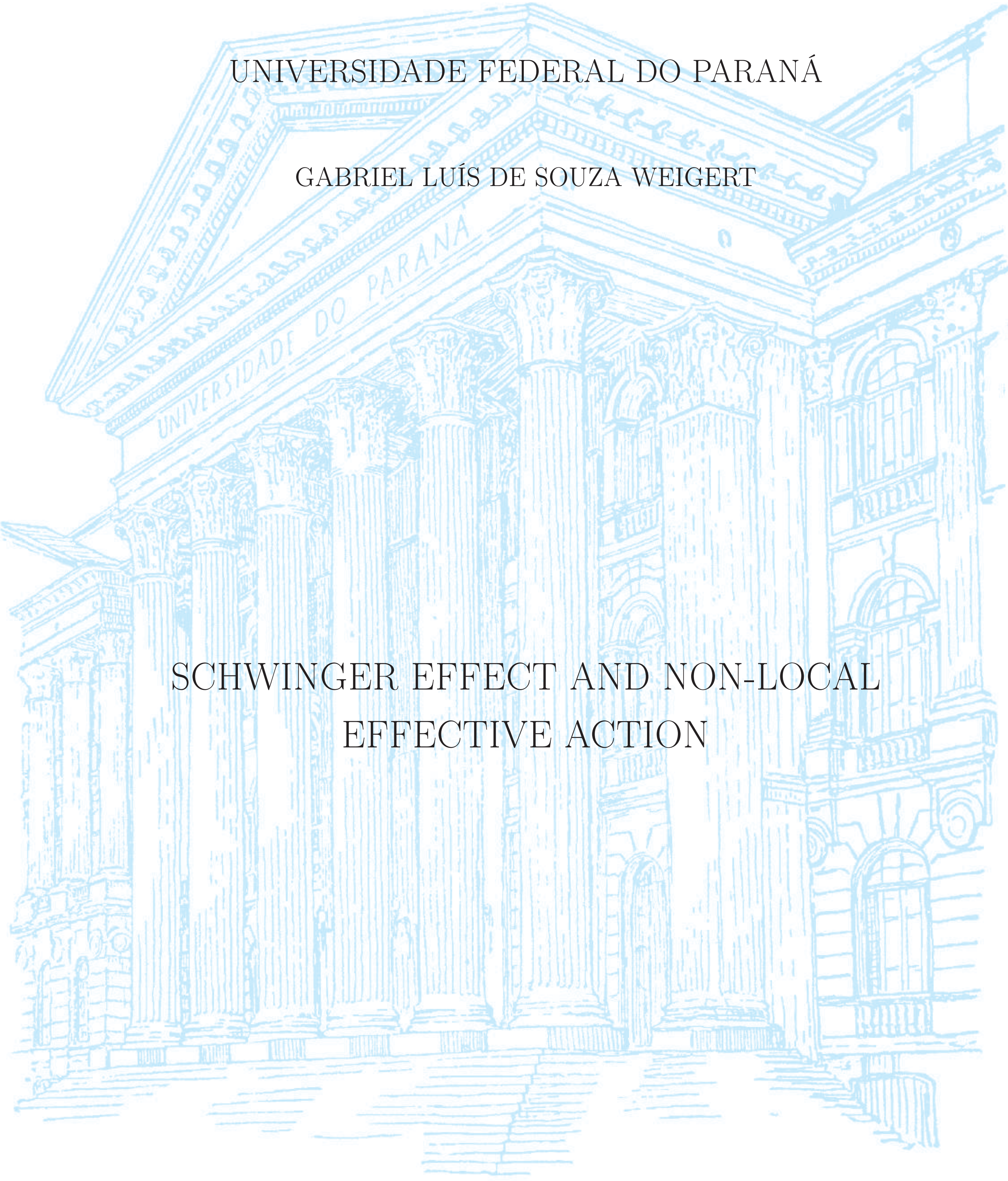

UNIVERSIDADE FEDERAL DO PARANÁ

GABRIEL LUÍS DE SOUZA WEIGERT

SCHWINGER EFFECT AND NON-LOCAL EFFECTIVE ACTION

CURITIBA, BRASIL
2024

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SCHWINGER EFFECT AND NON-LOCAL EFFECTIVE ACTION

Dissertação apresentada ao Curso de Pós-Graduação em Física do Setor de Ciências Exatas da Universidade Federal do Paraná, como requisito parcial para a obtenção do grau de Mestre em Física.

Orientador: Prof. Dr. Iberê Oliveira Kuntz de Souza

CURITIBA, BRASIL

2024

DADOS INTERNACIONAIS DE CATALOGAÇÃO NA PUBLICAÇÃO (CIP)
UNIVERSIDADE FEDERAL DO PARANÁ
SISTEMA DE BIBLIOTECAS – BIBLIOTECA DE CIÊNCIA E TECNOLOGIA

Weigert, Gabriel Luís de Souza
Schwinger effect and non-local effective action / Gabriel Luís de Souza Weigert. – Curitiba, 2024.
1 recurso on-line : PDF.

Dissertação (Mestrado) - Universidade Federal do Paraná, Setor de Ciências Exatas, Programa de Pós-Graduação em Física.

Orientador: Iberê Oliveira Kuntz de Souza

1. Eletrodinâmica quântica. 2. Vácuo. 3. Princípio de ação de Schwinger. I. Universidade Federal do Paraná. II. Programa de Pós-Graduação em Física. III. Souza, Iberê Oliveira Kuntz de. IV . Título.

Bibliotecário: Leticia Priscila Azevedo de Sousa CRB-9/2029

MINISTÉRIO DA EDUCAÇÃO
SETOR DE CIÊNCIAS EXATAS
UNIVERSIDADE FEDERAL DO PARANÁ
PRÓ-REITORIA DE PESQUISA E PÓS-GRADUAÇÃO
PROGRAMA DE PÓS-GRADUAÇÃO FÍSICA - 40001016020P4

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Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação FÍSICA da Universidade Federal do Paraná foram convocados para realizar a arguição da Dissertação de Mestrado de **GABRIEL LUÍS DE SOUZA WEIGERT** intitulada: **"Schwinger Effect and Non-Local Effetive Action"**, sob orientação do Prof. Dr. IBERE OLIVEIRA KUNTZ DE SOUZA, que após terem inquirido o aluno e realizada a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.
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AGRADECIMENTOS

Agradeço primeiramente a Deus por ter me dado vida, amor e salvação, bem como por ter me permitido ter saúde e determinação para a realização desta dissertação e mestrado.

Agradeço aos meus pais, Fabiola e Reinaldo, à minha avó Terezinha e à minha irmã Fernanda, por todo o apoio que me deram ao longo da vida e em todas as áreas, além de todo incentivo e orientação. Mesmo quando duvidei de como proceder, vocês estavam lá para me ajudar. Amo muito vocês.

Ao meu orientador, professor Iberê, agradeço por me proporcionar a oportunidade de trabalhar com a fascinante área de teorias quânticas de campo, por toda a paciência, e orientação tanto na pesquisa quanto na minha futura vida de pesquisador.

Aos meus amigos, Amanda, Felipe e Vitor. Estes últimos anos foram desafiadores. Enfrentamos muitos problemas e encontramos inúmeros obstáculos e infinitos pelo caminho, mas conseguimos superá-los (menos os infinitos...). Agradeço também pelos rodízios e experiências que passamos juntos, além das horas gastas com discussões e correções.

Agradeço aos professores Cristiano, Giovani, Carlos e Renato por aceitarem participar da avaliação deste trabalho.

Agradeço também à CAPES pelo suporte financeiro e ao Programa de Pós-Graduação em Física da Universidade Federal do Paraná.

"There are no real one-particle systems in nature, not even few-particle systems. The existence of virtual pairs and of pair fluctuations shows that the days of fixed particle numbers are over."

V. Weisskopf

RESUMO

Neste trabalho apresentamos uma análise dos efeitos de uma correção quântica em um laço originada de uma ação efetiva não-local na criação de partículas no vácuo. Este mecanismo de produção de pares, conhecido como efeito Schwinger, surge de correções de um laço, ou seja, proporcional à primeira ordem de $\hbar$, para as equações de Maxwell e caracteriza o decaimento do vácuo na presença de um campo elétrico de fundo. No presente estudo, investigamos a produção de pares incorporando uma correção não-local de um laço para campos de fundo caracterizados por uma onda plana e uma onda com corrente externa constante. Obtivemos as expressões para a probabilidade de produção de pares e taxa de decaimento do vácuo para ambos os campos de fundo. Observou-se que o vácuo quântico não apresenta sinal de instabilidade em um fundo de onda plana, resultando em uma taxa de produção de partículas nula, conforme também evidenciado no trabalho de Schwinger. Em contraste, a produção de pares ocorre para uma quadricorrente externa constante no espaço de momento que corresponde a uma quadricorrente externa pontual no espaço de posição. Notamos que os campos magnéticos variáveis no tempo, induzidos por uma densidade de corrente quando a densidade de carga tende a zero, induzem campos elétricos que, por sua vez, produzem pares através do mecanismo de Schwinger.

Palavras-chaves: Ação efetiva não-local; Efeito de Schwinger; Eletrodinâmica quântica; Produção de pares; Vácuo quântico.

ABSTRACT

In this work we present an analysis of the effects of a one-loop quantum correction originating from a non-local effective action on particle creation in the vacuum. This pair production mechanism, known as the Schwinger effect, arises from one-loop corrections, i.e. proportional to the first order of $\hbar$, to Maxwell's equations and characterises the vacuum decay in the presence of an electric field background. In the present study, we investigate pair production by incorporating a non-local one-loop correction for backgrounds characterised by a plane wave and a wave with constant external current. We obtained expressions for the probability of pair production and the vacuum decay rate for both backgrounds. It was observed that the quantum vacuum does not show any sign of instability in a plane wave background, resulting in a null particle production rate, as also evidenced in Schwinger's work. In contrast, pair production occurs for a constant external four-current in momentum space, which corresponds to a point-like four-current in position space. We noted that time-varying magnetic fields, induced by a current density when the charge density tends to zero, induce electric fields which, in turn, produce pairs via the Schwinger mechanism.

Keywords: Non-local effective action; Pair production; Quantum electrodynamics; Quantum vacuum; Schwinger effect.

Contents

1 Introduction

The $20th$ century saw the consolidation of two major physical theories: quantum mechanics (QM) and special relativity (SR). Whereas the former focuses on atomic and subatomic scales, the latter is concerned with the relation between space and time, especially under conditions approaching the speed of light.

Although each theory was successful within its domain, neither managed to effectively explain the dynamics of quantum particles in the relativistic regime. Consequently, no theory could reconcile the principles of both domains. This is because both theories exhibit inherent incompatibilities, including different interpretations of time and the asymmetry between time and space derivatives in the Schrödinger equation.

Dirac addressed the description of relativistic spin-1/2 particles as an attempt to reconcile both theories. His efforts led to the formulation of the so-called Dirac's equation and a theory partially compatible with both QM and SR [1]. However, his work also posed problems, notably the presence of negative energy eigenvalues. Consequently, this raises issues such as the absence of a ground state, implying that a positively charged electron could emit one or even an infinite number of photons and then transition into negative energy levels [2–4].

To address this issue, Dirac resorted to the Pauli exclusion principle, which prohibits two identical fermions from occupying the same quantum state simultaneously [3, 5]. Assuming that all negative energy levels would already be occupied, Pauli's principle would prevent an electron with a positive energy state from transitioning into negative energy states. However, this approach introduces other complications: representing a single relativistic particle would require an infinite number of occupied states. Moreover, this description cannot be extended to bosons as they do not obey the Pauli exclusion principle.

It is noteworthy that, despite his interpretation being incorrect, Dirac predicted the existence of the positron [1, 6], a particle with the same mass as the electron but with opposite charge. This prediction was experimentally confirmed in 1933 [7].

Despite Dirac's significant contributions to the development of a relativistic quantum theory, the presented problems were only resolved through the abandonment of the quantisation of position and momentum, and the replacement of wave functions with the so-called quantum fields, namely spacetime-dependent operators.

Similarly to quantisation in quantum mechanics, field quantisation occurs as fields, functions of spacetime, are promoted to operators. These are subsequently subjected to specific (anti)commutation relations in a process known as second quantisation or canonical (field) quantisation [2, 8–10]. However, as we shall explore in this dissertation, there exists another prevalent quantisation method known as functional quantisation, whereby fields undergo quantisation via path integration.

This quantisation process of fields resulted in a significant advancement in modern physics, leading to the theoretical framework known as quantum field theory (QFT). This theory successfully reconciled special relativity and quantum mechanics with the integration of concepts from field theory, thus solving the previously mentioned problems. It then culminated in the quantisation of light and its interaction with charged particles, leading to the emergence of the first QFT, which is known as quantum electrodynamics (QED).

To this day, QED, which extends the electromagnetic theory into the quantum domain, stands out as one of the most precise theories ever developed. Recent experimental measurements of the electron magnetic moment show an agreement within 1 part in 10^{12} with its theoretical prediction [11].

The success of QED heavily relies on approximation methods, such as perturbation theory. In this approach, interactions between fields are treated as small perturbations. This consequently simplifies the obtainment of observables of physical interest [4]. However, as we increasingly depend on this method, certain aspects of nature may remain unexplored. Thus, it is essential to develop methods to investigate the non-perturbative regime as well.

This is exemplified in the works of Euler and Heisenberg [12] and Weisskopf [13], which represent a significant advancement in the non-perturbative regime of QED. In their works, it is employed what is known as an effective Lagrangian, a mathematical object that encodes quantum corrections to the classical Lagrangian. Equipped with it, they established that the quantum vacuum is permeated with quantum fluctuations arising from the creation and annihilation of virtual[1] electron-positron pairs. These lead to nonlinear dynamics of the electromagnetic fields in the vacuum, thereby revealing previously unexplored properties [14], namely light-by-light scattering [15, 16], vacuum birefringence [17, 18], and the phenomenon of interest in this dissertation: pair production [19–23].

Pair production in non-perturbative QED is related to the instability of the vacuum and its potential of decaying in the presence of a intense external electric field[2], resulting in the creation of real electron-positron pairs (see Figure 1). Although it had been recognised by Euler and Heisenberg in their work [24], it was later reformulated within the framework of QED by Schwinger in his paper on gauge invariance and vacuum polarisation [25]. Thus,

[1] Here, "virtual" refers to particles that does not obey the dispersion relation and exist for only a short time as dictated by the uncertainty principle $\Delta E \Delta t = \hbar$, where E is the particle's energy and t is time.

[2] Note that this effect can occur with an external electric field that is not purely electric, specifically in the presence of an adjacent external external magnetic field [24].

this effect is now referred to as the Schwinger effect.

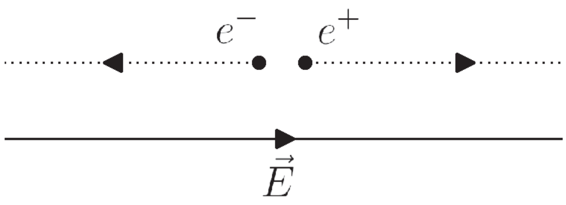


Figure 1 – Representation of the Schwinger effect, wherein electron-positron pairs emerge from a vacuum in response to an intense electric field. These particles, created in pairs, are then accelerated and separated by the electric field's influence. Note that the direction in which these charges are created does not necessarily match this representation. However, due to their charges, the particles tend to align or show a tendency to align with the external field.

Concerning experimental aspects, the Schwinger effect is manifested when the electric field strength reaches the theoretical critical value $E_c \sim 1.3 \times 10^{18}\, V/m$. Therefore, observing this effect remains difficult, posing challenges for investigating vacuum non-linearities (see the reviews in Refs.[21, 26]). Despite this obstacle, several experimental[3] proposals [28–30] have been suggested, and numerous current and forthcoming facilities are focused on studying this mechanism.

Regarding these facilities, we mention the LUXE (Laser Und XFEL Experiment) [31, 32], the Extreme Light Infrastructure (ELI) [33], and the Rutherford Appleton Laboratory [34], all located in Europe. Moving to the US, there are the Advanced Accelerator Experimental Tests II (FACET-II) [35], and the Zettawatt-Equivalent Ultrashort pulse laser System (ZEUS) at the University of Michigan [36]. Additionally, we find the Shanghai Coherent Light Facility (SCLF) [37] in China.

As the Schwinger effect incorporates a wide range of theoretical tools, it can serve as a theoretical laboratory to investigate not only the quantum vacuum in QED but also other phenomena. For instance, it has application in studies on the uncertainty principle [38], entanglement [39, 40], string theory [41], inflation [42–45], dark matter [46, 47], and gravitational pair production and black holes [48–51].

As noted, the Schwinger effect emerges from an effective action (or effective Lagrangian). In regimes where the field is weak but its derivatives are arbitrary, the effective action becomes non-local. These actions differ from the local ones by featuring an infinite number of derivatives or functions dependent on derivatives.

Non-local effective actions can be utilised to describe phenomena both within and beyond the Standard Model, encompassing applications in quantum gravity and cosmology [52–55], string theory [56, 57], conformal anomaly [58–60], quantum chromodynamics [61,

[3] An experimental layout can be found in [27].

62], electroweak interaction and Higgs mechanism [63, 64], and QED [65–67]. Therefore, this study aims to analyse the effects of quantum corrections originating from a non-local effective action within the context of an effective field theory. Specifically, we shall investigate these effects on the pair production mechanism under various electromagnetic field configurations.

In this dissertation, we adopt the notation of Peskin and Schroeder [8], employing natural units $\hbar = c = 1$ and Heaviside-Lorentz conventions. The latter means that the vacuum permittivity ε_0 and vacuum magnetic permeability μ_0 are set to 1. The former means that the units of measurement are such that [length] = [time] = [energy]$^{-1}$ = [mass]$^{-1}$. Furthermore, we work with the Minkowski metric with the convention $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. Here, Greek indices range from $\mu = 0, \ldots, 3$, and repeated indices imply summation.

This dissertation is divided into six chapters and can be delineated into two main parts: a literature review and the research that we conducted.

The first part comprises several chapters. In Chapter 2, we provide a general introduction to field quantisation through the functional formalism, i.e. via path integration. In Chapter 3, we apply the concepts established in the previous chapter to quantise gauge and spinor fields and then present the quantum theory of electromagnetism. Additionally, Chapter 4 introduces Schwinger's proper-time formulation and discusses the Schwinger effect.

The final part of this dissertation includes Chapter 5, which discusses the effective action containing a non-local quantum correction term. Here, we investigate the Schwinger effect incorporating this correction term for backgrounds characterised by a plane wave and a wave with a constant external current, presenting our results and analysis on the particle production rate and probability for both backgrounds. After that, we present the final considerations in Chapter 6, also offering directions for future research.

2 Functional Quantisation

As previously mentioned, field quantisation can be accomplished through the canonical approach, which holds historical significance as the first method for field quantisation. Moreover, this approach is commonly employed in quantum field theory courses. However, instead of utilising this formalism, this dissertation adopts the so-called functional quantisation, in which Feynman's path integrals are introduced. Rather than promoting fields to operators and subjecting them to (anti)commutation rules, quantisation is achieved by solving path integrals.

Following this quantisation procedure, we introduce the notion of effective action. This action is a modification of the classical action, encapsulating all the quantum properties of the system. Consequently, while the classical action yields classical equations of motion for a system, the effective action provides the equations of motion that describe the quantum dynamics.

In Section 2.1, we shall qualitatively derive the path integral for quantum mechanics, using the double-slit experiment as a motivating example. Following that, in Section 2.2, the same path integral will be formally derived for field theories, and a functional called the generating functional, which provides all the physical information about the theory, will be introduced. Afterwards, in Section 2.3, we will apply the concepts introduced in the preceding sections to quantise scalar fields as an example, also serving as a bridge to the subsequent sections. In Section 2.4, we shall present the so-called generating functional of connected diagrams. Finally, the effective action will be discussed in Section 2.5.

2.1 Path Integral in Quantum Mechanics

Consider a non-relativistic particle moving in one dimension, characterised by an initial position state $|x_i\rangle$ at an initial time t_i. The probability amplitude of finding it at a final position state $|x_f\rangle$, corresponding to a later time t_f, is expressed in the Schrödinger picture[1] as

$$U(x_f, t_f; x_i, t_i) = \langle x_f| e^{-i\frac{\hat{H}}{\hbar}(t_f - t_i)} |x_i\rangle_S \,, \tag{2.1}$$

where $\hat{H}$ is the Hamiltonian operator. Note that we shall use $\hbar \neq 1$ for now. In this approach, the expression between the states is the time evolution operator, which is responsible for evolving the initial state to the final state over the time interval $t_f - t_i$ [3].

An alternative method for computing transition amplitudes is conceived through the functional formalism. In this framework, these amplitudes are obtained through integration

[1] Note that this probability amplitude can be represented with other pictures.

over all possible amplitudes, also called paths, that a particle can present from its initial state $|x_i\rangle$ to its final state $|x_f\rangle$ using the so-called Feynman path integral [68].

We emphasise that these paths are not observable trajectories, but rather transition amplitudes. As we will see in the following construction, when considering a particle transitioning from an initial to a final state, the idea is that to calculate this transition fully, we need to consider all possible transitions that such a described particle might undergo.

To illustrate this formalism and provide a qualitative derivation of the path integral, let us use the double-slit experiment[2]. It is important to note that this derivation serves as a didactic way to introduce the concept of Feynman's path integral.

For the double-slit experiment, consider a single electron being emitted from a source S and directed towards a screen with two parallel slits, namely A_1 and A_2. After this electron passes through one of these slits, it is detected at the screen D, as depicted in Figure 2. The fundamental idea of the path integral lies in the fact that, according to

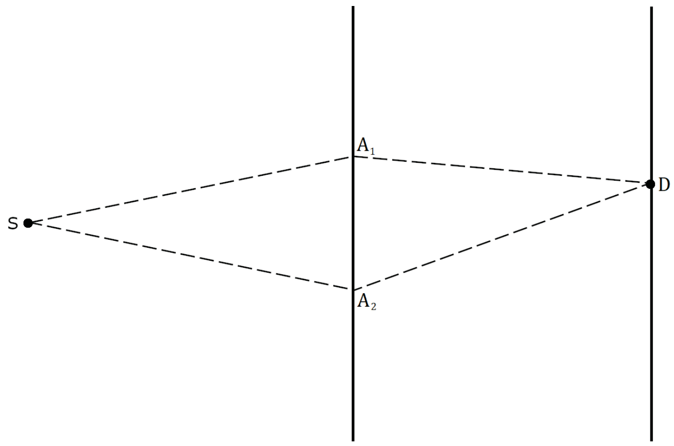


Figure 2 – Diagram of the double-slit experiment, where S corresponds to the electron source, A_1 and A_2 represent the parallel slits, D denotes the detector, and the dotted straight lines represent the possible trajectories or paths that the electron can go through. Note that these paths need not to be as shown here, as particles can follow any path from S to any of the slits and from them to D.

the superposition principle, the total amplitude for this electron to reach the detector D from the source S is the sum of two amplitudes: one for the electron passing through slit A_1 and then being detected at D, and the other for the electron passing through A_2 and then being detected at D (see Figure 2). Note that we must consider both amplitudes, as we do not know which path the particle will choose or how that path will unfold.

Qualitatively, the total transition amplitude can be expressed as

$$U_T = U_1(S \to A_1 \to D) + U_2(S \to A_2 \to D) = \sum_i U_i(S \to A_i \to D), \tag{2.2}$$

[2] Note that the derivation of the path integral is for generalised coordinates. However, we choose position as our generalised coordinates.

where U_i represents each of the amplitudes, and A_i represents each of the slits. Observe that U_T is a didactic representation or simplification of the total transition amplitude $U(x_f, t_f; x_i, t_i)$ in (2.1) and serves as a tool to understand the concept of path integral.

The need to sum all amplitudes can be understood with a basic notion of probability. The total probability of the particle leaving S and being detected at D is the sum of the probabilities of the particle leaving S, passing though A_1 and then being detected at D, and the probability of it leaving S, passing through A_2, and subsequently being detected at D.

As we introduce more screens with more slits, it becomes necessary to include the new amplitudes for the electron's additional paths between the screens. For instance, when we insert a screen with slits B_j between the screen with slits A_i and the detector, we must consider the amplitude of the electron passing through a slit A_i, then through a slit B_j, and finally reaching the detector (see Figure 3).

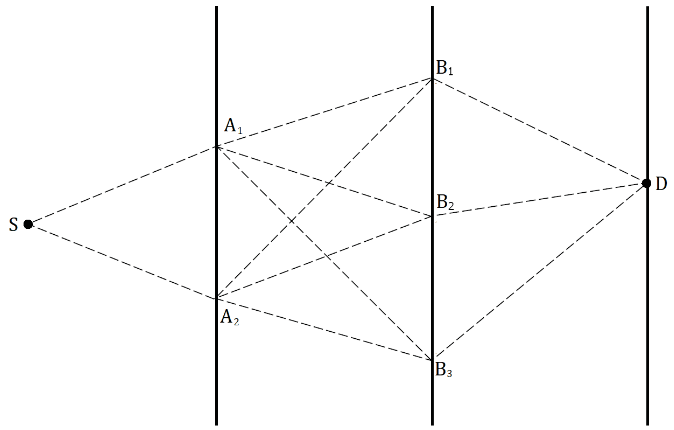


Figure 3 – Diagram of the double-slit experiment with the additional parallel slits B_1, B_2 and B_3. Observe that now we need to consider the additional paths that the electron can take from A_i to B_j and then to D.

Consequently, the total amplitude is given by

$$U_T = \sum_{i,j} U_{ij}(S \to A_i \to B_j \to D). \tag{2.3}$$

With the previous example in mind, we consider a more general system comprising an infinite number of screens with infinite slits between the source S and the detector D. In this configuration, the space between S and D can be regarded simply as empty space[3]. Therefore, to compute the total amplitude for a particle to travel between the initial and final states, corresponding to equation (2.1), we must consider all amplitudes or paths from the source to the detector [8–10], as shown in Figure 4. Mathematically, this can be

[3] Note that we could have considered the space between the source and the detector as empty even without using the idea of the double-slit experiment.

then expressed as

$$U(x_f, t_f; x_i, t_i) = \sum_{\text{all paths}} e^{\frac{i}{b}\alpha} = \int Dx\, e^{\frac{i}{b}\alpha[x]}. \tag{2.4}$$

Here, the phase $\alpha[x]$ differs for the different paths the particle can take, b is a constant, and Dx is the functional measure, with its integration meaning the integration over all possible continuous spatial paths or functions $x(t)$.

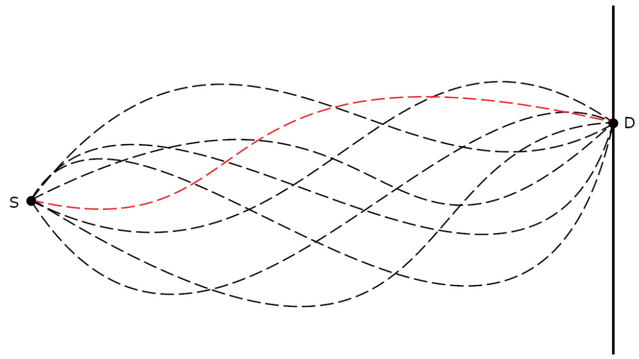


Figure 4 – As we consider a free particle propagating from S to D, we must account for the infinite amplitudes or path that it can take. Here, we represent some of these paths, illustrating the variety of trajectories, with the classical path shown as the red dotted line.

Having found an expression for the total amplitude, the challenge ahead lies in defining the phase to be integrated. To address this, we rely on two assumptions. Firstly, in the classical limit, the classical path x_c is the only path contributing to the total amplitude, according to the principle of stationary action. Secondly, considering the form of the integral (2.4), written as $I = \int dq\, e^{\frac{i}{b}f(q)}$, where b is a constant and $f(q)$ is a function dependent on the variable q, we can expect to employ the method of stationary phase [4, 8–10]. This method deals with integrals involving rapidly varying phases, a characteristic that becomes apparent in (2.4) as b approaches zero.

Given these assumptions, we set that $b = \hbar$ and that we are approaching the classical limit as $\hbar \to 0$. This decision of setting the constant as such is motivated by the behaviour of the exponential within the integrand of the path integral. As we take the limit to the classical regime, the rapid oscillations in the phase lead to cancellations that significantly impact the total amplitude [4, 8, 9, 69, 70]. Nevertheless, we note that the only path unaffected by these cancellations is the classical path since it remains stationary.

With these conditions in place, we apply the method of stationary phase. This method entails identifying the critical points of $f(q)$ [71], which, in our scenario, correspond to determining the critical points of the phase $\alpha[x]$ that coincide with the classical path x_c. In other words, we need to determine the functional that satisfies the following stationary condition:

$$\frac{\delta}{\delta x(t)} \left(\alpha[x]\right)\Big|_{x=x_c} = 0, \tag{2.5}$$

which is a functional derivative of the phase with respect to $x(t)$. Due to Hamilton's principle, this stationary condition holds only when we equate $\alpha[x(t)]$ to the classical action $S[x(t)]$. This functional is defined as

$$S[x] = \int_{t_i}^{t_f} dt\, L(\dot{x}, x),$$

where the Lagrangian $L(\dot{x}, x)$ for a given system is integrated from an initial time t_i to a final time t_f, and $\dot{x}$ represents the time derivative of the position x.

