}\partial_y^\alpha|_{y=x} \int_G \xi(z^{-1})_{mn} q_\alpha(z) \sum_{[\eta]\in\widehat{G}} d_\eta \sum_{j,k=1}^{d_\eta} \eta(z)_{jk} a(i,r,x,y,\eta)_{kj} dz \\
&+ \int_G \xi(z^{-1})_{mn} \sum_{[\eta]\in\widehat{G}} d_\eta \sum_{j,k=1}^{d_\eta} \eta(z)_{jk} R_N(i,r,z,\eta)_{kj} dz \\
&= \sum_{0\le|\alpha|<N} \frac{1}{\alpha!} \int_G \xi(z)^*_{mn} q_\alpha(z) \sum_{[\eta]\in\widehat{G}} d_\eta \operatorname{Tr}[\eta(z)\partial_y^\alpha|_{y=x} a(i,r,x,y,\eta)] dz \\
&+ \int_G \xi(z^{-1})_{mn} \sum_{[\eta]\in\widehat{G}} d_\eta \sum_{j,k=1}^{d_\eta} \eta(z)_{jk} R_N(i,r,z,\eta)_{kj} dz \\
&= \sum_{0\le|\alpha|<N} \frac{1}{\alpha!} \partial_y^\alpha \Delta_\xi^\alpha a(i,r,x,y,\xi)_{mn}|_{y=x} \\
&+ \int_G \xi(z^{-1})_{mn} \sum_{[\eta]\in\widehat{G}} d_\eta \sum_{j,k=1}^{n} \eta(z)_{jk} R_N(i,r,z,\eta)_{kj} dz, \qquad (6.12)
\end{aligned}$$

where $R_N(i,r,z,\eta)_{kj}$ is the remainder of the Taylor expansion in y of order $N \in \mathbb{N}$ of $a(i,r,x,y,\eta)_{kj}$ centered at x. Therefore, to prove the claim we only need to analyze these last series of integrals. For $(x,y) \in G \times G$, $1 \le i, r \le n$, define $k_{A,x,y}(i,r,\cdot)$ to be the distribution defined by

$$a(i,r,x,y,\xi) = \widehat{k_{A,x,y}}(i,r,\xi),\ [\xi] \in \widehat{G}.$$

Similarly, denote by $k_{\sigma,x}(i,r,\cdot)$ to be the right convolution kernels of A, that is, the distribution which satisfies

$$\sigma_A(i,r,x,\xi) = \widehat{k_{\sigma,x}}(i,r,\xi),\ [\xi] \in \widehat{G}.$$

Notice that

$$\begin{aligned}
Af(x)_r &= \int_G \sum_{i=1}^{n} f_i(y) k_{A,x,y}(i,r,y^{-1}x) dy \\
&= \int_G \sum_{i=1}^{n} f_i(xz^{-1}) k_{A,x,xz^{-1}}(i,r,z) dz,
\end{aligned}$$

where the first equality can be found in [25]. Also,

$$Af(x)_r = \int_G \sum_{i=1}^{n} f_i(y) k_{\sigma,x}(i,r,y^{-1}x)dy = \int_G \sum_{i=1}^{n} f_i(xz^{-1}) k_{\sigma,x}(i,r,z)dz,$$

therefore we have that

$$k_{A,x,y}(i,r,y^{-1}x) = k_{\sigma,x}(i,r,y^{-1}x),\ k_{A,x,xz^{-1}}(i,r,z) = k_{\sigma,x}(i,r,z),$$

in the sense of distributions, for all $1 \leq i, r \leq n$. Now, fix $1 \leq i, r \leq n$, and notice that

$$\begin{aligned}\sum_{[\eta]\in\widehat{G}} d_\eta \sum_{j,k=1}^{n} \eta(z)_{jk} R_N(i,r,z,\eta)_{kj} &= \sum_{[\eta]\in\widehat{G}} d_\eta \sum_{j,k=1}^{n} \eta(z)_{jk} [a(i,r,x,xz^{-1}\eta)_{kj} \\ &\quad - \partial_y|_{y=x} a(i,r,x,y,\eta)_{kj}] \\ &= k_{A,x,xz^{-1}}(i,r,z)_{kj} - \partial_y|_{y=x} k_{A,x,y}(i,r,z)_{kj} \\ &= k_{\sigma,x}(i,r,z) - \sum_{|\alpha|<N} q_\alpha(z) \partial_{z_1}^{\alpha}|_{z_1=e} k_{A,x,xz_1^{-1}}(i,r,z).\end{aligned}$$

Applying this to (6.12), our goal now is to prove that the Fourier transform of the expression above is in $S^{m-(\rho-\delta)N}_{\rho,\delta}((G \times \widehat{G}) \otimes \mathrm{End}(E_0))$ for every N sufficiently big, which we do as follows. For any multi-indices γ, β, let $M' \in 2\,\mathbb{N}_0$ such that

$$M' > \rho|\gamma| - \delta|\beta| - m + (\rho-\delta)N,$$

and

$$\frac{M'}{2} > -\rho|\gamma| + \delta|\beta| + m + 2(\rho-\delta)N. \tag{6.13}$$

Then

$$\begin{aligned}&\left\| \langle\xi\rangle^{\rho|\gamma|-\delta|\beta|-m+(\rho-\delta)N} \Delta^\gamma \partial_x^\beta \left[\sum_{[\eta]\in\widehat{G}} d_\eta \sum_{i=1}^{n} \sum_{j,k=1}^{n} \int_G \xi(z^{-1})_{mn} \eta(z)_{jk} R_N(i,r,z,\eta)_{kj} dz \right] \right\|_{op} \\ &\leq \left\| \langle\xi\rangle^{M'} \Delta^\gamma \partial_x^\beta \left[\sum_{[\eta]\in\widehat{G}} d_\eta \sum_{i=1}^{n} \sum_{j,k=1}^{n} \int_G \xi(z^{-1})_{mn} \eta(z)_{jk} R_N(i,r,z,\eta)_{kj} dz \right] \right\|_{op} \\ &\leq \|(\mathrm{Id}+\mathcal{L})^{\frac{M'}{2}} [q_\gamma(z) \partial_x^\beta [k_{\sigma,x}(i,r,z) - \sum_{|\alpha|<N} q_\alpha(z) \partial_{z_1}^{\alpha}|_{z_1=e} k_{A,x,xz_1^{-1}}(i,r,z)]\|_{L^1(G)_z} \\ &= \|(\mathrm{Id}+\mathcal{L})^{\frac{M'}{2}} [q_\gamma(z) \partial_x^\beta R^{k_{A,x,x}}_{x,N}(i,r,z)]\|_{L^1(G)_z},\end{aligned} \tag{6.14}$$

where

$$k_{\sigma,x}(i,r,z) = k_{A,x,xz^{-1}}(i,r,z) = \sum_{|\alpha|<N} q_\alpha(z)\partial_{z_1}^\alpha|_{z_1=e}k_{A,x,xz_1^{-1}}(i,r,z) + R_{x,N}^{k_{A,x,x}}(i,r,z),$$

for all $x,y,z \in G$ by the Taylor expansion theorem. Moreover, by [44], Lemma 7.4, the remainder satisfies the estimate

$$|R_{x,N}(i,r,z)| \leq C|z|^N \max_{|\alpha|\leq N} \|\partial_x^\alpha k_{A,x,x}(i,r,z)\|_{L^\infty(G)_x}.$$

