UNIVERSIDADE FEDERAL DO PARANÁ

HENRIQUE ANTONIO RODRIGUES KNOPKI

SEARCHING FOR PHYSICAL DESCRIPTIONS RELATIVE TO A QUANTUM SYSTEM

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RESUMO

Um dos conceitos-chave em que toda a física se baseia é o de referenciais bem definidos. Se a mecânica quântica é uma teoria universal, então deve haver uma maneira de incorporar completamente esse conceito na teoria. Diversos autores fornecem estruturas nas quais o próprio referencial é um sistema quântico. Neste trabalho, argumentamos a favor de uma delas usando o requisito de construir espaços de Hilbert com graus de liberdade relativos. Além disso, mostramos que, para mais de dois subsistemas, não é possível encontrar uma transformação de coordenadas que mude para o referencial de um deles de modo que todos os graus de liberdade sejam relativos a ele. No lado informacional da mecânica quântica, existem recursos quânticos, que são as características que conferem a ela a vantagem sobre a teoria clássica em várias tarefas. Foi demonstrado que, em geral, os recursos quânticos não são invariáveis sob transformações quânticas de referencial. Um desses recursos é uma certa noção de incompatibilidade de contexto, que, quando combinada com coerência e discordância quântica, resulta no conteúdo total de informação do estado, que é invariante sob transformações de referencial. Embora o fenômeno da incompatibilidade quântica esteja no cerne da mecânica quântica, existem múltiplas definições para ela, cada uma conceitualizando um aspecto de sua manifestação. As primeiras definições eram declarações puramente sobre medições, mas recentemente o estado do sistema também foi envolvido, dando origem ao termo: incompatibilidade de contexto. Especificamente, a incompatibilidade de contexto independente da teoria é o recurso que analisamos e encontramos que não é invariante. Ao comparar esta forma de incompatibilidade entre um referencial de massa infinita bem localizado e um referencial quântico, descobrimos que é possível obter mais incompatibilidade quando o observador é um sistema quântico, mas, inevitavelmente, à medida que o estado do referencial quântico se torna mais e mais delocalizado, o contexto tende a ser compatível.

Palavras-chaves: Incompatibilidade, recursos quânticos, referenciais quânticos.

ABSTRACT

One of the key concepts that all of physics rely on is of well established reference frames. If quantum mechanics is a universal theory, then there must be a way to embed this concept fully into the theory. Several authors provide frameworks in which the reference frame itself is a quantum system. In this work we argue in favor of one of them using the requirement of building Hilbert spaces with relative degrees of freedom. Furthermore, we show that, for more than two subsystems it is not possible to find a coordinate transformation that jumps to the reference frame of one of them such that all degrees of freedom are relative to it. On the informational side of quantum mechanics, there are quantum resources, which are the features that give it the advantage over classical theory in several tasks. It has been shown that, in general, quantum resources are not invariant under quantum reference frame transformations. One of these resources is a certain notion of context incompatibility, which, when combined with coherence and quantum discord, results in the total information content of the state, which is invariant under reference frame transformations. Although the phenomenon of quantum incompatibility lies at the heart of quantum mechanics, there are multiple definitions for it, each one conceptualizing an aspect of its manifestation. The first definitions were declarations purely about measurements, but recently the state of the system was also involved, giving rise to the term: context incompatibility. Specifically, the so called theory independent context incompatibility is the resource we analyzed and found that it is not invariant. When comparing this form of incompatibility between well-localized infinite-mass reference frame and a quantum reference frame, we find that it is possible to obtain more incompatibility when the observer is a quantum system but, inevitably, as the state of the quantum reference frame gets more and more delocalized, the context tends to be compatibile.

Key-words: Incompatibility, Quantum Resources, Quantum reference frames.

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1 INTRODUCTION

Quantum information theory provides means for a faster realization of information processing protocols in quantum computing relative to known protocols. It also enables the realization of completely new ones, powerful enough to break the nonquantum cryptography systems and even perform a teleportation of a quantum state. The quantum features that allow the existence of such powerful protocols are called *quantum resources* [1].

The most important quantum resources are *coherence*, which gives birth to the qubit. The latter serves as a resource for numerous computational algorithms [2], such as entanglement, which is a form of strong correlation that serves as a resource for quantum teleportation [3] [4] and, at last, quantum incompatibility, which serves as a resource necessary for the observation of Bell inequality violations [5, 6]. The most primitive one comes with the uncertainty principle, stating that two observables can only be measured with arbitrary precision simultaneously only when they commute [7]. Moreover, with the generalization of the measurement process to positive operator-valued measures POVMs, improvements for the concept of incompatibility became necessary. Some propositions include joint measurability [8] and nondisturbance [9]. The first one encompasses the idea that when two measurements cannot be performed at the same time they are incompatible and the second one states that two measurements are nondisturbing only when measuring one does not alter the probability distribution of the other.

The definitions of incompatibility mentioned above regard only the measurements and their algebraic structure. Nevertheless, a more recent approach [10] considers that as the system becomes more and more classical, there should be no incompatibility. And the manner in which quantum mechanics approaches classicality is through the quantum state, thus incompatibility of physical contexts arises. It is based on an information leakage protocol where Alice prepares a state measuring an observable and sends the system to Bob, promising the amount of information in the resulting state but, Eve, a spy, attempts to retrieve some information about the state by measuring another observable. If Bob notices a difference in the amount of information in the state he receives, the context formed by the inicial state and the observables is incompatible. Later on, the concept evolved to the so-called independent context incompatibility, defined in [11], encompasses the physical state and takes a step further, it concerns only probabilities, making it applicable not just to quantum mechanics, but also to any probabilistic theories, be it classical or not.

On the other side of physics, reference frames are indispensable entities, since every physical quantity needs to be defined relative to one of those for it to make sense. Quantum mechanics does not escape this requirement, it is also built upon a Newtonian inertial reference frame that is well-localized in momentum and position. In this sense, quantum mechanics apparently needs classical mechanics to be made sense of. Aharonov and Kaufherr [12] question the need of classical massive objects relative to which quantum states are defined. They provide a coordinate transformation to the quantum reference frame of a quantum particle.

Since Aharonov and Kaufherr's work, at least two other approaches to quantum reference frames have been developed. One approach that changes to the center of mass and relative coordinates [13, 14], just like the hydrogen atom solution in any quantum mechanics textbook, and another, that claims to be fully relational and independent of an external classical reference frame, with changes only to relative position [15]. In this work, we argue in favor of the first approach, particularly for the two-particle case, because it does not lead to Hilbert spaces defined over degrees of freedom pertaining to different reference frames, while the relational approach does. Moreover we show that, originating from a classical reference frame, it is impossible to find a coordinate transformation that leads to the reference frame of a quantum system such that all canonical degrees of freedom are defined relative to it.

Special relativity shows that quantities that were thought to be absolute such as time and distances, in fact are not the same, they depend on the observer. It has already been shown that indispensable quantum resources such as *coherence* and *entanglement* are not invariant under quantum reference frame transformations [15–17]. But a combination of quantum resources has been shown to be invariant under these transformations, namely coherence, quantum discord and the incompatibility of a physical context [18].

As our second result, we show that, in general, the theory independent context incompatibility of the context composed of position, momentum and the physical state is not invariant under quantum reference frame transformations. In fact, there are cases in which quantum reference frames dispose of more incompatibility than classical ones. But, as the quantum system serving as reference frame gets more delocalized relative to the starting reference frame, the context inevitably tends to become compatible.

2 THEORETICAL FOUNDATIONS

2.1 QUANTUM THEORY

2.1.1 State vector in a Hilbert Space

Classical physics granted perfect prediction of every experiment given the initial conditions and the interactions in play due to the presumed objective existence of trajectories. Later experiments, like Davisson and Germer's (1927) [19] which is a Young's double-slit experiment but, performed with electrons instead of light. This experiment confirmed the de Broglie hypothesis regarding the wave behavior of matter, demonstrating the intrinsic probabilistic nature of measurement outcomes. Quantum mechanics accounts for this randomness by promoting the position and momenta to Hermitian operators acting on a Hilbert space. Therefore, Quantum Mechanics postulates that a given physical quantity $\mathcal{A}$ is represented by a linear operator A that is Hermitian ($A = A^{\dagger}$). Bounded nondegenerate Hermitian operators have spectral decomposition, given by

$$A = \sum_i a_i \left|a_i\right\rangle\!\left\langle a_i\right|, \tag{2.1}$$

with real eigenvalues a_i that represent the possible results of a measurement of this physical quantity. Its eigenvectors $|a_i\rangle$ are orthogonal and the associated projectors satisfy the closure relation

$$\left\langle a_i \middle| a_j \right\rangle = \delta_{ij}, \tag{2.2a}$$

$$\sum_i \left|a_i\right\rangle\!\left\langle a_i\right| = \mathbb{1}. \tag{2.2b}$$

The probabilities of obtaining a certain outcome is encoded in the physical state of the system in which the measurement is performed. The probabilistic character of the measurement outcome of the observable is accounted for by the *superposition principle*, which states that a system can occupy any normalized linear combination of the eigenvectors of the observable being measured. Therefore we can assign a vector in a Hilbert space to every possible physical state. Once a measurement of $\mathcal{A}$ is performed and an outcome a_i is observed, the update of the physical state is given according to the physical requirement of *repeatability* of the measurement, i.e., a second ideal measurement of $\mathcal{A}$ must yield the same result. It follows that the state after the measurement is the eigenstate $|a_i\rangle$ of A:

$$|\Psi\rangle \xrightarrow{a_i} \frac{A_i\,|\Psi\rangle}{\sqrt{\langle\Psi|\,A_i\,|\Psi\rangle}}, \tag{2.3}$$

i.e., the normalized projection of $|\Psi\rangle$ in the i-th subspace of the A eigenbasis where $A_i = \left|a_i\right\rangle\!\left\langle a_i\right|$. The probability $p(a_i)$ of an outcome a_i to occur is given by *Born's* rule

$$p(a_i) = |\,\langle a_i|\Psi\rangle\,|^2, \tag{2.4}$$

that is, the modulus squared of the coefficients of the expansion of the state on the A eigenbasis $|\Psi\rangle = \sum_i c_i |a_i\rangle$, with $\sum_i |c_i|^2 = 1$. This aligns with the requirement that the probabilities of all outcomes sum up to 1, thus supporting the previously mentioned state normalization.

The expectation value of an observable A can be computed through its definition in statistics: the probability weighted average of the possible outcomes a_i

$$\begin{aligned}\langle A\rangle &= \sum_i p(a_i)a_i \\ &= \langle\Psi| \sum_i a_i A_i |\Psi\rangle \\ &= \langle\Psi| A |\Psi\rangle .\end{aligned} \tag{2.5}$$

The temporal evolution is generated by the Hamiltonian H, much like in classical mechanics, which is the observable corresponding to the total energy of the system considering that H is explicitly time-independent. The equation that dictates the evolution of a physical state $|\Psi(t)\rangle$ is also a postulate and is named the *Schrödinger equation*

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle . \tag{2.6}$$

In order to preserve normalization of the vector state, a unitary operator $U(t, t_0)$ can be conceived to evolve the state vector from time t_0 to t, i.e. $|\Psi(t)\rangle = U(t, t_0)|\Psi(t_0)\rangle$. This evolution leads to the differential equation

$$i\hbar\frac{\partial}{\partial t}U = HU. \tag{2.7}$$

If H is time-independent, indicating conservative dynamics, solving for U yields

$$U(t, t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}. \tag{2.8}$$

The last postulate states that the state of a system composed of two subsystems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with dimensions d_1 and d_2, respectively, is a normalized vector in the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ of dimension $d = d_1d_2$. A basis $\{|e_i\rangle\}$ of $\mathcal{H}_1$ and a basis $\{|f_j\rangle\}$ of $\mathcal{H}_2$ may form the basis $\{|e_i\rangle \otimes |f_j\rangle\}$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that any state vector $|\psi\rangle$ of the composite system can be expanded in this basis

$$|\psi\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} \alpha_{ij}|e_i\rangle|f_j\rangle , \tag{2.9}$$

where we chose not to use the tensor product symbol "⊗" from here on.

The time evolution can be represented in two alternative ways, the *Heisenberg* and *Schrödinger picture*. Here the state vector was considered as the time-dependent entity, defining the Schrödinger pictures but, since the physical predictions are calculated as in Eq. (2.5) the Heisenberg picture may as well be used:

$$\langle\psi(t)| A |\psi(t)\rangle = \langle\psi(0)| U(t, t_0)^\dagger A U(t, t_0) |\psi(0)\rangle = \langle\psi(0)| A(t) |\psi(0)\rangle , \tag{2.10}$$

where the evolution is applied in the observable as $A(t) = U^\dagger AU$ while the state $|\psi(0)\rangle$ stays static in time. Finally, the equation of motion for the Heisenberg operator $A(t)$ is

$$\frac{\partial A(t)}{\partial t} = \frac{i}{\hbar}[H, A(t)], \tag{2.11}$$

where H is time-independent.

2.1.2 Continuous degrees of freedom

Not all physical quantities are limited to discrete values, position and momentum are continuous variables and can take any value in the real numbers. Therefore, these observables must have an eigenbasis composed of an infinite set of eigenvectors resulting in an infinite-dimensional uncountable Hilbert space. The analogue of Eq. (2.2) for continuous variables such as position X is

$$X = \int x\,|x\rangle\langle x|\,dx, \tag{2.12a}$$

$$\langle x|x'\rangle = \delta(x - x'), \tag{2.12b}$$

$$\int |x\rangle\langle x|\,dx = \mathbb{1}, \tag{2.12c}$$

where $\delta(x - x')$ is *Dirac*'s delta distribution. Eq. (2.12a) is the spectral decomposition of X and Eq. (2.12c) represents the orthogonality between the eigenstates and the closure relation, respectively. One can expand a quantum state $|\Psi\rangle$ in terms of the position eigenstates

$$|\Psi\rangle = \int \Psi(x)\,|x\rangle\,dx, \tag{2.13}$$

where the linear combination coefficient $\Psi(x)$ is the *position wave function* of the system. The translation operator $T(d)$ acts on a position eigenstate as

$$T(d)\,|x\rangle = |x + d\rangle\,. \tag{2.14}$$

Acting on $|\Psi\rangle$, it yields the new wave function $\Psi(x - d)$. Therefore the translation operator $T(d)$ is given by

$$T(d) = \exp\left(-\frac{i}{\hbar}dP\right). \tag{2.15}$$

Taking the canonical momentum to be the generator of spatial translations, it satisfies the canonical commutation relation with the position operator

$$[X, P] = i\hbar\mathbb{1}. \tag{2.16}$$

We can also start building the Hilbert space with the momentum observable P using its own set of eigenstates.

$$P = \int p\,|p\rangle\langle p|\,dp, \tag{2.17a}$$

$$\langle p|p'\rangle = \delta(p - p'), \qquad \int |p\rangle\langle p|\,dp = \mathbb{1}. \tag{2.17b}$$

The state $|\Psi\rangle$ can be expanded in terms of momentum eigenstates as

$$|\Psi\rangle = \int \phi(p)\,|p\rangle\,dp, \tag{2.18}$$

where $\phi(p)$ is the *momentum wave function* of the system. The relationship between the momentum and position wave functions is obtained knowing the inner product between each element of the position and momentum basis, which is found imposing that P is the generator of translations:

$$\langle x|p\rangle = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\exp\left[\frac{ixp}{\hbar}\right]. \tag{2.19}$$

Using Eqs. (2.17b) and (2.18) we express the position wave function as the Fourier transform of the momentum wave function

$$\psi(x) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int \phi(p)\exp\left[\frac{ixp}{\hbar}\right]dp, \tag{2.20a}$$

$$\phi(p) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int \psi(x)\exp\left[\frac{-ixp}{\hbar}\right]dx. \tag{2.20b}$$

For a Gaussian wave function centered at a with variance Δ

$$\psi(x) = \left(\frac{1}{2\pi\Delta^2}\right)^{\frac{1}{4}}\exp\left[-\frac{(x-a)^2}{4\Delta^2}\right], \tag{2.21}$$

the corresponding momentum wave function is

$$\phi(p) = \left(\frac{2\Delta^2}{\pi\hbar^2}\right)^{\frac{1}{4}}\exp\left(-\frac{\Delta^2 p^2}{\hbar^2}\right)\exp\left(\frac{iap}{\hbar}\right), \tag{2.22}$$

an imaginary exponential modulated by a Gaussian of null mean momentum.

In three dimensions, each spatial direction has its own Hilbert space and the total state is treated as a composite system. This is due to the fact that translations in different directions must commute. The defining algebra of the position and momenta operators is, then

$$[X_j, P_k] = \delta_{jk} i\hbar \mathbb{1}, \tag{2.23}$$

where $j, k = 1, 2, 3$ labels three orthogonal spatial directions.

2.1.3 The Density Operator

To formulate an information theory based on quantum mechanics it is necessary to introduce a structure which allows the existence of subjective ignorance in the quantum state. This is accomplished assigning probabilities p_i to the possible states the system can be occupying as a reflection of the observer's lack of information about it. We describe this as an ensemble of quantum states $\{p_i, |\psi_i\rangle\}$, and lift the structure of the Hilbert space $\mathcal{H}$ to a

Hilbert space $\mathcal{S}(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H}^*$ composed of the bounded linear operators acting on $\mathcal{H}$, where * denotes the dual space. The quantum state is described by the density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{2.24}$$

formed by the *convex combination* of the projectors $|\psi_i\rangle\langle\psi_i|$ with non-negative coefficients $p_i \in \mathbb{R}$ such that $\sum_i p_i = 1$. In general, this convex combination is not unique and the number of terms in the sum can vary.

