UNIVERSIDADE FEDERAL DO PARANÁ

JOÃO PEDRO LASS KAMINSKI

RELATIVISTIC QUANTUM RESOURCES: ANALYSIS OF ENTANGLEMENT AND QUANTUM STEERING

CURITIBA

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Trabalho apresentado como requisito parcial para a obtenção do título de Mestre em Física pelo Programa de Pós Graduação em Física do Setor de Ciências Exatas da Universidade Federal do Paraná.

Orientadora: Ana Cristina Sprotte Costa

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No dia vinte e oito de fevereiro de dois mil e vinte e quatro às 14:00 horas, na sala Teams da Microsoft, Videoconferência, foram instaladas as atividades pertinentes ao rito de defesa de dissertação do mestrando **JOÃO PEDRO LASS KAMINSKI**, intitulada**: "Relativistic Quantum Resources: Analysis of Entanglement and Quantum Steering"**, sob orientação da Profa. Dra. ANA CRISTINA SPROTTE COSTA. A Banca Examinadora, designada pelo Colegiado do Programa de Pós-Graduação FÍSICA da Universidade Federal do Paraná, foi constituída pelos seguintes Membros: ANA CRISTINA SPROTTE COSTA (UNIVERSIDADE FEDERAL DO PARANÁ), JONAS MAZIERO (UNIVERSIDADE FEDERAL DE SANTA MARIA), IBERE OLIVEIRA KUNTZ DE SOUZA (UNIVERSIDADE FEDERAL DO PARANÁ). A presidência iniciou os ritos definidos pelo Colegiado do Programa e, após exarados os pareceres dos membros do comitê examinador e da respectiva contra argumentação, ocorreu a leitura do parecer final da banca examinadora, que decidiu pela APROVAÇÃO. Este resultado deverá ser homologado pelo Colegiado do programa, mediante o atendimento de todas as indicações e correções solicitadas pela banca dentro dos prazos regimentais definidos pelo programa. A outorga de título de mestre está condicionada ao atendimento de todos os requisitos e prazos determinados no regimento do Programa de Pós-Graduação. Nada mais havendo a tratar a presidência deu por encerrada a sessão, da qual eu, ANA CRISTINA SPROTTE COSTA, lavrei a presente ata, que vai assinada por mim e pelos demais membros da Comissão Examinadora.

CURITIBA, 28 de Fevereiro de 2024.

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TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação FÍSICA da Universidade Federal do Paraná foram convocados para realizar a arguição da Dissertação de Mestrado de **JOÃO PEDRO LASS KAMINSKI** intitulada: **"Relativistic Quantum Resources: Analysis of Entanglement and Quantum Steering"**, sob orientação da Profa. Dra. ANA CRISTINA SPROTTE COSTA, que após terem inquirido o aluno e realizada a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.
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This work is dedicated to the indomitable human spirit and the love that fuels it.

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*"Aristotle said a bunch of stuff that was wrong. Galileo and Newton fixed things up. Then Einstein broke everything again. Now, we've basically got it all worked out, except for **small stuff***, *big stuff, hot stuff, cold stuff,* ***fast stuff****, heavy stuff, dark stuff, turbulence, and the* ***concept of time****" .*

(Zach Weinersmith, Science: Abridged Beyond the Point of Usefulness)

**Emphases made by the author of this work.*

RESUMO

Avanços recentes na área de informação quântica parecem indicar que emaranhamento, um dos recursos quânticos mais fundamentais, não é uma quantidade covariante. Isto é comumente explicado como consequência da manifestação quântica de um fenômeno da relatividade especial, as rotações de Wigner, que surgem na composição de dois boosts de Lorentz não paralelos. Se dois observadores inerciais não concordarem sobre a quantidade de emaranhamento presente num sistema físico, eles podem concordar em quão direcionável (se direcionável) esse sistema é? Este trabalho se propõe a responder essa pergunta para partículas massivas de spin 1/2. Os resultados indicam que, apesar de direcionável, as correlações quânticas presentes no sistema são limitadas por cima pela quantidade de informação armazenável no espaço do grau de liberdade de menor dimensão, isto é, no spin. Isto motiva mais discussões sobre as supossições fundamentais utilizados para alcançar os resultados, suposições estas que são, muitas vezes, dadas como certas no formalismo em que esse trabalho se baseia. Uma discussão profunda sobre o tópico e possibilidades de novos caminhos a serem seguidos também se encontram neste trabalho.

Palavras-chaves: Relatividade; Transformações de Lorentz; Mecânica Quântica; Informação Quântica; Recursos Quânticos.

ABSTRACT

Recent advances in the field of quantum information suggest that entanglement, one of the most fundamental quantum resources, is not a covariant quantity. This is often attributed to the quantum manifestation of a known phenomenon of special relativity, the Wigner rotations, which arise from the composition of two nonparallel Lorentz boosts. If two inertial observers cannot agree on a system's amount of entanglement, may they agree on how steerable (if so) the system is? This work intends to answer this question for massive spin 1/2 particles. The results indicate that although steerable, the system's quantum correlations are upper-bounded by the amount of information storable in the space of the smaller dimensioned degree of freedom, namely the spin space. This motivates further discussion on the fundamental suppositions used to reach the results, suppositions that are, often, taken for granted in the formalism this work is based on. A deep discussion about the topic and suggestions of paths to be taken in the future can also be found within.

Key-words: Relativity; Lorentz Transformations; Quantum Mechanics; Quantum Information; Quantum Resources.

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SUMMARY

1 INTRODUCTION

The 20^{th} century gave physics two prodigal daughters, Quantum Mechanics (QM) and Relativity, both Special (SR) and General (GR). The former is widely regarded as the most successful theory of nature, surviving rigorous experimental scrutiny and giving rise to some of the most important technology of our time. QM, when put together with SR, gave rise to Quantum Field Theory (WEINBERG, 1995), being responsible for some of the most precise, already experimentally verified, predictions in history. Although very popular, there are many problems not still solved by such theories.

For instance, GR and QM are not yet fully integrated since no Quantum Gravity theory, that work in all energy regimes, has been successfully developed. This represents one of the biggest, if not the biggest, challenge on the frontier of human knowledge. As such, this work shall avoid problems where gravity cannot be disregarded. Also, even within QM it is not all roses, as it turns out, no specific interpretation of the theory has been singled out and what it truly says about the universe is still debated and controversial. In this controversy another connection between QM and relativity can be made, although indirect: the most avid critic of QM is also the most important figure to the development of Relativity. No other than Albert Einstein, together with Podolsky and Rosen (EPR) published a paper in 1935 (EINSTEIN et al., 1935) that questioned the completeness of QM. Intending to verify the validity of EPR's claim, John Stewart Bell derived his famous theorem (BELL, 1964) and ended up demonstrating one of the most fundamental yet non-trivial properties of QM: it cannot be simultaneously real, that is, with defined underlying values for all degrees of freedom, independent of measurement, and local, meaning that all information regarding a physical system is contained within it.

Another conundrum, of more recent appearance, materialized when researchers realized that, on one hand, SR requires that "The laws of physics must be the same in all inertial reference frames" while, on the other, it has been shown that entanglement, a very fundamental quantum resource, varies with Lorentz transformations[1]. This phenomenon was first demonstrated in the 2002 paper by Peres *et al.* (PERES et al., 2002), where the authors proved that, after a boost, the entropy of the reduced state of an electron's spin, varied. This fact indicates that the amount of entanglement between the spin and momentum degrees of freedom of a particle is dependent on the boost that connects different inertial frames. This is in contradiction with SR, at least if entanglement is to be considered as fundamental as any other, already known to be covariant quantity (ENGELBERT, 2022). Since then, many works investigated boost-

[1] Such transformations connect inertial frames of reference and are staples of special relativity.

induced entanglement, some for single-particle systems and others for more complex systems containing two or more particles, even extensions for similar phenomena in GR have been proposed (DUNNINGHAM et al., 2009; CZACHOR, 1997; AHN et al., 2003; STREITER et al., 2021; PETRECA et al., 2022; BASSO; MAZIERO, 2021a,b, 2022).

The main goal of this work is to introduce both theories up to the point where the previous conundrum can be understood, followed by a discussion about possible solutions and their very existence. Then, after understanding the conceptual difficulties associated with spin-momentum entanglement, we will use quantum steering, a measure of quantum correlations where one of the parties may be untrusted, to certify such correlations in the states of interest. For that purpose, a physical system will be modeled with the necessary characteristics for entanglement detection in relativistic systems. After demonstrating such a process, we will consider more realistic models for the physical system that are also compatible with both Reid's variance (REID, 1989) and the entropic uncertainty relation (SCHNEELOCH et al., 2013) criteria for steering. Each of the chosen criteria has its advantages and disadvantages, as will be demonstrated.

Many problems were encountered on the path towards obtaining the results. Steering was not detected using the variance criterion. The reasons for this were somewhat understood. On the other hand, steering was certified using the entropic uncertainty criterion and proven to be dependent on the state preparation parameters, although no clear analytical dependence on them was successfully established. In sum, in the relativistic regime, the statistical behavior of the physical system seems to change. We use the quantum resource formalism to demonstrate as much, formulating in the process a "relativistic quantum resource" formalism, more appropriate for the work at hand. Due to the intrinsic difficulties of the work, it closes with a discussion about such challenges and the way one might, as we did, go around them, without losing the ability to describe nature with rigor.

2 QUANTUM MECHANICAL FORMALISM

This chapter's purpose is to introduce the standard mathematical tools used in the construction of QM. Here lies the definition of later used concepts such as states, both pure and mixed, how to define them over a Hilbert space, and how to describe such spaces for both discrete and continuous degrees of freedom. Later, the definition of the relevant quantum resources for this work, entanglement and quantum steering as well as the criteria for their detection appear. Next, some examples that should make these higher concepts better understandable appear. Most of the concepts presented in this chapter are found in a much more extensively explored form in the textbook "Quantum Computation and Quantum Information" (NIELSEN; CHUANG, 2010).

2.1 QUANTUMNESS 101

In QM, all information contained in a physical system is in its *state*[1], hereafter denoted by $|\psi\rangle$. Here the "ket" notation was used to indicate that the state is represented as a vector in a Hilbert space, such space being defined as

Definition 1. A Hilbert space $\mathcal{H}$ is a complex inner product space that is complete with respect to the norm induced by the inner product. That is, $\mathcal{H}$ is a vector space over the field of complex numbers $\mathbb{C}$ equipped with an inner product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H}\to\mathbb{C}$. Here $\langle\cdot,\cdot\rangle$ satisfies the following properties for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathcal{H}$ and all $a\in\mathbb{C}$:

1. **Antilinearity in the second argument:** $\langle\mathbf{u}, a\mathbf{v}+b\mathbf{w}\rangle = a^*\langle\mathbf{u},\mathbf{v}\rangle + b^*\langle\mathbf{u},\mathbf{w}\rangle$.
2. **Conjugate symmetry:** $\langle\mathbf{u},\mathbf{v}\rangle = \langle\mathbf{v},\mathbf{u}\rangle^*$.
3. **Positive-definiteness:** $\langle\mathbf{u},\mathbf{u}\rangle\geq 0$ for all $\mathbf{u}\in\mathcal{H}$, and $\langle\mathbf{u},\mathbf{u}\rangle = 0$ *iff* $\mathbf{u}=\mathbf{0}$.

A space is complete if every Cauchy sequence[2] in it converges to one of its elements in the norm induced by the inner product.

[1] Here the word "contained" could be exchanged by "accessible" without loss of meaning. This is so because if a system has any information that is not accessible one cannot prove that such information exists, therefore, it is not of physical significance.

[2] A sequence $(a_n)_{n=1}^{\infty}$ of elements in a metric space (X,d) is called a Cauchy sequence if, for any positive number $\epsilon>0$, there exists a positive integer N such that $d(a_n,a_m)<\epsilon$ for all $n,m\geq N$. In other words, the terms of the sequence become arbitrarily close to each other as n and m becomes large.

Being QM a physical theory, it must allow the connection between the mathematical formalism and observable phenomena in the physical world. Such connection is done by the expectedly named *observables*, defined below.

Definition 2. In QM, an *observable* is a Hermitian operator $A = A^\dagger$ acting on the Hilbert space $\mathcal{H}$ of states. That is, for any state $|\psi\rangle \in \mathcal{H}$, the possible outcomes of measuring the observable A on $|\psi\rangle$ are the eigenvalues of A that we denote by λ. The probability of obtaining each eigenvalue is given by the Born rule:

- For an eigenvalue λ of A and eigenvectors $|\phi_i\rangle$ corresponding to λ, the probability of obtaining λ when A is measured on the prepared state $|\psi\rangle$, is given by

$$p(\lambda) = \sum_i |\langle\phi_i|\psi\rangle|^2. \tag{2.1}$$

  Here, if the state is not degenerate in relation to the eigenvalue, the sum is unneeded. A slight notation adjustment was done, when denoting the inner product between states, from here on out $\langle\cdot|\cdot\rangle$ will be used instead of $\langle\cdot,\cdot\rangle$.

2.1.1 Discrete Observables

Since A is Hermitian, its eigenvalues are all real. Also, its eigenvectors form a complete orthonormal basis of $\mathcal{H}$. So, any state $|\psi\rangle$ can be written as a linear combination of its eigenvectors:

$$|\psi\rangle = \sum_i c_i |\phi_\lambda\rangle . \tag{2.2}$$

In (2.2), c_i are complex coefficients.

Observables that live in countable spaces[3] such as A are called discrete. Since the basis $\{|\phi_\lambda\rangle\}$ of A is, by construction, orthogonal, it obeys

$$\langle\phi_j|\phi_i\rangle = \delta_{ji}. \tag{2.3}$$

And, since it is complete, we have

$$\sum_\lambda^d |\phi_\lambda\rangle\langle\phi_\lambda| = \mathbb{I}, \tag{2.4}$$

where d is the dimension of the space where A is defined on. Since all physical information is "extracted" from a physical system using observables, it is most common to represent any state on an observable's basis. As long as one chooses any observable of the same Hilbert space to represent the state, all information will be contained in

[3] Spaces with observables that have countable eigenvalues.

such state. But, sometimes, choosing the right basis helps in reading that information. However let us not get ahead of ourselves: Firstly let the definition of a projector be written in the following definition

Definition 3. Any state-vector paired with its dual, $|\psi\rangle\langle\psi|$ is called a *projector*. When projectors are written based on the eigenvectors of an operator, such as $\mathcal{A}_\lambda := |\phi_\lambda\rangle\langle\phi_\lambda|$, they are called projectors of that operator. Projectors are useful tools that obey certain properties.

- A projector $\mathcal{P}$ is idempotent, which means that applying the projector twice is equivalent to applying it once. Mathematically,

$$\mathcal{P}^2 = |\psi\rangle \underbrace{\langle\psi|\psi\rangle}_{=\mathbb{I}} \langle\psi| = |\psi\rangle\langle\psi| = \mathcal{P}. \tag{2.5}$$

- The eigenvalues of $\mathcal{A}_\lambda$ are either 0 or 1. This property allows the projector to select specific subspaces by projecting onto eigenspaces associated with eigenvalues equal to 1.

Geometrically, the action of a projector can be visualized as "projecting" a vector onto a subspace spanned by its eigenvectors associated with eigenvalues of 1, while orthogonalizing it to the subspace associated with eigenvalues of 0.

Going further into this work most operations will be written in terms of projectors. Therefore, familiarizing oneself with them is primal to a good understanding of it. In this sense, one may rewrite some of the previous relations using them:

$$A = \sum_i^d \lambda_i \mathcal{A}_i; \tag{2.6a}$$

$$\sum_i \mathcal{A}_i = \mathbb{I}; \tag{2.6b}$$

$$\mathcal{A}_i \mathcal{A}_j = \delta_{ij} \mathcal{A}_i; \tag{2.6c}$$

$$p(\lambda) = \langle\psi| \mathcal{A}_\lambda |\psi\rangle. \tag{2.6d}$$

The relation in (2.6c) is valid only for orthogonal projectors. Those are not the most general, but are the most useful. From now on all projectors are of such kind. One may also use the projectors to calculate the observable's mean value, such as $\langle A\rangle$ that is given by

$$\langle A\rangle = \sum_i^d \lambda_i p(\lambda_i) = \sum_i^d \langle\psi| \lambda_i \mathcal{A}_\lambda |\psi\rangle = \langle\psi| A |\psi\rangle. \tag{2.7}$$

Here we might do a small sanity check of this formalism. If QM is to be considered a consistent probabilistic theory, the chance of all possible outcomes of a measurement must add to unity. As it turns out

$$\begin{aligned}\sum_i^d p(\lambda_i) &\overset{(2.6d)}{=} \sum_i^d \langle\psi| \mathcal{A}_i |\psi\rangle \\ &= \langle\psi| \left(\sum_i^d \mathcal{A}_i\right) |\psi\rangle \\ &\overset{(2.6a)}{=} \langle\psi|\psi\rangle = 1,\end{aligned} \tag{2.8}$$

guaranteeing that, as far as statistical consistency is regarded, QM presents no problems.

2.1.2 Continuous Observables

This work focuses on systems characterized by both discrete and continuous degrees of freedom (DoF). In the standard formulation of QM, the mathematical framework primarily revolves around discrete observables, such as discrete energy levels or spin states. In the previous section, the basics for discrete DoF have been laid out. However, the physical systems of concern, particles in continuous position or momentum spaces, require a more comprehensive treatment that extends the mathematical tools to the continuous case.

The study of continuous degrees of freedom introduces various challenges compared to discrete ones. In the continuous case, observables associated with position, momentum, and other continuous quantities cannot be represented by discrete values or eigenstates as in the discrete case. Instead, these observables span an infinite-dimensional non-countable space (dim$(\mathcal{H}) = \infty$), making the development of new mathematical techniques necessary.

By extending the mathematical tools from discrete observables to the continuous case, one can gain a deeper understanding of the behavior and the properties of actual quantum systems. This extension allows one to explore phenomena such as wave-particle duality, interference, and superposition in systems characterized by continuous variables.

In this section, the mathematical formalism required to handle continuous observables in QM is laid down starting with position: Each component of a particle's position vector has its own Hilbert space. That is, if $\vec{R}$ is its position operator, then $\vec{R} : \mathcal{H}_r \to \mathcal{H}_r$ where $\mathcal{H}_r = \mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_z$. Let one begin with one of these spaces, $\mathcal{H}_y$ for instance, to then, further ahead, extend to the more general $\mathcal{H}_r$ space.

In $\mathcal{H}_y$, the operator that defines the y-axis position of a particle is simply denoted by Y and its eigenvectors are $|y\rangle$, that correspond to eigenvalues $Y = y$. As expected, one may construct an eigenvalue relation with those two:

$$Y\left|y\right\rangle = y\left|y\right\rangle, \tag{2.9}$$

where the eigenvalues y range from $-\infty$ to ∞. All eigenvectors of Y are still orthogonal and form an orthonormal basis[4], both requirements of QM. That is,

$$\langle y|y'\rangle = \delta(y - y'); \tag{2.10a}$$

$$\int_{-\infty}^{\infty} dy \left|y\right\rangle \left\langle y\right| = \mathbb{I}. \tag{2.10b}$$

Here $\delta(y - y')$ is the so called Dirac-delta function (actually a distribution). This function is defined as,

$$\delta(y - y') = \begin{cases} \infty, & \text{if } y = y' \\ 0, & \text{if } y \neq y' \end{cases}. \tag{2.11}$$

Based on this, the generic state $|\psi\rangle$ may be expanded in the position basis as

$$|\psi\rangle = \int_{-\infty}^{\infty} dy\ \psi(y) \left|y\right\rangle. \tag{2.12}$$

Here $\psi(y) := \langle y|\psi\rangle$ is just a coefficient, if all the coefficients in $-\infty < y < \infty$ can be associated with a function of y, then $\psi(y)$ is called the state's *wave function*.

One may write the Y operator using projectors as well, the difference is that such projectors can take the state into infinitesimally small spaces, so they must themselves be infinitesimal. To obtain them, one can go from discrete to continuous in the following manner,

$$A = \sum_{i}^{d} \lambda_i \left|\phi_i\right\rangle \left\langle\phi_i\right| \xrightarrow{d\to\infty} Y = \int_{-\infty}^{\infty} dy\ y\ \left|y\right\rangle \left\langle y\right|. \tag{2.13}$$

Being this the case, clearly, the continuous projectors take the form of $\left|y\right\rangle \left\langle y\right| dy$. Having this definition in hand, one might obtain the Born rule for the continuous case, being it defined on an infinitesimal probability $dp(y)$, *i.e.*

$$dp(y) = \left\langle\psi\right| (\left|y\right\rangle \left\langle y\right| dy) \left|\psi\right\rangle = \langle\psi|y\rangle \langle y|\psi\rangle\, dy = |\psi(y)|^2 dy. \tag{2.14}$$

Equipped with this infinitesimal probability, one can extend (2.14) for any interval, such as $y \in [a, b]$, by means of

$$p(y \in [a, b]) = \int_a^b |\psi(y)|^2\ dy. \tag{2.15}$$

By remembering the QM requirement that probabilities add to unity (or integrate to unity, in this case), it's clear that $\int_{-\infty}^{\infty} dy\ p(y) = 1$.

[4] Normality is required so probabilities can be infered from the state. Orthogonality is required so the states are physically distinguishable.

As it turns out, the position operator is not the only operator of interest in the so-called spatial Hilbert space. The momentum operator P also lives there and has quite an interesting relation with the aforementioned position operator. Before going further on the properties of P, let's talk about one of the first and most revolutionary discoveries of QM: Heisenberg's uncertainty principle.

2.1.2.1 Heisenberg's uncertainty principle

The principle is well encapsulated in the maxim "To know all about position is to know nothing about momentum, and vice-versa" (HEISENBERG, 1927). One can take this to mean that a system in a position eigenstate is in a complete superposition[5] of all possible momentum eigenstates of the same space. Or, in layman's terms, that a measurement of the system position erases any knowledge about its former momentum, being the degree of erasure proportional to the precision of the measurement.

Enough being said about it, how does the principle manifest mathematically? Let's find out, beginning with the definition of the standard deviation of an observable A:

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \rightarrow \Delta A^2 = \langle A^2 \rangle - \langle A \rangle^2 . \tag{2.16}$$

In plain English, ΔA^2 measures the average square difference from the mean $\langle A \rangle$. Now, we must first note that

$$\begin{aligned} \langle (A - \langle A \rangle)^2 \rangle &= \langle A^2 - 2A \langle A \rangle + A^2 \rangle \\ &= \langle A^2 \rangle - 2 \langle A \langle A \rangle \rangle + \langle A \rangle^2 \\ &= \langle A^2 \rangle - 2 \langle A \rangle \langle A \rangle + \langle A \rangle^2 \\ &= \langle A^2 \rangle - \langle A \rangle^2 . \end{aligned} \tag{2.17}$$

Therefore $\Delta A^2 = \langle (A - \langle A \rangle)^2 \rangle := \sigma_A^2$. The consequence of this fact that matters for this work is that, being A Hermitian ($A = A^\dagger$), we can define

$$(A - \langle A \rangle) |\psi\rangle = |k\rangle , \tag{2.18a}$$

$$\langle \psi | (A - \langle A \rangle) = \langle k | , \tag{2.18b}$$

such that

$$\langle (A - \langle A \rangle)^2 \rangle = \langle \psi | (A - \langle A \rangle)(A - \langle A \rangle) |\psi\rangle = \langle k | k \rangle . \tag{2.19}$$

Knowing that both the X and P operators are Hermitian, we can use the relations in (2.18) and (2.19) to write

$$\sigma_Y^2 = \langle (Y - \langle Y \rangle)^2 \rangle = \langle f | f \rangle ; \tag{2.20a}$$

$$\sigma_{PY}^2 = \langle (P_Y - \langle P_Y \rangle)^2 \rangle = \langle g | g \rangle . \tag{2.20b}$$

[5] Meaning equiprobability for all eigenstates of momentum.