Now applying the definition of $\mathcal{L}$ and the Leibniz's rule on (6.14), gives us

$$\begin{aligned}\|(\mathrm{Id}+\mathcal{L})^{\frac{M'}{2}}&[q_\gamma(z)\partial_x^\beta R_{x,N}^{k_{A,x,x}}(i,r,z)]\|_{L^1(G)_z} \\ &\lesssim \sum_{1\leq i_1\leq\cdots\leq i_d\leq d,\,|\lambda|\leq M'} \|X_{i_1,z}^{\lambda_1}\ldots X_{i_d,z}^{\lambda_d}[R_{x,N}^{q_\gamma(z)\partial_x^\beta k_{A,x,x}}(i,r,z)\|_{L^1(G)_z},\end{aligned} \tag{6.15}$$

where the notation $\lesssim$ denotes that the inequality holds up to a constant which independs on the functions involved. By estimates similar to the ones above, we have

$$|X_{i_1,z}^{\lambda_1}\ldots X_{i_d,z}^{\lambda_d}[R_{x,N}^{q_\gamma(z)\partial_x^\beta k_{A,x,x}}(i,r,z)]| \lesssim |z|^{N-|\lambda|} \max_{|\alpha|\leq N-|\lambda|} \|\partial_z^{\alpha+\lambda}(q_\gamma(z)\partial_x^\beta k_{A,x,x}(i,r,z))\|_{L^\infty(G)_x},$$

and since $\{\partial_z^{\alpha+\lambda}(q_\gamma(\cdot)\partial_x^\beta k_{A,x,x}(i,r,\cdot))\}_{i,r=1}^n$ are the right-convolution kernels of a pseudo-differential operator of order

$$s' = m + \delta|\beta| + \delta|\alpha| + \delta|\lambda| - \rho|\gamma|,$$

by Proposition 6.7 of [44] we have that

$$\|\partial_z^{\alpha+\lambda}(q_\gamma(z)\partial_x^\beta k_{A,x,x}(i,r,z))\|_{L^\infty(G)_z} \lesssim |z|^{-\frac{s'+d}{\rho}},$$

for $1\leq i,r\leq n$. Putting these previous estimates together, one obtains

$$\|(\mathrm{Id}+\mathcal{L})^{\frac{M'}{2}}[q_\gamma(z)\partial_x^\beta R_{x,N}^{k_{A,x,x}}(i,r,z)]\|_{L^1(G)_z} \lesssim \int_G |z|^{N-|\lambda|}|z|^{-\frac{s'+d}{\rho}}, \tag{6.16}$$

and the integral on the right-hand-side is finite if

$$\rho(|\lambda| - N) + s' + d < \rho d.$$

But indeed, for N big enough, we have $M' \leq (\rho-\delta)N$. By choosing M' so that $(\rho-\delta)N/2 \lesssim M'$ also, we have that $(\rho - \delta)N$ is proportional to M'. Hence, using that $|\alpha| + |\lambda| \leq N$ and (6.13) yields

$$\begin{aligned}
\rho(|\lambda| - N) + s' + d &= \rho|\lambda| - \rho N + m + \delta|\beta| + \delta|\alpha| + \delta|\lambda| - \rho|\gamma| + d \\
&< \rho|\lambda| - \rho N + m + \delta|\beta| + \rho|\alpha| + \rho|\lambda| - \rho|\gamma| + d \\
&= \rho|\lambda| - \rho(N - |\alpha| - |\lambda|) + m + \delta|\beta| - \rho|\gamma| + d \\
&\leq \rho|\lambda| + m + \delta|\beta| - \rho|\gamma| + d \\
&\leq \rho M' + \frac{M'}{2} - 2N(\rho - \delta) + d \leq \frac{3M'}{2} - 2N(\rho - \delta) + d \\
&\sim \frac{3(\rho - \delta)N}{2} - 2N(\rho - \delta) + d = -\frac{(\rho - \delta)N}{2} + d,
\end{aligned}$$

where in the second to last line we used the fact that $|\lambda| \leq M'$ from (6.15). By choosing $N \geq N_0 \geq 2d(1 - \rho)/(\rho - \delta)$, we get that $\rho(|\lambda| - N) + s' + d < \rho d$, which by (6.12), (6.16) and the previous inequalities proves that the remainder term in the asymptotic expansion is in $S^{m-(\rho-\delta)N}_{\rho,\delta}((G \times \widehat{G}) \otimes \mathrm{End}(E_0))$, for all such N. □

6.2.2 Even and odd functions on compact Lie groups

Recall that $f \in C^\infty(G)$ is called central if $f(xy) = f(yx)$, for every $x, y \in G$. Following [96], we will say $f \in C^\infty(G)$ is *even* if it is invariant under inversions, that is: $f(x^{-1}) = f(x)$ for every $x \in G$. Similarly, we will say that $f \in C^\infty(G)$ is *odd* if $f(x^{-1}) = -f(x)$, for every $x \in G$.

Lemma 6.2.10. *Let $f \in C^\infty(G)$ be central and even. Then for any left-invariant vector field X, the function Xf is odd. Moreover, if $g \in C^\infty$ is odd, then fg is also odd.*

Proof. Indeed, we have that

$$\begin{aligned} Xf(x^{-1}) &= \lim_{t\to 0}\frac{f(x^{-1}\exp(tX)) - f(x^{-1})}{t} \\ &= \lim_{t\to 0}\frac{f(\exp(-tX)x) - f(x)}{t} \\ &= \lim_{t\to 0}\frac{f(x\exp(-tX)) - f(x)}{t} \\ &= -Xf(x), \end{aligned}$$

for any $x \in G$, which proves the first claim. The second claim is immediate. □

Lemma 6.2.11. *Let $f \in C^\infty(G)$ be odd. Then $\int_G f(x)dx = 0$.*

Proof. Notice that $\int_G f(x)dx = \int_G f(x^{-1})dx = -\int_G f(x)dx$, so $\int_G f(x)dx = 0$. □

6.3 Sharp Gårding inequality on compact Lie groups

6.3.1 Main Results

Here we present the main result in [83], obtained by the author and Ruzhansky. Its proof will be then carried out through several lemmas.

Theorem 6.3.1. *Let G be a compact Lie group, $\dim(G) = d$, and E_0 a n-dimensional $\mathbb{C}$-vector space. Let $A = Op(\sigma_A) \in \Psi^m_{\rho,\delta}((G\times\widehat{G})\otimes End(E_0))$, $0\le\delta<\rho\le 1$, be such that its matrix-valued symbol $\sigma_A(x,\xi)$ is positive semi-definite for every $(x,[\xi])\in G\times\widehat{G}$, in the sense that $B^*\sigma_A(x,\xi)B \in \mathbb{C}^{d_\xi\times d_\xi}$ is positive semi-definite for all $B \in (\mathbb{C}^{d_\xi\times d_\xi})^{n\times 1}$. Equivalently, this means that for any $B_1,\dots,B_n \in \mathbb{C}^{d_\xi\times 1}$, the inequality*

$$\sum_{i,r=1}^{n} \overline{B_i}^T \sigma_A(i,r,x,\xi)B_r \ge 0,$$

holds for all $(x,[\xi])\in G\times\widehat{G}$. Then there exists $C>0$ such that, for every $u\in C^\infty(G,E_0)$ we have

$$\mathrm{Re}\langle Au,u\rangle_{L^2(G,E_0)} \ge -C\|u\|^2_{H^{(m-(\rho-\delta))/2}(G,E_0)}, \tag{6.17}$$

where $\langle\cdot,\cdot\rangle_{L^2(G,E_0)}$ denotes the canonical inner product on $L^2(G,E_0)$.