The purity $\mathcal{P}(\rho)$ of a density operator is defined to be a notion of the mixedness of a state, i.e., the certainty about the preparation of the state. That is accomplished using the squared norm of the density operator with the Hilbert-Schmidt norm $\langle A, B\rangle_{HS} = \mathrm{Tr}(A^\dagger B)$:

$$\mathcal{P}(\rho) = \mathrm{Tr}(\rho^2). \tag{2.25}$$

In the case of having maximal knowledge about the state it is defined as a *pure state* given by $\rho = |\psi\rangle\langle\psi|$. In the case of minimal knowledge about the state $\rho = \frac{\mathbb{1}}{d}$ is said to be the maximally mixed state. Hence, the upper bound of the purity is 1 and it is achieved by pure states, while the lower bound is $\frac{1}{d}$ and it is achieved by the maximally mixed state.

The two properties[1] that define a density operator are the *positivity* and the *unit trace.*

$$\mathrm{Tr}\,\rho = 1 \quad \text{(Unit trace)}, \tag{2.26a}$$

$$\langle\phi|\,\rho\,|\phi\rangle \geq 0, \quad \forall\,|\phi\rangle \in \mathcal{H} \quad \text{(Positivity)}. \tag{2.26b}$$

The postulates presented in section 2.1 can be translated to the density operator language. A physical quantity is still represented by Hermitian operators A. The probability $p(a_i)$ of obtaining the result a_i from a measurement of A is given by Born's rule

$$p(a_i) = \mathrm{Tr}(A_i\rho). \tag{2.27}$$

The state immediately after the measurement undergoes the collapse represented by the map

$$\phi_{a_i}(\rho) = \frac{A_i\rho A_i}{\mathrm{Tr}(A_i\rho)}. \tag{2.28}$$

An important case is the *nonselective measurement*, where the measurement is performed, but the result is forgotten or not even read. The density operator is then updated to a mixed state, where the ensemble is $\{p(a_i) = \mathrm{Tr}(A_i\rho), \phi_{a_i}(\rho)\}$ and the map is

$$\phi_A(\rho) = \sum_i p(a_i)\phi_{a_i}(\rho) = \sum_i A_i\rho A_i. \tag{2.29}$$

[1] Standard textbooks [20] also present Hermiticity as a defining property of the density operator, but here we take into consideration that positivity implies hermiticity [21].

The Schrödinger equation is replaced by the Liouville-von Neumann equation,

$$\frac{\partial}{\partial t}\rho = \frac{i}{\hbar}[H, \rho]. \tag{2.30}$$

The time evolution operator is the same as in section 2.1 and the state evolves from time t_0 to t as

$$\rho(t) = U(t, t_0)\rho U^\dagger(t, t_0). \tag{2.31}$$

2.1.4 POVMS and the measurement process

Measurements, as presented in the fashion of the previous sections are projective and the possible results are conclusive about the value of the measured physical quantity, i.e., orthogonal, for that reason they are called projection-valued measurements (PVMs). A closer look into the measurement process reveals that the experimental procedure that extracts the sought after information about the system involves physical interactions. To illustrate it, consider the Stern-Gerlach experiment depicted in figure 1. It involves a neutral particle with spin $\frac{1}{2}$ entering a non-homogeneous magnetic field $\boldsymbol{B}$ pointing in the z direction, or the direction chosen to perform the spin measurement. A classical description of this physical situation would reveal that the trajectory of this particle would be deflected upwards if the angular momentum of the particle were aligned with $\boldsymbol{B}$ and downwards if they were anti-aligned. The quantum description results in a superposition of spin up and down. It can be shown that the state $|\psi\rangle$ of the particle before the measurement is

$$|\psi\rangle = \left(\frac{|\uparrow\rangle\,|\psi_+\rangle + |\downarrow\rangle\,|\psi_-\rangle}{N}\right), \tag{2.32}$$

where $\langle\psi_-|\psi_+\rangle \approx 0$ and N is the normalization constant. Since the detectors are sufficiently distant, there is no significant overlap between $|\psi_+\rangle$ and $|\psi_-\rangle$, the first being a state completely localized in the upper half of the detector and the second on the bottom half. What is truly measured is the position degree of freedom, i.e., spin is measured indirectly through a PVM of position. In this case, the measurement is perfect, i.e., the correlation between the deflection (up or down) of the particle and the possible results of spin is perfect. If the detectors were placed closer to the apparatus, the overlap would be different from zero, because the particle could be detected nearer to the centerline, i.e., $\langle\psi_-|\psi_+\rangle \neq 0$. The result is an imperfect correlation between the detection of the particle above (below) $z = 0$ and spin up (down). This captures the essence of a *positive operator valued measurement* POVM, a measurement that is made in the system but does not necessarily retrieve a conclusive result of a physical quantity. In this case, this manifests itself in the unreliability of the measurement to accurately represent the spin observable.

The POVM can also extract information about other properties that are not physical observables. For instance, in a delayed choice Mach-Zehnder interferometer a POVM is performed such that its possible outcomes are the wave and particle character of a physical fig/stern gerlach.png

Figure 1 – Stern-Gerlach experiment where *a*) the detector is far away from the setting and *b*) the detector is close to the setting.

system, providing a counterexample to the famous Bohr's complementarity principle, which states that in an experiment a quantum system can exhibit either wave-like or particle-like characteristics [22].

This generalization of PVM to POVM is formalized by relaxing the requirement of orthogonality between the projector associated to the results of the measurements in Eq. (2.2) in the following manner.

A POVM F is represented by a set of n Hermitian operators $\{F_i\}$ called elements of the POVM, which follows the rules

1. Positivity: $\langle F_i \rangle \geq 0$ for all states;
2. Completeness: $\sum_i^n F_i = \mathbb{1}$.

The i-th element, after the POVM F is performed in the state ρ, occurs with probability $\mathrm{Tr}(F_i \rho)$, i.e., Born's rule is also valid and the state is updated as

$$\rho \xrightarrow{F_i} \frac{E_i^\dagger \rho E_i}{\mathrm{Tr}(F_i \rho)}, \tag{2.33}$$

where $F_i = E_i E_i^\dagger$. A POVM that follows these properties and in addition has orthogonal elements $F_i F_j = \delta_{ij} F_i$ is a PVM. This indicates that POVMs are in fact a more general measurement model since it has all the PVMs as a specific case.

POVMs can be implemented in different ways, each one of them is referred to as an instrument $\mathfrak{I}$ [23], which is a collection of maps $\{\mathfrak{I}_i(\rho)\}$ that preserve the positivity of ρ and does not increase the trace and hold $\mathrm{Tr}[\mathfrak{I}_i(\rho)] = \mathrm{Tr}[\rho F_i]$. The instrument $\mathfrak{I}$ updates the state to $\frac{\mathfrak{I}_i(\rho)}{\mathrm{Tr}[\mathfrak{I}_i(\rho)]}$, and if the outcome of the measurement is not revealed it is updated to the convex combination

$$\mathfrak{I}(\rho) = \sum_i \mathfrak{I}_i(\rho), \tag{2.34}$$

where the operator on the RHS of Eq. (2.34) has unitary trace and is a positive operator. Note that the $\{E_i\}$ in Eq. (2.33) represents a specific instrument called *Lüders* instrument [24].

2.1.5 The reduced state

Consider a system AB consisting of subsystems A and B. The density of the system acts on the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Given that the state of the system is ρ_{AB} but the observer that assigned this state to the system can only access the subsystem B, e.g., the subsystem A was sent to another laboratory that has no possibility of communication with the laboratory in which part B is located. The state ρ_B assigned to the subsystem B by the observer is given by the partial trace of the state of the system AB, eliminating the degrees of freedom that belongs to subsystem A:

$$\rho_B = \mathrm{Tr}_A(\rho_{AB}). \tag{2.35}$$

The same is true for the description of subsystem A, but the partial trace is performed over the space $\mathcal{H}_B$.

Initially, utilizing the partial trace operation to obtain the state of a subsystem seems arbitrary, but invoking the fact that the description assigned to a system does not alter its physical properties, the expectation value of a measurement of the observable O performed on subsystem B must not change if we describe only subsystem B or the whole system AB

$$\mathrm{Tr}_{AB}(O\rho_{AB}) = \mathrm{Tr}_B(O\rho_B). \tag{2.36}$$

Therefore, it is clear that the partial trace is a good candidate in order to describe the state of a subsystem ignoring subsystems inaccessible to the observer. In fact, the partial trace is not only a good candidate, in fact, it is the only one that satisfy the requirement of Eq. (2.36).

2.2 CLASSICAL AND QUANTUM ENTROPIES

2.2.1 Shannon's Entropy

In classical information theory, the quantity that measures the ignorance about a random variable X is Shannon's entropy [25]

$$H(X) = -\sum_{x\in X} p(x)\log p(x), \tag{2.37}$$

where $p(x) > 0$ represents the probability of obtaining the outcome x and they sum up to 1. Shannon's entropy is zero if and only if a certain outcome occurs with probability one, while is maximal and reaches $\log n$ if and only if every outcome occurs with equal probability $\frac{1}{n}$, where n is the number of elements in the sample space of X.

To understand the meaning of Shannon's entropy it is worthwhile to list properties that the information $I(p(x))$, obtained given that the outcome x occurred, should satisfy

1. $I(p(x)) \geq 0$. Observing a random variable never yields loss of information;

2. $I(p(x))$ decreases as $p(x)$ increases and is maximal if $p(x) = 1$. It means that little information is gained when an event that has high probability to occur is observed;

3. $I(1) = 0$. No information is gained from an outcome that is certain to occur;

4. $I(p(x, y)) \leq I(p(x)) + I(p(y))$, and the equality holds only for independent events, that is, those satisfying $p(x, y) = p(x)p(y)$. The information gained when observing independent events is the sum of the information gained by the observation of each event and if they are correlated, the information gain is always less.

It can be shown [26] that the function that satisfies these properties is

$$I(p(x)) = -\log(p(x)), \tag{2.38}$$

where the choice of the logarithm base determines the unit system and can be selected according to the specific application. *Bits* are defined by base 2, *nats* are define by base e, the Euler number. The sum of all $I(p(x))$ with weight $p(x)$, i.e. the mean information acquired by observing a random variable, is $H(X)$, as in Eq. (2.37). $H(X)$ follows the same properties that $I(p(x))$ satisfies.

2.2.2 Kullback-Leibler divergence

Crucial for the definitions of incompatibility that will be analyzed in this work is the relative entropy, also called Kullback-Leibler divergence $D(p||q)$ introduced in [27]. It captures the notion of the discrepancy between the probability distributions p and q over a sample space X. A way to construct the divergence of p from q is to imagine using the distribution q as a model for the real underlying distribution p. Therefore, we utilize the probabilities p_i to weight the summation of the difference between apparent information $I(q(x))$ and the real information $I(p(x))$, resulting in

$$\begin{aligned} D(p||q) &= \sum_{x \in X} -p(x)[\log q(x) - \log p(x)] \\ &= -H(X) \sum_{x \in X} -p(x) \log q(x) \geq 0. \end{aligned} \tag{2.39}$$

The relative entropy is zero if and only if the two probability distributions are identical, $p(x) = q(x)$. It is important to notice that the relative entropy is not symmetric, meaning $D(p||q) \neq D(q||p)$ in general, thus it can't represent a distance between probability distributions.

2.2.3 Von Neumann entropy

A possible generalization for quantum systems was given by von Neummann in [28]. The von Neumann entropy of a quantum state ρ is given by

$$S(\rho) = -\operatorname{Tr}(\rho \log \rho), \tag{2.40}$$

where by convention the base of the logarithm is equal to the dimension of the Hilbert space of the system, although it can be chosen arbitrarily as well, so that it would fix the unit system as in Shannon's entropy. The von Neumann entropy reduces to the Shannon entropy if we write the spectral decomposition of ρ in terms of its eigenvectors $|\lambda_i\rangle$ having eigenvalue λ_i

$$S(\rho) = -\sum_{i,j} \langle\lambda_i|\, \lambda_j \, |\lambda_j\rangle\langle\lambda_j| \log\left(\sum_k \lambda_k \, |\lambda_k\rangle\langle\lambda_k|\right) |\lambda_i\rangle \tag{2.41}$$

$$= -\sum_i \lambda_i \log \lambda_i = H(\Lambda), \tag{2.42}$$

where Λ is any observable that is simultaneously diagonalizable with ρ, i.e., $\Lambda = \sum_i a_i \, |\lambda_i\rangle\langle\lambda_i|$ and $H(\Lambda)$ is Shannon's entropy associated to the probability distribution of Λ on the state ρ.

Some properties of the von Neumann entropy are:

1. Non-negativity: $S(\rho) \geq 0$, equality if and only if ρ is pure;
2. Upper-bound: $S(\rho) \leq \log d$, equality is reached when $\rho = \frac{\mathbb{1}}{d}$, i.e., ρ is maximally mixed;
3. Invariance under unitary transformations: If U is unitary, then $S(U\rho U^\dagger) = S(\rho)$;
4. Subadditivity: For a composite system AB, the triangle inequality holds

$$|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B), \tag{2.43}$$

where the LHS is the *Araki-Lieb* inequality and the equality holds for pure states while in the RHS the equality holds for separable states $\rho_{AB} = \rho_A \otimes \rho_B$.

One of the main conceptual differences between quantum and classical systems is that in classical systems the entropy of the whole can never be smaller than the entropy of the parts, as seen in property 4 of Shannon's entropy, while for quantum systems, it is possible to have ignorance about the parts while having full knowledge of the whole as shown in Eq. (2.43).

Similar to the interpretation of Shannon's entropy, $S(\rho)$ carries the meaning of the ignorance about the system so to quantify the information $I(\rho)$ we have about the system, we sum the current information and the current ignorance resulting in the maximum information of the system

$$I(\rho) = \log d - S(\rho), \tag{2.44}$$

where $d = dim(\mathcal{H})$. An important measure of total correlations is the *quantum mutual information between parts* A and B

$$I_{A:B}(\rho) = S(\rho_A) + S(\rho_B) - S(\rho), \tag{2.45}$$

where ρ_A and ρ_B are the reduced states given by Eq. (2.35). This quantity may be understood as the amount of information that is encoded in A and also in B by summing all information of A and B and subtracting the total AB information, resulting in the hatched part in figure 2.

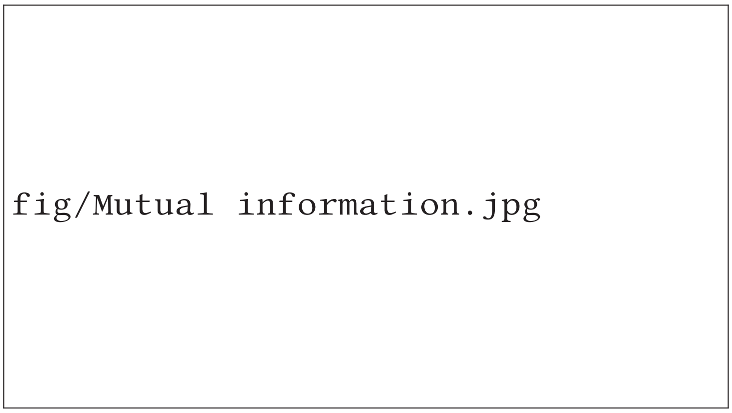

Figure 2 – Diagram depiction of the intersection of the information contained in part A and part B, which is the mutual information.

2.2.4 Quantum relative entropy

The quantum analogue of the Kullback-Leibler divergence is the quantum relative entropy of a state ρ from σ

$$S(\rho||\sigma) = \mathrm{Tr}\,\rho\log\rho - \mathrm{Tr}\,\rho\log\sigma, \tag{2.46}$$

which also encapsulates a notion of distance of ρ from σ in a sense of how indistinguishable they are in a unidirectional way, since $S(\rho||\sigma) \neq S(\sigma||\rho)$, in general. The nonnegativity is granted by *Klein's inequality*: $S(\rho||\sigma) \geq 0$, reviewed in [29].

The mutual information between parts has an intimate connection with relative entropy. Consider the same setting as in the previous section, now with $\sigma = \rho_A \otimes \rho_B$,

$$\begin{aligned} S(\rho||\rho_A \otimes \rho_B) &= \mathrm{Tr}[\rho\log\rho - \rho\log(\rho_A \otimes \rho_B)] \\ &= -S(\rho) - \mathrm{Tr}[\rho\log(\rho_A \otimes \mathbb{1})] + \mathrm{Tr}[\rho\log(\mathbb{1} \otimes \rho_B)] \\ &= S(\rho_A) + S(\rho_B) - S(\rho) = I_{A:B}(\rho). \end{aligned} \tag{2.47}$$

Therefore, the mutual information between parts A and B can be interpreted as the discrepancy in using $\rho_A \otimes \rho_B$ as a model for ρ.

2.2.5 Linear entropy

Expanding von Neumann's entropy with the Mercator series

$$\log(1 - x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n!}, \tag{2.48}$$

around the identity

$$-\mathrm{Tr}(\rho\log\rho) = -\mathrm{Tr}\left\{\rho[(\mathbb{1} - \rho) - \frac{1}{2}(\mathbb{1} - \rho)^2 + \frac{1}{6}(\mathbb{1} - \rho)^3 + \ldots]\right\}, \tag{2.49}$$

where the first-order term of $1 - \rho$ is defined as the linear entropy.

$$S_L(\rho) = 1 - \mathcal{P}(\rho). \tag{2.50}$$

It is important to notice that the linear entropy is not an approximation of the von Neumann entropy, but the fact that when the von Neumann entropy differs from zero, the linear entropy is also non-zero and vice versa grants us the possibility of using the linear entropy as a substitute in entropic based resource quantifiers. In fact, a monotonical relationship between the two is verified for a variety of parametrization of states of arbitrary dimension d [30], where the comparison can be made using d as the base of the logarithm and multiplying the linear entropy by $\frac{d}{d-1}$, in this way, the von Neumann and the linear entropy have the same maximum and minimum values.