Having the (2.20) equations in hand we can use the Cauchy-Schwarz inequality (MINCULETE, 2021) as

$$\sigma_Y^2 \sigma_{PY}^2 = \langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2. \tag{2.21}$$

Now by taking a better look at the $\langle f|g\rangle$ one may realize that

$$\begin{aligned}
\langle f|g\rangle &= \langle (Y - \langle Y\rangle)(P_Y - \langle P_Y\rangle)\rangle \\
&= \langle (YP_Y - Y\langle P_Y\rangle - \langle Y\rangle P_Y + \langle Y\rangle \langle P_Y\rangle)\rangle \\
&= \langle YP_y\rangle - \langle Y\rangle \langle P_Y\rangle - \langle Y\rangle \langle P_Y\rangle + \langle Y\rangle \langle P_Y\rangle \\
&= \langle YP_y\rangle - \langle Y\rangle \langle P_Y\rangle \\
&= \langle YP_Y\rangle - \langle P_Y\rangle \langle Y\rangle .
\end{aligned} \tag{2.22}$$

The last property we must import from mathematics is the following inequality that complex numbers obey:

$$z^* z \geq \left[\frac{1}{2i}(z - z^*)\right]^2. \tag{2.23}$$

Substituting z for $\langle f|g\rangle$ and z^* for $\langle g|f\rangle$ in (2.23), together with (2.21) one gets

$$\begin{aligned}
&\sigma_Y^2 \sigma_{P_Y}^2 = \langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2 = \langle g|f\rangle \langle f|g\rangle \geq \left[\frac{1}{2i}(\langle f|g\rangle - \langle g|f\rangle)\right]^2 \therefore \\
&\sigma_Y^2 \sigma_{P_Y}^2 \geq \left[\frac{1}{2i}(\langle f|g\rangle - \langle g|f\rangle)\right]^2.
\end{aligned} \tag{2.24}$$

The final step in the deduction of the general Heisenberg uncertainty principle, the so-called Robertson relation (ROBERTSON, 1929), is the realization that

$$\begin{aligned}
\langle f|g\rangle - \langle g|f\rangle &= (\langle YP_Y\rangle - \langle P_Y\rangle \langle Y\rangle) - (\langle P_Y Y\rangle - \langle P_Y\rangle \langle Y\rangle) \\
&= \langle YP_Y\rangle - \langle P_Y Y\rangle \\
&= \langle (YP_Y - P_Y Y)\rangle \\
&= \langle [Y, P_Y]\rangle .
\end{aligned} \tag{2.25}$$

such that (2.24) becomes

$$\sigma_Y^2 \sigma_{P_Y}^2 \geq \left[\frac{1}{2i}\langle [Y, P_Y]\rangle\right]^2. \tag{2.26}$$

If one adds to the previous relation the canonical commutation relation $[Y, P_Y] = i\hbar\mathbb{I}$, the most widely known form of the principle comes out:

$$\sigma_Y^2 \sigma_{P_Y}^2 \geq \left[\frac{1}{2i} i\hbar\right]^2 = \left(\frac{\hbar}{2}\right)^2 = \left(\frac{h}{4\pi}\right)^2 \to \sigma_Y \sigma_{P_Y} \geq \frac{h}{4\pi}. \tag{2.27}$$

Although the relation in (2.27) is more famous, the one in (2.26) is more useful because it is valid for any two operators that have the same restrictions we imposed on Y and P_Y, being such restrictions that they are Hermitian. Therefore any two Hermitian operators will obey (2.26).

2.1.3 Consequences for Momentum

This slight tangent into Heisenberg's uncertainty principle had as objective the demonstration of the fact that in QM observables that are not compatible[6] can live in the same Hilbert space and consequently act on the same states. This means that QM dictates that one cannot know a system's position and momentum with total precision at any time. This is in total contradiction with Classical Mechanics, where the dynamics of a physical system is exactly the process of total description of the time evolution of such DoF.

Furthermore, momentum and position are maximally incompatible, which means that total knowledge about one requires total ignorance about the other. As a consequence, the respective operator bases form a pair of Mutually Unbiased Bases (MUB)[7]. Other consequences of the facts just shown are that, if the $\psi(y)$ is the position wave-function, then a momentum wave-function $\overline{\psi}(p_y)$ can be obtained via the Fourier transform of $\psi(y)$:

$$\overline{\psi}(p_y) = \frac{1}{N}\int_{-\infty}^{\infty} dy\; e^{i\frac{p_y y}{\hbar}}\psi(y), \tag{2.28}$$

where $N = (2\pi\hbar)^{-1/2}$ guarantees normalization. Being the previous equation true, the relationship between the bases can be summed up as $\langle y|p_y\rangle = (2\pi\hbar)^{-1/2}e^{iyp_y/\hbar}$ (RIOS, 2018).

2.1.4 Composite Systems

Since each independent DoF has its own Hilbert space in QM, to fully describe a physical system one must be able to describe many DoF together. The way the theory does it is by composing the many spaces via the tensor product $\otimes$. If the system has a DoF A associated with the space $\mathcal{H}_A$, and another B with its space $\mathcal{H}_B$, then the system's Hilbert space, $\mathcal{H}_S$, can be fully described by $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$. Furthermore, if $\{|a_i\rangle\}$ is a basis of $\mathcal{H}_A$ and $\{|b_j\rangle\}$ a basis of $\mathcal{H}_B$, then $\{|a_i\rangle \otimes |b_j\rangle := |a_i\rangle\,|b_j\rangle = |a_i, b_j\rangle\}$ is a basis of $\mathcal{H}_S$. A property of the tensor product is such that if $dim(\mathcal{H}_A) = d_A$ and $dim(\mathcal{H}_B) = d_B$, then $dim(\mathcal{H}_S) = d_A d_B$. Equipped with all this information, one can write a general state of H_S as

$$|\psi\rangle = \sum_{i=1}^{d_A}\sum_{j=1}^{d_B} c_{ij}\,|a_i\rangle\,|b_j\rangle\,. \tag{2.29}$$

The importance of the coefficients c_{ij} cannot be overstated, being the case where $c_{ij} = c_i c_j$ only a special case. Being this true, the state is called separable and can be

[6] Compatibility here takes the meaning of being able to do sequential measurements where the former does not affect the latter. Mathematically this is represented by $[A, B] = 0$.

[7] A system prepared in an eigenstate of one of the bases has outcomes of measurement for all the vectors of the other basis predicted to occur with equal probability. Being this probability equal to one over the space dimension (1/d) .

described as $|\psi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B$. Generally, this is not the case, when at least one of the coefficients cannot be decomposed, the state will be called entangled. Entanglement is one of, if not the most important property of QM, at least according to one of its founders, Erwin Schrödinger, who wrote in 1935 in a letter addressed to Albert Einstein: "I would not call [entanglement] one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives (particles) have become entangled" (SCHRÖDINGER, 1935). One can expect to hear more about entanglement further ahead on this work when Quantum Resources are discussed.

The composition of the DoF of a system has been described for discrete spaces only. But nothing keeps one from extending it to the case where one or both $\mathcal{H}_A$ and $\mathcal{H}_B$ are continuous spaces. The standard example of this happening is the case of the position vector of a single particle in 3D, as previously mentioned, if $\mathcal{H}_r = \mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_z$ then a general position state can be written on the basis $\{|\boldsymbol{r}\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle = |x, y, z\rangle\}$ as

$$|\psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\ dy\ dz\ \psi(x, y, z)\,|x, y, z\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3r\ \psi(\boldsymbol{r})\,|\boldsymbol{r}\rangle\,. \tag{2.30}$$

Once again there is a coefficient that describes entanglement or separability between the DoF of the state. As it turns out, this time this coefficient is the 3D wave-function $\psi(x, y, z)$ being the state separable only when $\psi(x, y, z) = \phi(x)\phi(y)\phi(z)$. Since $|\boldsymbol{r}\rangle$ forms a base of $\mathcal{H}_r$ all base properties apply and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\ dy\ dz\,|x, y, z\rangle\langle x, y, z| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3r\ \ |\boldsymbol{r}\rangle\langle\boldsymbol{r}| = \mathbb{I}. \tag{2.31}$$

Furthermore, for an position eigenstate $|\boldsymbol{r}\rangle$ it follows that

$$|\boldsymbol{r}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3r'\ \ |\boldsymbol{r'}\rangle\langle\boldsymbol{r'}|\boldsymbol{r}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3r' f(\boldsymbol{r'}, \boldsymbol{r})\,|\boldsymbol{r'}\rangle = |\boldsymbol{r}\rangle\,. \tag{2.32}$$

For the previous equation to be valid, $\langle\boldsymbol{r'}|\boldsymbol{r}\rangle = f(\boldsymbol{r'}, \boldsymbol{r})$ must be equal to $\delta(\boldsymbol{r} - \boldsymbol{r'})$, the 3D extension of the Dirac function.

All the properties described to position in $\mathcal{H}_r$ are directly extendable to momentum in a manner not different from the one in the previous chapter,

$$|\psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp_x\ dp_y\ dp_z\ \overline{\psi}(p_x, p_y, p_z)\,|p_x, p_y, p_z\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3p\ \overline{\psi}(\boldsymbol{p})\,|\boldsymbol{p}\rangle\,; \tag{2.33a}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp_x\ dp_y\ dp_z\ \ |p_x, p_y, p_z\rangle\langle p_x, p_y, p_z| = \mathbb{I}; \tag{2.33b}$$

$$\langle\boldsymbol{p'}|\boldsymbol{p}\rangle = \delta(\boldsymbol{p} - \boldsymbol{p'}). \tag{2.33c}$$

No problem appears if the system is composed of more than one entity with its position DoF and/or any other discrete or continuous DoF[8]. For example, let's consider a case of major importance for this work, the description of a spin-1/2 particle. For a particle equipped with both momentum and spin, its state can be written as

$$|\psi\rangle = \sum_{i=1}^{d_s} \int_{-\infty}^{\infty} dp \; \psi_i(p) \, |s_i, p\rangle \; . \tag{2.34}$$

Here the momentum has been restricted to be one-dimensional for simplicity. The spin s can take any integer or half-integer value, being its dimension determined by $d_s = 2s + 1$. In the case of $s = 1/2$, equation (2.34) can be rewritten as

$$|\psi\rangle = \sum_{s=-1/2}^{s=1/2} \int_{-\infty}^{\infty} dp \; \psi_s(p) \, |s, p\rangle \; , \tag{2.35}$$

where $\psi_s(p) := \langle p, s|\psi\rangle$ represents a wave function with two discrete components. By evaluating their coefficients one will be able to account for the quantumness in the nature that they represent. For such, a way to detect and measure such non-classicalities must be presented. In that sense let's talk about entanglement and its different manifestations.

2.2 QUANTUM RESOURCES

Any useful characteristic of a physical system can be called a resource. If a system demonstrates to have qualities of that kind that turn out to be impossible to model using classical physics, then the system is said to possess *Quantum Resources*. Although many others exist, the focus of this work will be on entanglement and resources rarer than entanglement[9]. Special attention is brought to the fact that entanglement is the main resource for several quantum computation protocols. Therefore, it has clear importance as far as applications are of concern (HORODECKI et al., 2009). Also, historically, the first resource to be presented as changing in a relativistic context was entanglement (PERES et al., 2002). As it will be argued, steering, the next resource of interest, will serve mainly as entanglement detection with an untrustworthy player, although this characteristic is fundamental to the motivation to use it in this work, as shall be argued later.

The apex of resource rarity is Bell non-locality (BELL, 1964), the resource that arose from Bell's attempt to solve the polemic proposed in the EPR paper (EINSTEIN et al., 1935). For Bell non-locality, the correlations may be of any nature, that is, no

[8] As long as one is not concerned with changes of reference frame in which one of the entities is to be regarded as the new origin, as is the case in quantum reference frame transformations. The difficulties associated with such cases will be expanded upon later in this work.

[9] In this context, being rarer means being present in a smaller set of states.

assumption is made about the nature of a party's measurements. The framework Bell developed could serve to detect even post-quantum correlations if any were discovered (PLÁVALA, 2023). This work will not investigate the presence of this resource in the systems of interest because all interesting objections to the entanglement creation process that it focuses on are solved by the requirements of quantum steering alone. The search for Bell non-locality would only further complicate the process of obtaining results without adding much depth to them. The interesting quality of Bell non-locality of most concern for this work is that, for pure quantum states, it is equivalent to entanglement (YU et al., 2012). Since Steering is a kind of hybrid between Bell non-locality and entanglement, the same equivalence holds for it, that is, for pure states, the three resources are equivalent. Put simply, all pure entangled states are steerable and Bell non-local.

2.2.1 Entanglement

As previously mentioned, entanglement is the complementary concept to separability (DAS et al., 2016). With this simple requirement in mind, we can define it rigorously as follows.

Definition 4. Any state $|\psi\rangle$ living in a composite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, will be *separable* if and only if it can be written as the tensor product of two states living in each of the sub-spaces, that is,

$$|\psi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B \, , \tag{2.36}$$

where $|\phi\rangle_A \in \mathcal{H}_A$ and $|\phi\rangle_B \in \mathcal{H}_B$.
If a state is **NOT** separable it is *entangled*.

As it turns out, having a definition of entanglement on hands is not enough to fully understand the nature of the system. That is so because different states in the same Hilbert space can possess different amounts of entanglement, and separability alone is not capable of distinguishing among them. To evaluate a system's degree of entanglement new mathematical tools must be introduced, that of density matrices.

2.2.1.1 Density Matrices

There are different and more general ways to represent a quantum state than state-vectors. The *density matrix*, for instance, can model ignorance about the states in its formalism. Let us not rush into it, if the system is described by the state vector $|\psi\rangle$, then the density matrix associated with it is

$$\rho = |\psi\rangle \langle\psi| \, . \tag{2.37}$$

All density matrices have trace one and are positive operators whose general form is

$$\rho = \sum_i a_i \, |\psi_i\rangle \langle\psi_i| \,. \tag{2.38}$$

In principle, the sum in (2.38) can have infinite terms as long as it represents the statistical mixture of infinitely many different $|\psi_i\rangle$ states each with a_i probability associated with them. In practice, many different combinations of $|\psi_i\rangle$ and a_i can generate the same ρ such that they are not distinguishable from one another. The trivial case of a pure state such as the one in (2.37) happens when only one of the a_i is different from 0. If the states $|\psi_i\rangle$ are all orthogonal they can be at most d, being d the Hilbert space dimension. Being this the case as well as the statistical distribution of them being uniform, this means all $a_i = 1/d$, then the density matrix can be written as $\rho = \mathbb{I}/d$. In this case, we have the maximum mixture, or maximum ignorance state.

Since physicists assume nature to be quantum and QM says that physical systems are represented as pure states in a Hilbert space, how come some systems are better modeled by states like the one in (2.38)? As it turns out, one can understand this state as one where different states $|\psi_i\rangle$ are unreliably prepared with probability a_i. That is, the state is **definitely** in one of the $|\psi_i\rangle$ options but "we" (the observers) do not know in which one. This makes explicit the way density matrices can be used to subjective ignorance together with objective information.

These operators were introduced with the promise that they could help to evaluate the amount of steering a system possesses, but before doing so, let's understand how probabilities and mean values are obtained using this new formalism. The probability of measuring B and finding a value b_i is, using the trace operation

$$\text{Tr}(X) = \sum_{j=1}^{d_B} \langle b_j| \, X \, |b_j\rangle \,, \tag{2.39}$$

simply

$$p(b_i) = \text{Tr}(B_i \rho). \tag{2.40}$$

Here $B_i = |b_i\rangle \langle b_i|$ is the projector on state $|b_i\rangle$. In this sense, the mean value of a B measurement can be obtained via

$$\langle B\rangle = \text{Tr}(B\rho). \tag{2.41}$$

Now comes the interesting part, by taking the partial trace[10] over one of the tensor spaces of a composite system ρ_{AB}, one obtains the partial state of the rest[11]. By partial

[10] For a state in a $\{|a_i, b_j\rangle\}$ basis the definition on (2.40) is already a partial trace over B (MAZIERO, 2017).

[11] Although slightly circular this is sound logic. The partial state can be understood as the state containing the marginal probabilities when marginalizing over the probabilities of the traced DoF.

state, what is meant is the best possible description of the statistical behavior of this part of the system. Given a pure bipartite state on the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, one has that

$$\begin{aligned} \rho_{AB} &= |\psi\rangle\langle\psi| \\ &= \sum_{ij} c_{ij} |a_i, b_j\rangle \sum_{mn} c^*_{mn} \langle a_m, b_n| \\ &= \sum_{ijmn} c_{ij} c^*_{mn} |a_i\rangle\langle a_m| \otimes |b_j\rangle\langle b_n| . \end{aligned} \tag{2.42}$$

Then, being $\{|b_k\rangle\}$ a basis of $\mathcal{H}_B$, the partial state on A, ρ_A, will simply be

$$\begin{aligned} \rho_A =& \mathrm{Tr}_B(\rho_{AB}) \\ =& \sum_{k=1}^{d_B} \langle b_k| \rho_{AB} |b_k\rangle \\ =& \sum_{k=1}^{d_B} \langle b_k| \left(\sum_{ijmn} c_{ij} c^*_{mn} |a_i\rangle\langle a_m| \otimes |b_j\rangle\langle b_n| \right) |b_k\rangle \\ =& \sum_{k=1}^{d_B} \sum_{ijmn} c_{ij} c^*_{mn} |a_i\rangle\langle a_m| \langle b_k|b_j\rangle \langle b_n|b_k\rangle \\ =& \sum_{k=1}^{d_B} \sum_{ijmn} c_{ij} c^*_{mn} |a_i\rangle\langle a_m| \delta_{kj}\delta_{nk} \\ =& \sum_{k=1}^{d_B} \sum_{im} c_{ik} c^*_{mk} |a_i\rangle\langle a_m| \\ \neq& (|\phi\rangle\langle\phi|)_A . \end{aligned} \tag{2.43}$$

The intended meaning of the last line in equation (2.43) is that "in general" the partial state is not a quantum(pure) state. If the state is separable, $c_{ik} = c_i c_k$ and $c^*_{mk} = c^*_m c^*_k$, therefore

$$\begin{aligned} \rho_A &= \sum_{im} c_i c^*_m |a_i\rangle\langle a_m| \sum_{k=1}^{d_B} c_k c^*_k \\ &= \sum_{im} c_i c^*_m |a_i\rangle\langle a_m| \\ &= (|\phi\rangle\langle\phi|)_A . \end{aligned} \tag{2.44}$$

What the previous equations demonstrate is that a pure partial state is obtained from a pure global state if this global state is separable. Even though this is the case, it is a fact that

$$\begin{aligned} p(a_i) =& \mathrm{Tr}(A_i \otimes \mathbb{I}_B \rho_{AB}) \\ =& \mathrm{Tr}_A(\mathrm{Tr}_B(A_i \otimes \mathbb{I}_B \rho_{AB})) \\ =& \mathrm{Tr}_A(A_i \rho_A), \end{aligned} \tag{2.45}$$

meaning that the statistics of a measurement in one of the subsystems is preserved in its partial state. Since for a pure state, a suitable projective measurement can return results with probability one, it is possible to detect if one's partial state is pure or, not finding any measurement as such, mixed. This means that, under the assumption that the total state is pure, if this formalism is to be believed, measurements on the partial state serve to detect entanglement (entangled states will have **NO** partial projection with probability one).

With this strong condition in hand, one can measure the "mixedness" of the partial state to detect the degree of entanglement of the whole state. As it turns out, there is a great way to do so, by measuring the partial state's entropy!

2.2.1.2 Entropy

In classical information theory, entropy can be understood as a measure of ignorance about a statistical quantity. The standard derivation of this concept is the Shannon entropy H (SHANNON, 1948). For a random variable X that can take values x_i with $i = 1, 2...N$ each with probability p_i associated to them, the Shannon entropy is

$$H(X) = -\sum_{i=1}^{N} p_i \ln(p_i). \tag{2.46}$$

$H(X)$ quantifies the amount of information obtained by measuring or revealing the value of X. That is why it can be understood as the lack of knowledge or ignorance over the distribution. To visualize this, consider the case in which the value of x is always known, $x = a$ for instance. In this case we have total knowledge of x. The probability that $x = a$, $p(x = a) = 1$ and since it is a single outcome ($N = 1$), the entropy $H(X) = -1\ln(1) = 0$. Therefore, total knowledge implies null entropy. The opposite case, when X is maximally unknown, happens when, for a given number of possibilities N, the probability of any given x_i, p_i, is equal to $1/N$. This means that there is no preferential value for x and, conveniently, this is the case in which the entropy takes its maximum value.

Using the density matrix formalism, an analogous quantity can be defined for quantum states, that being the von Neumann entropy S (VON NEUMANN, 2013), defined as

$$S(\rho) = -\text{Tr}(\rho \ln(\rho)). \tag{2.47}$$

The connection between both quantities is made by using the diagonal representation of ρ so that (2.47) reduces to (2.46). For S to be equal to zero, ρ must be a pure state. On the other hand, if $\rho = \mathbb{I}/d$,then $S(\rho) = \ln(d)$, being this its maximum value.

One paying good attention will remember that a pure partial state is connected to a separable global state and a mixed partial state to an entangled global one. S being sensitive to state purity makes it a great candidate for a measure of entanglement and, as it turns out, it is. Defining the partial state entropy as

$$S_A(\rho_{AB}) := S(\rho_A) = -\text{Tr}(\rho_A \ln(\rho_A)), \tag{2.48}$$

one gets a smooth function that goes from zero, in the separable state case, to $\ln(d)$ for the maximally entangled state.

Lastly, it is important to spotlight the fact that the partial trace entropy is *symmetric*, meaning that, for pure states, $S_A(\rho_{AB}) = S_B(\rho_{AB})$. This is rapidly proven by taking into consideration the Araki-Lieb inequality (NIELSEN; CHUANG, 2010)

$$|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB}) \leq S(\rho_A) + S(\rho_A). \tag{2.49}$$

Since the global state is pure, $S(\rho_{AB}) = 0$ making the equality $S_A(\rho_{AB}) = S_B(\rho_{AB})$ unavoidable. As a consequence, for a pure global state that lives in a bipartite space in which each partition has different dimensions, as will be the case for the states of interest in this work, the entropy will be limited by the dimension of the smaller dimension state. Having the benefit of hindsight, this fact can be said to have great consequences for the species of phenomena being researched by this project.

Before advancing into more abstract concepts, let's take time to appreciate the meaning of the fact that the entropy of a partial state can be greater than the entropy of its global counterpart. If entropy is ignorance, the reasonable conclusion is that there can be more information about a system than the information contained in its parts. Furthermore, a direct consequence of this is that the correct description of the physics of a state accessible to an observer may, and in general does, depend on information not contained within his or her grasp. No connection between a state and the spatial position of the physical system it represents has been drawn. Therefore, in principle, the partial states may be as distant as it is feasible and still carry information about each other that a correct description of physics can't be done without. It is based on facts of this kind that QM is said to be a non-local theory.

Now, the formalism of entanglement has been introduced on the assumptions that the global state is pure and completely known; what may one conclude if one or both of these conditions are relaxed? Is it possible to detect entanglement only with information about the partial state? Questions as such will be addressed next.

2.2.2 Quantum Steering

Beginning with the same bipartite state ρ_{AB}, if two different observers Alice and Bob, receive the partial states ρ_A and ρ_B, respectively, working together they can do quantum-state-tomography[12] to discover both the partials and the global state they are given. Having modeled out the states, to measure entanglement becomes a trivial matter. But what if they do not work together?

[12] The process of doing a large number of measurements of different observables to determine the quantum state a source is producing.

If only Bob, for instance, does state-tomography in its partial state, and Alice affirms that she is doing measurements that somehow perturb Bob's state, as would be the case if the states were entangled, how can Bob falsify Alice's statement? Or, put differently, how may one observer certify quantum correlations, such as entanglement, when only one of the players in the protocol is to be trusted?

To these questions, quantum steering is the answer (WISEMAN et al., 2007; UOLA et al., 2020)! In a descriptive sense, steering is the ability of one party to alter, in some way, the physical system that another party possesses, but certified only with the latter party. For such a task, entanglement between both parties' states is a must. That is so because, as previously mentioned, all steerable states are entangled. With such a definition in mind, a protocol for steering detection can be laid out. If all local, where local means "that which is related to the partial state", statistics can be described by a quantum state that in no way depends on another party's influence, a so-called "local hidden state" then the state is said to be *unsteerable*. In mathematical terms, given the bipartite state ρ_{AB}, if Alice performs a choice x of measurement, obtaining a, after informing both the choice and results of its measurements to Bob, he is left with an unnormalized conditional state $\varrho_{a|x}$. The set of all $\varrho_{a|x}$ is called an assemblage and represents all of Bob's information about his partial state. Such is made evident when one realizes that

$$\sum_a \varrho_{a|x} = \rho_B = \text{Tr}_A(\rho_{AB}). \tag{2.50}$$

Equation (2.50) is valid for cases where Alice is in possession of a quantum state just like Bob. In general, Alice's portion can be of any nature and its local nature may be probed with any kind of measurement or none at all. Alice can, if she wishes to, just inform Bob about a made-up result on a given choice. That's why Alice is considered "untrusted" in this formalism.

