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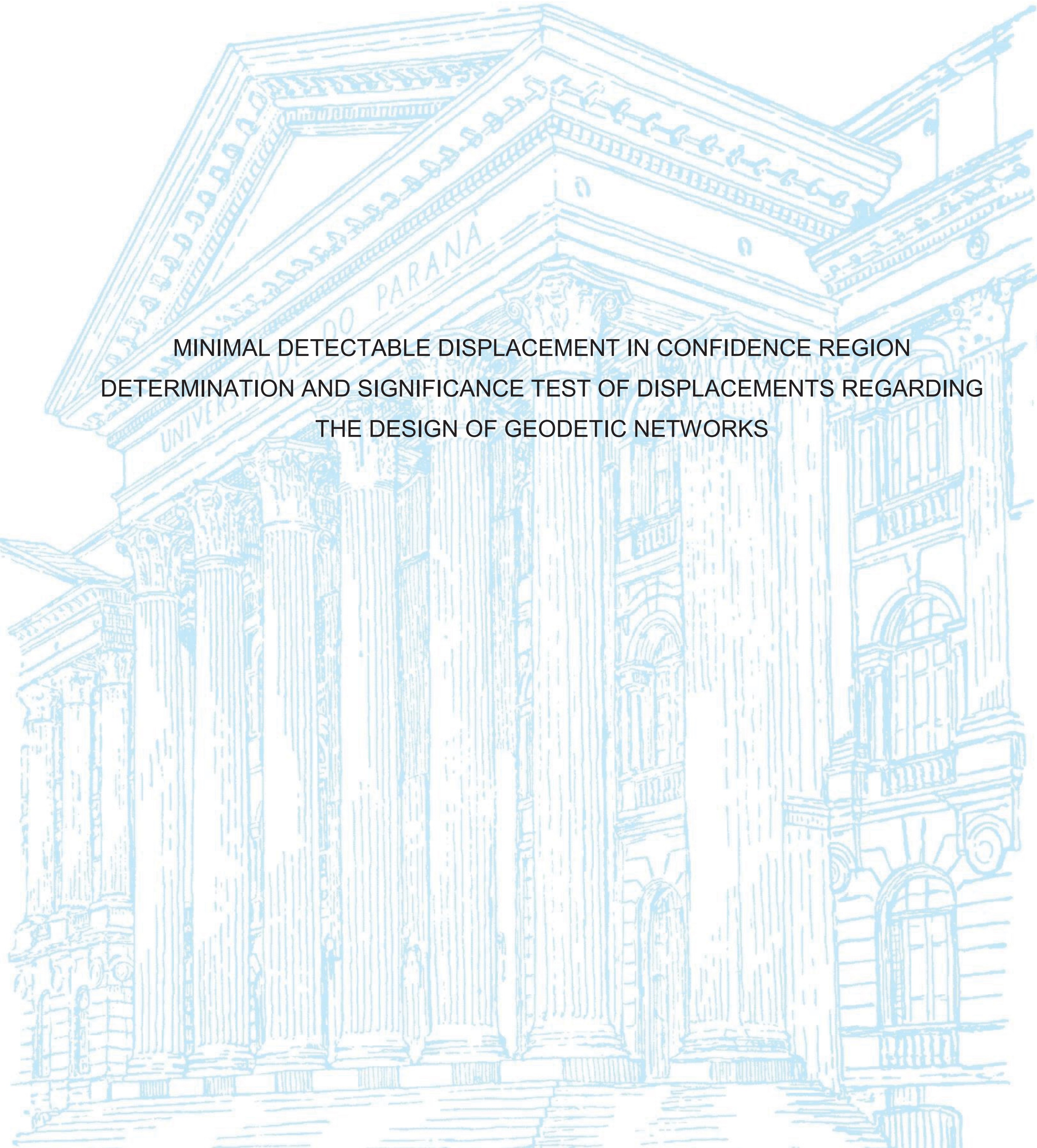

MINIMAL DETECTABLE DISPLACEMENT IN CONFIDENCE REGION DETERMINATION AND SIGNIFICANCE TEST OF DISPLACEMENTS REGARDING THE DESIGN OF GEODETIC NETWORKS

CURITIBA

2024

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MINIMAL DETECTABLE DISPLACEMENT IN CONFIDENCE REGION DETERMINATION AND SIGNIFICANCE TEST OF DISPLACEMENTS REGARDING THE DESIGN OF GEODETIC NETWORKS

Tese de Doutorado apresentado ao Programa de Pós-Graduação em Ciências Geodésicas, Setor de Ciências da Terra, Universidade Federal do Paraná, como requisito parcial à obtenção do título de Doutor em Ciências Geodésicas.

Orientador: Prof. Dr. Luís Augusto Koenig Veiga

Coorientador: Prof. Dr.Ivandro Klein

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RESUMO

O uso de redes geodésicas para o monitoramento estrutural é amplamente aplicado em projetos de engenharia e em outras áreas. Esta tese examina alguns aspectos relacionados às abordagens adotadas para determinar deslocamentos de pontos de monitoramento, essenciais para interpretar deformações de estruturas. Entre os aspectos mais relevantes estão as propriedades da rede, como geometria ou configuração, e a qualidade das observações. Adicionalmente, os métodos usados para a determinação de coordenadas, frequentemente associados à estimativa de mínimos quadrados (LSE), pré-análise por meio de análises de sensibilidade e testes de deslocamento (testes de congruência) têm um papel crucial no monitoramento geodésico. Este trabalho foca na investigação de algumas propriedades desses aspectos. Inicialmente, foi estudado e analisado o ajuste de rede sob a abordagem livre para dois métodos comuns: as restrições internas mínimas e o método baseado em inversões generalizadas, especialmente o método inverso de Moore-Penrose. Os resultados mostraram que ambos os métodos são equivalentes para ajuste de rede, e assim no passo de pré-análise. Posteriormente, a avaliação do teste de sensibilidade para análise de deslocamento foi explorada. Para isto, o teste de congruência foi modificado seguindo as propriedades de sensibilidade, em particular o uso do valor crítico associado ao parâmetro de não-centralidade ao invés do valor crítico do teste de congruência baseado no teste qui-quadrado. As análises consideraram a geometria da rede e as propriedades do modelo estocástico. Os principais achados da pesquisa mostraram a influência das propriedades da rede na capacidade de detecção de deslocamentos usando o teste de congruência modificado. Por fim, foi realizada uma análise do método apresentado por Prószyński e Łapiński em 2021, que integra as propriedades de sensibilidade no teste de congruência. Os principais achados relacionam-se à influência da configuração ou geometria da rede, modelo estocástico, dimensão espacial, valores de limiar e tipos de erros aceitos no método de Prószyński e Łapiński 2021. Por último, desenvolveu-se uma pré-análise de uma rede geodésica GNSS proposta para monitoramento, baseada nos conceitos supracitados.

Palavras-chave: Monitoramento geodésico, mínimos deslocamentos detectáveis, pre-analises

ABSTRACT

The use of geodetic networks for structural monitoring is widely applied in engineering projects and related areas. This thesis examines some aspects related to the approaches adopted to determine displacements of monitoring points, which are relevant for interpreting deformations of structures. Among the standout aspects are the network properties, such as geometry or configuration, and the quality of observations. Additionally, the methods used for coordinate determination, commonly associated with least square estimation (LSE), pre-analysis through sensitivity analyses, and displacement tests (congruence tests) play a key role in geodetic monitoring. This work focuses on investigating some properties of these aspects. Initially, network adjustment under the free approach was studied and analyzed for two common methods: the minimum inner constraints and the method based on generalized inverses, particularly the Moore-Penrose inverse method. The results showed both methods are equivalent for network adjustment, and thus in the pre-analysis step. Subsequently, the assessment of the sensitivity test for displacement analysis was explored. For this, the congruence test was modified following the sensitivity properties, particularly the use of the critical value associated with the non-centrality parameter instead of the critical value of the congruence test based on the chi-square test. The analyses considered network geometry and stochastic model properties. The main findings of the research showed the influence of network properties on the capacity to detect displacements using the modified congruence test. Finally, an analysis of the method presented by Prószyński and Łapiński in 2021, which integrates the sensitivity properties in the congruence test, was conducted. The main findings here relate to the influence of network configuration or geometry, stochastic model, spatial dimension, threshold values, and types of errors accepted in the Prószyński and Łapiński 2021 method. Lastly, a pre-analysis of a proposed GNSS geodetic network for monitoring was developed under the aforementioned concepts.

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Keywords: Geodetic monitoring, minimal detectable displacements (MDD), pre-analyses

TABLE OF FIGURES

Figure 1: Ellipsoids of confidence and sensitivity for $h < h^*$ 21
Figure 2: Ellipsoids of confidence and sensitivity for $h > h^*$ 21

TABLE OF CONTENTS

1 INTRODUCTION **10**
1.1 HYPOTHESIS 12
1.2 OBJECTIVE 13
1.2.1 Main Objective 13
1.2.2 Specific objectives 13
1.2.3 Materials and methods 13
1.3 STRUCTURE OF THE WORK 13
1.3.1 Literature review: 14
1.3.2 Developed papers section 14
1.3.3 Conclusions section: 14
2 LITERATURE REVIEW **15**
2.1 FREE NETWORK ADJUSTMENT OF GEODETIC NETWORKS 15
2.1.1 Minimum Inner Constraints Approach 16
2.1.2 Pseudo inverse 17
2.2 DEFORMATION ANALYSIS: GLOBAL CONGRUENCE TEST (GCT) 18
2.3 SENSITIVITY ANALYSIS 19
2.4 MINIMAL DETECTABLE DISPLACEMENTS (MDD) 19
2.5 CONFIDENCE REGION DETERMINATION SUPPORTED BY NETWORK SENSITIVITY CHARACTERISTICS 20
3 DEVELOPED PAPERS **22**
3.1 FREE NETWORK ADJUSTMENT: MINIMUM INNER CONSTRAINTS AND PSEUDO-INVERSE APPROACHES 22
3.2 INFLUENCE OF NETWORK CONFIGURATION AND STOCHASTIC MODEL ON THE DETERMINATION OF THE MINIMUM DETECTABLE DISPLACEMENTS (MDD) THROUGH SENSITIVITY ANALYSIS AND SIGNIFICANCE TEST 33
3.3 MINIMAL DETECTABLE DISPLACEMENT IN CONFIDENCE REGION DETERMINATION AND SIGNIFICANCE TEST OF DISPLACEMENTS REGARDING THE DESIGN OF GEODETIC NETWORKS 49
5 CONCLUSIONS **69**
REFERENCES 71

1 INTRODUCTION

The employment of geodetic networks for monitoring purposes is widely recognized as an effective method to quantify displacements caused by structural deformations. As outlined by Caspary and Rüeger (1987) geodetic monitoring is predominantly utilized in key areas including the observation of recent crustal movements, the study of slope creep, the tracking of glacier and shelf ice movements, the monitoring of ground subsidence, and the analysis of deformations in both man-made and natural structures. The primary objective of geodetic monitoring lies in substantiating hypotheses within disciplines such as geophysics, geology, glaciology, and engineering sciences. Additionally, it plays a crucial role in the assessment of engineering structures, specifically in Structural Health Monitoring (SHM), aiding in the protection against hazards to populations, and in determining liability for damages resulting from structural failures.

An important aspect concerning the properties of geodetic networks employed for monitoring is their capability to detect displacements, which is approached by the sensitivity analysis through the minimum detectable displacements (MDD) computation. This analysis involves several factors such as the positioning of monitoring targets on the structures or in proximity to them, a factor traditionally linked with network geometry. The quality of observations, integral to the stochastic model, is another key consideration. Furthermore, the methodology for calculating coordinate positions typically associated with the least squares estimator approach and how the sensitivity test is applied are essential components in the effective utilization of geodetic networks for monitoring purposes. The process of integrating the aforementioned elements to yield the most effective outcomes can be characterized as the optimization of geodetic monitoring networks. The best scenario for the displacements estimations through sensitivity analysis is defined.

The geometry and stochastic model influences could be exemplified by design order problems discussed by Schmitt (1985). Here certain aspects related to the geometry of the network and stochastic models could be applied to sensitivity analysis. In particular, the first-order design (FOD) and second-order design (SOD) pertain to the network's geometry and the precision of observations, respectively. Other aspects of the design order problem, in particular the datum definition or zero-order design (ZOD) have been studied (Even-Tzur, 2010). In this context, Grafarend and Sansò

(2005) present a comprehensive exploration of various optimization processes for design order problem, which can be extended to sensitivity analysis. Kuang (1996) presents the fundamentals of the pre-analysis approaches for the design of geodetic networks. Amiri-Simkooei, Asgari and Zaminpardaz (2012); Ogundare (2015) presents a review of design stage of geodetic networks. All these principles are applicable to the sensitivity analyzes to evaluate their influences on the displacements estimation (Kuang, 1991).

The sensitivity analysis is carried out by the least squares estimation (LSE). According to Amiri-Simkooei, Asgari and Zaminpardaz, (2012) the computation of the cofactor matrix of the parameters for different network dimensions provides a priori quality for the parameters, which permits the evaluation of the network in the design step. In the case of the sensitivity analyses, the LSE provides the cofactor matrix to compute the MDD and subsequently the network sensitivity. At this juncture, various approaches offered by the least squares estimator methods are available for conducting sensitivity analyses. Among these, the A-model and B-model stand out as common models (Teunissen, 2000), along with the constraints or combined model (Ghilani, 2017; Strang; Borre, 1997). Additionally, the free adjustment facilitated by inner constraints or pseudo-inverse approaches is explored (Meissl, 1982; Ogundare, 2018; Welsch, 1979)

Some aspects related to the geometry and stochastic model and their influence on the sensitivity analysis were presented by Kuang, (1991), in this study, these elements influence the sensitivity magnitude improving or worsening its detecting capacity. Regarding the least squares estimation method, the inner constraints or pseudo-inverse approach are commonly used to estimate the pre-analysis of the geodetic network and the sensitivity analysis (Kotsakis, 2013; Meissl, 1982; Aydin, 2014). The main idea besides the use of this model is to avoid the influence of control points in these analyses.

The application of the sensitivity analysis can be find in several works, Hsu and Hsiao, (2002) presented a study to determine the sensitivity of the GNSS network designed for crustal deformation analyses. Even-Tzur, (2010) analyzes the influence of the Datum definition in the sensitivity analysis. Küreç and Konak, (2014) present a study to investigate the possibilities of monitoring crustal movement, during collective evaluation of first-and second-order GPS densification networks. Yu et al., (2000) present the sensitivity analyses for volcano monitoring. Erdogan, Hekimoglu and Durdag (2017) present a study comparing the empirical and theoretical MDD on geodetic networks. Książek and Łapiński, (2022) present a study applied to the control network based on sensitivity analysis.

Regarding the theoretical bases of the sensitivity analysis, a modification of this was presented by Prószyński and Łapiński, (2021). This approach searches the integration of the accuracy and sensitivity criteria in a unique analysis supported by the MDD and the computation of the significance and sensitivity ellipsoids. The method called *Variance factor option (I)* is based in the determination of a specific value for type II probability error (β) such that critical value of the sensitivity and significance approaches fulfilled the equality $u_{h,\alpha_0} = \lambda_{\alpha_0\underline{\gamma_0},h}$ where u is the critical value for the significance test for h degrees of freedom and type I error probability (α_0) . While λ corresponds to the noncentrality parameter associated with the critical value of sensitivity analysis. In this method, the key element is the degrees of freedom h, given by the dimension of the displacement vector. In this model, aspects related to the network properties are not explored.

Based on the previous concepts, this work firstly is focused on the analysis of the least squares estimation method, in particular, the comparison of the inner constraint approach and pseudo-inverse method to solve the inversion of normal equation matrix for the network preanalysis, network adjustment and sensitivity analysis. Here the main contribution is the application of different analysis of these methods, in particular on the sensitivity analyses.

Subsequent studies focused on aspects of sensitivity analysis, particularly the method presented by Prószyński and Łapiński (2021). This research involved analyzing network properties such as geometry, redundancy, stochastic models, network dimension, and types of models (linear and nonlinear) within the context of the Prószyński and Łapiński approach. The primary contribution of this research lies in examining the influence of network properties, specifically, the optimization of the Prószyński and Łapiński method, which has not yet been addressed.

1.1 HYPOTHESIS

If the geodetic network properties, including geometry or configuration, redundancy, quality of observations, spatial dimension, and model type (linear or nonlinear), are modified, then the accuracy analysis based on sensitivity characteristics will not solely depend on the magnitude of the displacement vector but also of the geodetic network properties (network configuration, network dimension) and the accuracy of the observations.

1.2 OBJECTIVE

In the following items, the general and specific objectives proposed for the development of this work will be addressed.

1.2.1 Main Objective

Evaluate the influence of the properties of a geodetic network on the determination of confidence region supported by network sensitivity characteristics.