Now that we have established the phase as the classical action, the transition amplitude (2.1) in terms of a path integral reads

$$U(x_f, t_f; x_i, t_i) = \langle x_f|e^{-i\frac{\hat{H}}{\hbar}(t_f-t_i)}|x_i\rangle_S = \mathcal{C}\int_{x(t_i)=x_i}^{x(t_f)=x_f} Dx(t)\, e^{\frac{i}{\hbar}S[x]}. \tag{2.6}$$

In this equation, $\mathcal{C}$ is a constant, $Dx(t)$ is the functional measure, an object that accounts for all the possible paths in the configuration space that a particle can take, or all the possible amplitudes, connecting the initial and final points x_i and x_f, respectively. We stress that the functional and canonical quantisation formalism in QM are equivalent. This means that the solution of this path integral corresponds to the transition amplitude of a particle, as obtained in the operator-based approach[4].

2.1.1 Gaussian integrals

In the next section, as we solve path integrals, we shall encounter integrands that appear in the form of Gaussian functions. This indicates that we shall be dealing with classical actions that are quadratic in the fields. Given this characteristic, we shall address these Gaussian integrals here.

Consider a one-dimensional integral given by

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2+jx}, \tag{2.7}$$

where a and j are constants with respect to the integration variable[5] x. To solve this integral, we complete the square in the exponential term, obtaining

$$I = \int dx\, e^{-\frac{1}{2}a\left(x-\frac{j}{a}\right)^2+\frac{1}{2a}j^2}. \tag{2.8}$$

Note the decoupling of the variable x and the constant j. Moving forward, as we perform a change of variables $x \to x + \frac{j}{a}$, the integral changes to

$$I = e^{\frac{1}{2}\frac{j^2}{a}}\int dx\, e^{-\frac{1}{2}ax^2} = e^{\frac{1}{2}\frac{j^2}{a}} I_g. \tag{2.9}$$

4 A proof of the equivalence of equation (2.6) with Schrödinger equation can be found in [3, 68].

5 As we will encounter multiple integrals in this and subsequent sections and chapters, we will specify the integration limits only once.

The next step is to solve I_g.

To achieve this, we start by squaring the integral and introducing a dummy integration variable y for one of the integrals. This yields

$$I_g^2 = \left(\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2}\right)^2 = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}ay^2}. \tag{2.10}$$

As the integration of one variable does not affect the other, we can rewrite it as

$$I_g^2 = \int dx \int dy\, e^{-\frac{1}{2}a(x^2+y^2)} = 2\pi \int_0^{\infty} dr\, r\, e^{-\frac{1}{2}ar^2}. \tag{2.11}$$

Here, we transitioned to polar coordinates in the second equality, where 2π arises from the integration over a solid angle. This integral is solved by employing a change of variables $u = ar^2/2$, resulting in $I_g = \sqrt{2\pi/a}$. Therefore, (2.9) is solved, yielding

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2+jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{1}{2}\frac{j^2}{a}}. \tag{2.12}$$

Now consider an integral of N dimensions. In this generalisation, the constant a is promoted to a N by N matrix A_{ij}, namely $\mathbf{A}$, j becomes the vector $\mathbf{J}$, and x a vector x_i, also expressed as $\mathbf{x}$, where $i, j = 1, \cdots, N$. Therefore, the integral is now given as

$$I = \int_{-\infty}^{\infty} d\mathbf{x}\, e^{-\frac{1}{2}\mathbf{x}^\dagger \mathbf{A}\mathbf{x} + \mathbf{J}^\dagger \mathbf{x}}, \tag{2.13}$$

where $\mathbf{x}^\dagger \mathbf{A}\mathbf{x} = x_i A^{ij} x_j$, $\mathbf{J}^\dagger \mathbf{x} = J_i x^i$, and $d\mathbf{x} = \prod dx_i$ indicating N integrals.

To solve this integral, the approach is similar to the one we used previously for the one dimensional case. First, we need to decouple $\mathbf{J}^\dagger$ from $\mathbf{x}$. To do this, we perform a change of variables $\mathbf{x} \to \mathbf{x} + \mathbf{A}^{-1}\mathbf{J}$ [72], so that the integral (2.13) can be rewritten as

$$I = \int d\mathbf{x}\, e^{-\frac{1}{2}\mathbf{x}^\dagger \mathbf{A}\mathbf{x} + \mathbf{J}^\dagger \mathbf{x}} = e^{\frac{1}{2}\mathbf{J}^\dagger \mathbf{A}^{-1}\mathbf{J}} \int d\mathbf{x}\, e^{-\frac{1}{2}\mathbf{x}^\dagger \mathbf{A}\mathbf{x}}, \tag{2.14}$$

where $\mathbf{A}^{-1}$ corresponds to the inverse of the matrix $\mathbf{A}$. Diagonalising the matrix $\mathbf{A}$ simplifies the integral (2.14) into a product of one-dimensional integrals over the variables x_i [9, 10, 72]. The result of these integrals is expressed in (2.12). Therefore, we have

$$\int d\mathbf{x}\, e^{-\frac{1}{2}\mathbf{x}^\dagger \mathbf{A}\mathbf{x} + \mathbf{J}^\dagger \mathbf{x}} = \left(\frac{(2\pi)^N}{\det\mathbf{A}}\right)^{\frac{1}{2}} e^{\frac{1}{2}\mathbf{J}^\dagger \mathbf{A}^{-1}\mathbf{J}}. \tag{2.15}$$

Here, the determinant of $\mathbf{A}$ arises from the product of the eigenvalues a of the matrix $\mathbf{A}$ when solving N one-dimensional integrals [9, 72]. Additionally, we define a constant $\mathcal{C} = \sqrt{\frac{(2\pi)^n}{\det\mathbf{A}}}$, yielding

$$\int d\mathbf{x}\, e^{-\frac{1}{2}\mathbf{x}^\dagger \mathbf{A}\mathbf{x} + \mathbf{J}^\dagger \mathbf{x}} = \mathcal{C}\, e^{\frac{1}{2}\mathbf{J}^\dagger \mathbf{A}^{-1}\mathbf{J}}. \tag{2.16}$$

As we aim to solve path integrals, such as (2.6), it becomes necessary to generalise the result in (2.15). For this purpose, consider a time-dependent function $x(t)$ and a differential operator $\hat{A}$. Assuming that the classical action in (2.6) takes the quadratic form $S = \int_{-\infty}^{\infty} dt\, x(t)\hat{A}\, x(t)$, we arrive at the following path integral:

$$\int_{x_f}^{x_i} Dx(t)\; e^{\left[\frac{i}{2}\int dt\,dt' x(t)\hat{A}\,x(t') + i\int dt\; J(t)\,x(t)\right]} = \left(\frac{(2i\pi)^N}{\det\hat{A}}\right)^{\frac{1}{2}} e^{\left[-\frac{i}{2}\int dt\,dt' J(t)G(t-t')J(t')\right]}. \qquad (2.17)$$

Here, $\hat{A}\,G(t-t') = \delta(t-t')$, where $G(t-t')$ is a Green's function referred to as the propagator, being the inverse of the differential operator $\hat{A}$, and $\delta(t-t')$ represents a Dirac delta function. Note that the factor $(2i\pi)^N$ is divergent since we are integrating over an infinite number of degrees of freedom. However, we need not concern ourselves with this factor, as it will be cancelled out when computing n-point functions in Subsection 2.2.1.

2.2 Path Integral in Quantum Field Theory

Having qualitatively derived the path integral for quantum mechanics early on, we shall proceed by deriving it for fields. In essence, a field is a function that assigns spacetime points to amplitudes. For example, a scalar field assigns a single scalar value to each point in spacetime, while a vector field assigns a vector to each point in spacetime.

Mathematically, fields are characterised by their behaviour under Lorentz transformations. In addition, the action that describe the dynamics of these fields are built having in mind the invariance under such transformations, thereby consistent with special relativity. We can represent a generic field as

$$\phi(t, \mathbf{x}) = \phi(x), \qquad (2.18)$$

where we introduce the four-vector $x^\mu = (x^0, \mathbf{x})$, with $x^0 = t$ indicating the time component and $\mathbf{x} = (x^1, x^2, x^3)$ denoting the three-dimensional position vector. Note that by employing this four-vector as the function argument, we are suppressing the spacetime index μ for the sake of notation. It is noteworthy that fields may also have indices associated with them. For example, vector fields $A^\mu(x)$ possess a spacetime index μ while a spinor field $\psi^a(x)$ includes a spinorial index a, and so forth[6].

Before exploring the path integral for fields and consequently their quantisation, it is necessary to define the so-called vacuum state $|0\rangle$. This state corresponds to a state with no real particles and serves as the ground state of the quantum field theory. Additionally, considering a Fock space and the creation and destruction operators $\hat{a}^\dagger_{k_i}$ and $\hat{a}_{k_i}$, respectively, the vacuum is defined as the state that is annihilated by all destruction operators, and, by definition, has zero energy [2, 4]. This means that

$$\prod_i^n \hat{a}_{k_i}\,|0\rangle = 0. \qquad (2.19)$$

[6] See Section 3.1.1 and 3.1.2 for the definition of vector and spinor fields, respectively.

Furthermore, by applying a product of creation operators to the vacuum state, we obtain a multi-particle state, expressed as

$$|k_1, \cdots, k_n\rangle = \mathcal{M}\, \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_n} |0\rangle\,, \tag{2.20}$$

where each particle is represented by its momentum k_i, with $i = 1, \cdots, n$. In (2.20), $\mathcal{M}$ corresponds to a normalisation constant.

In QFT, our focus lies on the transition between vacuum states $|0\rangle$, representing a transition between ground states. These are crucial as they allow us to derive observables in QFT [2, 8, 10]. An important characteristic of such a transition is that the vacuum states here considered are distinct as they are evaluated at different times. Thus, consider an initial vacuum state $|0; t_f\rangle$, at time t_i and a final vacuum state $|0; t_f\rangle$ at time t_f, their transition amplitude is written as

$$\left\langle \begin{array}{c} \text{no particles} \\ \text{at } t_f \end{array} \middle| \begin{array}{c} \text{no particles} \\ \text{at } t_i \end{array} \right\rangle = \langle 0; t_f | 0; t_i\rangle_H = \langle 0| e^{-i\hat{H}(t_f - t_i)} |0\rangle_S\,, \tag{2.21}$$

Observe that in the first equality, the states are in the Heisenberg picture. In contrast, we are using the Schrödinger picture in the last equality, and the states there, although written similarly, represent different states. Additionally, it is important to mention that from now on, $\hbar = 1$ will be utilised unless stated otherwise.

To express this transition amplitude in terms of a path integral, similar to what we did for quantum mechanics, we first need to define the form of the Hamiltonian $\hat{H}$. For this derivation, we will focus on a real scalar field, a function that assigns a real number to each point in spacetime. The reason behind this choice lies in the simplicity of this field, as it allows us to avoid dealing with complexities related to gauge invariance, which we will encounter in Chapter 3 when quantising gauge fields.

Hence, the Hamiltonian that describes a real scalar field is given as

$$\begin{aligned} \hat{H}(t) &= \int_{-\infty}^{\infty} d^3x \left[\frac{1}{2}\hat{\pi}^2(x) + \frac{1}{2}(\nabla\hat{\phi}(x))^2 + \frac{1}{2}m^2\hat{\phi}^2(x) + \cdots \right] \\ &= \int d^3x \left[\frac{1}{2}\hat{\pi}^2 + V[\hat{\phi}] \right], \end{aligned} \tag{2.22}$$

where $\hat{\pi}(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \hat{\phi})}$ is the momentum density conjugate to the real scalar field $\hat{\phi}(x)$, similar to the conjugated momentum for the canonical coordinates q, and m is the mass of the field [8, 10]. Moreover, in the first line, the dots denote possible interaction terms, while in the second line, we have grouped the terms dependent on spatial derivatives, the mass term, and the interactions into a general potential defined as

$$V[\hat{\phi}] \equiv \frac{1}{2}(\nabla\hat{\phi}(x))^2 + \frac{1}{2}m^2\hat{\phi}^2(x) + \cdots\,. \tag{2.23}$$

The derivation of the path integral involves discretising the time interval over which the transition (2.21) occurs. This means dividing the interval between the initial and final times into n equal segments of infinitesimal length δt, such that $t_f - t_i = n\,\delta t$. Subsequently, we take the continuum limit $\delta t \to 0$ and $n \to \infty$ to obtain the continuous time interval.

Furthermore, this discretisation modifies how we use the field operators $\hat{\phi}(x)$ and $\hat{\pi}(x)$. This means that as time is divided into several segments, such operators will present intermediate configurations or eigenstates $|\phi_j(\mathbf{x})\rangle$ and $|\pi_j(\mathbf{x})\rangle$, with respective intermediate eigenvalues $\phi_j(\mathbf{x})$ and $\pi_j(\mathbf{x})$. Here, the subscript j represents an intermediate time t_j [68]. Given this discretisation, we can directly solve the matrix element (2.21) using the Hamiltonian (2.22) at an intermediate time t_j [10]. Without such, we would face difficulties due to the time dependence of the Hamiltonian operator.

Before proceeding, it is important to note that the complete set of $\hat{\phi}$ obeys the following eigenvalue equations[7]:

$$\hat{\phi}_j(\mathbf{x})\,|\phi_j\rangle = \phi_j(\mathbf{x})\,|\phi_j\rangle\,, \tag{2.24}$$

which is equivalent to the position operator $\hat{x}$ and its eigenstates $|x\rangle$ [10]. Likewise, the operator $\hat{\pi}$, analogous to the momentum operator $\hat{p}$, fulfils

$$\hat{\pi}_j(\mathbf{x})\,|\pi_j\rangle = \pi_j(\mathbf{x})\,|\pi_j\rangle\,. \tag{2.25}$$

For clarity in notation, we have opted not to explicitly denote the position dependence on the states.

To discretise the transition amplitude (2.21), we introduce intermediate states of $\hat{\phi}(x)$ at each intermediate time step, allowing us to write

$$\langle 0; t_f | 0; t_i\rangle = \int_{\phi_1(\mathbf{x})}^{\phi_n(\mathbf{x})} D\phi_1(\mathbf{x})\cdots D\phi_n(\mathbf{x})\,\langle 0|e^{-i\delta t\hat{H}(t_n)}|\phi_n\rangle\,\langle\phi_n|\cdots|\phi_1\rangle\,\langle\phi_1|e^{-i\delta t\hat{H}(t_i)}|0\rangle\,. \tag{2.26}$$

Here, $\hat{H}(t_n)$ and $\hat{H}(t_i)$ correspond to the Hamiltonian at the final and initial times, respectively. In this discretisation, each insertion of an intermediate field configuration corresponds to using the completeness relation of the states $|\phi_j\rangle$:

$$\int D\phi_j(\mathbf{x})\,|\phi_j\rangle\,\langle\phi_j| = \mathbb{1}_{\phi_j}. \tag{2.27}$$

We then observe that given the discretisation, each "piece" or infinitesimal path is equivalent to a matrix element separated by an infinitesimal time step δt.

Since we need to compute each piece, our first task is to find a way to express it. This can be achieved by utilising the completeness relation of the states $|\pi_j\rangle$:

$$\int D\pi_j(\mathbf{x})\,|\pi_j\rangle\,\langle\pi_j| = \mathbb{1}_{\pi_j}. \tag{2.28}$$

[7] It is important to mention that these eigenstates can be constructed using destruction operators a_{k_i}, such that $a_{k_i}|\phi\rangle = \phi_{k_i}|\phi\rangle$. A detailed discussion on this can be found in [72].

Hence, a matrix element can be expressed as

$$\begin{aligned}\langle\phi_{j+1}|e^{-i\delta t\hat{H}(t_j)}|\phi_j\rangle &= \langle\phi_{j+1}|\mathbb{1}_{\pi_j}e^{-i\delta t\hat{H}(t_j)}|\phi_j\rangle \\ &= \int D\pi_j(\mathbf{x})\,\langle\phi_{j+1}|\pi_j\rangle\,\langle\pi_j|e^{\left[-i\delta t\int d^3x\,\left(\frac{1}{2}\hat{\pi}^2+V[\hat{\phi}]\right)\right]}|\phi_j\rangle\,.\end{aligned} \tag{2.29}$$

Here, we substituted the Hamiltonian (2.22).

To compute the action of the Hamiltonian in the states $|\phi_j\rangle$ and $\langle\pi_j|$, we make use of the of the Campbell-Baker-Hausdorff formula [2, 70, 73], which reads

$$e^{\delta t(\hat{A}+\hat{B})} = e^{\delta t\hat{A}}e^{\delta t\hat{B}}e^{-\frac{1}{2}(\delta t)^2[\hat{A},\hat{B}]+\cdots},$$

with the operators $\hat{A}$ and $\hat{B}$. It is important to mention that since δt is infinitesimal, terms that depend on $(\delta t)^2$ and higher-order ones are ignored, as the exponentials go to 1. Also, by using the equation (2.25), the matrix element (2.29) is then written as

$$\langle\phi_{j+1}|e^{-i\delta t\hat{H}(t_j)}|\phi_j\rangle = \int D\pi_j(\mathbf{x})\,\langle\phi_{j+1}|\pi_j\rangle\,\langle\phi_j|\pi_j\rangle\,e^{\left[-i\delta t\int d^3x\,\left(\frac{1}{2}\pi_j^2(\mathbf{x})+V[\phi_j]\right)\right]}, \tag{2.30}$$

where we used a consequence of the eigenvalue equation (2.24):

$$V[\hat{\phi}_j(\mathbf{x})]\,|\phi_j\rangle = V[\phi_j(\mathbf{x})]\,|\phi_j\rangle\,. \tag{2.31}$$

Continuing, we need to compute $\langle\phi_{j+1}|\pi_j\rangle$ and $\langle\phi_j|\pi_j\rangle$. To achieve this, we utilise the generalisation of the relation $\langle p|x\rangle = e^{-ipx}$ from quantum mechanics to fields [10]:

$$\langle\pi|\phi\rangle \equiv \exp\left(-i\int_{-\infty}^{\infty} d^3x\;\pi(\mathbf{x})\phi(\mathbf{x})\right). \tag{2.32}$$

Therefore, the matrix element (2.30) is finally expressed as

$$\begin{aligned}\langle\phi_{j+1}|e^{-i\delta t\hat{H}(t_j)}|\phi_j\rangle =& e^{\left(-i\delta t\int d^3x\,V[\phi_j]\right)} \\ &\times\int D\pi_j(\mathbf{x})\,e^{\left\{i\int d^3x\,\left[\pi_j(\mathbf{x})(\phi_{j+1}(\mathbf{x})-\phi_j(\mathbf{x}))-\frac{1}{2}\delta t\pi_j^2(\mathbf{x})\right]\right\}}.\end{aligned} \tag{2.33}$$

Observe that we factored out the exponential of the potential functional. As this integral is Gaussian, it can be solved using the result (2.16).

By recognising that $A = i\delta t$ and $J = i(\phi_{j+1}(\mathbf{x}) - \phi_j(\mathbf{x}))$, we obtain, through the solution of the integral, the matrix element

$$\langle\phi_{j+1}|e^{-i\delta tH(t_j)}|\phi_j\rangle = e^{\left(-i\delta t\int d^3x\;V(\phi_j)\right)}\mathcal{C}_j\,e^{\left[-\int d^3x\;\frac{1}{2}(\phi_{j+1}(\mathbf{x})-\phi_j(\mathbf{x}))^2(i\delta t)^{-1}\right]}, \tag{2.34}$$

where $\mathcal{C}_j$ is a factor that depends on δt.

We can proceed by grouping the exponentials and factoring out $i\delta t$, resulting in

$$\langle\phi_{j+1}|e^{-i\delta tH(t_j)}|\phi_j\rangle = \mathcal{C}_j\;\exp\left\{i\delta t\int d^3x\;\left[\frac{1}{2}\left(\frac{\phi_{j+1}(\mathbf{x})-\phi_j(\mathbf{x})}{\delta t}\right)^2 - V(\phi_j)\right]\right\}. \tag{2.35}$$

It is interesting to observe that the term being integrated in the exponential is recognised to be the Lagrangian density in an intermediate time t_j,

$$\mathcal{L}[\phi_j, \partial_t \phi_j] = \frac{1}{2}\left(\frac{\phi_{j+1}(\mathbf{x}) - \phi_j(\mathbf{x})}{\delta t}\right)^2 - V(\phi_j). \tag{2.36}$$

After deriving the matrix element (2.35), the other pieces can be computed using the same method. Substituting these derived elements back into the discretised amplitude (2.26) gives us

$$\langle 0; t_f | 0; t_i \rangle = \lim_{\substack{\delta t \to 0 \\ n \to \infty}} \int D\phi_1(\mathbf{x}) \cdots D\phi_n(\mathbf{x}) \prod_{j=1}^{n} \mathcal{C}_j \, e^{i\delta t \int d^3x \, \mathcal{L}[\phi_j, \partial_t \phi_j]}. \tag{2.37}$$

Moving towards the continuum limit, we define the functional measure $D\phi(x)$, similar to the quantum mechanical case. This measure encapsulates all possible paths a field can take in the field configuration space, and it is defined as

$$D\phi(x) = \prod_{i,x} D\phi_i(x). \tag{2.38}$$

Therefore, the continuum limit of equation (2.37) results in the path integral

$$\langle 0; t_f | 0; t_i \rangle = \mathcal{C} \int_{\phi(\mathbf{x}, t_i) = \phi_i(\mathbf{x})}^{\phi(\mathbf{x}, t_f) = \phi_f(\mathbf{x})} D\phi(x) \, e^{iS[\phi]}, \tag{2.39}$$

where $\mathcal{C}$ is a divergent constant, as we took the continuum limit of $\mathcal{C}_j$, that can be absorbed by the functional measure $D\phi(x)$. In this equation, $\phi_i(\mathbf{x})$ and $\phi_f(\mathbf{x})$ denote the initial and final field configurations, respectively, and $S[\phi]$ is the classical action. This functional integral is a generalisation of the quantum mechanical path integral (2.6), providing the transition amplitude from an initial vacuum state $|0; t_i\rangle$ to a final vacuum state $|0; t_f\rangle$ through the integration over all possible field configurations between these states in the field configuration space.

As a result, the process of field quantisation emerges from solving this path integral for the specific field. It is worth mentioning that in QFT, we often use the expression "integrating out a field" to describe this process of field quantisation, achieved through solving the corresponding functional integral.

Before concluding this section, it is essential to define the classical action $S[\phi]$ as introduced in the path integral (2.39). Let us focus on the Lagrangian density (2.36). We can begin by reintroducing the terms dependent on spatial derivatives, the mass term, and interactions through the potential $V(\phi_j)$, defined in (2.23), and then take the continuum limit to obtain

$$\mathcal{L} = \frac{1}{2}\left(\partial_t \phi(t)\right)^2 - \frac{1}{2}\left(\nabla \phi(x)\right)^2 - \frac{1}{2} m^2 \phi^2(x) + \cdots. \tag{2.40}$$

Additionally, the first and second terms can be combined by defining a four-gradient, a derivative in $1 + 3$ dimensions of the form

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right) \qquad \text{and} \qquad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x_0}, -\nabla\right). \tag{2.41}$$

The derivative with respect to x^0 is a time derivative, whereas $\nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ denotes the gradient [4, 8, 10, 68]. Thus, the Lagrangian density becomes

$$\mathcal{L}(\phi, \partial_\mu\phi) = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2(x) + \cdots . \tag{2.42}$$

Making it possible to define the classical action $S[\phi]$ as

$$S[\phi] = \int_{t_i}^{t_f} dt\, L = \int_{t_i}^{t_f}\int_{-\infty}^{\infty} d^4x\, \mathcal{L}(\phi, \partial_\mu\phi), \tag{2.43}$$

where $\mathcal{L}$ is the Lagrangian density that, from now on, will be just called Lagrangian, and $d^4x = dt\, d^3x$. In general, it is common to set $t_i = -\infty$ and $t_f = +\infty$, allowing us to drop the time labels in (2.39) and to write $\langle 0|0\rangle$ instead of $\langle 0; t_f|0; t_i\rangle$. In this case, the integral over x in (2.43) spans all of spacetime. Although the derivation of the path integral was performed for a real scalar field, $\phi(x)$ can correspond to any given field, as we shall see in this and the next chapter.

2.2.1 n-point functions

In quantum field theories, an essential component in describing phenomena are the so-called n-point functions or correlation functions. These functions encapsulate information about the theory, enabling the computation of observables, such as scattering amplitudes, through the Lehmann-Symanzik-Zimmermann (LSZ) formula [2, 10].

These n-point functions correspond to vacuum expectation values of the product of n fields, denoted as $\langle 0|\hat{T}\left(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\right)|0\rangle$, where $\hat{T}$ is known as the time ordering operator [4, 8, 10, 68]. The operator $\hat{T}$ is responsible for ensuring that the field operators act on the state $|0\rangle$ in a specific time sequence, namely $t_f > t_1 > \cdots > t_n > t_i$. This means that the product of two bosonic fields is given as

$$\hat{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right) = \begin{cases} \hat{\phi}(x_1)\hat{\phi}(x_2) \text{ for } & x_1^0 > x_2^0 \\ \hat{\phi}(x_2)\hat{\phi}(x_1) \text{ for } & x_2^0 > x_1^0 \end{cases}. \tag{2.44}$$

To derive a general expression for computing these expectation values, we begin by obtaining the transition amplitude of the product of two fields. This is represented in the Heisenberg picture as

$$\langle 0|\hat{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle_H \quad \text{with} \quad t_f \geq t_1 > t_2 \geq t_i, \tag{2.45}$$

where t_1 and t_2 correspond to the times of the operators $\hat{\phi}(x_1)$ and $\hat{\phi}(x_2)$, respectively. The procedure for deriving this 2-point function is similar to that of obtaining the transition amplitude between two vacuum states, as we did in the last section.

Therefore, we again divide the interval from the initial to the final time into n segments of infinitesimal lengths δt. Moreover, we insert intermediate field configurations using the field completeness relation of the operator $\hat{\phi}(x)$, as given in (2.27). These steps correspond to

$$\begin{aligned}\langle 0|\hat{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle_H = \int D\phi_1(\mathbf{x})\cdots D\phi_n(\mathbf{x})\ &\langle 0|e^{-i\delta t\hat{H}(t_n)}|\phi_n\rangle\langle\phi_n|\cdots|\phi_3\rangle \\ &\times\langle\phi_3|e^{-i\delta t\hat{H}(t_2)}\hat{\phi}(x_2)|\phi_2\rangle\langle\phi_2|e^{-i\delta t\hat{H}(t_1)}\hat{\phi}(x_1)|\phi_1\rangle \\ &\times\langle\phi_1|e^{-i\delta t\hat{H}(t_f)}|0\rangle\,.\end{aligned}\tag{2.46}$$

Following this, we proceed by acting the operators $\hat{\phi}(x_1)$ and $\hat{\phi}(x_2)$ on the eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$, respectively, using the relation (2.24). This results in

$$\begin{aligned}\langle 0|\hat{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle_H = \int D\phi_1(\mathbf{x})\cdots D\phi_n(\mathbf{x})\ &\langle 0|e^{-i\delta t\hat{H}(t_n)}|\phi_n\rangle\langle\phi_n|\cdots|\phi_3\rangle \\ &\times\langle\phi_3|e^{-i\delta t\hat{H}(t_2)}\phi_2(\mathbf{x}_2)|\phi_2\rangle\langle\phi_2|e^{-i\delta t\hat{H}(t_1)}\phi_1(\mathbf{x}_1)|\phi_1\rangle \\ &\times\langle\phi_1|e^{-i\delta t\hat{H}(t_f)}|0\rangle\,.\end{aligned}\tag{2.47}$$

Note that the subscriptions 1 and 2 in the eigenvalues correspond to times t_1 and t_2, respectively. Factoring out these eigenvalues, we arrive at the integral

$$\begin{aligned}\langle 0|\hat{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle_H = \int D\phi_1(\mathbf{x})\cdots D\phi_n(\mathbf{Nx})\,&\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\,\langle 0|e^{-i\delta t\hat{H}(t_n)}|\phi_n\rangle \\ &\times\langle\phi_n|\cdots|\phi_3\rangle\langle\phi_3|e^{-i\delta t\hat{H}(t_2)}|\phi_2\rangle \\ &\times\langle\phi_2|e^{-i\delta t\hat{H}(t_1)}|\phi_1\rangle\langle\phi_1|e^{-i\delta t\hat{H}(t_f)}|0\rangle\,,\end{aligned}\tag{2.48}$$

which resembles (2.26). Here, each of the matrix elements within the integral must be computed. However, since we have already done this in the previous section, we will not repeat it here.