It is important to point out that the linear entropy does not satisfy the desired property of additivity but it is considerably simpler to calculate than the von Neumann entropy and the computational cost is drastically lower since it does not require diagonalization of density operators to be computed.

2.3 QUANTUM RESOURCES

Quantum resources are features of quantum theory that can be exploited in order to accomplish tasks in a more efficient way when compared to other information processing methods, such as the advantage that quantum computation has over classical computation, the promise of a worldwide quantum network to replace regular internet and quantum cryptography, which is far safer than the classical couterpart. Some of the core quantum resources that enable these accomplishments are *coherence* and *entanglement*. To characterize a quantum resource in terms of a formal structure [1], there must be four consistent definitions:

1. The *free states*, i.e., states that possess no resource;
2. The *free operations*, i.e., operations that never increase the amount of resource in the system;
3. A quantifier for the resource;
4. A task that is not possible to be accomplished without the resource.

2.3.1 Coherence

Coherence is understood as the amount of interference possible to be observed with a given observable A with eigenbasis $\{|a_i\rangle\}$ of a Hilbert space $\mathcal{H}$ of dimension d. Starting from the set of free states, or incoherent states, defined as all ρ^{free} that are diagonal in the eigenbasis of A, then the set of all incoherent states is

$$Inc = \left\{\rho^{free} \in \mathcal{S}(\mathcal{H}), \quad \text{such that } \rho^{free} = \sum_{i=1}^{d} \lambda_i |a_i\rangle\langle a_i| \right\}, \tag{2.51}$$

where λ_i are the populations of $|a_i\rangle$. In other words, if a state ρ is a mixture of eigenstates of A, it is an A-incoherent state. The free operations are any map $\phi(\rho) : Inc \longrightarrow Inc$, i.e., any operation that produces no elements outside the diagonal of ρ when written in the eigenbasis of A.

In [31], the coherence quantifier $C(\rho)$ is defined as the relative entropy between the state of interest $\rho = \sum_{i,j=1}^{d} \lambda_{ij} \left|a_i\right\rangle\!\left\langle a_j\right|$ and $\rho_d = \sum_{i=1}^{d} \lambda_{ii} \left|a_i\right\rangle\!\left\langle a_i\right|$, which is merely ρ with all terms outside the diagonal being zero

$$C_A(\rho) = S(\rho_d||\rho) = S(\rho_d) - S(\rho). \tag{2.52}$$

The state with the highest coherence has equal weight among all possible elements of the basis:

$$|\psi\rangle_{max} = \sum_{i=1}^{d} \frac{1}{\sqrt{d}} |a_i\rangle . \tag{2.53}$$

In fact, the procedure of eliminating the off-diagonal elements of ρ is equivalent of performing a nonselective measurement of A with the map of Eq. (2.29), being a way of understanding the fact that measurements destroy all coherence in that basis. In effect,

$$\phi_A(\rho) = \sum_{i,j,k=1}^{d} \lambda_{ij} \left|a_k\right\rangle\!\left\langle a_k\right| \left|a_i\right\rangle\!\left\langle a_j\right| \left|a_k\right\rangle\!\left\langle a_k\right| \tag{2.54}$$

$$= \sum_{k} \lambda_{kk} \left|a_k\right\rangle\!\left\langle a_k\right| = \rho_d. \tag{2.55}$$

2.3.2 Entanglement of pure bipartite states

The definition of bipartite entanglement for pure states is straightforward. Given two systems A and B and their state space $\mathcal{H}_A \otimes \mathcal{H}_B$, the vector state of the system is entangled if and only if it is not separable, i.e., cannot be written as a product state. If the state is separable, it is not entangled [4]. Note that if a state is entangled, by extracting information from one of the subsystems, information about the other is also retrieved, i.e. entanglement is a form of correlation between parts. The special feature about these correlations is that the knowledge about the whole can be maximal while the knowledge of the parts is not. Take the singlet state $|\psi\rangle$, for example,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B). \tag{2.56}$$

Measuring spin in particle A immediately grants the observer the knowledge that the spin of particle B is anti-parallel to the result obtained in part A. However, any nonquantum physical system can exhibit this kind of correlation. Here, the total spin is well defined while the spin of each part is not well defined in any direction, in fact the expression above is rotationally invariant, such that this strong correlation has no classical analogue. A way to quantify the amount of entanglement in a bipartite system in a pure state ρ_{AB} is by calculating the von

Neumann entropy of the reduced state,

$$E(\rho_{AB}) = S(\rho_A) = S(\rho_B). \tag{2.57}$$

Thanks to Eq. (2.43) the amount of entanglement is the same whether you calculate $S(\rho_A)$ or $S(\rho_B)$. This quantifier is called *entropy of entanglement.* This is an example of how one can have full knowledge about the composite system's state but maximum ignorance about the parts A and B since the reduced density operators are maximally mixed.

For continuous variables, such as the spatial degrees of freedom, consider a generic bipartite state

$$|\psi\rangle = \iint \psi(x_1, x_2) |x_1\rangle |x_2\rangle \, dx_1 dx_2. \tag{2.58}$$

$|\psi\rangle$ is separable if and only if the wave function is separable in the coordinates, i.e., $\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$, which results in two separate integrals

$$|\psi\rangle = \int \psi_1(x_1) |x_1\rangle \, dx_1 \int \psi_2(x_2) |x_2\rangle \, dx_2. \tag{2.59}$$

The von Neumann entropy often diverges when considering continuous variables, whereas the linear entropy never diverges. Therefore, the *linear entropy of entanglement* can be used as an entanglement quantifier

$$E_L(\rho_{AB}) = 1 - \mathcal{P}(\rho_A) = 1 - \mathcal{P}(\rho_B). \tag{2.60}$$

We do not have to worry about any situation where one of the entropies is zero while the other is not because when one is non zero, the other automatically is too.

2.3.3 Quantum Discord

Consider a system composed of parts A and B, entanglement is not the cause, in general, of all the correlations present in the system for there may be correlations due to statistical mixtures. Imagine the correlation of the z component of spin being zero in the singlet state given in Eq. (2.56) but instead in a statistical mixing manner

$$\rho_S = \frac{|\uparrow\rangle\langle\uparrow|_A \, |\downarrow\rangle\langle\downarrow|_B + |\downarrow\rangle\langle\downarrow|_A \, |\uparrow\rangle\langle\uparrow|_B}{2}. \tag{2.61}$$

Ollivier and Zurek define the content of Quantum Discord in [32] as the measure of the amount of information about part A that cannot be extracted by revealing the outcome of the measurement that was performed on part B. Keeping that in mind, consider that the quantum mutual information $I_{A:B}(\rho)$ presented in Eq. (2.45) is a good measure of all correlations established in the state ρ, afterall, it measures the information of part A that is also encoded in part B and vice versa. Now we find the information content of part A after a measurement of Π_B given that the k-th outcome occured, the reduced state is

$$\rho_{A|k} = \frac{\mathrm{Tr}_A \left(\Pi_{B_k} \rho\right)}{\mathrm{Tr}\, \Pi_{B_k} \rho}, \tag{2.62}$$

where Π_{B_k} are the projectors of the spectral decomposition of Π_B. The amount of ignorance about part A considering multiple rounds of the same experiment is, in average,

$$S(\rho_A|\Pi_B) = \sum_k \mathrm{Tr}\big(\Pi_{B_k}\rho\big)S(\rho_{A|k}), \tag{2.63}$$

so that the average information gained about part A after the measurement in part B is

$$J_{A:B}(\rho) = S(\rho_A) - S(\rho_A|\Pi_B). \tag{2.64}$$

The basis dependent Quantum Discord is defined as

$$D_B(\rho) = I_{A:B}(\rho) - J_{A:B}(\rho). \tag{2.65}$$

If Π_B is the observable that maximizes the information to be obtained from part A it turns into Quantum Discord per se, but in this work only the basis dependent quantum discord will be of interest.

In [33], Rulli and Sarandy symmetrize the notion of quantum discord by considering measurements in both parts of the system. This goes by noticing that $J_{A:B}(\rho)$ can be written in terms of a relative entropy

$$J_{A:B}(\Pi_B) = S(\rho_A) + S(\phi_{\Pi_B}(\rho_B)) - S(\phi_{\Pi_B}(\rho)) \tag{2.66}$$

$$= S(\phi_{\Pi_B}(\rho)||\rho_A \otimes \phi_{\Pi_B}(\rho_{\Pi_B})), \tag{2.67}$$

where ϕ_{Π_B} is the nonselective measurement and

$$S(\phi_{\Pi_B}(\rho)) = S(\phi_{\Pi_B}(\rho_B)) + S(\rho_A|\Pi_B). \tag{2.68}$$

Now, using the relationship between the relative entropy and mutual information in Eq. (2.47) we write

$$D_B(\rho) = I_{A:B}(\rho) - I_{A:B}(\phi_{\Pi_B}(\rho)). \tag{2.69}$$

The next step is considering the measurement of the observables $\Pi_A \otimes \mathbb{1}$ and $\mathbb{1} \otimes \Pi_B$ instead of only one measurement:

$$D_{AB}(\rho) = I_{A:B}(\rho) - I_{A:B}(\phi_{\Pi_A\Pi_B}(\rho)), \tag{2.70}$$

where $\phi_{\Pi_A\Pi_B}(\rho) = \phi_{\Pi_A}(\phi_{\Pi_B}(\rho))$ where the order of the measurements does not matter because they commute. Eq. (2.70) represents the AB-discord. In the same way as the unsymmetrized discord, to eliminate the dependece on the measured observables, symmetrized quantum discord is defined as Eq. (2.70) minimized over all observables.

2.4 INCOMPATIBILITY

Measurement incompatibility is a feature that Quantum Mechanics exhibits that makes it impossible to reduce the uncertainty associated to an observable to zero given that an incompatible observable was measured first. This becomes evident when the RHS of the famous Robertson-Schrödinger uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2} | \langle \psi | \, [A, B] \, | \psi \rangle \, |, \tag{2.71}$$

is non-zero [7], where A and B are observables acting on a Hilbert space $\mathcal{H}$ with dimension d and $\Delta O = \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$. A sufficient condition for the RHS to be non-zero is that A and B do not commute and, which is, in fact, one of the definitions of measurement incompatibility. If two observables $[A, B] \neq 0$, then A and B are said incompatible. In particular, if the basis $\{|a_i\rangle\}$ formed by the eigenvectors of A and the basis $\{|b_i\rangle\}$ formed by the eigenvetors of B follow the relation

$$| \langle a_i | b_j \rangle |^2 = \frac{1}{d}, \tag{2.72}$$

the two basis are said *mutually unbiased*, because the probability of transition between any eigenstate of A to B is equal and vice-versa. This is a good motivation to define that if their eigenbasis are mutually unbiased, A and B are said *maximally incompatible.* An important property that these observables satisfy is that sequential measurements of A and B destroy all information contained in any state ρ, thus rendering a maximally mixed state $\frac{\mathbb{1}}{d}$,

$$\phi_{BA}(\rho) = \sum_{i,j=1}^{d} |b_j\rangle\langle b_j| \, |a_i\rangle\langle a_i| \, \rho \, |a_i\rangle\langle a_i| \, |b_j\rangle\langle b_j| \tag{2.73}$$

$$= \frac{1}{d} \sum_{j=1}^{d} |b_j\rangle\langle b_j| \operatorname{Tr}\{\rho\} = \frac{\mathbb{1}}{d}, \tag{2.74}$$

where it was used the fact that the trace of a quantum state is 1, the completeness relation for B, and the unbiasedness of the basis.

Operationally, the definition of noncommutativity makes sense with the rationale of the uncertainty principle. Performing a measurement of B yields an eigenstate of B and, if A and B do not commute, the new state is not necessarily an eigenstate of A, which results in non-zero probabilities for the possible results of a measurement of A.

The notion of noncommutativity is not adequate to represent incompatibility with the rise of POVMs, leading to more sophisticated definitions of incompatibility that involve POVMs and even ones that reinvent the notion of incompatibility for PVMs. Some of them are: *joint measurability* [8], *nondisturbance* [9], *coexistence* [34], etc.

Joint measurability aims to capture the idea that if two POVMs, A with elements $\{A_x\}$ and B with elements $\{B_y\}$, can be performed by a *mother POVM* G with elements $\{G_{xy}\}$ such that the marginalization of the elements of G over the index of one retrieves the other POVM

$$A_x = \sum_y G_{xy}, \quad B_y = \sum_x G_{xy}. \tag{2.75}$$

It can be shown that PVMs that commute are always jointly measurable and if they don't commute they are not jointly measurable. Joint measurability has been shown in [35] to be a quantum resource for a specific class of quantum computing tasks.

Nondisturbance establishes the criterion that if there exists an instrument $\mathfrak{I}_A$ that implements POVM A and does not disturb the probability distribution of a POVM B over any state ρ

$$\mathrm{Tr}\left[B_y \mathfrak{I}_A(\rho)\right] = \mathrm{Tr}\left[B_y \rho\right] \forall \rho, \tag{2.76}$$

then it is said that A *can be measured without disturbing* B. Note that it is an asymmetric concept, meaning that if Eq. (2.76) holds it is not granted that B can be measured without disturbing A. Of course, if A and B commute and are PVMs, either one can be measured without disturbing the other and the reverse is also valid. In addition, if A and B are mutually nondisturbant and are PVMs, they commute.

The definitions above are all criteria based solely on the measurements and they do not concern the state. However, classical theories do not exhibit incompatibility and the way to approach classicality in quantum theory is through the quantum state, not the measurements performed on it. This is a fundamental aspect inducing the search for further notions of incompatibility, some that are able to apprehend the classical limit. Indeed, some authors have recently conceived a paradigm of incompatibility which encompasses the role of the quntum state [10], giving rise to the concept of *context incompatibility.*

The MSA (Martins-Savi-Angelo) context incompatibility is based on an information leakage protocol. Alice prepares a state by measuring an observable X in a state ρ and promises to Bob an amount of information $I(\phi_X(\rho))$, where $\phi_X(\cdot)$ is the non selective measurement map in Eq. (2.29). Eve, an eavesdropper, intercepts this state and attempts to extract information about it measuring an observable Y. Then, Bob actually receives the amount $I(\phi_{YX}(\rho))$ of information. If he detects any difference between the received and the promised amount of information, the context $\mathbb{C} = \{X, Y, \rho\}$ is incompatible. The quantification of how incompatible $\mathbb{C}$ is given by the amount of leaked information:

$$I_{MSA} = I(\phi_X(\rho)) - I(\phi_{YX}(\rho)) \tag{2.77}$$

$$= S(\phi_{YX}(\rho)) - S(\phi_X(\rho)), \tag{2.78}$$

which is zero, i.e., diagnoses compatibility of $\mathbb{C}$, if and only if $\phi_{YX}(\rho) = \phi_X(\rho)$, that is, when Eve does not extract any information (and therefore doesn't learn anything about the communication).

A generalization on this concept was made in [36] for POVMs but the structure of the protocol remains the same.

2.4.1 Theory Independent Context Incompatibility

Recently, the interest in exploring general probabilistic theories (GPTs) that generalize classical and quantum theory and encompass other theories has grown a lot, both to explore the possibilities of a more fundamental theory and to acquire a deeper understanding of quantum theory, rebuilding it from different sets of axioms. An extensive review of GPTs is found in [37].

A notion of incompatibility based solely on probabilities called *theory independent context incompatibility* (TICI) was recently developed in [11]. To embrace generality, TICI uses the concept of nonselective measurements, which are measurements done without revealing the result. Suppose a physical quantity $A = \{a_i\}$ is measured in a preparation ϵ yielding the result a_i, the probability of a measurement of $B = \{b_i\}$ yielding b_j is $p_\epsilon(b_j|a_i)$. Forgetting the measurement result is akin to summing over all a_i weighted by the probability $p_\epsilon(a_i)$ of a measurement of A yielding a_i in the preparation ϵ. So, the nonselective measurement is defined as the preparation $M_A(\epsilon)$ associated with the probability distribution

$$p_{M_A(\epsilon)}(b_j) = \sum_i p_\epsilon(b_j|a_i)p(a_i). \tag{2.79}$$

If a nonselective measurement of B in a preparation ϵ does not disturb the probability distribution of A and the inverse holds, that is,

$$p_\epsilon(a_i) = p_{M_B(\epsilon)}(a_i), \tag{2.80a}$$

$$p_\epsilon(b_i) = p_{M_A(\epsilon)}(b_i), \tag{2.80b}$$

then the context $\mathbb{C} = \{A, B, \epsilon\}$ is compatible.

Much like how much quantum discord is quantified in a physical state by the quantum relative entropy, a measure of how incompatible a certain context $\mathbb{C} = \{\rho, A, B\}$ is given by how much the measurement of one physical quantity disturbes the probability distribution of the other. This measure can be built by use of the Kullback-Leibler divergence (2.39) as follows

$$\mathcal{I}_{\mathbb{C}} = \frac{D(p_\epsilon(a_i)||p_{M_B(\epsilon)}(a_i)) + D(p_\epsilon(b_i)||p_{M_A(\epsilon)}(b_i))}{2}, \tag{2.81}$$

where the Kullback-Leibler divergence for both of the expressions in the criteria were taken into account where the violation of just one suffices to imply the incompatibility of the context $\mathbb{C}$. This quantifier is also consistent with the criteria in Eq. (2.4.1) recalling that the divergence $D(p||q)$ is zero if and only if $p = q$.

