UNIVERSIDADE FEDERAL DO PARANÁ

POLYANA BENK

# QUIVER REPRESENTATIONS AND REFLECTION FUNCTORS

CURITIBA

2023

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# QUIVER REPRESENTATIONS AND REFLECTION FUNCTORS

Dissertação apresentada como requisito parcial à obtenção do grau de Mestre em Matemática, no Curso de Pós-Graduação em Matemática, Setor de Ciências Exatas, da Universidade Federal do Paraná.

Orientador: Prof. Dr. Matheus Batagini Brito.
Coorientador: Prof. Dr. Edson Ribeiro Alvares.

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*Aos meus pais.*

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*"Algebra is generous; she often gives more than is asked of her."*

*(Jean Le Rond d'Alembert)*

# RESUMO

Esta dissertação tem como foco a teoria de representação de quivers. Essa teoria é um campo fascinante de estudo que tem conexões profundas com outras áreas da matemática, incluindo geometria algébrica, teoria de Lie e topologia. Também tem aplicações importantes em física, ciência da computação e outras áreas da ciência e engenharia. Uma ferramenta importante que será usada nesta dissertação são os sistemas de raízes, que são coleções de vetores em um espaço Euclidiano que codificam as simetrias de certas estruturas algébricas, como álgebras de Lie e grupos de Lie. No contexto da teoria da representação de quivers, os sistemas de raízes desempenham um papel importante na classificação de representações, fornecendo uma ferramenta poderosa para entender a estrutura desses objetos matemáticos. Começamos examinando os conceitos fundamentais da teoria da representação de quivers, incluindo álgebras de caminho, representações simples e indecomponíveis, o grupo de Grothendieck, módulos projetivos e a forma de Euler. Em seguida, estabelecemos várias propriedades chave de órbitas e funtores de reflexão, que são as duas ferramentas necessárias para provar o teorema de Gabriel. Este teorema nos dá uma bijeção entre representações indecomponíveis e raízes positivas do sistema de raízes. No entanto, este teorema só é válido para um caso especial de quivers, os chamados quivers de Dynkin. Para estender o escopo do teorema de Gabriel além dos quivers de Dynkin, investigaremos as representações preprojetivas e preinjetivas, bem como o quiver de Auslander-Reiten, que fornece informações sobre a estrutura de uma variedade maior de quivers.



**Palavras-chave:** Quivers; Sistema de raízes; Teoria de representação; Teorema de Gabriel.

# ABSTRACT

This dissertation focuses on the theory of quiver representation. Quiver representation theory is a fascinating field of study that has deep connections to other areas of mathematics, including algebraic geometry, Lie theory, and topology. It also has important applications in physics, computer science, and other areas of science and engineering. Root systems will be an important tool used in this dissertation, which are collections of vectors in a Euclidean space that encode the symmetries of certain algebraic structures, such as Lie algebras and Lie groups. In the context of quiver representation theory, root systems play an important role in the classification of representations, providing a powerful tool for understanding the structure of these mathematical objects. We begin by examining the fundamental concepts of quiver representation theory, including path algebras, simple and indecomposable representations, the Grothendieck group, projective modules, and the Euler form. Next, we establish several key properties of orbits and reflection functors, which are the two tools needed to prove Gabriel's theorem. This theorem gives us a bijection between indecomposable representations and positive roots of the root systems. However, this theorem only holds for a special case of quivers, the so-called Dynkin quivers. To extend the scope of Gabriel's theorem beyond Dynkin quivers, we will investigate preprojective and preinjective representations, as well as the quiver of Auslander-Reiten, which provide new insights into the structure of a wider range of quivers.

**Keywords:** Quivers; Root system; Representation Theory; Gabriel's theorem.

# LIST OF SYMBOLS

| Symbol | Definition | Page |
|---|---|---|
| $\Delta$ | Basis of a root system | 139 |
| $\mathrm{Rep}(\vec{Q})$ | Category of finite dimensional representation of a quiver | 19 |
| $C$ | Coxeter element | 160 |
| $C^{\pm}$ | Coxeter functors | 79 |
| $\dim(V)$ | Dimension of a representation $V$ | 25 |
| $\Omega$ | Edges of a quiver | 15 |
| $\mathsf{E}$ | Euclidean space | 132 |
| $\mathbb{K}$ | Ground field | 17 |
| $K(\mathcal{C})$ | Grothendieck group of a category $\mathcal{C}$ | 24 |
| $(\ ,\ )$ | Inner product | 132 |
| $l(s)$ | Length of a reflection $s$ | 148 |
| $w_0$ | Longest element of the Weyl group | 159 |
| $\theta$ | Maximal root of the Weyl group | 150 |
| $\mathsf{R}_{-}$ | Negative root system | 140 |
| $\mathbb{O}_x$ | Orbit of $x$ | 168 |
| $s_\alpha$ | Orthogonal reflection of the root $\alpha$ | 132 |
| $\mathbb{K}\vec{Q}$ | Path algebra of a quiver | 21 |
| $\mathsf{R}_{+}$ | Positive root system | 140 |
| $P(i)$ | Projective representation for the vertex $i$ | 26 |
| $\vec{Q}$ | Quiver | 15 |
| $\ell$ | Rank of a root system | 135 |
| $\Phi_i^{\pm}$ | Reflection functors of $i$ | 37 |
| $V$ | Representation of a quiver | 17 |
| $R(\vec{Q}, v)$ | Representation space | 168 |
| $L$ | Root lattice | 145 |
| $\mathsf{R}$ | Root system | 134 |
| $\Delta_T$ | Set of vertex of the Auslander-Reiten quiver | 122 |
| $S(i)$ | Simple representation for the vertex $i$ | 22 |
| $T$ | Slice | 103 |
| $G_x$ | Stabilizer group | 170 |
| $I$ | Vertices of a quiver | 15 |
| $\mathscr{C}$ | Weyl chamber | 144 |
| $\mathscr{W}$ | Weyl group | 135 |

# CONTENTS

# INTRODUCTION

In the 1930s, Andreas Speiser introduced the concept of a graph [20], in the sense of a collection of vertices and edges, in his study of algebraic number theory. Graphs were later used by Claude Chevalley in his work on Lie algebras [8], which are a type of algebraic structure used to study continuous symmetries. Chevalley defined a root system associated to a Lie algebra, which is essentially a collection of vectors that encode its symmetries. The combinatorial structure of these root systems was shown by Dynkin to be closely related to certain types of graphs [9], which are now called Dynkin diagrams.

In the 1960s, the mathematicians Gabriel [10] and Auslander [3] independently introduced the notion of a quiver, which is a directed graph that allows loops and multiple edges. Gabriel was the one that gave the name quivers to the structure he was working with, this name was proposed because he felt like the word graph was overused in algebra. They showed that quivers are closely related to representations of certain types of algebras, known as path algebras. In particular, the vertices of a quiver can be identified with the simple modules of the associated path algebra, and the arrows can be thought of as describing how these modules are related to each other. This identification allows one to study the structure of the path algebra by analyzing the quiver, and vice versa.

The study of quivers and quiver representations continued to develop in the following decades, with important contributions from many mathematicians. For example, in 1975, Auslander, along with Reiten, published the paper [4] in which they introduced a theory to study almost split sequences. This theory was then used to define the quivers known today as Auslander-Reiten quivers. The AR quiver encodes information about the indecomposable modules over an Artin algebra (a type of finite-dimensional algebra).

Another important contribution to the theory of quivers and quiver representations was made by Bongartz in his 1981 paper. In the paper [7], Bongartz associated quadratic forms to each basic algebra whose Gabriel quiver has no oriented cycle. He uses the works of Gabriel, Auslander, Reiten, Bernstein, Gelfand and Ponomarev as basis for his paper.

We have several applications for quivers and quivers representations. There are connections with several fields of mathematics, for example, Hall algebras, quantum groups, elliptic Lie algebras, and cluster algebra [11]. These connections of quivers and other mathematical fields highlight the richness and diversity of mathematics.

The study of quivers and quiver representations has led to many important applications in algebra, geometry, and physics, and continues to be an active area of research today. There are numerous books and papers on the subject, including the influential book [2] by Assem, Simson, and Skowroński, as well as the paper [15] by Keller, which introduced the notion of an A-infinity quiver and its connection to topology. Other important papers in the field include [19] by Ringel and [14] by Kac.

The main goal of this dissertation is to use the theory of root systems, from Lie algebras, and the representation theory to study representations of quivers. To fulfill that goal we present in this paper the basic definitions of quivers and of quiver representation, defining indecomposable, projective and injective representations, also path algebra, Grothendieck group and the Euler form. We give a few properties of orbits in the appendix and study in depth the reflection functors.

We have two sets of quivers that play an important role here, the Dynkin quivers and the Euclidean quivers. In the case of Dynkin quivers it is possible to find every indecomposable representation. In a more general case it is not possible to get all the indecomposable representations this way, but we can find a very large class of indecomposable representations.

The most important result in this dissertation is Gabriel's theorem, given by Gabriel in [10]. This theorem states that for a Dynkin quiver $\vec{Q}$, the map

$$\begin{aligned} \mathrm{Ind}(\vec{Q}) &\to \mathsf{R}_+ \\ [V] &\mapsto \dim(V) \end{aligned}$$

gives a bijection between the set $\mathrm{Ind}(\vec{Q})$ of isomorphism classes of nonzero indecomposable representations of $\vec{Q}$ and the set $\mathsf{R}_+$ of positive roots from a root system $\mathsf{R}$. In order to proof this theorem we need two tools that we will study in this dissertation, the geometry of orbits and the reflection functors. The latter being very useful not only for Dynkin quivers but also in the case where the quivers are not Dynkin.

Reflection functors are also called the BGP functors because they were introduced by Bernstein, Gelfand and Ponomarev in the paper [6]. These functors are operations we can do in a given quiver, such operations preserves the structure of the quiver, but may change the vertices and arrows themselves. Besides being an important tool to proof Gabriel's theorem we will use these functors to construct the preprojective and preinjective representations, and we do so by applying reflection functors to projective and injective representations.

After proving Gabriel's theorem we do an ordering of the positive roots for the root system $\mathsf{R}$ and use Gabriel's theorem in order to find all the indecomposable representations of a Dynkin quiver. That is not possible for all kinds of quivers. However, by introducing the preprojective representations we can find a large class of indecomposable representations. For that we need to have a graph $Q$ with no edge loops and an orientation for $Q$ with no oriented cycles.

Using these definitions we can construct the Auslander-Reiten quivers, in which we use the preprojective representations. We define the quivers $Q \times \mathbb{Z}$ and $\mathbb{Z}Q \subset Q \times \mathbb{Z}$, then we define a slice $T$ of $\mathbb{Z}Q$ and for this slice we define the preprojective Auslander-Reiten quiver $\vec{Q}_T$ as the subquiver of $\mathbb{Z}Q$ with the set of vertices

$$\Delta_T = \{q \in \mathbb{Z}Q \mid I_q^T \neq 0\} \subset \mathbb{Z}Q.$$

Using the Auslander Reiten quiver we can prove that the map

$$\Delta_T \to \operatorname{Ind}(\vec{Q})$$
$$q \mapsto I_q^T$$

is a bijection between the set $\Delta_T$ and the set of isomorphism classes of nonzero preprojective indecomposable representations of $\vec{Q}_T$.

In the appendix we study the finite and infinite root systems for the readers' convenience. The finite root system we study from an axiomatic perspective, i.e., we define a root system $\mathsf{R}$ as a subset of a euclidean space $\mathsf{E}$ that satisfies the following four axioms,

(R1) The set $\mathsf{R}$ is finite, spans $\mathsf{E}$ and does not contain 0.

(R2) If $\alpha \in \mathsf{R}$, the only multiples of $\alpha$ in $\mathsf{R}$ are $\pm\alpha$.

(R3) If $\alpha \in \mathsf{R}$, the reflection $s_\alpha$ leaves $\mathsf{R}$ invariant.

(R4) If $\alpha, \beta \in \mathsf{R}$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

We study the basis of a root system and show that every root system has a basis. One important group for the root system study is the Weyl group, that consistes of the composition of reflections. The Weyl chambers of a given root system $\mathsf{R}$ are the connected components of $E - \cup P_\alpha$, where $P_\alpha = \{\beta \in \mathsf{E}; (\beta, \alpha) = 0\}$ is a hyperplane. Then we define the Cartan matrices of a root system $\mathsf{R}$ and Dynkin diagrams. We can show how to recover the root system $\mathsf{R}$ from the Cartan matrices and Dynkin diagrams. The infinite root systems are used in chapter five, and we give some properties of it without proof.

This dissertation is organized as follows. The first chapter is for the basic theory of quivers. The second chapter defines the reflection functors. The third chapter is dedicated to the demonstration of Gabriel's theorem and also the characterization of all the indecomposable representations for a Dynkin quiver. In the fourth chapter we bring a large class of indecomposable representations for quivers that are not Dynkin, they are called the preprojective representations. Lastly, in the appendix A we have all the theory of root systems that we used during the dissertation. In the appendix B we have the theory of orbits that will be used mostly in the demonstration of Gabriel's theorem.

# Chapter 1

# BASIC THEORY

Quivers are directed graphs that have become an important tool in mathematics for studying various algebraic structures. They arise in many areas, including algebraic geometry, representation theory, Lie theory, and homological algebra. In this chapter, we will introduce the basic definitions of quivers and explore their properties. We will define what a quiver is, representations of quivers, projectives and injectives modules, the Euler form, and Dynkin and Euclidean graphs. We will also discuss the concepts of paths, cycles, and subquivers in quivers. By the end of this chapter, the reader will have a solid foundation in the fundamental concepts of quiver theory. We follow [16] for the most part of this work.

## 1.1 Basic definitions

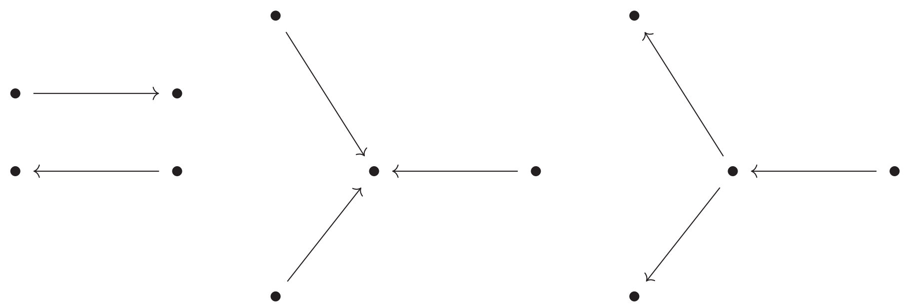
It is important to highlight that while quivers are relatively simple objects, they can present highly complex behavior. This is due to the fact that quivers have a rich algebraic structure associated with them, which allows for the study of many mathematical phenomena. For example, the representation theory of a quiver, which involves studying how to associate vector spaces to the vertices and linear operators to the edges, has deep connections to algebraic geometry and mathematical physics. Moreover, quivers can be used to define other important mathematical structures, such as cluster algebras and noncommutative algebraic geometry.

**Definition 1.1.1.** *A quiver $\vec{Q}$ is a directed graph. Formally, it can be described by a set of vertices $I$, a set of edges $\Omega$, and two maps $s, t : \Omega \to I$ which assign to every edge its source and target, respectively.*

For $i, j \in I$ we will use $h : i \to j$ for an edge $h$ with source $i$ and target $j$, which means that $s(h) = i$ and $t(h) = j$. Throughout this dissertation we will assume that the set of edges and vertices are finite and also unless stated otherwise we will assume that $\vec{Q}$ is connected.

We can forget the directions of edges in $\vec{Q}$, and then we get a graph $Q$. We can also think of $\vec{Q}$ as a graph $Q$ along with an orientation. This means that we choose for each edge of $Q$, which of the two endpoints is the source and which is the target. So we can write $\vec{Q} = (I, \Omega)$.

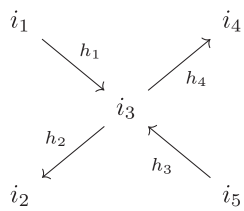
**Example 1.1.2.** *All the oriented graphs below are examples of quivers,*



**Example 1.1.3.** *If we want to describe a quiver more formally, we can say that* $\vec{Q}$ *is the quiver with the set of vertices* $I = \{i_1, i_2, i_3, i_4, i_5\}$*, the set* $\Omega = \{h_1, h_2, h_3, h_4\}$ *and the two maps*

$$\begin{aligned} s &: \Omega \to I & t &: \Omega \to I \\ s(h_1) &= i_1 & t(h_1) &= i_3 \\ s(h_2) &= i_3 & t(h_2) &= i_2 \\ s(h_3) &= i_5 & t(h_3) &= i_3 \\ s(h_4) &= i_3 & t(h_4) &= i_4. \end{aligned}$$

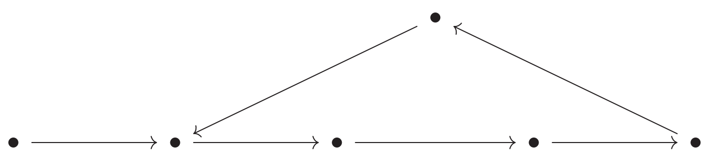
*Then this quiver has the following drawing*




**Definition 1.1.4.** *A quiver* $\vec{Q}$ *has an oriented cycle if there exists a subset of vertices* $\{i_1, \cdots, i_n\}$ *and a sequence of edges* $h_t : i_t \to i_{t+1}$ *with* $t = 1, \cdots, n$*, such that* $i_1 = i_n$*.*

It is worth noting that oriented cycles play a crucial role in the study of quivers and their associated algebraic structures. In particular, the presence or absence of oriented cycles in a quiver has important implications for the properties of its associated algebraic structure, such as its representation theory.

**Example 1.1.5.** *The quiver below has an oriented cycle,*

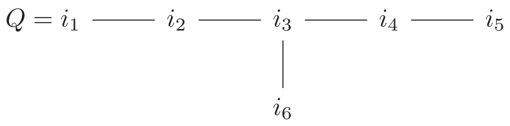


**Definition 1.1.6.** *A connected graph with* $n$ *vertices in* $I$ *is called a tree if it has* $n - 1$ *edges in* $\Omega$*.*

Trees are fundamental objects in mathematics that have numerous applications in different areas of study. In particular, trees arise in combinatorics, graph theory, algebraic geometry, and computer science, among others. The absence of cycles in trees gives them several nice properties, such as the existence of a unique path between any two vertices and the fact that they are minimally connected.

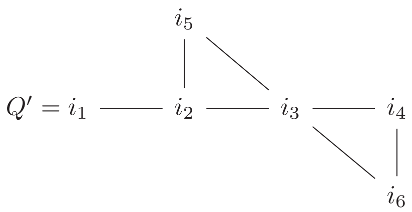
**Definition 1.1.7.** *The degree of a vertex in a tree is the number of vertices that are connected to it.*

**Example 1.1.8.** *The graph $Q$ given by*




*is a tree. The degree of vertices $i_1, i_5, i_6$ is $1$, the degree of vertex $i_3$ is $3$ and the degree of vertices $i_2, i_4$ is $2$.*

*The graph $Q'$ given by*




*is not a tree, since it has $6$ vertices and $7$ edges.*

Set the notation $\mathbb{K}$ for the ground field. Unless specified otherwise, all vectors spaces and linear operators are considered over the field $\mathbb{K}$.

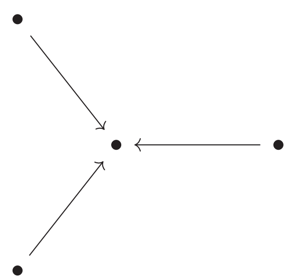
**Definition 1.1.9.** *A representation $V$ of a quiver $\vec{Q}$ is the following collection of data.*

*(1) For every vertex $i \in I$, a vector space $V_i$ over $\mathbb{K}$.*

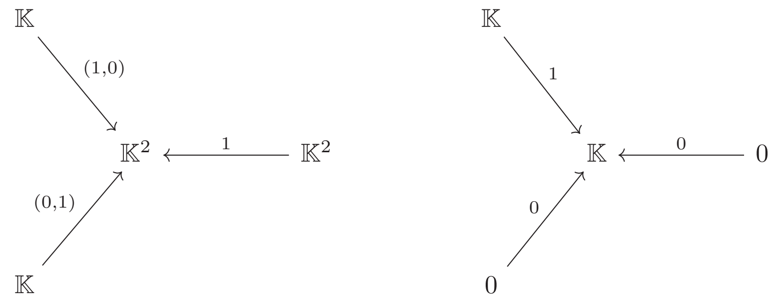
*(2) For every edge $h \in \Omega$ with $h : i \to j$, a linear operator $x_h : V_i \to V_j$.*

It is important to emphasize that this concept is at the heart of the theory of quivers and has deep connections to many other areas of mathematics. Representations of quivers provide a powerful tool for studying the algebraic structures associated with them, such as algebras and Lie algebras. Moreover, the representation theory of quivers has important applications in other fields, such as algebraic geometry and mathematical physics. By associating vector spaces to the vertices of a quiver and linear operators between these vector spaces to the edges, we can study the behavior of the algebraic structure in terms of the linear operators.

**Example 1.1.10.** *Given the quiver*



*we have the following examples of representations,*




*Where*

$$\left|\begin{array}{rr|rr|rr|rr} 1: & \mathbb{K}\to\mathbb{K} & 0: & 0\to\mathbb{K} & (1,0): & \mathbb{K}\to\mathbb{K}^2 & (0,1): & \mathbb{K}\to\mathbb{K}^2 \\ & x\mapsto x & & x\mapsto 0 & & x\mapsto (x,0) & & x\mapsto (0,x). \end{array}\right|$$

**Definition 1.1.11.** *Let $V, W$ be two representations of the quiver $\vec{Q}$. We define a morphism of representations $f : V \to W$ as a collection of linear operators $f_i : V_i \to W_i\ \forall\ i \in I$, which commute with the operators $x_h$, that is, if $h \in \Omega$ with $h : i \to j$ then $f_j x_h = x_h f_i$. Equivalently, the square below is commutative,*

$$\begin{array}{ccc} V_i & \xrightarrow{f_i} & W_i \\ {\scriptstyle x_h}\downarrow & & \downarrow{\scriptstyle x_h} \\ V_j & \xrightarrow[f_j]{} & W_j \end{array}$$

Morphisms $f : V \to W$ form a vector space with operations of sum and scalar multiplication. We will denote this vector space as $\mathrm{Hom}_{\vec{Q}}(V, W)$, or just $\mathrm{Hom}(V, W)$ when there is no ambiguity. We will use the notation $\mathrm{End}_{\vec{Q}}(V) = \mathrm{Hom}_{\vec{Q}}(V, V)$ for the algebra of endomorphisms of a representation $V$ and $\mathrm{Aut}_{\vec{Q}}(V) = \{f \in \mathrm{End}_{\vec{Q}}(V); f \text{ is invertible}\}$ for the group of automorphisms of $V$.

**Example 1.1.12.** *Consider the quiver $\vec{Q}$ given by*

$$\vec{Q} = 1 \xrightarrow{h_1} 2 \xleftarrow{h_2} 3.$$

*Let $V, W$ be two representations,*

$$V = \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{0} 0$$

$$W = \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K}.$$

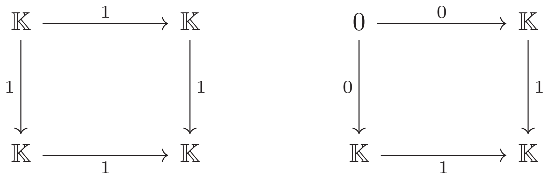
*Then we have the morphism $f : V \to W$ such that*

$$f_1 = 1, \quad f_2 = 1, \quad f_3 = 0.$$

*Indeed, this quiver has two edges, so we have the following two squares*

$$\begin{array}{ccc} V_1 & \xrightarrow{f_1} & W_1 \\ {\scriptstyle x_{h_1}}\downarrow & & \downarrow{\scriptstyle x_{h_2}} \\ V_2 & \xrightarrow[f_2]{} & W_2 \end{array} \qquad \begin{array}{ccc} V_3 & \xrightarrow{f_3} & W_3 \\ {\scriptstyle x_{h_2}}\downarrow & & \downarrow{\scriptstyle x_{h_2}} \\ V_2 & \xrightarrow[f_2]{} & W_2 \end{array}$$

*Notice that they are commutative.*

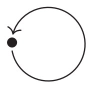



Throughout this dissertation, unless stated otherwise, we will only be considering finite dimensional representations, which means those representations where each space $V_i$ is finite dimensional.

**Definition 1.1.13.** *We will denote the category of finite dimensional representations of the quiver $\vec{Q}$ by $Rep(\vec{Q})$.*

There are many examples of representations of quivers that can help illustrate the fundamental concepts of this theory. In the following examples, we will explore how different representations of quivers can be used to study various algebraic structures and provide insights into the behavior of these structures.

**Example 1.1.14.** *Consider the Jordan quiver $\vec{Q}$ given by*



*A representation of this quiver is a pair $(V, x)$, where $V$ is a $\mathbb{K}$-vector space and $x : V \to V$ is a linear operator. Thus, classifying representations of $\vec{Q}$ is equivalent to classifying linear operators up to a change of basis, or matrices up to conjugate. This is a classical problem of linear algebra over an algebraically closed field, the classification is given by Jordan canonical form. Note that the answer depends on the ground field $\mathbb{K}$, if $\mathbb{K}$ is not algebraically closed, the answer is different.*

**Example 1.1.15.** *Let $\vec{Q}$ be a quiver given by*

$$1 \longrightarrow 2$$

*Then,*

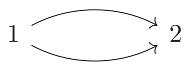
$$Rep(\vec{Q}) = \{(V_1, V_2, x); V_1, V_2 \text{ vector spaces, } x : V_1 \to V_2 \text{ a linear map}\}.$$

*In this case, it is known that up to a change of basis for $V_1, V_2$, any such operator can be brought to the form*

$$x = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

*where $I_r$ is the $r \times r$ unit matrix. Note that in this case, the classification does not depend on the field $\mathbb{K}$.*

**Example 1.1.16.** *Consider the Kronecker quiver given by*



*Classifying representations of this quiver is equivalent to classifying pairs of linear operators $x, y : V_1 \to V_2$. This is a significantly more difficult problem than classifying a single linear operator. In this case a complete classification is known but is rather complicated. For now, let us just consider classification of representations with $dim(V_1) = dim(V_2) = 1$. In this case, choosing a basis in $V_1, V_2$ we can treat $x, y$ as scalars. Therefore, isomorphism classes of representations are in bijection with $\mathbb{K}^2/\mathbb{K}^\times = \{0\} \cup \mathbb{P}^1(\mathbb{K})$. In particular, if the field $\mathbb{K}$ is finite, then there are finitely many isomorphism classes.*

The category $\mathrm{Rep}(\vec{Q})$ is endowed with the following operations.

- Direct sums: If $V, W \in \mathrm{Rep}(\vec{Q})$, we define their direct sum $V \oplus W \in \mathrm{Rep}(\vec{Q})$ by $(V \oplus W)_i = V_i \oplus W_i$, with the obvious definition of operators $x_h$.
- Subrepresentations and quotients: A subrepresentation $V \subset W$ is a collection of vector subspaces $V_i \subset W_i$ such that $x_h V \subset V$. More precisely, for any edge $h : i \to j$, we have $x_h(V_i) \subset V_j$. In this situation, we can also define the quotient representation $W/V$ by $(W/V)_i = W_i/V_i$, with the obvious definition of $x_h$.
- Kernel and image: for any morphism of representations $f : V \to W$, we define representations $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ by

$$(\mathrm{Ker}(f))_i = \mathrm{Ker}(f_i : V_i \to W_i) \qquad \text{and} \qquad (\mathrm{Im}(f))_i = \mathrm{Im}(f_i : V_i \to W_i).$$

It is easy to check that images and kernels defined as above satisfy properties such as $\mathrm{Im}(f) \simeq V/(\mathrm{Ker}(f))$, in other words $\mathrm{Rep}(\vec{Q})$ is an abelian category over $\mathbb{K}$.

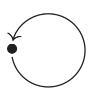
Using these notions, we can rewrite the results of Example 1.1.15 by saying that any representation is isomorphic to a direct sum of the following representations.

$$\mathbb{K} \xrightarrow{\ 1\ } \mathbb{K} \qquad\qquad 0 \xrightarrow{\ 0\ } \mathbb{K} \qquad\qquad \mathbb{K} \xrightarrow{\ 0\ } 0.$$

**Example 1.1.17.** *The quiver $\vec{Q}$ given by*

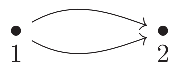

*is an example of a quiver of finite type.*

**Example 1.1.18.** *The Jordan quiver*



*is not of finite type. Over an algebraically closed field, for every $d$ there is exactly one one-parameter family of indecomposable representations with dimension $d$. Namely, $J_\lambda$, for $\lambda \in \mathbb{K}$, the Jordan block of size $d$ with eigenvalue $\lambda$.*

**Example 1.1.19.** *The Kronecker quiver*



*is also not of finite type. In Example 1.1.16 we constructed an infinite number of representations of dimension* $(1, 1)$.

## 1.2 Path algebra, simple and indecomposable representation

Path algebras are a powerful tool for studying the algebraic structures associated with quivers. They are defined as the free algebra generated by all paths in a quiver, where a path is a sequence of edges such that the source of each arrow coincides with the target of the next arrow. Path algebras provide a natural way to encode the algebraic relations between the edges of a quiver, which are essential for understanding the behavior of the algebraic structure associated with the quiver.

Given a quiver $\vec{Q}$, a path of length $l$ in $\vec{Q}$ is an element $p \in \Omega^l$ such that if

$$p = (h_l, \cdots, h_1), \text{ then } s(h_{i+1}) = t(h_i), \ 1 \leq i < l.$$

We define the source and target of a path $(h_l, \cdots, h_1)$ in the obvious way,

$$s(h_l, \cdots, h_1) = s(h_1) \quad \text{and} \quad t(h_l, \cdots, h_1) = t(h_l).$$

We can define a product of paths by

$$(h_l, \cdots, h_1)(h'_m, \cdots, h'_1) = \begin{cases} (h_l, \cdots, h_1, h'_m, \cdots, h'_1), & \text{if } s(h_1) = t(h'_m), \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Thus, a path of length $l$ can be written as a product of $l$ edges. It is also convenient to extend this definition allowing paths of length zero. Formally, we introduce elements $e_i$ with $s(e_i) = t(e_i) = i$ and extend the multiplication by

$$e_i p = \begin{cases} p, & t(p) = i \\ 0, & \text{otherwise,} \end{cases} \qquad p e_i = \begin{cases} p, & s(p) = i \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Note that, in particular, this implies

$$e_i e_j = \delta_{ij} e_i.$$

**Definition 1.2.1.** *The path algebra* $\mathbb{K}\vec{Q}$ *of a quiver* $\vec{Q}$ *is the algebra with basis given by all paths in* $\vec{Q}$, *including paths of length zero, and multiplication defined by* (1.1) *and* (1.2).

The following properties of the path algebra are simple to check from the definition.

(1) The path algebra $\mathbb{K}\vec{Q}$ is an associative algebra with unit $1 = \sum e_i$.

(2) The path algebra $\mathbb{K}\vec{Q}$ is naturally $\mathbb{Z}_+$-graded by path length, and $(\mathbb{K}\vec{Q})_0 = \bigoplus \mathbb{K}e_i$ is semisimple.

(3) The path algebra $\mathbb{K}\vec{Q}$ is finite dimensional if and only if the quiver $\vec{Q}$ contains no oriented cycles.

(4) Elements $e_i$ are indecomposable projectors. Recall that a projector $e$ is decomposable if it is possible to write $e$ as a sum of nonzero orthogonal projectors, i.e., $e = e' + e''$ with $(e')^2 = e', (e'')^2 = e''$, and $e'e'' = e''e' = 0$.

**Definition 1.2.2.** *A representation $V \in Rep(\vec{Q})$ is called*

- *simple or irreducible if it contains no nontrivial subrepresentations,*
- *semisimple if it is isomorphic to a direct sum of simple representations,*
- *indecomposable if it cannot be written as a direct sum of nonzero subrepresentations.*

**Example 1.2.3.** *For the quiver $\vec{Q}$ given by*

$$\bullet \longrightarrow \bullet$$

*the simple representations are*

$$\mathbb{K} \xrightarrow{\ 0\ } 0 \qquad\qquad 0 \xrightarrow{\ 0\ } \mathbb{K}.$$

*The representation*

$$\mathbb{K} \xrightarrow{\ 1\ } \mathbb{K}$$

*is indecomposable but not simple, since the representation*

$$\mathbb{K} \xrightarrow{\ 0\ } 0$$

*is a subrepresentation. It is also not semisimple because the direct sum of the simple representations is*

$$\mathbb{K} \xrightarrow{\ 0\ } \mathbb{K}.$$

Our first goal is to classify the simple representations of a quiver $\vec{Q}$, up to an isomorphism. When $\vec{Q}$ has no oriented cycles this classification is easy. For every $i \in I$ we can define the representation $S(i)$ by

$$S(i)_j = \begin{cases} \mathbb{K} & i = j \\ 0 & i \neq j \end{cases} \tag{1.3}$$

and all $x_h = 0$. It is clear that each $S(i)$ is simple and that they are pairwise nonisomorphic.

**Theorem 1.2.4.** *Let $\vec{Q}$ be a quiver without oriented cycles. Then $\{S(i); i \in I\}$ form a full list of simple representations of $\vec{Q}$ up to an isomorphism.*

*Proof.* Assume that $V$ is a simple representation. Consider the set $I' = \{i \in I; V_i \neq 0\}$. Since $\vec{Q}$ contains no oriented cycles, there must exist $i \in I'$ such that there are no edges $i \to j$ with $j \in I'$. Therefore, $V$ contains a subrepresentation $V'$ given by $V'_i = V_i$ and $V'_j = 0$ for $j \neq i$. Since $V$ is simple this implies that $V = V'$ and $\dim(V_i) = 1$. □

This theorem is a fundamental result in the theory of quiver representations, and it has many important consequences. For example, it implies that the number of non-isomorphic simple representations of $\vec{Q}$ is equal to the number of vertices of $\vec{Q}$.

**Theorem 1.2.5.** *Any finite dimensional representation of a quiver $\vec{Q}$ can be written as a direct sum of indecomposable representations. Such decomposition is unique up to reordering and up to an isomorphism.*

This result provides a powerful tool for understanding the structure of the representation category of a quiver. The uniqueness of the decomposition ensures that the indecomposable representations form a complete set of building blocks for all representations of the quiver, and it allows us to study the representation theory of the quiver by focusing on these simpler objects. Therefore, our goal now will be the classification of indecomposable representations of $\vec{Q}$.

**Definition 1.2.6.** *Given a quiver $\vec{Q}$. Define*

$$Ind(\vec{Q})=\{\text{isomorphism classes of nonzero indecomposable representations of } \vec{Q}\}.$$

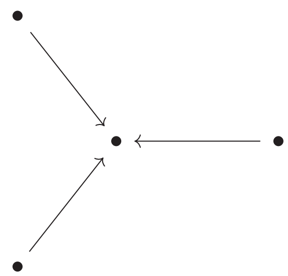
**Example 1.2.7.** *For the quiver $\vec{Q}$ given by*

$$1 \longrightarrow 2$$

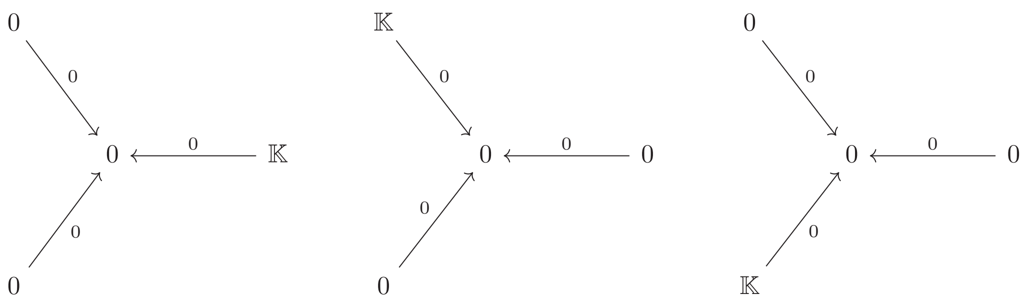
*the results of Section 1.1 show that indecomposable representations of $\vec{Q}$ are*

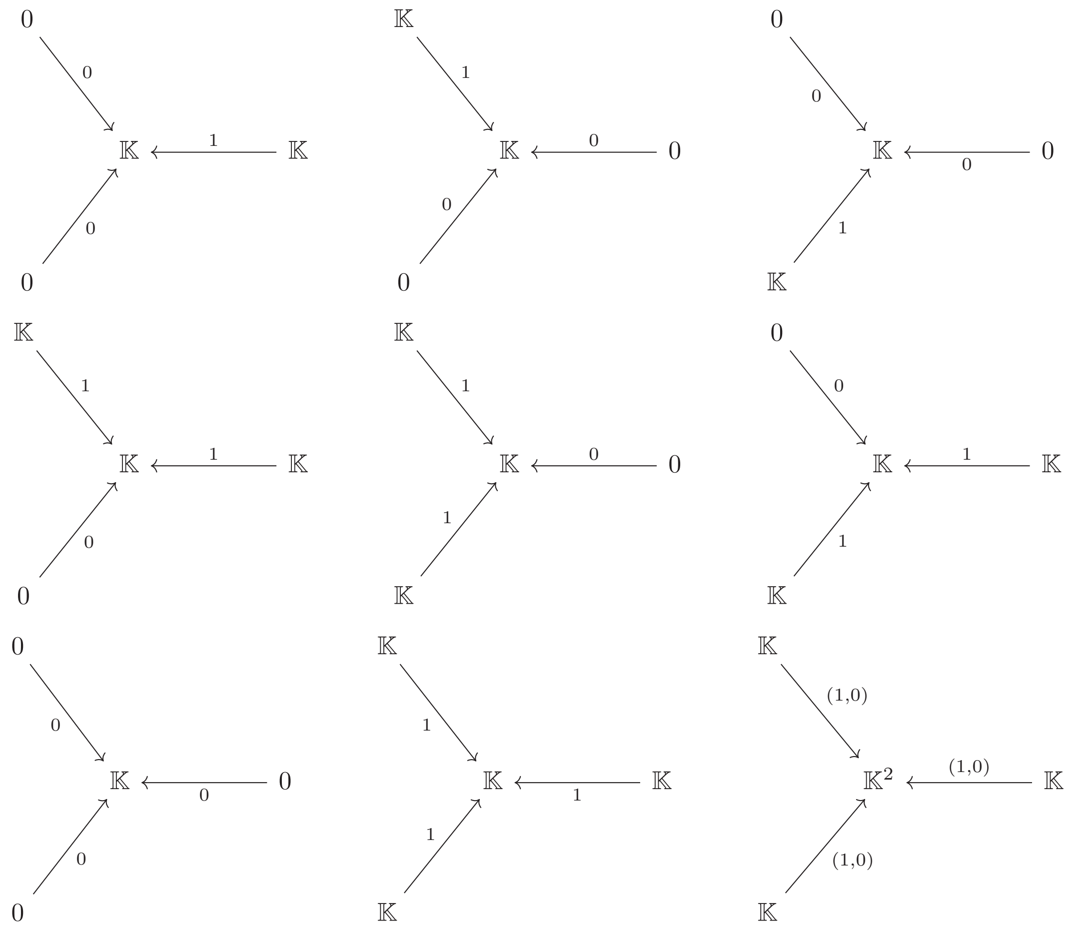
$$S(1) = \mathbb{K} \xrightarrow{\ 0\ } 0 \qquad S(2) = 0 \xrightarrow{\ 0\ } \mathbb{K} \qquad I = \mathbb{K} \xrightarrow{\ 1\ } \mathbb{K}$$

**Example 1.2.8.** *Let $\vec{Q}$ be the quiver given by*



*Classifying representations of this quiver is essentially equivalent to classifying triples of subspaces in a vector space. Consider the representations bellow*






*Each of these representations is indecomposable. Later we will show that this is a full list of indecomposable representations for the quiver $\vec{Q}$. Thus giving a complete answer to the problem of classifying triples of subspaces in a vector space.*

## 1.3 Grothendieck group and dimension vector

The Grothendieck group, named after the influential mathematician Alexander Grothendieck, is an algebraic structure that plays an important role in many areas of mathematics, including algebraic geometry, number theory, and representation theory. Informally, the Grothendieck group is a way of formally adding and subtracting objects in a given category, with the goal of obtaining a simpler and more manageable structure. In the context of quiver representations, the Grothendieck group provides a way of organizing and classifying the indecomposable representations of a quiver, and it allows us to study the representation theory of the quiver in a more systematic way.