Remark 6.3.2. In the previous statement and throughout the rest of this chapter, for a column-vector of matrices $B \in (\mathbb{C}^{d_\xi \times d_\xi})^{n\times 1}$ we denote by $B^* \in (\mathbb{C}^{d_\xi \times d_\xi})^{1\times n}$ the row-vector of matrices given by

$$B^* = (B_1^*, \ldots, B_n^*)\,.$$

First, notice that if $Q : H^{\frac{m-(\rho-\delta)}{2}}(G, E_0) \to H^{-\frac{m-(\rho-\delta)}{2}}(G, E_0)$ is a bounded linear operator, we have

$$\begin{aligned}\mathrm{Re}\langle Qu, u\rangle_{L^2(G,E_0)} &\geq -|\langle Qu, u\rangle_{L^2(G,E_0)}| \\ &\geq -\|Qu\|_{H^{-\frac{m-(\rho-\delta)}{2}}(G,E_0)} \|u\|_{H^{\frac{m-(\rho-\delta)}{2}}(G,E_0)} \\ &\geq -\|Q\|_{op}\|u\|^2_{H^{\frac{m-(\rho-\delta)}{2}}(G,E_0)}.\end{aligned}$$

Hence, Theorem 6.3.1 will follow once we show that A can be written as $A = P + Q$, where P is positive and Q is as above.

First, following the ideas in [96] and [25], we construct an auxiliary function $w_\xi : G \to \mathbb{C}$, for each $\xi \in \mathrm{Rep}(G)$.

We can assume G is a closed subgroup of $\mathrm{GL}(N, \mathbb{R})$ for some $N \in \mathbb{N}$. Then its Lie algebra $\mathfrak{g} \subset \mathbb{R}^{N\times N}$ is a d-dimensional vector subspace such that $[A, B] \doteq AB - BA \in \mathfrak{g}$, for every $A, B \in \mathfrak{g}$. Let $U \subset G$ and $V \subset \mathfrak{g}$ be neighbourhoods of the identity $\mathrm{Id}_N = e_G \in G$ and $0 \in \mathfrak{g}$, respectively, so that the matrix exponential mapping is a diffeomorphism $\exp : V \to U$. Without loss of generality, we may assume that V is the open ball $V = B(0, r) = \{z \in \mathbb{R}^d \,|\, |z| < r\}$, of radius $r > 0$. Let $\phi : [0, \infty) \to [0, \infty)$ be a smooth function such that the mapping

$$\begin{aligned}&\mathfrak{g} \to \mathbb{R}\\ &z \mapsto \phi(|z|)\end{aligned}$$

is supported in V and such that $\phi(s) = 1$ for all sufficiently small $s > 0$. For every $\xi \in \mathrm{Rep}(G)$ define

$$\begin{aligned}w_\xi &: G \to \mathbb{R}\\ &x \mapsto \phi(|\exp^{-1}(x)|\langle\xi\rangle^{\frac{\rho+\delta}{2}})\psi(\exp^{-1}(x))\langle\xi\rangle^{\frac{d(\rho+\delta)}{4}},\end{aligned}$$

where $\psi(y) \doteq C_0|\det D\exp(y)|^{-\frac{1}{2}} f(y)^{-\frac{1}{2}}$, for every $y \in \mathfrak{g} \cong \mathbb{R}^d$, $D\exp$ is the Jacobi matrix of the mapping $\exp$, f is the density with respect to the Lebesgue measure of the Haar measure on G pulled back to $\mathfrak{g} \cong \mathbb{R}^d$ by the exponential mapping, and with $C_0 = (\int_{\mathbb{R}^d} \phi(|z|)^2 dz)^{-\frac{1}{2}}$.

The following lemma states the main properties of w_ξ that will be used in this paper, its proof can be found in [25], but we also state it here with some slight changes for the sake of completeness.

Lemma 6.3.3. *The functions w_ξ defined above are smooth, for every $\xi \in Rep(G)$. Moreover, they satisfy the following properties:*

1. $w_\xi(e) = C_0 \langle \xi \rangle^{\frac{d(\rho+\delta)}{4}}$;
2. w_ξ *is central and inversion invariant;*
3. *dist*$(x, e) \sim |\exp^{-1}(x)| \lesssim \langle \xi \rangle^{-\frac{\rho+\delta}{2}}$ *on the support of* w_ξ;
4. $\|w_\xi\|_{L^2(G)} = 1$;
5. $(x, [\xi]) \mapsto w_\xi(x) Id_{d_\xi} \in S^{d(\rho+\delta)/4}_{\rho,(\rho+\delta)/2}(G \times \widehat{G})$,

for every $\xi \in Rep(G)$, where dist(x, e) denotes the geodesic distance from the group neutral element e to $x \in G$. Also, $|\cdot|$ denotes the central norm on $\mathfrak{g}$ given by $|X| = \int_G |uX^{-1}u^{-1}|_0 du$, where $|\cdot|_0$ denotes the Euclidean norm on $\mathfrak{g} \cong \mathbb{R}^d$ and the product is the product of matrices. Also, Id_{d_ξ} denotes the $d_\xi \times d_\xi$ identity matrix.

Proof. Notice that, due to the properties of ϕ, we immediately have that $w_\xi(e) = C_0 \langle \xi \rangle^{\frac{d(\rho+\delta)}{4}}$, w_ξ is central (since f is invariant under adjoint representation as a density of two bi-invariant measures) and inversion invariant, and $\operatorname{dist}(x, e) \leq r \langle \xi \rangle^{-\frac{(\rho+\delta)}{2}}$ on $\operatorname{supp} w_\xi$. As for the $\|w_\xi\|_{L^2(G)}$, we have

$$\begin{aligned}
\int_G |w_\xi(x)|^2 dx &= \langle \xi \rangle^{\frac{d(\rho+\delta)}{2}} \int_{\mathbb{R}^d} \phi(|Y| \langle \xi \rangle^{\frac{(\rho+\delta)}{2}})^2 |\psi(Y)|^2 |\det D\exp(Y)| f(Y) dY \\
&= \int_{\mathbb{R}^d} \phi(|Z|)^2 |\psi(Z \langle \xi \rangle^{-\frac{(\rho+\delta)}{2}})|^2 |\det D\exp(Z \langle \xi \rangle^{-\frac{(\rho+\delta)}{2}})| f(Z \langle \xi \rangle^{-\frac{(\rho+\delta)}{2}}) dZ \\
&= C_0^2 \int_{\mathbb{R}^d} \phi(|Z|)^2 dZ = 1,
\end{aligned}$$

where in the second line we applied the change of variables $Z = Y \langle \xi \rangle^{\frac{(\rho+\delta)}{2\kappa}}$, while in the third line we simply used the expression of ψ.