In possession of its assemblage, Bob will try to explain it with a hidden state σ_λ associated with probabilities $p(\lambda)$, with λ representing a hidden variable that parameterizes the system. Being this the model of its assemblage, all Alice can do is update the probability of the states such that

$$\varrho_{a|x} = p(a|x) \int d\lambda\; p(\lambda|a,x)\sigma_\lambda = \int d\lambda\; p(\lambda)p(a|x,\lambda)\sigma_\lambda. \tag{2.51}$$

If the statistics obtained by Bob fit into a model of the kind in (2.51), then the state is **not** steerable. The first integral on (2.51) coresponds to the case in which the information on Alice's side (choice x and result a) just gives new information about the distributions of the states in σ_λ. This has the effect of updating $p(\lambda)$ to the more restricted $p(\lambda|a,x)$ distribution. The second integral in equation (2.51) represents a situation where Alice is in possession of the $p(\lambda)$ distribution and decides to simulate the σ_λ states. With her simulation, she can obtain $\varrho_{a|x}$ drawing from the pool of local hidden states and, by

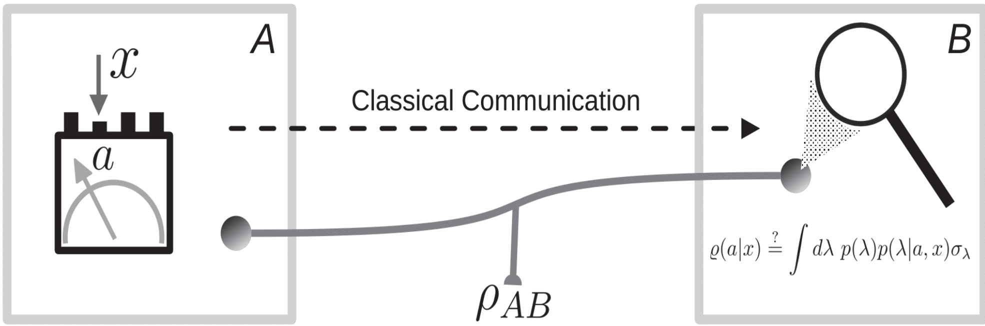

FIGURE 1 – Diagram of a standard steering-detection setup. Alice (A), on the left, and Bob (B), on the right, share a bipartite quantum state ρ_{AB}. Alice, using her black box, chooses to measure the observable x obtaining a as a result. Bob performs tomography in his local state and is informed of Alice's procedures via classical communication. Putting his and her results together, he produces the assemblage $\varrho(a|x)$ and questions if it may be explained by a local hidden state model to falsify steerability in the scenario.

choosing x, obtain the results a. Both interpretations can be made equivalent (UOLA et al., 2020). A diagram of a standard setup for steering certification can be found in figure 1. There, both parties receive quantum states since Nature is assumed to be quantum and the experimental procedure will use quantum measurements. This diagram is followed by the previously mentioned resource hierarchy (figure 2) that shows the steering position relative to entanglement.

2.2.2.1 Steering Criteria

Though not too great in complexity, the difficulty surrounding the concept of steering is to rigorously prove that no local hidden state is present given only the

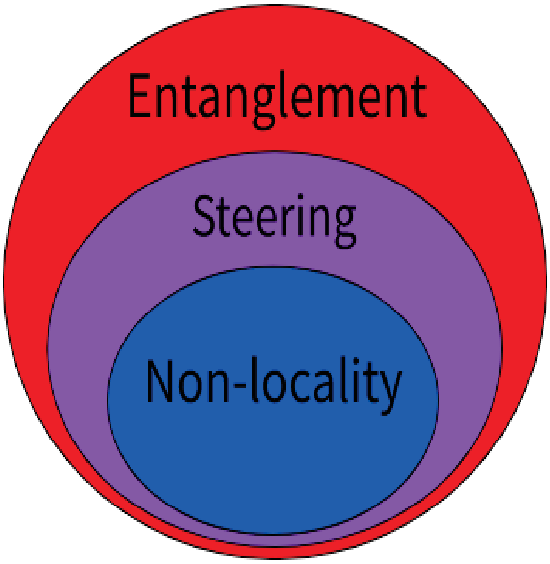

FIGURE 2 – Diagram of the quantum resource hierarchy for resources rarer than entanglement. The diagram format does not have a one-to-one correspondence to the number of states that possess each resource. It serves only to demonstrate that all non-local states are steerable and entangled, all steerable states are entangled, but not all entangled states are steerable or non-local. Just as not all steerable states are non-local. Thanks to the well-defined hierarchy, steering always serves as entanglement detection.

assemblage. Symmetries of the state may help reduce the complexity of the task, but, given a general state, there is no known way to construct a specific local hidden state model for it. Then, to make steering detection operationally feasible, the standard strategy is to assume that the system does obey (2.51) and construct around the assumption some kind of inequality that, when violated, will falsify the assumption. These inequalities are called "steering criteria" and represent the usual way that steerable states are found and measured in the literature (UOLA et al., 2020).

Although criteria for discrete states are more prevalent in the literature, there are also criteria appropriated for continuous DoF and even for the discrete to continuous case. In this work, the criteria of choice are "Reid's variance criterion" (REID, 1989) and the "entropic uncertainty criterion" (SCHNEELOCH et al., 2013), which will be introduced next. Before beginning, it is important to call attention to the fact that Reid was talking about more-than-classical correlations in her work before the concept of steering had been formally established. The term used by the author for her work's results was "Einstein-Podolsky-Rosen paradox", later the community came to interpret her proposed detection of this kind of paradox as the same as the detection of steering.

2.2.2.2 Reid's Variance Criterion

Considering a set of two continuous observables on Bob's side, made up of linear combinations of the usual canonical ones

$$\begin{aligned} B_{\beta 1} &= X_B \cos\beta_1 + P_B \sin\beta_1; \\ B_{\beta 2} &= X_B \cos\beta_2 + P_B \sin\beta_2. \end{aligned} \tag{2.52}$$

Between $B_{\beta 1}$ and $B_{\beta 2}$ an uncertainty principle may be established using the general formula (obtained on (2.26))

$$\sigma_X^2 \sigma_Y^2 \geq \left(\frac{1}{2i} \langle [X, Y] \rangle \right)^2, \tag{2.53}$$

where σ_X^2 is just the variance of X and $[X, Y]$ is the commutator of X and Y. By setting $X = B_{\beta 1}$ and $X = B_{\beta 2}$, one finds

$$\begin{aligned} \sigma_{B_{\beta 1}}^2 \sigma_{B_{\beta 2}}^2 \geq & \left(\frac{1}{2i} \langle [B_{\beta 1}, B_{\beta 2}] \rangle \right)^2 \\ = & \left(\frac{1}{2i} \langle [X_B \cos\beta_1 + P_B \sin\beta_1, X_B \cos\beta_2 + P_B \sin\beta_2] \rangle \right)^2 \\ = & \left(\frac{1}{2i} \langle \cos\beta_1 \sin\beta_2 [X_B, P_B] + \sin\beta_1 \cos\beta_2 [P_B, X_B] \rangle \right)^2 \\ = & \left(\frac{1}{2i} \langle \cos\beta_1 \sin\beta_2 (i)\mathbb{I} + \sin\beta_1 \cos\beta_2 (-i)\mathbb{I} \rangle \right)^2 \\ = & \left(\frac{1}{2} \langle \cos\beta_1 \sin\beta_2 \mathbb{I} - \sin\beta_1 \cos\beta_2 \rangle \, \mathbb{I} \right)^2 \\ = & \frac{1}{4} \sin(\beta_1 - \beta_2)^2. \end{aligned} \tag{2.54}$$

Here the canonical commutation relation was considered to be $[X, P] = i\mathbb{I}$. For simplicity, this condition can be obtained by setting $\hbar = 1$ or $P_B = P/\hbar$. Since those are observables on Bob's side, the side in which quantum measurements are to be taken, the bound of the inequality being constructed should only depend on the qualities of these observables. With the intent of maximizing this bound so that the criterion is as tight as possible, meaning, able to detect steering as often as possible, the condition $\beta_1 - \beta_2 = \pi/2$ is established guaranteeing that

$$\sigma_{B_{\beta 1}}^2 \sigma_{B_{\beta 2}}^2 \geq \frac{1}{4}. \tag{2.55}$$

The derivation of (2.55) is based on the work present in Paulo Muraro's thesis (FERREIRA, 2019). Given the relation in (2.55), Reid's variance criterion can be put forth as: If there are any choice of measurements on Alice's side A, such that

$$\sigma_{min}^2(B_{\beta 1}) \sigma_{min}^2(B_{\beta 2}) < \frac{1}{4}, \tag{2.56}$$

where $\sigma_{min}^2(B) = \int da \; p(a) \sigma^2(B|a)$, being a and b measurement results of Alice and Bob respectively and $\sigma^2(B|a)$ the conditional variance of Bob's over Alice's measurement of

A obtaining a as result, the state is steerable[13]. Otherwise, if $\sigma^2_{min}(B_{\beta 1})\sigma^2_{min}(B_{\beta 2}) \geq \frac{1}{4}$, **it is not**. A level of intuition about this may be gained by understanding the following argument: if X and P are incompatible, a maximal classical correlation between Alice's state and one of the DoF would imply no correlation with the other. Being this so, for the variance of BOTH DoF to be brought bellow their limits together, a more than classical correlation must be shared. Since the certification depends only on Bob's measurements, it qualifies as a steerign criterion.

Writing the condition explicitly one gets that, for steerable states and the right choice of measurements

$$\int da_1 p(a_1)\sigma^2(B_{\beta 1}|a_1) \int da_2 p(a_2)\sigma^2(B_{\beta 2}|a_2) < \frac{1}{4}. \tag{2.57}$$

By making assumptions about the nature of Alice's measurements the condition could be further refined. This will be done later by taking into consideration the part of the physical system being analyzed by Alice and its characteristics.

2.2.2.3 Entropic Uncertainty Criterion

Being QM a probabilistic theory, we can model a projective measurement observable to behave not differently from the random variable used in (2.46). Being this so, if $B_1 = \sum_{i=1}^{d_1} \lambda_i |\alpha_i\rangle\langle\alpha_i|$ and $B_2 = \sum_{i=1}^{d_2} \lambda_j |\beta_j\rangle\langle\beta_j|$, for a given state (SCHNEELOCH et al., 2013),

$$H(B_1) + H(B_2) \geq -\log_2(\Xi_B), \tag{2.58}$$

where $\Xi_B = \max_{i,j}(|\langle\alpha_i|\beta_j\rangle|^2)$ is the maximized overlap between the projective eigenstates. As it is clear to see, if the observable are the same, $\Xi_B = 1$, its maximum value, and $\log_2(\Xi_B) = 0$ so that both entropies can be zero simultaneously. If the observables are somewhat different, $\Xi_B < 1$ and $-\ln(\Xi_B) > 0$, meaning that both entropies are never zero at the same time. In the case of completely distinct, continuous observables, $-\log_2(\Xi_B) = \log_2(\pi e)$ (BIAŁYNICKI-BIRULA; MYCIELSKI, 1975).

[13] $\sigma^2(B|a)$ is the variance after Alice has informed Bob about A and a. Bob measures his variance normally but is sure to tag it with Alice's information to verify if she was able to successfully diminish it.

Now, let's consider the so-called conditional entropy

$$
\begin{aligned}
H(X|Y) &= \sum_{y\in Y} p(y) H(X|Y=y) \\
&= -\sum_{y\in Y} p(y) \sum_{x\in X} p(x|y)\log_2(p(x|y)) \\
&= -\sum_{y\in Y, x\in X} p(y)p(x|y)\log_2(p(x|y)) \\
&= -\sum_{y\in Y, x\in X} p(x,y)\log_2\left(\frac{p(x,y)}{p(y)}\right) \\
&= -\sum_{y\in Y, x\in X} p(x,y)[\log_2(p(x,y)) - \log_2(p(y))] \\
&= -\sum_{y\in Y, x\in X} p(x,y)\log_2(p(x,y)) + \sum_{y\in Y, x\in X} p(x,y)\log_2(p(y)) \\
&= -\sum_{y\in Y, x\in X} p(x,y)\log_2(p(x,y)) + \sum_{y\in Y} p(y)\log_2(p(y)) \\
&= H(X,Y) - H(Y).
\end{aligned}
\tag{2.59}
$$

This entropy has interesting properties. If X and Y are independent, $p(x,y) = p(x)p(y)$ and

$$
\begin{aligned}
H(X|Y) &= -\sum_{y\in Y, x\in X} p(x,y)\log_2(p(x,y)) + \sum_{y\in Y} p(y)\log_2(p(y)) \\
&= -\sum_{y\in Y, x\in X} p(x)p(y)\log_2(p(x)p(y)) + \sum_{y\in Y} p(y)\log_2(p(y)) \\
&= -\sum_{y\in Y, x\in X} p(x)p(y)[\log_2(p(x)) + \log_2(p(y))] + \sum_{y\in Y} p(y)\log_2(p(y)) \\
&= -\sum_{y\in Y, x\in X} p(x)p(y)\log_2(p(x)) - \sum_{y\in Y, x\in X} p(x)p(y)\log_2(p(y)) + \sum_{y\in Y} p(y)\log_2(p(y)) \\
&= -\sum_{x\in X} p(x)\log_2(p(x)) - \sum_{y\in Y,} p(y)\log_2(p(y)) + \sum_{y\in Y} p(y)\log_2(p(y)) \\
&= -\sum_{x\in X} p(x)\log_2(p(x)) = H(X).
\end{aligned}
\tag{2.60}
$$

This means that independence makes it so that Y cannot condition X. Such a fact will be of major importance when this kind of conditionability translates to correlations between both variables. Following this vein, if acquiring knowledge of Y gives the same amount of knowledge about X, that is, x and y can be ordered such that $p(x) = p(y)$ implying $p(x,y) = p(y)$, then

$$
\begin{aligned}
H(X|Y) &= -\sum_{y\in Y, x\in X} p(x,y)\log_2(p(x,y)) + \sum_{y\in Y} p(y)\log_2(p(y)) \\
&= -\sum_{y\in Y,} p(y)\log_2(p(y)) + \sum_{y\in Y} p(y)\log_2(p(y)) \\
&= 0.
\end{aligned}
\tag{2.61}
$$

This means that the conditional entropy goes to zero only when X and Y are completely correlated. Thus, by summing the conditional entropies relative to incompatible observables, and achieving diminishing values for both, one may only explain the correlations using a quantum (at least more than classical) framework. Since the bound is established over only one of the parties, this process serves to detect steering.

After this tangent on the properties of $H(X|Y)$, it should be clear that $H(X|Y)$ is, at maximum, equal to $H(X)$. Therefore, exchanging X for B and Y for A one can rewrite (2.58) as

$$H(B_1|A_1) + H(B_2|A_2) \geq -\log_2(\Xi_B), \tag{2.62}$$

where violations will occur only when the correlations between A and B are enough to decrease the left-hand-side of (2.62) bellow the bound determined using only B_1 and B_2. The interpretation of this process as a steering criterion becomes clear when one realizes that the B observables can be attributed to Bob who, by being trusted, establishes the bound, and the A observables are given to Alice, that will, through measurements, attempt to acquire enough information about Bob's state so as to violate the entropic uncertainty relation first established in (2.58). The way this criterion was derived in this work varies from its historical appearance (WALBORN et al., 2009). There the impossibility of local hidden states is taken as the focus rather than our "one-party-verified more-than-quantum correlations" method. The results are the same.

This criterion has many advantages, it is known to be tighter than others (SCHNEELOCH et al., 2013), meaning that it detects steering for all states other criteria do and then some more. It is also appropriate for hybrid continous-discrete systems, being this the type of system this work is interested in. The price paid for these advantages is added mathematical complexity when evaluating the conditional entropies. At times, no analytical solution can be obtained, in such cases, numerical methods may be utilized.

2.3 CHAPTER CLOSING REMARKS

After careful study of this chapter, one should be familiar with the basic concepts of quantum theory as well as the mathematical formalism that describes it. Furthermore, laid over this knowledge one should be able to understand and identify quantum resources, those more-than-classical correlations. Cpncepts like entanglement and steering, make QM a wonderful tool to, by deconstructing it, expand one's intuition about Nature and its phenomena. In the subject of intuition, it is not at all uncommon for an individual to take time to grasp some of the more abstract concepts of QM. Do not be dismayed if you find yourself on that camp, even the Nobel Laureate Richard Feynman, famous for his contributions in quantum theory, once said "If you think you understand quantum mechanics, you don't understand quantum mechanics". Familiarizing oneself with the heretofore-presented concepts may take time, but the insight about nature obtained is worth the effort.

If one has caught a taste for counter-intuitive concepts in theoretical physics, the next chapter will be to this one's liking. The theory of relativity has very little in common with QM besides being born in the same period. Presenting concepts that intuition often fails to capture may be their similarity.

3 RELATIVISTIC FORMALISM

In the following chapter, one can find the tools of special relativity (SR) that are needed in the derivation of this work results. Before mastering one's understanding about such theory, one must learn the mathematical tools it is based on, those of *Group theory*. Group theory is a very extensive area of study in mathematics and essential for the development of any sophisticated understanding of abstract algebra. For such reasons, this work does not intend to describe all there is to know about group theory, being content by just borrowing the useful concepts and ignoring the overwhelming rest. Much of the formalism hereby presented is based on references such as (TUNG, 1985) and (WEINBERG, 1995).

3.1 GROUP THEORY FORMALISM

3.1.1 Basics of Group Theory

Group theory is a branch of mathematics that studies the abstract structure of groups. A group is a set of elements equipped with a binary operation that combines any two elements, called *composition*, to form a third element of the same group. Such property can be written as

$$\forall\ X, Y \in \mathbb{G} \exists\ Z \coloneqq X \circ Y \in \mathbb{G}. \tag{3.1}$$

Here $\mathbb{G}$ is the group of interest and X, Y and Z are some of its elements. Also, $\circ$ represents composition. Although necessary, (3.1) is not enough to rigorously define a group. For that $\circ$ must also obey three properties:

1. ***Association.*** If X, Y and Z are elements of $\mathbb{G}$, then

$$(X \circ Y) \circ Z = X \circ (Y \circ Z). \tag{3.2}$$

2. ***Identity.*** A group must have an unique element $\mathbb{I}$, that when composed with any other element leaves it unaltered. That is, for $X \in \mathbb{G}$ it follows that

$$X \circ \mathbb{I} = X. \tag{3.3}$$

   $\mathbb{I}$ is called the *identity* or *neutral* element of $\mathbb{G}$.

3. ***Inverse.*** For every $X \in \mathbb{G}$ there exists $X^{-1} \in \mathbb{G}$, such that

$$X^{-1} \circ X = X \circ X^{-1} = \mathbb{I}. \tag{3.4}$$

   X^{-1} is called the inverse of X.

From properties 2 and 3 it follows that

$$\mathbb{I}^{-1} = \mathbb{I} \therefore \tag{3.5a}$$

$$X \circ \mathbb{I} = \mathbb{I} \circ X. \tag{3.5b}$$

Now equipped with the basic concepts of what a group is, one can start to narrow down into more useful cases. Since the intent of this chapter is to describe the formalism of SR, where Lorentz Transformations are of major importance, let us learn a bit about *transformation groups*.

3.1.2 Transformation groups

Transformation groups are closely related to symmetry groups: transformation groups frequently consist of all transformations that preserve a certain inherent structure (a symmetry). Recalling Noether's theorem, one can say that, in a physical system, the presence of a symmetry of the action is equivalent to a conservation law. Therefore transformation groups can be understood as the groups of operators that conserve quantities.

In the next section one can find the group that preserves space-time intervals and its properties. Such group is called the Poincaré group and it is the place where Lorentz Transformations live.

Before going further, it is important to bring attention to the fact that *Matrix Groups*[1] are a special kind of transformation group. This fact is majorly important for this work because, as will be seen, Lorentz transformations can be written as matrices.

3.1.3 Poincaré group

The Poincaré group and its subgroups are the major focus of this chapter.The Poincaré group is the group of *minkowski space-time isometries*[2]. This particular group has some very important properties that must be listed even before we define operation within it.

- ***Non-Commutativity.*** The Poincaré group is non-Abelian[3], that is, in it exist at least two elements a and b such that $a \circ b \neq b \circ a$.

1 A matrix group $\mathbb{G}$ consists of invertible matrices over a specified field $\mathbb{K}$, where matrix multiplication is a well-defined operation.

2 The minkowski space is a combination of spacial and time manifolds, the isometries referred are the invariance of *intervals* between two distinct events.

3 One can understand this property by considering that this group contains the famous three-dimensional rotation group $SO(3)$, where two members (rotations) do not, generally, commute.

- ***The cake is a Lie.*** The Poincaré group is a Lie-group. A direct consequence of this is that this group is both continuous and a differentiable manifold[4]. Put simply, Calculus applies!

- ***Isometric Interval.*** As said before, this is a group of transformations, but of a special kind. These transformations are the Minkowski isometries, in other words, they preserve *space-time intervals*.

3.1.4 Poincaré Symmetry

To define rigorously the members of a group it is quite useful to explore its transformations. From now on, the notation used for the Poincaré group is $\mathbb{P}$ and it has three types of transformations.

1. ***Translation.*** Displacements in both space and time ($\mathbf{P}$) form an Abelian Lie-subgroup of Poincaré's. That is, $\mathbb{P}$ is symmetric under 4-translation.[5]

2. ***Rotations.*** Rotations on space (and not time) ($\mathbf{R}$) are a symmetry of $\mathbb{P}$. Rotations also form a Lie-subgroup just like translations except by being not Abelian.

3. ***Boosts.*** Boosts ($\mathbf{K}$) are the only transformation of $\mathbb{P}$ that do not form a subgroup[6]. Anyhow, they are the transformations connecting any two uniformly moving bodies.

It is useful to separate and distinguish the three symmetries because the last two, $\mathbf{R}$ and $\mathbf{K}$, compose the *Lorentz group* ($\mathbb{L}$). Most phenomena of this work are described by $\mathbb{L}$ alone, it being a subgroup of $\mathbb{P}$ and simpler to work with.

The Poincaré symmetry is associated with ten generators, implying, by Noether's theorem, on ten conservation laws. As it turns out, it has one for the system's energy, one for each of its linear and angular momentum components, six in total, and lastly, three for the velocity of the center of mass (BARNETT, 2011).

As shown, the allegorical "habit hole" of group theory runs quite deep. Having listed enough characteristics of $\mathbb{P}$, let us move toward defining operation within it and to the derivation of its properties.

---

4 A manifold is a topological space that locally resembles Euclidean space. It is "flat" near to each point, no sharp edges.

5 We use 4-translation to name a displacement in four dimensions, three spatial and one temporal.

6 The importance of this fact cannot be overstated. Such fact will become evident when Wigner rotations are introduced further in the work.

3.2 FROM POINCARÉ TO LORENTZ

3.2.1 Definitions within Poincaré's group

Since this work interest is in space-time isometries, the fundamental element will be a 4-vector, that is, a four dimensional vector. For instance, the coordinates in this space are given by $x^\mu = (ct, \vec{x})$; here time is taken to be just another dimension, not so different from the three spatial ones. The components of such vector are noted by x^μ where μ goes from zero to three to account for the four dimensions. The choice of a Latin letter for the index, such as i or j instead of μ, would represent only the spatial part, that is, the one to three components. One should also know that summation is implied for repeated indices[7]. The last step is to define a "metric signature[8]". This can be achieved by choosing a metric tensor η. We shall proceed with the quite standard $\eta =$ *diag*$(1, -1, -1, -1)$.

Some terminology must also be introduced, when a 4-vector has upper indexed components, such as x^μ, it's called *contravariant*; if instead it has a low index such as p_μ, it is called *covariant*. Both forms are connected by the metric tensor via

$$k_\mu = \eta_{\mu\nu} k^\nu. \tag{3.6}$$

As an example, let us look at the covariant form of the coordinate vector:

$$x_\mu = \eta_{\mu\nu} x^\nu = (ct, -\vec{x})^T, \tag{3.7}$$

in matrix notation this is

$$\begin{Bmatrix} x_0 \\ -x_1 \\ -x_2 \\ -x_3 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Bmatrix} \begin{Bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{Bmatrix}. \tag{3.8}$$

The Mikowski space, $\mathbb{M}$ from now on, has as inner product $\langle \cdot, \cdot \rangle$ defined as

$$\langle x, y \rangle = x^0 y^0 - \mathbf{x} \cdot \mathbf{y} = \sum_{\mu,\nu} \eta_{\mu\nu} x^\mu y^\nu = x^T \eta y. \tag{3.9}$$

The previous equation contains the last example of explicit summation, from now on Einstein's notation takes over. The following equations will have both tensor and indexical notations.