In quantum theory, the preparation is the density operator $\rho \in \mathcal{H}$, the physical quantities are the observables, the nonselective measurement is given by Eq. (2.29) and the compatibility criteria translates to

$$p_\rho(a_i) = p_{\phi_B(\rho)}(a_i), \tag{2.82}$$

$$p_\rho(b_i) = p_{\phi_A(\rho)}(b_i). \tag{2.83}$$

Multiplying both equations by their respective projectors, A_i and B_i, summing over all i and recognizing the definition of the nonselective measurement map in Eq. (2.29), we get the criteria expressed in terms of the states

$$\phi_A(\rho) = \phi_{AB}(\rho), \tag{2.84}$$

$$\phi_B(\rho) = \phi_{BA}(\rho), \tag{2.85}$$

where $\phi_{AB}(\cdot) = \phi_A(\phi_B(\cdot))$ is the composition of the nonselective measurements. It is straightforward to verify that if $[A, B] = 0$, then the criteria in Eq.(2.84) are always satisfied, granting TICI a status of generalization of the concept of measurement incompatibility.

It is worth noting that for quantum theory the relative entropy would be the quantifier of the difference between the density operators before and after the measurement and, in fact, calculating the quantum analogue of Eq. (2.84),

$$\mathcal{I}_{\mathbb{C}} = \frac{S(\phi_A(\rho)||\phi_{AB}(\rho)) + S(\phi_B(\rho)||\phi_{BA}(\rho))}{2}, \tag{2.86}$$

results in complete equivalence to the quantifier in Eq. (2.81) if A and B are observables that act on $\mathcal{H}$. However, considering ρ to be bipartite, and the measurements acting on each partition, they are not equivalent.

2.5 DISCRETIZATION OF CONTINUOUS VARIABLES

The Kullback-Leibler divergence has some issues for continuous variables, such as the possibility of going to infinity or being negative, therefore the method of discretization of infinite dimensional Hilbert spaces introduced in [38] can be utilized to surpass these problems. The method discretizes continuous infinite dimensional Hilbert spaces and turns its dimension finite so that computing entropies and the Kullback-Leibler divergence becomes a feasible task.

The discretization method is the substitution of a continuous basis $\{|q\rangle\}$ with $q \in \mathbb{R}$ ranging from $-\infty$ to $+\infty$ of a basis $B_q = \{|q_j\rangle\}$ with $q_j \in \mathbb{R}$ such that $j = -\Delta, ..., 0, ..., \Delta$, where Δ is half the number of partitions, and $\delta_q = q_i - q_{i-1}$ for all i. By definition, the basis vectors are unnormalized

$$\langle q_i|q_j\rangle = \frac{\delta_{ij}}{\delta_q}, \tag{2.87}$$

where δ_{ij} is the Kronecker delta. The projector Π_j associated to the basis element $|q_j\rangle$ is defined as

$$\Pi_j = \delta_q |q_j\rangle\langle q_j|. \tag{2.88}$$

With that machinery a general state $|\psi\rangle$ can be expanded in the basis B_q as

$$|\psi\rangle = \sum_{j=-\Delta}^{\Delta} \Pi_j |\psi\rangle = \sum_{j=-\Delta}^{\Delta} \delta_q \langle q_j|\psi\rangle |q_j\rangle = \sum_{j=-\Delta}^{\Delta} \delta_q \psi(q_j) |q_j\rangle, \tag{2.89}$$

where $\psi(q_j)$ is the discretized wave function with $q_j = j\delta_q$. Therefore, for the position representation, the probability of finding a particle in a position x_j is $|\psi(x_j)|^2\delta_q$.

The amount of incompatibility in a context $\mathbb{C} = \{\rho, X, P\}$ reads

$$\mathcal{I}_{\mathbb{C}} = \frac{D(p_\rho(x_i))||p_{\phi_{P_1}(\rho)}(x_i)) + D(p_\rho(p_i)||p_{\phi_{X_1}(\rho)}(p_i))}{2}, \tag{2.90}$$

where ρ may be a multipartite state.

3 REFERENCE FRAMES AND COORDINATE SYSTEMS

A coordinate system is a set of parameters that uniquely labels an element of an underlying set, such as the events composing space-time (of Galileo, Minkowski, or other) points in a phase space or even vectors in a Hilbert space. The parameters used to describe the physical system of interest are conveniently chosen to simplify its mathematical description, e.g., spherical coordinates are the most suitable coordinates to describe a system that has spherical symmetry. Here we aim to describe a physical system by means of coordinate systems assigned by different reference frames and, most importanly, we want to get an intuition on how quantum systems describe the physical world.

3.1 NON-QUANTUM

3.1.1 Passive vs active picture for coordinates

There are two physically equivalent ways to transform coordinate systems, the active transformations, which acts on the object itself, therefore maintaining the coordinate system intact whilst the object changes, and there are passive transformations, which preserve the object while the coordinate system changes. Consider a vector $\boldsymbol{r}$ in $\mathbb{R}^3$ written in the $E = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ basis in which the coordinates are the coefficients $r_i^{(e)}$ of the linear combination

$$\boldsymbol{r} = \sum_i r_i^{(e)} \hat{\boldsymbol{e}}_i. \tag{3.1}$$

Performing a passive transformation into a basis $F = \{\hat{\boldsymbol{f}}_1, \hat{\boldsymbol{f}}_2, \hat{\boldsymbol{f}}_3\}$, in which $\hat{\boldsymbol{f}}_i = \sum_j M_{ji} \hat{\boldsymbol{e}}_j$, while the basis vectors transform according to M^T, the coordinates transform according to M^{-1}, i.e., $r^{(f)} = M^{-1} r^{(e)}$, in matrix notation, where the absence of bold vectors are to denote column matrices. Moving on to the active transformation we have

$$M^{-1} r^{(e)} = r'^{(e)}, \tag{3.2}$$

the distinction between the two being that, in the active transformation, we are interpreting the RHS as coordinates of a new vector $\boldsymbol{r'}$. Therefore, we need to reinterpret the RHS as coordinates of the same vector in a new basis in order to make the two approaches consistent. The importance of the last step will be clearer when dealing with Hilbert spaces in which the task is to find an operator that performs this coordinate transformation and, since an operator transforms a vector, we are dealing with an active transformation and to make it equivalent to the passive one, the action of the operator in a vector must be followed by a reinterpretation of the meaning of the labels assigned to the new vector as the new coordinate systems' parameters.

3.1.2 Coordinate transformations in Hilbert space

By choosing the observables responsible for building the Hilbert space that contains the possible states of the system of interest, that is the representation space, a coordinate system and therefore a reference frame is assumed. Take the Hilbert space $\mathcal{H}$ generated by the position observable X, for example, the parameter that labels each ket of the complete basis $\{|\boldsymbol{x}\rangle\}$ is the position relative to a reference frame previously assumed. A transformation to another reference frame in a Hilbert space is accomplished by applying a unitary operator suited to the case of interest.

3.1.2.1 Translation

Suppose we want to change from the reference frame S, as described above, to a frame S' that has its origin dislocated by a vector $\boldsymbol{d}$. The unitary operator that suits the case is the translation operator $T(\boldsymbol{d}) = \exp\left(-\frac{i}{\hbar}\boldsymbol{d}\cdot\boldsymbol{P}\right)$, where $\boldsymbol{P}$ is the momentum operator. Performing the transformation on the position operator we have

$$\begin{aligned} \boldsymbol{X}' &= T(\boldsymbol{d})XT^{\dagger}(\boldsymbol{d}) \\ &= \boldsymbol{X} - \boldsymbol{d}\mathbb{1}, \end{aligned} \tag{3.3}$$

while the transformation has no effect on the momentum operator, that is, $\boldsymbol{P}' = T(\boldsymbol{d})PT^{\dagger}(\boldsymbol{d}) = \boldsymbol{P}$, because it commutes with the translation operator. The transformation of a general state $|\psi\rangle = \int d\boldsymbol{x}\psi(\boldsymbol{x})\,|\boldsymbol{x}\rangle$, where $d\boldsymbol{x} = dxdydz$, results in

$$\begin{aligned} |\psi'\rangle &= T(\boldsymbol{d})\,|\psi\rangle \\ &= \int d\boldsymbol{x}\psi(\boldsymbol{x})\,|\boldsymbol{x}+\boldsymbol{d}\rangle \\ &= \int d\boldsymbol{x}\psi(\boldsymbol{x}-\boldsymbol{d})\,|\boldsymbol{x}\rangle\,. \end{aligned} \tag{3.4}$$

As expected, the wave function perceived by S' maintains its functional form but is dislocated by the vector $\boldsymbol{d}$. Note that, considering Eq. (3.4), the operator $\boldsymbol{X}$ now has the interpretation of the position relative to the origin of S', even not being transformed. In a similar manner, considering $\boldsymbol{X}'$ and the state $|\psi\rangle$, it also has the interpretation of position relative to S'. This equivalence can also be highlighted with the expressions of the mean value for the relative position

$$\langle\psi|\,X'\,|\psi\rangle = \langle X\rangle - d \tag{3.5a}$$

$$\langle\psi'|\,X\,|\psi'\rangle = \langle X\rangle - d. \tag{3.5b}$$

3.1.2.2 Boosts

In the case of S' moving with a velocity $\boldsymbol{v}$ relative to S, we can employ the unitary operator $B(\boldsymbol{v}) = \exp\left[\frac{i}{\hbar}v(t\boldsymbol{P} - m\boldsymbol{X})\right]$ to implement the usual Galilean transformation

$$\boldsymbol{X'} = \boldsymbol{X} - \boldsymbol{v}t, \tag{3.6a}$$

$$\boldsymbol{P'} = \boldsymbol{P} - m\boldsymbol{v}, \tag{3.6b}$$

and the state perceived by S' is

$$\begin{aligned} |\psi'\rangle &= B(\boldsymbol{v})\,|\psi\rangle \\ &= \exp\!\left(-\frac{mv^2t}{2\hbar}\right)\int d\boldsymbol{x}\psi(\boldsymbol{x})\exp\!\left(-\frac{i}{\hbar}m\boldsymbol{v}\cdot\boldsymbol{x}\right)|\boldsymbol{x}+\boldsymbol{v}t\rangle \\ &= \exp\!\left(-\frac{mv^2t}{2\hbar}\right)\int d\boldsymbol{x}\psi(\boldsymbol{x}-\boldsymbol{v}t)\exp\!\left[-\frac{i}{\hbar}m\boldsymbol{v}\cdot(\boldsymbol{x}-\boldsymbol{v}t)\right]|\boldsymbol{x}\rangle\,, \end{aligned} \tag{3.7}$$

where a phase now multiplies the original wave function, which is dislocated by $\boldsymbol{v}t$ as expected. Explicitly, the new wave function is given by

$$\psi'(x',t) = e^{-\frac{i}{\hbar}(mvx' + \frac{mv^2t}{2})}\psi(x',t). \tag{3.8}$$

This result was also obtained by Bargmann [39] by different means. The author analyzes the free particle Schrödinger equation

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\Psi(x,t), \tag{3.9}$$

through the Galilean transformation of the position coordinates $x' = x - vt$ and $t = t'$, which yields a modified equation with the addition of a potential,

$$i\hbar\partial_t\Psi(x',t) = \frac{\hbar}{2m}\partial_{x'}^2\Psi(x',t) - i\hbar v\partial_{x'}\Psi(x',t). \tag{3.10}$$

Finally, imposing the form invariance of Eq. (3.10) with respect to Eq.(3.9) we have a new wave function $\Psi'(x',t)$ that satisfies

$$i\hbar\partial_t\Psi'(x',t) = -\frac{\hbar^2}{2m}\partial_{x'}^2\Psi'(x',t), \tag{3.11a}$$

$$\Psi'(x',t) = e^{-\frac{i}{\hbar}(mvx' + \frac{mv^2t}{2\hbar})}\Psi(x',t). \tag{3.11b}$$

Comparing Eqs. (3.11b) and (3.8) we find that both methods yield the same wave function. In other words, applying the Galilean transformation with the unitary operator or performing the coordinate transformation is equivalent and the form invariance of the Schrödinger equation is granted both ways.

Notice that none of these transformations can accomplish the task of jumping to a reference frame in which the quantum system is always at the origin and at rest, it does not matter the choice of parameter of translation or boost. To do the job we need a more refined theory of reference frames.

3.2 QUANTUM REFERENCE FRAMES

To our knowledge, the first mention of the idea of quantum reference frames was regarding charge superselection rules [40] and the observability of the sign change of spinors under 2π rotations [41], both made by Aharonov and Susskind. The transformations between the quantum reference frames are first introduced in Aharonov and Kaufherr's work in [12] in order to solve a supposed "paradox" regarding quantum systems playing the role of finite-mass reference frames. In this section, we describe this approach followed by the contributions made by Angelo *et. al.* [14], Giacomini *et. al.* [15], Vanrietvelde *et. al.* [42, 43] and Savi *et. al.* [18].

3.2.1 Measuring device paradox

In classical physics, the extraction of information about a system can be done via an arbitrarily small exchange of energy between the system and the lab while in quantum mechanics, a quantum of action must be exchanged in the process of extraction according to the uncertainty principle in Eq. (2.71). Aharonov and Kaufherr emphasize that quantum mechanics relies on the existence of a classical reference frame, which has perfectly localized position and velocity. Although it seems to violate Heisenberg's uncertainty principle, it is possible to make sense of such classical feature within the quantum framework by requiring that the reference frame has infinite mass m in

$$\Delta\left[\frac{d}{dt}X\right]\Delta X \geq \frac{\hbar}{2m}, \tag{3.12}$$

which is a perfectly plausible hypothesis considering a heavy macroscopic laboratory as the reference frame relative to which the quantum system is described. Consider the measuring device to have finite mass, a measurement of the position of a particle relative to it yielding, e.g. x_0, would disturb its own position relative to the external reference frame after some time t as

$$\Delta X(t) = \Delta X(0) + \frac{\Delta P}{m}t, \tag{3.13}$$

which would disturb also the position of the particle relative to it such that this measurement would not be repeatable, contradicting the necessity of repeatability of a measurement implied by the postulate in Eq. (2.3). Therefore, the authors conclude that a measuring device with finite mass is inconsistent with quantum mechanics.

This inconsistency implies that quantum mechanics alone cannot describe the entire universe, it requires a classical theory to complete the description. Besides, all measuring devices have finite mass, and a measurement that involves amounts of energy capable of disturbing its position and momentum in principe should be possible to describe with a single theory.

This paradox can be presented in a more concrete and fundamental manner, as an inconsistency when describing particle 1, which is a free particle, relative to a finite-mass reference frame consisting of particle 0, which is playing the role of the measuring device. The starting reference frame will be referred to as the external reference frame and also has finite mass. The authors postulate, supported by special and general relativity, that no observer can perform an experiment that grants information about its own state of motion. Therefore, particle 0 cannot distinguish between being in an eigenstate of position or momentum. By consequence, the external reference frame should be able to measure the position of the center of mass of particle 0 and particle 1 while particle 0 measures particle 1's velocity simultaneously, that is,

$$\Delta V_1' = 0 \quad and \quad \Delta X_{CM} = 0. \tag{3.14}$$

The fact that these quantities may be measured simultaneously implies that

$$\left[V_1', X_{CM}\right] = 0, \tag{3.15}$$

where $X_{CM} = \frac{m_1 X_1 + m_0 X_0}{m_1 + m_0}$, $V_1' = \frac{P_1'}{m_1}$ and m_1 and m_0 are the masses of partice 1 and particle 0, respectively. The positions of the external reference frame and of particle 1, X_{ext}' and X_1', respectively, denoted by primed operators, are given relative to particle 0 and taken to be

$$X_{ext}' = -X_0, \tag{3.16a}$$

$$X_1' = X_1 - X_0, \tag{3.16b}$$

where X_1 and X_0 are the positions of particle 1 and 2, respectively, relative to the external reference frame. When testing the expected reasonable commutation relation given in Eq. (3.15) the authors found a result that differs from the expected one

$$\begin{aligned}\left[V_1', X_{CM}\right] &= \left[\frac{P_1'}{m_1}, \frac{m_1 X_1 + m_0 X_0}{m_1 + m_0}\right] \\ &= \left[\frac{P_1'}{m_1}, \frac{m_1 X_1'}{m_1 + m_0}\right] + \left[P_1', -X_{ext}'\right] \\ &= \frac{-i\hbar}{m_1 + m_0}\mathbb{1} \neq 0. \end{aligned} \tag{3.17}$$

In the limit of infinite-mass reference frame, $m_0 \to \infty$, Eq. (3.17) yields zero, in accordance with Eq. (3.15). The lack of agreement of the mathematical description and the good reasoning establishes the inconsistency of describing a system through a finite-mass reference frame.

A solution to the paradox is the introduction of a vector potential A that aims to restore the desired commutation relations and hopefully our intuition of the physical situation, that is,

$$mV_1' = P_1' + m_1 A'. \tag{3.18}$$

Form invariance of the vector potential and Eq. (3.18) under reference frame transformation yields

$$mV_1 = P_1 + m_1 A, \tag{3.19}$$

and, using the above equations and following extensive calculations available in [12] we find that the velocity that follows the desirable commutation relation is

$$V_1' = \frac{P_1'}{m_1} + \frac{P_0' + P_1'}{m_0}, \tag{3.20}$$

and, according to the external reference frame

$$V_1 = \frac{P_1}{m_1} + \frac{P_0 + P_1}{m_0}, \tag{3.21}$$

where we get the second term to vanish in the infinite mass reference frame limit ($m_0 \to \infty$).

The authors point that all the quantities involved in the solution are relational and the vector potential A accounts for the noninertiality of the finite-mass reference frame. Eq. (3.20) shows that a change in the velocity of particle 1 can be attributed to a change in the momentum of either particle 0 or particle 1 relative to the external reference frame. The first case is a consequence of the non-inertiality of particle 0.