1.2.2 Specific objectives

a) Evaluate network configuration or geometry of the geodetic network on the Prószyński and Łapiński, (2021) theory
b) Evaluate network dimension properties of the geodetic network on the Prószyński and Łapiński, (2021) theory
c) Evaluate observation quality of the geodetic network on the Prószyński and Łapiński, (2021) theory

1.2.3 Materials and methods

a) Define the least square approach to apply the sensitivity analysis on geodetic networks.
b) Compare the sensitivity analyses and congruence test.
c) Analyze the network properties and characteristics of the approach presented by Prószyński and Łapiński, (2021).
d) Apply the analysis to the designing of a geodetic monitoring network.

1.3 STRUCTURE OF THE WORK

The thesis is structured into five sections, exploring the research through a literature review, three articles embodying the proposed idea, and the conclusion. A summary of each section is presented below:

1.3.1 Literature review:

This section presents the theoretical basis of the research goal. Specifically, it covers the free adjustment network, which is content related to the first paper, sensitivity analysis and deformation test associated with the second paper, and finally, the method proposed by Prószyński and Łapiński (2021) related to the third paper.

1.3.2 Developed papers section

This section presents the papers developed during this work. The names of each paper are listed below:

Free network adjustment: Minimum inner constraints and Pseudo-inverse approaches.

Influence of network configuration and stochastic model on the determination of the minimum detectable displacements (MDD) through sensitivity analysis and significance test.

Minimal Detectable Displacement in confidence region determination and significance test of displacements regarding the design of geodetic networks.

1.3.3 Conclusions section:

This section presents the main conclusions of the research, integrating insights from the three papers developed.

2 LITERATURE REVIEW

2.1 FREE NETWORK ADJUSTMENT OF GEODETIC NETWORKS

The free adjustment is a least-squares estimation model in which no control points are defined for the network. This means that the observations are adjusted without a connection to the reference system (Ogundare, 2018). The use of the free adjustment approach aims to analyze the internal quality of the network, specifically whether the network observations are consistent and if they can provide high-quality indicators without the influence of two or more control points (Welsch, 1979).

The absence of control points in the free adjustment leads to the datum defect problem, which can be defined as the condition where there is no datum definition that allows the network to connect with a reference system. Mathematically, the datum defect is related to the rank defect problem, where the solution of the least-squares estimator (LSE) cannot be solved due to the impossibility of inverting the normal equation matrix using traditional methods. To analyze the free adjustment model, the classical LSE solution for a Gauss-Markov model will be presented as a starting point, as shown below.

$$\hat{x} = (A^T W A)^{-1} A^T W y. \quad (1)$$

Where $\hat{x}$ corresponds to the estimated parameters, A is the design or configuration matrix, W is the weight matrix, and y is the vector of observations. From eq.1 let $N = A^T W A$ and $u = A^T W A$ where N is the normal equation matrix and u is the vector of independent terms. Consequently, equation 1 can be reformulated as follows:

$$\hat{x} = (N)^{-1} u. \quad (2)$$

The structure of the normal equation matrix (N) depends on the A matrix and W matrix (design matrix and weight matrix respectively). It's important to note that the W matrix has a dimension of (n, n) where n corresponds to the number of observations.

Similarly, the A matrix has a dimension of (n, u) with u representing the number of unknowns in the system.

To successfully invert the normal equation matrix $N_{(u,u)}$ this matrix must be non-singular, which means that its determinant is not zero ($|N| \neq 0$). When N singular, inversion using traditional methods becomes impossible. In such instances, the datum defect, typically arising in models without control points (as in free adjustment), leads to this singularity in the N matrix (Deakin, 2005). To address the inversion of the normal equation matrix under the free adjustment approach, two methods are presented:

2.1.1 Minimum Inner Constraints Approach

The Minimum Inner Constraints model incorporates the minimal number of parameters necessary to define a reference system. In this model, known as the Minimum Constraints model, the external geometry is not considered during the adjustment procedure. Consequently, the shape and geometric size of the network are defined solely by its internal geometry (Ogundare, 2018). The normal equations (N) are given by:

$$N = A^T W A + G G^T \tag{3}$$

In the normal equation N, the term GG^T is added. The G matrix known as the constraint matrix, spans the null space of A and contains the inner datum parameters, which define the network's dimensionality. The configuration of the G matrix, as outlined by Kotsakis (2018), Ogundare (2018), and Setan (1995), incorporates considerations for rotation, translation, and scale within a 3D network, as follows:

$$G^t = \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & : & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & : & 0 & 0 & 1 & \cdots & 1 \\ 0 & Z_1 & -Y_1 & : & 0 & Z_2 & -Y_2 & \cdots & -Y_n \\ -Z_1 & 0 & X_1 & : & -Z_2 & 0 & X_2 & \cdots & X_n \\ Y_1 & -X_1 & 0 & : & Y_2 & -X_2 & 0 & \cdots & 0 \\ X_1 & Y_1 & Z_1 & : & X_2 & Y_2 & Z_2 & \cdots & Z_n \end{bmatrix} \tag{4}$$

The first three rows of the G matrix correspond to the 3D translations, the subsequent three rows to the 3D rotations, and the final row to the scale parameters. In both cases, X_1,Y_1,Z_1 represent the approximate coordinates of the network. The number of columns in the G matrix equals the number of parameters to be estimated. Therefore, for a level network requiring the estimation of five parameters, the translation of the vertical component in the G matrix is expressed as $G = [1\ 1\ 1\ 1\ 1]^T$ (Setan, 1995). The adjustment parameters are presented in Equation 5:

$$\hat{x} = N^{-1}A^TWAN^{-1}u \tag{5}$$

The cofactor matrix is given by:

$$Q_{\hat{x}} = N^{-1}A^TWAN^{-1} \tag{6}$$

2.1.2 Pseudo inverse

In free network adjustment, the inversion of the normal equation (N) matrix can be computed by the generalized inverses (Mälzer; Schmitt; Zippelt, 1979; Rao; Mitra, 1972). In particular, The Moore-Penrose inverse is the main inverse used in geodetic networks problems called minimum norm least squares g-inverse (Welsch, 1979). Thus, for $A \in \mathbb{R}^{n\,x\,u}$ and the linear system $A \cdot \vec{x} = \vec{y}$ with $\vec{x} \in \mathbb{R}^n; \vec{y} \in \mathbb{R}^m$. The Moore-Penrose pseudo-inverse provides a solution $\vec{x} = A^{\dagger}\vec{y}$, where $A^{\dagger}$ is a pseudo inverse of A . This matrix is unique and has the following properties (Equation 7):

$$\begin{aligned} &a.\, A \cdot A^{\dagger} \cdot A = A \\ &b.\, A^{\dagger} \cdot A \cdot A^{\dagger} = A^{\dagger} \\ &c.\, (A \cdot A^{\dagger})^t = A \cdot A^{\dagger} \\ &d.\, (A^{\dagger} \cdot A)^t = A^{\dagger} \cdot A \end{aligned} \tag{7}$$

For full rank matrices (rows or columns linearly independent) the pseudo inverse can be obtained for non-square matrix. Therefore, if $m < n$ (rows linearly independent), $A^{\dagger} = A^t \cdot (A \cdot A^t)^{-1}$ and for $m > n$ (columns linearly independent), $A^{\dagger} = (A^t \cdot A)^{-1} \cdot A^t$. When $A^{\dagger} = A^{-1}$. For matrices with deficient rank, the solution is commonly obtained by the Singular Value decomposition (SVD). Where A can be decomposed as $A = U\Sigma V^t$ and $A^\dagger = V\Sigma^\dagger U^t$ (Burdick, 2010). Thus, the solution for least squares is

$$\hat{x} = N^\dagger \cdot A^t \cdot W \cdot y \tag{8}$$

While the variance-covariance matrix is given by:

$$Q_{\hat{x}} = N^\dagger \tag{9}$$

2.2 DEFORMATION ANALYSIS: GLOBAL CONGRUENCE TEST (GCT)

Aydin (2014) develops the theoretical bases for deformation test. Here a displacement vector is given by:

$$\hat{d} = \hat{x}_2 - \hat{x}_1 \tag{10}$$

Where $\hat{x}_1$ and $\hat{x}_2$ are the least squares solution for the monitoring network parameters (point coordinates) in the first and second epoch respectively. The covariance matrix is computed as:

$$C_d = \sigma_0^2 Q_{\hat{d}} \tag{11}$$

Here Q_d is the cofactor matrix of the displacement vector, and σ_0^2 is the a-priori variance factor. Based on hypothesis testing, two hypotheses are formulated:

$$H_0 : E(\hat{d}) = 0 \text{ and } H_A : E(\hat{d}) \neq 0 \tag{12}$$

While the test statistic is given by:

$$\Phi = \hat{d}^T C_d^+ \hat{d} \,. \tag{13}$$

Which follows (central) χ^2-distribution with h degrees of freedom in H_0, is compared with the theoretical value of $\chi^2_{(\alpha,h)}$ corresponding to the α-significance level. If the test statistic Φ is smaller than the critical value, the null hypothesis H_0 is not rejected with the confidence level of $1-\alpha$, and it is concluded that there is no deformation between the two epochs. Otherwise, it is decided that displacement has occurred with the probability risk of a false positive given by α. This congruence test may be performed by considering the estimated variance factor. In this case, the test follows F (Fisher)-distribution. More information can be found in Aydin (2014).

2.3 SENSITIVITY ANALYSIS

The capacity of the network to detect displacements can be quantified by sensitivity analysis (Aydin, 2014). Based on the deformation test, a theoretical vector of expected displacement, denoted by Δ, is related to the alternative hypothesis defined as:

$$H_A : E\left(\hat{\mathrm{d}}\right) \neq 0 = \Delta \tag{14}$$

Here the non-rejection of H_A implies that the expected displacement values of the vector Δ can be detected by the monitoring network. The theoretical relationship of the non-centrality parameter (λ) and the test statistic (T) is given by:

$$T = \lambda = \Delta^T C_d^+ \Delta \tag{15}$$

The condition to define if the network is sensitive to displacements is given by:

$$\lambda \geq \lambda_0 \tag{16}$$

where λ_0 is the so-called lower bound of the non-centrality parameter which fulfills the given power of the test γ_0 being the complement of the type-II error probability β_0: $\gamma_0 = 1 - \beta_0$. Here λ_0 is obtained from Aydin and Demirel (2004).

2.4 MINIMAL DETECTABLE DISPLACEMENTS (MDD)

To evaluate the MDD, firstly a vector with the expected displacements is defined from a vector of directions g and a scale factor value denoted by b (Aydin, 2014). Thus, the condition $\Delta = bg$ is fulfilled. If $b = b_{min}$ $\left(\Delta_{min} = b_{min}\, g\right)$, then the determination of b_{min} is given by:

$$b_{min} = \sqrt{\lambda_0 \lambda_{max}} \quad (17)$$

where λ_{max} is the maximum eigenvalue of the covariance matrix of C_d. According to Küreç and Konak (2014) b_{min} is the best sensitivity level of the network. On the other hand, the worst sensitivity level of the network can be computed as:

$$b_{max} = \sqrt{\lambda_0 \lambda_{min}} \quad (18)$$

Where λ_{min} is the minimum eigenvalue of C_d. According to Hsu and Hsiao (2002) the average between b_{max} and b_{min} can be interpreted as the global sensitivity for the entire network.

To obtain the vector of directions of displacements, Aydin (2014) computed the (unity) eigenvectors corresponding to the maximum eigenvalue (λ_{max}) and the minimum eigenvalue ($\lambda_{\min}$) of the C_d matrix (Λ_{max} and Λ_{min} respectively). Thus: $\Lambda_{min} = b_{min}\Lambda_{max}$ and $\Lambda_{max} = b_{max}\Lambda_{min}$. The MDD in each i^{th} element of the vector $\hat{d}$ is given by the respective i^{th} element of Δ_{min} or Δ_{max}. For details, we suggest (AYDIN, 2014).

2.5 CONFIDENCE REGION DETERMINATION SUPPORTED BY NETWORK SENSITIVITY CHARACTERISTICS

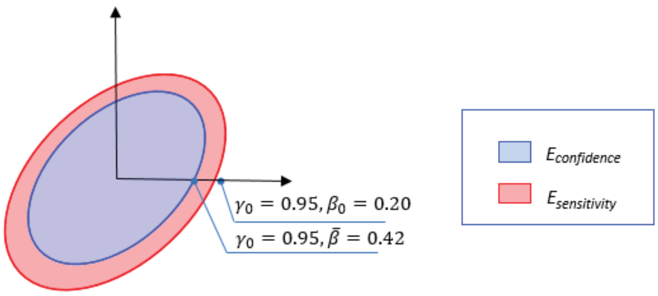
The theoretical basis for the confidence region determination supported by network sensitivity characteristics is developed for the scenarios where σ_0^2 is used. Here, a specific value for the power of the test γ_0 is determined through a value of β_0 coordinated with the stipulated level of significance α_0 and the h-dimensional displacement vector. This approach provides equality between the critical value for the significance test and the non-centrality parameter of the sensitivity test: $\Phi_{(h,\alpha_0)} = \lambda_{(h,\alpha_0,\beta_0)}$ (Prószyński; Łapiński, 2021).

Initially for $h = 1$ the relation $\lambda_{h,\alpha_0,\beta_0} > \Phi_{h,\alpha_0}$ is fulfilled. If the h value increases, both values $\lambda_{h,\alpha_0,\beta_0}$ and Φ_{h,α_0} increase. However, for a specific value of h (namely h^*), the relation $\Phi_{h,\alpha_0} = \lambda_{h,\alpha_0,\beta_0}$ is achieved (Prószyński; Łapiński, 2021). The results presented by the authors show that for $h^* = 7.3$ the above equality holds to $\alpha_0 = 0.05$ and $\beta_0 = 0.20$. In addition, for values greater than $h^* = 7.3$ (inflexion point), the relation

$\Phi_{h,\alpha_0} > \lambda_{h,\alpha_0,\beta_0}$ is fulfilled. Note that for different values for α_0 and β_0, the value for h^* also changes. Also the dimension of h should be a integer value.

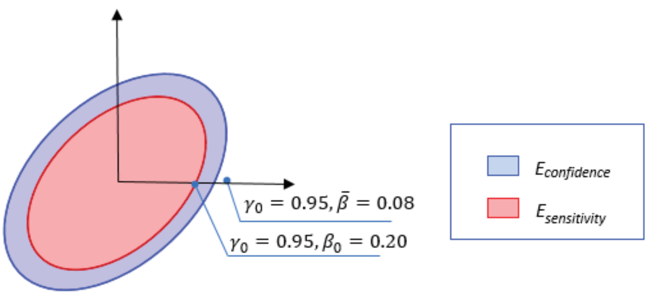
The approach proposed by Prószyński and Łapiński (2021) is based on the size comparison of three concentric ellipsoids; sensitivity ellipsoid, confidence ellipsoid, and significance ellipsoid. This comparison is carried out by global sensitivity. For the significance ellipsoid, the relation $\hat{d}^T C_d^+ \hat{d} = \Phi_{h,\alpha_0}$ is fulfilled for a α_0 significance level; for the confidence ellipsoid, $\hat{d}^T C_d^+ \hat{d} = \Phi_{h,CL}$ is fulfilled for a given confidence level (CL) and the sensitivity ellipsoid, $\hat{d}^T C_d^+ \hat{d} = \Phi_{h,\alpha_0,\beta_0}$. Considering $CL = 1-\alpha_0$, the significance ellipsoid turns into the confidence ellipsoid and the analysis focuses on the determination of $\Phi_{h,\alpha_0} = \lambda_{h,\alpha_0,\beta_0}$. In this case, two scenarios were defined: $h > h^*$ and $h < h^*$, where h^* is the value for h that satisfies the equality $\Phi_{h,\alpha_0} = \lambda_{h,\alpha_0,\beta_0}$. Then the confidence and sensitivity ellipsoids were determined, for $h < h^*$ the confidence ellipsoid is smaller than the sensitivity ellipsoid (Figure 1) while for $h > h^*$ the sensitivity ellipsoid is smaller than the confidence ellipsoid (Figure 2).

Figure 1: Ellipse of confidence and sensitivity for $h < h^*$.



Source: The author.

Figure 2: Ellipse of confidence and sensitivity for $h > h^*$.



Source: The author.

3 DEVELOPED PAPERS

For the composition of this thesis, 3 scientific articles were developed, of which 2 have been published, and last is in the revision phase.

3.1 FREE NETWORK ADJUSTMENT: MINIMUM INNER CONSTRAINTS AND PSEUDO-INVERSE APPROACHES

In this study, we conducted an in-depth examination of the two primary methodologies employed for the free adjustment of geodetic networks: the Minimum Inner Constraints method and the Pseudo Inverse technique. We provided a comprehensive theoretical framework for both methods before implementing them on a two-dimensional geodetic network. Our results demonstrated that, within the context of this specific network, both methods yield equivalent outcomes. Additionally, we delved into aspects of the iterative processes associated with these models, particularly focusing on evaluating their nonlinearity conditions. This paper makes a significant contribution to the field by elucidating the theoretical underpinnings of least square models, which are frequently utilized in sensitivity analyses and deformation testing within geodetic studies.