Finally, the generic element of the Poincaré group can be defined as Λ with coordinates Λ^μ_ν such that, any two inertial frames with position vectors given by x' (with

7 This is the so-called "Einstein notation".

8 A metric is a way to measure distances. In this context, the metric measures the distance between points in Minkowski space-time.

coordinates x'^{μ}) and x (with coordinates x^{ν})[9] are connected by means of

$$x^{\mu\prime} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}, \tag{3.10a}$$
$$x' = \Lambda x + a. \tag{3.10b}$$

In (3.10b), Λ is a 4x4 matrix defined by 16 coefficients (not all independent) and a is a 4-translation. The major property of these elements is that they leave the interval ($ds^2 = \langle dx, dx \rangle$) *invariant*:

$$ds'^2 := dx'^{\mu} dx'_{\mu} = dx^{\nu} dx_{\nu} = ds^2. \tag{3.11}$$

Written as a rigorous statement, $\Lambda \in \mathbb{P} \iff \forall x', x \in \mathbb{M}$, connected via (3.10), it follows that $\langle dx', dx' \rangle = \langle dx, dx \rangle$. Knowing this, for (3.10) to obey (3.11) it suffices that

$$\eta^{\mu\nu} \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} = \eta^{\alpha\beta}; \tag{3.12a}$$
$$\Lambda^T \eta \Lambda = \eta. \tag{3.12b}$$

Such sufiency can be proven by taking the definition (3.11) together with the property in (3.6):

$$\begin{aligned} ds'^2 &:= dx'^{\mu} dx'_{\mu} = \eta^{\nu\mu} dx'_{\nu} dx'_{\mu} = \eta^{\alpha\beta} dx_{\alpha} dx_{\beta} = dx^{\beta} dx_{\beta} = ds^2 \therefore \\ \eta^{\nu\mu} dx'_{\nu} dx'_{\mu} &= \eta^{\alpha\beta} dx_{\alpha} dx_{\beta} \xrightarrow{(3.10)} \eta^{\nu\mu} dx'_{\nu} dx'_{\mu} = \eta^{\nu\mu} \Lambda^{\alpha}_{\nu} \Lambda^{\beta}_{\mu} dx_{\alpha} dx_{\beta} = \eta^{\alpha\beta} dx_{\alpha} dx_{\beta} \therefore \\ \eta^{\alpha\beta} &= \eta^{\nu\mu} \Lambda^{\alpha}_{\nu} \Lambda^{\beta}_{\mu}, \end{aligned} \tag{3.13}$$

or, for tensor notation,

$$\begin{aligned} ds'^2 &= \langle dx', dx' \rangle = dx'^T \eta dx' = dx^T \eta dx = \langle dx, dx \rangle = ds^2 \therefore \\ dx'^T \eta dx' &= dx^T \Lambda^T \eta \Lambda dx = dx^T \eta dx \longrightarrow \Lambda^T \eta \Lambda = \eta. \end{aligned} \tag{3.14}$$

The theory demands that its elements satisfy the following properties.

1. ***Scalars.*** Any quantity that has the same value in **all** reference frames connected by Lorentz transformations is called an *scalar* or a *invariant*. The prime examples of this concept are the notion of interval, the speed of light in the vacuum, as well as fundamental properties of matter, such as charge and rest-mass.

2. ***Vector Transformation.*** The way a vector transform was defined in (3.10) for the contravariant case. Let us extend it for the covariant one:

$$\begin{aligned} x'_{\nu} &= \eta_{\nu\mu} x'^{\mu} \\ &= \eta_{\nu\mu} (\Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}) \\ &= \eta_{\nu\mu} \Lambda^{\mu}_{\nu} (\eta^{\nu\mu} \eta_{\nu\mu}) x^{\nu} + \eta_{\nu\mu} a^{\mu} = \Lambda^{\nu}_{\mu} x_{\mu} + a_{\nu} \\ &= \Lambda^{\nu}_{\mu} x_{\mu} + a_{\nu}. \end{aligned} \tag{3.15}$$

Hopefully, one infers from (3.15) that Λ^{ν}_{μ} is the transposed inverse of Λ^{μ}_{ν}.

[9] The different coloring done to 4-vectors serves as a visual aid to help identifying wich frame of reference they belong to. The same and more common "primed" notation is in use as well.

3. ***inner Product Invariance.*** The inner product is always done between a covariant and a contravariant vector through dual pairing. To know the product between vectors V and W with components V^μ and W^ν is to realize the process given by

$$\langle V, W\rangle = V_\nu W^\nu = \eta_{\nu\mu} V^\mu W^\nu = V^0W^0 - V^1W^1 - V^2W^2 - V^3W^3. \tag{3.16}$$

Further restrictions on $\mathbb{P}$ are still required. Since space-time translations, such as a^μ, maintain the intervals invariant[10], one can focus in the transformations where $a^\mu = 0$ without losing neither generality nor the power to describe interesting phenomena. The Poincaré subgroup with such restriction is called the *homogeneous Lorentz group*, or *Lorentz group* for short. This group is denoted as $\mathbb{L}$ as foreshadowed in section (3.1.4).

3.2.2 Definitions within Lorentz group

As previously noted, Lorentz group is a subgroup of Poincaré's ($\mathbb{L} \subset \mathbb{P}$). As a group, $\mathbb{L}$ obeys the properties (3.1) to (3.4). That is, given a set of $\Lambda s \in \mathbb{L}$, it follows that

$$\text{For } \Lambda_1, \Lambda_2 \in \mathbb{L},\ \Lambda_3 := \Lambda_1\Lambda_2 \in \mathbb{L}; \tag{3.17a}$$

$$\Lambda_3(\Lambda_2\Lambda_1) = (\Lambda_3\Lambda_2)\Lambda_1; \tag{3.17b}$$

$$\mathbb{I} = \Lambda_0 := \Lambda(v = 0); \tag{3.17c}$$

$$\Lambda^{-1}(v) = \Lambda(-v), \in \mathbb{L}, \tag{3.17d}$$

where v is the parameter that defines Λ.

This is still quite general, but one can achieve further restrictions within this group. By taking the determinant of (3.12b), with the fact that $det(\eta) = -1$, one obtains that

$$\begin{gathered} \det(\Lambda^T\eta\Lambda) = \det(\eta) = -1 \therefore \\ \det(\Lambda^T)\det(\eta)\det(\Lambda) = -1 \longrightarrow \det(\Lambda^T)\det(\Lambda) \therefore \\ \det(\Lambda)^2 = 1 \therefore \det(\Lambda) = \pm 1. \end{gathered} \tag{3.18}$$

By doing a similar process for the case when $\alpha = 0$ and $\beta = 0$ in (3.12a), one may note that

$$\begin{aligned} \eta^{\mu\nu}\Lambda^0_\mu\Lambda^0_\nu &= \eta^{00} = 1 \therefore \\ 1 &= \eta^{00}\Lambda^0_0\Lambda^0_0 + \eta^{11}\Lambda^0_1\Lambda^0_1 + \eta^2\Lambda^0_2\Lambda^0_2 + \eta^{33}\Lambda^0_3\Lambda^0_3; \\ 1 &= \Lambda^0_0\Lambda^0_0 - \Lambda^0_1\Lambda^0_1 - \Lambda^0_2\Lambda^0_2 - \Lambda^0_3\Lambda^0_3. \end{aligned} \tag{3.19}$$

This means that

$$(\Lambda^0_0)^2 = 1 + (\Lambda^0_1)^2 + (\Lambda^0_2)^2 + (\Lambda^0_3)^2 \geq 1. \tag{3.20}$$

As it turns out, $\mathbb{L}$ can be subdivided neatly based on the conditions defined in (3.18) and (3.20):

[10] The intervals are defined by differentials (ds^2), translations like a^μ do not contribute because $da^\mu = 0$.

- $\mathbb{L}_1 = \{\Lambda \in \mathbb{L} : \det(\Lambda) = 1\}$. This is the set of the *proper*[11] Lorentz transformations.
- $\mathbb{L}_{-1} = \{\Lambda \in \mathbb{L} : \det(\Lambda) = -1\}$. This is the set of the *improper*[12] Lorentz transformations.
- $\mathbb{L}^+ = \{\Lambda \in \mathbb{L} : \Lambda^0_0 \geq 1\}$. This is the set of the *orthochronous*[13] Lorentz transformations.
- $\mathbb{L}^- = \{\Lambda \in \mathbb{L} : \Lambda^0_0 \leq -1\}$. This is the set of the *anachronous*[14] Lorentz transformations.

Some of these sets might intersect. But only one intersection between them forms a group. The so-called *restricted* Lorentz group[15] $\mathbb{L}_1^+ = \mathbb{L}_1 \cap \mathbb{L}^+ = \{\Lambda \in \mathbb{L} : \det \Lambda = 1, \Lambda^0_0 \geq 1\}$. One great way to understand the fact that only $\mathbb{L}_1^+$ forms a group is the fact that, to be a group $\mathbb{I} \in \mathbb{L}_1^+$ is required. As it will be shown later, any infinitesimal transformation must be infinitesimally close to identity. Therefore such transformations are also part of $\mathbb{L}_1^+$. Since finite transformations are constructed as successive infinitesimal transformations, all finite $\Lambda \in \mathbb{L}_1^+$. Furthermore, it is useful to evaluate into which set, certain standard transformations live:

- **Parity Inversion** ($\hat{P}$). This operation flips the sign on all spatial coordinates $\therefore\ \hat{P} = diag(1, -1, -1, -1)$. By evaluating the conditions that define the sets previously defined, it is easy to identify that

$$\hat{P} \in \mathbb{L}_{-1}^+. \tag{3.21}$$

- **Time Inversion** ($\hat{\tau}$).This is the operations responsible for changing the sign on the time component $\therefore \hat{\tau} = diag(-1, 1, 1, 1)$. This fact identifies the transformation with a set by

$$\hat{\tau} \in \mathbb{L}_{-1}^-. \tag{3.22}$$

- **Time-Parity Inversion** (-$\mathbb{I} := \hat{P}\hat{\tau}$). Since both $\hat{P}, \hat{\tau} \in \mathbb{L}$, they obey composition. As it turns out their composition also forms a symmetry that reverses the sign of all components $\therefore \hat{\tau} = diag(-1, -1, -1, -1)$, making it easy to identify the fact that

$$-\mathbb{I} \in \mathbb{L}_1^-. \tag{3.23}$$

[11] So-called because they preserve spatial directions, no parity change.
[12] The negation of proper, invert spatial directions
[13] This means "preserving time's direction".
[14] This set has no standard name in literature. The author choose to name it so as a negation of orthochronous, that is, **not** preserving time's direction.
[15] A.K.A homogeneous Lorentz group or proper ortochronous Lorentz group.

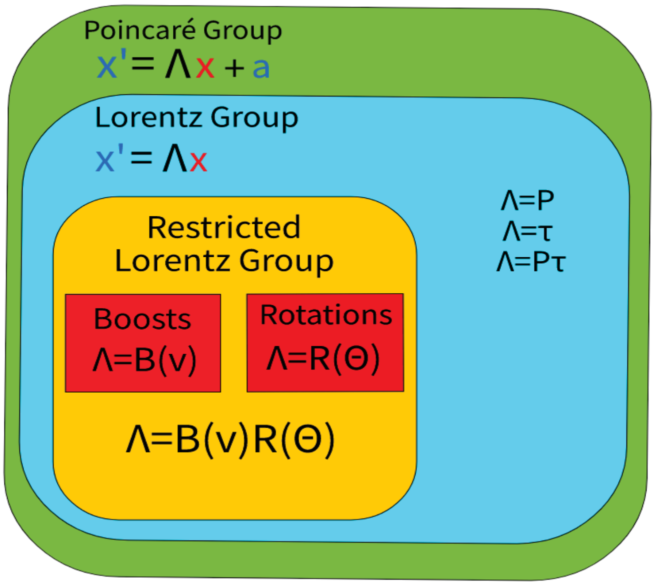

FIGURE 3 – The Poincaré group and its subgroups of importance presented on a Venn diagram. The most particular one, the Restricted Lorentz Group, is composed by Boosts and Rotations and can be extended to the Lorentz group via the symmetry related elements, $\hat{P}$, $\hat{\tau}$ and $\hat{P}\hat{\tau}$. Adding translations to the Lorentz group one obtains the most general group in this work, Poincaré's.

The earlier alluded fact that these transformations are important becomes apparent when one realizes that $\mathbb{L}_1^+ = \hat{P}\mathbb{L}_{-1}^+$, $\mathbb{L}_1^+ = \hat{\tau}\mathbb{L}_{-1}^-$ and $\mathbb{L}_1^+ = -\mathbb{I}\mathbb{L}_1^-$. As a consequence of the relations listed, one may realize that all the non-group forming sets are actually the cosets of $\mathbb{L}$ in relation to $\mathbb{L}_1^+$, that is

$$\mathbb{L} = \mathbb{L}_1^+ \cup \mathbb{L}_1^- \cup \mathbb{L}_{-1}^+ \cup \mathbb{L}_{-1}^-. \tag{3.24}$$

To distillate in a more didactically useful manner, the results of the previous section can be summed up in a diagram, such as the one showed in figure 3:

3.2.3 The Restricted Lorentz Group

Any member of $\mathbb{L}_1^+$, the Restricted Lorentz Group (RLG), can be written as two parts with distinct properties. They are *Boosts* $(\mathbf{B}(\mathbf{v}))$ and *Rotations* $(\mathbf{R}_{\hat{\mathbf{n}}}(\theta))$.

Boosts, as transformations, take vectors in a frame of reference into vectors in another frame associated with an observer traveling away from the former with a constant velocity $\mathbf{v}$. A boost could take, for example, a 4-vector in S, such as x^μ to another vector x'^μ in S', where S' is moving away from S with $\mathbf{v}$ and their spacial axes are aligned. The matrix representation of a general boost is given by

$$B(\mathbf{v}) = \begin{Bmatrix} \gamma & -\gamma\frac{v_x}{c} & -\gamma\frac{v_y}{c} & -\gamma\frac{v_z}{c} \\ -\gamma\frac{v_x}{c} & 1+(\gamma-1)\frac{v_x^2}{v^2} & (\gamma-1)\frac{v_xv_y}{v^2} & (\gamma-1)\frac{v_xv_z}{v^2} \\ -\gamma\frac{v_y}{c} & (\gamma-1)\frac{v_yv_x}{v^2} & 1+(\gamma-1)\frac{v_y^2}{v^2} & (\gamma-1)\frac{v_yv_z}{v^2} \\ -\gamma\frac{v_z}{c} & (\gamma-1)\frac{v_zv_x}{v^2} & (\gamma-1)\frac{v_zv_y}{v^2} & 1+(\gamma-1)\frac{v_z^2}{v^2} \end{Bmatrix}. \tag{3.25}$$

Different frames of reference may be required to have different orientations for their axes in SR. Therefore rotations are part of $\mathbb{L}_1^+$. Since they do nothing about the time component, SR's rotations are just three-dimensional rotations that operate like identity on time:

$$\begin{Bmatrix} 1 & 0 \\ 0 & \mathbf{R}_{\hat{\mathbf{n}}}(\theta) \end{Bmatrix}, \tag{3.26}$$

Where $\mathbf{R}_{\hat{\mathbf{n}}}(\theta)$ is an usual three-dimensional rotation, like the basic examples of

$$\mathbf{R}_{\hat{\mathbf{x}}}(\theta) = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{Bmatrix}, \quad \mathbf{R}_{\hat{\mathbf{y}}}(\theta) = \begin{Bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{Bmatrix},$$

$$\mathbf{R}_{\hat{\mathbf{z}}}(\theta) = \begin{Bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{Bmatrix}.$$

Boosts alone **DO NOT** form a group. We can't guarantee that $B(\mathbf{u})B(\mathbf{w}) = B(\mathbf{v})$. Therefore, a general transformation in the RLG is written as:

$$\Lambda = \mathbf{R}_{\hat{\mathbf{n}}}(\theta)B(\mathbf{v}). \tag{3.27}$$

This fact is much more than an oddity, from it a phenomenon arises that will be fundamental for this work, the so called *Thomas-Wigner Rotation*[16].

3.3 THOMAS-WIGNER ROTATION

Two boosts compound to a boost only when they are colinear, that is, $B(\mathbf{v})B(\mathbf{w}) = B(\mathbf{u})$ only if $\mathbf{v} \times \mathbf{w} = 0$. Being this the case, $\mathbf{u}$ will naturally be colinear to $\mathbf{v}$ and $\mathbf{w}$. Since speed in relativity is limited by the speed of light c, the composition between $\mathbf{v}$ and $\mathbf{w}$ is not additive, it is instead given by

$$u = \frac{v + w}{1 + \frac{vw}{c^2}}. \tag{3.28}$$

In non-relativistic classical physics, speeds are additive. The classical regime can be easily obtained by taking the limit $vw/c^2 \to 0$ in equation (3.28).

Alternatively, if boosts do not lie within the same line, they always compound to a boost and a rotation. We will note this special kind of rotation as $R_{\hat{n}}(\Omega)$ and call it Wigner's rotation. A representation of the phenomena appears in figure 4. As a logical statement, one can write this fact as: If $\mathbf{v} \times \mathbf{w} \neq 0$ then

$$B(\mathbf{v})B(\mathbf{w}) = B(\mathbf{u})R_{\hat{n}}(\Omega). \tag{3.29}$$

---

[16] The Wigner rotation, named after Eugene Wigner, a 20^{th} century Hungarian-American physicist, was actually discovered by Ludwik Silberstein, a Polish-American physicist of the same era.

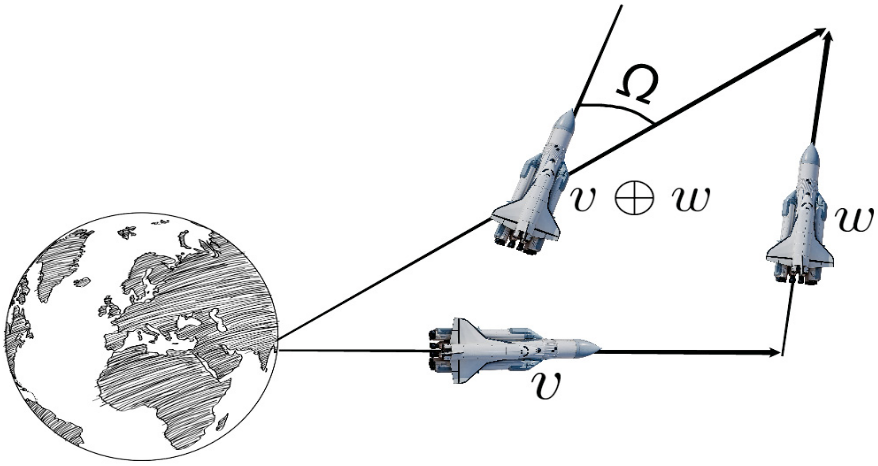

FIGURE 4 – Schematic representation of Wigner's rotation. The number first spaceship travels away from Earth with velocity v, the second spaceship travels away from the former ship with velocity w. Although Earth's axis are aligned with the first spaceship's axis and the first's with the second spaceship's, to Earth, ship number 2 travels away with velocity $v \oplus w$ that has an angle that differs of Ω from the angle related to w. That is the Wigner angle that appears from the Wigner rotation of the second spaceship.

The determination of the components of the matrix of rotation, $R_{\hat{n}}(\Omega)$, is difficult, but known. A great derivation of the components can be found in (O'DONNELL; VISSER, 2011). As with any rotation, to know $R_{\hat{n}}(\Omega)$ is to know $\hat{n}$, that is, the axis of rotation, as well as Ω the angle of rotation. The easiest of the two is undoubtedly the axis since it is just the axis perpendicular to the plane described by both boost velocities. Being $\hat{v}$ the direction of $B(\mathbf{v})$ and $\hat{w}$ the direction of $B(\mathbf{w})$, we can find $\hat{n}$ using

$$\hat{n} = \hat{v} \times \hat{w}, \tag{3.30}$$

where $\times$ is the usual external product.

To determine Ω one might first multiply (3.29) by $B^{-1}(\mathbf{u})$ from the left, obtaining

$$B^{-1}(\mathbf{u})B(\mathbf{v})B(\mathbf{w}) = R_{\hat{n}}(\Omega). \tag{3.31}$$

Next, to continue on the quest for Ω's value, one can write $\mathbf{u}$ using the relativistic composition of velocities, whose general formula is

$$\mathbf{u} = \frac{1}{1 - \frac{\mathbf{v}\cdot\mathbf{w}}{c^2}} \left[\frac{\mathbf{w}}{\gamma_v} - \mathbf{v} + \frac{\gamma_v}{1+\gamma_v} \frac{\mathbf{w}\cdot\mathbf{v}}{c^2}\mathbf{v} \right]. \tag{3.32}$$

Now, using (3.17d), it's easy to affirm that $B^{-1}(\mathbf{u}) = B(-\mathbf{u})$. The direct consequence of that is that, now, $R_{\hat{n}}(\Omega) = B(-\mathbf{u})B(\mathbf{v})B(\mathbf{w})$. Finally, a rotation matrix as defined in (3.26) obeys the trace-related equation

$$\text{Tr}(R_{\hat{n}}(\Omega)) = 2 + 2\cos\Omega. \tag{3.33}$$

Hiding the algebra under the rug, one gets

$$\cos\Omega = \frac{(\gamma_u + \gamma_v + \gamma_w + 1)^2}{(\gamma_u + 1)(\gamma_v + 1)(\gamma_w + 1)} - 1. \tag{3.34}$$

Equation (3.34) is general. A special case that is of great interest for this work is the rotation when the two boosts being composed are perpendicular, that is, when $|\hat{v} \times \hat{w}| = 1$. Being this the case, $\mathbf{w} \cdot \mathbf{v} = \mathbf{0}$ and (3.32) reduces to

$$\mathbf{u} = \frac{\mathbf{w}}{\gamma_v} - \mathbf{v}. \tag{3.35}$$

In cases such as this, we have

$$\cos\Omega = \frac{(\gamma_w + 1)(\gamma_v + 1)}{\gamma_w \gamma_v + 1} - 1. \tag{3.36}$$

In this work the general formula (3.34) will be used only when it proves useful to demonstrate the dependence of the rotation on the angle-between-boosts. Such cases are of less interest and therefore (3.36) will be the most useful of them both.

Although not mathematically complicated, the phenomenon just laid out has been described as paradoxical (MOCANU, 1992) and is almost impossible for someone without relativity-trained intuition to understand. Put as a sentence the apparent contradiction comes forth straightforwardly: If observer B moves away from A rapidly, and a third observer C moves rapidly away from B in a velocity perpendicular to the one that connects A and B, then A and B can agree on the direction of their axes. And so can B and C. But A and C cannot!

One familiar with logic will find that the previous sentence seems to violate the transitive property of equality. This evades being a straight logic contradiction only when one accounts for the fact that, if two sets of axes are "indistinguishable" they are not necessarily "equal" and, as it turns out, boost-connected axes are indistinguishable but cannot be equal thanks to the Wigner rotations.

Apparent paradoxes are no rarity in SR and definitely help "season" the theory to the palate of many a dreaming physicist. For those captivated by the previous paragraphs, a quick search on the topics of the twin paradox and barn paradox can prove an exquisite food for thought.

The next and last stop on the train of SR is the "Unitary Representation Station". In this station, the generators of the Lorentz transformation live, and understanding their function is fundamental to extend SR to the quantum world.

3.3.1 Unitary Representation of the Lorentz Group

A fact not explicitly mentioned before is that $\mathbb{P}$ is a *continuous group*[17] (and so is $\mathbb{L}$, since it is a subgroup of $\mathbb{P}$). A proper definition of a continuous group is as follows:

Definition 5. A *continuous group* is such that the composition of any two elements is determined by *analytical functions* of the elements being composed. That is, if $y = (y_1, ..., y_n)$ parameterize $\mathbb{G}$, this means that all elements of $\mathbb{G}$ are given by the same function $g(\cdot)$, of parameter y, then

$$g(y'') = g(y)g(y'), \tag{3.37}$$

such that y'' is an analytical function of y and y', that is, $y'' = f(y, y')$.

Since analytical functions are defined as being equivalent to a locally convergent power series, they must be describable by elements infinitesimally close to identity, the first term of the series. If the notation for an infinitesimal transformation is Λ_{inf}, then as argued just earlier

$$\Lambda_{inf} = \mathbb{I}_{4x4} + \Pi. \tag{3.38}$$

Here Π is also infinitesimal and can have its properties and form discovered from the previous properties of the transformations. If (3.38) is to be believed the composition of many Λ_{inf} generates the finite transform Λ, since any power series will be just an infinite linear combination of powers of $\mathbb{I}_{4x4} + \Pi$. Also, one familiar with power series could have noticed that (3.38) is just a truncation of e^{Π} as $\Pi \longrightarrow 0$. Both paths lead to the same fact, which is that (3.38) implies that

$$\Lambda = e^{\Pi}. \tag{3.39}$$

If one knows the form of the general finite transformation, one can start to enforce some known conditions on it, for instance

$$\Lambda^{-1}\Lambda = \mathbb{I} \therefore \Lambda^{-1}e^{\Pi} = \mathbb{I} \longrightarrow \Lambda^{-1} = e^{-\Pi}. \tag{3.40}$$

We also have that

$$\Lambda^T = (e^{\Pi})^T = e^{\Pi^T}, \tag{3.41}$$

consequently,

$$\Lambda^T\eta\Lambda = \eta \longrightarrow \eta\Lambda^T\eta = \Lambda^{-1} \longrightarrow \eta e^{\Pi^T}\eta = e^{-\Pi}. \tag{3.42}$$

[17] It was said they are Lie groups, and all Lie groups are continuous.