3.2.2 One dimensional solution

The authors also present a fully covariant solution; they provide a transformation to a quantum reference frame. It is given by a coordinate transformation to the relative coordinates of one of the particles. Consider a system consisting of n free particles with the Hamiltonian

$$H = \sum_{i=0}^{n} \frac{P_i^2}{2m}, \tag{3.22}$$

assigned by an external reference frame S and we wish to obtain the description of the system according to particle 0. The change in coordinates utilized was from the positions and momenta x_i and p_i, relative to S, to the positions and momenta Q_i and π_i relative to particle 0. In this case, we have

$$Q_0 = X_0, \quad \pi_0 = P_T = \sum_{i=0}^{n} P_i, \tag{3.23}$$

$$Q_i = X_i - X_0, \quad \pi_i = P_i, \quad i = 1, ..., n, \tag{3.24}$$

where $\left[Q_i, \pi_j\right] = \delta_{ij} i\hbar \mathbb{1}$ for $i, j = 1, ..., n$. The unitary transformation that performs this change in position and momenta coordinate is

$$U = \exp\left(\frac{i}{\hbar} \sum_{i=1}^{n} P_i X_0\right), \tag{3.25}$$

where $Q_n = U^\dagger X_n U$ and $\pi_n = U^\dagger P_n U$ for $i = 0, ..., n$. In order to find the Hamiltonian H' that particle 0 uses to calculate the dynamics, we should consider the form invariance of the Schrödinger equation of a general operator $A(X_n, P_n)$ in the Heisenberg picture in Eq. (2.11).

The calculations are done in appendix A. We have that $H'(Q_n, \pi_n) = UH(X_n, P_n)U^\dagger$, so, in Eq. (3.22) we get

$$H'(Q_n, \pi_n) = \frac{\left(\pi_0 - \sum_{i=1}^{n} \pi_i\right)^2}{2m_0} + \sum_{i=1}^{n} \frac{\pi_i^2}{2m_i}, \tag{3.26}$$

where we can consider the system to be in an eigenstate of total momentum with $\pi_0 = 0$ (in its own reference frame its momentum is zero) and

$$H'(Q_n, \pi_n) = \sum_{i=1}^{n} \frac{(\pi_i + m_i\Pi/m_0)^2}{2m_i} - \frac{M\Pi^2}{2m_0^2}, \tag{3.27}$$

where $\Pi = \sum_{i=1}^{n} \pi_i$ and $M = \sum_{i=0}^{n} m_i$. The elimination of particle 0 from the Hamiltonian comes with the expense of introducing the vector potential $A = \frac{\Pi}{m_0}$ to the momentum in the first term inside the parenthesis. If we calculate the Heisenberg velocity of the i-th particle v_i, we get

$$v_i = \frac{[Q_i, H']}{i\hbar} = \frac{\pi_i}{m_i} + \frac{\Pi}{m_0}. \tag{3.28}$$

Just as in the previous solution of the paradox in Eq. (3.20), the free particle does not move with velocity equal its momentum divided by its mass, but now an initial operator U that changes between quantum reference frames was introduced.

3.2.3 Center of mass transformation

In [13], the map to center of mass and relative coordinates of a closed system composed by particles 0 and 1 was utilized, just as in solving the hydrogen atom in quantum mechanics textbooks:

$$|a\rangle_0 |b\rangle_1 \mapsto \left|\frac{m_0 a + m_1 b}{m_0 + m_1}\right\rangle_{CM} |b - a\rangle_r. \tag{3.29}$$

In [14], the authors showed that this map can be written via the application of the unitary operator

$$T_{CM} = \exp\left(-\frac{i}{\hbar}\frac{m_1}{M} X_1 P_0\right) \exp\left\{\frac{i}{\hbar} X_0 P_1\right\}, \tag{3.30}$$

where $M = m_0 + m_1$. The analysis of these two contributions made by Angelo *et. al.* [14], unlike the one made by Aharonov and Kaufherr [12], also considers how quantum coordinate transformations act on state vectors and density operators, which provides a better understanding of the theory of quantum reference frames. This transformation acting on a state

$$|\psi\rangle = \int \psi(x_0, x_1) |x_0\rangle |x_1\rangle \, dx_0 dx_1, \tag{3.31}$$

yields the state $|\psi'\rangle = T_{CM} |\psi\rangle$

$$|\psi'\rangle = \int \tilde{\psi}(x_{CM}, x_{r_1}) |x_{CM}\rangle \left|x_{r_1}\right\rangle dx_{CM} dx_{r_1}, \tag{3.32}$$

where $x_{CM} = \frac{m_0 x_0 + m_1 x_1}{M}$, $x_{r_1} = x_1 - x_0$, and $\tilde{\psi}(x_{CM}, x_{r_1}) = \psi(x_{CM} - \frac{m_1}{M} x_{r_1}, x_{CM} + \frac{m_0}{M} x_{r_1})$. To obtain state of particle 1 relative to particle 0, they invoke the fact that the center of mass is a quantity defined relative to the external reference frame and particle 1 cannot discover its location much less alter it. Therefore the center of mass degree of freedom may be discarded via partial trace $\rho_{r_1} = \text{Tr}_{CM}(|\psi'\rangle\langle\psi'|)$

$$\rho_{r_1} = \int dx_{CM} \int dx_{r_1} d\bar{x}_{r_1} \tilde{\psi}(x_{CM}, x_{r_1}) \tilde{\psi}^*(x_{CM}, \bar{x}_{r_1}) \left| x_{r_1} \right\rangle\!\left\langle \bar{x}_{r_1} \right|, \tag{3.33}$$

resulting in the state that particle 0 assigns to particle 1.

3.2.3.1 Product of Gaussian states

From now on, states $|x_0\rangle$ refer to Gaussian wave packets centered at x_0 with variance Δ, that is

$$|x_0\rangle = \frac{1}{\sqrt{\Delta\sqrt{2\pi}}} \int dx \exp\left[\frac{(x - x_0)^2}{4\Delta^2}\right] |x\rangle \, . \tag{3.34}$$

Consider a system of two particles in a product of Gaussian states relative to the external frame:

$$\psi(x_0)\phi(x_1) \propto \exp\left[-\frac{(x_0 - a)^2}{2\Delta_0^2}\right] \exp\left[-\frac{(x_1 - b)^2}{2\Delta_1^2}\right], \tag{3.35}$$

where particle 0 is localized around a with uncertainty Δ_0 and particle 1 is localized around b with uncertainty Δ_1. Applying T_{CM} to Eq.(3.35) the wave-function of the system is

$$\Psi(x_{CM}, x_{r_1}) \propto \exp\left[-\frac{(x_{CM} - \alpha)^2}{4\Delta_{CM}^2}\right] \exp\left[-\frac{(x_{r_1} - \beta)^2}{4\Delta_{r_1}^2}\right] \exp\left[\gamma(x_{CM} - \alpha)(x_{r_1} - \beta)\right], \tag{3.36}$$

where the parameters are given by

$$\Delta_{CM}^2 = \Delta_0^2\Delta_1^2/(\Delta_0^2 + \Delta_1^2), \tag{3.37a}$$
$$\Delta_{r_1}^2 = M^2\Delta_0^2\Delta_1^2/(m_0^2\Delta_0^2 + m_1^2\Delta_1^2), \tag{3.37b}$$
$$\alpha = (m_0 a + m_1 b)/M, \tag{3.37c}$$
$$\beta = b - a, \tag{3.37d}$$
$$\gamma = (m_1\Delta_1^2 - m_0\Delta_0^2)/(M\Delta_0^2\Delta_1^2). \tag{3.37e}$$

The transformation led a product state to other Gaussian states for the center of mass and for the relative coordinates multiplied by an exponential. Note that if the system is prepared in such a way that $\gamma = 0$, that is $m_0\Delta_0^2 = m_1\Delta_1^2$, the transformed state is localized around α in the center of mass with variance Δ_{CM} and around β in the relative coordinate with variance Δ_{r_1}. In this case there is no entanglement between the new coordinates. When transforming product states the approximation $|a\rangle_0 |b\rangle_1 \approx |\alpha\rangle_{CM} |\beta\rangle_{r_1}$ is utilized to show that in principle a choice can be made such that the equality holds.

Note that, in general, a separable state can be transformed into an entagled state, indicating that quantum resources may not be invariant under quantum reference frame transformations. Although it might seem counter-intuitive, remember that entanglement is a form of correlation between subsystems. In this case, the positions of particle 0 and 1 relative to the external reference frame are uncorrelated but that does not prevent the position of the center of mass and relative position to be correlated. Therefore, a change in the amount of entanglement is expected when changing quantum reference frames because in the new frame entanglement is seen between other degrees of freedom, not the original ones.

3.2.4 Mach-Zehnder Interferometer

Unlike the double-slit experiment, the *Mach-Zehnder Interferometer* (MZI) [44], is a typical, and mathematically simple, example of setup wherein path interference can be observed. In this work, we use the qualitative result of the MZI, and hence, we omit the details and follow an approach similar to the one made in [45]. In part *I* of figure 3, a single particle

fig/MZI.png

Figure 3 – Mach-Zehnder Interferometer: In part *I* the particle enters the interferometer and its state turns into an equally weighted superosition of branches. In part *II* a relative phase of π between the branches through reflections, so that in part *III* constructive interference occurs in D_2 and destructive in D_2 yielding 100% of clicks in D_1.

is emitted and enters the interferometer encountering a beam-splitter, transforming the state into an equal weighted superposition between the upper path and the lower path. Whenever the wave-function is reflected, it gains a phase of $\frac{\pi}{2}$. Part *II* is where the upper branch of the wave-function gets a phase of $\frac{\pi}{2}$ relative to the other branch. In part *III*, the branches have a relative phase of π, which fulfill the condition to occur destructive interference between the branches of the wave function towards detector D_1 and constructive interference towards D_2 such that it clicks in 100% of times that the experiment is made.

3.2.5 Role of the center of mass

In classical mechanics, the momentum of the center of mass is a conserved quantity in closed systems since it is conserved throughout the dynamics. In quantum reference frames theory, it plays even a bigger role which can be intuitively understood through the following example. Consider two different physical situations, (a) and (b), depicted in figure 4(a) and 4(b), respectively. Experiment (a) is a regular MZI but cut in half. Experiment (b) involves no

fig/Superposition of MZI.jpg

Figure 4 – Distinct experiments involving Mach-Zehnder interferometers. In (a) the particle approaches in superposition as usual. In (b) the interferometer is in superposition.

superposition for the particle while the interferometer is in superposition. Both experiments are being described relative to an external classical reference frame. If we follow the possible paths and evaluate the probabilities, we conclude that in experiment (a) only one detector clicks while in experiment (b) both detectors are equally likely to click. One might think that due to the symmetry of the situation, i.e., the interferometer observing the particle approaching it in superposition in counterpart to the particle observing the interferometer in superposition, the two experiments would be physically equivalent. Let us analyze the physical states of these systems in order to check that. In (a), consider the distance from the mirrors to the center of the interferometer be L, then the state of the particle-interferometer system is

$$|\psi_a\rangle = |0\rangle_i \left(\frac{|L\rangle_p + |-L\rangle_p}{\sqrt{2}} \right), \tag{3.38}$$

where the subindex i labels the interferometer's states and p labels the particle's states. In experiment (b) the state of the system is

$$|\psi_b\rangle = \left(\frac{|L\rangle_i + |-L\rangle_i}{\sqrt{2}} \right) |0\rangle_p \,. \tag{3.39}$$

Since the probability of each detector clicking is a property of the physical system and the relationship between the particle and the interferometer, not of the choice of reference frame, if we switch to the reference frame of the particle in experiment (a), interference is expected and, switching to the reference frame of the interferometer in experiment (b) should yield no interference. It can be seen following the possible paths for the particle to take in figure 4. The transformed states are

$$|\psi_a\rangle \approx \frac{\left|\frac{-m_pL}{M}\right\rangle_{CM}|-L\rangle_{r_1} + \left|\frac{m_pL}{M}\right\rangle_{CM}|L\rangle_{r_1}}{\sqrt{2}}, \tag{3.40a}$$

$$|\psi_b\rangle \approx \frac{\left|\frac{-m_iL}{M}\right\rangle_{CM}|L\rangle_{r_1} + \left|\frac{m_iL}{M}\right\rangle_{CM}|-L\rangle_{r_1}}{\sqrt{2}}. \tag{3.40b}$$

Tracing the center of mass from Eqs. (3.40) would yield a mixture of $|-L\rangle$ and $|L\rangle$, which causes both detectors to click, thus being the correct description of (b) but incorrect for (a). Assuming that the interferometer is much heavier than the particle, i.e., $m_i \gg m_p$, then $\frac{m_pL}{M} \longrightarrow 0$ and $\frac{m_iL}{M} \longrightarrow L$, which results in

$$|\psi_a\rangle \approx |0\rangle_{CM}\left(\frac{|-L\rangle_{r_1} + |L\rangle_{r_1}}{\sqrt{2}}\right), \tag{3.41a}$$

$$|\psi_b\rangle \approx \frac{|-L\rangle_{CM}|L\rangle_{r_1} + |L\rangle_{CM}|-L\rangle_{r_1}}{\sqrt{2}}. \tag{3.41b}$$

Tracing the center of mass yields the relative states ρ_a and ρ_b

$$\rho_a \approx \frac{\left(|-L\rangle_{r_1} + |L\rangle_{r_1}\right)\left(\langle -L|_{r_1} + \langle L|_{r_1}\right)}{2}, \tag{3.42a}$$

$$\rho_b \approx \frac{|L\rangle\langle L|_{r_1} + |-L\rangle\langle -L|_{r_1}}{2}. \tag{3.42b}$$

Now the differences become crystal clear, the superposition of $|-L\rangle$ and $|L\rangle$ in Eq. (3.42a) indicates that interferece occurs, causing only one of the detectors to click and the mixture in Eq. (3.42b) indicates the absence of interference.

Thus, the center of mass plays the role of distinguishing such similar and apparently symmetric physical situations, for when it is entangled with the relative degrees of freedom, mixed statistics is attributed to the relative state, while when it factorizes, the relative state remains pure and therefore exhibits interference.

3.2.6 $N + 1$-particle system

One can also transition to the center of mass and relative coordinates,

$$x_{CM} = \frac{1}{M}\sum_i^N m_i x_i, \tag{3.43}$$

$$x_{r_i} = x_i - x_0, \tag{3.44}$$

where $M = \sum_{i=0}^{N} m_i$. The associated operator for the transformation is constructed by writing Eq. (3.43) as

$$x_{CM} = x_0 + \sum_{i=1}^{N}\frac{m_i}{M}x_{r_i}, \tag{3.45}$$

therefore first we should translate all x_i with $i > 0$ of x_0 and then multiply it by $\frac{m_i}{M}$ and sum it all to x_0. Writing in terms of translation operators

$$T_{CM} = \exp\left(-\frac{i}{\hbar}\sum_{i=1}^{N}\frac{m_i X_i}{M}\right)\exp\left(\frac{i}{\hbar}X_0\sum_{i=1}^{N}P_i\right). \tag{3.46}$$

Note that this change of coordinates is unitary and hence canonical. Consider $N = 2$, for example, acting on the positions and momenta operators as $T_{CM}^{\dagger} \cdot T_{CM}$ yields

$$\begin{aligned}
X_{CM} &= \frac{1}{M}\sum_{i}^{N} m_i X_i, \quad P_{CM} = \sum_{i=0}^{N} P_i,\\
X_{r_1} &= X_1 - X_0, \quad P_{r_{1,CM}} = m_1\left(\frac{P_1}{m_1} - \frac{P_{CM}}{M}\right),\\
X_{r_2} &= X_2 - X_0, \quad P_{r_{2,CM}} = m_2\left(\frac{P_2}{m_2} - \frac{P_{CM}}{M}\right),
\end{aligned} \tag{3.47}$$

where we interpret $P_{r_i,CM}$ as the momentum of particle i relative to the center of mass of the whole system. A canonical transformation T_P that leads to the momentum relative to the new reference frame, i.e., particle 0, which is given in [46] for $N = 2$ and leads the position and momentum operators to

$$\begin{aligned}
X_{CM} &= \frac{1}{M}\sum_{i}^{N} m_i X_i, \quad P_{CM} = \sum_{i=0}^{N} P_i,\\
X_{r_1,CM} &= c\left(X_1 - \frac{m_0 X_0 + m_2 X_2}{m_0 + m_2}\right), \quad P_{r_1} = \mu_{01}\left(\frac{P_1}{m_1} - \frac{P_0}{m_0}\right),\\
X_{r_2,CM} &= c(X_2 - \frac{m_0 X_0 + m_1 X_1}{m_0 + m_1}), \quad P_{r_2} = \mu_{02}\left(\frac{P_2}{m_2} - \frac{P_0}{m_0}\right),
\end{aligned} \tag{3.48}$$

where $\mu_{0i} = \frac{m_0 m_i}{m_0 + m_i}$ is the reduced mass between particle 0 and i and $c = \frac{m_0 m_1 m_2}{M \mu_{01}\mu_{02}}$. Whereas with T_{CM}, the momenta were relative to the center of mass, now the position operators are relative to the center of mass of the other two particles up to the constant c that depends on all masses. Here, the nonlocality that appeared in the Aharanov-Kaufherr formalism manifests itself in the form of the Hilbert space nonlocality of the translation operator, for instance, for particle 1:

$$\exp\left(-\frac{i}{\hbar}\delta P_{r_1}\right)|x_{CM}\rangle\left|x_{r_1}\right\rangle\left|x_{r_2}\right\rangle = |x_{CM}\rangle\left|x_{r_1} + \delta\right\rangle\left|x_{r_2} + \frac{\mu_{01}}{m_0}\delta\right\rangle. \tag{3.49}$$

Eq. (3.49) shows explicitly that displacing particle 1, relative to particle 0, causes a kickback in the quantum reference frame due to the interaction necessary for the displacement to happen. This kickback manifests itself in particle 2, even having not interacted with particle 0. Note the subtlelty here, the states utilized in Eq. (3.49) are in the coordinate system reached with T_{CM} while P_{r_2} is reached with T_P, i.e., the building blocks of the Hilbert space in question are the relative positions and the *momenta relative to the center of mass*, therefore this translation operator does not act merely shifting the position of particle 2 by δ. In fact, this is due to the fact that X_{r_j} and P_{r_k} are not canonically conjugated. Computing the commutation relations one gets $[X_{r_j}, X_{r_k}] = 0$, $[P_{r_j}, P_{r_k}] = 0$, $[X_{r_j}, P_{r_k}] = i\hbar\mathbb{1}$ for all j and k, but the crossed position and momentum gives

$$[X_{r_j}, P_{r_k}] = i\hbar\frac{m_k}{m_0 + m_k}\mathbb{1}. \tag{3.50}$$

Taking the limit of classical reference frame, $m_0 \to \infty$, the canonical commutation relations are regained. A nonintuitive consequence of Eq. (3.50) is that when describing particles 1 and 2 relative to another quantum system, it is not possible to assign a tensor product structure $\mathcal{H}_{r_1} \otimes \mathcal{H}_{r_2}$ to the spatial relative degrees of freedom. That is because a necessary condition to do so is that the operators that construct each Hilbert space must commute, which is not the case of the relative position of a particle and the momentum of the other particle. Therefore the Hilbert space built upon the relative degrees of freedom of the two spaces is joint. Two potential decompositions into tensor products are $\mathcal{H}^x_{r_1} \otimes \mathcal{H}^x_{r_2}$ and $\mathcal{H}^p_{r_1} \otimes \mathcal{H}^p_{r_2}$, where $\mathcal{H}^x_{r_i}$ and $\mathcal{H}^x_{r_3}$ are built upon the canonical conjugate operators given in Eq. (3.47) and $\mathcal{H}^p_{r_1}$ and $\mathcal{H}^p_{r_2}$ are built upon Eq. (3.48).