Let $\mathcal{C}$ be an abelian category. We can define the Grothendieck group $K(\mathcal{C})$, also called the $K$- group, as the abelian group generated by symbols $[M]$ with $M \in \mathcal{C}$ and relations $[A] - [B] + [C] = 0$ for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

In particular, this implies that $[A] = [B]$ if $A, B$ are isomorphic and that $[A \oplus B] = [A] + [B]$.

In this section we will discuss the Grothendieck group of $\mathrm{Rep}(\vec{Q})$, which will be simply denoted by $K(\vec{Q})$.

**Example 1.3.1.** *Let $\vec{Q} = \bullet$, so that $Rep(\vec{Q}) = \mathcal{V}ec$ is the category of finite dimensional vector spaces. Since every vector space is isomorphic to a direct sum of copies of $\mathbb{K}$, we see that in this case $K(\vec{Q}) = \mathbb{Z}$ is the free group generated by the element $[\mathbb{K}]$. One can also describe the isomorphism $K(\vec{Q}) \simeq \mathbb{Z}$ by $[M] \mapsto dim(M)$.*

The following theorem is a natural generalization of this result. For a representation $V$ of $\vec{Q}$, define its graded dimension $\dim(V) \in \mathbb{Z}^I$ by

$$(\dim(V))_i = \dim(V_i).$$

**Theorem 1.3.2.** *Let $\vec{Q}$ be a quiver without oriented cycles. Then the map*

$$\begin{aligned} dim : K(\vec{Q}) &\to \mathbb{Z}^I \\ [V] &\mapsto dim(V) \end{aligned}$$

*is an isomorphism.*

*Proof.* Since every representation has a composition series with simple factors, by Theorem 1.2.4 this implies that $[V] = \sum n_i[S(i)]$. Thus, the classes $[S(i)]$ generate $K(\vec{Q})$.

Next, note that dim is well-defined on $K(\vec{Q})$, since $\dim(S(i)) = e_i = (0, \dots, 1, \dots, 0)$ are independent in $\mathbb{Z}^I$, this implies that $[S(i)]$ are independent in $K(\vec{Q})$ and thus are free generators of the abelian group $K(\vec{Q})$. □

An important consequence of this theorem is that it allows us to study the representation theory of a quiver purely in terms of its underlying combinatorial structure, namely its vertices and edges. This is because the dimension vector of a representation is entirely determined by the number of times each vertex appears in the representation, which in turn is determined by the edges between the vertices. Therefore, the study of the representation theory of a quiver can be reduced to a purely combinatorial problem, involving only the structure of the quiver itself.

**Definition 1.3.3.** *A quiver $\vec{Q}$ is of finite type if for any $v \in \mathbb{Z}^I_+$, the number of isomorphism classes of indecomposable representations of dimension $v$ is finite.*

It is worth noting that this is a special class of quivers that has important applications in algebraic geometry, representation theory, and mathematical physics. This property leads to many interesting algebraic and geometric properties, such as the existence of a finite-dimensional algebra associated with the quiver. Moreover, quivers of finite type have connections with many other important mathematical objects, such as cluster algebras, module spaces of vector bundles, and one that is very important to us, Dynkin quivers.

## 1.4 Projective modules and the standard resolution

Projective modules are an important concept in algebraic geometry and representation theory that generalize the notion of free modules. Roughly speaking, a projective module is a module that behaves like a free module in many aspects, but with certain additional properties that make it more flexible and useful. One of the key features of projective modules is that they admit a dual notion of injective modules, which are modules that behave like direct summands of the dual of a projective module.

This duality is a fundamental concept in algebraic geometry and representation theory, and provides a powerful tool for understanding the structure of algebraic systems. The standard resolution is a tool for constructing projective modules, and is an important technique in the study of homological algebra and algebraic geometry.

Recall that a module $P$ over an associative algebra $A$ is projective if and only if the functor $\mathrm{Hom}(P, -)$ is exact. In particular, a direct summand of a free module is projective. Thus, for the path algebra $A = \mathbb{K}\vec{Q}$, we define for each $i \in I$ the representations $P(i)$ by

$$P(i)_j = \mathbb{K}^n \tag{1.4}$$

where $n$ is the number of paths from $i$ to $j$. The representation $P(i)$ is projective, since $A = \bigoplus_i P(i)$. Note that $P(i)$ can be infinite dimensional.

**Example 1.4.1.** *Let $\vec{Q}$ be the quiver given by*

$$1 \longrightarrow 2$$

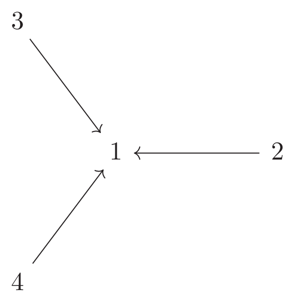
*the projective representations for each vertex are*

$$P(1) = \mathbb{K} \xrightarrow{\quad 1 \quad} \mathbb{K}$$

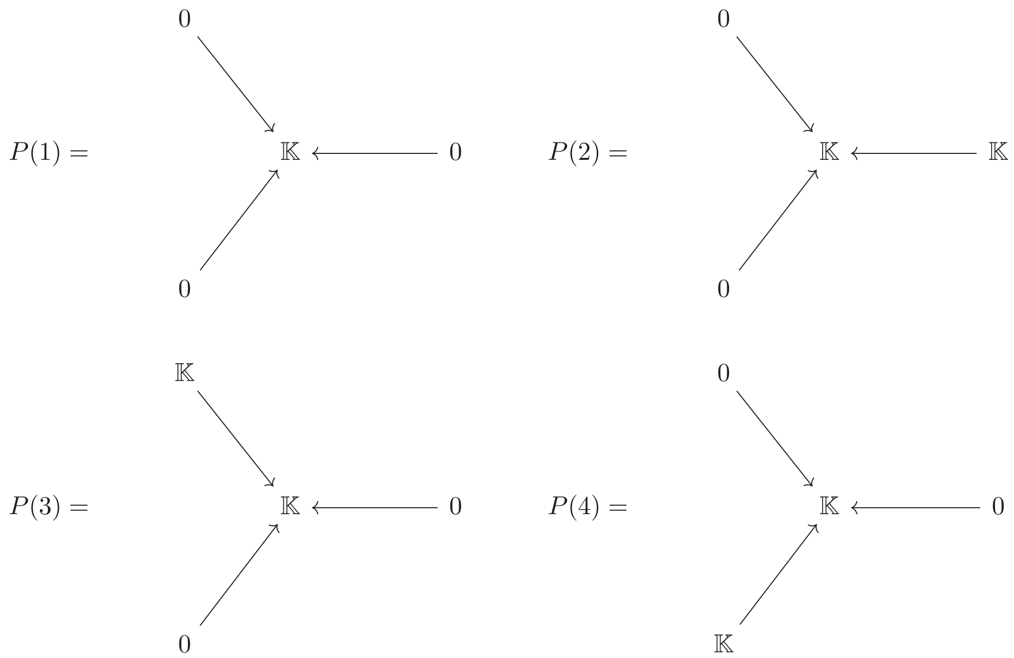
$$P(2) = 0 \xrightarrow{\quad 0 \quad} \mathbb{K}.$$

*Note that $P(2) = S(2)$.*

**Example 1.4.2.** *For the quiver $\vec{Q}$ given by*



*then the projective representations for all vertices are,*




**Theorem 1.4.3.** *For any representation $V \in Rep(\vec{Q})$, we have $Hom_{\vec{Q}}(P(i), V) = V_i$*

*Proof.* For $V = \{(V_i), (x_h)\}$ define the homomorphisms

$$\begin{array}{ccc} \Psi_i : V_i \to \mathrm{Hom}_{\vec{Q}}(P(i), V) & \text{and} & \Phi_i : \mathrm{Hom}_{\vec{Q}}(P(i), V) \to V_i \\ \Psi_i(v) = (p \mapsto x_p(v)) & & \Phi_i(f) = fe_i \end{array}$$

if $p$ is a path starting at $i$. Therefore, we have that

$$\Psi_i \Phi_i(f) = \Psi_i(fe_i) = (p \mapsto x_p(fe_i) = fp) = f$$

and

$$\Phi_i \Psi_i(v) = x_{e_i}(v) = v.$$

□

If $\vec{Q}$ has no oriented cycles, then $P(i)$ is finite dimensional and in the Grothendieck group we have

$$[P(i)] = \sum_j p_{ji}[S(j)],$$

where $p_{ji}$ is the number of paths $i \to j$. In particular, it is possible to order vertices so that the matrix $p_{ji}$ is upper triangular with ones on the diagonal and thus invertible. Therefore, the classes $[P(i)], i \in I$ also form a basis of $K(\vec{Q})$.

**Theorem 1.4.4.** *Assume that $\vec{Q}$ has no oriented cycles. Then the representations $\{P(i); i \in I\}$ form the full set of nonzero indecomposable projective objects in $Rep(\vec{Q})$ up to an isomorphism.*

*Proof.* A module is indecomposable if and only if 0 and 1 are the only idempotents in the ring of endomorphisms. By Theorem 1.4.3 we have that $\mathrm{Hom}(P(i), P(i)) = (P(i))_i = \mathbb{K}$, so $P(i)$ is indecomposable. To show that it is a full set of indecomposable projective, assume that $P$ is a projective module. Let

$$P' = \bigoplus_{i \in I} n_i P(i) \text{ where } n_i = \dim(\mathrm{Hom}(P, S(i))).$$

Then, it follows from Theorem 1.4.3 that $\mathrm{Hom}(P, S(i)) \simeq \mathrm{Hom}(P', S(i))$. Using projectivity of $P$ this implies that $\mathrm{Hom}(P, V) \simeq \mathrm{Hom}(P', V)$, for any $V \in \mathrm{Rep}(\vec{Q})$. Therefore, $P \simeq P'$. □

This theorem has important implications for the representation theory of quivers and their associated algebras. In particular, it allows for a complete classification of the indecomposable projective representations of a quiver algebra, which is crucial for understanding its structure.

We will use representations $P(i)$ to construct a standard resolution for any representation $V$ of $\vec{Q}$. First, we review some general theory. Let $A$ be an associative algebra with unit, then for any module $M$ we have

$$M \simeq A \otimes_A M = A \otimes_{\mathbb{K}} M/I,$$

where the subspace $I$ is generated by elements $ab \otimes m - a \otimes bm$ with $a, b \in A$ and $m \in M$. Moreover, it suffices to take elements $b$ from some set of generators of $A$. If $L \subset A$ is a subspace such that elements $l \in L$ generate $A$, then $I$ is spanned by $al \otimes m - a \otimes lm$ with $a \in A, l \in L$ and $m \in M$. Thus, we have an exact sequence of $A$-modules with all tensor products taken over $\mathbb{K}$

$$A \otimes L \otimes M \xrightarrow{d_1} A \otimes M \xrightarrow{d_0} M \longrightarrow 0$$

$$d_1(a \otimes l \otimes m) = al \otimes m - a \otimes lm$$

$$d_0(a \otimes m) = am.$$

This is a beginning of a free resolution of $M$. The next term would come from relations among generators $l \in L$, and the process continues as before.

This has a modification, if $A_0$ is a subalgebra, $L$ is a subspace such that $A_0 L \subset L, LA_0 \subset L$, and $A_0, L$ generate $A$, then we have an exact sequence

$$A \otimes_{A_0} L \otimes_{A_0} M \xrightarrow{d_1} A \otimes_{A_0} M \xrightarrow{d_0} M \longrightarrow 0 \tag{1.5}$$

with the morphisms defined in the same way as before. We can now apply this in the case $A = \mathbb{K}\vec{Q}$.

**Theorem 1.4.5.** *Let $\vec{Q}$ be a quiver with set of vertices $I$ and set of edges $\Omega$. For any $\mathbb{K}\vec{Q}$-module $V$, we have the following short exact sequence of $\mathbb{K}\vec{Q}$-modules*

$$0 \longrightarrow \bigoplus_{h \in \Omega} P(t(h)) \otimes \mathbb{K}h \otimes V_{s(h)} \xrightarrow{d_1} \bigoplus_{i \in I} P(i) \otimes V_i \xrightarrow{d_0} V \longrightarrow 0 \tag{1.6}$$

*where $P(i)$ is the projective module defined by 1.4, and the differentials are defined by*

$$d_1(p \otimes h \otimes v) = ph \otimes v - p \otimes x_h(v),$$
$$d_0(p \otimes v) = x_p(v).$$

*This resolution will be called the standard resolution of $V$.*

*Proof.* From (1.5) we can write

$$A \otimes_{A_0} L \otimes_{A_0} V \xrightarrow{\quad d_1 \quad} A \otimes_{A_0} V \xrightarrow{\quad d_0 \quad} V \xrightarrow{\qquad} 0.$$

If we consider $A_0 = \bigoplus \mathbb{K}e_i$ the subspace of paths of length zero. Then $L = \bigoplus_{h \in \Omega} \mathbb{K}h$ and that gives us

$$A \otimes_{A_0} V = \bigoplus Ae_i \otimes e_i V = \bigoplus P(i) \otimes V_i.$$

So we have

$$A \otimes_{A_0} L \otimes_{A_0} V \xrightarrow{\quad d_1 \quad} \bigoplus P(i) \otimes V_i \xrightarrow{\quad d_0 \quad} V \xrightarrow{\qquad} 0.$$

The exactness at $V$ and $\bigoplus P(i) \otimes V_i$ follows from the exactness of (1.5).

Now all that remains is to show that $d_1$ is injective. Assume

$$d_1 \left( \sum_n p_n \otimes h_n \otimes v_n \right) = \sum p_n h_n \otimes v_n - p_n \otimes x_{h_n}(v_n) = 0. \tag{1.7}$$

Let $l =$ maximal length of the paths $p_n$ appearing in this sum. Then taking the terms of length $l+1$ in (1.7) we get

$$\sum p_k h_k \otimes v_k = 0,$$

where the sum is taken over all $k$ such that $l(p_k) = l$.

On the other hand, if we take different $h_k$ then the paths of the form $p_k h_k$ are linearly independent, which leads to a contradiction. □

The standard resolution allows us to compute Ext groups between representations, which in turn gives us information about the dimension of Hom spaces and the classification of indecomposable representations. We will denote Ext functors in this category by $\mathrm{Ext}^i_{\vec{Q}}(V, W)$.

**Corollary 1.4.6.** *Let $\vec{Q}$ be a quiver without oriented cycles. For any $V, W \in Rep(\vec{Q})$, we have $Ext^i_{\vec{Q}}(V, W) = 0$ for any $i > 1$.*

*Proof.* We have that $\mathrm{Ext}^i_{\vec{Q}}(V, W)$ can be computed using a standard resolution of $V$. Every representation $V$ has a standard resolution of the form

$$0 \xrightarrow{\qquad} P_1 \xrightarrow{\qquad} P_0 \xrightarrow{\qquad} V \xrightarrow{\qquad} 0$$

and the corollary follows. □

**Example 1.4.7.** *Let $V = S(i)$ be a simple representation then the standard resolution* (1.6) *becomes*

$$0 \longrightarrow \bigoplus_{h:i\to j} P(j) \longrightarrow P(i) \longrightarrow S(i) \longrightarrow 0$$

*Using this and Theorem 1.4.3, we have, in particular*

$$dim(Hom(S(i), S(j))) = \delta_{ij}$$
$$dim(Ext^1(S(i), S(j))) = \text{ number of edges } i \to j \text{ in } \vec{Q}.$$

Categories satisfying $\text{Ext}^i(V, W) = 0$ for any $i > 1$ are called hereditary. They also admit another characterization given below.

**Corollary 1.4.8.** *If $P \in Rep(\vec{Q})$ is projective then any subrepresentation of $P$ is also projective.*

*Proof.* Let $V \subset P$ and $W = P/V$, so we have a short exact sequence

$$0 \longrightarrow V \longrightarrow P \longrightarrow W \longrightarrow 0$$

Then, for any $X \in \text{Rep}(\vec{Q})$ we have the corresponding long exact sequence of Ext functors

$$\cdots \longrightarrow \text{Ext}^1(W, X) \longrightarrow \text{Ext}^1(P, X) \longrightarrow$$

$$\longrightarrow \text{Ext}^1(V, X) \longrightarrow \text{Ext}^2(W, X) \longrightarrow \cdots$$

We have $\text{Ext}^1(P, X) = 0$ since $P$ is projective and $\text{Ext}^2(W, X) = 0$ by Corollary 1.4.6. Then we have that $\text{Ext}^1(V, X) = 0$, which implies that $V$ is projective and therefore any subrepresentation of $P$ is projective. □

The fact that subrepresentations of projective representations of a quiver $\vec{Q}$ with no oriented cycles are also projective is a fundamental property of projective modules. It means that projective modules have a strong level of self-containment, in the sense that any smaller piece of the module also inherits the property of being projective.

Much of the theory of projective modules has a counterpart in the theory of injective modules, the following theorem shows that.

**Theorem 1.4.9.** *Let $\vec{Q}$ be a quiver. Define, for any $i \in I$,*

$$Q(i) = (e_i A)^*, \tag{1.8}$$

*where $A = \mathbb{K}\vec{Q}$ is the path algebra and $*$ is the graded dual: $(e_i A)^* = \bigoplus_l (e_i A^l)^*$, where $A^l$ is the span of paths of length $l$. Note that $Q(i)$ has a natural structure of a left $A$-module. Then we have the following statements,*

*(1) For $V \in Rep(\vec{Q})$, we have natural isomorphisms $Hom_{\vec{Q}}(V, Q(i)) \simeq V_i^*$.*

*(2) If $\vec{Q}$ has no oriented cycles, then modules $Q(i), i \in I$, form a full set of indecomposable injective representations of $\vec{Q}$.*

*(3) Any representation $V$ of $\vec{Q}$ has an injective resolution of the form*

$$0 \longrightarrow V \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0.$$

## 1.5 Euler form

The Euler form is a fundamental tool in the study of representations of quivers. It provides a way to measure the noncommutativity of a quiver by assigning a value to each pair of representations. The fact that the Euler form only depends on the dimensions of the representations allows for a more efficient computation of the form, as it reduces the problem to linear algebra.

Given two representations $V, W \in \mathrm{Rep}(\vec{Q})$, define $\langle V, W\rangle \in \mathbb{Z}$ by

$$\langle V, W\rangle = \sum (-1)^i \dim(\mathrm{Ext}^i(V, W)) = \dim(\mathrm{Hom}(V, W)) - \dim(\mathrm{Ext}^1(V, W)), \tag{1.9}$$

since $\mathrm{Ext}^i(V, W) = 0$ for $i > 1$ by Corollary 1.4.6.

**Example 1.5.1.** *For $V = P(i)$ we have that $\langle P(i), W\rangle = dim(Hom(P(i), W)) - dim(Ext^1(P(i), W)) = dim(W_i)$, by Theorem 1.4.3. In particular,*

$$\langle P(i), S(j)\rangle = \delta_{ij}.$$

*Similarly, if $W = Q(i)$ is an indecomposable injective, then*

$$\langle V, Q(i)\rangle = dim(V_i).$$

**Theorem 1.5.2.** *The number $\langle V, W\rangle$ only depends on $dim(V), dim(W)$ and thus defines a bilinear form on $\mathbb{Z}^I$, called the Euler form. Moreover, for $v, w \in \mathbb{Z}^I$, we have*

$$\langle v, w\rangle = \sum_{i\in I} v_i w_i - \sum_{h\in\Omega} v_{s(h)} w_{t(h)}. \tag{1.10}$$

*Proof.* It follows from the long exact sequence of Ext functors that if we have a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

then $\langle A, W\rangle - \langle B, W\rangle + \langle C, W\rangle = 0$. Thus, we can use the standard resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0$$

of $V$ constructed in Theorem 1.4.5 to compute $\langle V, W\rangle$. Since $\langle P(i), W\rangle = \dim(W_i)$, this gives

$$\begin{aligned}\langle V, W\rangle &= \langle P_0, W\rangle - \langle P_1, W\rangle \\ &= \sum_{i\in I} \dim(V_i)\, \dim(W_i) - \sum_{h\in\Omega} \dim(V_{s(h)})\, \dim(W_{t(h)}).\end{aligned}$$

□

The formula given for the Euler form in terms of the dimensions and the edges of the quiver is also important, as it allows for a concrete computation of the form in specific examples. The Euler form is used in many contexts in representation theory, such as the definition of the Auslander-Reiten quiver, and the construction of tilting modules.

**Example 1.5.3.** *We have that* $\langle S(i), S(j)\rangle = \delta_{ij} -$ *(number of edges* $i \to j$*).*

Note that $\langle\ ,\ \rangle$ is not symmetric. We will frequently use the symmetrized Euler form

$$(v, w) = \langle v, w\rangle + \langle w, v\rangle = \sum_i v_i w_i (2 - 2n_{ii}) - \sum_{i \neq j} n_{ij} v_i w_j, \tag{1.11}$$

where $n_{ij}$ is the number of unoriented edges between $i$ and $j$ in the graph $Q$. We will also use the associated quadratic form, called the Tits form

$$q_{\vec{Q}}(v) = \frac{1}{2}(v, v) = \langle v, v\rangle\,. \tag{1.12}$$

Note that the symmetrized Euler form and the Tits form are independent of the orientation of $\vec{Q}$. In fact, they can be defined for any graph $Q$.

## 1.6 Dynkin and Euclidean graphs

We define Euclidean and Dynkin graphs based in the Tits form in (1.12). Recall that a quadratic form $q$ is positive definite (semidefinite) if

$$q_{\vec{Q}}(v) > 0 \text{ for all nonzero } v \in \mathbb{R}^I \quad \left(q_{\vec{Q}}(v) \geq 0 \text{ for all } v \in \mathbb{R}^I\right). \tag{1.13}$$

**Definition 1.6.1.** *Let $Q$ be a connected graph. If the associated Tits form $q_{\vec{Q}}$ defined by* (1.12) *is positive definite then we call $Q$ Dynkin. If the Tits form is positive semidefinite we call $Q$ Euclidean.*

In this theory we use only the diagrams that are simply laced, those are the ADE diagrams. Different from the root system theory, where we have some more diagrams, you can see that in Theorem A.10.5. In the rest of this work we will call Dynkin diagrams the ones from the next theorem.

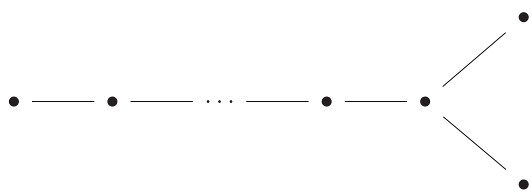
**Theorem 1.6.2.** *Let $Q$ be a connected graph.*

*(1) The graph $Q$ is Dynkin if and only if it is one of the graphs below, with $\ell$ vertices in each case.*

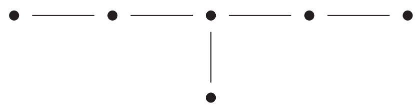
$A_\ell, \ell \geq 1$



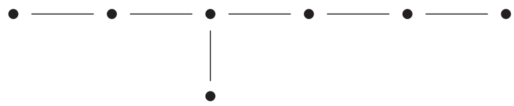
$D_\ell, \ell \geq 4$



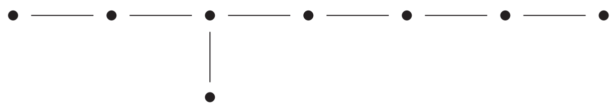
$E_6$



$E_7$

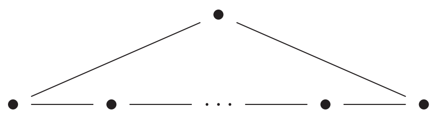


$E_8$

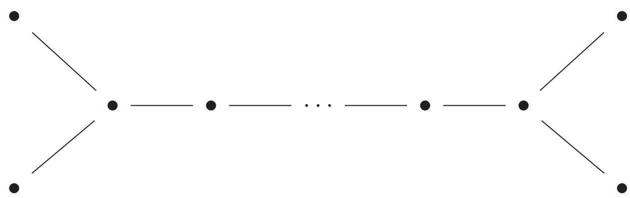


*(2) The graph $Q$ is Euclidean if and only if it is one of the graphs below, with $\ell + 1$ vertices in each case.*

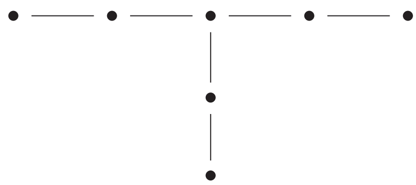
$\hat{A}_\ell, \ell \geq 1$



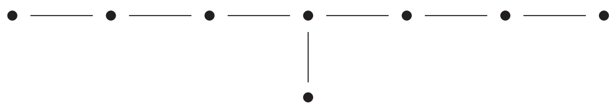
$\hat{D}_\ell, \ell \geq 4$



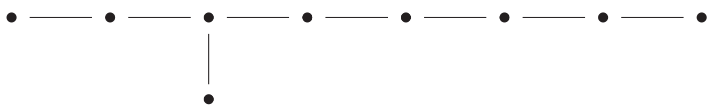
$\hat{E}_6$



$\hat{E}_7$



$\hat{E}_8$



Another way to characterize Dynkin graphs is with Gabriel's Theorem. The proof of this theorem is one of the most important goals of this work. The theorem is named after Pierre Gabriel, who proved it in his 1972 paper "Unzerlegbare Darstellungen I" [10], and provides a powerful tool for understanding the structure and properties of finite-dimensional algebras, and has been used to prove many other important results in algebraic and geometric settings. Overall, Gabriel's Theorem is a cornerstone of modern algebraic theory and a key result for anyone studying the structure of finite-dimensional algebras.

**Theorem 1.6.3** (Gabriel's Theorem)**.** *A connected quiver $\vec{Q}$ is of finite type if and only if the underlying graph $Q$ is Dynkin.*

# Chapter 2

# REFLECTION FUNCTORS

Reflection functors are an important tool in representation theory that were introduced by Bernstein, Gelfand, and Ponomarev in [6]. They are used to study the category of representations of a quiver, and have deep connections with the geometry of the quivers. The study of reflection functors has its roots in the work of Borel and Tits on algebraic groups, where the notion of reflection groups plays a central role.

In the context of representation theory, reflection functors are used to construct new representations from old ones, and to relate representations of different quivers. The study of reflection functors has led to many interesting results and connections with other areas of mathematics, such as algebraic geometry and Lie theory. In this chapter, we will explore the theory of reflection functors, their properties, and some of their applications in representation theory. In this chapter we consider $\vec{Q}$ as an arbitrary quiver with vertex set $I$ and edges set $\Omega$. We follow [16].

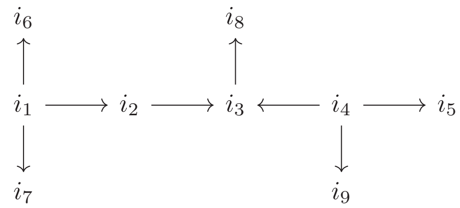
## 2.1 Definitions

**Definition 2.1.1.** *Let $i \in I$.*

- *The vertex $i$ is called a sink if there are no edges $h : i \to j$ for $h \in \Omega$.*
- *The vertex $i$ is called a source if there are no edges $h : j \to i$ for $h \in \Omega$.*

Sinks are vertices that have no outgoing edges, while sources are vertices that have no incoming edges. These types of vertices play a crucial role in the theory of quivers and their representations. For example, in the case of acyclic quivers (quivers with no cycles), the sink vertices correspond to the simple representations of the quiver, while the source vertices correspond to the projective representations. Moreover, the existence and non-existence of sinks and sources can have important consequences for the structure and behavior of a quiver and its representations.

**Example 2.1.2.** *Consider the quiver $\vec{Q}$ given by*

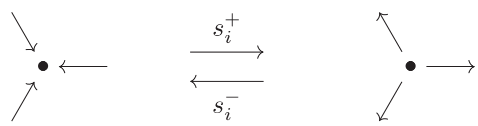


*then $I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9\}$. According to Definition 2.1.1 the vertices $i_5, i_6, i_7, i_8, i_9$ are sinks, the vertices $i_1, i_4$ are sources and the vertices $i_2, i_3$ are neither a sink nor a source.*

**Definition 2.1.3.** *Let $i \in I$ be a vertex for the quiver $\vec{Q}$.*

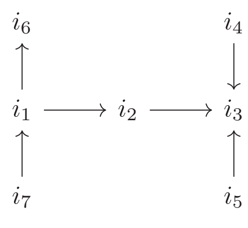
- *If $i$ is a sink for $\vec{Q}$ we define a new quiver $\vec{Q}' = s_i^+(\vec{Q})$ by reversing the orientation of all edges incident to $i$.*
- *If $i$ is a source for $\vec{Q}$ we define a new quiver $\vec{Q}' = s_i^-(\vec{Q})$ by reversing the orientation of all edges incident to $i$.*

*We can see this reversion of orientation in the next diagram.*

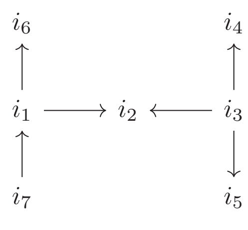


Note that if $i$ is a sink for $\vec{Q}$, then $i$ becomes a source for $\vec{Q}'$. Similarly, if $i$ is a source for $\vec{Q}$, then $i$ becomes a sink for $\vec{Q}'$. Moreover, we have that $s_i^- s_i^+(\vec{Q}) = \vec{Q}$.

**Example 2.1.4.** *Consider the quiver $\vec{Q}$ given by*



*then we have the quiver $\vec{Q}' = s_3^+(\vec{Q})$*



**Lemma 2.1.5.** *If $Q$ is a tree, as in Definition 1.1.6, with at least two vertices, then there exists a vertex $k \in I$ whose degree is 1, as in Definition 1.1.7.*

*Proof.* Since $Q$ has $n$ vertices with $n \geq 2$ then it has $n-1$ edges. The sum of all degrees of this tree is $2(n-1)$ since each edge connects two vertices. If no vertex is connected to exactly one other vertex then each vertex is connected to at least two vertices. So the sum of the degree would be greater or equal to $2n$. We know that $2n > 2(n-1)$, which is a contradiction. There must exist a vertex $k \in I$ that is connected to exactly one other vertex. □

**Lemma 2.1.6.** *If $Q$ is a tree and $\Omega, \Omega'$ are two orientations of $Q$, then $\Omega'$ can be obtained from $\Omega$ by a sequence of operations $s_i^{\pm}$.*

*Proof.* The proof goes by induction on the number of vertices. If we have two vertices, there's only two possibilities for the arrows

$$\Omega: \qquad 1 \longrightarrow 2$$

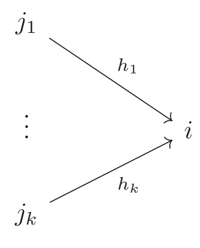
$$\Omega': \qquad 1 \longleftarrow 2$$

Then we have $\Omega' = s_1^-(\Omega)$ and $\Omega = s_1^+(\Omega')$.

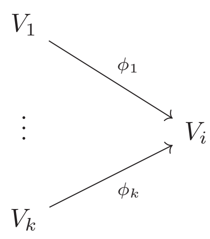
If $I$ has $n+1$ vertices, by Lemma 2.1.5 let $k \in I$ be a vertex which is connected to exactly one other vertex. Let $Q'$ be the graph obtained from $Q$ by removing the vertex $k$ and the incident edge $h$. The tree $Q'$ has $n$ vertices, and it has $n-1$ edges. The graph $Q'$ is also a tree. By the induction hypothesis, we can find a sequence of reflections $s_{i_1}^{\pm}, \cdots, s_{i_l}^{\pm}$ such that $s_{i_l}^{\pm} \cdots s_{i_1}^{\pm}\Omega = \Omega'$ on $Q'$. Multiplying if necessary by $s_k^{\pm}$, depending on whether $k$ is a source or a sink, we see that $\Omega'$ can be obtained from $\Omega$ by a sequence of operations $s_i^{\pm}$. □

This theorem is a fundamental result in the study of tree quivers and their orientations. This result has important consequences for the classification of representations of tree quivers and their mutations, as it allows us to understand the structure of the space of representations by studying a single orientation, and then transforming it to obtain all others.

Suppose that $i \in I$ is a sink for the quiver $\vec{Q}$ and $V$ is a representation of $\vec{Q}$. As $i$ is a sink we have that there exists $j_t \in I$ with $t = 1, \cdots, k$ such that




So, in the representation $V$ we have that



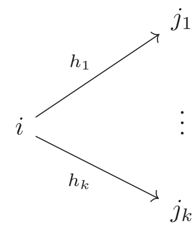
with $V_{j_t} = V_t$.

Therefore, we can define a map

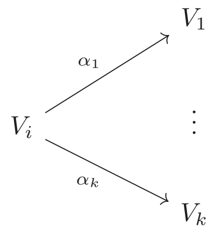
$$\phi : \bigoplus_{h_t : j_t \to i} V_t \to V_i, \tag{2.1}$$

where $\phi = \begin{pmatrix} \phi_1 & \cdots & \phi_k \end{pmatrix}$.

Now suppose that $i \in I$ is a source for the quiver $\vec{Q}$ and $V$ is a representation of $\vec{Q}$. As $i$ is a source we have that there exists $j_t \in I$ with $t = 1, \cdots, k$ such that




So, in the representation $V$ we have that




with $V_{j_t} = V_t$.

Therefore, we can write a map

$$\alpha : V_i \to \bigoplus_{h_t : i \to j_t} V_t. \tag{2.2}$$

Where $\alpha = \begin{pmatrix} \alpha_1 & \cdots & \alpha_k \end{pmatrix}$.

**Definition 2.1.7.**

*(i) Let $i \in I$ be a sink for $\vec{Q}$ and denote $\vec{Q}' = s_i^+(\vec{Q})$. Define the functor*

$$\Phi_i^+ : Rep(\vec{Q}) \to Rep(\vec{Q}')$$

*by*

$$(\Phi_i^+(V))_j = \begin{cases} V_j, & \text{if } j \neq i \\ Ker(\phi), & \text{if } j = i \end{cases}$$

*with $\phi$ as defined in* (2.1). *We define on $\Phi_i^+(V)$ the structure of representation of $\vec{Q}'$ as follows: for every edge $h : i \to k$ in $\vec{Q}'$ the corresponding operator $x_h$ is the composition of the canonical inclusion and projection*

$$Ker\left(\bigoplus V_k \to V_i\right) \to \left(\bigoplus V_k\right) \to V_k.$$

*Operators $x_h$ for edges not incident to $i$ are defined in the obvious way.*

*(ii) Let $i \in I$ be a source for $\vec{Q}$ and denote $\vec{Q}' = s_i^-(\vec{Q})$. Define the functor*

$$\Phi_i^- : Rep(\vec{Q}) \to Rep(\vec{Q}')$$

*by*

$$(\Phi_i^-(V))_j = \begin{cases} V_j, & \text{if } j \neq i \\ Coker(\alpha), & \text{if } j = i \end{cases}$$

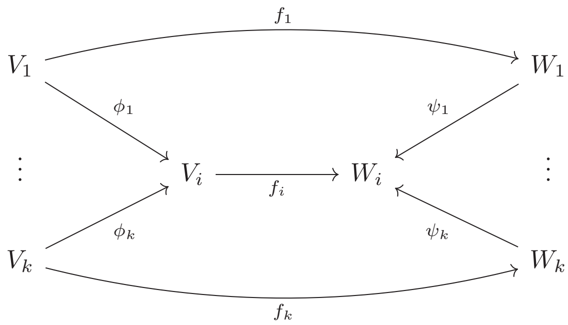
*with $\alpha$ as defined in (2.2). We define on $\Phi_i^-(V)$ the structure of representation of $\vec{Q}'$ as follows: for every edge $h : k \to i$ in $\vec{Q}'$ the corresponding operator $x_h$ is the composition of the canonical inclusion and projection*

$$V_k \to \left(\bigoplus V_k\right) \to Coker\left(V_i \to \bigoplus V_k\right).$$

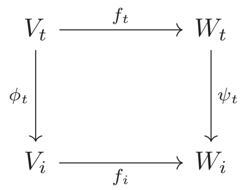
*Operators $x_h$ for edges not incident to $i$ are defined in the obvious way.*

It remains to define $\Phi_i^\pm$ on the morphisms. Consider $V, W$ two representations of $\vec{Q}$ with $f : V \to W$ a morphism between representations.

First we will show how to apply the functor $\Phi_i^+$ to the morphism $f$. Consider $i \in I$ a sink and $\phi_t : V_t \to V_i$ with $V_{j_t} = V_t$. In positions $j \neq i$ we have that $(\Phi_i^+(f))_j = f_j$. In position $i$ we have the following diagram




with the following squares being commutative for $1 \leq t \leq k$




(2.3)

From the commutativity of the square in (2.3) we have the equations

$$\begin{aligned} f_i\phi_1 &= \psi_1 f_1 \\ &\vdots \\ f_i\phi_k &= \psi_k f_k. \end{aligned} \tag{2.4}$$

So we can write the equations in (2.4) as

$$f_i\phi = \psi \begin{pmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_k \end{pmatrix}, \tag{2.5}$$

where $\phi = \begin{pmatrix} \phi_1 & \cdots & \phi_k \end{pmatrix}$ and $\psi = \begin{pmatrix} \psi_1 & \cdots & \psi_k \end{pmatrix}$.

Using (2.5) we can draw the commutative diagram with exact lines

$$\begin{array}{ccccccc}
0 & \longrightarrow & Ker(\phi) & \longrightarrow & \bigoplus_{i=1}^{k} V_j & \xrightarrow{\phi} & V_i \\
 & & & & \downarrow & & \downarrow f_i \\
0 & \longrightarrow & Ker(\psi) & \longrightarrow & \bigoplus_{i=1}^{k} W_j & \xrightarrow{\psi} & W_i
\end{array}$$

hence there exists a unique morphism $\overline{f}$ such that the diagram below is commutative

$$\begin{array}{ccccccc}
0 & \longrightarrow & Ker(\phi) & \longrightarrow & \bigoplus_{i=1}^{k} V_j & \xrightarrow{\phi} & V_i \\
 & & \downarrow \overline{f} & & \downarrow & & \downarrow f_i \\
0 & \longrightarrow & Ker(\psi) & \longrightarrow & \bigoplus_{i=1}^{k} W_j & \xrightarrow{\psi} & W_i
\end{array}$$

Then $(\Phi_i^+(V))_i = \mathrm{Ker}(\phi)$, $(\Phi_i^+(W))_i = \mathrm{Ker}(\psi)$ and $(\Phi_i^+(f))_i = \overline{f}$. Therefore, $\Phi_i^+(f) : \Phi_i^+(V) \to \Phi_i^+(W)$ is well-defined on the morphisms.

Since the diagram

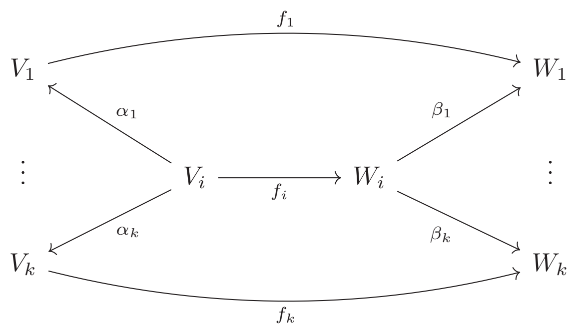
$$\begin{array}{ccccc}
Ker(\phi) & \longrightarrow & \bigoplus_{i=1}^{k} V_j & \longrightarrow & V_j \\
\downarrow \overline{f} & & \downarrow & & \downarrow f_j \\
Ker(\psi) & \longrightarrow & \bigoplus_{i=1}^{k} W_j & \longrightarrow & W_j
\end{array}$$

is commutative, then the square

$$\begin{array}{ccc}
Ker(\phi) & \longrightarrow & V_j \\
\downarrow \overline{f} & & \downarrow f_j \\
Ker(\psi) & \longrightarrow & W_j
\end{array}$$

is also commutative. Then $\overline{f}$ is a morphism between representations.