In this case,

$$e^{\eta\Pi^T\eta} = \mathbb{I} + \eta\Pi^T\eta + \frac{1}{2!}(\eta\Pi^T\eta)^2 + ... \tag{3.43a}$$
$$= \mathbb{I} + \eta\Pi^T\eta + \frac{1}{2!}\eta\Pi^T\eta\eta\Pi^T\eta + ... \tag{3.43b}$$
$$= \mathbb{I} + \eta\Pi^T\eta + \frac{1}{2!}\eta(\Pi^T)^2\eta + ... \tag{3.43c}$$
$$= \eta(\mathbb{I} + \Pi^T + \frac{1}{2!}(\Pi^T)^2 + ...)\eta \tag{3.43d}$$
$$= \eta e^{\Pi^T}\eta, \tag{3.43e}$$

resulting in

$$\eta e^{\Pi^T}\eta = e^{\eta\Pi^T\eta} = e^{-\Pi} \longrightarrow \eta\Pi^T\eta = -\Pi. \tag{3.44}$$

Quite a lot of information can be obtained about Π's components using (3.44) together with the signature η. For instance, one may note that

$$\Pi_{0i} = \Pi_{i0}, \tag{3.45a}$$
$$\Pi_{ij} = \Pi_{ji}, \tag{3.45b}$$
$$\Pi_{\mu\mu} = -\Pi_{\mu\mu} \therefore \tag{3.45c}$$
$$\Pi_{\mu\mu} = 0. \tag{3.45d}$$

With (3.45) in mind a general matrix representation of Π can be written as

$$\begin{aligned}
\Pi &= \begin{Bmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{10} & 0 & \omega_{12} & \omega_{13} \\ \omega_{20} & -\omega_{12} & 0 & \omega_{23} \\ \omega_{30} & -\omega_{13} & -\omega_{23} & 0 \end{Bmatrix} \\
&= \omega_{01}\begin{Bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix} + \omega_{02}\begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix} + \omega_{03}\begin{Bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Bmatrix} \\
&+ \omega_{12}\begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix} + \omega_{13}\begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Bmatrix} + \omega_{23}\begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{Bmatrix} \\
&= \omega_{01}M^{01} + \omega_{02}M^{02} + \omega_{03}M^{03} + \omega_{12}M^{12} + \omega_{13}M^{13} + \omega_{23}M^{23},
\end{aligned} \tag{3.46}$$

where the matrices M are the generators of the Lorentz group[18]. This matrices can be separated into two distinct classes, those that are symmetric, namely, M^{0i} and those

[18] As momentum is the generator of translation, the generators of Lorentz transformations are the components of the "relativistic angular momentum", it being the generalization of the classical angular momentum.

that are anti-symmetric M^{ij} with $i \neq 0$. What maybe be hard to visualize is the fact that these two classes are directly related to the two components of a general Lorentz transformation, that is, Boosts and Rotations, their generators being, respectively, M^{0i} and M^{ij}. For such a reason, it proves useful to redefine the notation used via

$$Q^i = M^{0i}, \quad J^i = \epsilon_{ijk} M^{jk}. \tag{3.47}$$

Using the new notation one might realize that the following commutation relations apply:

$$[Q^i, Q^j] = -\epsilon_{ijk} J^K, \tag{3.48a}$$

$$[J^i, J^j] = \epsilon_{ijk} J^k, \tag{3.48b}$$

$$[J^i, Q^j] = \epsilon_{ijk} Q^k. \tag{3.48c}$$

This demonstrates once again the fact that rotations, represented by their generators J, define a closed algebra and form a group[19]. Meanwhile, boost generators do not! Using (3.39) and (3.48) one might rewrite a general Lorentz transform as

$$\Lambda = e^{-(\vec{\omega}\cdot\mathbf{J}+\vec{b}\cdot\mathbf{Q})}, \tag{3.49}$$

where $\mathbf{J} = J^1\hat{x} + J^2\hat{y} + J^3\hat{z}$, $\mathbf{Q} = Q^1\hat{x} + Q^2\hat{y} + Q^3\hat{z}$, $\vec{\omega}$ represents a rotation around a $\hat{\omega}$ axis and $\vec{b}$ represents a boost on the $\hat{b}$ direction.

3.3.2 Examples

Considering a boost in the $\hat{x}$ direction, that is

$$\vec{\omega} = \vec{0} \qquad \text{and} \qquad \vec{b} = b\hat{x}, \tag{3.50}$$

consequently, we have

$$\Lambda = e^{-\vec{b}\cdot\mathbf{Q}} = \mathbb{I}_{4x4} - (\hat{x}\cdot\mathbf{Q})b + \frac{1}{2!}(\hat{x}\cdot\mathbf{Q})^2 b^2 - \frac{1}{3!}(\hat{x}\cdot\mathbf{Q})^3 b^3 + ... \tag{3.51}$$

[19] The $SO(3)$ group.

Using the fact that $(Q^i)^3 = Q^i$,

$$\begin{aligned}
\Lambda &= \mathbb{I}_{4x4} - (\hat{x}\cdot\mathbf{Q})(b + \frac{b^3}{3!} + \frac{b^5}{5!} + ...) + (\hat{x}\cdot\mathbf{Q})^2(\frac{b^2}{2!} + \frac{b^2}{4!} + \frac{b^6}{6!} + ...) \\
&= \mathbb{I}_{4x4} - (\hat{x}\cdot\mathbf{Q})\sinh(b) + (\hat{x}\cdot\mathbf{Q})^2(\cosh(b) - 1) \\
&= \begin{Bmatrix} \cosh(b) & -\sinh(b) & 0 & 0 \\ -\sinh(b) & \cosh(b) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} \\
&= \begin{Bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}
\end{aligned} \tag{3.52}$$

if $b = \tanh^{-1}(\beta)$.

The important takeaway from (3.52) is that $\vec{b} = tanh^{-1}(\beta)\hat{b}$ for the equality between the last two matrices written to be verified.

Let us do the same for a rotation around the $\hat{z}$ axis, for such

$$\vec{\omega} = \omega\hat{z} \qquad \text{and} \qquad \vec{b} = \vec{0}. \tag{3.53}$$

From this, it follows that:

$$\Lambda = e^{-\vec{\omega}\cdot\mathbf{J}} = \mathbb{I}_{4x4} - (\hat{z}\cdot\mathbf{J})\omega + \frac{1}{2!}(\hat{z}\cdot\mathbf{J})^2\omega^2 - \frac{1}{3!}(\hat{z}\cdot\mathbf{J})^3\omega^3 + ... \tag{3.54}$$

Using the fact that $(J^i)^2 = -\mathbb{I} \rightarrow (J^i)^3 = -(J^i)$,

$$\begin{aligned}
\Lambda &= \mathbb{I}_{4x4} - (\hat{z}\cdot\mathbf{J})(\omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} + ...) + (\hat{z}\cdot\mathbf{J})^2(\frac{\omega^2}{2!} - \frac{\omega^2}{4!} + \frac{\omega^6}{6!} + ...) \\
&= \mathbb{I}_{4x4} - (\hat{z}\cdot\mathbf{J})\sin(\omega) + (\hat{z}\cdot\mathbf{J})^2(\cos(\omega) - 1) \\
&= \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\omega) & -\sin(\omega) & 0 \\ 0 & \sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} \\
&= \mathbf{R}_{\hat{\mathbf{z}}}(\theta)
\end{aligned} \tag{3.55}$$

if $\omega = \theta$.

Both examples serve to illustrate how $\Lambda = e^{\Pi}$ is truly the general form of a finite Lorentz transformation, be it a boost, a rotation, or an amalgamation of both.

3.4 CHAPTER CLOSING REMARKS

Knowing the facts presented in this chapter, one ought to be ready to understand how different observers are connected in the universe. The details about how different their description of a physical system is, or better put, the *state* of one, shall be presented in the following chapters.

The essential takeaway from this chapter is that SR puts restrictions on frames of reference and, therefore, on observers. Also, it is within this theory that one finds the true definition of terms so often used in quantum information theory, such as Causality and Locality. Therefore, without accommodating SR's requirements, no physical theory can be said complete and representing of nature.

Having obtained the formalism to represent the consequences of Lorentz transformations for vectors and the unitary representation of such operations, the next step is to import such description into the realm of QM. This way the grave consequences that SR's Wigner rotations have for quantum resources can be demonstrated and measured.

4 RELATIVISTIC QUANTUM RESOURCES

Equipped with the formalism from the previous two chapters one can move forward towards a quantum mechanical representation of Lorentz transformation for spin-momentum states, as well as the consequences of such transformations for the quantum resources of such states.

This chapter shaw demonstrate that the fundamental quantum resources of interest for this work, entanglement and steering, are dependent on the frame of reference they are being evaluated on. Being this so, in the relativistic regime, the same state may have different statistical behavior. Let us see how.

4.1 RELATIVISTIC QUANTUM MECHANICS

To describe the physics of relativity in the formalism of QM, position and momentum, classically thought as vectors in $\mathbb{R}_3$ now must be extended to 4-vectors. The extension is made simply by attaching a Hilbert space, corresponding to the zeroeth component of the 4-vectors to the usual position space. This is, if originally $\mathcal{H}_r = \mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_z$, now, the 4-position space is simply

$$\mathcal{H}_\Lambda = \mathcal{H}_t \otimes \mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_z, \tag{4.1}$$

where $\mathcal{H}_t$ is the time component Hilbert space, and the Λ subscript on $\mathcal{H}_\Lambda$ serves only to mark the fact that it is a relativistic Hilbert space.

Although the notion of a time state and a time operator are controversial in QM, the polemic around it will be left for the "discussion" section. To ease the mind of the reader about such issues, two things can already be said. Firstly, a time component is **required** by SR because there, time is indistinguishable as a DoF from the usual three spatial ones. Secondly, the polemic nature of time in QM arises from the dynamics, that is so because the dynamics in QM are controlled by the time evolution operator which is itself a function of the Hamiltonian operator, the latter having time in a place of distinction from other DoF. The fact that time is detached from the spatial DoF can be made explicit by looking at Schrödinger's equation. The fact that differentiation on time occurs once and on position twice, demonstrates the asymmetry just mentioned. Schrödinger's equation as well as the dependence the time evolution has on the Hamiltonian are present in Appendix 1.

In the $\mathcal{H}_\Lambda$ space the canonical operators are X^μ and P^ν, their action on respective 4-position states, $|ct, x, y, z\rangle := |x^\mu\rangle$, and 4-momentum states, $|E/c, p_x, p_y, p_z\rangle := |p^\mu\rangle$, can be summed up as

$$X^\mu |x^\mu\rangle = x^\mu |x^\mu\rangle\,, \tag{4.2}$$

and

$$P^\mu \left|p^\mu\right\rangle = p^\mu \left|p^\mu\right\rangle . \tag{4.3}$$

4.1.1 Mass-Shell Constraint

In principle, the components of $|x^\mu\rangle$ can vary freely. The only requirement for them being to be covariant between two states connected via Lorentz transformation[1]. 4-momentum states on the other hand have a restriction. The $\mathcal{H}_t$ component $|E/c\rangle$ must be such that

$$p^\mu p_\mu = \frac{E^2}{c^2} - (p_x^2 + p_y^2 + p_z^2) = m_0^2 c^2. \tag{4.4}$$

This is so because the rest mass, m_0, remains constant for a given particle. Being $m_0 c^2$ a relativistic scalar (the inner product of the 4-momentum vector), no reference frame may measure for the same particle a $|p^\mu\rangle$ that is in disagreement with (4.4). Being this the case, the zeroeth component of the 4-momentum is $p_0 = \sqrt{\mathbf{p}^2 + m_0^2 c^2}$, with $\mathbf{p}^2 = (p_x^2 + p_y^2 + p_z^2)$. Knowing that, one may take a general state of 4-momenta

$$|\psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^4p\, \overline{\psi}(p^\mu) \left|p^\mu\right\rangle \tag{4.5}$$

and rewrite it, using a Dirac delta[2] function to represent the restriction as

$$\begin{aligned} |\psi\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^4p\ \delta(p^\mu p_\mu - m_0^2c^2)\overline{\psi}(p^\mu) \left|p^\mu\right\rangle \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3p dp^0\ \delta((p^0)^2 - \mathbf{p}^2 - m_0^2c^2)\overline{\psi}(p^0, p^i) \left|p^0, p^i\right\rangle \end{aligned} \tag{4.6}$$

Before integrating over dp^0, a new condition is added: the condition that only positive energies are permitted. The appearance of negative energies in the context of relativistic QM is a well-known fact and a meaningful one: in Dirac's equation (DIRAC, 1928), negative energy solutions led to the theoretical prediction of antiparticles, that were later discovered. In the present context though, only positive energies are permitted since we are modeling particles moving time-fowardly. More on this fact will be presented in the discussion section as well. A Heavside function, $\theta(p^0)$, defined as

$$\theta(x-b) = \begin{cases} 0, & \text{if } x < b \\ 1, & \text{if } x \geq b, \end{cases} \tag{4.7}$$

will take care of this,

$$\begin{aligned} |\psi\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp^i dp^0\ \delta((p^0)^2 - \mathbf{p}^2 - m_0^2c^2)\overline{\psi}(p^0, p^i)\theta(p^0) \left|p^0, p^i\right\rangle \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp^i \frac{1}{2\sqrt{\mathbf{p}^2 + m_0 c^2}}\ \overline{\psi}(\sqrt{\mathbf{p}^2 - m_0^2c^2}, p^i) \left|\sqrt{\mathbf{p}^2 - m_0^2c^2}, p^i\right\rangle . \end{aligned} \tag{4.8}$$

[1] The relation between the values of the time and position components will tell about the nature of the event they describe. That is so because it is with them that intervals are calculated.

[2] The filtration property of the delta function will guarantee integration only over states that obey the mass-shell restriction.

Since p^0 is no longer a free parameter, we can write the state free of any dependence on it by introducing the invariant integration measure $d\mu(p^i) := d^3p/(2\sqrt{\mathbf{p}^2 + m_0c^2})$. So

$$|\psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mu(p^i)\overline{\psi}(p^i)\,|p^i\rangle\,. \tag{4.9}$$

The properties of importance for the $\{|p^i\rangle\}$ basis are

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mu(p^i)\,|p^i\rangle\,\langle p^i| = \mathbb{I}, \tag{4.10}$$

and

$$\langle p^i|p^j\rangle = 2p_0\delta(p^i - p^j). \tag{4.11}$$

The momentum wave equation normalization is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mu(p^i)|\overline{\psi}(p^i)|^2 = 1. \tag{4.12}$$

4.1.2 The relativistic spin-momentum state

The states of interest of this work are momentum-spin states; in equation (4.9) a general momentum distribution was obtained. The addition of the spin DoF is made straightforwardly so that the relativistic version of equation (2.35) is obtained

$$|\psi\rangle = \sum_{s=-1/2}^{s=+1/2}\int_{-\infty}^{\infty} d\mu(p^i)\overline{\psi}_s(p^i)\,|s, p^i\rangle\,. \tag{4.13}$$

Now, a small caveat must be brought up: spin is defined as the intrinsic angular momentum of a particle. At rest, that is, $p^i = 0$, no orbital angular momentum can exist, so all that is left is spin. For this reason, spin is usually defined on such rest-frames-of-reference but, by writing a state of the kind $|s, p^i\rangle$, another reference frame is already being taken into consideration, the one that measures the values p^i for P^i. Naturally, for a single value of momentum, a single Lorentz transformation, being the appropriate boost, can access that frame, leaving spin unaltered since a single boost does not cause rotation.

Unfortunately, no single transformation can take the particle from rest, whose associated state is $|p^i = 0\rangle$ to the distribution in (4.13). Only a superposition of infinitely many, individually adjusted transformation operators could. Being this so, no single definition of spin could be associated with a momentum distribution, and the separability of both DoF, in the way done in (4.13) would be meaningless. So a hypothesis is laid out as follows.

Hypothesis. There exists a frame of reference different from rest in which spin is well-defined and independent from momentum, even if the state is not in a momentum eigenstate. Put straightforwardly, states such as

$$|\psi\rangle = \int_{-\infty}^{\infty} d\mu(p^i)\overline{\psi}_s(p^i)\,|p^i\rangle \otimes |s\rangle \tag{4.14}$$

exist.

This hypothesis is equivalent to the hypothesis that it is possible to perturb the momentum state without perturbing the spin. For instance, if a particle with a defined non-zero momentum and separable spin, that is, $|\psi\rangle = |p^i\rangle \otimes |s\rangle$, passes through a semi-transparent mirror with a 50% chance of reflection, then the state becomes $|\psi\rangle_{mirror} = \frac{1}{\sqrt{2}}(|p^i\rangle + |-p^i\rangle) \otimes |s\rangle$, a seemingly reasonable proposition. Being this the case, states like the one in (4.14) are preparable.

4.1.3 General State Transformation

Being in possession of the general spin-momentum state the next step is to define how a Lorentz transformation acts over it. For such, one may consider that the transformation $\Lambda p^\mu = p'^\mu$ is paralleled by the state transformation

$$U(\Lambda)|p^\mu\rangle = |p'^\mu\rangle, \tag{4.15}$$

where $U(\Lambda)$ is the unitary operator that implements the Λ transformation. For spin, the transformation is slightly more complex, starting in the rest frame $|s, k^\mu\rangle$, where $k^\mu = (p^0, 0, 0, 0)$ and spin is well defined, we go to the lab frame $|s, p^\mu\rangle := |s, p^\mu\rangle$ via

$$U(\Lambda_{p^\mu})\,|s, k^\mu\rangle = |s, p^\mu\rangle. \tag{4.16}$$

Now, in possession of the lab eigenstate, one may ask how any observer, with a frame connected to the lab by the transformation Λ, sees the vector. The formalism shows that what happens is

$$U(\Lambda)|s, p^\mu\rangle = \sum_{s'} R_{s's}(\Lambda, p'^\mu)|s, p'^\mu\rangle, \tag{4.17}$$

where $R_{s's}(\Lambda, p'^\mu) := \langle s'|R|s\rangle$ are the elements of the Wigner rotation matrix. It is important to note that the rotation depends on p'^μ. This is so because it supposedly acts on the rest-frame that is obtained via $U(\Lambda_{p'^\mu})^{-1}$. As previously discussed, although the rotation is modeled to occur at rest, we hypothesize that the degrees of freedom are accessible to the observer on the moving frame.

Before moving forward, it is important to show how the rotation represented by $R_{s's}(\Lambda, p'^\mu)$ acts on a spin-1/2 space, being it

$$R_{s's}(\Lambda, p'^\mu) = \cos\left(\frac{\Omega}{2}\right)\mathbb{I} + i\sin\left(\frac{\Omega}{2}\right)\hat{n}\cdot\vec{\sigma}. \tag{4.18}$$

Here Ω is the Wigner angle, $\hat{n}$ is the axis of rotation and $\vec{\sigma}$ is the standard Pauli vector, $\vec{\sigma} = \hat{i}\sigma_x + \hat{j}\sigma_y + \hat{k}\sigma_z$.

Alright, being the transformation for an eigenstate well understood one can begin to apply it to general states such as the one in equation (4.13), being $|\psi\rangle$ the state in the lab and $|\psi'\rangle$ the transformed state:

$$\begin{aligned} U(\Lambda)|\psi\rangle = |\psi'\rangle = U(\Lambda) \sum_{s=-1/2}^{s=+1/2} \int_{-\infty}^{\infty} d\mu(p^i)\overline{\psi}_s(p^i)|s, p^i\rangle \\ = \sum_{s'} \sum_{s=-1/2}^{s=+1/2} \int_{-\infty}^{\infty} d\mu(p^i)\overline{\psi}_s(p^i)R_{s's}(\Lambda, p'^{\mu})|s, p'^i\rangle. \end{aligned} \tag{4.19}$$

At this point, generality must give way to especifity if the phenomena is to be understood. Let's look at the effects of Lorentz transformation on less abstract states so the consequences may be better visualized.

4.2 STATES OF INTEREST

After presenting the formalism of relativistic QM, the continuation of this work comes in the form of applying it to concrete examples better paralleled by nature. Following that, one should be able to better understand its importance and usefulness.

4.2.1 Contrapropagating Momenta Eigenstates

The most trivial but interesting state in this context is the case in which the system travels either to the left or right with equal probability and the same velocity, that is, where the state is given by

$$|\psi\rangle = \frac{(|p^{\mu}\rangle + |-p^{\mu}\rangle)}{\sqrt{2}} \otimes |\sigma_x = +\rangle. \tag{4.20}$$

Here the spin could be defined in any direction as long as the state was separable, to work this out as an example the x-direction was chosen. To guarantee normalization of the state in (4.20), either momentum has to be taken to be discrete, as can be done considering that the experimental observation of momentum always happens in finite intervals, or take the path from its construction as a continuous DoF and use delta functions as wave functions to select eigenstates of momenta. Respectively, these approaches can be found in (FREIRE; ANGELO, 2019) and (ENGELBERT, 2022). One of this state's exquisite properties is the fact that it cannot be obtained from a single boost out of a rest frame, that is

$$|\psi\rangle \neq U(\Lambda)(|k^{\mu}\rangle \otimes |\sigma\rangle). \tag{4.21}$$

Actually, if it is true that $|\psi\rangle = T(|k^\mu\rangle \otimes |\sigma\rangle)$, than, the generic transformation T must be something of the kind of

$$T = \frac{U(\Lambda) + U(-\Lambda)}{\sqrt{2}}. \tag{4.22}$$

What (4.22) shows is that T is a boost superposition. Being such, it is not a classical frame of reference transformation. It is an all-different being that belongs to the genre of quantum reference frame (QRF) transformations. Sidelining the nuanced discussion around QRFs in the name of progress, one advances by noticing that

$$U(\Lambda)|\psi\rangle = |\psi'\rangle \xrightarrow{(4.18)} \frac{|p^\mu\rangle \otimes R(\Omega)|\sigma_x = +\rangle + |-p^\mu\rangle \otimes R(-\Omega)|\sigma_x = +\rangle}{\sqrt{2}}, \tag{4.23}$$

here $R(-\Omega)$ is the same rotation as $R(\Omega)$, but in the opposite direction. The dependence of the rotation on the boost and momentum has been made implicit by the Wigner angle Ω. Moving forward in the path that goes from generality to specificity, one must fix both Λ and p^μ to define the way $R(\Omega)$ operates. Considering that $p^\mu = (E/c, 0, 0, p_z)$ and that $\Lambda = B(v\hat{i})$, a boost in the x-direction, the rotation will occur around the y-axis, and the only momentum components to change under Λ will be p^0 and p^1. Therefore

$$|\psi'\rangle = |p^0, p^1, p^2\rangle \frac{|p_z\rangle \otimes R(\Omega)|\sigma_x = +\rangle + |-p_z\rangle \otimes R(-\Omega)|\sigma_x = +\rangle}{\sqrt{2}}. \tag{4.24}$$

In principle both $|p^2\rangle$ and $|p_z\rangle$ could be painted red since they remained unaltered under Λ. But, since the blue frame of reference will measure the same values, painting them blue is fine as well. Since the direction of rotation was controlled by the original z-momentum sign, the way $U(\Lambda)$ alters the spin state depends on each of the particle's momenta superposition branches. Such fact makes the operation of $U(\Lambda)$ over $|\psi\rangle$ not dissimilar to the operation of U_{cnot} over an informational QuBit, being the latter a standard entanglement creation operator widely known in literature (NIELSEN; CHUANG, 2010). In fact, for $\Lambda(v)$ with $v \to c$, $\Omega \to \pi/2$ and

$$|\psi'\rangle \xrightarrow{v \to c} \frac{|p_z\rangle \otimes |\sigma_z = +\rangle + |-p_z\rangle \otimes |\sigma_z = -\rangle}{\sqrt{2}}, \tag{4.25}$$

where the three separable momenta have been disregarded since the interest lies in entanglement. Using the usual computational basis notation for spin, $\{|\sigma_z = +\rangle = |0\rangle, |\sigma_z = -\rangle = |1\rangle, |\sigma_x = +\rangle = |+\rangle, |\sigma_x = -\rangle = |-\rangle\}$, we can rewrite the previous process as

$$\begin{aligned} |\psi\rangle &= \frac{(|p_z\rangle + |-p_z\rangle) \otimes |+\rangle}{\sqrt{2}}, \\ U(\Lambda(c))|\psi\rangle &= \frac{|p_z\rangle \otimes |1\rangle + |-p_z\rangle \otimes |0\rangle}{\sqrt{2}}. \end{aligned} \tag{4.26}$$

The way the $U(\Lambda(c))$ operator takes a separable state into an entangled one, on equation (4.26), is easily comparable with

$$\begin{aligned} |\phi\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \,, \\ U_{cnot}\, |\phi\rangle &= \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}}, \end{aligned} \tag{4.27}$$

being this the standard entanglement creation procedure in the context of quantum computation (WILLIAMS, 2010).