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INVESTIGACIÓN

Free network adjustment: Minimum inner constraints and Pseudo-inverse approaches

Ajuste de red libre: Enfoques de condiciones internas mínimas y pseudo inversa

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ABSTRACT:

The least squares technique is a classic procedure to compute the coordinates of a geodetic network. Different approaches of this method have been developed to perform the least squares adjustment and thus solve the linearized system that relates the observations (internal geometry) and the reference system (external geometry). The free adjustment is a model that does not use fix coordinates in the design matrix, thus the solution does not have connection with referential system or datum. Therefore, the rank deficiency problem or datum defect, which in terms of linear algebra defines a singular matrix in the system of normal equations, must be solved. Two mainly approaches of free adjustment are used to solve a geodetic network, the minimum inner constraints and pseudo-inverse technique. Both models provide results in an arbitrary reference system, therefore, the S-transformation is a typical procedure to transform the result to a known datum. This paper presents a review of both methods and the necessary methodology to perform a free network adjustment. Finally, an example was presented to analyze the equivalence between both methods. The results obtained were compared with an estimation realized through the constrained adjustment.

Keywords: Least squares, Free adjustment networks, Minimum inner constrains, Pseudo-inverse, S- transformation.

RESUMEN

El método de los mínimos cuadrados es un procedimiento clásico para calcular las coordenadas de una red geodésica. Se pueden utilizar diferentes modelos para realizar el ajuste por mínimos cuadrados y así resolver el sistema linealizado que relaciona las observaciones (geometría interna) y el sistema de referencia (geometría externa). Uno de los métodos es el ajuste libre, el cual es un modelo que no utiliza coordenadas fijas en la matriz de diseño, por lo que la solución no tiene conexión con el sistema de referencia o datum. Por lo tanto, el problema de la deficiencia de rango o datum en términos de alegra lineal define una matriz singular para el sistema de ecuaciones normales que tiene que ser resuelto para ajustar una red geodésica. Mediante este método se utilizan principalmente dos enfoques de ajuste libre, la técnica de restricción mínima interna y la técnica pseudo inversa. Ambos modelos proporcionan resultados en un sistema de referencia arbitrario, por lo que la S-transformación es un procedimiento típico para transformar los resultados a un datum o sistema de referencia conocido. En este trabajo se presenta una revisión de ambos métodos y la metodología necesaria para realizar un ajuste de red libre. Finalmente se presentó un ejemplo para analizar la equivalencia entre ambos métodos. Los resultados obtenidos se compararon con una estimación realizada a través del modelo de ajuste con constreñimientos.

Palabras clave: Mínimos cuadrados, ajuste libre de redes, restricción mínima interna, Pseudo-inversa, Transformación S.

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Introduction

The main technique for coordinates computation of a geodetic network is the least squares method. This method relates the internal geometry (observations) and the external geometry (parameters). An important step on the least square adjustment of networks is the definition of a referential system or datum that allows connecting the internal geometry with a reference system. An approach for the definition of the datum is the selection of control points belonging to the external geometry network, so these points are considered fix in the design matrix during the adjustment procedure (absolute constrains) (Mikhail & Ackermann 1976, Mikhail & Gracie 1981, Teunissen 2011, Gemael et al. 2015, Ghilani 2018, Ogundare 2019). For the network adjustment process, the stability of the control points that define the datum is relevant, because displacements in them or changing in their positions can generate influences on the coordinate comparison or in deformation network analyses. Thus, the selected control points must have a good stability.

There are alternative procedures to reduce the influence of the stability of the control points among them we find the free adjustment. A main characteristic of the free network adjustment is not to consider the influence of external factors, therefore the errors associate to the control points are not considered (Mälzer et al. 1979, Blaha 1982, 1982a, Papo 1985, Even-Tzur 2011, Even-Tzur 2015). Thus, the stability and consistency problems of the coordinates that define the datum do not affect the adjustment results (Even-Tzur 2006). This characteristic is useful for geodetic monitoring activity, where the deformation elements can be estimated only if the control points that define the datum do not change between the measurement epochs (Even-Tzur 2011). In general, the network is treated as free network when all stations are assumed as unstable, and hence a minimum trace datum is used trough of free adjustment (Setan 2001).

The absences of datum parameters in the adjustment procedure generates the datum defect problem. For network adjustment by least square procedure this situation means that inversion of the normal equation matrix (N) cannot be computed by traditional techniques. Thus, the adjustment solution can be obtained by specific methods. Perelmuter (1979), Papo & Perlmutter (1981), Teunissen (1981), Leick (1982) and Ogundare (2019), present the free adjustment for networks using "minimum inner constraints", where are fixed a minimum quantity of approximate coordinates that permit the datum definition for 1D, 2D, and 3D networks. This coordinates are added to design matrix A, thus de rank deficient is solved and the inversion of the normal equation matrix is possible. Rao (1972), Mittermayer (1972), Grafarend & Schaffrin (1974), Perelmuter (1979), Teunissen (1981), Meissl (1982) and Ogundare (2019), present another approach with generalized inverses, in particular the Moore-Penrose inverse, this method provides a mathematical solution to inversion of normal equation matrix. For Ogundare (2019), in the context of network adjustment by least squares the minimal inner constraints method provides similar results that Pseudo-inverse. The goal of this work is to present a review of both methods with the main characteristics and their application in a geodetic network.

Least square estimation

The least square estimation provides a solution for an equation system with redundancy measurements through of a mathematical model. Particularly, for geodetic applications, the solution of these systems provides the parameters, mainly coordinates and heights (Vanicek & Wells 1972, Mikhail & Ackermann 1976, Mikhail & Gracie 1981, Cross 1990, Krakiwsky 1994, Vanicek 1995, Strang & Borre 1997, Wells & Krakiwsky 1997, Camargo 2000, Nievergelt 2000, Aduol 2003, Teunissen 2011, Brinker & Minnick 2013, Gemael *et al*. 2015, Ghilani 2018, Ogundare 2019, Schaffrin & Snow 2019). The basic functional model is presented in Equation 1:

$$y_{mx1} = A_{mxn} \cdot dx_{nx1} \qquad (1)$$

Where y corresponds to the observation vector of *dx* dimension *(m x 1)*, A is the design matrix *(m x n)* and is the unknown parameters vector *(n x 1)*. The *y* vector is composed of surveying or geodetic measurements; therefore, this vector is contaminated by errors arising from the measurement's procedure. Thus, in order to reduce these errors on the results, the observation data is greater than the number of unknown parameters $(m > n)$. This condition, called redundancy, makes that system to be inconsistent and the unknown parameters can be estimated by different techniques. The network adjustment is the common geodetic procedure where the data of observations is redundant, therefore, the parameter estimation or adjustment process is necessary, the least squares solution is the main technique used for the network adjustment (Mikhail & Ackermann 1976, Mikhail & Gracie 1981, Teunissen 2011, Gemael *et al*. 2015, Ghilani 2018, O gundare 2019).

The least square solution is given by $dx = N^{-1} \cdot U$, where $N = (A^t \cdot W \cdot A)$ (normal equations) and $U = A^t \cdot W \cdot y$ (vector terms), W is the weight matrix $(m \times m)$. Therefore, the inversion of the N matrix is possible only if its determinant is different to zero ($|N| \neq 0$). Thus, the non-singular condition of N matrix means than the columns on the A matrix are not linearly dependent (Welsch 1979, Caspary *et al* 1987, Deakin 2005, Teunissen 2006, Ogundare 2019).

Geodetic network datum

For a geodetic network, the datum is defined as the parameters (coordinates) that permit the positioning of the network in an arbitrary referential system (Kuang 1996, Strang & Borre 1997, Ogundare 2019). In other words, the coordinates define the rotation, translation and scale of the system. The number of coordinates necessary to the definition and their dimensionality depended on the network type (1D, 2D, 3D) and the geodetic observables. For the observables, each of them can define rotation, translation or scale. The Table 1 presents the mainly geodetic observables and the datum element that define.

For the network type, Ghilani (2018), explains that to a 1D-network one vertical control point provides the datum definition (vertical translation). In addition for 2D and 3D classical networks, one control point (translation matrix) with same dimensionality of the network and one direction or azimuth (rotation matrix) are necessary, in both cases the scale is provided by the EDM sensor (observable). A particular case is the GNSS network, where the definition is done by one point, because the coordinates *x,y,z* provide the translation, the baseline components *dx, dy, dz* the orientations and scale (Ogundare 2019). Different sets of network configuration and parameters to define the survey geodetic network datum are presented in the Table 2.

According to the number of parameters that define the geodetic network we found two kinds of datums, over-constrained and minimum constrained. The over-constrained definition (more points than necessary to datum definition), provides a connection with referential system, that is an advantage. Conversely, the main problem for over-constraints definition is a stability and accuracy of controls points because the network accuracy can be affected by strains in the network geometry.

On the other hand, the minimum constrained datum is a solution without external influences. Therefore, the measurements or observations define the network geometry. A disadvantage is the absence of control points, this means in relative position for the coordinates. (Caspary *et al.*, 1987, Kuang 1996, Ogundare 2019).

Free adjustment

For the geodetic network, the internal geometry that is defined by observations of distances, directions or heights differences needs to be connected to a geodetic reference frame. For this, the external geometry composed of control coordinates are part of the least square adjustment process, commonly these coordinates are called constraints or fix parameters. This process permits to connect the observations with a geodetic reference frame (Deakin 2005, Teunissen 2006, Shahar & Even-Tzur 2014). For Deakin (2005) and Teunissen (2006), the observations provide partial definitions of a geodetic datum; therefore, the datum definition is done when the constraints parameters are used in the adjustment process.

The concept of free adjustment of geodetic networks is defined as the absence of fixed parameters in the adjustment process, in other words, there is no set of coordinates of the external geometry of the network during adjustment. Therefore, the elements of the internal geometry (observations) do not integrate the frame of reference during adjustment (Mittermayer 1972, Mälzer *et al.* 1979, Papo 1985, Deakin 2005, Teunissen 2006, Shahar & Even-Tzur 2014). The absences of datum parameters in the adjustment procedure generates the datum defect problem or rank deficient. In network adjustment by least square procedure this situation means that inversion of the normal equation matrix (N) cannot be obtained by traditional inverse procedure, because the matrix is singular, that is, the matrix has columns that are linear combination of the others. Two methods to compute the free adjustment network prevail: the minimal constrained and free adjustment through of generalized inverses. Both methods provide a solution to inversion of the normal equation matrix.

Minimum inner constraints model

The minimum inner constraints model incorporates a minimal amount of parameters necessary to define a referential system. Thus, the external geometry is not considered in

Table 1: Observations that define datum parameters, Adapted from Kuang (1996)

Observable	Translation (t)	Rotation (ω)	Scale (s)
Distances	-	-	s
Horizontal directions	-	-	-
Azimuth	-	ω-Z	-
Zenith directions	-	ω-X, ω-Y	-
GNSS/ Position	t-X, t-Y, t-Z	ω-X, ω-Y, ω-Z	s
2D position differences	-	ω-Z	s
Height differences	-	ω-Y, ω-Z	s

Table 2: Datum parameters, Adapted from Kuang (1996)

Network dimension	Observation type(s)	Network name	Datum parameters		
			Translation	Rotation	Scale
1	Height differences	Level network	1	--	--
2	Distances	Trilateration	1 0 0 1	y_i^0 $-x_i^0$	--
2	Angles	Triangulation	1 0 0 1	y_i^0 $-x_i^0$	y_i^0 x_i^0
3	Distance / Angles	3D network	1 0 0 0 1 0 0 0 1	$0 \quad -z_i^0 \quad y_i^0$ $z_i^0 \quad 0 \quad -x_i^0$ $-y_i^0 \quad x_i^0 \quad 0$	--

* The rotation matrix correspond to the vector representation of rotations (Zeng *et al.*, 2015)

the adjustment procedure, this model is called minimum constraints model. Therefore, the shape and the geometric size of the network is defined only by the internal geometry (Mikhail & Ackermann 1976, Mikhail & Gracie 1981, Snow 2002, Teunissen 2011, Ogundare 2019, Ghilani 2018). The normal equations (N) and the independent vector terms (U) to minimum inner constraints adjustment are presented following:

$$N = A^t \cdot W \cdot A + G \cdot G^t \qquad (2)$$

$$U = A^t \cdot W \cdot l \qquad (3)$$

In the normal equation N, the term $G \cdot G^t$ is added. The G matrix called constrained matrix span the null space of A and contains the inner datum parameters that define the dimensionality of the network. The configuration of the G matrix considers the rotation, translation and scale. Koch (1985), Setan (1995), Kuang (1996), Acar (2006), Rossikopoulos *et al.* (2016), Kotsakis (2018) and Ogundare (2019) presented the set of the G matrix for 3D network (Equation 4)

$$G^t = \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & : & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & : & 0 & 0 & 1 & \cdots & 1 \\ 0 & Z_1 & -Y_1 & : & 0 & Z_2 & -Y_2 & \cdots & -Y_n \\ -Z_1 & 0 & X_1 & : & -Z_2 & 0 & X_2 & \cdots & X_n \\ Y_1 & -X_1 & 0 & : & Y_2 & -X_2 & 0 & \cdots & 0 \\ X_1 & Y_1 & Z_1 & : & X_2 & Y_2 & Z_2 & \cdots & Z_n \end{bmatrix} \qquad (4)$$

The first three rows of the G matrix correspond to the 3D translations, the next three row correspond to the 3D rotations and the last row to the scale parameters. For both cases X_1, Y_1, Z_1 are approximate coordinates of the network. The number of columns of the G matrix is equal to the number of parameters to be estimated. Thus, for a level network with five parameters to estimate, the translation of vertical component is expressed in the G matrix as $G = [1\ 1\ 1\ 1\ 1]^t$ (Setan, 1995). The adjustment parameters are presented in the Equation 5:

$$dx = N^{-1} \cdot A^t \cdot W \cdot A \cdot N^{-1} \cdot U \qquad (5)$$

The variance covariance matrix is:

$$Q_x = N^{-1} \cdot A^t \cdot W \cdot A \cdot N^{-1} \qquad (6)$$

Generalized inverses

In free network adjustment, the inversion of the normal equation (N) matrix can be computed by the generalized inverses (Rao 1972, Grafarend *et al.* 1974, Mälzer 1979, Leick 1982, Meissl 1982). In particular, The Moore-Penrose inverse is the main inverse used in geodetic networks problems called "Minimum norm least squares g-inverse" (Welsch 1979). Thus, for $A \in \mathbb{R}^{mxn}$ and the linear system $A \cdot \vec{x} = \vec{y}$ with $\vec{x} \in \mathbb{R}^n$; $\vec{y} \in \mathbb{R}^m$. The Moore-Penrose pseudo-inverse provides a solution $\vec{x} = A^{\dagger}\vec{y}$, where $A^{\dagger}$ is a pseudo inverse of A. This matrix is unique and has the following properties (Equation 7):

$$\begin{aligned} a.\, & A \cdot A^{\dagger} \cdot A = A \\ b.\, & A^{\dagger} \cdot A \cdot A^{\dagger} = A^{\dagger} \\ c.\, & (A \cdot A^{\dagger})^t = A \cdot A^{\dagger} \\ d.\, & (A^{\dagger} \cdot A)^t = A^{\dagger} \cdot A \end{aligned} \qquad (7)$$

For full rank matrices (rows or columns linearly independent) the pseudo inverse can be obtained for non-square matrix. Therefore, if $m < n$ (rows linearly independent), $A^{\dagger} = A^t \cdot (A \cdot A^t)^{-1}$ and for $m > n$ (columns linearly independent), $A^{\dagger} = (A^t \cdot A)^{-1} \cdot A^t$. When $= n$, $A^{\dagger} = A^{-1}$. For matrices with deficient rank, the solution is commonly obtained by the Singular Value decomposition (SVD). Where A can be decomposed as $A = U \Sigma V^t$ and (Burdick, 2010).