We are now left with the proof of

$$(x,\xi) \mapsto w_\xi(x)\mathrm{Id}_{d_\xi} \in S^{\frac{d(\rho+\delta)}{4}}_{\rho,\frac{(\rho+\delta)}{2}}(G\times\widehat{G}). \tag{6.18}$$

From the compactness of G, proving (6.18) is equivalent to showing that, for every multi-index β and for any fixed $x \in G$, $(\partial^\beta w_\xi)(x)\mathrm{Id}_{d_\xi} \in S^{\frac{d(\rho+\delta)}{4}+\frac{(\rho+\delta)}{2}|\beta|}_{\rho,0}(G)$ (see Lemma 3.3 in [96]). First observe that

$$\partial^\beta w_\xi(x)\mathrm{Id}_{d_\xi} = \sum_{\alpha;|\alpha|\le|\beta|} C_{\alpha,\beta}\Big[\partial^\alpha\phi(|\exp^{-1}(x)|\langle\xi\rangle^{\frac{(\rho+\delta)}{2}})\Big]\partial^{\beta-\alpha}\psi(\exp^{-1}(x))\langle\xi\rangle^{\frac{d(\rho+\delta)}{4}}\mathrm{Id}_{d_\xi}$$

$$= \sum_{\alpha;|\alpha|\le|\beta|} C_{\alpha,\beta}\phi_\alpha(|\exp^{-1}(x)|\langle\xi\rangle^{\frac{(\rho+\delta)}{2}})\langle\xi\rangle^{\frac{(\rho+\delta)|\alpha|}{2}}\chi_{\beta-\alpha}(\exp^{-1}(x))\langle\xi\rangle^{\frac{d(\rho+\delta)}{4}}\mathrm{Id}_{d_\xi},$$

where $\phi_\alpha, \chi_{\beta-\alpha}$ are suitable functions such that $\phi_\alpha \in C_0^\infty(\mathbb{R})$ is constant near the origin, while $\chi_{\beta-\alpha} \in C_0^\infty(V)$. Since

$$\langle\xi\rangle^{\frac{n(\rho+\delta)}{4}+\frac{(\rho+\delta)|\alpha|}{2}}\mathrm{Id}_{d_\xi} \in S^{\frac{n(\rho+\delta)}{4}+\frac{(\rho+\delta)}{2}|\beta|}_{1,0}(G\times\widehat{G}),$$

for every $|\alpha| \le |\beta|$, then for every (fixed) $x \in G$,

$$\partial_x^\beta w_\xi(x)\mathrm{Id}_{d_\xi} \in S^{\frac{d(\rho+\delta)}{4}+\frac{(\rho+\delta)}{2}|\beta|}_{1,0}(G\times\widehat{G})$$

if $\phi_\alpha(|\exp^{-1}(x)|\langle\xi\rangle^{\frac{(\rho+\delta)}{2}})\chi_{\beta-\alpha}(\exp^{-1}(x))\mathrm{Id}_{d_\xi} \in S^0_{1,0}(G\times\widehat{G})$ for all α and β as above. Therefore, to complete the proof it is enough to check that these last terms are standard global symbols of order 0.

Now, given $x \in G$, it is easy to see that

$$\phi_\alpha(|\exp^{-1}(x)|\langle\xi\rangle^{\frac{(\rho+\delta)}{2\kappa}})\chi_{\beta-\alpha}(\exp^{-1}(x)) \le C, \tag{6.19}$$

for some constant $C > 0$. In fact, if x is such that $\exp^{-1}(x) = 0$, then ϕ_α is constant and the inequality follows. If, instead, $\exp^{-1}(x) \ne 0$, then, since ϕ_α is compactly supported in ξ, we get that the symbol in the left hand side of (6.19) is compactly supported, then smoothing, and the inequality follows. This concludes the proof of (6.18) from which the result follows. □

Lemma 6.3.4. *Let σ_A be the matrix valued symbol of A as in Theorem 6.3.1. Define the am-*

plitude

$$p(i,r,x,y,\xi) \doteq \int_G w_\xi(xz^{-1}) w_\xi(yz^{-1}) \sigma_A(i,r,z,\xi) dz \in \mathbb{C}^{d_\xi \times d_\xi}, \tag{6.20}$$

for every $1 \leq i,r \leq n$, $x,y \in G$, $[\xi] \in \widehat{G}$, *where* w_ξ *is as in Lemma 6.3.3. Then* $p \in \mathcal{A}^m_{\rho,\frac{\rho+\delta}{2}}((G \times \widehat{G}) \otimes End(E_0))$ *and the linear operator* $P : C^\infty(G,E_0) \to C^\infty(G,E_0)$ *given by*

$$(Pu)_r(x) = \int_G \sum_{i=1}^n \sum_{[\xi]\in\widehat{G}} d_\xi\, Tr(\xi(y^{-1}x) p(i,r,x,y,\xi) u_i(y)) dy, \quad 1 \leq r \leq n,$$

for every $x \in G$, *is a well defined pseudo-differential operator of order* m, *and also a positive linear operator on* $L^2(G,E_0)$.

Proof. First, we verify that $p \in \mathcal{A}^m_{\rho,\frac{\rho+\delta}{2}}((G\times\widehat{G})\otimes \mathrm{End}(E_0))$. Indeed, note that by the Leibniz's rule, for any multi-indices α, β, γ, the matrix $\partial_x^\beta \partial_y^\gamma \Delta_\xi^\alpha p(i,r,x,y,\xi)$ is given by a sum of terms of the form

$$\int_G (\Delta_\xi^\eta \partial_x^\beta w_\xi(xz^{-1}))(\Delta_\xi^\lambda \partial_y^\gamma w_\xi(yz^{-1}))(\Delta_\xi^\mu \sigma_A(i,r,z,\xi)) dz,$$

where $|\eta+\lambda+\mu| \geq |\alpha|$ range over a finite set. Also, due to properties of w_ξ from Lemma 6.3.3, we have

$$\|(\Delta_\xi^\eta \partial_x^\beta w_\xi(xz^{-1}))(\Delta_\xi^\lambda \partial_y^\gamma w_\xi(yz^{-1}))\|_{op} \lesssim \langle\xi\rangle^{d\frac{(\rho+\delta)}{2} - \rho(|\eta|+|\lambda|) + \frac{(\rho+\delta)}{2}(|\beta|+|\gamma|)}.$$

Therefore, since $\mathrm{supp}(w_\xi)$ is contained in a set of measure proportional to $\langle\xi\rangle^{-d\frac{(\rho+\delta)}{2}}$, we can estimate

$$\begin{aligned}\left\| \int_G (\Delta_\xi^\eta \partial_x^\beta w_\xi(xz^{-1}))(\Delta_\xi^\lambda \partial_y^\gamma w_\xi(yz^{-1}))(\Delta_\xi^\mu \sigma_A(i,r,z,\xi)) dz \right\|_{op} \\ \lesssim \langle\xi\rangle^{d\frac{(\rho+\delta)}{2} - \rho(|\eta|+|\lambda|) + \frac{(\rho+\delta)}{2}(|\beta|+|\gamma|) - d\frac{(\rho+\delta)}{2} + m - \rho|\mu|} \\ \leq \langle\xi\rangle^{m-\rho|\alpha| + \frac{(\rho+\delta)}{2}(|\beta|+|\gamma|)},\end{aligned}$$

and so $\|\partial_x^\beta \partial_y^\gamma \Delta_\xi^\alpha p(i,r,x,y,\xi)\|_{op} \lesssim \langle\xi\rangle^{m-\rho|\alpha|+\frac{(\rho+\delta)}{2}(|\beta|+|\gamma|)}$ also, which proves the first claim.