3.2.7 The relational approach

More recent approaches [15, 47] have the philosophy that different observers may assign different states to the same physical system such that no state has absolute meaning, in addition they are related by a reference frame transformation. Much like Aharonov and Kaufherr, Giacomini *et. al.* treat the reference frame as a finite-mass physical system [15] but, in addition, they pursuit total covariance, which we shall refer to here as relationality. That is, they want to make no reference whatsoever to an external classical reference frame by developing a theory of quantum reference frames that is fully embedded in quantum mechanics. Instead, all quantities must refer to a physical system and be defined relative to another physical system. To obtain it, they postulate the universality of quantum mechanics, which imposes that the reference frame should also be describable by other components of the system as a quantum system, i.e. it may be in superposition of states or even entangled with other parts of the whole.

Let's consider a system of two particles, labeled from 1 to 2. Given the state of the system $|\psi\rangle^{(R)}_{12}$ with respect to the laboratory R, the goal is to find a transformation S that promotes one of the particles, say particle 1, to a reference frame that describes R and particle 2 with a quantum state $|\psi\rangle^{(1)}_{2R}$, as depicted in figure 5. The subscript indices are the components of the system to be described and the superscript index denotes the component relative to which the rest of the system is being described.

This can be accomplished by modifying the Aharonov-Kaufherr transformation given by Eq. (3.25)

$$S_x = \pi_{R1} \exp\left(\frac{i}{\hbar} P_2 X_R\right), \tag{3.51}$$

where the parity-swap operator $\pi_{R1} : \mathcal{H}_1 \to \mathcal{H}_R$ acts according to $\pi^\dagger_{R1} X_1 \pi_{R1} = -X_R$, reversing the direction of the position vector and changing the associated Hilbert space, therefore

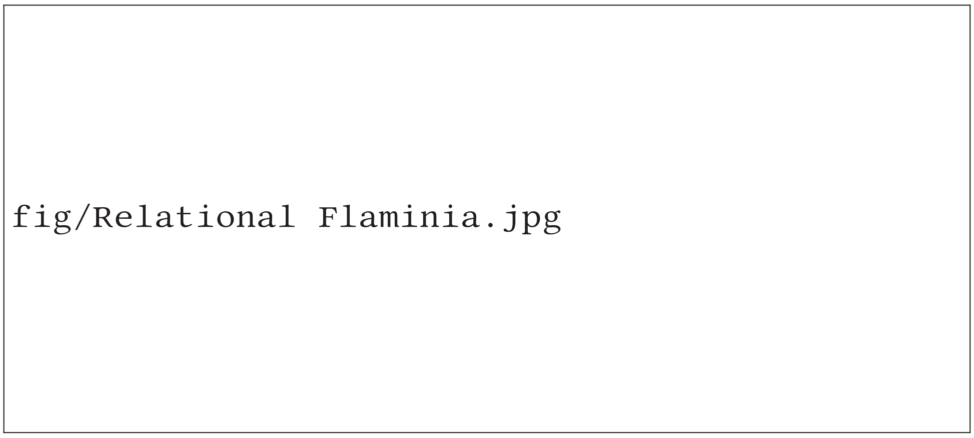

Figure 5 – The reference frame describing the positions of particles 1 and 2 and the change to the perspective of particle 1, which describes the initial reference frame R and particle 2.

obtaining the sought after description of the initial reference frame R relative to particle 1. The transformation S_x acts on vectors that belong to the Hilbert space of the system composed by particles 1 and 2, $\mathcal{H}_1 \otimes \mathcal{H}_2$, and returns vectors belonging to the Hilbert space of the system composed of particle 2 and R, i.e. a state that describes particle 2 and R relative to particle 1 in $\mathcal{H}_2 \otimes \mathcal{H}_R$. The transformation S_x acting on a general state $|\psi\rangle_{12}^{(R)}$ gives

$$\begin{aligned} |\psi\rangle_{2R}^{(1)} &= S_x |\psi\rangle_{12}^{(R)} \\ &= S_x \int \psi_{12}^{(R)}(x_1, x_2) |x_1\rangle_1 |x_2\rangle_2 \, dx_1 dx_2 \\ &= \int \psi_{2R}^{(1)}(x_{r_2}, x_R) \left|x_{r_2}\right\rangle_2 |x_R\rangle_R \, dx_{r_2} dx_R, \end{aligned} \tag{3.52}$$

where the substitution $x_R = -x_1$, $x_{r_2} = x_2 - x_1$ has been made and the wave functions that the reference frames assign to the systems are related by $\psi_{2R}^{(1)}(x_{r_2}, x_R) = \psi_{12}^{(R)}(-x_R, x_{r_2} - x_R)$. For a pure density operator, it is true that $\rho_{2R}^{(1)} = S_x |\psi\rangle_{2R}^{(1)} \langle\psi|_{2R}^{(1)} S_x^\dagger$, and for an ensemble of density operators, it also holds due to the linearity of the map $S_x \cdot S_x^\dagger$, that

$$\rho_{2R}^{(1)} = S_x \rho_{12}^{(R)} S_x^\dagger, \tag{3.53}$$

where the subscripts and superscripts carry the same meaning as for pure states. A first example that highlights the fundamental difference between the approaches of Angelo *et. al.* [14] and Giacomini *et. al.* [15] is a reference frame describing only one particle. While the transformation T_{CM} does not affect a one-particle state $|\psi\rangle_1^{(R)}$, the transformation S_x acts as

$$S_x |\psi\rangle = \int \psi_1^{(R)}(x_1) |x_1\rangle \, dx_1 \tag{3.54}$$

$$= \int \psi_1^{(R)}(-x_R) |x_R\rangle \, dx_R, \tag{3.55}$$

such that the transformed wave function is $\psi_R^{(1)}(x_R) = \psi_1^{(R)}(-x_R)$, i.e., it is the exact same functional form but mirrored. That symmetry is worth highliting: if an observer prepares a system in a quantum state $\psi(x)$, one can say that the system prepares the observer in a quantum state $\psi(-x)$.

Other important examples to gain intuition about this transformation are depicted in figure 6. In figure 6(a), the reference frame R observes particle 1 in a superposition of states

fig/Exemplos TransformaÃğÃčo da Flaminia.jpg

Figure 6 – Wave-function of the system in reference frames R and of particle 1. In a) particle 1 is in a superposition of sharply localized stated in position a and b while particle 2 is in a general state. In b) particle 1 and particle 2 are entangled such that in each branch of the superposition they are sharply localized and a distance L of each other.

that are sharply localized in a and b, and particle 2 in a general state:

$$|\Psi\rangle_{12}^{(R)} = \frac{1}{\sqrt{2}} \int \left[\delta_\epsilon(x_1 - a) + \delta_\epsilon(x_1 - b)\right] |x_1\rangle_1 \, dx_1 \int \psi(x_2) \, |x_2\rangle_2 \, dx_2, \tag{3.56}$$

where $\delta_\epsilon(x)$ is a Gaussian model for Dirac's delta with arbitrarily small variance ϵ presented in appendix B. Transforming it with S_x yields

$$|\Psi\rangle_{2R}^{(1)} = \frac{1}{\sqrt{2}} \iint \left[\delta_\epsilon(x_R + a) + \delta_\epsilon(x_R + b)\right] \psi(x_{r_2} - x_R) \left|x_{r_2}\right\rangle_2 |x_R\rangle_R \, dx_{r_2} dx_R \tag{3.57}$$

$$= \frac{1}{\sqrt{2}} \int \left|x_{r_2}\right\rangle_2 (\psi(x_{r_2} - a) \, |a\rangle_R + \psi(x_{r_2} - b) \, |b\rangle_R) dx_{r_2}, \tag{3.58}$$

where the second step is for better visualization of the fact that this is an entangled state where the position $-a(-b)$ of R relative to particle 1 is correlated with a translation of $-a(-b)$ of the wave function of particle 2, maintaining its functional form.

Another example is that of entangled particles at a fixed distance L from each other as depicted in figure 6(b):

$$|\Psi\rangle_{12}^{(R)} = \frac{1}{\sqrt{2}} (|a\rangle_1 \, |a + L\rangle_2 + |b\rangle_1 \, |b + L\rangle_2). \tag{3.59}$$

This state is transformed with S_x to

$$|\Psi\rangle_{2R}^{(1)} = \frac{1}{\sqrt{2}} \, |L\rangle_{r_2} \, (|-a\rangle + |-b\rangle)_R, \tag{3.60}$$

which represents a superposition of positions $-a$ and $-b$ for R completely uncorrelated with particle 2 that is well localized at a distance L.

3.2.7.1 The dynamics

Based on the assumption that the Schrödinger equation (2.30) holds in the quantum reference frame R, i.e., it has enough mass so that it approximates a classical reference frame while still being described by quantum mechanics, we have

$$i\hbar\frac{d\rho_{12}^{(R)}}{dt} = [H_{12}^{(R)}, \rho_{12}^{(R)}]. \tag{3.61}$$

Transforming the state according to Eq. (3.53), the Hamiltonian must be transformed as

$$H_{2R}^{(1)} = S_x H_{12}^{(R)} S_x^\dagger + i\hbar\frac{dS_x}{dt}S_x^\dagger, \tag{3.62}$$

in order to maintain form invariance of the Schrödinger equation. In this way, starting from a massive quantum reference frame it is possible to jump to another quantum reference frame and the dynamics will still be given unitarily by the Schrödinger equation. As the operator S_x is not time-dependent, transforming dynamics with this unitary transformation yields the same results as Aharonov and Kaufherr's in subsection 3.2.1.

3.2.8 Perspective neutral framework

The approach of Vanrietvelde *et. al.* [42] aims to employ Mach's principle, which abandons the notion of an absolute inertial reference frame in favor of a completely relational description of Physics [48]. Einstein himself took this principle quite seriously throughout the development of the theory of general relativity. The result of this consideration was not to postulate Mach's principle, but to postulate the equivalence principle, which is a similar statement. This reasoning led to the following Lagrangian of a system of N particles of unit mass:

$$L = \frac{1}{2}\sum_{i=0}^{N}\dot{q}_i^2 - \frac{1}{2N}\left(\sum_{i=0}^{N}\dot{q}_i\right)^2 - V(\{q_i - q_j\}_{i\neq j}^N), \tag{3.63}$$

in which the second term is the kinetic energy of the center of mass, V is an interaction that is function of the relative positions of the particles and q_i are positions relative to a completely arbitrary reference frame. This is the Lagrangian in the center of mass reference frame but expressed in generalized coordinates. One can see this by making the point transformation $x_i = q_i - \sum_{i=1}^{N} m_i q_i/M$ and obtaining

$$L = \frac{1}{2}\sum_{i=0}^{N}\dot{x}_i^2 - V(\{x_i - x_j\}_{i\neq j}^N). \tag{3.64}$$

The Lagrangian in Eq. (3.64) is fully relational in the sense that there is gauge freedom in the generalized coordinates $(q_i, \dot{q}_i)$, meaning that the Lagrangian is invariant under arbitrary translations $(q_i + f(t), \dot{q}_i + \dot{f}(t))$, where $f(t)$ is an arbitrary function of time. These generalized coordinates come with the expense of a constraint in the conjugated momenta, namely that the momentum of the center of mass P is null and defines a constraint surface

$$\sum_{i=0}^{N} p_i = 0. \tag{3.65}$$

P is also a generator of global translations, which is a gauge transformation in this case and $\{P, H\} = 0$ holds for the Hamiltonian H following from the Lagrangian

$$H = \frac{1}{2}\sum_{i}^{N} p_i^2 + V(\{q_i - q_j\}_{i\neq j}). \tag{3.66}$$

The total Hamiltonian, which is the sum of H with all gauge generators [49, 50] is given by

$$H_{tot} = \frac{1}{2}\sum_{i=0}^{N} p_i^2 + V(\{q_i - q_j\}_{i\neq j}) + \lambda\sum_{i=0}^{N} p_i, \tag{3.67}$$

where λ is an arbitrary time function, a Lagrange multiplier that represents the gauge freedom of the system.

This Hamiltonian is the *perspective neutral structure*: it will be shown that fixing a gauge determines a reference frame or, equivalently, a perspective. It is actually not surprising since the gauge freedom is translational, it could be interpreted from the start as a freedom in the origin of the coordinate system. Demanding that particle 0, the one we are promoting to reference frame, stays at the origin at all times, we have

$$q_0 = 0, \tag{3.68}$$

wich implies $\dot{q}_i = 0$. The Hamilton equations for H_{tot} reads

$$\dot{q}_i = \frac{\partial H_{tot}}{\partial p_i} = \frac{p_i}{m} + \lambda, \tag{3.69a}$$

$$\dot{p}_i = -\frac{\partial H_{tot}}{\partial q_i} = -\frac{\partial V}{\partial q_i}, \tag{3.69b}$$

which fixes the gauge with

$$\lambda = -\frac{p_0}{m_0}. \tag{3.70}$$

The Hamiltonian with the fixed gauge in the perspective of particle 0 is

$$H_0 = \frac{1}{2}\sum_{i\neq 0}\left(\frac{1}{m_i} + \frac{1}{m_0}\right)p_i^2 + \sum_{\substack{i\neq j\\ i,j\neq 0}}\frac{p_i p_j}{m_0} + V(\{q_i\}_{i\neq 0}). \tag{3.71}$$

It is worth noting that this Hamiltonian is not quadratic in the momenta as a consequence of the noninertiality of particle 0 and, taking the limit $m_0 \to \infty$ it reduces to the Hamiltonian given in Eq. (3.67) but without particle 0, which describes all the other particles with quadratic momentum, i.e. from an inertial perspective. Therefore, fixing the gauge is equivalent to jumping to the perspective of a particle of the system.

Quantizing the Hamiltonian in Eq. (3.66) for three particles gives

$$H = \frac{1}{2}\left(\frac{P_0^2}{m_0} + \frac{P_1^2}{m_1} + \frac{P_2^2}{m_2}\right) + V(X_2 - Q_1, Q_2 - Q_0, Q_1 - Q_0), \tag{3.72}$$

and a general state $|\phi\rangle^{free}$ can be decomposed in the momentum basis as

$$|\phi\rangle^{free} = \iiint \phi^{free}(p_0, p_1, p_2)\, |p_0\rangle\, |p_1\rangle\, |p_2\rangle\, dp_0 dp_1 dp_2. \tag{3.73}$$

The analogue of the constraint surface (3.65) is the subspace composed by the states $|\phi\rangle^{con}$ that satisfy

$$P\,|\phi\rangle^{con} = (P_1 + P_2 + P_3)\,|\phi\rangle^{con} = 0, \tag{3.74}$$

i.e. is the constrained subspace of eigenstates of total momentum $P = P_1 + P_2 + P_3$ with eigenvalue zero. There are three ways in which we can write the same state

$$\begin{aligned} |\phi\rangle^{con} &= \iint \phi_{12}^{(0)}(p_1, p_2)\, |-p_1 - p_2\rangle_0\, |p_1\rangle_1\, |p_2\rangle_2\, dp_1 dp_2 \\ &= \iint \phi_{02}^{(1)}(p_0, p_2)\, |p_0\rangle_0\, |-p_0 - p_2\rangle_1\, |p_2\rangle_2\, dp_0 dp_2 \\ &= \iint \phi_{01}^{(2)}(p_0, p_1)\, |p_0\rangle_0\, |p_1\rangle_1\, |-p_0 - p_1\rangle_2\, dp_0 dp_1, \end{aligned} \tag{3.75}$$

where $\phi_{12}^{(0)}(p_1, p_2) = \phi^{free}(-p_1 - p_2, p_1, p_2)$, $\phi_{02}^{(1)}(p_0, p_2) = \phi^{free}(p_0, -p_2 - p_0, p_2)$ and $\phi_{01}^{(2)}(p_0, p_1) = \phi^{free}(p_0, p_1, -p_0 - p_1)$. Here, the redundant Hilbert space, i.e., the correspondent to the variable that was eliminated, is discarded using the partial trace, resulting in

$$\begin{aligned} |\phi\rangle_{12}^{(0)} &= \iint \phi_{12}^{(0)}(p_1, p_2)\, |p_1\rangle_1\, |p_2\rangle_2\, dp_1 dp_2, \\ |\phi\rangle_{02}^{(1)} &= \iint \phi_{02}^{(1)}(p_0, p_2)\, |p_0\rangle_0\, |p_2\rangle_2\, dp_0 dp_2, \\ |\phi\rangle_{01}^{(2)} &= \iint \phi_{01}^{(2)}(p_0, p_1)\, |p_0\rangle_0\, |p_1\rangle_1\, dp_0 dp_1. \end{aligned} \tag{3.76}$$

Each decomposition is considered a different perspective, first particle 0, then particle 1 and then particle 2. To change from the perspective of particle 0 to particle 1 the operator $\pi_{01} \exp\left(\frac{i}{\hbar} Q_1 P_2\right)$, which is exactly the same as the transformation found by Giacomini *et. al.* [15] with slightly different notation accomplishes the task. This equivalence is no surprise since both approaches are intended to be fully relational. The proof follows from direct application:

$$\begin{aligned} \pi_{01} \exp\left(\frac{i}{\hbar} Q_1 P_2\right) |\phi\rangle_{02}^{(1)} &= \iint \phi^{free}(-p_1 - p_2, p_1, p_2)\, |-p_2 - p_1\rangle_0\, |p_2\rangle_2\, dp_1 dp_2 \\ &= \iint \phi_{01}^{(2)}(p_0, p_1)\, |p_0\rangle_0\, |p_1\rangle_1\, dp_0 dp_1 \\ &= |\phi\rangle_{01}^{(2)} \end{aligned} \tag{3.77}$$

where the last steps are merely recognizing that $p_0 = -p_2 - p_1$.