After computing each matrix element using (2.35), organising each computed element, and taking the continuous limit as done in (2.37), we obtain the path integral

$$\langle 0|\hat{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle_H = \mathcal{C}\int D\phi(x)\ \phi(x_1)\,\phi(x_2)\,e^{iS[\phi]}.\tag{2.49}$$

Here, $\mathcal{C}$ is a divergent constant, as in (2.39). This result is easily generalised to a n-point function with

$$\langle 0|\hat{T}\left(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\right)|0\rangle = \mathcal{C}\int D\phi\ \phi(x_1)\cdots\phi(x_n)\,e^{iS[\phi]}.\tag{2.50}$$

In this result, we note that on the left side of the equal sign, the fields are operators[8], justifying the need for the time-ordering operator. In contrast, on the right side of the

[8] Note that in QFT, field operators are not assigned to observables as in quantum mechanics. They need not to be Hermitian. Here, observables are calculated with the n-point functions and the LSZ formula.

equal sign, the path integral already takes care of this ordering, as the fields $\phi(x_1)\cdots\phi(x_n)$ are only functions of spacetime [70], and their integration can be carried out in any chosen order.

Furthermore, we can introduce sources $J(x)$ to the path integral, thereby defining a mathematical object known as generating functional

$$Z[J] = \langle 0|0\rangle_J = \int D\phi \; e^{i\left(S[\phi]+\int d^4x \; J(x)\phi(x)\right)}, \tag{2.51}$$

which generates all n-point functions of a quantum field theory[9]. The sources we introduced serve as a tool and have no physical significance here, being used to compute all the n-point functions, the connected and disconnected diagrams[10], through functional differentiations with respect to $J(x)$, as we shall explore shortly. Moreover, we note that when $J = 0$, corresponding to $Z[0]$, we return to the path integral (2.39).

Additionally, (2.51) establishes an analogy between quantum field theory and statistical mechanics. Both the generating functional and partition function share a similar structure involving integration over all possible configurations with an exponential statistical weight [8]. In the context of QFT, this weight corresponds to the classical action, while in statistical mechanics, it represents the total energy of the system in a respective microstate [74].

The n-point functions are given by the functional differentiation of the generating functional (2.51) with respect to the sources $J(x)$. Before we start deriving this alternative way of obtaining the n-point functions, we need to introduce the so-called functional derivative

$$\frac{\delta}{\delta J(x)} J(x') = \delta(x-x'), \tag{2.52}$$

where $\delta(x-x')$ is the four-dimensional, unless stated otherwise, Dirac delta. This derivative implies that

$$\frac{\delta}{\delta J(x)} \int_{-\infty}^{\infty} d^4x' \; J(x')\phi(x') = \int_{-\infty}^{\infty} d^4x' \; \phi(x')\,\delta(x-x') = \phi(x), \tag{2.53}$$

which will be important in the following example.

Using these two relations, we differentiate the generating functional (2.51) with respect to $J(x_1)$, giving

$$\begin{aligned} \frac{\delta}{\delta J(x_1)} Z[J]\bigg|_{J=0} &= \mathcal{C} \int D\phi \; e^{i\left(S[\phi]+\int d^4x \; J(x)\phi(x)\right)} \, i\frac{\delta}{\delta J(x_1)} \int d^4x \; J(x)\phi(x) \\ &= i\mathcal{C} \int D\phi \; \phi(x_1)\, e^{i\left(S[\phi]+\int d^4x \; J(x)\phi(x)\right)} \\ &= i\mathcal{C} \, \langle 0|\hat{\phi}(x_1)|0\rangle \, . \end{aligned} \tag{2.54}$$

[9] It is important to mention that this generator can also be defined in quantum mechanics.

[10] The contrast between these diagrams will become apparent as we explore the scalar interacting theory in Subsection 2.3.2.

This expression represents the 1-point function. Differentiating again, now with respect to $J(x_2)$, yields

$$\begin{aligned}
&\frac{\delta}{\delta J(x_2)}\left(\frac{\delta}{\delta J(x_1)}Z[J]\right)\Bigg|_{J=0} \\
&\qquad = i^2\mathcal{C}\int D\phi\;\phi(x_1)\,e^{i\left(S[\phi]+\int d^4x\;J(x)\phi(x)\right)}\,\frac{\delta}{\delta J(x_2)}\int d^4x\;J(x)\phi(x) \\
&\qquad = i^2\mathcal{C}\int D\phi\;\phi(x_1)\,\phi(x_2)\,e^{i\left(S[\phi]+\int d^4x\;J(x)\phi(x)\right)} \\
&\qquad = i^2\mathcal{C}\,\langle 0|\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle\,,
\end{aligned} \tag{2.55}$$

the 2-point function. As we will see in the following section, this function is called a propagator and its meaning will be explained there. Based on previous results, we observed a recurring pattern in the differentiation of the generating functional. This pattern involves incorporating fields into the integrand of $Z[J]$, thus increasing the number of fields to the expectation value.

However, it is important to remember that $Z[J]$ contains a divergent constant $\mathcal{C}$. To get ride of it, we can define a normalised correlation function of n fields with

$$\langle 0|\hat{T}\left(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\right)|0\rangle = \frac{(-i)^n}{Z[J]}\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\Bigg|_{J=0}, \tag{2.56}$$

where the added term $Z[J]$ in the denominator is responsible for normalising the expectation value of the vacuum, allowing the constant $\mathcal{C}$ to be cancelled out. Note that even though this function is normalised, we are using the same notation as for the non-normalised n-point function. From now on, the n-point functions to be calculated are the normalised ones. Furthermore, after the differentiations, we set the source $J(x)$ to zero since it is a tool used in differentiating $Z[J]$. In this way, the normalised expectation value of n fields also reads

$$\langle 0|\hat{T}\left(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\right)|0\rangle = \frac{\int D\phi(x)\;\phi(x_1)\cdots\phi(x_n)e^{iS[\phi]}}{\int D\phi(x)\;e^{iS[\phi]}}. \tag{2.57}$$

2.3 Scalar Field Theory

The first field theory we shall quantise in this dissertation is the scalar field theory. This theory elucidates the dynamics of scalar fields, i.e. fields that are invariant under Lorentz transformations. Due to this invariance, these fields describe spin-zero particles. In addition, the reason for choosing to begin with this field theory resembles the criterion adopted for the selection of the Hamiltonian in the path integral derivation: simplicity.

2.3.1 Scalar free theory

We shall begin by considering the non-interacting case, where the scalar fields do not couple to themselves or any other field. This theory is referred to as free scalar theory and is described by the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}m^2\varphi^2. \tag{2.58}$$

Here, $\varphi = \varphi(x)$ represents the scalar field in $1+3$ dimensions, and m is the mass of the field. The first term on the right-hand side is referred to as a kinetic term, while the second one is known as the mass term[11]. Furthermore, the subscript in $\mathcal{L}_0$ is merely an indication that it is in a free theory. From now on, this is what the subscript means unless stated otherwise.

Recall that quantisation in the functional approach is given by the solution of path integral (2.51) corresponding to $Z[0]$. This means that to quantise the free scalar field and obtain the associated n-point functions, we must solve the path integral

$$Z_0[0] = \int D\varphi \, \exp\left[i \int d^4x \, \left(\frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}m^2\varphi^2\right)\right]. \tag{2.59}$$

To solve this, we compare it with the Gaussian integral (2.16) and notice that the action must take on a quadratic form

$$S[\varphi] = \int d^4x \, \varphi(x) \, \mathbf{A} \, \varphi(x). \tag{2.60}$$

Upon comparison, we also observe that $Z_0[0] = \mathcal{C}$, which confirms our choice of normalisation in (2.56).

However, rather than directly solving (2.59), our focus will be on the generating functional with $J \neq 0$. The reason for this will become more evident throughout this subsection and in the next. Thus, we proceed by carrying out an integration by parts of the fields φ with respect to x, implicitly assuming that these fields decay rapidly enough at infinity, i.e. there are no boundary terms at infinity, resulting in

$$Z_0[J] = \int D\varphi \, \exp\left[i \int d^4x \, \left(-\frac{1}{2}\varphi \partial^\mu \partial_\mu \varphi - \frac{1}{2}m^2\varphi^2 + J\varphi\right)\right]. \tag{2.61}$$

We now proceed by expressing the action in quadratic form, which results in the generating functional

$$Z_0[J] = \int D\varphi \, \exp\left\{i \int d^4x \, \left[-\frac{1}{2}\varphi\left(\Box + m^2\right)\varphi + J\varphi\right]\right\}, \tag{2.62}$$

where we used the d'Alembert operator, also known as the box operator, $\Box = \partial^\mu \partial_\mu$. After performing these steps, we see that this integral has the same form as the result (2.15), thus enabling the identification of the operator $\mathbf{A} = i\left(\Box + m^2\right)$.

Since the result depends on the inverse $\mathbf{A}^{-1}$, we can define a Green's function $G(x-x')$ such that the inverse is expressed as $\mathbf{A}^{-1} = G(x-x')$, namely

$$\left(\Box + m^2\right) G(x-x') = -i\delta(x-x'). \tag{2.63}$$

[11] Solving the Euler-Lagrange equations for these fields revels that this term corresponds to the mass m in the equations of motion.

The simple way to solve this involves writing the Fourier transformations of the Green's function and the Dirac delta from the position to the momentum space. These transformations are given as

$$G(x-x') = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4}\, G(k)e^{-ik(x-x')} \quad \text{and} \quad \delta(x-x') = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-x')}, \tag{2.64}$$

where $G(k)$ is the Green's function in momentum space. Here, we introduce the four-momentum $k^\mu = (k^0, \mathbf{k})$, where k^0 corresponds to energy E, and $\mathbf{k} = (k^1, k^2, k^3)$ represents the three-dimensional momentum vector. Replacing these relations back to (2.63) results in

$$\begin{aligned}(\Box + m^2)\int \frac{d^4k}{(2\pi)^4}\, G(k)e^{-ik(x-x')} &= \int \frac{d^4k}{(2\pi)^4}\, G(k)e^{-ik(x-x')}\left[(-ik)^2 + m^2\right] \\ &= \int \frac{d^4k}{(2\pi)^4}\, G(k)e^{-ik(x-x')}\left(-k^2 + m^2\right) \\ &= -i\int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-x')}.\end{aligned} \tag{2.65}$$

In this equation, the term $(-ik)^2$ arises from the second derivative of the exponential. From these integrals, we obtain that

$$\left(-k^2 + m^2\right) G(k) = -i, \tag{2.66}$$

and then

$$G(k) = \frac{i}{k^2 - m^2}. \tag{2.67}$$

This Green's function is the propagator in momentum space and corresponds to the amplitude of a particle with momentum k in momentum space. In the free scalar theory, this propagator represents a spin-zero particle in momentum space with mass m, emerging from the quantisation of scalar fields $\varphi(x)$.

Returning to position space, we find the propagator to be

$$\mathbf{A}^{-1} = G(x-x') = \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-x')}. \tag{2.68}$$

This Green's function is known as the Feynman propagator $G_F(x-x')$, representing the amplitude for a free particle to travel between the points x and x'. Note that we introduced a factor $i\epsilon$, with $\epsilon > 0$ and infinitesimal, that shifts the poles $k^0 = \pm E_p = \pm\sqrt{(\mathbf{k}^2 + m)}$ off the real axis [8, 10, 68, 73], as illustrated in Figure 5. This integration can occur by closing the contour in the lower half-plane, corresponding to a particle propagating from the point x' to x when $x^0 > x'^0$, thus yielding the Green's function $G(x-x')$. Conversely, if the integration closes the upper half-plane, the particle propagates from point x to x' with $x'^0 > x^0$, giving the Green's function $G(x'-x)$. Therefore, the Feynman propagator is obtained by combining these two solutions

$$G_F(x-x') = \theta(x^0 - x'^0)\, G(x-x') + \theta(x'^0 - x^0)\, G(x'-x), \tag{2.69}$$

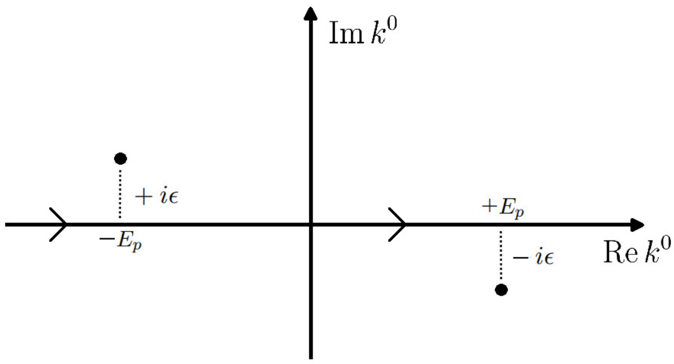


Figure 5 – Illustration of the integral poles shifting by the factor $i\epsilon$.

where $\theta(x)$ is the Heaviside step function.

Since we obtained $\mathbf{A}^{-1}$, this term can be substituted into the result (2.15), hence solving the Gaussian integral (2.62) and leading to

$$Z_0[J] = Z_0[0] \exp\left(-\frac{1}{2}\int d^4x\, d^4x'\; J(x) G_F(x-x') J(x')\right). \tag{2.70}$$

With this solution, we can obtain all the n-point functions of the theory described by the Lagrangian (2.58), meaning that we have achieved the quantisation of these fields. Furthermore, we observe the decoupling[12] of the field φ and the source J, which will facilitate obtaining the n-point functions by differentiating the generating functional (2.70) with (2.56).

The factor $Z[0]$ that corresponds to (2.59) does not need to be specified, as it cancels out when calculating the n-point functions with (2.56). However, from it, we can derive a property that holds for any functional integral that is quadratic in the fields. Indeed, the integration of (2.59) gives

$$\begin{aligned} Z_0[0] &= \int D\varphi \exp\left[i\int d^4x \left(\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2\right)\right] \\ &= C\left[\det i\left(\Box + m^2\right)\right]^{-\frac{1}{2}}, \end{aligned} \tag{2.71}$$

where we used (2.15) and defined the constant $C = \sqrt{(2\pi)^n}$. By considering any bosonic field, we obtain the determinant of a differential operator $\mathbf{A}$ related to that field, similar to the result above. This operator is diagonal and has eigenvalues a_i such that

$$\det\mathbf{A} = \prod_{i=1}^{\infty} a_i. \tag{2.72}$$

We can proceed by writing these eigenvalues in exponential form, that is

$$\det\mathbf{A} = \prod_{i=1}^{\infty} \exp(\log a_i) = \exp\left(\sum_i^{\infty} \log a_i\right) = \exp\left(\mathrm{Tr}\log\mathbf{A}\right), \tag{2.73}$$

[12] Note that we no longer have terms like $J\varphi$.

where log is a natural logarithm unless stated otherwise. Furthermore, Tr corresponds to the functional trace over all spacetime indices of $\mathbf{A}$. In position space, the trace corresponds to

$$\operatorname{Tr}\log\mathbf{A} = \int_{-\infty}^{\infty} d^4x\,\langle x|\log\mathbf{A}|x\rangle\,. \tag{2.74}$$

This property will be necessary as we explore effective actions in Section 2.5.

Redirecting our attention to the derivation of correlation functions, we aim to obtain the vacuum expectation value of the product of two fields

$$\langle 0|\hat{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle = \frac{(-i)^2}{Z_0[J]}\frac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0}\,. \tag{2.75}$$

We start by differentiating (2.56) once with respect to $J(x_1)$:

$$\begin{aligned}\frac{\delta}{\delta J(x_1)}Z_0[J] &= Z_0[0]\frac{\delta}{\delta J(x_1)}\exp\left(-\frac{1}{2}\int d^4x\,d^4x'\;J(x)G_F(x-x')J(x')\right)\\ &= -\frac{1}{2}Z_0[J]\frac{\delta}{\delta J(x_1)}\left(\int d^4x\,d^4x'\;J(x)G_F(x-x')J(x')\right),\end{aligned} \tag{2.76}$$

and then proceed by using the relation (2.52), which results in

$$\begin{aligned}\frac{\delta}{\delta J(x_1)}Z_0[J] &= -\frac{1}{2}Z_0[J]\left\{\int d^4x\,d^4x'\,G_F(x-x')\left[\delta(x_1-x)J(x')+J(x)\delta(x_1-x')\right]\right\}\\ &= -\frac{1}{2}Z_0[J]\left[\int d^4x'\,G_F(x_1-x')J(x')+\int d^4x\,G_F(x-x_1)J(x)\right].\end{aligned} \tag{2.77}$$

Note that the vacuum expectation value for one scalar field, a 1-point function for this field, is $\langle 0|\varphi(x)|0\rangle = 0$, which is associated with the phenomenon known as spontaneous symmetry breaking. Since this topic is beyond the scope of this work, refer to the references [4, 8, 10].

Continuing, differentiating once more, this time with respect to $J(x_2)$, yields

$$\begin{aligned}&\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}Z_0[J]\\ &\quad= -\frac{1}{2}\frac{\delta}{\delta J(x_2)}\left[Z_0[J]\left(\int d^4x'\,G_F(x_1-x')J(x')+\int d^4x\,G_F(x-x_1)J(x)\right)\right]\\ &\quad= -\frac{1}{2}\left[Z_0[J]\frac{\delta}{\delta J(x_2)}\left(\int d^4x'\,G_F(x_1-x')J(x')+\int d^4x\,G_F(x-x_1)J(x)\right)\right]\\ &\quad= -\frac{1}{2}Z_0[J]\left(\int d^4x'\,G_F(x_1-x')\delta(x_2-x')+\int d^4x\,G_F(x-x_1)\delta(x_2-x)\right)\\ &\quad= -\frac{1}{2}Z_0[J]\left(G_F(x_1-x_2)+G_F(x_2-x_1)\right)\\ &\quad= -Z_0[J]\,G_F(x_1-x_2).\end{aligned} \tag{2.78}$$

Here, we used the product rule from the second to the third line. However, since the derivative of $Z_0[J]$ includes sources, this term will be zero when substituted back into

(2.75) and $J = 0$ is applied. Therefore, we omit it in the third and subsequent lines. The other steps follow the same as the derivative with respect to $J(x_1)$. However, in the last line, we used the symmetry property of Green's functions

$$G_F(x_1 - x_2) = G_F(x_2 - x_1). \tag{2.79}$$

Substituting the result (2.78) back into (2.75) leads to

$$\begin{aligned} \langle 0|\hat{T}\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle &= -\frac{(-i)^2}{Z_0[J]} Z_0[J] G_F(x_1 - x_2)\bigg|_{J=0} \\ &= G_F(x_1 - x_2). \end{aligned} \tag{2.80}$$

In this equation, we note that the 2-point function corresponds to the scalar free field Feynman propagator (2.68). Therefore, the transition amplitude from one vacuum to another with the product of two field corresponds to the amplitude of a particle propagating between two points in spacetime.

Moreover, this result can be visually depicted through Feynman diagrams, which serve as graphical tools for calculating amplitudes in quantum field theory. Consequently, after obtaining all the diagrams, one can describe all phenomena related to a particular quantum field theory.

Thus, we represent the correlation function of two fields, a two-point propagator, for a free scalar field as

$$x_1 \;\text{———————}\; x_2 \quad = G_F(x_1 - x_2). \tag{2.81}$$

This diagram is known as a tree-level diagram. As we consider interactions, other types of diagrams will be represented. Among these are the so-called loop diagrams, which scale proportionally with $\hbar$ and the coupling constant λ and are equivalent to quantum corrections resulting from the quantisation of field interactions.

2.3.2 Scalar interacting theory

After quantising the free theory, we focus on addressing the interacting theory. Here, additional complexity is introduced to the Lagrangian (2.58) by including a self-interaction term φ^4. This theory is described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\varphi^4, \tag{2.82}$$

where $\mathcal{L}_0$ represents the free Lagrangian (2.58), and λ denotes a dimensionless coupling constant governing the self-interaction term [8, 68, 70]. Note that, due to the self-interaction term, the particles described by these fields interact with each other. For instance, two bosons created from these fields interact with one another. As we will observe in this subsection, these interactions lead to quantum corrections to the free theory.

Given that we are employing field quantisation via the functional approach, our initial task involves formulating the generating functional $Z[J]$ for this theory, followed by the computation of n-point functions. Thus, by recalling the generating functional (2.51), we express it as

$$Z[J] = \int D\varphi \, \exp\left[-i\frac{\lambda}{4!}\int d^4x \, \varphi^4(x)\right] e^{iS_0[\varphi,J]}, \tag{2.83}$$

where $S_0[\varphi, J]$ denotes the free classical action with sources J, this means

$$S_0[\varphi, J] = \int d^4x \, \left[-\frac{1}{2}\varphi\left(\Box + m^2\right)\varphi + J\varphi\right]. \tag{2.84}$$

Due to the interaction term, we cannot solve this integral exactly, as we cannot express the action in quadratic form and utilise the result (2.16). Hence, to quantise this theory and derive the n-point functions, we will utilise an approximate method known as perturbation theory.

In perturbation theory, we assume the coupling constant λ to be sufficiently small, enabling a Taylor expansion around zero [8, 10, 68]. To proceed, we first acknowledge that

$$\frac{1}{i}\frac{\delta}{\delta J(x)} \to \varphi(x), \tag{2.85}$$

which is related to the fact that differentiating $Z[J]$ with respect to $J(x)$ leads to the appearance of fields $\varphi(x)$ [2, 68], as observed in equations (2.56) and (2.57). Hence, this allows us to represent the quartic interaction term as

$$\frac{\lambda}{4!}\varphi^4 \to \frac{\lambda}{4!}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4. \tag{2.86}$$

Consequently, the generating functional (2.83) is rewriten as

$$Z[J] = \int D\varphi \, \exp\left[-i\frac{\lambda}{4!}\int d^4x \, \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\right] e^{iS_0[\varphi,J]}. \tag{2.87}$$

As the exponential term is now independent of the field $\varphi(x)$, it can be factored out from the path integral. This leads to the condensed expression

$$Z[J] = \exp\left[-i\frac{\lambda}{4!}\int d^4x \, \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\right] Z_0[J], \tag{2.88}$$

where we observe that the remaining terms, once the interaction term is factored out, correspond to the free-generating functional $Z_0[J]$.

By expanding the exponential with respect to the coupling constant λ, we obtain the power series

$$\begin{aligned}\exp\left[-i\frac{\lambda}{4!}\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\right] \approx 1 - i\frac{\lambda}{4!}\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4 \\ +\frac{1}{2!}\left(-i\frac{\lambda}{4!}\right)^2\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\int d^4x'\left(\frac{1}{i}\frac{\delta}{\delta J(x')}\right)^4 + \cdots .\end{aligned} \tag{2.89}$$

Each order of λ corresponds to a quantum correction to the free theory. This correspondence is evident, as the first term of the expansion, which is 1, represents the free theory when substituted back into $Z[J]$. Subsequently, each successive term in the expansion corresponds to a higher-order correction. Furthermore, these corrections manifest diagrammatically as loops, corresponding to interactions between particles, hence the name "loop correction".

Since our focus now solely lies on the physical implications, and for the sake of simplicity, we shall explore interactions up to the first order in λ, thus, one-loop corrections. The concepts arising from these interactions will serve as a foundation for higher-order corrections and further studies in QFT. Additionally, since certain computations resemble those of the free field, we shall omit the intermediate steps corresponding to the differentiations to avoid repetition.

The generating functional with interactions up to the first order in λ is given as

$$Z[J] \approx \exp\left[1 - i\frac{\lambda}{4!}\int d^4x\left(\frac{\delta}{\delta J(x)}\right)^4\right] Z_0[J]. \tag{2.90}$$

Note that we used $(1/i)^4 = 1$. By expressing the derivative explicitly, we have

$$\begin{aligned}Z[J] \approx & Z_0[0]\exp\left[1 - i\frac{\lambda}{4!}\int d^4x\left(\frac{\delta^4}{\delta J^4(x)}\right)\right] \\ & \times \exp\left(-\frac{1}{2}\int d^4x'\, d^4x''\; J(x')G_F(x'-x'')J(x'')\right),\end{aligned} \tag{2.91}$$

and thus, upon performing the differentiations, we obtain the generating functional

$$\begin{aligned}Z[J] \approx & Z_0[0]\Bigg[1 - i\frac{\lambda}{8}\, G_F(0)\, G_F(0)\int d^4x \\ & - i\frac{\lambda}{4}\, G_F(0)\int d^4x\left(\int d^4x_3 G_F(x-x_3)\, J(x_3)\int d^4x_4\, G_F(x-x_4)\, J(x_4)\right) \\ & - i\frac{\lambda}{4!}\int d^4x\Bigg(\int d^4x_3\, G_F(x-x_3)\, J(x_3)\int d^4x_4\, G_F(x-x_4)\, J(x_4) \\ & \times\int d^4x_5\, G_F(x-x_5)\, J(x_5)\int d^4x_6\, G_F(x-x_6)\, J(x_6)\Bigg)\Bigg] \\ & \times\exp\left(-\frac{1}{2}\int d^4x\, d^4x' J(x)G_F(x-x')J(x')\right).\end{aligned} \tag{2.92}$$

In this equation, we note that the integral in the second term on the right-hand side corresponds to the volume of the entire spacetime, thus it is divergent. However, this term can be absorbed by $Z_0[0]$, given that this term cancels out when computing n-point functions.

Moreover, $G_F(0)$ is also divergent, indicating that quantum corrections diverge. Consequently, methods such as renormalisation would be required to address this infinity as we calculate observables. For further details on the topic, refer to [4, 8–10, 75]. However, since our focus is not on computing these observables, this divergence can be ignored.

Thus, by rewriting the generating functional to "remove" the divergent term, we obtain

$$
\begin{aligned}
Z[J] \approx & Z_0[0] \\
& \times \left[1 - i\frac{\lambda}{4} G_F(0) \int d^4x \left(\int d^4x_3 G_F(x-x_3)\, J(x_3) \int d^4x_4\, G_F(x-x_4)\, J(x_4) \right)\right. \\
& - i\frac{\lambda}{4!} \int d^4x \left(\int d^4x_3\, G_F(x-x_3)\, J(x_3) \int d^4x_4\, G_F(x-x_4)\, J(x_4) \right. \\
& \quad \left.\left. \times \int d^4x_5\, G_F(x-x_5)\, J(x_5) \int d^4x_6\, G_F(x-x_6)\, J(x_6) \right)\right] \\
& \times \exp\left(-\frac{i}{2} \int d^4x\, d^4x'\, J(x) G_F(x-x') J(x') \right).
\end{aligned}
\tag{2.93}
$$

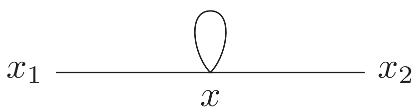
Having this generating functional at our disposal, we can compute any n-point function up to one-loop diagram using the relation (2.56).

Therefore, let us begin with a two-point function. By using the relation (2.56), we obtain the following vacuum expectation value of two fields:

$$
\langle 0 | \hat{T}\left(\hat{\varphi}(x_1)\, \hat{\varphi}(x_2)\right) | 0 \rangle = G_F(x_1 - x_2) - i\frac{\lambda}{2} G_F(0) \int d^4x\, G_F(x-x_1)\, G_F(x-x_2), \tag{2.94}
$$

where we observe that the first term corresponds to the free propagator (2.80), while the second term corresponds to quantum corrections due to interactions between fields. It is noteworthy that the factor $-i\lambda\, G_F(0)$ represents interactions. To be precise, they symbolise vertices where propagators intersect and interact.

Thus, we have a one-loop correction represented as



$$
= -i\frac{\lambda}{2} G_F(0) \int d^4x\, G_F(x-x_1)\, G_F(x-x_2). \tag{2.95}
$$

From a purely pictorial standpoint, a loop diagram corresponds to a particle moving from x_1 to x, subsequently decaying into a virtual particle and its virtual antiparticle that traverse the loop and ultimately annihilate into a real particle moving from x to x_2.

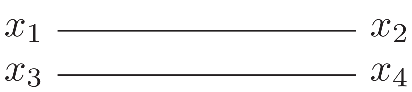
To proceed, we will explore the four-point correlation function. Therefore, upon performing the differentiations, we arrive at the expectation value of the product of the four fields

$$
\begin{aligned}
&\langle 0|\hat{T}\left(\hat{\varphi}(x_1)\,\hat{\varphi}(x_2)\,\hat{\varphi}(x_3)\,\hat{\varphi}(x_4)\right)|0\rangle = \\
&+\Big(G_F(x_1-x_2)\,G_F(x_3-x_4)+G_F(x_1-x_3)\,G_F(x_2-x_4)+G_F(x_1-x_4)\,G_F(x_2-x_3)\Big) \\
&-i\lambda\int d^4x\,\Big(G_F(x-x_1)\,G_F(x-x_2)\,G_F(x-x_3)\,G_F(x-x_4)\Big) \\
&-i\frac{\lambda}{2}\,G_F(0)\int d^4x\,\Big(G_F(x_1-x_2)\,G_F(x-x_3)\,G_F(x-x_4) \\
&\qquad+G_F(x_1-x_3)\,G_F(x-x_2)\,G_F(x-x_4)+G_F(x_1-x_4)\,G_F(x-x_2)\,G_F(x-x_3) \\
&\qquad+G_F(x_2-x_3)\,G_F(x-x_1)\,G_F(x-x_4)+G_F(x_2-x_4)\,G_F(x-x_1)\,G_F(x-x_2) \\
&\qquad+G_F(x_3-x_4)\,G_F(x-x_1)\,G_F(x-x_2)\Big).
\end{aligned}
\tag{2.96}
$$

As evidenced here, while computing an expectation value with more points, we encounter additional and distinct combinations of Green's functions compared to (2.95). Furthermore, by considering more points and higher orders in λ, we obtain more complex corrections. However, since our goal is merely to gain a better understanding of the fundamentals of QFT, this correlation function is already sufficient.

