

UNIVERSIDADE FEDERAL DO PARANÁ

MORONI MENESSES BRUCH BORA

HOPF K-ALGEBRAS, HOPF ALGEBRAS, AND HOPF CATEGORIES: HOPF
ALGEBRAIC STRUCTURES IN INCREASING ORDER OF GENERALIZATION.

CURITIBA

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ALGEBRAIC STRUCTURES IN INCREASING ORDER OF GENERALIZATION.

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vezes em que só há escuridão.

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*“Não vos amoldeis às estruturas deste mundo,
mas transformai-vos pela renovação da mente,
a fim de distinguir qual é a vontade de Deus:
o que é bom, o que Lhe é agradável, o que é perfeito.
(Bíblia Sagrada, Romanos 12, 2)*

RESUMO

Há três conceitos de álgebras de Hopf em ordem crescente de abstração: $\mathbb{K}$ -álgebras de Hopf, álgebras de Hopf em categorias monoidais trançadas e categorias de Hopf. Essa dissertação apresenta definições, construções e resultados fundamentais para cada um destes casos.

Palavras-chaves: $\mathbb{K}$ -álgebras de Hopf; álgebras de Hopf; categorias de Hopf; categorias monoidais; categoria enriquecidas

ABSTRACT

There are three concepts of Hopf algebras in a crescent order of abstraction: Hopf $\mathbb{K}$ -algebras, Hopf algebras in braided monoidal categories and Hopf categories. This dissertation presents fundamental definitions, constructions and results in each one of those cases.

Keywords: Hopf $\mathbb{K}$ -algebras; Hopf algebras; Hopf categories; monoidal categories; enriched categories.

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1 INTRODUCTION

The main objective of this dissertation is to present and relate three structures: Hopf $\mathbb{K}$ -algebras, Hopf algebras, and Hopf categories. While the concept of Hopf $\mathbb{K}$ -algebras generalizes fields and vector spaces, the categorical concept of Hopf algebras generalizes Hopf $\mathbb{K}$ -Algebras, groups, monoids, and rings. Meanwhile, the concept of Hopf categories generalizes Hopf algebras and groupoids. Note that these three concepts we are working with are in a growing order of generalization. The relations we present are about such generalizations and common properties of these three concepts. In the final chapter, we will study the concept of Hopf categories to finitely generated projective modules.

Hopf $\mathbb{K}$ -algebras have their origins in a work by Heinz Hopf in algebraic topology in 1941 but also in later works of Pierre Cartier, Jean Dieudonné, Armand Borel and Gerhard Hochschild in the 1950's in the theory of algebraic groups. In these first decades, Hopf algebras come usually associated with Lie groups, algebraic groups or Lie algebras; for instance, the group algebra of a group G is always a Hopf algebra, as we will see in Example 2.5.4; we will also see examples which do not fit in this setting, as the Sweedler algebra (Example 2.5.3). Finite-dimensional Hopf $\mathbb{K}$ -algebras also satisfy the important property that the dual space of a Hopf algebra has a canonical Hopf algebra structure (Proposition 2.5.11). The more general notion of a Hopf algebra in a braided monoidal category seemingly appears in the 1990's in a work by Shan Majid, and this approach unifies Hopf $\mathbb{K}$ -algebras and groups. As we show in this dissertation, several fundamental properties of Hopf $\mathbb{K}$ -algebras still hold in this more general setting. In particular, a Hopf algebra in this general sense may not have a dual Hopf algebra, but the definition of Hopf algebra is self-dual, in the sense that a Hopf algebra in the opposite monoidal category is also a Hopf algebra in the original monoidal category. Much more recently, a new development was presented in (BATISTA; CAENEPEEL; VERCRUYSSSE, 2016), the Hopf categories over a monoidal category $\mathcal{V}$, or Hopf $\mathcal{V}$ -categories. These new objects now unify groupoids and Hopf algebras and require a new foundation, the theory of enriched categories over monoidal categories. Moreover, the definition of Hopf category is not self-dual anymore: a Hopf $\mathcal{V}$ -category over the opposite monoidal category $\mathcal{V}^{op}$ is what is called a dual Hopf $\mathcal{V}$ -category in (BATISTA; CAENEPEEL; VERCRUYSSSE, 2016), and these concepts are quite distinct. However, when the underlying monoidal category is the category $\mathcal{M}_k^f$ of finitely generated projective modules over a commutative ring k , we recover both the dual of a finite-dimensional $\mathbb{K}$ -algebra and the self-duality of the definition of Hopf algebra (Theorem 5.2.3).

In the second chapter, we begin by introducing $\mathbb{K}$ -algebras, that are vector spaces which also have a compatible ring structure. Then we move to $\mathbb{K}$ -coalgebras, which can be seen as dual structures of $\mathbb{K}$ -algebras; in fact, the dual vector space of a $\mathbb{K}$ -algebra has a canonical structure of $\mathbb{K}$ -algebra, and the dual of a (finite-dimensional) $\mathbb{K}$ -algebra has a canonical structure of $\mathbb{K}$ -coalgebra. We finally introduce $\mathbb{K}$ -bialgebras, vector spaces which are simultaneously $\mathbb{K}$ -algebras and $\mathbb{K}$ -coalgebras, plus some compatibilities, and Hopf $\mathbb{K}$ -algebras, which are bialgebras possessing an antipode, a particular $\mathbb{K}$ -linear endomorphism that is an antimorphism of algebras and coalgebras. We give examples of all structures defined in this chapter.

In the third chapter, we present algebras, coalgebras and bialgebras in a (braided) monoidal category, in order to construct the concept of Hopf algebras. We will see that Hopf algebras in the category of $\mathbb{K}$ -vector spaces coincide with the Hopf $\mathbb{K}$ -algebras of the second chapter, but we will also show that Hopf algebras in the category of sets correspond to groups. The main objective of this chapter is to generalize the concepts introduced in the previous chapter, in such manner that the reader may realize the generalization and common properties between the concepts of Hopf algebras and Hopf $\mathbb{K}$ -algebras in a categorical approach.

In the fourth chapter, we present enriched categories (which generalize algebras), enriched categories over coalgebras (which generalize coalgebras and bialgebras), Hopf categories (which generalize Hopf algebras) and the dual concepts of such structures. The main objective of this chapter is to present the fundamentals of the theory of Hopf categories, extending results which were proved for Hopf algebras in the second and third chapters.

Then, in the last chapter, the key results of the previous chapter are used to develop the theory of Hopf categories applied to the category of finitely generated projective modules over a commutative ring k . We prove that there is a biequivalence between $\mathcal{M}_k^f$ -categories and dual $\mathcal{M}_k^f$ -categories, and such correspondence induces a biequivalence between Hopf- $\mathcal{M}_k^f$ categories and dual Hopf $\mathcal{M}_k^f$ -categories.

The reader is assumed to have familiarity with linear algebra, including the main concepts of multilinear algebra, such as tensors and tensor mappings. The reader is also assumed to have familiarity with the basics of group theory and ring theory. We shall present all the category theory and the module theory needed for this dissertation.

2 HOPF $\mathbb{K}$ -ALGEBRAS

Hopf Algebras are structures which generalize the concept of inverses (such for in groups), by a concept called antipode. Our interest in this second chapter is to introduce the concept of Hopf $\mathbb{K}$ -algebras, which is a special case of Hopf algebras (that generalizes Hopf $\mathbb{K}$ -algebras and groups as we shall view in the next chapter). However, before studying Hopf $\mathbb{K}$ -algebras, it is necessary to study the structures of algebras and coalgebras. In this chapter, we may refer to $\mathbb{K}$ -algebras just as 'algebras', $\mathbb{K}$ -coalgebras just as 'coalgebras', $\mathbb{K}$ -bialgebras as bialgebras and Hopf $\mathbb{K}$ -algebras just as Hopf-algebras.

For this chapter the main reference is (FERREIRA; MURAKAMI, 2020). However, all this content may be also found in (DASCALESCU; NASTASESCU; RAIANU, 2000).

2.1 $\mathbb{K}$ -ALGEBRAS AND $\mathbb{K}$ -COALGEBRAS.

We begin with the usual definition of a $\mathbb{K}$ -algebra:

Definition 2.1.1. *Let A be a $\mathbb{K}$ -vector space. An $\mathbb{K}$ -algebra structure on A consists of a bilinear function $m : A \times A \rightarrow A$ and an element $1_A \in A$ such that*

$$m(m(a, b), c) = m(a, m(b, c)),$$

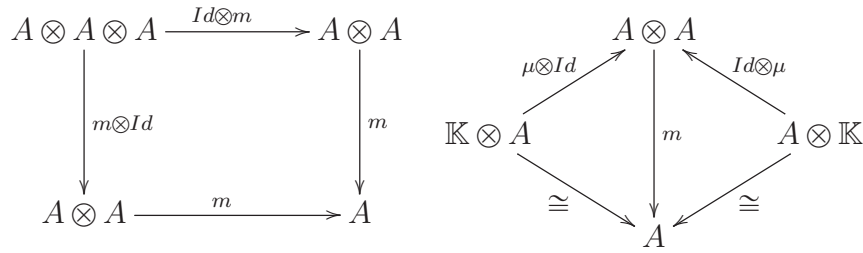
$$m(1_A, a) = m(a, 1_A) = a, \quad \forall a, b, c \in A.$$

*The first equation is called **associativity** and the second one is called **unit property**.*

It is not difficult to verify that a $\mathbb{K}$ -algebra is also an associative ring with unit.

The next definition fits better within the context of Hopf $\mathbb{K}$ -algebras and it is equivalent to the above one.

Definition 2.1.2. (*$\mathbb{K}$ -algebra -second definition*) *Let A be a $\mathbb{K}$ -vector space. A structure of $\mathbb{K}$ -algebra on A is a triple (A, m, μ) such that $m : A \otimes A \rightarrow A$ and $\mu : \mathbb{K} \rightarrow A$ are both linear maps and the following diagrams commute:*



We call the map m the product and the map μ the unit map. We call the first diagram the associativity diagram, and we call the second one the unit property diagram.

Some comments are useful at this point. First of all, as we have seen in Definition 2.1.1, it is possible to define algebras without resorting to tensor products. However, the tensor product is crucial to define the dual structure of an algebra. Second, the isomorphisms mentioned in the second commutative diagram are specifically the canonical isomorphisms $1 \otimes x \mapsto x$ and $x \otimes 1 \mapsto x$.

When we refer to algebras in the usual way (i.e, the first definition given), there is an element called the unit of the algebra, the element 1_A , which can be shown to be unique. But we can also think of the unit as a $\mathbb{K}$ -linear function. Basically, the unit function is the unique $\mathbb{K}$ -linear map $\mu : \mathbb{K} \rightarrow A$ such that $\mu(1) = 1_A$ when we see A as an algebra in the usual notion. Note also that the product $m : A \times A \rightarrow A$ induces a $\mathbb{K}$ -linear map $m : A \otimes A \rightarrow A$ which, due to the associativity of m , satisfies the conditions of Definition 2.1.2. In this manner we see that every $\mathbb{K}$ -algebra with respect to Definition 2.1.1 corresponds to a $\mathbb{K}$ -algebra as in Definition 2.1.2. It can be shown that the converse also holds.

Example 2.1.1. A first example of $\mathbb{R}$ -algebra is the field $\mathbb{C}$ over the field $\mathbb{R}$ with the usual product and unit.

Example 2.1.2. Another example of $\mathbb{R}$ -algebra is the skew field of quaternions with basis $\{1, i, j, k\}$ with 1 being the unit and i, j, k such that $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji, jk = i = -kj$ and $ki = j = -ik$. It is easy to verify that this product is associative. This is an example of an algebra with noncommutative product.

Example 2.1.3. The ring of polynomials $\mathbb{K}[t]$ with unit as the constant polynomial 1 and the product as usual is an example of a $\mathbb{K}$ -algebra.

Example 2.1.4. Another example of algebra is the algebra of square matrices $M_n(\mathbb{K})$. $M_n(\mathbb{K})$ is a $\mathbb{K}$ -vector space, the product of matrices is associative, and the unit is the identity matrix I_n .

Example 2.1.5. Let V be a $\mathbb{K}$ -vector space and let $\text{End}(V)$ be the space of linear operators of V , also called the space of linear endomorphisms of V . $\text{End}(V)$ is a $\mathbb{K}$ -algebra with product given by the composition of linear operators and unit given by the identity operator I_V .

Example 2.1.6. Let V be a $\mathbb{K}$ -vector space and define recursively $V^0 = \mathbb{K}$, $V^1 = V$ and $V^n = V^{n-1} \otimes V$ for all $n \in \mathbb{N}$. Then the $\mathbb{K}$ -vector space $T(V) = \bigoplus_{n \in \mathbb{N}} V^n$ is a $\mathbb{K}$ -algebra, with product and unit defined as:

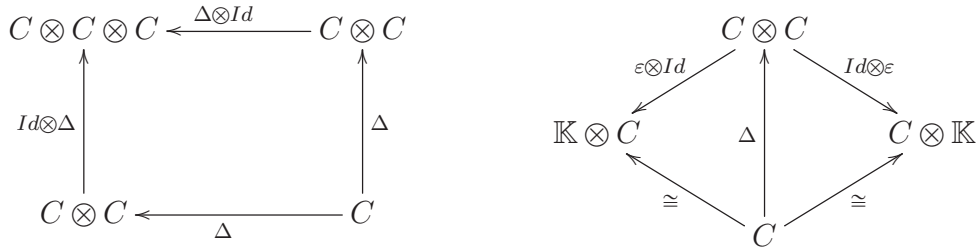
$$(v_1 \otimes \cdots \otimes v_n)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$$

$$1_{T(V)} = 1_{\mathbb{K}}.$$

This is the definition of the product on the generators of the components of $T(V)$; the definition for the whole space follows distributively. $T(V)$ is the **tensor algebra** of V . This algebra is also called the **free algebra over $\mathcal{B}$** if $\mathcal{B}$ is a basis for V .

Given a definition which is written in terms of mappings and equalities of mappings expressed as commutative diagrams, the dual definition is obtained by reversing all arrows (see for instance (MACLANE, 1988), pages 31-32). In this sense the definition of a $\mathbb{K}$ -coalgebra is, essentially, the dual definition of an $\mathbb{K}$ -algebra, as follows:

Definition 2.1.3. ($\mathbb{K}$ -Coalgebra) Let C be a $\mathbb{K}$ -vector space. A $\mathbb{K}$ -coalgebra is a triple (C, Δ, ε) such that $\Delta : C \rightarrow C \otimes C$ and $\varepsilon : C \rightarrow \mathbb{K}$ are both linear maps and the following diagrams commute:



We call the map Δ the coproduct and the map ε the counit map. The first diagram is the coassociativity diagram, and the second one is the counit map property diagram.

We remark that the isomorphisms in the counit map property diagram are specifically the isomorphisms $x \mapsto 1 \otimes x$ and $x \mapsto x \otimes 1$. Later, these isomorphisms will be seen as left and right units in a monoidal category (Chapter 3).

Now, a first example of coalgebra:

Example 2.1.7. We may define a structure of coalgebra on $\mathbb{K}$ in terms of the basis of $\mathbb{K}$ as a $\mathbb{K}$ -vector space, as follows:

$$\Delta(1) = 1 \otimes 1,$$

$$\varepsilon(1) = 1.$$

Observe that we have:

$$\begin{aligned}
 (\Delta \otimes Id) \circ \Delta(1) &= (\Delta \otimes Id)(1 \otimes 1) \\
 &= 1 \otimes 1 \otimes 1 \\
 &= 1 \otimes \Delta(1) \\
 &= (Id \otimes \Delta) \circ \Delta(1).
 \end{aligned}$$

All the equalities follow from the definition of Δ . Hence, Δ is coassociative. Now, we prove the counit property. In fact, we have:

$$\begin{aligned}
 (\varepsilon \otimes Id) \circ \Delta(1) &= (\varepsilon \otimes Id)(1 \otimes 1) \\
 &= \varepsilon(1) \otimes 1 \\
 &= 1 \otimes 1.
 \end{aligned}$$

The first equality follows from the definition of Δ while the second equality follows from the application of the tensor mapping $\varepsilon \otimes Id$, and the last equality follows from the definition of ε .

But, analogously, we also have:

$$\begin{aligned}
 (Id \otimes \varepsilon) \circ \Delta(1) &= (Id \otimes \varepsilon)(1 \otimes 1) \\
 &= Id \otimes \varepsilon(1) \\
 &= 1 \otimes 1.
 \end{aligned}$$

Example 2.1.8. Let V be a $\mathbb{K}$ -vector space with basis $\mathcal{B} = \{v_1, \dots, v_n\}$. Then, we can define a coalgebra structure by $\Delta(v_i) = v_i \otimes v_i, \varepsilon(v_i) = 1$ for all $i \in \{1, \dots, n\}$. In fact, since every linear map is determined by its restriction to a basis of its domain, we may prove that Δ is coassociative by evaluation at basis vectors. In fact,

$$\begin{aligned}
 (\Delta \otimes Id) \circ \Delta(v_i) &= (\Delta \otimes Id)(v_i \otimes v_i) \\
 &= \Delta(v_i) \otimes v_i \\
 &= v_i \otimes v_i \otimes v_i \\
 &= v_i \otimes \Delta(v_i) \\
 &= (Id \otimes \Delta)(v_i \otimes v_i) \\
 &= (Id \otimes \Delta) \circ \Delta(v_i).
 \end{aligned}$$

The first, third, fourth and last equalities follow from the definition of Δ , while the second and fifth equalities follow from the application of the tensor of linear mappings Id and Δ . Hence, we have proved that Δ is coassociative. Now, for the counit property, observe that:

$$\begin{aligned}
 (\varepsilon \otimes Id) \circ \Delta(v_i) &= (\varepsilon \otimes Id)(v_i \otimes v_i) \\
 &= \varepsilon(v_i) \otimes v_i \\
 &= 1 \otimes v_i.
 \end{aligned}$$

The first equality follows from the definition of Δ , the second equality is the application of tensor mapping $\varepsilon \otimes Id$ and the last equality follows from the definition of ε .

Analogously, also observe that:

$$\begin{aligned} (Id \otimes \varepsilon) \circ \Delta(v_i) &= (Id \otimes \varepsilon)(v_i \otimes v_i) \\ &= v_i \otimes \varepsilon(v_i) \\ &= v_i \otimes 1. \end{aligned}$$

Hence, we have proved that the counit property holds.

Observe that, in the above example, we have defined the coalgebra structure in terms of a basis of our $\mathbb{K}$ -vector space. In most examples, we shall define the coproduct and the counit of a coalgebra on a basis.

Example 2.1.9. Let V be a $\mathbb{K}$ -vector space with basis $\{v_n; n \geq 0\}$. We shall define a structure of coalgebra on V by setting $\Delta(v_n) = \sum_{i=0}^n v_i \otimes v_{n-i}$ and $\varepsilon(v_n) = \delta_{0,n}$.

In fact, for proving the coassociativity, we evaluate:

$$\begin{aligned} (\Delta \otimes Id) \circ \Delta(v_n) &= (\Delta \otimes Id) \left(\sum_{i=0}^n v_i \otimes v_{n-i} \right) \\ &= \sum_{i=0}^n \Delta(v_i) \otimes v_{n-i} \\ &= \sum_{i=0}^n \sum_{k=0}^i v_k \otimes v_{i-k} \otimes v_{n-i} \\ &= \sum_{i_1+i_2+i_3=n} v_{i_1} \otimes v_{i_2} \otimes v_{i_3}. \end{aligned}$$

The first and third equalities follow from the definition of Δ , while the second equality follows from the application of the tensor mapping $\Delta \otimes Id$, and the last equality is a rearrangement of the terms.

On the other hand, observe that:

$$\begin{aligned}
(Id \otimes \Delta) \circ \Delta(v_n) &= (Id \otimes \Delta) \left(\sum_{i=0}^n v_i \otimes v_{n-i} \right) \\
&= (Id \otimes \Delta) \left(\sum_{i=0}^n v_{n-i} \otimes v_i \right) \\
&= \sum_{i=0}^n v_{n-i} \otimes \Delta(v_i) \\
&= \sum_{i=0}^n \sum_{k=0}^i v_{n-i} \otimes v_k \otimes v_{i-k} \\
&= \sum_{i_1+i_2+i_3=n} v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \\
&= (\Delta \otimes Id) \circ \Delta(v_n).
\end{aligned}$$

The first and fourth equalities follow from the definition of Δ , while the second and fifth equalities are just a rearrangement of the terms, the third equality is the application of the tensor mapping $Id \otimes \Delta$, and the last equality follows from the previous computation. Hence, Δ is coassociative.

Now, we prove the counit property. Evaluating at v_n ,

$$\begin{aligned}
(\varepsilon \otimes Id) \circ \Delta(v_n) &= (\varepsilon \otimes Id) \left(\sum_{i=0}^n v_i \otimes v_{n-i} \right) \\
&= \sum_{i=0}^n \varepsilon(v_i) \otimes v_{n-i} \\
&= \sum_{i=0}^n \delta_{0,i} \otimes v_{n-i} \\
&= 1 \otimes v_n.
\end{aligned}$$

The first equality follows from the definition of Δ , while the second equality follows from the application of the tensor mapping $\varepsilon \otimes Id$, the third equality follows from the definition of ε , and the last equality follows from the definition of $\delta_{0,i}$.

Analogously, for the second equality of the counit property, also evaluating at v_n , we have:

$$\begin{aligned}
(Id \otimes \varepsilon) \circ \Delta(v_n) &= (Id \otimes \varepsilon) \left(\sum_{i=1}^n v_i \otimes v_{n-i} \right) \\
&= \sum_{i=1}^n v_i \otimes \varepsilon(v_{n-i}) \\
&= \sum_{i=1}^n v_i \otimes \delta_{0,n-i} \\
&= v_n \otimes 1.
\end{aligned}$$

Hence, the counit property holds, and V is a $\mathbb{K}$ -coalgebra.

Example 2.1.10. Let A be a $\mathbb{K}$ -vector space with basis $\mathcal{B} = \{v_i, i \in I\} \cup \{u\}$, where I is a nonempty set (which can be finite or infinite). We may define a structure of coalgebra in A as follows:

$$\begin{aligned}\Delta(v_i) &= v_i \otimes u + u \otimes v_i, \\ \Delta(u) &= u \otimes u, \\ \varepsilon(v_i) &= 0, \\ \varepsilon(u) &= 1.\end{aligned}$$

As before, we shall prove the coassociativity of Δ and the counit property of ε by evaluating operators at basis vectors. Starting with the coassociativity

$$\begin{aligned}(\Delta \otimes Id) \circ \Delta(v_i) &= (\Delta \otimes Id)(v_i \otimes u + u \otimes v_i) \\ &= \Delta(v_i) \otimes u + \Delta(u) \otimes v_i \\ &= (v_i \otimes u + u \otimes v_i) \otimes u + \Delta(u) \otimes v_i \\ &= (v_i \otimes u + u \otimes v_i) \otimes u + u \otimes u \otimes v_i \\ &= v_i \otimes u \otimes u + u \otimes v_i \otimes u + u \otimes u \otimes v_i.\end{aligned}$$

The first, third and fourth equalities follow from the definition of Δ , while the second equality is just an application of the tensor mapping $\Delta \otimes Id$, and the last equality is an application of the multilinearity of the universal property of tensor vector spaces.

But, also observe that:

$$\begin{aligned}(Id \otimes \Delta) \circ \Delta(v_i) &= (Id \otimes \Delta)(v_i \otimes u + u \otimes v_i) \\ &= v_i \otimes \Delta(u) + u \otimes \Delta(v_i) \\ &= v_i \otimes u \otimes u + u \otimes (v_i \otimes u + u \otimes v_i) \\ &= v_i \otimes u \otimes u + u \otimes v_i \otimes u + u \otimes u \otimes v_i \\ &= (\Delta \otimes Id) \circ \Delta(v_i).\end{aligned}$$

The first, and third equalities follow from the definition of Δ , the second is the application of the tensor mapping $Id \otimes \Delta$, while the fourth equality follows from the application of the multilinearity of the universal property of the tensor, and the last equality follows from the previous computation.

Now, also observe that:

$$\begin{aligned}
(\Delta \otimes Id) \circ \Delta(u) &= (\Delta \otimes Id)(u \otimes u) \\
&= \Delta(u) \otimes u \\
&= u \otimes u \otimes u \\
&= u \otimes \Delta(u) \\
&= (Id \otimes \Delta)(u \otimes u) \\
&= (Id \otimes \Delta) \circ \Delta(u).
\end{aligned}$$

The first, third, fourth and last equalities follow from the definition of Δ , while the second and fifth equalities follow from the application of the tensor mappings $\Delta \otimes Id$ and $Id \otimes \Delta$, respectively.

Hence, Δ is coassociative.

Now, observe that:

$$\begin{aligned}
(\varepsilon \otimes Id) \circ \Delta(v_i) &= (\varepsilon \otimes Id)(v_i \otimes u + u \otimes v_i) \\
&= \varepsilon(v_i) \otimes u + \varepsilon(u) \otimes v_i \\
&= 1 \otimes v_i.
\end{aligned}$$

The first equality follows from the definition of Δ , while the second equality follows from the application of the tensor mapping $\varepsilon \otimes Id$, and the last equality follows from the definition of ε .

Also, observe that:

$$\begin{aligned}
(Id \otimes \varepsilon) \circ \Delta(v_i) &= (Id \otimes \varepsilon)(v_i \otimes u + u \otimes v_i) \\
&= v_i \otimes \varepsilon(u) + u \otimes \varepsilon(v_i) \\
&= v_i \otimes 1.
\end{aligned}$$

The first equality follows from the definition of Δ , while the second equality follows from the application of the tensor mapping $Id \otimes \varepsilon$, and the last equality follows from the definition of ε .

In the same manner, we have that

$$\begin{aligned}
(\varepsilon \otimes Id) \circ \Delta(u) &= (\varepsilon \otimes Id)(u \otimes u) \\
&= \varepsilon(u) \otimes u \\
&= 1 \otimes u.
\end{aligned}$$

The first equality follows from the definition of Δ for u , while the second equality is just an application of $\varepsilon \otimes Id$, and the third equality follows from the definition of ε . Analogously, we have that $Id \otimes \varepsilon \circ \Delta(u) = u \otimes 1$.

2.2 PROPERTIES OF $\mathbb{K}$ -ALGEBRAS AND $\mathbb{K}$ -COALGEBRAS

Before we can go on, we must introduce a map which will allow us to define tensor products of algebras, tensor products of coalgebras, bialgebras and then, Hopf Algebras. This functions "interchange places" in a tensor product.

Definition 2.2.1. (*Twist map*) Let V, W be given $\mathbb{K}$ -vector spaces. We define:

$$\tau : V \otimes W \rightarrow W \otimes V,$$

$$\tau(v \otimes w) = w \otimes v,$$

to be the **twist map** of V and W .

The existence of the twist map follows from the universal property of the tensor product: for proving it, we must begin with the bilinear function $f : V \times W \rightarrow W \otimes V$ defined as $f(v, w) = w \otimes v$ which then induces the twist function $\tau : V \otimes W \rightarrow W \otimes V$ due to the universal property of tensor products. The next definition is a first application of the twist function:

Definition 2.2.2. (*Commutative algebra*) Let A be a given $\mathbb{K}$ -algebra, and let $m : A \otimes A \rightarrow A$ be its product map. We say this product is **commutative** if:

$$m \circ \tau = m,$$

where $\tau : A \otimes A \rightarrow A \otimes A$ is the twist map on $A \otimes A$. We also say that the algebra A is commutative.

We can easily see that this definition of commutativity for an $\mathbb{K}$ -algebra is equivalent to the usual notion of commutativity for a ring. In fact, the algebra A is commutative as a ring if $xy = yx$ for all $x, y \in A$, which means that

$$m \circ \tau(x \otimes y) = m(y \otimes x) = m(x \otimes y).$$

Example 2.2.1. Examples of commutative algebras are the $\mathbb{R}$ -algebra $\mathbb{C}$ and, for a field $\mathbb{K}$, the $\mathbb{K}$ -algebra of polynomials $\mathbb{K}[t]$.

Example 2.2.2. An example of non-commutative algebra is the algebra of square matrices $M_n(\mathbb{K})$.

The next definition is obtained by dualizing the notion of commutativity:

Definition 2.2.3. (Cocommutative coalgebra) Let C be a given $\mathbb{K}$ -coalgebra, and $\Delta : C \rightarrow C \otimes C$ its coproduct map. We say this coproduct is **cocommutative** if:

$$\tau \circ \Delta = \Delta,$$

where $\tau : C \otimes C \rightarrow C \otimes C$ is the twist function on $C \otimes C$. We also say that C is a cocommutative coalgebra.

All the examples of $\mathbb{K}$ -coalgebras which appeared so far are examples of cocommutative coalgebras. Later we shall present an example of a coalgebra such that its coproduct is noncocommutative.

One point to observe is that the notation of the product in an algebra may be simplified: for two vectors v, w , we may just write $v \cdot w$ or vw . There is no simple solution, in terms of notation, for the coproduct of an element of a coalgebra C . However, R. Heyneman and M. Sweedler had solved this problem by introducing the **Sweedler notation**. Here is how we represent the coproduct of an element $c \in C$ using such notation:

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}.$$

This notation may look confusing in a first contact, but it becomes easier to understand if we consider the following useful fact: the coproduct takes a vector in C to a vector in the tensor product $C \otimes C$, and this vector need not to be a homogeneous tensor. Hence, given $c \in C$, we may write

$$\Delta(c) = \sum_{j=1}^n c_{(j,1)} \otimes c_{(j,2)},$$

and we obtain the Sweedler notation by dropping the index “ j ” in the summands and switching “ j ” by “ (c) ” under the summation symbol.

Observe that, since Δ is coassociative, i.e. $(\Delta \otimes Id) \circ \Delta(c) = (Id \otimes \Delta) \circ \Delta(c)$, we have the equalities:

$$\begin{aligned} & \sum_{(c)} \sum_{(c_{(1)})} c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \\ & = (\Delta \otimes Id) \circ \Delta(c) = (Id \otimes \Delta) \circ \Delta(c) \\ & = \sum_{(c)} \sum_{(c_{(2)})} c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}. \end{aligned}$$

So, we can define $\Delta_2 : C \rightarrow C \otimes C \otimes C$ as $\Delta_2(c) = (\Delta \otimes Id) \circ \Delta(c) = (Id \otimes \Delta) \circ \Delta(c)$, and we are allowed to introduce a further simplification of the Sweedler notation for Δ_2 :

$$\Delta_2(c) = (\Delta \otimes Id) \circ \Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}.$$

For the counit map, we have:

$$\begin{aligned} c \otimes 1 &= \sum_{(c)} c_{(1)} \otimes \varepsilon(c_{(2)}), \\ 1 \otimes c &= \sum_{(c)} \varepsilon(c_{(1)}) \otimes c_{(2)}. \end{aligned}$$

But, abusing notation, we shall usually make the identifications $c = c \otimes 1$ and $c = 1 \otimes c$ by the canonical isomorphisms $C \simeq C \otimes \mathbb{K}$ and $C \simeq \mathbb{K} \otimes C$, and just write:

$$c = \sum_{(c)} c_{(1)} \otimes \varepsilon(c_{(2)}) = \sum_{(c)} \varepsilon(c_{(1)}) \otimes c_{(2)}.$$

We may generalize Δ_2 to include more applications of the coproduct, and so define a linear map $\Delta_n : C \rightarrow C^{\otimes (n+1)}$ recursively:

$$\Delta_n(c) = (\Delta \otimes Id^{n-1}) \otimes \Delta_{n-1}(c).$$

The next result provides expressions for Δ_n that constitute a generalization of the coassociative law:

Proposition 2.2.1. *Let (C, Δ, ε) a coalgebra. Then, for all $n \geq 2$ and $0 \leq p \leq n-1$:*

$$\Delta_n = (id^p \otimes \Delta \otimes id^{n-1-p}) \circ \Delta_{n-1}.$$

Proof. The proof of this comes from induction in p and in n , as follows:

For the proof, we are going to do it by induction, beginning with $n = 2$. But for $n = 2$, the proposition is already proved, since the coassociativity holds.

So, let us assume that the result holds for some $n \geq 2$. We are going to prove that it holds for $n+1$, proving it by induction in p . By the definition of Δ_{n+1} , we have:

$$\Delta_{n+1} = (\Delta \otimes Id^n) \otimes \Delta_n.$$

So it holds for $p = 0$.

Assume it holds for a chosen $p, 0 < p < n+1$. We will show it also holds for $p+1$.

So, we have:

$$\Delta_{n+1} = (id^p \otimes \Delta \otimes id^{n-p}) \circ \Delta_n.$$

But we can also write:

$$\Delta_n = (id^p \otimes \Delta \otimes id^{n-p-1}) \circ \Delta_{n-1}.$$

Therefore,

$$\begin{aligned}
\Delta_{n+1} &= (Id^p \otimes (\Delta \otimes Id) \otimes Id^{n-p-1}) \circ (Id^p \otimes \Delta \otimes Id^{n-p-1}) \circ \Delta_{n-1} \\
&= (Id^p \circ Id^p) \otimes (\Delta \otimes Id) \circ \Delta \otimes (Id^{n-p-1} \circ Id^{n-p-1}) \circ \Delta_{n-1} \\
&= (Id^p \circ Id^p) \otimes (Id \otimes \Delta) \circ \Delta \otimes (Id^{n-p-1} \circ Id^{n-p-1}) \circ \Delta_{n-1} \\
&= (Id^{p+1} \otimes \Delta) \otimes Id^{n-p-1} \circ \Delta_{n-1}.
\end{aligned}$$

And that follows from coassociativity. Hence, we have proved that the proposition holds for all p such that $0 \leq p \leq n+1$. $\square$

2.3 DUALITY FOR $\mathbb{K}$ -ALGEBRAS AND $\mathbb{K}$ -COALGEBRAS

Since the definitions of algebras and coalgebras are dual to each other, if we have a finite-dimensional algebra, we may define the dual finite-dimensional coalgebra, and vice-versa. The following definitions and propositions are essential for these constructions.

Let V be a $\mathbb{K}$ -vector space and $f \in Hom(V, \mathbb{K}) = V^*$ be given. Let us observe that it is possible to define a bilinear form:

$$\langle , \rangle : V^* \times V \rightarrow \mathbb{K}, \quad \langle f, v \rangle = f(v).$$

Let us also consider, given vector spaces V, W , the linear map given by:

$$\iota_{V,W} : V^* \otimes W^* \rightarrow (V \otimes W)^*, \quad \iota_{V,W}(f \otimes g)(v \otimes w) = \langle f, v \rangle \langle g, w \rangle.$$

First, we need to justify that this transformation is well defined. In what follows, when there is no risk of confusion, we will drop the indices and write ι for $\iota_{V,W}$.

Defining $\iota(f \otimes g) : (V \otimes W) \rightarrow \mathbb{K}$ by $\iota(f \otimes g)(v \otimes w) = \langle f, v \rangle \langle g, w \rangle$, we need to prove that this functional is well-defined. But, if we define the bilinear form $f \times g : V \times W \rightarrow \mathbb{K}$ by $(f \times g)(v, w) = \langle f, v \rangle \langle g, w \rangle$, there is a unique $\iota(f \otimes g) : V \otimes W \rightarrow \mathbb{K}$ such that $\iota(f \otimes g)(v \otimes w) = (f \times g)(v, w)$. By existence and uniqueness, ι is well defined. By construction, ι is also linear. Hence the existence and uniqueness of such a function is proved. A remarkable fact is that ι is one-to-one.

Proposition 2.3.1. *Let V, W be $\mathbb{K}$ -vector spaces. Then, the map $\iota : V^* \otimes W^* \rightarrow (V \otimes W)^*$ defined as:*

$$\iota(f \otimes g)(v \otimes w) = \langle f, v \rangle \langle g, w \rangle.$$

is one-to-one.

Proof. In order to see that ι is one-to-one, we will compute its kernel. Let

$$\sum_{i=1}^n f_i \otimes g_i \in \text{Ker}(\iota) \subset V^* \otimes W^*$$

and assume that not all f_j are zero.

If $\{f_1, \dots, f_n\}$ is linearly dependent (L.D), then there's a functional f_j such that $f_j = \sum_{i \neq j} \alpha_i f_i$ and then we can rewrite the expression above as:

$$\sum_{i \neq j} f_i \otimes g_i + \left(\sum_{i \neq j} \alpha_i f_i \right) \otimes g_j = \sum_{i \neq j} f_i \otimes (g_i + \alpha_i g_j).$$

We may repeat this process until we have obtained an *L.I.* set, so we may assume that $\{f_1, \dots, f_n\}$ is linearly independent (L.I) from the start. So, for such $\{f_1, \dots, f_n\}$ there are $v_1, \dots, v_n \in V$ such that $f_i(v_j) = \delta_{ij}$.

But $\sum_{i=1}^n f_i \otimes g_i \in \text{Ker}(\iota)$, so, for each v_j and each $w \in W$ we have:

$$\sum_{i=1}^n \langle f_i, v_j \rangle \langle g_i, w \rangle = 0,$$

or

$$\langle f_j, v_j \rangle \langle g_j, w \rangle = 0, \quad \forall j = 1, \dots, n.$$

Since the expression above holds for every $w \in W$ and $\langle f_j, v_j \rangle = f_j(v_j) = 1$, we have $g_j = 0$.

As $g_1 = g_2 = \dots = g_n = 0$, it follows that:

$$\sum_{i=1}^n f_i \otimes g_i = \sum_{i=1}^n f_i \otimes 0 = 0.$$

Since the element above in $\text{Ker}(\iota)$ is arbitrary, we have $\text{Ker}(\iota) = \{0\}$. So, ι is one-to-one. $\square$

Note that the above proof did not assume that V and W are finite-dimensional. However, if V and W are finite-dimensional, we have that ι is bijective, since in this case both $V^* \otimes W^*$ and $(V \otimes W)^*$ have the same finite dimension, which is $\dim(V) \cdot \dim(W)$.

The dual vector space of a coalgebra has a natural algebra structure, as we prove in the following.

Definition 2.3.1. (*Transpose map*) If $\phi : V \rightarrow W$ is a linear map, then the **transpose** of ϕ is the linear map $\phi^* : W^* \rightarrow V^*$ defined as:

$$\langle \phi^*(f), v \rangle = \langle f, \phi(v) \rangle.$$

We recall that, for the field $\mathbb{K}$, we have the isomorphism $\Phi : \mathbb{K} \rightarrow \mathbb{K}^*$ given as:

$$\langle \Phi(\alpha), \beta \rangle = \alpha\beta.$$

Its inverse is the isomorphism $\Phi^{-1} : \mathbb{K}^* \rightarrow \mathbb{K}$ given by:

$$\Phi^{-1}(f) = \langle f, 1 \rangle.$$

Lemma 2.3.1. *(The dual algebra of a coalgebra) If (C, Δ, ε) is a coalgebra with co-product Δ and counit ε , then C^* is a algebra, with product and unit given by $m = \Delta^* \circ \iota$ and $\mu = \varepsilon^* \circ \Phi$.*

Proof. Let $f, g \in C^*$ be given, and let us denote $m(f \otimes g) = f * g$. We need to prove that this product is associative and that μ is a unit. For any $c \in C$:

$$\langle f * g, c \rangle = \langle \Delta^*(\iota(f \otimes g)), c \rangle.$$

By the definition of m , and the definition of transpose map, we may write:

$$\langle \Delta^*(\iota(f \otimes g)), c \rangle = \langle \iota(f \otimes g), \Delta(c) \rangle.$$

Using sigma notation, we may write

$$\langle \iota(f \otimes g), \Delta(c) \rangle = \langle \iota(f \otimes g), \sum_{(c)} c_{(1)} \otimes c_{(2)} \rangle.$$

And, since $\langle, \rangle$ is linear, we get

$$\langle f * g, c \rangle = \langle \iota(f \otimes g), \sum_{(c)} c_{(1)} \otimes c_{(2)} \rangle = \sum_{(c)} \langle \iota(f \otimes g), c_{(1)} \otimes c_{(2)} \rangle = \sum_{(c)} \langle f, c_{(1)} \rangle \langle g, c_{(2)} \rangle.$$

Now, we shall prove the coassociativity using this expression. In fact, evaluating at $c \in C$, we have:

$$\begin{aligned} \langle (f * g) * h, c \rangle &= \sum_{(c)} \langle (f * g), c_{(1)} \rangle \langle h, c_{(2)} \rangle \\ &= \sum_{(c)} \langle f, c_{(1)} \rangle \langle g, c_{(2)} \rangle \langle h, c_{(3)} \rangle. \end{aligned}$$

But we also have:

$$\begin{aligned} \langle f * (g * h), c \rangle &= \sum_{(c)} \langle f, c_{(1)} \rangle \langle (g * h), c_{(2)} \rangle \\ &= \sum_{(c)} \langle f, c_{(1)} \rangle \langle g, c_{(2)} \rangle \langle h, c_{(3)} \rangle \\ &= \langle (f * g) * h, c \rangle. \end{aligned}$$

For the proof of the counit property, note that, for $\alpha \in \mathbb{K}$ and $c \in C$, it follows from Definition 2.3.1 and the definition of Φ that:

$$\begin{aligned}\langle \mu(\alpha), c \rangle &= \langle \varepsilon^*(\Phi(\alpha)), c \rangle \\ &= \langle \Phi(\alpha), \varepsilon(c) \rangle \\ &= \alpha \varepsilon(c).\end{aligned}$$

In particular, we obtain:

$$\langle \mu(1), c \rangle = \varepsilon(c).$$

Hence, given $f \in C^*$ and $c \in C$, we have that

$$\begin{aligned}\langle f * \mu(1), c \rangle &= \sum_{(c)} \langle f, c_{(1)} \rangle \langle \mu(1), c_{(2)} \rangle \\ &= \sum_{(c)} \langle f, c_{(1)} \varepsilon(c_{(2)}) \rangle \\ &= \langle f, \sum_{(c)} c_{(1)} \varepsilon(c_{(2)}) \rangle \\ &= \langle f, c \rangle \\ &= f(c)\end{aligned}$$

Analogously, we obtain $\langle \mu(1) * f, c \rangle = f(c)$. It follows that μ is the unit map in C^* and $1_{C^*} = \mu(1) = \varepsilon$. $\square$

Corollary 2.3.1. *The counit of a $\mathbb{K}$ -coalgebra is unique.*

Proof. Let $\varepsilon, \varepsilon'$ be two counit maps of C with coproduct Δ . Then, due to Lemma 2.3.1, C^* is an algebra and $\mu' = \varepsilon^* \circ \Phi$ and $\mu'' = \varepsilon'^* \circ \Phi$ are units for C^* . However, in an algebra, the unit map is unique. Hence, $\mu' = \mu''$.

Then, for $c \in C$:

$$\varepsilon'(c) = \langle \mu''(1), c \rangle = \langle \mu'(1), c \rangle = \varepsilon(c).$$

Hence, the counit is unique. $\square$

The next result is analogous to the previous one; now we start with an algebra and we construct a coalgebra structure on its dual; however, the algebra must be finite-dimensional.

Proposition 2.3.2. *(The dual coalgebra of a finite-dimensional algebra) Let (A, m, μ) be a finite-dimensional algebra. $(A^*, \Delta, \varepsilon)$ is a coalgebra with the coproduct given by $\Delta = \iota^{-1} \circ m^*$ and counit map given by $\varepsilon = \Phi^{-1} \circ \mu^*$.*

Although the statement of this proposition is analogous to the previous one, its proof is not. Until now we haven't used the fact that the dimension we are dealing with is finite. But now, we are going to need this, since we need the existence of the inverse map $\iota^{-1} : (A \otimes A)^* \rightarrow A^* \otimes A^*$.

Proof. Let $\Delta : A^* \rightarrow A^* \otimes A^*$ be the linear mapping $\Delta = \iota^{-1} \circ m^*$ and $\varepsilon : A^* \rightarrow \mathbb{K}$ be the functional $\varepsilon = \Phi^{-1} \circ \mu^*$. Consider the linear mapping $\theta : A^* \otimes A^* \otimes A^* \rightarrow (A \otimes A \otimes A)^*$ given by:

$$\theta(f \otimes g \otimes h)(a \otimes b \otimes c) = \langle f, a \rangle \langle g, b \rangle \langle h, c \rangle$$

First, observe that, by definition of $\Delta = \iota^{-1} \circ m^*$ in the dual coalgebra, we have that:

$$\begin{aligned} \langle \Delta(f), a \otimes b \rangle &= \sum_{(f)} \langle f_{(1)}, a \rangle \langle f_{(2)}, b \rangle \\ &= f(ab) \\ &= \langle f, ab \rangle \end{aligned}$$

Observe that $\theta = \iota_{A, A \otimes A} \circ (Id \otimes \iota_{A, A})$, is a composition of one-to-one functions. Hence, θ is a one-to-one function. Then we get, for $f \in A^*$:

$$\begin{aligned} (\Delta \otimes Id) \circ \Delta(f) &= (\Delta \otimes Id) \left(\sum_{(f)} f_{(1)} \otimes f_{(2)} \right) \\ &= \sum_{(f)} \sum_{(f_{(1)})} f_{(1)(1)} \otimes f_{(1)(2)} \otimes f_{(2)} \end{aligned}$$

and

$$\begin{aligned} (Id \otimes \Delta) \circ \Delta(f) &= (Id \otimes \Delta) \left(\sum_{(f)} f_{(1)} \otimes f_{(2)} \right) \\ &= \sum_{(f)} \sum_{(f_{(2)})} f_{(1)} \otimes f_{(2)(1)} \otimes f_{(2)(2)} \end{aligned}$$

Applying θ to the first expression and evaluating at $a \otimes b \otimes c \in A \otimes A \otimes A$, we get:

$$\begin{aligned} \left\langle \theta \left(\sum_{(f)} \sum_{(f_{(1)})} f_{(1)(1)} \otimes f_{(1)(2)} \otimes f_{(2)} \right), a \otimes b \otimes c \right\rangle &= \sum_{(f)} \sum_{(f_{(1)})} \langle f_{(1)(1)}, a \rangle \langle f_{(1)(2)}, b \rangle \langle f_{(2)}, c \rangle \\ &= \sum_{(f)} \langle f_{(1)}, ab \rangle \langle f_{(2)}, c \rangle \\ &= f(abc). \end{aligned}$$

Doing the same for the second expression, we have:

$$\begin{aligned}
 \left\langle \theta \left(\sum_{(f)} f_{(1)} \otimes f_{(2)(1)} \otimes f_{(2)(2)} \right), a \otimes b \otimes c \right\rangle &= \sum_{(f)} \sum_{(f)_{(2)}} \langle f_{(1)}, a \rangle \langle f_{(2)(1)}, b \rangle \langle f_{(2)(2)}, c \rangle \\
 &= \sum_{(f)} \langle f_{(1)}, a \rangle \langle f_{(2)}, bc \rangle \\
 &= f(abc) \\
 &= \left\langle \theta \left(\sum_{(f)} \sum_{(f_{(1)})} f_{(1)(1)} \otimes f_{(1)(2)} \otimes f_{(2)} \right), a \otimes b \otimes c \right\rangle.
 \end{aligned}$$

Since θ is one-to-one, and by the previous equalities, we conclude that

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta.$$

Hence coassociativity is proven.

For the proof of the counit property, note that:

$$\varepsilon(f) = \Phi^{-1}(\mu^*(f)) = (\mu^*(f))(1) = f(\mu(1)) = f(1_A),$$

hence

$$\begin{aligned}
 \left\langle \sum_f \varepsilon(f_{(1)}) f_{(2)}, a \right\rangle &= \sum_{(f)} \varepsilon(f_{(1)}) \langle f_{(2)}, a \rangle \\
 &= \sum_{(f)} \langle f_{(1)}, 1_A \rangle \langle f_{(2)}, a \rangle \\
 &= f(1_A a) = f(a).
 \end{aligned}$$

Since this holds for every $a \in A$ we conclude that

$$\sum_{(f)} \varepsilon(f_{(1)}) f_{(2)} = f.$$

The other side is analogous. Hence, the counit property is proven. $\square$

2.4 MORPHISMS OF $\mathbb{K}$ -ALGEBRAS AND $\mathbb{K}$ -COALGEBRAS

In linear algebra, we do not deal only with vector spaces, but also with linear mappings between them, which we may also call morphisms of vector spaces, which preserve the operations of vector sum and scalar multiplication. Then, when we deal with algebras and coalgebras, we also deal with morphisms of algebras and coalgebras, as we define in this section.

Definition 2.4.1. (*Morphism of $\mathbb{K}$ -algebras*)

Let (A, m_1, μ_1) and (B, m_2, μ_2) be both $\mathbb{K}$ -algebras. Given a linear map $f : A \rightarrow B$, we say f is a **morphism of $\mathbb{K}$ -algebras** if the following diagrams commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m_1} & A \\ f \otimes f \downarrow & & \downarrow f \\ B \otimes B & \xrightarrow{m_2} & B \end{array} \qquad \begin{array}{ccc} A & \xleftarrow{\mu_1} & \mathbb{K} \\ f \downarrow & \swarrow \mu_2 & \\ B & & \end{array}$$

Note that, in the definition of morphisms of $\mathbb{K}$ -algebras, the first diagram states that $f(ab) = f(a)f(b)$ and the second one means that $f(1_A) = 1_B$.

Then, let us give some examples of morphisms of algebras.

Example 2.4.1. Some inclusions are morphisms of algebras. For example, the inclusion $i : \mathbb{R} \rightarrow \mathbb{C}$ is a morphism of algebras. Another example is the unit morphism in the algebra of square matrices $\mu : \mathbb{K} \rightarrow M_n(\mathbb{K})$

Example 2.4.2. Another example of morphism of algebras is $P(\alpha) : \mathbb{K}[t] \rightarrow \mathbb{K}$ with $\alpha \in \mathbb{K}$ given by $P(\alpha)(a_0 + a_1t + \cdots + a_nt^n) = a_0 + a_1\alpha + \cdots + a_n\alpha^n$. Basically, it is the polynomial evaluated at α .

Example 2.4.3. Recall from Example 2.1.6 that if V is a $\mathbb{K}$ -vector space then we have the tensor algebra

$$T(V) = \bigoplus_{n=0}^{\infty} V^n,$$

where $V^0 = \mathbb{K}$, $V^1 = V$ and, recursively, $V^n = V^{n-1} \otimes V$. The tensor algebra $T(V)$ satisfies the following universal property: given another $\mathbb{K}$ -algebra A and a linear mapping $f : V \rightarrow A$, there exists a unique morphism of algebras $\bar{f} : T(V) \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{i} & T(V) \\ & \searrow f & \downarrow \bar{f} \\ & & A \end{array}$$

In the above diagram, $i : V \rightarrow T(V)$ is the inclusion mapping. In fact, let $\mathcal{B} = \{v_i\}$ be a basis for V . Then, f is determined by the set $\{f(v_i)\}$. Since the set $\{i(v_i)\}$ generates $T(V)$ as an algebra, if there exists a morphism of algebras $\bar{f} : T(V) \rightarrow A$ such that $\bar{f}(i(v_i)) = f(v_i)$ for every i then $\bar{f}$ is unique. Now the existence of such a morphism can be shown using properties of tensor products and the definition of the product of $T(V)$. This is just an idea of how to prove that the universal property holds. For a detailed proof, see (RADFORD, 2011).

The next definition is dual to the previous one, as follows:

Definition 2.4.2. (*Morphism of $\mathbb{K}$ -coalgebras*) Let $(C, \Delta_1, \varepsilon_1)$ and $(D, \Delta_2, \varepsilon_2)$ be both coalgebras. Given a linear map $f : C \rightarrow D$, we say that f is a **morphism of $\mathbb{K}$ -coalgebras** if the following diagrams commute:

$$\begin{array}{ccc} D \otimes D & \xleftarrow{\Delta_D} & D \\ f \otimes f \uparrow & & \uparrow f \\ C \otimes C & \xleftarrow{\Delta_C} & C \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{\varepsilon_D} & \mathbb{K} \\ f \uparrow & \nearrow \varepsilon_C & \\ C & & \end{array}$$

The main examples of morphisms of $\mathbb{K}$ -coalgebras are related to bialgebras, and then we shall discuss such examples later.

We can obtain a better intuition on the meaning of the above definition by rewriting it on elements, using the sigma notation. In sigma notation, the first diagram states that, for arbitrary $c \in C$:

$$\sum_{(f(c))} f(c)_{(1)} \otimes f(c)_{(2)} = \sum_{(c)} f(c_{(1)}) \otimes f(c_{(2)}).$$

And the second diagram states that $\varepsilon_D(f(c)) = \varepsilon_C(c)$.

As we may define morphism of coalgebras, it is natural to think about obtaining other coalgebra structures, such as quotient coalgebras, for example. Some constructions of coalgebras in terms morphisms are the constructions of subcoalgebras and quotient coalgebras.