Next, notice that

$$\begin{aligned}\langle Pu, u\rangle_{L^2(G,E_0)} &= \int_G \langle Pu(x), u(x)\rangle_{E_0} dx \\ &= \int_G \int_G \sum_{r,i=1}^{n} \sum_{[\xi]\in\widehat{G}} d_\xi \,\mathrm{Tr}[\xi(y)^*\xi(x)p(i,r,x,y,\xi)u_i(y)]\, dy\, \overline{u_r(x)}\, dx\,.\end{aligned}$$

Substituting $p(i,r,x,y,\xi)$ by its definition in the previous expression, one obtains

$$\int_G \int_G \sum_{[\xi]\in\widehat{G}} \sum_{r,i=1}^{n} \mathrm{Tr}[\xi(x) \int_G w_\xi(xz^{-1})w_\xi(yz^{-1})\sigma_A(i,r,z,\xi)\, dz\, u_i(y)\xi(y)^*]\, dy\, \overline{u_r(x)}\, dx\,. \tag{6.21}$$

Let

$$M(i,z,\xi) \doteq \int_G w_\xi(yz^{-1})\xi(yz^{-1})^* u_i(y)dy \in \mathbb{C}^{d_\xi\times d_\xi}.$$

Then (6.21) can be written as

$$\int_G \sum_{i,r=1}^{n} \sum_{[\xi]\in\widehat{G}} d_\xi \,\mathrm{Tr}[M(r,z,\xi)^*\xi(z)\sigma_A(i,r,z,\xi)\xi(z)^*M(i,z,\xi)]dz,$$

which is non-negative by our hypothesis since

$$\begin{aligned}&\mathrm{Tr}[M(i,z,\xi)^*\xi(z)\sigma_A(i,r,z,\xi)\xi(z)^*M(r,z,\xi)] \\ &= \sum_{k=1}^{d_\xi} f_k^* M(i,z,\xi)^*\sigma_A(i,r,z,\xi)M(r,z,\xi)f_k \geq 0,\end{aligned}$$

as $\xi(z)^*M(r,z,\xi)f_k \in \mathbb{C}^{d_\xi\times 1}$, where $\{f_k\}_{k=1}^{d_\xi}$ is any orthonormal basis of column vectors in $\mathbb{C}^{d_\xi}$. □

Lemma 6.3.5. *Following the previous notation, we have that* $(i,r,x,[\xi]) \mapsto p(i,r,x,x,\xi) - \sigma_A(x,\xi)$ *is the symbol of a pseudo-differential operator bounded from* $H^s(G,E_0)$ *to* $H^{s-(m-(\rho-\delta))}(G,E_0)$*, for any* $s\in\mathbb{R}$.

Proof. By Theorem 6.2.5, it is enough to show that

$$(i,r,x,[\xi]) \mapsto p(i,r,x,x,\xi) - \sigma_A(i,r,x,\xi) \in S^{m-(\rho-\delta)}_{\rho,\delta}((G\times\widehat{G})\otimes \mathrm{End}(E_0)).$$

Notice that, by Lemma 6.3.3,

$$\begin{aligned} p(i,r,x,x,\xi) - \sigma_A(i,r,x,\xi) &= \int_G w_\xi(z)^2 \sigma_A(i,r,xz^{-1},\xi)dz - \sigma_A(i,r,x,\xi) \\ &= \int_G w_\xi(z)^2 (\sigma_A(i,r,xz^{-1},\xi) - \sigma_A(i,r,x,\xi))dz. \end{aligned}$$

Using the Taylor expansion of $\sigma_A(i,r,xz^{-1},\xi)$ at x we can write

$$\sigma_A(i,r,xz^{-1},\xi) = \sigma_A(i,r,x,\xi) + \sum_{|\gamma|=1} \partial_x^\gamma \sigma_A(i,r,x,\xi) q_\gamma(z) + R_x(i,r,z,\xi),$$

where $R_x \in S^{m+2\delta}_{\rho,\delta}((G \times \widehat{G}) \otimes \mathrm{End}(E_0))$ is the Taylor remainder of order 2 of $\sigma_A(\cdot,\cdot,x,\cdot)$. Notice that we can choose the "polynomials" q_γ so that they are odd for all $|\gamma| = 1$, and using that w_ξ is even, we can conclude that

$$\int_G w_\xi^2(z) q_\gamma(z) dz = 0,$$

for all $|\gamma| = 1$. Hence, for multi-indices α, β:

$$\Delta_\xi^\alpha \partial_x^\beta (p(i,r,x,x,\xi) - \sigma_A(i,r,x,\xi)) = \Delta_\xi^\alpha \int_G w_\xi(z)^2 \partial_x^\beta R_x(i,r,z,\xi)\, dz\,.$$

Applying the Leibniz rule, we can write the expression above as a sum of terms of the form

$$\int_G \partial_x^\beta \Delta_\xi^{\alpha_1} R_x(i,r,z,\xi) \Delta_\xi^{\alpha_2} w_\xi(z) \Delta_\xi^{\alpha_3} w_\xi(z) dz,$$

where $|\alpha_1 + \alpha_2 + \alpha_3| \geq |\alpha|$ range over a finite set. By Lemma 7.4 of [44], the remainder satisfies

$$\|\Delta_\xi^{\alpha_1} \partial_x^\beta R_x(i,r,z,\xi)\|_{op} \leq C|z|^2 \max_{|\gamma|\leq 2} \sup_{x\in G} \|\Delta_\xi^{\alpha_1} \partial_x^\gamma \partial_x^\beta \sigma_A(i,r,x,\xi)\|_{op} \lesssim |z|^2 \langle\xi\rangle^{m-\rho|\alpha_1|+\delta(2+|\beta|)},$$

for all $z \in G$, $|z| \lesssim \langle\xi\rangle^{-\frac{\rho+\delta}{2}}$ on $\mathrm{supp}(w_\xi)$ by Lemma 6.3.3, and $|\mathrm{supp}(w_\xi)| \lesssim \langle\xi\rangle^{-d\frac{(\rho+\delta)}{2}}$, we conclude that

$$\left\| \int_G \partial_x^{\beta} \Delta_\xi^{\alpha_1} R_x(i,r,z,\xi) \Delta_\xi^{\alpha_2} w_\xi(z) \Delta_\xi^{\alpha_3} w_\xi(z) dz \right\|_{op}$$
$$\lesssim \langle \xi \rangle^{-\rho-\delta+m-\rho|\alpha_1|+\delta(2+|\beta|)+d\frac{(\rho+\delta)}{4}-\rho|\alpha_2|+d\frac{(\rho+\delta)}{4}-\rho|\alpha_3|-d\frac{(\rho+\delta)}{2}}$$
$$\leq \langle \xi \rangle^{m-(\rho-\delta)-\rho|\alpha|+\delta|\beta|},$$

which implies

$$\|\Delta_\xi^{\alpha} \partial_x^{\beta} (p(i,r,x,x,\xi) - \sigma_A(i,r,x,\xi))\|_{op} \lesssim \langle \xi \rangle^{m-(\rho-\delta)-\rho|\alpha|+\delta|\beta|}$$

proving the claim. □

Lemma 6.3.6. *Let σ_P be the symbol of the pseudo-differential operator P defined in Lemma 6.3.4. Then the pseudo-differential operator with symbol $\{\sigma_P(i,r,x,\xi) - p(i,r,x,x,\xi)\}_{i,r=1}^n$ is bounded from $H^s(G,E_0)$ to $H^{s-(m-(\rho-\delta))}(G,E_0)$, for any $s \in \mathbb{R}$.*