The solution for least squares is

$$dx = N^{\dagger} \cdot A^{t} \cdot W \cdot y \quad (8)$$

While the variance-covariance matrix is given by:

$$Q_x = N^{\dagger} \quad (9)$$

S-transformation

The datum independence on the free network adjustment turns necessary the transformation of results of each epoch to a common datum for the particular analysis as deformation or densification, also by defects in the network configuration or practical limitations (such as obstruction of the line of sight or destruction of points) (Setan 1995, Setan & Singh 2001). Thus, the S-transformation technique permits the datum re-definition between referential systems or epochs (Baarda 1981, Gründig *et al.* 1985, Caspary *et al.* 1987, Setan 1995, Setan & Singh 2001, Teunissen 2006, Acar *et al.* 2008, Doganalp *et al.* 2010, Even-Tzur 2012). For the S-transformation, the estimation of parameters (dx) and the cofactor matrix Qx are necessary (Baarda 1981, Gründig *et al.* 1985, Caspary *et al.* 1987, Erol *et al.* 2006, Teunissen 2006, Acar *et al.* 2008, Doganalp *et al.* 2010, Guo 2012, Even-Tzur 2012, Schmitt 2013). The equations for the transformation are presented:

$$x_j = S_j \cdot x_i$$
$$Q_{xj} = S_j \cdot Q_{xj} \cdot S_j^t \quad (10)$$
$$S_j = (I - G' \cdot (G'^{t} \cdot I_j \cdot G')^{-1} \cdot G'^{t} \cdot I_j$$

Where:

x_j: Transformed parameters between referential system

Q_{xj}: Transformed cofactor matrix between referential system

S_j: Corresponds to the transformation matrix

I_j: Corresponds the diagonal matrix for defining the base after S-transformation, the diagonal elements can be one for elements that participate into datum definition or zero for other points

I: Identity matrix

G': Corresponds to the inner constraint matrix; this matrix is composed by rotation, translation and scale.

For a level network composed of four points, the S-transformation can be explained through an example. For this, we considered the transformation between two referential systems (Caspary *et al.* 1987, Setan 1995):

Ordinary minimum constraints with station 1 chosen as the datum point

Minimum trace where all stations are used for datum definition

The G matrix can be defined by scale constraint, therefore $G = [1\ 1\ 1\ 1]^t$ and for case (a) $I_{ja} = [1\ 0\ 0\ 0]^t$ and for case (b) $I_{jb} = [1\ 1\ 1\ 1]^t$. The identity matrix has a dimension of 4x4. Thus, the transformation from (a) to (b) is defined by:

$$x_{ja} = S_{jb} \cdot x_i$$
$$S_{jb} = (I - G \cdot (G^t \cdot I_{jb} \cdot G)^{-1} \cdot G^t \cdot I_{jb} \quad (11)$$

Application

As an example, the free adjustment was applied in the downstream geodetic network of Salto Caxias hydroelectric power station located in the Parana state, Brazil (Figure 1). The external geometry of this network have four (4) stations while the internal geometry is composed of six (6) distances, twelve (12) angles and one (1) azimuth observation (Table 3) (Granemann, 2005).

The minimum inner constraints method and pseudo-inverse approach were applied in network adjustment according to section 5 and 6 respectively. Additionally the constrained adjustment was calculated with the P1 point as fixed and oriented to point P3 (90º). The stochastic model of the observations corresponds to measures of variability (standard deviation), therefore the weight matrix was defined by the inverse of variance of the observations.

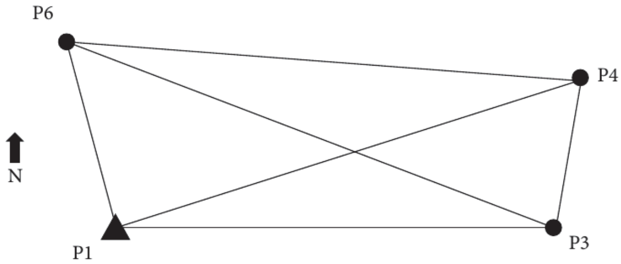

Figure 1: 2D network of Salto Caxias

The results for the constrained adjustment are presented in the Table 4, in this case, the point P1 is the control point or absolute constraint.

For the minimum inner constraints method the G matrix has a dimension of 2x8 and is composed only of translation parameters:

$$G^t = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad (12)$$

For free adjustment procedure applied to nonlinear models, the determination of the initial coordinates and iterative process are a critical step. Tsutomu (1986) & Katsumi (1990) related the influence of the determination of initial coordinates and the free adjustment results. Kotsakis (2012) explains the relation between the stability of the

Table 3: Network observation of downstream geodetic network of Salto Caxias hydroelectric power station

Network observations								
Line	**Distance (m)**	**σ (mm)**	**Angle**	**Value**	**σ ('')**	**Angle**	**Value**	**σ ('')**
P1 - P3	232.809	1.0	P3 P1 P4	75° 49' 39.36''	1.4	P4 P6 P3	18° 17' 10.68''	1.5
P6 - P4	653.555	4.0	P4 P1 P3	17° 6' 24.84''	1.7	P4 P6 P1	83° 58' 5.88''	2.0
P4 - P3	205.711	3.0	P3 P1 P6	267° 03' 55.44''	1.2	P1 P4 P6	20° 12' 16.2''	2.8
P3 - P1	581.863	3.0	P1 P6 P4	276° 01' 53.76''	2.0	P3 P4 P1	56° 17' 1.32''	1.8
P1 - P4	670.340	4.0	P6 P4 P3	283° 30' 42.1''	2.2	P1 P3 P6	21° 23' 0.6''	0.9
P6 - P3	637.678	3.0	P4 P3 P1	253° 23' 29.7''	1.1	P6 P3 P4	85° 13' 28.92''	0.8
			Az P1 P3	90° 0'0''	1.0			

Table 4: Constrained adjustment results

Point	**Constrained least squares**			
	X (m)	**σ (m)**	**Y (m)**	**σ (m)**
P1	1000.000	Control point	1000.000	Control point
P3	1581.8635	0.0018	1000.0000	0.0001
P4	1640.6799	0.0017	1197.1894	0.0019
P6	988.0811	0.0009	1232.5038	0.0009

network and the iterative convergent solution. Thus, the free adjustment applied to networks with nonlinear models should have a special treatment to represent the network geometry. In this work, we used the procedure presented by Tsutomu (1986), therefore the initial coordinates correspond to the adjustment coordinates obtained by the constrained adjustment of the same network with the point P3 fixed.

In Figure 2 from Kotsakis (2012), AB line is a distance, the grey line represents the probable positions of point B with respect to point A and the red line represents the possible positions of B with respect to the reference system. The B point is the real location of this coordinate. B1 and B2 points are two different initial coordinates for iterative process. In this case, the network stability is better if the B1 is the initial coordinate due to the proximity between the coordinates B1 and B. For Ipsen (2011), this situation can be explained due to B1 or B2 is far from B. the method may not converge. This means the solution does not represent the network geometry. In other words, for rank- deficient models the convergent solution is not necessarily unique.

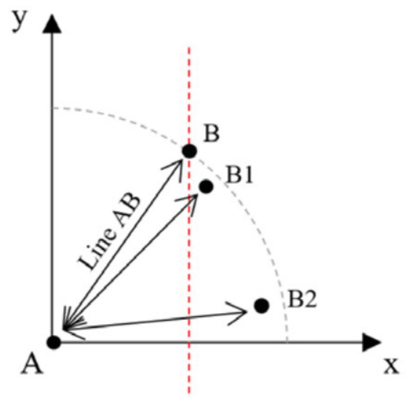

Figure 2: Adapted from Kotsakis (2012), position of initial coordinates to free adjustment

The adjustment parameters and their precisions for both approaches are presented in Table 5:

The global test (chi-square) was applied to each adjustment, the results are presented in Table 6 to confidence level of 95% with (n-u)=(19-8)=11 degrees of freedom for free adjustments and (n-u)=(19-6)=13 degrees of freedom for the constrained adjustment.

Table 5: Least square solution by Pseudo-inverse approach and Minimum inner constraints method

Point	Pseudo inverse approach				Minimum inner constraints			
	X (m)	σ (m)	Y (m)	σ (m)	X (m)	σ (m)	Y (m)	σ (m)
P1	999.9963	0.0011	999.9966	0.0016	999.9963	0.0009	999.9966	0.0005
P3	1581.8598	0.0010	999.9966	0.0015	1581.8598	0.0009	999.9966	0.0005
P4	1640.6762	0.0011	1197.1861	0.0022	1640.6762	0.0009	1197.1861	0.0014
P6	988.0775	0.0011	1232.5005	0.0017	988.0775	0.0009	1232.5005	0.0008

The S-transformation was applied to both adjustment processes, so one control point was selected (P1) and considered as constrained parameter, its coordinates are (1000.00m, 1000.00m). The measured bearings and the scale by the measured distances define the orientation of the network. The dimension of the G' and I matrix is 2x8. Both matrix are presented following:

$$G^t = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad (13)$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

The adjustment vector parameters transformed through S-Transformation is obtained through:

$$\hat{x} = x_o + x_j \quad (15)$$

Where x_0 corresponds to initial vector of the parameters, the variance covariance matrix was obtained with equation 5. The results and their precision are presented in the Table 7:

Finally, differences between the free adjustments and constrained adjustment are summarized in the Table 8.

Table 6: Global test to each adjustment approach

Method	Estimate	Critical value (95%)	Status
Pseudo inverse	6.1648	19.67510	Pass
Minimum inner constraints	6.1648	19.67510	Pass
Parametric adjustment	6.9851	24.73560	Pass

Table 7: S-transformation results to Pseudo-inverse approach and Minimum inner constraints

Point	Pseudo inverse approach				Minimum inner constraints			
	X (m)	σ (m)	Y (m)	σ (m)	X (m)	σ (m)	Y (m)	σ (m)
P1	1000.000	0.0000	1000.000	0.0000	1000.000	0.0000	1000.000	0.0000
P3	1581.8635	0.0018	1000.000	0.0028	1581.8635	0.0018	1000.000	0.0000
P4	1640.6799	0.0020	1197.1894	0.0036	1640.6799	0.0017	1197.1894	0.0019
P6	988.0811	0.0015	1232.5038	0.0009	988.0811	0.0009	1232.5038	0.0009

Table 8: Summary of differences between free adjustments and constrained adjustment

Method / adjustment elements	Pseudo-inverse	Minimum inner constraints	Constrained adjustment
Coordinate unknowns	8	8	6
Datum defect	X,Y Coordinates and orientation	X,Y Coordinates and orientation	X,Y Coordinates and orientation
Datum definition	Free	Free	Fix
Degrees of freedom	11	11	13
Posterior variance	0.5604	0.5604	0.5373
χ^2 estimate	6.1648	6.1648	6.9851
Critical value of χ^2	19.6751	19.6751	24.7356
Global test (one – tailed)	Pass	Pass	Pass

Conclusions

Two approaches for free adjustment computations were presented, the pseudo-inverse method that provides a mathematical solution to compute the inverse of normal equation matrix (N) and therefore maintains the classical formulation to the least squares. On the other hand, the minimum inners constraints method needs the addition of the G matrix, which contains the minimum parameters to datum definition. Thus, the G matrix spans null space to the design matrix A, consequently the lack of information of the network datum or the rank deficiency is solved and the inversion of the normal equation matrix is done.

For the example presented, both methods provide equivalent results for the parameters, and global test, therefore according to Ogundare (2019) it was verified the similarity of both methods. The S-transformation is necessary to transform datum from the arbitrary referential system (provided by the free adjustment) to the reference datum. The results of the S-transformation have equivalent results for both methods.

One of the main differences between both methods (free and parametric) is related to the definition of the network geometry, which in the case of free adjustment is obtained without the need to set coordinates in the design matrix. Therefore, the network geometry is defined in an arbitrary system from the observations themselves. This feature is useful for evaluating the quality of a network adjustment.

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3.2 INFLUENCE OF NETWORK CONFIGURATION AND STOCHASTIC MODEL ON THE DETERMINATION OF THE MINIMUM DETECTABLE DISPLACEMENTS (MDD) THROUGH SENSITIVITY ANALYSIS AND SIGNIFICANCE TEST

This research explores the impact of various factors on the detection of minimum detectable displacements (MDD) in geodetic networks. Specifically, it examines how the network's configuration, the choice of stochastic model and the application of a local or global approach influence MDD determination. The study introduces a methodology that incorporates sensitivity analysis and significance testing, integrating sensitivity attributes to establish confidence regions based on MDD according to Prószyński and Łapiński, (2021)

A key aspect of this research involves assessing the correspondence between the critical value in a significance test and the non-centrality parameter obtained from a chi-square distribution. This assessment is crucial for calculating concentric ellipsoids, which represent both sensitivity and accuracy. The study meticulously analyzes how alterations in the network configuration, the selected stochastic model, and the type of analysis (local or global) affects the interplay between sensitivity and accuracy.

The findings underscore the importance of these factors in geodetic network design and analysis. By highlighting the significant role these elements play, the research offers vital insights for developing robust and effective geodetic networks in practical scenarios.

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Influence of network configuration and stochastic model on the determination of the minimum detectable displacements (MDD) through sensitivity analyses and significance test

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Abstract:

This study investigates the influence of geodetic network configuration, stochastic model, and the approach local or global on the determination of minimum detectable displacements (MDD) using sensitivity analyses and significance tests. The proposed approach integrates sensitivity characteristics to establish confidence regions based on MDD. Also, we examine the equality between the critical value of a significance test and the non-centrality parameter derived from a chi-square distribution to compute concentric ellipsoids representing sensitivity and accuracy. The analyses were focused to evaluate how variations in network configuration, stochastic model, and the type of analysis (if global or local) affect the relationship between sensitivity and accuracy. Our results showed the importance of considering these factors, providing valuable insights for robust network design and analysis in practical applications.

Keywords: Statistical tests; deformation analysis; minimal detectable displacements; geodetic monitoring

1. Introduction

In the pre-analysis of the geodetic monitoring networks, the sensitivity analysis provides valuable insights related to the capacity of the geodetic network to detect deformations. The sensitivity analysis is applied for specific probability levels based on the minimum detectable displacement (MDD) of monitoring points (Even-Tzur, 2010). Here, some aspects such as the network configuration, stochastic model, and the sensitivity analyses type, namely, global (for the entire network), and local (for specific points) play a key role. Thus, if the computed MDD exceeds the desired threshold based on the specified probability levels and the number of points tested simultaneously, it indicates a need for design improvement. This can be achieved by adding new observations and points, reducing their standard deviation, changing the probability levels or the approach (global or local). The computation of MDD is essential in geodetic monitoring as it accounts for both Type I errors (false positive) and Type II errors (false negative). These error types define a false alarm, namely, a deformation incorrectly detected and an undetected deformation. The last one is a critical condition for geodetic monitoring (Carvajal et al., 2022).

In addition to the sensitivity analysis, the network accuracy is also analyzed in the pre-analysis or design stage. Here, the thresholds for confidence and significance tests are defined to provide the best network design according to requirements (Prószyński & Łapiński, 2021). As in the sensitivity

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analysis, in accuracy analysis, the configuration of the network, the stochastic model, and the probability levels are aspects can influence the results. In this context according to (Prószyński & Łapiński, 2021), the analysis of accuracy and sensitivity are traditionally applied separately due to the lack of a theoretical basis to consider a unique analysis. Therefore, the same authors(Prószyński & Łapiński, 2021a)(Prószyński & Łapiński, 2021a)(Prószyński & Łapiński, 2021a) provide a theoretical basis to consider the confidence region and significance test in sensitivity analysis in a unified approach based on MDD determination.

The method called variance factor (I) supports the confidence region in a network sensitivity characteristic. For this, the method provides equality between the critical value of the significance test of displacements $\Phi_{h,\alpha}$ (associated with confidence region) and the non-centrality parameter $\lambda_{h,\alpha,\beta}$ based on χ^2-distribution for a specific value for the power of test γ_0 determined by a Type II error probability β_0 ($\gamma_0 = 1 - \beta_0$), coordinated with the stipulated Type I error probability or level of significance α_0 and the h-dimensional displacement vector such that $\Phi_{h,\alpha_0} = \lambda_{h,\alpha_0,\beta_0}$. After the equality determination, the MDDs are computed and represented as concentric ellipsoids where the MDD corresponds to the semi-major axis (Cüneyt Aydin, 2014). Here the relation between accuracy and sensitivity depends only on probabilistic concepts and does not consider aspects such as the network configuration and the stochastic model.

In this study were carried out several experiments to analyze the relationship between accuracy and sensitivity. Thus, initially, the sensitivity analysis characteristics were included in the deformation detection analysis through the global congruence test. Furthermore, we presented aspects related to the network configuration, stochastic model, and simultaneous displacements (multivariate and univariate approaches). Here, our results showed that the configuration network, stochastic model, and the type of analysis, namely, global or local influence the deformation analysis (MDD value) for both approaches. Finally, we presented experiments to analyze the influence of the network configuration, and the stochastic model in the relation between sensitivity and significance analysis presented by (Prószyński & Łapiński, 2021).