The following definition is intended not only to remind the well-known concepts of ideals and subalgebras of a $\mathbb{K}$ -algebra, but also to present these structures in a manner that is related to the dual definitions of a "coideal" and a "subcoalgebra" of a $\mathbb{K}$ -coalgebra.

Definition 2.4.3. Let (A, m, μ) be an $\mathbb{K}$ -algebra. Then:

- A subspace I of A such that $m(I \otimes A + A \otimes I) \subset I$ is called an **ideal**.
- A subspace B of A such that $m(B \otimes B) \subset B$ and $\mu(1) \in B$ is called a **subalgebra** of A .

Example 2.4.4. Let $\mathbb{H}$ be the $\mathbb{R}$ -algebra of quaternions. Then, the field of complex numbers $\mathbb{C} = \{a + bi; a, b \in \mathbb{R}\}$ is a subalgebra of $\mathbb{H}$. It follows that $\mathbb{H}$ is also a $\mathbb{C}$ -vector space; however, $\mathbb{H}$ is not a $\mathbb{C}$ -algebra because its product is not $\mathbb{C}$ -bilinear.

Example 2.4.5. Let $M_n(\mathbb{K})$ be the algebra of square matrices of order n . Then, the subspace of upper triangular matrices is a subalgebra of $M_n(\mathbb{K})$.

Example 2.4.6. Every $\mathbb{K}$ -algebra can be identified with a subalgebra of an algebra of linear endomorphisms of a $\mathbb{K}$ -vector space. In fact, consider, for a $\mathbb{K}$ -algebra A , the linear transformation given by:

$$\begin{aligned} T : A &\rightarrow \text{End}(A), \\ a &\mapsto T_a : b \mapsto ab. \end{aligned}$$

Note that $T(ab)(c) = T_{ab}(c) = (ab)c = a(bc) = T_a(T_b(c))$, $T(1) = Id$, and therefore T is a morphism of algebras. If $T(a) = 0$ then $T_a(b) = 0$ for every b , which implies that $ab = 0$ for every b , and then $a = a1_A = T_a(1_A) = 0$. Hence, T is injective. Thus, we can see T as an inclusion of algebras.

Example 2.4.7. Let $\mathbb{K}[t]$ be the $\mathbb{K}$ -algebra of polynomials in $\mathbb{K}$. Then, given a polynomial $p(t) \in \mathbb{K}[t]$, the set $p(t)\mathbb{K}[t] = \{p(t)q(t); q(t) \in \mathbb{K}[t]\}$ is an ideal of $\mathbb{K}[t]$.

Example 2.4.8. Let A be a $\mathbb{K}$ -algebra and $\{I_l; l \in L\}$ be a collection of ideals in A . Then, $I = \bigcap_{l \in L} I_l$ is also an ideal in A . In fact, since I is an intersection of subspaces of A , it is also a subspace of A . If $x \in I$ then x belongs to each ideal I_l . By the definition of ideal, if $a \in A$ then $xa, ax \in I_l$ for every ideal I_l and hence, $xa, ax \in I$. Hence, I is an ideal of A .

Example 2.4.9. Given a subset $X \subset A$, the **ideal generated by X** is the ideal:

$$\langle X \rangle = \bigcap_{X \subset I, I \text{ is an ideal of } A} I.$$

The next definition is dual to the definitions of subalgebra and ideal, as follows:

Definition 2.4.4. Let (C, Δ, ε) be a $\mathbb{K}$ -coalgebra. Then:

- A subspace I of C such that $\Delta(I) \subset C \otimes I + I \otimes C$ and $\varepsilon(I) = \{0\}$ is called a **coideal**.
- A subspace D of C such that $\Delta(D) \subset D \otimes D$ is called a **subcoalgebra** of C .

Example 2.4.10. In the coalgebra introduced in Example 2.1.8, which is a space V with basis $\mathcal{B} = \{v_1, \dots, v_n\}$ and coalgebra structure defined by $\Delta(v_i) = v_i \otimes v_i$ and $\varepsilon(v_i) = 1$ for all $i \in \{1, \dots, n\}$, we have that any subspace generated by a subset of $\mathcal{B}$ is a subcoalgebra. In fact, observe that if V' is such a subspace and $v_i \in V'$, we have that:

$$\Delta(v_i) = v_i \otimes v_i \in V' \otimes V',$$

which means, since the v_i 's in V' generate V' , that $\Delta(V') \subset V' \otimes V'$.

Example 2.4.11. In Example 2.1.9, we introduced a coalgebra which is a $\mathbb{K}$ -vector space V with basis $\mathcal{B} = \{v_n, n \geq 0\}$, $\Delta(v_n) = v_n \otimes v_{n-i}$ and $\varepsilon(v_n) = \delta_{0,n}$, we claim that the subspace V' generated by $\mathcal{B}' = \{v_n, n > 0\}$ is a coideal of V . In fact, for $v_n, n > 0$, we have that:

$$\begin{aligned}\Delta(v_n) &= \sum_{i=0}^n v_i \otimes v_{n-i} \\ &= \sum_{i=1}^n v_i \otimes v_{n-i} + v_0 \otimes v_n \in V \otimes V' + V' \otimes V.\end{aligned}$$

The first equality follows from the definition of Δ , while the second equality follows from a separation of the first and last terms of the sum, and the belonging follows from the fact that $v_i \in V'$ for $i > 0$ and $V' \subset V$. Now, for every v_n with $n > 0$, we have that $\varepsilon(v_n) = \delta_{n,0} = 0$. Hence, since v_n with $n > 0$ generates V' , we have that $\varepsilon(V') = \{0\}$. Hence, V' is a coideal of V .

In what follows we will investigate the kernel and image of a morphism of $\mathbb{K}$ -coalgebras, we will define quotient coalgebras and prove an analogous of the First Isomorphism Theorem for rings in the context of $\mathbb{K}$ -coalgebras. We begin by proving a fundamental result in linear algebra which describes the kernel of a tensor product of $\mathbb{K}$ -linear maps.

Proposition 2.4.1. *Let V_1, V_2, W_1, W_2 be $\mathbb{K}$ -vector spaces and $f : V_1 \rightarrow V_2$ and $g : W_1 \rightarrow W_2$ be both linear transformations. Then:*

$$\text{Ker}(f \otimes g) = \text{ker}(f) \otimes W_1 + V_1 \otimes \text{ker}(g).$$

Proof. Let $(v_\alpha)_{\alpha \in A_1}$ be a basis of $\text{ker}(f)$, and let us complete it with $(v_\alpha)_{\alpha \in A_2}$ to form a basis of V_1 . Then, by properties of linear mappings, $(f(v_\alpha))_{\alpha \in A_2}$ is a linearly independent subset of V_2 . Consider, in the same manner, $(w_\beta)_{\beta \in B_1}$ a basis of $\text{ker}(g)$, and complete it with $(w_\beta)_{\beta \in B_2}$ to be a basis of W_1 . Then, by properties of linear mappings, we have also that $(g(w_\beta))_{\beta \in B_2}$ is a linearly independent subset of W_2 . Then, consider

$$q = \sum_{\alpha \in A_1 \cup A_2, \beta \in B_1 \cup B_2} c_{\alpha\beta} v_\alpha \otimes w_\beta \in \text{ker}(f \otimes g).$$

Then:

$$(f \otimes g)(q) = \sum_{\alpha \in A_1 \cup A_2, \beta \in B_1 \cup B_2} c_{\alpha\beta} f(v_\alpha) \otimes g(w_\beta) = 0.$$

Since $(f(v_\alpha))_{\alpha \in A_2}$ and $(g(w_\beta))_{\beta \in B_2}$ are L.I., then $(f(v_\alpha) \otimes g(w_\beta))_{\alpha \in A_2, \beta \in B_2}$ is L.I. Since for $\alpha \in A_1$ and $\beta \in B_1$ $f(v_\alpha) = 0$ and $g(w_\beta) = 0$ by construction, we have that $c_{\alpha\beta} = 0$ for all

$\alpha \in A_2, \beta \in B_2$. Then, $q \in \ker(f) \otimes W_1 + V_1 \otimes \ker(g)$ and we obtain that

$$\ker(f \otimes g) \subset \ker(f) \otimes W_1 + V_1 \otimes \ker(g).$$

The other inclusion is straightforward. Hence, $\ker(f \otimes g) = \ker(f) \otimes W_1 + V_1 \otimes \ker(g)$. $\square$

The next proposition is important for understanding the meaning of coideals and subcoalgebras.

Proposition 2.4.2. *Let $f : C \rightarrow D$ be a morphism of $\mathbb{K}$ -coalgebras. Then $\text{im}(f)$ is a subcoalgebra of D and $\ker(f)$ is a coideal of C .*

Proof. Denote Δ_C, Δ_D the coproducts, and $\varepsilon_C, \varepsilon_D$ the counities of C and D , respectively. Consider $w = f(v) \in \text{im}(f)$. Then, since f is a morphism of $\mathbb{K}$ -coalgebras, we have $\Delta_D \circ f(v) = (f \otimes f) \circ \Delta_C(v) \in \text{im}(f) \otimes \text{im}(f)$. So, $\Delta_D(w) \in \text{im}(f) \otimes \text{im}(f)$. Then $\Delta_D(\text{im}(f)) \subset \text{im}(f) \otimes \text{im}(f)$. Hence, $\text{im}(f)$ is a subcoalgebra of D .

For the proof that $\ker(f)$ is a coideal, consider $c \in \ker(f)$. Since f is a morphism of $\mathbb{K}$ -coalgebras, and $c \in \ker(f)$, we have

$$0 = \Delta_D \circ f(c) = (f \otimes f) \circ \Delta_C(c).$$

Hence $\Delta_C(c) \in \ker(f \otimes f)$. By Proposition 2.4.1,

$$\ker(f \otimes f) = C \otimes \ker(f) + \ker(f) \otimes C.$$

Hence, $\Delta_C(c) \in C \otimes \ker(f) + \ker(f) \otimes C$.

Since $c \in \ker(f)$ is arbitrary,

$$\Delta_C(\ker(f)) \subset C \otimes \ker(f) + \ker(f) \otimes C.$$

To conclude the proof that $\ker(f)$ is a coideal of C , we need to prove that $\varepsilon_C(\ker(f)) = 0$. However, since f is a morphism of $\mathbb{K}$ -coalgebras, we have, for $c \in \ker(f)$:

$$0 = \varepsilon_D \circ f(c) = \varepsilon_C(c).$$

Therefore, $\varepsilon(\ker(f)) = \{0\}$. Hence, $\ker(f)$ is a coideal and the proposition is proven. $\square$

In $\mathbb{K}$ -vector spaces, we can construct quotient spaces by taking kernels of linear transformations. In $\mathbb{K}$ -algebras, we can also construct quotient algebras by taking kernels of algebra morphisms, which are ideals. Now we are going to prove the existence of the quotient algebra and the Theorem of Homomorphisms for $\mathbb{K}$ -algebras, and afterwards we will prove analogous results for $\mathbb{K}$ -coalgebras.

Theorem 2.4.1 (Theorem of Homomorphisms for $\mathbb{K}$ -algebras). *Let A be a $\mathbb{K}$ -algebra, I an ideal and $\pi : A \rightarrow A/I$ the canonical projection of $\mathbb{K}$ -vector spaces. Then*

- *There are unique linear maps $\overline{m} : A/I \otimes A/I \rightarrow A/I$ and $\overline{\mu} : \mathbb{K} \rightarrow A/I$ such that $(A/I, \overline{m}, \overline{\mu})$ is a $\mathbb{K}$ -algebra, and $\pi : A \rightarrow A/I$ is a morphism of $\mathbb{K}$ -algebras.*
- *If $f : A \rightarrow B$ is a morphism of $\mathbb{K}$ -algebras such that $I \subset \ker(f)$, then there is a unique morphism of $\mathbb{K}$ -algebras $F : A/I \rightarrow B$ such that $F \circ \pi = f$, $\text{im}(F) = \text{im}(f)$, and moreover F being one-to-one is equivalent to $I = \ker(f)$.*

Proof. For the first item, given $a \in A$, we denote $\pi(a) = \overline{a}$, and denote $m(a \otimes b) = ab$. Consider the function $m' : A/I \times A/I \rightarrow A/I$ defined as $m'(\overline{a}, \overline{b}) = \overline{ab}$. Observe that:

$$\begin{aligned} (m'((\overline{a} + \overline{a'}), (\overline{b} + \overline{b'}))) &= (\overline{a} + \overline{a'})(\overline{b} + \overline{b'}) \\ &= (\overline{a})(\overline{b}) + \overline{ab'} + \overline{a'b} + \overline{a'b'} \\ &= m'(\overline{a}, \overline{b}) + m'(\overline{a}, \overline{b'}) + m'(\overline{a'}, \overline{b}) + m'(\overline{a'}, \overline{b'}). \end{aligned}$$

Also, observe that, for $\alpha, \beta \in \mathbb{K}$:

$$\begin{aligned} m'(\alpha\overline{a}, \beta\overline{b}) &= \overline{\alpha a \beta b} \\ &= \pi(\alpha a \beta b) \\ &= \alpha \beta \pi(ab) \\ &= \alpha \beta m'(a, b) \end{aligned}$$

Hence, this function is bilinear, and then it induces a linear mapping $\overline{m} : A/I \otimes A/I \rightarrow A/I$ which, since it is induced by a product, is associative. Moreover, $\overline{\mu} = \pi \circ \mu$ is the unit map for A/I . Hence, we have already that π preserves units. Observe, then, that:

$$\begin{aligned} \overline{m} \circ (\pi \otimes \pi)(a \otimes b) &= \overline{m}(\pi(a) \otimes \pi(b)) \\ &= \overline{m}(\overline{a} \otimes \overline{b}) \\ &= \overline{ab} \\ &= \pi \circ m(a \otimes b). \end{aligned}$$

The first equality is just an application of $\pi \otimes \pi$, while the second equality follows from the notation we use, just like the third and last equalities, we have discussed in the beginning of the proof. Hence, $\pi : A \rightarrow A/I$ is a morphism of algebras.

Now, assume that $\overline{m}' : A/I \otimes A/I \rightarrow A/I$ and $\overline{\mu}' : \mathbb{K} \rightarrow A/I$ is an algebra structure in A/I such that $\pi : A \rightarrow A/I$ is a morphism of $\mathbb{K}$ -algebras. Hence:

$$\begin{aligned}
\overline{m}(\bar{a} \otimes \bar{b}) &= \overline{m} \circ (\pi \otimes \pi)(a \otimes b) \\
&= \pi \circ m(a \otimes b) \\
&= \overline{m'} \circ (\pi \otimes \pi)(a \otimes b) \\
&= \overline{m'} \circ (\bar{a} \otimes \bar{b}),
\end{aligned}$$

and so $\overline{m} = \overline{m'}$. Also, $\overline{\mu'} = \pi \circ \mu = \mu'$. Hence, such structure is unique.

For the second item, observe that, since π is a projection and $\ker(f) \subset I$, there is a unique linear mapping $F : A/I \rightarrow B$ such that $F \circ \pi = f$. Let us prove that this is a morphism of algebras. In fact, for every element $\bar{a} \in A/I$, there exists $a \in A$ such that $\pi(a) = \bar{a}$. So, observe that:

$$\begin{aligned}
F \circ \overline{m}(\bar{a} \otimes \bar{b}) &= F \circ \overline{m} \circ (\pi \otimes \pi)(a \otimes b) \\
&= F \circ \pi \circ m_A(a \otimes b) \\
&= f \circ m(a \otimes b) \\
&= m_B(f \otimes f)(a \otimes b) \\
&= m_B(f(a) \otimes f(b)) \\
&= m_B(F \circ \pi(a) \otimes F \circ \pi(b)) \\
&= m_B \circ (F \otimes F)(\pi(a) \otimes \pi(b)) \\
&= m_B \circ (F \otimes F)(\bar{a} \otimes \bar{b}).
\end{aligned}$$

The first and last equalities follow from the notation for $\pi(a)$ and $\pi(b)$, while the second equality follows from the fact that π is a morphism of $\mathbb{K}$ -algebras, the third and sixth equalities follow from the definition of F , the fourth equality follows from the fact that f is a morphism of $\mathbb{K}$ -algebras, the fifth equality is a simple application of $f \otimes f$, and the seventh equality follows from application of $\pi \otimes \pi$. Since F is the unique mapping satisfying $F \circ \pi = f$, such morphism of $\mathbb{K}$ -algebras is unique. Observe that $I \subset \ker(f)$ by hypothesis and suppose that F is one-to-one. Then, $\ker(F) = 0$, and, since $\ker(\pi) = I$ and $F \circ \pi = f$, we have that $\ker(f) \subset I$. Supposing that $I = \ker(f)$, if $F(\bar{a}) = 0$, this means that $F \circ \pi(a) = 0$ and then $f(a) = 0$ and hence $a \in I$, which means that $\bar{a} = 0$. Hence, F is one-to-one. $\square$

The next theorem shows that the quotient space of a $\mathbb{K}$ -coalgebra by a coideal has a canonical coalgebra structure, and that a Theorem of Homomorphisms also holds for $\mathbb{K}$ -coalgebras.

Theorem 2.4.2 (Theorem of Homomorphism for $\mathbb{K}$ -coalgebras). *Let C be a $\mathbb{K}$ -coalgebra, I a coideal and $\pi : C \rightarrow C/I$ the canonical projection of $\mathbb{K}$ -vector spaces. Then:*

- *There are unique linear maps $\bar{\Delta} : C/I \rightarrow C/I \otimes C/I$ and $\bar{\varepsilon} : C/I \rightarrow \mathbb{K}$ such that $(C/I, \bar{\Delta}, \bar{\varepsilon})$ is a $\mathbb{K}$ -coalgebra, and $\pi : C \rightarrow C/I$ is a morphism of $\mathbb{K}$ -coalgebras.*
- *If $f : C \rightarrow D$ is a morphism of $\mathbb{K}$ -coalgebras such that $I \subset \ker(f)$, then there is a unique morphism of $\mathbb{K}$ -coalgebras $F : C/I \rightarrow D$ such that $F \circ \pi = f$, $\text{im}(F) = \text{im}(f)$ and F being one-to-one is equivalent to $I = \ker(f)$.*

Proof. For $c \in C$, let us denote $\pi(c) = \bar{c}$.

For the first item, the proof follows like this:

Observe, since I is a coideal, that $\Delta(I) \subset I \otimes C + C \otimes I$. Then:

$$\begin{aligned} (\pi \otimes \pi) \circ \Delta(I) &\subset (\pi \otimes \pi)(I \otimes C + C \otimes I) \\ &= \pi(I) \otimes \pi(C) + \pi(C) \otimes \pi(I) \\ &= \{0\}, \end{aligned}$$

and the last equality holds since $\pi(I) = \{0\}$. Then, $I \subset \ker((\pi \otimes \pi) \circ \Delta)$.

Therefore, we may define a unique linear transformation $\bar{\Delta} : C/I \rightarrow C/I \otimes C/I$ such that $\bar{\Delta} \circ \pi = (\pi \otimes \pi) \circ \Delta$. So, we may write $\bar{\Delta}(\bar{c}) = \sum_{(c)} \bar{c}_{(1)} \otimes \bar{c}_{(2)}$. Our aim now is to prove that $\bar{\Delta}$ is coassociative. Observe that

$$\begin{aligned} (\bar{\Delta} \otimes Id) \circ \bar{\Delta} \circ \pi &= (\bar{\Delta} \otimes Id) \circ (\pi \otimes \pi) \circ \Delta \\ &= (\bar{\Delta} \circ \pi \otimes Id \circ \pi) \circ \Delta \\ &= ((\pi \otimes \pi) \circ \Delta \otimes \pi \circ Id) \circ \Delta \\ &= (\pi \otimes \pi \otimes \pi) \circ (\Delta \otimes Id) \circ \Delta \\ &= (\pi \otimes \pi \otimes \pi) \circ (Id \otimes \Delta) \circ \Delta \\ &= (\pi \circ Id \otimes (\pi \otimes \pi) \circ \Delta) \circ \Delta \\ &= (Id \otimes \pi \otimes \bar{\Delta} \circ \pi) \circ \Delta \\ &= (Id \otimes \bar{\Delta}) \circ (\pi \otimes \pi) \circ \Delta \\ &= (Id \otimes \bar{\Delta}) \circ \bar{\Delta} \circ \pi. \end{aligned}$$

The first, third, seventh, and last equalities hold since $\Delta \circ \pi = (\pi \otimes \pi) \circ \Delta$. Moreover, the fifth hold by the coassociativity of Δ . Since $\text{im}(\pi) = C/I$, the coassociativity of $\bar{\Delta}$ is proven for every $\bar{c} \in C/I$. Since π is surjective, $\bar{\Delta}$ is coassociative.

Now, our aim is to prove the existence of a counit $\bar{\varepsilon}$. However, observe that, since I is a coideal, we have $\varepsilon(I) = \{0\}$. So, there is a unique $\bar{\varepsilon} : C/I \rightarrow \mathbb{K}$ such that $\bar{\varepsilon} \circ \pi = \varepsilon$, and therefore $\bar{\varepsilon}(\bar{c}) = \varepsilon(c)$.

Moreover, $\bar{\varepsilon}$ satisfies the counit property with respect to $\bar{\Delta}$:

$$\begin{aligned}
 \sum_{(c)} \bar{\varepsilon}(\overline{c_{(1)}}) \overline{c_{(2)}} &= \sum_{(c)} \varepsilon(c_{(1)}) \overline{c_{(2)}} \\
 &= \pi \left(\sum_{(c)} \varepsilon(c_{(1)}) c_{(2)} \right) \\
 &= \pi(c) \\
 &= \bar{c}.
 \end{aligned}$$

Analogously, $\sum_{(c)} \overline{c_{(1)}} \bar{\varepsilon}(\overline{c_{(2)}}) = \bar{c}$.

Then, $\bar{\varepsilon}$ is a counit.

Since, by construction, $\bar{\Delta} \circ \pi = (\pi \otimes \pi) \circ \Delta$ and $\bar{\varepsilon} \circ \pi = \varepsilon$ we conclude that π is a morphism of $\mathbb{K}$ -coalgebras. This coalgebra structure is unique since $\bar{\Delta}$ and $\bar{\varepsilon}$ are unique.

For the second part, let $f : C \rightarrow D$ be a morphism of $\mathbb{K}$ -coalgebras. Then, in fact, there is a unique linear map $F : C/I \rightarrow D$ such that $F \circ \pi = f$ because $\ker(f) \subset I$. Then, $F(\bar{c}) = f(c)$.

The proof that F is a morphism of $\mathbb{K}$ -coalgebras follows from the definition of F and from the fact that f is a morphism of $\mathbb{K}$ -coalgebras. In fact,

$$\begin{aligned}
 (\Delta_D \circ F)(\bar{c}) &= \Delta_D(F(\bar{c})) \\
 &= \Delta_D(f(c)) \\
 &= \sum_{(c)} f(c)_{(1)} \otimes f(c)_{(2)} \\
 &= \sum_{(c)} f(c_{(1)}) \otimes f(c_{(2)}) \\
 &= \sum_{(c)} F(\overline{c_{(1)}}) \otimes F(\overline{c_{(2)}}) \\
 &= (F \otimes F) \sum_{(c)} \overline{c_{(1)}} \otimes \overline{c_{(2)}} \\
 &= (F \otimes F) \circ \bar{\Delta}(\bar{c}).
 \end{aligned}$$

The first equality is just the application of F . The second and fifth equalities hold because $F(\bar{c}) = f(c)$. The fourth equality holds since f is a morphism of coalgebras. The third and the last equalities hold by Sweedler notation of coproducts. The sixth equality holds by the definition and linearity of the tensor product of linear maps.

Also, observe that:

$$\begin{aligned}
 (\varepsilon_D \circ F)(\bar{c}) &= \varepsilon_D(F(\bar{c})) \\
 &= \varepsilon_D(f(c)) \\
 &= \varepsilon_C(c) \\
 &= \bar{\varepsilon}(\bar{c}).
 \end{aligned}$$

The second equality holds since $F(\bar{c}) = f(c)$. The third equality holds since f is a morphism of $\mathbb{K}$ -coalgebras. The last equality holds by the construction of $\bar{\varepsilon}$.

So, it is proven that F is a morphism of coalgebras.

By the definition of F as the unique $\mathbb{K}$ -linear map such that $F \circ \pi = f$, the remaining affirmations are straightforward. $\square$

Proposition 2.4.3. *Let $f : C \rightarrow D$ be a morphism of coalgebras, then the transpose map $f^* : D^* \rightarrow C^*$ is a morphism of algebras.*

Proof. In the above proposition, D^* and C^* are $\mathbb{K}$ -algebras as in Lemma 2.3.1. First, let us prove that f^* is multiplicative. Consider $d^*, e^* \in D^*$ and $c \in C$. Then:

$$\begin{aligned}
 \langle f^*(m(d^* \otimes e^*)), c \rangle &= \langle m(d^* \otimes e^*), f(c) \rangle \\
 &= \langle \Delta^*(\iota(d^* \otimes e^*)), f(c) \rangle \\
 &= \langle (\iota(d^* \otimes e^*), \Delta(f(c))) \rangle \\
 &= \langle (\iota(d^* \otimes e^*)), \sum_{(f(c))} f(c)_{(1)} \otimes f(c)_{(2)} \rangle \\
 &= \langle (\iota(d^* \otimes e^*)), \sum_{(c)} f(c_{(1)}) \otimes f(c_{(2)}) \rangle \\
 &= \sum_{(c)} \langle d^*, f(c_{(1)}) \rangle \langle e^*, f(c_{(2)}) \rangle \\
 &= \sum_{(c)} \langle f^*(d^*), c_{(1)} \rangle \langle f^*(e^*), c_{(2)} \rangle \\
 &= \langle \iota(f^*(d^*) \otimes f^*(e^*)), \sum_{(c)} c_{(1)} \otimes c_{(2)} \rangle \\
 &= \langle \iota(f^*(d^*) \otimes f^*(e^*)), \Delta(c) \rangle \\
 &= \langle \Delta^*(\iota(f^*(d^*) \otimes f^*(e^*))), c \rangle \\
 &= \langle m(f^*(d^*) \otimes f^*(e^*)), c \rangle.
 \end{aligned}$$

The first, third, seventh, and tenth equalities hold by Definition 2.3.1. The second and the last equalities hold by the definition of m as in Lemma 2.3.1. The fourth and ninth

equalities hold by Sweedler notation. The fifth equality holds since f is a morphism of coalgebras. The sixth and eighth equalities hold by definition of ι . Hence, f is multiplicative.

For the proof that f preserves the unit, observe that:

$$\begin{aligned}
 \langle f^*(\mu_{D^*}(1)), c \rangle &= \langle \mu_{D^*}(1), f(c) \rangle \\
 &= \langle \varepsilon_D^*(\Phi(1)), f(c) \rangle \\
 &= \langle \Phi(1), \varepsilon_D(f(c)) \rangle \\
 &= \varepsilon_D(f(c)) \\
 &= \varepsilon_C(c) \\
 &= \langle \Phi(1), \varepsilon_C(c) \rangle \\
 &= \langle \varepsilon_C^* \Phi(1), c \rangle \\
 &= \langle \mu_{C^*}(1), c \rangle.
 \end{aligned}$$

The first, third and seventh equalities hold by Definition 2.3.1. The second and the last equalities hold by the definition of unity given in Lemma 2.3.1. The fourth and sixth equalities hold by the definition of Φ . The fifth equality holds since f is a morphism of coalgebras.

Hence, f^* is a morphism of algebras and the proposition is proven. $\square$

We will close this section relating ideals and subalgebras of a dual $\mathbb{K}$ -algebra C^* to subcoalgebras and coideals of the $\mathbb{K}$ -coalgebra C . We begin by recalling the definitions of annihilators of subsets of V and V^* . Let V be a $\mathbb{K}$ -vector space and let $S \subset V$ be a subset of V . Then

$$S^\perp = \{v^* \in V^* : \langle v^*, S \rangle = \{0\}\} \subset V^*$$

and, for $T \subset V^*$ a subset of V^* :

$$T^\perp = \{v \in V : \langle T, v \rangle = \{0\}\} \subset V.$$

It is straightforward that $S^\perp$ and $T^\perp$ are subspaces of V^* and V respectively. With such definitions, we have the following proposition:

Proposition 2.4.4. *Let V, W be $\mathbb{K}$ -vector spaces and let $X \subset V^*$ and $Y \subset W^*$ be subspaces of their duals V^* and W^* . Then:*

$$(\iota(X \otimes Y))^\perp = X^\perp \otimes W + V \otimes Y^\perp.$$

Proof. Let $u \in X^\perp \otimes W + V \otimes Y^\perp$. Then, we have that $\iota(x \otimes y)(u) = 0$ for all $x \in X, y \in Y$. Since $X \otimes Y$ is generated as $\mathbb{K}$ -vector space by homogeneous tensors $x \otimes y$, we have

that $u \in (\iota(X \otimes Y))^\perp$.

Conversely, let us write $V = X^\perp \oplus V'$, $W = Y^\perp \oplus W'$. Since the tensor distributes over direct sums, we have that:

$$V \otimes W = (X^\perp \otimes Y^\perp) \oplus (V' \otimes Y^\perp) \oplus (X^\perp \otimes W') \oplus (V' \otimes W').$$

Then, let $u \in (\iota(X \otimes Y))^\perp$. Due to the previous expression, we may write $u = u_1 + u_2 + u_3 + u_4$ with $u_1 \in X^\perp \otimes Y^\perp$, $u_2 \in V' \otimes Y^\perp$, $u_3 \in X^\perp \otimes W'$, $u_4 \in V' \otimes W'$. Since $u \in (\iota(X \otimes Y))^\perp$, for $x \in X, y \in Y$ given:

$$0 = \iota(x \otimes y)(u) = 0 + 0 + 0 + \iota(x \otimes y)(u_4).$$

Suppose that $u_4 \neq 0$. Then, we may write $u_4 = \sum_i v_i \otimes w_i$ with $\{v_i, i \in I\}, \{w_i, i \in I\}$ being L.I. Let us define functionals $v_i^* \in V^*$ such that $\langle v_i, v_j \rangle = \delta_{i,j}$ and $v_i^*(X^\perp) = 0$, and $w_i^* \in W^*$ such that $\langle w_i^*, w_j \rangle = \delta_{i,j}$ and $w_i^*(Y^\perp) = 0$.

Then, we have that:

$$0 = \iota(v_1^* \otimes w_1^*)(u) = \iota(v_1^* \otimes w_1^*)(u_4) = 1,$$

which leads to contradiction.

Then, $u_4 = 0$. Hence:

$$u = u_1 + u_2 + u_3 \in (X^\perp \otimes Y^\perp) \oplus (V' \otimes Y^\perp) \oplus (X^\perp \otimes W') \subset X^\perp \otimes W + V \otimes Y^\perp.$$

□

The next proposition shows the duality between the concepts of ideals and subcoalgebras, and the concept of coideals and subalgebras.

Proposition 2.4.5. *Let C be a finite-dimensional $\mathbb{K}$ -coalgebra and V a subspace of C . Then:*

- *V being a subcoalgebra of C is equivalent to $V^\perp$ being an ideal of C^* .*
- *V being a coideal of C is equivalent to $V^\perp$ being a subalgebra of C^* .*

Proof. For the first item, suppose that V is a subcoalgebra of C . Then, there is an inclusion $i : V \rightarrow C$ which is a morphism of $\mathbb{K}$ -coalgebras. Hence, the transpose map $i^* : C^* \rightarrow V^*$ is a morphism of $\mathbb{K}$ -algebras by Proposition 2.4.3 and $V^\perp$ is its kernel by the definition of $V^\perp$. Then, $V^\perp$ is an ideal, since it is a kernel of a morphism of $\mathbb{K}$ -algebras. Then, V being a subcoalgebra of C implies $V^\perp$ being an ideal of C^* .

Conversely, suppose $V^\perp$ is an ideal of C^* . Then, take $x \in (V^\perp)^\perp = V$ (this equality holds because C is finite-dimensional). Let us write

$$\Delta(x) = \sum_i y_i \otimes z_i.$$

Assume, by contradiction, that $\Delta(x) \notin V \otimes C$; we may suppose that $z_1, \dots, z_n$ are L.I., and that $y_1 \notin V$. Therefore, since $(V^\perp)^\perp = V$, there is $\alpha \in V^\perp$ such that $\langle \alpha, y_1 \rangle \neq 0$. Let $\beta \in C^*$ be such that $\langle \beta, z_i \rangle = \delta_{1,i}$. Since $V^\perp$ is an ideal and $\alpha \in V^\perp$, then $m(\alpha \otimes \beta) \in V^\perp$. Then, observe that:

$$\begin{aligned} 0 &= \langle m(\alpha \otimes \beta), x \rangle \\ &= \langle \Delta^* \circ \iota(\alpha \otimes \beta), x \rangle \\ &= \langle \iota(\alpha \otimes \beta), \Delta(x) \rangle \\ &= \langle \iota(\alpha \otimes \beta), \sum_i y_i \otimes z_i \rangle \\ &= \sum_i \langle \alpha, y_i \rangle \langle \beta, z_i \rangle \\ &= \langle \alpha, y_1 \rangle \langle \beta, z_1 \rangle \neq 0. \end{aligned}$$

The first equality holds since $m(\alpha \otimes \beta) \in V^\perp$. The second equality holds by the definition of m given in Lemma 2.3.1. The third equality holds by the Definition 2.3.1 (Transpose map). The fourth equality holds by the way we've written $\Delta(x)$. The fifth equality holds by the definition of ι and the last equality holds since $\langle \beta, z_i \rangle = \delta_{1,i}$. The inequality holds since $\langle \alpha, y_1 \rangle \neq 0$ as we've supposed, and $\langle \beta, z_1 \rangle = 1$ since $\langle \beta, z_i \rangle = \delta_{1,i}$.

Then, necessarily, $\Delta(x) \in V \otimes C$.

By an analogous argument, $\Delta(x) \in C \otimes V$.

It follows that

$$\Delta(x) \in V \otimes C \cap C \otimes V \subset V \otimes V,$$

and then $\Delta(x) \in V \otimes V$.

Hence, V is a subcoalgebra, as we wanted to prove.

For the second item, take V a coideal of C . Observe that

$$\begin{aligned} \langle m(V^\perp \otimes V^\perp), V \rangle &= \langle (\Delta^* \circ \iota)(V^\perp \otimes V^\perp), V \rangle \\ &= \langle \iota(V^\perp \otimes V^\perp), \Delta(V) \rangle \\ &\subseteq \langle \iota(V^\perp \otimes V^\perp), V \otimes C + C \otimes V \rangle \\ &= \langle \iota(V^\perp \otimes V^\perp), V \otimes C \rangle + \langle \iota(V^\perp \otimes V^\perp), C \otimes V \rangle \\ &= \{0\}. \end{aligned}$$

The first equality holds by the definition of m given in Lemma 2.3.1. The second equality holds by Definition 2.3.1 (Transpose map). The inclusion holds since V is a coideal and so $\Delta(V) \subset V \otimes C + C \otimes V$. The third equality holds by linearity and the last equality holds by the definitions of ι and $V^\perp$. Then, we have $m(V^\perp \otimes V^\perp) \subset V^\perp$.

Since V is a coideal, we have that:

$$\begin{aligned} \{0\} &= \varepsilon(V) \\ &= \langle \Phi(1), \varepsilon(V) \rangle \\ &= \langle \varepsilon^* \circ \Phi(1), V \rangle \\ &= \langle 1_{C^*}, V \rangle. \end{aligned}$$

Then, $1_{C^*} \in V^\perp$. Hence, $V^\perp$ is a subalgebra of C^* .

Now, suppose $V^\perp$ is a subalgebra of C^* . We are going to prove that $V^{\perp\perp} = V$ is a coideal of C . First of all, since $V^\perp$ is a subalgebra of C^* , $1_{C^*} \in V^\perp$ and then, we get:

$$\begin{aligned} \{0\} &= \langle 1_{C^*}, V \rangle \\ &= \langle \varepsilon^* \circ \Phi(1), V \rangle \\ &= \langle \Phi(1), \varepsilon(V) \rangle \\ &= \varepsilon(V). \end{aligned}$$

The first equality holds since $1_{C^*} \in V^\perp$, the second equality holds by the definition of unit given in Lemma 2.3.1, the third equality holds by the Definition 2.3.1 (Transpose map) applied to ε , and the last equality holds by the definition of Φ . Also, observe that:

$$\begin{aligned} \langle \iota(V^\perp \otimes V^\perp), \Delta(V) \rangle &= \langle (\Delta^* \circ \iota)(V^\perp \otimes V^\perp), V \rangle \\ &= \langle m(V^\perp \otimes V^\perp), V \rangle \\ &\subset \langle V^\perp, V \rangle = 0. \end{aligned}$$

The first equality holds by the Definition 2.3.1; the second one holds by the definition of m given in Lemma 2.3.1. The inclusion holds since we are supposing $V^\perp$ is a subalgebra of C^* and the last equality holds by the definition of $V^\perp$. Then, we have $\Delta(V) \subset (\iota(V^\perp \otimes V^\perp))^\perp$, and by the previous proposition, it is included in $C \otimes V + V \otimes C$. Hence, V is a coideal of C .

□

In the above proposition, we have used the following lemma, which can be found in (DASCALESCU; NASTASESCU; RAIANU, 2000) in the page 24.

Lemma 2.4.1. *Let V and W be $\mathbb{K}$ -vector spaces, and $X \subset V$, $Y \subset W$ be vector subspaces. Then $(V \otimes Y) \cap (X \otimes W) = X \otimes Y$.*

2.5 $\mathbb{K}$ -BIALGEBRAS AND HOPF $\mathbb{K}$ -ALGEBRAS.

Until now, we have studied algebras and coalgebras, but not a structure which is an algebra and a coalgebra on the same vector space. Now we are going to introduce algebras which also are coalgebras, in such a manner that the algebra and coalgebra structures are related to each other. Such structures are called Bialgebras. In order to define $\mathbb{K}$ -bialgebras we need first to prove that the tensor product of $\mathbb{K}$ -algebras is a $\mathbb{K}$ -algebra, and analogously, the tensor product of $\mathbb{K}$ -coalgebras is a $\mathbb{K}$ -coalgebra.

First of all, recall the twist map from Definition 2.2.1: if V and W are $\mathbb{K}$ -vector spaces then the twist function $\tau_{V \otimes W} : V \otimes W \rightarrow W \otimes V$ is defined on generators by $\tau_{V \otimes W}(v \otimes w) = w \otimes v$. We will need it to understand what follows.

Proposition 2.5.1. *Let A, B be both $\mathbb{K}$ -algebras, then, $A \otimes B$ is also a $\mathbb{K}$ -algebra, with product given by:*

$$m_{A \otimes B} = (m_A \otimes m_B) \circ (Id_A \otimes \tau_{B \otimes A} \otimes Id_B).$$

And the unit is given by:

$$\mu_{A \otimes B} = \mu_A \otimes \mu_B.$$

Proof. Denoting $m_A(a \otimes b) = ab$ and $m_B(c \otimes d) = cd$, we have that $m_{A \otimes B}(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$. Observe that, given $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$, we have:

$$\begin{aligned} m_{A \otimes B}(m_{A \otimes B} \otimes Id)(a_1 \otimes b_1 \otimes a_2 \otimes b_2 \otimes a_3 \otimes b_3) &= \\ &= m_{A \otimes B}(m_{A \otimes B}(a_1 \otimes b_1 \otimes a_2 \otimes b_2) \otimes a_3 \otimes b_3) \\ &= m_{A \otimes B}(m_A \otimes m_B \circ (Id \otimes \tau_{B \otimes A} \otimes Id)(a_1 \otimes b_1 \otimes a_2 \otimes b_2)) \otimes a_3 \otimes b_3 \\ &= m_{A \otimes B}((m_A \otimes m_B)((a_1 \otimes a_2 \otimes b_1 \otimes b_2) \otimes a_3 \otimes b_3)) \\ &= m_{A \otimes B}(a_1 a_2 \otimes b_1 b_2 \otimes a_3 \otimes b_3) \\ &= (m_A \otimes m_B) \circ (Id \otimes \tau_{B \otimes A} \otimes Id)(a_1 a_2 \otimes b_1 b_2 \otimes a_3 \otimes b_3) \\ &= (m_A \otimes m_B)(a_1 a_2 \otimes a_3 \otimes b_1 b_2 \otimes b_3) \\ &= a_1 a_2 a_3 \otimes b_1 b_2 b_3. \end{aligned}$$

The first equality holds by the definition of tensor map. The second and fifth equalities follow the definition of $m_{A \otimes B}$. The third and sixth equalities follow from the definition of Twist mapping. The fourth and the last equalities follow by application of the map $m_A \otimes m_B$.

By an analogous argument,

$$m_{A \otimes B}(Id \otimes m_{A \otimes B})(a_1 \otimes b_1 \otimes a_2 \otimes b_2 \otimes a_3 \otimes b_3) = a_1 a_2 a_3 \otimes b_1 b_2 b_3.$$

Then, $m_{A \otimes B}$ is associative.

For the property of the unit, observe that, for $a \otimes b \in A \otimes B$:

$$\begin{aligned}
 m_{A \otimes B}(\mu_A(1) \otimes \mu_B(1) \otimes a \otimes b) &= (m_A \otimes m_B) \circ (Id \otimes \tau \otimes Id)(\mu_A(1) \otimes \mu_B(1) \otimes a \otimes b) \\
 &= (m_A \otimes m_B)(\mu_A(1) \otimes a \otimes \mu_B(1) \otimes b) \\
 &= m_A(\mu_A(1) \otimes a) \otimes m_B(\mu_B(1) \otimes b) \\
 &= a \otimes b.
 \end{aligned}$$

The first equality follows from the definition of $m_{A \otimes B}$, the second equality follows from Definition 2.2.1 (Twist mapping), the third equality follows from the definition of tensor map and the last one follows from the unit property applied to the $\mathbb{K}$ -algebras A and B . From an analogous argument, it follows that:

$$m_{A \otimes B}(a \otimes b \otimes \mu_A(1) \otimes \mu_B(1)) = a \otimes b.$$

Hence, we have proved that $A \otimes B$ is a $\mathbb{K}$ -algebra. □

The following proposition is the analogous for coalgebras to the previous one.

Proposition 2.5.2. *Let C, D be $\mathbb{K}$ -coalgebras. Then $C \otimes D$ is a $\mathbb{K}$ -coalgebra, with coproduct and counit morphisms given by:*

$$\begin{aligned}
 \Delta_{C \otimes D}(c \otimes d) &= (Id \otimes \tau_{C \otimes D} \otimes Id) \circ (\Delta_C \otimes \Delta_D)(c \otimes d) \\
 \varepsilon_{C \otimes D}(c \otimes d) &= \varepsilon_C(c) \varepsilon_D(d).
 \end{aligned}$$

Proof. Let $c \otimes d \in C \otimes D$ be given. Then, we have:

$$\begin{aligned}
(\Delta_{C \otimes D} \otimes Id) \circ \Delta_{C \otimes D}(c \otimes d) &= (\Delta_{C \otimes D} \otimes Id) \circ (Id \otimes \tau \otimes Id) \circ (\Delta_C \otimes \Delta_D)(c \otimes d) \\
&= (\Delta_{C \otimes D} \otimes Id) \circ (Id \otimes \tau \otimes Id) \\
&\quad \left(\sum_c c_{(1)} \otimes c_{(2)} \right) \otimes \left(\sum_d d_{(1)} \otimes d_{(2)} \right) \\
&= (\Delta_{C \otimes D} \otimes Id) \circ (Id \otimes \tau \otimes Id) \left(\sum_{(c),(d)} c_{(1)} \otimes c_{(2)} \otimes d_{(1)} \otimes d_{(2)} \right) \\
&= (\Delta_{C \otimes D} \otimes Id) \left(\sum_{(c),(d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)} \right) \\
&= (((Id \otimes \tau \otimes Id) \circ (\Delta_C \otimes \Delta_D)) \otimes Id) \\
&\quad \left(\sum_{(c),(d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)} \right) \\
&= \sum_{(c),(d)} ((Id \otimes \tau \otimes Id) \circ (\Delta_C \otimes \Delta_D))(c_{(1)} \otimes \\
&\quad d_{(1)}) \otimes c_{(2)} \otimes d_{(2)} \\
&= \sum_{(c),(d)} ((Id \otimes \tau \otimes Id)) \left(\sum_{(c_{(1)})} \sum_{(d_{(1)})} c_{(1)(1)} \otimes c_{(1)(2)} \otimes \right. \\
&\quad \left. d_{(1)(1)} \otimes d_{(1)(2)} \otimes c_{(2)} \otimes d_{(2)} \right) \\
&= \sum_{(c),(d)} (Id \otimes \tau \otimes Id) \left(\sum_{(c_{(1)}),(d_{(1)})} c_{(1)(1)} \otimes c_{(1)(2)} \otimes \right. \\
&\quad \left. d_{(1)(1)} \otimes d_{(1)(2)} \right) \otimes c_{(2)} \otimes d_{(2)} \\
&= \sum_{(c),(d)} \left(\sum_{(c_{(1)}),(d_{(2)})} c_{(1)(1)} \otimes d_{(1)(1)} \otimes c_{(1)(2)} \otimes d_{(1)(2)} \right) \otimes c_{(2)} \otimes d_{(2)} \\
&= \sum_{(c),(d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)} \otimes c_{(3)} \otimes d_{(3)}.
\end{aligned}$$

The first and fifth equalities follow from the definition of $\Delta_{C \otimes D}$, the second and seventh equalities follow from sigma notation applied to the coproduct, the fourth and the ninth equalities follow from Definition 2.2.1 (Twist mapping) and the last equality follows from sigma notation and the coassociativity applied to the coalgebras C and D . By an analogous argument, we get the same result for $(Id \otimes \Delta_{C \otimes D}) \circ \Delta_{C \otimes D}$. Hence, $\Delta_{C \otimes D}$ is coassociative. In Sweedler notation, we have:

$$\Delta_{C \otimes D}(c \otimes d) = \sum_{(c \otimes d)} (c \otimes d)_{(1)} \otimes (c \otimes d)_{(2)} = \sum_{(c)} \sum_{(d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)},$$

$$\Delta_{C \otimes D}(c \otimes d) = \sum_{(c \otimes d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}.$$

This is the Sweedler notation for $\Delta_{C \otimes D}$. Using such notation, we can prove the counit property. In fact, evaluating, we have that:

$$\begin{aligned} (\varepsilon_{C \otimes D} \otimes Id) \circ \Delta_{C \otimes D}(c \otimes d) &= (\varepsilon_{C \otimes D} \otimes Id) \left(\sum_{(c \otimes d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)} \right) \\ &= \sum_{(c \otimes d)} \varepsilon_{C \otimes D}(c_{(1)} \otimes d_{(1)})(c_{(2)} \otimes d_{(2)}) \\ &= \sum_{(c \otimes d)} \varepsilon_C(c_{(1)})\varepsilon_D(d_{(1)})(c_{(2)} \otimes d_{(2)}) \\ &= \sum_{(c \otimes d)} \varepsilon_C(c_{(1)})c_{(2)} \otimes \varepsilon_D(d_{(1)})d_{(2)} \\ &= c \otimes d. \end{aligned}$$

The first equality holds by the previous observation about the Sweedler notation for $\Delta_{C \otimes D}$. The second equality holds by the definition of tensor map. The third equality holds by the definition of $\varepsilon_{C \otimes D}$. The fourth equality follows from a simple scalar manipulation in tensor product, and the last equality follows from the counit property of ε_C and ε_D . By an analogous argument, we have that $(Id \otimes \varepsilon_{C \otimes D}) \circ \Delta_{C \otimes D}(c \otimes d) = c \otimes d$. Hence, the counit property is proven and the proposition holds. $\square$

Recall Example 2.1.7, *i.e.*, the coalgebra structure on the field $\mathbb{K}$. The coproduct of $\mathbb{K}$ is given by $\Delta_{\mathbb{K}}(1_{\mathbb{K}}) = 1_{\mathbb{K}} \otimes 1_{\mathbb{K}}$ and the counit map is the identity map. Hence, since $\mathbb{K}$ is a field, $\mathbb{K}$ is itself a $\mathbb{K}$ -algebra and it is a coalgebra.

Since we have proven that the tensor of algebras/coalgebras are also algebras/coalgebras, if $(H, m, \mu, \Delta, \varepsilon)$ is an algebra with product given by m , unit morphism given by μ , and it is also a coalgebra with Δ as the coproduct, and ε as the counit morphism, we may consider the case where m, μ are morphisms of algebras, or Δ, ε are morphisms of coalgebras. The next theorem deals with such considerations, and it is essential to define $\mathbb{K}$ -bialgebras.

Theorem 2.5.1. *Let H be a $\mathbb{K}$ -vector space with maps $m : H \otimes H \rightarrow H, \mu : \mathbb{K} \rightarrow H, \Delta : H \rightarrow H \otimes H, \varepsilon : H \rightarrow \mathbb{K}$ such that (H, m, μ) is a $\mathbb{K}$ -algebra and (H, Δ, ε) is a $\mathbb{K}$ -coalgebra. Then, the following conditions are equivalent:*

- m and μ are morphisms of coalgebras.
- Δ and ε are morphisms of algebras

Recall the structures for tensor coalgebras and algebras stated in the previous propositions. The maps m and μ are referred as a morphism of coalgebras with such coalgebras structures, and the maps Δ and ε are referred as a morphisms of algebras in such algebras.

Proof. First of all, if Δ is a morphism of algebras then, by Definition 2.4.1 (Morphism of Algebras), and by the structure of algebra in $H \otimes H$, Δ being a morphism of algebras is equivalent to the following diagrams being commutative:

$$\begin{array}{ccc}
 H \otimes H & \xrightarrow{m} & H \xrightarrow{\Delta} H \otimes H \\
 \downarrow \Delta \otimes \Delta & & \uparrow m \otimes m \\
 H \otimes H \otimes H \otimes H & \xrightarrow{id \otimes \tau \otimes id} & H \otimes H \otimes H \otimes H
 \end{array}
 \quad
 \begin{array}{ccc}
 H & \xrightarrow{\Delta} & H \otimes H \\
 \mu \uparrow & & \uparrow \mu \otimes \mu \\
 \mathbb{K} & \xrightarrow{\cong} & \mathbb{K} \otimes \mathbb{K}
 \end{array}$$

And also by Definition 2.4.1 (Morphism of algebras), ε being a morphism of algebras is equivalent to the following diagrams being commutative:

$$\begin{array}{ccc}
 H \otimes H & \xrightarrow{\varepsilon \otimes \varepsilon} & \mathbb{K} \otimes \mathbb{K} \\
 \downarrow m & & \downarrow \cong \\
 H & \xrightarrow{\varepsilon} & \mathbb{K}
 \end{array}
 \quad
 \begin{array}{ccc}
 & H & \\
 \mu \nearrow & & \searrow \varepsilon \\
 \mathbb{K} & \xrightarrow{id} & \mathbb{K}
 \end{array}$$

Also, by Definition 2.4.2 (morphism of coalgebras), and due to the structure of coalgebra in H and $H \otimes H$, m being a morphism of coalgebras is equivalent to the first and third diagrams to commute, while μ being a morphism of coalgebras is equivalent to the second and fourth diagrams to commute. So, if Δ and ε are morphisms of algebras, then the four diagrams above commute, and then m and μ are morphisms of coalgebras, and if μ and m are morphisms of coalgebras, then also the four diagrams commute, and then Δ and ε are morphisms of algebras. Hence, the two conditions are equivalent and the proposition is proven. $\square$

Then, we can finally define a $\mathbb{K}$ -bialgebra structure, as follows:

Definition 2.5.1 (Definition of a $\mathbb{K}$ -bialgebra). *A $\mathbb{K}$ -vector space H with an $\mathbb{K}$ -algebra and a $\mathbb{K}$ -coalgebra structure $(H, m, \mu, \Delta, \varepsilon)$ satisfying one of the equivalent conditions in Theorem 2.5.1 is called a $\mathbb{K}$ -bialgebra.*

Now, we shall discuss some examples of $\mathbb{K}$ -bialgebras:

Example 2.5.1. Let G be a group and $\mathbb{K}G$ be the $\mathbb{K}$ -vector space with basis given by the group G . Hence, since every element $x \in \mathbb{K}G$ may be written as $x = \sum_{g \in G} \alpha_{g,x} g$, we may define a structure of $\mathbb{K}$ -algebra on $\mathbb{K}G$ by:

$$m(x \otimes y) = m\left(\sum_{g \in G} \alpha_{g,x} g \otimes \sum_{h \in G} \alpha_{h,y} h\right) = \sum_{g \in G} \sum_{h \in G} \alpha_{g,x} \alpha_{h,y} gh,$$

$$\mu(1) = e,$$

where e is the neutral element of G . We may define a structure of coalgebra on $\mathbb{K}G$ by:

$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1.$$

for $g \in G$.