Having proven that the Lorentz boost can take separable states into entangled ones, one may ask how to account for its amount. Let's apply to this state the entanglement measure based on the partial state entropy, that we have introduced in equation (2.48), using

$$R(\Omega)\,|+\rangle = \cos(\Omega/2)\,|0\rangle + \sin(\Omega/2)\,|1\rangle \,, \tag{4.28}$$

we rewrite the state in (4.24) as

$$|\psi'\rangle = \frac{|p_z\rangle \otimes (\cos(\Omega/2)|0\rangle + \sin(\Omega/2)|1\rangle) + |{-p_z}\rangle \otimes (\cos(\Omega/2)|0\rangle - \sin(\Omega/2)|1\rangle)}{\sqrt{2}}, \tag{4.29}$$

such that, on a $\{|p_z, 0\rangle\,, |p_z, 1\rangle\,, |{-p_z}, 0\rangle\,, |{-p_z}, 1\rangle\}$ basis[3], we obtain

$$\begin{aligned} \rho &= |\psi'\rangle\,\langle\psi'| \\ &= \frac{1}{2}\begin{Bmatrix} \cos^2\left(\frac{\Omega}{2}\right) & \cos\left(\frac{\Omega}{2}\right)\sin\left(\frac{\Omega}{2}\right) & \cos^2\left(\frac{\Omega}{2}\right) & -\cos\left(\frac{\Omega}{2}\right)\sin\left(\frac{\Omega}{2}\right) \\ \cos\left(\frac{\Omega}{2}\right)\sin\left(\frac{\Omega}{2}\right) & \sin^2\left(\frac{\Omega}{2}\right) & \cos\left(\frac{\Omega}{2}\right)\sin\left(\frac{\Omega}{2}\right) & -\sin^2\left(\frac{\Omega}{2}\right) \\ \cos^2\left(\frac{\Omega}{2}\right) & \cos\left(\frac{\Omega}{2}\right)\sin\left(\frac{\Omega}{2}\right) & \cos^2\left(\frac{\Omega}{2}\right) & -\cos\left(\frac{\Omega}{2}\right)\sin\left(\frac{\Omega}{2}\right) \\ -\cos\left(\frac{\Omega}{2}\right)\sin\left(\frac{\Omega}{2}\right) & -\sin^2\left(\frac{\Omega}{2}\right) & -\cos\left(\frac{\Omega}{2}\right)\sin\left(\frac{\Omega}{2}\right) & \sin^2\left(\frac{\Omega}{2}\right) \end{Bmatrix}. \end{aligned} \tag{4.30}$$

Now, taking the partial trace over momentum, one obtains

$$\begin{aligned} \rho_s &= \mathrm{Tr}_p(|\psi'\rangle\,\langle\psi'|) \\ &= \frac{1}{2}\begin{Bmatrix} \cos^2\left(\frac{\Omega}{2}\right) + \sin^2\left(\frac{\Omega}{2}\right) & \cos^2\left(\frac{\Omega}{2}\right) - \sin^2\left(\frac{\Omega}{2}\right) \\ \cos^2\left(\frac{\Omega}{2}\right) - \sin^2\left(\frac{\Omega}{2}\right) & \cos^2\left(\frac{\Omega}{2}\right) + \sin^2\left(\frac{\Omega}{2}\right) \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} 1 & \cos(\Omega) \\ \cos(\Omega) & 1 \end{Bmatrix}. \end{aligned} \tag{4.31}$$

Being this matrix defined on the spin-z basis $\{|0\rangle\,, |1\rangle\}$. The diagonal form of this matrix is

$$\rho_s = \frac{1}{2}\begin{Bmatrix} 1 + \cos(\Omega) & 0 \\ 0 & 1 - \cos(\Omega) \end{Bmatrix}, \tag{4.32}$$

[3] Since the momentum space is infinite, no finite matrix could rigorously represent this density matrix, but all terms not represented in (4.30) are zero and would not alter the partial state after the trace over the continuous space.

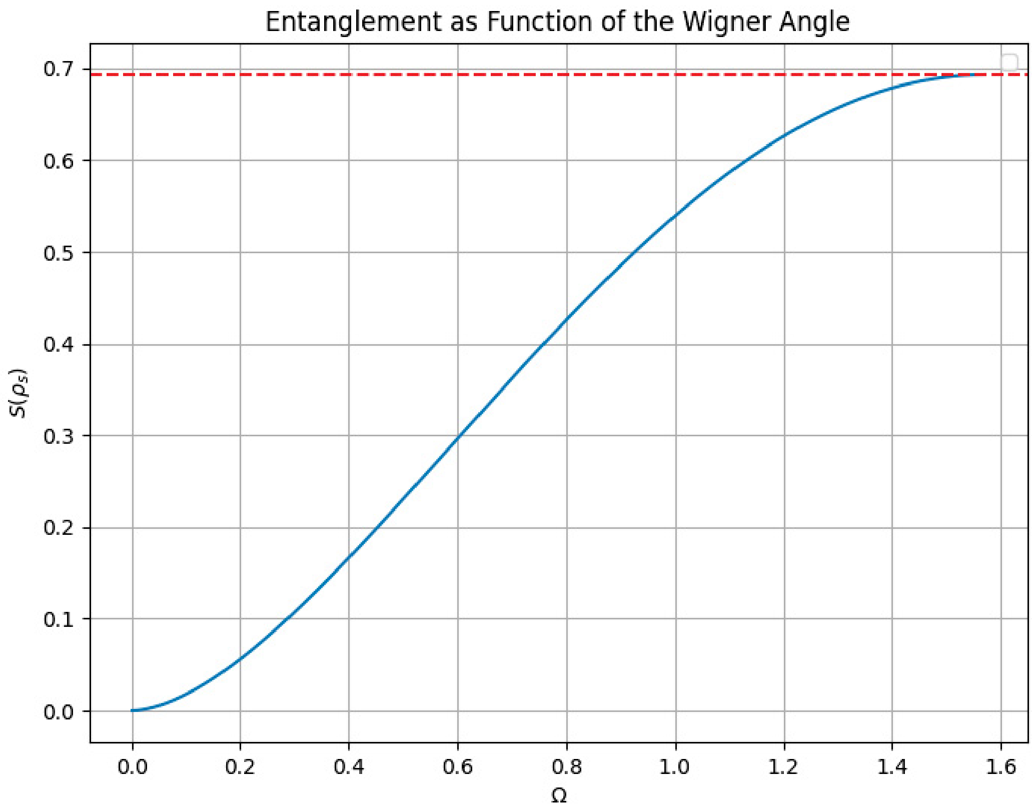

FIGURE 5 – Plot of the dependence of the partial state entropy of a spin-momentum system on the Wigner angle, in blue. The upper bound for a two-dimensional partial state entropy appears in red. The state for $\Omega = 0$ is separable and, therefore, the entropy is null. For angles greater than zero, the entropy increases accounting for entanglement. For $\Omega = \pi/2$ the entropy reaches its maximum of $S(\rho_s) = \ln(2)$, and the system has become fully entangled.

on the spin-x basis $\{|+\rangle, |-\rangle\}$. Having a diagonal representation in hands, the von Neumann entropy of the partial state is simply

$$\begin{aligned} S(\rho_s) &= -\sum_{i=1}^{2} p_i \ln(p_i) \\ &= -\left(\left(\frac{1+\cos(\Omega)}{2}\right)\ln\left(\frac{1+\cos(\Omega)}{2}\right) + \left(\frac{1-\cos(\Omega)}{2}\right)\ln\left(\frac{1-\cos(\Omega)}{2}\right)\right). \end{aligned} \tag{4.33}$$

To better visualize this behavior, the plot of the function in (4.33) appears in figure 5.

Being it clear that entanglement increases as the Wigner angle tends to $\pi/2$, reaching its maximum exactly for this angle, the natural next step for this work would be to verify if this state violates a steering criterion and, if so, for which angle or other parameter of preparation the violation first occurs. Unfortunately, although, for pure states such as the one being studied, all steerable states are entangled, and vice-versa (WISEMAN et al., 2007), not all criteria will detect it as such. The chosen criterion for this work, Reid's variance criterion, for instance, will not. That is so because the criterion depends on the variance of both momentum and position, and states like the one in (4.23) have finite momentum variance but infinite position variance. This is a consequence of the fact that an eigenstate in momentum is a maximally delocalized state in position, as discussed in chapter 2. Furthermore, the amount of variance reduction that spin measurements can induce is limited by its (small) dimension[4], being it so, it can never take an infinite dispersion state into one that violates the Heisenberg uncertainty principle and Reid's criterion, therefore.

Another disadvantage of this simple state is that it is not very realistically implementable. Take the short discussion of the previous paragraph about the position variance for instance. The state is totally positionally undefined, how will one "find" the system within the walls of a laboratory to measure it? Also, no experimental instrument has infinite precision as to access the very fine-grained structure of continuous momentum, much more likely the system will be found in a small interval of momenta, being it so, preparations such as the one in (4.23) are idealized and not implementable.

Both problems just mentioned can be worked on by having a different preparation for the system, let us look into it.

4.2.2 Gaussian Momenta Superposition

Rarely it is the case that a system in a momenta superposition state has its z-momenta being either completely greater or lesser than zero, like the one is (4.24). The more nuanced and likely case is the one where the momentum is a Gaussian distribution around a center value that itself can be greater or lesser than zero (superpositions of distribution around different centers are also allowed). Now, as long as the Gaussian distributions are either very narrow (small dispersion) or centered around a value distant from zero or, ideally, both, there will be a vanishing (although never null) chance that, when making a measurement over a "negative" distribution, an experimentalist will obtain a positive result and vice-versa. A general form for states obeying the properties just listed are

$$|\pm p_0\rangle = \int_{-\infty}^{\infty} \frac{\Delta^{1/2}}{\pi^{1/4}} \, dp \; e^{-\frac{\Delta^2}{2}(p\mp p_0)^2} e^{-ipx_0} \, |p\rangle \, . \tag{4.34}$$

There is much information about the state in this definition, $\pm p_0$ are the Gaussian centers for both positive and negative momenta distribution. They are equally far from the origin. Both momentum distributions are associated with a single position, one whose Gaussian center is x_0. This is to be understood as the particle having a reasonably well-defined position independent of its momentum, for example, being contained inside a laboratory. Finally, Δ is the inverse of the momentum dispersion, meaning that a very

[4] This too will be elaborated in the discussion section.

sharp Gaussian will have large values for Δ. All these parameters are set in arbitrary units.

Now, the interest lies in states of systems that have both positive and negative momenta in superposition, therefore the description that will be used is a superposition of both $|p_0\rangle$ and $|-p_0\rangle$, as follows

$$|\psi\rangle = \frac{\Delta^{1/2}}{(2\sqrt{\pi}(1+e^{-\Delta^2 p_0^2}))^{1/2}} \int_{-\infty}^{\infty} dp\ (e^{-\frac{\Delta^2}{2}(p-p_0)^2} + e^{-\frac{\Delta^2}{2}(p+p_0)^2})e^{-ipx_0}|p\rangle \otimes |+\rangle. \quad (4.35)$$

Notice that the spin-state appeared already and that the normalization of the superposition is different than just $1/\sqrt{2}$ times each Gaussian state normalization, that is thanks to the interference between both Gaussians and is expected. The "usual" value appears in the regime where $\Delta^2 p_0^2 \to \infty$ such that $e^{-\Delta^2 p_0^2} \to 0$. Also of importance is the fact that only one of the four Hilbert spaces that are part of the full 4-position space is being represented. That is so because spin-momentum entanglement is being created only on the axis of the momentum superposition perpendicular to the boost. This is a choice and depends on preparation, but being the state in equation (4.35) preparable, and it is, this choice greatly simplifies the calculations yet to be done[5].

Although calculations could be done, the separability of the spin and position states guarantees that no action done by one party could alter the other's dispersion and, therefore, no steering can occur. Simply put, no entanglement means no steering.

Now the case in which the state is somewhat rotated will be considered so a point where steerability appears, if so, based on either the rotation angle or the boost parameter (since they are uniquely connected), may be found. If $0 < v < c$, $0 < \Omega < \pi/2$ and following the rotation rules for spin-states, considering a boost-induced-rotation around the y-axis, as is the case in which the state is in a x-momentum state and the boost is on the z-direction, or vice-versa, that is controlled by the Gaussian centers[6]. The state after the Lorentz transformation is

$$\begin{aligned}|\psi'\rangle = \frac{\Delta^{1/2}}{(2\sqrt{\pi}(1+\cos(\Omega)e^{-\Delta^2 p_0^2}))^{1/2}} \int_{-\infty}^{\infty} dp\ e^{-ipx_0}(e^{-\frac{\Delta^2}{2}(p-p_0)^2}(\alpha|+\rangle + \beta|-\rangle) \\ + e^{-\frac{\Delta^2}{2}(p+p_0)^2}(\alpha|+\rangle - \beta|-\rangle))|p\rangle.\end{aligned} \quad (4.36)$$

Here $\alpha = \cos(\Omega/2)$ and $\beta = \sin(\Omega/2)$. The appearance of $\cos(\Omega)$ in the normalization factor is due to how the rotation was conducted (controlled by p_0 and not p). Calling

5 The reasonability of this choice comes from the fact that, given any superposition of any two momentum 3-vectors, they will always live in either a line, when parallel or anti-parallel, or a plane, when not. Being the case that they are in the same line, a single Hilbert space is already enough to describe them, if in a plane, the two Hilbert spaces required to describe the vectors can be chosen such that the superposition lies in only one of them. In other words, coherence is present only in one of the two.

6 The most rigorous case would have the rotation controlled by each infinitesimal interval of momentum. This process would produce an all-too-complicated spin distribution that would be mathematically unapproachable. Anyway, as $e^{-\Delta^2 p_0^2} \to 0$ both approaches have the same meaning.

$\Delta^{1/2}/(2\sqrt{\pi}(1+\cos(\Omega)e^{-\Delta^2 p_0^2}))^{1/2} = N$, and changing basis, first to z-spin and then to position, one gets new versions of the state in (4.36), firstly

$$|\psi'\rangle = \frac{N}{\sqrt{2}}\int_{-\infty}^{\infty} dp\, e^{-ipx_0}(e^{-\frac{\Delta^2}{2}(p-p_0)^2}((\alpha+\beta)|0\rangle + (\alpha-\beta)|1\rangle) \\ + e^{-\frac{\Delta^2}{2}(p+p_0)^2}((\alpha-\beta)|0\rangle + (\alpha+\beta)|1\rangle))|p\rangle, \tag{4.37}$$

and then

$$|\psi'\rangle = \frac{N}{\Delta}\int_{-\infty}^{\infty} dx\, e^{\frac{-(x-x_0)^2}{2\Delta^2}}(e^{ip_0(x-x_0)}(\alpha|+\rangle + \beta|-\rangle) \\ + e^{-ip_0(x-x_0)}(\alpha|+\rangle - \beta|-\rangle))|x\rangle. \tag{4.38}$$

With these representations of the state in hand, one last step towards certifying steering for it is still lacking, we must define the procedure each party, Alice and Bob, will perform. Since steering has only one trusted party, usually Bob, the bound is defined over his observables. Being the bound of both Reid's and the entropic uncertainty criterion based on, respectively, variance of the continuous DoF and their entropic uncertainty relation, the designated observables for Bob are P and X, leaving Alice with the spin-observables, namely σ_x and σ_z. The exclusion of σ_y occurs because the rotation is set to occur around de y-axis, leaving the σ_y state unchanged[7].

Another reason to give Alice the spin measurements is that, in the literature, the relativistic spin is considered ill-defined (GIACOMINI et al., 2019b; SALDANHA; VEDRAL, 2007), being it so the choice to leave Alice with our model of spin measurements leaves leeway to substitute it by any procedure that reproduces the statistics we expect, without being the yet-undefined relativistic spin observable. A discussion on this topic is sure to appear in the next chapter.

4.2.2.1 Reid's Variance Criterion

Having chosen Bob's and Alice's observables, the bound presented in equation (2.56) can be explicitly rewritten as

$$\delta^2_{min}(X_B)\delta^2_{min}(P_B) = \\ \sum_{\epsilon=\pm} p(\sigma_n^{(A)}=\epsilon)\; \delta^2(X_B|\sigma_n^{(A)}=\epsilon) \sum_{\epsilon'=\pm} p(\sigma_m^{(A)}=\epsilon')\; \delta^2(P_B|\sigma_m^{(A)}=\epsilon') < \frac{1}{4}, \tag{4.39}$$

where $\delta^2_{min}(X_B) = \int\, dx_A\; p(x_A)\; \delta^2(X_B|x_A)$ has been adapted for spin measurements as $\delta^2_{min}(X_B) = \sum_{\epsilon=\pm}\; p(\sigma_n^{(A)}=\epsilon)\; \delta^2(X_B|\sigma_n^{(A)}=\epsilon)$.

Let us begin by calculating the probabilities of a measurement performed by Alice. From now on all results are related to the boosted frame of reference, making the

[7] By unchanged the desired meaning is that, although phases may change, the probabilities of measurement of spin-y states remain unchanged and no entanglement is created with momentum.

color coding redundant. For the task of calculating the probabilities one can use Born's rule, remembering equation (2.1), such that

$$\begin{aligned}
p(\sigma_x = +) = &\, |\,\langle +|\psi\rangle\,|^2 \\
=&\frac{N^2}{\Delta^2}\int_{-\infty}^{\infty} dx\; e^{\frac{-(x-x_0)^2}{\Delta^2}}\alpha^2(e^{2ip_0(x-x_0)}) + e^{-2ip_0(x-x_0)} + 2) \\
=&\frac{(1+e^{-\Delta^2 p_0^2})(1+\cos(\Omega))}{2(1+\cos(\Omega)e^{-\Delta^2 p_0^2})} \\
:=&p_{x+}.
\end{aligned} \tag{4.40}$$

The state after measurement is

$$|\psi_{x+}\rangle = \frac{N}{p_{x+}^{1/2}\Delta}\int_{-\infty}^{\infty} dx\; e^{\frac{-(x-x_0)^2}{2\Delta^2}} 2\cos(p_0(x-x_0))\alpha\,|+\rangle \otimes |x\rangle\,. \tag{4.41}$$

We can do the exact same process to obtain

$$\begin{aligned}
p(\sigma_x = -) = &\, |\,\langle -|\psi\rangle\,|^2 \\
=&\frac{N^2}{\Delta^2}\int_{-\infty}^{\infty} dx\; e^{\frac{-(x-x_0)^2}{\Delta^2}}\beta^2(-e^{2ip_0(x-x_0)}) + -e^{-2ip_0(x-x_0)} + 2) \\
=&\frac{(1-e^{-\Delta^2 p_0^2})(1-\cos(\Omega))}{2(1+\cos(\Omega)e^{-\Delta^2 p_0^2})} \\
:=&p_{x-},
\end{aligned} \tag{4.42}$$

and

$$|\psi_{x-}\rangle = \frac{N}{p_{x-}^{1/2}\Delta}\int_{-\infty}^{\infty} dx\; e^{\frac{-(x-x_0)^2}{2\Delta^2}} 2i\sin(p_0(x-x_0))\beta\,|-\rangle \otimes |x\rangle\,. \tag{4.43}$$

Next, we can calculate the conditional position dispersion for these post-measurement states:

$$\begin{aligned}
p(\sigma_x^{(A)} = +)\delta^2(X_B|\sigma_x^{(A)} = +) = &\; p(\sigma_x^{(A)} = +)(\langle\psi_{x+}|X_B^2|\psi_{x+}\rangle - \langle\psi_{x+}|X_B|\psi_{x+}\rangle^2) \\
= &\;\frac{\alpha^2 N^2}{\Delta^2}\int_{-\infty}^{\infty} dx\; x^2\; e^{\frac{-(x-x_0)^2}{\Delta^2}} 4\cos^2(p_o(x-x_0)) \\
&-\left(\frac{\alpha^2 N^2}{p_{x+}^{1/2}\Delta^2}\int_{-\infty}^{\infty} dx\; x\; e^{\frac{-(x-x_0)^2}{\Delta^2}} 4\cos^2(p_o(x-x_0))\right)^2 \\
= &\;\frac{\alpha^2((1+\mathcal{C})(x_0^2+\Delta^2/2) - \Delta^4 p_0^2\cdot\mathcal{C})}{1+\cos(\Omega)\cdot\mathcal{C}} - \frac{2\alpha^4 x_0^2(1+\mathcal{C})}{(1+\cos(\Omega))(1+\cos(\Omega)\cdot\mathcal{C})}.
\end{aligned} \tag{4.44}$$

Here, $\mathcal{C} = e^{-\Delta^2 p_0^2}$ just for ease of notation. One can straightforwardly do the same for the case in which Alice gets a spin-down measurement,

$$\begin{aligned} p(\sigma_x^{(A)}=-)\delta^2(X_B|\sigma_x^{(A)}=-) &= p(\sigma_x^{(A)}=-)(\langle\psi_{x-}|X_B^2|\psi_{x-}\rangle - \langle\psi_{x-}|X_B|\psi_{x-}\rangle^2) \\ &= \frac{\beta^2N^2}{\Delta^2}\int_{-\infty}^{\infty} dx\; x^2\; e^{\frac{-(x-x_0)^2}{\Delta^2}} 4\sin^2(p_o(x-x_0)) \\ &- \left(\frac{\alpha^2N^2}{p_{x-}^{1/2}\Delta^2}\int_{-\infty}^{\infty} dx\; x\; e^{\frac{-(x-x_0)^2}{\Delta^2}} 4\sin^2(p_o(x-x_0))\right)^2 \\ &= \frac{\beta^2((1-\mathcal{C})(x_0^2+\Delta^2/2)+\Delta^4p_0^2\cdot\mathcal{C})}{1+\cos(\Omega)\cdot\mathcal{C}} - \frac{2\beta^4x_0^2(1-\mathcal{C})}{(1-\cos(\Omega))(1+\cos(\Omega)\cdot\mathcal{C})}. \end{aligned} \tag{4.45}$$

Now one must do the same process for measurements in $\sigma_z^{(A)}$ and the dispersion on momentum,

$$\begin{aligned} p(\sigma_z=+) = |\langle 0|\psi\rangle|^2 &= \frac{N^2}{2}\int_{-\infty}^{\infty} dp\; (e^{-\Delta^2(p-p_0)^2}(\alpha+\beta)^2 \\ &+ e^{-\Delta^2(p+p_0)^2}(\alpha-\beta)^2 + 2e^{\frac{\Delta^2}{2}((p-p_0)^2+(p+p_0)^2)}(\alpha^2-\beta^2)) \\ &= \frac{1}{2}. \end{aligned} \tag{4.46}$$

Curiously, the state is symmetric in a way that $|\langle 1|\psi\rangle|^2 = |\langle 0|\psi\rangle|^2$. That is so because the integral on $e^{-\Delta^2(p-p_0)^2}$ is the same as the one on $e^{-\Delta^2(p+p_0)^2}$. Not needing to calculate it, one advances. The post-measurement states are

$$|\psi_{z+}\rangle = N\int_{-\infty}^{\infty} dp\; e^{-ipx_0}(e^{-\frac{\Delta^2}{2}(p-p_0)^2}(\alpha+\beta) + e^{-\frac{\Delta^2}{2}(p+p_0)^2}(\alpha-\beta))\,|0\rangle\otimes|p\rangle\,, \tag{4.47}$$

and

$$|\psi_{z-}\rangle = N\int_{-\infty}^{\infty} dp\; e^{-ipx_0}(e^{-\frac{\Delta^2}{2}(p-p_0)^2}(\alpha-\beta) + e^{-\frac{\Delta^2}{2}(p+p_0)^2}(\alpha+\beta))\,|1\rangle\otimes|p\rangle\,. \tag{4.48}$$

Now all is ready for the dispersion calculation. Luckily the same symmetry that gave equal probabilities for the $\sigma_z=+$ and $\sigma_z=-$ cases will give the same dispersion for both cases:

$$\begin{aligned} p(\sigma_z^{(A)}=-)\delta^2(P_B|\sigma_z^{(A)}=-) &= p(\sigma_z^{(A)}=-)(\langle\psi_{z-}|P_B^2|\psi_{x-}\rangle - \langle\psi_{z-}|P_B|\psi_{z-}\rangle^2) \\ &= \frac{N^2}{2}\int_{-\infty}^{\infty} dp\; p^2\;(e^{-\Delta^2(p-p_0)^2}(\alpha+\beta)^2 + e^{-\Delta^2(p+p_0)^2}(\alpha+\beta)^2 + 2e^{-\frac{\Delta^2}{2}((p-p_0)^2+(p+p_0)^2)}) \\ &- \left(\frac{N^2}{2}\int_{-\infty}^{\infty} dp\; p\;(e^{-\Delta^2(p-p_0)^2}(\alpha+\beta)^2 + e^{-\Delta^2(p+p_0)^2}(\alpha+\beta)^2 + 2e^{-\frac{\Delta^2}{2}((p-p_0)^2+(p+p_0)^2)})\right)^2 \\ &= \frac{1}{2}\left(\frac{1}{2\Delta^2} + \frac{p_0^2}{1+\cos(\Omega)\cdot\mathcal{C}} - \frac{p_0^2\sin^2(\Omega)}{(1+\cos(\Omega)\cdot\mathcal{C})^2}\right). \end{aligned} \tag{4.49}$$