2. Theoretical basis

2.1 Deformation analysis: Global Congruence Test (GCT)

The theoretical approach initially considers the deformation test. Here a displacement vector is given by:

$$\hat{\mathrm{d}} = \mathrm{x}_2 - \mathrm{x}_1 \qquad (1)$$

Where x_1 and x_2 are the least squares solution for the monitoring network parameters (point coordinates) in the first and second epoch respectively. The covariance matrix is computed as:

$$C_d = \sigma_0^2 Q_d \quad (2)$$

Here Q_d is the cofactor matrix of the displacement vector, and σ_0^2 is the a-priori variance factor. Based on hypothesis testing, two hypotheses are formulated:

$$H_0: E(\hat{d}) = 0 \text{ and } H_A: E(\hat{d}) \neq 0 \quad (3)$$

While the test statistic is given by:

$$\Phi = \hat{d}^T C_d^+ \hat{d} \quad (4)$$

Which follows (central) χ2-distribution with h degrees of freedom in H_0, is compared with the theoretical value of $\chi^2_{\alpha,h}$ corresponding to the α-significance level. If the test statistic Φ is smaller than the threshold value, the null hypothesis H_0 is not rejected with the confidence level of $1 - \alpha$, and it is concluded that there is no deformation between the two epochs. Otherwise, it is decided that displacement has occurred with the probability risk of a false positive given by α. This congruence test may be performed by considering the estimated variance factor. In this case, the test follows F (Fisher)-distribution. Nonetheless, this case is outside the scope of this paper. More information can be found in (Cüneyt Aydin, 2014).

2.2 Sensitivity analysis

The capacity of the network to detect displacements can be quantified by sensitivity analysis (Aydin, 2014). Based on the deformation test, a theoretical vector of expected displacement, denoted by Δ, is related to the alternative hypothesis defined as:

$$H_A: E(\hat{d}) \neq 0 = \Delta \quad (5)$$

Here the non-rejection of H_A implies that the expected displacement values of the vector Δ can be detected by the monitoring network. The theoretical relationship of the non-centrality parameter (λ) and the test statistic (T) is given by:

$$T = \lambda = \Delta^T {C_d}^+ \Delta \quad (6)$$

The condition to define if the network is sensitive to displacements is given by:

$$\lambda \geq \lambda_0 \quad (7)$$

where λ_0 is the so-called lower bound of the non-centrality parameter which fulfills the given power of the test γ_0 being the complement of the type-II error probability β_0: $\gamma_0 = 1 - \beta_0$. Here λ_0 is obtained from (C. Aydin & Demirel, 2005).

2.3 Minimal detectable displacements (MDD)

To evaluate the MDD, firstly a vector with the expected displacements is defined from a vector of directions g and a scale factor value denoted by b (Aydin, 2014). Thus, the condition $\Delta = b\mathrm{g}$ is fulfilled. If $b = b_{min}$ ($\Delta_{min} = b_{min}\mathrm{g}$), then the determination of b_{min} is given by:

$$b_{min} = \sqrt{\lambda_0 \lambda_{max}} \tag{8}$$

where λ_{max} is the maximum eigenvalue of the covariance matrix of C_d . According to (Küreç & Konak, 2014) b_{min} is the best sensitivity level of the network. On the other hand, the worst sensitivity level of the network can be computed as:

$$b_{max} = \sqrt{\lambda_0 \lambda_{min}} \tag{9}$$

where λ_{min} is the minimum eigenvalue of C_d. According to (Hsu & Hsiao, 2002) the average between b_{max} and b_{min} can be interpreted as the global sensitivity for the entire network.

To obtain the vector of directions of displacements, (Aydın, 2014) computed the (unity) eigenvectors corresponding to the maximum eigenvalue (λ_{max}) and the minimum eigenvalue (λ_{min}) of the C_d matrix (Λ_{max} and Λ_{min} respectively). Thus: $\Delta_{min} = b_{min}\Lambda_{max}$ and $\Delta_{max} = b_{max}\Lambda_{min}$. The MDD in each i^{th} element of the vector $\hat{\mathrm{d}}$ is given by the respective i^{th} element of Δ_{min} or Δ_{max}. For details, we suggest (Aydin, 2014).

2.4 Global and Local sensitivity analysis

The sensitivity analysis can be carried out under a global or local approach. Here an important analysis related to simultaneous and unitary displacements arises. If the global analysis is applied the MDD represents simultaneous displacements of all the points that make up the monitoring network. Conversely, if the local sensitivity is applied the MDD is computed for a specific point. These conditions imply that the h-dimensional vector of displacements changes and therefore the non-centrality parameter also. Here, for 2D and 3D networks, the local sensitivity can be computed under multivariate (simultaneous displacements) or univariate approaches. For example, a multivariate analysis for a 3D point with $\alpha_0 = 5\%$ and $\gamma_0 = 80\%$ implies $\lambda_{(\gamma_0=80\%,\alpha_0=5\%,h=3)} = 10.9$. On another hand, if the univariate approach is used, the non-centrality parameter is computed as $\lambda_{(\gamma_0=80\%,\alpha_0=5\%,h=1)} = 7.85$. These conditions are fulfilled for the significance test also ($\chi^2_{(\alpha_0=5\%,\ h=3)} = 7.81$ and $\chi^2_{(\alpha_0=5\%,\ h=1)} = 3.84$) (Bandeira et al., 2021).

Thus, the multivariate and univariate approaches are related to aspects of detectability, where the multivariate approach has more difficulty to detect deformation due to critical values being calculated from stochastic models without covariance (e.g., GNSS). For the univariate approach, the neglect of covariance can be generated false positives and false negatives in addition to providing only a displacement magnitude and not their directions. Based on this, the multivariate approach is recommended (Bandeira et al., 2021).

2.6 Confidence region determination supported by network sensitivity characteristics

The theoretical basis for the confidence region determination supported by network sensitivity characteristics is developed for the scenarios where σ_0^2 is used. Here, a specific value for the power of test γ_0 is determined through a value of β_0 coordinated with the stipulated level of significance α_0 and the h-dimensional displacement vector. This approach provides equality between the critical value for the significance test and the non-centrality parameter of the sensitivity test:$\Phi_{h,\alpha_0} = \lambda_{h,\alpha_0,\beta_0}$ (Prószyński & Łapiński, 2021).

Initially for $h = 1$ the relation $\lambda_{h,\alpha_0,\beta_0} > \Phi_{h,\alpha_0}$ is fulfilled. If the h value increases, both values $\lambda_{h,\alpha_0,\beta_0}$ and Φ_{h,α_0} also increase. However, for a specific value of h (namely h^*), the relation $\Phi_{h,\alpha_0} = \lambda_{h,\alpha_0,\beta_0}$ is achieved (Prószyński & Łapiński, 2021). The results presented by the authors show that for $h^* = 7.3$ the above equality holds to $\alpha_0 = 0.05$ and $\beta_0 = 0.20$. In addition, for values greater than $h^* = 7.3$, the relation $\Phi_{h,\alpha_0} > \lambda_{h,\alpha_0,\beta_0}$ is fulfilled. Note that for different values for α_0 and β_0, the value for h^* also changes.

The approach proposed by Prószyński & Łapiński (2021) is based on the size comparison of three concentric ellipsoids; sensitivity ellipsoid, confidence ellipsoid, and significance ellipsoid. This comparison is carried out by global sensitivity. For the significance ellipsoid, the relation $\hat{\mathrm{d}}^T \mathrm{C}_d^+ \hat{\mathrm{d}} = \Phi_{h,\alpha_0}$ is fulfilled for a α_0 significance level; for the confidence ellipsoid, $\hat{\mathrm{d}}^T \mathrm{C}_d^+ \hat{\mathrm{d}} = \Phi_{h,CL}$ is fulfilled for a given confidence level (CL) and for the sensitivity ellipsoid, $\hat{\mathrm{d}}^T \mathrm{C}_d^+ \hat{\mathrm{d}} = \lambda_{h,\alpha_0,\beta_0}$. Considering $CL = 1 - \alpha_0$, the significance ellipsoid turns into the confidence ellipsoid and the analysis focuses on the determination of $\Phi_{h,\alpha_0} = \lambda_{h,\alpha_0,\beta_0}$. In this case, two scenarios were defined: $h > h^*$ and $h < h^*$, where h^* is the value for h that satisfies the equality $\Phi_{h,\alpha_0} = \lambda_{h,\alpha_0,\beta_0}$. Then the confidence and sensitivity ellipsoids were determined, for $h < h^*$ the confidence ellipsoid is smaller than the sensitivity ellipsoid (Figure 1) while for $h > h^*$ the sensitivity ellipsoid is smaller than the confidence ellipsoid (Figure 2).

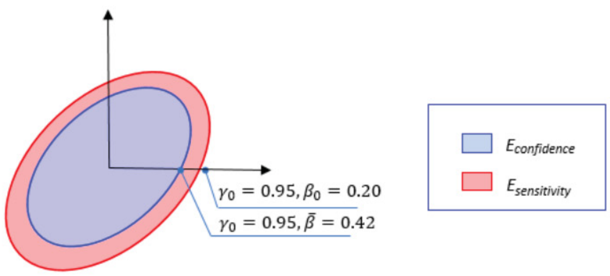

Figure 1: Ellipsoids of confidence and sensitivity for $h < h^*$adapted from: (Prószyński & Łapiński, 2021)

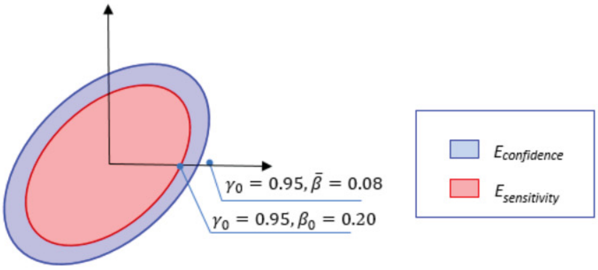

Figure 2: Ellipsoids of confidence and sensitivity for $h > h^*$ adapted from (Prószyński & Łapiński, 2021)

3. Experiments and analyses

In this section, we conducted several experiments to evaluate the behavior of the MDD under different scenarios, such as network configuration, number of observations, local or global MDD, as well as different stochastic models. For this, we utilized two leveling networks: Network A, which comprises 6 points as depicted in Figure 3, and Network B, which consists of 9 points as shown in Figure 5. The number of observations for each network is denoted by n, while h represents the dimensionality of the displacement vector. Table 1 provides a description of each experiment.

Table 1: Summary of experiments

Experiment	Description
Experiment 1	Local MDD ($h = 1$) for network A with $n = 11$
Experiment 2	Local MDD ($h = 1$) for network A improved redundancy ($n = 15$)
Experiment 3	Test of local MDD values for $n = 15$ on network A with $n = 11$
Experiment 4	MDD for three simultaneous points ($h = 3$) on network A with $n = 15$
Experiment 5	Global MDD ($h = 6$) on network A with $n = 11$
Experiment 6	Global MDD ($h = 9$) on network B with $n = 12$
Experiment 7	Global MDD for network A with $n = 15$ and network B with $n = 20$
Experiment 8	Global MDD for network A with $n = 15$ and network B with $n = 20$ with gradients of different precisions

For the first experiment, Figure 3 shows a leveling monitoring network with 11 height differences, and a standard deviation of 1 mm from (Nowel, 2018). To evaluate the MDD, a trial and error methodology presented by (Bandeira et al., 2021) was applied for each point of the network (local sensitivity approach). In this step, in addition to the application of the significance test, which considers only false positives in H_0 ; the sensitivity analysis was applied also, namely, the occurrences of false positives in H_0 and false negatives in H_A were considered, in both cases $\mathrm{Q}_d = 2 \cdot \mathrm{Q}_x$. In the last case, the critical value is determined by the non-centrality parameter $\lambda_{(\gamma_0,\alpha_0,\mathrm{h})}$ instead of $\chi^2_{(\alpha_0,\mathrm{h})}$

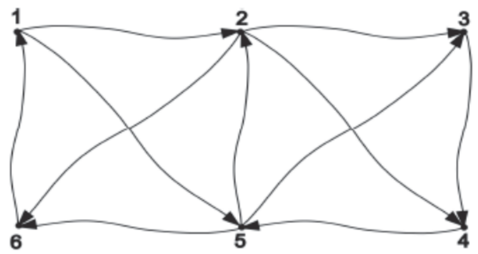

Figure 3: leveling network from (Nowel, 2018)

The results of the first experiment show that the MDD values are lower for the significance approach (Table 2). These results are equivalent to the theoretical basis presented in (Prószyński & Łapiński, 2021) since $h < h^*$. Another relevant aspect of this experiment is related to points 2 and 5. These points have a small magnitude for MDD in comparison with the points 1,3,4,6. The main difference between these two groups is the number of observation connections which are 5 and 3 respectively.

Table 2: Local MDD for significance and sensitivity approach based on the GCT

Point	$\boldsymbol{d}_{(\boldsymbol{i},\mathbf{1})}$ (mm) (Significance analysis)	$\boldsymbol{d}_{(\boldsymbol{i},\mathbf{1})}$ (mm) (Sensitivity analysis)
1	1.5	2.1
2	1.1	1.5
3	1.5	2.1
4	1.5	2.1
5	1.1	1.5
6	1.5	2.1

To evaluate the role of the network configuration, it was added observations between points 1-3,1-4,3-6,4-6 with 1mm of standard deviation. Figure 4 shows the new network configuration.

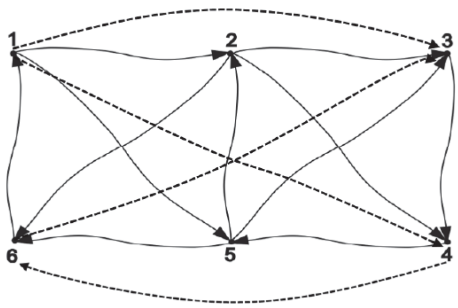

Figure 4: 1D network adapted from example 1

From the new configuration, all the points have the same local MDD. Thus, the local MDD is 1.1 mm and 1.5 mm for the significance and sensitivity respectively. Therefore, here the configuration of the network influences the MDD values. To evaluate this condition, the new local MDD found in the network with 15 observations were tested in the first configuration (11 leveling differences). The results are presented in Table 3

Table 3: Results for tested of MDD obtained in the second configuration inserted in the first configuration

Deformation considered for significance analysis				Deformation considered for sensitivity analysis			
Point	$\boldsymbol{d}_{(i,1)}$ (mm)	${d^T}_{(i,1)} \cdot \Sigma^{-1}_{QQ(i,i)} \cdot d_{(i,1)}$	$\chi^2_{(\alpha_0=5\%,d=1)}$	Point	$\boldsymbol{d}_{(i,1)}$ (mm)	${d^T}_{(i,1)} \cdot \Sigma^{-1}_{QQ(i,i)} \cdot d_{(i,1)}$	$\lambda_{(\gamma_0=80\%,\alpha_0=5\%,h=1)}$
1	1.1	2.2926	3.8415	1	1.5	4.2632	7.8488
2	1.1	**4.3560**	3.8415	2	1.5	**8.1000**	7.8488
3	1.1	2.2926	3.8415	3	1.5	4.2632	7.8488
4	1.1	2.2926	3.8415	4	1.5	4.2632	7.8488
5	1.1	**4.3560**	3.8415	5	1.5	**8.1000**	7.8488
6	1.1	2.2926	3.8415	6	1.5	4.2632	7.8488

From Table 3, the detection was successful for points 2 and 5 for both tests. Note that for these points the local of MDD was the same as in the first experiment. For the rest of the points, the test under the significance and sensitivity could not identify the deformations. These results showed the importance of the network configuration in the design stage.

Subsequently, simultaneous displacements were tested. For this, were defined three displacements in the network with 15 leveling differences and 1 mm of standard deviation. In this case, the critical value for significance and sensitivity analysis becomes $\chi^2_{(\alpha_0=5\%,h=3)} = 7.81$ and $\chi^2_{(\alpha_0=5\%,d=3)} = 10.9$, for both cases $\gamma_0 = 80\%$. Here two MDDs were computed, in the direction of the largest and in the direction of the smallest variances. The results are presented in Tables 4 and 5 for significance and sensitivity analysis respectively.

Table 4: MDDs for simultaneous displacement for significance analysis

Point	MDDs (largest variance direction)			MDDs (smallest variance direction)		
	$\boldsymbol{d}_{(i,j,k)}$ (mm)	$\boldsymbol{d}^t \cdot \boldsymbol{\Sigma}_{QQ}^{-1} \cdot \boldsymbol{d}$	$\chi^2_{(\alpha_0=5\%,d=3)}$	$\boldsymbol{d}_{(i,j,k)}$ (mm)	$\boldsymbol{d}^t \cdot \boldsymbol{\Sigma}_{QQ}^{-1} \cdot \boldsymbol{d}$	$\chi^2_{(\alpha_0=5\%,d=3)}$
1,3,5	1.4, 0.2, 0.2	9.4800	7.8147	0.6, -0.1, 1.3	9.4800	7.8147
2,4,6	1.4, 0.2, 0.2	9.4800	7.8147	0.7, -0.2, 1.2	9.4800	7.8147

Table 5: MDDs for simultaneous displacement for sensitivity analysis

Point	MDDs (largest variance direction)			MDDs (smallest variance direction)		
	$\boldsymbol{d}_{(i,j,k)}$ (mm)	$\boldsymbol{d}^t \cdot \boldsymbol{\Sigma}_{QQ}^{-1} \cdot \boldsymbol{d}$	$\lambda_{(\gamma_0=80\%,\alpha_0=5\%,h=3)}$	$\boldsymbol{d}_{(i,j,k)}$ (mm)	$\boldsymbol{d}^t \cdot \boldsymbol{\Sigma}_{QQ}^{-1} \cdot \boldsymbol{d}$	$\lambda_{(\gamma_0=80\%,\alpha_0=5\%,h=3)}$
1,3,5	1.5, 0.3, 0.3	11.8200	10.903	0.7, -0.2, 1.4	11.8200	10.903
2,4,6	1.5, 0.3, 0.3	11.8200	10.903	0.8, -0.1, 1.3	11.8200	10.903

Form Tables 4 and 5 note that the local MDD from Table 3 can be less or greater than the global MDD (univariate or multivariate analysis). These results show the importance of the type of analysis (if local or global) to compute the MDD values in both significance and sensitivity approaches.