Proof. As in the proof of the previous lemma, it is enough to prove that

$$(i,r,x,[\xi]) \mapsto \sigma_P(i,r,x,\xi) - p(i,r,x,x,\xi) \in S^{m-(\rho-\delta)}_{\rho,\frac{(\rho+\delta)}{2}}((G\times\widehat{G}) \otimes \mathrm{End}(E_0)).$$

By Proposition 6.2.9, we have the asymptotic expansion

$$\sigma_P(i,r,x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \Delta_\xi^{\alpha} \partial_y^{\alpha} p(i,r,x,y,\xi)|_{y=x},$$

with the properties specified in Proposition 6.2.9. Recalling the formula (6.20), notice that after a change of variables $z^{-1}x \mapsto z$ one obtains

$$\Delta_\xi^{\alpha} \partial_y^{\alpha} p(i,r,x,y,\xi)|_{y=x} = \Delta_\xi^{\alpha} \int_G w_\xi(z) \partial_z^{\alpha} w_\xi(z) \sigma_A(i,r,z^{-1}x,\xi)\, dz\,.$$

Let $N \in \mathbb{N}$ to be chosen later. Define

$$S_N(i,r,x,\xi) = \sigma_P(i,r,x,\xi) - p(i,r,x,x) - R_N(i,r,x,\xi),$$

where $R_N \in S^{m-\frac{(\rho+\delta)}{2}(N+1)}_{\rho,\delta}((G\times\widehat{G}) \otimes \mathrm{End}(E_0))$ is the remainder term given by the asymptotic expansion above. Applying the Taylor expansion to the formula above yields

$$\begin{aligned}
S_N(i,r,x,\xi) &= \sum_{1\le|\alpha|\le N} \Delta_\xi^\alpha \int_G w_\xi(z)\partial_z^\alpha w_\xi(z)\sigma_A(i,r,x,\xi)\,dz \\
&\quad + \sum_{1\le|\alpha|\le N} \Delta_\xi^\alpha \int_G w_\xi(z)\partial_z^\alpha w_\xi(z) R_{\sigma,1}(i,r,z^{-1}x,\xi)\,dz - R_N(i,r,x,\xi) \\
&\doteq I(i,r,x,\xi) + J(i,r,x,\xi) - R_N(i,r,x,\xi),
\end{aligned}$$

where $R_{\sigma,1}$ is the remainder in the Taylor expansion of order 1 of σ_A, centered at x. Therefore, if we can prove that I, J and R_N all belong to the specified symbol class, we will have proved the lemma. Let us examine each one at a time. First, we see that the first term in the sums in I (of order 1) vanishes since

$$\int_G w_\xi(z)\partial_z^\alpha w_\xi(z)\,dz = 0,$$

for $|\alpha| = 1$ because the functions w_ξ and $\partial^\alpha w_\xi$ are even and odd, respectively, by Lemmas 6.2.10, 6.2.11 and 6.3.3. In particular, I is given by

$$\begin{aligned}
&\sum_{2\le|\alpha|\le N} \Delta_\xi^\alpha \int_G w_\xi(z)\partial_z^\alpha w_\xi(z)\,dz\,\sigma_A(i,r,x,\xi) \\
&\qquad = \sum_{2\le|\alpha|\le N}\sum_{\kappa,\lambda,\mu} C_{\kappa,\lambda,\mu} \int_G (\Delta_\xi^\kappa w_\xi(z))(\Delta_\xi^\lambda \partial_z^\alpha w_\xi(z))\,dz\,\Delta_\xi^\mu \sigma_A(i,r,x,\xi),
\end{aligned}$$

for some constants $C_{\kappa,\lambda,\mu}$, where the sum is taken over a finite set satisfying $|\kappa+\lambda+\mu| \ge |\alpha|$, where here we have used the "Leibniz's" rule for difference operators, that is, Proposition 2.2.15. Recalling that by Lemma 6.3.3 $(x,[\xi]) \mapsto w_\xi(x)\mathrm{Id}_{d_\xi}$ is in $S^{d(\rho+\delta)/4}_{\rho,(\rho+\delta)/2}(G\times\widehat{G})$, we get that

$$\int_G |(\Delta_\xi^\kappa w_\xi(z))(\Delta_\xi^\lambda \partial_z^\alpha w_\xi(z))|\,dz \lesssim \langle\xi\rangle^{d\frac{(\rho+\delta)}{2}-\rho(|\kappa|+|\lambda|)+|\alpha|\frac{(\rho+\delta)}{2}} \langle\xi\rangle^{-d\frac{(\rho+\delta)}{2}} = \langle\xi\rangle^{-\rho(|\kappa|+|\lambda|)+|\alpha|\frac{(\rho+\delta)}{2}}, \tag{6.22}$$

where we have taken into account that the support of $z\mapsto w_\xi(z)$ is contained in a set of measure $\sim \langle\xi\rangle^{-d\frac{(\rho+\delta)}{2}}$, by Lemma 6.3.3, and that taking differences in ξ does not increase the support in z. Thus we get

$$\begin{aligned}\|I(i,r,x,\xi)\|_{op} &\lesssim \sum_{2\leq|\alpha|\leq N}\sum_{\kappa,\lambda,\mu} C_{\kappa,\lambda,\mu}\langle\xi\rangle^{-\rho(|\kappa|+|\lambda|)+|\alpha|\frac{(\rho+\delta)}{2}}\langle\xi\rangle^{m-\rho|\mu|}\\ &\lesssim \sum_{2\leq|\alpha|\leq N}\langle\xi\rangle^{m-\rho|\alpha|+|\alpha|\frac{(\rho+\delta)}{2}} \lesssim \langle\xi\rangle^{m-(\rho-\delta)},\end{aligned}$$

since

$$-\rho|\alpha| + |\alpha|\frac{(\rho+\delta)}{2} = -|\alpha|\frac{(\rho-\delta)}{2} \leq -(\rho-\delta).$$