Finally, with all relations in hand, we can check in what conditions the inequality in (4.39) may occur:

$$
\begin{aligned}
&\delta^2_{min}(X_B)\delta^2_{min}(P_B) = \sum_{\epsilon=\pm} p(\sigma_n^{(A)}=\epsilon)\ \delta^2(X_B|\sigma_n^{(A)}=\epsilon)\ \sum_{\epsilon'=\pm} p(\sigma_m^{(A)}=\epsilon')\ \delta^2(P_B|\sigma_m^{(A)}=\epsilon')\\
&= p(\sigma_x^{(A)}=-)\ \delta^2(X_B|\sigma_x^{(A)}=-)\ (p(\sigma_z^{(A)}=+)\ \delta^2(P_B|\sigma_z^{(A)}=+) +\ p(\sigma_z^{(A)}=-)\ \delta^2(P_B|\sigma_z^{(A)}=-)\\
&+ p(\sigma_x^{(A)}=+)\ \delta^2(X_B|\sigma_x^{(A)}=+)\ (p(\sigma_z^{(A)}=+)\ \delta^2(P_B|\sigma_z^{(A)}=+) +\ p(\sigma_z^{(A)}=-)\ \delta^2(P_B|\sigma_z^{(A)}=-)\\
&= \left(\frac{\alpha^2((1+\mathcal{C})(x_0^2+\Delta^2/2)-\Delta^4p_0^2\cdot\mathcal{C})}{1+\cos(\Omega)\cdot\mathcal{C}} - \frac{2\alpha^4x_0^2(1+\mathcal{C})}{(1+\cos(\Omega))(1+\cos(\Omega)\cdot\mathcal{C})}\right)\\
&\times\left(\frac{1}{2\Delta^2}+\frac{p_0^2}{1+\cos(\Omega)\cdot\mathcal{C}}-\frac{p_0^2\sin^2(\Omega)}{(1+\cos(\Omega)\cdot\mathcal{C})^2}\right)\\
&+ \left(\frac{\beta^2((1-\mathcal{C})(x_0^2+\Delta^2/2)+\Delta^4p_0^2\cdot\mathcal{C})}{1+\cos(\Omega)\cdot\mathcal{C}} - \frac{2\beta^4x_0^2(1-\mathcal{C})}{(1-\cos(\Omega))(1+\cos(\Omega)\cdot\mathcal{C})}\right)\\
&\times\left(\frac{1}{2\Delta^2}+\frac{p_0^2}{1+\cos(\Omega)\cdot\mathcal{C}}-\frac{p_0^2\sin^2(\Omega)}{(1+\cos(\Omega)\cdot\mathcal{C})^2}\right).
\end{aligned}
\tag{4.50}
$$

Let us keep going, some mathematical manipulation is needed to simplify the results, no bother, one advances,

$$
\begin{aligned}
&\delta^2_{min}(X_B)\delta^2_{min}(P_B) =\\
&= \left\{\frac{1}{1+\cos(\Omega)\cdot\mathcal{C}}\left[\mathcal{C}(\beta^2-\alpha^2)(\Delta^4p_0^2-(x_0^2+\Delta^2/2))-2x_0^2\left(\frac{\alpha^4(1+\mathcal{C})}{1+\cos(\Omega)}+\frac{\beta^4(1-\mathcal{C})}{1-\cos(\Omega)}\right)\right]\right\}\\
&\times\left\{\frac{1}{2\Delta^2}+\frac{p_0^2}{1+\cos(\Omega)\cdot\mathcal{C}}-\frac{p_0^2\sin^2(\Omega)}{(1+\cos(\Omega)\cdot\mathcal{C})^2}\right\},
\end{aligned}
\tag{4.51}
$$

using $\alpha^2-\beta^2=\cos(\Omega), \alpha^4=\frac{(1+\cos(\Omega))^2}{4}$, and $\beta^4=\frac{(1-\cos(\Omega))^2}{4}$, plus some rearranging, we get

$$
\begin{aligned}
&\delta^2_{min}(X_B)\delta^2_{min}(P_B) = \left(\frac{\Delta^2}{2}-\frac{\mathcal{C}\Delta^4p_0^2\cos(\Omega)}{(1+\cos(\Omega)\cdot\mathcal{C})}\right)\left(\frac{1}{2\Delta^2}+\frac{p_0^2}{1+\cos(\Omega)\cdot\mathcal{C}}-\frac{p_0^2\sin^2(\Omega)}{(1+\cos(\Omega)\cdot\mathcal{C})^2}\right)\\
&= \frac{1}{4}+\frac{\Delta^2p_0^2}{2}\left(\frac{1-\cos(\Omega)\mathcal{C}}{1+\cos(\Omega)\mathcal{C}}-\frac{\sin^2(\Omega)+\Delta^2p_0^2\cos(\Omega)\mathcal{C}}{(1+\cos(\Omega)\mathcal{C})^2}+\frac{2\Delta^2p_0^2\sin^2(\Omega)\cos(\Omega)\mathcal{C}}{(1+\cos(\Omega)\mathcal{C})^3}\right).
\end{aligned}
\tag{4.52}
$$

After what seemed to be an unending calculation we reached the value of the minimized dispersion as a function of the Wigner angle and the preparation parameters p_0 and Δ. A plot of equation (4.52) appears in figure 6.

As shown in figure 6, the lowest value the variance in equation (4.52) is exactly $1/4$ for the maximally entangled case, that is, for $\Omega=\pi/2$.

So, no absolute violation of the criterion was found, although unfortunate, it is not problematic as it will shortly be argued. Before following such a path let us consider the more general observables for Bob, like the ones defined in (2.52). In this case, the

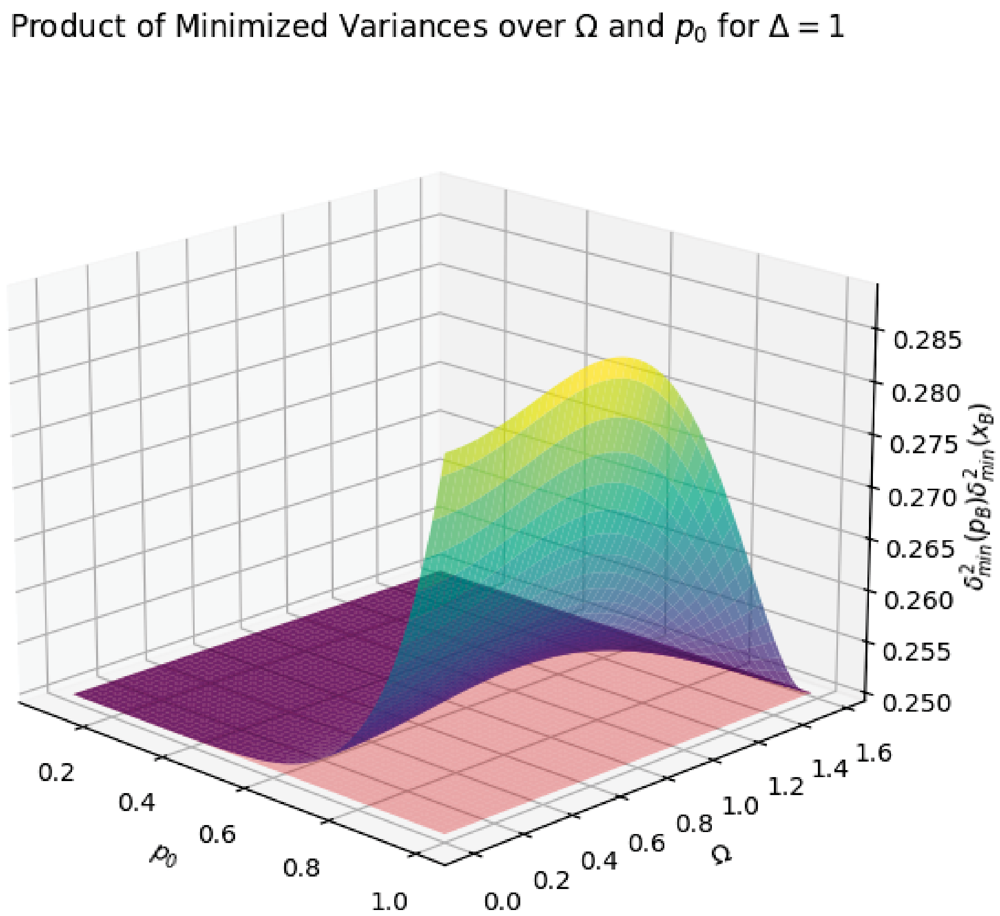

FIGURE 6 – Plot of the product of the minimized variances after spin measurements by Alice. The state preparation was done with $\Delta = 1$. The plane in which the variance equals $1/4$ is plotted in red. No violation occurs, Reid's criterion wasn't able to certify steering for any preparation-boost combination.

minimized variance is rewritten as

$$\delta^2_{min}(B_{\beta 1})\delta^2_{min}(B_{\beta 2}) = \sum_{\epsilon=\pm} p(\sigma_n^{(A)} = \epsilon)\, \delta^2(B_{\beta 1}|\sigma_n^{(A)} = \epsilon) \sum_{\epsilon'=\pm} p(\sigma_m^{(A)} = \epsilon')\, \delta^2(B_{\beta 2}|\sigma_m^{(A)} = \epsilon'). \tag{4.53}$$

Fixing $\sigma_n = \sigma_x$ and $\sigma_m = \sigma_z$, in the same way just performed, we get, as $e^{-\Delta^2 p_0^2} \to 0$, for simplicity

$$\delta^2_{min}(B_{\beta 1}) = \frac{1}{2}\left\{\frac{\sin^2(\beta_1)(1+2\Delta^2 p_0^2)}{\Delta^2} + \cos^2(\beta_1)\left(\Delta^2 + 2x_0^2 - \frac{\alpha^4 4x_0^2}{(1+\cos(\Omega))} - \frac{\beta^4 4x_0^2}{(1-\cos(\Omega))}\right)\right\}. \tag{4.54}$$

Good care must be taken not to mix up β_1, the angle of the observable superposition, and β that is simply $\cos(\Omega/2)$. Moving on,

$$\delta^2_{min}(B_{\beta 2}) = \cos^2(\beta_2)\frac{\Delta^2}{2} + \sin^2(\beta_2)\left(\frac{1}{2\Delta^2} + p_0^2(1-\sin^2(\Omega))\right). \tag{4.55}$$

Multiplying both reduced variances and using the necessary condition that $\beta_1 - \beta_2 = \pi/2$,

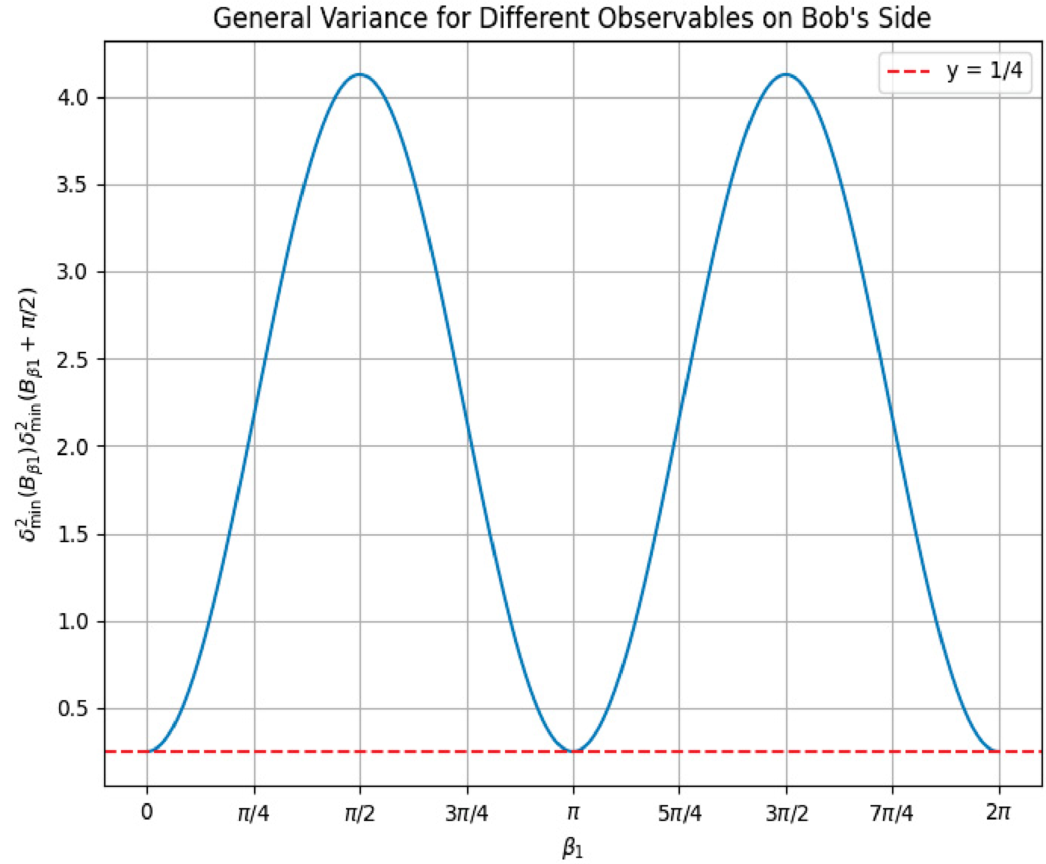

FIGURE 7 – Plot of the product of the minimized variances after spin measurements by Alice for preparation with $\Delta = p_0 = 2$. Here the observable $B_{\beta 1}$ is the generalized form of Bob's observables. The line in which the variance equals $1/4$ is plotted in red. The minimal variance occurs for values of β_1 that are whole multiples of π.

we get

$$\delta^2_{min}(B_{\beta 1})\delta^2_{min}(B_{\beta 1+\pi/2}) = \frac{1}{2}\left(\frac{\sin^2(\beta_1)(1+2\Delta^2 p_0^2)}{\Delta^2} + \cos^2(\beta_1)\left(\Delta^2 + 2x_0^2 - \frac{(\alpha^4)4x_0^2}{(1+\cos(\Omega))} - \frac{(\beta^4)4x_0^2}{(1-\cos(\Omega))}\right)\right) \times \left(\cos^2(\beta_1+\pi/2)\frac{\Delta^2}{2} + \sin^2(\beta_1+\pi/2)\left(\frac{1}{2\Delta^2} + p_0^2(1-\sin^2(\Omega))\right)\right), \tag{4.56}$$

whose plot appears on figure 7.

The fact that the minimal variance occurs for choices of β_1 that are multiples of π shows that the best observables to choose from are exactly P_B and X_B. Thus, the variance given in (4.52) is the best possible one. Before advancing a commentary must be done, given different choices of preparation parameters as well as for different Wigner angles, the minimal and maximal variances showed in figure 7 change. Nonetheless, the minimal values are always for β_1 being a whole multiple of π. The fact that the variance in equation (4.52) is the best achievable truly stands.

At this point, one may be satisfied and pack his or her things and go home, but not us. It is known that, for pure states, entanglement and steering are equivalent. So how may one detect steering, in the present context, without finding a violation of the chosen inequality? We propose that the fact that the variance depends on the Wigner angle and, consequently, on the degree of entanglement between spin and momentum, already indicates a degree of control on Bob's state, by Alice. Of course, this work's interest lies on the quantitative description of these phenomena rather than the qualitative one. Being this the case, not all hope is lost.

If one considers the variance of the state without spin measurements, this is, when Alice does nothing, and compares it to the case in which Alice acts, any change in the variance that Bob may obtain must be caused by Alice. Being this so, what we propose to do is to establish a "control" in the experiment to serve as background to the "active" case. The way this is done is quite simple, one just has to calculate

$$\delta^2(B_{\beta_1})\delta^2(B_{\beta_2}), \tag{4.57}$$

this being the variance without any spin influence, and compare it with $\delta^2_{min}(B_{\beta_1})\delta^2_{min}(B_{\beta_2})$. As it turns out, using the same state of interest, previously defined in (4.36), one obtains

$$\begin{aligned}\delta^2(B_{\beta_1})\delta^2(B_{\beta_2}) = &\left(\cos^2(\beta_1)\left(\frac{\Delta^2}{2} - \frac{\Delta^4 p_0^2\cos(\Omega)\mathcal{C}}{1+\cos(\Omega)\mathcal{C}}\right) + \sin^2(\beta_1)\left(\frac{1}{2\Delta^2} - \frac{p_0^2}{1+\cos(\Omega)\mathcal{C}}\right)\right.\\ &\left.+ \cos(\beta_1)\sin(\beta_1)\left(\frac{i\Delta^2 p_0^2\cos(\Omega)\mathcal{C}}{2(1+\cos(\Omega)\mathcal{C})}\right)\right)\left(\cos^2(\beta_2)\left(\frac{\Delta^2}{2} - \frac{\Delta^4 p_0^2\cos(\Omega)\mathcal{C}}{1+\cos(\Omega)\mathcal{C}}\right)\right.\\ &\left.+ \sin^2(\beta_2)\left(\frac{1}{2\Delta^2} - \frac{p_0^2}{1+\cos(\Omega)\mathcal{C}}\right) + \cos(\beta_2)\sin(\beta_2)\left(\frac{i\Delta^2 p_0^2\cos(\Omega)\mathcal{C}}{2(1+\cos(\Omega)\mathcal{C})}\right)\right).\end{aligned} \tag{4.58}$$

This expression, for the most interesting case in which, $e^{-\Delta^2 p_0^2} \to 0$ and $\beta_1 - \beta_2 = \pi/2$ reduces to

$$\begin{aligned}\delta^2(B_{\beta_1})\delta^2(B_{\beta_2}) = &\cos^4(\beta_1)\left(\frac{1}{4} + \frac{\Delta^2 p_0^2}{2}\right) + \sin^4(\beta_1)\left(\frac{1}{4} + \frac{\Delta^2 p_0^2}{2}\right)\\ &+ \cos^2(\beta_1)\sin^2(\beta_1)\left(\frac{\Delta^4}{4} + \frac{1}{4\Delta^4} + \frac{p_0^2}{\Delta^2 + p_0^4}\right).\end{aligned} \tag{4.59}$$

But we mustn't forget that the most important context, the one with minimal variance, is the one in which β_1 is a whole multiple of π, and that, therefore

$$\delta^2(X_B)\delta^2(P_B) = \frac{1}{4} + \frac{\Delta^2 p_0^2}{2}. \tag{4.60}$$

Now, having established a control variance, we can check if the spin-minimized variance can reach smaller values, for such we can just take the difference between equations (4.52), in the regime where $\mathcal{C} \to 0$, and equation (4.60), getting

$$\begin{aligned}\delta^2_{min}(X_B)\delta^2_{min}(P_B) - \delta^2(X_B)\delta^2(P_B) &= \left(\frac{1}{4} + \frac{\Delta^2 p_0^2(1-\sin^2(\Omega))}{2}\right) - \left(\frac{1}{4} + \frac{\Delta^2 p_0^2}{2}\right)\\ &= -\sin^2(\Omega).\end{aligned} \tag{4.61}$$

This simple and elegant solution is also easily interpreted. As the Wigner angle increases, entanglement also increases, and the ability that Alice has, by measuring spin, to decrease Bob's product of variances increases. Noticeably, the difference between variances does not depend on anything but the Wigner angle, a fact that is of interest because it guarantees preparation independence. The only care that must be taken is to be sure that the approximations used are valid. Well, when it comes to that, the only thing truly necessary is that $\mathcal{C} = e^{-\Delta^2 p_0^2} \to 0$. This condition is not all restrictive, since both Δ and p_0 are separately controllable and, being the function exponential, small increments in either one of the parameters guarantee extreme decreases in the function value.

Although steering and entanglement are equivalent for pure states, and we are working with one, this bound was not able to distinguish our quantum-correlated (entangled) state from one with classical correlations that could be described as a local hidden state model on Bob's side and therefore no steerable. We didn't concern ourselves with finding such a model, but undoubtedly the classical correlation that could justify our results would have to depend on the Wigner angle in the way displayed in equation (4.61). To guarantee steerability *per se*, either different, more appropriate observables are chosen, such that violation is seen with the $1/4$ bound, or another, stronger, criterion is implemented. We propose to do the latter.

4.2.2.2 Entropic Uncertainty Criterion

Having failed to rigorously certify steering using the previous criterion our hopes lay with the proven-to-be-tighter entropic uncertainty criterion (SCHNEELOCH et al., 2013). A part of the calculations done previously will prove to be useful in this new evaluation, but putting one foot in front of the other, let's begin by defining the criterion for the specific context of interest, this is, where Alice measures spin-x and spin-z and Bob does X and P measurements:

$$h(P_B|\sigma_z^A) + h(X_B|\sigma_x^A) \geq \log_2(\pi e). \tag{4.62}$$

Here,

$$h(X_B|\sigma_x^A) = -\int_{-\infty}^{\infty} dx \sum_{\epsilon=\pm} p(X_B = x, \sigma_x^A = \epsilon) \ln(p(X_B = x|\sigma_x^A = \epsilon)) \tag{4.63}$$

is just the generalization of the conditional entropy for hybrid continuous-discrete observables.

To obtain the necessary conditional probabilities $p(X_B|\sigma_x^A = \epsilon)$ and $p(P_B|\sigma_z^A = \epsilon)$ one may use the general conditional probability rule

$$p(A|B) = \frac{p(A, B)}{p(B)}. \tag{4.64}$$

Fortunately, we already know, from the calculations made in the last criterion, the values of $p(\sigma_z^A = \pm 1) = 1/2$ and $p(\sigma_z^A = \pm) = p_{x\pm}$. So we must only calculate the joint infinitesimal probabilities $dp(P_B, \sigma_z^A)$ and $dp(X_B, \sigma_x^A)$. The probabilities here are infinitesimal because they will be integrated to count for the whole distribution as first demonstrated in (4.63). To calculate these infinitesimal entropies we simply use the Born rule. For instance

$$\begin{aligned}\frac{dp(P_B, \sigma_z^A = +)}{p(\sigma_z^A = +)} &= \frac{|\langle p, 0|\psi\rangle|^2 dp}{1/2} \\ &= N^2[(\alpha+\beta)e^{-\frac{\Delta^2}{2}(p-p_0)^2} + (\alpha-\beta)e^{-\frac{\Delta^2}{2}(p+p_0)^2}]^2 dp.\end{aligned} \tag{4.65}$$

Similarly

$$\begin{aligned}\frac{dp(P_B, \sigma_z^A = -)}{p(\sigma_z^A = -)} &= \frac{|\langle p, 1|\psi\rangle|^2 dp}{1/2} \\ &= N^2[(\alpha-\beta)e^{-\frac{\Delta^2}{2}(p-p_0)^2} + (\alpha+\beta)e^{-\frac{\Delta^2}{2}(p+p_0)^2}]^2 dp\end{aligned} \tag{4.66}$$

and

$$\begin{aligned}\frac{dp(X_B, \sigma_x^A = +)}{p(\sigma_x^A = +)} &= \frac{|\langle x, +|\psi\rangle|^2 dx}{p_{x+}} \\ &= \frac{1}{p_{x+}} \frac{\cos^2\left(\frac{\Omega}{2}\right) e{-\frac{(x-x_0)^2}{\Delta^2}}}{\Delta\sqrt{\pi}\,(1+\cos(\Omega)\mathcal{C})} \left(1+\cos\left(2p_0(x-x_0)\right)\right) dx.\end{aligned} \tag{4.67}$$

Also

$$\begin{aligned}\frac{dp(X_B, \sigma_x^A = -)}{p(\sigma_x^A = -)} &= \frac{|\langle x, -|\psi\rangle|^2 dx}{p_{x-}} \\ &= \frac{1}{p_{x-}} \frac{\sin^2\left(\frac{\Omega}{2}\right) e^{-\frac{(x-x_0)^2}{\Delta^2}}}{\Delta\sqrt{\pi}\,(1+\cos(\Omega)\mathcal{C})} \left(1-\cos\left(2p_0(x-x_0)\right)\right) dx.\end{aligned} \tag{4.68}$$

Great, having the probabilities in hand, the next step is to evaluate the integrals in the form established in (4.63). Unfortunately, no analytical solution was achieved by the standard methods of integration. Being this an oddly specific integral, pertaining to a particular class of states, in a particular context (this particular criterion), the integrals of interest could not even be found in integral tables. Being this the case, numerical integration became necessary, and the results for different preparations appear in figures 8, 9, and 10.