The next experiments are focused to evaluate (Prószyński & Łapiński, 2021) method. Thus, we used the leveling network from Figure 3 with a standard deviation of 1mm for each height difference (11 observations). The inner-constrained approach is applied in the adjustment (see Ogundare, 2018). The covariance matrix for the deformation vector $\hat{\mathrm{d}}$ was obtained by $\mathrm{C}_d = 2\mathrm{Q}_x$. In case $h = rank(\mathrm{C}_d) = 5$. The values for the non-centrality parameter were obtained from (Aydin & Demirel, 2004). Table 6 shows the results.

Table 6: non-centrality parameter for sensitivity and confidence for leveling network of Figure 3

Ellipsoid	α_0	β	$\lambda_{h,\alpha,b}$
Sensitivity	0.05	0.20	12.828
Confidence	0.05	0.27	11.070

For the computation of the global sensitivity values (see Eqs. 4 and 5), we used the maximum and minimum eigenvalues of C_d and the non-centrality parameters of Table 6. The results are presented in Table 7:

Table 7: Global sensitivity values for the third experiment

Ellipsoid	b_{min} (mm)	b_{max} (mm)	Average (mm)
Sensitivity	2.06	3.58	2.82
Confidence	1.92	3.32	2.62

By analyzing Table 7, we note that the size of the sensitivity ellipsoid is higher than the confidence ellipsoid as expected, once that $h = 5 < h^* = 7.3$ (Figure 1). After, a new experiment was developed with a leveling network with 9 points and 12 leveling differences with a standard deviation of 1 mm (Figure 5).

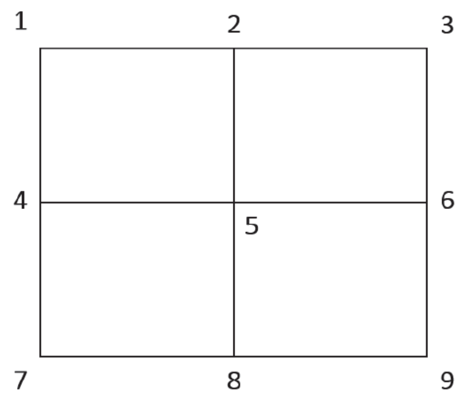

Figure 5: Leveling network with 9 points

The adjustment procedure, the determination of non-centrality parameter, and global sensitivity values were carried out according to the previous experiment. The results are presented in Table 8.

Table 8: non-centrality parameter for sensitivity and confidence ellipsoids for leveling network with 9 points and 12 height differences

Ellipsoid	α_0	β	$\lambda_{h,\alpha,b}$
Sensitivity	0.05	0.20	15.022
Confidence	0.05	0.18	15.507

The global sensitivity values are presented in Table 9.

Table 9: Global sensitivity values for leveling network with 9 point and 12 height differences

Ellipsoid	b_{min} (mm)	b_{max} (mm)	Average (mm)
Sensitivity	2.24	5.48	3.86
Confidence	2.27	5.67	3.97

The results in this experiment show that the non-centrality parameter (and thus the respective ellipsoid) obtained for the confidence approach was higher than the sensitivity approach, according to $h = 8 > h^* = 7.3$. The difference between these experiments and those presented in (Prószyński & Łapiński, 2021) is that here we addressed geodetic network applications rather than a theoretical analysis without displacement values.

To evaluate the network configuration influence, observations were included in both networks analyzed. Figures 4 and 6 showed the new configuration for each network.

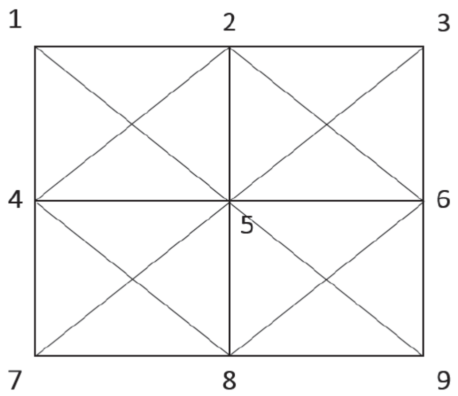

Figure 6: 1D network adapted from example 2

In both cases, the new observations have a standard deviation of 1 mm. Note that the non-centrality parameters are the same as Tables 6 and 8 respectively since we do not change the number of network points (and thus the value of h). The new global sensitivity values are presented in Tables 10 and 11.

Table 10: Global sensitivity values for the first Network (6 network points) with new observations

Ellipsoid	b_{min} (mm)	b_{max} (mm)	Average (mm)
Sensitivity	2.07	2.07	2.07
Confidence	1.92	1.92	1.92

Table 11: Global sensitivity values for the second network with new observations

Ellipsoid	b_{min} (mm)	b_{max} (mm)	Average (mm)
Sensitivity	1.83	3.64	2.73

Confidence	1.86	3.69	2.77

By Analyzing Tables 7 and 10 or Tables 9 and 11, we note that the global sensitivity values decrease for both networks adding new observations, especially the b_{max} values for the first network. We can also note that $b_{min} = b_{max}$ for the network when all points are tied with each other as in Figure 4 and all observations have the same precision (the same is not true for the second network as shown in Figure 6). Furthermore, the differences between the global sensitivity values of the sensitivity and the confidence approach decrease for both networks.

This kind of analysis provides interesting tools for the pre-analysis or design of deformation networks, being not covered before in the theoretical experiments of Prószyński & Łapiński (2021). To evaluate the role of the stochastic model, the new observations in Figure 4 were now defined with a standard deviation of 2 mm. The results are presented in Table 12.

Table 12: Global sensitivity values for network of Figure 4 with new observations with a standard deviation of 2 mm

Ellipsoid	b_{min} (mm)	b_{max} (mm)	Average (mm)
Sensitivity	2.07	2.93	2.50
Confidence	1.92	2.72	2.32

The same case was considered with the new observations in Figure 6. Each new observation has now a standard deviation of 2 mm and the results are presented in Table 13.

Table 13: Global sensitivity values for the second network with new observations with a standard deviation of 2 mm

Ellipsoid	b_{min} (mm)	b_{max} (mm)	Average (mm)
Sensitivity	2.18	4.76	3.47
Confidence	2.21	4.83	3.52

By Analyzing Tables 10 and 12 or Tables 11 and 13, we note that the global sensitivity values increase if the standard deviation of the new observations increases from 1 mm to 2 mm as expected. However, the global sensitivity values decrease in relation to the original case without new observations for both networks. Besides that, the differences between the global sensitivity values of the sensitivity and the confidence approach decrease again for both networks (see Tables 7 and 12 or Tables 8 and 13). Therefore, these experiments also show the role of the stochastic model in this kind of analysis, especially when designing deformation networks.

Therefore, the addition of new observations reduces the MDD values, even if these observations are of poorer precision than the previous ones. Furthermore, increasing the network's redundancy reduces the discrepancies between the results of significance and sensitivity analysis.

4. Conclusions

In this work we have studied the relationship between significance and sensitivity in MDDs computation. First, under the GCT approach, we compare the detectability of significance and sensitivity analysis Here we found that the network configuration, stochastic model, and the type of analysis, i.e., if global or local influences on the MDD values.

Also, we analyzed the (Prószyński & Łapiński, 2021) method under the same conditions. Thus, the influence of network configuration and stochastic model on the *variance factor method (I)*, which jointly analyzes aspects of sensitivity and accuracy in the pre-analysis of geodetic networks showed that If the network and stochastic model improvement, namely, the addition of more observations and better standard deviations for the observations, provides on average better values for MDD and reduces the magnitude between the semi-major axis of the sensitivity and significance ellipsoids. These results presented provides key information for the optimization of geodetic network design.

Therefore, the geodesist must be aware of the following issue: only the occurrence of false positives will be considered (significance analysis) or also the occurrence of false negatives (sensitivity analysis). It should be noted that in the case of geodetic monitoring, the occurrence of false negatives (undetected deformations) is generally more critical than the occurrence of false positives (“false alarm"). For future studies, we recommend analyze some properties of the MDD directions for the smallest and largest directions for simultaneous displacements (Table 4 and Table 5) and the design or pre-analysis of a real monitoring geodetic network considering all the aspects addressed here.

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AUTHOR´S CONTRIBUTION

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3.3 MINIMAL DETECTABLE DISPLACEMENT IN CONFIDENCE REGION DETERMINATION AND SIGNIFICANCE TEST OF DISPLACEMENTS REGARDING THE DESIGN OF GEODETIC NETWORKS

This research is centered on evaluating the effectiveness of confidence region determination, particularly through the lens of sensitivity characteristics in the context of a priori analysis for geodetic monitoring networks. The study delves into several key factors: the dimensionality of the displacement vector, the structural setup of the network (including the number of observations), the spatial dimensions of network points, and the functional model utilized.

The findings of this investigation reveal that these elements play a pivotal role in influencing the overall Minimal Detectable Displacement (MDD) values when using confidence and sensitivity methods. This, in turn, affects the network's ability to accurately identify displacements. A notable observation is that an increase in the number of observations within the network tends to minimize the differences in displacement detection between the confidence and sensitivity methods.

Moreover, the study highlights that geodetic networks, even those with identical parameter dimensionality, can exhibit different levels of displacement detection efficiency based on their spatial dimensions (1D, 2D, or 3D). To provide a practical application of these findings, the research includes a case study on the design of a GNSS (Global Navigation Satellite System) network for geodetic monitoring. This case study is modeled on the approach outlined by Prószyński and Łapiński (2021), demonstrating the real-world implications and utility of the study's insights.

Minimal Detectable Displacement in confidence region determination and significance test of displacements regarding the design of geodetic networks

Abstract:

This study aims to investigate the properties of confidence region determination supported by the sensitivity characteristics method, with a focus on the a priori analysis of geodetic monitoring networks. Various aspects such as displacement vector dimensionality, network configuration (number of observations), spatial dimension of network points, and its functional model were examined to achieve this objective. Our results demonstrate that these aspects have a significant impact on the global Minimal Detectable Displacement (MDD) values for confidence and sensitivity approaches, thereby influencing the network's capacity to detect displacements. Specifically, increasing the number of observations reduces the discrepancies between the displacements detected by the confidence and sensitivity approaches. Furthermore, we show that geodetic networks with the same parameter dimensionality but different spatial dimensions (1D, 2D or 3D) exhibit varying capabilities for detecting displacements. Lastly, we present an example of GNSS (Global Navigation Satellite System) network design for geodetic monitoring under the approach of Prószyński & Łapiński (2021).

1. Introduction

The a priori analyses of accuracy and sensitivity are important indicators for geodetic monitoring networks. While the first provides information about the network's positional quality, the second indicates the capacity of the network to detect displacements (G Even-Tzur, 2002; Gilad Even-Tzur, 1999, 2010). According to (Prószyński & Łapiński, 2021) commonly both indicators are presented in the form of a confidence ellipsoid and a sensitivity ellipsoid for each network point to represent the quality and sensitivity respectively. Here the axes of these ellipsoids are computed from Minimal Detectable Displacement (MDD) values (Aydın, 2014).

According to (Prószyński & Łapiński, 2021) the relation between accuracy and sensitivity is concerned only with the size comparison of ellipsoids due to a lack of theoretical basis. This condition extends to the significance tests of computed displacements where sensitivity characteristics are not considered. Under this scenario, (Prószyński & Łapiński, 2021) provide a theoretical basis to connect the sensitivity to confidence and significance analysis in a unique approach based on MDD.

The method proposed by (Prószyński & Łapiński, 2021) called variance factor option (I) provides the size of significance, sensitivity and confidence ellipsoids from MDD values which are computed from an inequality between the critical values of displacement $\phi_{h,a}$ and the noncentrality parameter $\lambda_{h,\alpha,\beta}$. For this, the h-dimensionality vector of displacements and the probability levels, namely, significance level or Type I error probability (α_0), Type II error or false negative probability (β_0), and power of test ($\gamma_0 = 1 - \beta_0$) are coordinated to obtain a relation that permits include the sensitivity characteristics on confidence and significance analysis. If the significance level is chosen as $\alpha_0 = 1 - CL$, being CL the confidence level for the confidence regions of the network points, then the confidence and significance approaches given the same MDD values. This assumption is considered throughout the whole paper. Note that, the relation focuses on the h-dimensionality vector of displacements which provides two main scenarios, $\lambda_{h,\alpha,\beta} > \phi_{h,a}$ and $\lambda_{h,\alpha,\beta} < \phi_{h,a}$ where the inflection point is given by $h = 7.3$ ($\lambda_{h,\alpha,\beta} = \phi_{h,a}$) for $\alpha_0 = 5\%$ and $\beta_0 = 20\%$. Consequently, the sensitivity ellipsoid will exceed the dimensions of both the confidence and significance ellipsoids if $h > 7.3$ or vice-versa.

However, the research of (Prószyński & Łapiński, 2021) does not analyze aspects such as displacement vector dimensionality (global or local MDD values), network spatial dimension (1D, 2D, 3D), network configuration or redundancy and functional model (e.g., if linear or nonlinear), neither their role in the a priori analysis or design of geodetic monitoring networks. Although these aspects have been studied for sensitivity analysis, (Caspary &

Rüeger, 1987; Shan-long Kuang, 1991; J. O. Ogundare, 2015; Schmitt, 1982) ,their impact on a unified approach for confidence and sensitivity MDD values has not been analyzed yet. Thus, this work presents an analysis under the approach of (Prószyński & Łapiński, 2021) focused on the design of geodetic monitoring networks. The findings found in this research show that these aspects have a significant impact on the global sensitivity values for confidence and sensitivity approaches, thereby influencing the network's capacity to detect displacements. In addition, we also provide an example of GNSS (Global Navigation Satellite System) network design for monitoring purposes considering MDD values for both confidence and sensitivity ellipsoids.

2. Sensitivity analyses of geodetic monitoring networks

For planning purposes, the design of geodetic monitoring networks can be addressed through sensitivity analysis, which assesses the network's ability to detect displacements via the MDD or deformation parameters (Aydin, 2014; duchnowski, 2011; Küreç & Konak, 2020). Therefore, the network optimization can be based on a threshold displacement that will define whether the network is suitable for its purposes or must be improved. The optimization criteria for sensitivity analysis considers the displacement vector, denoted as $\hat{d}$ which is analyzed under the null $H_0 : E(\hat{d}) = 0$ and the alternative hypothesis $H_A : E(\hat{d}) \neq 0$ that implies deformation between two epochs (Aydın, 2014; Even-Tzur, 2010; Prószyński & Łapiński, 2021). The statistical test is given by:

$$T = \hat{d}^T C_{\hat{d}}^{+} \hat{d} \sim \chi^2_{h,\lambda} \quad (1)$$

In eq.1 λ corresponds to the noncentrality parameter of the expected displacement vector that the network can detect and $C_{\hat{d}}^{+}$ denotes the generalized inverse of the covariance matrix $C_{\hat{d}}$ of the displacement vector $\hat{d}$. Thus, the alternative hypothesis becomes $H_A : E(\hat{d}) \neq 0 = \Delta$ and $T = \lambda = \hat{\Delta}^T C_{\hat{d}}^{+} \hat{\Delta}$. The network will be sensitive to the expected displacements if $\lambda \geq \lambda_0$ where λ_0 corresponds to the lower bound of the noncentrality parameter given for α_0, γ_0 and h-dimensional vector, where h is the rank of $C_{\hat{d}}$ (Aydin, 2014; Książek & Łapiński, 2022).

Sensitivity analysis can be applied using either a local or a global approach, which entails evaluating unitary or simultaneous displacements. In the case of global sensitivity analysis, it is applied to the entire network, while local sensitivity analysis focuses on specific points. For example, for a 3D point in a local analysis, $h = 3$, while for a 2D point $h = 2$ and for a 1D point $h = 1$.