Next we will show how to apply the functor $\Phi_i^-$ to the morphism $f$. Consider $i \in I$ a source and $\alpha_t : V_i \to V_t$ with $V_{j_t} = V_t$. In positions $j \neq i$ we have that $(\Phi_i^-(f))_j = f_j$. In position $i$ we have the following diagram



with the following squares being commutative for $1 \leq t \leq k$

$$
\begin{array}{ccc}
V_t & \xrightarrow{f_t} & W_t \\
\uparrow{\scriptstyle \alpha_i} & & \uparrow{\scriptstyle \beta_i} \\
V_i & \xrightarrow{f_i} & W_i
\end{array}
\tag{2.6}
$$

From the commutativity of the square in (2.6) we have the equations

$$
\begin{aligned}
\beta_1 f_i &= f_1 \alpha_1 \\
&\vdots \\
\beta_k f_i &= f_k \alpha_k .
\end{aligned}
\tag{2.7}
$$

So we can write the equations in (2.7) as

$$
\beta f_i = \begin{pmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_k \end{pmatrix} \alpha,
\tag{2.8}
$$

where $\alpha = \begin{pmatrix} \alpha_1 & \cdots & \alpha_k \end{pmatrix}$ and $\beta = \begin{pmatrix} \beta_1 & \cdots & \beta_k \end{pmatrix}$.

Using (2.8) we can draw the commutative diagram with exact lines

$$
\begin{array}{ccccccc}
V_i & \xrightarrow{\alpha} & \bigoplus_{j=1}^{k} V_j & \longrightarrow & Coker(\alpha) & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_i} & & \downarrow & & & & \\
W_i & \xrightarrow[\beta]{} & \bigoplus_{j=1}^{k} W_j & \longrightarrow & Coker(\beta) & \longrightarrow & 0
\end{array}
$$

hence there exists a unique morphism $\overline{f}$ such that the diagram below is commutative

$$
\begin{array}{ccccccc}
V_i & \xrightarrow{\alpha} & \bigoplus_{j=1}^{k} V_j & \longrightarrow & Coker(\alpha) & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_i} & & \downarrow & & \downarrow{\scriptstyle \overline{f}} & & \\
W_i & \xrightarrow[\beta]{} & \bigoplus_{j=1}^{k} W_j & \longrightarrow & Coker(\beta) & \longrightarrow & 0
\end{array}
$$

Then $(\Phi_i^-(V))_i = \text{Coker}(\alpha)$, $(\Phi_i^-(W))_i = \text{Coker}(\beta)$ and $(\Phi_i^-(f))_i = \overline{f}$. Therefore, $\Phi_i^-(f) : \Phi_i^-(V) \to \Phi_i^-(W)$ is well-defined on the morphisms.

Since the diagram

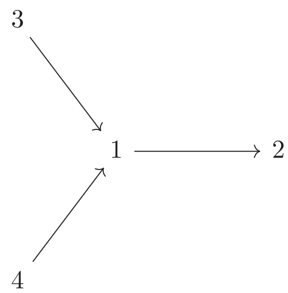
$$
\begin{array}{ccccc}
V_j & \longrightarrow & \bigoplus_{j=1}^{k} V_j & \longrightarrow & Coker(\alpha) \\
\downarrow{\scriptstyle f_j} & & \downarrow & & \downarrow{\scriptstyle \overline{f}} \\
W_j & \longrightarrow & \bigoplus_{j=1}^{k} W_j & \longrightarrow & Coker(\beta)
\end{array}
$$

is commutative, then the square

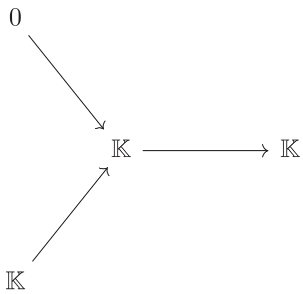
$$\begin{array}{ccc} V_j & \longrightarrow & Coker(\alpha) \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ W_j & \longrightarrow & Coker(\beta) \end{array}$$

is also commutative. Then $\overline{f}$ is a morphism between representations.

**Example 2.1.8.** *Consider the quiver $\vec{Q}$ given by*

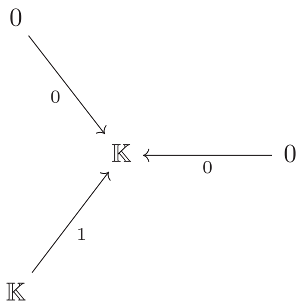



*and consider the representation $V$ given by*

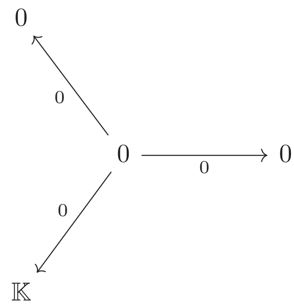



*We can apply the reflection functors, and we would have,*

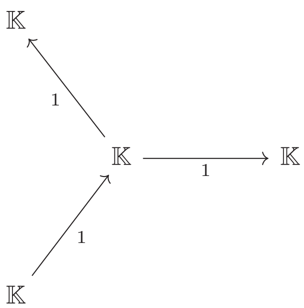
$\Phi_2^+(V) =$




$\Phi_1^-(\Phi_2^+(V)) =$




$\Phi_3^-(V) =$

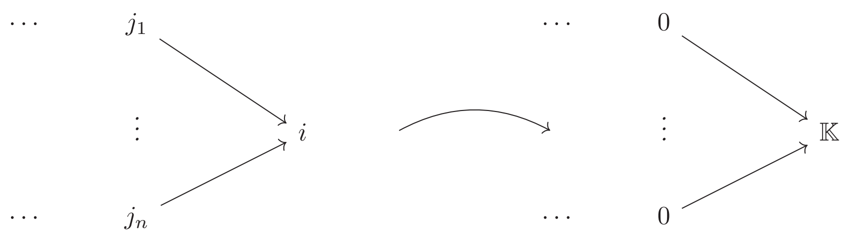


*which is true because in the first case we have* $Ker(\mathbb{K} \to \mathbb{K}) = 0$*, in the second case we have* $Coker(\mathbb{K} \to \mathbb{K}) = 0$ *and in the third case we have* $Coker(0 \to \mathbb{K}) = \mathbb{K}$*.*

**Remark 2.1.9.**

*(1) If* $i \in I$ *is a sink then* $\Phi_i^+(S(i)) = 0$*.*

*By* (1.3) *we have* $(S(i))_j = 0$ *for all* $j \neq i$ *and* $(S(i))_i = \mathbb{K}$*. Since* $i$ *is a sink we can let* $h_t : j_t \to i$ *be all the arrows that end in* $i$*. That part of the quiver is shown below*



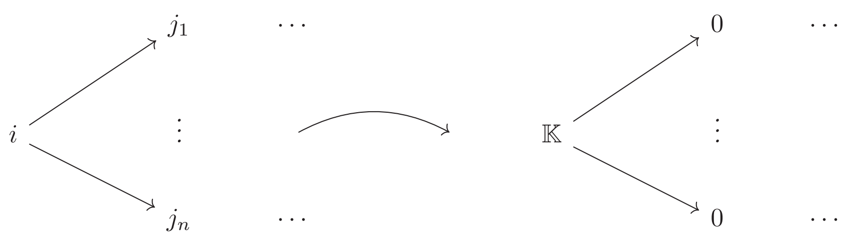
*For all* $j \neq i$ *we have that* $(\Phi_i^+(S(i)))_j = 0$ *and*

$$(\Phi_i^+(S(i)))_i = Ker\left(\bigoplus_{t=1}^{n}(S(i))_{j_t} \to (S(i))_i\right) = Ker\left(\bigoplus_{t=1}^{n} 0 \to \mathbb{K}\right) = Ker(0 \to \mathbb{K}) = 0$$

*therefore* $(\Phi_i^+(S(i)))_j = 0$ *for all* $j \in I$ *and* $\Phi_i^+(S(i)) = 0$*.*

*(2) If* $i \in I$ *is a source then* $\Phi_i^-(S(i)) = 0$*.*

*By* (1.3) *we have* $(S(i))_j = 0$ *for all* $j \neq i$ *and* $(S(i))_i = \mathbb{K}$*. Since* $i$ *is a source we can let* $h_t : i \to j_t$ *be all the arrows that begin in* $i$*. That part of the quiver is shown below*



*For all* $j \neq i$ *we have that* $(\Phi_i^-(S(i)))_j = 0$ *and*

$$(\Phi_i^-(S(i))_i = Coker\left((S(i))_i \to \bigoplus_{t=1}^{n}(S(i))_{j_t}\right) = Coker\left(\mathbb{K} \to \bigoplus_{t=1}^{n} 0\right) = Coker(\mathbb{K} \to 0) = \frac{0}{0} = 0$$

*so* $(\Phi_i^-(S(i))_j = 0$ *for all* $j \in I$ *and* $\Phi_i^-(S(i)) = 0$*.*

## 2.2 Properties of the reflection functors

The reflection functors have interesting properties, some of them we need to prove Gabriel's Theorem in Chapter 3 and others we use to study preprojective representations in Chapter 4. This section is dedicated to these properties. Consider the notation $R^n\Phi$ for the derived functors of a left exact functor $\Phi$, and $L^n\Phi$ for the derived functors of a right exact functor $\Phi$.

**Theorem 2.2.1.** *Let $\vec{Q}$ be a quiver and $i \in I$.*

*(1) The functor $\Phi_i^+$ is left exact.*

*(2) The functor $\Phi_i^-$ is right exact.*

*Proof.* (1) Let $V, W, U \in \operatorname{Rep}(\vec{Q})$ such that

$$0 \longrightarrow V \xrightarrow{\ f\ } W \xrightarrow{\ g\ } U \tag{2.9}$$

is an exact sequence. We shall prove that

$$0 \longrightarrow \Phi_i^+(V) \xrightarrow{\Phi_i^+(f)} \Phi_i^+(W) \xrightarrow{\Phi_i^+(g)} \Phi_i^+(U) \tag{2.10}$$

is an exact sequence.

For every $j \neq i$ we have that

$$(\Phi_i^+(V))_j = V_j, (\Phi_i^+(W))_j = W_j, (\Phi_i^+(U))_j = U_j, (\Phi_i^+(f))_j = f_j \text{ and } (\Phi_i^+(g))_j = g_j,$$

so the sequence is exact.

For $i$ we have the commutative diagram with exact lines below

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
0 & \longrightarrow & Ker(\phi) & \xrightarrow{\ a\ } & \bigoplus V_j & \xrightarrow{\ \phi\ } & V_i \\
 & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle f_i} \\
0 & \longrightarrow & Ker(\psi) & \xrightarrow{\ b\ } & \bigoplus W_j & \xrightarrow{\ \psi\ } & W_i \\
 & & \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle g_i} \\
0 & \longrightarrow & Ker(\eta) & \xrightarrow{\ c\ } & \bigoplus U_j & \xrightarrow{\ \eta\ } & U_i
\end{array} \tag{2.11}$$

To finish the argument we need to show that the sequence

$$0 \longrightarrow Ker(\phi) \xrightarrow{\ \overline{f}\ } Ker(\psi) \xrightarrow{\ \overline{g}\ } Ker(\eta)$$

is exact. In order to do that we need to show that the sequence

$$0 \longrightarrow \bigoplus V_j \xrightarrow{\ \alpha\ } \bigoplus W_j \xrightarrow{\ \beta\ } \bigoplus U_j$$

is also exact, where

$$\alpha = \begin{pmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_k \end{pmatrix} : \bigoplus V_j \to \bigoplus W_j \quad \text{and} \quad \beta = \begin{pmatrix} g_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_k \end{pmatrix} : \bigoplus W_j \to \bigoplus U_j.$$

We prove that $\alpha$ is injective and that $\mathrm{Ker}(\beta)=\mathrm{Im}(\alpha)$.

If $\alpha(v_1,\cdots,v_k)=(0,\cdots,0)$ then

$$\begin{pmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_k \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}=\begin{pmatrix} f_1(v_1) \\ \vdots \\ f_k(v_k) \end{pmatrix}=\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\begin{cases} f_1(v_1)=0 \\ \quad\vdots \\ f_k(v_k)=0. \end{cases}$$

The function $f_t$ is injective for every $t\in\{1,\cdots,k\}$, so $v_1=\cdots=v_k=0$ which means that

$$(v_1,\cdots,v_k)=(0,\cdots,0)$$

and $\alpha$ is injective.

If $(w_1,\cdots,w_k)\in\mathrm{Ker}(\beta)$ then $\beta(w_1,\cdots,w_k)=(0,\cdots,0)$ and we can write

$$\begin{pmatrix} g_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_k \end{pmatrix}\begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}=\begin{pmatrix} g_1(w_1) \\ \vdots \\ g_k(w_k) \end{pmatrix}=\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

And hence we have the equations

$$\begin{cases} g_1(w_1)=0 \\ \quad\vdots \\ g_k(w_k)=0. \end{cases}$$

This means that $w_t\in\mathrm{Ker}(g_t)$. From (2.9) we have that $\mathrm{Ker}(g_t)=\mathrm{Im}(f_t)$ and hence we can write $w_t=f_t(v_t)$ for some $v_t\in V_t$. Then

$$\begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}=\begin{pmatrix} f_1(v_1) \\ \vdots \\ f_k(v_k) \end{pmatrix}=\begin{pmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_k \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}=\alpha(v_1,\cdots,v_k)$$

which gives $\mathrm{Ker}(\beta)\subset\mathrm{Im}(\alpha)$.

If $(w_1,\cdots,w_k)\in\mathrm{Im}(\alpha)$ then we can write $(w_1,\cdots,w_k)=\alpha(v_1,\cdots,v_k)$ for some $v_t\in V_t$ and then

$$(w_1,\cdots,w_k)=(f_1(v_1),\cdots,f_k(v_k)).$$

We can write this equation as a system of equations,

$$\begin{cases} f_1(v_1)=w_1 \\ \quad\vdots \\ f_k(v_k)=w_k. \end{cases}$$

This means that $w_t \in \mathrm{Im}(f_t) = \mathrm{Ker}(g_t)$ and $g_t(w_t) = 0$, so we have

$$\beta(w_1, \cdots, w_k) = (g_1(w_1), \cdots, g_k(w_k)) = (0, \cdots, 0)$$

and $(w_1, \cdots, w_k) \in \mathrm{Ker}(\beta)$. This shows that $\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$.

Now we can prove that the sequence

$$0 \longrightarrow Ker(\phi) \xrightarrow{\overline{f}} Ker(\psi) \xrightarrow{\overline{g}} Ker(\eta)$$

is exact. For this we need to prove that $\overline{f}$ is injective and that $\mathrm{Ker}(\overline{g}) = \mathrm{Im}(\overline{f})$.

If $\overline{f}(v) = 0$ for $v \in \mathrm{Ker}(\phi)$ then from the diagram (2.11) we have that $b\overline{f}(v) = \alpha a(v) = 0$. Since $\alpha$ and $a$ are injective it follows that $v = 0$ and therefore $\overline{f}$ is injective.

If $w \in \mathrm{Ker}(\overline{g})$ then

$$\overline{g}(w) = 0 \quad \Rightarrow \quad c\overline{g}(w) = \beta b(w) = 0 \quad \Rightarrow \quad b(w) \in \mathrm{Ker}(\beta) = \mathrm{Im}(\alpha).$$

then there exists $v \in \oplus V_j$ such that $\alpha(v) = b(w)$ and

$$\psi\alpha(v) = f_i\phi(v) = \psi b(w) = 0 \quad \Rightarrow \quad \phi(v) \in \mathrm{Ker}(f_i).$$

Since $f_i$ is injective we have $\phi(v) = 0$ and $v \in \mathrm{Ker}(\phi) = \mathrm{Im}(a)$. There exists $u \in \mathrm{Ker}(\phi)$ such that $a(u) = v$ then

$$b\overline{f}(u) = \alpha a(u) = \alpha(v) = b(w) \quad \Rightarrow \quad b(\overline{f}(u) - w) = 0 \quad \Rightarrow \quad \overline{f}(u) - w \in \mathrm{Ker}(b).$$

Since $b$ is injective we have $\overline{f}(u) = w$ and $w \in \mathrm{Im}(\overline{f})$.

If $w \in \mathrm{Im}(\overline{f})$ then there exists $v \in \mathrm{Ker}(\phi)$ such that $\overline{f}(v) = w$. Then

$$b\overline{f}(v) = \alpha a(v) = b(w) \quad \Rightarrow \quad 0 = \beta\alpha a(v) = \beta b(w) = c\overline{g}(w) \quad \Rightarrow \quad \overline{g}(w) \in \mathrm{Ker}(c).$$

Since $c$ is injective we have $\overline{g}(w) = 0$ and $w \in \mathrm{Ker}(\overline{g})$. Therefore, $\mathrm{Ker}(\overline{g}) = \mathrm{Im}(\overline{f})$.

From the above discussion we have the following commutative diagram with exact lines and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Ker(\phi) & \xrightarrow{a} & \bigoplus V_j & \xrightarrow{\phi} & V_i \\
 & & \downarrow \overline{f} & & \downarrow \alpha & & \downarrow f_i \\
0 & \longrightarrow & Ker(\psi) & \xrightarrow{b} & \bigoplus W_j & \xrightarrow{\psi} & W_i \\
 & & \downarrow \overline{g} & & \downarrow \beta & & \downarrow g_i \\
0 & \longrightarrow & Ker(\eta) & \xrightarrow{c} & \bigoplus U_j & \xrightarrow{\eta} & U_i
\end{array}$$

And the sequence in (2.10) is exact and the functor $\Phi_i^+$ is left exact.

(2) The second item is done analogously.

□

These properties are useful in studying the representation theory of quivers, as they allow us to relate representations of a quiver to representations of certain subquivers or quotient quivers obtained by applying the reflection functors. Furthermore, they provide a connection between quiver representations and modules over certain algebras associated to the quiver, known as path algebras.

**Theorem 2.2.2.** *Retain the notation of the section.*

*(1)* $R^n\Phi_i^+(V) = 0$ *for all* $n > 1$*, and* $R^1\Phi_i^+(V) = 0$ *if and only if* $V$ *satisfies the following condition*

$$\left(\bigoplus_{h:k\to i} V_k\right) \to V_i \qquad \textit{is surjective.} \tag{2.12}$$

*(2)* $L^n\Phi_i^-(V) = 0$ *for all* $n > 1$*, and* $L^1\Phi_i^-(V) = 0$ *if and only if* $V$ *satisfies the following condition*

$$V_i \to \left(\bigoplus_{h:i\to k} V_k\right) \qquad \textit{is injective.} \tag{2.13}$$

*Proof.* (1) By Theorem 1.4.9 any representation $V \in \text{Rep}(\vec{Q})$ has an injective resolution, which means that there exists a short exact sequence

$$0 \longrightarrow V \xrightarrow{\ \delta\ } I_0 \xrightarrow{\ d_o\ } I_1 \longrightarrow 0$$

where $I_0, I_1$ are injective representations.

We have that $R^i\Phi_i^+(V) = H^i(R\Phi_i^+(V))$. By Theorem 2.2.1 we get the exact sequence

$$0 \longrightarrow \Phi_i^+(V) \xrightarrow{\ \Phi_i^+(\delta)\ } \Phi_i^+(I_0) \xrightarrow{\ \Phi_i^+(d_0)\ } \Phi_i^+(I_1) \tag{2.14}$$

and we can write the complex

$$0 \longrightarrow \Phi_i^+(I_0) \xrightarrow{\ \Phi_i^+(d_0)\ } \Phi_i^+(I_1) \longrightarrow 0.$$

Note that

$$R^0\Phi_i^+(V) = \text{Ker}(\Phi_i^+(d_0)), \qquad R^1\Phi_i^+(V) = \frac{\Phi_i^+(I_1)}{\text{Im}(\Phi_i^+(d_0))}, \qquad R^n\Phi_i^+(V) = 0\ \forall\ n > 1.$$

For $R^0\Phi_i^+(V)$ we know that the sequence (2.14) is exact and hence

$$R^0\Phi_i^+(V) = \text{Ker}(\Phi_i^+(d_0)) = \text{Im}(\Phi_i^+(\delta)) = \Phi_i^+(V)$$

since $\Phi_i^+(\delta)$ is injective and

$$\text{Im}(\Phi_i^+(\delta)) = \frac{\Phi_i^+(V)}{\text{Ker}(\Phi_i^+(\delta))}.$$

For $R^1\Phi_i^+(V)$ we need to study what happens in $j \neq i$ and in $i$ separated. For $j \neq i$ we have

$$0 \longrightarrow (\Phi_i^+(V))_j = V_j \xrightarrow{\delta_j} (\Phi_i^+(I_0))_j = (I_0)_j \xrightarrow{(d_0)_j} (\Phi_i^+(I_1))_j = (I_1)_j \longrightarrow 0$$

and $\mathrm{Im}((\Phi_i^+(d_0))_j) = \mathrm{Im}((d_0)_j) = (\Phi_i^+(I_1))_j = (I_1)_j$ then $(R^1\Phi_i^+(V))_j = 0$.

Now for $i$, consider $j_1, \cdots, j_n \in I$ to be all the vertex such that $h_k : j_k \to i$. Then we have

$$\phi : \bigoplus_{t=1}^{n} V_{j_t} \to V_i, \quad \psi : \bigoplus_{t=1}^{n} (I_0)_{j_t} \to (I_0)_i \quad \text{and} \quad \eta : \bigoplus_{t=1}^{n} (I_1)_{j_t} \to (I_1)_i.$$

So we have the commutative diagram with exact lines and columns,

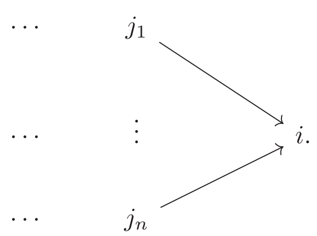
$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (\Phi_i^+(V))_i & \xrightarrow{a} & \bigoplus V_{j_t} & \xrightarrow{\phi} & V_i \\
 & & \downarrow{\scriptstyle (\Phi_i^+(\delta))_i} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta_i} \\
0 & \longrightarrow & (\Phi_i^+(I_0))_i & \xrightarrow{b} & \bigoplus (I_0)_{j_t} & \xrightarrow{\psi} & (I_0)_i \\
 & & \downarrow{\scriptstyle (\Phi_i^+(d_0))_i} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle (d_0)_i} \\
0 & \longrightarrow & (\Phi_i^+(I_1))_i & \xrightarrow{c} & \bigoplus (I_1)_{j_t} & \xrightarrow{\eta} & (I_1)_i \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Coker}((\Phi_i^+(d_0))_i) & & 0 & & 0
\end{array} \tag{2.15}$$

We first show that $\psi$ is surjective. Since the representation $I_0$ is injective we can write $I_0$ as the direct sum of injective indecomposable representations, i.e.,

$$I_0 = \bigoplus_{h=1}^{m} I_h \text{ for some } m$$

where $I_h = (e_h A)^*$ are the linear combination of paths ending in $h \in I$.

By Definition 2.1.7 $i$ is a sink for $\vec{Q}$, so we have




For $h \neq i$ the indecomposable component $I_h$ is zero since there are no paths beginning in $i$ and ending in $h$, so $\bigoplus (I_0)_h = 0$. If $(I_0)_i$ is zero of course $\psi$ is surjective. If $(I_0)_i$ is different from zero we need to have the indecomposable component $I_i$ in the sum of $I_0$. For $I_i$ we have

$$\begin{array}{cccc}
\cdots & \mathbb{K}^{t_1} & \searrow & \\
\cdots & \vdots & & \mathbb{K} \\
\cdots & \mathbb{K}^{t_n} & \nearrow &
\end{array}$$

where $t_k$ is the number of paths from $j_k$ to $i$. So

$$\psi : \bigoplus_{k=1}^{n} \mathbb{K}^{t_k} \to \mathbb{K}$$

and $\psi$ is surjective. Therefore, $\psi$ is surjective in all cases. The diagram (2.15) becomes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\Phi_i^+(V))_i & \xrightarrow{a} & \bigoplus V_{j_t} & \xrightarrow{\phi} & V_i & & \\
 & & \downarrow{\scriptstyle (\Phi_i^+(\delta))_i} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta_i} & & \\
0 & \longrightarrow & (\Phi_i^+(I_0))_i & \xrightarrow{b} & \bigoplus (I_0)_{j_t} & \xrightarrow{\psi} & (I_0)_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\Phi_i^+(d_0))_i} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle (d_0)_i} & & \\
0 & \longrightarrow & (\Phi_i^+(I_1))_i & \xrightarrow{c} & \bigoplus (I_1)_{j_t} & \xrightarrow{\eta} & (I_1)_i & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker}((\Phi_i^+(d_0))_i) & & 0 & & 0 & &
\end{array}$$

By the Snake Lemma there exists a morphism

$$\gamma : V_i \to \operatorname{Coker}((\Phi_i^+(d_0))_i)$$

such that the sequence

$$(\Phi_i^+(V))_i \xrightarrow{a} \bigoplus V_{j_t} \xrightarrow{\phi} V_i \xrightarrow{\gamma} \operatorname{Coker}((\Phi_i^+(d_0))_i) \longrightarrow 0 \longrightarrow 0$$

is exact. Then we have that

$$\phi \text{ is surjective} \iff \operatorname{Coker}((\Phi_i^+(d_0))_i) = (R^1\Phi_i^+(V))_i = 0.$$

If $\phi$ is surjective then $\operatorname{Im}(\phi) = V_i$. Since the sequence is exact we have that $V_i = \operatorname{Im}(\phi) = \operatorname{Ker}(\gamma)$, which means that for every $v \in V_i$ we have $\gamma(v) = 0$ and that is all the domain. So $\gamma = 0$, $\operatorname{Im}(\gamma) = 0$ and $\gamma$ is also surjective so $\operatorname{Coker}((\Phi_i^+(d_0))_i) = \operatorname{Im}(\gamma) = 0$.

If $\operatorname{Coker}((\Phi_i^+(d_0))_i) = 0$ then $\phi$ is surjective, since the sequence is exact.

(2) The second item is done analogously.

□

Based in the Theorem 2.2.2 we will make the next definition.

**Definition 2.2.3.**

*(i) We denote by* $Rep^{i,-}$ *the full subcategory of* $Rep(\vec{Q})$ *consisting of representations* $V$ *that satisfy*

$$\left(\bigoplus_{h:k\to i} V_k\right) \to V_i \qquad \textit{is surjective.}$$

*(ii) We denote by* $Rep^{i,+}$ *the full subcategory of* $Rep(\vec{Q})$ *consisting of representations* $V$ *that satisfy*

$$V_i \to \left(\bigoplus_{h:i\to k} V_k\right) \qquad \textit{is injective.}$$

For the proof of the next important result we need a lemma on commutative diagrams.

**Lemma 2.2.4.** *If we have a commutative diagram of modules and linear applications*

$$\begin{array}{ccccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \\ L' & \xrightarrow[f']{} & M' & \xrightarrow[g']{} & N' & & \end{array}$$

*with the first line exact and* $g'f' = 0$*, then there exists only one* $w : N \to N'$ *such that* $wg = g'v$*.*

*Proof.* Let $x \in N$, then there exists $y \in M$ such that $g(y) = x$ since $g$ is surjective, and $g'v(y) \in N'$, so define $g'v(y) = y_x$, and

$$w : N \to N'$$
$$x \mapsto y_x.$$

The map $w$ is well-defined, indeed if there exists $y^1, y^2 \in M$ such that $g(y^1) = g(y^2) = x$ then by definition

$$w(x) = y^1_x = g'v(y^1)$$
$$w(x) = y^2_x = g'v(y_2)$$

therefore $y^1_x = y^2_x$.

The map $w$ is a linear application by how we defined it. Let us see that $wg = g'v$, given $y \in M$ write $g(y) = x$ then we have $wg(y) = w(x) = y_x = g'v(y)$, so $wg(y) = g'v(y)$.

If there exists $w' : N \to N'$ linear application such that $w'g = g'v$ then $w'g = g'v = wg$ and since $g$ is surjective $w = w'$. □

The next theorem brings more properties of the functors $\Phi_i^{\pm}$ and using the Definition 2.2.3 we have an equivalence between $\text{Rep}^{i,-}(\vec{Q})$ and $\text{Rep}^{i,+}(\vec{Q}')$. Recall that the simple reflections $s_i = s_{\alpha_i} : \mathbb{Z}^I \to \mathbb{Z}^I$ are as defined in Appendix A.

**Theorem 2.2.5.** *Let $i$ be a sink for $\vec{Q}$ and $\vec{Q}' = s_i^+(\vec{Q})$. Consider $Rep^{i,\pm}$ as defined in Definition 2.2.3.*

*(1) Functors $\Phi_i^+, \Phi_i^-$ are adjoint to each other, which means that for any $W \in Rep(\vec{Q}')$, $V \in Rep(\vec{Q})$, we have a natural isomorphism*

$$Hom_{\vec{Q}}(\Phi_i^-(W), V) \simeq Hom_{\vec{Q}'}(W, \Phi_i^+(V)).$$

*(2) For any $V \in Rep(\vec{Q}), \Phi_i^+(V) \in Rep^{i,+}(\vec{Q}')$. Similarly, for any $W \in Rep(\vec{Q}'), \Phi_i^-(W) \in Rep^{i,-}(\vec{Q})$.*

*(3) The restrictions*

$$\Phi_i^+ : Rep^{i,-}(\vec{Q}) \to Rep^{i,+}(\vec{Q}')$$
$$\Phi_i^- : Rep^{i,+}(\vec{Q}') \to Rep^{i,-}(\vec{Q})$$

*are inverse to each other and thus give an equivalence of categories $Rep^{i,-}(\vec{Q}) \simeq Rep^{i,+}(\vec{Q}')$.*

*(4) If $V \in Rep^{i,-}(\vec{Q})$, then $dim(\Phi_i^+(V)) = s_i(dim(V))$.*

*Similarly, if $V' \in Rep^{i,+}(\vec{Q}')$, then $dim(\Phi_i^-(V')) = s_i(dim(V'))$.*

*Proof.*

(1) Let

$$\phi : \text{Hom}_{\vec{Q}}(\Phi_i^-(W), V) \to \text{Hom}_{\vec{Q}'}(W, \Phi_i^+(V)),$$

and take $\delta : W \to \Phi_i^+(V)$. We want to find out which morphism $\gamma : \Phi_i^-(W) \to V$ is such that $\phi(\gamma) = \delta$. For each $j \neq i$ we have that $\delta_j : W_j \to (\Phi_i^+(V))_j = V_j$, therefore we have $\gamma_j = \delta_j$.

Now for $i$ we have

$$\delta_i : W_i \to (\Phi_i^+(V))_i = \text{Ker}(v) \text{ where } v : \bigoplus_{h:j\to i} V_j \to V_i.$$

We can draw the commutative diagram with exact lines

$$\begin{array}{ccccccccc} & & W_i & \xrightarrow{w} & \bigoplus W_j & \longrightarrow & Coker(w) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_i} & & \downarrow & & & & \\ 0 & \longrightarrow & Ker(v) & \longrightarrow & \bigoplus V_j & \xrightarrow[v]{} & V_i & & \end{array}$$

By Lemma 2.2.4 there exists only one morphism $\gamma_i : \text{Coker}(w) \to V_i$ such that the square is commutative

$$\begin{array}{ccccccccc} & & W_i & \xrightarrow{w} & \bigoplus W_j & \longrightarrow & Coker(w) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_i} & & \downarrow & & \downarrow{\scriptstyle \gamma_i} & & \\ 0 & \longrightarrow & Ker(v) & \longrightarrow & \bigoplus V_j & \xrightarrow[v]{} & V_i & & \end{array}$$

Therefore we define a $\gamma : \Phi_i^-(W) \to V$ such that $\phi(\gamma) = \delta$. For $\gamma_k = \delta_k$ the following diagram is commutative

$$\begin{array}{ccc} W_k & \longrightarrow & Coker(w) \\ {\scriptstyle \gamma_k}\downarrow & & \downarrow{\scriptstyle \gamma_i} \\ V_k & \longrightarrow & V_i \end{array}$$

since the next diagram is commutative

$$\begin{array}{ccccc} W_k & \longrightarrow & \bigoplus W_j & \longrightarrow & Coker(w) \\ {\scriptstyle \gamma_k}\downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma_i} \\ V_k & \longrightarrow & \bigoplus V_j & \longrightarrow & V_i \end{array}$$

We have that $\gamma$ is a morphism between $\Phi_i^-(W)$ and $V$ with $\phi(\gamma) = \delta$. Then we constructed $\phi$ to be surjective.

If we let $\phi(\gamma) = \phi(\gamma') = \delta$ then $\gamma_j = \delta_j = \gamma'_j$ for all $j \neq i$. For $i$ we have that $\gamma_i$ is the only morphism that makes the following diagram commutate

$$\begin{array}{ccccccccc} & & W_i & \xrightarrow{w} & \bigoplus W_j & \longrightarrow & Coker(w) & \longrightarrow & 0 \\ & & {\scriptstyle \delta_i}\downarrow & & \downarrow & & \vdots{\scriptstyle \gamma_i} & & \\ 0 & \longrightarrow & Ker(v) & \longrightarrow & \bigoplus V_j & \xrightarrow[v]{} & V_i & & \end{array}$$

The morphism $\gamma'_i$ also makes that diagram commutative, so $\gamma_i = \gamma'_i$ and $\gamma = \gamma'$. Which makes $\phi$ an injective morphism.

Then $\phi$ is an isomorphism.

(2) For every representation $V \in \text{Rep}(\vec{Q})$ we have the exact sequence

$$0 \longrightarrow Ker(\phi) \longrightarrow \bigoplus V_j \xrightarrow{\phi} V_i.$$

We have that $(\Phi_i^+(V))_i = \text{Ker}(\phi)$ so

$$(\Phi_i^+(V))_i = \text{Ker}(\phi) \to \bigoplus V_j$$

is injective and then $\Phi_i^+(V) \in \text{Rep}^{i,+}(\vec{Q}')$.

For every representation $W \in \text{Rep}(\vec{Q}')$ we have the exact sequence

$$W_i \xrightarrow{\psi} \bigoplus W_j \longrightarrow \text{Coker}(\psi) \longrightarrow 0.$$

We have that $(\Phi_i^-(W))_i = \text{Coker}(\psi)$ so

$$\bigoplus W_j \to \text{Coker}(\psi) = (\Phi_i^-(W))_i$$

is surjective and then $\Phi_i^-(W) \in \text{Rep}^{i,-}(\vec{Q})$.

(3) We want to see that

$$Rep^{i,-}(\vec{Q}) \underset{\Phi_i^-}{\overset{\Phi_i^+}{\rightleftarrows}} Rep^{i,+}(\vec{Q}')$$

are inverse to each other and they are equivalences of categories.

We will see that $\Phi_i^-\Phi_i^+ = Id$ and $\Phi_i^+\Phi_i^- = Id$ is analogous.

Let us apply $\Phi_i^-\Phi_i^+$ to a representation. We want to see that $\Phi_i^-\Phi_i^+(V) = V$. Of course $(\Phi_i^-\Phi_i^+(V))_j = V_j$ with $j \neq i$. For $i$ we have that $(\Phi_i^+(V))_i = \mathrm{Ker}(\phi)$ where $\phi : \bigoplus V_j \to V_i$. We have the commutative diagram with exact lines

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ker(\phi) & \xrightarrow{\delta} & \bigoplus V_j & \xrightarrow{\phi} & V_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & \\
0 & \longrightarrow & Ker(\phi) & \xrightarrow[\delta]{} & \bigoplus V_j & \longrightarrow & Coker(\delta) & \longrightarrow & 0
\end{array}$$

hence there exists $\beta : V_i \to \mathrm{Coker}(\delta)$ such that the square commutes. Then we have the commutative diagram below

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & Ker(\beta) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ker(\phi) & \xrightarrow{\delta} & \bigoplus V_j & \xrightarrow{\phi} & V_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & Ker(\phi) & \xrightarrow[\delta]{} & \bigoplus V_j & \longrightarrow & Coker(\delta) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & Coker(\beta) & &
\end{array}$$

By the Snake Lemma we have that $\beta$ is an isomorphism, so $\mathrm{Coker}(\delta) \simeq V_i$. Therefore, $(\Phi_i^-\Phi_i^+(V))_i = \mathrm{Coker}(\delta) \simeq V_i$.

(4) If $V \in \mathrm{Rep}^{i,-}(\vec{Q})$ then the sequence

$$0 \longrightarrow \mathrm{Ker}(\phi) \longrightarrow \bigoplus V_j \xrightarrow{\phi} V_i \longrightarrow 0$$

is exact. The dimension vector of $\Phi_i^+(V)$ is

$$\begin{aligned}
\dim(\Phi_i^+(V)) &= \big(\dim(\Phi_i^+(V))_1, \cdots, \dim(\Phi_i^+(V))_i, \cdots, \dim(\Phi_i^+(V))_\ell\big) \\
&= (\dim(V_1), \cdots, \dim(\mathrm{Ker}(\phi)), \cdots, \dim(V_\ell))
\end{aligned}$$

where $\phi : \bigoplus V_j \to V_i$. By the exact sequence we have,

$$\dim(\mathrm{Ker}(\phi)) = \sum \dim(V_j) - \dim(V_i).$$

On the other hand we have that

$$s_i(\alpha_j) = \alpha_j - (\alpha_j, \alpha_i)\alpha_i$$

where $(\alpha_j, \alpha_i) = 2\delta_{ij} - A_{ij}$ and $A_{ij}$ is the number of edges between $i$ and $j$. So

$$s_i(\alpha_j) = \begin{cases} \alpha_j + A_{ij}\alpha_i, & \text{if } j \neq i \\ -\alpha_i, & \text{if } j = i. \end{cases}$$

For any given vector we can write

$$(x_1, \cdots, x_\ell) = x_1\alpha_1 + \cdots + x_\ell\alpha_\ell.$$

The reflection of this vector is

$$s_i(x_1, \cdots, x_\ell) = x_1 s_i(\alpha_1) + \cdots + x_\ell s_i(\alpha_\ell).$$

If $x_j = \dim(V_j)$ we know that $x_j s_j(\alpha_j) = -x_j\alpha_j$. We know that $i$ is a sink so either it does not have any edges between $i$ and $j$ which leads us to $x_j s_i(\alpha_j) = x_j\alpha_j$ or there exists edges from $j$ to $i$. Then $x_j s_i(\alpha_j) = x_j(\alpha_j + \alpha_i) = x_j\alpha_j + x_j\alpha_i$.

Which gives us

$$s_i(\dim(V)) = \left( \dim(V_1), \cdots, \sum_{h:j\to i} \dim(V_j) - \dim(V_i), \cdots, \dim(V_\ell) \right)$$

and therefore $\dim(\Phi_i^+(V)) = s_i(\dim(V))$.

The second statement is done analogously.

□

**Corollary 2.2.6.** *If $i$ is a sink for $\vec{Q}$, $\vec{Q}' = s_i^+\vec{Q}$, $V \in Rep^{i,-}(\vec{Q})$ and $W \in Rep(\vec{Q})$ then*

$$Hom_{\vec{Q}}(V, W) = Hom_{\vec{Q}'}(\Phi_i^+(V), \Phi_i^+(W)).$$

*Similarly, if $i$ is a source for $\vec{Q}$, $\vec{Q}' = s_i^-\vec{Q}$, $V \in Rep(\vec{Q})$ and $W \in Rep^{i,+}(\vec{Q})$ then*

$$Hom_{\vec{Q}}(V, W) = Hom_{\vec{Q}'}(\Phi_i^-(V), \Phi_i^-(W)).$$

*Proof.* This immediately follows from Theorem 2.2.5. By item (3) if $V \in \mathrm{Rep}^{i,-}(\vec{Q})$ then $V = \Phi_i^-(X)$ for some $X \in \mathrm{Rep}^{i,+}(\vec{Q}')$. By item (1) it follows that

$$\mathrm{Hom}_{\vec{Q}}(V, W) = \mathrm{Hom}_{\vec{Q}}(\Phi_i^-(X), W) = \mathrm{Hom}_{\vec{Q}'}(X, \Phi_i^+(W)) = \mathrm{Hom}_{\vec{Q}'}(\Phi_i^+(V), \Phi_i^+(W)).$$

The second part is proved in the same way. □

By Theorem 1.2.5 every representation can be written as the direct sum of indecomposable representations, which makes indecomposable representations an important type of representation. In light of this fact, the next theorem investigates the behavior of indecomposable representations under the application of the functor $\Phi_i^{\pm}$.