3.2.9 Quantum resource covariance

As we have seen until now, coherence and entanglement are not invariant under reference frame transformations but can a combination of both form a scalar invariant quantity, much like spatial separation Δx and temporal separations Δt between events a and b form an invariant $(\Delta x)^2 - c^2(\Delta t)^2$? That is the question that Savi *et. al.* raised and answered in [18]. Their strategy to tackle the problem is to begin with a known invariant under reference frame transformations, the total information $I(\rho)$ content of a state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, where $dim(\mathcal{H}_A \otimes \mathcal{H}_B) = d = d_A d_B$ and $dim(\mathcal{H}_{A(B)}) = d_{A(B)}$. Note that any reference frame transformation T is unitary $T^\dagger T = TT^\dagger = \mathbb{1}$ and leaves the informational content of the system invariant $I(T\rho T^\dagger) = I(\rho)$. The goal here is to decompose $I(\rho)$ into known quantum resources, the procedure is accomplished in three steps

1. Perform a measurement of the observable $A = \sum_{i=1}^{d_A} a_i A_i$ in part A and compute the information decrease $I(\phi_A(\rho)) - I(\rho)$;
2. Perform a measurement of the observable $B = \sum_{i=1}^{d_B} b_i B_i$ in part B and compute the information decrease $I(\phi_{BA}(\rho)) - I(\phi_A(\rho))$;
3. Eliminate the rest of the information measuring observables $\tilde{A} \otimes \tilde{B}$ that form a mutually unbiased basis with $A \otimes B$, leading to the maximally mixed state (a state with no remaining information).

The decrease in information in step 1 can be recognized as the negative of the sum of the A-coherence in ρ_A and the one-way quantum discord quantified using Eqs. (2.52) and (2.69), respectively:

$$\begin{aligned} -[C_A(\rho_a) + D_A(\rho)] &= -[S(\phi_A(\rho_A)) - S(\rho_A) - S(\mathrm{Tr}_A\, \phi_A(\rho)) - S(\phi_A(\rho_A)) \\ &\qquad + S(\phi_A(\rho)) + S(\rho_A) + S(\rho_B) - S(\rho)] \\ &= S(\rho) - S(\phi_A(\rho)) \\ &= I(\phi_A(\rho)) - I(\rho), \end{aligned} \tag{3.78}$$

where the fact that $\mathrm{Tr}_A\, \phi_A(\rho) = \rho_B$ and $\mathrm{Tr}_A\, \phi_A(\rho) = \phi_A(\rho_A)$ was utilized.

The decrease in information in step 2 can be explained in the same manner,

$$-[C_B(\rho_B) + D_B(\phi_A(\rho))] = I(\phi_{BA}) - I(\phi_A(\rho)), \tag{3.79}$$

which resultis in a decrease in information so far given by

$$I(\phi_{BA}) - I(\rho) = -[C_A(\rho_a) + C_B(\rho_B) + D_{AB}(\rho)]. \tag{3.80}$$

In step 3, an observable $\tilde{A} \otimes \tilde{B}$ that forms a MUB with $A \otimes B$ is measured yielding the maximally mixed state $\frac{\mathbb{1}}{d}$. The decrease in information is

$$I(\phi_{\tilde{A}\tilde{B}AB}(\rho)) - I(\phi_{AB}(\rho)) = I_{MSA}, \tag{3.81}$$

where I_{MSA} is the context incompatibility for the context $\{\rho, A \otimes B, \tilde{A} \otimes \tilde{B}\}$ presented in section 2.4. Therefore, we have the information decomposed in three quantum features of the context: coherence of each part according to the corresponding observable, A or B; quantum discord of the joint observable and the context incompatibility of the context formed by the joint observable $A \otimes B$ and an observable that forms a MUB with it.

$$I(\rho) = C_A(\rho_a) + C_B(\rho_B) + D_{AB}(\rho) + I_{MSA}. \tag{3.82}$$

Covariance is ensured by the invariance of information under unitary transformations. Transforming the context yields the same decomposition for the other reference frame characterized by the context $\{\rho', A' \otimes B', \tilde{A}' \otimes \tilde{B}'\}$.

4 RESULTS

In this chapter, we compare the approaches of Angelo *et. al.* [14] and Giacomini *et. al.* [15] showing that they lead to different physical descriptions relative to a quantum system, and favoring one of them in the end. Furthermore, we demonstrate that a transformation that results in the degrees of freedom relative to one particle does nos exists when the system is composed of more than two particles. Finally, we analyze the theory-independent context incompatibility under reference frame transformations to find that it is not invariant.

4.1 THE DEGREES OF FREEDOM ACCESSIBLE TO OTHER OBSERVER

To analyze a physical system using the tools available to other observer S', i.e., change reference frames. It's fair that we perform a coordinate transformation T on the observables available to the first observer S to obtain the observables available to S' according to the interpretation obtained in section 3.1.2. In view of Born's rule we can make use of the cyclicity of the trace to adopt two different points of view, the active and the passive version of the transformation in analogy to section 3.1.1. In the passive version only the observables O are transformed, alluding to the basis change:

$$\begin{cases} O' = T^\dagger O T \\ \rho' = \rho, \end{cases} \tag{4.1}$$

while in the active version only the state is transformed in allusion to the transformation of the vector,

$$\begin{cases} O' = O \\ \rho' = T\rho T^\dagger. \end{cases} \tag{4.2}$$

The active transformation is to be interpreted as an observer in the S' reference frame assigning a quantum state to the system by means of the degrees of freedom available to him.

The active and the passive versions are physically equivalent since the trace, which is the mathematical operation that produces physical predictions, is cyclical:

$$p'(o_i) = \mathrm{Tr}\Big(T^\dagger O_i T \rho\Big) \quad \text{(Passive version)}, \tag{4.3a}$$

$$p'(o_i) = \mathrm{Tr}\Big(O_i T \rho T^\dagger\Big) \quad \text{(Active version)}, \tag{4.3b}$$

where O_i are the orthogonal projectors that compose the observable O. Eqs. (4.3a) and (4.3b) should be interpreted in the same manner, the probability of a measurement of the observable O' performed by the observer S' yielding the result i referring to the i-th eigenstate of O'.

4.1.1 Time evolution

Suppose a physical system undergoes a unitary evolution U_t, then the probability of obtaining the result o_i after some time t can be calculated in the Schrödinger or Heisenberg picture as

$$p(o_i|t) = \mathrm{Tr}(O_i\rho_t) \tag{4.4a}$$
$$= \mathrm{Tr}(O_{i,t}\rho), \tag{4.4b}$$

where $O_{i,t} = U_t^\dagger O_i U_t$ and $\rho_t = U_t\rho U_t^\dagger$ represent the time evolution of the observable in the Heisenberg picture and of the state operator in the Schrödinger picture, respectively. Applying a transformation T from the current reference frame S to a reference frame S' in the Schrödinger picture, Eq. (4.4a), we get

$$\begin{aligned} p(o_i'|t) &= \mathrm{Tr}\left(T^\dagger O_i T\rho_t\right) \\ &= \mathrm{Tr}\left(U_t^\dagger T^\dagger O_i T U_t\rho\right) \\ &= \mathrm{Tr}\left(T_t^\dagger O_{i,t} T_t\rho\right), \end{aligned} \tag{4.5}$$

where $T_t = U_t^\dagger T U_t$ is the Heisenberg picture of the frame transformation. Hence, we have that the reference frame transformation is consistent with the perspective of the Schrödinger picture, and it suffices to use also the Heisenberg picture of the frame transformation. Notice that if we intend to change reference frames in the Heisenberg picture we must transform using the Heisenberg picture of the reference frame transformation operators to obtain Eq. (4.5).

4.1.2 Collapse

Suppose a measurement of an observable O performed on a state ρ_t at a time t relative to a reference frame S and the outcome o_i was produced. The collapse observed by the reference frame S', connected to S through the transformation T is

$$\begin{aligned} T\phi_{o_i}(\rho_t)T^\dagger &= \frac{TO_i\rho_t O_i T^\dagger}{\mathrm{Tr}(O_i\rho_t)} \\ &= \frac{TO_iT^\dagger T\rho_t T^\dagger TO_iT^\dagger}{\mathrm{Tr}(TO_iT^\dagger T\rho_t T^\dagger)} \\ &= \frac{\tilde{O}_i\rho_t'\tilde{O}_i}{\mathrm{Tr}\left(\tilde{O}_i\rho_t'\right)}, \end{aligned} \tag{4.6}$$

where $\tilde{O}_i = TO_iT^\dagger$. Notice that the projectors of O are transformed with the inverse transformation in comparison to the passive version. It happens because the eigenbasis of the observable O has the meaning of the possible states to which the physical state might collapse in a measurement and, therefore, their description according to S' must be obtained in the same way as the description of the states.

4.1.3 Different descriptions

If we apply the transformation S_x to the position and momenta coordinates we obtain

$$S_x^\dagger X_1 S_x = -X_R, \qquad S_x^\dagger P_1 S_x = -(P_R + P_2), \tag{4.7a}$$

$$S_x^\dagger X_2 S_x = X_2 - X_R, \quad S_x^\dagger P_2 S_x = P_2. \tag{4.7b}$$

In Eqs. (4.7) the positions are relative to particle 1 but the momenta are still relative to the initial reference frame R. Thus, the new coordinates do not provide a fully covariant description, the position basis being privileged to be relative. This results in an ambiguity in the notion of describing physics relative to a quantum reference frame. One needs to choose what physical quantity will be described in a relative manner. An example of another choice is the eigenbasis of relative momenta [15] according to the transformation given by Giacomini *et. al.*

$$S_p^\dagger X_1 S_p =, \qquad S_p^\dagger P_1 S_p = -P_R, \tag{4.8a}$$

$$S_p^\dagger X_2 S_p = X_2, \qquad S_p^\dagger P_2 S_p = P_2 - P_R. \tag{4.8b}$$

The center of mass transformation T_{CM} given in Eq. (3.30) acts in the position and momenta coordinates as

$$T_{CM}^\dagger X_0 T_{CM} = \frac{m_0 X_0 + m_1 X_1}{m_0 + m_1} = X_{CM}^{(R)}, \quad T_{CM}^\dagger X_1 T = X_1 - X_0, \tag{4.9}$$

$$T_{CM}^\dagger P_0 T_{CM} = P_0 + P_1 = P_{CM}^{(R)}, \qquad T_{CM}^\dagger P_1 T_{CM} = \mu_{01}\left(\frac{P_1}{m_1} - \frac{P_0}{m_0}\right). \tag{4.10}$$

These transformations aim to provide the physics relative to a new reference frame but they are fundamentally different. To emphasize this point, we investigate the physical system depicted in figure 7, where a well-localized body composed of particles 0 and 1 decays, separating the particles in such a way that the total momentum is conserved and by consequence the position of the center of mass is preserved relative to an external reference frame R, whose origin is located on the center of mass of the system and assigns to the system the state

$$|\psi\rangle = \frac{|x\rangle_0 |x-d\rangle_1 + |-x\rangle_0 |-x+d\rangle_1}{\sqrt{2}}. \tag{4.11}$$

Performing the two transformations and defining $|\psi\rangle_{CM,r_1} = T_{CM}|\psi\rangle$ and $|\psi\rangle_{1R}^{(0)} = S_x|\psi\rangle$ we have

$$|\psi\rangle_{CM,r_1} = |0\rangle_{CM}\left(\frac{|d\rangle + |-d\rangle}{\sqrt{2}}\right)_{r_1}, \tag{4.12a}$$

$$|\psi\rangle_{1R}^{(0)} = \frac{|x\rangle_R |d\rangle_{r_1} + |-x\rangle_R |-d\rangle_{r_1}}{\sqrt{2}}. \tag{4.12b}$$

fig/Decaying Body.jpg

Figure 7 – The gradient of color means that the particles are in superposition between the situation where particle 1 is on the right side of the center of mass and the situation where it is on the left of the center of mass.

Computing the density operators and tracing out the degrees of freedom of the reference frame R in Eq. (4.12b) and the CM in Eq. (4.12a) to attempt to eliminate the external reference frame we have

$$\rho_{r_1} = \frac{(|d\rangle + |-d\rangle)_{r_1}(\langle d| + \langle -d|)_{r_1}}{2}, \tag{4.13a}$$

$$\rho_1^{(0)} = \frac{|d\rangle\langle d|_1 + |-d\rangle\langle -d|_1}{2}. \tag{4.13b}$$

At first sight both descriptions seem to agree, measurements of the relative position of particle 1 will result around $-d$ and d with equal probability. There is, however, a fundamental difference in the reduced density operators: the first is a pure state while the second one is mixed. This means that one (or none) of them must be the true description of the system relative to particle 0.

The solution to this conundrum is to look at the building blocks of the relative Hilbert spaces. $\mathcal{H}_{r_1}$ is built upon the canonical pair $(X_1 - X_0, \mu_{01}\left(\frac{P_1}{m_1} - \frac{P_0}{m_0}\right))$, i.e., the relative position and relative momentum while the canonical pair that builds $\mathcal{H}_1$ is $(X_1 - X_0, P_1)$, the relative position but the momentum defined relative to the old reference frame. Therefore $\mathcal{H}_1$ is not defined solely on the physical quantities that are accessible to particle 0, it is a hybrid reference frame, while $\mathcal{H}_{r_1}$ is purely relative and hence not hybrid.

It is worth noting that the total Hilbert space $\mathcal{H}_{CM} \otimes \mathcal{H}_{r_1}$ is also hybrid, but the degrees of freedom relative to the old reference frame, namely the center of mass, can be traced out, while the hybridness in $\mathcal{H}_R \otimes \mathcal{H}_1$ cannot be traced out since both $\mathcal{H}_R$ and $\mathcal{H}_1$ are hybrid. It is also worth acknowledging that both descriptions are equally valid and give the right predictions provided they are interpreted correctly.

4.1.4 There is no transformation for the relative degrees of freedom

Suppose there is a unitary transformation T that leads each position and momentum of a part of the system relative to R in its position and momentum relative to particle 0, that is,

$$\begin{aligned} T^\dagger X_1 T &= X_1 - X_0, \quad T^\dagger P_1 T = \mu_{10}\left(\frac{P_1}{m_1} - \frac{P_0}{m_0}\right), \\ T^\dagger X_2 T &= X_2 - X_0, \quad T^\dagger P_2 T = \mu_{20}\left(\frac{P_2}{m_2} - \frac{P_0}{m_0}\right), \\ T^\dagger X_N T &= X_N - X_0, \quad T^\dagger P_N T = \mu_{N1}\left(\frac{P_N}{m_N} - \frac{P_0}{m_0}\right). \end{aligned} \tag{4.14}$$

The commutation relation of the position and momentum transforms as

$$T^\dagger [X_j, P_k] T = i\hbar\delta_{jk} T^\dagger T, \tag{4.15}$$

which, using the unitarity of T, leads to

$$[X_j - X_0, \mu_{j0}\left(\frac{P_j}{m_j} - \frac{P_0}{m_0}\right)] = i\hbar\delta_{jk}\mathbb{1}. \tag{4.16}$$

$$\frac{m_j}{m_1 + m_j} i\hbar\mathbb{1} = i\hbar\delta_{jk}\mathbb{1} \tag{4.17}$$

Eq. (4.16) contradicts the commutation relation between the relative position of a particle j and the relative momentum of a particle k with $j \neq k$, which is not zero. Therefore, a unitary transformation T that satisfies Eq. (4.14) cannot exist.