Due to Example 2.1.8, this is in fact a structure of $\mathbb{K}$ -coalgebra in $\mathbb{K}G$. Hence, $\mathbb{K}G \otimes \mathbb{K}G$ has also a structure of $\mathbb{K}$ -algebra and $\mathbb{K}$ -coalgebra, given by Proposition 2.5.1 and Proposition 2.5.2. Observe that Δ is a morphism of $\mathbb{K}$ -algebras. In fact, we have:

$$\begin{aligned} \Delta(gh) &= gh \otimes gh \\ &= (g \otimes g)(h \otimes h) \\ &= \Delta(g)\Delta(h) \end{aligned}$$

for $g, h \in G$.

The first and last equalities follow from the definition of Δ , while the second equality follows from the definition of product in $\mathbb{K}G \otimes \mathbb{K}G$. Also, observe that since $1_{\mathbb{K}G} = e \in G$,

$$\begin{aligned} \Delta(1_{\mathbb{K}G}) &= 1_{\mathbb{K}G} \otimes 1_{\mathbb{K}G} \\ &= 1_{\mathbb{K}G \otimes \mathbb{K}G}. \end{aligned}$$

The first equality follows from the definition of Δ , while the second equality follows from the definition of unit of the tensor product of algebras in Proposition 2.5.1.

Hence, Δ is a morphism of algebras.

In fact, note that, for $g, h \in G$:

$$\varepsilon(gh) = 1 = 1 \cdot 1 = \varepsilon(g)\varepsilon(h)$$

The first and last equalities follow from the definition of ε and the fact that G is a group. Now, observe that ε is also a morphism of algebras. In fact, we have:

$$\begin{aligned} (\varepsilon \otimes Id) \circ \Delta(g) &= (\varepsilon \otimes Id)(g \otimes g) \\ &= (\varepsilon(g) \otimes g) \\ &= 1 \otimes g. \end{aligned}$$

The first equality follows from the definition of Δ , while the second equality follows from the application of the tensor mapping $\varepsilon \otimes Id$, and the last equality follows from the definition of ε . By an analogous argument, we have that $(Id \otimes \varepsilon) \circ \Delta(g) = g \otimes 1$. Hence, $\mathbb{K}G$ is a $\mathbb{K}$ -bialgebra.

For the next example, recall the universal property of a free algebra $T(V)$ on a $\mathbb{K}$ -vector space (see Examples 2.1.6 and 2.4.3): for every $\mathbb{K}$ -algebra A and every linear mapping $f : V \rightarrow A$ there is a unique morphism of algebras $\bar{f} : T(V) \rightarrow A$ such that $f = \bar{f} \circ i$ where i is the inclusion $i : V \rightarrow T(V)$.

Example 2.5.2 (Free bialgebra over a coalgebra). Let (C, δ, ϵ) be a $\mathbb{K}$ -coalgebra and consider the free algebra $T(C)$. We may define a structure of $\mathbb{K}$ -coalgebra in $T(C)$ in such a way that $T(C)$ is a $\mathbb{K}$ -bialgebra. In fact, recall that $\delta : C \rightarrow C \otimes C$ is a linear mapping, and also recall the canonical inclusion $i_{C \otimes C} : C \otimes C \rightarrow T(C) \otimes T(C)$. Then, $i_{C \otimes C} \circ \delta$ is a linear mapping of C into $T(C) \otimes T(C)$, which is a $\mathbb{K}$ -algebra due to Proposition 2.5.1. Then, we may define $\Delta : T(C) \rightarrow T(C) \otimes T(C)$ as the unique $\mathbb{K}$ -algebra mapping such that $\Delta \circ i_C = i_{C \otimes C} \circ \delta$.

We shall do the same to ϵ . In fact, $\epsilon : C \rightarrow \mathbb{K}$ is a linear mapping, $\mathbb{K}$ is a $\mathbb{K}$ -algebra, and then we may define a morphism of algebras $\varepsilon : T(C) \rightarrow \mathbb{K}$ in the same manner as Δ .

Now, our aim is to prove the coassociativity of Δ and ε . In fact, note that we have an equality of linear mappings $(\delta \otimes Id) \circ \delta = (Id \otimes \delta) \circ \delta : C \rightarrow C \otimes C \otimes C$ and also observe the inclusion $i_{C \otimes C \otimes C} : C \otimes C \otimes C \rightarrow T(C) \otimes T(C) \otimes T(C)$. Then, there is a unique algebra morphism $F : T(C) \rightarrow T(C) \otimes T(C) \otimes T(C)$ such that $F \circ i_C = i_{C \otimes C \otimes C}(\delta \otimes Id) \circ \delta$. But note that $(\Delta \otimes Id) \circ \Delta$ also obeys this rule. Analogously, $(Id \otimes \Delta) \circ \Delta$ satisfies it too. So, $(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta$ since this morphism is unique.

With an analogous argument, we obtain that ε satisfies the counit property. Since Δ and ε are both morphisms of algebras, we have that $T(C)$ is a $\mathbb{K}$ -bialgebra.

The next definitions and propositions are consequences of Theorem 2.5.1 and Definition 2.5.2, as follows:

Definition 2.5.2 (Morphism of $\mathbb{K}$ -bialgebras). Let B, B' be both $\mathbb{K}$ -bialgebras. Then $f : B \rightarrow B'$ is a **morphism of $\mathbb{K}$ -bialgebras** if f is both a morphism of $\mathbb{K}$ -algebras and a morphism of $\mathbb{K}$ -coalgebras.

Definition 2.5.3 (Bi-ideals). Let B be a bialgebra. Then, $I \subset B$ is a **bi-ideal** if it is both an ideal and a coideal.

The next example we will give is not just an example of bi-ideal, but also of a noncocommutative and noncommutative $\mathbb{K}$ -bialgebra, and an example of morphism of

bialgebras, as follows.

Example 2.5.3. (Sweedler Algebra) Let $\mathbb{K}$ be a field of characteristic distinct from 2. Let (C, δ, ϵ) be a $\mathbb{K}$ -coalgebra with basis $\mathcal{B} = \{g, u, x\}$, with a $\mathbb{K}$ -coalgebra structure defined as

$$\begin{aligned}\delta(g) &= g \otimes g, \\ \delta(u) &= u \otimes u, \\ \delta(x) &= x \otimes g + u \otimes x,\end{aligned}$$

as the coproduct and ϵ defined as

$$\begin{aligned}\epsilon(g) &= \epsilon(u) = 1, \\ \epsilon(x) &= 0.\end{aligned}$$

Note that δ is obviously coassociative for g and u , and also observe that:

$$\begin{aligned}(\delta \otimes Id) \circ \delta(x) &= (\delta \otimes Id)(x \otimes g + u \otimes x) \\ &= \delta(x) \otimes g + \delta(u) \otimes x \\ &= (x \otimes g + u \otimes x) \otimes g + u \otimes u \otimes x \\ &= x \otimes g \otimes g + u \otimes x \otimes g + u \otimes u \otimes x.\end{aligned}$$

The first and third equality follow from the definition of δ , while the second equality follows from the application of the tensor mapping $(\delta \otimes Id)$, and the last equality follows from the properties of tensors.

Also, observe that:

$$\begin{aligned}(Id \otimes \delta) \circ \delta(x) &= (Id \otimes \delta)(x \otimes g + u \otimes x) \\ &= x \otimes \delta(g) + u \otimes \delta(x) \\ &= x \otimes g \otimes g + u \otimes (x \otimes g + u \otimes x) \\ &= x \otimes g \otimes g + u \otimes x \otimes g + u \otimes u \otimes x \\ &= (\delta \otimes Id) \circ \delta(x).\end{aligned}$$

The first and third equalities follow from the definition of δ , while the second equality follows from the application of the tensor mapping $Id \otimes \delta$, the fourth equality follows from the properties of the tensor, and the last equality follows from the previous computations. Hence, we have that δ is coassociative. Now, observe that:

$$\begin{aligned}(\epsilon \otimes Id) \circ \delta(g) &= (\epsilon \otimes Id)(g \otimes g) \\ &= \epsilon(g) \otimes g \\ &= 1 \otimes g.\end{aligned}$$

The first equality follows from the definition of δ , while the second equality follows from the application of the tensor mapping $\epsilon \otimes Id$ and the last equality follows from the definition of ϵ . Analogously, $(\epsilon \otimes Id) \circ \delta(u) = 1 \otimes u$, $(Id \otimes \epsilon) \circ \delta(g) = g \otimes 1$ and $(Id \otimes \epsilon) \circ \delta(u) = u \otimes 1$.

Now let us check the counit property at x .

$$\begin{aligned} (\epsilon \otimes Id) \circ \delta(x) &= (\epsilon \otimes Id)(x \otimes g + u \otimes x) \\ &= \epsilon(x) \otimes g + \epsilon(u) \otimes x \\ &= 0 + 1 \otimes x = 1 \otimes x. \end{aligned}$$

The first equality follows from the definition of δ , while the second equality follows from the application of the tensor mapping $(\epsilon \otimes Id)$; the third equality follows from the definition of ϵ , and the last equality is straightforward.

Analogously, one obtain $(Id \otimes \epsilon) \circ \delta(x) = x \otimes 1$. Therefore C is a coalgebra.

By Example 2.5.2, the free algebra $T(C)$ is a $\mathbb{K}$ -bialgebra. Let us denote its coproduct as Δ and its counit as ε . Consider the ideal I of $T(C)$ generated by

$$\{u - 1, g^2 - 1, x^2, xg + gx\}$$

We claim that this ideal is also a coideal, and hence, a bi-ideal.

In order to do this, consider the canonical projection $\pi : T(C) \rightarrow T(C)/I$; since $\ker(\pi) = I$ and

$$\ker(\pi \otimes \pi) = \ker(\pi) \otimes T(C) + T(C) \otimes \ker(\pi) = I \otimes T(C) + T(C) \otimes I,$$

proving that $\Delta(I) \subset I \otimes T(C) + T(C) \otimes I$ is equivalent to proving that $(\pi \otimes \pi) \circ \Delta(I) = 0$. Moreover, since Δ is multiplicative, it is enough to show that the generators of I are sent to zero by $(\pi \otimes \pi) \circ \Delta$. Analogously, since ε is multiplicative, we will show that $\varepsilon(I) = 0$ by showing that ε sends the generators of I to zero.

Observe that:

$$\begin{aligned} (\pi \otimes \pi) \circ \Delta(u - 1) &= (\pi \otimes \pi)(\Delta(u) - \Delta(1)) \\ &= (\pi \otimes \pi)(u \otimes u - 1 \otimes 1) \\ &= \pi(u) \otimes \pi(u) - \pi(1) \otimes \pi(1) \\ &= \bar{1} \otimes \bar{1} - \bar{1} \otimes \bar{1} = 0. \end{aligned}$$

The first equality follows from the linearity of Δ , while the second equality follows from the definition of Δ , the third equality follows from the application of the tensor mapping $\pi \otimes \pi$ and the fact that this transformation is a morphism of $\mathbb{K}$ -algebras, the fourth equality follows from the definition of I , and the last equality is straightforward.

Now we take $g^2 - 1$.

$$\begin{aligned}
(\pi \otimes \pi) \circ \Delta(g^2 - 1) &= (\pi \otimes \pi)(\Delta(g^2) - \Delta(1)) \\
&= (\pi \otimes \pi)((\Delta(g))^2 - 1 \otimes 1) \\
&= (\pi \otimes \pi)((g \otimes g)^2 - 1 \otimes 1) \\
&= (\pi \otimes \pi)(g^2 \otimes g^2 - 1 \otimes 1) \\
&= (\pi(g^2) \otimes \pi(g^2) - 1 \otimes 1) \\
&= \bar{1} \otimes \bar{1} - \bar{1} \otimes \bar{1} \\
&= 0.
\end{aligned}$$

The first equality follows from the linearity of Δ , while the second equality follows from the fact that Δ is a morphism of $\mathbb{K}$ -algebras, and the third equality follows from the definition of Δ , the fourth equality follows from the structure of product as in Proposition 2.5.1, the fifth equality is just an application of the tensor $\pi \otimes \pi$, the sixth equality follows from the definition of I and the last equality is straightforward.

Then, also observe that:

$$\begin{aligned}
(\pi \otimes \pi) \circ \Delta(x^2) &= (\pi \otimes \pi) \circ \Delta(x)^2 \\
&= (\pi \otimes \pi)(x \otimes g + u \otimes x)^2 \\
&= (\pi \otimes \pi)((x \otimes g)^2 + (x \otimes g)(u \otimes x) + (u \otimes x)(x \otimes g) + (u \otimes x)^2) \\
&= (\pi \otimes \pi)(x^2 \otimes g^2 + xu \otimes gx + ux \otimes xg + u^2 \otimes x^2) \\
&= \pi(x^2) \otimes \pi(g^2) + \pi(xu) \otimes \pi(gx) + \pi(ux) \otimes \pi(xg) + \pi(u^2) \otimes \pi(x^2) \\
&= \pi(xu) \otimes \pi(gx) + \pi(ux) \otimes \pi(xg) \\
&= \pi(x)\pi(u) \otimes \pi(gx) + \pi(u)\pi(x) \otimes \pi(xg) \\
&= \pi(x) \otimes \pi(gx) + \pi(x) \otimes \pi(xg) \\
&= \pi(x) \otimes (\pi(gx) + \pi(xg)) \\
&= \pi(x) \otimes (\pi(gx + xg)) \\
&= 0.
\end{aligned}$$

The first equality follows from the fact that Δ is a morphism of $\mathbb{K}$ -algebras, while the second equality follows from the definition of Δ , the third and fourth equalities follow from computations, the fifth equality follows from the application of $\pi \otimes \pi$, the sixth and eighth equalities follow from the definition of I , the eighth equality follows from the fact that π is a morphism of algebras, the ninth equality follows from the universal property of the tensor, and the tenth follows from the linearity of π . Finally, let us apply $(\pi \otimes \pi) \circ \Delta$

to $xg + gx$.

$$\begin{aligned}
(\pi \otimes \pi) \circ \Delta(xg + gx) &= (\pi \otimes \pi)(\Delta(xg) + \Delta(gx)) \\
&= (\pi \otimes \pi)(\Delta(x)\Delta(g) + \Delta(g)\Delta(x)) \\
&= (\pi \otimes \pi)((x \otimes g + u \otimes x)(g \otimes g) + (g \otimes g)(x \otimes g + u \otimes x)) \\
&= (\pi \otimes \pi)(xg \otimes g^2 + ug \otimes xg + gx \otimes g^2 + gu \otimes gx) \\
&= \pi(xg) \otimes \pi(g^2) + \pi(ug) \otimes \pi(xg) + \pi(gx) \otimes \pi(g^2) + \pi(gu) \otimes \pi(gx) \\
&= (\pi(xg) + \pi(gx)) \otimes \pi(g^2) + \pi(g) \otimes (\pi(xg) + \pi(gx)) \\
&= (\pi(xg + gx)) \otimes \pi(g^2) + \pi(g) \otimes (\pi(xg + gx)) \\
&= 0.
\end{aligned}$$

The first equality follows from the linearity of Δ , while the second equality follows from the fact that Δ is a morphism of $\mathbb{K}$ -algebras, the third equality follows from the definition of Δ and the fourth equality follows from the structure of product as in Proposition 2.5.1. The fifth equality is just an application of $\pi \otimes \pi$ (π is a morphism of $\mathbb{K}$ -algebras) and from the definition of I , the sixth equality follows from the universal property of tensorial product, the seventh equality follows from the linearity of π and the last equality follows from the definition of I . Now, our aim is to prove that $\varepsilon(I) = 0$, also by evaluating on generators. To begin, it is clear that

$$\varepsilon(x^2) = \varepsilon(x)^2 = 0$$

since ε is a morphism of $\mathbb{K}$ -algebras and $\varepsilon(x) = 0$.

Moving on,

$$\begin{aligned}
\varepsilon(u - 1) &= \varepsilon(u) - \varepsilon(1) \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

The first equality follows from the linearity of ε and the second from the definition of ε . Now evaluating at $g^2 - 1$,

$$\begin{aligned}
\varepsilon(g^2 - 1) &= \varepsilon(g^2) - \varepsilon(1) \\
&= \varepsilon(g)^2 - 1 \\
&= 1^2 - 1 \\
&= 0.
\end{aligned}$$

The first equality follows from the linearity of ε , while the second equality follows from the fact that ε is a morphism of $\mathbb{K}$ -algebras and the third equality follows from the definition of ε .

Also, note that:

$$\begin{aligned}\varepsilon(xg + gx) &= \varepsilon(xg) + \varepsilon(gx) \\ &= \varepsilon(x)\varepsilon(g) + \varepsilon(g)\varepsilon(x) \\ &= 0 + 0 = 0.\end{aligned}$$

The first equality follows from the linearity of ε , while the second equality follows from the fact that ε is a morphism of algebras, and the third equality follows from the definition of ε .

Since ε is a morphism of algebras, we have that $\varepsilon(I) = \{0\}$.

Hence, I is a coideal and then, a bi-ideal, and the morphism $\pi : T(C) \rightarrow T(C)/I$ is a morphism of bialgebras.

The $\mathbb{K}$ -bialgebra $T(C)/I$ defined above is the **Sweedler algebra** over the field $\mathbb{K}$. It is an example of a noncommutative and noncocommutative $\mathbb{K}$ -bialgebra.

This algebra is usually denoted by H_4 , and it may be defined by generators and relations in the following form:

$$H_4 = \mathbb{K}\langle g, x : g^2 = 1, x^2 = 0, xg = -gx \rangle.$$

The coproduct Δ is defined on generators as

$$\Delta(x) = x \otimes g + 1 \otimes x, \quad \Delta(g) = g \otimes g$$

and, therefore, ε on the generators is given by $\varepsilon(g) = 1, \varepsilon(x) = 0$.

Proposition 2.5.3. *Let B be a $\mathbb{K}$ -bialgebra. Then, if I is a subspace of B , B/I being a bialgebra is equivalent to I being a bi-ideal.*

When we say that B/I is a bialgebra, we mean that B/I is a bialgebra with respect to the canonical structures of algebra and coalgebra of B/I : $\Delta_{B/I}(\pi(a)) = (\pi \otimes \pi)\Delta_B(a)$ and $\varepsilon_{B/I}(\pi(a)) = \varepsilon_B(a)$, and also $m_{B/I}(\pi(a) \otimes \pi(b)) = \pi m_B(a \otimes b)$ and $\mu_{B/I}(\pi(a)) = \pi(\mu_B(1))$ for all $a, b \in B$.

Proof. Assume that B/I is a bialgebra. Then $\pi : B \rightarrow B/I$ is a morphism of $\mathbb{K}$ -bialgebras. Then, since π is a morphism of algebras, I is an ideal, and since π is a morphism of coalgebras, I is a coideal. Hence, I is a bi-ideal.

Assume now that I is a bi-ideal. In particular, it follows that π is a morphism of $\mathbb{K}$ -bialgebras. Denote $m_B(a \otimes b) = ab$. Then:

$$\begin{aligned}
 \Delta_{B/I}(\pi(a)\pi(b)) &= \Delta_{B/I}(\pi(ab)) \\
 &= (\pi \otimes \pi) \circ \Delta_B(ab) \\
 &= (\pi \otimes \pi)(\Delta_B(a)\Delta_B(b)) \\
 &= (\pi \otimes \pi)(\Delta_B(a)) \cdot (\pi \otimes \pi)(\Delta_B(b)) \\
 &= \Delta_{B/I}(\pi(a))\Delta_{B/I}(\pi(b))
 \end{aligned}$$

The first equality follows from the fact that π is a morphism of $\mathbb{K}$ -algebras. The second and the last equalities follow from the definition of $\Delta_{B/I}$. The third equality follows from the fact that Δ is a morphism of $\mathbb{K}$ -algebras, and the fourth equality follows from the fact that $\pi \otimes \pi$ is a morphism of $\mathbb{K}$ -algebras. Hence, $\Delta_{B/I}$ is a morphism of algebras. Also, observe that:

$$\begin{aligned}
 \varepsilon_{B/I}(\pi(a)\pi(b)) &= \varepsilon_B(ab) \\
 &= \varepsilon_B(a)\varepsilon_B(b) \\
 &= \varepsilon_{B/I}(\pi(a))\varepsilon_{B/I}(\pi(b)).
 \end{aligned}$$

The first and last equalities follow from the expression given for $\varepsilon_{B/I}$, and since π is a morphism of $\mathbb{K}$ -algebras, the second expression follows from the fact that B is a $\mathbb{K}$ -bialgebra and then ε_B is a morphism of algebras.

Then, we have that $\Delta_{B/I}$ and $\varepsilon_{B/I}$ are both morphisms of algebras, and by Definition 2.5.1(Bialgebras) we have that B/I is a bialgebra. $\square$

It is straightforward that the canonical projection $\pi : B \rightarrow B/I$ is a morphism of bialgebras, as in Definition 2.5.2 (Morphism of $\mathbb{K}$ -bialgebras).

The next definitions and propositions refer to specific types of elements in coalgebras and bialgebras. They are the grouplike elements and primitive elements. The first name is related to an example of coalgebras involving groups.

Definition 2.5.4 (Grouplike and primitive elements). *Let C be a $\mathbb{K}$ -coalgebra. Then:*

- *An element $c \in C$ is called **grouplike** if $c \neq 0$ and $\Delta(c) = c \otimes c$. The set of grouplike elements of C is denoted by $G(C)$.*
- *Let $g, h \in G(C)$. Then, $c \in C$ is **(g,h)-primitive** if $\Delta(c) = c \otimes g + h \otimes c$. If C is a bialgebra and $g = h = 1$, then c is just called a **primitive element**. The set of*

(g, h) -primitive elements of C is denoted by $P_{g,h}(C)$ and if C is a bialgebra and $g = h = 1$, then we denote it by $P(C)$.

Note that if $c \in G(C)$, then $\varepsilon(c) = 1$, while if $c \in P_{g,h}(C)$, then $\varepsilon(c) = 0$. In fact, let $c \in G(C)$. Applying the counit property to c we obtain:

$$1 \otimes c = (\varepsilon \otimes Id) \circ \Delta(c) = (\varepsilon \otimes Id)(c \otimes c) = \varepsilon(c) \otimes c.$$

The first equality follows from the counit property, the second follows from the fact that $c \in G(C)$ and the third is just the application of $\varepsilon \otimes Id$ in $c \otimes c$. Since $c \neq 0$, from $1 \otimes c = \varepsilon(c) \otimes c$ we conclude that $\varepsilon(c) = 1$. Also, observe that, if $c \in P_{g,h}(C)$, we have:

$$\begin{aligned} 1 \otimes c &= (\varepsilon \otimes Id) \circ \Delta(c) \\ &= (\varepsilon \otimes Id)(c \otimes g + h \otimes c) \\ &= \varepsilon(c) \otimes g + \varepsilon(h) \otimes c \\ &= \varepsilon(c) \otimes g + 1 \otimes c. \end{aligned}$$

The first equality is the counit property applied to c . The second equality holds since $c \in P_{g,h}(C)$. The third equality follows from the application of $\varepsilon \otimes Id$. The last equality holds since h and g are grouplike elements and then $\varepsilon(g) = \varepsilon(h) = 1$. Then, we have $\varepsilon(c) \otimes g = 0$. Since $g \neq 0$, we have that $\varepsilon(c) = 0$.

The next proposition is a property of the set of the grouplike elements of a given coalgebra.

Proposition 2.5.4. *Let C be a $\mathbb{K}$ -coalgebra. Then, any subset of $G(C)$ is $L.I.$*

Proof. Let C be a $\mathbb{K}$ -coalgebra and consider $G(C)$. Assume $G(C)$ is not a $L.I$ set. Since any element $c \in G(C)$ is non-null, any unitary subset $\{g\} \subset G(C)$ is $L.I$. Let $n \in \mathbb{N}$ be the smallest natural number such that any subset of $G(C)$ with n elements is $L.I$, but a $L.D$ subset of $G(C)$ with $n + 1$ elements exists. So, let $\{g, g_1, \dots, g_n\}$ a $L.D$ subset of $G(C)$ with $n + 1$ elements listed in such a way that we may write

$$g = \sum_{i=1}^n \alpha_i g_i.$$

Since n is minimal, $\alpha_i \neq 0$ for all $i \in \{1, \dots, n\}$.

Since $g \in G(C)$, we have:

$$\Delta(g) = g \otimes g = \left(\sum_{i=1}^n \alpha_i g_i \right) \otimes \left(\sum_{j=1}^n \alpha_j g_j \right) = \sum_{i,j=1}^n \alpha_i \alpha_j g_i \otimes g_j.$$

The first equation follows from the fact that $g \in G(C)$. The second equation follows from the expression for g , and the last equality is a property of tensor products.

However, by the linearity of g , we also have:

$$\Delta(g) = \Delta\left(\sum_{i=1}^n \alpha_i g_i\right) = \sum_{i=1}^n \alpha_i \Delta(g_i) = \sum_{i=1}^n \alpha_i g_i \otimes g_i.$$

The first equality follows from the expression for g , the second equality follows from the linearity of Δ , and the last equality follows since $g_1, \dots, g_n$ are grouplike elements.

Then, we have that:

$$\sum_{i,j=1}^n \alpha_i \alpha_j g_i \otimes g_j = \sum_{i=1}^n \alpha_i g_i \otimes g_i.$$

Since there is no *L.D* subsets of $G(C)$ with n elements, $\{g_1, \dots, g_n\}$ is a *L.I* set, and then $\{g_1 \otimes g_1, g_1 \otimes g_2, \dots, g_n \otimes g_n\}$ is also a *L.I* set. Hence, $\alpha_i \alpha_j = \delta_{i,j}$. Since $\alpha_i \neq 0$ for all $i \in \{1, \dots, n\}$, then $n = 1$. Hence, $g = \alpha_1 g_1$. Finally, given that g and g_1 are grouplike elements, $1 = \epsilon(g) = \epsilon(\alpha_1 g_1) = \alpha_1 \epsilon(g_1) = \alpha_1$, hence $\alpha_1 = 1$ and $g = g_1$, which is a contradiction with the hypothesis that $\{g, g_1\}$ has $n + 1 = 2$ elements. Hence there is no *L.D.* subset of grouplike elements and the proposition is proved. $\square$

The next proposition shows that the set of grouplike has an additional structure when our $\mathbb{K}$ -coalgebra is, in fact, a $\mathbb{K}$ -bialgebra. This additional structure is that of a monoid.

Definition 2.5.5. Let M be a set and $m : G \times G \rightarrow G$ be a function satisfying the following conditions:

- m is associative. That is, $m \circ (m \times Id) = m \circ (Id \times m)$.
- There exists $e \in M$ such that, for any element $g \in M$, we have $m(e, g) = m(g, e) = g$.

Then the pair (G, m) is called a **monoid** and m is the product of the monoid.

Proposition 2.5.5. Let B be a $\mathbb{K}$ -bialgebra. Then, $G(B)$ is a monoid with m_B restricted to $G(B)$ as the product.

In this case, $m(g, h) = m_B(g \otimes h)$.

Proof. Let $g, h \in G(B)$ and denote $m(g, h) = gh$. Then:

$$\Delta(gh) = \Delta(g)\Delta(h) = (g \otimes g)(h \otimes h) = gh \otimes gh.$$

The first equality holds since B is a bialgebra and then Δ is a morphism of algebras. The second equality holds since $g, h \in G(B)$ and the last equality is just the product in $B \otimes B$. Then, gh is grouplike or $gh = 0$. However,

$$\varepsilon(gh) = \varepsilon(g)\varepsilon(h) = 1 \cdot 1 = 1.$$

The first equality holds since B is a bialgebra and then ε is a morphism of algebras. The second equality holds since $g, h \in G(B)$. Since ε is linear, we have that $gh \neq 0$. Hence, gh is grouplike. Since Δ is a morphism of algebras, we have that $\Delta(1) = 1 \otimes 1$. Since m_B is associative and 1 is its neutral element, $G(B)$ is a monoid. $\square$

Before we can finally define Hopf $\mathbb{K}$ -algebras, we need the definition of the convolution product, which is given as follows:

Definition 2.5.6. Let C be a $\mathbb{K}$ -coalgebra and A a $\mathbb{K}$ -algebra and let $f, g : C \rightarrow A$ be both linear mappings. Then, the **convolution product** of f and g , denoted by $f * g : C \rightarrow A$, is defined as

$$f * g = m_A \circ (f \otimes g) \circ \Delta_C.$$

The following result states that $\text{Hom}(C, A)$ is a $\mathbb{K}$ -algebra with this product.

Proposition 2.5.6. Let C be a $\mathbb{K}$ -coalgebra and A a $\mathbb{K}$ -algebra. Then $(\text{Hom}(C, A), m_{\text{Hom}(C, A)}, \mu_{\text{Hom}(C, A)})$ is a $\mathbb{K}$ -algebra, with $m_{\text{Hom}(C, A)}$ and $\mu_{\text{Hom}(C, A)}$ defined as

$$m_{\text{Hom}(C, A)}(f \otimes g) = f * g,$$

$$\mu_{\text{Hom}(C, A)}(1) = \mu_A \circ \varepsilon_C.$$

Proof. First, let us prove the associativity of $m_{\text{Hom}(C, A)}$. In fact, we have, for $f, g, h \in \text{Hom}(C, A)$:

$$\begin{aligned}
m_{Hom(C,A)} \circ (Id \otimes m_{Hom(C,A)})(f \otimes g \otimes h) &= m_{Hom(C,A)}(f \otimes m_{Hom(C,A)}(g \otimes h)) \\
&= f * (g * h) \\
&= m_A \circ (f \otimes (m_A \circ (g \otimes h) \circ \Delta_C)) \circ \Delta_C \\
&= m_A \circ (f \otimes Id) \circ (Id \otimes m_A) \circ (Id \otimes (g \otimes h)) \\
&\quad \circ (Id \otimes \Delta_C) \circ \Delta_C \\
&= m_A \circ (Id \otimes m_A) \circ (f \otimes g \otimes h) \circ (Id \otimes \Delta_C) \circ \Delta_C \\
&= m_A \circ (m_A \otimes Id) \circ (f \otimes g \otimes h) \circ (\Delta_C \otimes Id) \circ \Delta_C \\
&= m_A \circ (m_A \circ (f \otimes g) \otimes h) \circ (\Delta_C \otimes Id) \circ \Delta_C \\
&= m_A \circ (m_A \circ (f \otimes g) \circ \Delta_C) \otimes h \circ \Delta_C \\
&= m_A \circ (f * g \otimes h) \circ \Delta_C \\
&= (f * g) * h \\
&= m_{Hom(C,A)} \circ (m_{Hom(C,A)} \otimes Id)(f \otimes g \otimes h).
\end{aligned}$$

The first, fourth, fifth, seventh and eighth equalities follow from properties of tensor products. The second and the last equalities follows from the definition of $m_{Hom(C,A)}$. The third, ninth and tenth equalities follow from Definition 2.5.6 (Convolution product). The sixth equality follow from the associativity of m_A and the coassociativity of Δ_C . Then, $m_{Hom(C,A)}$ is associative. Observe that, in Sigma notation and for $c \in C$, we have:

$$f * g(c) = \sum_{(c)} f(c_{(1)})g(c_{(2)}).$$

Now, let us prove the unit property. In fact, observe that, for $f \in Hom(C, A)$ given, we have:

$$\begin{aligned}
m_{Hom(C,A)} \circ (f \otimes (\mu_A \circ \varepsilon_C)) &= m_A \circ (f \otimes (\mu_A \circ \varepsilon_C)) \circ \Delta_C \\
&= m_A \circ ((Id \circ f \circ Id) \otimes \mu_A \circ \varepsilon_C) \circ \Delta_C \\
&= m_A \circ (Id \otimes \mu_A) \circ f \circ (Id \otimes \varepsilon_C) \circ \Delta_C \\
&= f.
\end{aligned}$$

The first equality follows from the definition of $m_{Hom(C,A)}$, the second and third equalities follow from simple tensor product properties and notation abuse ($C \otimes \mathbb{K} = C$), and the last equality follows from the unit property of A and the unit property of C . Hence, we have the counit property proven and then, the proposition holds. $\square$

Then, we can define the following:

Definition 2.5.7. Let C be a $\mathbb{K}$ -coalgebra and A a $\mathbb{K}$ -algebra. Then, $(\text{Hom}(C, A), *, \mu \circ \varepsilon)$ is the **convolution algebra** of C and A .

Let H be a $\mathbb{K}$ -bialgebra. Hence, H is both a $\mathbb{K}$ -algebra and a $\mathbb{K}$ -coalgebra. So, we may consider $\text{Hom}(H, H)$ as the convolution algebra of H . This is essential for the definition of Hopf $\mathbb{K}$ -algebras.

Definition 2.5.8 (Antipode and Hopf $\mathbb{K}$ -algebras). Let $(H, m, \mu, \Delta, \varepsilon)$ be a $\mathbb{K}$ -bialgebra. If there exists $S : H \rightarrow H$ such that:

$$S * Id = \mu \circ \varepsilon$$

$$Id * S = \mu \circ \varepsilon$$

then S is called an **antipode**, and $H = (H, m, \mu, \Delta, \varepsilon, S)$ is a **Hopf $\mathbb{K}$ -algebra**.

In Sweedler Notation, it means that, for $h \in H$, we have:

$$\sum_{(h)} S(h_{(1)})h_{(2)} = \sum_{(h)} h_{(1)}S(h_{(2)}) = 1\varepsilon(h)$$

Example 2.5.4. Recall Example 2.5.1, where we have the $\mathbb{K}$ -vector space $\mathbb{K}G$ with basis given by a group G , and an algebra structure where the product is induced by the multiplication in G , the unit is the neutral element, and the coalgebra structure is defined in such a way that every element of G is a grouplike element.

We may define the antipode $S : \mathbb{K}G \rightarrow \mathbb{K}G$ as:

$$S(g) = g^{-1}.$$

Observe that S is, in fact, an antipode for $\mathbb{K}G$. In fact, we have:

$$\begin{aligned} m \circ (S \otimes Id) \circ \Delta(g) &= m \circ (S \otimes Id)(g \otimes g) \\ &= m(S(g) \otimes g) \\ &= S(g)g \\ &= g^{-1}g \\ &= e \\ &= \mu(1) \\ &= \mu \circ \varepsilon(g). \end{aligned}$$

The first equality follows from the definition of Δ , while the second equality follows from an application of the tensor mapping $S \otimes Id$, the third equality is just the notation for product, the fourth equality follows from the definition of S , the fifth equality follows from the structure of group, the sixth equality follows from the definition of unit,

and the last equality follows from the definition of ε . By analogous computations, we also have that

$$m \circ (Id \otimes S) \circ \Delta(g) = \mu \circ \varepsilon(g).$$

Example 2.5.5. Recall Example 2.5.3 (Sweedler algebra). The Sweedler algebra H_4 can be defined by generators and relations in the following manner:

$$H_4 = \mathbb{K}\langle g, x; g^2 = 1, x^2 = 0, xg = -gx \rangle,$$

with coproduct and counit defined as

$$\Delta(g) = g \otimes g,$$

$$\Delta(x) = x \otimes 1 + g \otimes x,$$

$$\varepsilon(g) = 1,$$

$$\varepsilon(x) = 0.$$

As we have shown in Example 2.5.3, the coalgebra structure is well-defined in such a way that $(H_4, m, \mu, \Delta, \varepsilon)$ is a $\mathbb{K}$ -bialgebra. Moreover, it is easy to see that $\{1, g, x, gx\}$ is a basis for H_4 .

Let us define the antipode S as the linear mapping $S : H_4 \rightarrow H_4$ given by

$$S(1) = 1, \quad S(g) = g, \quad S(x) = -gx, \quad S(gx) = x.$$

We will check that the defining equalities for the antipode hold for each element in the mentioned basis of H_4 .

Observe that:

$$\begin{aligned} (S * Id)(1) &= m \circ (S \otimes Id) \circ \Delta(1) \\ &= m \circ (S \otimes Id)(1 \otimes 1) \\ &= m(S(1) \otimes 1) \\ &= S(1)1 \\ &= 1 \\ &= \mu(1) \\ &= \mu \circ \varepsilon(1). \end{aligned}$$

The first equality follows from the definition of convolution product, while the second equality follows from the fact that Δ is a morphism of algebras, the third equality is just an application of the tensor $S \otimes Id$, the fourth equality follows from notation for product,

the fifth equality follows from the definition of S , the sixth follows from unit property, the seventh equality follows from the definition of μ , while the last equality follows from the definition of ε . In an analogous way, we have that $(Id * S)(1) = \mu \circ \varepsilon(1)$.

Now, also observe that:

$$\begin{aligned}
 (S * Id)(g) &= m \circ (S \otimes Id) \circ \Delta(g) \\
 &= m \circ (S \otimes Id)(g \otimes g) \\
 &= m(S(g) \otimes g) \\
 &= S(g)g \\
 &= g^2 \\
 &= 1 \\
 &= \mu(1) \\
 &= \mu \circ \varepsilon(g).
 \end{aligned}$$

The first equality follows from the definition of convolution product, while the second equality follows from the definition of Δ . The third equality is just an application of the tensor mapping $S \otimes Id$, the fourth equality is just the notation of product, the fifth equality follows from the definition of S , while the sixth equality from the defining relation for g^2 in the definition of H_4 . The seventh equality follows from the definition of μ , and the last equality follows from the definition of ε . In an analogous way, we have that $Id * S(g) = \mu \circ \varepsilon(g)$.

Now, observe that:

$$\begin{aligned}
 (S * Id)(x) &= m \circ (S \otimes Id) \circ \Delta(x) \\
 &= m \circ (S \otimes Id)(x \otimes 1 + g \otimes x) \\
 &= m(S(x) \otimes 1 + S(g) \otimes x) \\
 &= S(x)1 + S(g)x \\
 &= -gx + gx \\
 &= 0 \\
 &= \mu(0) \\
 &= \mu \circ \varepsilon(x).
 \end{aligned}$$

The first equality follows from the definition of convolution product, the second equality follows from the definition of Δ , the third equality follows from linearity and the application of the tensor mapping $S \otimes Id$ and the fourth equality follows from product notation. The fifth equality follows from the definition of antipode, the sixth equality is straightforward,

the seventh equality follows from the linearity of μ , and the last equality follows from the definition of ε .

But, also observe that:

$$\begin{aligned}
 (Id * S)(x) &= m \circ (Id \otimes S) \circ \Delta(x) \\
 &= m \circ (Id \otimes S)(x \otimes 1 + g \otimes x) \\
 &= m(x \otimes S(1) + g \otimes S(x)) \\
 &= xS(1) + gS(x) \\
 &= x \cdot 1 + g(-gx) \\
 &= x - g^2x \\
 &= x - x \\
 &= 0 \\
 &= \mu(0) \\
 &= \mu \circ \varepsilon(x).
 \end{aligned}$$

The first equality follows from the definition of convolution product, while the second equality follows from the definition of Δ , the third equality is just an application of $Id \otimes S$ and linearity, the fourth equality is just the product notation; the fifth equality follows from the definition of S , the sixth and seventh equalities follow from the algebra structure in H_4 , the eighth equality is straightforward, the ninth equality follows from the linearity of μ , and the last equality follows from the definition of ε .

Also, observe that:

$$\begin{aligned}
 (S * Id)(gx) &= m \circ (S \otimes Id) \circ \Delta(gx) \\
 &= m \circ (S \otimes Id)(\Delta(g)\Delta(x)) \\
 &= m \circ ((S \otimes Id)(g \otimes g)(x \otimes 1 + g \otimes x)) \\
 &= m \circ (S \otimes Id)(gx \otimes g + g^2 \otimes gx) \\
 &= m \circ (S \otimes Id)(gx \otimes g + 1 \otimes gx) \\
 &= m(S(gx) \otimes g + S(1) \otimes gx) \\
 &= S(gx)g + S(1)gx \\
 &= xg + gx \\
 &= 0 \\
 &= \mu(0) \\
 &= \mu(\varepsilon(x)\varepsilon(g)) \\
 &= \mu \circ \varepsilon(xg).
 \end{aligned}$$

The first equality follows from the definition of convolution product, while the second equality follows from the fact that Δ is a morphism of algebras, the third equality follows

from the definition of Δ , the fourth equality follows from the structure of product in $H_4 \otimes H_4$ just like in Proposition 2.5.1, the fifth equality follows from the fact that $g^2 = 1$, the sixth equality follows from the application of the tensor mapping $S \otimes Id$, while the seventh equality follows from notation for product. The eighth equality follows from the definition of antipode, the ninth equality follows from the fact that $xg = -gx$, the tenth equality follows from the linearity of μ , the eleventh equality follows from the definition of ε in x , and the last equality follows from the fact that ε is a morphism of algebras.

Now, observe that:

$$\begin{aligned}
 (Id * S)(gx) &= m \circ (Id \otimes S) \circ \Delta(gx) \\
 &= m \circ (Id \otimes S)(\Delta(g)\Delta(x)) \\
 &= m \circ (Id \otimes S)((g \otimes g)(x \otimes 1 + g \otimes x)) \\
 &= m \circ (Id \otimes S)(gx \otimes g + g^2 \otimes gx) \\
 &= m \circ (Id \otimes S)(gx \otimes g + 1 \otimes gx) \\
 &= m(gx \otimes S(g) + 1 \otimes S(gx)) \\
 &= gxS(g) + 1 \cdot S(gx) \\
 &= (gx)g - 1 \cdot x \\
 &= 0 \\
 &= \mu(0) \\
 &= \mu(\varepsilon(g)\varepsilon(x)) \\
 &= \mu \circ \varepsilon(gx).
 \end{aligned}$$

The first equality follows from the definition of convolution product, while the second equality follows from the fact that Δ is a morphism of algebras, the third equality follows from the definition of Δ , the fourth and ninth equalities follow from the $\mathbb{K}$ -algebra structures in H_4 and $H_4 \otimes H_4$ due to the Proposition 2.5.1, and the fifth equality follows from the fact that $g^2 = 1$. The sixth equality follows from an application of the tensor mapping $Id \otimes S$, the seventh equality is just a notation for the product, the eighth equality follows from the definition of S , the tenth equality is straightforward, the eleventh equality follows from the linearity of μ , the twelfth equality follows from the fact that $\varepsilon(x) = 0$, and the last equality follows from the fact that ε is a morphism of algebras.

Hence, we have that S is an antipode for H_4 .

For the next examples, we need to define Lie Algebras.

Definition 2.5.9. A Lie Algebra $(V, [\cdot, \cdot])$ is a $\mathbb{K}$ -vector space V together with a bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ satisfying the following properties:

$$\bullet [x, y] = -[y, x], x, y \in V.$$

$$\bullet [x, [y, z]] = -[z, [x, y]] - [y, [z, x]]$$

Example 2.5.6. Let A be a $\mathbb{K}$ -algebra. Then, A is a Lie Algebra with bracket given by $[x, y] = xy - yx, x, y \in A$.

Then, we may consider the following example of a Hopf $\mathbb{K}$ -algebra.

Example 2.5.7. Let us consider $U(V)$ the enveloping universal algebra of a Lie Algebra V , defined in the following manner: $U(V) = T(V)/I$, where I is the ideal of $T(V)$ generated by the elements of the form $xy - yx - [x, y]$.

So, $U(V)$ is a $\mathbb{K}$ -algebra.

Observe that $U(V)$ has the structure of a coalgebra in the following manner:

We define, for a basis $\{v_i\}$ of V $\Delta(v_i) = 1 \otimes v_i + v_i \otimes 1$ and $\varepsilon(v_i) = 0$, in such way that Δ and ε are morphisms of algebras. Then, we have a bialgebra. If we define $S(x) = -x$ for $x \in V$ in such way that S is a antimorphism of algebras, we have the antipode.

The next definitions and propositions are essential for understanding the antipode as a generalization for inverses.

Definition 2.5.10. Let A and B be both $\mathbb{K}$ -algebras. Then a linear mapping $T : A \rightarrow B$ is an **antimorphism of $\mathbb{K}$ -algebras** if $T(ab) = T(b)T(a)$ and $T(1_A) = 1_B$ for all $a, b \in A$. And let C, D be both $\mathbb{K}$ -coalgebras. Then a linear mapping $T : C \rightarrow D$ is an **antimorphism of coalgebras** if $\sum_{(T(c))} T(c)_{(1)} \otimes T(c)_{(2)} = \sum_{(c)} T(c_{(2)}) \otimes T(c_{(1)})$ for all $c \in C$ and $\varepsilon_D T = \varepsilon_C$.

Definition 2.5.11. Let H_1, H_2 be both Hopf $\mathbb{K}$ -algebras. A linear map $f : H_1 \rightarrow H_2$ is a **morphism of Hopf $\mathbb{K}$ -algebras** if f is a morphism of bialgebras.

Proposition 2.5.7. Let H be a Hopf $\mathbb{K}$ -algebra with antipode S . So, S is an antimorphism of $\mathbb{K}$ -algebras and an antimorphism of $\mathbb{K}$ -coalgebras.

Proof. First of all, observe that, since ε is a morphism of algebras, we have $\varepsilon(1) = 1$ and, since Δ is also a morphism of algebras, we have $\Delta(1) = 1 \otimes 1$.

Now, observe that:

$$\begin{aligned} 1 &= 1 \cdot 1 \\ &= \varepsilon(1)1 \\ &= \mu \circ \varepsilon(1) \\ &= (Id * S)(1) \\ &= 1S(1) \\ &= S(1). \end{aligned}$$

The second equality follows since $\varepsilon(1) = 1$, the third follows from the definition of μ , the fourth from the definition of antipode, the fifth follows from the fact $\Delta(1) = 1 \otimes 1$ and the first and sixth equalities are straightforward.

Now, let us consider the functions $M, N, P : H \otimes H \rightarrow H$ defined as:

$$\begin{aligned} M(g \otimes h) &= gh, \\ N(g \otimes h) &= S(h)S(g), \\ P(g \otimes h) &= S(gh). \end{aligned}$$

Since $M, N, P \in \text{Hom}(H \otimes H, H)$, we can consider the convolution product of $\text{Hom}(H \otimes H, H)$. Defining $\bar{\varepsilon} = \varepsilon_{H \otimes H}$ we have that $\mu \circ \bar{\varepsilon}$ is the unit of this algebra. Note that we just need to prove that $P * M = \mu \circ \bar{\varepsilon} = M * N$, since if these equalities hold then we can conclude that:

$$P = P * (M * N) = (P * M) * N = N.$$

Observe, for $g, h \in H$:

$$\begin{aligned} (M * N)(g \otimes h) &= \sum_{(g), (h)} M(g_{(1)} \otimes h_{(1)}) N(g_{(2)} \otimes h_{(2)}) \\ &= \sum_{(g), (h)} g_{(1)} h_{(1)} S(h_{(2)}) S(g_{(2)}) \\ &= \sum_{(g)} g_{(1)} S(g_{(2)}) \varepsilon(h) \\ &= \varepsilon(g) \varepsilon(h) 1 = \mu \circ \bar{\varepsilon}(g \otimes h). \end{aligned}$$

The first equality follows from the definition of convolution product, the second follows from the definition of M and N . The third and fourth equalities follow from Definition 2.5.8 (Antipode) applied to h and g respectively, while the last equality follows from the definition of $\bar{\varepsilon}$ and μ . So, we have it proven that $M * N = \mu \circ \bar{\varepsilon}$.

Let us observe also for $g, h \in H$:

$$\begin{aligned} (P * M)(g \otimes h) &= \sum_{(g), (h)} P(g_{(1)} \otimes h_{(1)}) M(g_{(2)} \otimes h_{(2)}) \\ &= \sum_{(g), (h)} S(g_{(1)} h_{(1)}) g_{(2)} h_{(2)} \\ &= \sum_{(gh)} S((gh)_{(1)}) (gh)_{(2)} \\ &= (S * Id)(gh) \\ &= \varepsilon(gh) 1 \\ &= \varepsilon(g) \varepsilon(h) 1 = \mu \circ \bar{\varepsilon}(g \otimes h). \end{aligned}$$

The first and fourth equalities follow from Definition 2.5.7 (Convolution algebra), while the second equality follows from the definitions of P and M . The third and sixth equalities follow from the fact that H is Hopf $\mathbb{K}$ -algebra. The fifth equality follows from Definition 2.5.8 (Antipode) and the last equality follows from the definition of μ and $\bar{\varepsilon}$. So, we have proved that the antipode is an antimorphism of $\mathbb{K}$ -algebras. Now, let us prove that S is an antimorphism of $\mathbb{K}$ -coalgebras. Define $\bar{\mu} = \mu_{H \otimes H}$ and consider $N, P : H \rightarrow H \otimes H$ defined as:

$$\begin{aligned} N(h) &= \sum_{(h)} S(h_{(2)}) \otimes S(h_{(1)}), \\ P(h) &= \Delta(S(h)). \end{aligned}$$

Observe that it suffices to prove that $P * \Delta = \Delta * N = \mu \circ \varepsilon$, since if these equalities hold then $P = P * (\Delta * N) = (P * \Delta) * N = N$, in the convolution algebra $\text{Hom}(H, H \otimes H)$. First, observe that, for $h \in H$:

$$\begin{aligned} P * \Delta(h) &= \sum_{(h)} P(h_{(1)}) \Delta(h_{(2)}) \\ &= \sum_{(h)} \Delta(S(h_{(1)})) \Delta(h_{(2)}) \\ &= \sum_{(h)} \Delta(S(h_{(1)}) h_{(2)}) \\ &= \Delta \left(\sum_{(h)} S(h_{(1)}) h_{(2)} \right) \\ &= \Delta((S * Id)(h)) \\ &= \Delta(\mu \circ \varepsilon(h)) \\ &= \Delta(1) \varepsilon(h) \\ &= 1 \otimes 1 \varepsilon(h) = \bar{\mu} \circ \varepsilon(h). \end{aligned}$$

The first and fifth equalities follow from Definition 2.5.7 while the second equality follows from the definition of P , the third and eighth follow from the fact that H is a Hopf $\mathbb{K}$ -algebra and then Δ is a morphism of algebras, the sixth equality follows from Definition 2.5.8, and the last equality follows from the definition of $\bar{\mu}$. But, also, observe that, for $h \in H$:

$$\begin{aligned}
\Delta * N(h) &= \sum_{(h)} \Delta(h_{(1)}) N(h_{(2)}) \\
&= \sum_{(h)} \left(h_{(1)(1)} \otimes h_{(1)(2)} \right) \left(S(h_{(2)(2)}) \otimes S(h_{(2)(1)}) \right) \\
&= \sum_{(h)} (h_{(1)} \otimes h_{(2)}) (S(h_{(4)}) \otimes S(h_{(3)})) \\
&= \sum_{(h)} h_{(1)} S(h_{(4)}) \otimes h_{(2)} S(h_{(3)}) \\
&= \sum_{(h)} h_{(1)} S(h_{(3)}) \otimes ((Id * S)(h_{(2)})) \\
&= \sum_{(h)} h_{(1)} S(h_{(3)}) \otimes \mu \varepsilon(h_{(2)}) \\
&= \sum_{(h)} h_{(1)} S(h_{(3)}) \otimes \varepsilon(h_{(2)}) 1 \\
&= \sum_{(h)} h_{(1)} \varepsilon(h_{(2)}) S(h_{(3)}) \otimes 1 \\
&= \sum_{(h)} h_{(1)} S(h_{(2)}) \otimes 1 \\
&= (Id * S)(h) \otimes 1 \\
&= \mu \circ \varepsilon(h) \otimes 1 \\
&= \varepsilon(h) (1 \otimes 1) \\
&= \bar{\mu} \circ \varepsilon(h).
\end{aligned}$$

The first, fifth and tenth equalities follow from Definition 2.5.7, while the second equality follows from the definition of N . The third equality follows from the Sweedler notation, the fourth is the definition of product for $H \otimes H$, the sixth and eleventh equalities follow from Definition 2.5.8. The seventh and twelfth equalities follow from the definition of μ , the ninth equality follows from the counit property, and the last equality follows from the definition of $\bar{\mu}$. So, we have proven that S is an antimorphism of $\mathbb{K}$ -coalgebras. $\square$

Note that, if H is a Hopf $\mathbb{K}$ -algebra with antipode S , then the grouplike elements of H form a group with m as the product. In fact, m is associative, 1_H is a neutral element, and by the definition of antipodes we have, for $g \in G(H)$:

$$\begin{aligned}
1 &= \mu \circ \varepsilon(g) = (S * Id)(g) \\
&= m \circ (S \otimes Id) \circ \Delta(g) \\
&= S(g)g.
\end{aligned}$$

That follows since g is grouplike and so $\varepsilon(g) = 1$. But also observe that, in an analogous way, $gS(g) = 1$. So, $S(g) = g^{-1}$. But also observe that, since S is an antimorphism of $\mathbb{K}$ -coalgebras, we have:

$$\Delta(S(g)) = S(g) \otimes S(g).$$

So, $S(g)$ is a grouplike element and then $G(H)$ forms a group. This can be expressed in the next proposition:

Proposition 2.5.8. *Let $(H, m, \mu, \Delta, \varepsilon, S)$ be a Hopf $\mathbb{K}$ -algebra. Then, $G(H)$ is a group with m as operation, 1_H as neutral element and inverse given by $S(g) = g^{-1}$.*

The next proposition relates antipodes and the twist mapping (Definition 2.2.1). We recall that given $\mathbb{K}$ -vector spaces V, W , the associated twist map $\tau : V \otimes W \rightarrow W \otimes V$ is defined on generators by $\tau(v \otimes w) = w \otimes v$.