**Theorem 2.2.7.** *Let $V \in Rep(\vec{Q})$ be a nonzero indecomposable representation and let $i \in I$ be a sink for $\vec{Q}$, when we apply $\Phi_i^+$, or be a source for $\vec{Q}$, when we apply $\Phi_i^-$.*

*(1) We have that $V \simeq S(i)$ if and only if $\Phi_i^{\pm}(V) = 0$.*

*(2) If $V \not\simeq S(i)$ then $V \in Rep^{i,\mp}(\vec{Q})$ and $\Phi_i^{\pm}(V)$ is a nonzero indecomposable representation of $\vec{Q}' = s_i^{\pm}(\vec{Q})$. In this case, $V \simeq \Phi_i^{\mp}(\Phi_i^{\pm}(V))$.*

*Proof.* (1) The first part is in Remark 2.1.9.

If $\Phi_i^+(V) = 0$ then $(\Phi_i^+(V))_j = V_j = 0$ for all $j \neq i$ and $(\Phi_i^+(V))_i = \text{Ker}(\phi) = 0$ where $\phi : \bigoplus V_j \to V_i$ and $\bigoplus V_j = 0$ since all $V_j = 0$. We have that $V_i \simeq \mathbb{K}$ otherwise the representation would not be indecomposable. Therefore, $V \simeq S(i)$.

(2) If $i \in I$ is a sink and $V \notin \text{Rep}^{i,-}(\vec{Q})$ then

$$\bigoplus_{h:j\to i} V_j \to V_i$$

is not surjective. So we can write $V_i = \text{Im}(\phi) \bigoplus \mathbb{K}^r$ and $V$ can be rewritten as

$$V = V' \bigoplus S(i)^r$$

where $V'_j = V_j$ for $j \neq i$ and $V'_i = \text{Im}(\phi)$, which is a contradiction since $V$ is indecomposable. Therefore, $V \in \text{Rep}^{i,-}(\vec{Q})$.

From Corollary 2.2.6 we have that

$$\text{Hom}_{\vec{Q}}(V, V) = \text{Hom}_{\vec{Q}'}(\Phi_i^+(V), \Phi_i^+(V)),$$

so $\text{End}(V) = \text{End}(\Phi_i^+(V))$. As $V$ is indecomposable then $\text{End}(V)$ is a local ring and by the equality so is $\text{End}(\Phi_i^+(V))$, therefore $\Phi_i^+(V)$ is indecomposable. If $V$ is indecomposable and $\Phi_i^+(V) = 0$ then $V \simeq S(i)$, but by hypothesis $V \not\simeq S(i)$ so $\Phi_i^+(V) \neq 0$. Therefore, by the proof of item (3) of Theorem 2.2.5, $V \simeq \Phi_i^- \Phi_i^+(V)$.

□

**Corollary 2.2.8.** *Let $V \in Rep(\vec{Q})$ be a nonzero indecomposable representation and let $i \in I$ be a sink, respectively a source, for $\vec{Q}$. We have that $\Phi_i^{\pm}(V)$ is a nonzero indecomposable if and only if $s_i(dim(V)) \geq 0$.*

*Proof.* If $\Phi_i^+(V)$ and $V$ are nonzero indecomposable representations then by Theorem 2.2.7 (1) $V \not\simeq S(i)$. By Theorem 2.2.7 (2) we have that $V \in \text{Rep}^{i,-}(\vec{Q})$. By Theorem 2.2.5 (4), $\dim(\Phi_i^+(V)) = s_i(\dim(V))$. We have that

$$\dim(\Phi_i^+(V))_j = \dim(V_j) \geq 0.$$

In the position $i$ we have

$$\dim(\Phi_i^+(V))_i = \dim(\text{Ker}(\phi)) = -\dim(V_i) + \sum \dim(V_j) \geq 0$$

since $\phi$ is surjective. Therefore, $s_i(\dim(V)) \geq 0$.

If $s_i(\dim(V)) \geq 0$ we saw that

$$s_i(\dim(V)) = \begin{cases} \dim(V_j), & \text{if } j \neq i \\ -\dim(V_i) + \sum_{h:k\to i} \dim(V_k), & \text{if } j = i \end{cases}$$

so, $\dim(V_j) \geq 0$ and $-\dim(V_i) + \sum_{h:k\to i} \dim(V_k) \geq 0$. If $V \simeq S(i)$ then

$$\dim(V_j) = 0 \text{ for all } j \neq i \quad \Rightarrow \quad \sum_{h:k\to i} \dim(V_k) = 0$$

and $\dim(V_i) = 1$ therefore

$$-\dim(V_i) + \sum_{h:k\to i} \dim(V_k) = -1 + 0 = -1 \ngeq 0$$

which is a contradiction. So $V \not\simeq S(i)$ and $V$ is a nonzero indecomposable representation. By Theorem 2.2.7 $\Phi_i^{\pm}(V)$ is a nonzero indecomposable representation. □

**Corollary 2.2.9.** *If $i$ is a sink for $\vec{Q}, \vec{Q}' = s_i^+\vec{Q}$, $V \in Rep(\vec{Q})$ and $W \in Rep^{i,-}(\vec{Q})$, then*

$$Ext^1_{\vec{Q}}(V, W) = Ext^1_{\vec{Q}'}(\Phi_i^+(V), \Phi_i^+(W)).$$

*Similarly, if $i$ is a source for $\vec{Q}, \vec{Q}' = s_i^-\vec{Q}$, $V \in Rep^{i,+}(\vec{Q})$ and $W \in Rep(\vec{Q})$, then*

$$Ext^1_{\vec{Q}}(V, W) = Ext^1_{\vec{Q}'}(\Phi_i^-(V), \Phi_i^-(W)).$$

*Proof.* We will prove the first statement and the second can be done analogously. It suffices to consider the case when $V$ is indecomposable. Since $i$ is a sink, $S(i) = P(i)$ and if $V = S(i) = P(i)$ then

$$0 = \mathrm{Ext}^1_{\vec{Q}}(V, W) = \mathrm{Ext}^1_{\vec{Q}'}(\Phi_i^+(V), \Phi_i^+(W)) = 0.$$

Thus, we can assume that $V \not\simeq S(i)$. According to Theorem 2.2.7, $V \in \mathrm{Rep}^{i,-}(\vec{Q})$. Let

$$0 \longrightarrow W \longrightarrow A \longrightarrow V \longrightarrow 0$$

be an extension representing some class in $\mathrm{Ext}^1(V, W)$. Note that since $R^1\Phi_i^+(V) = R^1\Phi_i^+(W) = 0$, a long exact sequence of derived functors implies that $R^1\Phi_i^+(A) = 0$, so $A \in \mathrm{Rep}^{i,-}(\vec{Q})$.

Since $W \in \mathrm{Rep}^{i,-}(\vec{Q})$, we have $R^1\Phi_i^+(W) = 0$, so the sequence

$$0 \longrightarrow \Phi_i^+(W) \longrightarrow \Phi_i^+(A) \longrightarrow \Phi_i^+(V) \longrightarrow 0$$

is also exact, this gives a map $\mathrm{Ext}^1_{\vec{Q}}(V, W) \to \mathrm{Ext}^1_{\vec{Q}'}(\Phi_i^+(V), \Phi_i^+(W))$.

A similar argument shows that $\Phi_i^-$ gives a map $\mathrm{Ext}^1_{\vec{Q}'}(\Phi_i^+(V), \Phi_i^+(W)) \to \mathrm{Ext}^1_{\vec{Q}}(V, W)$. The fact that these two maps are inverse to each other is easy to see. □

Combining Corollary 2.2.6 and Corollary 2.2.9, we get the following result.

**Theorem 2.2.10.** *If $i$ is a source for $\vec{Q}, \vec{Q}' = s_i^- \vec{Q}$ and $V_1, V_2 \in Rep(\vec{Q})$ are nonzero indecomposable representations such that $\Phi_i^-(V_1), \Phi_i^-(V_2)$ are nonzero, then*

$$Hom_{\vec{Q}}(V_1, V_2) = Hom_{\vec{Q}'}(\Phi_i^-(V_1), \Phi_i^-(V_2)),$$
$$Ext^1_{\vec{Q}}(V_1, V_2) = Ext^1_{\vec{Q}'}(\Phi_i^-(V_1), \Phi_i^-(V_2)).$$

The next theorem is a result about the action of reflection functors on projective indecomposable representations. Projective indecomposable representations are an important class of representations, and understanding how reflection functors act on them is crucial for understanding the behavior of these functors more generally.

**Theorem 2.2.11.** *If $\vec{Q}' = s_k^+(\vec{Q})$ and $P'(i)$ is the indecomposable projective representation of $\vec{Q}'$ relative to vertex $i \neq k$ then $\Phi_k^-(P'(i)) = P(i)$ is an indecomposable projective representation of $\vec{Q}$.*

*Proof.* We have that $P'(i)$ is the projective related to the vertex $i$ for the quiver $s_k^+(\vec{Q})$ with $k \neq i$. Note that $k$ is a source for $s_k^+(\vec{Q})$.

Le us denote by $P'(i)_j$ the $\mathbb{K}$-vector space of the representation $P'(i)$ in vertex $j$. By definition $P'(i)_j$ is the vector space that has as basis the paths that start in $i$ and finish in $j$.

If $j \neq k$, by definition

$$\left(\Phi_k^-(P'(i))\right)_j = P'(i)_j.$$

Besides that, as $j \neq k$ the paths of $\vec{Q}$ and $\vec{Q}'$ are the same and

$$P(i)_j = P'(i)_j.$$

Since $k$ is a source for $\vec{Q}'$ then $P'(i)_k = 0$ and by definition

$$\left(\Phi_k^-(P'(i))\right)_k = \text{Coker}\left(0 \to \bigoplus_{k \to j} P'(i)_j\right) = \bigoplus_{k \to j} P'(i)_j.$$

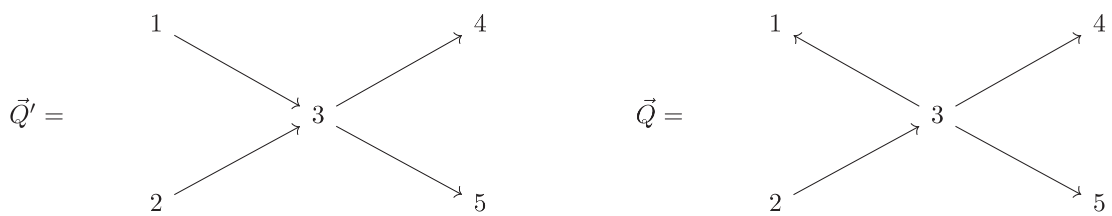
For each arrow $k \to j$ we have a vector space generated by paths from $i$ to $j$. Therefore,

$$P(i)_k = \bigoplus_{k \to j} P'(i)_j.$$

□

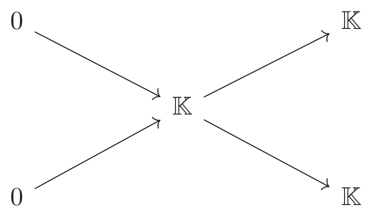
We can see how this works in some examples. The first example is with only one path and the second example will have more than one path.

**Example 2.2.12.** *Consider the quivers $\vec{Q}' = s_1^+(\vec{Q})$ and $\vec{Q}$*

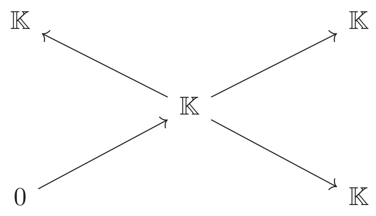


*The projective representations for the vertex* $3$ *are*

$P'(3)$



$P(3)$



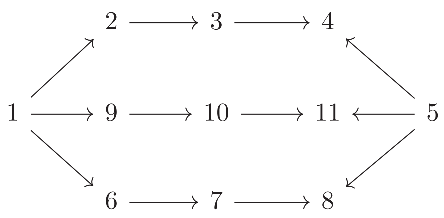
*note that* $\Phi_1^-(P'(3)) = P(3)$ *since*

$$\textit{Coker}\ (0 \to \mathbb{K}) = \frac{\mathbb{K}}{0} = \mathbb{K}$$

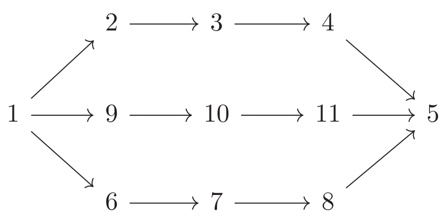
*and the other vector spaces remain the same.*

**Example 2.2.13.** *Consider the quivers* $\vec{Q}' = s_5^+(\vec{Q})$ *and* $\vec{Q}$

$\vec{Q}' =$

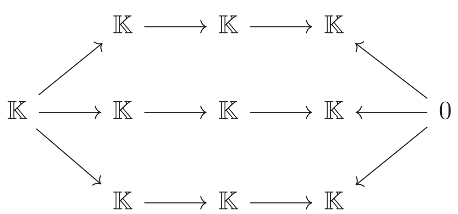


$\vec{Q} =$

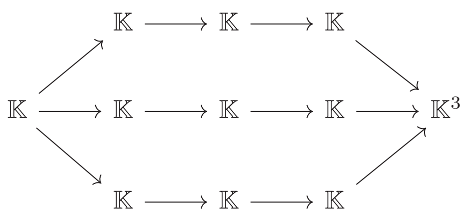


*The projective representations for the vertex* $1$ *are*

$P'(1) =$



$P(1) =$



*note that* $\Phi_5^-(P'(1)) = P(1)$ *since*

$$\textit{Coker}\ (0 \to \mathbb{K}^3) = \frac{\mathbb{K}^3}{0} = \mathbb{K}^3$$

*and the other vector spaces remain the same.*

# Chapter 3

# DYNKIN QUIVERS

In this chapter we will study representations of Dynkin quivers. The main goal is to prove a Theorem due to Gabriel that can be found in [10], which states that a quiver has a finite number of indecomposable representations if and only if it is Dynkin. After that we will classify all the indecomposable representations for a Dynkin quiver.

In this chapter we consider $\vec{Q}$ to be a simply laced Dynkin quiver with set of vertices $I$ and set of edges $\Omega$. We recall the notation of root systems that we determined in the Appendix A.

- $\mathsf{R}$ is the root system of $\vec{Q}$;
- $\alpha_i$ are the simple roots, for $i \in \{1, \cdots, \ell\}$ and $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ is a basis for the root system with $\ell = \mid I \mid$;
- $L$ is the root lattice for the root system $\mathsf{R}$, with the identification $L \simeq \mathbb{Z}^I$ given by
$$\sum_{i=1}^{\ell} n_i \alpha_i \mapsto (n_1, \cdots, n_\ell);$$
- $\mathsf{R}_+ = \mathsf{R} \cap L_+$ is the set of positive roots and $\mathsf{R}_- = \mathsf{R} \cap L_-$ is the set of negative roots, with $\mathsf{R} = \mathsf{R}_+ \cup \mathsf{R}_-$;
- $s_\alpha(\beta) = \alpha - \langle \beta, \alpha \rangle \alpha$ are the reflections for $\alpha \in \mathsf{R}$ and $\beta \in L$, with $\langle \beta, \alpha \rangle = \frac{(\beta,\alpha)}{(\alpha,\alpha)}$. If $\alpha \in \Delta$ then $s_\alpha$ is called a simple reflection;
- $\mathscr{W} = \text{span}\{s_\alpha;\ \alpha \in \Delta\}$ is the Weyl group for the root system $\mathsf{R}$.

Note that in the Dynkin case both $\mathscr{W}$ and $\mathsf{R}$ are finite. With the identification for the root lattice with $\mathbb{Z}^I$ we are writing the elements of $L$ in $\mathbb{Z}^I$ with $\{\alpha_1 \cdots, \alpha_\ell\}$ as a basis and that gives us

$$K(\vec{Q}) \simeq L$$
$$[V] \mapsto \dim(V) = \sum_{i=1}^{\ell} (\dim(V_i))\alpha_i.$$

Then the class of simple representations $[S(i)]$ are identified with the simple roots $\alpha_i$.

## 3.1 Coxeter element

The Coxeter elements are named after the mathematician Harold Scott MacDonald Coxeter, who first introduced them in his seminal work on Coxeter groups. In this context, they are often used to measure the complexity of the group, and have many important applications in geometric and algebraic contexts.

For us in this work recall that if $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ is an ordering for the basis of the root system $\mathsf{R}$ then a Coxeter element $C \in \mathscr{W}$ as defined in Definition A.11.2 is

$$C = s_\ell \cdots s_1 \quad \text{with} \quad s_{\alpha_i} = s_i.$$

**Definition 3.1.1.** *For a positive semidefinite bilinear form ( , ), the set*

$$rad( \ , \ ) = \{x \in \mathbb{Z}^n; (x, y) = 0 \ \forall \ y \in \mathbb{Z}^n\}$$

*is called the radical of the form ( , ), and its elements are called radical vectors.*

**Theorem 3.1.2.** *Let $C \in \mathscr{W}$ be a Coxeter element and $x \in L \otimes_{\mathbb{Z}} \mathbb{R}$. Then $Cx = x$ if and only if $x$ is in the radical of the form ( , ).*

*Proof.* Assume that $C = s_\ell \cdots s_1$ and $Cx = x$ then

$$\begin{aligned} Cx &= x \\ s_\ell \cdots s_1(x) &= x \\ s_{\ell-1} \cdots s_1(x) &= s_\ell(x) \\ s_{\ell-1} \cdots s_1(x) - x &= s_\ell(x) - x \\ s_{\ell-1} \cdots s_1(x) - x &= x - \langle x, \alpha_\ell \rangle \alpha_\ell - x \\ s_{\ell-1} \cdots s_1(x) - x &= - \langle x, \alpha_\ell \rangle \alpha_\ell. \end{aligned}$$

In the right-hand side of the equation we have a multiple of $\alpha_\ell$. In the left-hand side we have a linear combination of $\alpha_1, \cdots, \alpha_{\ell-1}$, and $x$. Each simple root appear once in the writing of $C$, then there is no $\alpha_\ell$ in the left-hand side. We can write

$$s_{\ell-1} \cdots s_1(x) - x = x + a_{\ell-1}\alpha_{\ell-1} + \cdots + a_1\alpha_1 - x = a_{\ell-1}\alpha_{\ell-1} + \cdots + a_1\alpha_1 = - \langle x, \alpha_\ell \rangle \alpha_\ell.$$

That means that $a_i = 0$ for all $i = \{1, \cdots, \ell - 1\}$ and $\langle x, \alpha_\ell \rangle = 0$ which implies that $(x, \alpha_\ell) = 0$.

If we proceed this way we will get $(x, \alpha_i) = 0$ for all $i = \{1, \cdots, \ell\}$. Since $\Delta$ is a basis then $(x, \alpha) = 0$ for all $\alpha \in L \otimes_{\mathbb{Z}} \mathbb{R}$. Therefore, $x$ is in the radical of the form ( , ).

If $x$ is in the radical of the form ( , ) then $(x, \alpha) = 0$ for all $\alpha \in L$. Then $(x, \alpha_i) = 0$ for all $\alpha_i \in \Delta$. So

$$s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i = x \ \forall \ i = \{1, \cdots, \ell\}.$$

Therefore, $Cx = s_\ell \cdots s_1(x) = x$. □

**Corollary 3.1.3.** *Let $Q$ be Dynkin. Then:*

*(1) The operator $C-1$ is invertible in $L \otimes_{\mathbb{Z}} \mathbb{R}$.*

*(2) For any $\alpha \in L, \alpha \neq 0$, there exists $k > 0$ such that $C^k(\alpha) \not\geq 0$.*

*(3) For any $\alpha \in L, \alpha \neq 0$, there exists $k < 0$ such that $C^k(\alpha) \not\geq 0$.*

*Proof.* (1) Since $\vec{Q}$ is Dynkin we know that $( \ , \ )$ is positive definite by Definition 1.6.1. If $(C-1)(x) = 0$ then $C(x) - x = 0$ and $C(x) = x$. By Theorem 3.1.2 we know that $x$ is in the radical of the form $( \ , \ )$. So $(x, \alpha) = 0$ for all $\alpha \in L \otimes_{\mathbb{Z}} \mathbb{R}$. Therefore, $x = 0$ and then $C - 1$ is invertible.

(2) Let $h$ be the order of $C$. Then $C^h = 1$ and

$$0 = \frac{C^h - 1}{C - 1} = C^{h-1} + \cdots + C + 1.$$

If $\alpha \neq 0$ then

$$0 = \left(C^{h-1} + \cdots + C + 1\right)(\alpha) = C^{h-1}(\alpha) + \cdots + C(\alpha) + \alpha. \tag{3.1}$$

This sum is made coordinate by coordinate so if all $C^k(\alpha) \geq 0$ for $\alpha > 0$ then the sum would be

$$C^{h-1}(\alpha) + \cdots + C(\alpha) + \alpha > 0.$$

Therefore, there exists $k > 0$ such that $C^k(\alpha) \not\geq 0$.

(3) The argument for this item is very similar to item (2). We can start from (3.1) and multiply both sides by $C^{-h+1}$ to obtain

$$0 = C^{-h+1}\left(C^{h-1}(\alpha)\right) + \cdots + C^{-h+1}\left(C(\alpha)\right) + C^{-h+1}(\alpha) = \alpha + \cdots + C^{-h+2}(\alpha) + C^{-h+1}(\alpha),$$

now all the powers are negative. This sum is made coordinate by coordinate so if all $C^k(\alpha) \geq 0$ for $\alpha > 0$ then the sum would be

$$\alpha + \cdots + C^{-h+2}(\alpha) + C^{-h+1}(\alpha) > 0.$$

Therefore, there exists $k < 0$ such that $C^k(\alpha) \not\geq 0$.

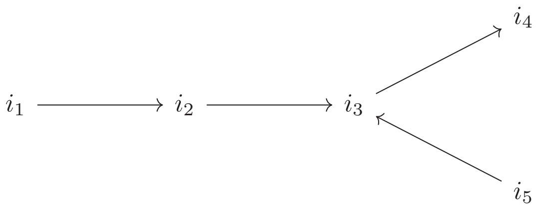
□

This theorem is an important result in the study of Coxeter elements in Dynkin quivers. It shows that the Coxeter element has some interesting properties in this setting. The invertibility of $C - 1$ in $L \otimes_{\mathbb{Z}} \mathbb{R}$ is a key fact that allows us to establish the other two parts of the theorem, which describe the behavior of $C$ on the root lattice $L$. These results help us to understand the structure of the Dynkin quiver and its associated root system in more detail.

**Definition 3.1.4.** *Given a sequence of indices $i_1, \cdots, i_k \in I$, respectively a reduced expression $w = s_{i_k} \cdots s_{i_1}$ for an element $w \in \mathscr{W}$. We call the sequence of indices or the reduced expression adapted to an orientation $\Omega$ of $Q$ if*

$$\begin{aligned}
&i_1 \text{ is a sink for } \vec{Q},\\
&i_2 \text{ is a sink for } s^+_{i_1}(\vec{Q}),\\
&\vdots\\
&i_k \text{ is a sink for } s^+_{i_{k-1}} \cdots s^+_{i_1}(\vec{Q}).
\end{aligned}$$

**Example 3.1.5.** *Consider the quiver $\vec{Q}$ with orientation $\Omega$ given by*




*We have that $i_4, i_3, i_2 \in I$ is a sequence of indices adapted to the orientation $\Omega$, since*

$$\begin{aligned}
&i_4 \text{ is a sink for } \vec{Q},\\
&i_3 \text{ is a sink for } s^+_{i_4}(\vec{Q}),\\
&i_2 \text{ is a sink for } s^+_{i_3} s^+_{i_4}(\vec{Q}).
\end{aligned}$$

Let $I = \{i_1, \cdots, i_\ell\}$ be an ordering of the vertices adapted to $\Omega$. That gives us an ordering for the basis $\Delta = \{\alpha_{i_1}, \cdots, \alpha_{i_\ell}\}$. As in Definition A.11.2 we have the corresponding Coxeter element $C = s_{i_\ell} \cdots s_{i_1}$ adapted to $\vec{Q}$.

In this case, we can define a sequence of quivers

$$\vec{Q}, \quad s^+_{i_1}(\vec{Q}), \quad s^+_{i_2} s^+_{i_1}(\vec{Q}), \quad \cdots, \quad C^+(\vec{Q}) = s^+_{i_\ell} \cdots s^+_{i_1}(\vec{Q}).$$

With the last quiver in this sequence being the same as the first one, $C^+(\vec{Q}) = \vec{Q}$. Indeed, for every edge $h$ its orientation was reversed twice.

**Lemma 3.1.6.** *Let $Q$ be Dynkin, $v \in \mathbb{Z}^I_+$ and $v \neq 0$. Then there exists $k \in \mathbb{Z}_+$ and a sequence $j_1 \cdots, j_{k+1} \in I$ adapted to $\vec{Q}$ such that $v \geq 0, s_{j_1}(v) \geq 0, \cdots, s_{j_k} \cdots s_{j_1}(v) \geq 0$ but $s_{j_{k+1}} s_{j_k} \cdots s_{j_1}(v) \ngeq 0$.*

*Proof.* Choose an ordering of vertices $I = \{i_1, \cdots, i_\ell\}$ as in Definition 3.1.4. Consider the sequence

$$\begin{aligned}
&v, s_{i_1}(v), \cdots, s_{i_\ell} \cdots s_{i_1}(v) = C(v),\\
&s_{i_1} C(v), \cdots, C^2(v),\\
&\vdots
\end{aligned}$$

Since this sequence contains $C^t(v)$ for all $t \geq 0$, by Corollary 3.1.3 this sequence must contain an element $C^t(v) \not\geq 0$. Let us take the minimum $t$ such that $C^t(v) \not\geq 0$. Therefore, somewhere between $C^{t-1}$ and $C^t$ we have the index $j_{k+1}$ that makes $s_{j_{k+1}} s_{j_k} \cdots s_{j_1}(v) \not\geq 0$. □

**Corollary 3.1.7.** *If $v$ is a positive root and $j_1, \cdots, j_{k+1}$ are indices as in Lemma 3.1.6 then*

$$v = s_{j_1} \cdots s_{j_k}(\alpha_{j_{k+1}}).$$

*Proof.* Write $\alpha' = s_{j_k} \cdots s_{j_1}(v)$. We have that $\alpha'$ is a positive root and $s_{j_{k+1}}(\alpha') \not\geq 0$ by Lemma 3.1.6. By Lemma A.6.3 we have that $s_{j_{k+1}}$ permutes all the positive roots except for $\alpha_{j_{k+1}}$. It follows that $\alpha' = \alpha_{j_{k+1}}$ and hence $v = s_{j_1} \cdots s_{j_k}(\alpha_{j_{k+1}})$. □

## 3.2 Gabriel's Theorem

Gabriel's Theorem is a fundamental result in the representation theory of finite-dimensional algebras, which provides a complete classification of the indecomposable representations of a given quiver. This theorem gives a powerful tool to analyze the structure of the module category, which is essential in many areas of mathematics, including algebraic geometry, algebraic topology, and mathematical physics. In this section, we will explore the statement of Gabriel's Theorem as in Theorem 1.6.3, as well as its proof, and discuss some of its consequences and applications.

**Proposition 3.2.1.** *Let $\vec{Q}$ be a Dynkin quiver. Consider the map*

$$\begin{aligned} \phi : K(\vec{Q}) &\to L \\ [V] &\mapsto dim(V) \\ [S(i)] &\mapsto \alpha_i. \end{aligned}$$

*The map $\phi$ is an isomorphism and $\phi(Ind(\vec{Q})) = \mathsf{R}_+$, which means that we have a bijection between the set $Ind(\vec{Q})$ of isomorphism classes of nonzero indecomposable representations of $\vec{Q}$ and the set $\mathsf{R}_+$ of positive roots. We will denote by $I_\alpha$ the indecomposable representation of graded dimension $\alpha \in \mathsf{R}_+$.*

*Proof.* The idea of this proof is to construct indecomposable representations by applying reflections functors. Much alike one constructs all roots by applying simple reflections to simple roots. However, there is a trick when we want to construct representations, we can only apply the reflection functor $\Phi_i^{\pm}$ if $i$ is a sink or a source for $\vec{Q}$ as seen in Definition 2.1.7. This justifies the Definition 3.1.4.

Let $V$ be a nonzero indecomposable representation of $\vec{Q}$. Denote by $v = \dim(V)$ and let $j_1, \cdots, j_{k+1}$ be a sequence of indices as in Lemma 3.1.6. By Corollary 2.2.8 we see that

$$\Phi^+_{j_1}(V) \text{ is a nonzero indecomposable representation of } s^+_{j_1}(\vec{Q}),$$
$$\Phi^+_{j_2}\Phi^+_{j_1}(V) \text{ is a nonzero indecomposable representation of } s^+_{j_2}s^+_{j_1}(\vec{Q}),$$
$$\vdots$$
$$V' = \Phi^+_{j_k}\cdots\Phi^+_{j_1}(V) \text{ is a nonzero indecomposable representation of } \vec{Q}' = s^+_{j_k}\cdots s^+_{j_1}(\vec{Q}).$$

However, since $s_{j_{k+1}}(dim(V')) \ngeq 0$ then $\Phi_{j_{k+1}}(V') = 0$. By Theorem 2.2.7 we must have $V' \simeq S(j_{k+1})$. Thus, by Theorem 2.2.5, we see that we must have

$$V \simeq \Phi^-_{j_1}\cdots\Phi^-_{j_k}(S(j_{k+1})).$$

In particular, this implies that $\dim(V) = s_{j_1}\cdots s_{j_k}(\alpha_{j_{k+1}})$ is a root and thus a positive root. The representation $V$ is uniquely determined by $\dim(V)$, since given $\dim(V) = s_{j_1}\cdots s_{j_k}(\alpha_{j_{k+1}})$ we have that only exists one representation $V$ with that dimension.

If $v$ is a positive root then choose a sequence $j_1, \cdots, j_{k+1}$ as in Lemma 3.1.6. Then by Corollary 3.1.7 we have that $v = s_{j_1}\cdots s_{j_k}(\alpha_{j_{k+1}})$. If we define $V$ by

$$V \simeq \Phi^-_{j_1}\cdots\Phi^-_{j_k}(S(j_{k+1})),$$

then $V$ will be an indecomposable representation with $\dim(V) = v$. □

With this proposition we will finally demonstrate Gabriel's theorem, which we restate for the reader's convenience.

**Theorem.** *A connected quiver $\vec{Q}$ is of finite type if and only if the underlying graph $Q$ is Dynkin.*

*Proof.* Assume that $\vec{Q}$ is of finite type. By Definition 1.6.1 it is enough to show that the associated Tits form $q$ is positive defined, i.e., $q(v) > 0$ for all $v \in \mathbb{R}^I$ by (1.13).

By Theorem 1.2.5 every representation is a direct sum of indecomposable representations. This implies that for any $v \in \mathbb{Z}^I_+$, there is a finite number of isomorphism classes of representations of dimension $v$. By Theorem B.1.1 we have that the action of the group $GL(v)$ on the representation space $R(\vec{Q}, v)$ has only finitely many orbits. Then there must exist an orbit $\mathbb{O}_x$ with $\dim(\mathbb{O}_x) = \dim(R(v))$, by Proposition B.2.1 (5).

By Proposition B.2.1 (3),

$$\dim(\mathbb{O}_x) = \dim(GL(v)) - \dim(G_x),$$

where $G_x$ is the stabilizer of $x$ in $GL(v)$. Then

$$\dim(G_x) = \dim(GL(v)) - \dim(\mathbb{O}_x) = \dim(GL(v)) - \dim(R(v)).$$

On the other hand, the subgroup of scalar matrices $\mathbb{K}^\times$ acts trivially in $R(v)$ therefore $\dim(G_x) \geq 1$. So we have

$$\begin{aligned} \dim(GL(v)) - \dim(R(v)) &\geq 1, \\ \sum v_i^2 - \sum_h v_{s(h)}v_{t(h)} &\geq 1, \\ q(v) = \langle v, v\rangle &\geq 1. \end{aligned} \tag{3.2}$$

Thus, $q$ is positive defined for any nonzero $v \in \mathbb{Z}_+^I$.

Now consider $v \in \mathbb{Z}^I$, write $v = v_+ - v_-$ with $v_+, v_- \in \mathbb{Z}_+^I$ and with disjoint support. So $q(v_+) \geq 1$, $q(v_-) \geq 1$ by (3.2). By (1.12) we can write

$$\begin{aligned} 2q(v) = (v, v) = (v_+ - v_-, v_+ - v_-) &= (v_+, v_+) - 2(v_+, v_-) + (v_-, v_-) \\ q(v) &= \frac{1}{2}(v_+, v_+) - (v_+, v_-) + \frac{1}{2}(v_-, v_-) \\ &= q(v_+) - (v_+, v_-) + q(v_-) \geq 2 - (v_+, v_-). \end{aligned}$$

And by (1.11) we can write

$$(v_+, v_-) = \sum_{i\in I}(v_+)_i(v_-)_i(2 - 2n_{ii}) - \sum_{i\neq j} n_{ij}(v_+)_i(v_-)_j \tag{3.3}$$

with $n_{ij}$ the number of edges between $i$ and $j$. In particular $n_{ij} \geq 0$.

Since $v_+, v_-$ have disjoint support we have that $(v_+)_i(v_-)_i = 0 \ \forall \ i \in I$. So

$$\sum_{i\in I}(v_+)_i(v_-)_i(2 - 2n_{ii}) = 0.$$

Therefore, we can rewrite (3.3) as

$$(v_+, v_-) = -\sum_{i\neq j} n_{ij}(v_+)_i(v_-)_j$$

Since $v_+, v_- \in \mathbb{Z}_+^I$ we have $(v_+)_i(v_-)_j \geq 0$ for all $i, j \in I$, and then

$$(v_+, v_-) = -\sum_{i\neq j} n_{ij}(v_+)_i(v_-)_j \leq 0.$$

Therefore, $-(v_+, v_-) \geq 0$ and hence $q(v) \geq 2 - (v_+, v_-) \geq 2$. Thus, $q$ is positive defined for any nonzero $v \in \mathbb{Z}^I$.

If $v \in \mathbb{Q}^I$ then we can write

$$v_i = \frac{a_i}{b_i} \text{ for all } i \in I \text{ and let } d = \sum_{j\in I} b_j \in \mathbb{Z}.$$

So,

$$dv = (dv_1, \cdots, dv_\ell) = \left(\sum_{j\neq 1} b_j a_1, \cdots, \sum_{j\neq \ell} b_j a_\ell\right) \in \mathbb{Z}^I,$$

and hence $q(dv) \geq 2$ by the previous case. Since

$$q(dv) = \frac{1}{2}(dv, dv) = \frac{d^2}{2}(v, v) = d^2 q(v) \geq 2.$$

It follows that $q(v) \geq \frac{2}{d^2} > 0$.

Therefore, we have a quadratic form defined in $\mathbb{Q}^I$. This quadratic form comes from a symmetric bilinear form. Since it is positive defined in $\mathbb{Q}^I$, there exists an orthonormal basis $\{v_1, \cdots, v_\ell\}$ of $\mathbb{Q}^I$ with respect to this bilinear form. Hence, for each $x \in \mathbb{R}^I$ we can write $x = \sum x_i v_i$ and

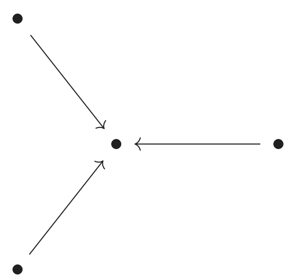
$$q(x) = \sum q(v_i) x_i^2.$$

Since $q(v_i) > 0$ and $x_i^0 \geq 0$ we have that $q(x) > 0$ if $x \neq 0$.

Assume that $\vec{Q}$ is Dynkin. The set of isomorphism classes of nonzero indecomposable representation of $\vec{Q}$ is in bijection with the set of positive roots by Proposition 3.2.1. In the Dynkin quiver the set of positive roots is finite, so there is a finite number of isomorphism classes of nonzero indecomposables. That implies that a Dynkin quiver is of finite type. □

This theorem provides a fundamental characterization of quivers of finite type. This is a powerful result because it allows us to identify the finite type quivers by simply looking at their graphs, which are much easier to visualize and manipulate than the representations themselves. The theorem also highlights the deep connection between the geometry of root systems and the representation theory of quivers, as the Dynkin graphs play a crucial role in both fields.

**Example 3.2.2.** *Let $\vec{Q}$ be the quiver given by*



*The corresponding graph is the Dynkin diagram of type $D_4$. We saw in Example A.10.2 that the corresponding root system has* 12 *positive roots. On the other hand, we had given* 12 *nonisomorphic indecomposable representations of this quiver in Example 1.2.8. Thus, Proposition 3.2.1 implies that the list given in Example 1.2.8 is the full list of possible nonzero indecomposable representations. In particular, this gives a complete answer to the problem of classifying triples of subspaces in a vector space.*

## 3.3 Ordering of positive roots

In the previous section we showed that for any positive root $\alpha \in \mathsf{R}_+$ there is a unique, up to an isomorphism, indecomposable representation $I_\alpha$ of $\vec{Q}$. This representation can be constructed by applying a sequence of reflection functors $\Phi_i^-$ to a simple representation. In this section we refine this result, showing how one can get all indecomposable representations from a single sequence of reflection functors. In particular, it will give us a natural order on the set of positive roots.

With the way of writing $w_0$ given in Theorem A.11.5 we can prove the following theorem that will help write all the indecomposable representations. The Theorem 3.3.1 was proved by Lusztig in [17]. Consider $\Omega_0$ the orientation of $Q$ where every $i \in I_0$ is a sink and every $i \in I_1$ is a source.

**Theorem 3.3.1.** *Given an orientation $\Omega$ of a Dynkin graph $Q$, there exists a reduced expression $w_0 = s_{i_k} \cdots s_{i_1}$ which is adapted to $\Omega$.*

*Proof.* By Theorem A.11.5 we know that $w_0 = \cdots c_0 c_1 c_0$ is the reduced expression adapted to the orientation $\Omega_0$. Suppose that $w_0 = s_{i_r} \cdots s_{i_1}$ is the reduced expression adapted to the orientation $\Omega$.

Let us show that $w_0 = s_j s_{i_r} \cdots s_{i_2}$ is a reduced expression adapted to $s_{i_1}^+ \Omega$, where $\alpha_j = -w_0(\alpha_{i_1})$. So we can write

$$\alpha_j = -w_0(\alpha_{i_1}) = -s_{i_k} \cdots s_{i_1}(\alpha_{i_1}) = s_{i_r} \cdots s_{i_2}(\alpha_{i_1}).$$

If we write $\beta_0 = s_{i_r} \cdots s_{i_2}(\beta)$ we have

$$\begin{aligned} s_j s_{i_r} \cdots s_{i_2}(\beta) &= s_j(\beta_0) = \beta_0 - \langle \beta_0, \alpha_j \rangle \alpha_j = s_{i_r} \cdots s_{i_2}(\beta) - \langle s_{i_r} \cdots s_{i_2}(\beta), s_{i_r} \cdots s_{i_2}(\alpha_{i_1}) \rangle s_{i_r} \cdots s_{i_2}(\alpha_{i_1}) \\ &= s_{i_r} \cdots s_{i_2}(\beta) - \langle \beta, \alpha_{i_1} \rangle s_{i_r} \cdots s_{i_2}(\alpha_{i_1}) = s_{i_r} \cdots s_{i_2}(\beta - \langle \beta, \alpha_{i_1} \rangle \alpha_{i_1}) = s_{i_r} \cdots s_{i_2} s_{i_1}(\beta) \\ &= w_0(\beta). \end{aligned}$$

Therefore, $s_j s_{i_r} \cdots s_{i_2}$ is an expression for $w_0$, and it has the same amount of elements as $s_{i_r} \cdots s_{i_1}$, so it is a reduced expression for $w_0$.