4.2 CONTEXT INCOMPATIBILITY UNDER REFERENCE FRAME TRANSFORMATIONS

Considering the context $\mathbb{C} = \{\rho, X_1, P_1\}$ where ρ is the state of a system composed of particles 0 and particle 1 in a classical reference frame and the Hilbert space discretization is employed with interval δ_q, in addition, all sums are over the whole domain of the discretization. The criteria of compatibility in Eq. (2.84) read

$$\phi_{X_1}(\rho) = \phi_{X_1 P_1}(\rho), \tag{4.18a}$$

$$\phi_{P_1}(\rho) = \phi_{P_1 X_1}(\rho), \tag{4.18b}$$

The RHS of Eqs. (4.18a) and (4.18b) are always the maximally mixed state, since X_1 and P_1 form a MUB, see Eq. (2.73). For the LHS we have

$$\phi_{X_1}(\rho) = \sum_{x_0, x_0', x_1} \delta_q \psi(x_0, x_1)\psi^*(x_0', x_1) \left|x_0, x_1\right\rangle\!\left\langle x_0', x_1\right|, \tag{4.19a}$$

$$\phi_{P_1}(\rho) = \sum_{x_0, x_0', p_1} \delta_q \phi(x_0, p_1)\phi^*(x_0', p_1) \left|x_0, p_1\right\rangle\!\left\langle x_0', p_1\right|, \tag{4.19b}$$

where $\phi(x_0, p_1)$ is the discrete Fourier transform of $\psi(x_0, x_1)$. It's clear that the context $\mathbb{C} = \{\rho, X_1, P_1\}$ exhibits incompatibility since the states in Eq. (4.19) are not maximally mixed in general.

In the reference frame of particle 0 the context $\mathbb{C}$ transforms into $\mathbb{C}' = \{\rho_{r_1}, X_1, P_1\}$ and the criteria for compatibility reads

$$\phi_{X_1}(\rho_{r_1}) = \phi_{X_1P_1}(\rho_{r_1}), \tag{4.20a}$$

$$\phi_{P_1}(\rho_{r_1}) = \phi_{P_1X_1}(\rho_{r_1}), \tag{4.20b}$$

where, once again, the RHS of Eqs. (4.20a) and (4.20b) are the maximally mixed state, $\rho_{r_1} = \mathrm{Tr}_{CM}(T_{CM}\rho T_{CM}^{\dagger})$ and

$$\phi_{X_1}(\rho_{r_1}) = \sum_{x_{CM},x_{r_1}} \delta_q \tilde{\psi}(x_{CM}, x_{r_1})\tilde{\psi}^*(x_{CM}, x_{r_1}) \left|x_{r_1}\right\rangle\!\left\langle x_{r_1}\right|, \tag{4.21a}$$

$$\phi_{P_1}(\rho_{r_1}) = \sum_{x_{CM},x_{r_1}} \delta_q \tilde{\phi}(x_{CM}, p_{r_1})\tilde{\phi}^*(x_{CM}, p_{r_1}) \left|p_{r_1}\right\rangle\!\left\langle p_{r_1}\right|, \tag{4.21b}$$

where $\tilde{\psi}(x_{CM}, x_{r_1}) = \psi(x_{CM} - \frac{m_1}{M}x_{r_1}, x_{CM} + \frac{m_0}{M}x_{r_1})$ and $\tilde{\phi}(x_{CM}, p_{r_1})$ is the Fourier transform of $\tilde{\psi}$ in the x_{r_1} coordinate.

To quantify the amount of incompatibility we use the Kullback-Leibler divergence based quantifier with the Hilbert space discretization introduced in Eq. (2.90). In order to do that, we calculate the probability distributions of position for the states ρ and $\phi_{P_1}(\rho)$ given by

$$p_\rho(x_i) = \sum_{x_0} \psi(x_0, x_i)\psi^*(x_0, x_i), \tag{4.22}$$

$$p_{\phi_{P_1}(\rho)}(x) = \frac{1}{\Delta}, \tag{4.23}$$

where the Hilbert space of particle 1 was discretized in Δ segments of lenght δ_q . For the probability distribution in momentum of the states ρ and $\phi_{X_1}(\rho)$

$$p_\rho(p_i) = \sum_{x_0} \phi(x_0, x_i)\phi^*(x_0, x_i), \tag{4.24}$$

$$p_{\phi_{P_1}(\rho)}(x) = \frac{1}{\Delta}. \tag{4.25}$$

The homogeneous probabilities $\frac{1}{\Delta}$ arise because the probability distribution of an observable that forms a MUB with the one previously measured is always homogeneous. Therefore, the amount of incompatibility in the context $\mathbb{C}$ is given by

$$I_{\mathbb{C}} = \sum_{i=1}^{\Delta} \frac{p_\rho(x_i)\log\big(p_\rho(x_i)\Delta\big) + p_\rho(p_i)\log\big(p_\rho(p_i)\Delta\big)}{2}, \tag{4.26}$$

while the amount of incompatibility in C' is given by

$$I_{\mathbb{C}'} = \sum_{i=1}^{\Delta} \frac{p_{\rho_{r_1}}(x_i)\log\Big(p_{\rho_{r_1}}(x_i)\Delta\Big) + p_{\rho_{r_1}}(p_i)\log\Big(p_{\rho_{r_1}}(p_i)\Delta\Big)}{2}, \tag{4.27}$$

where, from Eqs. (4.21a), the probabilities are homogeneous for the states that already have been measured and the remaining probabilities are given by

$$p_{\rho_{r_1}}(x_i) = \sum_{x_{CM}} \tilde{\psi}(x_{CM}, x_i)\tilde{\psi}^*(x_{CM}, x_i), \tag{4.28}$$

$$p_{\rho_{r_1}}(p_i) = \sum_{x_{CM}} \tilde{\phi}(x_{CM}, p_i)\tilde{\phi}^*(x_{CM}, p_i). \tag{4.29}$$

Now we calculate numerically the amount of incompatibility in each context because the discretization method is a good approximation only for large Δ. To do that, the state must be specified, so we present three case studies of interest. All the discrete Fourier transforms were also numerically calculated.

4.2.1 Product of Gaussian states

Consider that the new reference frame is not entangled with particle 1 and both are in Gaussian states centered in a with variance Δ_0 and b with variance Δ_1, respectively

$$\Psi(x_0, x_1) = \left(\frac{1}{2\pi\Delta_0^2\Delta_1^2}\right)^{\frac{1}{4}} \exp\left[-\frac{(x_0 - a)^2}{4\Delta_0^2}\right] \exp\left[-\frac{(x_1 - b)^2}{4\Delta_1^2}\right], \tag{4.30}$$

Transforming to the center of mass and relative coordinates we have

$$\tilde{\Psi}(x_{CM}, x_{r_1}) = \frac{1}{\sqrt{2\pi\Delta_0\Delta_1}} \exp\left[-\frac{(x_{CM} - \alpha)^2}{4\Delta_{CM}^2}\right] \exp\left[-\frac{(x_{r_1} - \beta)^2}{4\Delta_{r_1}^2}\right] \times$$
$$\times \exp\left[\gamma(x_{CM} - \alpha)(x_{r_1} - \beta)\right], \tag{4.31}$$

where the parameters α, β, Δ_{CM} and Δ_{r_1} are given by Eq. (3.37). In figure 8 the incompatibility of context $\mathbb{C}$, associated to the state in Eq. (4.30) and the incompatibility of context $\mathbb{C}'$, which is associated to the transformed state are calculated varying Δ_0. The parameters chosen to define the state were $a = 10$, $b = 0$, $\Delta_1 = 4$ and both masses equal to 1, all in arbitrary units.

As expected, the incompatibility of $\mathbb{C}$ does not vary with Δ_0 because the probability distributions for particle 1 do not depend on the state of particle 0. The incompatibility of $\mathbb{C}'$, for a well-localized quantum reference frame approaches the incompatibility of $\mathbb{C}$. That happens because for small Δ_0 the coordinate transformation approaches a mere translation of origin, causing the probability distributions to maintain its profile and, therefore, the incompatibility.

But when the QRF gets more and more delocalized relatively to the external reference frame, it sees particle 1 more and more delocalized as well, i.e., the probability distributions approach homogeneity as shown in figure 8, and, therefore, the context $\mathbb{C}$ tends to compatibility when the state approaches the maximally mixed state, see Eq. (4.27), which is expected by the criteria in Eq. (4.20). It is good to point out that the reduced state ρ_{r_1} is not pure in general, fig/pgentropy.png

fig/pgprob.png

Figure 8 – Left panel is the plot of incompatibility of the contexts $\mathbb{C}$ and $\mathbb{C}'$ against Δ_0 and right panel is the plot of position distributions of probability of particle 1 relative to particle 0 varying Δ_0.

therefore an argument such as "if probability distribution is homogeneous for position it must be well-localized for momentum", does not hold in this situation.

We don't see incompatibility actually reaching zero since the functional form of the states is fixed and only Δ_0 is changed, i.e., increasing Δ_0 even more would exceed the fixed range Δ of the simulation and the state would lose its normalization.

4.2.2 Product state; reference frame in superposition of Gaussian states

Now we put particle 0 in superposition of Gaussian states and particle 1 in a Gaussian state, forming a product state, whose wave function reads

$$\Psi(x_0, x_1) = N\left\{\exp\left[-\frac{(x_0 - a)^2}{4\Delta_0^2}\right] + \exp\left[-\frac{(x_0 + a)^2}{4\Delta_0^2}\right]\right\}\exp\left[-\frac{(x_1 - b)^2}{4\Delta_1^2}\right]. \tag{4.32}$$

The parameters chosen were the same as in the previous case and the normalization factor is

$$N = \left(\frac{1}{2\pi\Delta_0^2\Delta_1^2}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2 + 2\exp\left(-\frac{a^2}{2\Delta_0^2}\right)}}. \tag{4.33}$$

Transforming to the center of mass and relative coordinates we get a superposition of states that are similar to Eq. (4.31).

Figure 9 is also a plot of incompatibility of $\mathbb{C}$ and $\mathbb{C}'$ varying Δ_0 and the probability distribution of position of particle 1 relative to particle 0. We can see that while the external reference frame assigns a Gaussian distribution for the position of particle 1, the new QRF, which is in a superposition of well-localized states assigns a profile of two peaks centered at $x = 10$ and $x = -10$, therefore we would not expect that the incompatibility would be equal for small Δ_0. When Δ_0 becomes comparable to the separation between the Gaussians, the two peaks merge and the analysis becomes the same as the one made for the previous case.

fig/psgentropy.png

fig/psgprob.png

Figure 9 – Plot of incompatibility of the contexts $\mathbb{C}$ and $\mathbb{C}'$ against Δ_0 and plot of position distributions of probability of particle 1 relative to particle 0 varying Δ_0.

4.2.3 Entangled state

Here we considered the entangled state in Eq. (4.11)

$$\Psi(x_0, x_1) = N\left\{ \exp\left[-\frac{(x_0 - a)^2}{4\Delta_0^2}\right] \exp\left[-\frac{(x_1 + b)^2}{4\Delta_1^2}\right] + \right.$$
$$\left. + \exp\left[-\frac{(x_0 + a)^2}{4\Delta_0^2}\right] \exp\left[-\frac{(x_1 - b)^2}{4\Delta_1^2}\right]\right\}, \tag{4.34}$$

where $a = 10$ and $b = 15$ and the normalization factor is

$$N = \left(\frac{1}{2\pi\Delta_0^2\Delta_1^2}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2 + 2\exp\left(-\frac{a^2}{2\Delta_0}\right)\exp\left(-\frac{b^2}{2\Delta_1^2}\right)}}. \tag{4.35}$$

In figure 10 the incompatibilities of $\mathbb{C}$ and $\mathbb{C}'$ are calculated varying Δ_0 and the position probability distributions are plotted. As in the first case, the position probability distribution for

fig/egentropy.png

fig/egprob.png

Figure 10 – Plot of incompatibility of the contexts $\mathbb{C}$ and $\mathbb{C}'$ against Δ_0 and plot of position distributions of probability of particle 1 relative to particle 0 varying Δ_0.

small Δ_0 is similar to the position probability distribution relative to the exernal reference frame. As Δ_0 grows, the incompatibility decreases and when the two peaks merge, the incompatibility decreases slower than before, happening around $\Delta_0 = 11.25$, as we can see in the figure. The rest of the analysis (large Δ_0), is the same as the two previous cases.

An interesting feature about entangled states is that one can build a state in which the incompatibility in $\mathbb{C}'$ is greater than the incompatibility in $\mathbb{C}$. Consider a state such that in all Gaussian product terms, the center of the Gaussian of particle 1 is always dislocated of b to the right of particle 0.

$$\Psi(x_0, x_1) = N\Bigg\{ \exp\left[-\frac{(x_0 - a)^2}{4\Delta_0^2}\right] \exp\left[-\frac{(x_1 - a - b)^2}{4\Delta_1^2}\right] + \\ + \exp\left[-\frac{(x_0 + a)^2}{4\Delta_0^2}\right] \exp\left[-\frac{(x_1 + a - b)^2}{4\Delta_1^2}\right]\Bigg\}, \tag{4.36}$$

with $a = 10$ and $b = 10$. Figure 11 shows incompatibilities of $\mathbb{C}$ and $\mathbb{C}'$ and the position probability distributions relative to particle 0 varying Δ_0. Particle 0 sees only a Gaussian

fig/aegentropy.png

fig/aegprob.png

Figure 11 – Plot of incompatibility of the contexts $\mathbb{C}$ and $\mathbb{C}'$ against Δ_0 and plot of position distributions of probability of particle 1 relative to particle 0 varying Δ_0.

distribution, which is a less distributed probability distribution than the two Gaussian peaks seen by the external reference frame. The incompatibility therefore is greater in $\mathbb{C}'$ than in $\mathbb{C}$ for small Δ_0.

5 CONCLUSION

There are different approaches to quantum reference frames, the center of mass and relative coordinates of Angelo *et. al.* [14] and the fully relational approach of Giacomini *et. al.* [15] of a quantum particle. They yield different descriptions of the same physical system. We find that it is because the fully relational approach utilizes a canonical transformation that yields the relative positions but the momenta are still defined relative to the old reference frame while the approach by Angelo *et. al.* transforms both position and momentum to the relative degrees of freedom in the two particle case. Therefore, we favor the truly relative description to be the physics relative to a quantum system.

When asking the question whether there is a unitary transformation that produces the relative degrees of freedom for more than two particles, we found that it does not exist. However, it doesn't mean that we cannot use the relative observables in a passive picture to find the physics relative to a quantum reference frame. It just means that given a state of more than two particles, there is no clear connection between the wave function relative to the initial reference frame and the wave function relative to the quantum system. It also does not mean that such a wave function also does not exist. The relational and the perspective neutral approaches exhibit the relative positions only and leaves the momenta untouched, which can lead to erroneous conclusions if wrongly interpreted as completely relative. Our results shed some light in this hybridness of the description by showing that the reduced relative state is fundamentally different not in the analysis of position probabilities but in the analysis of the noncommuting observables such as momentum, that reveals the coherence or it's absence in position basis.

We find that the theory-independent incompatibility is not invariant under quantum reference frame transformations, corroborating the findings of Savi and Angelo [18]. The analysis of the theory-independent incompatibility under quantum reference frame transformations for the spatial degrees of freedom of a particle showed that when the quantum reference frame is delocalized, it results in a broader probability distribution for the particle's position, making the context less compatible and reaching compatibility only for the maximally mixed state, as expected since position and momentum form a MUB. An interesting feature is that it is possible to jump to a reference frame in which the probability distribution is more localized and, therefore disposes of more incompatibility but still, as the reference frame gets more delocalized, the incompatibility goes to zero.

The question we pose after these findings is whether there is a way of finding a quantum frame's description of many particles, how it is built and what is the relation between the perspectives of the particles. Furthermore we ask if, in light of the informational invariant composed of coherence, correlations, and context incompatibility, is there a combination of features that, when combined with the theory-independent context incompatibility, also forms an invariant under reference frame transformation.

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Appendix

APPENDIX A – HAMILTONIAN FROM A FINITE-MASS REFERENCE FRAME

Considering the transformation of generic operators that are function of the position and momenta in the external reference frame

$$A' = U^\dagger A(X_n, P_n)U = A(Q_n, \pi_n). \tag{A.1}$$

If we consider that the Schrödinger equation for the Heisenberg operators must be form invariant we have

$$\begin{aligned} i\hbar\dot{A}' &= i\hbar(\dot{U}^\dagger AU + U^\dagger \dot{A}U + U^\dagger A\dot{U}) && \text{(A.2)}\\ &= (U^\dagger H - HU^\dagger)AU + U(AH - HA)U^\dagger + U^\dagger A(UH - HU) && \text{(A.3)}\\ &= U^\dagger[A, UHU^\dagger]U && \text{(A.4)}\\ &= U^\dagger[A, \bar{H}]U, && \text{(A.5)} \end{aligned}$$

where $\bar{H}(X_n, P_n) = UH(X_n, P_n)U^\dagger$, which implies that

$$i\hbar\dot{A}(Q_n, \pi_n) = [A(Q_n, \pi_n), \bar{H}(Q_n, \pi_n)], \tag{A.6}$$

and, hence $H' = \bar{H}(Q_n, \pi_n)$.

APPENDIX B – DIRAC'S DELTA MODEL

Consider a function

$$\delta_\epsilon(x) = \left(\frac{1}{2\pi\epsilon^2}\right)^{\frac{1}{4}} \exp\left(-\frac{x^2}{4\epsilon^2}\right), \tag{B.1}$$

it's can be a model for Dirac's delta once the limit $\epsilon \longrightarrow 0$ is taken. It suffices to prove that it satisfies the defining property of Dirac's delta, consider a function $phi(x)$ and the integral

$$\int_{-\infty}^{+\infty} \delta_\epsilon^2(x-a)\phi(x)dx = \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi\epsilon^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(x-a)^2}{2\epsilon^2}\right)\phi(x)dx, \tag{B.2}$$

doing the substitution $y = \frac{x-a}{\epsilon}$ we have

$$\int_{-\infty}^{+\infty} \left(\frac{1}{2\pi\epsilon^2}\right)^{\frac{1}{2}} \exp\left(-\frac{y^2}{2}\right)\phi(\epsilon y + a)dx \overset{\epsilon \longrightarrow 0}{=} \phi(a). \tag{B.3}$$

Therefore, it serves as a good wave function for a well localized particle.