Proposition 2.5.9. *Let $(B, m, \mu, \Delta, \varepsilon)$ be a bialgebra. Then:*

- B^{op} and B^{cop} are $\mathbb{K}$ -bialgebras.
- S being an antipode for B is equivalent to S being an antipode for $(B^{op})^{cop}$.

Denoting the product of B as m , remember that B^{op} is the same $\mathbb{K}$ -vector space B with same coproduct, unit and counit, but with product $m_{B^{op}} = m_B \circ \tau$, where τ is the twist function as in Definition 2.2.1. Analogously, B^{cop} is the same $\mathbb{K}$ -vector space B with same product, unit and counit, but $\Delta_{B^{cop}} = \tau \circ \Delta_B$.

Proof. Let us first show that B^{op} is a $\mathbb{K}$ -bialgebra. First of all, we note that

$$m_{B^{op} \otimes B^{op}} = m_{(B \otimes B)^{op}} = m_{B \otimes B} \circ \tau_{B \otimes B, B \otimes B}.$$

In fact, denoting $m_B(a \otimes b) = ab$ and $m_{B^{op}}(a \otimes b) = a \cdot b = ba$, given $a_1, a_2, b_1, b_2 \in B$, we have that:

$$\begin{aligned} m_{B^{op} \otimes B^{op}}((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)) &= a_1 \cdot a_2 \otimes b_1 \cdot b_2 \\ &= a_2 a_1 \otimes b_2 b_1 \\ &= (a_2 \otimes b_2)(a_1 \otimes b_1) \\ &= m_{(B \otimes B)^{op}}((a_1 \otimes b_1)(a_2 \otimes b_2)) \\ &= m_{B \otimes B} \circ \tau_{B \otimes B, B \otimes B}((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)). \end{aligned}$$

The first equality follows from the definition of $m_{B^{op}}$, while the second equality follows from the definition of m_B , the third equality follows from the definition of $m_{B \otimes B}$ and the

last equality follows from Definition 2.2.1.

Now, let us denote $m_{B^{op} \otimes B^{op}}((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)) = (a_1 \otimes b_1) \cdot (a_2 \otimes b_2)$. Then, our aim is to prove that Δ and ε are morphisms of $\mathbb{K}$ -algebras with $m_{B^{op}}$ and $\mu_{B^{op}} = \mu_B$ as the product and the unit respectively.

For $a, b \in B$, observe that:

$$\begin{aligned}\Delta(a \cdot b) &= \Delta(ba) \\ &= \Delta(b)\Delta(a) \\ &= \Delta(a) \cdot \Delta(b).\end{aligned}$$

The first and last equalities follow from the definition of $\cdot$, while the second follows from the fact that Δ is a morphism of $\mathbb{K}$ -algebras. Since it is straightforward that $\Delta(1) = 1 \otimes 1$, we get that Δ is a morphism of $\mathbb{K}$ -algebras.

The calculations for ε are similar. Considering $a, b \in B$, we have:

$$\varepsilon(a \cdot b) = \varepsilon(ba) = \varepsilon(b)\varepsilon(a) = \varepsilon(a)\varepsilon(b).$$

The first equality follows from the definition of the opposite product $\cdot$, while the second equality follows from the fact that ε is a morphism of algebras for B , and the last equality follows from the fact that $\mathbb{K}$ is a field and then its product is commutative. Also, $\varepsilon(1) = 1$ also holds. So, we have proven that B^{op} is a $\mathbb{K}$ -bialgebra. Now, let us prove that B^{cop} is a $\mathbb{K}$ -bialgebra. In fact, let us observe that

$$\Delta_{B^{cop} \otimes B^{cop}}(a \otimes b) = \sum_{(a), (b)} a_{(2)} \otimes b_{(2)} \otimes a_{(1)} \otimes b_{(1)}.$$

In fact, for $a, b \in B$, we have:

$$\begin{aligned}\Delta_{B^{cop} \otimes B^{cop}}(a \otimes b) &= (Id \otimes \tau \otimes Id)(\Delta_{B^{cop}} \otimes \Delta_{B^{cop}})(a \otimes b) \\ &= (Id \otimes \tau \otimes Id) \left(\sum_{(a), (b)} a_{(2)} \otimes a_{(1)} \otimes b_{(2)} \otimes b_{(1)} \right) \\ &= \sum_{(a), (b)} a_{(2)} \otimes b_{(2)} \otimes a_{(1)} \otimes b_{(1)}.\end{aligned}$$

The first equality follows from the definition of $\Delta_{B^{cop} \otimes B^{cop}}$, while the second follows from the definition of $\Delta_{B^{cop}}$ and the third follows from the definition of τ , (i.e Definition 2.2.1). For the proof that B^{cop} is a $\mathbb{K}$ -bialgebra, we prove that $m_{B^{cop}} = m_B$ and $\mu_{B^{cop}} = \mu_B$ are

morphisms of $\mathbb{K}$ -coalgebras. So, for $a, b \in B$, observe that:

$$\begin{aligned}
 (m_{B^{cop}} \otimes m_{B^{cop}}) \circ \Delta_{B^{cop} \otimes B^{cop}}(a \otimes b) &= (m_{B^{cop}} \otimes m_{B^{cop}}) \left(\sum_{(a),(b)} a_{(2)} \otimes b_{(2)} \otimes a_{(1)} \otimes b_{(1)} \right) \\
 &= \sum_{(a),(b)} a_{(2)} b_{(2)} \otimes a_{(1)} b_{(1)} \\
 &= \sum_{(ab)} (ab)_{(2)} \otimes (ab)_{(1)} \\
 &= \Delta_{B^{cop}}(ab) \\
 &= \Delta_{B^{cop}} \circ m_{B^{cop}}(a \otimes b).
 \end{aligned}$$

The first equality follows from the expression that we've obtained for $\Delta_{B^{cop} \otimes B^{cop}}$, while the second equality is an application of $m_{B^{cop}} \otimes m_{B^{cop}}$, the third follows from the fact that $m_{B^{cop}}$ is a morphism of $\mathbb{K}$ -coalgebras for B , the fourth follows from the definition of $\Delta_{B^{cop}}$ and the last equality follows from application of $m_{B^{cop}}$ to $(a \otimes b)$. Clearly, $\varepsilon_{B^{cop} \otimes B^{cop}}(a \otimes b) = \varepsilon(a)\varepsilon(b) = \varepsilon(b)\varepsilon(a) = \varepsilon(ba) = \varepsilon(a \cdot b) = \varepsilon(m_{B^{cop}}(a \otimes b))$. Therefore, $m_{B^{cop}}$ is a morphism of $\mathbb{K}$ -coalgebras. Now let us prove that μ is a morphism of coalgebras.

In fact, observe that:

$$\Delta_B \circ \mu(1) = \Delta_B(1_B) = 1_B \otimes 1_B = (\mu \otimes \mu)(1 \otimes 1) = (\mu \otimes \mu) \circ \Delta_{\mathbb{K}}(1).$$

The first and third equalities follow from the definition of μ , while the second follow from the fact that Δ_B is a morphism of $\mathbb{K}$ -algebras and the last equality follows from the definition of $\Delta_{\mathbb{K}}$. Also:

$$\varepsilon_B(\mu(1)) = \varepsilon_B(1) = 1 = \varepsilon_{\mathbb{K}}(1)$$

Hence, it is proven that μ is a morphism of $\mathbb{K}$ -coalgebras. Now, let us prove the second statement. In fact, suppose that B has an antipode S . Since $(\Delta^{op})^{cop} = \tau \circ \Delta$ and $(m^{op})^{cop} = m \circ \tau$ we have that:

$$m \circ \tau \circ (S \otimes Id) \circ \tau \circ \Delta = m \circ (Id \otimes S) \circ \tau \circ \Delta = m \circ (S \otimes Id) \circ \Delta.$$

The first equality follows from Definition 2.2.1, and the second equality follows from the hypothesis that S is an antipode for B . Then, we have proven that

$$m \circ \tau \circ (S \otimes Id) \circ \tau \circ \Delta = m \circ (S \otimes Id) \circ \Delta.$$

By analogous arguments, we prove that

$$m \circ \tau \circ (Id \otimes S) \circ \tau \circ \Delta = m \circ (Id \otimes S) \circ \Delta.$$

Hence, it follows that S being an antipode for B is equivalent to S being an antipode for $(B^{op})^{cop}$. $\square$

We have proven that, if B is a $\mathbb{K}$ -bialgebra, then B^{op} and B^{cop} are also $\mathbb{K}$ -bialgebras, and that B is a Hopf $\mathbb{K}$ -algebra if and only if $(B^{op})^{cop}$ also is, with the same antipode. Note that we have not shown that either B^{op} or B^{cop} is a Hopf algebra when B is, which may not be true. In fact, given a Hopf algebra H , the next proposition provides an equivalent condition on the antipode for H^{op} and H^{cop} to be Hopf algebras.

Proposition 2.5.10. *Let H be a Hopf algebra with antipode S . Then, the following are equivalent:*

- a) H^{op} is a Hopf algebra.
- b) H^{cop} is a Hopf algebra.
- c) S is bijective.

Proof. Let $H' = H^{op}$ and $H'' = H^{cop}$. Observe that, by the previous proposition H^{op} is a Hopf Algebra if and only if $((H^{op})^{op})^{cop} = H^{cop}$ is a Hopf Algebra with the same antipode. Now let us prove that c) implies b). Then, consider $\bar{S}$ such that $S \circ \bar{S} = \bar{S} \circ S = Id$ (that exists since we suppose S is bijective). $\bar{S}$ is also an antimorphism of algebras and coalgebras, since $\bar{S}$ is the inverse of a bijective morphism of algebras and coalgebras. Also, observe that, for all $h \in H$:

$$\begin{aligned} \sum_{(h)} \bar{S}(h_{(2)})h_{(1)} &= \sum_{(h)} (\bar{S}(h_{(2)})(\bar{S}(S(h_{(1)})))) \\ &= \bar{S} \left(\sum_{(h)} S(h_{(1)})h_{(2)} \right) \\ &= \bar{S}(\varepsilon(h)1) = \varepsilon(h)1. \end{aligned}$$

The first equality follows from the definition of $\bar{S}$, while the second and last equalities follow from the fact that $\bar{S}$ is an antimorphism of algebras, and the third follows from Definition 2.5.8. Let $\circledast$ denote the convolution product for H^{cop} . Then, $\bar{S} \circledast Id = \mu \circ \varepsilon$. By analogous arguments, we can prove that $Id \circledast \bar{S} = \mu \circ \varepsilon$. Hence, H^{cop} is a Hopf $\mathbb{K}$ -algebra.

Now, let us prove that b) implies c). Take T as the antipode for H^{cop} and let us verify that $T \circ S = Id$. In fact, observe that, for $h \in H$:

$$\begin{aligned}
T(S(h)) &= T\left(\sum_{(h)} S(\varepsilon(h_{(2)}))h_{(1)}\right) \\
&= \sum_{(h)} T(S(\varepsilon(h_{(2)}))h_{(1)}) \\
&= \sum_{(h)} \varepsilon(h_{(2)})T(S(h_{(1)})) \\
&= \sum_{(h)} 1\varepsilon(h_{(2)})T(S(h_{(1)})) \\
&= \sum_{(h)} \mu \circ \varepsilon(h_{(2)})T(S(h_{(1)})) \\
&= \sum_{(h)} h_{(3)}T(h_{(2)})T(S(h_{(1)})) \\
&= \sum_{(h)} h_{(2)}T(\varepsilon(h_{(1)})1) \\
&= \sum_{(h)} \varepsilon(h_{(1)})h_{(2)} = h.
\end{aligned}$$

The first and last equalities follow from the property of counit, while the sixth equality follows from the definition of T , the seventh follows from the fact that T is an antimorphism of algebras and Definition 2.5.8, and the eighth equality follows from the fact that T is an antimorphism of algebras only.

Hence, b) implies c). The proof also shows that the antipode of both H^{op} and H^{cop} is the inverse of the antipode of H . $\square$

Then, we need a last proposition, that will be also proved in the end of this dissertation in a more general way (*i.e*, for Hopf categories of modules).

Proposition 2.5.11. *Let $(H, m, \mu, \Delta, \varepsilon, S)$ be a Hopf $\mathbb{K}$ -algebra and that $\dim(H) < \infty$. Then, we have that its dual H^* is also a Hopf $\mathbb{K}$ -algebra, with antipode given by $T = S^*$.*

Proof. By Lemma 2.3.1 (The dual algebra of a coalgebra) and Proposition 2.3.2, we know that H^* is a $\mathbb{K}$ -algebra and a $\mathbb{K}$ -coalgebra. Since H is a $\mathbb{K}$ -bialgebra, m and μ are

morphisms of algebras. Then,

$$\begin{aligned}
\Delta_{H^*} \circ m_{H^*} &= \iota^{-1} \circ m^* \circ \Delta^* \circ \iota \\
&= \iota^{-1} \circ (\Delta \circ m)^* \circ \iota \\
&= \iota^{-1} \circ (m_{H \otimes H} \circ (\Delta_H \otimes \Delta_H))^* \circ \iota \\
&= (\Delta_H \otimes \Delta_H)^* \circ (m_{H \otimes H}) \\
&= (\Delta_H \otimes \Delta_H)^* \circ \iota^{-1} \circ \iota \circ (m_{H \otimes H})^* \\
&= m_{H^* \otimes H^*} \circ (\Delta_{H^*} \otimes \Delta_{H^*}).
\end{aligned}$$

Also, observe that:

$$\begin{aligned}
\Delta_{H^*} \circ \mu_{H^*} &= \iota^{-1} \circ m^* \circ \varepsilon^* \circ \Phi \\
&= \iota^{-1} \circ (\varepsilon \circ m)^* \circ \Phi \\
&= \iota^{-1} \circ (\varepsilon \otimes \varepsilon)^* \circ \Phi \\
&= \mu_{H \otimes H}
\end{aligned}$$

Now, observe that:

$$\begin{aligned}
\Delta_{H^*} \circ (Id \otimes S^*) \circ m_{H^*} &= \Delta^* \circ \iota \circ (Id \otimes S^*) \circ \iota^{-1} \circ m^* = \\
&= \Delta^* \circ (Id \otimes S)^* \circ m^* \\
&= (m \circ (Id \otimes S) \circ \Delta)^* \\
&= (\mu \circ \varepsilon)^* \\
&= \varepsilon^* \circ \mu^* \\
&= \varepsilon^* \circ \Phi \circ \Phi^{-1} \circ \mu^* \\
&= \mu_{H^*} \circ \varepsilon_{H^*}.
\end{aligned}$$

The first equality follows from the definitions of Δ_{H^*} and m_{H^*} , while the third and fifth equalities follow from the property of dual transformations that $(f \circ)^* = g^* \circ f^*$, the fourth equality follows from the fact that S is an antipode, and the last equality follows from the definition of μ_{H^*} and ε_{H^*} .

The other antipode equality for S^* follows by analogous computations.

Hence H^* is a Hopf algebra with S^* as antipode. □

Then, by Example 2.5.1, and the proposition above, we have that the dual of the group algebra $\mathbb{K}G$ of a finite group is a Hopf $\mathbb{K}$ -algebra. The next example gives explicitly the structure of this Hopf $\mathbb{K}$ -algebra.

Example 2.5.8 (Dual group algebra). By Example 2.5.1, we have that $\mathbb{K}G$ is Hopf $\mathbb{K}$ -algebra, and by the previous proposition, we have that $(\mathbb{K}G)^*$ is also a Hopf $\mathbb{K}$ -algebra. The product in $(\mathbb{K}G)^*$ is given by $\overline{m} = \Delta^* \circ \iota$, where Δ is the coproduct in $\mathbb{K}G$. Consider the dual basis $\{p_g, g \in G\}$. Given $g, h, l \in G$,

$$\begin{aligned} \langle \Delta^* \circ \iota(p_g \otimes p_h), l \rangle &= \langle \iota(p_g \otimes p_h), \Delta(l) \rangle \\ &= \langle \iota(p_g \otimes p_h), l \otimes l \rangle \\ &= \langle p_g, l \rangle \langle p_h, l \rangle \\ &= \delta_{g,l} \delta_{h,l} \\ &= \delta_{g,h} \langle p_g, l \rangle. \end{aligned}$$

The first equality follows from Definition 2.3.1, the second equality follows from the application of Δ , the third equality follows from the definition of functional in a tensor of $\mathbb{K}$ -vector spaces, the last two equalities follow from the definition of dual basis. Hence:

$$p_g \cdot p_h = \delta_{g,h} p_g$$

The unit is $1_{(\mathbb{K}G)^*} = \sum_{g \in G} p_g$, since

$$\left(\sum_{g \in G} p_g \right) \cdot p_h = \sum_{g \in G} (p_g \cdot p_h) = \sum_{g \in G} \delta_{g,h} p_g = p_h = p_h \cdot \left(\sum_{g \in G} p_g \right)$$

for every $h \in G$ and the unit is unique.

Now, let us compute $\overline{\Delta}$, i.e, the coproduct in $(\mathbb{K}G)^*$. In fact, we have that $\overline{\Delta} = \iota^{-1} \circ m^*$. Computing for $g, k, l \in G$, we have that:

$$\begin{aligned} \langle m^*(p_g), k \otimes l \rangle &= \langle p_g, m(k \otimes l) \rangle \\ &= \langle p_g, kl \rangle \\ &= \delta_{g,kl} \\ &= \left\langle \iota \left(\sum_{\substack{x,y \in G \\ xy=g}} p_x \otimes p_y \right), k \otimes l \right\rangle. \end{aligned}$$

The first equality follows from the definition of transpose mapping, the third equality follows from the definition of the dual basis, and the last one follows from the definition of the dual basis and the mapping ι . Hence:

$$(\iota^{-1} \circ m^*)(p_g) = \sum_{\substack{x,y \in G \\ xy=g}} p_x \otimes p_y = \sum_{h \in G} p_{gh^{-1}} \otimes p_h.$$

Let us prove that the counit is given by $\bar{\varepsilon}(p_g) = \delta_{e,g}$, (e being the neutral element in G). In fact, we have that:

$$\begin{aligned} (\bar{\varepsilon} \otimes Id) \circ \bar{\Delta}(p_g) &= (\bar{\varepsilon} \otimes Id) \left(\sum_{h \in G} p_{gh^{-1}} \otimes p_h \right) \\ &= \sum_{h \in G} \bar{\varepsilon}(p_{gh^{-1}}) \otimes p_h \\ &= 1 \otimes p_g. \end{aligned}$$

The first equality follows from the definition of $\bar{\Delta}$, and the third equality follows from the definition of $\bar{\varepsilon}$ and the fact that $gh^{-1} = e$ only when $h = g$. Also observe that

$$\begin{aligned} (Id \otimes \bar{\varepsilon}) \circ \bar{\Delta}(p_g) &= (Id \otimes \bar{\varepsilon}) \left(\sum_{h \in G} p_{gh^{-1}} \otimes p_h \right) \\ &= \sum_{h \in G} p_{gh^{-1}} \otimes \bar{\varepsilon}(p_h) \\ &= p_g \otimes 1. \end{aligned}$$

The first equality follows from the definition of $\bar{\Delta}$, while the third equality follows from the fact that $e = e^{-1}$ because e is the neutral element in G . This is the counit since the counit is unique. By duality, the antipode $\bar{S}$ is given by $\bar{S}(p_g) = p_g \circ S = p_{g^{-1}}$. Indeed, for $g, h \in G$:

$$\begin{aligned} \langle \bar{S}(p_g), h \rangle &= \langle S^*(p_g), h \rangle \\ &= \langle p_g, S(h) \rangle \\ &= \langle p_g, h^{-1} \rangle \\ &= \delta_{g, h^{-1}} \\ &= \delta_{g^{-1}, h} \\ &= \langle p_{g^{-1}}, h \rangle \end{aligned}$$

Hence, $\bar{S}(p_g) = p_{g^{-1}}$

3 HOPF ALGEBRAS

3.1 CATEGORICAL INTRODUCTION

In the previous chapter we discussed Hopf algebras as $\mathbb{K}$ -vector spaces, the Hopf $\mathbb{K}$ -algebras. In this chapter we will discuss Hopf algebras from a categorical point of view (*i.e.*, discuss Hopf Algebra objects in a monoidal category). The main references for this chapter are (MACLANE, 1988), (JOHNSON; YAU, 2021) and (ALVES; BATISTA, 2014). First of all, let us review some basic points of category theory. We also use (ETINGOF et al., 2016).

Definition 3.1.1 (Category). *A **category** $\mathcal{C}$ consists of a class of objects $\mathcal{C}_0$, for each pair of objects $A, B \in \mathcal{C}_0$ a set of morphisms $Hom(A, B)$ and a composition $\circ$, which is a collection of operations for each triple of objects:*

$$\circ : Hom(A, B) \times Hom(B, C) \rightarrow Hom(A, C),$$

$$\circ(g, f) = f \circ g,$$

such that the following statements hold, for $A, B, C, D \in \mathcal{C}_0$:

- *$\circ$ is associative, that is:*

$$(f \circ g) \circ h = f \circ (g \circ h),$$

for all $f \in Hom(C, D)$, $g \in Hom(B, C)$, $h \in Hom(A, B)$.

- *For each object $B \in \mathcal{C}_0$, there is a morphism $1_B \in Hom(B, B)$, called identity morphism of B , such that for any objects $A, C \in \mathcal{C}_0$ and morphisms $f \in Hom(A, B)$ and $g \in Hom(B, C)$ we have:*

$$1_B \circ f = f,$$

$$g \circ 1_B = g.$$

- *If $(A, B) \neq (C, D)$, $Hom(A, B) \cap Hom(C, D) = \emptyset$.*

When we discuss categories, we discuss mathematical objects from the point of view of the morphisms between them. Some examples of categories are:

- *Sets* is the category of sets, where the objects are the sets, the morphisms in $Hom(A, B)$ for two sets A, B are the set of functions $\{f : A \rightarrow B\}$. The identity functions are the identity morphisms.

- $\mathbb{K} - Vect$ is the category of $\mathbb{K}$ -vector spaces, where the objects are the $\mathbb{K}$ -vector spaces, and for two vector spaces A, B , $Hom(A, B)$ is the $\mathbb{K}$ -vector space of linear maps from A to B , the composition is the composite of two linear mappings and the identity morphisms are the identity maps.
- $\mathbb{K} - Alg$ denotes the category of $\mathbb{K}$ -algebras. The objects are the $\mathbb{K}$ -algebras, the morphisms are the $\mathbb{K}$ -algebra morphisms, the composition is the composition of algebra morphisms, and the identity morphisms are the identity maps.
- $\mathbb{K} - Coalg$ denotes the category of $\mathbb{K}$ -coalgebras. The objects are the $\mathbb{K}$ -coalgebras, the morphisms are the $\mathbb{K}$ -coalgebra morphisms, the composition is the composition of coalgebra morphisms, and the identity morphisms are the identity maps.
- $\mathbb{K} - Bialg$ denotes the category of $\mathbb{K}$ -bialgebras. The objects are the $\mathbb{K}$ -bialgebras, the morphisms are the $\mathbb{K}$ -bialgebras morphisms, the composition is the composition of bialgebra morphisms, and the identity morphisms are the identity maps.
- $\mathbb{K} - Hopf$ denotes the category of Hopf $\mathbb{K}$ -algebras. The objects are the Hopf $\mathbb{K}$ -algebras, the morphisms are the morphisms of Hopf $\mathbb{K}$ -Algebras, the composition is the composition of Hopf algebra morphisms, and the identity morphisms are their identity maps.
- Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can define the *product category* $\mathcal{C} \times \mathcal{D}$, where the collection of objects is $(\mathcal{C} \times \mathcal{D})_0 = \{(A, B), A \in \mathcal{C}_0, B \in \mathcal{D}_0\}$, and the collection of morphisms are $Hom((A_1, B_1), (A_2, B_2)) = \{(f, g), f \in Hom(A_1, A_2), g \in Hom(B_1, B_2)\}$.
The composition is defined as $(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ f_2, g_1 \circ g_2)$ and the identity for (A, B) is $(1_A, 1_B)$ for all $(A, B) \in (\mathcal{C}, \mathcal{D})_0$.
This example is essential to define monoidal categories, and to define Hopf algebras and to extend the concept of Hopf algebra beyond the category of $\mathbb{K}$ -vector spaces.
- For a $\mathcal{C}$ a category, we may consider the category $\mathcal{C}^{op}$ defined in the following manner: The objects are the same as in $\mathcal{C}$, but for each morphism $f \in Hom(A, B)$ with $A, B \in \mathcal{C}$, it corresponds to a morphism $f^{op} \in Hom(B, A)$, but with $B, A \in \mathcal{C}_0^{op}$. For $f^{op} \in Hom(B, A)$ and $g^{op} \in Hom(C, B)$ (i.e., $f \in Hom(A, B)$ and $g \in Hom(B, C)$ in $\mathcal{C}$) we define $f^{op} \circ g^{op} = (g \circ f)^{op} \in Hom(C, A)$ in $\mathcal{C}^{op}$ (observe that

$g \circ f \in \text{Hom}(A, C)$) and hence such composition is well-defined). We also define $(1_A)^{op} = 1_A \in \text{Hom}(A, A)$. This example is called the **opposite category** of $\mathcal{C}$.

Now we shall introduce a last example of category, the category $\mathcal{Cat}$, that is, the category of categories.

The objects are the categories, and the morphisms are functors, which we are going to define as follows:

Definition 3.1.2 (Functors). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, with compositions $\circ_{\mathcal{C}}$ and $\circ_{\mathcal{D}}$. Then a **functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is defined as it follows:*

- A function $F : \mathcal{C}_0 \rightarrow \mathcal{D}_0$.
- For each pair of objects $A, B \in \mathcal{C}$ we have a function $F_{A,B} = F$ such that:
 - $F_{A,B} : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$ such that $F(1_A) = 1_{F(A)}$ and,
 - $F(f \circ g) = F(f) \circ F(g)$ if F is **covariant**, and
 - $F_{A,B} : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(B), F(A))$ such that $F(1_A) = 1_{F(A)}$ and $F(f \circ g) = F(g) \circ F(f)$ if F is **contravariant**.

We shall denote $F(A) = FA$ if A is an object of $\mathcal{C}$.

Then, we have the example $\mathcal{Cat}$, which is the category whose the objects are categories and the morphisms are the functors between them.

Then, let us give some examples of functors:

Example 3.1.1. Let $\mathcal{C}$ be a given category, let $X \in \mathcal{C}_0$ be an object, and let $\text{Hom}(X, _) : \mathcal{C} \rightarrow \text{Sets}$ be a functor defined as:

- For an object $Y \in \mathcal{C}_0$, $\text{Hom}(X, _)(Y) = \text{Hom}(X, Y)$.
- For $f \in \text{Hom}(Y, Z)$, we have an associated function $\text{Hom}(X, f) : \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z)$ which for a $g \in \text{Hom}(X, Y)$ we have $\text{Hom}(X, f)(g) = f \circ g$.
- We define the composition in the following manner: For $g : Y \rightarrow Z$ and $f : Z \rightarrow W$ in $\mathcal{C}$, we define $\text{Hom}(X, f) \circ \text{Hom}(X, g) : \text{Hom}(X, Y) \rightarrow \text{Hom}(X, W)$ as the composition of $\text{Hom}(X, f)$ and $\text{Hom}(X, g)$ as functions, i.e, for $h \in \text{Hom}(X, Y)$, $\text{Hom}(X, f) \circ \text{Hom}(X, g)(h) = \text{Hom}(X, f)(g \circ h) = f \circ (g \circ h)$. Since the composition in category $\mathcal{C}$ is associative, we have that $f \circ (g \circ h) = (f \circ g) \circ h$ and then $\text{Hom}(X, f) \circ \text{Hom}(X, g) = \text{Hom}(X, f \circ g)$.

This is a covariant functor. One may also consider the other Hom functor $\text{Hom}(_, X)$, which is a contravariant functor.

The point now is that for two categories $\mathcal{C}$ and $\mathcal{D}$ in Cat , we can consider $Hom(\mathcal{C}, \mathcal{D})$ as a category, whose objects are the functors $F : \mathcal{C} \rightarrow \mathcal{D}$ since we can define morphisms between functors, which we call natural transformations, as follows:

Definition 3.1.3 (Natural transformations). *Let $F, G \in Hom(\mathcal{C}, \mathcal{D})$ with $\mathcal{C}, \mathcal{D}$ objects in Cat . Then, a **natural transformation** $\theta : F \rightarrow G$ consists of a collection of morphisms $\theta_X : FX \rightarrow GX \in \mathcal{D}$ such that for each pair of objects $X, Y \in \mathcal{C}_0$ and for every morphism $f : X \rightarrow Y \in \mathcal{C}$ the following diagram commutes:*

$$\begin{array}{ccc} FX & \xrightarrow{\theta_X} & GX \\ \downarrow Ff & & \downarrow Gf \\ FY & \xrightarrow{\theta_Y} & GY. \end{array}$$

We shall define two different compositions between natural transformations: vertical composition and horizontal composition. We shall use more frequently, in this chapter, the vertical composition. However, we cannot ignore horizontal composition, which will be reviewed in the third part of this dissertation.

Definition 3.1.4 (Vertical composition and horizontal composition). *Suppose $\theta : F \rightarrow G$ is a natural transformation between functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$.*

- *Suppose $\phi : G \rightarrow H$ is a natural transformation to another functor $H : \mathcal{C} \rightarrow \mathcal{D}$. The **vertical composition***

$$\phi\theta : F \rightarrow H$$

is the natural transformation defined as:

$$(\phi\theta)_X = \phi_X \circ \theta_X : FX \rightarrow HX, X \in \mathcal{C}_0.$$

- *Suppose $\theta' : F' \rightarrow G'$ is a natural transformation for functors $F', G' : \mathcal{D} \rightarrow \mathcal{E}$ with $\mathcal{E}$ being another category. Then, the **horizontal composition***

$$\theta' * \theta : F' \circ F \rightarrow G' \circ G$$

is the natural transformation defined for each $X \in \mathcal{C}_0$, as either component of the following commutative diagram:

$$\begin{array}{ccc} F'FX & \xrightarrow{\theta'_{FX}} & G'FX \\ \downarrow F'\theta_X & & \downarrow G'\theta_X \\ F'GX & \xrightarrow{\theta'_{GX}} & G'GX \end{array}$$

The next definition is a definition for isomorphisms between two objects in a category, as follows:

Definition 3.1.5. Let $\mathcal{C}$ be a category and let A, B be both objects in $\mathcal{C}_0$ and let $f \in \text{Hom}(A, B)$ be a morphism. Then, f is an **isomorphism** if there exists $f^{-1} \in \text{Hom}(B, A)$ such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \text{Id}_A & \downarrow f^{-1} \\ & & A \end{array} \quad \begin{array}{ccc} B & \xrightarrow{f^{-1}} & A \\ & \searrow \text{Id}_B & \downarrow f \\ & & B \end{array}$$

Then f^{-1} is called an **inverse** of the morphism f .

If $\text{Hom}(A, B)$ has an isomorphism, then A and B are **isomorphic**.

In the majority of the examples in the next, morphisms are functions and isomorphisms are bijective functions. In the category of $\mathbb{K}$ -vector spaces, $\mathbb{K}$ -algebras, $\mathbb{K}$ -coalgebras, $\mathbb{K}$ -bialgebras and the category of Hopf $\mathbb{K}$ -algebras they are bijective morphisms since they have an inverse. It is a fact that identity maps are always isomorphisms.

Now that we have defined and viewed functors and natural transformations, we shall define monoidal categories, as follows:

Definition 3.1.6 (Monoidal categories). A **monoidal category** is a tuple $(\mathcal{C}, \otimes, I, \alpha, \ell, r)$ consisting of:

- A category $\mathcal{C}$,
- A functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ called the **monoidal product**,
- An object $I \in \mathcal{C}_0$ called **unit**,
- a natural isomorphism:

$$\alpha : \otimes \circ (\otimes \times \text{Id}) \rightarrow \otimes \circ (\text{Id} \times \otimes)$$

called **associator**,

- natural isomorphisms:

$$\ell : I \otimes \text{Id} \rightarrow \text{Id},$$

$$r : \text{Id} \otimes I \rightarrow \text{Id},$$

called **left unit isomorphism** and **right unit isomorphism**, respectively. These data are required to satisfy the following diagrams:

• **The middle unit axiom:**

$$\begin{array}{ccc}
 (X \otimes I) \otimes Y & \xrightarrow{\alpha_{X,I,Y}} & X \otimes (I \otimes Y) \\
 \downarrow r_X \otimes Y & & \downarrow X \otimes \ell_Y \\
 X \otimes Y & \xrightarrow{Id} & X \otimes Y
 \end{array}$$

• **Pentagon Axiom:**

$$\begin{array}{ccc}
 & (W \otimes X) \otimes (Y \otimes Z) & \\
 \alpha_{W \otimes X, Y, Z} \nearrow & & \searrow \alpha_{W, X, Y \otimes Z} \\
 ((W \otimes X) \otimes Y) \otimes Z & & W \otimes (X \otimes (Y \otimes Z)) \\
 \downarrow (\alpha_{W, X, Y}) \otimes Z & & \uparrow W \otimes (\alpha_{X, Y, Z}) \\
 (W \otimes (X \otimes Y)) \otimes Z & \xrightarrow{\alpha_{W, X \otimes Y, Z}} & W \otimes ((X \otimes Y) \otimes Z)
 \end{array}$$

There are some points to explain about this definition. First, when we denote $\otimes \times Id$ as a functor (and then $\otimes \circ (\otimes \times Id)$ is the vertical composition of $\otimes$ with $\otimes \times Id$) we mean the functor $\otimes \times Id : (\mathcal{C} \times \mathcal{C}) \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ which is defined by:

$$\begin{aligned}
 (\otimes \times Id)((A, B), C) &= \otimes(A, B) \times C = (A \otimes B) \times C, \\
 (\otimes \times Id)(f, g, h) &= (f \otimes g, h)
 \end{aligned}$$

for all $f : A \rightarrow A', g : B \rightarrow B', h : C \rightarrow C'$.

Then, the composition $\otimes \circ (\otimes \times Id)$ takes $((A, B), C)$ into $(A \otimes B) \otimes C$. A similar definition holds for $Id \times \otimes$. Then the associator as an isomorphism represents a weaker version of associativity for monoidal products. The monoidal product is defined in such a manner because the main example of monoidal product (tensor $\mathbb{K}$ -vector spaces) is not strictly associative, that is, for U, V, W $\mathbb{V}$ -vector spaces we have an isomorphism $U \otimes (V \otimes W) \simeq (U \otimes V) \otimes W$, not an equality. However, we will define strict monoidal categories, and in this definition the associativity in monoidal categories is true.

Similarly, the tensor $Id \otimes I$ takes an object $A \in \mathcal{C}$ into the object $A \otimes I$ and a morphism $f \in Hom(A, B)$ into $f \otimes Id_I$.

Second, in general we shall identify the identity of an object with the object in terms of notation, as in $Id_A = A$.

Now, we shall view some examples of monoidal categories.

Example 3.1.2. A first example is of tensor vector spaces. We know that given A, B $\mathbb{K}$ -vector spaces, we have the tensor vector space $A \otimes B$. We can define in $Vect$ the monoidal product $\otimes : Vect \times Vect \rightarrow Vect$ as:

$$\otimes(A, B) = A \otimes B$$

The associators are the well-known isomorphisms (remember that, for $\mathbb{K}$ -vector spaces A, B and C , there is a standard isomorphism $f : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$). The

unit object is $\mathbb{K}$ and the left and right units are the isomorphisms related to $\mathbb{K}$ (the multiplication by scalars)).

Example 3.1.3. Another example of monoidal structure can be found in $Sets$. For two sets A, B , we may define the monoidal product as the cartesian product in $Sets$. Then, for $A, B \in Sets_0$, we may write $A \otimes B = A \times B$. The associator for $((A, B), C)$ and $(A, (B, C))$ is the function $\alpha_{A,B,C}$ which is given by $\alpha_{A,B,C}((a, b), c) = (a, (b, c))$. The unit object is a singleton $\{*\}$ which represents a set with just one element. The right unit and left unit for A are $r(A, \{*\}) = A$ and $\ell(\{*\}, A) = A$.

Another example of monoidal products is a generalization of the previous one, that is related to the **categorical product**, which will be useful to study several important examples of this dissertation.

Definition 3.1.7. Let $\mathcal{C}$ be a category and let $\{A_i\}_{i \in I}$ be a family of objects of $\mathcal{C}$. Then, a **categorical product** of such collection is an object $\prod\{A_i\}$ together with a collection of morphisms $\{\pi_i : \prod\{A_j\} \rightarrow A_i\}_{i \in I}$, called **projection morphisms**, such that for each object $C \in \mathcal{C}_0$ and each collection of morphisms $\{f_i : C \rightarrow A_i\}$ there exists a unique morphism $\prod f_i : C \rightarrow \prod\{A_i\}_{i \in I}$ such that $f_i = \pi_i \circ \prod f_i$ for $i \in I$.

In terms of commuting diagrams:

$$\begin{array}{ccc}
 & \prod\{A_i\} & \\
 \pi_i \swarrow & & \uparrow \prod f_i \\
 A_i & & C \\
 f_i \swarrow & & \\
 & &
 \end{array}$$

In what follows, assume that the category $\mathcal{C}$ has categorical products of every family of objects of $\mathcal{C}$. We will show that the categorical product in $\mathcal{C}$ induces a structure of monoidal category on $\mathcal{C}$, which unit object we shall define later.

Observe that a categorical product is unique up to isomorphism. In fact, let $\prod\{A_i\}$ and $\prod^*\{A_i\}$ be both categorical products for $\{A_i\}$, with projection morphisms $\pi_i : \prod\{A_i\} \rightarrow A_i$ and $\pi_i^* : \prod^*\{A_i\} \rightarrow A_i$. By the definition of categorical product, there

exists a unique morphism $\prod \pi_i^* : \prod^* \{A_i\} \rightarrow \prod \{A_i\}$.

$$\begin{array}{ccc}
 & \prod \{A_i\} & \\
 \pi_i \swarrow & \uparrow \text{---} \prod \pi_i^* & \\
 A_i & & \\
 \pi_i^* \swarrow & & \\
 & \prod^* \{A_i\} &
 \end{array}$$

There is also a unique morphism $\prod^* \pi_i : \prod \{A_i\} \rightarrow \prod^* \{A_i\}$. Observe that the composition $\prod \pi_i^* \circ \prod^* \pi_i : \prod \{A_i\} \rightarrow \prod \{A_i\}$ is such that $\pi_i \circ (\prod \pi_i^* \circ \prod^* \pi_i) = \pi_i$ for all i , just like the same is true for $Id_{\prod \{A_i\}}$, that is, we have that

$$\pi_i \circ \left(\prod \pi_i^* \circ \prod^* \pi_i \right) = \pi_i \quad \text{and} \quad \pi_i \circ Id_{\prod \{A_i\}} = \pi_i, \forall i.$$

By the definition of categorical product there exists a unique morphism $f : \prod \{A_i\} \rightarrow \prod \{A_i\}$ such that $\pi_i \circ f = \pi_i$ for all i , and hence $\prod \pi_i^* \circ \prod^* \pi_i = Id_{\prod \{A_i\}}$. The same reasoning applied to $\prod^* \{A_i\}$ yields $\prod^* \pi_i \circ \prod \pi_i^* = Id_{\prod^* \{A_i\}}$.

When we consider the product of two objects A, B , we shall denote $A \times B$.

Observe that for objects $A_1, A_2, A_3 \in \mathcal{C}_0$ for a category $\mathcal{C}$, we have that

$$(A_1 \times A_2) \times A_3$$

is a product of A_1, A_2, A_3 . In fact, note that there are projections $\pi_1 = \pi_{1A_1 \times A_2} \circ \pi_{1(A_1 \times A_2) \times A_3}, \pi_2 = \pi_{2A_1 \times A_2} \circ \pi_{1(A_1 \times A_2) \times A_3}$ and $\pi_3 = \pi_{2(A_1 \times A_2) \times A_3}$. For an object $C \in \mathcal{C}_0$, suppose there exist morphisms $f_1 : C \rightarrow A_1, f_2 : C \rightarrow A_2, f_3 : C \rightarrow A_3$. Then there is a unique morphism $f_1 \times f_2 : C \rightarrow A_1 \times A_2$ such that $f_1 = \pi_{1A_1 \times A_2} \circ f_1 \times f_2, f_2 = \pi_{2A_1 \times A_2} \circ f_1 \times f_2$, by the definition of categorical product. Also, by the definition of categorical product, we have that there is a unique morphism $(f_1 \times f_2) \times f_3 : C \rightarrow (A_1 \times A_2) \times A_3$ such that $f_1 \times f_2 = \pi_{1(A_1 \times A_2) \times A_3} \circ (f_1 \times f_2) \times f_3$. And $f_3 = \pi_3 \circ (f_1 \times f_2) \times f_3$, which we may consider $f_1 \times f_2 \times f_3 : C \rightarrow A_1 \times A_2 \times A_3$.

Then, we have that $f_1 = \pi_{1A_1 \times A_2} \circ \pi_{1(A_1 \times A_2) \times A_3} \circ (f_1 \times f_2) \times f_3, f_2 = \pi_{2A_1 \times A_2} \circ \pi_{1(A_1 \times A_2) \times A_3} \circ (f_1 \times f_2) \times f_3,$

$$f_3 = \pi_{2(A_1 \times A_2) \times A_3} \circ (f_1 \times f_2) \times f_3.$$

Therefore, $(A_1 \times A_2) \times A_3$ is a categorical product for A_1, A_2, A_3 . Analogously, we have that $A_1 \times (A_2 \times A_3)$ is a categorical product for $A_1, A_2, A_3 \in \mathcal{C}_0$.

Then, there is an isomorphism

$$i_{A_1, A_2, A_3} : (A_1 \times A_2) \times A_3 \rightarrow A_1 \times (A_2 \times A_3)$$

for each $A_1, A_2, A_3 \in \mathcal{C}_0$. By the construction of this isomorphism, we can consider all objects in the diagram below as a categorical product of A_1, A_2, A_3 and A_4 , and then

there exists a unique canonical isomorphism $i : ((A_1 \times A_2) \times A_3) \times A_4 \rightarrow A_1 \times (A_2 \times (A_3 \times A_4))$, which means that the following diagram commutes (and we will prove it):

$$\begin{array}{ccc}
 & (A_1 \times A_2) \times (A_3 \times A_4) & \\
 i_{A_1 \times A_2, A_3, A_4} \nearrow & & \nwarrow i_{A_1, A_2, A_3 \times A_4} \\
 ((A_1 \times A_2) \times A_3) \times A_4 & & A_1 \times (A_2 \times (A_3 \times A_4)) \\
 \downarrow i_{A_1, A_2, A_3 \times A_4} & & \uparrow A_1 \times i_{A_2, A_3, A_4} \\
 (A_1 \times (A_2 \times A_3)) \times A_4 & \xrightarrow{i_{A_1, A_2 \times A_3, A_4}} & A_1 \times ((A_2 \times A_3) \times A_4)
 \end{array}$$

Observe that the diagram above is the pentagon diagram applied to $\times$.

Let us prove that this diagram commutes. The proof may be found in (BUSCHI SERGIO, s.d.).

We know that the category $(Sets, \times, \{*\})$ is monoidal. Consider, for an object $X \in \mathcal{C}$, the functor $Hom(X, _)$. Observe that, for given objects $Y, Z \in \mathcal{C}$, $Hom(X, Y) \times Hom(X, Z)$ is isomorphic to $Hom(X, Y \times Z)$. In fact, $Hom(X, Y \times Z)$ is a categorical product in the category image of $Hom(X, _)$, and since such category is contained in $Sets$, $Hom(X, Y) \times Hom(X, Z)$ also works as a categorical product for the image of $Hom(X, _)$. Then, both objects are categorical products for X and Y in the category of the image $H(X, _)$, and hence both are isomorphic.

More precisely, if $p_Y : Y \times Z \rightarrow Y$ and $p_Z : Y \times Z \rightarrow Z$ are the projections of the product $Y \times Z$, then we have a function $\varphi_{Y,Z} : Hom(X, Y \times Z) \rightarrow Hom(X, Y) \times Hom(X, Z)$ given by $f \mapsto (p_Y \circ f, p_Z \circ f)$. Conversely, by the definition of categorical product, given morphisms $g \in Hom(X, Y)$, $h \in Hom(X, Z)$, there exists a unique morphism $f \in Hom(X, Y \times Z)$ such that $p_Y \circ f = g$ and $p_Z \circ f = h$. This shows that $\varphi_{Y,Z}$ is bijective. Then, it is an isomorphism, and we have proved that $Hom(X, _)$ preserves products.

Recall that all objects in the image of $Hom(X, _)$ are sets. Hence, since $Hom(X, _)$ preserves products, the image of the pentagon diagram by $Hom(X, _)$ commutes, since $(Sets, \times, \{*\})$ is a monoidal category. However, for given $f, g : A \rightarrow B \in \mathcal{C}$, if $Hom(X, f) = Hom(X, g)$ for each $X \in \mathcal{C}$, then $f = g$, since we may consider $X = A$ and then $f = Hom(A, f)(1_A) = Hom(A, g)(1_A) = g$. Therefore we conclude that the original pentagon diagram in $\mathcal{C}$ commutes.

Now, it remains to define a unit object for a category with products. We define terminal and initial objects, and prove that there is a terminal object in such category, which will be the unit of its monoidal structure.

Definition 3.1.8 (Terminal object, Initial object). *Let $\mathcal{C}$ be a category. Then*

- A **terminal object** I is an object in $\mathcal{C}_0$ such that for every object $C \in \mathcal{C}_0$ there is an unique morphism $t_C : C \rightarrow I$.

- A **initial object** A is an object in $\mathcal{C}_0$ such that for every object $C \in \mathcal{C}_0$ there is an unique morphism $i_C : A \rightarrow C$.

Observe that if a category has products then it has a terminal object, which corresponds to the product of the empty family of objects. Remember that a categorical product of n objects $\{A_1, \dots, A_n\}$ is defined as an object $A_1 \times \dots \times A_n$ such that for any object C and morphisms $f_i : C \rightarrow A_i$ there is an unique morphism $f_1 \times \dots \times f_n$ such that $f_i = \pi_i \circ f_1 \times \dots \times f_n$. So, a product of 0 objects is a object I such that for any object C there is an unique morphism $f : C \rightarrow I$, just like the definition of terminal object. Hence, there are isomorphisms $\ell : I \times A \rightarrow A$ and $r : A \times I \rightarrow A$ for each object $A \in \mathcal{C}_0$. For objects A_1, A_2, B_1, B_2 and morphisms $f_1 : A_1 \rightarrow B_1$ and $f_2 : A_2 \rightarrow B_2$, we may define $f_1 \times f_2$ as the unique morphism such that the following diagram commutes:

$$\begin{array}{ccccc} A_1 & \xleftarrow{\pi_1} & A_1 \times A_2 & \xrightarrow{\pi_2} & A_2 \\ \downarrow f_1 & & \downarrow f_1 \times f_2 & & \downarrow f_2 \\ B_1 & \xleftarrow{\pi_1} & B_1 \times B_2 & \xrightarrow{\pi_2} & B_2 \end{array}$$

Observe, by construction, that for $f_3 : A_3 \rightarrow B_3$, we have that

$$i_{B_1, B_2, B_3} \circ (f_1 \times f_2) \times f_3 = f_1 \times (f_2 \times f_3) \circ i_{A_1, A_2, A_3}.$$

So, we may define a monoidal structure on $\mathcal{C}$ by taking the monoidal product $\times : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ as $\times(A, B) = A \times B$ for $A, B \in \mathcal{C}_0$ and $\times(f_1, f_2) = f_1 \times f_2$, and a terminal object I of $\mathcal{C}$ as the unit.

Then, we shall define strict monoidal category, which will be widely used in the next chapter.

Definition 3.1.9 (Strict monoidal categories). *Let $(\mathcal{C}, \otimes, 1, \alpha, \ell, r)$ be a monoidal category. If α, ℓ and r are identity natural transformations, then the category is said to be a **strict monoidal category**.*

Example 3.1.4. An example of strict monoidal Category is the category of functors from a category $\mathcal{C}$ into itself. Given two functors $F, G : \mathcal{C} \rightarrow \mathcal{C}$, we can consider $F \otimes G = F * G$ (the horizontal composition). In fact, given functors $F, G, H : \mathcal{C} \rightarrow \mathcal{C}$, it can be verified by the construction of commutative diagrams that $(F * G) * H = F * (G * H)$. We also notice that $F * Id_{\mathcal{C}} = F = Id_{\mathcal{C}} * F$ for all $F \in Hom(\mathcal{C}, \mathcal{C})$. Hence, the associators, left and right units are all identity natural identities (identity morphisms). The morphisms in such category are the vertical composition of morphisms.

And then, we shall define braided monoidal category. This is a generalization for the twist function we have seen in the previous chapter.

Definition 3.1.10 (Braided monoidal category). *A monoidal category $(\mathcal{C}, \otimes, I)$ is called a **braided monoidal category** if there is a natural isomorphism $\sigma : \otimes \rightarrow \otimes \circ \tau$, where τ is defined as the functor $\tau : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ that takes the object (M, N) to (N, M) , and the morphism (f, g) to (g, f) such that the following diagrams commute for $M, N, P \in \mathcal{C}_0$:*

$$\begin{array}{ccc}
 (M \otimes N) \otimes P & \xrightarrow{\alpha_{M,N,P}} & M \otimes (N \otimes P) \\
 \downarrow \sigma_{M,N \otimes P} & & \downarrow \sigma_{M,N \otimes P} \\
 (N \otimes M) \otimes P & & (N \otimes P) \otimes M \\
 \downarrow \alpha_{N,M,P} & & \downarrow \alpha_{N,P,M} \\
 N \otimes (M \otimes P) & \xrightarrow{N \otimes \sigma_{M,P}} & N \otimes (P \otimes M)
 \end{array}$$

and

$$\begin{array}{ccc}
 M \otimes (N \otimes P) & \xrightarrow{\alpha_{M,N,P}^{-1}} & (M \otimes N) \otimes P \\
 \downarrow M \otimes \sigma_{N,P} & & \downarrow \sigma_{(M \otimes N),P} \\
 M \otimes (P \otimes N) & & P \otimes (M \otimes N) \\
 \downarrow \alpha_{M,P,N}^{-1} & & \downarrow \alpha_{P,M,N}^{-1} \\
 (M \otimes P) \otimes N & \xrightarrow{\sigma_{M,P} \otimes N} & (P \otimes M) \otimes N
 \end{array}$$

In a strict monoidal category, the two diagrams above are:

$$\begin{array}{ccc}
 M \otimes N \otimes P & \xrightarrow{\sigma_{M,N \otimes P}} & N \otimes P \otimes M \\
 \searrow \sigma_{M,N \otimes P} & & \nearrow N \otimes \sigma_{M,P} \\
 & N \otimes M \otimes P & \\
 \nearrow \sigma_{M \otimes N, P} & & \searrow \sigma_{M,P \otimes N} \\
 M \otimes N \otimes P & \xrightarrow{\sigma_{M \otimes N, P}} & P \otimes M \otimes N \\
 \searrow M \otimes \sigma_{N,P} & & \nearrow \sigma_{M,P \otimes N} \\
 & M \otimes P \otimes N &
 \end{array}$$

Example 3.1.5. A standard example is the category of $\mathbb{K}$ -vector spaces with monoidal products as tensor products, and we take the braidings as the twist functions, just as we have defined in the previous chapter.

Example 3.1.6. Another example can be found in $(Sets, \times, \{*\})$ with the flip function $f_{A,B} : A \times B \rightarrow B \times A$ defined as $f((a, b)) = (b, a)$.

And then, a generalization of this second example appears in categorical products. Let $\mathcal{C}$ a category with products be given. Then, for two objects A, B in $\mathcal{C}$, we have

the objects $A \times B$ and $B \times A$ (i.e, the categorical products of A and B in a interchanging order). For the object $A \times B$, there are the projection morphisms $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ and for the object $B \times A$ there are the projection morphisms $\pi'_1 : B \times A \rightarrow B$ and $\pi'_2 : B \times A \rightarrow A$ (it follows from the definition of categorical product). Then, there exists the morphism $\pi_2 \times \pi_1 : A \times B \rightarrow B \times A$ (that is an isomorphism, and its inverse is $\pi'_2 \times \pi'_1 : B \times A \rightarrow A \times B$). It is easy to show that this class of morphisms satisfy the braiding axioms and then, we have a braiding for a category with products.