The expected vector displacements Δ is composed by a vector of directions (g) and a scale factor (b) such that $\Delta = bg$. Thus, the condition of the MDD is given where $b = b_{min}$ that implies the best sensitivity of the network, and therefore the relation $\Delta = \Delta_{min} = b_{min}g$ is fulfilled. According to (Aydın, 2014; García-Asenjo et al., 2023) the b_{min} is given by $b_{min} = \sqrt{\lambda_0 \lambda_{max}}$ where λ_{max} corresponds to the maximum eigenvalue of the covariance matrix $C_{\hat{d}}$. On another hand, the worst sensitivity of the network is given by $b_{max} = \sqrt{\lambda_0 \lambda_{min}}$ where λ_{min} corresponds to the minimum eigenvalue of the covariance matrix $C_{\hat{d}}$ (Aydin, 2014; Erdogan et al., 2017; Hsu & Hsiao, 2002; Küreç Nehbit & Konak, 2014).

3. Confidence Region and Network Sensitivity Characteristics: Variance factor option (I)

For the significance tests of displacements, the alternative hypothesis ($H_A\colon E(\hat{d}) \neq 0$) has a statistical test $T = \phi = \hat{d}^T C_{\hat{d}}^{+} \hat{d} \sim \chi^2_{\alpha_0,h}$ with u_{h,α_0} as critical value (Aydın, 2014). While under the sensitivity analysis, the critical value is given by $\lambda_0 = \lambda_{\alpha_0,\gamma_0,h}$. Thus, the main difference is the negligence of Type II Error in the significance approach ($\gamma_0 = 1 - \beta_0$). The basis of the variance factor option (I) method considers that the inequality $u_{h,\alpha_0} = \lambda_{\alpha_0,\gamma_0,h}$ is fulfilled for a certain value of $h = rank(C_{\hat{d}})$. For this, the type II probability error given by $\beta_0 = 1 - \gamma_0$ is modified to $\underline{\beta_0} = 1 - \underline{\gamma}_0$ which is computed from a specific value of significance α_0 such that $u_{h,\alpha_0} = \lambda_{\alpha_0 \underline{\gamma_0},h}$ is achieved (Prószyński & Łapiński, 2021). In this case, we say that $\underline{\beta_0}$ is coordinated with α_0.

From the equality aforementioned and based on MDD values, the significance, confidence, and sensitivity ellipsoids are computed. Here, the probability levels of Type I error (α_0), Type II error (β_0), power of the test ($\gamma_0 = 1 - \beta_0$), and the confidence level (CL) define each ellipsoid. Thus, the significance ellipsoid is obtained from $\hat{d}^T C_{\hat{d}}^{+} \hat{d} = \Phi_{h,\alpha_0}$; the confidence ellipsoid is obtained from $\hat{d}^T C_{\hat{d}}^{+} \hat{d} = \Phi_{h,CL}$; and the sensitivity ellipsoid is obtained from $\hat{d}^T C_{\hat{d}}^{+} \hat{d} = \lambda_{h,\alpha_0,\beta_0}$. From these definitions, some relations permit to reduce the number of ellipsoids analyzed. In particular, the relation between confidence and significance ellipsoids can be reduced considering $CL = 1 - \alpha_0$ that it involves $u_{h,CL} = u_{h,\alpha_0}$. Hence, the analysis solely relies on the relationship $u_{h,\alpha_0} = \lambda_{\alpha_0 \underline{\gamma}_0,h}$ (Prószyński & Łapiński, 2021). Thus, from now on confidence and significance analysis (or ellipsoids) will be synonyms.

According to (Prószyński & Łapiński, 2021) the equality $u_{h,\alpha_0} = \lambda_{\alpha_0 \underline{\gamma}_0,h}$ is achieved for $h = 7.3 = h^*$ for $\alpha_0 = 5\%$ and $\beta_0 = 20\%$. Thus, two scenarios are obtained: $h > h^*$ and $h < h^*$, which defined the relation between the sensitivity ellipsoid and the confidence ellipsoid (Figure 1). In Figure 1, for $h < h^*$ the confidence ellipsoid is smaller than the sensitivity ellipsoid (a) while for $h > h^*$ the sensitivity ellipsoid is smaller than the confidence ellipsoid (b). The value of h^* depends on the values for α_0 and β_0. Here, we assume the a priori variance factor to be known and the opposite case is outside the scope of this paper. Details are found in (Prószyński & Łapiński, 2021).

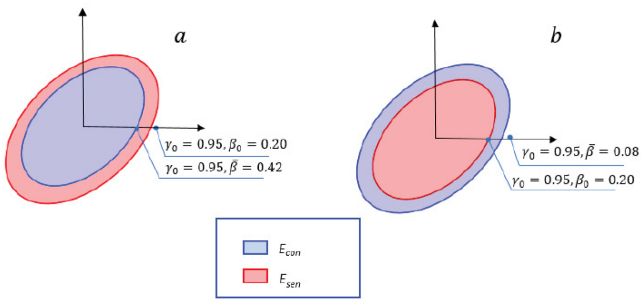

Figure 1: Sensitivity and confidence ellipsoids adapted of (Prószyński & Łapiński, 2021)

4. Experimental setup

Under the aforementioned relations, the objective of the experiments is to analyze the influence of the following aspects on sensitivity and significance analyses: 1) network configuration or redundancy; 2) spatial dimension of the network; and 3) functional model of the network. Additionally, an example of GNSS monitoring network design is also presented, considering the recent approach of Prószyński and Łapiński (2021).

5.1 Network configuration or redundancy

To analyze the effect of network configuration or redundancy, we considered a leveling network with 8 points, where the number of connections per point varies from 2 (no redundancy) to 7 (maximum redundancy), as illustrated in Figure 2. All height differences were assumed to have a standard deviation of 1 mm. For the global MDD analysis with $rank(C_{\hat{a}}) = 7$, the significance level was set as $\alpha_0 = 5\%$, with $\underline{\beta} = 21\%$ for the significance analysis and $\beta = 20\%$ or $\beta = 5\%$ for the sensitivity analysis. The results of the maximum and minimum MDD values are shown in Table 1. For the local MDD analysis, with $h = 1$, the significance level was set as $\alpha_0 = 5\%$, with $\underline{\beta} = 50\%$ for the significance analysis and $\beta = 45\%$ or $\beta = 20\%$ for the sensitivity analysis. The results of the local MDD values are shown in Table 2.

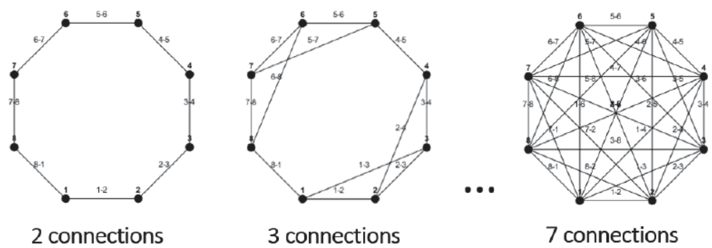

Figure 2: Sequential connections per point of the leveling network

5.2 Spatial dimension of the network

To analyze the effect of the network's spatial dimension, we considered a one-dimensional (1D) leveling network with 10 points, a two-dimensional (2D) triangulateration network with 6 points, and a three-dimensional (3D) GNSS network with 4 points, all with $h = rank(C_{\hat{a}}) = 9$ and no observation redundancy: $r = n - u = 0$ (Figure 3). Linear observations for all networks were assumed to have a standard deviation of 1 mm. Angular observations for the 2D network were assumed to have a standard deviation of 0.001 / 100 rad to have equivalent weights with linear observations considering distances of 100 m between network points. For the global MDD values, the significance level was set as $\alpha_0 = 5\%$, , with $\underline{\beta} = 16\%$ for the significance analysis and $\beta = 4\%$ or $\beta = 20\%$ for the sensitivity analysis. The results of the maximum and minimum MDD values are shown in Table 3. Subsequently, a one-dimensional (1D) leveling network with 22 points, a two-dimensional (2D) triangulateration network with 12 points, and a three-dimensional (3D) GNSS network with 8 points were considered, all with $h = rank(C_{\hat{a}}) = 21$ and no observation redundancy: $r = n - u = 0$ (Figure 4). For the global MDD values, the significance level was set as $\alpha_0 = 5\%$, with $\underline{\beta} = 4\%$ for the significance analysis and $\beta = 16\%$ or $\beta = 20\%$ for the sensitivity analysis. The results of the maximum and minimum MDD values are shown in Table 4.

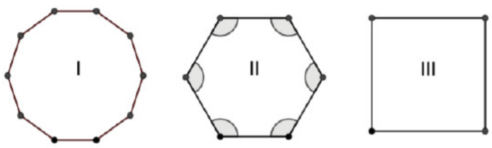

Figure 3: equal h-dimensional networks (h=9): I correspond to a 1D leveling network, II corresponds to a 2D horizontal network and III corresponds to a 3D GNSS network

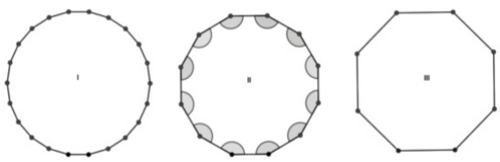

Figure 4: equal h-dimensional networks (h=21): I correspond to a 1D leveling network, II corresponds to a 2D horizontal network and III corresponds to a 3D GNSS network

5.3 Functional model of the network

To analyze the effect of the network's functional model, we reconsidered the 2D networks from the previous experiments (Figures 3 and 4, here referred as "Case I"), with the significance level $\alpha_0 = 5\%$ and $\beta = 20\%$ for sensitivity analysis. However, instead of using angle and distance equations, we now considered a linearized model based on coordinate differences, similar to those of the 1D and 3D networks. (see, e.g. (Ghilani, 2017; S Kuang, 1996; J. Ogundare, 2015). For this, we assumed that the new observations (coordinate differences Δx and Δy) have a standard deviation of 1 mm each (referred as "Case II"). Additionally, we considered the networks with both null redundancy and maximum redundancy (Figures 5 and 6). The results are presented in Table 5 for the 2D network with 6 points and in Table 6 for the 2D network with 12 points.

Finally, a last analysis was conducted regarding the functional model: considering a 2D trilateration network (i.e. distance equations only), a 2D triangulateration network (angle and distance equations), and a 3D GNSS network (coordinate difference equations) as in Figure 7. In all networks, $h = rank(C_{\hat{a}}) = 9$, and linear observations have a standard deviation of 1 mm, while angular observations in the triangulateration network have a standard deviation of 0.001 / 100 rad. In the trilateration and GNSS networks, all points are interconnected, while in the triangulateration network, there are 6 fewer distances than in the trilateration network, but with the inclusion of six internal angles, to have the same number of redundant observations in both cases. For the global MDD values, the significance level was set as $\alpha_0 = 5\%$, with $\underline{\beta} = 16\%$ for the significance analysis and $\beta = 20\%$ for the sensitivity analysis. The results of the maximum and minimum MDD values are shown in Table 7.

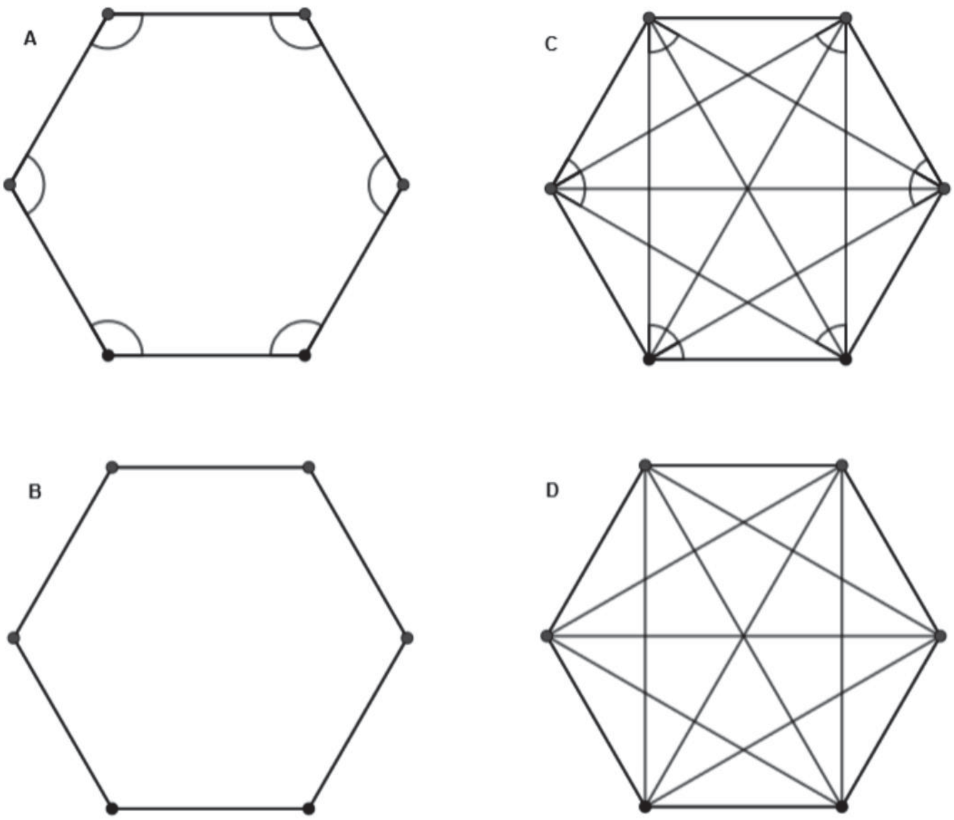

Figure 5: A and B correspond to a 2D network of "Case I" without and with redundancy (respectively), C and D correspond to a 2D network of "Case II" without and with redundancy (respectively), $h = rank\left(C_{\hat{a}}\right) = 9$ in all cases

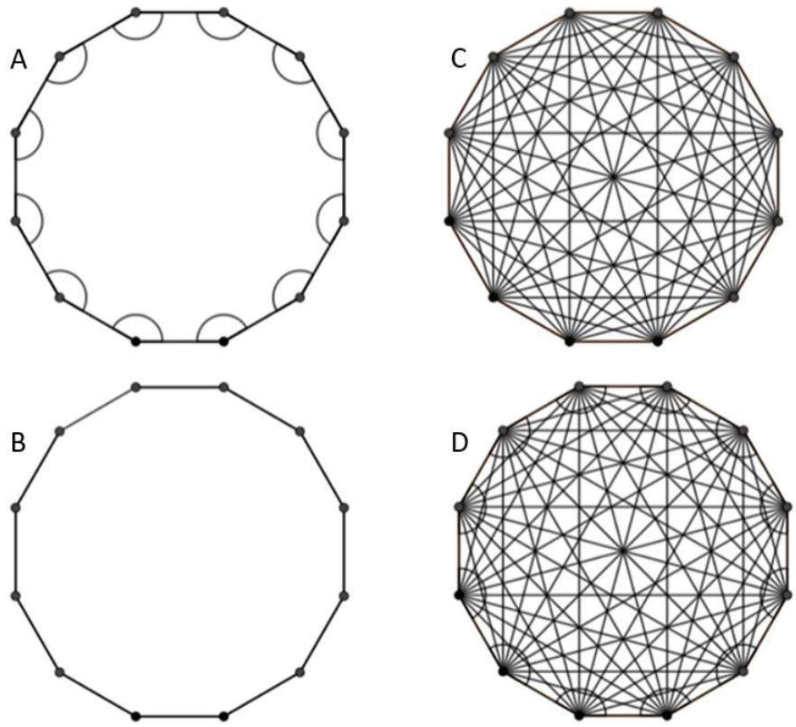

Figure 6: A and B correspond to a 2D network of "Case I" without and with redundancy (respectively), C and D correspond to a 2D network of "Case II" without and with redundancy (respectively), $h = rank\left(C_{\hat{d}}\right) = 21$ in all cases

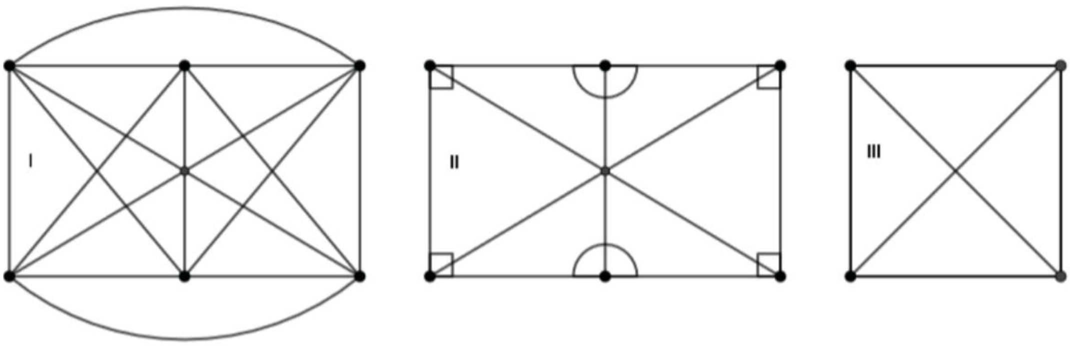

Figure 7: I correspond to a 2D trilateration network, II corresponds to a 2D triangulateration network and III corresponds to a 3D GNSS network, $h = rank\left(C_{\hat{d}}\right) = 9$ in all cases

5.4 Example of geodetic monitoring network design

To demonstrate the practical application of the recent approach by Prószyński and Łapiński (2021), we present an example of geodetic monitoring network design. In this case, the simulated GNSS network has 6 points, and it was stipulated that the local MDD value of each network point should not exceed 20 mm (with $h = 3$ in a local analysis). To obtain a realistic a priori covariance matrix with variances and covariances for the 3D components of the same baseline, the very recent approach by (Koch et al., 2023) was applied, considering the case of precise ephemerides. Two variables were analyzed: the time span of each tracking session (1h or 2h) and the number of connections or 3D baselines per point (Figure 8). The significance level was set as $\alpha_0 = 5\%$, with $\underline{\beta} = 36\%$ for the significance analysis and $\beta = 20\%$ or $\beta = 5\%$ for the sensitivity analysis. The results of the maximum local MDD values are shown in Table 8. Considering a GNSS receiver at each network point, we have 6-1=5 independent baselines per tracking session, and the optimal solutions in terms of cost to meet the 20 mm threshold value are presented in Table 9.