Applying a similar argument to $\Delta_\xi^\gamma\partial_x^\beta I(i,r,x,\xi)$ we obtain the respective decay estimates, which allow us to conclude that $I \in S^{m-(\rho-\delta)}_{\rho,\frac{(\rho+\delta)}{2}}((G\times\widehat{G})\otimes \mathrm{End}(E_0))$, as desired. We now consider the term J. In this case,

$$\|J(i,r,x,\xi)\|_{op} \lesssim \sum_{1\leq|\alpha|\leq N}\sum_{\kappa,\lambda,\mu}\int_G \|(\Delta_\xi^\kappa w_\xi(z))(\Delta_\xi^\lambda\partial_z^\alpha w_\xi(z))(\Delta_\xi^\mu R_{\sigma,1}(i,r,z^{-1}x,\xi)\|_{op}\,dz$$

where again we have used the Leibniz's rule for difference operators, and the middle sum is over a finite set satisfying $|\kappa+\lambda+\mu| \geq |\alpha|$. Using Lemma 6.3.3 and the estimates for the remainder in the Taylor expansion, we get that

$$\begin{aligned}\|J(i,r,x,\xi)\|_{op} &\lesssim \sum_{1\leq|\alpha|\leq N}\sum_{\kappa,\lambda,\mu}\int_{\mathrm{supp}(w_\xi)}\langle\xi\rangle^{d\frac{(\rho+\delta)}{2}-\rho(|\kappa|+|\lambda|)+|\alpha|\frac{(\rho+\delta)}{2})}|z|^1\,dz\times\\ &\times \max_{|\gamma|\leq 1}\|\|\Delta_\xi^\mu\partial_y^\gamma\sigma_A(i,r,y,\xi)\|_{op}\|_{L^\infty(G)_y}\\ &\lesssim \sum_{1\leq|\alpha|\leq N}\langle\xi\rangle^{d\frac{(\rho+\delta)}{2}-\rho(|\alpha|)+|\alpha|\frac{(\rho+\delta)}{2}-\frac{(\rho+\delta)}{2}-d\frac{(\rho+\delta)}{2}+m+\delta}\\ &\lesssim \langle\xi\rangle^{m-\frac{(\rho-\delta)}{2}-\frac{(\rho-\delta)}{2}} = \langle\xi\rangle^{m-(\rho-\delta)}.\end{aligned}$$

Again, applying a similar argument to $\Delta_\xi^\gamma\partial_x^\beta J(i,r,x,\xi)$ we obtain the respective decay estimates, which allow us to conclude that $J \in S^{m-(\rho-\delta)}_{\rho,\frac{(\rho+\delta)}{2}}((G\times\widehat{G})\otimes\mathrm{End}(E_0))$, as desired. It only remains necessary to study the remainder R_N. In this case, by Proposition 6.2.9, we have that $R_N \in S^{m-\frac{(\rho+\delta)}{2}(N+1)}_{\rho,\frac{\rho+\delta}{2}}((G\times\widehat{G})\otimes\mathrm{End}(E_0))$. By taking N sufficiently large, we obtain that R_N belongs to the desired symbol class, which concludes the proof. □

Proof of Theorem 6.3.1. Let $Q = A - P$, with the operator P as in Lemma 6.3.4. Let $u \in$

$C^\infty(G, E_0)$. Then $A = P + Q$ and the positivity of P implies

$$\begin{aligned}\mathrm{Re}\langle Au, u\rangle_{L^2(G,E_0)} &= \mathrm{Re}\langle Pu, u\rangle_{L^2(G,E_0)} + \mathrm{Re}\langle Qu, u\rangle_{L^2(G,E_0)}\\ &\geq \mathrm{Re}\langle Qu, u\rangle_{L^2(G,E_0)}.\end{aligned}$$

Let now $P_0 = \mathrm{Op}(\{p(i, r, x, x, \xi)_{i,r=1}^n\})$. Writing $Q = (A - P_0) + (P_0 - P)$, we have

$$\sigma_{A-P_0}(i, r, x, \xi) = \sigma_A(i, r, x, \xi) - p(i, r, x, x, \xi)$$

and

$$\sigma_{P_0-P}(i, r, x, \xi) = p(i, r, x, x, \xi) - \sigma_P(i, r, x, \xi).$$

Consequently, both $A - P_0$ and $P_0 - P$ are bounded from $H^{\frac{m-(\rho-\delta)}{2}}(G, E_0)$ to $H^{-\frac{m-(\rho-\delta)}{2}}(G, E_0)$ by Lemmas 6.3.5 and 6.3.6, respectively. It follows that Q is also bounded between these spaces so that

$$\begin{aligned}|\,\mathrm{Re}\langle Qu, u\rangle_{L^2(G,E_0)}| &\leq \|Qu\|_{H^{-\frac{m-(\rho-\delta)}{2}}(G,E_0)}\|u\|_{H^{\frac{m-(\rho-\delta)}{2}}(G,E_0)}\\ &\lesssim \|u\|^2_{H^{\frac{m-(\rho-\delta)}{2}}(G,E_0)},\end{aligned}$$

completing the proof of Theorem 6.3.1. □

6.4 Sharp Gårding inequality on homogeneous spaces

We now recall the Fourier analysis on homogeneous vector bundles as introduced in [27]. But first, we introduce the setting of compact homogeneous manifolds, following [104].

Let $p : E \to X$ be a vector bundle. A continuous map $s : X \to E$ is called a section of E if for all $x \in X$, $p(s(x)) = p(x)$. We denote by $\Gamma(E)$ the set of all sections of E. If X, E are smooth manifolds, we also define $\Gamma^\infty(E)$ the set of all *smooth* sections of E. If X is orientable, the space $L^q(E)$, $1 \leq q < \infty$, is then defined as the completion of the set of all smooth sections $s \in \Gamma^\infty(E)$ such that

$$\|s\|_{L^q(E)} \doteq \left(\int_X \|s(x)\|^q_{E_x}\,dx\right)^{\frac{1}{q}} < \infty. \tag{6.23}$$

Now, consider G be a compact Lie group and K a closed subgroup of G. Let $M = G/K$

be equipped with its natural compact manifold topology. There exists a natural left action of G on M given by $g \cdot hK = ghK$, for every $g, h \in G$. We say that a vector bundle $p : E \to M$ is a homogeneous vector bundle over M if G acts on E on the left and this action satisfies:

1. $g \cdot E_x = E_{gx}$, for all $x \in M$, $g \in G$;

2. The previously induced mappings from E_x to E_{gx} are linear.

There is natural left action of G on $\Gamma(E)$, $G \times \Gamma(E) \to \Gamma(E)$ given by

$$(g \cdot s)(x) = g \cdot s(g^{-1}x),$$

for all $x \in X$, $g \in G$. For a homogeneous vector bundle $p : E \to M$, let $E_0 = p^{-1}(K)$, be the fiber at the identity coset. As shown in [22], there exists $\tau \in \mathrm{Hom}(K, \mathrm{End}(E_0))$, and a natural right action of K on $G \times E_0$ by $(g, v) = (gk, \tau^{-1}(k)v)$. We denote by $G \times_\tau E_0$ the quotient $(G \times E_0)/K$ under this action. Bott also shows that $G \times_\tau E_0$ admits a natural homogeneous vector bundle structure, and that in fact there exists $\tau \in \mathrm{Hom}(K, \mathrm{End}(E_0))$ such that

$$E \cong G \times_\tau E_0,$$

as vector bundles. Now consider the vector subspace $C^\infty(G, E_0)^\tau \subset C^\infty(G, E_0)$ given by

$$C^\infty(G, E_0)^\tau = \{f \in C^\infty(G, E_0) | \forall g \in G, \forall k \in K, f(gk) = \tau(k)^{-1} f(g)\},$$

and likewise $L^2(G, E_0)^\tau \subset L^2(G, E_0)$ by

$$L^2(G, E_0)^\tau = \{f \in L^2(G, E_0) | f(gk) = \tau(k)^{-1} f(g) \text{ for a. e. } g, k \in G\}.$$