Now we need to show that this reduced expression is adapted to $s_{i_1}^+ \Omega$. We have that

$$\begin{aligned} w_0 &= s_j s_{i_r} \cdots s_{i_2} = s_j s_{i_r} \cdots s_{i_2} s_{i_1} s_{i_1} = s_j w_0 s_{i_1} \\ s_j &= w_0 s_{i_1} w_0^{-1} \end{aligned}$$

Then by [17, p. 462] we have that $w_0 = s_j s_{i_r} \cdots s_{i_2}$ is adapted to $s_{i_1}^+ \Omega$.

By Lemma 2.1.6 since every orientation can be found from $\Omega_0$ by reflections then there exists a reduced expression adapted to any $\Omega$. □

If we choose a reduced expression for $w_0$ as

$$w_0 = s_{i_r} \cdots s_{i_1}, \quad r = |\mathsf{R}_+|. \tag{3.4}$$

Then we can write $\mathsf{R}_+ = \{\gamma_1, \cdots, \gamma_r\}$, where

$$\begin{aligned} \gamma_1 &= \alpha_{i_1}, \\ \gamma_2 &= s_{i_1}(\alpha_{i_2}), \\ &\vdots \\ \gamma_r &= s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r}). \end{aligned} \tag{3.5}$$

Indeed, we can see that $\gamma_j \in \mathsf{R}_+$ for $1 \leq j \leq k$. Otherwise, we would have that $\gamma_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ is negative. Then by Lemma A.6.5 we have that for some index $1 \leq t \leq j$

$$\gamma_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) = s_{i_1} \cdots s_{i_{t-1}} s_{i_{t+1}} \cdots s_{i_{j-1}}(\alpha_{i_j}).$$

So we can write $\gamma_j$ with fewer elements, and then we could write $w_0$ with fewer elements and that is a contradiction since the expression for $w_0$ is already reduced. Therefore, $\gamma_j \in \mathsf{R}_+$.

Besides that we can see that $\gamma_j \neq \gamma_m$ for all $j \neq m$. Otherwise, we would have that

$$\gamma_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) = s_{i_1} \cdots s_{i_{m-1}}(\alpha_{i_m}) = \gamma_m.$$

Supposing $t > j$ we can do

$$\alpha_{i_j} = s_{i_j} \cdots s_{i_{m-1}}(\alpha_{i_m})$$

applying $s_{i_j}$ in both sides we have,

$$-\alpha_{i_j} = s_{i_{j+1}} \cdots s_{i_{m-1}}(\alpha_{i_m}) \in \mathsf{R}_-.$$

Then we have the same contradiction from the last part. Therefore, we have that $\mathsf{R}_+ = \{\gamma_1, \cdots, \gamma_r\}$.

What we showed and the way we defined the set $\{\gamma_1, \cdots, \gamma_r\}$ gives us some properties

$$\begin{aligned} \gamma_j &\in \mathsf{R}_+ \\ s_{i_k} \cdots s_{i_1}(\gamma_j) &\in \mathsf{R}_+ \quad \text{for all} \quad k = 1, \cdots, j-1 \\ s_{i_j} \cdots s_{i_1}(\gamma_j) &\in \mathsf{R}_-. \end{aligned}$$

Let us now choose a reduced expression for $w_0$ adapted to $\vec{Q}$ and let $\gamma_1, \cdots, \gamma_r$ be as defined in (3.5). From the previous discussion we can construct all indecomposable representations of $\vec{Q}$ by

$$\begin{aligned} I_1 =& S(i_1), & \dim I_1 = \gamma_1 = \alpha_{i_1}, \\ I_2 =& \Phi^-_{i_1}(S(i_2)), & \dim I_2 = \gamma_2 = s_{i_1}(\alpha_{i_2}), \\ &\vdots \\ I_r =& \Phi^-_{i_1} \cdots \Phi^-_{i_{r-1}}(S(i_r)), & \dim I_r = \gamma_r = s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r}). \end{aligned} \tag{3.6}$$

**Example 3.3.2.** *Consider the quiver $\vec{Q}$ given by*

$$1 \longrightarrow 2.$$

*Then the adapted reduced expression for $w_0$ is $w_0 = s_2 s_1 s_2$. The corresponding roots are*

$$\begin{aligned} \gamma_1 &= \alpha_2 \\ \gamma_2 &= s_2(\alpha_1) = \alpha_1 + \alpha_2 \\ \gamma_3 &= s_2 s_1(\alpha_2) = \alpha_1. \end{aligned}$$

*Therefore, the corresponding indecomposables are*

$$I_1 = S(2) = \ 0 \longrightarrow \mathbb{K} \ ,$$

$$I_2 = \Phi_2^-( \ \mathbb{K} \longleftarrow 0 \ ) = \ \mathbb{K} \xrightarrow{1} \mathbb{K} \ = P(1),$$

$$I_3 = \Phi_2^- \Phi_1^-( \ 0 \longrightarrow \mathbb{K} \ ) = \Phi_2^-( \ \mathbb{K} \xleftarrow{1} \mathbb{K} \ ) = \ \mathbb{K} \longrightarrow 0 \ = S(1).$$

**Theorem 3.3.3.** *Let $I_1, \cdots, I_r$ be indecomposable representations, defined as by* (3.6). *Then*

$$Hom_{\vec{Q}}(I_a, I_b) = 0 \quad \textit{for } a > b,$$
$$Ext^1_{\vec{Q}}(I_a, I_b) = 0 \quad \textit{for } a \leq b.$$

*Proof.* By Theorem 2.2.5 and Remark 2.1.9, if $i$ is a sink for $\vec{Q}$ then for any $V \in \mathrm{Rep}(s_i^+\vec{Q})$ we have that

$$\mathrm{Hom}(\Phi_i^-(V), S(i)) = \mathrm{Hom}(V, \Phi_i^+ S(i)) = \mathrm{Hom}(V, 0) = 0.$$

Using Theorem 2.2.10, this gives for $a > b$

$$\begin{aligned}\mathrm{Hom}(I_a, I_b) &= \mathrm{Hom}(\Phi_{i_1}^- \cdots \Phi_{i_b}^- \cdots \Phi_{i_{a-1}}^-(S(i_a)), \Phi_{i_1}^- \cdots \Phi_{i_{b-1}}^-(S(i_b))) \\ &= \mathrm{Hom}(\Phi_{i_b}^- \cdots \Phi_{i_{a-1}}(S(i_a)), S(i_b)) = 0.\end{aligned}$$

The second part is proved similarly. Using as the starting point the equality $\mathrm{Ext}^1(S(i), V) = 0$ for all $V$ and for $i$ a source. Which is immediate since in this case $S(i) = P(i)$ is projective. □

In the next chapter we will show that the nonzero indecomposable representations $I_1, \cdots, I_r$ constructed as above are in bijection with the vertices of a certain quiver, called the Auslander-Reiten quiver of $\vec{Q}$.

# Chapter 4

# COXETER FUNCTOR AND PREPROJECTIVE REPRESENTATIONS

When the quiver $\vec{Q}$ is not Dynkin it is much harder to describe all the indecomposable representations as we did in the Dynkin case. We can, however, use the reflection functors to construct a large class of indecomposable representations of $\vec{Q}$, those are called preprojective representations. We will also show that in the Dynkin case all the representations are preprojective. In this chapter $Q$ is a graph with no edge loops, all the orientations of $Q$ we consider will be such that there are no oriented cycles.

So far we worked with graphs that had a corresponding finite root system, that is not the case in this chapter. Here we can have infinity root systems as well. Therefore, we need to remind the reader the notation given in Section A.12. We denote by $\mathsf{R}^{re}$ the set of real roots, by $\mathsf{R}^{im}$ the set of imaginary roots. We also denote by $\mathsf{R}^{re}_+$ the positive roots and $\mathsf{R}^{re}_-$ the negative roots with $\mathsf{R}^{re} = \mathsf{R}^{re}_+ \cap \mathsf{R}^{re}_-$. For Dynkin quivers, by Theorem A.7.1 ($c$), we have that every root is a real root. For this chapter recall that in Theorem 1.6.2 we showed which graphs are Dynkin and which are Euclidean.

In Section A.12 we define a root system based on generalized Cartan matrices. If we have a graph $Q$ with no edge loops and let $A = (a_{ij})_{1\leq i,j\leq \ell}$ be the matrix defined by

$$a_{ij} = \text{ number of edges between } i, j,$$

then the matrix $C_Q = 2 - A$ is the generalized Cartan matrix for the graph $Q$.

## 4.1 Coxeter functor

In this section we want to define a Coxeter functor, this functor is related to the Coxeter element, and we defined this element in Definition A.12.5. Consider $\vec{Q}$ to be a quiver without oriented cycles and denote by $C = s_{i_\ell} \cdots s_{i_1}$ a Coxeter element adapted to $\vec{Q}$, with $\ell =| I |$. If we consider the corresponding sequence of reflections $C^+ = s^+_{i_\ell} \cdots s^+_{i_1}$, then $C^+\vec{Q} = \vec{Q}$ since the orientation of every edge is reversed twice.

For any $k = 1, \cdots, \ell$ we can define roots

$$p_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad \text{and} \quad q_k = s_{i_\ell} \cdots s_{i_{k+1}}(\alpha_{i_k}). \tag{4.1}$$

**Lemma 4.1.1.** *The elements $p_k$ and $q_k$, as in* (4.1)*, are distinct positive real roots, and we have that*

*(a)* $C(p_k) = -q_k$;

*(b)* $C^{-1}(q_k) = -p_k$;

*(c)* $\{p_1, \cdots, p_\ell\} = \{\alpha \in \mathsf{R}^{re}_+; C\alpha \in \mathsf{R}^{re}_-\}$.

*(d)* $\{q_1, \cdots, q_\ell\} = \{\alpha \in \mathsf{R}^{re}_+; C^{-1}\alpha \in \mathsf{R}^{re}_-\}$.

*Proof.* In the same way as made in (3.5) and using Lemma A.12.8, we have that $p_k$ and $q_k$ are distinct positive real roots.

To prove item $(a)$ and $(b)$ we have that,

$$C(p_k) = s_{i_\ell} \cdots s_{i_1}(s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})) = s_{i_\ell} \cdots s_{i_k}(\alpha_{i_k}) = -s_{i_\ell} \cdots s_{i_{k+1}}(\alpha_{i_k}) = -q_k,$$
$$C^{-1}(q_k) = s_{i_1} \cdots s_{i_\ell}(s_{i_\ell} \cdots s_{i_{k+1}}(\alpha_{i_k})) = s_{i_1} \cdots s_{i_k}(\alpha_{i_k}) = -s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = -p_k.$$

For the equality of sets of item $(c)$ we have that if $q_k \in R^{re}_+$ then $C(p_k) = -q_k \in R^{re}_-$, which means that $\{p_1, \cdots, p_\ell\} \subset \{\alpha \in \mathsf{R}^{re}_+; C\alpha \in \mathsf{R}^{re}_-\}$.

On the other hand we have that $C = s_{i_\ell} \cdots s_{i_1}$ is a reduced expression for $C$, so by Lemma A.12.7, it sends $\ell$ elements of $R^{re}_+$ into $R^{re}_-$ then $\{\alpha \in \mathsf{R}^{re}_+; C\alpha \in \mathsf{R}^{re}_-\}$ is a set with $\ell$ elements and $\{p_1, \cdots, p_\ell\}$ also has $\ell$ elements then we have the equality.

The equality in item $(d)$ is obtained analogously. □

The corresponding indecomposable representation associated with $p_k$ is

$$P_k = \Phi^-_{i_1} \cdots \Phi^-_{i_{k-1}}(S(i_k)), \quad \text{where} \quad \dim(P_k) = p_k. \tag{4.2}$$

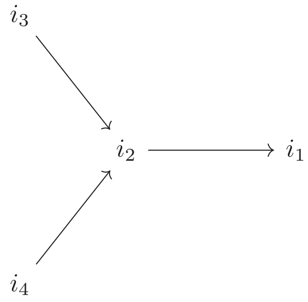
Note that here $S(i_k)$ is the simple representation of the quiver $s^+_{i_{k-1}} \cdots s^+_{i_1}\vec{Q}$. Similarly, the indecomposable representation associated with $q_k$ is

$$Q_k = \Phi^+_{i_\ell} \cdots \Phi^+_{i_{k+1}}(S(i_k)), \quad \text{where} \quad \dim(Q_k) = q_k. \tag{4.3}$$

Note that here $S(i_k)$ is the simple representation of the quiver $s^+_{i_{k+1}} \cdots s^+_{i_\ell}\vec{Q}$.

Next we have some examples of these representations. We will give an example with a Dynkin quiver and another with a Euclidean quiver.

**Example 4.1.2.** *Consider the graph $D_4$ with orientation given by*




*We have that $I = \{i_1, i_2, i_3, i_4\}$ is an ordering of the vertices adapted to the orientation since*

$$i_1 \text{ is a sink for } \vec{Q}$$
$$i_2 \text{ is a sink for } s_1^+(\vec{Q})$$
$$i_3 \text{ is a sink for } s_2^+ s_1^+(\vec{Q})$$
$$i_4 \text{ is a sink for } s_3^+ s_2^+ s_1^+(\vec{Q})$$

*with $s_{i_k}^+ = s_k^+$ for $k = 1, 2, 3, 4$. Therefore, we have the Coxeter element adapted to $\vec{Q}$ as $C = s_4 s_3 s_2 s_1$.*

*We can write the roots $p_k$ and $q_k$ for this quiver as*

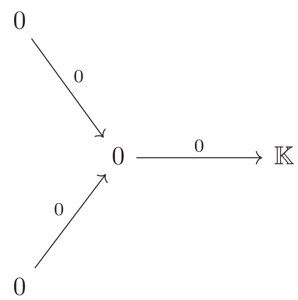
$$\begin{aligned} p_1 &= \alpha_{i_1} & q_1 &= s_4 s_3 s_2(\alpha_{i_1}) \\ p_2 &= s_1(\alpha_{i_2}) & q_2 &= s_4 s_3(\alpha_{i_2}) \\ p_3 &= s_1 s_2(\alpha_{i_3}) & q_3 &= s_4(\alpha_{i_3}) \\ p_4 &= s_1 s_2 s_3(\alpha_{i_4}) & q_4 &= \alpha_{i_4}. \end{aligned}$$

*Then we can write the corresponding indecomposable representation associated with $p_k$ and $q_k$ as*

$$\begin{aligned} P_1 &= S(i_1) & Q_1 &= \Phi_{i_4}^+ \Phi_{i_3}^+ \Phi_{i_2}^+(S(i_1)) \\ P_2 &= \Phi_{i_1}^-(S(i_2)) & Q_2 &= \Phi_{i_4}^+ \Phi_{i_3}^+(S(i_2)) \\ P_3 &= \Phi_{i_1}^- \Phi_{i_2}^-(S(i_3)) & Q_3 &= \Phi_{i_4}^+(S(i_3)) \\ P_4 &= \Phi_{i_1}^- \Phi_{i_2}^- \Phi_{i_3}^-(S(i_4)) & Q_4 &= S(i_4). \end{aligned}$$

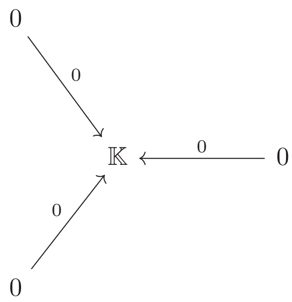
*And we can actually find these representations, first let us see the $P_k$ indecomposable representations. Representation $P_1$ is also a simple representation,*

$P_1 :$

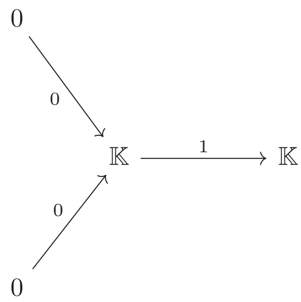


*The representation* $P_2$ *can be calculated as*

$S(i_2):$

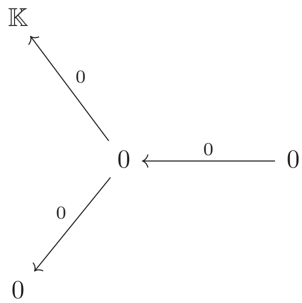



$P_2:$

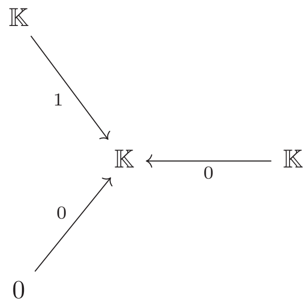



*The representation* $P_3$ *can be calculated as*

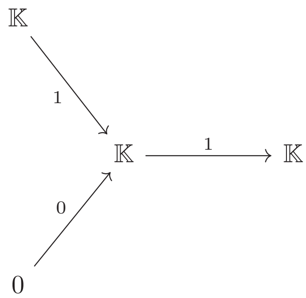
$S(i_3):$




$\Phi^{-}_{i_2}(S(i_3)):$

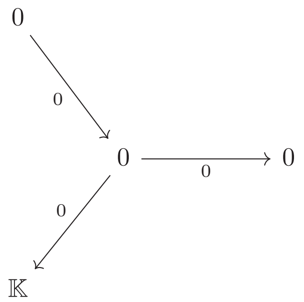



$P_3:$

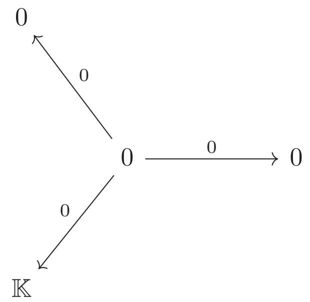


*The representation $P_4$ can be calculated as*

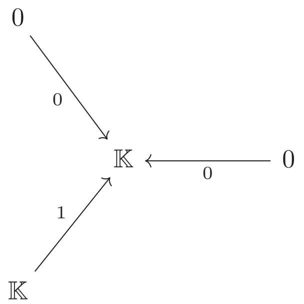
$S(i_4)$ :



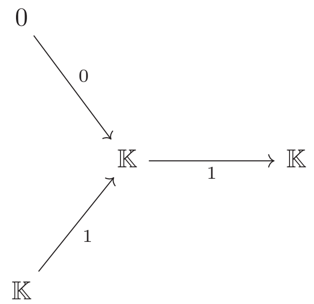
$\Phi^-_{i_3}(S(i_4))$ :



$\Phi^-_{i_2}\Phi^-_{i_3}(S(i_4))$ :

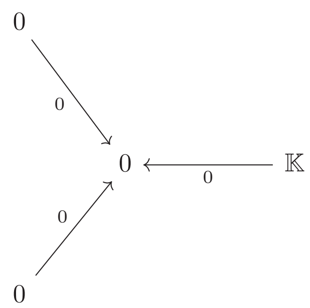


$P_4$ :

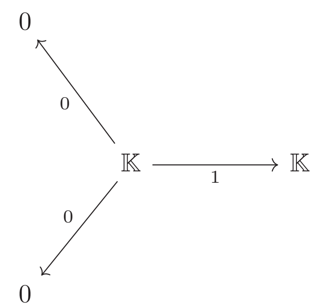


*Now we will see the $Q_k$ indecomposable representations. The representation $Q_1$ can be calculated as*

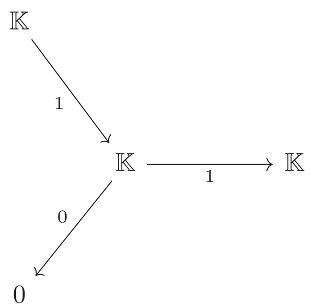
$S(i_1)$ :



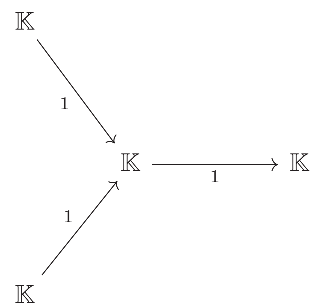
$\Phi^+_{i_2}(S(i_1))$ :



$\Phi^+_{i_3}\Phi^+_{i_2}(S(i_1))$ :

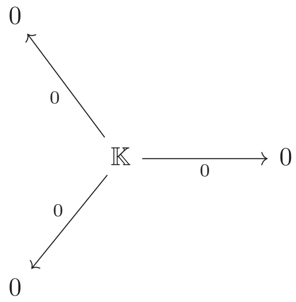


$Q_1$ :

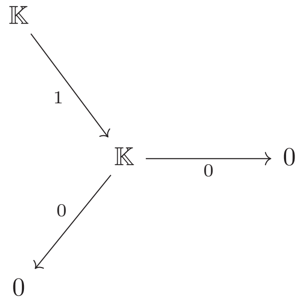


*The representation $Q_2$ can be calculated as*

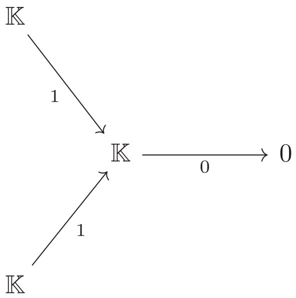
$S(i_2):$




$\Phi^+_{i_3}(S(i_2)):$

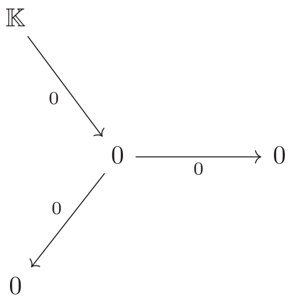



$Q_2:$

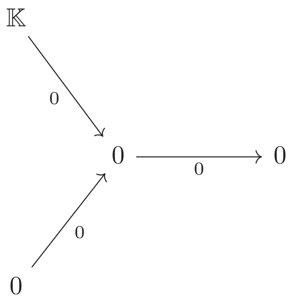



*The representation $Q_3$ can be calculated as*

$S(i_3):$

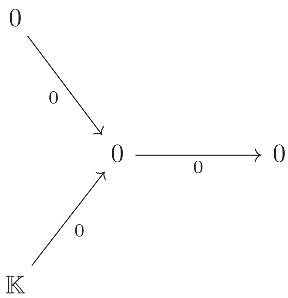



$Q_3:$



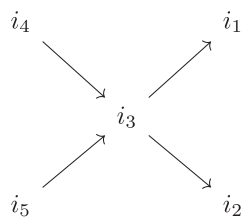

*The representation $Q_4$ is also a simple representation*

$Q_4:$




**Example 4.1.3.** *Consider the graph $\hat{D}_4$ with orientation given by*

$\vec{Q} =$



*We have that $I = \{i_1, i_2, i_3, i_4, i_5\}$ is an ordering of the vertices adapted to the orientation, since*

$$
\begin{aligned}
&i_1 \text{ is a sink for } \vec{Q}\\
&i_2 \text{ is a sink for } s_1^+(\vec{Q})\\
&i_3 \text{ is a sink for } s_2^+ s_1^+(\vec{Q})\\
&i_4 \text{ is a sink for } s_3^+ s_2^+ s_1^+(\vec{Q})\\
&i_5 \text{ is a sink for } s_4^+ s_3^+ s_2^+ s_1^+(\vec{Q})
\end{aligned}
$$

*with $s_{i_k}^+ = s_k^+$ for $k = 1, 2, 3, 4, 5$. Therefore, we have the Coxeter element adapted to $\vec{Q}$ as $C = s_5 s_4 s_3 s_2 s_1$.*

*We can write the roots $p_k$ and $q_k$ for this quiver as*

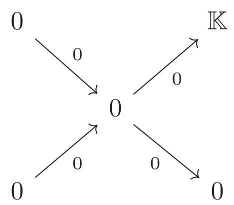
$$
\begin{aligned}
p_1 &= \alpha_{i_1} & q_1 &= s_5 s_4 s_3 s_2(\alpha_{i_1})\\
p_2 &= s_1(\alpha_{i_2}) & q_2 &= s_5 s_4 s_3(\alpha_{i_2})\\
p_3 &= s_1 s_2(\alpha_{i_3}) & q_3 &= s_5 s_4(\alpha_{i_3})\\
p_4 &= s_1 s_2 s_3(\alpha_{i_4}) & q_4 &= s_5(\alpha_{i_4})\\
p_5 &= s_1 s_2 s_3 s_4(\alpha_{i_5}) & q_5 &= \alpha_{i_5}.
\end{aligned}
$$

*Then we can write the corresponding indecomposable representation associated with $p_k$ and $q_k$ as*

$$
\begin{aligned}
P_1 &= S(i_1) & Q_1 &= \Phi_{i_5}^+ \Phi_{i_4}^+ \Phi_{i_3}^+ \Phi_{i_2}^+(S(i_1))\\
P_2 &= \Phi_{i_1}^-(S(i_2)) & Q_2 &= \Phi_{i_5}^+ \Phi_{i_4}^+ \Phi_{i_3}^+(S(i_2))\\
P_3 &= \Phi_{i_1}^- \Phi_{i_2}^-(S(i_3)) & Q_3 &= \Phi_{i_5}^+ \Phi_{i_4}^+(S(i_3))\\
P_4 &= \Phi_{i_1}^- \Phi_{i_2}^- \Phi_{i_3}^-(S(i_4)) & Q_4 &= \Phi_{i_5}^+(S(i_4))\\
P_5 &= \Phi_{i_1}^- \Phi_{i_2}^- \Phi_{i_3}^- \Phi_{i_4}^-(S(i_5)) & Q_5 &= S(i_5).
\end{aligned}
$$

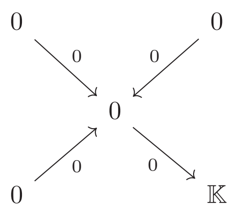
*And we can actually find these representations, first let us see the $P_k$ indecomposable representations. Representation $P_1$ is also a simple representation,*

$P_1 :$

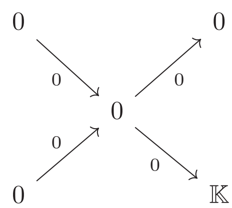



*The representation $P_2$ can be calculated as*

$S(i_2) :$

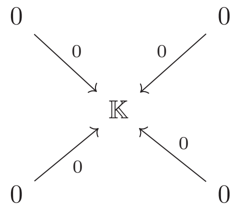



$P_2 :$

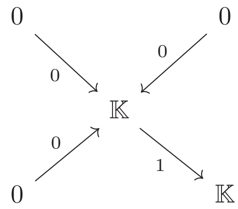


*The representation $P_3$ can be calculated as*

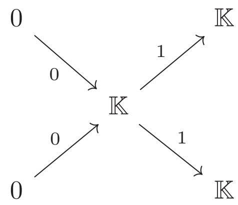
$S(i_3):$



$\Phi^{-}_{i_2}(S(i_3)):$

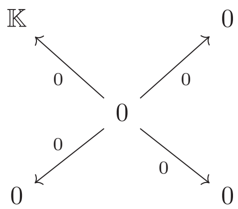


$P_3:$

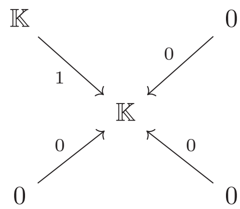


*The representation $P_4$ can be calculated as*

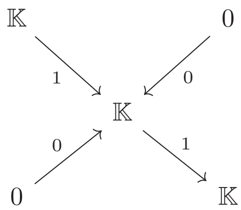
$S(i_4):$



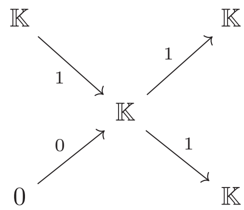
$\Phi^{-}_{i_3}(S(i_4)):$



$\Phi^{-}_{i_2}\Phi^{-}_{i_3}(S(i_4)):$

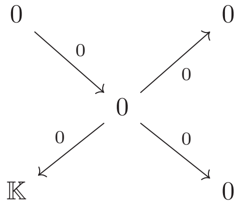


$P_4:$

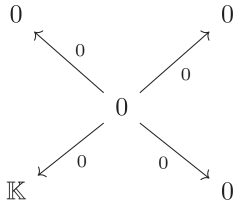


*The representation $P_5$ can be calculated as*

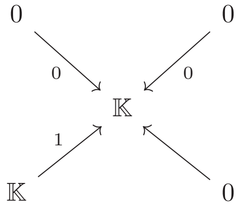
$S(i_5)$ :



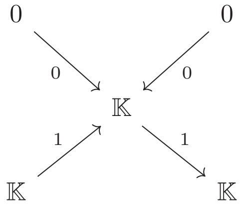

$\Phi^-_{i_4}(S(i_5))$ :



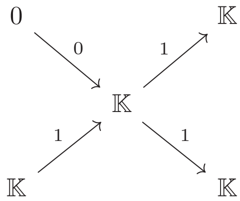

$\Phi^-_{i_3}\Phi^-_{i_4}(S(i_5))$ ::




$\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_4}(S(i_5))$ ::

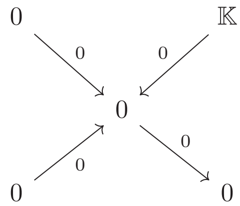



$P_5$ :

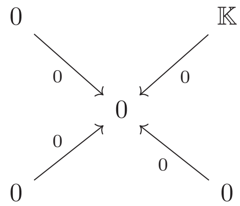



*Now we will see the $Q_k$ indecomposable representations. The representation $Q_1$ can be calculated as*

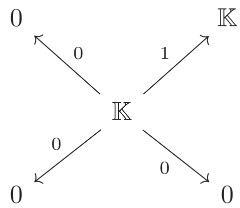
$S(i_1)$ :



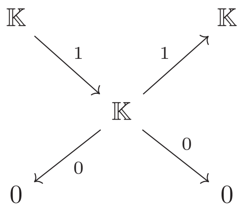

$\Phi^+_{i_2}(S(i_1))$ :



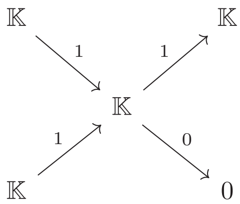

$\Phi^+_{i_3}\Phi^+_{i_2}(S(i_1))$ ::




$\Phi^+_{i_4}\Phi^+_{i_3}\Phi^+_{i_2}(S(i_1))$ ::

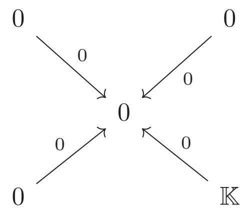



$Q_1$ :

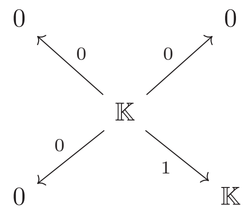


*The representation $Q_2$ can be calculated as*

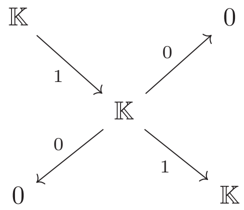
$S(i_2):$



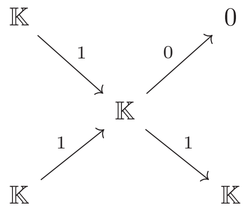
$\Phi^+_{i_3}(S(i_2)):$



$\Phi^+_{i_4}\Phi^+_{i_3}(S(i_2))::$

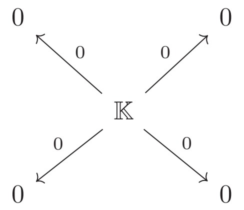


$Q_2:$

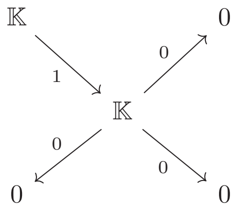


*The representation $Q_3$ can be calculated as*

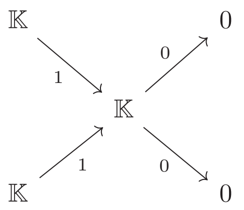
$S(i_3):$



$\Phi^+_{i_4}(S(i_3)):$

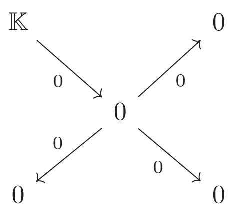


$Q_3:$

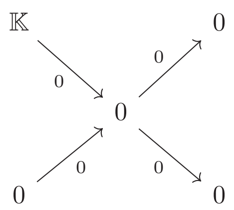


*The representation $Q_4$ can be calculated as*

$S(i_4):$

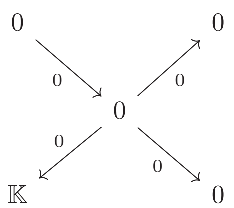


$Q_4:$



*The representation $Q_5$ is also a simple representation*

$Q_5:$



**Lemma 4.1.4.** *For representations $P_k$ as in (4.2) and nonzero indecomposable projective representations $P(i_k)$ as in (1.4) we have that*

$$P_k = \Phi^-_{i_1} \cdots \Phi^-_{i_{k-1}}(S(i_k)) = P(i_k).$$

*Similarly, for representations $Q_k$ as in (4.3) and nonzero indecomposable injective representations $Q(i_k)$ as in (1.8) we have that*

$$Q_k = \Phi^+_{i_\ell} \cdots \Phi^+_{i_{k+1}}(S(i_k)) = Q(i_k).$$

*Proof.* If $\vec{Q}' = s^+_k(\vec{Q})$ and $P'(i)$ is the projective indecomposable representation of $\vec{Q}'$ with $i \neq k$ then by Theorem 2.2.11 we have that $\Phi^-_k(P'(i)) = P(i)$ is a projective indecomposable representation for $\vec{Q}$. We will use the notation $P(i_k)$ for the projective indecomposable representation for $\vec{Q}$ and $P(i_k)^t$ for the projective indecomposable representation for $s^+_{i_{k-1}} \cdots s^+_{i_1}\vec{Q}$. Then we have that

$$\begin{array}{ll}
P_1 := S(i_1) = P(i_1) & i_1 \text{ is a sink for } \vec{Q} \\
P_2 := \Phi^-_{i_1}(S(i_2)) = \Phi^-_{i_1}(P(i_2)^2) = P(i_2) & i_2 \text{ is a sink for } s^+_{i_1}\vec{Q} \\
P_3 := \Phi^-_{i_1}\Phi^-_{i_2}(S(i_3)) = \Phi^-_{i_1}\Phi^-_{i_2}(P(i_3)^3) = \Phi^-_{i_1}(P(i_3)^2) = P(i_3) & i_3 \text{ is a sink for } s^+_{i_2}s^+_{i_1}\vec{Q} \\
\quad\vdots & \quad\vdots \\
P_\ell := \Phi^-_{i_1} \cdots \Phi^-_{i_{\ell-1}}(S(i_\ell)) = \Phi^-_{i_1} \cdots \Phi^-_{i_{\ell-1}}(P(i_\ell)^\ell) = P(i_\ell) & i_\ell \text{ is a sink for } s^+_{i_{\ell-1}} \cdots s^+_{i_1}\vec{Q}.
\end{array}$$

We can do it analogously to show that $Q_k = Q(i_k)$. □

This theorem relates the representation theory of a quiver to its orientation. It shows that for certain representations $P_k$ and $Q_k$, which are constructed recursively using the orientation of the quiver, there exist corresponding indecomposable projective and injective representations $P(i_k)$ and $Q(i_k)$ that can also be constructed recursively using the orientation of the quiver.

**Definition 4.1.5.** *The Coxeter functor $C^+$ associated with the Coxeter element $C = s_{i_\ell} \cdots s_{i_1}$ is*

$$C^+ = \Phi^+_{i_\ell} \cdots \Phi^+_{i_1} : Rep(\vec{Q}) \to Rep(\vec{Q}). \tag{4.4}$$

*Similarly, we can also define $C^-$ associated with the Coxeter element $C^{-1} = s_{i_1} \cdots s_{i_\ell}$ is*

$$C^- = \Phi^-_{i_1} \cdots \Phi^-_{i_\ell} : Rep(\vec{Q}) \to Rep(\vec{Q}). \tag{4.5}$$

**Theorem 4.1.6.** *Let $V$ be an indecomposable representation of $\vec{Q}$. The representations $C^+(V)$ and $C^-(V)$ are also indecomposable, possibly zero. If $V$ and $C^+(V)$ are nonzero, then $C^-C^+(V) \simeq V$. Similarly, if $V$ and $C^-(V)$ are nonzero, then $C^+C^-(V) \simeq V$.*

*Proof.* By Theorem 2.2.7 we have that if $V$ is an indecomposable representation of $\vec{Q}$ then $\Phi^\pm_i(V) = 0$ or $\Phi^\pm_i(V)$ is a nonzero indecomposable representation. Therefore, we have that either $C^\pm(V) = 0$ or by applying successive times the second condition on the composition of $C^\pm$ we have that $C^\pm(V)$ is a nonzero indecomposable representation. □

One important case for us is when we apply $C^{\pm}$ to an indecomposable representation $V$ and get $C^{\pm}(V) = 0$. The next result is for these cases.

**Theorem 4.1.7.** *Let $V$ be a nonzero indecomposable representation of $\vec{Q}$. Then the following conditions are equivalent,*

*(i)* $C^{+}(V) = 0$.

*(ii)* $V \simeq P_k$ *for some* $k = 1, \cdots, \ell$.

*(iii)* $V$ *is projective.*

*Proof.* If $C^{+}(V) = 0$ then $\Phi^{+}_{i_\ell} \cdots \Phi^{+}_{i_1}(V) = 0$ and there exists $k$ such that $\Phi^{+}_{i_{k-1}} \cdots \Phi^{+}_{i_1}(V) \simeq S(i_k)$ for some $k \leq \ell$, by Theorem 2.2.7 item $(a)$. Otherwise, we would have that $C^{+}(V)$ is a nonzero indecomposable representation, by Theorem 2.2.7 item $(b)$, and that is a contradiction since $C^{+}(V) = 0$. Therefore,

$$\Phi^{+}_{i_{k-1}} \cdots \Phi^{+}_{i_1}(V) \simeq S(i_k)$$

and

$$V \simeq \Phi^{-}_{i_1} \cdots \Phi^{-}_{i_{k-1}}(S(i_k)) = P_k$$

for some $k = 1, \cdots, \ell$.

If $V \simeq P_k$ for some $k = 1, \cdots, \ell$ then, by Lemma 4.1.4, $P_k$ is always a projective representation so $V$ is projective.

We know that $\{P(i_k); i_k \in I\}$ is the set of all projective indecomposable representation by Theorem 1.4.4. If $V$ is a projective indecomposable representation and $P_k = P(i_k)$ then $V \simeq P_k$ for some $k \leq \ell$. As we can write $P_k = \Phi^{-}_{i_1} \cdots \Phi^{-}_{i_{k-1}}(S(i_k))$ we have that

$$\begin{aligned} C^{+}(V) = C^{+}(P_k) &= \Phi^{+}_{i_\ell} \cdots \Phi^{+}_{i_1}(P_k) = \Phi^{+}_{i_\ell} \cdots \Phi^{+}_{i_1}\left(\Phi^{-}_{i_1} \cdots \Phi^{-}_{i_{k-1}}(S(i_k))\right) \\ &= \Phi^{+}_{i_\ell} \cdots \Phi^{+}_{i_1}\Phi^{-}_{i_1} \cdots \Phi^{-}_{i_{k-1}}(S(i_k)) = \Phi^{+}_{i_\ell} \cdots \Phi^{+}_{i_k}(S(i_k)) = \Phi^{+}_{i_\ell} \cdots \Phi^{+}_{i_{k+1}}(0) = 0, \end{aligned}$$

since $\Phi^{+}_{i_k}(S(i_k)) = 0$ by Remark 2.1.9. □

**Theorem 4.1.8.** *Let $V$ be a nonzero indecomposable representation of $\vec{Q}$. Then the following conditions are equivalent,*

*(i)* $C^{-}(V) = 0$.

*(ii)* $V \simeq Q_k$ *for some* $k = 1, \cdots, \ell$.