In the next section, we shall define algebras and coalgebras in a more general way.

3.2 ALGEBRAS AND COALGEBRAS

In the previous chapter, we have seen the definitions of algebras and coalgebras using arrows diagrams, but the objects in such definitions were $\mathbb{K}$ -vector spaces. This more general way to define algebras and coalgebras does not use vector spaces as objects, and then, generalizes those previous definitions.

Definition 3.2.1 (Algebra). *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. An **algebra object** in $\mathcal{C}$, or simply an algebra in $\mathcal{C}$, is a triple (A, m, μ) consisting of a object A , a morphism $m : A \otimes A \rightarrow A$ and a morphism $\mu : I \rightarrow A$ such that the following diagrams commute:*

$$\begin{array}{ccccc}
 (A \otimes A) \otimes A & \xrightarrow{\alpha} & A \otimes (A \otimes A) & \xrightarrow{A \otimes m} & A \otimes A \\
 \downarrow m \otimes A & & & & \downarrow m \\
 A \otimes A & \xrightarrow{m} & & & A \\
 & & A \otimes A & & \\
 & \swarrow \mu \otimes A & & \nwarrow A \otimes \mu & \\
 I \otimes A & & & & A \otimes I \\
 & \searrow \ell & & \swarrow r & \\
 & & A & &
 \end{array}$$

The first diagram is called **associativity diagram** and the second is called **unit property diagram**.

Example 3.2.1. Observe that except for the associator, the diagrams above are very similar to the diagrams of $\mathbb{K}$ -algebra in the previous chapter. In $\mathbb{K}$ -vector spaces, an algebra A is a $\mathbb{K}$ -algebra, and the associator is the standard isomorphism between $(A \otimes A) \otimes A$ and $A \otimes (A \otimes A)$.

Example 3.2.2. Another example of algebra is in $(Sets, \times, \{*\})$. Let (A, m, μ) be an algebra in $Sets$. For elements $a, b, c \in A$, we use the notation $m(a \times b) = ab$ and $\mu(*) = 1$. Therefore have that $(ab)c = a(bc)$ and $1a = a1 = a$. So, an algebra in $Sets$ is the same as a monoid.

Another example of algebra can be found in any monoidal category, and it is the unit object I . We will prove that in the next proposition.

Proposition 3.2.1. *Let $(\mathcal{C}, \otimes, I)$ a monoidal category. Then I is an algebra, with multiplication being $m_I = \ell_I = r_I$ and $\mu_I = I$.*

Proof. Before proving that I is an algebra, we need to prove that, for $(\mathcal{C}, \otimes, I, \alpha, \ell, r)$ a monoidal category, the following diagram commutes for $X, Y \in \mathcal{C}_0$:

$$\begin{array}{ccc}
 (X \otimes Y) \otimes I & \xrightarrow{\alpha} & X \otimes (Y \otimes I) \\
 \searrow r & & \swarrow X \otimes r \\
 & X \otimes Y & \\
 (I \otimes X) \otimes Y & \xrightarrow{\alpha} & I \otimes (X \otimes Y) \\
 \searrow \ell \otimes Y & & \swarrow \ell \\
 & X \otimes Y &
 \end{array}$$

We prove the commutativity of the first diagram. We are going to prove it by proving the commutativity of the following diagram:

$$\begin{array}{ccccc}
 & ((X \otimes Y) \otimes I) \otimes I & \xrightarrow{r} & (X \otimes Y) \otimes I & \\
 & \downarrow \alpha \otimes I & & \downarrow \alpha & \\
 r \otimes I \swarrow & (X \otimes (Y \otimes I)) \otimes I & \xrightarrow{r} & X \otimes (Y \otimes I) & \searrow r \\
 & \downarrow (X \otimes r) \otimes I & & \downarrow X \otimes r & \\
 (X \otimes Y) \otimes I & \xrightarrow{r} & & & X \otimes Y
 \end{array}$$

Observe that the triangular diagram in the left corner coincides with the diagram we want to prove tensored by I . By the naturality of r , we prove that this first diagram commutes.

Also, due to the naturality of r , the square on the top, the trapezium in the middle and the outer trapezium commute. Now, let us prove that the right triangle commutes. In fact, since the arrows are all isomorphisms, we may see that:

$$\begin{aligned}
(X \otimes r_Y)a_{X,Y,I} &= (X \otimes r_Y)r_{X \otimes (Y \otimes I)}(\alpha_{X,Y,I} \otimes I)r_{(X \otimes Y) \otimes I}^{-1} \\
&= r_{X,Y}((X \otimes r_Y) \otimes I)(\alpha_{X,Y,I} \otimes I)r_{(X \otimes Y) \otimes I}^{-1} \\
&= r_{X \otimes Y}(r_{(X \otimes Y) \otimes I} \otimes I)r_{(X \otimes Y) \otimes I}^{-1} \\
&= r_{X \otimes Y}.
\end{aligned}$$

The first equality follows from the commutativity of the top square. The second equality follows from the commutativity of the trapezium in the bottle. The third equality holds since the triangle in the left corner commutes. Finally, the commutativity of the outer trapezium. Hence, the commutativity of the first diagram is proven. The commutativity of the second diagram is analogous. Now, we need a second step: we claim that, in a monoidal category $(\mathcal{C}, \otimes, I)$ we have that:

$$\begin{aligned}
\ell_{I \otimes X} &= I \otimes \ell_X, \\
r_{I \otimes X} &= r_X \otimes I, \\
\ell_I &= r_I.
\end{aligned}$$

Observe that by naturality of ℓ applied to the morphism $\ell_X : I \otimes X \rightarrow X$, the following diagram commutes:

$$\begin{array}{ccc}
I \otimes (I \otimes X) & \xrightarrow{I \otimes \ell_X} & I \otimes X \\
\downarrow \ell_{I \otimes X} & & \downarrow \ell_X \\
I \otimes X & \xrightarrow{\ell_X} & X
\end{array}$$

Hence, we have that $\ell_X \ell_{I \otimes X} = \ell_X (I \otimes \ell_X)$. Since ℓ_X is an isomorphism, we have that $\ell_{I \otimes X} = I \otimes \ell_X$. The second equation is proven analogously. So, let us prove the third equation. Note, by the previous proof, that the following diagram commutes:

$$\begin{array}{ccc}
(I \otimes I) \otimes I & \xrightarrow{\alpha} & I \otimes (I \otimes I) \\
\downarrow \ell_{I \otimes I} & \swarrow \ell_{I \otimes I} & \\
I \otimes I & &
\end{array}$$

and using what we have proven until now, we have that

$$\ell_I \otimes I = \ell_{I \otimes I} \alpha = (I \otimes \ell_I) \alpha.$$

And then, by the middle unit axioms, we have:

$$(I \otimes \ell_I) \alpha = r_I \otimes I.$$

and then, by the naturality of α we have that:

$$\ell_I \otimes I = r_I \otimes I.$$

And since $r : Id \otimes I \rightarrow Id_C$ is a natural isomorphism, we have that $\ell_I = r_I$, as we wanted to prove. Now, our aim is to prove that I is in fact an algebra with such morphisms. Call $r_I = \ell_I = m$. Then, by the definition of unit, we have that:

$$m(I \otimes m)\alpha = m(I \otimes r_I)\alpha = m(r_I \otimes I) = m(m \otimes I).$$

The first and last equalities follow from the definition of m , while the second equality follows from what we already know by middle unit axioms (i, e , $(I \otimes \ell_I)\alpha = r_I \otimes I$). Now, we need to prove the unit axiom. However, observe that:

$$m(\mu \otimes I) = m(I \otimes I) = m = \ell_I.$$

The first equality follows from the definition of the unit morphism in I , the second follows from the fact that $\otimes$ is a functor and then sends identity into identity and the last equality follows from the definition of m . Analogously, we have that $m(I \otimes \mu) = r_I$.

Hence, we have proven that I is an algebra. $\square$

Definition 3.2.2 (Coalgebra). *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A **coalgebra object**, or simply a **coalgebra** in $\mathcal{C}$, is a triple (C, Δ, ε) consisting of an object C , a morphism $\Delta : C \rightarrow C \otimes C$ and a morphism $\varepsilon : C \rightarrow I$ such that the following diagrams commute:*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \Delta \otimes C \\ C \otimes C & \xrightarrow{C \otimes \Delta} C \otimes (C \otimes C) \xrightarrow{\alpha^{-1}} (C \otimes C) \otimes C & \\ & & \\ & C & \\ & \swarrow \ell^{-1} \quad \searrow r^{-1} & \\ I \otimes C & & C \otimes I \\ & \swarrow \varepsilon \otimes C \quad \searrow C \otimes \varepsilon & \\ & C \otimes C & \end{array}$$

The morphism Δ is called **coproduct** and the morphism ε is called **counit**. The first diagram is called **coassociativity diagram**, while the second diagram is called **counit property diagram**.

As in the case of algebras, if the category is strict monoidal, we may write the coassociativity diagram as:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \Delta \otimes C \\ C \otimes C & \xrightarrow{C \otimes \Delta} & C \otimes C \otimes C \end{array}$$

Example 3.2.3. A standard example of coalgebras is a $\mathbb{K}$ -coalgebra. Another example can be found in $(Sets, \times, \{*\})$. Take a set C and for each element $c \in C$ define $\Delta(c) = (c, c)$ and $\varepsilon(c) = *$.

In fact, Δ is coassociative and let us compute that:

$$\begin{aligned} (\varepsilon \times C) \circ \Delta(c) &= (\varepsilon \times C)(c, c) \\ &= (*, c) \\ &= \ell^{-1}(c) \end{aligned}$$

The first equality follows from the definition of Δ , while the second follows from the definition of ε and the third equality follows from the definition of ℓ in $(Sets, \times, \{*\})$. By an analogous argument, we have that $(C \times \varepsilon) \circ \Delta = r^{-1}$.

Also, observe that such structure of coalgebra in $(Sets, \times, \{*\})$ is unique. In fact, first observe that, since any counit must be defined as $\varepsilon' : C \rightarrow \{*\}$, and $\{*\}$ is a terminal object, then ε is the unique function we may define from C to $\{*\}$. Also, observe that, for any $c \in C$, we know that exists c', c'' such that $\Delta(c) = (c', c'')$.

Then, due to the counit property:

$$(*, c'') = (\varepsilon \times C)\Delta(c) = (*, c).$$

All equalities follow from the counit property. Then, $c'' = c$. Analogously, $c' = c$. Hence, each object in $(Sets, \times, \{*\})$ has a unique coalgebra structure.

Example 3.2.4. A more general example of coalgebra is related to the categorical products. Just like the example of $Sets$, this structure is unique when the monoidal structure is given by a categorical product. In fact, let $C \in \mathcal{C}_0$ in a category with categorical product be given. Every object C in a category with categorical products is a coalgebra with $\Delta = Id_C \times Id_C$ and $\varepsilon : C \rightarrow I$ being the unique morphism possible from C to I . Such coalgebra structure is unique for C . In fact, it is straightforward that Δ is coassociative. Then, we shall prove that ε is a counit, and we will do it in some steps. First, we prove the associativity of the following diagram:

$$\begin{array}{ccc} C \times C & & \\ \Delta \uparrow & \searrow C \times \varepsilon & \pi_1 \\ C & \xrightarrow{r} & C \times I \xrightarrow{r} C \\ & \nearrow r^{-1} & \nearrow C \end{array}$$

Our aim is to prove the commutativity of the left triangle. The outer triangle commutes by the definition of Δ , while the top triangle commutes by the definition of r and the down triangle commutes by the definition of an inverse.

Hence, the left triangle commutes. Also, by analogous arguments, we have that $(\varepsilon \times C) \circ \Delta = \ell^{-1}$. Now, we need to prove that the structure is unique. Again, we need to prove the commutativity of the previous diagram, but now we do not assume the commutativity of the outer triangle. Instead, we assume the commutativity of the left triangle. Observe that the down triangle commutes by the definition of inverse,

and the top triangle commutes by the definition of r . Then, since all three triangles inside commute, the outer triangle commutes. By analogous arguments, we have that $\pi_2 \circ \Delta = C$. Then, by the uniqueness in the categorical product, we obtain that $\Delta = C \times C$.

Example 3.2.5. Let $(C, \otimes, I)$ be a monoidal category. The unit object is a coalgebra with $\Delta = Id_I \otimes Id_I$ and $\varepsilon = I$.

In fact, by Proposition 3.2.1, we have $\ell_I = r_I$. Hence, $\Delta : I \rightarrow I \otimes I$ given by $\Delta = \ell_I^{-1} = r_I^{-1}$ and $\varepsilon : I \rightarrow I$ given by $\varepsilon = I$ define a coalgebra in I . Since the defining diagrams for a coalgebra are dual to those that define an algebra, and both Δ and ε are the inverses of the morphisms that define the algebra structure on I , it follows that I is a coalgebra.

In $\mathbb{K}$ -algebras and $\mathbb{K}$ -coalgebras, we had morphisms of algebras and morphisms of coalgebras. Here, we also have a general definition of morphisms of algebras and morphisms of coalgebras, as follows.

Definition 3.2.3. Let (A, m_A, μ_A) and (B, m_B, μ_B) be both algebras in a monoidal category $\mathcal{C}$. Then, $f : A \rightarrow B$ is a **morphism of algebras** if the following diagrams commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ \downarrow m_A & & \downarrow m_B \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow \mu_A & \nearrow \mu_B & \\ I & & \end{array}$$

If a morphism f satisfies the first diagram, we say f is **multiplicative**. If f satisfies the second diagram, then f **preserves the unit**.

Due to the previous chapter, we know that morphisms of $\mathbb{K}$ -algebras are an example of morphisms of algebras in the monoidal category $\mathbb{K} - Vect$. Then, we have that such definition is a generalization of the first definition of morphisms of algebras.

Example 3.2.6. Since algebras in $Sets$ are the same as monoids, the morphisms of algebras are the same as morphisms of monoids.

The next definition is dual to the previous one and generalizes morphisms of coalgebras as we have defined.

Definition 3.2.4. Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be both coalgebras objects in a monoidal category $\mathcal{C}$. A **morphism of coalgebras** is a morphism $f : C \rightarrow D$ such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow \Delta_C & & \downarrow \Delta_D \\ C \otimes C & \xrightarrow{f \otimes f} & D \otimes D \end{array} \quad \begin{array}{ccc} C & \xrightarrow{f} & D \\ \varepsilon_C \downarrow & \swarrow \varepsilon_D & \\ I & & \end{array}$$

Note that the diagrams in this definition and the diagrams in the definition of morphisms of $\mathbb{K}$ -coalgebras of the previous chapter are similar.

Example 3.2.7. In $(Sets, \times, \{*\})$, every function is a morphism of coalgebras. In fact, let $f : C \rightarrow D$ be a function. Then, for $c \in C$, we evaluate:

$$\begin{aligned} (f \times f) \circ \Delta(c) &= (f \times f)(c, c) \\ &= (f(c), f(c)) \\ &= \Delta \circ f(c). \end{aligned}$$

The first equality follows from the definition of Δ in C , the second equality follows from the application of the function $f \times f$, and the last equality follows from the definition of Δ in D . Also, observe that $\varepsilon \circ f(c) = * = \varepsilon(c)$, and then f is a morphism of coalgebras.

For the next propositions we consider $\mathcal{C}$ as a braided monoidal category. Our aim now is to define bialgebras in a more general way, as we just did to algebras and coalgebras.

Proposition 3.2.2. Let $(\mathcal{C}, \otimes, I, \sigma)$ be a strict braided monoidal category.

- If A and B are both algebras in $\mathcal{C}$, then $A \otimes B$ is also an algebra, with product defined as:

$$m_{A \otimes B} = (m_A \otimes m_B)(A \otimes \sigma_{B,A} \otimes B),$$

and the unit defined as:

$$\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ \ell_I^{-1} = (\mu_A \otimes \mu_B) \circ r_I^{-1}.$$

- If C and D are both coalgebras in $\mathcal{C}$, then $C \otimes D$ is also a coalgebra, with coproduct defined as:

$$\Delta_{A \otimes B} = (A \otimes \sigma_{B,A}^{-1} \otimes B) \circ (\Delta_A \otimes \Delta_B),$$

and the counit defined as:

$$\ell_I \circ \varepsilon_{A \otimes B} = r_I \circ \varepsilon_A \otimes \varepsilon_B.$$

Proof. First, due to the naturality of the braiding the following relations hold:

$$\sigma_{B,A}(m_B \otimes A) = (A \otimes m_B)\sigma_{B \otimes B,A},$$

$$\sigma_{B,A}(B \otimes m_A) = (m_A \otimes B)\sigma_{B,A \otimes A}.$$

For proving the associativity, observe that:

$$\begin{aligned} m_{A \otimes B}(m_{A \otimes B} \otimes A \otimes B) &= m_{A \otimes B}(m_A \otimes m_B \otimes A \otimes B)(A \otimes \sigma_{B,A} \otimes B \otimes A \otimes B) \\ &= (m_A \otimes m_B)(A \otimes \sigma_{B,A} \otimes B)(m_A \otimes m_B \otimes A \otimes B) \\ &\quad (A \otimes \sigma_{B,A} \otimes B \otimes A \otimes B) \\ &= (m_A \otimes m_B)(m_A \otimes A \otimes m_B \otimes B)(A \otimes A \otimes \sigma_{B \otimes B,A} \otimes B) \\ &\quad (A \otimes \sigma_{B,A} \otimes B \otimes A \otimes B) \\ &= (m_A \otimes m_B)(A \otimes m_A \otimes B \otimes m_B)(A \otimes A \otimes \sigma_{B \otimes B,A} \otimes B) \\ &\quad (A \otimes \sigma_{B,A} \otimes B \otimes A \otimes B) \\ &= (m_A \otimes m_B)(A \otimes m_A \otimes B \otimes m_B)(A \otimes A \otimes \sigma_{B,A} \otimes B \otimes B) \\ &\quad (A \otimes A \otimes B \otimes \sigma_{B,A} \otimes B)(A \otimes \sigma_{B,A} \otimes B \otimes A \otimes B) \\ &= (m_A \otimes m_B)(A \otimes m_A \otimes B \otimes m_B)(A \otimes \sigma_{B,A \otimes A} \otimes B \otimes B) \\ &\quad (A \otimes B \otimes A \otimes \sigma_{B,A} \otimes B) \\ &= (m_A \otimes m_B)(A \otimes \sigma_{B,A} \otimes B \otimes m_B)(A \otimes B \otimes m_A \otimes B \otimes B) \\ &\quad (A \otimes B \otimes A \otimes \sigma_{B,A} \otimes B) \\ &= (m_A \otimes m_B)(A \otimes \sigma_{B,A} \otimes B)(A \otimes B \otimes m_A \otimes m_B) \\ &\quad (A \otimes B \otimes A \otimes \sigma_{B,A} \otimes B) \\ &= m_{A \otimes B}(A \otimes B \otimes m_{A \otimes B}). \end{aligned}$$

The first, second and last equalities follow from the definition of $m_{A \otimes B}$, while the third, and seventh equalities follow from the naturality of σ and the expressions above. The fourth equality follows from the associativity of m_A and m_B and fifth and sixth equalities follow from the commuting diagram in the definition of braided monoidal category. Since the definition of $\Delta_{C \otimes D}$ is dual to the definition of $m_{C \otimes D}$, the coassociativity of $\Delta_{C \otimes D}$ follows analogously.

For the unit property, we evaluate:

$$\begin{aligned}
 m_{A \otimes B}(A \otimes B \otimes \mu_{A \otimes B}) &= (m_A \otimes m_B)(A \otimes \sigma_{B,A} \otimes B)(A \otimes B \otimes \mu_{A \otimes B}) \\
 &= (m_A \otimes m_B)(A \otimes \sigma_{B,A} \otimes B)(A \otimes B \otimes \mu_A \otimes \mu_B) \\
 &= (m_A \otimes m_B)(A \otimes \mu_A \otimes B \otimes \mu_B) \\
 &= (m_A(A \otimes \mu_A) \otimes m_B(B \otimes \mu_B)) \\
 &= A \otimes B.
 \end{aligned}$$

The first equality follows from the definition of $m_{A \otimes B}$, while the second follows from the definition of $\mu_{A \otimes B}$, the third follows by the naturality of σ , the fourth follows from the fact that $\otimes$ is an functor and the last equality is just the unit property applied to A and B .

It is analogous that $m_{A \otimes B}(\mu_{A \otimes B} \otimes A \otimes B) = A \otimes B$, and due to the duality of the property of counit, it also follows analogously that the counit property for $\varepsilon_{A \otimes B}$ also holds. $\square$

The next theorem is a generalization of the proposition we used to define bialgebras in the previous chapter.

Theorem 3.2.1. *Let $(\mathcal{C}, \otimes, I, \sigma)$ be a strict braided monoidal category and let $(H, m, \mu, \Delta, \varepsilon)$ be an object in $\mathcal{C}$ that is simultaneously an algebra and a coalgebra. Then, the following affirmatives are equivalent:*

- *The maps Δ and ε are morphisms of algebras.*
- *The maps m and μ are morphisms of coalgebras.*

Proof. First, remember that $m_{I \otimes I} = \ell_I = r_I$, $\Delta_I = \ell_I^{-1} = r_I^{-1}$ and $\mu_I = \varepsilon_I = I$ and, since the category is strict monoidal, and so $I \otimes I = I$, we obtain $\Delta_I = m_I = I$. Then, consider the following four diagrams:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{m} & H & \xrightarrow{\Delta} & H \otimes H \\
 \downarrow \Delta \otimes \Delta & & & & \uparrow m \otimes m \\
 H \otimes H \otimes H \otimes H & \xrightarrow{H \otimes \sigma_{H,H} \otimes H} & H \otimes H \otimes H \otimes H & & \\
 \downarrow \Delta_I & \Delta_I & \downarrow \mu \otimes \mu & \downarrow \varepsilon \otimes \varepsilon & \downarrow \varepsilon \\
 I \xrightarrow{\Delta_I} I \otimes I & H \otimes H \xrightarrow{m_{I \otimes I}} H & I \xrightarrow{\mu} H & I \xrightarrow{\varepsilon} H & \\
 \downarrow \mu & \downarrow \mu & \downarrow \mu & \downarrow \mu & \\
 H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{m} & I
 \end{array}$$

Since $\Delta_I = m_{I \otimes I} = I$, we have that, Δ being a morphism of algebras is equivalent to the two first diagrams to commute, while ε being a morphism of algebras is equivalent to the third and last diagrams to commute. However, m being a morphism of coalgebras is equivalent to the first and third diagrams to commute, while μ being a

morphism of coalgebras is equivalent to the second and fourth diagrams to commute. $\square$

Definition 3.2.5 (Bialgebras). *If $(H, m, \mu, \Delta, \varepsilon)$ is an algebra and a coalgebra simultaneously and it satisfies one of the conditions of the theorem above, then, H is called a **bialgebra**.*

Example 3.2.8. In *Sets*, for example, algebras are monoids, and every set C is a coalgebra with $\Delta(c) = (c, c)$ and $\varepsilon(c) = *$.

The braiding in *Sets* is the flip map and then, if M is a monoid, the product on $M \times M$ is given by $(m, n)(m', n') = (m m', n n')$ and the unit of $M \times M$ is given by $(1, 1)$. It is immediate that Δ and ε are always morphisms of algebras, and hence every monoid is a bialgebra in *Sets*.

Then, as we have defined in a more general way a bialgebra, we may also define morphisms of bialgebras, as follows.

Definition 3.2.6. *Let B, H be both bialgebras. Then, $f : B \rightarrow H$ is a **morphism of bialgebras** if it is a morphism of algebras and coalgebras.*

The examples of morphisms of bialgebras given in the previous chapter are examples of morphism of bialgebras as in Definition 2.5.2. However, there is also the example in $(\text{Sets}, \times, \{*\})$, as follows.

Example 3.2.9. In $(\text{Sets}, \times)$, since every function is a morphism of coalgebras, if A, B are algebras in *Sets* (i.e, monoids), they are also bialgebras. Then, every morphism of algebras between A and B are also morphisms of bialgebras, since they are also morphisms of coalgebras, since every function in *Sets* is a morphism of coalgebras.

Now, our aim is to generalize Hopf algebras, as we did for $\mathbb{K}$ -algebras, $\mathbb{K}$ -coalgebras and $\mathbb{K}$ -bialgebras. In the previous chapter we have first defined the convolution product. However, in this section, for a coalgebra C and an algebra A in a braided monoidal category $\mathcal{C}$, we can not be certain that $\text{Hom}(C, A)$ is an algebra in $\mathcal{C}$ with the convolution product since $\text{Hom}(C, A)$ may not even be an object of $\mathcal{C}$.

Then, we shall to define Hopf algebras directly, as follows:

Definition 3.2.7. *Let $(H, M, \mu, \Delta, \varepsilon)$ be a bialgebra. If there exists a morphism $S : H \rightarrow H$ such that:*

$$m \circ (S \otimes H) \circ \Delta = m \circ (H \otimes S) \circ \Delta = \mu \circ \varepsilon,$$

*then H is a **Hopf algebra**, and S is an **antipode** for H .*

In the previous chapter we have studied Hopf $\mathbb{K}$ -algebras, which are the Hopf algebras in the category of $\mathbb{K}$ -vector spaces. There we have also studied some important Hopf algebras associated to groups. The theory developed in this present chapter shows that the connection is deeper, as we see in the next example.

Example 3.2.10. In $(Sets, \times, \{*\})$, a Hopf algebra is the same as a group. In fact, we have that an antipode S is such that for $c \in C$ being C a set (monoid, bialgebra), we have:

$$\begin{aligned} 1 &= m \circ (S \times C) \circ \Delta(c) = m \circ (S \times C)(c, c) \\ &= m(S(c), c) \\ &= S(c)c. \end{aligned}$$

The first equality follows from the definition of antipode, while the second equality follows from the definition of Δ , the third is the application of $S \times C$ and the last is a notation for m . For analogous reasons, we have that $cS(c) = 1$. Hence, a Hopf algebra in $Sets$ is the same as a group.

We finish this chapter showing that a fundamental property of the antipode of a Hopf $\mathbb{K}$ -algebra still holds in this general setting; the antipode of a Hopf algebra in a braided monoidal category is an antimorphism of algebras and coalgebras.

Proposition 3.2.3. *Let $(H, m_H, \mu_H, \Delta_H, \varepsilon_H, S)$ be a Hopf algebra. Then S is an antimorphism of algebras and coalgebras. That is:*

$$\begin{aligned} S \circ m &= m \circ (S \otimes S) \circ \sigma, \\ \Delta \circ S &= \sigma \circ (S \otimes S) \circ \Delta. \end{aligned}$$

Proof. First, let us observe we cannot do the same as in Proposition 2.5.7, because $Hom(H \otimes H, H)$ may not be an object in the category and, even if it was an object, we cannot resort to doing computations on elements and using the Sweedler notation.

Then, let us consider the morphisms $f, g, h : H \otimes H \rightarrow H$ given by the formulas:

$$\begin{aligned} f &= m_H \circ (S \otimes S) \circ \sigma_{H,H} \\ g &= S \circ m_H \\ h &= m_H^3 \circ (f \otimes H \otimes H \otimes g) \circ \Delta_{H \otimes H}^2. \end{aligned}$$

Here, $\Delta_{H \otimes H}^2$ is the morphism $(\Delta_{H \otimes H} \otimes H \otimes H) \circ \Delta_{H \otimes H} = (H \otimes H \otimes \Delta_{H \otimes H}) \circ \Delta_{H \otimes H}$ (this equality is true due to the coassociativity of $\Delta_{H \otimes H}$), m_H^2 is the morphism $m_H \circ$

$(m_H \otimes H) = m_H \circ (H \otimes m_H)$ (this equality is true due to the associativity of m_H) and $m_H^3 = m_H \circ (m_H^2 \otimes H) = m_H \circ (H \otimes m_H^2)$ (this equality is true due to the associativity of m_H). Observe that f, g are morphisms that are similar to those in Proposition 2.5.7. However, since we have not the structure of algebra in $\text{Hom}(H \otimes H, H)$, we consider the morphism h . Then, observe that:

$$\begin{aligned}
m_H^2 \circ (H \otimes H \otimes g) \circ \Delta_{H \otimes H} &= m_H^2 \circ (H \otimes H \otimes (S \circ m_H)) \circ \Delta_{H \otimes H} \\
&= m_H \otimes (m_H \otimes H) \circ (H \otimes H \otimes (S \circ m_H)) \circ \Delta_{H \otimes H} \\
&= m_H \circ (m_H \otimes (S \circ m_H)) \circ \Delta_{H \otimes H} \\
&= m_H \circ (H \otimes (S \circ m_H)) \circ (m_H \otimes H \otimes H) \circ \Delta_{H \otimes H} \\
&= m_H \circ (H \otimes S) \circ (m_H \otimes m_H) \circ \Delta_{H \otimes H} \\
&= m_H \circ (H \otimes S) \circ \Delta_H \circ m_H \\
&= \mu_H \circ \varepsilon_H \circ m_H \\
&= \mu_H \circ \varepsilon_{H \otimes H}.
\end{aligned}$$

The first equality follows from the definition of g , while the second follows from the definition of m_H ; the third, fourth and fifth equalities follow from the fact that $\otimes$ is a functor, the sixth and last equalities follow from the fact that H is a bialgebra, and the seventh equality follows from the definition of S . And so, we can compute that:

$$\begin{aligned}
h &= m_H^3 \circ (f \otimes H \otimes H \otimes g) \circ \Delta_{H \otimes H}^2 \\
&= m_H \circ (H \otimes m_H^2) \circ (f \otimes H \otimes H \otimes g) \circ \Delta_{H \otimes H}^2 \\
&= m_H \circ (f \otimes m_H^2 \circ (H \otimes H \otimes g)) \circ \Delta_{H \otimes H}^2 \\
&= m_H \circ (f \otimes m_H^2 \circ (H \otimes H \otimes g)) \circ (H \otimes H \otimes \Delta_{H \otimes H}) \circ \Delta_{H \otimes H} \\
&= m_H \circ (f \otimes m_H^2 \circ (H \otimes H \otimes g) \circ \Delta_{H \otimes H}) \circ \Delta_{H \otimes H} \\
&= m_H \circ (f \otimes \mu_H \circ \varepsilon_{H \otimes H}) \circ \Delta_{H \otimes H} \\
&= m_H \circ (f \otimes \mu_H) \circ (H \otimes H \otimes \varepsilon_{H \otimes H}) \circ \Delta_{H \otimes H} \\
&= m_H \circ (f \otimes \mu_H) \\
&= f.
\end{aligned}$$

The first equality is the definition of h , the second equality is the associativity applied to m^3 , the third, fifth and seventh equalities follow from the fact that $\otimes$ is a functor, the fourth follows from the definition of $\Delta_{H \otimes H}^2$, the sixth equality follows from the last computation, the eighth equality follows from the counit property for $H \otimes H$ and the last equality follows from the unit property for H .

Now, our aim is to prove that $h = g$. Evaluating:

$$\begin{aligned}
& m_H^2 \circ (f \otimes H \otimes H) \circ \Delta_{H \otimes H} \\
&= m_H^2 \circ (m_H \circ (S \otimes S) \circ \sigma) \otimes H \otimes H \circ \Delta_{H \otimes H} \\
&= m_H^2 \circ (m_H \otimes H \otimes H) \circ ((S \otimes S) \otimes H \otimes H) \circ (\sigma_{H,H} \otimes H \otimes H) \circ \Delta_{H \otimes H} \\
&= m_H^2 \circ (m_H \otimes H \otimes H) \circ ((S \otimes S) \otimes H \otimes H) \circ (\sigma_{H,H} \otimes H \otimes H) \circ (H \otimes \sigma_{H,H} \otimes H) \circ (\Delta_H \otimes \Delta_H) \\
&= m_H^2 \circ (m_H \otimes H \otimes H) \circ ((S \otimes S) \otimes H \otimes H) \circ ((\sigma_{H,H} \otimes H) \circ ((H \otimes \sigma_{H,H}) \otimes H) \circ (\Delta_H \otimes \Delta_H)) \\
&= m_H^2 \circ (m_H \otimes H \otimes H) \circ ((S \otimes S) \otimes H \otimes H) \circ (\sigma_{H \otimes H, H} \otimes H) \circ (\Delta_H \otimes \Delta_H).
\end{aligned}$$

The first equality follows from the definition of f , the second, fourth, and the last equality follows from the axioms of braiding and the third equality follows from the definition of $\Delta_{H \otimes H}$.

Then, we have the commutativity of the following diagram due to the naturality of σ :

$$\begin{array}{ccc}
(H \otimes H) \otimes H & \xrightarrow{\sigma_{H \otimes H, H}} & H \otimes (H \otimes H) \\
\downarrow (S \otimes H) \otimes S & & \downarrow S \otimes (S \otimes H) \\
(H \otimes H) \otimes H & \xrightarrow{\sigma_{H \otimes H, H}} & H \otimes (H \otimes H) \\
\downarrow m \otimes H & & \downarrow H \otimes m \\
H \otimes H & \xrightarrow{\sigma_{H, H}} & H \otimes H
\end{array}$$

Then, we keep evaluating:

$$\begin{aligned}
&=^6 m_H^2 \circ (m_H \otimes H \otimes H) \circ ((S \otimes S) \otimes H \otimes H) \circ (\sigma_{H \otimes H, H} \otimes H) \circ (\Delta_H \otimes \Delta_H) \\
&=^7 m_H^2 \circ (((m_H \otimes H) \circ (S \otimes S) \circ \sigma_{H \otimes H, H}) \otimes H) \circ (\Delta_H \otimes \Delta_H) \\
&=^{*,8} m_H^2 \circ ((\sigma_{H,H} \circ (m \otimes H) \circ (S \otimes H \otimes S)) \otimes H) \circ (\Delta_H \otimes \Delta_H) \\
&=^9 m_H^2 \circ ((\sigma_{H,H} \otimes H \otimes H) \circ ((m \otimes H) \otimes H \otimes H) \circ (S \otimes H \otimes S \otimes H) \circ (\Delta_H \otimes \Delta_H)) \\
&=^{10} m_H^2 \circ ((\sigma_{H,H} \circ (m \otimes H) \circ (S \otimes H)) \otimes S \otimes H) \circ (\Delta_H \otimes \Delta_H) \\
&=^{11} m_H^2 \circ ((\sigma_{H,H} \circ m \circ (S \otimes H) \circ \Delta_H) \otimes S \otimes H) \circ (H \otimes H \otimes \Delta_H) \\
&=^{12} m_H^2 \circ (\sigma_{H,H} \circ (\mu \circ \varepsilon) \otimes (S \otimes H)) \circ (H \otimes H \otimes \Delta_H) \\
&=^{13} m_H^2 \circ (\sigma_{H,H} \otimes H) \circ (\mu \otimes H \otimes H) \circ (S \otimes H) \circ (\varepsilon \otimes \Delta) \\
&=^{14} m_H^2 \circ (H \otimes \mu_H \otimes H) \circ (S \otimes H) \circ (\varepsilon_H \otimes \Delta_H) \\
&=^{15} m_H \circ (m_H \otimes H) \circ (H \otimes \mu_H \otimes H) \circ (S \otimes H) \circ (\varepsilon_H \otimes \Delta_H) \\
&=^{16} m_H \circ (m_H \circ (H \otimes \mu_H) \otimes H) \circ (S \otimes H) \circ (\varepsilon_H \otimes \Delta_H) \\
&=^{17} m_H \circ (H \otimes H) \circ (S \otimes H) \circ (\varepsilon_H \otimes \Delta_H) \\
&=^{18} m_H \circ (S \otimes H) \circ \Delta_H \circ (\varepsilon_H \otimes H) \\
&=^{19} \mu_H \circ \varepsilon_H \circ (\varepsilon_H \otimes H) \\
&=^{20} \mu_H \circ \varepsilon_{H \otimes H}.
\end{aligned}$$

The eleventh and the eighteen equalities follow from the definition of S , the thirteenth equality follows from the naturality of the braiding, the fourteenth equality follows from the associativity applied to m_H , the sixteenth equality follows from the unit property in H , the last equality follows from the property of ε_H , and the seventh follows from the commutativity the previous diagram. Then, evaluating:

$$\begin{aligned}
 f & \stackrel{1}{=} h \stackrel{2}{=} m_H^3 \circ (f \otimes H \otimes H \otimes g) \circ \Delta_{H \otimes H}^2 \\
 & \stackrel{3}{=} m_H \circ (m_H^2 \otimes H \otimes H \otimes H) \circ (f \otimes H \otimes H \otimes g) \circ \Delta_{H \otimes H}^2 \\
 & \stackrel{4}{=} m_H \circ ((m_H^2 \circ (f \otimes H \otimes H)) \otimes g) \circ \Delta_{H \otimes H}^2 \\
 & \stackrel{5}{=} m_H \circ (m_H^2 \circ (f \otimes H \otimes H) \otimes g) \circ (\Delta_{H \otimes H} \otimes H \otimes H) \circ \Delta_{H \otimes H}^2 \\
 & \stackrel{6}{=} m_H \circ (m_H^2 \circ (f \otimes H \otimes H) \circ \Delta_{H \otimes H} \otimes g) \circ \Delta_{H \otimes H}^2 \\
 & \stackrel{7}{=} m_H \circ (\mu_H \circ \varepsilon_{H \otimes H} \otimes g) \circ \Delta_{H \otimes H} \\
 & \stackrel{8}{=} m_H \circ (\mu_H \otimes H) \circ (\varepsilon_{H \otimes H} \otimes g) \circ \Delta_{H \otimes H} \\
 & \stackrel{9}{=} (\varepsilon_{H \otimes H} \otimes g) \circ \Delta_{H \otimes H} \\
 & \stackrel{10}{=} g.
 \end{aligned}$$

The first equality follows from what we have proven about f and h , while the second equality is the definition of h , the third equality is the associativity applied to m , the fourth, sixth and eighth equalities follow from the fact that $\otimes$ is a functor, the fifth equality follows from the coassociativity applied to Δ , the seventh equality follows from the previous computation, the ninth equality follows from the unit property in H and the last equality follows from the counit property applied to g . Hence, $f = g$, that is, the antipode is an antimorphism of algebras:

$$S \circ m = m \circ (S \otimes S) \circ \sigma.$$

Now, we consider $f, g, h : H \rightarrow H \otimes H$ given by the formulas:

$$\begin{aligned}
 f &= \sigma_{H,H} \circ (S \otimes S) \circ \Delta_H, \\
 g &= \Delta_H \circ S, \\
 h &= m_{H \otimes H}^2 \circ (g \otimes H \otimes H \otimes f) \circ \Delta_H^3.
 \end{aligned}$$

First, we evaluate:

$$\begin{aligned}
m_{H \otimes H} \circ (g \otimes H \otimes H) \circ \Delta_H^2 &=^1 m_{H \otimes H} \circ (\Delta_H \circ S \otimes H \otimes H) \circ \Delta_H^2 \\
&=^2 m_{H \otimes H} \circ (\Delta_H \circ S \otimes H \otimes H) \circ (H \otimes \Delta_H) \circ \Delta_H \\
&=^3 m_{H \otimes H} \circ (\Delta_H \circ S \otimes \Delta_H) \circ \Delta_H \\
&=^4 m_{H \otimes H} \circ (\Delta_H \otimes \Delta_H) \circ (S \otimes H) \circ \Delta_H \\
&=^5 \Delta_H \circ m_H \circ (S \otimes H) \circ \Delta_H \\
&=^6 \Delta_H \circ \mu_H \circ \varepsilon_H \\
&=^7 \mu_{H \otimes H} \circ \varepsilon_H.
\end{aligned}$$

The first equality follows from the definition of g , while the second equality follows from the coassociativity applied to Δ_H , the third and fourth equalities follow from the fact that $\otimes$ is a functor, the fifth and last equalities follow from the fact that H is a bialgebra and then Δ_H is a morphism of algebras, and the sixth equality follows from the definition of antipode. Now, we compute that:

$$\begin{aligned}
h &= m_{H \otimes H}^2 \circ (g \otimes H \otimes H \otimes f) \circ \Delta_H^3 \\
&= m_{H \otimes H} \circ (m_{H \otimes H} \otimes H \otimes H) \circ (g \otimes H \otimes H \otimes f) \circ \Delta_H^3 \\
&= m_{H \otimes H} \circ (m_{H \otimes H} \circ (g \otimes H \otimes H)) \otimes f \circ \Delta_H^3 \\
&= m_{H \otimes H} \circ (m_{H \otimes H} \circ (g \otimes H \otimes H)) \otimes f \circ (\Delta_H^2 \otimes H) \circ \Delta_H \\
&= m_{H \otimes H} \circ (m_{H \otimes H} \circ (g \otimes H \otimes H) \circ \Delta_H^2 \otimes f) \circ \Delta_H \\
&= m_{H \otimes H} \circ (\mu_{H \otimes H} \circ \varepsilon_H \otimes f) \circ \Delta_H \\
&= m_{H \otimes H} \circ (\mu_{H \otimes H} \otimes f) \circ (H \otimes \varepsilon_H) \circ \Delta_H \\
&= m_{H \otimes H} \circ (\mu_{H \otimes H} \otimes f) \\
&= f.
\end{aligned}$$

The first equality is the definition of h , the second equality is the associativity applied to $m_{H \otimes H}^2$, the third, fifth and seventh equalities follows from the fact that $\otimes$ is a functor, the fourth equality is the coassociativity applied to Δ_H^3 , the sixth equality follows from the previous computation, the eighth equality follows from the counit property, and the last

equality is the unit property. Now, observe that:

$$\begin{aligned}
& m_{H \otimes H} \circ (H \otimes H \otimes f) \circ \Delta_H^2 \\
&= m_{H \otimes H} \circ (H \otimes H \otimes (\sigma_{H,H} \circ (S \otimes S) \otimes \Delta_H)) \circ \Delta_H^2 \\
&=^3 (m_H \otimes m_H) \circ (H \otimes \sigma_{H,H} \otimes H) \circ (H \otimes H \otimes (\sigma_{H,H} \circ (S \otimes S) \otimes \Delta_H)) \circ \Delta_H^2 \\
&= (m_H \otimes m_H) \circ (H \otimes \sigma_{H,H} \otimes H) \circ (H \otimes H \otimes \sigma_{H,H}) \circ (H \otimes H \otimes S \otimes S) \\
&\circ (H \otimes H \otimes \Delta_H) \circ \Delta_H^2 \\
&=^5 (m_H \otimes m_H) \circ (H \otimes \sigma_{H,H} \otimes H) \circ (H \otimes H \otimes \sigma_{H,H}) \circ (S \otimes S) \circ \Delta_H^3 \\
&= (m_H \otimes m_H) \circ (H \otimes ((\sigma_{H,H} \otimes H) \circ (H \otimes \sigma_{H,H}))) \circ (S \otimes S) \\
&\circ \Delta_H^3 \\
&= (m_H \otimes m_H) \circ (H \otimes \sigma_{H, H \otimes H}) \circ (S \otimes S) \\
&\circ \Delta_H^3 \\
&= (m_H \otimes m_H) \circ (H \otimes \sigma_{H, H \otimes H}) \circ (H \otimes H \otimes S \otimes S) \circ (H \otimes H \otimes \Delta_H) \\
&\circ \Delta_H^2 \\
&=^8 (m_H \otimes m_H) \circ (H \otimes ((\sigma_{H, H \otimes H}) \circ (H \otimes S \otimes S) \circ (H \otimes \Delta_H))) \circ \Delta_H^2 \\
&= (m_H \otimes m_H) \circ (H \otimes ((S \otimes S \otimes H) \circ (H \otimes \Delta_H) \circ \sigma_{H,H})) \circ \Delta_H^2
\end{aligned}$$

The first equality follows from the definition of f , the second equality follows from the definition of $m_{H \otimes H}$, the third, fifth, and eighth equalities follow from the fact that $\otimes$ is a functor, the fourth and seventh equalities follow from the coassociativity applied to Δ_H^3 , the sixth equality follows from axiom of braiding, the ninth equality follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
H \otimes H & \xrightarrow{\sigma_{H,H}} & H \otimes H \\
\downarrow \Delta_H \otimes H & & \downarrow H \otimes \Delta_H \\
H \otimes H \otimes H & & H \otimes H \otimes H \\
\downarrow S \otimes S \otimes H & & \downarrow S \otimes H \otimes S \\
H \otimes H \otimes H & \xrightarrow{\sigma_{H, H \otimes H}} & H \otimes H \otimes H
\end{array}$$

and so we keep computing:

$$\begin{aligned}
&=^{10} (m_H \otimes m_H) \circ (H \otimes ((S \otimes H \otimes S) \circ (H \otimes \Delta_H) \circ \sigma_{H,H})) \circ \Delta_H^2 \\
&=^{11} (m_H \otimes m_H) \circ ((H \otimes S \otimes H \otimes S) \circ (H \otimes H \otimes \Delta_H) \circ (H \otimes \sigma_{H,H})) \circ \Delta_H^2 \\
&=^{12} (m_H \otimes H) \circ (H \otimes S \otimes (m_H \circ (H \otimes S) \circ \Delta_H)) \circ (H \otimes \sigma_{H,H}) \circ \Delta_H^2 \\
&=^{13} (m_H \otimes H) \circ (H \otimes S \otimes (\mu_H \circ \varepsilon_H)) \circ (H \otimes \sigma_{H,H}) \circ \Delta_H^2 \\
&=^{14} (m_H \otimes H) \circ (H \otimes S \otimes \mu_H) \circ (H \otimes H \otimes I) \circ (H \otimes H \otimes \varepsilon_H) \circ (H \otimes \sigma_{H,H}) \circ \Delta_H^2 \\
&=^{15} (m_H \otimes \mu_H) \circ (H \otimes S \otimes I) \circ (H \otimes H \otimes \varepsilon_H) \\
&\quad \circ (H \otimes \sigma_{H,H}) \circ \Delta_H^2 \\
&=^{16} (H \otimes \mu_H) \circ (m_H \otimes I) \circ (H \otimes S) \circ (H \otimes H \otimes \varepsilon_H) \circ (H \otimes \sigma_{H,H}) \circ \Delta_H^2 \\
&=^{17} (H \otimes \mu_H) \circ (m_H \otimes I) \circ (H \otimes S) \circ (H \otimes \varepsilon_H \otimes H) \circ \Delta_H^2 \\
&=^{18} (H \otimes \mu_H) \circ m_H \circ (H \otimes S) \circ (H \otimes \varepsilon_H \otimes H) \circ (H \otimes \Delta_H) \circ \Delta_H \\
&=^{19} (H \otimes \mu_H) \circ m_H \circ (H \otimes S) \circ \Delta_H \\
&=^{20} (H \otimes \mu_H) \circ \mu_H \circ \varepsilon_H \\
&=^{21} (H \otimes \mu_H) \circ (\mu_H \otimes I) \circ \varepsilon_H \\
&=^{22} (\mu_H \otimes \mu_H) \circ \varepsilon_H \\
&=^{23} \mu_{H \otimes H} \circ \varepsilon_H.
\end{aligned}$$

The tenth, eleventh, thirteenth, fourteenth and sixteenth equalities follow from the fact that $\otimes$ is a functor the twelfth and twentieth equalities follow from the definition of antipode, the seventeenth equality follows from the naturality of σ , the eighteenth equality follows from the coassociativity applied to Δ_H , the nineteenth equality follows from the counit property, the twentieth-first and twenty-second equality follow from the fact that the category $\mathcal{V}$ is strict monoidal, and the last equality follows from the definition of $\mu_{H \otimes H}$.

Finally:

$$\begin{aligned}
f &\stackrel{1}{=} h \stackrel{2}{=} m_{H \otimes H}^2 \circ (g \otimes H \otimes H \otimes f) \circ \Delta_H^3 \\
&\stackrel{3}{=} m_{H \otimes H} \circ (H \otimes H \otimes m_{H \otimes H}) \circ (g \otimes H \otimes H \otimes f) \circ \Delta_H^3 \\
&\stackrel{4}{=} m_{H \otimes H} \circ (g \otimes m_{H \otimes H} \circ (H \otimes H \otimes f)) \circ \Delta_H^3 \\
&\stackrel{5}{=} m_{H \otimes H} \circ (g \otimes m_{H \otimes H} \circ (H \otimes H \otimes f)) \circ (H \otimes \Delta_H^2) \circ \Delta_H \\
&\stackrel{6}{=} m_{H \otimes H} \circ (g \otimes (m_{H \otimes H} \circ (H \otimes H \otimes f) \circ \Delta_H^2)) \circ \Delta_H \\
&\stackrel{7}{=} m_{H \otimes H} \circ (g \otimes (\mu_{H \otimes H} \circ \varepsilon_H)) \circ \Delta_H \\
&\stackrel{8}{=} m_{H \otimes H} \circ (g \otimes \mu_{H \otimes H}) \circ (H \otimes \varepsilon_H) \circ \Delta_H \\
&\stackrel{9}{=} m_{H \otimes H} \circ (g \otimes \mu_{H \otimes H}) \\
&\stackrel{10}{=} g.
\end{aligned}$$

The first equality follows from what we had proved, the second equality is the definition of h , the third equality follows from the associativity applied to $m_{H \otimes H}$, the fourth, sixth and eighth equalities follow from the fact that $\otimes$ is a functor. The fifth equality follows from the coassociativity applied to Δ_H^3 , the seventh equality follows from the previous computation, the ninth equality follows from the counit property in H and the last equality is just the unit property applied to g . Now, we need to prove that S preserves units and counits. In fact, observe that:

$$\begin{aligned}
 \mu_H &=^1 (\varepsilon_H \otimes H) \circ \Delta_H \circ \mu_H \\
 &=^2 (\varepsilon_H \otimes H) \circ (\mu_{H \otimes H}) \\
 &=^3 (\varepsilon_H \otimes H) \circ (\mu_H \otimes \mu_H) \\
 &=^4 (\varepsilon_H \circ \mu_H \otimes \mu_H) \\
 &=^5 \mu_H \circ \varepsilon_H \circ \mu_H \\
 &=^6 m_H \circ (H \otimes S) \circ \Delta_H \circ \mu_H \\
 &=^7 m_H \circ (H \otimes S) \circ (\mu_H \otimes \mu_H) \\
 &=^8 m_H \circ (\mu_H \otimes (S \circ \mu_H)) \\
 &=^9 S \circ \mu_H.
 \end{aligned}$$

The first equality follows from the counit property, the second and seventh equalities follow from the fact that Δ_H is a morphism of algebras, the third equality follows from the definition of $\mu_{H \otimes H}$, the fourth and eighth equalities follow from the fact that $\otimes$ is a functor, the fifth equality follows from the fact that the category $\mathcal{V}$ is strict monoidal, the sixth equality follows from the definition of S and the last equality is just the unit property applied to $S \circ \mu_H$. Hence, S preserves unities. Now, also observe that:

$$\begin{aligned}
 \varepsilon_H &= \varepsilon_H \circ \mu_H \circ \varepsilon_H \\
 &= \varepsilon_H \circ m_H \circ (H \otimes S) \circ \Delta_H \\
 &= \varepsilon_{H \otimes H} \circ (m_H \circ (H \otimes S) \circ \Delta_H) \\
 &= (\varepsilon_H \otimes \varepsilon_H) \circ (H \otimes S) \circ \Delta_H \\
 &= (\varepsilon_H \otimes \varepsilon_H \circ S) \circ \Delta_H \\
 &= \varepsilon_H \circ S.
 \end{aligned}$$

The first equality follows from the last diagram in Theorem 3.2.1, the second equality follows from the definition of S , the third equality follows from the fact that $m_{H \otimes H}$ is a morphism of coalgebras, the fourth equality follows from the definition of $\varepsilon_{H \otimes H}$, the fifth equality follows from the fact that $\otimes$ is a functor, and the last equality is the counit property applied to $\varepsilon_H \circ S$. Hence, we have proved that S is an antimorphism of algebras and coalgebras. $\square$

4 HOPF CATEGORIES

In this section, we assume that all monoidal categories are strict. In fact, Mac Lane's Theorem states that every monoidal category is equivalent to a strict monoidal category through an adjoint equivalence (for more details see (JOHNSON; YAU, 2021), page 13). The first part of this chapter has the aim to give all necessary categorical theory to understand the generalization of Hopf algebras: Hopf categories. In the next chapter, we will generalize algebras, coalgebras, bialgebras and Hopf algebras in such context. The main reference for the first part of this chapter are (JOHNSON; YAU, 2021) and (BATISTA; CAENEPEEL; VERCRUYSSSE, 2016).