It can be shown that the bijection $\chi_\tau : \Gamma^\infty(E) \to C^\infty(G, E_0)^\tau$, given by

$$\chi_\tau(s)(g) \doteq g^{-1} \cdot s(gK),$$

for every $g \in G$, extends to a surjective isometry from $L^2(E)$ into $L^2(G, E_0)^\tau$. Therefore we identify $\Gamma^\infty(E)$ and $L^2(E)$ with $C^\infty(G, E_0)^\tau$ and $L^2(G, E_0)^\tau$, respectively. The Sobolev space $H^s(E)$ for $s \in \mathbb{R}$ is then defined as the completion of the set of smooth sections under the norm

$$\|u\|_{H^s(E)} \doteq \|\chi_\tau u\|_{H^s(G,E_0)}.$$

Now let $\tilde{A} : \Gamma^\infty(E) \to \Gamma^\infty(E)$ be a continuous linear operator. Then $\tilde{A}$ induces a continuous linear map $A : C^\infty(G, E_0)^\tau \to C^\infty(G, E_0)^\tau$ by

$$A = \chi_\tau \circ \tilde{A} \circ \chi_\tau^{-1}.$$

If also $A \in \Psi^m_{\rho,\delta}((G \times \widehat{G}) \otimes \mathrm{End}(E_0))$, we say that $\tilde{A} \in \Psi^m_{\rho,\delta}(E)$, and define its symbol by $\sigma_{\tilde{A}} \doteq \sigma_A$, where σ_A is the matrix-valued symbol defined in (6.1). The quantization formula (6.2) then implies

$$\tilde{A}s(gK) = \chi_\tau^{-1}\left(\sum_{i,r=1}^{d_\tau}\sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}\left(\xi(x)\sigma_{\tilde{A}}(i,r,g,\xi)\widehat{\chi_\tau s}(i,\xi)\right)\otimes e_i\right),$$

where $\{e_i\}_{i=1}^{d_\tau}$ is an orthonormal basis of E_0. As a consequence of Theorem 6.3.1, we obtain the following result.

Corollary 6.4.1. *Let $p : E \to M = G/K$ be a homogeneous vector bundle over a compact homogeneous manifold M, where $K < G$ are compact Lie groups, $E \cong G \times_\tau E_0$. Let $\tilde{A} \in \Psi^m_{\rho,\delta}(E)$, $0 \leq \delta < \rho \leq 1$, be such that its matrix-valued symbol $\sigma_A(x,\xi)$ is positive semi-definite for every $(x,[\xi]) \in G \times \widehat{G}$, in the sense of Theorem 6.3.1. Then there exists $C > 0$ such that*

$$\mathrm{Re}\langle \tilde{A}s, s\rangle_{L^2(E)} \geq -C\|s\|^2_{H^{\frac{m-(\rho-\delta)}{2}}(E)},$$

for every $s \in \Gamma^\infty(E)$, where $\langle\cdot,\cdot\rangle_{L^2(E)}$ denotes the canonical inner product on $L^2(E)$ and $H^r(E)$ denotes the Sobolev space over E of order $r \in \mathbb{R}$.

Proof. Let $A = \mathrm{Op}(\sigma_A)$ be the vector-valued operator associated with $\tilde{A}$. For $s \in \Gamma^\infty(E)$, set $u = \chi_\tau s$. Theorem 6.3.1 implies that there exists $C > 0$ such that

$$\begin{aligned}\mathrm{Re}\langle Au, u\rangle_{L^2(G,E_0)} &\geq -C\|u\|^2_{H^{\frac{m-(\rho-\delta)}{2}}(G,E_0)}\\ &= -C\|s\|^2_{H^{\frac{m-(\rho-\delta)}{2}}(E)}.\end{aligned}$$

The result then follows from the fact that

$$\begin{aligned}\mathrm{Re}\langle \tilde{A}s, s\rangle_{L^2(E)} &= \mathrm{Re}\langle \chi_\tau^{-1} A \chi_\tau \chi_\tau^{-1} u, \chi_\tau^{-1} u\rangle_{L^2(E)} \\ &= \mathrm{Re}\langle A\chi_\tau \chi_\tau^{-1} u, u\rangle_{L^2(G,E_0)} = \mathrm{Re}\langle Au, u\rangle_{L^2(G,E_0)},\end{aligned}$$

where we have used that $\chi_\tau^{-1} : L^2(G, E_0)^\tau \to L^2(E)$ is an isometry. □

Finally, note that if $E = G \times_{\mathbb{1}} \mathbb{C} \cong M$ is the trivial bundle (where $\mathbb{1} : K \to \mathbb{C}$ is the trivial representation), then we can identify $\Gamma^\infty(E) \cong C^\infty(M)$, and $\chi_{\mathbb{1}} : C^\infty(M) \to C^\infty(G)^K$ is just the projective lifting defined by

$$\chi_{\mathbb{1}} f(g) \equiv \dot{f}(g) \doteq f(gK),$$

for every $f \in C^\infty(M)$, and $g \in G$. The pseudo-differential operator classes and Sobolev spaces of M are then defined likewise. As an immediate consequence of the previous result, we obtain the following corollary.

Corollary 6.4.2. *Let $M = G/K$ be a compact homogeneous manifold, where $K < G$ are compact Lie groups. Let $\tilde{A} \in \Psi^m_{\rho,\delta}(M)$, $0 \leq \delta < \rho \leq 1$, be such that its matrix valued symbol $\sigma_A = \sigma_A(x,\xi) \in S^m_{\rho,\delta}(G)$ is positive semi-definite for every $(x, [\xi]) \in G \times \widehat{G}$. Then there exists $C > 0$ such that*

$$\mathrm{Re}\langle \tilde{A}u, u\rangle_{L^2(M)} \geq -C\|u\|^2_{H^{\frac{m-(\rho-\delta)}{2}}(M)},$$

for every $u \in C^\infty(M)$.

Part III

Conclusion

Chapter 7

Conclusion

We conclude this thesis by summarizing its main contributions and potential future research.

In the first part we presented a concise notation for the Fourier analysis on a general product of compact Lie groups. We then provided an application of such notation in the study of global hypoellipticity and global solvability of a broad class of first-order differential operators defined on a product of compact Lie groups. We remark that this application is of theoretical importance since it generalizes certain previous results and further abstracts much of the theory already established on the torus.

In the second part we generalized the validity of the important and useful "sharp Gårding" inequality on compact Lie groups from the scalar-valued setting to the vector-valued setting. This new result certainly will be useful in generalizing many other results in analysis from the scalar-valued to the vector-valued setting. This of course is of importance as there are many important uses for vector-valued functions in mathematics and physics. We also proved the new result that such a "Gårding" inequality also holds in compact homogeneous spaces. This is quite relevant as many important manifolds are included in these spaces, such as the n-dimensional sphere, for any $n \in \mathbb{N}$. Finally, we also generalized the concept of amplitudes and amplitude operators from the scalar-valued compact Lie group setting to the vector-valued setting. This could be useful in obtaining new results on pseudo-differential operators in this setting, which have many applications in mathematics and physics.

With respect to potential further future projects in these subjects, perhaps the most natural next step would be to study the global hypoellipticity and global solvability of left-invariant pseudo-differential operators on the product of compact Lie groups. The reason for that is, as observed in Remark 2.2.20, that these operators also act as Fourier multipliers in this setting. Therefore similar techniques could possibly be used to study these operators. Another possibility is to study the many other types of global hypoellipticity and global solvability present in the literature to the same class of operators considered in this thesis. In the context of the sharp Gårding inequality, there are many opportunities for further research in its applications in analysis, such as local solvability (see [105]) and in proving well-posedness of certain Cauchy problems (see [72], [83]).

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