An important characteristic of the diagrams resulting from this four-point correlation function is their classification into connected and disconnected types. In brief, connected diagrams are characterised by their components being interconnected within the diagram. Meanwhile, the disconnected ones consist of components that remain unconnected to each other [68].

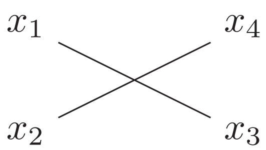
As an initial example of a disconnected diagram, let us consider the first term on the second line on the right-hand side of (2.96). This term is illustrated as



$$= G_F(x_1-x_2)\,G_F(x_3-x_4). \tag{2.97}$$

Here, we observe the presence of two Green's functions, one propagating from x_1 to x_2 and another from x_3 to x_4. Furthermore, as evident, these propagators do not have components that connect as they are parallel. Hence, such a diagram is categorised as disconnected.

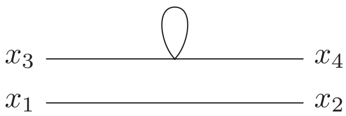
Meanwhile, the third line of (2.96) corresponds to a vertex, which is illustrated as



$$= -i\lambda\int d^4x\,\Big(G_F(x-x_1)\,G_F(x-x_2)\,G_F(x-x_3)\,G_F(x-x_4)\Big). \tag{2.98}$$

Such a diagram is classified as connected since its constituents are interconnected, indicated by their intersection at x as observed in the integral. This diagram represents interactions between propagators and is used to the construction of more complex diagrams.

Upon examining the fourth line, we identify yet another disconnected diagram. This becomes clearer as we look at its diagram



$$= -i\frac{\lambda}{2}\, G_F(0) \int d^4x\, G_F(x_1 - x_2)\, G_F(x - x_3)\, G_F(x - x_4). \tag{2.99}$$

Here, there is a free propagator from x_1 to x_2 and a loop from x_3 to x_4. However, they are not connected as none of their parts meet among themselves. Take note that $G_F(x_1 - x_2)$ is just a constant with respect to the integral, which corresponds to a loop with the presence of the factor $-i\frac{\lambda}{2}\, G_F(0)$. The remaining terms in the subsequent lines follow a similar representation and meaning.

2.4 Connected Diagrams

The motivation behind classifying the diagrams above stems from the fact that only the connected diagrams are relevant for computing physical scattering amplitudes [8–10, 68, 70]. Therefore, we introduce the so-called generating functional of connected diagrams

$$W[J] = -i \log Z[J]. \tag{2.100}$$

To explore how this generating functional generates connected Green's functions, let us begin by considering that $Z[J]$ depends on a generic field $\phi(x)$. With this premise, we proceed by differentiating (2.100) with respect to a source $J(x_1)$ and then comparing it with the derivative of $Z[J]$ and the n-point functions (2.56).

This derivative and comparison lead us to observe that

$$\left.\frac{\delta W[J]}{\delta J(x_1)}\right|_{J=0} = \left.\frac{-i}{Z[J]}\frac{\delta Z[J]}{\delta J(x_1)}\right|_{J=0} = \langle 0|\phi(x_1)|0\rangle\,, \tag{2.101}$$

which represents the vacuum expectation value of the field $\phi(x)$. It is worth noting that, in this case, this value is zero. However, it may be a non-zero constant when there spontaneous symmetry breaking takes place [4, 8, 68].

We proceed by differentiating $W[J]$ once more, this time with respect to the source $J(x_2)$. This then gives

$$-i\left.\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} = (-i)^2 \left.\left(\frac{1}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x_1)\,\delta J(x_2)} - \frac{1}{Z^2[J]}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)}\right)\right|_{J=0}, \tag{2.102}$$

where the product rule has been applied. By employing (2.56), we recognise that the first term on the right side corresponds to the two-point Green's function, which may include connected and disconnected diagrams. Conversely, the second term represents two disconnected one-point correlation functions, similar to (2.97). Thus, we have

$$\begin{aligned} -i\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0} &= \langle 0|\hat{T}(\hat{\psi}(x_1)\hat{\psi}(x_2))|0\rangle - \langle 0|\hat{\psi}(x_1)|0\rangle\,\langle 0|\hat{\psi}(x_2)|0\rangle \\ &= \langle 0|\hat{T}(\hat{\psi}(x_1)\hat{\psi}(x_2))|0\rangle_{\text{connected}}\,. \end{aligned} \tag{2.103}$$

In this outcome, we observe that the presence of disconnected diagrams is eliminated from the n-point function that initially comprised both connected and disconnected diagrams. Consequently, only connected diagrams are obtained, consistent with our expectations.

Furthermore, this finding can be generalised to connected correlation functions of n points using the following differentiation relation [8, 9, 68, 70]:

$$\langle 0|\hat{T}\left(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\right)|0\rangle_{\text{connected}} = (-i)^{n-1}\frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0}\,. \tag{2.104}$$

This generating functional will be crucial in the next section, as it is necessary to define a modification of the classical action that contains all the information on the system, namely the effective action.

2.5 1PI Effective Action

As discussed in the previous section, the generating functional $W[J]$ generates all connected n-point functions in the theory. However, we omitted the possibility that these functions may be composed of two or more connected diagrams, which we shall call one-particle irreducible (1PI) diagrams.

To grasp this concept, let us examine the following diagram corresponding to a two-loop correction, a quantum correction proportional to λ^2

. (2.105)

This diagram is classified as one-particle (1P) reducible because by cutting between the loops, we observe that it can be constructed from two one-loop diagrams (2.95), which can be classified as one-particle irreducible (1PI) diagrams. In other words, we note that some diagrams are composed of "more fundamental" diagrams, or rather, from a basis of 1PI diagrams [10, 68, 70].

To derive the 1PI effective action, consider a generic field $\phi(x)$, whose dynamics are described by a classical action $S[\phi]$. When calculating the vacuum expectation value for a single field, this value may be zero, as in the case of the scalar theory (2.77), or non-zero, in the context of spontaneous symmetry breaking. We know that these outcomes vary as they may or may not depend on the source $J(x)$.

Bearing this in mind, we can express this value as

$$\frac{\delta W[J]}{\delta J(x)} = \frac{1}{i}\frac{\delta}{\delta J(x)}\log(Z[J]) = \langle 0|\phi(x)|0\rangle^J \equiv \phi_c(x), \tag{2.106}$$

where the superscript J in the expectation value indicates non-zero sources. Also, we denote this value as $\phi_c(x)$, commonly referred to as the mean-field or classical field. The reason behind the latter designation comes from the fact that when we take the classical limit, $\hbar \to 0$, this expectation value corresponds to a classical field in a classical field theory.

Furthermore, in (2.106), we notice that the mean-field and the source are conjugates, allowing us to perform a Legendre transformation on $W[J]$ and define the so-called 1PI effective action

$$\Gamma\left[\phi_c(x)\right] \equiv W[J] - \int_{-\infty}^{\infty} d^4x\ J(x)\phi_c(x). \tag{2.107}$$

As we shall further explore, this functional reproduces all the 1PI diagrams of the theory and thus encompasses all of the physics of a quantum theory. Additionally, it emerges as a modification of the classical action incorporating all loop corrections [8, 10, 68, 70].

To gain a better understanding of this effective action, we can examine it by differentiating it with respect to $\phi_c(x)$, yielding

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = \frac{\delta W[J]}{\delta\phi_c(x)} - \int d^4x' \left(\phi_c(x')\frac{\delta J(x')}{\delta\phi_c(x)} + J(x')\frac{\delta\phi_c(x')}{\delta\phi_c(x)}\right), \tag{2.108}$$

where we applied the product rule to the second term on the right-hand side of the equation (2.107). To proceed further, we apply the chain rule to the first term on the right side of (2.108) and utilise the definition of the mean-field (2.106). These steps result in

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = \int d^4x'\ \frac{\delta W[J]}{\delta J(x')}\frac{\delta J(x')}{\delta\phi_c(x)} - \int d^4x'\ \frac{\delta W[J]}{\delta J(x')}\frac{\delta J(x')}{\delta\phi_c(x)} - \int d^4x'\ J(x')\delta(x-x'). \tag{2.109}$$

Here, we simplified the last term using the functional derivative relation (2.52).

From this result, we obtain that

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J(x), \tag{2.110}$$

which has the same structure as the classical Euler-Lagrange equation for a system in the presence of an external source. However, this effective action includes, in addition to classical properties, all quantum corrections that can arise from the quantisation of the system. In other words, by solving the Euler-Lagrange equation for this action, we obtain the equations governing the dynamics of the quantum theory.

We can better observe this correspondence with the Euler-Lagrange equation as we take the limit $J(x) \to 0$, which results in $\phi_c \to \phi_c =$ constant. Then (2.110) reduces to

$$\left.\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)}\right|_{\phi_c(x)=\phi_c} = 0, \tag{2.111}$$

the stationary-action principle, yielding the quantum equations of motion for the field ϕ_c. Furthermore, as we take the classical limit $\hbar \to 0$, ϕ_c becomes a classical field, and the equation above becomes the stationary-action principle for a classical action.

Now that we understand that the effective action $\Gamma[\phi_c]$ is an object analogous to the classical action $S[\phi]$, with the difference that the former is encompassed by quantum corrections, we shall explore how this action leads to n-point functions for a QFT. To accomplish this, we differentiate (2.110) with respect to $J(x)$. This leads to

$$\frac{\delta}{\delta J(x_1)}\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -\delta(x-x_1), \tag{2.112}$$

where we used the relation (2.52). Here, we realise that we can rewrite (2.112) as

$$\int d^4x' \,\frac{\delta\phi_c(x')}{\delta J(x_1)}\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(x')\delta\phi_c(x)} = -\delta(x-x_1). \tag{2.113}$$

Using the definition of the mean-field (2.106), we express the integral above as

$$\int d^4x' \,\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x')}\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(x')\delta\phi_c(x)} = -\delta(x-x_1). \tag{2.114}$$

For this equation to hold, we must have

$$\left(\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(x')\delta\phi_c(x)}\right)^{-1} = -\frac{\delta^2 W[J]}{\delta J(x)\delta J(x')}. \tag{2.115}$$

As we take the limit where $J = 0$ and $\phi_c(x) = \phi_c$, we utilise (2.104) and then observe that the right-hand side of the equation corresponds to the connected two-point propagator

$$\left(\left.\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(x)\delta\phi_c(x')}\right|_{\phi_c}\right)^{-1} = G(x-x'). \tag{2.116}$$

This propagator is now known as the dressed propagator and corresponds to the propagator filled with quantum corrections given by the quantisation of interactions between fields. To observe this, we can parameterise the propagator [68, 70] as

$$G = \frac{1}{G_F^{-1} - \Sigma}. \tag{2.117}$$

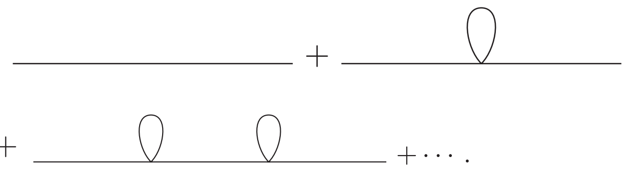
Note that here, we are not explicitly expressing the dependence on x and x' for the sake of notation. The term G_F corresponds to the free propagator, and Σ is a self-energy term that represents a one-loop diagram (2.95).

Upon expanding this dressed propagator, we obtain

$$G = G_F + G_F \, \Sigma \, G_F + G_F \, \Sigma \, G_F \, \Sigma \, G_F + \cdots \, , \tag{2.118}$$

and the diagrams

$$G(x_1 - x_2) =$$



(2.119)

This outcome illustrates that the propagator derived from the second derivative of the effective action and its subsequent expansion corresponds precisely to a series of 1PI diagrams, namely the diagrams (2.81) and (2.95). Hence, acquiring an effective action for a field theory results in capturing all information from it.

To compute n-point correlation functions, the result (2.116) can be generalised to

$$i \frac{\delta^n \Gamma[\phi_c]}{\delta \phi_c(x_1) \cdots \delta \phi_c(x_n)} = G_{1\mathrm{PI}}(x_1, \cdots, x_n), \tag{2.120}$$

as we did for the generating functionals $Z[J]$ and $W[J]$.

2.5.1 Background field method

Having understood the concept of effective action, let us now explore the so-called background field method [10, 70, 76, 77], which serves as a means to derive this effective action for any field ϕ.

Essentially, the central idea of this method is to expand a generic field $\phi(x)$ into two fields: the background field $\phi_c(x)$, which represents the classical part of the theory we aim to quantise; and the fluctuations $\varphi(x)$, which, upon quantisation, will account for quantum corrections to the classical part. Thus, with this in mind, the field $\phi(x)$ is decomposed as

$$\phi(x) = \phi_c(x) + \varphi(x), \tag{2.121}$$

and its classical action is expressed as

$$S[\phi(x)] = S[\phi_c(x) + \varphi(x)]. \tag{2.122}$$

To obtain the quantisation of fluctuations $\varphi(x)$, we employ the functional approach to quantisation, thereby writing the generating functional $Z[J]$ as

$$Z[J] = \int D\varphi(x) \, e^{i\left[S[\phi_c(x)+\varphi(x)] + \int d^4x \, J(x)(\phi_c(x)+\varphi(x))\right]}. \tag{2.123}$$

Here, we are integrating only over the field $\varphi(x)$ as it is the one we want to quantise. Thus the field $\phi_c(x)$ is constant with respect to this integration.

Moving forward, we expand the exponential argument around the background field $\phi_c(x)$, resulting in

$$\begin{aligned} I &= S[\phi_c(x) + \varphi(x)] + \int d^4x\, J(x)(\phi_c(x) + \varphi(x)) \\ &= S[\phi_c] + \int d^4x\, J\phi_c + \int d^4x \left(\frac{\delta S[\phi]}{\delta\phi(x)}\bigg|_{\phi_c} + J \right) \varphi(x) \\ &\quad + \frac{1}{2} \int d^4x\, d^4x' \varphi(x) \left(\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(x')}\bigg|_{\phi_c} \right) \varphi(x') + \cdots . \end{aligned} \tag{2.124}$$

In this expansion, we observe that the third term in the second line becomes zero, as it is a solution of the equation of motion of the classical action $S[\phi]$, similar to what we have seen in (2.111). Hence, we can rewrite the expansion as

$$I = S[\phi_c] + \int d^4x\, J\phi_c + \frac{1}{2} \int d^4x\, d^4x' \varphi(x) \left(\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(x')}\bigg|_{\phi_c} \right) \varphi(x') + \cdots . \tag{2.125}$$

As we are interested in corrections up to one loop, we will not include the higher-order terms $(\cdots)$ in the next steps. Nevertheless, it is worth mentioning that this method can be employed to find effective actions with these higher-order corrections. However, some difficulties associated with solving the path integral for these terms may arise.

As we substitute this result into the generating functional (2.123), we obtain

$$Z[J] = e^{i(S[\phi_c] + i\int d^4x\, J\phi_c)} \int D\varphi \exp\left[\frac{i}{2} \int d^4x\, d^4x' \varphi(x) \left(\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(x')}\bigg|_{\phi_c} \right) \varphi(x') \right], \tag{2.126}$$

where we have factored out the terms that do not depend on the fluctuations $\varphi(x)$, as they remain constant with respect to integration. Furthermore, recognising that this integral is Gaussian, we apply the result (2.15), yielding

$$Z[J] = e^{i(S[\phi_c] + \int d^4x\, J\phi_c)} \left[\det\left(\frac{1}{i} \frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(x')}\bigg|_{\phi_c} \right) \right]^{-\frac{1}{2}}, \tag{2.127}$$

the quantisation of fluctuations φ with a background field ϕ_c.

To derive the effective action $\Gamma[\phi_c]$ from this result, we utilise the property of the determinant (2.73), leading to

$$\left[\det\left(\frac{1}{i} \frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(x')}\bigg|_{\phi_c} \right) \right]^{-\frac{1}{2}} = \exp\left[-\frac{1}{2} \mathrm{Tr}\log\left(\frac{1}{i} \frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(x')}\bigg|_{\phi_c} \right) \right]. \tag{2.128}$$

Here, we observe that the factor $1/i$ inside $\mathrm{Tr}\log$ contributes only an additive constant, thus we discard it. Upon substituting this outcome back into (2.127), we derive the generating functional up to one-loop corrections

$$Z[J] = \exp i \left[S[\phi_c] + \frac{i}{2} \mathrm{Tr} \log \left(\left. \frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(x')} \right|_{\phi_c} \right) + \int d^4x \, J\phi_c \right]. \tag{2.129}$$

Now that we have the generating functional, we can efficiently substitute it into the generating functional of connected diagrams (2.100) and then perform a Legendre transformation, thus obtaining the effective action

$$\Gamma[\phi_c] = S[\phi_c] + \frac{i}{2} \hbar \mathrm{Tr} \log \left(\left. \frac{\delta^2 S[\phi]}{\delta\phi(x)\, \delta\phi(x')} \right|_{\phi_c} \right). \tag{2.130}$$

Here, we observe that this functional appears as a modification of the classical action, as the first term on the right-hand side corresponds to the classical action, while the second term represents one-loop quantum corrections, i.e. proportional[13] to the first order of $\hbar$, arising from the quantisation of the fluctuations φ. Therefore, upon deriving such an effective action for the theory under our investigation, we encapsulate all the quantum properties of the system up to the one-loop level.

[13] We reintroduced $\hbar$ here for clarity.

3 Quantum Electrodynamics

Quantum Electrodynamics (QED) stands as the extension of electrodynamics to the quantum regime. It is responsible for describing the behaviour of photons, electrons and positrons, along with their interactions. As previously discussed, it is the pioneering theory in reconciling quantum mechanics and special relativity, representing one of the most precise theories ever formulated.

We shall begin by introducing some notions of Lie groups, representations, and Lie algebra in Section 3.1. There, we shall explore gauge invariance in the bosonic and fermionic sectors of QED, as well as define the fields present in these sectors. Additionally, the classical field equations of motion will be obtained from the QED Lagrangian in Section 3.2. Moving on to Sections 3.3 and 3.4, we shall quantise the so-called gauge and Dirac fields, respectively.

3.1 Symmetries

Symmetries, denoting the preservation or invariance of a system's dynamics under a group of transformations, hold significant importance in quantum field theories. One reason for its importance stems from the fact that Lagrangians in QFT are built from selecting the symmetries we want the system of interest to possess.

For example, we exclusively deal with Lagrangians that remain invariant under Lorentz transformations in QFT, thus ensuring the theory's validity in relativistic regimes. Another instance is found in quantum electrodynamics, where the focus lies on gauge transformations, i.e local and continuous transformations, under which QED's Lagrangian is invariant.

The reason that we explore this invariance, known as gauge invariance, is because it is an essential characteristic of classical electromagnetism. Consequently, it is expected to also manifest in its quantum counterpart.

To be more precise, gauge invariance is not a symmetry per se, in the traditional sense of the word, but rather an inherent ambiguity within the theory. This means that different mathematical expressions describe the same physical configuration, leading to some problems with quantisation, as we will further discuss in this chapter. However, for the sake of nomenclature, we shall also call it symmetry.

From a mathematical standpoint, the transformations associated with this invariance are based on the so-called Lie groups, which consist of elements represented by g that depend on a set of continuous parameters α_a, where $a = 1, \cdots, n$ and n is the group dimension [4, 78, 79].

Furthermore, the term "local transformation" refers to the fact that the parameters α_a depend on spacetime coordinates and are thus expressed as $\alpha^a(x)$. In other words, under a local transformation, the system changes by a different amount at every point in spacetime. In contrast, a global transformation acts the same way at all points.

In summary, groups and their elements are abstract objects and do not inherently possess physical meaning. Therefore, in order to utilise them in physics and give them meaning, we must first define the so-called group representations.

A representation is an operation that assigns to a generic and abstract element g of a group G, a linear operator $D_R(g)$, acting on a vector space V [4, 78–80]. This operation ensures that the representation of the product of two group elements corresponds to the product of the representation of each group element

$$D_R(g_1 \cdot g_2) = D_R(g_1) D_R(g_2). \tag{3.1}$$

Note that representations preserve the group structure, meaning that the representation $D_R(g)$ can be expressed in various forms as long as the underlying group structure, namely the table for group operations, remains unchanged. Consequently, the operation of an element g, given a representation $D_R(g)$, on a vector ϕ^i, $i = 1, \cdots, n$, in the vector space V, is given by

$$\phi^i \to \left(D_R(g)\right)^i{}_j\, \phi^j, \tag{3.2}$$

where for a Lie group its elements are defined as $g = g(\alpha^1, \cdots, \alpha^n) \equiv g(\alpha)$.

As the notion of representation has been established, a representation for the Lie groups must be identified. Assuming that the parameter α_a is infinitesimal, that is, $\alpha_a \ll 1$, $D_R(g)$ is expressed as a Taylor series

$$\begin{aligned} D_R(g(\alpha_a)) &= \mathbb{1} + \left.\frac{\partial D_R(g(\alpha^a))}{\partial \alpha^a}\right|_{\alpha=0} d\alpha_a + O(\alpha^2) \\ &= \mathbb{1} + iT^a_R\, d\alpha_a + O(\alpha^2). \end{aligned} \tag{3.3}$$

Here, we define the so-called generators of the Lie group in the representation R,

$$T^a_R \equiv -i \left.\frac{\partial D_R}{\partial \alpha^a}\right|_{\alpha=0}, \tag{3.4}$$

with the inclusion of the factor i to ensure that the generator is Hermitian. This object is responsible for generating all transformations within a group. Further, equation (3.3) can be represented by an exponential functional of the form

$$D_R(g(\alpha_a)) = e^{-i\alpha_a T^a_R}, \tag{3.5}$$

which in the case of a local transformation, we set that $\alpha_a = \alpha_a(x)$.

An essential feature of these Lie group generators is that they form a structure called a Lie algebra. This structure is a vector space V equipped with the brackets operation $[\cdot,\cdot]$ defined as

$$[\cdot,\cdot] \,:\, V \times V \;\to V, \tag{3.6}$$

which follows the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0. \tag{3.7}$$

Here, A, B and C are matrices [4].

In addition, the Lie algebra can be expressed[1] by

$$[T_R^a, T_R^b] = i f^{ab}{}_c\, T_R^c, \tag{3.8}$$

where the antisymmetric[2] tensor denoted by $f^{ab}{}_c$ is referred to as structure constants and is responsible for defining the Lie algebra. As it is representation-independent, we have the flexibility to choose any representation for a given group. This choice is subject to the condition that they satisfy the commutation relation (3.8), dictated by the structure constants obtained from the group operation table. Thus, with the structure constants and the generators established, all the transformations of a group can be computed.

An important detail to note is that in the case of abelian symmetry transformations, where the elements of the group commute among themselves, we have $[T_R^a, T_R^b] = 0$, resulting in the vanishing of the structure constants.

3.1.1 Bosonic sector

As we previously reviewed some concepts of group theory, we shall explore in this subsection the symmetries within the bosonic sector of QED, particularly those under gauge transformations, i.e. local and continuous transformations.

The bosonic sector of QED is described by vector fields $A^\mu(x)$, also known as gauge fields. In simple terms, a vector field assigns a vector to each point in spacetime. A more precise definition is then provided by the way they transform under Lorentz transformations $\Lambda^\mu{}_\nu$.

Let us recall that a four-vector x^μ transforms under a Lorentz transformation $\Lambda^\mu{}_\nu$, preserving the length $g_{\mu\nu}x^\mu x^\nu$, as follows:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu, \tag{3.9}$$

where x'^μ is the transformed four-vector. This transformation is then extended to a vector field $A^\mu(x)$ with

$$A^\mu(x) \to A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \tag{3.10}$$

[1] The derivation of this relation can be motivated by the product of two different transformations $D_R(g(\alpha_a))$, and can be found in [4, 80].

[2] This means that permutations of indices introduce a negative sign to the tensor.

Here, the coordinates x corresponding to the transformed points are kept fixed, and Λ^{-1} denotes the inverse Lorentz transformation. Consequently, we observe that $A'^{\mu}(x)$, representing the transformed vector field evaluated at the transformed spacetime points, equals, via the Lorentz transformation, the value of the original field evaluated at the points before the transformation, thus $(\Lambda^{-1}x)$.

Given the way vector fields transform under a Lorentz transformation, as seen in (3.10), they represent in QED the bosonic spin-1 particles such as photons. As we introduce another field in the subsequent subsection, we will see that this field transforms differently and thus describes other particles, such as electrons and positrons, which have spin-1/2.

Having defined the vector fields $A^{\mu}(x)$, we can now investigate their behaviour under gauge transformations and the implications on the Lagrangian that governs the bosonic sector in QED. This Lagrangian, known as Maxwell's Lagrangian, is expressed as

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{3.11}$$

where $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$ represents the electromagnetic antisymmetric tensor, responsible for describing the electromagnetic field in spacetime. In the context of QED, the gauge fields $A^{\mu}(x)$ generalise the scalar electric potential ϕ and the magnetic vector $\mathbf{A}$ so that $A^{\mu} = (\phi, \mathbf{A})$, and are hence also known as the four-potential. [4, 8, 10, 81].

Concerning the gauge transformations in QED, these are based on the $U(1)$ group, a continuous group of 1×1 unitary matrices with the group operation of matrix multiplication [78, 79]. Within this group, the group element $U(x)$ is represented by an exponential phase (3.5) as

$$U(x) = e^{-i\alpha(x)}. \tag{3.12}$$

Since it has dimension 1, $U(1)$ possesses only one generator, conventionally set to 1. In addition, it is an abelian group, meaning the order in which its transformations are applied does not matter.

Under these transformations, a gauge field transforms as

$$A_{\mu} \to A_{\mu} + \partial_{\mu}\alpha(x), \tag{3.13}$$

which corresponds to a change of the potential vector by the gradient of any arbitrary function $\alpha(x)$. In terms of the transformations $U(x)$, we have

$$A_{\mu}(x) \to U(x) A_{\mu}(x) U^{-1}(x) + i \left(\partial_{\mu} U(x)\right) U^{-1}(x). \tag{3.14}$$

By applying these transformations to the field strength tensor $F_{\mu\nu}$, we observe the following:

$$F_{\mu\nu} \to \partial_{\mu}\left(A_{\nu} + \partial_{\nu}\alpha(x)\right) - \partial_{\nu}\left(A_{\mu} + \partial_{\mu}\alpha(x)\right) = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = F_{\mu\nu}. \tag{3.15}$$

This implies that $F_{\mu\nu}$ is invariant under these transformations, as is the Lagrangian (3.11). In other words, electromagnetic fields, and consequently Maxwell's equations, remain unaffected by a redefinition of the form of equation (3.13). They exhibit invariance under gauge transformations.

3.1.2 The Dirac field

Before we begin with the examination of the impacts of gauge transformations on the fermionic sector of quantum electrodynamics, we need to define the fields that compose it. In this sector, we are interested in the so-called spinor fields, being responsible for describing spin-1/2 particles, namely the electron and the positron.

In contrast to vector fields, which straightforwardly transform under Lorentz transformations as seen in (3.10), spinor fields transform under the so-called spinorial representation of dimension 2 of the Lorentz group.

To understand the significance of a dimension 2 representation in this context, let us revisit the connection between the spin s of a particle and the dimension n of a rotation group representation in quantum mechanics. This relation is expressed by

$$n = 2s + 1. \tag{3.16}$$

As we observe, in order to describe spin-1/2 particles such as the electron, there exists the necessity of finding a dimension 2 representation of the rotation group [3, 70]. This representation is proportional to the Pauli matrices σ_i

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.17}$$

which defines the Lie algebra of rotations

$$[J^i, J^j] = i\epsilon^{ijk} J^k. \tag{3.18}$$

Here, $J^i = \sigma^i/2$ is the rotation generator, also known as the angular-momentum operator [3], and ϵ^{ijk} represents the Levi-Civita symbol.

With this representation in hand and the understanding that the Lorentz group generalises the rotation group, it becomes evident that obtaining a dimension 2 representation of the Lorentz group is required, as such representation will enable us to define the fields that accurately describe spin-1/2 particles.

Therefore, let us begin by defining the so-called Dirac matrices, also referred to as gamma matrices γ^μ. These are 4×4 matrices that generalise the concept of Pauli matrices to four dimensions (3.17). They are defined to satisfy the anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}_4. \tag{3.19}$$

Here, $\mathbb{I}_4$ denotes the 4×4 identity matrix [4, 8–10].

Furthermore, these matrices can be represented in various forms, with one of the most common being the Weyl representation, also known as the chiral representation. In this representation, they take the form

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{3.20}$$

where $\mathbb{I}_2$ is the 2×2 identity matrix and σ_i are the Pauli matrices. Before we proceed, it may initially seem incorrect to use these matrices as they are 4×4, considering that we are seeking a 2×2 representation. However, as we shall see shortly, this does not lead to an error once we define the field that transforms under this representation and its meaning.