4.1 PRELIMINARY DEFINITIONS AND PROPOSITIONS

In this chapter we shall deal with bicategories, 2-categories, lax functors, equivalences, biequivalences, 2-equivalences and monoidal functors. First, we need to define equivalence. The definition we will be using is equivalent to the original definition given in (JOHNSON; YAU, 2021) and the reader may see such equivalence in such reference, which will be also the main reference to this chapter.

Definition 4.1.1 (Equivalence). *A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called an **equivalence** if it obeys the following conditions:*

1- F is one-to-one on morphisms for each set of morphisms between two objects.

*2- F is **essentially surjective** on objects. This means that if A is an object in $\mathcal{D}_0$, then there is an object $B \in \mathcal{D}_0$, an isomorphism $i : A \rightarrow B$ and an object $C \in \mathcal{C}_0$ such that $FC = B$.*

Example 4.1.1. Let $\mathcal{C}$ be a category with just one object C and one morphism 1_C , and a category $\mathcal{D}$ with two objects D_1 and D_2 , the morphisms 1_{D_1} and 1_{D_2} , and two isomorphisms $\alpha : D_1 \rightarrow D_2$ and $\beta : D_2 \rightarrow D_1$ such that $\alpha^{-1} = \beta$.

The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent and we have an equivalence defined as $i : \mathcal{C} \rightarrow \mathcal{D}$, $i(C) = D_1$ and $i(1_C) = 1_{D_1}$. It is one-to-one on the morphisms, and essentially surjective.

Example 4.1.2. Consider $Mat(\mathbb{K})$ the category of matrices in $\mathbb{K}$ defined in the following manner: The objects are the natural numbers, and the morphisms from a number n to a number m are the $m \times n$ matrices. Composition of morphisms in $Mat(\mathbb{K})$ is matrix multiplication. Then, consider the category $vect$ of the $\mathbb{K}$ -vector spaces with finite dimension. Then, we may define an equivalence between $Mat(\mathbb{K})$ and $vect$ $i :$

$Mat(\mathbb{K}) \rightarrow vect$, $i(n) = \mathbb{K}^n$ and i takes an $m \times n$ matrix A to the corresponding linear map $T : \mathbb{K}^n \rightarrow \mathbb{K}^m$ in $vect$ given by left multiplication by A (identifying $\mathbb{K}^n$ with $Mat_{1,n}(\mathbb{K})$).

Definition 4.1.2. A **bicategory** is a tuple

$$(B, 1, c, a, l, r)$$

consisting of the following data.

- B contains a collection of objects. Given objects $X, Y \in B$, there exists a category $B(X, Y)$. The objects in B are called **0-cells**. The objects in $B(X, Y)$ for each $X, Y \in B$ are called **1-cells**, and its morphisms are called **2-cells**. The collection of 0-cells is denoted B_0 , while the collection of 1-cells is denoted B_1 and the collection of 2-cells is denoted B_2 .
- 1 is the category with just one object with the identity being the unique morphism.
- The composition and identity morphisms in $B(X, Y)$ are called **vertical composition** and **identity 2-cells**. An invertible morphism in $B(X, Y)$ is called **invertible 2-cell** and its inverse is called **vertical inverse**. For a 1-cell f its identity is denoted 1_f .
- For each object X in B :

$$1_X : 1 \rightarrow B(X, X)$$

is a functor. We identify the morphism 1_X with the 1-cell $1_X \in B(X, X)$ called **identity 1-cell** of X .

- For each triple of objects $X, Y, Z \in B$,

$$c_{XYZ} : B(Y, Z) \times B(X, Y) \rightarrow B(X, Z)$$

is a functor, called the **horizontal composition**. For 1-cells $f \in B(X, Y)$ and $g \in B(Y, Z)$ and 2-cells $\alpha \in B(X, Y)$ and $\beta \in B(Y, Z)$ we use the notations:

$$c_{XYZ}(g, f) = g \circ f$$

$$c_{XYZ}(\beta, \alpha) = \beta * \alpha$$

- For objects $W, X, Y, Z \in B$,

$$a_{WXYZ} : c_{WYZ}(c_{XYZ} \times Id_{B(W, X)}) \rightarrow c_{WYZ}(Id_{B(Y, Z)} \times c_{WXY})$$

is a natural isomorphism, called the **associator**, between functors

$$B(Y, Z) \times B(X, Y) \times B(W, X) \rightarrow B(W, Z).$$

- For each pair of object $X, Y \in B$:

$$\begin{array}{ccc}
 c_{XY}Y(1_Y \times Id_{B(X,Y)}) & & c_{XX}Y(Id_{B(X,Y)} \times 1_X) \\
 \searrow \ell_{XY} & & \swarrow r_{XY} \\
 & Id_{B(X,Y)} &
 \end{array}$$

are natural isomorphisms, called, respectively, **left unitor** and **right unitor**.

- **Unity Axiom:** The middle unity diagram

$$\begin{array}{ccc}
 (g \circ 1_Y) \circ f & \xrightarrow{a} & g \circ (1_Y \circ f) \\
 \searrow r_g * 1_f & & \swarrow 1_g * \ell_f \\
 & g \circ f &
 \end{array}$$

commutes in $B(X, Z)$ for $f \in B(X, Y)$ and $g \in B(Y, Z)$.

- **Pentagon Axiom:** The diagram:

$$\begin{array}{ccccc}
 & & (k \circ h) \circ (g \circ f) & & \\
 & \nearrow a_{k \circ h, g, f} & & \nwarrow a_{k, h, g \circ f} & \\
 ((k \circ h) \circ g) \circ f & & & & k \circ (h \circ (g \circ f)) \\
 \downarrow a_{k, h, g} * 1_f & & & & \uparrow 1_k * a_{h, g, f} \\
 (k \circ (h \circ g)) \circ f & \xrightarrow{a_{k, h \circ g, f}} & & & k \circ ((h \circ g) \circ f)
 \end{array}$$

for $f \in B(V, W), g \in B(W, X), h \in B(X, Y)$ and $k \in B(Y, Z)$ is commutative in $B(V, Z)$.

The main example of bicategory is also an example of 2-category, as follows:

Definition 4.1.3 (2-categories). A **2-category** is a bicategory $(B, 1, c, a, l, r)$ in which the associator, the left unitor and the right unitor are all identity natural transformations.

Example 4.1.3. Let Cat be the category of categories: its objects are categories, its morphisms are functors, and its composition is composition of functors. Let C be a subcategory of Cat . Then C is a 2-category whose 0-cells are its objects (which are categories), its 1-cells are functors (contained in C) between objects of C , and 2-cells are the natural transformations between such functors. The vertical composition and horizontal composition are the vertical compositions and horizontal compositions between natural transformations and the identity 1-cells are the identity functors.

Definition 4.1.4. (Lax Functor) Suppose that $(B, 1, c, a, \ell, r)$ and $(B', 1', c', a', \ell', r')$ are bicategories. A **lax functor**

$$(F, F^2, F^0) : B \rightarrow B'$$

from B to B' is a triple consisting of the following data:

- $F : B_0 \rightarrow B'_0$ is a function of objects.
- For each pair of objects $X, Y \in B$, it is equipped with a **local functor**:

$$F : B(X, Y) \rightarrow B'(FX, FY).$$

- For all objects $X, Y, Z \in B$, it is equipped with natural transformations:

$$\begin{array}{ccc} B(Y, Z) \times B(X, Y) & \xrightarrow{c} & B(X, Z) \\ \downarrow F \times F & \nearrow F^2 & \downarrow F \\ B'(FY, FZ) \times B'(FX, FY) & \longrightarrow & B'(FX, FZ) \end{array}$$

and

$$F_0 : 1'_{FX} \rightarrow F \circ 1_X$$

- We also have the commutativity of the following diagrams for 1-cells $f \in (W, X), g \in B(X, Y)$ and $h \in B(Y, Z)$:

$$\begin{array}{ccc} (Fh \circ Fg) \circ Ff & \xrightarrow{a'} & Fh \circ Fg \circ Ff \\ \downarrow F^2_{h,g} * 1_{Ff} & & \downarrow Fh * F^2_{g,h} \\ F(hg) \circ Ff & & Fh \circ F(gf) \\ \downarrow F^2_{hg,f} & & \downarrow F^2_{h,gf} \\ F((hg)f) & \xrightarrow{Fa} & F(h(gf)) \\ 1'_{FX} \circ Ff \xrightarrow{\ell'} Ff & & Ff \circ 1'_{FW} \xrightarrow{r'} Ff \\ \downarrow F^0 * 1_{Ff} & \uparrow F\ell & \downarrow 1_{Ff} * F^0_W \\ F1_X \circ Ff \xrightarrow{F^2_{1_X, f}} F(1_X \circ f) & & Ff \circ F1_W \xrightarrow{F^2_{f, 1_W}} F(f \circ 1_W) \end{array}$$

A **pseudofunctor** is a lax functor $(F, F^2, F^0) : B \rightarrow B'$ where F_2 and F_0 are natural transformations. A **strict functor** is a pseudofunctor when F_2 and F_0 are identity functors. A **bifunctor** is a strict functor between 2-categories.

If B and B' are 2-categories, a **2-isomorphism** is a strict functor $F : B \rightarrow B'$ such that there exists a strict functor $G : B' \rightarrow B$ such that $G \circ F = Id_B$ and $F \circ G = Id_{B'}$. Recall that a *Hom* set is the set of morphisms between two objects.

Definition 4.1.5. A pseudofunctor of bicategories $F : B \rightarrow C$ is a **biequivalence** if:

- F is essentially surjective on 0-cells, i.e, every 0-cell in $c \in C_0$ is isomorphic to a 0-cell $F(b)$ with $b \in B_0$;
- F is essentially full in 1-cells, i.e, F is essentially surjective on 1-cells;
- F is fully faithful in 2-cells, that is, F is surjective on 2-cells, and F is injective in *Hom* sets for 2-cells.

If B and C are 2-categories then we call a biequivalence a **2-equivalence**.

Then we also need the definition of monoidal functor. We will state the definition for monoidal categories in general, that is, without assuming strictness.

Definition 4.1.6. For monoidal categories $\mathcal{C}$ and $\mathcal{D}$, a **monoidal functor** $(F, \phi_0, \phi_2) : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- A functor $F : \mathcal{C} \rightarrow \mathcal{D}$,
- A natural transformation

$$\phi_2 : (F \otimes F) \circ \otimes \rightarrow \otimes \circ (F \otimes F),$$

- A morphism

$$\phi_0 : I_{\mathcal{D}} \rightarrow FI_{\mathcal{C}}$$

such that the following diagrams commute for all $X, Y, Z \in \mathcal{C}_0$:

$$\begin{array}{ccc}
 (FX \otimes FY) \otimes FZ & \xrightarrow{\alpha_{\mathcal{D}}} & FX \otimes (FY \otimes FZ) \\
 \downarrow \phi_2 \otimes FZ & & \downarrow FX \otimes \phi_2 \\
 F(X \otimes Y) \otimes FZ & & FX \otimes (F(Y \otimes Z)) \\
 \downarrow \phi_2 & & \downarrow \phi_2 \\
 F((X \otimes Y) \otimes Z) & \xrightarrow{F\alpha_{\mathcal{C}}} & F(X \otimes (Y \otimes Z))
 \end{array}$$

$$\begin{array}{ccc}
 I_{\mathcal{D}} \otimes FX & \xrightarrow{\ell_{FX}^{\mathcal{D}}} & FX \\
 \downarrow \phi_0 \otimes FX & & \uparrow F\ell_X^{\mathcal{C}} \\
 FI_{\mathcal{C}} \otimes FX & \xrightarrow{\phi_2} & F(I_{\mathcal{C}} \otimes X)
 \end{array}$$

$$\begin{array}{ccc}
 FX \otimes I_{\mathcal{D}} & \xrightarrow{r_{FX}^{\mathcal{D}}} & FX \\
 \downarrow FX \otimes \phi_0 & & \uparrow Fr_X^{\mathcal{C}} \\
 FX \otimes FI_{\mathcal{C}} & \xrightarrow{\phi_2} & F(X \otimes I_{\mathcal{C}})
 \end{array}$$

The first diagram is called **associativity**, while the second diagram is called **left unity** and the third diagram is called **right unity**. When ϕ_2 and ϕ_0 are isomorphisms, then it is called **strong monoidal functor**. If ϕ_2 and ϕ_0 are identities, then it is called **strict monoidal functor**.

4.2 HOPF CATEGORIES

All this section have as the main reference (BATISTA; CAENEPEEL; VERCRUYSSSE, 2016). First, we shall define enriched categories, which generalize categories and also algebras in a monoidal category. Then, we shall state some notation and

construction. Let $(\mathcal{V}, \otimes, I)$ be a strict monoidal category and let X be a non-empty class. We can construct a new monoidal category $\mathcal{V}(X)$. In this new monoidal category, the objects are a family of objects in $\mathcal{V}_0$ indexed by $X \times X$:

$$M = (M_{x,y})_{x,y \in X}.$$

A morphism $\phi : M \rightarrow N$ in $\mathcal{V}(X)$ consists of a family of morphisms $\phi_{x,y} : M_{x,y} \rightarrow N_{x,y}$ indexed by $X \times X$. Observe that for defining objects or morphisms in $\mathcal{V}(X)$, we need to define it for all collections indexed by $X \times X$. Then, the monoidal product in $\mathcal{V}(X)$ may be defined as:

$$(M \otimes N)_{x,y} = M_{x,y} \otimes N_{x,y}.$$

The unit object is $I_{x,y} = I$ for $x, y \in X$. The monoidal product of two morphisms ϕ, ψ are given by $(\phi \otimes \psi)_{x,y} = \phi_{x,y} \otimes \psi_{x,y}$. Recall that, since the monoidal category $\mathcal{V}$ is strict, we have that the associator, left units and right units are identity morphisms.

We may also define the opposite functor $(-)^{op} : \mathcal{V}(X) \rightarrow \mathcal{V}(X)$, which will be important later on, as $(M)_{x,y}^{op} = M_{y,x}$ and, for $\phi : M \rightarrow N$ in $\mathcal{V}(X)$, $(\phi)^{op} : M^{op} \rightarrow N^{op}$ is the family $(\phi)_{x,y}^{op} = \phi_{y,x}$.

Definition 4.2.1 (Enriched category). *A $\mathcal{V}$ -category A or a category A enriched in $\mathcal{V}$ consists of an object in $\mathcal{V}(X)$ together with two classes of morphisms, namely:*

- *The product morphisms m defined as:*

$$m_{x,y,z} : A_{x,y} \otimes A_{y,z} \rightarrow A_{x,z}$$

defined for each $x, y, z \in X$.

- *Unit morphisms $\mu_x : I_{x,x} \rightarrow A_{x,x}$ defined for each $x \in X$.*

Also, these data must satisfy the following equalities:

$$m_{x,y,t} \circ (A_{x,y} \otimes m_{y,z,t}) = m_{x,z,t} \circ (m_{x,y,z} \otimes A_{z,t})$$

$$m_{x,x,y} \circ (\mu_x \otimes A_{x,y}) = A_{x,y} = m_{x,y,y} \circ (A_{x,y} \otimes \mu_y)$$

There are two remarks to be made. First, enriched categories generalize two mathematical structures: categories and algebras. In fact, it generalizes categories because a *Sets*-category is a just a (usual) category: for a category $\mathcal{C}$, we may see X as the class of objects $\mathcal{C}_0$, $A_{x,y}$ as its class of morphisms from x to y ($x, y \in \mathcal{C}_0$, and $m_{x,y,z}$ as the composition). The first equality says that the composition is associative, and the second equality guarantees the existence of an identity morphism. It also generalizes

algebras since if X is a singleton then a $\mathcal{V}$ -category is the same as an algebra in $\mathcal{V}$. Second, since m is associative, we may denote

$$\begin{aligned} m_{x,y,z,t}^2 &= m_{x,y,t} \circ (A_{x,y} \otimes m_{y,z,t}) = m_{x,z,t} \circ (m_{x,y,z} \otimes A_{z,t}), \\ m_{x,y,z,t,u}^3 &= m_{x,y,u} \circ (A_{x,y} \otimes m_{y,z,t,u}^2) = m_{x,t,u} \circ (m_{x,y,z,t}^2 \otimes A_{t,u}). \end{aligned}$$

As we have defined a generalization for algebras and categories, we can also define a generalization for both algebra morphisms and functors, as follows.

Definition 4.2.2. Let A and B be $\mathcal{V}$ -categories with index classes X and Y respectively. Then a $\mathcal{V}$ -**functor** $f : A \rightarrow B$ consists of the following data: for each $x \in X$, we have $f(x) \in Y$ and morphisms

$$f_{x,y} : A_{x,y} \rightarrow B_{f(x),f(y)}$$

in $\mathcal{V}$ such that the following diagrams commute for all $x, y, z \in X$:

$$\begin{array}{ccc} A_{x,y} \otimes A_{y,z} & \xrightarrow{m_{x,y,z}} & A_{x,z} \\ \downarrow f_{x,y} \otimes f_{y,z} & & \downarrow f_{x,z} \\ B_{f(x),f(y)} \otimes B_{f(y),f(z)} & \xrightarrow{m_{f(x),f(y),f(z)}} & B_{f(x),f(z)} \end{array} \quad \begin{array}{ccc} I & \xrightarrow{\mu_x} & A_{x,x} \\ & \searrow \mu_{f(x)} & \downarrow f_{x,x} \\ & & B_{f(x),f(x)} \end{array}$$

For categories, the first diagram generalizes the statement that a $\mathcal{V}$ -functor preserves composition, and the second diagram generalizes the statement that a $\mathcal{V}$ -functor preserves identities. For algebras, the first diagram generalizes the statement that a morphism of algebras is multiplicative, while the second diagram generalizes the statement that a morphism of algebras preserves units. For categories, there is the concept of natural transformation, and it also has a generalization as follows.

Definition 4.2.3. Let A, B be $\mathcal{V}$ -categories and $f, g : A \rightarrow B$ be $\mathcal{V}$ -functors. A $\mathcal{V}$ -**natural transformation** $\alpha : f \Rightarrow g$ consists of a class of morphisms $\alpha_x : I \rightarrow B_{g(x),f(x)}$ such that the following diagram commutes for all $x, y \in X$:

$$\begin{array}{ccc} A_{x,y} & \xrightarrow{g_{x,y} \otimes \alpha_y} & B_{g(x)g(y)} \otimes B_{g(y),f(y)} \\ \downarrow \alpha_x \otimes f_{x,y} & & \downarrow m_{g(x),g(y),f(y)} \\ B_{g(x),f(x)} \otimes B_{f(x),f(y)} & \xrightarrow{m_{g(x),f(x),f(y)}} & B_{g(x),f(y)} \end{array}$$

There are two points to explain about the definition above. First, since the category $\mathcal{V}$ is strict monoidal, $A_{x,y} \otimes I = A_{x,y} = I \otimes A_{x,y}$. Second, this definition generalizes natural transformations. In fact, consider two categories A, B , that is, two *Sets*-categories; let $f, g : A \rightarrow B$ be functors and let x, y be objects of A . Then we have that a morphism $\alpha_x : \{*\} \rightarrow B_{g(x),f(x)}$ can be identified with an element of $B_{g(x),f(x)}$. If the family $\{\alpha_x; x \in X\}$ is a natural transformation in the sense of Definition 4.2.2, then we

have the equality

$$m_{g(x),f(x),f(y)}(\alpha_x, f_{x,y}) = m_{g(x),g(y),f(y)}(g_{x,y}, \alpha_y),$$

where $m_{u,v,t}$ is just composition of morphisms in B . Given $h \in A_{x,y}$ we obtain the equality

$$f_{x,y}(h) \circ \alpha_x = m_{g(x),f(x),f(y)}(\alpha_x, f_{x,y})(h) = m_{g(x),g(y),f(y)}(g_{x,y}, \alpha_y)(h) = g_{x,y}(h) \circ \alpha_y,$$

which is exactly the defining equality for a natural transformation. It can be shown that with these definitions we have a 2-category whose 0-cells are the $\mathcal{V}$ -categories.

Definition 4.2.4 (Composition of $\mathcal{V}$ -functors). *For $\mathcal{V}$ -functors $f : A \rightarrow B$ and $g : B \rightarrow C$, we define the **composition** of f and g as:*

$$(g \circ f)_{x,y} : A_{x,y} \rightarrow C_{(g \circ f)(x), (g \circ f)(y)}$$

$$(g \circ f)_{x,y} = g_{f(x), f(y)} \circ f_{x,y}.$$

Definition 4.2.5 (Vertical composition and horizontal Composition). *Let $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ be both $\mathcal{V}$ -natural transformations. We define the **vertical composition** of α and β by the rule:*

$$(\beta \circ \alpha)_x = m_{h(x),g(x),f(x)}(\beta_x \otimes \alpha_x).$$

*Now consider $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ being $\mathcal{V}$ -natural transformations. Then, the **horizontal composition** of α and β is defined by the rule:*

$$(\alpha * \beta)_x = m_{g'(f'(x)),g'(f(x)),g(f(x))}((g'_{f'(x),f(x)} \circ \alpha_x) \otimes \beta_{f(x)})$$

$$= m_{g'(f'(x)),g(f'(x)),g(f(x))}(\beta_{f'(x)} \otimes (g_{f'(x),f(x)} \otimes \alpha_x)).$$

It can be shown that with these definitions we have a 2-category whose 0-cells are the $\mathcal{V}$ -categories.

Proposition 4.2.1. *Let $\mathcal{V}$ be a monoidal category. Then $\mathcal{V}$ -categories form a 2-category where the $\mathcal{V}$ -categories are the 0-cells, $\mathcal{V}$ -functors are 1-cells and $\mathcal{V}$ -natural transformations are 2-cells.*

This 2-category will be denoted by ${}_{\mathcal{V}}Cat$.

Proposition 4.2.2. *A monoidal functor $F : \mathcal{V} \rightarrow \mathcal{W}$ induces a bifunctor $\overline{F} : {}_{\mathcal{V}}Cat \rightarrow {}_{\mathcal{W}}Cat$. If F is a strong monoidal equivalence of categories, then the induced bifunctor is a biequivalence.*

When we refer to the induced bifunctor $\overline{F}$, we mean it preserves compositions of $\mathcal{V}$ functors and $\mathcal{V}$ -natural transformations.

Proof. From the functor F , we shall define such induced functor and prove it is a bifunctor. Let A be a $\mathcal{V}$ -category. Define $\overline{F}(A)_{x,y} = F(A_{x,y})$ for the induced functor. Then, we have $\overline{F}$ defined on objects. Then, we shall describe $\overline{F}$ in 1-cells and 2-cells. Since F is a monoidal functor, consider the natural transformations ϕ_2 and ϕ_0 as in the definition of monoidal functor. Then, for each object $\overline{F}(A)$, we can define multiplication and units in this manner:

$$\begin{aligned} m'_{x,y,z} &= F(m_{x,y,z}) \circ \phi_2, \\ \mu'_x &= F(\mu_x) \circ \phi_0. \end{aligned}$$

Observe that both mappings are well-defined, since ϕ_2 takes $F(A_{x,y}) \otimes F(A_{y,z})$ into $F(A_{x,y} \otimes A_{y,z})$ and $F(m_{x,y,z})$ takes $F(A_{x,y}) \otimes F(A_{y,z})$ into $F(A_{x,z})$. However, also observe that $F(A_{x,y}) \otimes F(A_{y,z}) = \overline{F}(A)_{x,y} \otimes \overline{F}(A)_{y,z}$ and $F(A_{x,z}) = \overline{F}(A)_{x,z}$.

Now, our aim is to prove that m' is associative. In fact, we evaluate:

$$\begin{aligned} m'_{x,y,t} \circ (\overline{F}(A)_{x,y} \otimes m'_{y,z,t}) &= F(m_{x,y,t}) \circ \phi_2 \circ (F(A_{x,y}) \otimes F(m_{y,z,t}) \circ \phi_2) \\ &= F(m_{x,y,t}) \circ \phi_2 \circ (F(A_{x,y}) \otimes F(m_{y,z,t})) \circ (F(A_{x,y}) \otimes \phi_2) \\ &= F(m_{x,y,t}) \circ F(A_{x,y} \otimes m_{y,z,t}) \circ \phi_2 \circ (F(A_{x,y}) \otimes \phi_2) \\ &= F(m_{x,y,t} \circ (A_{x,y} \otimes m_{y,z,t})) \circ \phi_2 \circ (F(A_{x,y}) \otimes \phi_2) \\ &= F(m_{x,z,t} \circ (m_{x,y,z} \otimes A_{z,t})) \circ \phi_2 \circ (F(A_{x,y}) \otimes \phi_2) \\ &= F(m_{x,z,t}) \circ F(m_{x,y,z} \otimes A_{z,t}) \circ \phi_2 \circ (\phi_2 \otimes F(A_{z,t})) \\ &= F(m_{x,z,t}) \circ \phi_2 \circ (F(m_{x,y,z}) \otimes F(A_{z,t})) \circ (\phi_2 \otimes F(A_{z,t})) \\ &= F(m_{x,z,t}) \circ \phi_2 \circ (F(m_{x,y,z}) \circ \phi_2 \otimes F(A_{z,t})) \\ &= F(m_{x,z,t}) \circ \phi_2 \circ (F(m_{x,y,z}) \circ \phi_2 \otimes \overline{F}(A)_{z,t}) \\ &= m'_{x,z,t} \circ (m'_{x,y,z} \circ \overline{F}(A)_{z,t}). \end{aligned}$$

The first and last equalities follow from the definition of m' and F . The second and eighth equalities is a manipulation of compositions and identities, while the third and seventh equalities follow from the naturality of ϕ_2 the fourth and sixth equalities follow from the fact that F is a functor and from the associativity of ϕ_2 . The fifth equality follows from the associativity of m and the ninth equality follows from the definition of $\overline{F}$ in objects. Hence, we have proved the associativity. Now, let us prove the unit property. For the

first equality of the defining equalities for the unit property (Definition 4.2.1), we have:

$$\begin{aligned}
m'_{x,x,y} \circ (\mu'_x \otimes \overline{F}(A)_{x,y}) &= F(m_{x,x,y}) \circ \phi_2 \circ (F(\mu_x) \circ \phi_0 \otimes F(A_{x,y})) \\
&= F(m_{x,x,y}) \circ \phi_2 \circ (F(\mu_x) \otimes F(A_{x,y})) \circ (\phi_0 \otimes F(A_{x,x})) \\
&= F(m_{x,x,y}) \circ F(\mu_x \otimes A_{x,y}) \circ \phi_2 \circ (\phi_0 \otimes F(A_{x,x})) \\
&= F(m_{x,x,y} \circ (\mu_x \otimes A_{x,y})) \circ \phi_2 \circ (\phi_0 \otimes F(A_{x,x})) \\
&= F(A_{x,y}) \circ \phi_2 \circ (\phi_0 \otimes F(A_{x,x})) \\
&= F(A_{x,y}) \\
&= \overline{F}(A)_{x,y}.
\end{aligned}$$

The first and last equalities follow from the definition of m' , μ' and $\overline{F}$ for objects. The second equality is a manipulation of compositions, the third equality follows from the naturality of ϕ_2 , the fourth follows from the fact that F is a functor, the fifth equality follows from the unit property of μ and the sixth follows from the second diagram of Definition 4.1.6. The other side is analogous. Then, we have proved the unit property. Now, we shall define $\overline{F}$ for $\mathcal{V}$ -functors. Let $f : A \rightarrow B$ be a $\mathcal{V}$ -functor. We may define:

$$\overline{F}(f)_{x,y} = F(f_{x,y}) : \overline{F}(A)_{x,y} \rightarrow \overline{F}(B)_{f(x),f(y)}$$

Then, there are two statements we need to prove: That $\overline{F}(f)_{x,y}$ is a $\mathcal{W}$ -functor, and that $\overline{F}$ preserves compositions and identities. In fact, evaluating:

$$\begin{aligned}
&m'_{\overline{F}(f)(x), \overline{F}(f)(y), \overline{F}(f)(z)} \circ \overline{F}(f)_{x,y} \otimes \overline{F}(f)_{y,z} = \\
&= F(m_{\overline{F}(f)(x), \overline{F}(f)(y), \overline{F}(f)(z)}) \circ \phi_2 \circ (F(f_{x,y}) \otimes F(f_{y,z})) \\
&= F(m_{f(x), f(y), f(z)}) \circ F(f_{x,y} \otimes f_{y,z}) \circ \phi_2 \\
&= F(m_{f(x), f(y), f(z)} \circ (f_{x,y} \otimes f_{y,z})) \circ \phi_2 \\
&= F(f_{x,z} \circ m_{x,y,z}) \circ \phi_2 \\
&= F(f_{x,z}) \circ F(m_{x,y,z}) \circ \phi_2 \\
&= \overline{F}(f)_{x,y} \circ m'_{x,y,z}.
\end{aligned}$$

The first and last equalities follow from the definition of $\overline{F}$ for f and the definition of m' , while the second equality follows from the naturality of ϕ_2 together with the definition of $\overline{F}(f)(x)$ for $x \in X$; the third and fifth equalities follow from the fact that F is a functor, and the fourth equality follows from the first diagram in the definition of $\mathcal{V}$ -functor and the fact that f is a $\mathcal{V}$ -functor. Hence, the first diagram for $\mathcal{W}$ -functors is proven. Now, let

us prove that $\overline{F}$ preserves units. In fact, evaluating:

$$\begin{aligned}
 \overline{F}(f)_{x,x} \circ \mu'_x &= f_{x,x} \circ \mu_x \circ \phi_0 \\
 &= \mu_{f(x)} \circ \phi_0 \\
 &= \mu'_{f(x)} \\
 &= \mu'_{\overline{F}(f)(x)}.
 \end{aligned}$$

The first, third and last equalities follow from the definition of $\overline{F}(f)$ and μ' , while the second equality follows from the second diagram in the definition of $\mathcal{V}$ -functor. So, the second diagram is proven. Now let us prove that $\overline{F}$ preserves compositions and identities. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be both $\mathcal{V}$ -functors. Then we have:

$$\begin{aligned}
 \overline{F}(g \circ f)_{x,y} &= F((g \circ f)_{x,y}) \\
 &= F(g_{f(x),f(y)} \circ f_{x,y}) \\
 &= F(g_{f(x),f(y)}) \circ F(f_{x,y}) \\
 &= \overline{F}(g)_{f(x),f(y)} \circ \overline{F}(f)_{x,y} \\
 &= \overline{F}(g)_{\overline{F}(f)(x),\overline{F}(f)(y)} \circ \overline{F}(f)_{x,y}.
 \end{aligned}$$

The first, fourth and last equalities follow from the definition of $\overline{F}$ for $\mathcal{V}$ -functors, while the second equality follows from the definition of composition of $\mathcal{V}$ -functors, and the third equality follows from the fact that F is a functor. Since F is a functor, $\overline{F}$ preserves identities. Then, we need to define $\overline{F}$ for $\mathcal{V}$ -natural transformations. So, for $\alpha : f \Rightarrow g$ a $\mathcal{V}$ -natural transformation with $f, g : A \rightarrow B$, we define:

$$\overline{F}(\alpha)_x = F(\alpha_x) \circ \phi_0.$$

First, we shall prove that $\overline{F}(\alpha)$ is a $\mathcal{W}$ -natural transformation, and then prove that $\overline{F}$ preserves vertical compositions. In fact, evaluating:

$$\begin{aligned}
 m'_{\overline{F}(g)(x),\overline{F}(g)(y),\overline{F}(f)(y)} \circ \overline{F}(g)_{x,y} \otimes \overline{F}(\alpha)_y &= \\
 &= F(m_{g(x),g(y),f(y)}) \circ \phi_2 \circ F(g_{x,y}) \otimes (F(\alpha_y) \circ \phi_0) \\
 &= F(m_{g(x),g(y),f(y)}) \circ \phi_2 \circ (F(g_{x,y}) \otimes F(\alpha_y)) \circ (B_{g(x),g(y)} \otimes \phi_0) \\
 &= F(m_{g(x),g(y),f(y)}) \circ F(g_{x,y} \otimes \alpha_y) \circ \phi_2 \circ (B_{g(x),g(y)} \otimes \phi_0) \\
 &= F(m_{g(x),g(y),f(y)}) \circ (g_{x,y} \otimes \alpha_y) \circ \phi_2 \circ (B_{g(x),g(y)} \otimes \phi_0) \\
 &= F(m_{g(x),f(x),f(y)}) \circ (\alpha_x \otimes f_{x,y}) \circ \phi_2 \circ (B_{g(x),g(y)} \otimes \phi_0) \\
 &= F(m_{g(x),f(x),f(y)}) \circ F(\alpha_x \otimes f_{x,y}) \circ \phi_2 \circ (B_{g(x),g(y)} \otimes \phi_0) \\
 &= F(m_{g(x),f(x),f(y)}) \circ \phi_2 \circ (F(\alpha_x) \otimes F(f_{x,y})) \circ (B_{g(x),g(y)} \otimes \phi_0) \\
 &= F(m_{g(x),f(x),f(y)}) \circ \phi_2 \circ F(\alpha_x) \circ \phi_0 \otimes F(f_{x,y}) \\
 &= m'_{g(x),f(x),f(y)} \circ \overline{F}(\alpha)_x \otimes \overline{F}(f)_{x,y}.
 \end{aligned}$$

The first and last equalities follow from the definition of m' , $\bar{F}$ for g, f and α . The second equality follows from the fact that $\otimes$ is a functor, the third and seventh equalities follow from the naturality of ϕ_2 , the fourth and sixth equalities follow from the fact that F is a functor, the fifth equality follows from the definition of $\mathcal{V}$ -functor for α , and the eighth equality follows from the last diagrams in the definition of monoidal functor. Hence, we have proved that $\bar{F}(\alpha)$ is a $\mathcal{W}$ -natural transformation. Now, we need to prove that it preserves vertical compositions and horizontal compositions.

In fact, let $\alpha : f \rightarrow g$ and $\beta : g \rightarrow h$ be $\mathcal{V}$ -natural transformations.

$$\begin{aligned}\bar{F}((\beta \circ \alpha))_x &= \bar{F}(m_{h(x),g(x),f(x)} \circ (\beta_x \otimes \alpha_x)) \\ &= m'_{h(x),g(x),f(x)}(F(\beta_x) \otimes F(\alpha_x)) \\ &= (\bar{F}(\beta))_x \circ (\bar{F}(\alpha))_x.\end{aligned}$$

The first and last equalities follow from the definition of vertical composition, while the second is just the definition of $\bar{F}$ because we have proved and the fact that F is a functor. Then, we have proved that $\bar{F}$ preserves vertical compositions. Now, let us prove that if F is a strong monoidal equivalence, then $\bar{F}$ is a biequivalence.

In fact, observe that, for $f, g : A \rightarrow B$, if $\bar{F}(f_{x,y}) = \bar{F}(g_{x,y})$ for all $x, y \in X$, then $F(f_{x,y}) = F(g_{x,y})$ for all $x, y \in X$, which means that $f_{x,y} = g_{x,y}$ for all $x, y \in X$. This implies that $f = g$. Hence, $\bar{F}$ is injective in 1-cells. Also, observe that $\bar{F}$ is essentially surjective in 0-cells. In fact, observe that, for a family of objects $A = A_{x,y}$, for each pair $(x, y) \in X \times X$, there is an object $B_{x,y}$ such that $A_{x,y}$ is isomorphic to $B_{x,y}$. Consider $B_{x,y}$ as a 0-cell, we have that $\bar{F}$ is essentially surjective in 0-cells.

Also, observe that, if $\bar{F}(\alpha_x) = \bar{F}(\beta_x)$ for two natural $\mathcal{V}$ -transformations, we have that $F(\alpha_x) = F(\beta_x)$. Since F is a strong monoidal equivalence, it follows that $\bar{F}$ is injective in 2-cells. Then, we have that $\bar{F}$ is a biequivalence. $\square$

Before we define a first generalization of coalgebras, we need to discuss the tensor of two $\mathcal{V}$ -categories. So let us do this with the next proposition.

Proposition 4.2.3. *Let A and B be both $\mathcal{V}$ -categories and $\mathcal{V}$ be a strict monoidal braided category. Then $A \otimes B$ is also a $\mathcal{V}$ -category, with multiplication and unit defined as:*

$$\begin{aligned}m_{x,y,z}^{A \otimes B} &: A_{x,y} \otimes B_{x,y} \otimes A_{y,z} \otimes B_{y,z} \rightarrow A_{x,z} \otimes B_{x,z}, \\ m_{x,y,z}^{A \otimes B} &= (m_{x,y,z} \otimes m_{x,y,z}) \circ (A_{x,y} \otimes \sigma_{B_{x,y}, A_{y,z}} \otimes B_{y,z}); \\ \mu_x^{A \otimes B} &: I \rightarrow A_{x,x} \otimes B_{x,x}, \\ \mu_x^{A \otimes B} &= \mu_x^A \otimes \mu_x^B.\end{aligned}$$

Proof. The proof of this proposition is entirely analogous to the proof of Proposition 3.2.2 and therefore it will be omitted. $\square$

And now, we shall define a particular type of $\mathcal{V}$ -functor, which is a $\mathcal{V} - X$ -functor.

Definition 4.2.6. *Let A and B be $\mathcal{V}$ -categories over the same class X . Then $f : A \rightarrow B$ is a $\mathcal{V} - X$ -**functor** if $f(x) = x$ for all $x \in X$.*

When X is a unitary set, and therefore A and B are just two algebras in $\mathcal{V}$, then a $\mathcal{V} - X$ functor $f : A \rightarrow B$ is the same as a $\mathcal{V}$ -functor, which is the same as a morphism of algebras in $\mathcal{V}$. When $\mathcal{V} = \mathbf{Sets}$, a $\mathcal{V}$ -functor is a (usual) functor $f : A \rightarrow B$, and a $\mathcal{V} - X$ functor is a functor which is the identity function on objects.

We shall denote by $\mathcal{C}(\mathcal{V})$ the category of coalgebras in $\mathcal{V}$. It follows from Proposition 3.2.2 and Example 3.2.5 that if $\mathcal{V}$ is a braided monoidal category then $\mathcal{C}(\mathcal{V})$ is a monoidal category, so we may consider $\mathcal{C}(\mathcal{V})$ -categories. A $\mathcal{C}(\mathcal{V})$ -category is a generalization of bialgebras, as we will show in the next result.

Proposition 4.2.4. *Let X be a class and let $\mathcal{V}$ be a braided strict monoidal category. A $\mathcal{C}(\mathcal{V})$ -category with indexing class X is the same as an object in $\mathcal{V}(X)$ which has a structure of $\mathcal{V}$ -category, and also a structure of coalgebra in $\mathcal{V}(X)$, such that the morphisms $\Delta_{x,y}$ and $\varepsilon_{x,y}$ define $\mathcal{V}$ - X -functors $\Delta : A \rightarrow A \otimes A$ and $\varepsilon : A \rightarrow I$.*

The proof is analogous to the proof of the propositions that helped to define bialgebras in the previous chapters.

Proof. Since every object in $\mathcal{C}(\mathcal{V})$ is a coalgebra, for each object $A \in \mathcal{V}(X)$, we can consider the morphism $\Delta_{x,y} = \Delta_{A_{x,y}} : A_{x,y} \rightarrow A_{x,y} \otimes A_{x,y}$ as the family of morphisms which defines the morphism $\Delta : A \rightarrow A \otimes A$. Analogously, we can define $\varepsilon_{x,y} = \varepsilon_{A_{x,y}} : A_{x,y} \rightarrow I_{x,y} (= I)$. Then, consider the following four diagrams:

$$\begin{array}{ccc}
 A_{x,y} \otimes A_{y,z} & \xrightarrow{m_{x,y,z}} & A_{x,z} \\
 \downarrow \Delta_{x,y} \otimes \Delta_{y,z} & & \downarrow \Delta_{x,z} \\
 A_{x,y} \otimes A_{x,y} \otimes A_{y,z} \otimes A_{y,z} & \xrightarrow{m_{x,y,z}^{A \otimes A}} & A_{x,z} \otimes A_{x,z}
 \end{array} \tag{4.1}$$

$$\begin{array}{ccc}
 I & \xrightarrow{\mu_x} & A_{x,x} \\
 \searrow \mu_x \otimes \mu_x & & \downarrow \Delta_{x,x} \\
 & & A_{x,x} \otimes A_{x,x}
 \end{array} \tag{4.2}$$

$$\begin{array}{ccc}
A_{x,y} \otimes A_{y,z} & \xrightarrow{m_{x,y,z}} & A_{x,z} \\
\downarrow \varepsilon_{x,y} \otimes \varepsilon_{y,z} & & \downarrow \varepsilon_{x,z} \\
I \otimes I & \xrightarrow{=} & I
\end{array}
\tag{4.3}$$

$$\begin{array}{ccc}
I & \xrightarrow{\mu_x} & A_{x,x} \\
& \searrow & \downarrow \varepsilon_{x,x} \\
& & I
\end{array}
\tag{4.4}$$

Observe that Δ is a $\mathcal{V} - X$ -functor if the first and second diagrams commute for all $x, y, z \in X$ and ε is a $\mathcal{V} - X$ -functor if the third and last diagrams commute for all $x, y, z \in X$. Also note that $m_{x,y,z}$ is a coalgebra map if and only if the first and third diagrams commute, and $\mu_{x,y,z}$ is a coalgebra map if and only if the second and last diagrams commute. Hence, a $\mathcal{C}(\mathcal{V})$ -category A is the same as a $\mathcal{V}$ -category with $\mathcal{V} - X$ -functors $\Delta : A \rightarrow A \otimes A$ and $\varepsilon : A \rightarrow I$. Since Δ is defined from coassociative morphisms and for each pair of indices x, y $\varepsilon_{x,y}$ respects the counit property, we have that Δ is coassociative and ε respects the counit property. $\square$

Observe that in fact $\mathcal{C}(\mathcal{V})$ -categories generalize bialgebras, because bialgebras correspond to $\mathcal{C}(\mathcal{V})$ -categories with one object. Observe, also, that a $\mathcal{V}$ -functor f is a $\mathcal{C}(\mathcal{V})$ -functor if all $f_{x,y}$ are morphisms of coalgebras.

Then, since we have defined a generalization for algebras, coalgebras and bialgebras, we can also define a generalization for opposite algebras and co-opposite coalgebras, also using the braiding in $\mathcal{V}$. The next proposition will be essential for these generalizations.

Proposition 4.2.5. *Let $\mathcal{V}$ be a strict monoidal category. Then:*

- *Let A be a $\mathcal{V}$ -category and its opposite A^{op} be its opposite $\mathcal{V}$ -category. Then, A^{op} is also an $\mathcal{V}$ -category, with product and unit defined as:*

$$\begin{aligned}
m_{x,y,z}^{op} : A_{x,y}^{op} \otimes A_{y,z}^{op} &= A_{y,x} \otimes A_{z,y} \rightarrow A_{x,z}^{op} = A_{z,x}, \\
m_{x,y,z}^{op} &= m_{z,y,x} \circ \sigma_{A_{y,x}, A_{z,y}}, \\
\mu_{x,x}^{op} &= \mu_{x,x}.
\end{aligned}$$

- Let C be a coalgebra in $\mathcal{V}(X)$. Then C has another coalgebra structure, the coopposite coalgebra C^{cop} , where $C^{cop} = C$ as an object of $\mathcal{V}(X)$ (i.e, $C_{x,y}^{cop} = C_{x,y}$), and its coproduct and counit are defined as:

$$\begin{aligned}\Delta_{x,y}^{cop} &: C_{x,y} \rightarrow C_{x,y} \otimes C_{x,y}, \\ \Delta_{x,y}^{cop} &= \sigma_{C_{x,y}, C_{x,y}}^{-1} \circ \Delta_{x,y} \\ \varepsilon_{x,y}^{cop} &= \varepsilon_{x,y}.\end{aligned}$$

Proof. First, we shall prove the associativity of m^{op} and the unit property applied to μ^{op} . In fact, observe that, for $x, y, z, t \in X$:

$$\begin{aligned}m_{t,y,x}^{op} \circ (m_{t,z,y}^{op} \otimes A_{y,x}^{op}) &= m_{x,y,t} \circ \sigma \circ (m_{y,z,t} \circ \sigma \otimes A_{x,y}) \\ &= m_{x,y,t} \circ (A_{x,y} \otimes m_{y,z,t} \circ \sigma) \circ \sigma \\ &= m_{x,y,t} \circ (A_{x,y} \otimes m_{y,z,t}) \circ (A_{x,z} \otimes \sigma) \circ \sigma \\ &= m_{x,z,t} \circ (m_{x,y,z} \otimes A_{z,t}) \circ (A_{x,z} \otimes \sigma) \circ \sigma \\ &= m_{x,z,t} \circ (m_{x,y,z} \otimes A_{z,t}) \circ (A_{x,z} \otimes \sigma) \circ (\sigma \otimes A_{z,t}) \circ (A_{x,z} \otimes \sigma) \\ &= m_{x,z,t} \circ (m_{x,y,z} \otimes A_{z,t}) \circ \sigma \circ (A_{x,z} \otimes \sigma) \\ &= m_{x,z,t} \circ \sigma \circ (A_{z,t} \otimes m_{x,y,z}) \circ (A_{x,z} \otimes \sigma) \\ &= m_{x,z,t} \circ \sigma \circ (A_{z,t} \otimes m_{x,y,z} \circ \sigma).\end{aligned}$$

The first equality follows from the definition of m^{op} and A^{op} , while the second and seventh equalities follow from the naturality of σ , the third and last equalities follow from the fact that $\otimes$ is a functor, the fourth equality follows from the associativity of m , and the fifth and sixth equalities follow from the diagrams of σ in Definition 3.1.10 (Braided monoidal category). Hence, the associativity is proven.

For the unit property, observe that, for all $x, y \in X$:

$$\begin{aligned}m_{x,x,y}^{op} \circ (\mu_{x,x}^{op} \otimes A_{x,y}^{op}) &= m_{y,x,x} \circ \sigma \circ (\mu_{x,x} \otimes A_{y,x}) \\ &= m_{y,x,x} \circ (A_{y,x} \otimes \mu_{x,x}) \circ \sigma \\ &= A_{y,x} \\ &= A_{x,y}^{op}.\end{aligned}$$

The first and last equalities follow from the definitions of m^{op} , μ^{op} and A^{op} , while the second equality follows from the naturality of σ , and the third is the unit diagram applied to m and μ .

Analogously, we have that $m_{x,y,y}^{op} \circ (A_{x,y}^{op} \otimes \mu_{y,y}^{op}) = A_{x,y}^{op}$.

Hence, we have proved the unit property.

By an analogous proof, we have that $\Delta_{x,y}$ and $\varepsilon_{x,y}$ are coproduct and counit respectively i.e, Δ is coassociative and ε obeys the counit property. $\square$

Then, we have a definition that generalizes opposite algebras and coopposite coalgebras. After the following proposition, we shall define a generalization of Hopf algebras, *i.e*, the Hopf Categories.

Proposition 4.2.6. *Let $\mathcal{V}$ be a strict braided monoidal category and let A be a $\mathcal{C}(\mathcal{V})$ -category. Then $(A^{op})^{cop}$ is also a $\mathcal{C}(\mathcal{V})$ -category.*

Note that $A^{opcop} = (A^{op})^{cop}$.

Proof. We need to prove the four diagrams in the proof of Proposition 4.2.4 commute for m^{op} , μ^{op} , Δ^{op} and ε^{op} . The first diagram has the following form:

$$\begin{array}{ccc}
 A_{y,x} \otimes A_{z,y} & \xrightarrow{m_{x,y,z}^{op}} & A_{z,x} \\
 \downarrow \Delta_{y,x}^{cop} \otimes \Delta_{z,y}^{cop} & & \downarrow \Delta_{z,x}^{cop} \\
 A_{y,x} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{z,y} & \xrightarrow{m^{A^{op} \otimes A^{op}}} & A_{z,x} \otimes A_{z,x}
 \end{array} \quad (4.5)$$

First, observe that, for $A, B, C, D \in \mathcal{V}$, we have:

$$\sigma_{A \otimes B, C \otimes D} = (C \otimes \sigma_{A,D} \otimes B) \circ (\sigma_{A,C} \otimes \sigma_{B,D}) \circ (A \otimes \sigma_{B,C} \otimes D).$$

In fact, by the axioms of braiding, we have that:

$$\begin{aligned}
 & (C \otimes \sigma_{A,D} \otimes B) \circ (\sigma_{A,C} \otimes \sigma_{B,D}) \circ (A \otimes \sigma_{B,C} \otimes D) \\
 &= (C \otimes (\sigma_{A,D} \otimes B) \circ (A \otimes \sigma_{B,D}) \circ ((\sigma_{A,C} \circ B) \circ (A \otimes \sigma_{B,C}) \otimes D) \\
 &= (C \otimes \sigma_{A \otimes B, D}) \circ (\sigma_{A \otimes B, C} \otimes D) \\
 &= \sigma_{A \otimes B, C \otimes D}.
 \end{aligned}$$

The first equality follows from the fact that $\otimes$ is a functor, while all other equalities follow from the axioms of braiding.

Before proving the commutativity of the diagram above, we need to see the commutativity of the following diagram:

$$\begin{array}{ccc}
 A_{y,x} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{z,y} & \xrightarrow{\sigma} & A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} \\
 \downarrow \sigma^{-1} \otimes \sigma^{-1} & & \downarrow \sigma^{-1} \otimes \sigma^{-1} \\
 A_{y,x} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{z,y} & \xrightarrow{\sigma} & A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} \\
 \downarrow A \otimes \sigma \otimes A & & \downarrow A \otimes \sigma^{-1} \otimes A \\
 A_{y,x} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{z,y} & \xrightarrow{\sigma \otimes \sigma} & A_{z,y} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{y,x}
 \end{array} \quad (4.6)$$

However, this diagrams commutes by the naturality of σ and the following computation:

$$\begin{aligned}
 & (A \otimes \sigma \otimes A) \circ (\sigma \otimes \sigma) \circ (A \otimes \sigma \otimes A) \\
 &= (A \otimes \sigma \otimes A) \circ (\sigma \otimes A \otimes A) \circ (A \otimes A \otimes \sigma) \circ (A \otimes \sigma \otimes A) \\
 &= \sigma
 \end{aligned}$$

The first and second equalities follow from the axioms of braiding.

Now, for proving the commutativity of the main diagram (4.5), we shall proving it by extending inside the diagram as follows:

$$\begin{array}{ccccc}
 A_{y,x} \otimes A_{z,y} & \xrightarrow{\sigma} & A_{z,y} \otimes A_{y,x} & \xrightarrow{m_{z,y,x}} & A_{z,x} \\
 \downarrow \Delta_{y,x} \otimes \Delta_{z,y} & & \downarrow \Delta_{z,y} \otimes \Delta_{y,x} & & \downarrow \Delta_{z,x} \\
 A_{y,x} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{z,y} & \xrightarrow{\sigma} & A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} & & \\
 \swarrow = & & \downarrow A_{z,y} \otimes \sigma \otimes A_{y,x} & & \\
 A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} & \xleftarrow{A_{z,y} \otimes \sigma^{-1} \otimes A_{z,y}} & A_{z,y} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{y,x} & \xrightarrow{m_{z,y,x} \otimes m_{z,y,x}} & A_{z,x} \otimes A_{z,x} \\
 \downarrow \sigma^{-1} \otimes \sigma^{-1} & & \downarrow \sigma^{-1} & & \downarrow \sigma^{-1} \\
 A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} & \xrightarrow{A_{z,y} \otimes \sigma^{-1} \otimes A_{y,x}} & A_{z,y} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{y,x} & \xrightarrow{m_{z,y,x} \otimes m_{z,y,x}} & A_{z,x} \otimes A_{z,x}
 \end{array}$$

The top left square and bottom right square (as below) commute due to the naturality of σ :

$$\begin{array}{ccc}
 A_{y,x} \otimes A_{z,y} & \xrightarrow{\sigma} & A_{z,y} \otimes A_{y,x} \\
 \downarrow \Delta_{y,x} \otimes \Delta_{z,y} & & \downarrow \Delta_{z,y} \otimes \Delta_{y,x} \\
 A_{y,x} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{z,y} & \xrightarrow{\sigma} & A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} \\
 A_{z,y} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{y,x} & \xrightarrow{m_{z,y,x} \otimes m_{z,y,x}} & A_{z,x} \otimes A_{z,x} \\
 \downarrow \sigma^{-1} & & \downarrow \sigma^{-1} \\
 A_{z,y} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{y,x} & \xrightarrow{m_{z,y,x} \otimes m_{z,y,x}} & A_{z,y} \otimes A_{y,x}
 \end{array}$$

the lower left square (as below) commutes due to the axioms of braiding:

$$\begin{array}{ccc}
 A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} & \xleftarrow{A_{z,y} \otimes \sigma^{-1} \otimes A_{z,y}} & A_{z,y} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{y,x} \\
 \downarrow \sigma^{-1} \otimes \sigma^{-1} & & \downarrow \sigma^{-1} \\
 A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} & \xrightarrow{A_{z,y} \otimes \sigma^{-1} \otimes A_{y,x}} & A_{z,y} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{y,x}
 \end{array}$$

the triangle (as below) commutes since $\otimes$ is a functor:

$$\begin{array}{ccc}
 & A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} & \\
 & \downarrow A_{z,y} \otimes \sigma \otimes A_{y,x} & \\
 A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} & \xleftarrow{A_{z,y} \otimes \sigma^{-1} \otimes A_{z,y}} & A_{z,y} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{y,x}
 \end{array}$$

and the pentagon (as below) commutes due by the diagram of $\mathcal{C}(\mathcal{V})$ in Proposition 4.2.4:

$$\begin{array}{ccc}
A_{z,y} \otimes A_{y,x} & \xrightarrow{m_{z,y,x}} & A_{z,x} \\
\downarrow \Delta_{z,y} \otimes \Delta_{y,x} & & \downarrow \Delta_{z,x} \\
A_{z,y} \otimes A_{z,y} \otimes A_{y,x} \otimes A_{y,x} & & \\
\downarrow A_{z,y} \otimes \sigma \otimes A_{y,x} & & \\
A_{z,y} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{y,x} & \xrightarrow{m_{z,y,x} \otimes m_{z,y,x}} & A_{z,x} \otimes A_{z,x}
\end{array}$$

From the commutativity of the whole diagram, it follows that:

$$\Delta_{z,x}^{op} \otimes m_{x,y,z}^{op} = (m_{z,y,x} \otimes m_{z,y,x}) \circ (A_{z,y} \otimes \sigma_{A_{y,x} \otimes A_{y,x}}^{-1} \otimes A_{y,x}) \circ A_{A_{y,x} \otimes A_{y,x}, A_{z,y} \otimes A_{z,y}} \circ (\Delta_{y,x} \otimes \Delta_{y,x}),$$

and, by diagram (4.6), we have that it proves that our first diagram commutes.