*(iii)* $V$ *is injective.*

*Proof.* The proof is analogous to Theorem 4.1.7. □

**Corollary 4.1.9.** *If $V$ is a representation with $V \in Rep(\vec{Q})$.*

*(a)* $C^+(V) = 0$ *if and only if* $V$ *is projective.*

*(b)* $C^-(V) = 0$ *if and only if* $V$ *is injective.*

*Proof.* (a) If $V$ is any representation of $\vec{Q}$ we can write

$$V = \bigoplus_{i=1}^{n} V_i$$

with each $V_i$ an indecomposable subrepresentation of $V$.

If $C^+(V) = 0$ then

$$C^+(V) = C^+\left(\bigoplus_{i=1}^{n} V_i\right) = \bigoplus_{i=1}^{n} C^+(V_i) = 0.$$

As the direct sum is the null representation then each $C^+(V_i) = 0$ for all $i$, by Theorem 4.1.7 we have that each $V_i$ is projective for every $i$ so $V$ is projective.

If $V$ is projective then $V_i$ is projective for all $i$ and by Theorem 4.1.7 we have that $C^+(V_i) = 0$. Therefore,

$$C^+(V) = C^+\left(\bigoplus_{i=1}^{n} V_i\right) = \bigoplus_{i=1}^{n} C^+(V_i) = 0.$$

(b) It is analogous to the first one.

□

## 4.2 Preprojective and preinjective representations

In this section we choose an orientation of $Q$ without oriented cycles, we also choose a Coxeter element $C$ adapted to this orientation, and we let $C^{\pm}$ be the corresponding Coxeter functors as defined in Definition 4.1.5. The goal here is to prove an analogous to Proposition 3.2.1 that establishes a bijection between positive roots and indecomposable representations. This is not true in general, for instance the Kronecker quiver has infinite representations of a given dimension, but the theorem can be true in certain cases.

**Definition 4.2.1.** *If* $V$ *is an indecomposable representation of* $\vec{Q}$ *then* $V$ *is said to be*

- *preprojective if* $(C^+)^n(V) = 0$ *for* $n \gg 0$*;*
- *preinjective if* $(C^-)^n(V) = 0$ *for* $n \gg 0$*;*
- *regular if for any* $n \in \mathbb{Z}_+$*, we have* $(C^+)^n(V) \neq 0$ *and* $(C^-)^n(V) \neq 0$*.*

A representation $V \in \mathrm{Rep}(\vec{Q})$ is called preprojective, preinjective or regular if each of its indecomposable summands is of the corresponding type. We have a similar definition for the roots.

**Definition 4.2.2.** *If* $\alpha \in R_+^{re}$ *is a real positive root then* $\alpha$ *is said to be*

- *preprojective if* $C^n(\alpha) \ngtr 0$ *for some* $n > 0$*;*

- *preinjective if* $C^n(\alpha) \ngtr 0$ *for some* $n < 0.$*;*

**Example 4.2.3.** *Roots* $p_k$ *are preprojective and roots* $q_k$ *are preinjective since,*

$$C(p_k) = -q_k, \quad C^n(p_k) \ngtr 0 \text{ for } n = 1$$

*and*

$$C^{-1}(q_k) = -p_k, \quad C^n(q_k) \ngtr 0 \text{ for } n = -1.$$

**Remark 4.2.4.** *In the Dynkin case, from Corollary 3.1.3* (2)*, we have that for any positive root* $\alpha$ *there exists* $k > 0$ *such that* $C^k\alpha \ngtr 0$ *then every positive root is preprojective. From Corollary 3.1.3* (3)*, we have that for any positive root* $\alpha$ *there exists* $k < 0$ *such that* $C^k\alpha \ngtr 0$ *then every positive root is preinjective.*

**Theorem 4.2.5.** *Let* $C = s_{i_\ell} \cdots s_{i_1}$ *be a Coxeter element adapted to* $\vec{Q}$ *and* $C^{\pm}$ *the corresponding Coxeter functors.*

*(1) If* $V$ *is a nonzero preprojective indecomposable representation, then*

$$V \simeq (C^-)^n(P(i)) \quad \text{for some} \quad n \geq 0 \text{ and } i \in I.$$

*In this case, dim*$(V)$ *is a preprojective positive real root. Conversely, for every preprojective positive real root* $\alpha$*, there is a unique indecomposable representation* $V_\alpha$ *of graded dimension* $\alpha$ *and this representation is preprojective.*

*(2) If* $V$ *is a nonzero preinjective indecomposable representation then*

$$V \simeq (C^+)^n(Q(i)) \quad \text{for some} \quad n \geq 0 \text{ and } i \in I.$$

*In this case, dim*$(V)$ *is a preinjective positive real root. Conversely, for every preinjective positive real root* $\alpha$*, there is a unique indecomposable representation* $V_\alpha$ *of graded dimension* $\alpha$ *and this representation is preinjective.*

*Proof.* (1) If $V$ is a nonzero preprojective indecomposable representation then there exists $n+1$ such that $(C^+)^{n+1}(V) = 0$. Let us choose $n+1$ to be the minimum that satisfy this condition, so

$$(C^+)^n(V) \neq 0 \text{ and } C^+((C^+)^n(V)) = 0.$$

By Theorem 4.1.7 we have that

$$(C^+)^n(V) \simeq P_k \text{ for some } k = 1, \cdots, \ell.$$

If $V$ and $C^+(V)$ are nonzero then $C^-C^+(V) \simeq V$ by Theorem 4.1.6. As we have that $V$ is nonzero and

$(C^+)^t(V)$ is nonzero for all $t \le n$ then

$$C^-(C^+)^n(V) \simeq (C^+)^{n-1}(V)$$
$$C^-(C^+)^{n-1}(V) \simeq (C^+)^{n-2}(V)$$
$$\vdots$$
$$C^-(C^+)^2(V) \simeq C^+(V)$$
$$C^-C^+(V) \simeq V$$

and by recursion we get $(C^-)^n(C^+)^n(V) \simeq V$ which means $(C^-)^n(P_k) \simeq V$. By Lemma 4.1.4, $P_k = P(i_k)$ then $V \simeq (C^-)^n(P(i_k))$. For the dimension we have that

$$\dim(V) = \dim((C^-)^n(P_k)) = C^{-n}\dim(P_k) = C^{-n}(p_k),$$

where the second equality holds by Theorem 2.2.5 (4) and the third equality is true by (4.2). Since $\dim(V) = C^{-n}(p_k)$ then $C^{-n}(p_k)$ is a positive real root. By Definition 4.2.2 we have that $C^t(p_k) \not> 0$ for some $t > 0$, since $p_k$ is a preprojective positive real root. Then there exists $t + n > 0$ such that

$$C^{t+n}(C^{-n}(p_k)) = C^{t+n-n}(p_k) = C^t(p_k) \not> 0.$$

Therefore, $\dim(V)$ is a preprojective positive real root.

On the other hand, if $\alpha \in R_+^{re}$ and $\alpha$ is a preprojective root then we know that there exists $n > 0$ such that $C^n(\alpha) \not> 0$ by Definition 4.2.2. Let us choose $n+1$ to be the maximum that satisfy this condition, so

$$\alpha > 0, \quad C(\alpha) > 0, \quad \cdots \quad , C^n(\alpha) > 0 \quad \text{and} \quad C(C^n(\alpha)) = C^{n+1}(\alpha) \not> 0, \tag{4.6}$$

which means that $C(C^n(\alpha)) \in \mathsf{R}_-^{re}$. By the equality in item $(b)$ of Lemma 4.1.1 we have that $C^n(\alpha) = p_k$ for some $k = 1, \cdots, \ell$, then $\alpha = C^{-n}(p_k)$. Consider $V_\alpha = (C^-)^n(P(i_k))$. We have that

$$\dim(V_\alpha) = \dim((C^-)^n(P(i_k))) = C^{-n}(p_k) = \alpha,$$

where the second equality is true by Theorem 2.2.5. Since $P(i_k)$ is an indecomposable preprojective representation, and we are applying a sequence of $\Phi_i^-$ to this representation then by Theorem 2.2.7 we have that $V_\alpha$ is an indecomposable representation. We have that

$$p_k = C^n(\alpha), \quad C^{-1}p_k = C^{n-1}(\alpha), \quad \cdots \quad , C^{-n}p_k = \alpha$$

are all positive, from (4.6), then

$$P(i_k), \quad C^-P(i_k), \quad \cdots \quad , (C^-)^n(P(i_k))$$

are all nonzero. Also,

$$(C^+)^{n+1}(V_\alpha) = (C^+)^{n+1}(C^-)^n(P(i_k)) = C^+(P(i_k)) = 0$$

and $V_\alpha$ is preprojective. Therefore, $V_\alpha$ is a nonzero indecomposable preprojective representation. The representation $V_\alpha$ is unique since if there exists another representation $V'$ with dimension $\alpha$ then by the first part we have that $V' \simeq (C^-)^n(P(i)) = V_\alpha$.

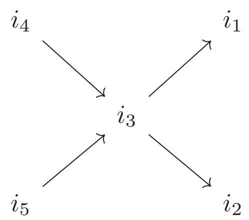
(2) The second one is made analogously.

□

This theorem is a fundamental result in the representation theory of quivers of finite type. It provides a complete characterization of the indecomposable preprojective and preinjective representations in terms of the Coxeter element and Coxeter functors. It also establishes a correspondence between positive roots and indecomposable representations, which is an important tool for studying the structure of the representation category.

**Example 4.2.6.** *Consider the graph $\hat{D}_4$ with orientation given by*

$\vec{Q} =$




*We have that $I = \{i_1, i_2, i_3, i_4, i_5\}$ is an ordering of the vertices adapted to the orientation, since*

$$\begin{aligned}
&i_1 \text{ is a sink for } \vec{Q}\\
&i_2 \text{ is a sink for } s_1^+(\vec{Q})\\
&i_3 \text{ is a sink for } s_2^+ s_1^+(\vec{Q})\\
&i_4 \text{ is a sink for } s_3^+ s_2^+ s_1^+(\vec{Q})\\
&i_5 \text{ is a sink for } s_4^+ s_3^+ s_2^+ s_1^+(\vec{Q})
\end{aligned}$$

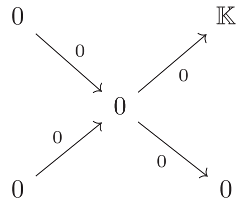
*with $s_{i_k}$ for $k = 1, 2, 3, 4, 5$. Therefore, we have the Coxeter element adapted to $\vec{Q}$ as $C = s_5 s_4 s_3 s_2 s_1$ and the Coxeter functors*

$$\begin{aligned}
C^+ &= \Phi_{i_5}^+ \Phi_{i_4}^+ \Phi_{i_3}^+ \Phi_{i_2}^+ \Phi_{i_1}^+\\
C^- &= \Phi_{i_1}^- \Phi_{i_2}^- \Phi_{i_3}^- \Phi_{i_4}^- \Phi_{i_5}^-
\end{aligned}$$

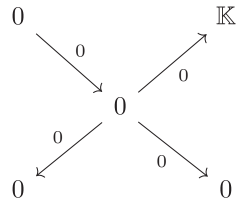
*For ease of notation we will write $P(i_k) = P_k$, $Q(i_k) = Q_k$ and $\Phi_{i_k}^\pm = \Phi_k^\pm$ for $k = 1, 2, 3, 4, 5$. In example 4.1.3 we calculate the five projective indecomposable representations. Now we want to apply $C^-$ in the projective indecomposable representations to find some preprojective indecomposable representations in the orbits of the projective ones as we saw in Theorem 4.2.5. In the $\hat{D}_4$ case we will apply $C^-$ until $n = 4$.*

*We will start with the projective $P_1$. For $n = 1$ we have,*

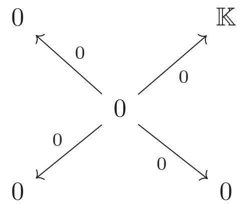
$P_1$ :



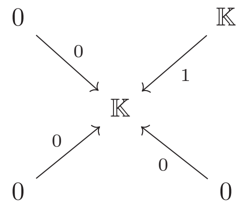

$\Phi_5^-(P_1)$ :



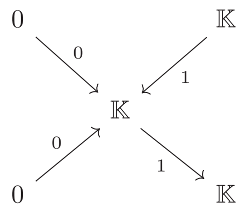

$\Phi_4^-\Phi_5^-(P_1)$ :



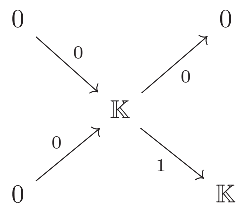

$\Phi_3^-\Phi_4^-\Phi_5^-(P_1)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(P_1)$ :

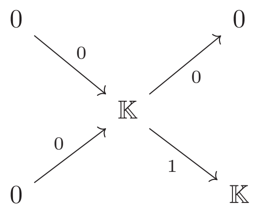



$C^-(P_1)$ :

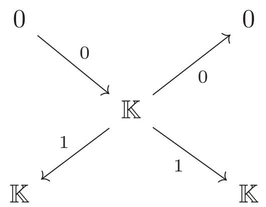



*Consider $C^-(P_1) = V_1$. For $n = 2$ we have,*

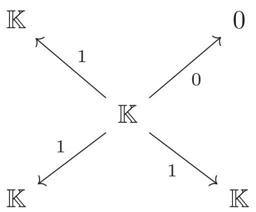
$V_1$ :



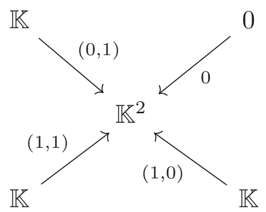

$\Phi_5^-(V_1)$ :



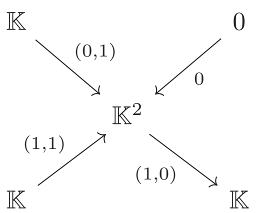

$\Phi_4^-\Phi_5^-(V_1)$ :



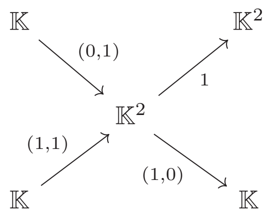

$\Phi_3^-\Phi_4^-\Phi_5^-(V_1)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(V_1)$ :

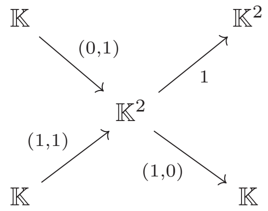



$C^-(V_1)$ :

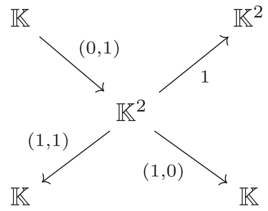


*Consider* $C^-(V_1) = (C^-)^2(P_1) = V_2$. *For* $n = 3$ *we have,*

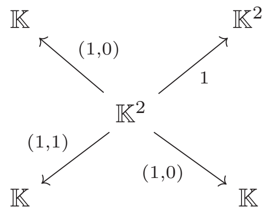
$V_2$ :



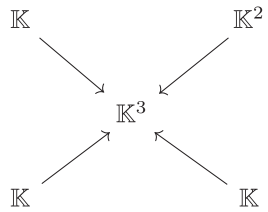

$\Phi_5^-(V_2)$ :



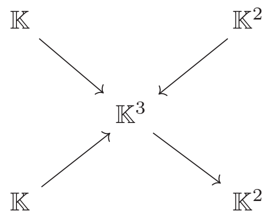

$\Phi_4^-\Phi_5^-(V_2)$ :



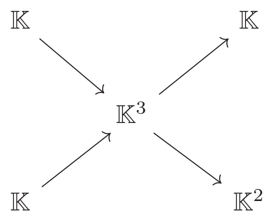

$\Phi_3^-\Phi_4^-\Phi_5^-(V_2)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(V_2)$ :

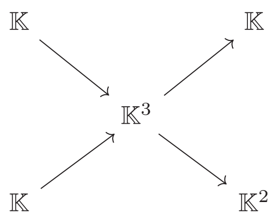



$C^-(V_2)$ :

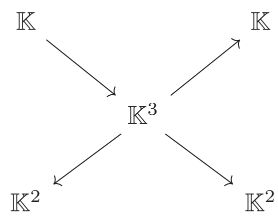



*Consider* $C^-(V_2) = (C^-)^3(P_1) = V_3$. *For* $n = 4$ *we have,*

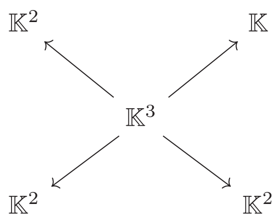
$V_3$ :



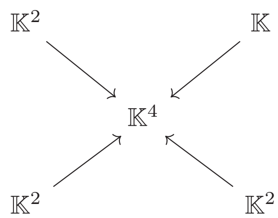

$\Phi_5^-(V_3)$ :



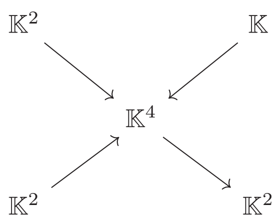

$\Phi_4^-\Phi_5^-(V_3)$ :



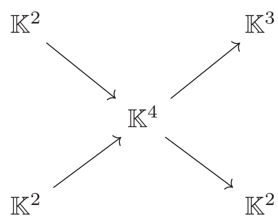

$\Phi_3^-\Phi_4^-\Phi_5^-(V_3)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(V_3)$ :

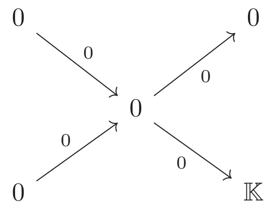



$C^-(V_3)$ :

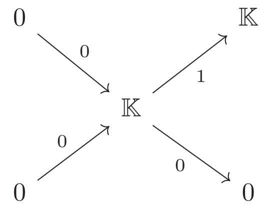


*Now for the projective $P_2$ we have,*

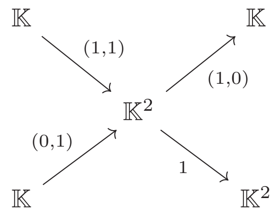
$P_2$ :



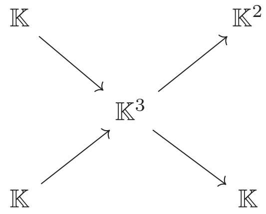

$C^-(P_2)$ :



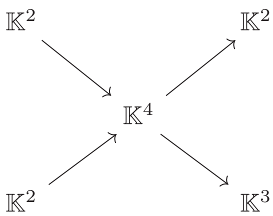

$(C^-)^2(P_2)$ :




$(C^-)^3(P_2)$ :

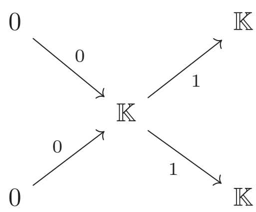



$(C^-)^4(P_2)$ :

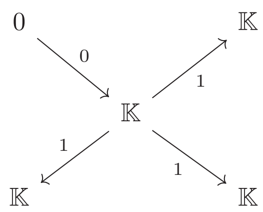



*Now for the projective $P_3$. For $n = 1$ we have,*

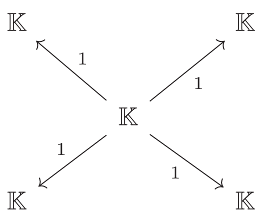
$P_3$ :



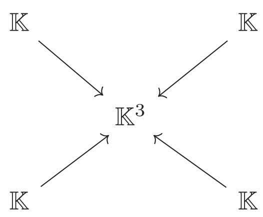

$\Phi_5^-(P_3)$ :



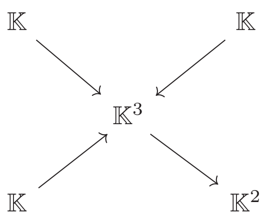

$\Phi_4^-\Phi_5^-(P_3)$ :



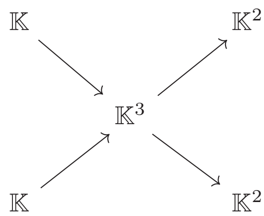

$\Phi_3^-\Phi_4^-\Phi_5^-(P_3)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(P_3)$ :

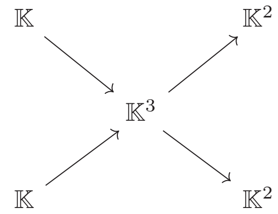



$C^-(P_3)$ :

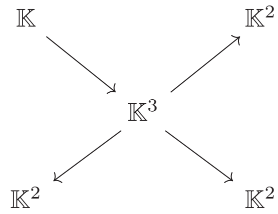


*Consider* $C^{-}(P_3) = V_1$*. For* $n = 2$ *we have,*

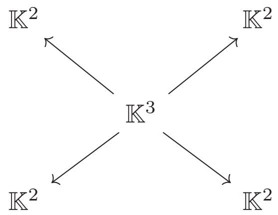
$V_1$ :



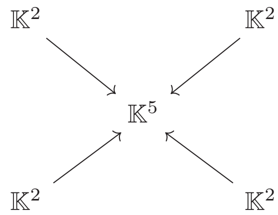

$\Phi_5^-(V_1)$ :



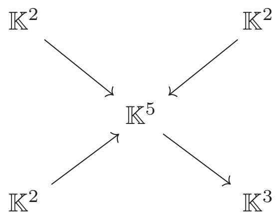

$\Phi_4^-\Phi_5^-(V_1)$ :



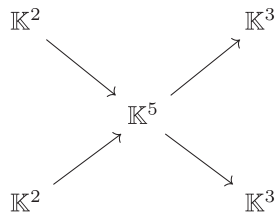

$\Phi_3^-\Phi_4^-\Phi_5^-(V_1)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(V_1)$ :

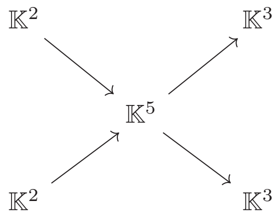



$C^-(V_1)$ :

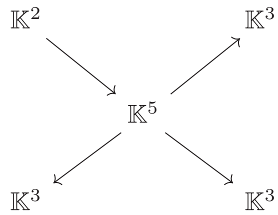



*Consider* $C^{-}(V_1) = (C^-)^2(P_3) = V_2$*. For* $n = 3$ *we have,*

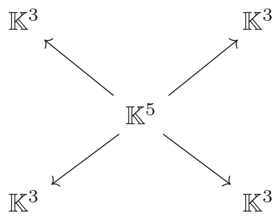
$V_2$ :



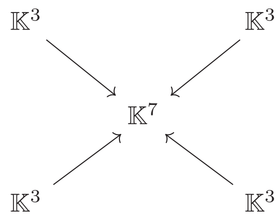

$\Phi_5^-(V_2)$ :



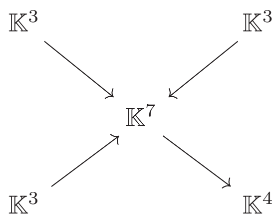

$\Phi_4^-\Phi_5^-(V_2)$ :



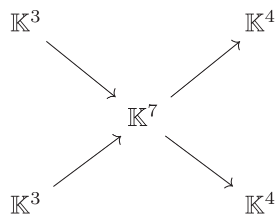

$\Phi_3^-\Phi_4^-\Phi_5^-(V_2)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(V_2)$ :

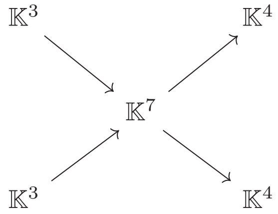



$C^-(V_2)$ :

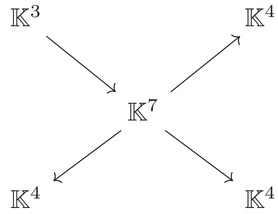


*Consider $C^{-}(V_2) = (C^{-})^3(P_3) = V_3$. For $n = 4$ we have,*

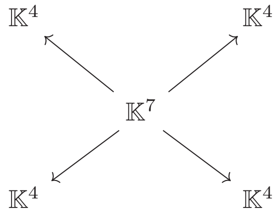
$V_3$ :



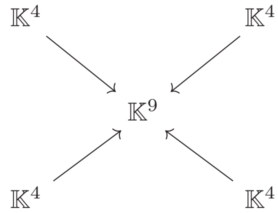

$\Phi_5^-(V_3)$ :



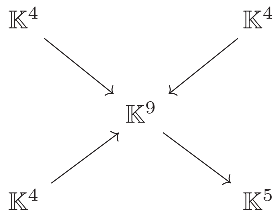

$\Phi_4^-\Phi_5^-(V_3)$ :



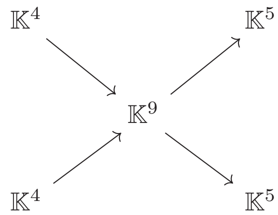

$\Phi_3^-\Phi_4^-\Phi_5^-(V_3)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(V_3)$ :

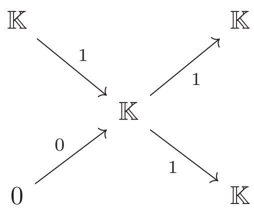



$C^-(V_3)$ :

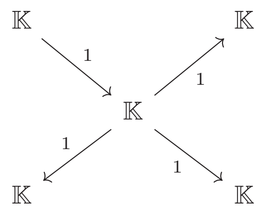



*Now for the projective $P_4$. For $n = 1$ we have,*

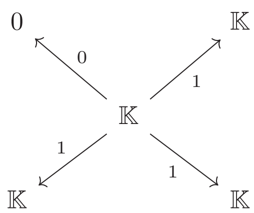
$P_4$ :



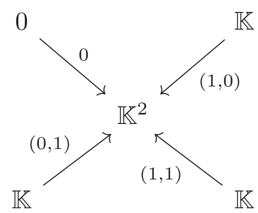

$\Phi_5^-(P_4)$ :



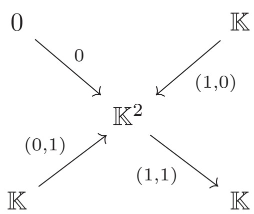

$\Phi_4^-\Phi_5^-(P_4)$ :



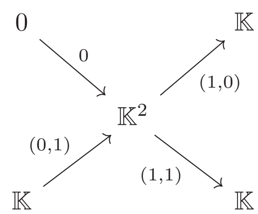

$\Phi_3^-\Phi_4^-\Phi_5^-(P_4)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(P_4)$ :

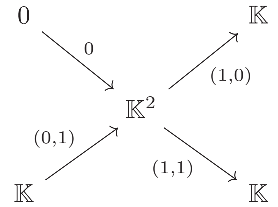



$C^-(P_4)$ :

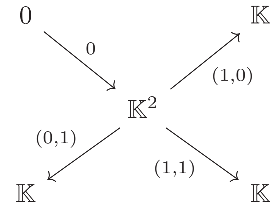


*Consider* $C^-(P_4) = V_1$. *For* $n = 2$ *we have,*

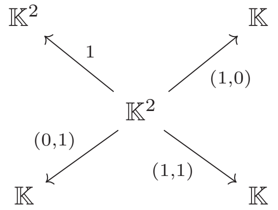
$V_1 :$



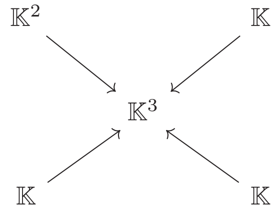

$\Phi_5^-(V_1) :$



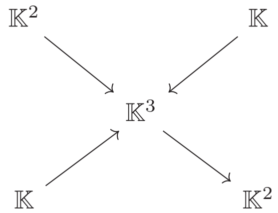

$\Phi_4^-\Phi_5^-(V_1) :$



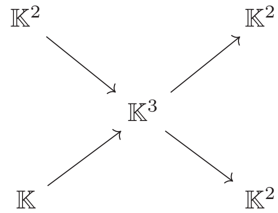

$\Phi_3^-\Phi_4^-\Phi_5^-(V_1) :$




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(V_1) :$

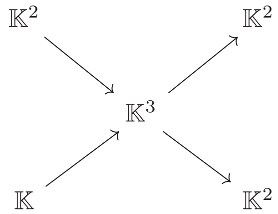



$C^-(V_1) :$

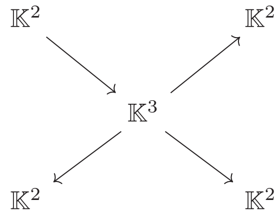



*Consider* $C^-(V_1) = (C^-)^2(P_4) = V_2$. *For* $n = 3$ *we have,*

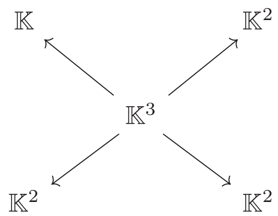
$V_2 :$



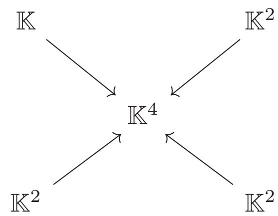

$\Phi_5^-(V_2) :$



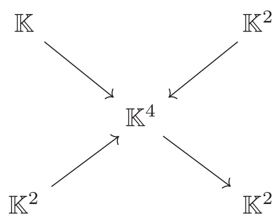

$\Phi_4^-\Phi_5^-(V_2) :$



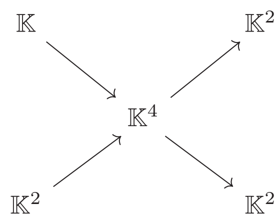

$\Phi_3^-\Phi_4^-\Phi_5^-(V_2) :$




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(V_2) :$

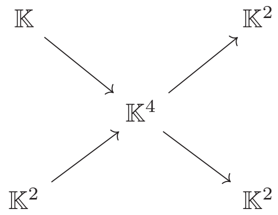



$C^-(V_2) :$

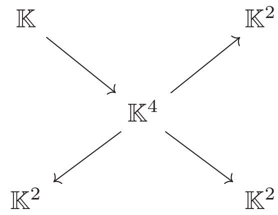


*Consider* $C^-(V_2) = (C^-)^3(P_4) = V_3$. *For* $n = 4$ *we have,*

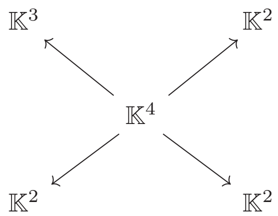
$V_3$ :



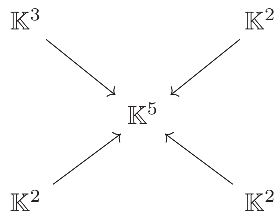

$\Phi_5^-(V_3)$ :



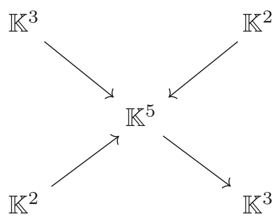

$\Phi_4^-\Phi_5^-(V_3)$ :



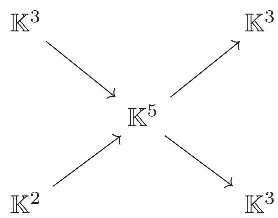

$\Phi_3^-\Phi_4^-\Phi_5^-(V_3)$ :




$\Phi_2^-\Phi_3^-\Phi_4^-\Phi_5^-(V_3)$ :

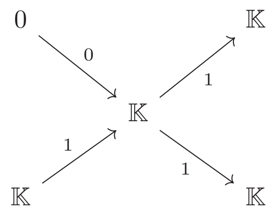



$C^-(V_3)$ :

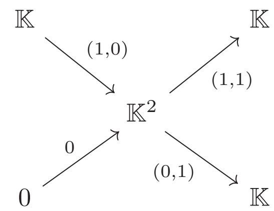



*Now for the projective* $P_5$ *we have,*

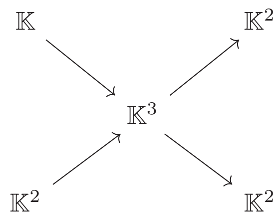
$P_5$ :



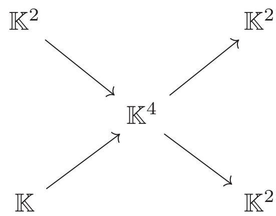

$C^-(P_5)$ :



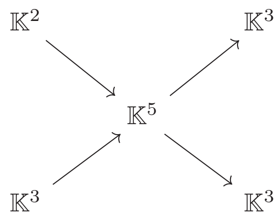

$(C^-)^2(P_5)$ :




$(C^-)^3(P_5)$ :

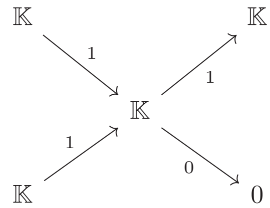



$(C^-)^4(P_5)$ :

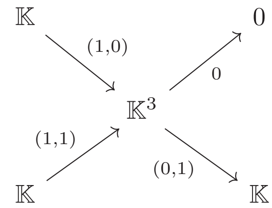



*In example 4.1.3 we calculate the five injective indecomposable representations. In an analogous way to the* *preprojective indecomposable representations we can calculate the preinjective indecomposable representations. For the injective representation $Q_1$ we have,*

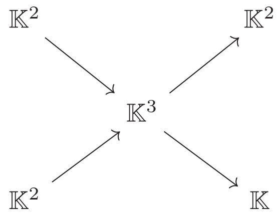
$Q_1$ :



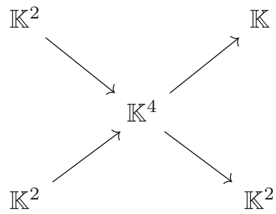

$C^+(Q_1)$ :



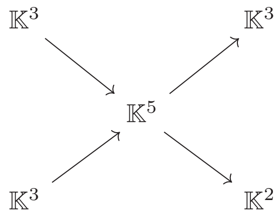

$(C^+)^2(Q_1)$ :




$(C^+)^3(Q_1)$ :

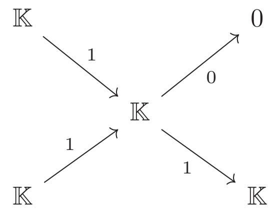



$(C^+)^4(Q_1)$ :

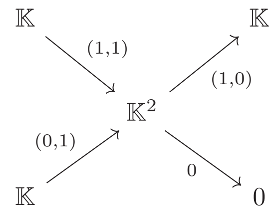



*For the injective representation $Q_2$ we have,*

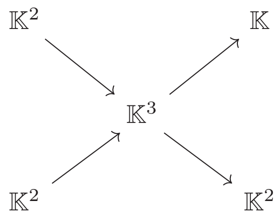
$Q_2$ :



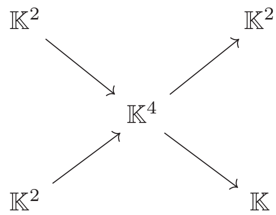

$C^+(Q_2)$ :



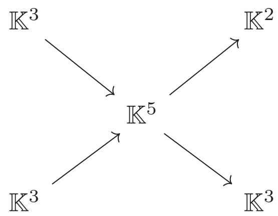

$(C^+)^2(Q_2)$ :




$(C^+)^3(Q_2)$ :

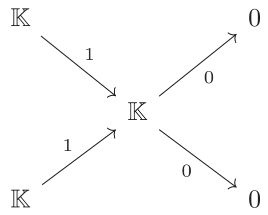



$(C^+)^4(Q_2)$ :

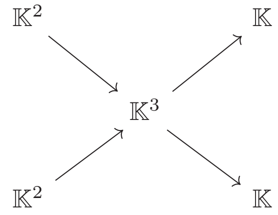


*For the injective representation $Q_3$ we have,*

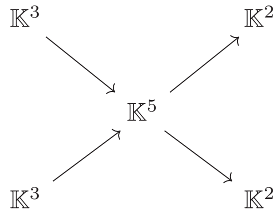
$Q_3$ :



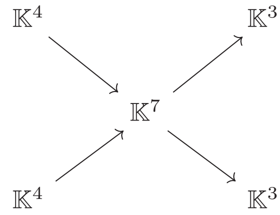

$C^+(Q_3)$ :



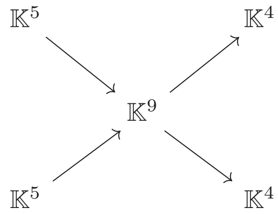

$(C^+)^2(Q_3)$ :




$(C^+)^3(Q_3)$ :

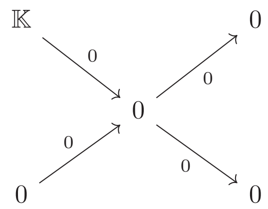



$(C^+)^4(Q_3)$ :

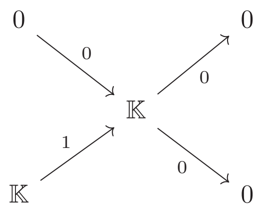



*For the injective representation $Q_4$ we have,*

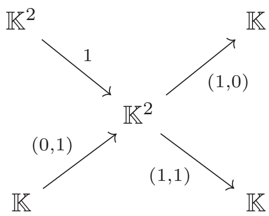
$Q_4$ :



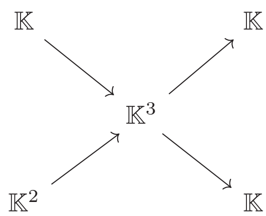

$C^+(Q_4)$ :



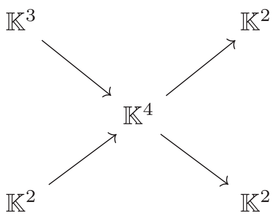

$(C^+)^2(Q_4)$ :



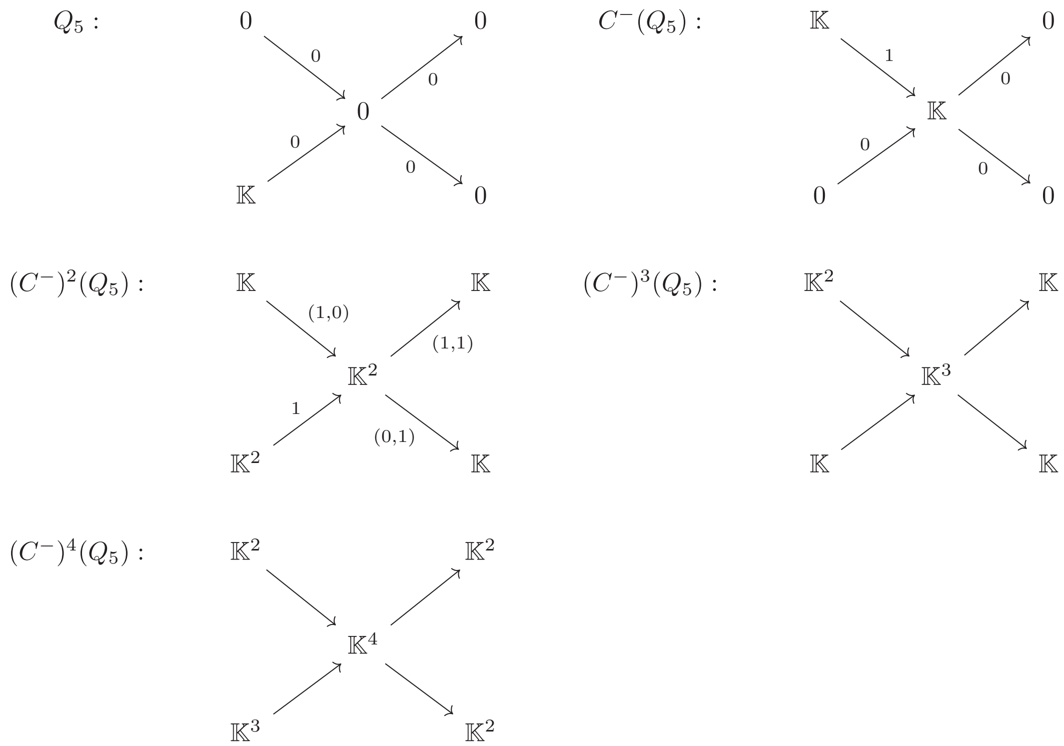

$(C^+)^3(Q_4)$ :




$(C^+)^4(Q_4)$ :



*For the injective representation $Q_5$ we have,*



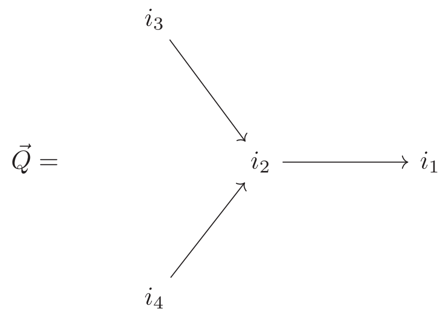

**Corollary 4.2.7.** *If $\vec{Q}$ is a Dynkin quiver, then any indecomposable representation is both preprojective and preinjective, and there are no nonzero regular representations.*

*Proof.* By Remark 4.2.4 we have that every positive root is preprojective and preinjective. Also in the Dynkin case, by Proposition 3.2.1 we have that the nonzero indecomposable representations are in bijection with the positive roots, so by Theorem 4.2.5 every indecomposable representation is both preprojective and preinjective. Therefore, it does not exist any nonzero indecomposable regular representations.