As we need to focus on finding the spinorial representation of the Lorentz algebra and its associated transformations, it is crucial to recall that the Lorentz algebra is written as

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho} + g^{\mu\sigma} M^{\nu\rho}), \tag{3.21}$$

where the tensors $M^{\mu\nu}$ are the generators of the Lorentz group [4, 8, 10, 70]. By defining these tensors in a particular representation, we can then define the transformations associated with these generators and the fields that transform under this representation.

With this in mind, we define the following representation of the Lorentz algebra:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\gamma^\mu\gamma^\nu - \frac{i}{2}g^{\mu\nu}, \tag{3.22}$$

where we consider the gamma matrices that we defined in (3.19) since we are interested in a representation corresponding to spin-1/2.

To demonstrate that such a representation satisfies the Lorentz algebra (3.21), we first need to evaluate

$$[S^{\mu\nu}, \gamma^\rho] = \left[\frac{i}{4}[\gamma^\mu, \gamma^\nu], \gamma^\rho\right]. \tag{3.23}$$

By considering $\mu \neq \nu$, this relation simplifies to

$$\begin{aligned} [S^{\mu\nu}, \gamma^\rho] &= \frac{i}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] \\ &= \frac{i}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{i}{2}\gamma^\rho\gamma^\mu\gamma^\nu. \end{aligned} \tag{3.24}$$

In the first line, we utilised (3.22), where the metric $g^{\mu\nu}$ vanishes as the indices are not equal, while, in the second line, we computed the commutation relation. We proceed by rewriting the action by adding and subtracting a factor $\frac{i}{2}\gamma^\mu\gamma^\rho\gamma^\nu$, which gives

$$\begin{aligned} [S^{\mu\nu}, \gamma^\rho] = &\frac{i}{2}\gamma^\mu\gamma^\nu\gamma^\rho + \frac{i}{2}\gamma^\mu\gamma^\rho\gamma^\nu - \frac{i}{2}\gamma^\mu\gamma^\rho\gamma^\nu \\ &- \frac{i}{2}\gamma^\rho\gamma^\mu\gamma^\nu - \frac{i}{2}\gamma^\mu\gamma^\rho\gamma^\nu + \frac{i}{2}\gamma^\mu\gamma^\rho\gamma^\nu. \end{aligned} \tag{3.25}$$

Here, we observe that we can group the first two terms of each line into anticommutation relations, so

$$[S^{\mu\nu}, \gamma^{\rho}] = \frac{i}{2}\gamma^{\mu}\{\gamma^{\rho}, \gamma^{\nu}\} - \frac{i}{2}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu} - \frac{i}{2}\{\gamma^{\rho}, \gamma^{\mu}\}\gamma^{\nu} + \frac{i}{2}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}. \tag{3.26}$$

By applying (3.19), we find that

$$[S^{\mu\nu}, \gamma^{\rho}] = i(\gamma^{\mu}g^{\nu\rho} - \gamma^{\nu}g^{\mu\rho}). \tag{3.27}$$

With this relation in hand, we can now compute $[S^{\mu\nu}, S^{\rho\sigma}]$. When $\sigma \neq \rho$, we have

$$\begin{aligned} [S^{\mu\nu}, S^{\rho\sigma}] &= \frac{i}{2}[S^{\mu\nu}, \gamma^{\rho}\gamma^{\sigma}] \\ &= \frac{i}{2}[S^{\mu\nu}, \gamma^{\rho}]\gamma^{\sigma} + \frac{i}{2}\gamma^{\rho}[S^{\mu\nu}, \gamma^{\sigma}]. \end{aligned} \tag{3.28}$$

From the first to the second line, we used the property $[A, BC] = [A, B]C + B[A, C]$, where A, B, and C are operators. To continue, we employ relation (3.27) and obtain

$$[S^{\mu\nu}, S^{\rho\sigma}] = -\frac{1}{2}\gamma^{\mu}\gamma^{\sigma}g^{\rho\nu} + \frac{1}{2}\gamma^{\nu}\gamma^{\sigma}g^{\mu\rho} - \frac{1}{2}\gamma^{\rho}\gamma^{\mu}g^{\sigma\nu} + \frac{1}{2}\gamma^{\rho}\gamma^{\nu}g^{\sigma\mu}. \tag{3.29}$$

From here, we can further simplify by using (3.22). Thus, we have

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\rho\mu}S^{\nu\sigma} - g^{\sigma\nu}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}). \tag{3.30}$$

Here, we used the antisymmetry property $S^{\alpha\beta} = -S^{\beta\alpha}$ in the last two terms on the right side. This proves that such a representation is a representation of the Lorentz group. To be more precise, this representation is the so-called spinorial representation of the Lorentz algebra.

Given this representation, we can express the following Lorentz transformation using the exponential form (3.5):

$$S(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right), \tag{3.31}$$

where $\omega_{\mu\nu}$ represents an antisymmetric tensor. This tensor specifies the type of Lorentz transformation being applied, whether it is a boost, a rotation, or a combination of both [4, 8].

With this transformation in hand, we can define the fields upon which it operates. These fields are understood as spinor or Dirac fields, denoted by $\psi^a(x)$. They are four-component fields indexed by $a = 1, \ldots, 4$, which is known as a spinor index. When subjected to a Lorentz transformation, it transforms as

$$\psi^a(x) \to \psi'^a(x) = S(\Lambda)^a{}_b\psi^b(\Lambda^{-1}x). \tag{3.32}$$

It is worth noting that, throughout this work, we shall not explicitly indicate these indices. An important feature of this field is that it does not describe just one spin-1/2 particle but rather two spin-1/2 particles, namely, a particle and its antiparticle. Therefore, we do not encounter issues with the representation being 4×4.

This becomes more evident when we represent the Dirac matrices in the chiral form, using the so-called Weyl spinors, as follows:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{3.33}$$

In this form, the spinor is divided into two components: $\psi_L(x)$, referred to as a left-handed Weyl spinor, and $\psi_R(x)$ a right-handed Weyl spinor [4, 8]. The subscriptions L and R refer to the directions of the spin and linear momentum, where the former corresponds to parallel and the latter antiparallel.

3.1.3 Fermionic sector

With the spinor fields $\psi(x)$ defined to describe spin-1/2 particles, we can finally explore their behaviour under gauge transformations and the consequences for the Lagrangian governing the fermionic sector in QED. This Lagrangian, commonly referred to as Dirac's Lagrangian, is given as

$$\mathcal{L}_D = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi. \tag{3.34}$$

Here, γ^μ represents the Dirac matrices, m stands for the mass of the fermionic fields, and $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$ and $\psi(x)$ denote the Dirac spinors[3]. As we shall explore in the next section, this Lagrangian leads to Dirac's equation. However, we must be aware that here we are dealing with fields and not wave functions as initially regarded by Dirac [1] in his attempt to develop a relativistic quantum theory of the electron.

Considering that the gauge transformations in QED are based on the $U(1)$ group, we begin by defining the transformation for the Dirac spinors as

$$\psi(x) \to U_e(x)\psi(x) \qquad \text{and} \qquad \bar{\psi}(x) \to U_e^{-1}(x)\bar{\psi}(x). \tag{3.35}$$

In simple terms, spinors ψ transform equivalently to local phase rotations. Here, contrary to (3.12), the transformation is defined as $U(x)_e = e^{-ie\alpha(x)}$, where the parameter e is the group generator and is interpreted as the charge of the field $\psi(x)$. In QED, this parameter corresponds to the electron charge.

Replacing this transformation in the Lagrangian yields

$$\mathcal{L}_D \to e^{ie\alpha(x)}\bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)e^{-ie\alpha(x)}\psi = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + e\bar{\psi}\gamma^\mu \partial_\mu \alpha(x) \neq \mathcal{L}_D, \tag{3.36}$$

which tells us that the Lagrangian is not invariant. Consequently, the theory described by (3.34) is not a gauge theory since it is not invariant under local transformations. In

[3] $\bar{\psi}$ is defined in such a way that the quantity $\bar{\psi}\psi$ is a Lorentz scalar [8].

addition, we note that it is at least invariant under global transformations of the $U(1)$ group

$$\mathcal{L}_D \to e^{ie\alpha}\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)e^{-ie\alpha}\psi = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi = \mathcal{L}_D. \tag{3.37}$$

However, as we are interested in obtaining a fermionic gauge theory, some changes in the Lagrangian must cancel the new extra term that arises in (3.36).

This issue is solved by coupling the Dirac fields with the gauge fields A_μ, establishing the interaction between electromagnetic fields and matter. This method is named minimal coupling and introduces an object D_μ referred to as covariant derivative [4, 8, 10, 70], which replaces the usual derivative in the Dirac's equation. Thus, this new derivative is defined as

$$D_\mu \equiv \partial_\mu + ieA_\mu(x). \tag{3.38}$$

By substituting this covariant derivative into Dirac's Lagrangian, we obtain

$$\mathcal{L}_D = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu, \tag{3.39}$$

where we observe that the last term corresponds to the coupling of the Dirac and gauge fields.

Since we are introducing or modifying a term in the Lagrangian, it is necessary to define how it transforms under the group $U(1)$. Thus, this transforms as

$$D_\mu\psi \to U_e(x)D_\mu\psi, \tag{3.40}$$

not forgetting that the vector field within the derivative transforms according to (3.14). Therefore, transforming both the gauge and the spinor fields using

$$\psi(x) \to e^{-ie\alpha(x)}\psi(x) \qquad \text{and} \qquad A_\mu \to A_\mu + \partial_\mu\alpha(x), \tag{3.41}$$

in the Dirac Lagrangian (3.39) leads to

$$\begin{aligned}
\mathcal{L}_D \to\ & e^{ie\alpha(x)}\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)e^{-ie\alpha(x)}\psi - e\bar{\psi}\gamma^\mu\psi\left(A_\mu + \partial_\mu\alpha(x)\right)\\
&= +\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + e\bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\alpha(x)\\
&\quad - e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu - e\bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\alpha(x)\\
&= \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu\\
&= \mathcal{L}_D.
\end{aligned} \tag{3.42}$$

Finally, we observe that the Lagrangian becomes invariant to local transformations of the $U(1)$ group.

Now that we know that the Dirac and Maxwell Lagrangians are invariant under local transformations based on the group $U(1)$, we unite them into a single Lagrangian, which describes both the bosonic and fermionic sectors of QED, in addition to the interaction of their fields. This Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\not{D} - m\right)\psi, \tag{3.43}$$

where the covariant derivative is written in Feynman slash notation and corresponds to $\not{D} = \gamma^\mu D_\mu$, and $D_\mu = \partial_\mu + ieA_\mu(x)$. Rewriting the Lagrangian with the gauge covariant derivative in its explicit form, we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu, \tag{3.44}$$

which, again, the first term corresponds to electromagnetic fields, the second to Dirac spinors, and the third is an interaction term between the electromagnetic and Dirac fields. Furthermore, from this last term, we can define a current $J^\mu = \bar{\psi}\gamma^\mu\psi$, which is conserved, that is $\partial_\mu J^\mu = 0$, as long as $\psi(x)$ satisfies Dirac's equation [8].

3.2 Equations of Motion

Since we have yet to quantise the gauge and Dirac fields, the Lagrangian (3.44) merely provides a classical description of the dynamics of these fields. Bearing this in mind, our focus in this section lies in examining the equations of motion that can be derived from (3.44) and their meaning. To obtain these equations of motion, we shall use

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\Phi\right)}\right) - \frac{\partial\mathcal{L}}{\partial\Phi} = 0, \tag{3.45}$$

the generalisation of the Euler-Lagrange equation for fields, with $\Phi = \Phi(x)$ corresponding to any field [80, 82].

3.2.1 Maxwell's equations

Let us begin with the gauge fields A_μ, which are responsible for describing the bosonic sector of QED. As we will demonstrate here, the Euler-Lagrange equations for these fields lead to the derivation of the covariant Maxwell's equations.

Expressing the Lagrangian (3.44) in terms of these fields and their derivatives by computing the contraction of the electromagnetic tensors, we find

$$\mathcal{L} = \frac{1}{2}(\partial^\mu A^\nu\partial_\nu A_\mu - \partial_\nu A_\mu\partial^\nu A^\mu) - e\bar{\psi}\gamma^\mu\psi A_\mu. \tag{3.46}$$

Note that we omitted the terms containing only Dirac fields, as derivatives of these fields yield zero.

With this Lagrangian at our disposal, we proceed to solve the first term of equation (3.45):

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)}\right) = \partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \partial_\nu F^{\mu\nu}. \tag{3.47}$$

Note that we used here the product rule and the fact that [83]:

$$\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\alpha \delta^\nu_\beta.$$

Solving for the second term of equation (3.45) involves finding the derivative of the Lagrangian with respect to the gauge fields A_μ so that

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -e\bar{\psi}\gamma^\mu \psi. \tag{3.48}$$

Substituting these into the Euler-Lagrange equation gives

$$\partial_\nu F^{\mu\nu} + e\bar{\psi}\gamma^\mu\psi = 0. \tag{3.49}$$

To obtain the conventional form of covariant Maxwell's equations, we use the definition of conserved current from Subsection 3.1.3 and move it to the right side of the equation. This step leads to

$$\partial_\nu F^{\mu\nu} = -eJ^\mu. \tag{3.50}$$

Here, we notice that the derivative is contracted with the index ν of the electromagnetic tensor. To contract it with the index μ, we recall that the electromagnetic tensor is antisymmetric, meaning that $F^{\mu\nu} = -F^{\nu\mu}$. Consequently, when we exchange the indices μ and ν, we get

$$\partial_\mu F^{\mu\nu} = eJ^\nu, \tag{3.51}$$

where the change of indices changed the negative sign of the source. Hence, we arrive at the inhomogeneous Maxwell's equations in covariant form[4], corresponding to Gauss's law and Ampère's law.

The other Maxwell's equations, the homogeneous ones, are derived from (3.51) using the definition of the dual field-strength tensor $\tilde{F}^{\mu\nu}$ provided by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{3.52}$$

In this equation, $\epsilon^{\mu\nu\rho\sigma}$ is defined as a fully antisymmetric tensor of rank four [81]. In simpler terms, this tensor is the generalised Levi-Civita symbol for four dimensions. To proceed, we differentiate the dual field-strength tensor in the same way as in the equation (3.51). So one obtains

$$\begin{aligned}\partial_\mu \tilde{F}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} \\ &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\sigma A_\rho.\end{aligned} \tag{3.53}$$

From the first to the second line, we used $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$, and applied the distributive property.

[4] Remember that we are using the Heaviside-Lorentz convention, so $\mu_0 = \varepsilon_0 = 0$.

Given that the contraction of antisymmetric tensors, like $\epsilon^{\mu\nu\rho\sigma}$, with symmetric tensors, such as $\partial_\mu \partial_\rho$, yields zero, we find

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{3.54}$$

This is known as the homogeneous Maxwell's equations, corresponding to Faraday's law of induction and Gauss's law of magnetism. Therefore, by combining (3.51) and (3.54), we arrive at the comprehensive classical description of the electromagnetic theory [81].

To demonstrate that the contraction of a symmetric tensor $A_{\mu\nu}$ and an antisymmetric tensor $B^{\mu\nu}$ equals zero, we express their contraction as

$$A_{\mu\nu} B^{\mu\nu} = -A_{\nu\mu} B^{\nu\mu} = -A_{\mu\nu} B^{\mu\nu}. \tag{3.55}$$

In the first equality, we swap the indices, resulting in $B^{\nu\mu}$ introducing a factor of -1. Transitioning from the middle to the last equality, we perform index changes $\mu \to \nu$ and $\nu \to \mu$. The result we obtain implies that $A_{\mu\nu} B^{\mu\nu} = 0$. Hence, the contraction of symmetric and antisymmetric tensors is zero.

3.2.2 Dirac's equation

Upon deriving the equations of motion from the bosonic sector of QED, we now turn our attention to the other sector and derive Dirac's equation. As we did in the last subsection, we rewrite the Lagrangian (3.44) in terms of the Dirac fields ψ and $\bar{\psi}$ and their derivatives

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - \bar{\psi} m \psi - e \bar{\psi} \gamma^\mu \psi A_\mu. \tag{3.56}$$

In this expression, we have omitted terms that solely involve gauge fields.

Let us compute the first term of the equation (3.45) for both fields, where the first equation will always correspond to the field ψ and the second to the field $\bar{\psi}$:

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \psi \right)} \right) = \partial_\nu \left(\bar{\psi} i \gamma^\nu \right) = i \gamma^\nu \partial_\nu \bar{\psi} \quad \text{and} \quad \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \bar{\psi} \right)} \right) = 0. \tag{3.57}$$

Now, calculating the second term of the Euler-Lagrange equation, which is a simple differentiation with respect to the fields, we have

$$\frac{\partial \mathcal{L}}{\partial \psi} = -\bar{\psi} m - e \bar{\psi} \gamma^\mu A_\mu \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i \gamma^\mu \partial_\mu \psi - m \psi - e \gamma^\mu \psi A_\mu. \tag{3.58}$$

Substituting these results into (3.45) leads to

$$i \gamma^\nu \partial_\nu \bar{\psi} + m \bar{\psi} + e \bar{\psi} \gamma^\mu A_\mu = 0 \quad \text{and} \quad - i \gamma^\mu \partial_\mu \psi + m \psi + e \gamma^\mu \psi A_\mu = 0. \tag{3.59}$$

To put these results in the usual form of Dirac's equation, we move the last term of each equation to the right side of the equal sign and multiply the second equation by

-1. Following these steps, we obtain

$$i\gamma^\nu\partial_\nu\bar{\psi} + m\bar{\psi} = -e\bar{\psi}\gamma^\mu A_\mu \quad \text{and} \quad i\gamma^\mu\partial_\mu\psi - m\psi = e\gamma^\mu\psi A_\mu. \tag{3.60}$$

Additionally, in the second equation, we change the indices from $\gamma^\mu\partial_\mu$ to $\gamma^\nu\partial_\nu$, and factor out the fields in both equations. These operations result in

$$(i\gamma^\nu\partial_\nu + m)\bar{\psi} = -\bar{\psi}\gamma^\mu A_\mu \quad \text{and} \quad (i\gamma^\nu\partial_\nu - m)\psi = \gamma^\mu\psi A_\mu, \tag{3.61}$$

Dirac's equations in the presence of electromagnetic fields described by A_μ.

For $A_\mu = 0$, these equations yield Dirac's equations, one of the early attempts to formulate a relativistic quantum theory of the electron. Note that in this attempt, ψ was regarded as a wave function rather than a field as we derived. However, this formulation came with several problems, including the issue of negative energy eigenstates.

3.3 Quantisation of Gauge Fields

The process of quantisation in the functional formalism, as discussed previously, involves integrating out the fields that we aim to quantise. This procedure corresponds to computing the n-point functions, or Green's functions, of the theory using both equations (2.57) and (2.56). Considering this, as we focus on the quantisation of gauge fields $A_\mu(x)$, the following functional integral must be solved

$$Z[J] = \int DA \exp\left(iS[A] + i\int d^4x\ J^\mu(x)A_\mu(x)\right). \tag{3.62}$$

In this expression, $J^\mu(x)$ denotes the sources, $S[A]$ stands for the classical action dependent on the gauge fields A^μ, and $DA \equiv DA^0DA^1DA^2DA^3$.

Following a similar approach to our treatment of scalar fields, the action for the gauge fields should be expressed in quadratic form as in (2.60), namely

$$S[A] = \int d^4x\ A_\mu\, O^{\mu\nu}\, A_\nu. \tag{3.63}$$

The reason for writing the action in this form is so that we can use the result (2.16) and then obtain the generating functional

$$Z[J_\mu] = Z[0] \exp\left(-\frac{1}{2}\int d^4x\, d^4x'\ J_\mu(x)D_F^{\mu\nu}(x-x')J_\nu(x')\right), \tag{3.64}$$

where $D_F^{\mu\nu}$ corresponds to the inverse of the operator $O^{\mu\nu}$. Note that specifying $Z[0]$ is unnecessary since this term is cancelled by the denominator in (2.56).

Therefore, we express the action for the gauge fields (3.11) that we shall quantise as

$$\begin{aligned} S[A] &= \int d^4x\ \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\ &= \frac{1}{2}\int d^4x\ \left(\partial_\nu A_\mu\partial^\nu A^\mu - \partial_\nu A_\mu\partial^\mu A^\nu\right). \end{aligned} \tag{3.65}$$

Following this, we perform an integration by parts in the second line, which leads to the action in the quadratic form

$$\begin{aligned} S[A] &= \frac{1}{2}\int d^4x\ A_\mu(x)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu(x) \\ &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)A_\nu(-k). \end{aligned} \tag{3.66}$$

In this equation, from the first to the second line, we applied a Fourier transformation from position to moment space[5] using

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4}\ e^{-ik\cdot x}A^\mu(k), \tag{3.67}$$

with $A^\mu(k)$ representing the gauge field in momentum space. The reason for this step is to simplify the next operations to be performed, as we are now working with functions rather than operators such as the derivative ∂_μ. From this action in quadratic form, we observe that

$$O^{\mu\nu}(x) = (\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu) \qquad \text{and} \qquad O^{\mu\nu}(k) = (-k^2 g^{\mu\nu} + k^\mu k^\nu). \tag{3.68}$$

Since we have obtained the quadratic form of the action, $D_F^{\nu\rho}(x - x')$ must be determined so that we can find the generating functional (3.64). Similar to the scalar field in the momentum space, $D_F^{\nu\rho}(k)$ must satisfy

$$\left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right) D_F^{\nu\rho}(k) = i\delta_\mu{}^\rho. \tag{3.69}$$

However, this equation has no solution because $O^{\mu\nu}$ in both the position and momentum spaces is singular due to gauge invariance. This becomes evident when considering eigenfunctions $A_\mu(k) = k_\mu\alpha(k)$, where α is any scalar function. By inserting these into (3.69), we derive

$$\begin{aligned} \left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right) k^\nu\alpha(k) &= \left(-k^2 g_{\mu\nu}k^\nu + k_\mu k_\nu k^\nu\right)\alpha(k) \\ &= \left(-k^2 k_\mu + k_\mu k^2\right)\alpha(k) \\ &= 0. \end{aligned} \tag{3.70}$$

Consequently, it presents eigenvalues equal to zero, indicating that its determinant is also null, hence it is not invertible.

If we apply this result to (3.66), we find that the path integral is badly defined, as we are integrating over a continuous infinity of physically equivalent field configurations, leading to an overcounting of these configurations due to gauge invariance. Here, we can better understand the ambiguity in the description of gauge fields in our theory. Since there are infinitely many different possible choices for the function $\alpha(x)$, we encounter an infinite number of field configurations that produce the same dynamics, as seen in (3.15).

[5] This transformation is shown in more detail in Chapter 5.

Essentially, the path integral treats A_μ and its gauge transformation A'_μ as two distinct field configurations. However, they should be considered the same as they correspond to the same action, given the gauge invariance, and therefore represent the same physical field configuration. To resolve this ambiguity, we employ a technique called gauge fixing.

The gauge fixing method developed by Faddeev and Popov in [84] involves eliminating redundant terms from the path integral, thereby solving it for distinct physical configurations [8–10, 70]. In essence, the concept aims to express a path integral as something similar to

$$N \int D\alpha \int D\bar{A}\, e^{iS[\bar{A}]}, \tag{3.71}$$

where N is a constant, the integral in $D\bar{A}$ corresponds to an integration over all gauge fields that are not physically equivalent, whereas the one in $D\alpha$ corresponds to an integration over all other possible configurations that can be generated from $\alpha(x)$, through the gauge transformations (3.13).

As the derivation of this technique falls outside the scope of this dissertation, we present that it leads to a modified classical action that takes the following form:

$$S[A] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right), \tag{3.72}$$

where the second term being integrated represents the gauge fixing term, ensuring that $O^{\mu\nu}$ remains non-singular. This allows for the computation of the path integral and, consequently, the quantisation of the gauge fields A_μ. Here, ξ is an arbitrary constant that parameterises any gauge choice. This choice depends on the specifics of the problems being studied and can either facilitate or complicate the calculations that rely on it.

Proceeding, we reorganise the terms of the action (3.72) and integrate them by parts. Afterwards, we performed a Fourier transform to momentum space, resulting in

$$\begin{aligned} S[A] &= \frac{1}{2}\int d^4x\, A_\mu(x)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu + \frac{1}{\xi}\partial^\mu\partial^\nu\right)A_\nu(x) \\ &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}A_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu - \frac{1}{\xi}k^\mu k^\nu\right)A_\nu(-k). \end{aligned} \tag{3.73}$$

Therefore, we see that the addition of the gauge-fixing term leads to a modification of the equation (3.69), which is changed to

$$\left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k_\mu k_\nu\right] D_F^{\nu\rho}(k) = i\delta_\mu{}^\rho. \tag{3.74}$$

The solution to this equation is straightforward by proposing an ansatz

$$D_F^{\nu\rho}(k) = Ag^{\nu\rho} + Bk^\nu k^\rho, \tag{3.75}$$

where A and B are functions of k^2. After substituting the proposed ansatz into (3.74) and expanding the terms, we obtain

$$-k^2 g_{\mu\nu} A g^{\nu\rho} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu A g^{\nu\rho} - k^2 g_{\mu\nu} B k^\nu k^\rho + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu B k^\nu k^\rho = i\delta_\mu{}^\rho. \quad (3.76)$$

By using the identity $g_{\mu\nu} g^{\nu\rho} = \delta_\mu{}^\rho$, along with other simplifications, we arrive at

$$-Ak^2 \delta_\mu{}^\rho + A k_\mu k^\rho - \frac{1}{\xi} A k_\mu k^\rho - \frac{1}{\xi} B k^2 k_\mu k^\rho = i\delta_\mu{}^\rho. \quad (3.77)$$

To find the coefficients, we compare them to both sides of the equation. Thus, for the coefficient A we recognise that $-Ak^2\delta_\mu{}^\rho = i\delta_\mu{}^\rho$, leading to $A = -\frac{i}{k^2}$. Substituting this value and isolating B yields

$$B = -\frac{\xi}{k^2 k_\mu k^\rho} \frac{i}{k^2} k_\mu k^\rho + \frac{1}{k^2 k_\mu k^\rho} \frac{i}{k^2} k_\mu k^\rho = \frac{i}{k^4}(1 - \xi). \quad (3.78)$$

Therefore, we find that the functions A and B are given by

$$A = -\frac{i}{k^2} \quad \text{and} \quad B = \frac{i}{k^4}(1 - \xi). \quad (3.79)$$

These coefficients can be replaced in (3.75) leading to

$$D_F^{\nu\rho}(k) = -\frac{i}{k^2} g^{\nu\rho} + \frac{i}{k^4}(1 - \xi) k^\nu k^\rho. \quad (3.80)$$

We can factor out $-\frac{i}{k^2}$, rename the indices, and add a term $i\epsilon$ which tends to 0 in the denominator, as we did in (2.68), which is necessary to specify the integral contour to return to position space. Thus, obtaining

$$D_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1 - \xi)\frac{k^\mu k^\nu}{k^2}\right). \quad (3.81)$$

Here, the constant ξ can assume any value as mentioned before, although the most commonly used in QED are the Landau gauge $\xi = 0$ and the Feynman gauge $\xi = 1$ [4, 8].

In QED, as we consider interactions with matter, the quantisation of the gauge fields $A^\mu(k)$ corresponds to the quantisation of the electromagnetic fields. The result we obtained is known as the photon propagator. This expression describes the propagation of a photon, an excitation of the quantised fields $A^\mu(k)$ with momentum k.

In position space, such a propagator corresponds to the amplitude for a photon propagating between the points x and x'. Through a Fourier transformation, and by setting $\xi = 1$, the photon propagator in position space is given as

$$D_F^{\mu\nu}(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\mu\nu}}{k^2 + i\epsilon} e^{-ik(x-x')}. \quad (3.82)$$

Such a propagator can be represented as a Feynman diagram:

$D_F^{\mu\nu}(x_1 - x_2) = x_1$ x_2 . (3.83)

As we substitute this result into (3.64), we arrive at the generating functional for gauge fields.

3.4 Quantisation of Dirac Fields

Before exploring the generating functional for the spinor fields ψ and $\bar{\psi}$ and then proceeding with their quantisation, we must first discuss the concept of anticommuting numbers and their properties.

As Dirac fields obey Fermi-Dirac statistics, they exhibit canonical anticommutation relations, unlike the bosonic fields previously discussed, which follow Bose-Einstein statistics and thus commute. This statistical distinction dictates that the temporal ordering of field operators in (2.56) entails a negative sign upon interchange of two operators [4, 8, 10, 70].

Consequently, certain concepts and notions introduced in Chapter 2, such as the sources $J(x)$ and the Gaussian integral result (2.15), require redefinition to accommodate this anticommutation property. Additionally, it is pertinent to mention that the path integral holds for any field, whether it is bosonic or fermionic.