So, let us prove that all the other three diagrams in the proof of Proposition 4.2.4

commute.

In fact, the second diagram has the following form:

$$\begin{array}{ccc}
I & \xrightarrow{\mu_x^{cop}} & A_{x,x} \\
\searrow \mu_x^{cop} \otimes \mu_x^{cop} & & \downarrow \Delta_{x,x}^{cop} \\
& & A_{x,x} \otimes A_{x,x}
\end{array}$$

From the second diagram in Proposition 4.2.4, applying σ , we have that:

$$\Delta_{A_{x,x}}^{cop} \circ \mu_x^{cop} = \sigma_{A_{x,x}, A_{x,x}} \circ \Delta \circ \mu_x = \sigma_{A_{x,x}, A_{x,x}} \circ (\mu_x \otimes \mu_x).$$

Our aim is to prove that $\sigma_{A_{x,x}, A_{x,x}} \circ (\mu_x \otimes \mu_x) = \mu_x \otimes \mu_x$.

By the naturality of σ ,

$$\sigma_{A_{x,x}, A_{x,x}} \circ (\mu_x \otimes \mu_x) = (\mu_x \otimes \mu_x) \circ \sigma_{I,I}.$$

However, since $I \otimes I = I$, we have that $\sigma_{I,I} = \sigma_{I \otimes I, I}$, then we have that:

$$\begin{aligned}
\sigma_{I \otimes I, I} &= (\sigma_{I,I} \otimes I) \circ (I \otimes \sigma_{I,I}) \\
&= \sigma_{I,I} \circ \sigma_{I,I} \\
&= \sigma_{I,I}^2.
\end{aligned}$$

The first equality follows from the axioms of braiding, while the second follows from the fact the category is strict and the third is straightforward.

Since σ is a natural isomorphism, we have that $\sigma_{I,I} = I \otimes I = I$.

Hence, the second diagram is proven.

The third diagram has the form:

$$\begin{array}{ccc}
A_{y,z} \otimes A_{x,y} & \xrightarrow{m_{x,y,z}^{op}} & A_{x,z} \\
\downarrow \varepsilon_{y,z} \otimes \varepsilon_{x,y} & & \downarrow \varepsilon_{x,z} \\
I \otimes I & \xrightarrow{=} & I
\end{array}$$

In fact, for an analogous reason to the previous diagram, we prove that it commutes. The fourth diagram is the same in the proof of Proposition 4.2.4, and hence commutes. $\square$

As a consequence to Proposition 4.2.6, taking X as a singleton, we conclude that if B is a bialgebra in a braided strict monoidal category $\mathcal{V}$ then B^{op} is an algebra, B^{cop} is a coalgebra and $(B^{op})^{cop}$ is a bialgebra in $\mathcal{V}$.

Hence, we may finally define a generalization for Hopf algebras with the generalization of antipodes, as follows:

Definition 4.2.7. A **Hopf $\mathcal{V}$ -category** is a $\mathcal{C}(\mathcal{V})$ -category A together with a morphism $S : A^{op} \rightarrow A^{cop}$ in $\mathcal{V}(X)$ such that:

$$m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} = \mu_x \circ \varepsilon_{x,y} : A_{x,y} \rightarrow A_{x,x},$$

$$m_{y,x,y} \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} = \mu_y \circ \varepsilon_{x,y} : A_{x,y} \rightarrow A_{y,y}$$

for all $x, y \in X$.

There are, again, two main examples of Hopf $\mathcal{V}$ -categories: the Hopf $\mathcal{V}$ -categories where X is a singleton, which are Hopf algebras in the category $\mathcal{V}$; and the Hopf $\mathcal{V}$ -categories where $\mathcal{V}$ is the monoidal category $(Sets, \times, \{*\})$.

From the previous chapter, we have proved that every set A has a unique structure of coalgebra defined as $\Delta(a) = (a, a)$ and $\varepsilon(a) = *$ for all $a \in A$. Hence, the categories $Sets$ and $\mathcal{C}(Sets)$ are identical.

Now, let H be a Hopf category in $Sets$. Recall that a $Sets$ -category is just a usual category, and hence H is a category with $Hom(x, y) = H_{y,x}$ and composition given by m . Take $a \in H_{x,y}$. Then, by the definition of coalgebra in $Sets$, we have that $aS_{x,y}(a) = 1_x$ and $S_{x,y}(a)a = 1_y$. Hence, we have that a Hopf category in $Sets$ is the same as a groupoid.

This can be summarized in the following proposition.

Proposition 4.2.7. Let $\mathcal{V} = (Sets, \times, \{*\})$. Then a Hopf $\mathcal{V}$ -category is the same as a groupoid.

The next lemma generalizes Proposition 3.2.3, which says that the antipode of a Hopf algebra in a monoidal category is an antimorphism of algebras and coalgebras; its proof is also a generalization of the original proof.

Lemma 4.2.1. Let A be a Hopf $\mathcal{V}$ -category. Then, the following statements hold, for all $x, y, z \in X$:

$$S_{x,z} \circ m_{x,y,z} = m_{z,y,x} \circ (S_{y,z} \otimes S_{x,y}) \circ \sigma_{A_{x,y}, A_{y,z}},$$

$$\Delta_{y,x} \circ S_{x,y} = \sigma_{A_{y,x}, A_{y,x}} \circ (S_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y}.$$

Before we prove the lemma, let us fix some notation. For an object A in a $\mathcal{C}(\mathcal{V})$ -category, $A_{x,y}$ and $A_{y,z}$ are both coalgebras. We recall that, as we proved in the previous chapter, in Proposition 3.2.2, the object $A_{x,y} \otimes A_{y,z}$ is also a coalgebra whose coproduct, which we will denote by $\Delta_{x,y,z}$, is defined as

$$\Delta_{x,y,z} = (A_{x,y} \otimes \sigma_{A_{x,y}, A_{y,z}} \otimes A_{y,z}) \circ \Delta_{x,y} \otimes \Delta_{y,z}$$

and counit (which we will denote by $\varepsilon_{x,y,z}$) as

$$\varepsilon_{x,y,z} = \varepsilon_{x,y} \otimes \varepsilon_{y,z}.$$

Hence, the first diagram in Proposition 4.2.4 can be restated as

$$\Delta_{x,z} \otimes m_{x,y,z} = (m_{x,y,z} \otimes m_{x,y,z}) \circ \Delta_{x,y,z}.$$

Then, we have the proof:

Proof. Consider the morphisms $f, g, h : A_{x,y} \otimes A_{y,z} \rightarrow A_{z,x}$ defined as:

$$\begin{aligned} f &= m_{z,y,x} \circ (S_{y,z} \otimes S_{x,y}) \circ \sigma_{A_{x,y}, A_{y,z}}, \\ g &= S_{x,z} \circ m_{x,y,z}, \\ h &= m_{z,y,x,z,x}^3 \circ (f \otimes A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta_{x,y,z}^2. \end{aligned}$$

First, we compute that:

$$\begin{aligned} m_{x,y,z,x}^2 \circ (A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta_{x,y,z} &= m_{x,z,x} \circ (m_{x,y,z} \otimes A_{z,x}) \circ (A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta_{x,y,z} \\ &= m_{x,z,x} \circ (m_{x,y,z} \otimes g) \circ \Delta_{x,y,z} \\ &= m_{x,z,x} \circ (m_{x,y,z} \otimes (S_{x,z} \circ m_{x,y,z})) \circ \Delta_{x,y,z} \\ &= m_{x,z,x} \circ (A_{x,z} \otimes S_{x,z}) \circ (m_{x,y,z} \otimes m_{x,y,z}) \circ \Delta_{x,y,z} \\ &= m_{x,z,x} \circ (A_{x,z} \otimes S_{x,z}) \circ \Delta_{x,z} \circ m_{x,y,z} \\ &= \mu_x \circ \varepsilon_{x,z} \circ m_{x,y,z} \\ &= \mu_x \circ \varepsilon_{x,y,z}. \end{aligned}$$

The first equality follows from associativity for m^2 , while the second and the fourth equalities follow from the fact that $\otimes$ is a functor, the third equality follows from the definition of g , the fifth and last equalities follow from the diagrams in the proof of Proposition 4.2.4, and the sixth equality follows from the definition of Hopf $\mathcal{V}$ -category.

Hence, we can compute that:

$$\begin{aligned}
h &= m_{z,y,x,z,x}^3 \circ (f \otimes A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta_{x,y,z}^2 \\
&= m_{z,x,x} \circ (A_{z,x} \circ m_{x,y,z,x}^2) \circ (f \otimes A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta_{x,y,z}^2 \\
&= m_{z,x,x} \circ (f \otimes (m_{x,y,z,x}^2 \circ (A_{x,y} \otimes A_{y,z} \otimes g))) \circ \Delta_{x,y,z}^2 \\
&= m_{z,x,x} \circ (f \otimes (m_{x,y,z,x}^2 \circ (A_{x,y} \otimes A_{y,z} \otimes g))) \circ (A_{x,y} \otimes A_{y,z} \circ \Delta_{x,y,z}) \circ \Delta_{x,y,z} \\
&= m_{z,x,x} \circ (f \otimes (m_{x,y,z,x}^2 \circ (A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta_{x,y,z})) \circ \Delta_{x,y,z} \\
&= m_{z,x,x} \circ (f \otimes (\mu_x \circ \varepsilon_{x,y,z})) \circ \Delta_{x,y,z} \\
&= m_{z,x,x} \circ (f \otimes \mu_x) \circ (A_{x,y} \otimes A_{y,z} \otimes \varepsilon_{x,y,z}) \circ \Delta_{x,y,z} \\
&= f \circ (A_{x,y} \otimes A_{y,z} \otimes \varepsilon_{x,y,z}) \circ \Delta_{x,y,z} \\
&= f.
\end{aligned}$$

The first equality follows from the definition of h , while the second equality follows from the associativity for m^3 , the third, fifth and seventh equalities follow from the fact that $\otimes$ is a functor, the fourth equality is the coassociativity for Δ^2 , the sixth equality follows from the previous computation, the eighth equality is the unit property applied to f and the last equality follows from the counit property of $\varepsilon_{x,y,z}$. Now, our aim is to prove that $h = g$. In fact, observe that:

$$\begin{aligned}
& m_{z,x,y,z}^2 \circ (f \otimes A_{x,y} \otimes A_{y,z}) \circ \Delta_{x,y,z} \\
&=^1 m_{z,x,y,z}^2 \circ ((m_{z,y,x} \circ (S_{y,z} \otimes S_{x,y}) \circ \sigma_{A_{x,y}, A_{y,z}}) \otimes A_{x,y} \otimes A_{y,z}) \circ \Delta_{x,y,z} \\
&=^2 m_{z,x,y,z}^2 \circ ((m_{z,y,x} \otimes A_{x,y} \otimes A_{y,z}) \circ (S_{y,z} \otimes S_{x,y}) \circ \sigma_{A_{x,y}, A_{y,z}}) \otimes A_{x,y} \otimes A_{y,z} \circ \Delta_{x,y,z} \\
&=^3 m_{z,y,x,y,z}^3 \circ (S_{y,z} \otimes S_{x,y} \otimes A_{x,y} \otimes A_{y,z}) \circ \sigma_{A_{x,y}, A_{y,z}} \otimes A_{x,y} \otimes A_{y,z} \circ \Delta_{x,y,z} \\
&=^4 m_{z,y,x,y,z}^3 \circ (S_{y,z} \otimes S_{x,y} \otimes A_{x,y} \otimes A_{y,z}) \circ (\sigma_{A_{x,y}, A_{y,z}} \otimes A_{x,y} \otimes A_{y,z}) \circ (A_{x,y} \otimes \sigma_{A_{x,y}, A_{y,z}} \otimes A_{y,z}) \circ \\
&\Delta_{x,y} \otimes \Delta_{y,z}) \\
&=^5 m_{z,y,x,y,z}^3 \circ (S_{y,z} \otimes S_{x,y} \otimes A_{x,y} \otimes A_{y,z}) \circ (\sigma_{A_{x,y} \otimes A_{x,y}, A_{y,z}} \otimes A_{y,z}) \circ (\Delta_{x,y} \otimes \Delta_{y,z}) \\
&=^6 m_{z,y,y,z}^2 \circ (A_{z,y} \otimes m_{y,x,y}) \circ (S_{y,z} \otimes S_{x,y} \otimes A_{x,y} \otimes A_{y,z}) \circ (\sigma_{A_{x,y} \otimes A_{x,y}, A_{y,z}} \otimes A_{y,z}) \circ \\
&(\Delta_{x,y} \otimes \Delta_{y,z}) \\
&=^7 m_{z,y,y,z}^2 \circ ((A_{z,y} \otimes m_{y,x,y}) \circ (S_{y,z} \otimes S_{x,y} \otimes A_{x,y}) \circ \sigma_{A_{x,y} \otimes A_{x,y}, A_{y,z}}) \otimes A_{z,y} \circ \\
&(\Delta_{x,y} \otimes \Delta_{y,z}) \\
&=^8 m_{z,y,y,z}^2 \circ (\sigma_{A_{y,y}, A_{z,y}} \circ (m_{y,x,y} \otimes A_{y,z}) \circ (S_{x,y} \otimes A_{x,y} \otimes S_{y,z})) \otimes A_{y,z} \circ (\Delta_{x,y} \otimes \Delta_{y,z}) \\
&=^9 m_{z,y,y,z}^2 \circ ((\sigma_{A_{y,y}, A_{z,y}} \circ (m_{y,x,y} \otimes A_{y,z}) \circ (S_{x,y} \otimes A_{x,y} \otimes S_{y,z})) \otimes A_{y,z}) \circ (\Delta_{x,y} \otimes A_{y,z} \otimes A_{y,z}) \circ \\
&(A_{x,y} \otimes \Delta_{y,z}) \\
&=^{10} m_{z,y,y,z}^2 \circ ((\sigma_{A_{y,y}, A_{z,y}} \circ (m_{y,x,y} \otimes A_{y,z}) \circ ((S_{x,y} \otimes A_{x,y}) \circ \Delta_{y,z}) \circ A_{y,z}) \circ (A_{x,y} \otimes \Delta_{y,z}) \\
&=^{11} m_{z,y,y,z}^2 \circ ((\sigma_{A_{y,y}, A_{z,y}} \circ (\mu_y \otimes A_{z,y} \otimes A_{z,y}) \circ (S_{y,z} \otimes A_{y,z}) \circ (\varepsilon_{x,y} \otimes \Delta_{y,z})) \\
&=^{12} m_{z,y,y,z}^2 \circ (A_{z,y} \otimes \mu_y \otimes A_{z,y}) \circ (S_{y,z} \otimes A_{y,z}) \circ (\varepsilon_{x,y} \otimes \Delta_{y,z}) \\
&=^{13} m_{z,y,z} \circ (m_{z,y,y} \circ A_{z,y}) \circ (A_{z,y} \otimes \mu_y \otimes A_{z,y}) \circ (S_{y,z} \otimes A_{y,z}) \circ (\varepsilon_{x,y} \otimes \Delta_{y,z}) \\
&=^{14} m_{z,y,z} \circ ((m_{z,y,y} \circ (A_{z,y} \otimes \mu_y)) \otimes A_{y,z}) \circ (S_{y,z} \otimes A_{y,z}) \circ (\varepsilon_{x,y} \otimes \Delta_{y,z}) \\
&=^{15} m_{z,y,z} \circ (S_{y,z} \otimes A_{y,z} \circ (\varepsilon_{x,y} \otimes \Delta_{y,z})) \\
&=^{16} m_{z,y,z} \circ (S_{y,z} \otimes A_{y,z}) \circ \Delta_{y,z} \circ (\varepsilon_{x,y} \otimes A_{y,z}) \\
&=^{17} \mu_z \circ \varepsilon_{y,z} \circ (\varepsilon_{x,y} \otimes A_{y,z}) \\
&=^{18} \mu_z \circ (\varepsilon_{x,y} \otimes \varepsilon_{y,z}) \\
&=^{19} \mu_z \circ \varepsilon_{x,y,z}.
\end{aligned}$$

The first equality follows from the definition of f , while the second, seventh, ninth, tenth and fourteenth equalities follow from the fact that $\otimes$ is a functor. The third and sixth equalities follow from the associativity applied to m^3 , the fourth equality follows from the definition of $\Delta_{x,y,z}$, and the fifth equality follows from the axioms of braiding, and the eleventh equality follows from the definition of antipode and triviality of I . The twelfth equality follows from the naturality of the braiding and the thirteenth equality follows from the associativity applied to m^2 . The fifteenth equality follows from the unit property, and the sixteenth and eighteenth equalities follow from the nullity of I . The seventeenth equality follows from the definition of antipode, the last equality follows from the definition

of $\varepsilon_{x,y,z}$ and the eighth equality follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
 (A_{x,y} \otimes A_{x,y}) \otimes A_{y,z} & \xrightarrow{\sigma_{A_{x,y} \otimes A_{x,y}, A_{y,z}}} & A_{y,z} \otimes (A_{x,y} \otimes A_{x,y}) \\
 \downarrow (S_{x,y} \otimes A_{x,y}) \otimes S_{y,z} & & \downarrow S_{y,z} \otimes (S_{x,y} \otimes A_{x,y}) \\
 (A_{y,x} \otimes A_{x,y}) \otimes A_{z,y} & \xrightarrow{\sigma_{A_{y,x} \otimes A_{x,y}, A_{z,y}}} & A_{z,y} \otimes (A_{y,x} \otimes A_{x,y}) \\
 \downarrow m_{y,x,y} \otimes A_{z,y} & & \downarrow A_{z,y} \otimes m_{y,x,y} \\
 A_{y,y} \otimes A_{z,y} & \xrightarrow{\sigma_{A_{y,y}, A_{z,y}}} & A_{z,y} \otimes A_{y,y}
 \end{array}$$

The whole diagram commutes by the naturality of σ . And then, finally:

$$\begin{aligned}
 f = h &= m_{z,y,x,z,x}^3 \circ (f \otimes A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta_{x,y,z}^2 \\
 &= m_{z,z,x} \circ (m_{z,y,x,z}^2 \otimes A_{z,x}) \circ (f \otimes A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta_{x,y,z}^2 \\
 &= m_{z,z,x} \circ (m_{z,x,y,z}^2 \circ (f \otimes A_{x,y} \otimes A_{y,z}) \otimes g) \circ (\Delta_{x,y,z} \otimes A_{x,y} \otimes A_{y,z}) \circ \Delta_{x,y,z} \\
 &= m_{z,z,x} \circ (m_{z,x,y,z}^2 \circ (f \otimes A_{x,y} \otimes A_{y,z}) \circ \Delta_{x,y,z} \otimes g) \circ \Delta_{x,y,z} \\
 &= m_{z,z,x} \circ ((\mu_z \circ \varepsilon_{x,y,z}) \circ g) \circ \Delta_{x,y,z} \\
 &= m_{z,z,x} \circ (\mu_z \otimes A_{z,x}) \circ g \circ (\varepsilon_{x,y,z} \otimes A_{x,y} \otimes A_{y,z}) \circ \Delta_{x,y,z} \\
 &= g.
 \end{aligned}$$

The first equality we have already proven, the second equality is the definition of h , the third equality follows from the associativity applied to m^3 , the fourth and fifth equalities follow from the fact that $\otimes$ is a functor and the coassociativity of Δ , the sixth equality follows from the previous computation, the seventh equality follows from the fact that the category $\mathcal{V}$ is strict monoidal, and the last equality is the unit and counit property. Hence, the first statement is proven.

Let us prove the second statement in a similar way. In fact, let us define $f, g, h : A_{x,y} \rightarrow A_{y,x} \otimes A_{y,x}$ by the formulas:

$$\begin{aligned}
 f &= \sigma_{A_{y,x}, A_{y,x}} \circ (S_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y}, \\
 g &= \Delta_{y,x} \circ S_{x,y}, \\
 h &= (m_{y,x,y,x}^{A \otimes A})^2 \circ (g \otimes A_{x,y} \otimes A_{x,y} \otimes f) \circ \Delta_{x,y}^3.
 \end{aligned}$$

First, we compute that:

$$\begin{aligned}
m_{y,x,y}^{A \otimes A} \circ (g \otimes A_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y}^2 &= m_{y,x,y}^{A \otimes A} \circ ((\Delta_{y,x} \circ S_{x,y}) \otimes A_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y}^2 \\
&= m_{y,x,y}^{A \otimes A} \circ ((\Delta_{y,x} \circ S_{x,y}) \otimes A_{x,y} \otimes A_{x,y}) \circ (A_{x,y} \otimes \Delta_{x,y}) \circ \Delta_{x,y} \\
&= m_{y,x,y}^{A \otimes A} \circ ((\Delta_{y,x} \circ S_{x,y}) \otimes \Delta_{x,y}) \circ \Delta_{x,y} \\
&= m_{y,x,y}^{A \otimes A} \circ (\Delta_{y,x} \otimes \Delta_{x,y}) \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
&= \Delta_{y,y} \circ m_{y,x,y} \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
&= \Delta_{y,y} \circ \mu_y \circ \varepsilon_{x,y} \\
&= \mu_y^{A \otimes A} \circ \varepsilon_{x,y}.
\end{aligned}$$

The first equality follows from the definition of g , the second equality follows from the coassociativity applied to Δ^2 , the third and fourth equality follow from the fact that $\otimes$ is a functor, the fifth and the equality follow from the diagrams in the proof of Proposition 4.2.4, and the sixth equality follows from the definition of antipode. Then, we can compute that:

$$\begin{aligned}
h &\stackrel{1}{=} (m_{y,x,y,x}^{A \otimes A})^2 \circ (g \otimes A_{x,y} \otimes A_{x,y} \otimes f) \circ \Delta_{x,y}^3 \\
&\stackrel{2}{=} m_{y,x,y}^{A \otimes A} \circ (m_{y,x,y}^{A \otimes A} \otimes A_{x,y} \otimes A_{x,y}) \circ (g \otimes A_{x,y} \otimes A_{x,y} \otimes f) \circ \Delta_{x,y}^3 \\
&\stackrel{3}{=} m_{y,x,y}^{A \otimes A} \circ (m_{y,x,y}^{A \otimes A} \circ (g \otimes A_{x,y} \otimes A_{x,y}) \otimes f) \circ \Delta_{x,y}^3 \\
&\stackrel{4}{=} m_{y,x,y}^{A \otimes A} \circ (m_{y,x,y}^{A \otimes A} \circ (g \otimes A_{x,y} \otimes A_{x,y}) \otimes A_{x,y}) \circ (\Delta_{x,y}^2 \otimes f) \circ \Delta_{x,y} \\
&\stackrel{5}{=} m_{y,x,y}^{A \otimes A} \circ (m_{y,x,y}^{A \otimes A} \circ (g \otimes A_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y}^2 \otimes f) \circ \Delta_{x,y} \\
&\stackrel{6}{=} m_{y,x,y}^{A \otimes A} \circ (\mu_y^{A \otimes A} \circ \varepsilon_{x,y} \otimes f) \circ \Delta_{x,y} \\
&\stackrel{7}{=} m_{y,x,y}^{A \otimes A} \circ (\mu_y^{A \otimes A} \otimes f) \circ (\varepsilon_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
&\stackrel{8}{=} m_{y,x,y}^{A \otimes A} \circ (\mu_y^{A \otimes A} \otimes f) \\
&\stackrel{9}{=} f.
\end{aligned}$$

The first equality is the definition of h , the second equality is the associativity applied to $(m^{A \otimes A})^2$, the third, fifth and seventh equalities follow from the fact that $\otimes$ is a functor, the fourth equality follow from the coassociativity applied to Δ^3 , the sixth equality follows from the previous computation, the eighth equality follows from the counit property and the last equality follows from the unit property applied to f . Then, we evaluate:

$$\begin{aligned}
& m_{x,y,x}^{A \otimes A} \circ (A_{x,y} \otimes A_{x,y} \otimes f) \circ \Delta_{x,y}^2 \\
&= m_{x,y,x}^{A \otimes A} \circ (A_{x,y} \otimes A_{x,y} \otimes (\sigma_{A_{y,x}, A_{y,x}} \circ (S_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y})) \circ \Delta_{x,y}^2 \\
&= m_{x,y,x} \otimes m_{x,y,x} \circ (A_{x,y} \otimes \sigma_{A_{x,y}, A_{x,y}} \otimes A_{x,y}) \circ \\
& (A_{x,y} \otimes A_{x,y} \otimes (\sigma_{A_{y,x}, A_{y,x}} \circ (S_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y})) \circ \Delta_{x,y}^2 \\
&= m_{x,y,x} \otimes m_{x,y,x} \circ (A_{x,y} \otimes \sigma_{A_{x,y}, A_{x,y}} \otimes A_{x,y}) \circ \\
& (A_{x,y} \otimes A_{x,y} \otimes \sigma_{A_{y,x}, A_{y,x}}) \circ (A_{x,y} \otimes A_{x,y} \otimes S_{x,y} \otimes S_{x,y}) \circ \\
& (A_{x,y} \otimes A_{x,y} \otimes \Delta_{x,y}) \circ \Delta_{x,y}^2 \\
&= m_{x,y,x} \otimes m_{x,y,x} \circ (A_{x,y} \otimes \sigma_{A_{x,y} \otimes A_{x,y}, A_{x,y}}) \circ (A_{x,y} \otimes A_{x,y} \\
& \otimes S_{x,y} \otimes S_{x,y}) \circ (A_{x,y} \otimes A_{x,y} \otimes \Delta_{x,y}) \circ \Delta_{x,y}^2 \\
&= m_{x,y,x} \otimes m_{x,y,x} \circ (A_{x,y} \otimes (\sigma_{A_{x,y} \otimes A_{x,y}, A_{x,y}} \\
& \circ (A_{x,y} \otimes S_{x,y} \otimes S_{x,y}) \circ (A_{x,y} \otimes \Delta_{x,y}))) \circ \Delta_{x,y}^2 \\
&= m_{x,y,x} \otimes m_{x,y,x} \circ (A_{x,y} \otimes ((S_{x,y} \otimes A_{x,y} \otimes S_{x,y}) \\
& (A_{x,y} \otimes \Delta_{x,y}) \circ \sigma_{A_{x,y}, A_{x,y}})) \circ \Delta_{x,y}^2 \\
&= m_{x,y,x} \otimes m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y} \otimes A_{x,y} \otimes S_{x,y}) \circ \\
& (A_{x,y} \otimes A_{x,y} \otimes \Delta_{x,y}) \circ (A_{x,y} \otimes A_{x,y} \otimes \sigma_{A_{x,y}, A_{x,y}}) \circ \Delta_{x,y}^2 \\
&= m_{x,y,x} \otimes (A_{x,y} \otimes A_{y,x}) \\
& \circ (A_{x,y} \otimes S_{x,y} \otimes (m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y})) \circ (A_{x,y} \otimes \sigma_{A_{x,y}, A_{x,y}}) \\
& \circ \Delta_{x,y}^2 \\
&= m_{x,y,x} \otimes (A_{x,y} \otimes A_{y,x}) \\
& \circ (A_{x,y} \otimes S_{x,y} \circ (\mu_x \circ \varepsilon_{x,y})) \circ (A_{x,y} \circ \sigma_{A_{x,y}, A_{x,y}}) \circ \Delta_{x,y}^2 \\
&= m_{x,y,x} \otimes (\mu_x) \circ (A_{x,y} \otimes S_{x,y}) \circ (A_{x,y} \otimes A_{x,y} \otimes \varepsilon_{x,y}) \\
& \circ (A_{x,y} \otimes \sigma_{A_{x,y}, A_{x,y}}) \circ \Delta_{x,y}^2 \\
&= (A_{x,x} \otimes \mu_x) \circ m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ (A_{x,y} \otimes \varepsilon_{x,y} \otimes A_{x,y}) \\
& \circ (A_{x,y} \otimes \Delta_{x,y}) \circ \Delta_{x,y} \\
&= (A_{x,x} \otimes \mu_x) \circ m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} \\
&= (A_{x,y} \otimes \mu_x) \circ \mu_x \circ \varepsilon_{x,y} \\
&= (\mu_x \otimes \mu_x) \circ \varepsilon_{x,y} \\
&= \mu_x^{A \otimes A} \circ \varepsilon_{x,y}
\end{aligned}$$

The first equality follows from the definition of f , while the second equality follows from the definition of $m^{A \otimes A}$, the third, fifth, seventh and eighth equalities follow from the fact that $\otimes$ is a functor, the fourth equality follow from the axioms of braiding, the ninth and twelfth equalities follow from the definition of S , the tenth and thirteenth equalities follow from the fact that $\mathcal{V}$ is a strict monoidal category, the eleventh equality follows from

the counit property, the last equality follows from the definition of $\mu^{A \otimes A}$, and the sixth equality follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
 A_{x,y} \otimes A_{x,y} & \xrightarrow{\sigma_{A_{x,y}, A_{x,y}}} & A_{x,y} \otimes A_{x,y} \\
 \downarrow \Delta_{x,y}, A_{x,y} & & \downarrow A_{x,y} \otimes \Delta_{x,y} \\
 (A_{x,y} \otimes A_{x,y}) \otimes A_{x,y} & \xrightarrow{\sigma_{A_{x,y} \otimes A_{x,y}, A_{x,y}}} & A_{x,y} \otimes (A_{x,y} \otimes A_{x,y}) \\
 \downarrow A_{x,y} \otimes S_{x,y} \otimes S_{x,y} & & \downarrow S_{x,y} \otimes A_{x,y} \otimes S_{x,y} \\
 (A_{x,y} \otimes A_{y,x}) \otimes A_{x,y} & \xrightarrow{\sigma_{A_{x,y}, A_{y,x} \otimes A_{y,x}}} & A_{y,x} \otimes (A_{x,y} \otimes A_{y,x})
 \end{array}$$

The diagram commutes due the naturality of σ . Then, we have that:

$$\begin{aligned}
 f &= h = (m^{A \otimes A})_{y,x,y,x}^2 \circ (g \otimes A_{x,y} \otimes A_{x,y} \otimes f) \circ \Delta_{x,y}^3 \\
 &= m_{y,x,y}^{A \otimes A} \circ (A_{y,x} \otimes A_{y,x} \otimes m_{x,y,x}^{A \otimes A}) \circ (g \otimes A_{x,y} \otimes A_{x,y} \otimes f) \circ \Delta_{x,y}^3 \\
 &= m_{y,x,y}^{A \otimes A} \circ (g \otimes m_{x,y,x}^{A \otimes A} \circ (A_{x,y} \otimes A_{x,y} \otimes f)) \circ \Delta_{x,y}^3 \\
 &= m_{y,x,y}^{A \otimes A} \circ (g \otimes m_{x,y,x}^{A \otimes A} \circ (A_{x,y} \otimes A_{x,y} \otimes f)) \circ (A_{x,y} \circ \Delta_{x,y}^2) \circ \Delta_{x,y} \\
 &= m_{x,y,x}^{A \otimes A} \circ (g \otimes m_{x,y,x}^{A \otimes A} \circ (A_{x,y} \otimes A_{x,y} \otimes f) \circ \Delta_{x,y}^2) \circ \Delta_{x,y} \\
 &= m_{x,y,x}^{A \otimes A} \circ (g \otimes \mu_x^{A \otimes A} \circ \varepsilon_{x,y}) \circ \Delta_{x,y} \\
 &= m_{x,y,x}^{A \otimes A} \circ (g \otimes \mu_x^{A \otimes A}) \circ (A_{x,y} \otimes \varepsilon_{x,y}) \circ \Delta_{x,y} \\
 &= m_{x,y,x}^{A \otimes A} \circ (g \otimes \mu_x^{A \otimes A}) \\
 &= g.
 \end{aligned}$$

We have already proven the first equality, while the second equality follows from the associativity applied to m^2 , and the third follows from definition of g and the fourth follows from definition of f ; the fifth, seventh and ninth equalities follow from the fact that $\otimes$ is a functor, the sixth equality follows from the coassociativity applied to Δ^2 , the eighth equality follows from the previous computation, the tenth equality follows from the counit property and the last equality follows from the unit property applied to g . Hence, $f = g$ and the proof is complete. $\square$

Note that we have not yet fully extended Proposition 3.2.3 to Hopf categories. The next proposition gives this generalization, as follows.

Proposition 4.2.8. *Let A be a Hopf $\mathcal{V}$ -category. The antipode $S : A \rightarrow (A^{op})^{cop}$ is a $\mathcal{C}(\mathcal{V}) - X$ -functor.*

Proof. First, we shall prove that $S_{x,y}$ is a morphism in $\mathcal{C}(\mathcal{V})$ for each $x, y \in X$. Then, we

shall prove the commutativity of the following diagrams:

$$\begin{array}{ccc}
 A_{x,y} & \xrightarrow{\Delta_{x,y}} & A_{x,y} \otimes A_{x,y} \\
 S_{x,y} \downarrow & & \downarrow S_{x,y} \otimes S_{x,y} \\
 A_{y,x} & \xrightarrow{\Delta_{x,y}^{cop}} & A_{y,x} \otimes A_{y,x}
 \end{array}
 \qquad
 \begin{array}{ccc}
 A_{x,y} & \xrightarrow{\varepsilon_{x,y}} & I \\
 S_{x,y} \downarrow & \nearrow \varepsilon_{y,x} & \\
 A_{y,x} & &
 \end{array}$$

The commutativity of the diagram on the left follows from the previous proposition. For the second one, we proceed as follows:

$$\begin{aligned}
 \varepsilon_{x,y} &= \varepsilon_{x,x} \circ \mu_x \circ \varepsilon_{x,y} \\
 &= \varepsilon_{x,y} \circ m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} \\
 &= (\varepsilon_{x,y} \otimes \varepsilon_{y,x}) \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} \\
 &= \varepsilon_{x,y} \otimes (\varepsilon_{y,x} \circ S_{x,y}) \circ \Delta_{x,y} \\
 &= (\varepsilon_{x,y} \otimes (\varepsilon_{y,x} \circ S_{x,y})) \Delta_{x,y} \\
 &= (I \otimes (\varepsilon_{y,x} \circ S_{x,y})) \circ (\varepsilon_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
 &= \varepsilon_{y,x} \circ S_{x,y} \circ (\varepsilon_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
 &= \varepsilon_{y,x} \circ S_{x,y}.
 \end{aligned}$$

The first equality follows from the last diagram in Proposition 4.2.4, the second equality follows from the definition of S , the third equality follow from the diagrams in Proposition 4.2.4, the fourth equality follows from the fact that $\otimes$ is a functor, the fifth equality follows from simple composition, the the seventh equality follows from the fact that $\mathcal{V}$ is a strict monoidal category, and the last equality follows from the counit property. Hence, the second diagram commutes.

Now, our aim is to show that S is a $\mathcal{C}(\mathcal{V})$ -functor. Hence, we need to prove that the following diagrams commute:

$$\begin{array}{ccc}
 A_{x,y} \otimes A_{y,z} & \xrightarrow{m_{x,y,z}} & A_{x,z} \\
 S_{x,y} \otimes S_{y,z} \downarrow & & \downarrow S_{x,z} \\
 A_{y,x} \otimes A_{z,y} & \xrightarrow{m_{x,y,z}^{op}} & A_{z,x}
 \end{array}
 \qquad
 \begin{array}{ccc}
 I & \xrightarrow{\mu_x} & A_{x,x} \\
 \mu_x \searrow & & \downarrow S_{x,x} \\
 & & A_{x,x}
 \end{array}$$

The commutativity of the first diagram follows from the previous proposition, after realizing that $m_{x,y,z}^{op} = m_{z,y,x} \circ \sigma_{A_{y,x}, A_{z,y}}$ and then observing that:

$$(S_{y,z} \otimes S_{x,y}) \circ \sigma_{A_{x,y}, A_{y,z}} = \sigma_{A_{y,x}, A_{z,y}} \circ (S_{x,y} \otimes S_{y,z}),$$

resulting from the naturality of σ . Then, the commutativity of the second diagram goes

as follows:

$$\begin{aligned}
\mu_x &= (\varepsilon_{x,x} \otimes A_{x,x}) \circ \Delta_{x,x} \circ \mu_x \\
&= (\varepsilon_{x,x} \otimes A_{x,x}) \circ (\mu_x \otimes \mu_x) \\
&= \varepsilon_{x,x} \circ \mu_x \otimes \mu_x \\
&= (m_{x,x,x} \circ (A_{x,x} \otimes S_{x,x}) \circ \Delta_{x,x}) \otimes \mu_x \\
&= (m_{x,x,x} \circ (A_{x,x} \otimes S_{x,x}) \circ \Delta_{x,x}) \circ \mu_x \\
&= m_{x,x,x} \circ (A_{x,x} \otimes S_{x,x}) \circ \mu_x \otimes \mu_x \\
&= m_{x,x,x} \circ (\mu_x \otimes S_{x,x} \circ \mu_x) \\
&= S_{x,x} \circ \mu_x.
\end{aligned}$$

The first equality follows from the counit property, the second and sixth equalities follow from the diagrams in Proposition 4.2.4, the third and seventh diagrams follow from the fact that $\otimes$ is a functor, the fourth equality follows from the definition of S , the fifth equality follows from the fact that $\mathcal{V}$ is a strict monoidal category, and the last equality follows from the unit property applied to $S_{x,x} \circ \mu_x$. $\square$

Then let us give a generalization of a morphism of Hopf algebras.

Definition 4.2.8. Let A and B be Hopf $\mathcal{V}$ -categories. A $\mathcal{C}(\mathcal{V})$ -functor is called a **Hopf $\mathcal{V}$ -functor** if:

$$S_{f(x),f(y)}^B \circ f_{x,y} = f_{y,x} \circ S_{x,y}^A$$

for all $x, y \in X$.

Hopf $\mathcal{V}$ -categories constitute a 2-category, as follows:

Definition 4.2.9. ${}_{\mathcal{V}}\text{HopfCat}$ is the 2-category where the objects are the Hopf $\mathcal{V}$ -Categories, the functors are the Hopf functors and the natural transformations are the natural transformations defined between Hopf functors.

The next result shows that strong monoidal functors preserve coalgebras.

Lemma 4.2.2. Let $F : \mathcal{V} \rightarrow \mathcal{W}$ be a strong monoidal functor. Then F induces a monoidal strong functor $\overline{F} : \mathcal{C}(\mathcal{V}) \rightarrow \mathcal{C}(\mathcal{W})$.

Proof. For a $\mathcal{V}$ -coalgebra C with coproduct Δ and counit ε , we can define a structure of coalgebra for $F(C)$ defined as:

$$\begin{aligned}
\Delta_{F(C)} &: \phi_2^{-1} \circ F(\Delta) : F(C) \rightarrow F(C \otimes C) \rightarrow F(C) \otimes F(C), \\
\varepsilon_{F(C)} &: \phi_0^{-1} \circ F(\varepsilon) : F(C) \rightarrow F(I_{\mathcal{V}}) \rightarrow I_{\mathcal{W}}.
\end{aligned}$$

In fact, observe that:

$$\begin{aligned}
(\Delta_{F(C)} \otimes F(C)) \circ \Delta_{F(C)} &= (\phi_2^{-1} \circ F(\Delta) \otimes F(C)) \circ \phi_2^{-1} \circ F(\Delta) \\
&= (\phi_2^{-1} \otimes F(C)) \circ (F(\Delta) \otimes F(C)) \circ \phi_2^{-1} \circ F(\Delta) \\
&= (\phi_2^{-1} \otimes F(C)) \circ \phi_2^{-1} \circ F(\Delta \otimes C) \circ F(\Delta) \\
&= (\phi_2^{-1} \otimes F(C)) \circ \phi_2^{-1} \circ (F((\Delta \otimes C) \circ \Delta)) \\
&= (\phi_2^{-1} \otimes F(C)) \circ \phi_2^{-1} \circ F(C \otimes \Delta) \circ F(\Delta) \\
&= (F(C) \otimes \phi_2^{-1}) \circ \phi_2^{-1} \circ F(C \otimes \Delta) \circ F(\Delta) \\
&= (F(C) \circ \phi_2^{-1}) \circ (F(C) \otimes F(\Delta)) \circ \phi_2^{-1} \circ F(\Delta) \\
&= (F(C) \otimes \phi_2^{-1} \circ F(\Delta)) \circ \phi_2^{-1} \circ F(\Delta) \\
&= (F(C) \otimes \Delta_{F(C)}) \circ \Delta_{F(C)}.
\end{aligned}$$

The first and last equalities follow from the definition of $\Delta_{F(C)}$, the second and eighth equalities follow from the fact that $\otimes$ is a functor, the third and seventh equalities follow from the naturality of ϕ_2^{-1} , the fourth and sixth equalities follow from the fact that F is a functor, and the fifth equality follows from the coassociativity of Δ .

Now, let us prove the counit property. In fact, we have:

$$\begin{aligned}
(\varepsilon_{F(C)} \otimes F(C)) \circ \Delta_{F(C)} &= (\varepsilon_{F(C)} \otimes F(C)) \circ \phi_2^{-1} \circ F(\Delta) \\
&= (\phi_0^{-1} \circ F(\varepsilon) \otimes F(C)) \circ \phi_2^{-1} \circ F(\Delta) \\
&= (\phi_0^{-1} \otimes F(C)) \circ (F(\varepsilon) \otimes F(C)) \circ \phi_2^{-1} \circ F(\Delta) \\
&= (\phi_0^{-1} \otimes F(C)) \circ \phi_2^{-1} \circ F(\varepsilon \otimes C) \circ F(\Delta) \\
&= (\phi_0^{-1} \otimes F(C)) \circ \phi_2^{-1} \circ F((\varepsilon \otimes C) \circ \Delta) \\
&= (\phi_0^{-1} \otimes F(C)) \circ \phi_2^{-1} \\
&= F(C).
\end{aligned}$$

The first equality follows from the definition of $\Delta_{F(C)}$, while the second equality follows from the definition of $\varepsilon_{F(C)}$, the third equality follows from the fact that $\otimes$ is a functor, the fourth equality follows from the naturality of ϕ_2 , the fifth equality follows from the fact that F is a functor, the sixth equality follows from the counit property of C , and the last equality follows from properties of the axioms of monoidal functors.

Hence, the strong monoidal functor $F : \mathcal{V} \rightarrow \mathcal{W}$ induces a strong monoidal functor $\overline{F} : C(\mathcal{V}) \rightarrow C(\mathcal{W})$. $\square$

The next proposition is related to Proposition 4.2.2.

Proposition 4.2.9. *Let $F : \mathcal{V} \rightarrow \mathcal{W}$ be a strong monoidal functor. Then, F induces bifunctors $F : {}_{\mathcal{C}(\mathcal{V})}Cat \rightarrow {}_{\mathcal{C}(\mathcal{W})}Cat$ and $F : {}_{\mathcal{V}}HopfCat \rightarrow {}_{\mathcal{W}}HopfCat$.*

Proof. We proved, in the previous lemma, that the strong monoidal functor $F : \mathcal{V} \rightarrow \mathcal{W}$ induces a strong monoidal functor $\overline{F} : C(\mathcal{V}) \rightarrow C(\mathcal{W})$. By Proposition 4.2.2, the latter induces a bifunctor $\overline{F} : {}_{C(\mathcal{V})}Cat \rightarrow {}_{C(\mathcal{W})}Cat$. Following the construction in Proposition 4.2.9, the multiplication morphisms are defined as:

$${}_{C(\mathcal{W})}m_{x,y,z} = F({}_{C(\mathcal{V})}m_{x,y,z}) \circ \phi_2 : F(A_{x,y}) \otimes F(A_{y,z}) \rightarrow F(A_{x,y} \otimes A_{y,z}) \rightarrow F(A_{x,z}),$$

and the unit morphisms are:

$${}_{C(\mathcal{V})}\mu_x = F({}_{C(\mathcal{W})}\mu_x) \circ \phi_0 : I_{C(\mathcal{V})} \rightarrow FI_{C(\mathcal{V})} = F(A_{x,x})$$

We recall that $F(A_{x,y}) = F(A)_{x,y}$, as in the construction of Proposition 4.2.2.

Then, let A be a Hopf $\mathcal{V}$ -category. We claim that the morphisms

$$F(S_{x,y}) : F(A_{x,y}) \rightarrow F(A_{y,x})$$

define an antipode for $F(A)$. Then, we need to prove the two statements of Definition 4.2.7 for this one. The first statement for this case has the form:

$$F({}_{C(\mathcal{V})}m_{x,y,x}) \circ \phi_2 \circ (F(A_{x,y}) \otimes F(S_{x,y})) \circ \phi_2^{-1} \circ F({}_{C(\mathcal{V})}\Delta_{x,y}) = F({}_{C(\mathcal{V})}\mu_x) \circ \phi_0 \circ \phi_0^{-1} \circ F({}_{C(\mathcal{V})}\varepsilon_{x,y}).$$

In fact, observe that:

$$\begin{aligned} & F({}_{C(\mathcal{V})}m_{x,y,x}) \circ \phi_2 \circ (F(A_{x,y}) \otimes F(S_{x,y})) \circ \phi_2^{-1} \circ F({}_{C(\mathcal{V})}\Delta_{x,y}) \\ &= F({}_{C(\mathcal{V})}m_{x,y,x}) \circ (F(A_{x,y} \otimes S_{x,y})) \circ \phi_2 \circ \phi_2^{-1} \circ F({}_{C(\mathcal{V})}\Delta_{x,y}) \\ &= F({}_{C(\mathcal{V})}m_{x,y,x}) \circ (F(A_{x,y} \otimes S_{x,y})) \circ F({}_{C(\mathcal{V})}\Delta_{x,y}) \\ &= F({}_{C(\mathcal{V})}m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ {}_{C(\mathcal{V})}\Delta_{x,y}) \\ &= F({}_{C(\mathcal{V})}\mu_x \circ {}_{C(\mathcal{V})}\varepsilon_{x,y}) \\ &= F({}_{C(\mathcal{V})}\mu_x) \circ F({}_{C(\mathcal{V})}\varepsilon_{x,y}) \\ &= F({}_{C(\mathcal{V})}\mu_x) \circ \phi_0 \circ \phi_0^{-1} \circ F({}_{C(\mathcal{V})}\varepsilon_{x,y}). \end{aligned}$$

The first equality follows from the naturality of ϕ_2 , the second and last equalities follow from the definition of inverse, the fifth equality follows from the fact that $\otimes$ is a functor and the fourth equality follows from the fact that S is an antipode for A . The proof of the other statement is similar. $\square$

Proposition 4.2.10. *Let A, B be Hopf $\mathcal{V}$ -categories. If $f : A \rightarrow B$ is a $C(\mathcal{V})$ -functor, then it is also a Hopf functor.*

Proof. Let us consider the following morphisms $k, g, h : A_{x,y} \rightarrow F_{f(y),f(x)}$ defined by the formulas:

$$k = S_{f(x),f(y)} \circ f_{x,y}; \quad g = f_{y,x} \circ S_{x,y}; \quad h = m_{f(y),f(x),f(y),f(x)}^2 \circ (k \otimes f_{x,y} \otimes g) \circ \Delta_{x,y}^2.$$

Let us compute that:

$$\begin{aligned}
m_{f(y),f(y),f(x)} \circ (f_{x,y} \otimes g) \circ \Delta_{x,y} &= m_{f(y),f(y),f(x)} \circ (f_{x,y} \otimes (f_{y,x} \circ S_{x,y})) \circ \Delta_{x,y} \\
&= m_{f(y),f(y),f(x)} \circ (f_{x,y} \otimes f_{y,x}) \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} \\
&= f_{x,x} \circ m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} \\
&= f_{x,x} \circ \mu_x \circ \varepsilon_{x,y} = \mu_{f(x)} \circ \varepsilon_{x,y}
\end{aligned}$$

The first equality follows from the definition of g , while the second equality follows from the fact that $\otimes$ is a functor, the third and last equalities follow from the fact that f is a $\mathcal{C}(\mathcal{V})$ -functor, and the fourth equality follows from the definition of antipode. Then, we may also compute that:

$$\begin{aligned}
h &= m_{f(y),f(x),f(y),f(x)}^2 \circ (k \otimes f_{x,y} \otimes g) \circ \Delta_{x,y}^2 \\
&= m_{f(y),f(x),f(y)} \circ (F_{f(y),f(x)} \otimes m_{f(y),f(y),f(x)}) \circ (k \otimes f_{x,y} \otimes g) \circ \Delta_{x,y}^2 \\
&= m_{f(y),f(x),f(y)} \circ (k \otimes m_{f(y),f(y),f(x)} \circ (f_{x,y} \otimes g)) \circ \Delta_{x,y}^2 \\
&= m_{f(y),f(x),f(y)} \circ (k \otimes m_{f(y),f(y),f(x)} \circ (f_{x,y} \otimes g)) \circ (A_{x,y} \otimes \Delta_{x,y}) \circ \Delta_{x,y} \\
&= m_{f(y),f(x),f(y)} \circ (k \otimes m_{f(y),f(y),f(x)} \circ (f_{x,y} \otimes g) \circ \Delta_{x,y}) \circ \Delta_{x,y} \\
&= m_{f(y),f(x),f(y)} \circ (k \otimes (\mu_{f(x)} \circ \varepsilon_{x,y})) \circ \Delta_{x,y} \\
&= m_{f(y),f(x),f(y)} \circ (k \otimes \mu_{f(x)}) \circ (A_{x,y} \otimes \varepsilon_{x,y}) \circ \Delta_{x,y} \\
&= (k \otimes \varepsilon_{x,y}) \circ \Delta_{x,y} \\
&= k
\end{aligned}$$

The first equality follows from the definition of h , the second equality follows from the associativity of m^2 , the third, fifth and seventh equalities follow from the fact that $\otimes$ is a functor, the fourth equality follows from the coassociativity of Δ^2 . Then, the sixth equality follows from the previous computation, the eighth equality follows from the property of unit, and the last equality follows from the property of counit. But then, we may also compute that:

$$\begin{aligned}
m_{f(y),f(x),f(y)} \circ (k \otimes f_{x,y}) \circ \Delta_{x,y} &= m_{f(y),f(x),f(y)} \circ ((S_{f(x),f(y)} \circ f_{x,y}) \otimes f_{x,y}) \circ \Delta_{x,y} \\
&= m_{f(y),f(x),f(y)} \circ (S_{f(x),f(y)} \otimes F_{f(x),f(y)}) \circ (f_{x,y} \otimes f_{x,y}) \circ \Delta_{x,y} \\
&= m_{f(y),f(x),f(y)} \circ (S_{f(x),f(y)} \otimes F_{f(x),f(y)}) \circ \Delta_{f(x),f(y)} \circ f_{x,y} \\
&= \mu_{f(y)} \circ \varepsilon_{f(x),f(y)} \circ f_{x,y} \\
&= \mu_{f(y)} \circ \varepsilon_{x,y}.
\end{aligned}$$

The first equality follows from the definition of k , while the second equality follow from the fact that $\otimes$ is a functor, and the third and last equalities follow from the fact that f is a $\mathcal{C}(\mathcal{V})$ -functor. The fourth equality follows from the definition of antipode. Then, we may

also compute that:

$$\begin{aligned}
k &= h = m_{f(y),f(x),f(y),f(x)}^2 \circ (k \otimes f_{x,y} \otimes g) \circ \Delta_{x,y}^2 \\
&= m_{f(y),f(y),f(x)} \circ (m_{f(y),f(x),f(y)} \otimes F_{f(y),f(x)}) \circ (k \otimes f_{x,y} \otimes g) \circ \Delta_{x,y}^2 \\
&= m_{f(y),f(y),f(x)} \circ ((m_{f(y),f(x),f(y)} \circ (k \otimes f_{x,y})) \otimes g) \circ \Delta_{x,y}^2 \\
&= m_{f(y),f(y),f(x)} \circ ((m_{f(y),f(x),f(y)} \circ (k \otimes f_{x,y})) \otimes g) \circ (\Delta_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
&= m_{f(y),f(y),f(x)} \circ ((m_{f(y),f(x),f(y)} \circ (k \otimes f_{x,y}) \circ \Delta_{x,y}) \otimes g) \circ \Delta_{x,y} \\
&= m_{f(y),f(y),f(x)} \circ ((\mu_{f(y)} \circ \varepsilon_{x,y}) \otimes g) \circ \Delta_{x,y} \\
&= m_{f(y),f(y),f(x)} \circ (\mu_{f(y)} \otimes g) \circ (\varepsilon_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
&= g \circ (I \otimes \varepsilon_{x,y}) \circ \Delta_{x,y} \\
&= g.
\end{aligned}$$

The first equality follows from previous computation, while the second equality follows from the definition of h . The third equality follows from the associativity of m^2 , the fourth, sixth and eighth equalities follow from the fact that $\otimes$ is a functor, the fifth equality follows from the coassociativity of Δ , the seventh equality follows from the previous computation, the ninth equality follows from the property of unit and the last equality follows from the property of counit. Hence, $k = g$, *i.e*

$$S_{f(x),f(y)} \circ f_{x,y} = f_{y,x} \circ S_{x,y}.$$

Hence, the proposition is proved. □

4.3 DUAL HOPF CATEGORIES

In this section we introduce the concept of dual Hopf category, which is the dualized version of a Hopf category in the sense that a dual Hopf $\mathcal{V}$ -category is in fact a Hopf $\mathcal{V}^{op}$ -category. The main references for this part are (BATISTA; CAENEPEEL; VERCRUYSSSE, 2016) and (JOHNSON; YAU, 2021). First, we shall define an opposite monoidal category, as follows:

Definition 4.3.1. Consider $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a strict monoidal category. Then, the **opposite monoidal category** $\mathcal{V}^{op} = (\mathcal{V}^{op}, \otimes^{op}, I)$ is the monoidal category whose objects are the same objects of $\mathcal{V}$, whose morphisms are given by:

$$Hom_{\mathcal{V}^{op}}(M, N) = Hom_{\mathcal{V}}(N, M),$$

composition is the opposite composition, and the opposite monoidal product is defined as $M \otimes^{op} N = N \otimes M$ and $f \otimes^{op} g = g \otimes f$.