A representation is regular if every indecomposable summand is regular and no nonzero indecomposable representation is regular in the Dynkin case. So there are no nonzero regular representation. □

In the Dynkin case this means that we can find all the preprojective and preinjective indecomposable representations. We will show this in an example.

**Example 4.2.8.** *Consider the quiver $D_4$ given by*



*We have that* $I = \{i_1, i_2, i_3, i_4\}$ *is an ordering of the vertices adapted to the orientation, since*

$$\begin{aligned} &i_1 \text{ is a sink for } \vec{Q} \\ &i_2 \text{ is a sink for } s_1^+(\vec{Q}) \\ &i_3 \text{ is a sink for } s_2^+ s_1^+(\vec{Q}) \\ &i_4 \text{ is a sink for } s_3^+ s_2^+ s_1^+(\vec{Q}) \end{aligned}$$

*with* $s_{i_k}$ *for* $k = 1, 2, 3, 4$. *Therefore, we have the Coxeter element adapted to* $\vec{Q}$ *as* $C = s_4 s_3 s_2 s_1$ *and the Coxeter functors*

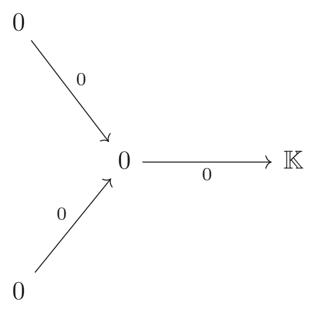
$$\begin{aligned} C^+ &= \Phi^+_{i_4} \Phi^+_{i_3} \Phi^+_{i_2} \Phi^+_{i_1} \\ C^- &= \Phi^-_{i_1} \Phi^-_{i_2} \Phi^-_{i_3} \Phi^-_{i_4} \end{aligned}$$

*For ease of notation we will write* $P(i_k) = P_k$, $Q(i_k) = Q_k$ *and* $\Phi^{\pm}_{i_k} = \Phi^{\pm}_k$ *for* $k = 1, 2, 3, 4$. *In example 4.1.2 we calculate the four projective indecomposable representations and the four injective indecomposable representations.*

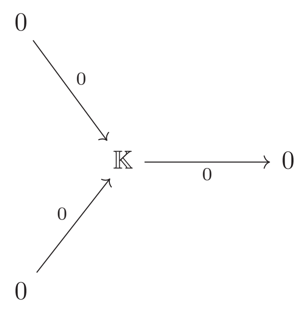
*In the Dynkin case by Corollary 4.2.7 all indecomposable representations are both preprojective and preinjective. So we can apply* $C^-$ *to every projective indecomposable representation as many times as needed until we get an injective indecomposable representation.*

*For the projective* $P_1$ *we have,*

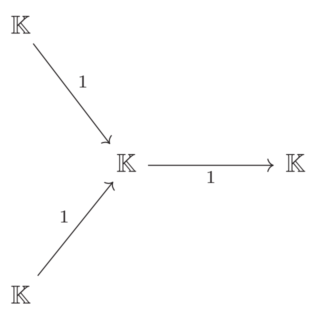
$P_1$ :




$C^-(P_1)$ :



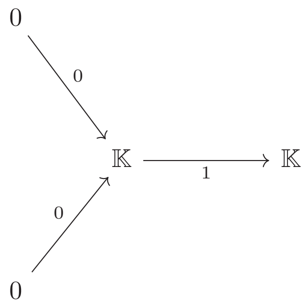

$(C^-)^2(P_1)$ :



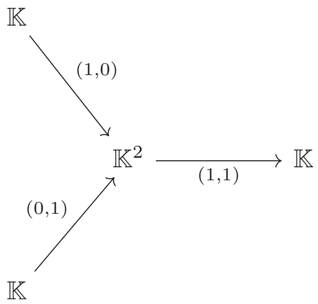

*and this means that* $(C^-)^2(P_1) = Q_1$.

*For the projective $P_2$ we have,*

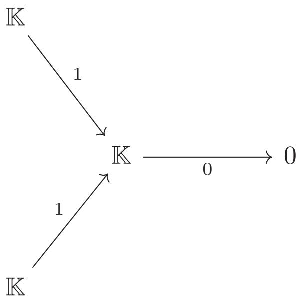
$P_2$ :




$C^-(P_2)$ :



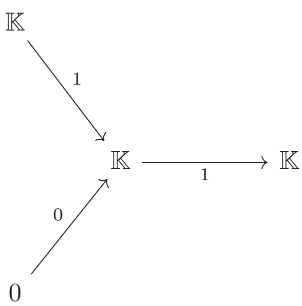

$(C^-)^2(P_2)$ :



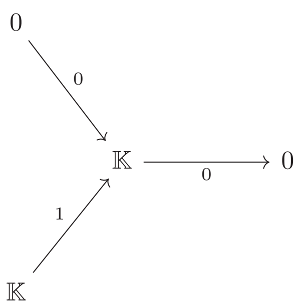

*and this means that* $(C^-)^2(P_2) = Q_2$.

*For the projective $P_3$ we have,*

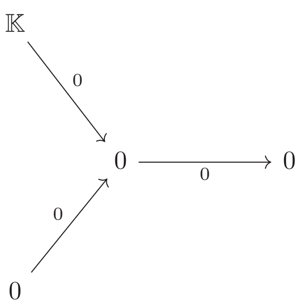
$P_3$ :




$C^-(P_3)$ :



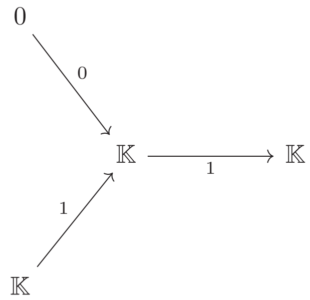

$(C^-)^2(P_3)$ :



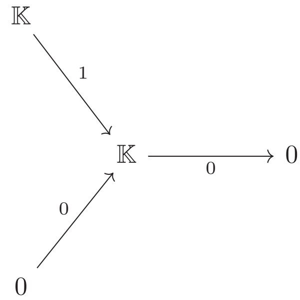

*and this means that* $(C^-)^2(P_3) = Q_3$.

*For the projective $P_4$ we have,*

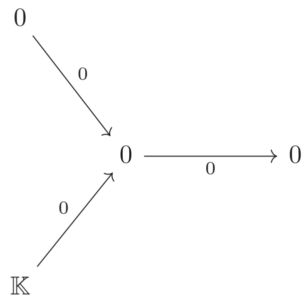
$P_4$ :



$C^-(P_4)$ :



$(C^-)^2(P_4)$ :



*and this means that* $(C^-)^2(P_4) = Q_4$.

## 4.3 Auslander-Reiten quiver: Combinatorics

In this section we will study some special types of quivers to introduce a tool that makes the combinatorics of orientations reversal operations $s_i^{\pm}$ much more transparent. The goal is to define the Auslander-Reiten (AR) quiver. This quiver is usually defined in terms of almost split sequences, that can be seen in [5]. We will define the AR quiver in a more elementary combinatorial construction.

From now on we assume that $Q$ is a connected bipartite graph with $I = I_0 \cup I_1$, which means that for each edge of $Q$ one vertex is in $I_0$ and the other one is in $I_1$. This partition is possible if and only if $Q$ has no cycles of odd length, in particular, any tree graph is bipartite. We will also assume that $Q$ is not the graph of only one vertex $\bullet$. For the example 4.2.6 we can write the partition of $I = \{i_1, i_2, i_3, i_4, i_5\}$ as $I_0 = \{i_1, i_2, i_4, i_5\}$ and $I_1 = \{i_3\}$.

We define the parity of $i \in I$ with the notation $p(i) \in \mathbb{Z}_2$ by

$$p(i) = \begin{cases} 0, & i \in I_0, \\ 1, & i \in I_1. \end{cases}$$

The quiver $Q \times \mathbb{Z}$ is defined by the set of vertices $I \times \mathbb{Z}$ and oriented edges

$$(i, n) \to (j, n+1) \text{ and } (j, n) \to (i, n+1)$$

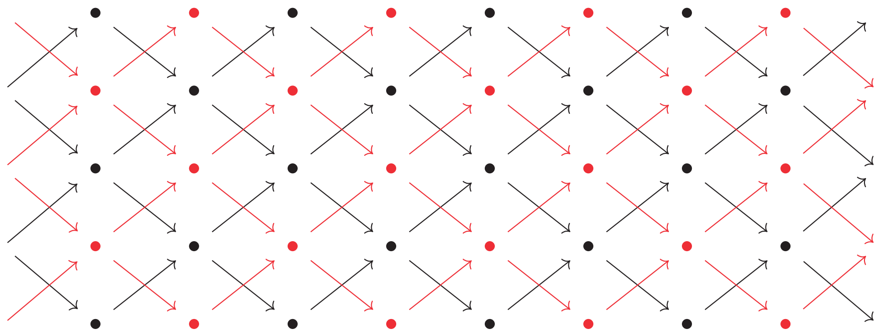
for every edge $i - j$ in $Q$ and every $n \in \mathbb{Z}$. It is easy to check that $Q \times \mathbb{Z}$ is not connected, $(i, n)$ is connected with $(j, m)$ if and only if $m - n \equiv d(i, j) \bmod(2)$ where $d(i, j)$ is the length of a path connecting $i$ and $j$ in $\vec{Q}$.

The simplest example to represent these concepts is with the graphs $A_\ell$. We will also give more complex examples with the $D_\ell$ graph, which is a Dynkin graph, and with the $\hat{D}_\ell$ graph, which is a Euclidean graph.

**Example 4.3.1.** *For the graph of type* $A_5$

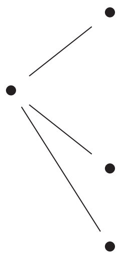


*we have the quiver* $Q \times \mathbb{Z}$

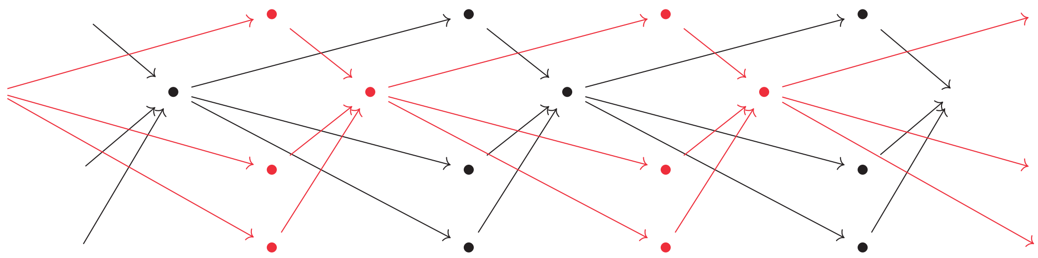


*We can see that this quiver is not connected, in fact there are two connected components, one in black and one in red.*

**Example 4.3.2.** *For the graph of type* $D_4$

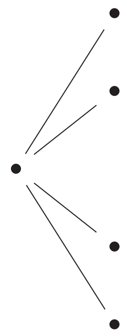


*we have the quiver* $Q \times \mathbb{Z}$

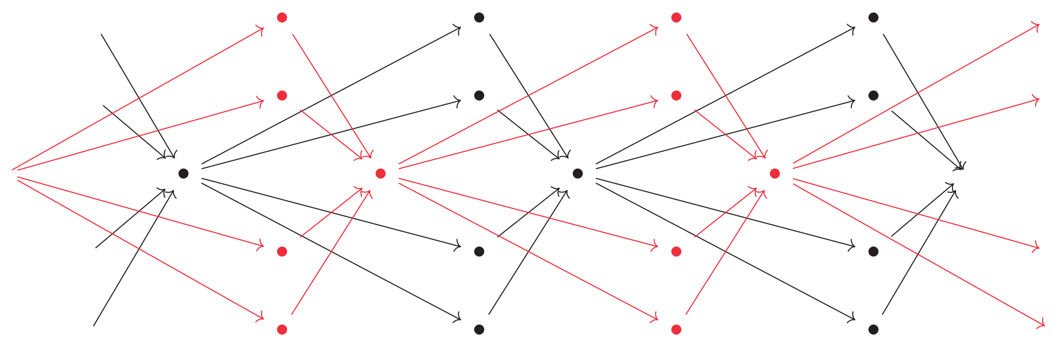


*In this example we can also see that this quiver is not connected, we have two connected components, one in black and one in red.*

**Example 4.3.3.** *For the graph of type* $\hat{D}_4$



*we have the quiver* $Q \times \mathbb{Z}$



*In this example we can also see that this quiver is not connected, we have two connected components, one in black and one in red.*

Define the quiver,

$$\mathbb{Z}Q = \{(i, n); p(i) + n \equiv 0 \text{ mod}(2)\} \subset Q \times \mathbb{Z}.$$

This quiver is called the translation quiver, and it is easy to see that $\mathbb{Z}Q$ is connected unless $Q$ is a single vertex.

**Example 4.3.4.** *In example 4.3.1 we have the* $Q \times \mathbb{Z}$ *quiver for the graph* $A_5$*, if we label the vertices of that graph we have,*

$$i_1 \text{ ——— } i_2 \text{ ——— } i_3 \text{ ——— } i_4 \text{ ——— } i_5$$

*Set* $I_0 = \{i_2, i_4\}$ *and* $I_1 = \{i_1, i_3, i_5\}$*. We have that*

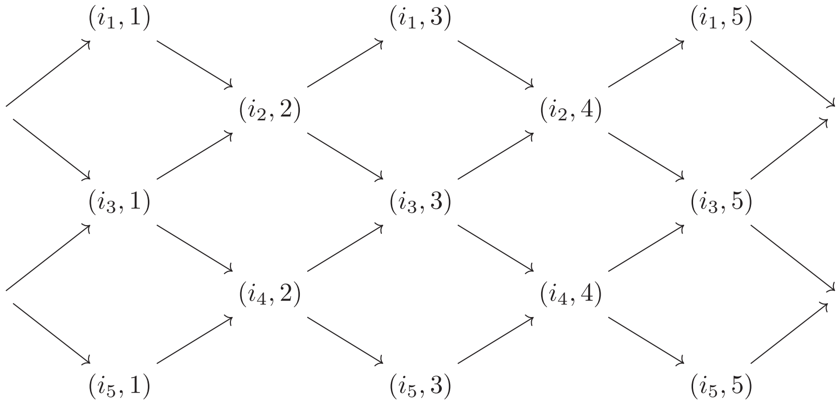
$$p(i_1) + 1 = 1 + 1 = 2 \equiv 0 \ mod(2) \Rightarrow (i_1, 1) \in \mathbb{Z}Q$$

$$p(i_1) + 2 = 1 + 2 = 3 \not\equiv 0 \ mod(2) \Rightarrow (i_1, 2) \notin \mathbb{Z}Q$$

$$p(i_2) + 1 = 0 + 1 = 1 \not\equiv 0 \ mod(2) \Rightarrow (i_2, 1) \notin \mathbb{Z}Q$$

$$p(i_2) + 2 = 0 + 2 = 2 \equiv 0 \ mod(2) \Rightarrow (i_2, 2) \in \mathbb{Z}Q.$$

*Hence, we have the $\mathbb{Z}Q$ quiver,*




The quiver $\mathbb{Z}Q$ has a canonical automorphism $\tau$ defined by

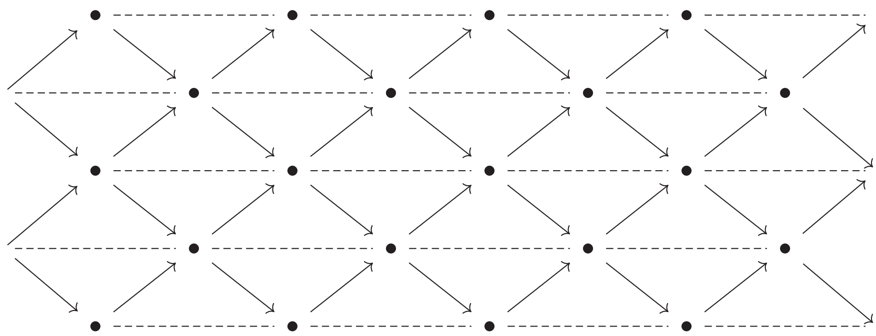
$$\begin{aligned} \tau : \mathbb{Z}Q &\to \mathbb{Z}Q \\ (i, n) &\mapsto (i, n-2). \end{aligned} \tag{4.7}$$

In the next three examples the dashed line represents that $\tau(i, n) = (i, n-2)$.

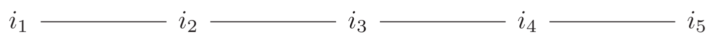
**Example 4.3.5.** *For the graph of type $A_5$*



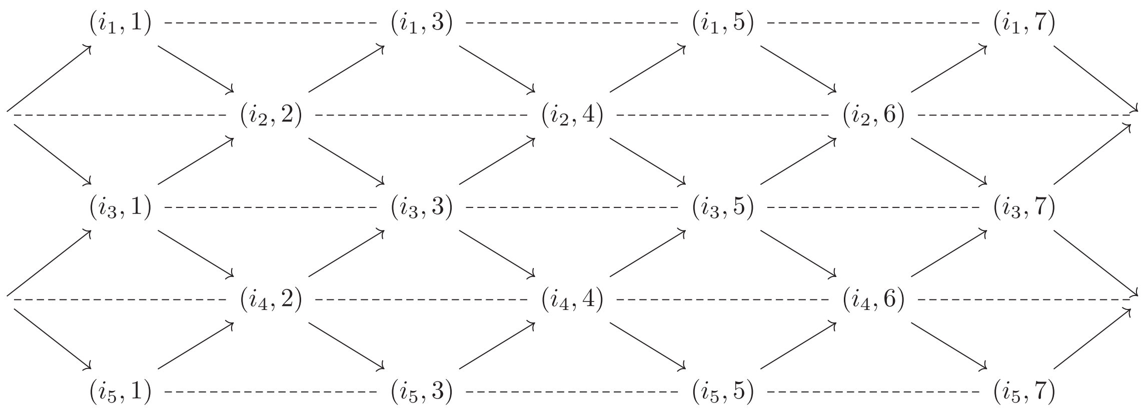
*we have the quiver $\mathbb{Z}Q$*



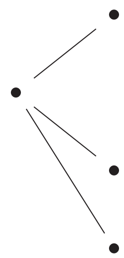
*To understand the way we named the vertices in $\mathbb{Z}Q$ we can name the vertices of the $A_5$ graph as*



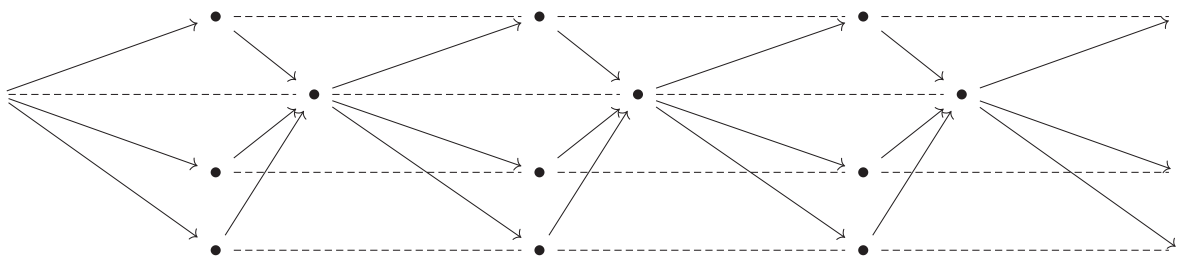
*then we have the $\mathbb{Z}Q$ quiver with the label we give to each vertex*



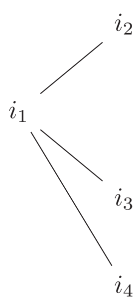

**Example 4.3.6.** *For the graph of type $D_4$*



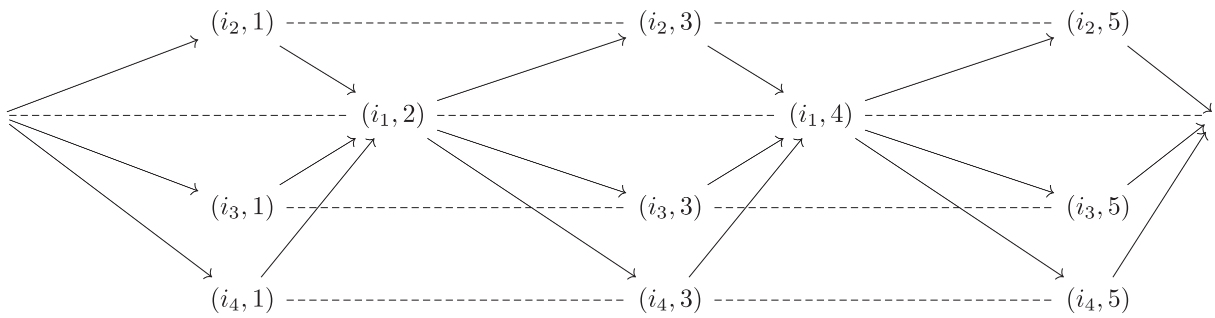
*we have the quiver $\mathbb{Z}Q$*



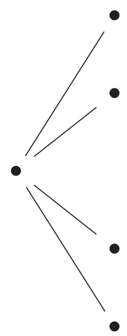
*To understand the way we named the vertices in $\mathbb{Z}Q$ we can name the vertices of the $D_4$ graph as*



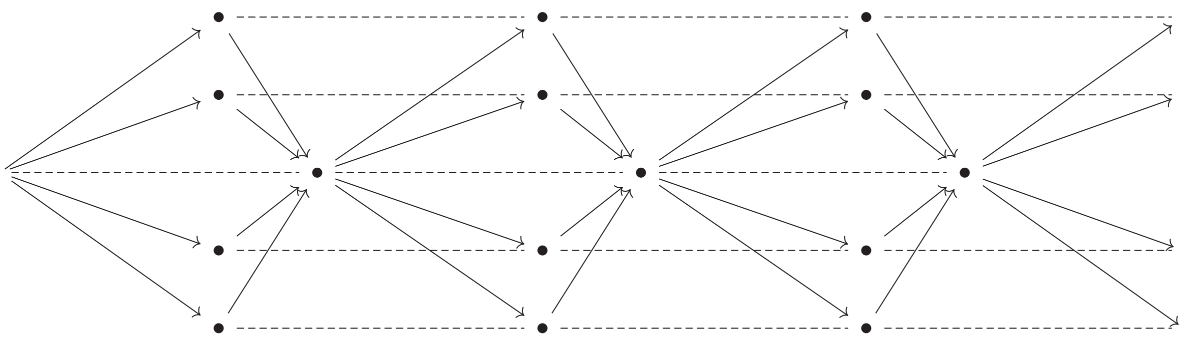
*then we have the $\mathbb{Z}Q$ quiver with the label we give to each vertex*



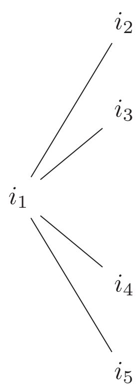

**Example 4.3.7.** *For the graph of type $\hat{D}_4$*



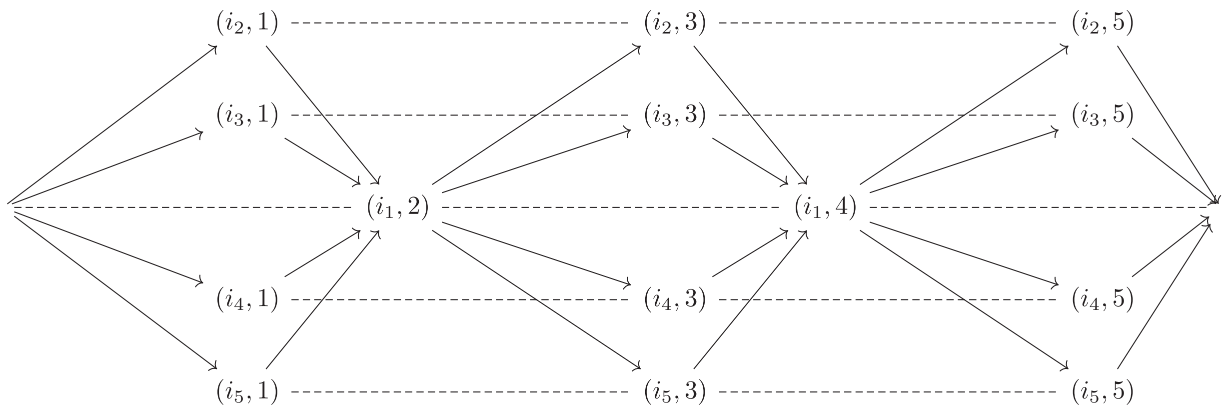
*we have the quiver $\mathbb{Z}Q$*



*To understand the way we named the vertices in $\mathbb{Z}Q$ we can name the vertices of the $\hat{D}_4$ graph as*



*then we have the $\mathbb{Z}Q$ quiver with the label we give to each vertex*




We can observe a natural projection of the quiver $\mathbb{Z}Q$ onto $Q$. By taking the sections of these projections, we obtain embeddings of $Q$ in $\mathbb{Z}Q$. The following definition clarifies this notion.

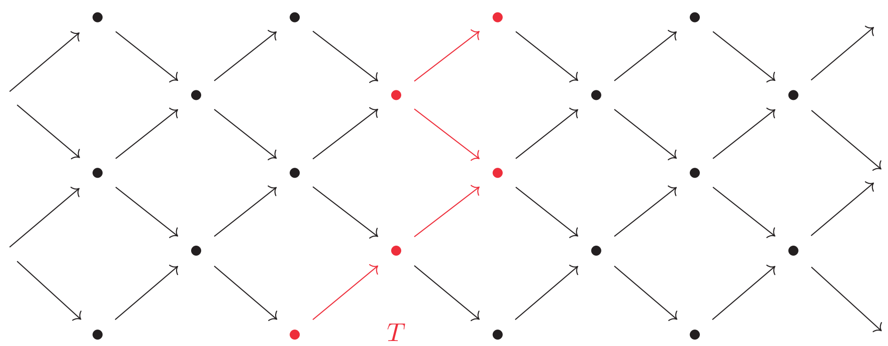
**Definition 4.3.8.** *A subset $T \subset \mathbb{Z}Q$ is called a slice if, for any $i \in I$ there is an unique vertex $q = (i, h_i) \in T$ and whenever vertices $i, j$ are connected by an edge in $Q$, we have $h_i = h_j \pm 1$.*

If we choose a slice $T \subset \mathbb{Z}Q$ that gives us an orientation $\Omega_T$ for $Q$. Which gives a quiver $\vec{Q}_T$ with underlying graph $Q$. We will orient every edge $e$ connecting vertices $i, j$ in $Q$ by

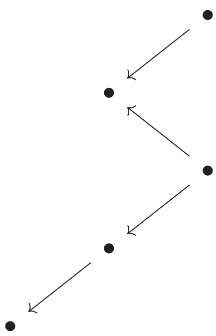
$$\begin{aligned} e : i \to j \quad &\text{if} \quad h_i = h_j + 1, \\ e : j \to i \quad &\text{if} \quad h_i = h_j - 1. \end{aligned}$$

The orientation $\Omega_T$ is exactly the opposite to the orientation we would get considering $T$ as a subquiver of $\mathbb{Z}Q$.

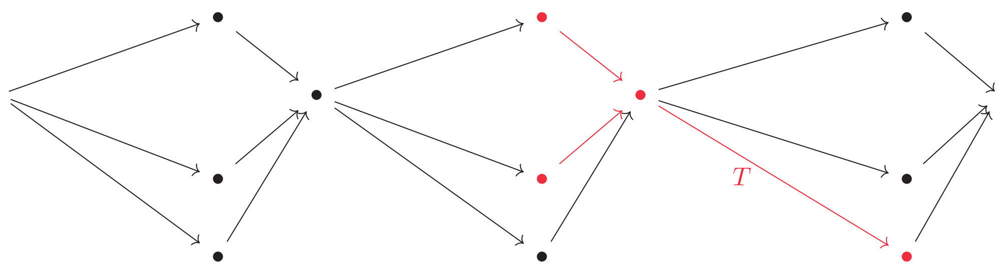
**Example 4.3.9.** *The next figure shows a slice $T$ in red for the quiver $\mathbb{Z}Q$ of graph $A_5$*



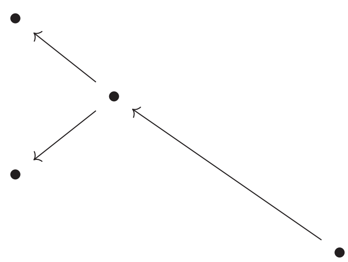
*Then the quiver with orientation given by this slice is $\vec{Q}_T$ given by,*



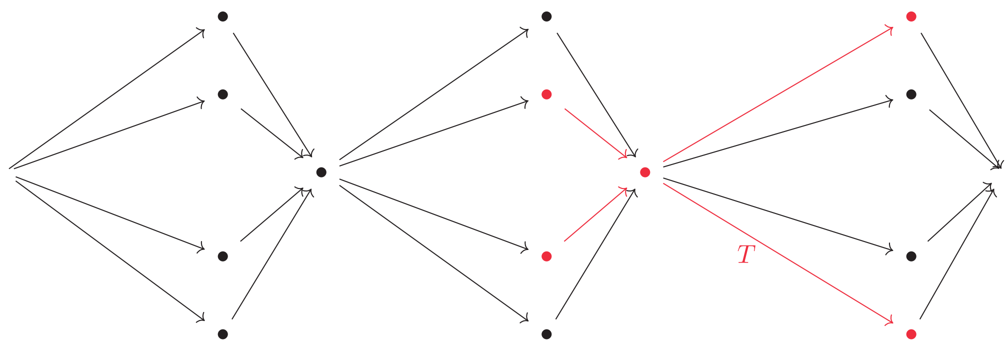
**Example 4.3.10.** *The next figure shows a slice $T$ in red for the quiver $\mathbb{Z}Q$ of graph $D_4$*



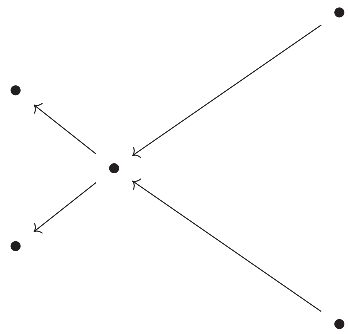

*Then the quiver with orientation given by this slice if $\vec{Q}_T$ given by*



**Example 4.3.11.** *The next figure shows a slice $T$ in red for the quiver $\mathbb{Z}Q$ of graph $\hat{D}_4$*



*Then the quiver with orientation given by this slice if $\vec{Q}_T$ given by*



**Lemma 4.3.12.** *If $Q$ is a tree, any orientation of $Q$ can be obtained from a slice in $\mathbb{Z}Q$.*

*Proof.* Given an orientation $\Omega$ of $Q$. If $q = (i, h_i) \in \mathbb{Z}Q$. Let us define

$$\zeta : I \to \mathbb{Z}$$
$$i \mapsto h_i.$$

Note that

$$\zeta(i) = \zeta(j) + 1 \text{ if } i \to j \text{ in } \Omega.$$

Then for every vertex $i \in I$ we have that if there exists $j \in I$ such that $i \to j$ then $(i, \zeta(i)) \to (j, \zeta(j))$ in $\mathbb{Z}Q$ and $(i, \zeta(i)), (j, \zeta(j)) \in T$. This way we have a slice $T$ of $\mathbb{Z}Q$. □

This theorem is a useful result in the study of quivers, as it allows us to easily generate all possible orientations of a tree quiver by considering its slices. It also provides a nice connection between the algebraic and combinatorial structures of the quiver, as it shows that the orientations of a tree quiver are intimately related to the elements of the associated path algebra.

**Lemma 4.3.13.** *Let $Q$ be a tree. Two slices $T, T'$ give the same orientation if and only if $T' = \tau^k(T)$ for some $k \in \mathbb{Z}$.*

*Proof.* If $T, T'$ are two slices of $\mathbb{Z}Q$ that give the same orientation of $Q$. Let $i \in I$ a vertex of $Q$. By Definition 4.3.8 there exists unique $h_i, h'_, \in \mathbb{Z}$ such that $(i, h_i) \in T$ and $(i, h'_i) \in T'$. Since $\mathbb{Z}Q$ is connected then

$$h_i - h'_i \simeq d(i, i) \bmod(2).$$

The graph $Q$ does not have edge loops, so $d(i, i) = 0$. Therefore, $h_i = h'_i + 2t$ for some $t$. We can assume that $t \geq 0$. Then by the definition of $\tau$ in (4.7) we have that

$$\tau^t(i, h_i) = (i, h'_i).$$

Consider an arrow in $T$, i.e., there exists $i, j \in I$ and an edge $i - j$ in $Q$ and an arrow $\alpha : (i, h_i) \to (j, h_i + 1)$ in $T$. Since $T'$ has the same orientation of $T$ we have an arrow $\beta : (i, h'_i) \to (j, h'_i + 1)$. Then $\tau^t(\alpha) = \beta$. Hence, $\tau^t$ is a quiver morphism and $\tau^t(T) = T'$.

If $T' = \tau^k(T)$ for some $k \in \mathbb{Z}$ then for every $(i, h_i) \in T$ we have that $(i, h_i - 2k) \in T'$. If $(i, h_i) \to (j, h_j)$ in the slice $T$ then $(i, h_i - 2k) \to (j, h_j - 2k)$ is in the slice $T'$. Therefore, $T$ and $T'$ give the same orientation for $Q$. □

This lemma has important consequences for understanding the combinatorics of slices and orientations of a tree quiver. It allows us to identify certain slices as equivalent, and can be used to simplify calculations involving Auslander-Reiten quivers. The operations $s_i^{\pm}$, which relate two different orientations of a quiver $\vec{Q}$ have their counterpart for the slices.

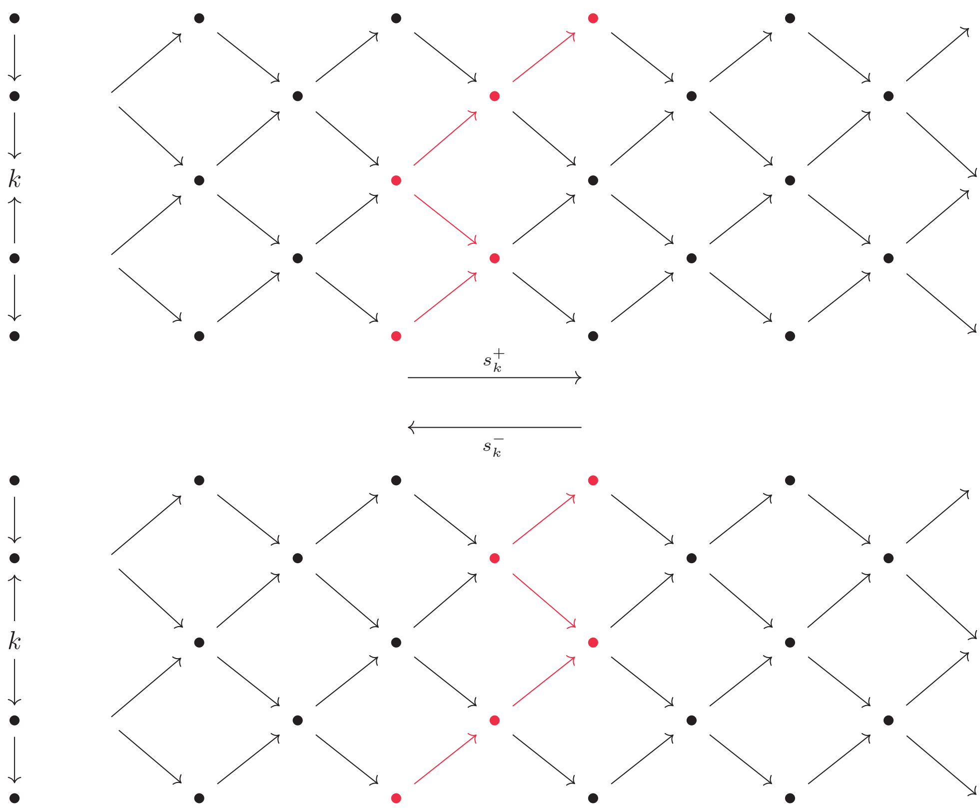
**Definition 4.3.14.** *Let $T = \{(i, h_i)\}$ be a slice and let $k \in I$ be a sink for the corresponding orientation $\Omega_T$, which means that the function $i \mapsto h_i$ has a local minimum at $k$. We define a new slice $T' = s_k^+ T$ by $T' = \{(i, h_i')\}$, where*

$$h_i' = \begin{cases} h_i + 2 & \text{if } i = k, \\ h_i & \text{if } i \neq k. \end{cases}$$

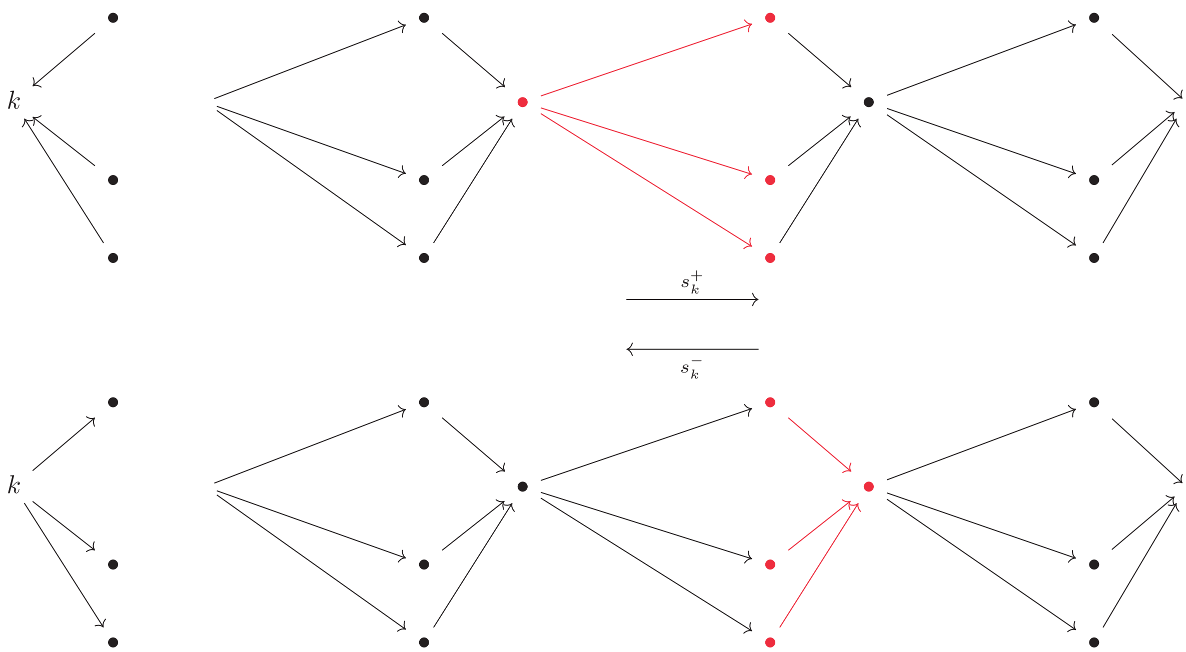
*Similarly, if $k \in I$ is a source for the orientation $\Omega_T$, which means that the function $i \mapsto h_i$ has a local maximum at $k$. We define a new slice $T' = s_k^- T$ by $T' = \{(i, h_i')\}$, where*

$$h_i' = \begin{cases} h_i - 2 & \text{if } i = k, \\ h_i & \text{if } i \neq k. \end{cases}$$

**Example 4.3.15.** *The next graph illustrates the operations $s_k^{\pm}$.*



**Example 4.3.16.** *The next graph illustrates the operations $s_k^{\pm}$.*




The operations for the slices agree with the operations $s_i^{\pm}$ for orientations,

$$\Omega_{s_i^{\pm}T} = s_i^{\pm}\Omega_T. \tag{4.8}$$

Any two orientations can be obtained from each other by a sequence of operations $s_i^{\pm}$, by Lemma 2.1.6.