3.4.1 Grassmann numbers

These particular numbers are known as Grassmann numbers, or variables, and are defined by their anticommutative property. This means that two Grassmann numbers η_i and η_j satisfy the condition

$$\eta_i \eta_j = -\eta_j \eta_i, \tag{3.84}$$

where the subscripts i and j are employed to distinguish between the Grassmann numbers [8–10, 70]. From this, it is straightforward to see that the square of a Grassmann number equals zero

$$\eta^2 = \eta\eta = -\eta\eta = 0. \tag{3.85}$$

With these properties established, the most general function with one Grassmann number η is given by a Taylor expansion up to the first power of η, namely

$$f(\eta) = a + b\eta, \quad a, b \in \mathbb{R}. \tag{3.86}$$

Note that the constants a and b could be replaced by complex numbers, but unlike real numbers, they do not commute with Grassmann numbers.

We must define the differentiation operation for these numbers. Accordingly, a function $f(\eta)$ can be differentiated with respect to η using the following defined rules:

$$\frac{\partial}{\partial \eta}\eta = 1 \quad \text{and} \quad \frac{\partial}{\partial \eta} a = 0. \tag{3.87}$$

Additionally, the derivative operator exhibits the anticommutation property, similar to Grassmann numbers [8, 69, 70]. In other words, when considering the product of two derivatives, we have

$$\frac{\partial}{\partial \eta_i}\frac{\partial}{\partial \eta_j} = -\frac{\partial}{\partial \eta_j}\frac{\partial}{\partial \eta_i}. \tag{3.88}$$

This property also affects the usual differentiation properties. For example, the product rule is expressed as

$$\frac{\partial}{\partial \eta_i}(\eta_j \eta_k) = \left(\frac{\partial \eta_j}{\partial \eta_i}\right) \eta_k - \eta_j \left(\frac{\partial \eta_k}{\partial \eta_i}\right) = \delta_{ij}\eta_k - \delta_{ik}\eta_j, \tag{3.89}$$

in which δ_{ij} and δ_{ik} are Kronecker deltas.

Before moving on to the definition of the integration of Grassmann numbers, we can perform an example in which we have a function of two Grassmann variables $g(\eta_i, \eta_j) = a + b\eta_i + c\eta_j + e\eta_i\eta_j$, where the coefficients are real numbers [69]. Differentiating $g(\eta_i, \eta_j)$ with respect to n_i leads to

$$\frac{\partial}{\partial \eta_i} g(\eta_i, \eta_j) = b + e\eta_j, \tag{3.90}$$

while differentiating $g(\eta_i, \eta_j)$ with respect to η_j leads to

$$\frac{\partial}{\partial \eta_j} g(\eta_i, \eta_j) = c + e\frac{\partial}{\partial \eta_j}(\eta_i \eta_j) = c - e\eta_i, \tag{3.91}$$

where we utilised the product rule (3.89). Therefore, we note that, unlike the usual differentiation rules, a negative sign appears in the second term, given the anticommutation property.

Since we have established the operation of differentiation for Grassmann numbers, it is natural that we also define the integration operation as well. The integral of Grassmann numbers is referred to as the Berezin integral and follows two axioms. Firstly, it needs to be linear, so that

$$\int_{-\infty}^{\infty} d\eta \; \alpha \, f(\eta) = \alpha \int_{-\infty}^{\infty} d\eta \; f(\eta), \tag{3.92}$$

where α is a constant, $f(\eta)$ is given by (3.86). Secondly, the integral of a total derivative is null [9, 70]. This means that

$$\int_{-\infty}^{\infty} d\eta \; \frac{\partial}{\partial \eta} f(\eta) = 0. \tag{3.93}$$

From these axioms, we define the following properties:

$$\int_{-\infty}^{\infty} d\eta \; \eta = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} d\eta = 0, \tag{3.94}$$

Due to these properties, the integration of our most general function (3.86) gives

$$\int d\eta \; f(\eta) = \int d\eta \; a + b \int d\eta \; \eta = b. \tag{3.95}$$

In this integration, we notice that the outcome matches that of differentiating a Grassmann number, thus

$$\int d\eta \; f(\eta) = \frac{\partial}{\partial \eta} f(\eta). \tag{3.96}$$

This statement can be illustrated with the following example.

Given a function of two Grassmann variables, $g(\eta_i, \eta_j) = a + b\eta_i + c\eta_j + e\eta_i\eta_j$ like the one that was used as an example for differentiation, the integrating with respect to η_i leads to

$$\int d\eta_i\ (a + b\eta_i + c\eta_j + e\eta_i\eta_j) = b + e\int d\eta_i\ \eta_i\eta_j = b + e\eta_j\int d\eta_i\ \eta_i = b + e\eta_j. \tag{3.97}$$

Note that η_j changed position with both η_i and $d\eta_i$, thus changing the sign twice. However, since $(-1)^2 = 1$, the sign remains positive. Integrating $g(\eta_i, \eta_j)$ with respect to η_j gives

$$\int d\eta_j\ (a + b\eta_i + c\eta_j + e\eta_i\eta_j) = c + e\int d\eta_j\ \eta_i\eta_j = c - e\eta_i\int d\eta_j\ \eta_j = c - e\eta_i. \tag{3.98}$$

Comparing these results with those of (3.90) and (3.91), we observe that the relation (3.96) holds.

In addition to defining the integration and differentiation operations, we need to generalise the notion of complex numbers to Grassmann numbers. Therefore, complex Grassmann numbers are written as

$$\eta \equiv \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2) \quad \text{and} \quad \eta^* \equiv \frac{1}{\sqrt{2}}(\eta_1 - i\eta_2), \tag{3.99}$$

where η^* is the complex conjugated of η [8, 70]. Regarding the integration of the numbers, we adopt the convention

$$\int d\eta^* d\eta\ (\eta\eta^*) = 1. \tag{3.100}$$

This convention is related to the order of integration of Grassmann numbers. Here, we choose to integrate first with respect to $d\eta$ and then with respect to $d\eta^*$. This means the innermost integral is evaluated first. If we were to perform these integrations in the reverse order, the integration would result in -1. The same thing happens in the derivative (3.4.1), which we choose to differentiate the Grassmann number first.

As we have already seen, the functional quantisation process involves integrating an exponential with the classical action. Taking this into consideration, our attention naturally turns to integrating functions involving complex Grassmann numbers in this structure. Thus, our goal is to evaluate the following integral:

$$I_M = \left(\prod_i \int d\eta_i^* d\eta_i\right) e^{-\boldsymbol{\eta}^* \mathbf{B} \boldsymbol{\eta}}, \tag{3.101}$$

where we introduce the Grassmann-valued n-component vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_n)$ and its complex conjugate $\boldsymbol{\eta}^*$, along with a Hermitian matrix denoted as $\mathbf{B}$ with eigenvalues b_i.

Before performing the multidimensional integral, let us first solve a simpler one-dimensional case, which is

$$I = \int d\eta^* d\eta\ e^{-\eta^* a \eta}. \tag{3.102}$$

This integral is solved by expanding the exponential using a Taylor series and then applying the convention (3.100) so that

$$I = \int d\eta^* d\eta \, (1 - \eta^* a \eta) = a \int d\eta^* d\eta \; \eta\eta^* = a. \tag{3.103}$$

Now we can address (3.101). To solve this integral, we employ unitary transformations of the form $\eta_i \to U_{ij}\eta_j$ to diagonalise the matrix $\mathbf{B}$ [8]. Applying this transformation to (3.101) yields

$$\begin{aligned} I_M &= \left(\prod_i \int d\eta_i^* d\eta_i\right) \exp\left(-\sum_i \eta_i^* b_{ij} \eta_i\right) \\ &= \prod_i \int d\eta_i^* d\eta_i \; \exp\left(-\eta_i^* b_{ij} \eta_i\right). \end{aligned} \tag{3.104}$$

As we use the result from the simpler integral (3.103), we hence obtain

$$\int d\boldsymbol{\eta} \, d\bar{\boldsymbol{\eta}} \; e^{\bar{\boldsymbol{\eta}} \mathbf{B} \boldsymbol{\eta}} = \prod_i b_i = \det \mathbf{B}. \tag{3.105}$$

With this established, the Gaussian integral (2.15) utilised in the quantisation of scalar and gauge fields (bosonic fields) can be generalised to fermionic fields as follows:

$$\int D\psi \, D\bar{\psi} \; e^{\bar{\psi}\mathbf{B}\psi + \bar{\boldsymbol{\eta}}\psi + \bar{\psi}\boldsymbol{\eta}} = \mathcal{N} \det \mathbf{B} \; e^{-\bar{\boldsymbol{\eta}} \, \mathbf{B}^{-1} \boldsymbol{\eta}}, \tag{3.106}$$

where $\mathcal{N}$ is a constant result of the integration [8–10, 69, 70, 72]. Taking these results into consideration, we can now write a generating functional for fermions, as detailed in the subsequent subsection.

3.4.2 Functional integral for fermions

As previously mentioned, the spinors $\psi(x)$ and $\bar{\psi}(x)$ obey Fermi statistics and thus exhibit the anticommutation property. To further emphasise this property, we can represent them as a linear combination of complex functions $\phi_i(x)$ and Grassmann numbers ψ_i [8, 69], yielding

$$\psi(x) = \sum_i \psi_i \phi_i(x). \tag{3.107}$$

As we express them in this manner, these fields satisfy the anticommutation relation (3.84), similar to Grassmann numbers. Based on this definition, along with the others established throughout this section, we can express the generating functional for free spinor fields as

$$Z[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} \, \exp\left[i \int d^4x \, \bar{\psi}(x) \left(i\partial\!\!\!/ - m\right) \psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right], \tag{3.108}$$

with Grassmann-valued sources $\bar{\eta}(x)$ and $\eta(x)$. Solving this integral results in the quantisation of Dirac fields. So, our goal here is to evaluate the integral (3.108), which presents the quadratic form

$$S[\psi] = \int d^4x \; \bar{\psi}(x) \, O(x) \psi(x). \tag{3.109}$$

Here, $O(x) = \left(i\partial\!\!\!/ - m\right)$, and its inverse is defined as $B_F(x - x')$.

We can proceed by employing the relation (3.106), which solves the integral (3.108), leading to

$$Z[\eta, \bar{\eta}] = Z[0,0] \exp\left(-\int d^4x\, d^4x'\; \bar{\eta}(x) B_F(x - x')\eta(x')\right). \tag{3.110}$$

In contrast to the quantisation of gauge fields, we will specify the fermionic generating functional for $\bar{\eta} = \eta = 0$, that is $Z[0, 0]$,

$$\begin{aligned} Z[0,0] &= \int D\psi D\bar{\psi}\, \exp\left[i \int d^4x\, \bar{\psi}(x) \left(i\partial\!\!\!/ - m\right) \psi(x)\right] \\ &= \mathcal{N} \det\left(i\partial\!\!\!/ - m\right). \end{aligned} \tag{3.111}$$

Here, we used the result (3.105), and $\mathcal{N}$ corresponds to a constant. The reason for this computation will be better explained as we examine the effective Lagrangian for fermions in Section 4.2.

The last step to have quantised spinor fields is the computation of the inverse $B_F(x - x')$. Similar to how we approached the quantisation of scalar and vector fields, we need to solve

$$\left(i\partial\!\!\!/ - m\right) B_F(x - x') = i\delta(x - x'). \tag{3.112}$$

To solve this equation, we perform the following Fourier transform for the momentum space:

$$B_F(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} B_F(k) \quad \text{and} \quad \delta(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-x')}, \tag{3.113}$$

where the function $B_F(k)$ is the operator $B_F(x - x')$ in momentum space. Substituting these equations back to (3.112) and substituting $\partial\!\!\!/ = \gamma^\mu \partial_\mu$, we have

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m) \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} S(k) &= \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} \left(\gamma^\mu k_\mu - m\right) S(k) \\ &= i \int \frac{d^4k}{(2\pi)^4}. \end{aligned} \tag{3.114}$$

From these integrals, we obtain

$$\left(k\!\!\!/ - m\right) B_F(k) = i, \tag{3.115}$$

with $k\!\!\!/ = \gamma^\mu k_\mu$. Therefore,

$$B_F(k) = \frac{i}{k\!\!\!/ - m}. \tag{3.116}$$

This result is known as the propagator of a fermion in momentum space. In other words, the excitations of the quantised fermion fields ψ and $\bar{\psi}$ produce fermions in momentum space with momentum k and mass m.

In position space, by performing a Fourier transformation, this propagator corresponds to the probability amplitude of a fermion propagating between the points x to x' as we can see in

$$B_F(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-x')} \frac{i}{\not{k} - m + i\epsilon}. \tag{3.117}$$

In this expression, we added a term $i\epsilon$ that tends to 0, as we did in (2.68), to specify the contour of the integration. Such a propagator can be represented as a Feynman diagram:

$$B_F(x_1 - x_2) = x_1 \longrightarrow x_2 \quad . \tag{3.118}$$

As we substitute this expression into (3.110), we obtain the generator of all n-point functions for spinor fields.

After having explored the quantisation of free gauge fields in Section 3.3 and free spinorial fields in this section, we can formulate the generating functional for interactive QED as

$$Z[A, \psi, \bar{\psi}] = \int DA\, D\psi\, D\bar{\psi}\, \exp\left(iS_0[A, \psi, \bar{\psi}]\right) \exp\left(1 - ie \int d^4x\, \bar{\psi}\gamma_\mu \psi A_\mu + \cdots\right). \tag{3.119}$$

In this equation, we have utilised the perturbation method by considering the coupling constant e small. Also, $S_0[A, \psi, \bar{\psi}]$ represents the free classical action for the Lagrangian (3.44) with source terms $J(x)$, $\eta(x)$ and $\bar{\eta}(x)$.

Upon solving the integrals in (3.119), we achieve the complete quantisation of both the electromagnetic and Dirac fields, thereby establishing the quantum theory of electromagnetism. However, due to the presence of the interaction term, there are no techniques to solve this integral exactly. Consequently, in the next chapter, we shall quantise Dirac fields, treating the electromagnetic fields as classical, in order to examine their interactions.

4 Schwinger's Proper-Time Formulation

The effective action plays a crucial role in QFT by encapsulating quantum corrections to the classical action. In quantum electrodynamics, for instance, these corrections can emerge from integrating out massive fermionic fields in the presence of an electromagnetic background, leading to modifications in the dynamics of these fields.

Note that the effective action we shall explore here is somewhat different from the 1PI effective action in Section 2.5. It is known as the low energy or Wilsonian effective action[1]. While the 1PI effective action refers to the Legendre transformation of the generating functional of connected diagrams $W[J]$, the Wilsonian effective action results from the integration of only a few degrees of freedom. In this case, as we are only integrating out the massive fermionic fields, our focus is solely on the system's low energy scale rather than encompassing the entirety of it, as we would with the 1PI effective action.

One phenomenon stemming from the corrections arising from the quantisation of these fermionic fields is the Schwinger effect, which predicts the creation of real electron-positron pairs in the vacuum when subjected to an intense electric field. This effect plays an important role in understanding the vacuum in quantum electrodynamics, as it sheds light on the vacuum's instability and possible decay.

In Section 4.1, we shall begin by introducing the Schwinger proper-time formalism. This will be followed by a derivation of the effective action for fermions in Section 4.2. Proceeding this, the derivation of the Euler-Heisenberg effective action will be covered in Section 4.3. Finally, we will discuss the Schwinger effect in Section 4.4.

4.1 Proper-Time Formalism

In order to study the Schwinger effect, it is necessary to first explore the so-called Schwinger proper-time formalism. This is a technique for evaluating loop integrals in quantum field theory and revolves around the following identity:

$$\frac{i}{A+i\varepsilon} = \int_0^\infty ds\, e^{is(A+i\varepsilon)}, \tag{4.1}$$

where the parameter s known as proper-time is introduced. It is important to be aware that this integral is convergent as long as $\varepsilon > 0$ and $A \in \mathbb{R}$ [10, 25, 87].

As a first example, to familiarise ourselves with this formalism and derive some physical meaning from a concept that we have already explored, we shall revisit the

[1] For more on the topic, refer to [85, 86].

Feynman propagator for a scalar field using this approach. Let us begin by recalling that this propagator is expressed as

$$D_F(x-x') = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-x')} \frac{i}{k^2 - m^2 + i\varepsilon}. \tag{4.2}$$

This expression was derived in Subsection 2.3.1 through the quantisation of massive scalar fields.

By employing the identity (4.1) and noting that $A = k^2 - m^2$, we arrive at

$$D_F(x-x') = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-x')} \int_0^\infty ds\; e^{is(k^2-m^2+i\varepsilon)}. \tag{4.3}$$

As we are interested in gaining physical intuition from this formalism through this propagator, we shall leave the resolution of the momentum integral for now.

To proceed, we will apply some concepts from non-relativistic quantum mechanics to this propagator. As a first step, we introduce the one-particle Hilbert space spanned by position eigenstates, denoted as $|x\rangle$ [10, 87, 88]. These eigenstates obey the following eigenvalue equation:

$$\hat{x}^\mu \,|x\rangle = x^\mu \,|x\rangle\,, \tag{4.4}$$

where $\hat{x}^\mu$ is the position operator and x^μ is its eigenvalue. Note that Schwinger's method does not specify whether the temporal coordinate is quantised, i.e. promoted to a "time operator" or a "proper-time operator". The introduction of this four-vector operator $\hat{x}^\mu$ and its eigenstates is part of the method and serves only as an intermediate step. Likewise, a momentum operator $\hat{k}$ with eigenvalues k^μ satisfies

$$\hat{k}^\mu \,|k\rangle = k^\mu \,|k\rangle\,, \tag{4.5}$$

where $|k\rangle$ is a momentum eigenstate. In addition, we recall that momentum and position eigenvalues obey

$$\langle k|x\rangle = e^{ikx}. \tag{4.6}$$

Having these equations established, we observe that the first exponential in (4.3) can be rewritten in terms of position and momentum eigenstates using the relation (4.6), leading to

$$D_F(x-x') = \int \frac{d^4k}{(2\pi)^4} \int_0^\infty ds\;\; \langle x'|k\rangle \,\langle k|x\rangle \, e^{is(k^2-m^2+i\varepsilon)}. \tag{4.7}$$

To simplify the momentum integration, we note that

$$e^{isk^2}\,\langle k|x\rangle = \langle k|e^{-is\hat{H}}|x\rangle\,, \tag{4.8}$$

where we define the Hamiltonian[2] as $\hat{H} = -\hat{k}^2$. Additionally, we can utilise the completeness relation for momentum eigenstates

$$(2\pi)^{-4}\int d^4k\;|k\rangle\,\langle k| = \mathbb{1}. \tag{4.9}$$

[2] Note that in this definition of the Hamiltonian, we are not including the mass m.

Thus, the propagator (4.7) can be further expressed as

$$D_F(x-x') = \int_0^\infty ds\, e^{-s\varepsilon}\, e^{-ism^2}\, \langle x'|e^{-is\hat{H}}|x\rangle = \int_0^\infty ds\, e^{-s\varepsilon}\, e^{-ism^2}\, \langle x';0|x;s\rangle\,, \tag{4.10}$$

where $|x;s\rangle \equiv e^{-is\hat{H}}\,|x\rangle$, showing that states can be represented in both the Heisenberg and Schrödinger pictures. Here, the interpretation of the propagator shifts slightly from what we observed in Section 2.3.1, as it now represents the amplitude for a particle to propagate from one position, denoted by x, to another, x', over a specific proper-time interval s, integrated over all possible proper-time intervals [10].

Returning to the propagator (4.3), its Gaussian integral with respect to the momentum k can be solved exactly using the result (2.15). To begin, we separate the terms that are multiplied with k and those that are not into two exponentials:

$$D_F(x-x') = \int_0^\infty ds \int \frac{d^4k}{(2\pi)^4}\, e^{isk^2+i(x-x')k}\, e^{(-ism^2+i^2s\varepsilon)}. \tag{4.11}$$

In addition, we identify that $A = -2isg^{\mu\nu}$. This leads through (2.15) to

$$D_F(x-x') = \frac{1}{16\pi^4}\int_0^\infty ds\, \sqrt{\frac{(2\pi)^4}{\det(-2isg^{\mu\nu})}} e^{-\frac{1}{2}i(x-x')^2\frac{i}{2is}}\, e^{(-ism^2-s\varepsilon)}, \tag{4.12}$$

the propagator in position space with respect to the positions x and x' in the proper-time representation.

We can organise the terms of this result using some properties. First, by employing the property $\det(c\mathbf{B}) = c^n \det\mathbf{B}$, where c is a scalar and $\mathbf{B}$ is an $n\times n$ matrix, and noting that the determinant of the metric is -1. Thus, the propagator simplifies to

$$D_F(x-x') = \frac{1}{16\pi^4}\int_0^\infty ds\, \sqrt{\frac{16\pi^4}{-16s^4}} e^{-i\frac{(x-x')^2}{4s}}\, e^{(-ism^2-s\varepsilon)}. \tag{4.13}$$

Observe that we have simplified certain terms within the exponentials. We then obtain the free propagator of a scalar field in terms of proper-time:

$$D_F(x-x') = \frac{-i}{16\pi^2}\int_0^\infty \frac{ds}{s^2}\, e^{-i\left[\frac{(x-x')^2}{4s}+sm^2-i\varepsilon s\right]}. \tag{4.14}$$

Solving this integral yields the propagator in position space. However, as our focus does not lie on scalar fields, we will refrain from solving this integral.

4.2 Effective Lagrangian for Fermions

To obtain the effective Euler-Heisenberg Lagrangian and thus explore the pair production mechanism, we first need to derive an effective action for QED resulting from the quantisation of massive fermionic fields.

As we have already performed the integration of Dirac fields in Section 3.4, we shall generalise the result (3.111) by introducing interactions with electromagnetic fields, replacing the usual derivative with a covariant one, expressed in (3.38). Therefore, the path integral for fermionic fields and its result are expressed as

$$\int D\psi\, D\bar{\psi}\, \exp\left(i\int d^4x\, \bar{\psi}\left(i\not{D}-m\right)\psi\right) = \mathcal{C}\det\left(i\not{D}-m\right). \tag{4.15}$$

Here, we employed the relation (3.106), with $\mathcal{C}$ representing a constant. Henceforth, we will ignore it, as it will only be an additive constant in the subsequent steps. Furthermore, it is important to note that we are integrating only over the fermionic fields, adopting the gauge fields $A^\mu(x)$ as a background since we did not solve the path integral for these fields. This approach is adopted because our focus lies within an energy scale small enough to avoid the need for the quantisation of electromagnetic fields and consideration of interactions with photons.

As previously demonstrated in Section 2.3.1, the determinant of an operator can be expressed in an exponential form. Therefore, (4.15) is now expressed as

$$\int D\psi\, D\bar{\psi}\, \exp\left(i\int d^4\, \bar{\psi}\left(i\not{D}-m\right)\psi\right) = \exp \mathrm{Tr}[\mathrm{tr}(\log(i\not{D}-m))], \tag{4.16}$$

where tr denotes a trace obtained by integrating over the states $|x\rangle$, while Tr represents a Dirac trace, corresponding to a trace over the Dirac matrices γ^μ.

With this representation, we construct an effective action $\Gamma[A]$ that includes both the classical gauge fields and the quantised Dirac fields, expressed above, as

$$i\Gamma[A] = i\int d^4x\, \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) + \mathrm{Tr}[\mathrm{tr}(\log(i\not{D}-m))]. \tag{4.17}$$

To incorporate the quantum correction term into the integral and derive the effective Lagrangian, we write the trace tr explicitly, leading to

$$i\Gamma[A] = i\int d^4x\, \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\mathrm{Tr}\left[\langle x|\log(i\not{D}-m)|x\rangle\right]\right). \tag{4.18}$$

Then, we identify the effective Lagrangian as

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\mathrm{Tr}\left[\langle x|\log(i\not{D}-m)|x\rangle\right]. \tag{4.19}$$

Before we proceed, let us briefly analyse this effective Lagrangian. As discussed earlier, the effective Lagrangian encodes quantum corrections to the classical Lagrangian. In our case, the first term on the right-hand side of the equation describes the classical dynamics of electromagnetic fields. In contrast, the second term arises from quantising fermionic fields, representing these quantum corrections.

An important effect arising from the quantisation of these fermionic fields is the creation and annihilation of virtual electron-positron pairs. By quantising these fields, the vacuum of our theory is now permeated by the so-called quantum fluctuations, a product of these processes of production and annihilation.

The intriguing aspect here is that, due to the presence of an electromagnetic background, these virtual pairs can polarise. Consequently, the vacuum begins to behave as a polarisable medium, which induces modifications in the dynamics of classical electromagnetic fields, giving rise to phenomena discussed in Chapter 1, such as pair production, which is of interest within this dissertation.

Having derived the effective Lagrangian (4.19), our focus turns to expressing it in the proper-time representation. For this, we differentiate[3] it with respect to m^2, yielding

$$\frac{d}{dm^2}\mathcal{L}_{\text{eff}} = -\frac{i}{2m}\text{Tr}\left[\langle x|\frac{i}{\not{D}+im}|x\rangle\right]. \tag{4.20}$$

To simplify this expression, we divide and multiply the term between states by $(i\not{D}+m)$, resulting in

$$\begin{aligned}\frac{d}{dm^2}\mathcal{L}_{\text{eff}} &= -\frac{i}{2m}\text{Tr}\left[\langle x|\frac{i}{\not{D}+im}\frac{(i\not{D}+m)}{(i\not{D}+m)}|x\rangle\right]\\ &= -\frac{i}{2m}\text{Tr}\left[\langle x|\frac{m}{\not{D}^2+m^2}+\frac{\not{D}}{\not{D}^2+m^2}|x\rangle\right].\end{aligned} \tag{4.21}$$

Since the factor $1/m$ is a constant with respect to the trace, we can move it into the trace and obtain

$$\frac{d}{dm^2}\mathcal{L}_{\text{eff}} = -\frac{i}{2}\text{Tr}\left[\langle x|\frac{1}{\not{D}^2+m^2}+\frac{\not{D}}{\not{D}^2 m+m^3}|x\rangle\right]. \tag{4.22}$$

Furthermore, it is worth noting that the Dirac trace of the second term between the states is zero due to the property that the trace of any odd number of gamma matrices is null [4, 8, 10]. These operations lead to

$$\frac{d}{dm^2}\mathcal{L}_{\text{eff}} = \frac{i}{2}\text{Tr}\left[\langle x|\frac{1}{-\not{D}^2-m^2}|x\rangle\right]. \tag{4.23}$$

After completing these steps, we use the identity (4.1) and express the above equation in the proper-time formalism:

$$\frac{d}{dm^2}\mathcal{L}_{\text{eff}} = \frac{1}{2}\int_0^\infty ds\; e^{-ism^2}\;\text{Tr}\left[\langle x|e^{-i\not{D}^2 s}|x\rangle\right]. \tag{4.24}$$

Note that we separated the mass term and derivative into two exponentials in this representation. Besides, since the exponential of m is a scalar and does not depend on either x or gamma matrices, we do not need to compute its traces.

Upon integrating the equation above over m^2 and reinstating the bosonic term that was eliminated during differentiation, we arrive at the effective Lagrangian in the

[3] Recall that $\frac{dg}{df} = \frac{dg}{dx}(\frac{df}{dx})^{-1}$, where g and f are functions of x.

proper-time representation

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\int_0^\infty \frac{ds}{s}\, e^{-ism^2}\, \mathrm{Tr}\left[\langle x|e^{-i\not{D}^2 s}|x\rangle\right]. \tag{4.25}$$

Observe that in this integration, a factor of i/s arises. By comparing with (4.10), we identify that the operator $\not{D}^2$ acts as the Hamiltonian of the system. To gain a better understanding of this operator, we need to obtain its form.

4.2.1 The Hamiltonian $\not{D}^2$

Let us begin by expressing the product of two covariant derivatives contracted with Dirac matrices as

$$\not{D}^2 = \gamma^\mu D_\mu \gamma^\nu D_\nu = (\gamma^\mu\partial_\mu + ie\gamma^\mu A_\mu)(\gamma^\nu\partial_\nu + ie\gamma^\nu A_\nu). \tag{4.26}$$

Here, the gauge field is dependent on the position operator, that is $A_\mu = A_\mu(\hat{x})$, given the fact that this field acts on the position states $|x\rangle$ as expressed in the proper-time integral (4.25).

To use the distributive property in this product, we will first act these derivatives on a function ψ dependent on x. After that, we shall evaluate the product and subsequently omit the function as we are only interested in the derivative. Following these steps, we obtain

$$\not{D}^2\psi = \gamma^\mu\partial_\mu\gamma^\nu\partial_\nu\psi + ie\gamma^\mu\gamma^\nu(\partial_\mu A_\nu)\psi + ie\gamma^\mu A_\nu\gamma^\nu\partial_\mu\psi + ie\gamma^\mu A_\mu\gamma^\nu\partial_\nu\psi - e^2\gamma^\mu A_\mu\gamma^\nu A_\nu\psi, \tag{4.27}$$

and, therefore,

$$\not{D}^2 = \gamma^\mu\partial_\mu\gamma^\nu\partial_\nu + ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu + ie\gamma^\mu A_\nu\gamma^\nu\partial_\mu + ie\gamma^\mu A_\mu\gamma^\nu\partial_\nu - e^2\gamma^\mu A_\mu\gamma^\nu A_\nu. \tag{4.28}$$

Since Dirac matrices commute with both fields and derivatives, we can rearrange the given equation, bearing in mind that these matrices may not commute among themselves. This rearrangement leads to

$$\not{D}^2 = \partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu + ieA_\nu\gamma^\mu\gamma^\nu\partial_\mu + ieA_\mu\gamma^\mu\gamma^\nu\partial_\nu - e^2A_\mu A_\nu\gamma^\mu\gamma^\nu. \tag{4.29}$$

Due to contractions involving the gamma matrices, we can simplify this equation to find a more compact form.