The integrals were done in the p and x space from -1000 to 1000. Given the choice of the parameters, being relatively small, $\Delta = p_0 = x_0 = 2$ in their greatest, this contemplates most of the integral space. This is so because the Gaussian distributions decay quickly, even for big dispersions such as the one when $\Delta = 0.1$.

For small values of the parameters, no steering was detectable. This is likely caused by the indistinguishability of the left and right Gaussians of momentum. Being

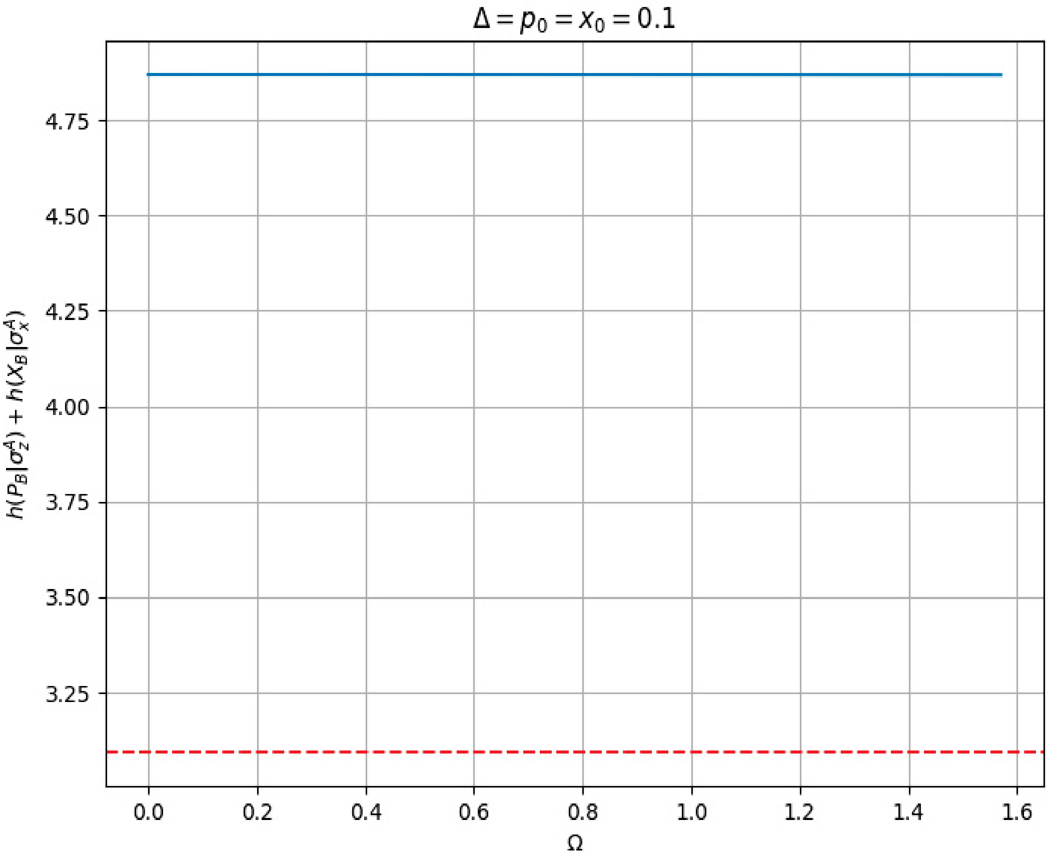

FIGURE 8 – Sum of Bob's conditional entropies over measurements done by Alice for preparation parameters $\Delta = p_0 = x_0 = 0.1$ as a function of the Wigner angle Ω. The red line indicates the criterion bound $\log_2 \pi e$. No violation of the entropic criterion is seen.

this so, spin does not entangle to a specific part of the wave function and is not capable of carrying much information about it. For greater parameter values, steering was certified after a given Wigner angle that depends on the preparation parameters. A great avenue of investigation to be followed is the search for the dependence of such a critical angle on the preparation parameters. The inability to deduce such dependence is the major weakness of the numerical integration approach.

4.2.3 Chapter Closing Remarks

As the chapter reaches its end the message it wishes to tell is that Lorentz transformations, the basis of SR, can be represented and act on quantum states and not without consequences. A very significant consequence is the fact that this kind of transformation acts over states with momentum coherence in a way that may entangle this DoF with spin. Being the process of entanglement creation also described in the chapter and exemplified within.

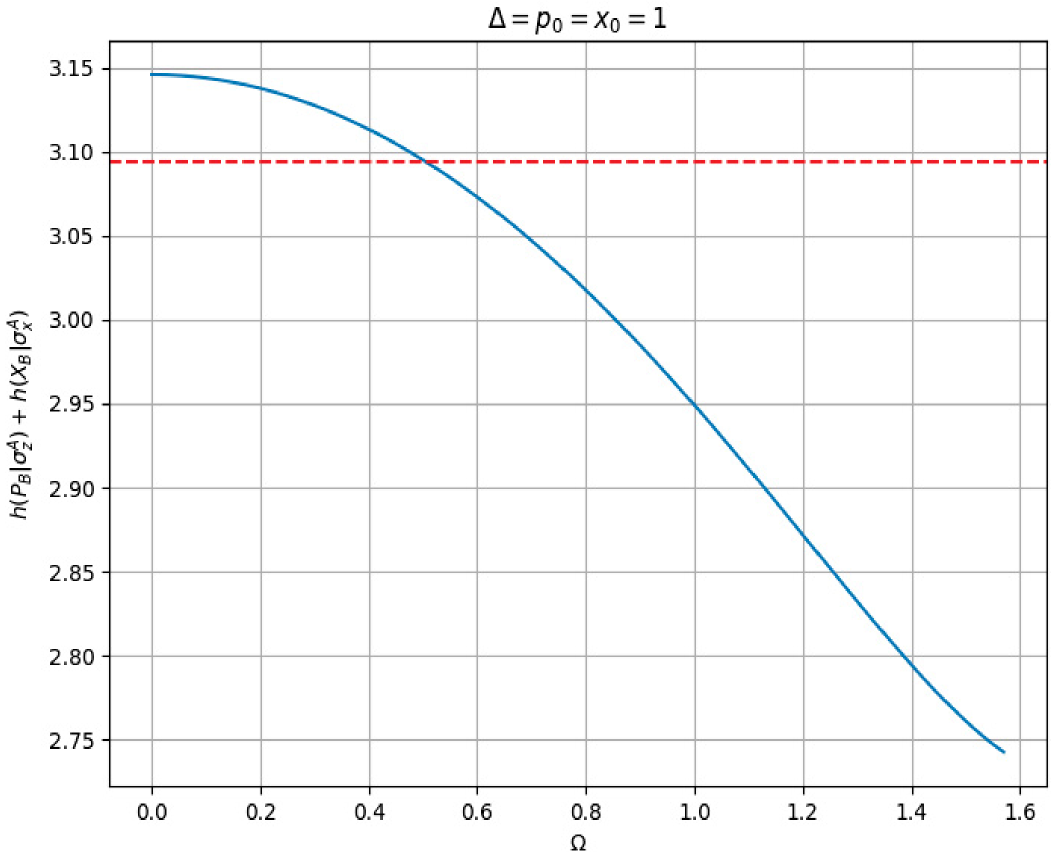

FIGURE 9 – Sum of Bob's conditional entropies over measurements done by Alice for preparation parameters $\Delta = p_0 = x_0 = 1$ as a function of the Wigner angle Ω. The red line indicates the criterion bound $\log_2 \pi e$. The entropic criterion gets violated and steering is certified.

Following the previously established quantum resource hierarchy, the chapter set out to discover if known steering criteria, Reid's variance criterion and the entropic uncertainty relation criterion, would be able to detect steering for the state of interest. The state used for entanglement detection was overly simplistic and unrealistic in a manner that would not allow detection by Reid's criterion. The system model was then improved by considering the superposition of Gaussian states of momentum.

Using the new state, the spin minimized product of variances was calculated but did not violate the definite barrier established by Heisenberg's uncertainty principle. Guided by the fact that, for pure states, steering and entanglement are equivalent, the decision was taken to compare the control variance, which is independent of Alice's influence, with the minimized one. From the comparison, a process of variance decrease was identified and showed to be dependent on the Wigner angle, which was previously related to the degree of entanglement between spin and momentum, as expected. Unfortunately, since the bound was not violated, the criterion was not capable

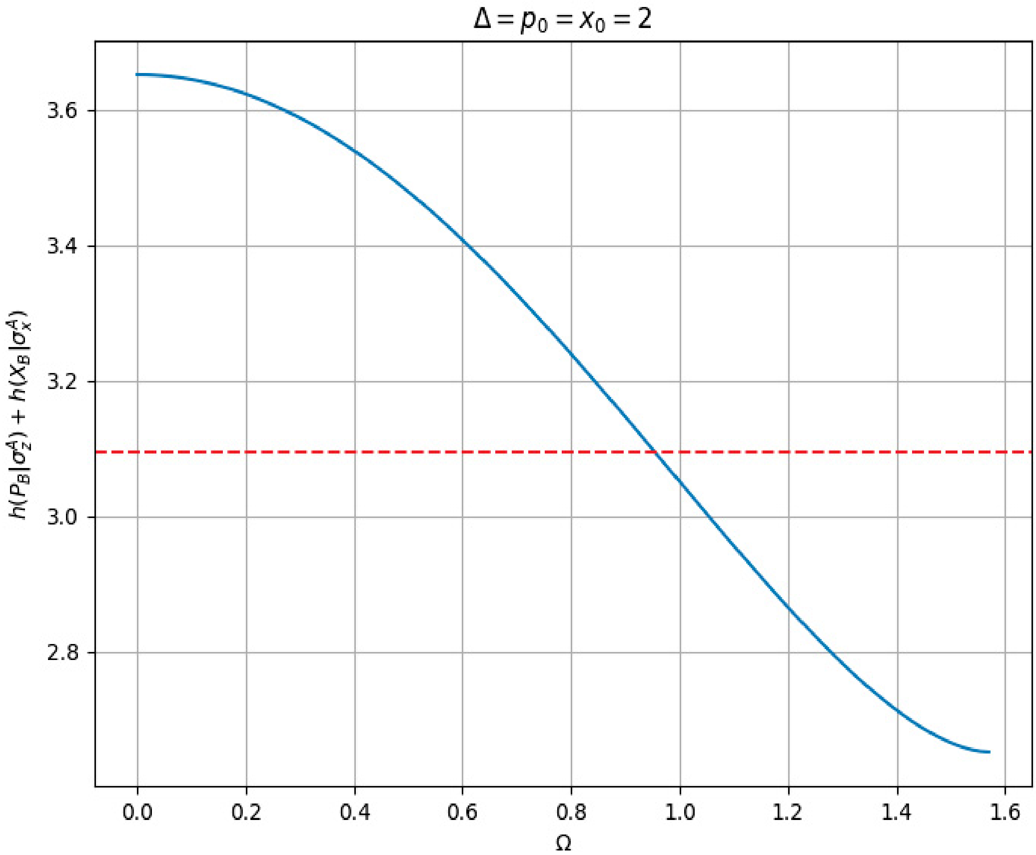

FIGURE 10 – Sum of Bob's conditional entropies over measurements done by Alice for preparation parameters $\Delta = p_0 = x_0 = 2$ as a function of the Wigner angle Ω. The red line indicated the criterion bound $\log_2 \pi e$. The entropic criterion gets violated and steering is certified.

of distinguishing entanglement from a classical correlation. The entropic uncertainty criterion, on the other hand, was capable of detecting steering for a critical Wigner angle depending on preparation parameters. Consequently, not all boosted observers would be able to certify entanglement. The ones with such capacity are those whose boost velocity exceeds the one related to the critical angle. To obtain results with this criterion, the required integrals had to be evaluated numerically.

Many approximations had to be taken and many subtle concepts were taken as understood for the work to advance. The nuanced nature of such concepts was recognized and will be discussed in the following chapter. The path taken to reach the work's results is reasonable and many caveats still standing will find resolution shortly.

5 DISCUSSION AND CONCLUSION

This work is a short amalgamation of concepts, both old and new, from two of the most successful descriptions of nature ever devised by the human mind. Since the work within QM and SR itself is not over, no hope was had that this work would be able to avoid or solve many of the difficulties associated with conjoining both theories. Be that as it may, we still advanced, taking positions as they seemed reasonable to us. It is in the hope that any reader shares the opinion about the reasonability of the workarounds we proposed, and then used, that this chapter is written.

To better organize this closing chapter, it will be separated into topics, divided by subsections, about some of the contentious matters of the work. Many topics are related and separated only arbitrarily to improve the ability of one to refer to a specific problem later.

5.1 DISCUSSION

The majority of the points of contempt present in this work appear by contradicting requirements of QM and SR, but not all. Being this so, let us begin with a devious concept within QM itself, that of *quantum reference frames*.

5.1.1 Quantum Reference Frames

Changes of reference frame are important parts of all physical theories, being physics assumed to be universal and unchanging in time[1], a theory must be able to accommodate the descriptions of other observers within the same universe, or, better put, under the same laws. The process of jumping from one's perspective to another's is a change, or transformation, of frame of reference. In SR these are the all too well know Lorentz transformations Λ that, as demonstrated are paralleled in QM by $U(\Lambda)$. Now, within QM, other types of reference frame transformations exist (GIACOMINI et al., 2019a).

The concept of Quantum Reference Frames was slightly touched on in the previous chapter, but it is no shallow concept, although it can be represented in a few words: Quantum theory is very accustomed to conceding physical systems, particles for instance, states of nature that have no defined reality, a concept pinacled by superposition. But, being the objects of the theory physical systems, and assuming that the observer modeling nature is itself a physical system, what keeps the system from having a description of the observer in a manner not different than the one the observer had of

[1] Or at least able to predict its own changes in time and space.

the system originally? If one has a description of an electron in a state of superposition, may the electron have a description of them?

If within QM there are unitary operators that take a system composed of observer and object[2], originally modeled in the observer frame, into the object's frame, without breaking the most important parts of the theory, such as the canonical structure[3], then the question of the previous paragraph will be answered.

Since this is an open topic in today's physics, when we faced it, namely, when the fact that spin is usually defined at rest and that no classical rest frame exists for momentum superposition was encountered, we opted to leave as a hypothesis that the spin structure is preserved in a momentum superposition, arguing then why such hypothesis was reasonable.

If spin is inexorably associated with rest, as it might well be, then, a truly fundamental description of the nature of such DoF will undoubtedly pass through the understanding of quantum reference frames.

5.1.2 Relativistic Spin Observable

Being exposed to the fact that spin is usually defined at rest, one might ask why that is so.

The answer is twofold. Firstly, at rest, no orbital angular momentum can exist. Therefore, all angular momentum must be intrinsic. Being this the case, at rest, total angular momentum and spin are one and the same, at least if the particle is the only physical entity in this universe. Secondly and more significant for this particular discussion, spin interactions[4] occur when a particle is put in an inhomogeneous magnetic field. However, the electromagnetic field itself is frame-dependent, being one of the facts that led to the discovery of SR itself. Then, for the dynamic to be covariant, if the field transforms, something about the spin interaction must transform as well.

The search for the correct description of the relativistic spin observable is still on as this is written. As previously outlined, some authors believe that spin has no momentum-independent meaning (GIACOMINI et al., 2019b), and if the problem is complicated in the case of well-defined rest and momentum-possessing frames, it gets much more complicated when quantum reference frames are of concern, as discussed in the previous subsection.

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[2] Here some way for the observer to model itself is assumed. A "physical" observer may be just a particle in the origin. The discussion of what qualities "physical" reference frames and/or observers must have is also prominent in the literature.

[3] In the sense of the canonical commutation relations.

[4] Other types of spin interactions exist, the one being described here is the one associated with spin measurements, that is, with the modeling of a Stern-Gerlach experiment.

5.1.3 Time in QM

There are many difficulties associated with time in QM, some are not problematic as far as this work is concerned. For instance, time evolution in QM is completely time-reversible. Therefore, the derivation of an arrow of time is a hard task, one that many set out to do (MACCONE, 2009; BASSO et al., 2023). Since this work is not concerned with dynamics, our problem with time is of a different nature.

There is no known time observable and no time state is writable, although many have been proposed (GOTŌ et al., 1980; SRINIVAS; VIJAYALAKSHMI, 1981; OPPENHEIM et al., 1999). For QM time is only a parameter of evolution fixed out of the dynamic by its Hamiltonian formalism, even for cases of time-dependent Hamiltonians (MUGA et al., 2007). A possible attempt to defend QM is that, if it permitted time-states it would allow for time superposition and time interference, concepts as hard to intuit as any, not that lack of intuability was reason to stop QM before.

Relativity, on the other hand, is based on 4-vectors of both position and momentum that are simultaneously well defined. In QM the uncertainty principle associates a limit of precision for incompatible observables such as 3-momentum and 3-position but, by extending the formalism to 4-vectors, an uncertainty principle arises between time and energy, and energy states are writable, being them the states associated exactly with the Hamiltonian operator. As of yet, the formalism does not go full circle.

This incongruity is avoided within this work when arguments are given on the reason to ignore the zeroeth component of the momentum 4-state. Since we end up being concerned only with the 3-momentum, rationally, only the 3-position is taken into consideration. But, in the formalism, no concept serves as an objection to time-based entanglement or steering. Finally, even the workaround found to discard the energy space in the 4-momentum state is not that set in stone, as shall be elaborated on next.

5.1.4 Negative Mass

More controversy can be found in the fact that, in the same way, a perfectly well-defined momentum state is associated with a completely undefined position. If a particle is found at rest (3-momentum and 3-position equal to zero) and at a given time, measured with enough precision, it must have a completely undefined rest mass. Being it permitted to have negative energies, it might need to have negative masses as well, being that whatever it is. Never mind the previously mentioned mass-shell constraint.

A way to ease the mind troubled by the previously discussed controversy is to bring attention to the fact that, in Dirac's equation (DIRAC, 1928), one of the best-known descriptions of spin-1/2 massive particles, deemed by some as the "real seed of modern physics" (ZICHICHI, 2000), negative solutions are known and have been interpreted as the solutions associated with antiparticles. But even then, these interpretations are commonly done over unrealistic states of perfectly defined rest, for instance. Physics describes the nature of a spin-1/2 particle beautifully, as long as it knows not where or when to find such a particle!

5.1.5 End of Discussion!

Much more could be said on the topics surrounding troubles in QM, SR, and their intersection. This work dares not, for instance, to go into the "Is quantum theory a theory of nature or information?" debate. And then there is also the supermassive elephant in the room, the fact that SR is not even the final word in classical physics[5], being generalized by general relativity, where commonly, not even the number of particles is a frame-independent parameter[6]. But then again, if this work dares not to attempt to solve smaller problems, quantum gravity is very out of scope.

Having followed the necessary tangents out of the important concepts needed to achieve our results, let us look inward again and talk a little about what they tell us.

5.2 CONCLUSION

This research project set out to understand and then measure the way in which Lorentz transformations altered the amount of resources a quantum state possesses. The curiosity about the topic appeared when the view that momentum and spin could become entangled via such transformations appeared in the literature (PERES et al., 2002). For such reasons the systems of interest were those represented by spin-momentum states, it is possible that similar effects could be found in states of different nature, although none is known. Having chosen the system, the next choice would have to be what resources to evaluate. The choice to go up the resource hierarchy was partially motivated by the resource's usefulness in applications, mostly in the field of quantum information, and then also motivated by the polemic around the spin-momentum separability problem, where the more loose requirements of only one party trustability fitted well with the not completely known nature of the relativistic spin observable. The chosen formalism does not require quantum measurements of spin in the standard way. Instead, Alice's measurements may be of any nature, as long as they demonstrate correlations in a manner similar to our model, our results stand. A small caveat should be brought to attention, since steering is to be evaluated for two

[5] Many definitions of "classical physics" fit this description. In this context, what is being said is that general relativity is the most successful local and realistic theory of nature, being the two fundamental characteristics of the definition of "classical" used.

[6] In specific cases the same pathology appears in SR.

DoF of the same particle, a locality loophole[7] will remain open. Allthough true, quantum correlations of similar nature have been certified in a loophole free manner (STORZ et al., 2023; CHRISTENSEN et al., 2013). Being this so, there is no reason to object to the assumption that a measurement of one of the DoF will not perturb the other in a classical information transfer way[8]. Having in hand an object, namely, spin-momentum systems, and an objective, entanglement, and steering detection, the project advanced toward the production of the results.

Beginning by reproducing the provocative results of boost-induced entanglement, we then realized that the simple model of the physical system was a) too simplistic to be implementable[9] and b) would never allow for steering detection by one of the chosen criteria. Then, the system's modeling evolved and the criteria were applied. For Reid's criterion, no absolute violation was found since the variance never went below the one established in Heisenberg's uncertainty principle. As it turns out, the fact that the reduced state entropy is limited by the dimension of the smaller space, spin in this case, makes it so the amount of information one can codify in one space, about the other, is limited. Spin measurements by Alice, in the optimal, maximally entangled, case, could, at best, select in which of the two Gaussian in superposition Bob's state would end up in. Being Gaussians states of minimal variance, that is, with variance equal to $1/4$, the results become reasonably explained. Now, we also demonstrated that although the final state is not one of violating variance, it still is one of diminished variance. Alice can take Bob's state from one of unbounded variance, as long as p_0 and Δ are great enough, to one of minimal variance, depending only on the degree of entanglement between both states, as measured by the Wigner angle. In this sense, at the very least, a process of creation of at-least-classical correlation was established, being the case that, as Ω increased, the amount of Alice's information about Bob's state became greater reaching the very limits of the bound although never surpassing them.

Concerned with the failure to certify entanglement for a pure and entangled state, another, tighter, criterion was implemented. Using the entropic uncertainty relation criterion we were able to find violations of the bound for certain preparations. The Wigner angle where violations first appear depends on the preparation parameters. For preparations that didn't display steering, a reasonable explanation was given suggesting that even entanglement creation would be hindered, being then, naturally, not enough to violate the criterion.

As the previous section should have left clear, this work proposes more questions than it answers. The questions of most interest, those who might inspire future

[7] This loophole is a consequence of the fact that, if the interval that separates Alice's and Bob's measuring events is not space-like, information about a measurement could affect the other's results.
[8] That is, via sub-luminal information transportation.
[9] Meaning experimentally feasible.

works are the quest for quantities that, together with the known-to-change-resources, form relativistic invariants. For entanglement, there are good candidates in coherence and other correlations (SAVI; ANGELO, 2021), but for steering there is no known complementary resource. Also, if relativity is to be believed. In the sense that the laws of nature are equal in all frames of reference. If spin becomes entangled to momentum for a boosted observer, it means that the transformed spin observable in the reference frame where spin and momentum were originally separable must also be entangled with the transformed momentum. This means that, given a system in the likeness of the one that was used in this work, for a different DoF, one that is the transformation of the standard spin observable, entanglement should already be present. By hypothesis, the nature of such DoF might be accessible by studying the dynamics of a transformed spin measurement. Basically, the dynamics of the system under the electromagnetic field of a Stern-Gerlach experiment that is located in a frame of reference traveling away from the system in a frame that would model the system as spin-momentum entangled.

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APÊNDICE 1 – POSTULATES OF QUANTUM MECHANICS

1. **State Postulate:** The state of a quantum system is described by a normalized state vector $|\psi\rangle$ in a complex Hilbert space.
2. **Measurement Postulate:** When a measurement is performed on a quantum system described by the state vector $|\psi\rangle$, the outcome is one of the eigenvalues of the corresponding observable. The probability of obtaining a specific eigenvalue is given by the squared magnitude of the projection of $|\psi\rangle$ onto the corresponding eigenvector.
3. **Evolution Postulate:** The time evolution of a quantum system is described by the Schrödinger equation:
$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$
, where $\hbar$ is the reduced Planck's constant, $\hat{H}$ is the Hamiltonian operator, and $|\psi\rangle$ is the state vector at timr t.
4. **Superposition Postulate:** The state vector of a quantum system can exist in a superposition of multiple states. If $|\psi_1\rangle$ and $|\psi_2\rangle$ are valid state vectors, then any linear combination $\alpha|\psi_1\rangle + \beta|\psi_2\rangle$, where α and β are complex numbers satisfying the normalization condition, is also a valid state vector.
5. **Composite Systems Postulate:** The state space of a composite quantum system is the tensor product of the individual state spaces of the constituent systems.
6. **Entanglement Postulate:** The composite system can exist in an entangled state, where the state of the whole cannot be expressed as a simple product of the individual states of its parts.
7. **Measurement Collapse Postulate:** After a measurement is made on a quantum system, the system collapses into an eigenstate corresponding to the measured eigenvalue.

Note that in the Schrödinger equation, the time derivative is with respect to t, and $\hat{H}$ represents the Hamiltonian operator, which is associated with the total energy of the system.

Other presentations of these postulates exist. This is just a quite usual presentation of them.