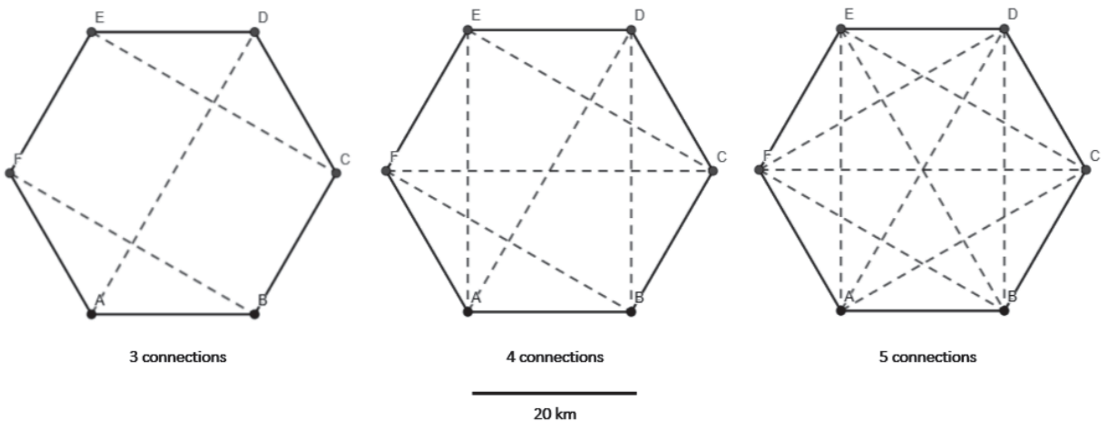

Figure 8: GNSS monitoring network with 3, 4 and 5 connections per point

5. Discussions:

Analyzing the results from Section 5.1 (Tables 1 and 2), we can conclude that the global max MDD values are much more affected than those of global min MDD or local MDD values when increasing the network redundancy. Convergence occurs at 5 connections per point for local MDD values and at 6 connections per point for global min MDD values. That is, a network with 5 connections per point (20 total observations) or with 7 connections per point (28 total observations) provides the same local MDD values (Table 2). Additionally, the local MDD values are smaller than those of global min MDD for the same number of connections per point, even when the false negative probability is lower. Besides that, increasing the network redundancy reduces the differences between the respective MDDs in significance and sensitivity analysis. For example, for two connections per point, the difference is 1.7 mm for global max MDD values; while for seven connections per point, this difference drops to only 0.4 mm (Table 1). Moreover, increasing the network redundancy also reduces the differences between global min and max MDD values, with both becoming equal when all points are interconnected (Table 1). Overall, the main conclusion here is: increasing network redundancy is more effective in reducing the global max MDD values than for global min MDD or local MDD values.

Analyzing the results from Section 5.2 (Tables 3 and 4), we can again conclude that the max MDD values are more affected than the min MDD values. In case, the larger the spatial dimension of the network, the smaller the max MDD value for a given $h = rank(C_{\hat{d}})$. However, the min MDD values are equal for 1D and 3D networks, being larger than those of 2D networks. A hypothesis is the fact that the 2D network has a nonlinear model with observations of different units (angles and distances). By linearizing the mathematical model of the 2D network to make it similar to the mathematical models of the 1D and 3D networks, i.e., only linear observations of equal precision, the min MDD values of the 2D network become equal to the min MDD values of the 1D and 3D networks. This fact confirms the previous hypothesis (see the respective results of “Case II” in Tables 5 and 6). Additionally, the differences between the max MDD values for networks of different spatial dimensions increase with the value of h. For example, for $h = 9$ and $\beta = 20\%$, the difference between the max MDD values of the 1D and 3D networks is 5.1 mm (Table 3); while for $h = 21$ and $\beta = 20\%$, this difference increases to 14.5 mm (Table 4). Therefore, considering the results of Sections 5.1 and 5.2 for network design purposes, the strategy of increasing the network redundancy should be more efficient for 1D networks, which generally have higher values of max MDD, than for 3D networks, which generally have lower values of max MDD.

Analyzing the results from Section 5.3 (Tables 5 and 6), we conclude that when the networks have no redundancy, Cases I and II present the same maximum MDD values. However, when the networks have redundancy, the maximum MDD values are lower for Case II compared to Case I. On the other hand, the minimum MDD values are higher for Case II compared to Case I. It should be noted that Case II is only hypothetical for 2D networks, but these experiments demonstrate that the underlying functional model also influences the MDD values.

Analyzing the results of Tables 1 and 7, we note that when observations have equal precision and all points are interconnected, the values of min MDD and max MDD are equal for networks with a linear functional model such as leveling (1D) and GNSS (3D) networks, and the same also occurs for the hypothetical Case II in 2D networks (Tables 5 and 6). However, this does not occur for networks with a nonlinear functional model such as trilateration networks, where the values of maximum MDD are significantly higher (Table 7). The hypothesis here is that the trilateration network has only one type of observation equation (horizontal distance) for two types of unknowns (x, y coordinates in the 2D plane). By removing 6 distances from the trilateration network and adding 6 angles, keeping the same number of redundant observations (Figure 7), we conclude that the max MDD values significantly reduce (see Table 7). This occurs because now we have two types of observation equations (horizontal distances and angles) for two types of unknowns (x, y coordinates in the 2D plane). Therefore, although trilateration is widely used in geodetic monitoring because of the effects of atmospheric refraction on the measured (García-Asenjo et al., 2023;

Masoud et al., 2021; Mohammad et al., 2023), its functional model is worse for displacement detection than triangulateration, even with the same number of redundant observations.

Analyzing the results of Section 5.4 (Tables 8 and 9), we note that displacements of 20 mm can be detected between 3 hours of time span (at a false negative probability of around 36%) and 6 hours of time span (at a false negative probability of around 5%). This is the first study to show a direct relationship between time span and the false negative probability for a GNSS monitoring network design. Therefore, this example demonstrates the practical importance of the recent approach of Prószyński and Łapiński (2021) in the a priori analysis of geodetic monitoring networks, where the optimal solution is obtained by not only absolute MDD values, but also considering the respective false negative probability.

6. Conclusions

In this contribution, we demonstrated how the theoretical discussions of Prószyński and Łapiński (2021) can be applied in the a priori analysis of geodetic monitoring analysis, investigating several aspects in numerical experiments that are not covered in the aforementioned research. Our main findings can be sumarized as follows:

- increasing network redundancy is more effective in reducing the global max MDD values than for global min MDD or local MDD values;
- increasing the network redundancy reduces the differences between the respective MDDs in significance and sensitivity analysis;
- the larger the spatial dimension of the network, the smaller the max MDD value for a given $h = rank(C_{\hat{d}})$;
- the differences between the max MDD values for networks of different spatial dimensions increase with the value of h for both significance and sensitivity analysis;
- the strategy of increasing the network redundancy should be more efficient for 1D networks, which generally have higher values of max MDD, than for 3D networks, which generally have lower values of max MDD;

- although trilateration is widely used in geodetic monitoring because of the effects of atmospheric refraction on the measured angles, its functional model is worse for displacement detection than triangulateration, even with the same number of redundant observations.

In addition, we also present an numerical example of GNSS monitoring network design where a direct relationship between time span and the false negative probabiltiy is demonstrated. Thus, this numerical experiments show the practical importance of the recent approach of Prószyński and Łapiński (2021) in the a priori analysis of geodetic monitoring networks.

For future studies, we recommend the design or a priori analysis of a real geodetic monitoring network considering the aspects highlighted here. We also recommend strategies to improve (reduce) the MDD values of trilaterations networks in both significance and sensitivity analysis.

7. Appendix

Connections per point	Significance ($\underline{\beta} = 21\%$)		Sensitivity ($\beta = 20\%$)		Sensitivity ($\beta = 5\%$)	
	MIN	MAX	MIN	MAX	MIN	MAX
2	2.6	6.9	2.7	7.0	3.3	8.6
3	2.3	3.7	2.3	3.8	2.9	4.7
4	2.2	3.3	2.2	3.3	2.7	4.1
5	2.0	2.6	2.0	2.7	2.4	3.3
6	1.9	2.2	1.9	2.2	2.3	2.7
7	1.9	1.9	1.9	1.9	2.3	2.3

Table 1: MDD values (mm) for the number of connections (global analysis with $\alpha_0 = 5\%$ and $h = rank\left(C_{\tilde{d}}\right) = 7$)

Connections per point	Significance ($\underline{\beta} = 50\%$)	Sensitivity ($\beta = 45\%$)	Sensitivity ($\beta = 20\%$)
2	1.4	1.5	2.0
3	1.2	1.3	1.7
4	1.1	1.2	1.6
5	1.0	1.0	1.4
6	1.0	1.0	1.4
7	1.0	1.0	1.4

Table 2: MDD values (mm) for the number of connections (local analysis with $\alpha_0 = 5\%$ and $h = 1$)

Spatial dimension	Sensitivity ($\beta = 4\%$)		Significance ($\underline{\beta} = 16\%$)		Sensitivity ($\beta = 20\%$)	
	MIN	MAX	MIN	MAX	MIN	MAX
1D	4.4	14.4	2.9	9.4	2.8	9.1
2D	2.5	8.9	1.7	5.8	1.6	5.6
3D	4.4	6.3	2.9	4.1	2.8	4.0

Table 3: MDD values (mm) for geodetic networks (global analysis with $\alpha_0 = 5\%$ and $h = rank\left(C_{\hat{d}}\right) = 9$)

Spatial dimension	Significance ($\underline{\beta} = 4\%$)		Sensitivity ($\beta = 16\%$)		Sensitivity ($\beta = 20\%$)	
	MIN	MAX	MIN	MAX	MIN	MAX
1D	4.0	28.4	3.6	24.9	3.3	23.0
2D	2.1	15.6	1.8	13.6	1.7	12.6
3D	4.0	10.6	3.6	9.2	3.3	8.5

Table 4: MDD values (mm) for geodetic networks (global analysis with $\alpha_0 = 5\%$ and $h = rank\left(C_{\hat{d}}\right) = 21$)

	Case I		Case II	
	MIN	MAX	MIN	MAX
Without redundancy	1.6	5.6	2.8	5.6
With redundancy	1.3	3.0	2.3	2.3

Table 5: MDD values (mm) for 2D horizontal networks (global analysis with $\alpha_0 = 5\%$, $\beta = 20\%$ and $h = rank\left(C_{\hat{d}}\right) = 9$)

	Case I		Case II	
	MIN	MAX	MIN	MAX
Without redundancy	1.7	12.6	3.3	12.6
With redundancy	1.0	2.6	1.9	1.9

Table 6: MDD values (mm) for 2D horizontal networks (global analysis with $\alpha_0 = 5\%$, $\beta = 20\%$ and $h = rank\left(C_{\hat{d}}\right) = 21$)

	Trilateration 2D		Triangulateration 2D		GNSS 3D	
	MIN	MAX	MIN	MAX	MIN	MAX
Significance ($\underline{\beta} = 16\%$)	2.5	5.6	1.8	4.4	2.9	2.9
Sensitivity ($\beta = 20\%$)	2.4	5.4	1.7	4.2	2.8	2.8

Table 7: MDD values (mm) for geodetic networks of Fig. 6 (global analysis with $\alpha_0 = 5\%$ and $h = rank(C_d) = 9$)

Connections Per point	Time session: 1 hour			Time session: 2 hours		
	Significance ($\underline{\beta} = 36\%$)	Sensitivity ($\beta = 20\%$)	Sensitivity ($\beta = 5\%$)	Significance ($\underline{\beta} = 36\%$)	Sensitivity ($\beta = 20\%$)	Sensitivity ($\beta = 5\%$)
3	24.0	28.4	35.6	16.8	19.9	24.9
4	20.0	23.7	29.7	14.0	16.5	20.8
5	17.6	20.8	26.1	12.3	14.5	18.2

Table 8: Maximum MDD values (mm) for GNSS monitoring network (local analysis with $\alpha_0 = 5\%$ and $h = 3$)

Significance ($\underline{\beta} = 36\%$)	Sensitivity ($\beta = 20\%$)	Sensitivity ($\beta = 5\%$)
5 connections per point and 3 sessions with 1 hour each (3 hours in total)	3 connections per point and 2 sessions with 2 hours each (4 hours in total)	5 connections per point and 3 sessions with 2 hours each (6 hours in total)

Table 9: Optimal solutions for the GNSS monitoring network with a threshold of 20 mm for MDD values (local analysis with $\alpha_0 = 5\%$ and $h = 3$)

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5 CONCLUSIONS

The thesis presented was oriented to analyze the least squares estimation (LSE) the geodetic network proprieties on sensitivity analysis. Initially, the first paper described the free adjustment through the pseudo inverse approach and minimum inner constraints. Both methods are usually used to compute the sensitivity analyses and focus of this paper is make a comparison between both methods. The first method, the pseudo-inverse approach computes the inverse of the normal equation matrix (N), based on the generalized inverses theory. On the other hand the minimum inner constraints modify the classical solution of the LSE added the G matrix called *inner constraints matrix* to the normal equation matrix to solve the inverse problem. As presented in this paper, the results obtained to 2D geodetic network were equivalents. Therefore, for sensitivity analysis both methods are applicable to determinate the network sensitivity through the minimum detectable displacements (MDD).

In the second paper, we explored the interplay between significance and sensitivity in computing Minimum Detectable Displacements (MDD). Initially, employing the Global Congruence Test (GCT) approach, we assessed the efficacy of significance and sensitivity analyses in detecting changes. Our findings indicate that the network configuration, the stochastic model used, and the type of analysis (whether global or local) significantly affect the MDD values. We also examined the methodology proposed by Prószyński and Łapiński (2021) under similar conditions. This involved investigating the influence of network configuration and stochastic model on the *variance factor method (I)*, which considers both sensitivity and accuracy aspects in the pre-analysis of geodetic networks. Our results showed that improvements in the network and stochastic model, such as adding more observations and using better standard deviations for these observations, generally lead to improved MDD values and a decreased discrepancy between the semi-major axes of the sensitivity and significance ellipsoids. These findings are crucial for optimizing geodetic network designs. Therefore, these aspects should consider whether to focus solely on false positives (significance analysis) or also include false negatives (sensitivity analysis). Particularly in geodetic monitoring, the risk of false negatives (unnoticed deformations) often outweighs the concern of false positives (false alarms).

The third paper we applied the theoretical concepts of Prószyński and Łapiński (2021) to the a priori analysis of geodetic monitoring networks, conducting numerical experiments to explore areas not covered in their research. Thus, the main findings include that increasing network redundancy effectively lowers global maximum MDD values, outperforming its impact on global minimum and local MDD values, and simultaneously reduces the discrepancy in MDDs observed in significance and sensitivity analyses. Networks with larger spatial dimensions tend to exhibit smaller maximum MDD values for a specified $rank(h)$ of the $C_{\hat{d}}$ matrix. However, as the spatial dimensions of networks vary, the discrepancy in their maximum MDD values escalates alongside the value of h, affecting both significance and sensitivity analyses. This enhancement in redundancy proves particularly advantageous for 1D networks, which generally show higher max MDD values, in contrast to 3D networks where these values are typically lower. Despite its frequent use in geodetic monitoring, trilateration's functional model falls short in displacement detection compared to triangulation, even when the number of redundant observations is equal, largely due to the atmospheric refraction's influence on measured angles. In addition, we presented a numerical example of GNSS monitoring network design, demonstrating a direct correlation between the time span and the probability of false negatives. These experiments underscore the practical relevance of Prószyński and Łapiński's (2021) approach in the a priori analysis of geodetic monitoring networks. For future research, we suggest designing or conducting a priori analysis of geodetic monitoring networks, considering the aspects outlined in this paper. We also recommend exploring strategies to improve (reduce) MDD values for significance and sensitivity analyses.

The findings found in this work provides guidelines to apply the Prószyński and Łapiński's method on different types of geodetic networks in a priori analysis of sensitivity. Here also we analyze the least square model (free adjustment) used to perform the sensitivity analysis.

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