The first simplification arises from the initial term. Using the fact that derivatives commute among themselves, we get

$$\begin{aligned}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu &= \frac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + \frac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu \\ &= \frac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + \frac{1}{2}\partial_\nu\partial_\mu\gamma^\nu\gamma^\mu.\end{aligned} \tag{4.30}$$

We can group the gamma matrices into an anticommutation operation, then obtaining

$$\begin{aligned}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu &= \frac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + \frac{1}{2}\partial_\mu\partial_\nu\gamma^\nu\gamma^\mu \\ &= \frac{1}{2}\partial_\mu\partial_\nu\{\gamma^\mu,\gamma^\nu\}.\end{aligned} \tag{4.31}$$

In addition, as the last term of (4.29) present gauge fields that commute among themselves, we can perform the last steps and get

$$A_\mu A_\nu\gamma^\mu\gamma^\nu = \frac{1}{2}A_\mu A_\nu\{\gamma^\mu,\gamma^\nu\}. \tag{4.32}$$

By substituting these simplifications back to (4.29), we obtain

$$\not{D}^2 = \frac{1}{2}\partial_\mu\partial_\nu\{\gamma^\mu,\gamma^\nu\}+ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu+ieA_\nu\gamma^\mu\gamma^\nu\partial_\mu+ieA_\mu\gamma^\mu\gamma^\nu\partial_\nu-e^2\frac{1}{2}A_\mu A_\nu\{\gamma^\mu,\gamma^\nu\}. \tag{4.33}$$

This given equation can be further simplified by interchanging the indices in the fourth term on the right side, swapping μ with ν and vice versa, as they are contracted indices. This manipulation yields

$$\not{D}^2 = \frac{1}{2}\partial_\mu\partial_\nu\{\gamma^\mu,\gamma^\nu\}+ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu+ieA_\nu\gamma^\mu\gamma^\nu\partial_\mu+ieA_\nu\gamma^\nu\gamma^\mu\partial_\mu-e^2\frac{1}{2}A_\mu A_\nu\{\gamma^\mu,\gamma^\nu\}. \tag{4.34}$$

Furthermore, we observe that now the third and fourth terms can be grouped using the anticommutator of the gamma matrices, which gives us

$$\not{D}^2 = \frac{1}{2}\partial_\mu\partial_\nu\{\gamma^\mu,\gamma^\nu\} + ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu + ieA_\nu\{\gamma^\mu,\gamma^\nu\}\partial_\mu - e^2\frac{1}{2}A_\mu A_\nu\{\gamma^\mu,\gamma^\nu\}. \tag{4.35}$$

We can simplify the anticommutations in equation (4.35) further, as these operations are governed by the relation (3.19). Utilising this, we have contractions with the metric $g^{\mu\nu}$, resulting in index changes, as observed in

$$\not{D}^2 = \partial_\mu\partial^\mu + 2ieA_\mu\partial^\mu - e^2A_\mu A^\mu + ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu. \tag{4.36}$$

Our task now is to find a way to rewrite the last term. We can split it into two terms, as done in (4.30). This yields

$$\begin{aligned}ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu &= \frac{ie}{2}(\gamma^\mu\gamma^\nu\partial_\mu A_\nu + \gamma^\mu\gamma^\nu\partial_\mu A_\nu) \\ &= \frac{ie}{2}(\gamma^\mu\gamma^\nu\partial_\mu A_\nu + \gamma^\nu\gamma^\mu\partial_\nu A_\mu).\end{aligned} \tag{4.37}$$

To proceed, we can rewrite the relation (3.19) as $\gamma^\nu\gamma^\mu = 2g^{\mu\nu} - \gamma^\mu\gamma^\nu$, resulting in

$$ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu = \frac{ie}{2}\left[\gamma^\mu\gamma^\nu\partial_\mu A_\nu + (2g^{\mu\nu} - \gamma^\mu\gamma^\nu)\partial_\nu A_\mu\right], \tag{4.38}$$

and then

$$ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu = \frac{ie}{2}\left[\gamma^\mu\gamma^\nu\partial_\mu A_\nu - \gamma^\mu\gamma^\nu\partial_\nu A_\mu + 2g^{\mu\nu}\partial_\nu A_\mu\right]. \tag{4.39}$$

As a next step, we factor out the gamma matrices of the first and second terms between square brackets and obtain

$$\begin{aligned} ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu &= \frac{ie}{2}\left[\gamma^\mu\gamma^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) + 2g^{\mu\nu}\partial_\nu A_\mu\right] \\ &= \frac{ie}{2}\left[\gamma^\mu\gamma^\nu F_{\mu\nu} + 2\partial_\mu A^\mu\right]. \end{aligned} \tag{4.40}$$

To simplify this derivative even further, we split the first term between square brackets into two:

$$ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu = \frac{ie}{2}\left[\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + 2\partial_\mu A^\mu\right]. \tag{4.41}$$

Since the electromagnetic tensor is antisymmetric, that is $F_{\mu\nu} = -F_{\nu\mu}$, we rewrite the second term on the right side as

$$\begin{aligned} ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu &= \frac{ie}{2}\left[\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu F_{\nu\mu} + 2\partial_\mu A^\mu\right] \\ &= \frac{ie}{2}\left[\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} - \frac{1}{2}\gamma^\nu\gamma^\mu F_{\mu\nu} + 2\partial_\mu A^\mu\right]. \end{aligned} \tag{4.42}$$

Here, we realise that the electromagnetic tensor can be factored out, and the Dirac matrices can be grouped in an anticommutation operation. This then results in

$$ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu = \frac{ie}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} + ie\partial_\mu A^\mu. \tag{4.43}$$

Furthermore, we introduce the antisymmetric tensor $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, which corresponds, albeit without a 1/2 factor, to the spinorial representation of the Lorentz group, as demonstrated in (3.22). Consequently, the transformation of the term under consideration becomes

$$ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu = \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} + ie\partial_\mu A^\mu. \tag{4.44}$$

In this relation, the first term on the right-hand side indicates a magnetic dipole moment term [10], which arises from the coupling, or interaction, of fermionic particles with electromagnetic fields.

With this term established, let us return to (4.36) and make all the necessary substitutions. Thus, we obtain

$$\not{D}^2 = \partial_\mu\partial^\mu + 2ieA_\mu\partial^\mu + ie\partial_\mu A^\mu - e^2 A_\mu A^\mu + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}. \tag{4.45}$$

Before we conclude, we observe that the first four terms on the right side of the equality correspond to the contraction of two covariant derivatives:

$$D_\mu D^\mu = \partial_\mu\partial^\mu + ie\partial_\mu A^\mu + 2ieA_\mu\partial^\mu - e^2 A_\mu A^\mu. \tag{4.46}$$

Therefore, (4.45) can finally be expressed as

$$\not{D}^2 = D_\mu D^\mu + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}. \tag{4.47}$$

Consequently, it is written as

$$\not{D}^2 = -(\hat{k}_\mu - eA_\mu(\hat{x}))^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}, \tag{4.48}$$

where we used the operator notation by using the substitution $\partial_\mu \to -i\hat{k}_\mu$. Note that we have factored out $-i$ from the product of covariant derivatives.

Substituting this Hamiltonian back into the effective Lagrangian (4.25) in the proper-time representation yields

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\int_0^\infty \frac{ds}{s}\, e^{-ism^2}\,\text{Tr}\left[\langle x|e^{i\left[(\hat{k}-eA(\hat{x}))^2-\frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}\right]s}|x\rangle\right]. \tag{4.49}$$

Here, the first term of the Hamiltonian, corresponding to the product of two covariant derivatives, signifies the translation generator from one state to another in the presence of electromagnetic fields due to the terms $eA(\hat{x})$, thereby accounting for interactions among fermionic particles and the background field. Considering that we are working with spin-1/2 particles/fields and the interactions presented above, we observe the emergence of a magnetic dipole moment term $\frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}$.

Moreover, it is interesting to note that if we were to consider scalar fields instead of fermionic fields, there would be no terms dependent on the Dirac matrices. Consequently, the only contribution from the Hamiltonian (4.48) would arise from the first term since the second term would be null.

As we shall explore in Section 4.4, when an intense electromagnetic field is considered, the polarised vacuum undergoes a process of decay, meaning that the virtual particles become real in a mechanism known as the Schwinger effect [25, 87, 89, 90]. This significant aspect of the quantum vacuum was initially investigated by Euler and Heisenberg and described through their effective Lagrangian.

4.3 Euler-Heisenberg Effective Lagrangian

In the previous section, we established the effective Lagrangian governing the interaction between fermionic fields and background electromagnetic fields, and elucidated that the vacuum in QED is a polarisable medium due to virtual pair production.

In order to explore more properties of the quantum vacuum, the subsequent task involves computing the traces specified in (4.49), for which we shall assume that $F_{\mu\nu}$ remains constant, i.e. constant electromagnetic fields.

This consideration is made because it represents a solvable case and leads to the significant contributions from Euler and Heisenberg [12], as well as Weisskopf [13]. Their work introduces a non-perturbative, renormalised, one-loop effective Lagrangian for QED in a classical electromagnetic background of constant field strength, which offers several important insights into the vacuum structure and non-perturbative QED.

As the required computations are lengthy, we shall outline some essential steps in deriving the Euler-Heisenberg effective Lagrangian. For a comprehensive derivation, we refer to the original papers mentioned above as well as Schwinger's in [25]. For more on the topic, see also [10, 87, 88, 90].

The central idea, carried out by Schwinger, for computing $\langle y|e^{-i\hat{H}s}|x\rangle$ in (4.49) revolves around solving the operator differential equation

$$i\partial_s \langle y;0|x;s\rangle = i\partial_s \langle y|e^{-i\hat{H}s}|x\rangle = \langle y|e^{-i\hat{H}s}\hat{H}|x\rangle\,, \tag{4.50}$$

where s denotes the proper-time parameter, and the Hamiltonian $\hat{H}$ is given by (4.48). For this, it is first necessary to express the Hamiltonian in terms of the Heisenberg-picture position operators $\hat{x}^\mu(0)$ and $\hat{x}^\mu(s)$, thus transforming this equation into a differential equation of functions.

Following Schwinger's approach, we introduce the operator

$$\hat{\Pi}_\mu = \hat{k}_\mu(x) - eA_\mu(\hat{x}), \tag{4.51}$$

so that the Hamiltonian (4.48) can be written as

$$\hat{H}(s) = -\hat{\Pi}_\mu(s)\,\hat{\Pi}^\mu(s) + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}. \tag{4.52}$$

It is worth noting that since this operator depends on the momentum operator $\hat{k}^\mu(s)$, it obeys the following commutation relation between position and momentum operators in four dimensions:

$$[\hat{x}^\mu(s), \hat{\Pi}^\mu(s)] = -ig^{\mu\nu}. \tag{4.53}$$

Having solved the Heisenberg equations for the operators $\hat{x}^\mu(s)$ and $\hat{\Pi}(s)$, the Hamiltonian is determined and has the form

$$\begin{aligned}\hat{H} =& -x_\mu(s)K^{\mu\nu}x_\nu(s) + 2x_\mu(s)K^{\mu\nu}x_\nu(0) - x_\mu(0)K^{\mu\nu}x_\nu(0) - \frac{i}{2}\mathrm{tr}[eF_{\mu\nu}\coth(esF^{\mu\nu})]\\ &-\frac{e}{2}\mathrm{tr}(\sigma^{\mu\nu}F_{\mu\nu}),\end{aligned} \tag{4.54}$$

where

$$K^{\mu\nu} \equiv \frac{e^2F^{\mu\nu}F_{\mu\nu}}{4\sinh^2(eF^{\mu\nu}s)}. \tag{4.55}$$

Note that this tensor is symmetric, since $\sinh^2 x$ is an even function. With this Hamiltonian, we express the differential equation (4.50) as

$$i\partial_s \langle x;0|x;s\rangle = -\mathrm{tr}\left[\frac{i}{2}eF_{\mu\nu}\coth(esF^{\mu\nu}) - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right]\langle x;0|x;s\rangle\,. \tag{4.56}$$

Observe that $x = y$ as presented in (4.49) and that this differential equation is now in terms of functions rather than operators since it depends on the eigenvalues of the previously mentioned operators through the action of the Hamiltonian on the states $|x\rangle$.

The solution to this differential equation leads the following effective Lagrangian [10, 25]:

$$\begin{aligned}\mathcal{L}_{\text{eff}} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{-ism^2}\,\mathrm{Tr}\exp\left[-\frac{1}{2}\mathrm{tr}\log\left(\frac{\sinh(esF^{\mu\nu})}{esF_{\mu\nu}}\right) + i\frac{es}{2}\mathrm{tr}(\sigma_{\mu\nu}F^{\mu\nu})\right],\end{aligned} \tag{4.57}$$

whose traces are computed with the use of the identity $1/2\{\sigma^{\mu\nu},\sigma^{\alpha\beta}\} = g^{\mu\alpha}g^{\nu\beta}\mathbb{I}_4 - g^{\nu\alpha}g^{\mu\beta}\mathbb{I}_4 + i\gamma^5\epsilon^{\mu\nu\alpha\beta}$, where $\mathbb{I}_4$ represents the 4×4 identity matrix, $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, and $\epsilon^{\mu\nu\alpha\beta}$ denotes the Levi-Civita symbol for four dimensions. In addition, the following function is introduced:

$$X = \sqrt{\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}}, \tag{4.58}$$

which is composed of Lorentz scalars.

Upon computing traces over spacetime indices, tr, and traces over Dirac matrices, Tr, the effective action (4.57) becomes the so-called Euler-Heisenberg effective Lagrangian

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}_{\text{EH}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2}{32\pi^2}\int_0^\infty \frac{ds}{s}\, e^{is\varepsilon}e^{-sm^2}\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{4.59}$$

which is divergent due to the term $\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)}F_{\mu\nu}\tilde{F}^{\mu\nu}$. The divergent behaviour is observed by expanding this term perturbatively in e, yielding

$$\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)}F_{\mu\nu}\tilde{F}^{\mu\nu} = -\frac{4}{e^2s^2} - \frac{2}{3}F_{\mu\nu}F^{\mu\nu} + \frac{e^2s^2}{45}\left[(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right] + \cdots. \tag{4.60}$$

Here, we observe that the first two terms on the right side lead to a divergent result upon integration in (4.59). Therefore, to address these divergences, we subtract them with counterterms [10], yielding the renormalised Euler-Heisenberg effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{EH}} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{e^2}{32\pi^2}\int_0^\infty \frac{ds}{s}\, e^{is\varepsilon}e^{-sm^2}\left[\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)}F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{4}{e^2 s^2} + \frac{2}{3}F_{\mu\nu}F^{\mu\nu}\right],\end{aligned} \tag{4.61}$$

which describes the dynamics of electromagnetic fields modified by quantum corrections, arising from the integration of massive fermionic fields over a constant electromagnetic background $F_{\mu\nu}$.

A notable feature of this effective Lagrangian is its non-perturbative nature in the coupling constant e. This enables the exploration of the non-perturbative regime of QED and, consequently, the effects presented in Chapter 1, particularly the Schwinger effect. As we will see in the next section, this effect is associated with a non-zero imaginary contribution arising from the ill-defined integral within the Euler-Heisenberg effective Lagrangian due to poles on the real axis.

To learn more about other physical implications of this effective Lagrangian, refer to [10, 24, 88] in addition to the works presented in the Chapter 1. Furthermore, an alternative derivation of the Euler-Heisenberg effective Lagrangian (4.61) is provided by sums over Landau levels [10]. Essentially, these levels correspond to quantised energy levels of a particle in a uniform magnetic field [91]. The reason to utilise this comes from the form of the Hamiltonian (4.48) and the nature of the system we are studying, which involves quantised fermionic particles interacting with electromagnetic fields.

4.4 Schwinger Effect

To begin the analysis of the Euler-Heisenberg effective Lagrangian, expressed in (4.61), we first reformulate it in terms of the electric and magnetic fields, denoted as $E = |\mathbf{E}|$ and $B = |\mathbf{B}|$, respectively.

Since solving the integral in this effective Lagrangian may be challenging for various configurations of electric and magnetic fields, we will first assume that the fields are parallel[4] and constant. Throughout our discussion, we shall consider the limit where $B \to 0$, rendering the configuration of parallel fields irrelevant. This assumption simplifies the expressions for the contractions of electromagnetic tensors, which are now expressed as

$$F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2) = 2(B^2 - E^2) \qquad \text{and} \qquad \tilde{F}_{\mu\nu}F^{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B} = -4EB. \tag{4.62}$$

Moreover, the scalar function of the electric and magnetic fields (4.58) becomes

$$X^2 = (B + iE)^2. \tag{4.63}$$

With these relations in place, the effective Lagrangian (4.61) is rewritten as

$$\begin{aligned}\mathcal{L}_{EH} =& \frac{1}{2}(E^2 - B^2) \\ &+ \frac{e^2}{32\pi^2}\int_0^\infty \frac{ds}{s}\, e^{i\varepsilon s}\, e^{-m^2 s}\left[-4EB\frac{\cos(esE)\cosh(esB)}{\sin(esE)\sinh(esB)} + \frac{4}{e^2 s^2} + \frac{4}{3}(B^2 - E^2)\right],\end{aligned} \tag{4.64}$$

where we have employed $\cosh(Xs) = \frac{1}{2}\left(e^{-Xs} + e^{Xs}\right)$, along with relations between trigonometric and complex functions. After some simplifications, we arrive at

$$\begin{aligned}\mathcal{L}_{EH} =& \frac{1}{2}(E^2 - B^2) \\ &- \frac{e^2}{8\pi^2}\int_0^\infty \frac{ds}{s}\, e^{i\varepsilon s}\, e^{-m^2 s}\left[E\cot(esE)B\coth(esB) - \frac{1}{e^2 s^2} - \frac{B^2 - E^2}{3}\right].\end{aligned} \tag{4.65}$$

It is worth observing that the integral exhibits singularities in the function $\cot(esE)$, associated with the electric field E. Conversely, the function $\coth(esB)$, linked to the

[4] Note that this is an idealisation, a first approximation, to understand the Schwinger effect and obtain some results. In addition, refer to [24] for perpendicular magnetic and electric background fields.

magnetic field B, does not display singularities for $s > 0$. Hence, we will focus on the scenario where $B \to 0$, indicating a purely electric background field. As we will see throughout this section, such singularities lead to an imaginary contribution to the effective Lagrangian, which correlates with pair production and vacuum decay in QED.

Upon taking this limit, the effective Lagrangian (4.65) simplifies to

$$\mathcal{L}_{EH} = \frac{1}{2}E^2 - \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{i\varepsilon s}\, e^{-m^2 s}\left[esE\cot(esE) - 1 + \frac{(esE)^2}{3}\right], \tag{4.66}$$

where we have utilised

$$\lim_{B\to 0} B\coth(esB) = \frac{1}{es}, \tag{4.67}$$

and factored the term $\frac{1}{e^2 s^2}$. In the same way as the integral in (4.65), we observe that the Lagrangian exhibits poles for the electric field when the proper-time parameter s is equal to $s_n = \frac{\pi}{eE}n$ for $n = 1, 2, \ldots$. Given these poles, we shall resort to the method of residues to solve the integral.

In addition, it is important to note that there is no pole at $s = 0$, as can be seen from expanding the integrand at small s

$$I = -\frac{1}{45}s + \frac{1}{45}s^2 - \frac{5}{378}s^3 + \cdots. \tag{4.68}$$

In this expansion, I represents the integrand, and we have set $E = e = m = 1$ and $\epsilon \to 0$ to simplify obtaining the Taylor series.

To solve the integral in (4.66), we extend its integration limits to $(-\infty, +\infty)$ and then deform the integration contour to include the imaginary axis and the contributions from each pole [10, 87, 90], as depicted in Figure 6. Since this integral has poles on the

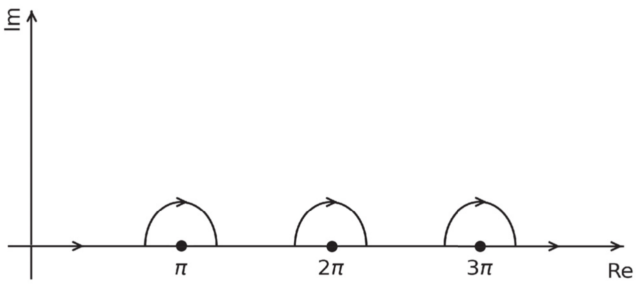


Figure 6 – Integration contour, including the imaginary axis and considering the poles in the real axis. Note that we are not showing the dependence on e and E.

real axis, its integration results in a non-zero imaginary contribution. Subsequently, we will observe that these contributions are associated with vacuum instability and pair production.

The motivation for employing this approach to evaluate the integral (4.66) along the contour illustrated in Figure 6 arises from the use of identity

$$\frac{1}{x \pm i\epsilon} = P\frac{1}{x} \mp i\pi\delta(x). \tag{4.69}$$

Here, P denotes the Cauchy principal value, and the term $i\epsilon$ is employed to avoid hitting the singularities [25]. In essence, an integral with a pole on the real axis at $z = z_0$, with this identity, can be expressed as

$$I = \int_{-\infty}^{\infty} dz \, \frac{f(z)}{z - z_0 + i\epsilon} = \left[\int_{-\infty}^{z_0 - \epsilon} dz + \int_{z_o + \epsilon}^{\infty} dz \right] \frac{f(z)}{z - z_0} - i\pi f(z_0), \tag{4.70}$$

where $f(z)$ is a complex function. The integrals in the last equality correspond to the Cauchy principal value, and the third term represents the residue calculated at z_0. It is important to note that it is imaginary, and accounts for half of the residue. For more on the topic, refer to [90]. Therefore, our attention will be focused on this imaginary contribution, and thus we shall compute $\operatorname{Im} \mathcal{L}_{EH}$.

Employing the residue theorem and the identity (4.69), we find

$$\operatorname{Im} \mathcal{L}_{EH} = -\frac{1}{8\pi^2} \int_{\infty}^{\infty} \frac{ds}{s^2} \, e^{-m^2 s^2} \left[eE \cot(esE) \right] = \frac{\pi}{8\pi^2} \sum_{n=1} \operatorname{Res}[f(s), s_n]. \tag{4.71}$$

Additionally, as we are integrating over the contour clockwise, there is a negative sign on the residue term, which cancels out with the negative sign in the first equality. Since we are dealing with simple poles, this residue is then given by

$$\operatorname{Im} \mathcal{L}_{\mathrm{EH}} = \frac{eE}{8\pi} \sum_{n=1} \lim_{s \to \frac{\pi}{eE} n} \left(s - \frac{\pi}{eE} n \right) \frac{1}{s^2} \, e^{-m^2 s^2} \cot(esE). \tag{4.72}$$

After solving the limit, we find that

$$\operatorname{Im} \mathcal{L}_{\mathrm{EH}} = \frac{e^2 E^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{m^2 \pi}{eE} n \right), \tag{4.73}$$

meaning that the effective Euler-Heisenberg Lagrangian for the case of a pure constant electric field exhibits a non-perturbative imaginary contribution. We can observe this non-perturbative nature by noting that we cannot perform a Taylor expansion around the coupling constant e.

The importance of this result lies in its connection to the vacuum instability[5]. To observe this, we express the vacuum-to-vacuum transition, also known as the vacuum-vacuum persistence amplitude [89], through the definitions of the connected diagram generating functional (2.100) and its relation with the effective action Γ as

$$\left\langle \begin{matrix} \text{no particles} \\ \text{at } t_f = \infty \end{matrix} \middle| \begin{matrix} \text{no particles} \\ \text{at } t_i = -\infty \end{matrix} \right\rangle = \langle 0 | 0 \rangle = e^{i[\mathrm{Re}(\Gamma) + i \mathrm{Im}(\Gamma)]}. \tag{4.74}$$

Note that this transition amplitude is computed between different vacuum states: one at a final time t_f and another at an initial time t_i. As these states are different, the transition

5 A parallel with unstable particles can be drawn here, as such particles exhibit imaginary contributions in their propagators, causing such functions to decay. For more on the topic, refer to [4, 8, 10].

amplitude between them is not necessarily 1, but it is an exponential of the effective action.

By squaring this amplitude, we obtain, as interpreted by Schwinger [25], the probability of no particle pair being produced:

$$|\langle 0|0\rangle|^2 = e^{-2\mathrm{Im}(\Gamma)} = e^{-2V_4 \mathrm{Im}\mathcal{L}_{EH}}, \tag{4.75}$$

where V_4 represents the volume of spacetime, obtained by the integral over x from the action. Consequently, the probability of pair production and vacuum decay is given by

$$P = 1 - e^{-2\mathrm{Im}(\Gamma)} = 1 - e^{-2V_4 \mathrm{Im}\mathcal{L}_{EH}}. \tag{4.76}$$

In this equation, the factor $2\mathrm{Im}\,\mathcal{L}_{\mathrm{EH}}$ is interpreted as the decay rate per unit of time and per unit of volume for the creation of an electron-positron pair by an external electric field.

Since this rate is non-zero in our scenario, as seen in (4.73), we find that the vacuum state in QED is susceptible to decay due to the creation of on-shell[6] electron-positron pairs by an intense external electric field. In other words, the presence of the background electric field accelerates and separates the pairs of virtual particles, leading to the emergence of real electron and positron pairs, as illustrated in Figure 1. Such production occurs as the work done by the external field to separate the virtual particles by a Compton wavelength is of the order of the binding energy of the pair, which corresponds to $2m$ [90, 92]. Consequently, this leads to an exceedingly small effect, achievable only when the electric field E approaches a critical value $E_c = m^2/e \approx 10^{18}\, V/m$ [10, 24, 93].

Bearing this in mind, we can illustrate the scenario where only a magnetic field exists, as shown in Figure 7. In essence, a magnetic field fails to separate and accelerate the virtual particle pairs as an electric field does, rendering the creation of asymptotic particle pairs impossible. Therefore, as seen in the effective action (4.61), where the pure magnetic field is not associated with any pole nor with the figure below, we observe the impossibility of pair production.

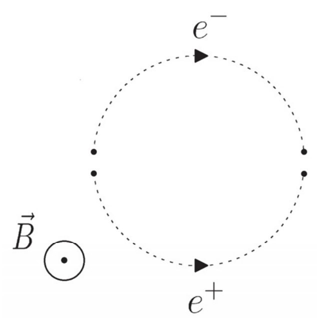


Figure 7 – Representation of the impossibility of producing real particles in a pure magnetic background. As the virtual particles are exposed to a magnetic field, they undergo circular motion, which consequently leads them to their annihilation instead of separation and acceleration.

[6] This means that the particles obey the relativistic dispersion relation.

5 Non-Local Correction

In this chapter, we present our developments for the analysis of a non-local effective action within the context of the Schwinger mechanism. Here, we investigate pair production for backgrounds characterised by a plane wave and by the presence of a constant external current.

To asses whether these backgrounds lead to imaginary contributions and consequently pair production rates, we shall compute the effective action on-shell. This involves substituting these backgrounds, which are solutions of the classical equation of motion, into the non-local effective action and integrating[1] over them.

In Section 5.1, we will introduce the non-local effective action to be examined in this chapter. After that, we shall investigate the impact of a non-local correction, resulting from non-local effective action, on the pair production mechanics in Section 5.2. This section is subdivided into two subsections: the first (5.2.1) explores pair production in a background defined by plane waves, while the second (5.2.2) investigates pair production within a background characterised by waves featuring a constant external current.

5.1 Non-Local Effective Action

In Section 2.5.1, we derived an expression for the effective action using the background field method. This method involves decomposing the field intended for quantisation into a background field, treated classically, and fluctuations, which are quantised. Through this process, we arrived at the effective action up to one-loop correction (2.130), which can be represented as

$$\Gamma = S + \Gamma_{\text{one-loop}}. \tag{5.1}$$

Here, S represents the classical action, and $\Gamma_{\text{one-loop}}$ is the one-loop correction arising from quantising fields of interest. This correction takes the form

$$\Gamma_{\text{one-loop}} = \frac{i}{2}\mathrm{Tr}\log\left(\frac{\delta^2 S}{\delta\phi(x)\,\delta\phi(x')}\right). \tag{5.2}$$

As discussed in Section 4.2, this one-loop correction can be computed using Schwinger's proper-time method. However, in this study, we adopt an alternative approach.

This approach, developed by Barvinsky, Vilkovisky, and collaborators, enables the computation of loop[2] corrections, and consequently general effective actions, through an expansion in powers of generalised curvatures, including field strengths and potentials [63,

[1] Recall that the effective action is described by an integral of the effective Lagrangian.
[2] In this dissertation, we focus solely on the one-loop correction, as a first approximation.