If $(\mathcal{V}, \otimes, 1, \sigma)$ is a strict braided monoidal category, then $(\mathcal{V}, \otimes^{op}, 1, \sigma^{-1})$ is a strict braided monoidal category, due to the dualization of the axioms of braiding. Then an algebra in $\mathcal{V}^{op}$ is a coalgebra in $\mathcal{V}$, a coalgebra in $\mathcal{V}^{op}$ is an algebra in $\mathcal{V}$, but a bialgebra in $\mathcal{V}^{op}$ still is a bialgebra in $\mathcal{V}$ and a Hopf algebra in $\mathcal{V}^{op}$ still is a Hopf algebra in $\mathcal{V}$. In this sense, we may say that the concepts of bialgebras and Hopf algebras in braided monoidal categories are self-dual. In comparison, the same does not hold for Hopf categories: A Hopf $\mathcal{V}^{op}$ -category does not, in general, correspond to a Hopf $\mathcal{V}$ -category; the "dualization" of the axioms for a Hopf $\mathcal{V}$ -category leads to the new concept of dual Hopf $\mathcal{V}$ -category.

We begin by defining dual $\mathcal{V}$ -categories.

Definition 4.3.2. A **dual $\mathcal{V}$ -category** C consists of an object in $\mathcal{V}(X)$ together with two classes of morphisms, namely

- The coproduct morphisms Δ defined as:

$$\Delta_{x,y,z} : C_{x,z} \rightarrow C_{x,y} \otimes C_{y,z}$$

defined for each $x, y, z \in X$;

- Counit morphisms $\varepsilon_x : C_{x,x} \rightarrow I$ for each $x \in X$;

and these data need to satisfy the following equalities:

$$(C_{x,y} \otimes \Delta_{y,z,t}) \circ \Delta_{x,y,t} = (\Delta_{x,y,z} \otimes C_{z,t}) \circ \Delta_{x,z,t},$$

$$(\varepsilon_x \otimes C_{x,y}) \circ \Delta_{x,x,y} = C_{x,y} = (C_{x,y} \otimes \varepsilon_y) \circ \Delta_{x,y,y}.$$

A first example of such definition is a coalgebra in the monoidal category $\mathcal{V}$, by taking the index set as a set with just one element. A dual $\mathcal{V}$ -category is essentially the same as a $\mathcal{V}^{op}$ -category (see Proposition 4.3.1 in the following). We may also define a dual $\mathcal{V}$ -functor, as follows:

Definition 4.3.3. Let C, D be dual $\mathcal{V}$ -categories, where $C \in \mathcal{V}(X)$ and $D \in \mathcal{V}(Y)$. A dual $\mathcal{V}$ -functor $f : C \rightarrow D$ consists of a function $f : X \rightarrow Y$ and a family of morphisms $f_{x,y} : D_{f(x),f(y)} \rightarrow C_{x,y}$ indexed by $X \times X$ such that:

$$(f_{x,y} \otimes f_{y,z}) \circ \Delta_{f(x),f(y),f(z)} = \Delta_{x,y,z} \circ f_{x,z}$$

$$\varepsilon_x \circ f_{x,x} = \varepsilon_{f(x)}.$$

The main point of the introduction of dual $\mathcal{V}$ -categories is that, on the contrary to what happens for Hopf algebras, the concept of Hopf category is not self-dual. Then, the

dual concept of $\mathcal{V}$ -functors and coalgebra morphisms are examples of dual $\mathcal{V}$ -functors. In fact, $\mathcal{V}^{op}$ -functors correspond to dual $\mathcal{V}$ -functors. Then, we shall define the dual concept of $\mathcal{V}$ -transformations as follows:

Definition 4.3.4. Let $f, g : C \rightarrow D$ be both dual $\mathcal{V}$ -functor. A **dual $\mathcal{V}$ -natural transformation** $\alpha : f \rightarrow g$ consists of morphisms $\alpha_x : D_{f(x),g(x)} \rightarrow I$ in $\mathcal{V}$ such that:

$$(f_{x,y} \otimes \alpha_y) \circ \Delta_{f(x),f(y),g(y)} = (\alpha_x \otimes g_{x,y}) \circ \Delta_{f(x),g(x),g(y)}.$$

Dual $\mathcal{V}$ -categories form a 2-category.

Definition 4.3.5. Let $\mathcal{V}$ be a strict monoidal category. The 2-category ${}^{\mathcal{V}}Cat$ of the dual $\mathcal{V}$ -categories is a tuple with the following data $({}^{\mathcal{V}}Cat_0, {}^{\mathcal{V}}Cat_1, {}^{\mathcal{V}}Cat_2)$:

The collection ${}^{\mathcal{V}}Cat_0$ is the collection of 0-cells, which are the dual $\mathcal{V}$ -categories. The collection ${}^{\mathcal{V}}Cat_1$ is the collection of the 1-cells, which are the dual $\mathcal{V}$ -functors. The collection ${}^{\mathcal{V}}Cat_2$ is the collection of 2-cells, which are the dual $\mathcal{V}$ -natural transformations. Compositions for dual $\mathcal{V}$ -functors and for dual $\mathcal{V}$ -natural transformations are as follows:

- Let $f : C \rightarrow D$ and $g : D \rightarrow E$ be dual $\mathcal{V}$ -functors. Then, $g \circ f$ is defined by the formula:

$$(g \circ f)_{x,y} = f_{x,y} \circ g_{f(x),f(y)} = E_{(g \circ f)(x), (g \circ f)(y)} \rightarrow C_{x,y}.$$

- Now let $f' : C \rightarrow D$ and $g' : D \rightarrow E$ be both dual $\mathcal{V}$ -functors and let $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ be dual $\mathcal{V}$ -natural transformations. Then we define:

$$(\alpha * \beta)_x = (\beta_{f(x)} \otimes (\alpha_x \circ g'_{f(x),f'(x)})) \circ \Delta_{(g \circ f)(x), (g' \circ f)(x), (g' \circ f')(x)}$$

$$((\alpha_x \circ g_{f(x),f'(x)}) \otimes \beta_{f'(x)}) \circ \Delta_{(g \circ f)(x), (g \circ f')(x), (g' \circ f')(x)}.$$

- Now let f, g, h be dual $\mathcal{V}$ -functors, and let $\alpha : f \Rightarrow g, \beta : g \Rightarrow h$ be dual $\mathcal{V}$ -transformations. Then, the vertical composition is defined as follows:

$$(\beta \circ \alpha)_x = (\alpha_x \otimes \beta_x) \circ \Delta_{f(x),g(x),h(x)} : B_{f(x),h(x)} \rightarrow I.$$

Proposition 4.3.1. Let $\mathcal{V}$ be a strict monoidal category. Then the 2-categories ${}^{\mathcal{V}}Cat$ and ${}_{\mathcal{V}^{op}}Cat$ are 2-isomorphic.

Proof. We will define a bifunctor $F : {}^{\mathcal{V}}Cat \rightarrow {}_{\mathcal{V}^{op}}Cat$. Take a dual $\mathcal{V}$ -category C , with underlying class X , and consider $A_{x,y} = C_{y,x}$ in $\mathcal{V}(X)$, (i.e, $A = C^{op}$) for every $x, y \in X$. For each $z, y, z \in X$ we have the $\mathcal{V}$ -morphisms:

$$\Delta_{x,y,z} : C_{x,z} \rightarrow C_{x,y} \otimes C_{y,z}.$$

Since $A = C^{op}$, we may view those as:

$$\Delta_{x,y,z} : A_{z,x} \rightarrow A_{z,y} \otimes^{op} A_{y,x}$$

because $A_{z,y} = C_{y,z}$, $A_{y,x} = C_{x,y}$ and $A_{y,x} \otimes A_{z,y} = A_{z,y} \otimes^{op} A_{y,x}$.

Hence, we may define a product m on A by considering $\Delta_{x,y,z}$ as a morphism in $\mathcal{V}^{op}$:

$$m_{z,y,x}^C = (\Delta_{x,y,z}^A)^{op} : A_{y,z} \otimes^{op} A_{x,y} = C_{z,y} \otimes C_{y,x} \rightarrow C_{z,x} \rightarrow A_{x,z}.$$

Our aim is to prove that m satisfies associativity and that $\mu_x = \varepsilon_x^{op} : I \rightarrow A_{x,x}$ satisfies the unit axiom.

Since Δ satisfies coassociativity, we can take its dual (i.e, we apply the opposite functor $(_)^{op}$ to the coassociativity equalities), as follows:

$$((\Delta_{x,y,z} \otimes C_{z,w}) \circ \Delta_{x,z,w})^{op} = ((C_{x,y} \otimes \Delta_{y,z,w}) \circ \Delta_{x,y,w})^{op},$$

and it implies that

$$(\Delta_{x,z,w})^{op} \circ (\Delta_{x,y,z}^{op} \otimes (C_{z,w})^{op}) = (\Delta_{x,y,w})^{op} \circ ((C_{x,y})^{op} \otimes (\Delta_{y,z,w})^{op}),$$

and then

$$m_{w,z,x} \circ (m_{z,y,x} \otimes A_{w,z}) = m_{w,y,x} \circ (A_{y,x} \otimes m_{w,z,y}),$$

therefore:

$$m_{w,z,x} \circ (A_{w,z} \otimes^{op} m_{z,y,x}) = m_{w,y,x} \circ (m_{w,z,y} \otimes^{op} A_{y,x}).$$

All that follows from duality applied to the coassociativity of Δ .

Hence, m is associative.

Now, we shall prove the unit property. In fact, we can deduce it by duality of the counit property, as follows:

$$((\varepsilon_x \otimes C_{x,y}) \circ \Delta_{x,x,y})^{op} = ((C_{x,y} \otimes \varepsilon_{y,x}) \circ \Delta_{x,y,y})^{op},$$

and it implies that

$$\Delta_{x,x,y}^{op} \circ (\varepsilon_x^{op} \otimes C_{x,y}^{op}) = \Delta_{x,y,y}^{op} \circ (C_{x,y}^{op} \otimes \varepsilon_y^{op}),$$

and it implies

$$m_{y,x,x} \circ (A_{y,x} \otimes^{op} \mu_x) = m_{y,y,x} \circ (\mu_y \otimes^{op} A_{x,y}).$$

All that follows from duality. Hence, we have proved the unit property. Therefore the functor is defined on 0-cells. Let us define such 2-functor for 1-cells. Let C, D be dual $\mathcal{V}$ -categories, let $f : C \rightarrow D$ be a dual $\mathcal{V}$ -functor and let us denote $F(C) = A, F(D) = B$. For all $x, y \in X$, we have $\mathcal{V}$ -morphisms:

$$f_{x,y} : D_{f(x),f(y)} = B_{f(y),f(x)} \rightarrow C_{x,y} = A_{y,x}.$$

Define $g : F(C) \rightarrow F(D)$ as $g(x) = f(x)$ for $x \in X$, and $g_{y,x} = f_{x,y}^{op} \in \text{Hom}_{\mathcal{V}^{op}}(C_{x,y}, D_{f(x),f(y)})$. Let us prove that g is a $\mathcal{V}^{op}$ -functor in $(A, \otimes^{op}, I)$: Observe that, since f is a dual $\mathcal{V}$ -morphism, applying the opposite functor $(_)^{op}$ we have:

$$((f_{x,y} \otimes f_{y,z}) \circ \Delta_{f(x),f(y),f(z)})^{op} = (\Delta_{x,y,z} \circ f_{x,z})^{op},$$

from which it follows that

$$(\Delta_{f(x),f(y),f(z)})^{op} \circ (f_{x,y} \otimes f_{y,z})^{op} = f_{x,z}^{op} \circ \Delta_{x,y,z}^{op},$$

and we obtain

$$(m_{f(z),f(y),f(x)}) \circ (g_{z,y} \otimes^{op} g_{y,x}) = g_{z,x} \circ m_{z,y,x}.$$

Hence we have proven that g is a $\mathcal{V}^{op}$ -morphism.

Therefore, we have the 2-functor F defined for 1-cells.

Finally, let us define such a functor for 2-cells. In fact, let $f, f' : C \rightarrow D$ be both dual $\mathcal{V}$ -functors and let $\alpha : f \rightarrow f'$ be a dual $\mathcal{V}$ -natural transformation. Observe that, for all $x \in X$, we have a $\mathcal{V}$ -morphism $\alpha_x : B_{f'(x),f(x)} \rightarrow I$. For each $x \in X$, let us define $(\alpha^{op})_x = \alpha_x^{op}$ and let us observe that this is a $\mathcal{V}^{op}$ -natural transformation $\alpha^{op} : F(f') \rightarrow F(f)$. The argument follows by duality, just how it followed until here. Since the functor is just the duality applied to ${}_{\mathcal{V}}\text{Cat}$, the inverse functor is just the dual. Hence, the proof is complete. $\square$

Corollary 4.3.1. *Let X be a class, and let $\mathcal{V}$ be a strict monoidal category. Then the 2-categories ${}_{\mathcal{V}^{op}(X)}\text{Cat}$ and ${}^{\mathcal{V}(X)}\text{Cat}$ are isomorphic.*

Let $\mathcal{V}$ be a braided strict monoidal category. Hopf $\mathcal{V}$ -categories were defined as $\mathcal{C}(\mathcal{V})$ -categories, where $\mathcal{C}(\mathcal{V})$ is the category of coalgebras in $\mathcal{V}$. In order to introduce dual Hopf $\mathcal{V}$ -categories we will need to consider dual $\mathcal{A}(\mathcal{V})$ -categories, where $\mathcal{A}(\mathcal{V})$ is the category of algebras in $\mathcal{V}$, which is a monoidal category since we have already proved in the previous chapter that the tensor product of two algebras is again an algebra. For an algebra $A \in \mathcal{V}(X)$, its product is denoted by $m_{x,y} : A_{x,y} \otimes A_{x,y} \rightarrow A_{x,y}$ and its unit is denoted by $\mu_{x,y} : I \rightarrow A_{x,y}$. Observe that in a $\mathcal{V}$ -category, the product is defined in 3 indices and the coproducts in the coalgebras is defined in 2 indices. In dual $\mathcal{V}$ -categories, the product is defined with 2 indices, and the coproduct is defined with 3 indices. Then, we define dual Hopf categories as follows:

Definition 4.3.6 ((dual Hopf $\mathcal{V}$ -category)). *Let C be a dual $\mathcal{A}(\mathcal{V})$ -category. C is called a dual Hopf $\mathcal{V}$ -category if there exist morphisms $S_{x,y} : C_{y,x} \rightarrow C_{x,y}$ in $\mathcal{V}$ such that:*

$$m_{x,y} \circ (C_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y,x} = \mu_{x,y} \circ \varepsilon_x$$

$$m_{y,x} \circ (S_{y,x} \circ C_{y,x}) \circ \Delta_{x,y,x} = \mu_{y,x} \circ \varepsilon_x$$

Hopf algebras in $\mathcal{V}$ are examples of dual Hopf $\mathcal{V}$ -categories. In the next chapter we will show that if $\mathcal{V}$ is the monoidal category of finitely generated projective modules over a commutative ring k then every Hopf $\mathcal{V}$ -category gives rise to a dual Hopf $\mathcal{V}$ -category, and conversely. However, it is necessary to prove that this definition makes sense. Then, we present the following proposition, which dualizes Proposition 4.2.4:

Proposition 4.3.2. *Let $\mathcal{V}$ be a strict braided monoidal category and X be a class. Then an $\mathcal{A}(\mathcal{V})$ -category C with underlying class X is an object in $\mathcal{V}(X)$ which is a $\mathcal{V}$ -category such that exists $\mathcal{V} - X$ -functors $m : A \otimes A \rightarrow A$ and $\mu : I \rightarrow A$.*

The proof of such proposition is analogous to the Proposition 4.2.4, as we may see:

Proof. Since every object in $\mathcal{V}$ can be viewed as an algebra, for each object $A \in \mathcal{V}(X)$, we can consider the morphism $m_{x,y} = m_{A_{x,y}} : A_{x,y} \otimes A_{x,y} \rightarrow A_{x,y}$ as the family of morphisms which defines the morphism $m : A \otimes A \rightarrow A$. Analogously, we can define $\mu_{x,y} = \mu_{A_{x,y}} : I \rightarrow A_{x,y}$. Then, consider the following four diagrams:

$$\begin{array}{ccc}
 A_{x,y} \otimes A_{y,z} & \xleftarrow{\Delta_{x,y,z}} & A_{x,z} \\
 \uparrow m_{x,y} \otimes m_{y,z} & & \uparrow m_{x,z} \\
 A_{x,y} \otimes A_{x,y} \otimes A_{y,z} \otimes A_{y,z} & \xleftarrow{\Delta_{A \otimes A}} & A_{x,z} \otimes A_{x,z} \\
 \uparrow \mu_{x,y} \otimes \mu_{y,z} & & \uparrow \mu_{x,z} \\
 I \otimes I & \xleftarrow{=} & I
 \end{array}
 \qquad
 \begin{array}{ccc}
 I & \xleftarrow{\varepsilon_x} & A_{x,x} \\
 \swarrow \varepsilon_x \otimes \varepsilon_x & & \uparrow m_{x,x} \\
 A_{x,x} \otimes A_{x,x} & &
 \end{array}$$

Observe that m and is a dual $\mathcal{V} - X$ -functor if the first and second diagrams commute for all $x, y, z \in X$ and μ is a dual $\mathcal{V} - X$ -functor if the third and last diagrams commute for all $x, y, z \in X$. Also note that $\Delta_{x,y,z}$ is an algebra map if and only if the first and third diagrams commute, and μ is an algebra map if and only if the second and last diagrams commute. Hence, a dual $\mathcal{A}(\mathcal{V})$ -category A is the same as a dual $\mathcal{V}$ -category with $\mathcal{V} - X$ -functors $m : A \otimes A \rightarrow A$ and $\mu : I \rightarrow A$. $\square$

Proposition 4.3.3. *The categories ${}_{\mathcal{C}(\mathcal{V})}\text{HopfCat}$ and ${}_{\mathcal{A}(\mathcal{V}^{op})}\text{HopfCat}$ are 2-isomorphic.*

Proof. In fact, by Proposition 4.3.1, and by the fact that $\mathcal{A}(\mathcal{V}^{op})$ is isomorphic to $\mathcal{C}(\mathcal{V})$, we have that there exists an isomorphism $F : \mathcal{A}(\mathcal{V}^{op}) \rightarrow \mathcal{C}(\mathcal{V})$ and then it induces an 2-isomorphism:

$$\overline{F} : {}_{\mathcal{A}(\mathcal{V}^{op})}\text{HopfCat} \rightarrow {}_{\mathcal{C}(\mathcal{V})}\text{HopfCat}$$

By Proposition 4.3.1, ${}_{\mathcal{A}(\mathcal{V}^{op})}\text{HopfCat}$ is 2-isomorphic to ${}_{\mathcal{A}(\mathcal{V}^{op})}\text{HopfCat}$. Then, we have that ${}_{\mathcal{C}(\mathcal{V})}\text{HopfCat}$ and ${}_{\mathcal{A}(\mathcal{V}^{op})}\text{HopfCat}$ are 2-isomorphic. $\square$

5 DUALITY FOR HOPF $\mathcal{M}_k^f$ -CATEGORIES

In this chapter, we shall discuss Hopf categories and dual Hopf categories enriched over a category of finitely generated projective modules. First, we shall develop the theory we need for defining projective modules, and then we apply the existence of dual bases for projective modules to the theory of Hopf categories. We remark that we will consider only finitely generated projective modules over a *commutative* ring k , since in this case we obtain a monoidal category with respect to the tensor product of modules. For a noncommutative algebra A one would need to consider A -bimodules in order to work with tensor products. The main references for this chapter are (BATISTA; CAENEPEEL; VERCRUYSSSE, 2016) and (ASSEM, 1997).

5.1 PRELIMINARY DEFINITIONS AND PROPOSITIONS

In this section, we shall discuss some concepts and results regarding projective modules over a commutative ring which will be needed in the sequel. The main reference for this section is (ASSEM, 1997).

Definition 5.1.1. *Let k be a commutative ring and M be a nonempty set in a tuple $(M, +, \cdot)$ with $+: M \times M \rightarrow M$ and $\cdot: k \times M \rightarrow M$ satisfying the following properties:*

1. $(M, +)$ is an abelian group.
2. $(ab) \cdot x = a \cdot (bx)$ for all $a, b \in k$ and $x \in M$.
3. $1 \cdot x = x$ for all $x \in M$.
4. $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in k$ and $x \in M$.
5. $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $a \in k$ and $x, y \in M$.

is called a **k -module**.

Here are some examples of modules:

Example 5.1.1. If $\mathbb{K}$ is a field then every $\mathbb{K}$ -vector space is a $\mathbb{K}$ -module.

Example 5.1.2. Let k be a commutative ring. Every ideal I of k is a k -module, and the quotient ring k/I is also a k -module with

$$a \cdot (x + I) = ax + I.$$

In particular, for every $m \geq 1$ the ring $\mathbb{Z}_m = \frac{\mathbb{Z}}{m\mathbb{Z}}$ is a $\mathbb{Z}$ -module.

Example 5.1.3. If n divides m , we can consider $\mathbb{Z}_n$ as a $\mathbb{Z}_m$ -module, with $(k + m\mathbb{Z})(l + n\mathbb{Z}) = kl + n\mathbb{Z}$.

Example 5.1.4. Let G be an additive abelian group. Then G has a canonical structure of $\mathbb{Z}$ -module given by “exponentiation” written additively, that is: given $g \in G$, we define $0 \cdot g = 0, 1 \cdot g = g$ and, recursively, $(n + 1) \cdot g = n \cdot g + g$ for $n \geq 1$; for negative integers, if $n > 0$ then $(-n) \cdot g = n(-g)$. It can be shown that this is a $\mathbb{Z}$ -module structure on G .

Definition 5.1.2. Let $M_1, M_2, \dots, M_n$ be k -modules. Then their direct sum is the cartesian product $M_1 \times \dots \times M_n$ with the following k -module structure:

$$\begin{aligned} (m_1, \dots, m_n) + (m'_1, \dots, m'_n) &= (m_1 + m'_1, \dots, m_n + m'_n), \\ a \cdot (m_1, \dots, m_n) &= (a \cdot m_1, \dots, a \cdot m_n), \end{aligned}$$

for $a \in k$ and $m_i \in M_i, i = 1, 2, \dots, n$.

Definition 5.1.3. A k -**submodule** of a k -module M is a subgroup N of M that is closed for multiplication by scalars: if $a \in k$ and $m \in N$ then $a \cdot m \in N$.

It is easy to see that a k -submodule is a k -module.

A **generating set** of a k -module M is a subset $X \subset M$ such that, for every $m \in M$, there exist $m_1, \dots, m_n \in X$ and $a_1, \dots, a_n \in k$ such that $m = \sum_{j=1}^n a_j m_j$. We say that M is **finitely generated** if M has a finite generating set.

A subset X of a k -module M is **linearly independent** (L.I.) if it is not empty and for every $m_1, \dots, m_n \in X$ and $a_1, \dots, a_n \in k$, if $\sum_{j=1}^n a_j m_j = 0$ then $a_1 = \dots = a_n = 0$.

A subset B of a k -module M is a **basis** of M if it is an L.I. generating set for M . Equivalently, every element $m \in M$ has a unique expression $m = \sum_{j=1}^n a_j m_j$ with $m_1, \dots, m_n \in B$. A k -module which has a basis is called a **free module**.

Contrary to what happens in vector spaces, a k -module may not have a basis.

Example 5.1.5. $\mathbb{Z}_n$ is a finitely-generated $\mathbb{Z}$ -module – for instance, $X = \{\bar{1}\}$ is a generating set. However, $\mathbb{Z}_n$ has no basis over $\mathbb{Z}$ because it has no L.I. subset, since for every element $\bar{a}$ of $\mathbb{Z}_n$ one has that $n \cdot \bar{a} = \bar{0}$.

Then, we may define morphisms of k -modules, as follows:

Definition 5.1.4. Let k be a commutative ring, and M, N be k -modules. Then, $f : M \rightarrow N$ is a **morphism of k -modules** if for all $x, y \in M$ and $a, b \in k$:

$$f(a \cdot x + b \cdot y) = a \cdot f(x) + b \cdot f(y).$$

A morphism of k -modules $f : M \rightarrow N$ is an **isomorphism** if there exists another morphism of k -modules $g : N \rightarrow M$ such that $f \circ g = Id_N$ and $g \circ f = Id_M$.

It can be shown that a morphism of k -modules is an isomorphism if and only if it is a bijection. Let us give some examples of morphisms of k -modules.

Example 5.1.6. Every linear mapping between $\mathbb{K}$ -vector spaces is a morphism of $\mathbb{K}$ -modules.

Example 5.1.7. Every group morphism between abelian groups is a morphism of $\mathbb{Z}$ -modules.

Example 5.1.8. Suppose n divides m . Then $f : \mathbb{Z}_m \rightarrow \mathbb{Z}_n$ defined as $f(\bar{k}) = \bar{k}$ is a morphism of $\mathbb{Z}_m$ -modules.

Observe that, if $f : M \rightarrow N$ is a morphism of k -modules, then the kernel of f (as a morphism of groups) is also a k -submodule of M , since if $m \in M$ and $a \in k$ then

$$f(a \cdot m) = a \cdot f(m) = 0.$$

The same happens to its image: the image of f is an additive subgroup of N , and

$$a \cdot f(m) = f(a \cdot m) \in \text{Im}(f).$$

Definition 5.1.5. Let k be a commutative ring. Then k -modules and morphisms of k -modules form the category of k -modules, which will be denoted by $\mathcal{M}_k$.

Now, we may define projective modules.

Definition 5.1.6. A k -module P is said to be **projective** if there exists a set $\{m_i \in P, i \in I\}$ and a set $\{f_i \in \text{Hom}(P, k), i \in I\}$ (called **dual basis**) such that for every $x \in P$, $f_i(x)$ is only nonzero for finitely many i , and $x = \sum f_i(x)m_i$.

Note that, in the above definition, if we assume that $\{m_i \in P, i \in I\}$ is L.I., then such set is a basis for such k -module M and this module is free.

Example 5.1.9. If $B = \{m_i; i \in I\}$ is a basis for a module M then we have a corresponding dual basis: we may define $f_i : M \rightarrow k$ by $f_i(\sum_{j \in J} a_j m_j) = a_i$ if $i \in J$ and 0 otherwise, for all finite subsets J of I . This function is well-defined because every $m \in M$ can be expressed as a linear combination of elements of B in a unique way. Hence every free k -module is a projective k -module.

Hence, the generating set for a projective module with a dual basis is a generalization of the concept of basis for a module.

However, note that the generating set for a projective module is not always L.I. Here follows an example of a projective module which is not a free module.

Example 5.1.10. Let m be a positive integer, let $n > 1$ and suppose that $m = dn$ with d and n coprime. Since n divides m , as we saw in Example 5.1.3, $\mathbb{Z}_n$ is a $\mathbb{Z}_m$ -module by $(r + m\mathbb{Z}) \cdot (k + n\mathbb{Z}) = rk + n\mathbb{Z}$. By Bézout's identity, there exist $a, b \in \mathbb{Z}$ such that

$$ad + bn = 1,$$

and hence $ad \equiv 1 \pmod{n}$. Now we have the map

$$f : \mathbb{Z}_n \rightarrow \mathbb{Z}_m, \quad f(k + n\mathbb{Z}) = kd + m\mathbb{Z}.$$

This map is well-defined because if $k + n\mathbb{Z} = l + n\mathbb{Z}$ then $m = nd \mid (k - l)d$, which implies that $kd + m\mathbb{Z} = ld + m\mathbb{Z}$. It is easy to see that it is a morphism of $\mathbb{Z}_m$ -modules. Finally, given $k + n\mathbb{Z}$, we have

$$k + n\mathbb{Z} = (ad)k + n\mathbb{Z} = (kd + m\mathbb{Z}) \cdot (a + n\mathbb{Z}) = f(k + n\mathbb{Z}) \cdot (a + n\mathbb{Z}),$$

and hence $\mathbb{Z}_n$ is generated by $\{a + n\mathbb{Z}\}$ as a $\mathbb{Z}_m$ -module, with dual basis $\{f : \mathbb{Z}_n \rightarrow \mathbb{Z}_m\}$. Clearly $\mathbb{Z}_n$ is not a free $\mathbb{Z}_m$ -module, since in this case it would have at least m elements, but it has n elements with $n < m$.

Hence, for every decomposition $m = nd$ with $n > 1, d > 1$, and n and d coprime, we have that $\mathbb{Z}_n$ is a projective $\mathbb{Z}_m$ -module which is not a free $\mathbb{Z}_m$ -module.

We remark that Definition 5.1.6 is an alternative definition for projective modules which is more convenient for this work. Usually, one says that a k -module P is projective if the functor $\text{Hom}(P, -) : \mathcal{M}_k \rightarrow \mathcal{M}_k$ is exact. Theorem 3.5 of (ROTMAN, 2009) shows the equivalence of these two definitions.

Then, we need a last definition: tensor product of k -modules. The definition is analogous to the definition of tensor $\mathbb{K}$ -vector spaces, as follows.

Definition 5.1.7. Let M, N be both k -modules. Then the **tensor module** of M, N is a k -module $M \otimes N$ with a bilinear map $i : M \times N \rightarrow M \otimes N$ such that for every k -module L and every bilinear map $\phi : M \times N \rightarrow L$, there is a unique linear function $\Phi : M \otimes N \rightarrow L$ such that $\Phi \circ i = \phi$.

As in the case of vector spaces, the tensor module $M \otimes N$ is unique modulo isomorphisms. It may be constructed as the quotient module of the free k -module

F with basis $M \times N$ by the smallest submodule $J \subset F$ such that the inclusion map $j : M \times N \rightarrow F$ induces a bilinear map $i : M \times N \rightarrow F/J$.

The equivalence class of (m, n) in $M \otimes N = F/J$ is denoted by $m \otimes n$. By construction, every element in $M \otimes N$ has the form

$$\sum_i m_i \otimes n_i$$

and the following equalities hold for $m, m_1, m_2 \in M, n, n_1, n_2 \in N$ and $a \in k$:

- $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$.
- $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$.
- $am \otimes n = m \otimes an = a(m \otimes n)$

If a projective k -module has a finite generating set then we say it is a **finitely generated projective k -module**. It can be shown that there is a corresponding dual basis for this generating set.

Definition 5.1.8. *The category $\mathcal{M}_k^f$ is the category of finitely generated projective k -modules. Its objects are finitely projective modules, the morphisms are k -module morphisms.*

Following the same computations which are done for vector spaces, it can be shown that the category of k -modules has a monoidal structure given by the tensor product.

Proposition 5.1.1. *The category of finitely generated projective modules has a monoidal structure $(\mathcal{M}_k^f, \otimes, k)$.*

Proof. We just have to prove that the tensor product of two finitely generated projective modules is also a finitely generated projective module.

If P, Q are projective modules, where P has a generating set $\{p_i; i \in I\}$ with dual basis $\{f_i \in \text{Hom}(P, k); i \in I\}$ and Q has a generating set $\{q_j; j \in J\}$ with dual basis $\{g_j \in \text{Hom}(Q, k); j \in J\}$, then $P \otimes Q$ has a generating set $\{p_i \otimes q_j; (i, j) \in I \times J\}$, with dual basis $\{(f_i \otimes g_j) \in \text{Hom}(P \otimes Q, k); (i, j) \in I \times J\}$, where $(f_i \otimes g_j)(u \otimes v) = f_i(u)g_j(v)$. In fact, given a generator $u \otimes v \in P \otimes Q$, there exist finite subsets $I' \subset I$ and $J' \subset J$ such that $u = \sum_{i \in I'} f_i(u)p_i$, $v = \sum_{j \in J'} g_j(v)q_j$, and then $u \otimes v = \sum_{(i,j) \in I' \times J'} (f_i \otimes g_j)(u \otimes v) p_i \otimes q_j$.

In particular, if both P and Q are finitely generated projective modules then $P \otimes Q$ also is.

□

5.2 DUAL HOPF $\mathcal{M}_k^f$ -CATEGORIES ARE HOPF $\mathcal{M}_k^f$ -CATEGORIES

Let $\mathcal{V} = (\mathcal{M}_k^f, \otimes, k)$ be the category of finitely generated projective modules over a commutative ring k , and let $\mathcal{V}^{op} = (\mathcal{M}_k^{fop}, \otimes^{op}, k)$ be its opposite monoidal category. We claim that the functor

$$(-)^* : \mathcal{M}_k^f \rightarrow \mathcal{M}_k^{fop}$$

that takes a module M into its dual $M^* = \text{Hom}(M, k)$ is an equivalence of categories.

The proof of this statement is analogous of the proof that the category of vector spaces with finite basis is isomorphic to its dual. In fact, take P a finitely generated projective module. Let $\{v_i; i = 1, \dots, n\}$ be a set of generators for P with dual basis $\{f_i \in \text{Hom}(P, k); i = 1, \dots, n\}$. Then, for each f_i , we may consider $\phi_i : P^* \rightarrow k$ defined as $\phi_i(g) = g(v_i)$. Since $\{f_i\}$ is a dual basis for P , for each $v \in P$ we have that $v = \sum_i f_i(v)v_i$.

If we apply g to this equality, we get:

$$\begin{aligned} g(v) &= \sum_i f_i(v)g(v_i) \\ &= \sum_i f_i(v)\phi_i(g) \\ &= \left(\sum_i \phi_i(g)f_i \right) (v) \end{aligned}$$

Hence, $g = \sum_i \phi_i(g)f_i$ and it follows that $\{f_i; i = 1, \dots, n\}$ is a generating set for P^* , with dual basis $\{\phi_i; i = 1, \dots, n\}$. This functor is essentially surjective on objects. In fact, we shall prove that $P \simeq P^{**} = (P^*)^*$ for every finitely generated projective module P . An isomorphism is given as

$$\phi^P : P \rightarrow P^{**}, \quad \phi^P(v)(g) = g(v),$$

for every $g \in P^*$ and $v \in P$. Let us simplify the notation and write $\phi_v = \phi^P(v)$. Observe that ϕ^P is surjective by the computations above: given $g \in P^*$ we have that

$$g = \sum_i \phi_{v_i}(g)f_i,$$

and therefore, given $\psi \in P^{**}$,

$$\psi(g) = \psi \left(\sum_i \phi_{v_i}(g)f_i \right) = \sum_i \phi_{v_i}(g)\psi(f_i) = \left(\sum_i \psi(f_i)\phi_{v_i} \right) (g),$$

therefore $\psi = \phi^P(\sum_i \psi(f_i)v_i)$. The mapping ϕ^P is also injective because if $\phi_v(g) = 0$ for all $v \in P$ then $g(v) = 0$, and hence $g = 0$.

Now we shall prove that $(-)^*$ is a strong monoidal functor:

$$((-)^*, \phi_0, \phi_2) : (\mathcal{M}_k^f, \otimes, k) \rightarrow (\mathcal{M}_k^{fop}, \otimes^{op}, k).$$

Let $\phi_0 : k \rightarrow k^*$ be the canonical isomorphism which is the inverse of $\psi : k^* \rightarrow k, \psi(f) = f(1)$. There is a natural isomorphism:

$$\phi_2 : \otimes^{op} \circ ((-)^*, (-)^*) \rightarrow (-)^* \circ \otimes$$

Let us define this isomorphism. For two finitely projective generated modules M and N , we need an isomorphism:

$$\phi_2(M, N) : M^* \otimes^{op} N^* \rightarrow (M \otimes N)^*$$

in $\mathcal{M}_k^{fop}$ or an isomorphism:

$$\phi_2(M, N) : (M \otimes N)^* \rightarrow N^* \otimes M^*$$

in $\mathcal{M}_k^f$. Let us then consider the mapping:

$$\begin{aligned} \iota : N^* \otimes M^* &\rightarrow (M \otimes N)^*, \\ \langle \iota(n^* \otimes m^*), m \otimes n \rangle &= \langle n^*, n \rangle \langle m^*, m \rangle. \end{aligned}$$

Observe that, since N and M are both finitely generated projective modules, we have that ι is invertible. The inverse of ι is:

$$\iota^{-1}(\mu) = \sum_{i,j} \langle \mu, m_i \otimes n_j \rangle n_j^* \otimes m_i^*.$$

where $\{m_i^*\}$ and $\{n_i^*\}$ are the dual basis for generating sets $\{m_i\}$ and $\{n_i\}$ of M and N respectively.

In fact, observe that:

$$\begin{aligned} \langle \iota(\iota^{-1}(\mu)), m \otimes n \rangle &= \left\langle \iota \left(\sum_{i,j} \langle \mu, m_i \otimes n_j \rangle n_j^* \otimes m_i^* \right), m \otimes n \right\rangle \\ &= \sum_{i,j} \langle \mu, m_i \otimes n_j \rangle \langle n_j^* \otimes m_i^*, m \otimes n \rangle \\ &= \sum_{i,j} \langle \mu, m_i \otimes n_j \rangle \langle n_j^*, n \rangle \langle m_i^*, m \rangle \\ &= \langle \mu, \sum_i \langle m_i^*, m \rangle m_i \otimes \sum_j \langle n_j^*, n \rangle n_j \rangle \\ &= \langle \mu, m \otimes n \rangle. \end{aligned}$$

The equalities follow by the bilinearity of $\langle -, - \rangle$ and the definitions of the maps ι, ι^{-1} and the defining property of the dual bases of M and N .

Conversely, given $m^* \in M^*, n^* \in N^*$:

$$\begin{aligned}
 \iota^{-1}\iota(n^* \otimes m^*) &= \sum_{i,j} \langle \iota(n^* \otimes m^*), m_i \otimes n_j \rangle n_j^* \otimes m_i^* \\
 &= \sum_{i,j} \langle m^*, m_i \rangle \langle n^*, n_j \rangle m_i^* \otimes n_j^* \\
 &= \sum_i \sum_j \langle m^*, m_i \rangle \langle n^*, n_j \rangle m_i^* \otimes n_j^* \\
 &= \sum_i \langle m^*, m_i \rangle m_i^* \otimes \sum_j \langle n^*, n_j \rangle n_j^* \\
 &= m^* \otimes n^*.
 \end{aligned}$$

Again, the equalities follow by the bilinearity of $\langle -, - \rangle$, and the property of dual basis. So, we may define $\phi_2(M, N)$ as ι^{-1} . As we can see, $((-)^*, \phi_0, \phi_2)$ is a strong monoidal functor, and it follows from Proposition 4.3.1 that $\mathcal{M}_k^{fop}Cat$ and $\mathcal{M}_k^fCat$ are 2-isomorphic. Combining these two biequivalences, we have the result:

Theorem 5.2.1. *Let k be a commutative ring. The functor $(-)^* : \mathcal{M}_k^f \rightarrow \mathcal{M}_k^{fop}$ induces a biequivalence:*

$$\mathcal{M}_k^fCat \rightarrow \mathcal{M}_k^{fop}Cat.$$

Proof. Let us describe this biequivalence in 0-cells. First of all, we apply the duality functor $(-)^* : \mathcal{M}_k^fCat \rightarrow \mathcal{M}_k^{fop}Cat$ (such functor is the induced functor by $(-)^* : \mathcal{M}_k^f \rightarrow \mathcal{M}_k^{fop}$, as in Proposition 4.2.2). In 0-cells, $(-)^*(A_{x,y}) = (A^*)_{x,y} = A_{x,y}^*$. Now, we shall compute the product and the unit map of A^* . For that, we need to follow the proof of the Proposition 4.2.2. The product is:

$$(m_{x,y,z})^* \circ \phi_2(A_{x,y}, A_{y,z}) : A_{y,z}^* \otimes A_{x,y}^* \rightarrow (A_{x,y} \otimes A_{y,z})^* \rightarrow A_{x,z}^*,$$

and the unit map is $\mu_x^* : k \rightarrow A_{x,x}^*$ in $\mathcal{M}_k^{fop}$. Note that in this case $(m_{x,y,z})^*$ is defined as the product in $\mathcal{M}_k^{fop}$. Then, for A^* , we apply the construction in the proof of the previous proposition which takes A to the dual category C as $C_{x,y} = A_{y,x}^*$. The coproduct is:

$$\Delta_{z,y,x} : \phi_2(A_{x,y}, A_{y,z}) \circ m_{x,y,z}^* : C_{z,x} \rightarrow C_{z,y} \otimes C_{y,x},$$

and the counit map is:

$$\varepsilon_x : \mu_x^* : C_{x,x} \rightarrow k.$$

Note that $m_{x,y,z}^*$ is the dual of the product in $\mathcal{M}_k^f$, and not the product in $\mathcal{M}_k^{fop}$. Now, let us describe the construction of the inverse on 0-cells. Given $C \in \mathcal{M}_k^fCat$, let us introduce the notation:

$$\Delta_{x,y,z}(c) = c_{(1,y)} \otimes c_{(2,y)} \in C_{x,y} \otimes C_{y,z},$$

where summation is implied. Let $A \in \mathcal{M}_k^{fop} Cat$ be defined as $A_{x,y} = C_{y,x}^*$. The product $(m_{x,y,z})^* : A_{x,y} \otimes A_{y,z} \rightarrow A_{x,z}$ is defined as:

$$\langle fg, c \rangle = \langle f, c_{(2,y)} \rangle \langle g, c_{(1,y)} \rangle$$

for $f \in A_{x,y}$, $g \in A_{y,z}$ and $c \in C_{z,x}$. The elements of the unit are $\varepsilon_x = A_{x,x}$

Note, then, that is how we define the product in $\mathcal{M}^{fop}$. Let us show that the product is associative. Take $f \in A_{x,y}$, $g \in A_{y,z}$, $h \in A_{z,w}$ and $c \in C_{x,w}$. Then:

$$\begin{aligned} \langle (fg)h, c \rangle &= \langle fg, c_{(2,y)} \rangle \langle h, c_{(1,z)} \rangle \\ &= \langle f, c_{(3,y)} \rangle \langle g, c_{(2,z)} \rangle \langle ah, c_{(1,w)} \rangle \\ &= \langle f, c_{(2,y)} \rangle \langle gh, c_{(1,w)} \rangle \\ &= \langle f(gh), c \rangle. \end{aligned}$$

Then, the theorem is proved. □

By the proof of Proposition 4.2.9, we know that a strong monoidal functor $F : \mathcal{V} \rightarrow \mathcal{W}$ induces an equivalence of categories $\overline{F} : \mathcal{C}(\mathcal{V}) \rightarrow \mathcal{C}(\mathcal{W})$. Hence, the functor $(-)^*$ induces an equivalence of categories $(-)^* : \mathcal{C}(\mathcal{M}_k^f) \rightarrow \mathcal{C}(\mathcal{M}_k^{fop})$. Observing that the categories $\mathcal{C}(\mathcal{M}_k^{fop})$ and $\mathcal{A}(\mathcal{M}_k^{fop})$ are isomorphic, we obtain an equivalence of categories:

$$(-)^* : \mathcal{C}(\mathcal{M}_k^f) \rightarrow \mathcal{A}(\mathcal{M}_k^{fop}).$$

Let us compute the algebra structure in the dual C^* of a coalgebra C . The structure of coalgebra in $\mathcal{M}_k^{fop}$ is the composition:

$$\phi_2(C, C)^{-1} \circ \Delta^* : C^* \rightarrow (C \otimes C)^* \rightarrow C^* \otimes C^*.$$

And m is the composition:

$$\begin{aligned} m &= \Delta^* \circ \iota : C^* \otimes C^* \rightarrow (C \otimes C)^* \rightarrow C^* \\ (c^* d^*, a) &= \langle c^*, a_{(2)} \rangle \langle d^*, c_{(1)} \rangle. \end{aligned}$$

Then, we claim that there exists a strong monoidal equivalence:

$$((-)^*, \phi_0, \phi_2) : (\mathcal{C}(\mathcal{M}_k^f), \otimes, k) \rightarrow (\mathcal{A}((\mathcal{M}_k^f)^{op}), \otimes^{op}, k).$$

In fact, ϕ_0 is the identity of k , and

$$\phi_2(C, D) : D^* \otimes C^* \rightarrow (C \otimes D)^*$$

in $\mathcal{A}(\mathcal{M}_k^f)^{op}$ is the inverse of the transformation $\iota : (C \times D)^* \rightarrow C^* \otimes D^*$ defined previously. It follows from Proposition 4.2.9 that $(-)^*$ induces a biequivalence

$$(-)^* : {}_{\mathcal{C}(\mathcal{M}_k^f)}Cat \rightarrow {}_{\mathcal{A}(\mathcal{M}_k^f)^{op}}Cat.$$

We know by Proposition 4.3.1 that ${}_{\mathcal{A}(\mathcal{M}_k^f)^{op}}Cat$ is 2-isomorphic to ${}_{\mathcal{A}(\mathcal{M}_k^f)}Cat$. Hence, it follows that:

Theorem 5.2.2. *Let k be a commutative ring. We have a biequivalence:*

$${}_{\mathcal{C}(\mathcal{M}_k^f)}Cat \rightarrow {}_{\mathcal{A}(\mathcal{M}_k^f)}Cat$$

Let us provide some details on how this biequivalence is constructed at the level of 0-cells. Let A be 0-cell in ${}_{\mathcal{C}(\mathcal{M}_k^f)}Cat$, that is, a $\mathcal{C}(\mathcal{M}_k^f)$ -category. We apply the duality functor $(-)^*$, sending A to A^* , with $(A^*)_{x,y} = A_{x,y}^*$. Then, we take the construction in Proposition 4.3.1, which sends A^* to C , with $C_{x,y} = A_{y,x}^*$. From Theorem 5.2.1, we know the dual k -linear structure in $\mathcal{C}$: each $C_{x,y}$ is a k -coalgebra, with opposite convolution as product, and $1_{x,y} = \varepsilon_{y,x}$ as unit element. Let us give what is the inverse construction. Let (C, Δ, ε) be a dual $\mathcal{A}(\mathcal{M}_k^f)$ -category. Each $A_{x,y} = C_{y,x}^*$ is a coalgebra with coproduct:

$$\Delta(a) = \sum_{i,j} \langle a, c_i c_j \rangle c_j^* \otimes c_i^*,$$

where $\{c_i\}$ is a generating set for $C_{y,x}$ with dual basis $\{c_i^*\}$ and $c_i c_j$ is the product in $C_{y,x}$.

Then, let us remember the last definition of the previous section: Hopf Dual $\mathcal{V}$ -Categories. Recall from Definition 4.3.5 that a dual $\mathcal{A}(\mathcal{V})$ -category C is a dual Hopf $\mathcal{V}$ -category if there exist morphisms $S_{x,y} : C_{y,x} \rightarrow C_{x,y}$ in $\mathcal{V}$ such that:

$$m_{x,y} \circ (C_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y,x} = \mu_{x,y} \circ \varepsilon_x$$

$$m_{y,x} \circ (S_{y,x} \circ C_{y,x}) \circ \Delta_{x,y,x} = \mu_{y,x} \circ \varepsilon_x.$$

and S is called the antipode of the dual Hopf category. We present now the last theorem of this dissertation, which extends the analogous result proved for Hopf $\mathbb{K}$ -algebras (Proposition 2.5.11).

Theorem 5.2.3. *Let k be a commutative ring. In the biequivalence from the previous theorem, Hopf $\mathcal{M}_k^f$ -categories correspond to dual Hopf $\mathcal{M}_k^f$ -categories.*

Proof. Assume that C is a dual Hopf $\mathcal{M}_k^f$ -category with antipode S , and let A be the corresponding Hopf $\mathcal{M}_k^f$ -category. Let us define:

$$T_{x,y} = S_{y,x}^* : A_{x,y} \rightarrow A_{y,x}.$$

Then, we claim that family of morphisms $T = \{T_{x,y}\}$ is an antipode for A . We need to prove for $a \in A$ that:

$$a_{(1)}T_{x,y}(a_{(2)}) = \varepsilon_{x,y}(h)1_x$$

$$T_{x,y}(a_{(1)})a_{(2)} = \varepsilon_{x,y}(h)1_y.$$

Observe that the two formulas are reduced to:

$$a_{(1)}T_{x,y}(a_{(2)}) = \langle a, 1_{y,x} \rangle \varepsilon_x$$

$$T_{y,x}(a_{(1)})a_{(2)} = \langle a, 1_{x,y} \rangle \varepsilon_y.$$

For all $c \in C$, we have that:

$$\begin{aligned} \langle a_{(1)}T_{x,y}(a_{(2)}), c \rangle &= \langle a_{(1)}, c_{(2,y)} \rangle \langle T_{x,y}(a_{(2)}), c_{(1,y)} \rangle \\ &= \langle a_{(1)}, c_{(2,y)} \rangle \langle a_{(2)}, S_{y,x}(c_{(1,y)}) \rangle \\ &= \langle a, S_{y,x}(c_{(1,y)})c_{(2,y)} \rangle \\ &= \langle a, 1_{y,x} \rangle \langle \varepsilon_x, c \rangle. \end{aligned}$$

The first equality follows from the definition of product in A , while the second equality follows from the definition of T , the third equality follows from the definition of coproduct in C , and the last equality follows from the definition of S .

Also, we have that:

$$\begin{aligned} \langle T_{y,x}(a_{(1)})a_{(2)}, c \rangle &= \langle T_{y,x}(a_{(1)}), c_{(1)} \rangle \langle a_{(2)}, c_{(2)} \rangle \\ &= \langle a_{(1)}, S_{x,y}(c_{(1)}) \rangle \langle a_{(2)}, c_{(2)} \rangle \\ &= \langle a, c_{(2)}S_{x,y}(c_{(1)}) \rangle \\ &= \langle a, 1_{x,y} \rangle \langle \varepsilon_x, c \rangle. \end{aligned}$$

The first equality follows from the definition of product in A , while the second equality follows from the definition of T , the third equality follows from the definition of the coproduct in C , and the last equality follows from the definition of S . $\square$

Note that these results are not necessarily valid for non-commutative rings. In non-commutative rings, we do not have the commutativity of the unit diagram.

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