94–96]. Within this expansion, a one-loop correction term manifests itself as a non-local term at the second order in the field strength. In the context of this work, "non-local" refers to a term composed of non-polynomial functions or that depends on derivatives, specifically the d'Alembertian operator $\Box$, or terms that involve an infinite number of derivatives. This term takes the form

$$\Gamma_{\text{one-loop}} \sim \int d^4x \; \Omega^{\mu\nu}\gamma(\Box)\Omega_{\mu\nu}. \tag{5.3}$$

In this expression, $\Omega^{\mu\nu}$ represents a field strength, which in this study, we consider to be the electromagnetic tensor $F^{\mu\nu}$, and $\gamma(\Box)$ denotes the so-called form factor, which contains the information in the quantum regime related to the specified quantum correction.

Given the complexity of the subject, which extends beyond the scope of this dissertation, we will refrain from demonstrating the derivation of these terms and instead refer the reader to the references provided in this section.

As a first approximation, we shall consider the massless limit or the limit where the field mass is small, meaning that the fields in the classical Lagrangian are massless. Taking this into consideration, the renormalised non-local correction term, computed by Barvinsky and Vilkovisky [94–96], appears as

$$\Gamma_{\text{one-loop}} \sim \int d^4x \; \Omega^{\mu\nu}\log\left(\frac{\Box}{\mu^2}\right)\Omega_{\mu\nu}, \tag{5.4}$$

where μ is a renormalisation scale that has dimension of energy, introduced through the process of dimensional regularisation to derive this correction term and the corresponding effective action, namely (5.5).

In this dissertation, we are interested in analysing the effects of an effective action that incorporates a one-loop correction (5.3) with the form factor specified in (5.4) on the Schwinger effect. To achieve this, we have the following general effective action up to one loop [60]:

$$\Gamma[A] = \int d^4x \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \alpha F^{\mu\nu}\log\left(\frac{\Box}{\mu^2}\right)F_{\mu\nu}\right]. \tag{5.5}$$

Here, the first term to be integrated corresponds to Maxwell's Lagrangian and accounts for classical electromagnetic fields while the second term is the non-local quantum correction. The coefficient α depends on the number and types of fields integrated out, thus quantised [63]. For the purpose of this work, α will be considered a free parameter, which enables the choice of fields to be introduced into the theory. Naturally, a future analysis of the effects of such a coefficient on pair production will be necessary.

Note that, since we can choose the quantised fields in this effective action, the quantum fluctuations arising from the creating of annihilation of virtual particles can manifest as any particle pair of choice. For instance, if we consider the quantisation of spin-1/2 particles, these quantum fluctuations are manifested by electron-positron pairs.

In addition, note that the effective action (5.5) presents a similar form to Eq. (4.17), where the first term corresponds to the background fields and the second to the quantum corrections, which alter the dynamics of the classical background fields.

5.1.1 Non-local effective action in momentum space

To analyse pair production for different configurations of the background field, through the computation of $2\mathrm{Im}\,\mathcal{L}$, let us first express the effective action (5.5) in momentum space. This shift is motivated by the convenience of working with the momentum k^μ rather than operators such as the derivatives ∂^μ.

We start by employing the Fourier transformation for the gauge field $A^\mu(x)$ to transform the first term of the effective action, which corresponds to the Maxwell's Lagrangian. This transformation is given by

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} A^\mu(k), \tag{5.6}$$

where $A^\mu(k)$ denotes the gauge field in momentum space. Substituting this expression into the electromagnetic tensor $F^{\mu\nu}$ gives

$$F^{\mu\nu} = \partial^\mu \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} A^\nu(k) - \partial^\nu \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} A^\mu(k). \tag{5.7}$$

By applying derivatives to exponentials, we obtain factors of the form $-ik^\mu$, leading to the following:

$$F^{\mu\nu} = -i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \left(k^\mu A^\nu(k) - k^\nu A^\mu(k)\right). \tag{5.8}$$

It is important to note that in this expression, we are not explicitly indicating the integration with respect to x, as we shall in the next step.

We proceed by writing the contraction $F^{\mu\nu}F_{\mu\nu}$ in the momentum space and indicating integration with respect to x, yielding

$$\begin{aligned}\int d^4x\, F^{\mu\nu}F_{\mu\nu} = -\int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4} e^{-ix\cdot(k'+k)} \left(k^\mu A^\nu(k) - k^\nu A^\mu(k)\right) \\ \times \left(k'_\mu A_\nu(k') - k'_\nu A_\mu(k')\right).\end{aligned} \tag{5.9}$$

By employing the identity

$$\int d^4x\, e^{-i(k+k')\cdot x} = (2\pi)^4 \delta(k+k'), \tag{5.10}$$

and subsequently integrating with respect to k, we obtain

$$\int d^4x\, F^{\mu\nu}F_{\mu\nu} = -\int \frac{d^4k}{(2\pi)^4} \left[k^\mu A^\nu(k) - k^\nu A^\mu(k)\right]\left[-k_\mu A_\nu(-k) + k_\nu A_\mu(-k)\right]. \tag{5.11}$$

Notice that this integration results in the change of variables $k' \to -k$ due to the delta. After performing the product within the integrand of (5.11) and factoring out the gauge fields using the relation $A_\mu g^{\mu\nu} = A^\nu$, we arrive at

$$\int d^4x\, F^{\mu\nu} F_{\mu\nu} = 2 \int \frac{d^4k}{(2\pi)^4} A_\mu(k) \left(k^2 g^{\mu\nu} - k^\nu k^\mu\right) A_\nu(-k). \tag{5.12}$$

Thus, the first term of the effective action is given by

$$\int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right] = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} A_\mu(k) \left(-k^2 g^{\mu\nu} + k^\nu k^\mu\right) A_\nu(-k). \tag{5.13}$$

Before representing the quantum correction term, the second term in the effective action, in momentum space, we need to establish a way to express the logarithmic operator in this space as well. One approach is to write it as

$$\log\left(\frac{\Box}{\mu^2}\right) = \int_0^\infty ds \left(\frac{2s}{-\Box - s^2} + \frac{2s}{s^2 + \mu^2}\right), \tag{5.14}$$

where s is just a parameter for integration. As we transition to momentum space, we have

$$\log\left(\frac{\Box}{\mu^2}\right) = \int_0^\infty ds \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \left(\frac{2s}{k^2 - s^2} + \frac{2s}{s^2 + \mu^2}\right). \tag{5.15}$$

Integrating now with respect to the parameter s leads to

$$\log\left(\frac{\Box}{\mu^2}\right) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \log\left(\frac{-k^2}{\mu^2}\right). \tag{5.16}$$

After expressing this operator in this manner, we can multiply it with an electromagnetic tensor, as given in (5.8). This product results in

$$\log\left(\frac{\Box}{\mu^2}\right) F_{\mu\nu} = -i \int \frac{d^4k}{(2\pi)^4} \log\left(\frac{-k^2}{\mu^2}\right) \left[\, k_\mu A_\nu(k) - k_\nu A_\mu(k)\right] e^{-ik\cdot x}. \tag{5.17}$$

When contracted with another electromagnetic tensor, as the correction term in the effective action (5.5), we obtain

$$\begin{aligned} \int d^4x\, \alpha F^{\mu\nu} \log\left(\frac{\Box}{\mu^2}\right) F_{\mu\nu} = \int d^4x\, \alpha \int \frac{d^4k'}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \left[\, -k'^\mu A^\nu(k') + k'^\nu A^\mu(k')\right] \\ \times \log\left(\frac{-k^2}{\mu^2}\right) \left[\, k_\mu A_\nu(k) - k_\nu A_\mu(k)\right] e^{-ix\cdot(k'+k)}. \end{aligned} \tag{5.18}$$

We can simplify this equation by utilising the identity (5.10) and integrating with respect to k, resulting in a change of variables $k \to -k'$. This then yields

$$\begin{aligned} \int d^4x\, \alpha F^{\mu\nu} \log\left(\frac{\Box}{\mu^2}\right) F_{\mu\nu} = 2\alpha \int \frac{d^4k}{(2\pi)^4} \log\left(\frac{-k^2}{\mu^2}\right) \\ \times \left[k^\mu A^\nu(k) k_\mu A_\nu(-k) \;-\; k^\mu A^\nu(k) k_\nu A_\mu(-k)\right]. \end{aligned} \tag{5.19}$$

We can proceed by following the same steps used to derive (5.12). This leads to

$$\int d^4x\, \alpha F^{\mu\nu} \log\left(\frac{\square}{\mu^2}\right) F_{\mu\nu} = -\frac{4}{2}\alpha \int \frac{d^4k}{(2\pi)^4} \log\left(\frac{-k^2}{\mu^2}\right) A_\mu(k) \left(-k^2 g^{\mu\nu} + k^\nu k^\mu\right) A_\nu(-k). \tag{5.20}$$

With the terms (5.13) and (5.20) at hand, we can rewrite the effective action (5.5) in momentum space as

$$\Gamma[A] = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} A_\mu(k) \left\{ (-k^2 g^{\mu\nu} + k^\nu k^\mu) \left[1 - 4\alpha \log\left(\frac{-k^2}{\mu^2}\right)\right] \right\} A_\nu(-k). \tag{5.21}$$

Therefore, with this formulation of the effective action, we are now able to analyse the Schwinger effect and derive the vacuum decay rates for different configurations of the electromagnetic background.

5.2 Wave Equation

To analyse the phenomenon of pair production in electromagnetic backgrounds, it is necessary first to find the classical equation describing the dynamics of these backgrounds. We can start with the covariant Maxwell's equation

$$\partial_\mu F^{\mu\nu} = eJ^\nu, \tag{5.22}$$

derived in the Subsection 3.2.1. Here, $J^\nu = (\rho, \mathbf{j})$ represents the (external) four-current that generalises the notion of electric charge density ρ and electric current density $\mathbf{j}$.

We can express the electromagnetic tensor in terms of gauge fields A_μ, such that equation (5.22) is written as

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = eJ^\nu. \tag{5.23}$$

Upon applying the distributive property and performing the contraction of the derivative ∂_μ with the other terms, we arrive at

$$\square A^\nu - \partial_\mu \partial^\nu A^\mu = eJ^\nu. \tag{5.24}$$

To simplify this differential equation, we can make use of gauge invariance, thus imposing the Lorentz gauge condition

$$\partial_\mu A^\mu = 0. \tag{5.25}$$

This then reduces (5.24) to the wave equation for electromagnetic fields in the presence of an external source:

$$\square A^\mu(x) = eJ^\mu(x). \tag{5.26}$$

In the following subsections, we will work with the system where the source is null, thus reducing this differential equation to the plane wave equation. Additionally, we will also consider the case where the source is constant in momentum space.

5.2.1 Plane wave

The most basic scenario we can investigate is provided by a background characterised by a monochromatic plane wave. This field configuration is obtained by solving equation (5.26) without external sources, i.e. $J^\mu = 0$. This case yields the following equation:

$$\Box A^\mu = 0. \tag{5.27}$$

This equation is accompanied by the plane wave solution

$$A^\mu(x) = \bar{A}^\mu \, e^{-ik\cdot x}, \tag{5.28}$$

where $\bar{A}_\mu$ is the position-independent wave amplitude. We can verify this solution by substituting it back into (5.27). Upon doing so, we observe that

$$\bar{A}^\mu \Box \, e^{-ik\cdot x} = 0. \tag{5.29}$$

Applying the d'Alembertian operator on the exponential leads to

$$-\bar{A}^\mu k^2 e^{-ik\cdot x} = 0. \tag{5.30}$$

Here the factor k^2 originates from the operation of the d'Alembertian operator on the exponential term. This factor is crucial as we must have $k^2 = 0$ for this solution to hold true. Also it signifies that the solution is light-like, indicating that $k^2 = m^2 = 0$.

In the scenario of a background characterised by a monochromatic plane wave, as described by equation (5.28), we find that the effective action (5.21) becomes identically zero:

$$\Gamma[A] = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \left\{ \left(-k^2 A_\mu g^{\mu\nu} A_\nu \; + \; A_\mu k^\mu k^\nu A_\nu\right) \left[1 - 4\alpha \log\left(\frac{-k^2}{\mu^2}\right)\right] \right\} = 0. \tag{5.31}$$

This occurs due to the specific gauge choice (5.25) that we adopted in deriving equation (5.26). In momentum space, this gauge corresponds to the contraction

$$k_\mu A^\mu = 0. \tag{5.32}$$

As a result, the terms in equation (5.31) associated with this gauge choice become zero. Additionally, we must recall that $k^2 = 0$, as we utilised this condition to solve equation (5.27). Consequently, the terms in the integral corresponding to this condition also become zero. Note that we used

$$\lim_{k^2 \to 0} k^2 \log(k^2) = 0.$$

For this effective action, the decay probability (4.76) is

$$P = 0. \tag{5.33}$$

Consequently the vacuum for the plane wave background is stable, rendering no pair production. This result aligns with Schwinger's findings in [25], where he computed the pair production rate for a single monochromatic plane wave background, yielding the same null result.

A reason for the production rate being zero for a plane wave background is that such a background solution to the wave equation is a free solution, meaning that it lacks interactions. If there is no interactions of this background with the quantised particles in the vacuum, there is no polarisation and consequently no pair production.

We can further confirm the result that the pair production probability is null by examining the Euler-Heisenberg effective Lagrangian (4.61), where we recall that the integration of fermions in a background electromagnetic field is expressed in terms of the Lorentz invariants $F^{\mu\nu}F_{\mu\nu}$ and $\tilde{F}^{\mu\nu}F_{\mu\nu}$. In the case of the monochromatic plane wave, these invariants[3] are

$$F^{\mu\nu}F_{\mu\nu} = 2A_\mu \left(\Box\, g^{\mu\nu} - \partial^\mu\partial^\nu\right) A_\nu = 0 \qquad \text{and} \qquad \tilde{F}^{\mu\nu}F_{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\partial_\rho(A_\sigma F_{\mu\nu}) = 0. \tag{5.34}$$

Here, we used the Lorentz gauge choice (5.25), the time-like condition $k^2 = 0$ and the symmetry properties of the tensor $\epsilon^{\mu\nu\rho\sigma}$. Since these Lorentz invariants are zero, so is the imaginary contribution of the Euler-Heisenberg effective Lagrangian (4.61).

5.2.2 Constant external current

To investigate the non-local effective action in the presence of an external source, we must first solve the wave equation (5.26). This can be easily done by transitioning to momentum space, and thus obtaining

$$A^\mu(k) = -e\,\frac{J^\mu(k)}{k^2}. \tag{5.35}$$

As we substitute this result into (5.21), considering the gauge choice (5.32), we find

$$\Gamma[A] = \frac{e^2}{2}\int \frac{d^4k}{(2\pi)^4}\left[-\frac{J_\mu(k)J^\mu(-k)}{k^2} + 4\alpha\frac{J_\mu(k)J^\mu(-k)}{k^2}\log\left(\frac{-k^2}{\mu^2}\right)\right]. \tag{5.36}$$

We emphasise that this effective action is valid for any background described by an external current J^μ, unlike the Euler-Heisenberg effective Lagrangian (4.61), which considers only constant electromagnetic fields as the background. For our study, we will assume that the external current $J^\mu(k)$ is independent of k and constant. Therefore, we need to solve the integral

$$\Gamma[A] = \frac{e^2}{32\pi^4}J_\mu J^\mu \int d^4k\left[-\frac{1}{k^2} + \frac{4\alpha}{k^2}\log\left(\frac{-k^2}{\mu^2}\right)\right]. \tag{5.37}$$

It is important to note that here we are performing an integration in momentum space. If we were in position space, such a constant external current could be interpreted as a

[3] A demonstration of the second equation can be found in Appendix A.

constant point-like current. Furthermore, we will solve these integrals in Euclidean space, enabling us to evaluate the integral using four-dimensional spherical coordinates.

To transition this integral from Minkowski space to Euclidean space, we shall employ the so-called Wick rotation. Essentially, this technique is a continuation to the complex plane, which involves rotating the first component of the four-vector in the complex plane by an angle of $-\pi/2$ such that we have

$$k_0 \to ik_E^0 \qquad \text{and} \qquad \mathbf{k} \to \mathbf{k}_E. \tag{5.38}$$

Here, the subscript E denotes Euclidean space [8, 10]. Therefore, as the function that we are integrating presents poles along the real axis, after the Wick rotation, these poles are repositioned onto the imaginary axis, as depicted in Figure 8. Additionally, it is worth noting that the spatial components $\mathbf{k}$ remain unchanged during this rotation.

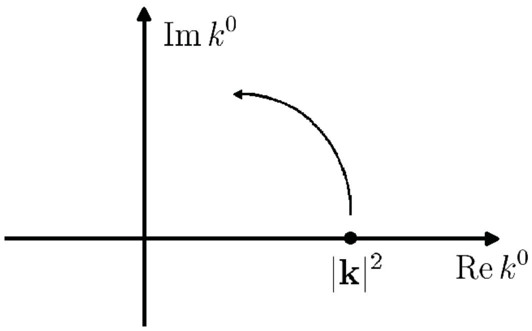


Figure 8 – Representation of Wick rotation, where poles along the real axis are rotated and thus repositioned onto the imaginary axis.

Hence, given a four-momentum k^μ, it undergoes the rotation described in (5.38), resulting in k^2 being expressed as

$$k^2 = (k^0)^2 - |\mathbf{k}|^2 = (ik_E^0)^2 - |\mathbf{k}_E|^2 = -k_E^2. \tag{5.39}$$

Notice that with this rotation, the Minkowski metric with signature $(+,-,-,-)$, which we previously employed, transforms into the Euclidean metric with signature $(+,+,+,+)$. Additionally, it is important to observe that $d^4k = id^4k_E$. Consequently, the integral (5.37) can be reformulated in Euclidean space as

$$\Gamma[A] = -\frac{e^2}{32\pi^4} J_E^2\, i \int_{-i\infty}^{i\infty} d^4k_E \left[\frac{1}{k_E^2} - \frac{4\alpha}{k_E^2}\log\left(\frac{k_E^2}{\mu^2}\right)\right]. \tag{5.40}$$

Note that as a result of the rotation, the integration limits also undergo a change, and the source is also in the Euclidean space, $J^2 = -J_E^2 = -[\rho_E^2 + \mathbf{j}^2]$.

To evaluate this integral, we perform a transformation to 4-dimensional spherical coordinates. In this transformation, we use the integration measure $d^4x = r^3 \sin^2\omega\, \sin\theta\, d\phi d\theta d\omega dr$.

Expressing it in terms of the four-dimensional solid angle $d\Omega_4$, we have $d^4x = d\Omega_4\, dr\, r^3$. Thus, the integral (5.40) is rewritten as

$$\Gamma[A] = -\frac{e^2}{32\pi^4} J_E^2\, i \int d\Omega_4 \int_0^{i\infty} dk_E\, k_E^3 \left[\frac{1}{k_E^2} - \frac{4\alpha}{k_E^2} \log\left(\frac{k_E^2}{\mu^2}\right)\right]. \tag{5.41}$$

The integration over the solid angle in 4 dimensions corresponds to $2\pi^2$, with this factor being simplified by the term $1/32\pi^4$, thus leading to the following effective action in Euclidean space and spherical coordinates:

$$\Gamma[A] = -\frac{ie^2}{16\pi^2} J_E^2 \int_0^{i\infty} dk_E \left[k_E - 4\alpha k_E \log\left(\frac{k_E^2}{\mu^2}\right)\right]. \tag{5.42}$$

Before proceeding with the integration, it is important to observe that these integrals are all divergent. This divergence arises due to integration from 0 to ∞. To address this issue, we can introduce a cutoff Λ on the upper limit, effectively limiting the momentum from above. Given the Wick rotation, the cutoff is represented as $\Lambda_E = i\infty$, and the effective action becomes

$$\Gamma[A] = -\frac{ie^2}{16\pi^2} J_E^2 \int_0^{\Lambda_E} dk_E \left[k_E - 4\alpha\, k_E \log\left(\frac{k_E^2}{\mu^2}\right)\right]. \tag{5.43}$$

Additionally, the cutoff introduced here is a physical energy scale that determines the range of validity of our theory. This implies that when exploring an energy scale surpassing this cutoff, it becomes necessary to formulate a more fundamental theory to precisely describe phenomena beyond this limit.

As the integration is straightforward, we will refrain from performing it step by step here. The important point to note is that to integrate the second term, we perform a change of variables $u = k_E^2/\mu^2$. Thus, we obtain the effective action

$$\Gamma[A] = -\frac{ie^2}{16\pi^2} J_E^2 \left[\frac{k_E^2}{2} - \frac{4\alpha}{2} k_E^2 \log\left(\frac{k_E^2}{\mu^2}\right) + \frac{4\alpha}{2} k_E^2\right]_0^{\Lambda_E}. \tag{5.44}$$

By utilising

$$\lim_{k_E \to 0} k_E^2 \log(k_E^2) = 0,$$

and performing the necessary simplifications, we get

$$\Gamma[A] = -\frac{ie^2}{32\pi^2} J_E^2\, \Lambda_E^2 \left[1 + 4\alpha - 4\alpha \log\left(\frac{\Lambda_E^2}{\mu^2}\right)\right]. \tag{5.45}$$

To proceed, we can recall that the cutoff Λ_E is very large, as we can consider that $\Lambda_E \to \infty$. Given this, we can observe that the first two terms within square brackets grow slower compared to the last term. Therefore, in the limit where the cutoff is large, we can consider that the only significant contribution to the pair production rate comes from the last term in (5.45). Thus, we have

$$2\text{Im}\,\Gamma[A] = \alpha \frac{e^2}{4\pi^2} J_E^2\, \Lambda_E^2 \log\left(\frac{\Lambda_E^2}{\mu^2}\right), \tag{5.46}$$

the production rate for a constant external current background. Note that this outcome can be treated perturbatively in e, differently from Schwinger's result in (4.73). It relies on the parameter α, which depends on the number and type of quantised fields, the constant external source J_E^2, and the cutoff Λ_E, dependent on the energy scale.

It is interesting to note that although the effective action (5.5) is already renormalised and depends on the parameter μ, it revels unconventional divergences highlighted by the cutoff Λ_E when its imaginary part is calculated, as seen in (5.46). These divergences can be traced back to the on-shell calculations of (5.36) and the choice of the external current J^μ. As the integration spans all momenta, from $-\infty$ to ∞, it is natural for divergences to appear, unless we have an external current function whose integral converges to some value.

Regarding a renormalisation process, we could argue that the production rate, namely $2\mathrm{Im}\,\mathcal{L}$, does not depend on the cutoff Λ_E. Since this production rate is a physical observable, representing also the probability of pair production through (4.76), it should not depend on Λ_E. Thus, we can impose that

$$\frac{d\mathrm{Im}\,\mathcal{L}}{d\Lambda_E^2} = 0, \tag{5.47}$$

and therefore utilising the renormalisation group. Consequently, the dependence on Λ_E would come from the external current J_E^μ, the electric charge e, or the parameter α, in such a way that (5.47) holds. One idea would be to introduce the dependence on the electric charge e. However, in this approach, this parameter would depend on two energy scales μ and Λ_E, such that $e = e\,(\mu, \Lambda_E)$. However, the concept of a renormalised parameter in the effective action or Lagrangian that depends on two energy scales or two running parameters needs yet to be analysed and discussed. Nevertheless, we are unsure whether this is the path to follow, whether it is viable, or how to proceed by introducing the Λ_E dependence on J_E^μ or α_E. Naturally, each option will probably have different implications for the theory and its renormalisation, and it is crucial to study how each choice affects the consistency and predictions of the model.

It is important to mention that renormalisation processes and methods are common in QFT and are well documented in fundamental texts such as [8, 10, 70, 75]. However, a study on a potential second renormalisation is necessary.

Another analysis of the production rate (5.46) concerns the constant external current J_E^2. Given that the external current depends solely on the current density, specifically $J_E^2 = \mathbf{j}^2$, and considering that our background has no charge density ($\rho = 0$), we observe that the production rate (5.46) remains non-zero. This indicates that even in the absence of electric fields from the charge density, pair production occurs.

However, it is important to remember that as the external current is constant in momentum space, it corresponds, in position space, to a point-like external current (a delta function) in position and time. This external current can lead to the generation of both magnetic and electric fields. Specifically, since the magnetic fields produced by the current density are not constant in time due to a rapid variation in the external current, electric fields are induced. These electric fields can, in turn, lead to pair production. If the current density were constant, the background field would be purely magnetic, and pair production would not occur as described by Schwinger's formalism.

6 Conclusions

In this work, we investigated the effects of a one-loop quantum correction derived from a non-local effective action on pair production in the vacuum. This mechanism, also known as the Schwinger effect, exemplifies the instability of the quantum vacuum in QED in the presence of an intense external electric field and demonstrates the so-called vacuum decay.

The analysis presented here focused on two backgrounds: the plane wave and a wave with a constant external current. To obtain the pair production rates, we considered a non-local effective action resulting from the quantisation of generic fields coupled with background electromagnetic fields through a parameter α.

In the scenario where the background field was composed of plane waves, we obtained a null production rate, indicating that the vacuum state remains stable under this background. Such a background does not produce imaginary contributions to the effective action, thus preventing vacuum decay, as observed by Schwinger. Conversely, in the scenario featuring a constant external current, we observed a non-zero pair production rate, indicating instability of the vacuum in the presence of this background.

As an extension of this work, we suggest an exploration of the free parameter α, given that this coefficient relies on the number and type of quantised fields in the theory. We expect changes in pair production rates and, consequently, in the vacuum decay probability. Another suggestion is to investigate the second renormalisation process that can occur in order to remove the cutoff dependence of the production rate for the constant external current.

Furthermore, we propose an examination of pair production, accounting for the non-local quantum correction, across other backgrounds. These include backgrounds defined by constant electromagnetic fields, similar to Schwinger's investigation. We anticipate that in this background, the imaginary contribution of the effective action will arise from the correction term, as the integration of the contraction of electromagnetic tensors does not present any poles. Other backgrounds that could be explored include inhomogeneous ones, which have been examined in the literature [21, 24].

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APPENDIX A – Lorentz invariant for a monochromatic plane wave

Here we want to demonstrate how $\tilde{F}^{\mu\nu}F_{\mu\nu} = 0$ for plane waves. To do so, we employ the definition of the dual field-strength tensor (3.52) and express this contraction as

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}F_{\mu\nu}. \tag{A.1}$$

By expressing the dual electromagnetic tensor in terms of the gauge fields, we obtain

$$\begin{aligned}\tilde{F}^{\mu\nu}F_{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right)F_{\mu\nu}\\ &= \epsilon^{\mu\nu\rho\sigma}\left(\partial_\rho A_\sigma\right)F_{\mu\nu},\end{aligned} \tag{A.2}$$

where we used the antisymmetric property of the Levi-Civita symbol from the first to the second line. Additionally, we observe that this equation can be rewritten using the product rule $\partial_\rho(A_\sigma F_{\mu\nu}) = (\partial_\rho A_\sigma)F_{\mu\nu} + A_\sigma\partial_\rho F_{\mu\nu}$, yielding

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\left[\partial_\rho(A_\sigma F_{\mu\nu}) - A_\sigma\partial_\rho F_{\mu\nu}\right]. \tag{A.3}$$

We can further simplify this equation by noticing that the last term of this equation is null due to the contraction between antisymmetric and symmetric tensors, as done in (3.54). Therefore, we have

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\partial_\rho(A_\sigma F_{\mu\nu}). \tag{A.4}$$

Upon substituting the plane wave solution (5.28) into equation (A.4), we obtain

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}\partial_\rho(A_\sigma F_{\mu\nu}) &= \epsilon^{\mu\nu\rho\sigma}\partial_\rho\left\{\bar{A}_\sigma e^{-ikx}\left[\partial_\mu\left(\bar{A}_\nu e^{-ikx}\right) - \partial_\nu\left(\bar{A}_\mu e^{-ikx}\right)\right]\right\}\\ &= -i\epsilon^{\mu\nu\rho\sigma}\partial_\rho\left[\bar{A}_\sigma e^{-ikx}\left(k_\mu\bar{A}_\nu e^{-ikx} - k_\nu\bar{A}_\mu e^{-ikx}\right)\right]\\ &= -i\epsilon^{\mu\nu\rho\sigma}\partial_\rho\left[\bar{A}_\sigma\left(k_\mu\bar{A}_\nu e^{-2ikx} - k_\nu\bar{A}_\mu e^{-2ikx}\right)\right].\end{aligned} \tag{A.5}$$

From the first to the second line, we performed the differentiations and factored out $-i$, and from the second to the third line, we multiplied the exponentials. Also, recall that $\bar{A}_\rho$ denotes the position-independent wave amplitude, which remains constant under differentiations. Upon differentiation with respect to ∂_ρ, we proceed to

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}\partial_\rho(A_\sigma F_{\mu\nu}) &= -2\epsilon^{\mu\nu\rho\sigma}k_\rho\bar{A}_\sigma\left(k_\mu\bar{A}_\nu - k_\nu\bar{A}_\mu\right)e^{-2ikx}\\ &= 0.\end{aligned} \tag{A.6}$$

Note that the factor $-2ik_\rho$ arises from the differentiation of the exponentials and that we have again a contraction of symmetric and antisymmetric tensors, yielding a null outcome.