**Definition 4.3.17.** *Let $T = \{(i, h_i)\} \subset \mathbb{Z}Q$ be a slice.*

*(a) We say that a vertex $q = (j, n) \in \mathbb{Z}Q$ is above (respectively strictly above), $T$ and write $q \succeq T$ (respectively $q \succ T$), if we have $n \geq h_j$, (respectively $n > h_j$). We say that a slice $T'$ is above a slice $T$ if every vertex $q \in T'$ is above $T$.*

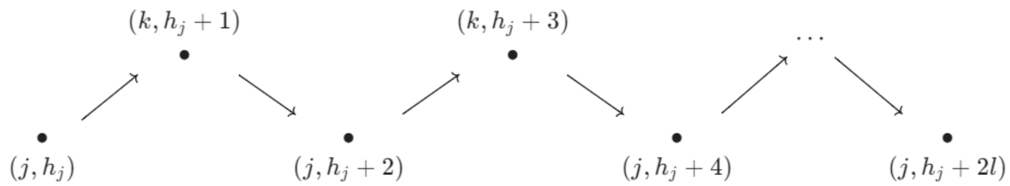
*(b) We say that a vertex $q = (j, n) \in \mathbb{Z}Q$ is below (respectively strictly below), $T$ and write $q \preceq T$ (respectively $q \prec T$), if we have $n \leq h_j$ (respectively $n < h_j$). We say that a slice $T'$ is below a slice $T$ if every vertex $q \in T'$ is below $T$.*

**Proposition 4.3.18.** *Let $T = \{i, h_i\}$ be a slice of $\mathbb{Z}Q$.*

*(a) We have that $p \succeq T$ if and only if there exists a path in $\mathbb{Z}Q$ from some vertex $q \in T$ to $p$.*

*(b) If $p \succ T$ and $q \in T$ then there are no paths from $p$ to $q$ in $\mathbb{Z}Q$.*

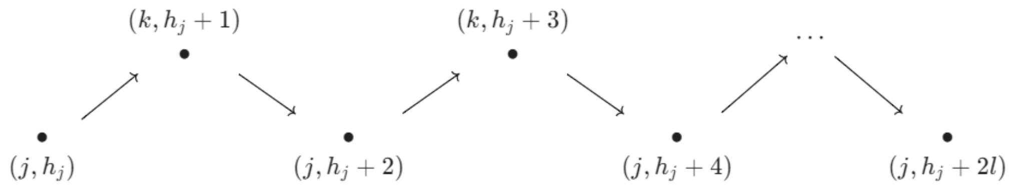
*Proof.* (a) If $p \succeq T$ we can write $p = (j, n)$ and there exists $q = (j, h_j) \in T$ such that $n \geq h_j$. Let $k$ be such that $j - k$ in $Q$, which must exist otherwise $Q$ would be only one vertex, or it would not be connected. Then $n = h_j + 2l$ and




Therefore, there is a path in $\mathbb{Z}Q$ from some $q \in T$ to $p$.

If we write $p = (j, n)$ and $q = (j, h_j) \in T$ such that it exists a path from $q$ to $p$ then of course $n \geq h_j$ and $p \succeq T$.

(b) Write $p = (j, n)$. Since $p \succ T$ we have that there exists $q' = (j, h_j) \in T$ such that $n > h_j$. If $j - k$ in $Q$ we have that $n = h_j + 2l$ and




this means we can have a path from $q'$ to $p$ but not one from $p$ to $q'$. Then we can not have a path from $p$ to any other $q \in T$.

$\square$

**Theorem 4.3.19.** *If $T, T'$ are two slices such that $T' \succeq T$, then one can obtain $T'$ by applying a sequence of operations $s_i^+$ to $T$. Moreover, this sequence is defined uniquely up to interchanging $s_i^+ s_j^+ \leftrightarrow s_j^+ s_i^+$ for $i, j$ not connected in $Q$.*

*Proof.* Define

$$d(T, T') = \mid \{q \in \mathbb{Z}Q; T \preceq q \prec T'\} \mid .$$

We proceed by induction on $d(T, T')$.

If $d(T, T') = 0$ then there is no vertex between $T$ and $T'$, which means that $T = T'$.

Now suppose that $d(T, T') > 0$. Then we know that there exists a vertex $q = (i, h_i)$ such that $T \preceq q \prec T'$. If $i \to j$ in $\Omega_T$ then there exists $q' = (j, h_j)$ with $h_i = h_j + 1$, we want to show that $q' \prec T'$. In fact in these conditions we have that there exists $(i, h_i') \in T'$ with $h_i < h_i'$, so it exists $(j, h_j') \in T'$ such that $h_i' = h_j' \pm 1$. Therefore, we have

$$h_i < h_i' \quad \Rightarrow \quad h_j + 1 < h_j' \pm 1$$

and we have that

$$\begin{array}{ll|ll} & h_j+1 < h'_j+1 & & h_j+1 < h'_j-1 \\ \Rightarrow & h_j < h'_j & \Rightarrow & h_j < h'_j-2 < h'_j \\ \Rightarrow & q'=(j,h_j) \prec T' & \Rightarrow & q'=(j,h_j) \prec T'. \end{array}$$

Then amongst all vertex that are strictly below $T'$ there exists a sink for $\Omega_T$. Indeed, as $Q$ does not have oriented cycles there exists a sink in $\vec{Q}_T$. Let $q=(i,h_i) \prec T'$ if it is not a sink, exists $j$ such that $i \to j$ and $q'=(j,h_j)$ is also strictly below $T'$ from what we have seen above. Since we have finitely many points between $T$ and $T'$ then this have to end with a sink.

Let $(k,h_k) \in T$ such that $(k,h_k) \prec T'$ and $k$ is a sink in $\Omega_T$. We can apply $s_k^+$ in $T$, since $k$ is a sink, and we have $\overline{T} = s_k^+(T)$ that takes the vertex $(k,h_k)$ in $(k,h_k+2)$ and keeps all the other vertex the same. So $\overline{T} \preceq T'$ since $(k,h_k+2)$ may be in $T'$ or not, but the rest of the slice stayed the same. Therefore, $d(\overline{T},T') = d(T,T')-1$ and by the induction hypothesis we have that $T'$ can be obtained from $\overline{T}$ by a sequence of operations $s_i^+$. Then

$$T' = s_{i_1}^+ \cdots s_{i_n}^+(\overline{T}) = s_{i_1}^+ \cdots s_{i_n}^+ s_k^+(T).$$

In this way $T'$ is obtained from $T$ by a sequence of operations $s_i^+$.

For the uniqueness, we will also do induction in $d(T,T')$. If $d(T,T')=0$ then $T=T'$.

If $d(T,T')>0$ and we write

$$T' = s_{i_k}^+ \cdots s_{i_1}^+(T) = s_{j_n}^+ \cdots s_{j_1}^+(T).$$

Then $i_1$ is a sink for $T$, but we can only move $(i_1,h_{i_1})$ by applying $s_{i_1}^+$ which means that the second sequence must contain $i_1$. This means that there exists $j_a$ such that $j_a=i_1$ for some $a$. Choose $a$ to be the smallest possible. So the sequence becomes

$$s_{j_n}^+ \cdots s_{j_1}^+ = s_{j_n}^+ \cdots s_{j_{a+1}}^+ s_{i_1}^+ s_{j_{a-1}}^+ \cdots s_{j_1}^+.$$

As $a$ is the smallest possible we have that $i_1$ is a sink for

$$s_{j_1}^+(T), s_{j_2}^+ s_{j_1}^+(T), \cdots, s_{j_{a-1}}^+ \cdots s_{j_1}^+(T).$$

Thus, as every $j \to i$ then $j$ can not be a sink itself, so $j_1, \cdots, j_{a-1}$ are not connected in $Q$, so we can permute and rewrite

$$T' = s_{j_n}^+ \cdots s_{j_{a+1}}^+ s_{i_1}^+ s_{j_{a-1}}^+ \cdots s_{j_1}^+(T) = s_{j_n}^+ \cdots s_{j_{a+1}}^+ s_{j_{a-1}}^+ \cdots s_{j_1}^+ s_{i_1}^+(T)$$

and

$$T' = s_{i_k}^+ \cdots s_{i_2}^+(\overline{T}) = s_{j_n}^+ \cdots s_{j_{a-1}}^+ \cdots s_{j_1}^+(\overline{T})$$

with $\overline{T} = s_{i_1}^+(T)$ and these sequences have $d(\overline{T},T') = d(T,T')-1$ since we took out the element $s_{i_1}^+$. So $T'$ is written in a unique way up to interchanging permutations in $\overline{T}$ and $T$. $\square$

This theorem is a crucial result in the study of the Auslander-Reiten quiver of a representation-infinite algebra. It provides a way to obtain any slice in the Auslander-Reiten quiver starting from a fixed slice, by applying a sequence of standard mutations. Furthermore, it ensures the uniqueness of this sequence up to certain commutation relations. This result is often used to show that the Auslander-Reiten quiver is a connected quiver.

**Corollary 4.3.20.** *For any two orientation $\Omega, \Omega'$ of a tree $Q$, one can obtain $\Omega'$ by applying a sequence of operations $s_i^+$ to $\Omega$. One can also get $\Omega'$ by applying a sequence of operations $s_i^-$ to $\Omega$.*

*Proof.* By Lemma 4.3.12 we have that any orientation of $Q$ can be obtained from a slice in $\mathbb{Z}Q$, so we have that there exists $T, T'$ slices such that $\Omega = \Omega_T$ and $\Omega' = \Omega_{T'}$. We can suppose that $T' \succeq T$ and by Theorem 4.3.19 we can write

$$T' = s_{i_n}^+ \cdots s_{i_1}^+(T).$$

Therefore, by (4.8) we have that

$$\Omega' = \Omega_{T'} = \Omega_{s_{i_n}^+ \cdots s_{i_1}^+(T)} = s_{i_n}^+ \cdots s_{i_1}^+(\Omega_T) = s_{i_n}^+ \cdots s_{i_1}^+(\Omega).$$

The second statement is made analogously. □

This corollary provides an important property of the operations $s_i^+$ and $s_i^-$ in the context of tree orientations. It shows that any two orientations of a tree can be related to each other by applying a sequence of these operations, either in the positive or negative direction. This result is fundamental in the study of Auslander-Reiten theory, as it allows us to understand the structure of the Auslander-Reiten quiver associated to a tree quiver in terms of the possible orientations of the tree. Furthermore, the uniqueness up to interchanging certain operations is an important observation, as it allows us to identify which sequences of operations lead to the same orientation.

## 4.4 Auslander-Reiten quiver: Representation theory

We introduced the quiver $\mathbb{Z}Q$ and the notion of slice as a convenient tool for visualizing sink to source transformations. In this section we will introduce the Auslander Reiten quiver and some properties of preprojective representations.

**Theorem 4.4.1.** *Let $Q$ be a bipartite graph. Then one can assign to every pair $(T, q)$, where $T$ is a slice of $\mathbb{Z}Q$ and $q$ is a vertex of $\mathbb{Z}Q$, a representation $I_q^T$ of $\vec{Q}_T$ so that the following conditions hold:*

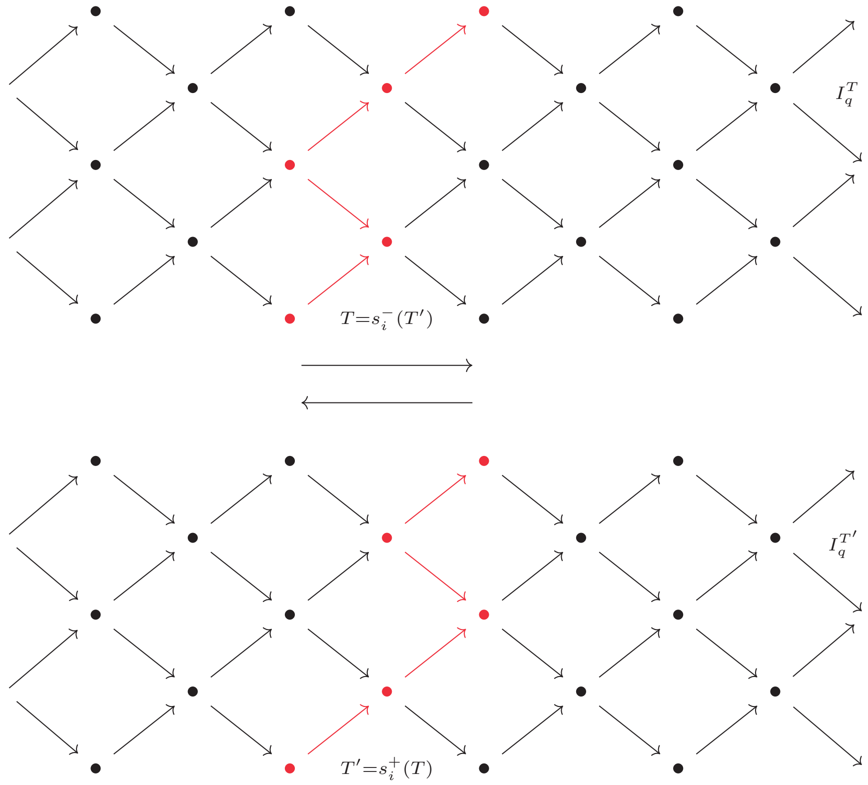
*(a) If $q$ is strictly below $T$, then $I_q^T = 0$.*

*(b) If $q = (i, h_i) \in T$ is a sink for $T$ then $I_q^T = S(i)$.*

*(c) If $T' = s_i^+ T$ and $q \succeq T'$ then $I_q^T = \Phi_i^- I_q^{T'}, I_q^{T'} = \Phi_i^+ I_q^T$.*

*Moreover, these conditions determine $I_q^T$ uniquely up to isomorphism.*

Condition $(c)$ of Theorem 4.4.1 is illustrated in the Figure below.




*Proof.* First note that if such representations $I_q^T$ exists, they are unique. Given $q \succeq T$ and choose a slice $T'$ such that $T' \succeq T$ that contains $q$ as a sink. By Theorem 4.3.19 we can write $T' = s_{i_n}^+ \cdots s_{i_1}^+(T)$ for some sequence $s_{i_n}^+ \cdots s_{i_1}^+$. Then we have

$$I_q^T = \Phi_{i_1}^- \cdots \Phi_{i_n}^-(S(i))$$

since $I_q^{T'} = S(i)$ where $S(i)$ is the simple representation of $\vec{Q}_{T'}$. This proves uniqueness of $I_q^T$.

For every pair $(T, q)$ with $q \succeq T$ we have a slice $T' \succeq T$ that contains $q$ such that $q$ is a sink of $T'$ and a sequence of reflections $s_{i_n} \cdots s_{i_1}$ such that $T' = s_{i_n}^+ \cdots s_{i_1}^+(T)$. Define $I_q^T$ by

$$I_q^T = \Phi_{i_1}^- \cdots \Phi_{i_n}^-(S(i))$$

in this definition the representations does not depend on the choice of $T'$ neither the reflections $s_{i_n}^+ \cdots s_{i_1}^+$. Indeed, by Theorem 4.3.19, for some choice of $T'$ the sequence of reflections is unique up to interchanging $s_i^+$ with $s_j^+$ when $i, j$ are not connected in $Q$. But in this case the functors $\Phi_i^-$ and $\Phi_j^-$ commute, that means that there exists a functorial isomorphism

$$\Phi_i^- \Phi_j^- \simeq \Phi_j^- \Phi_i^-.$$

Of course if $i$ and $j$ are not connected in $Q$ the vector space and the morphism of one does not affect the other. Then $I_q^T$ does not depend on the choice of reflections.

Now let us show that it does not depend on the choice of the slice $T'$. If $T' = \{(j, h'_j)\}$ and $T'' = \{(j, h''_j)\}$ are two slices that have $q = (i, h_i)$ as a sink, then the two slices coincide at $i$ and at every $j$ such that $j$ is connected with $i$ in $Q$. In this case, by Theorem 4.3.19 we have that with a sequence of operations $s_k^+$ we can get to $T''$ where $k$ is not connected to $i$. But $T'$ and $s_k^+(T')$ give the same representation $I_q^T$, indeed if $T' = s_{i_n}^+ \cdots s_{i_1}^+(T)$ and $T'' = s_k^+ s_{i_n}^+ \cdots s_{i_1}^+(T)$ we have

$$I_q^T = \Phi_{i_1}^- \cdots \Phi_{i_n}^-(S(i)) = \Phi_{i_1}^- \cdots \Phi_{i_n}^- \Phi_k^-(S(i))$$

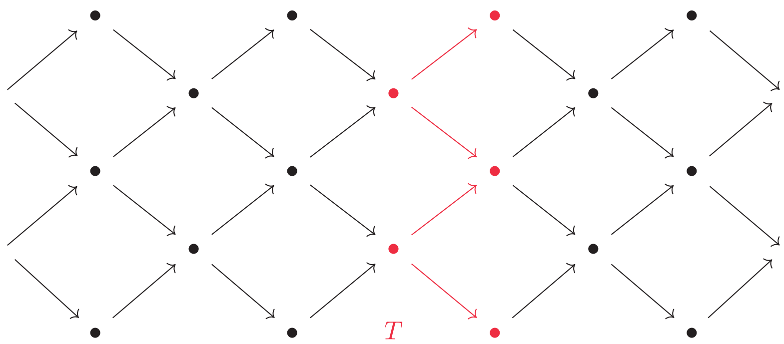
since $\Phi_k^-(S(i)) = S(i)$ since $k \neq i$ and $k$ is not connected to $i$.

Therefore, the representation $I_q^T$ as defined just depends on $q$ and $T$. And this definition satisfies the conditions. ☐

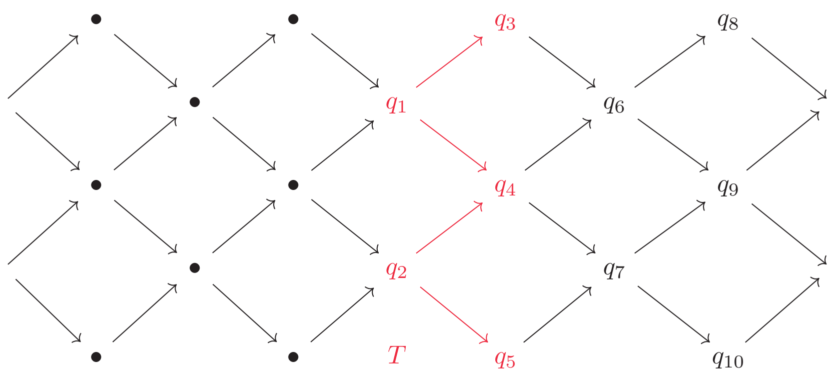
**Example 4.4.2.** *Consider the graph of type $A_5$*

$$i_1 \;\text{———}\; i_2 \;\text{———}\; i_3 \;\text{———}\; i_4 \;\text{———}\; i_5.$$

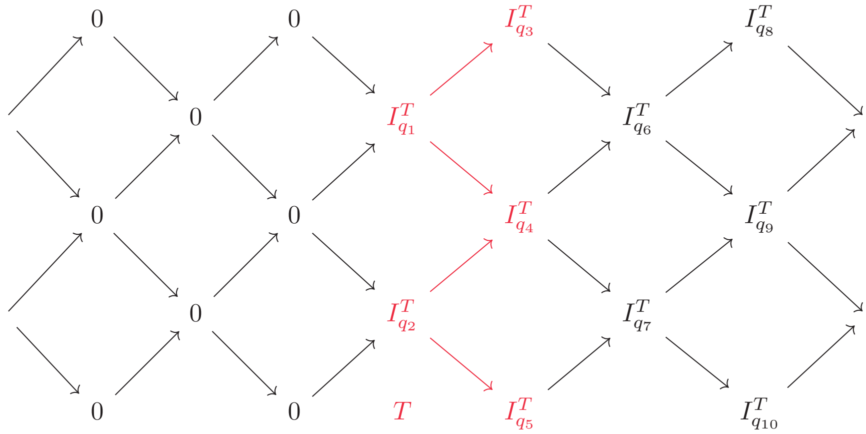
*We have the quiver $\mathbb{Z}Q$ with slice $T$ in red*




*By Theorem 4.4.1 we have a representation $I_q^T$ for every vertex $q \in \mathbb{Z}Q$. For every $q$ strictly below $T$ we have that $I_q^T = 0$. See below the vertices that we are going to calculate the associated representation $I_q^T$,*



*Therefore we have the following representations in the quiver* $\mathbb{Z}Q$




*Let us find the representations in the vertices that belong to the slice* $T$*. For the vertices* $q \in T$ *such that* $q$ *is a sink for* $T$ *we have the simple representations, so* $I^T_{q_1} = S(i_2)$ *and* $I^T_{q_2} = S(i_4)$ *with* $S(i_2), S(i_4)$ *relative to the quiver* $\vec{Q}_T$*. The quiver* $\vec{Q}_T$ *and the representations can be seen as follows,*

$$\vec{Q}_T = \qquad \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

$$S(i_2) = \qquad 0 \longrightarrow \mathbb{K} \longleftarrow 0 \longrightarrow 0 \longleftarrow 0$$

$$S(i_4) = \qquad 0 \longrightarrow 0 \longleftarrow 0 \longrightarrow \mathbb{K} \longleftarrow 0.$$

*To find the representations for the vertices that are not a sink we need to use other slices. Let* $T_1 = s^+_{i_2}(T)$, $T_2 = s^+_{i_2}s^+_{i_4}(T)$ *and* $T_3 = s^+_{i_4}(T)$ *then we have the representations*

$$I^T_{q_3} = \Phi^-_{i_2}(S(i_1)), \quad I^T_{q_4} = \Phi^-_{i_2}\Phi^-_{i_4}(S(i_3)), \quad \text{and} \quad I^T_{q_5} = \Phi^-_{i_4}(S(i_5)).$$

*With representation* $I^T_{q_3}$ *as,*

$$\vec{Q}_{T_1} = \qquad \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

$$S(i_1) = \qquad \mathbb{K} \longleftarrow 0 \longrightarrow 0 \longrightarrow 0 \longleftarrow 0$$

$$I^T_{q_3} = \qquad \mathbb{K} \longrightarrow \mathbb{K} \longleftarrow 0 \longrightarrow 0 \longleftarrow 0$$

*representation* $I_{q4}^T$ *as,*

$$\vec{Q}_{T_2} = \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$S(i_3) = \quad 0 \longleftarrow 0 \longrightarrow \mathbb{K} \longleftarrow 0 \longrightarrow 0$$

$$\Phi_{i_4}^-(S(i_3)) = \quad 0 \longleftarrow 0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{K} \longleftarrow 0$$

$$I_{q4}^T = \quad 0 \longrightarrow \mathbb{K} \longleftarrow \mathbb{K} \longrightarrow \mathbb{K} \longleftarrow 0.$$

*and representation* $I_{q5}^T$ *as,*

$$\vec{Q}_{T_3} = \quad \bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$S(i_5) = \quad 0 \longrightarrow 0 \longleftarrow 0 \longleftarrow 0 \longrightarrow \mathbb{K}$$

$$I_{q5}^T = \quad 0 \longrightarrow 0 \longleftarrow 0 \longrightarrow \mathbb{K} \longleftarrow \mathbb{K}$$

*For the vertices that do not belong to the slice $T$ we need longer slices in order to find the representations for each vertex. Consider* $T_4 = s_{i_2}^+ s_{i_1}^+ s_{i_4}^+ s_{i_3}^+(T)$, $T_5 = s_{i_2}^+ s_{i_4}^+ s_{i_3}^+ s_{i_5}^+(T)$, $T_6 = s_{i_2}^+ s_{i_1}^+ s_{i_4}^+ s_{i_3}^+ s_{i_2}^+(T)$, $T_7 = s_{i_2}^+ s_{i_4}^+ s_{i_3}^+ s_{i_1}^+ s_{i_2}^+ s_{i_5}^+ s_{i_4}^+(T)$, *and* $T_8 = s_{i_2}^+ s_{i_4}^+ s_{i_3}^+ s_{i_5}^+ s_{i_4}^+(T)$ *then we have the representations*

$$I_{q6}^T = \Phi_{i_2}^- \Phi_{i_1}^- \Phi_{i_4}^- \Phi_{i_3}^-(S(i_2)), \quad I_{q7}^T = \Phi_{i_2}^- \Phi_{i_4}^- \Phi_{i_3}^- \Phi_{i_5}^-(S(i_4)), \quad I_{q8}^T = \Phi_{i_2}^- \Phi_{i_1}^- \Phi_{i_4}^- \Phi_{i_3}^- \Phi_{i_2}^-(S(i_1)),$$
$$I_{q9}^T = \Phi_{i_2}^- \Phi_{i_4}^- \Phi_{i_3}^- \Phi_{i_1}^- \Phi_{i_2}^- \Phi_{i_5}^- \Phi_{i_4}^-(S(i_3)), \quad \text{and} \quad I_{q10}^T = \Phi_{i_2}^- \Phi_{i_4}^- \Phi_{i_3}^- \Phi_{i_5}^- \Phi_{i_4}^-(S(i_5)).$$

*With representation* $I_{q6}^T$ *as,*

$$\vec{Q}_{T_4} = \quad \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$S(i_2) = \quad 0 \longrightarrow \mathbb{K} \longleftarrow 0 \longrightarrow 0 \longrightarrow 0$$

$$I_{q6}^T = \quad \mathbb{K} \longrightarrow \mathbb{K} \longleftarrow \mathbb{K} \longrightarrow \mathbb{K} \longleftarrow 0$$

*representation* $I_{q7}^T$ *as,*

$$\vec{Q}_{T_5} = \quad \bullet \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

$$S(i_4) = \quad 0 \longleftarrow 0 \longleftarrow 0 \longrightarrow \mathbb{K} \longleftarrow 0$$

$$I_{q7}^T = \quad 0 \longrightarrow \mathbb{K} \longleftarrow \mathbb{K} \longrightarrow \mathbb{K} \longleftarrow \mathbb{K}$$

*representation $I^T_{q_8}$ as,*

$$\vec{Q}_{T_6} = \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$S(i_1) = \quad \mathbb{K} \longleftarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

$$I^T_{q_8} = \quad 0 \longrightarrow 0 \longleftarrow \mathbb{K} \longrightarrow \mathbb{K} \longleftarrow 0$$

*representation $I^T_{q_9}$ as,*

$$\vec{Q}_{T_7} = \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$S(i_3) = \quad 0 \longleftarrow 0 \longrightarrow \mathbb{K} \longleftarrow 0 \longrightarrow 0$$

$$I^T_{q_9} = \quad \mathbb{K} \longrightarrow \mathbb{K} \longleftarrow \mathbb{K} \longrightarrow \mathbb{K} \longleftarrow \mathbb{K}$$

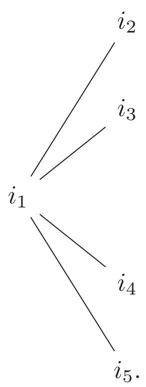
*and representation $I^T_{q_{10}}$ as,*

$$\vec{Q}_{T_8} = \quad \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

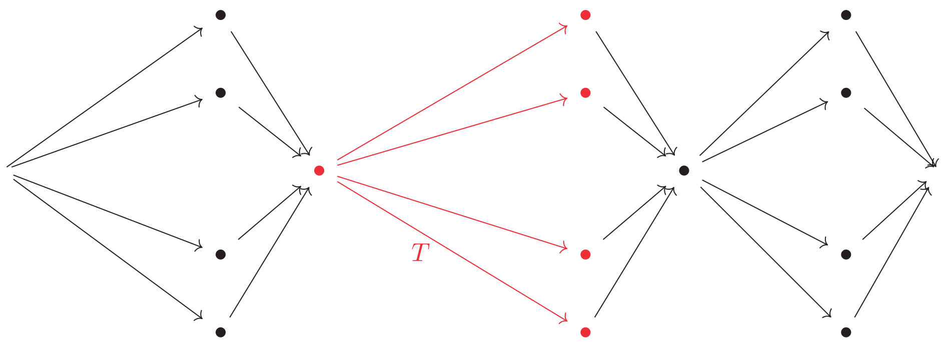
$$S(i_5) = \quad 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longrightarrow \mathbb{K}$$

$$I^T_{q_{10}} = \quad 0 \longrightarrow \mathbb{K} \longleftarrow \mathbb{K} \longrightarrow 0 \longleftarrow 0.$$

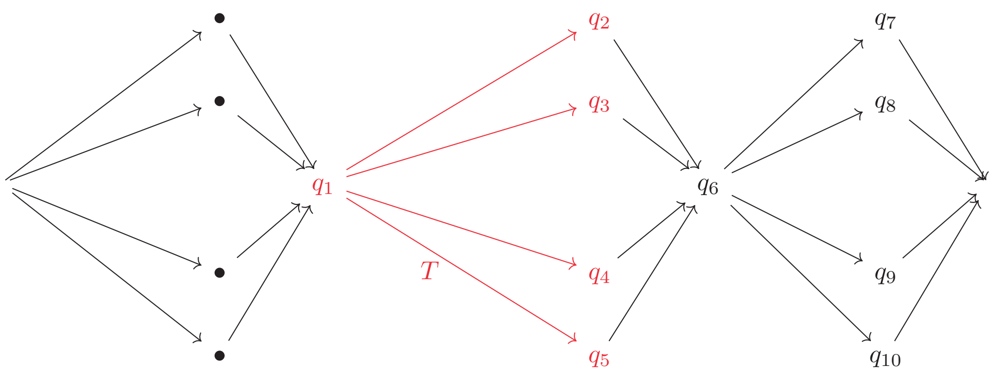
**Example 4.4.3.** *Consider the graph of type $\hat{D}_4$*



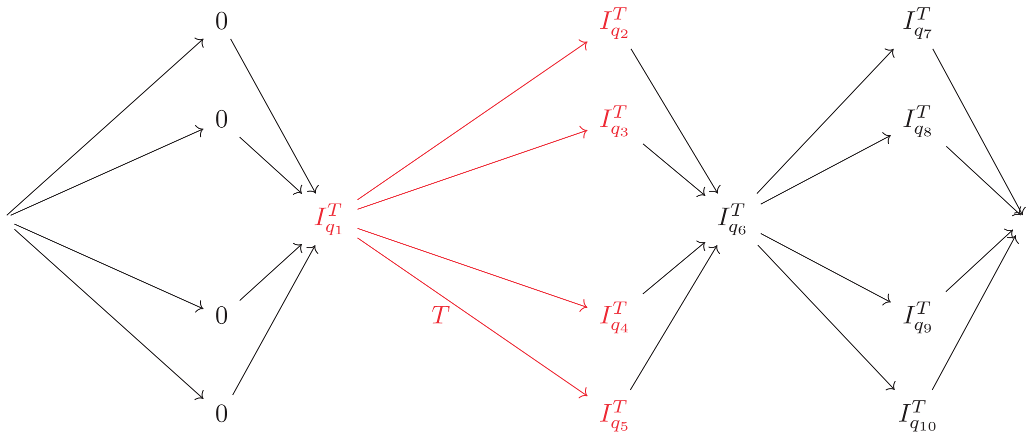
*We have the quiver $\mathbb{Z}Q$ with the slice $T$ in red*



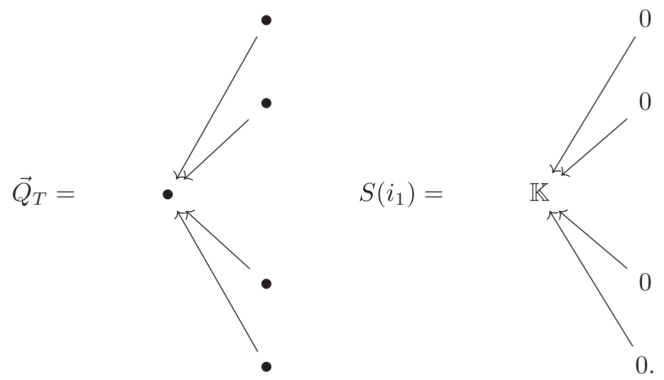

*In the same way as in the Example 4.4.2 we have a representation $I_q^T$ for every vertex $q \in \mathbb{Z}Q$. For every $q$ strictly below $T$ we have that $I_q^T = 0$. See below the vertices that we are going to calculate the associated representation $I_q^T$,*




*Therefore we have the following representations in the quiver $\mathbb{Z}Q$*



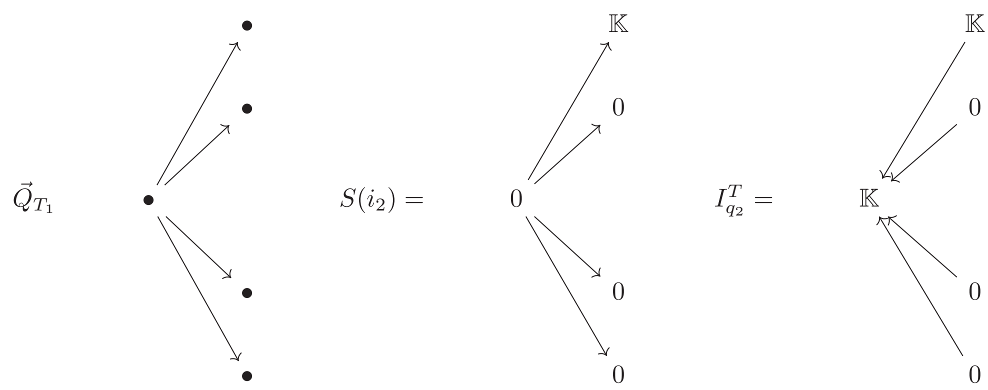

*Let us find the representations in the vertices that belong to the slice $T$. For the vertices $q \in T$ such that $q$ is a sink for $T$ we have the simple representations, so $I_{q_1}^T = S(i_1)$, with $S(i_1)$ relative to the quiver $\vec{Q}_T$. The* *quiver* $\vec{Q}_T$ *and the representation can be seen as follows,*



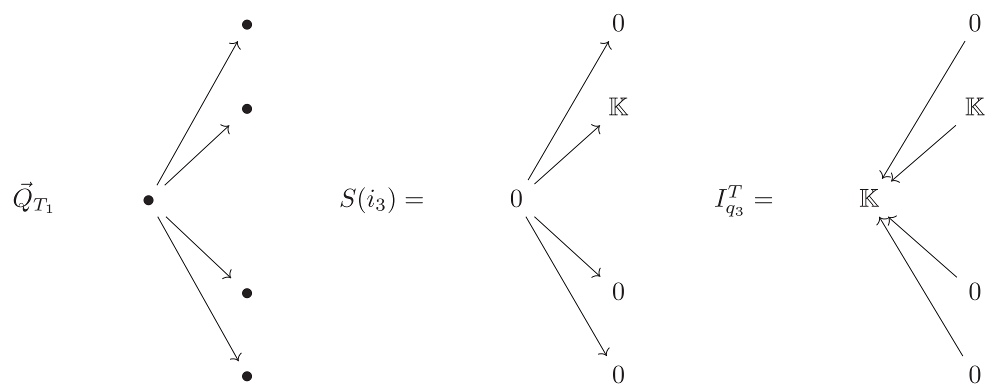

*To find the representations for the vertices that are not a sink we need to use other slices. Let* $T_1 = s^+_{i_1}(T)$ *then we have the representations*

$$I^T_{q_2} = \Phi^-_{i_1}(S(i_2)), \quad I^T_{q_3} = \Phi^-_{i_1}(S(i_3)), \quad I^T_{q_4} = \Phi^-_{i_1}(S(i_4)), \quad \text{and} \quad I^T_{q_5} = \Phi^-_{i_1}(S(i_5)).$$

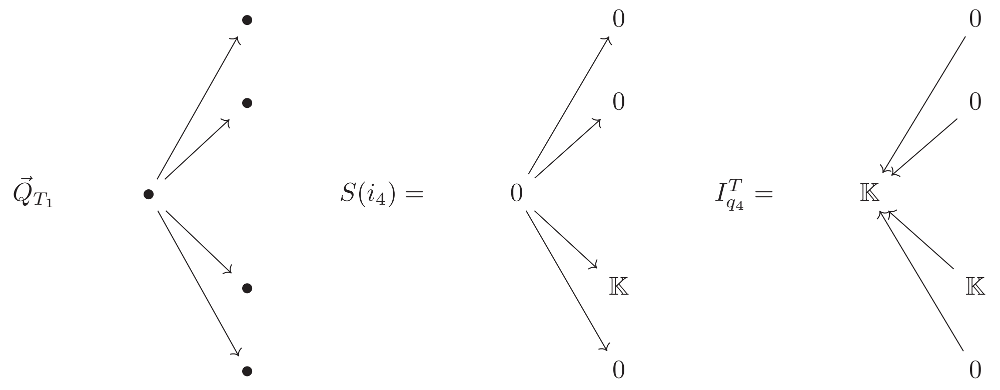
*With representation* $I^T_{q_2}$ *as,*



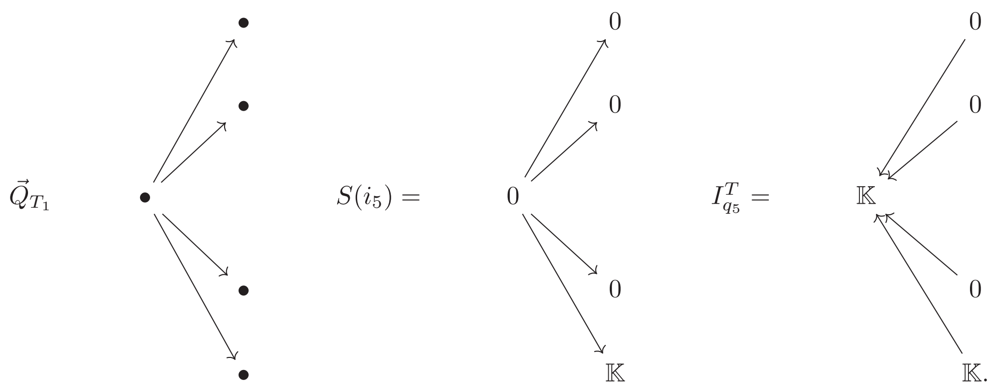

*representation* $I^T_{q_3}$ *as,*



*representation $I^T_{q_4}$ as,*



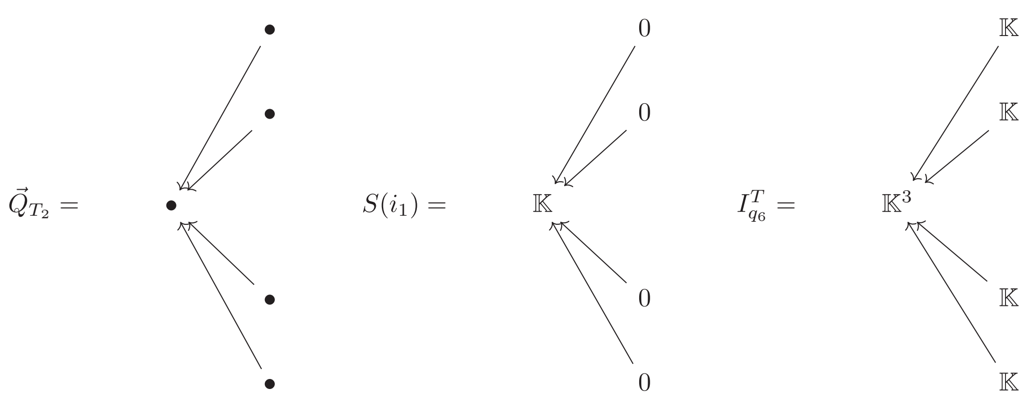

*and representation $I^T_{q_5}$ as,*



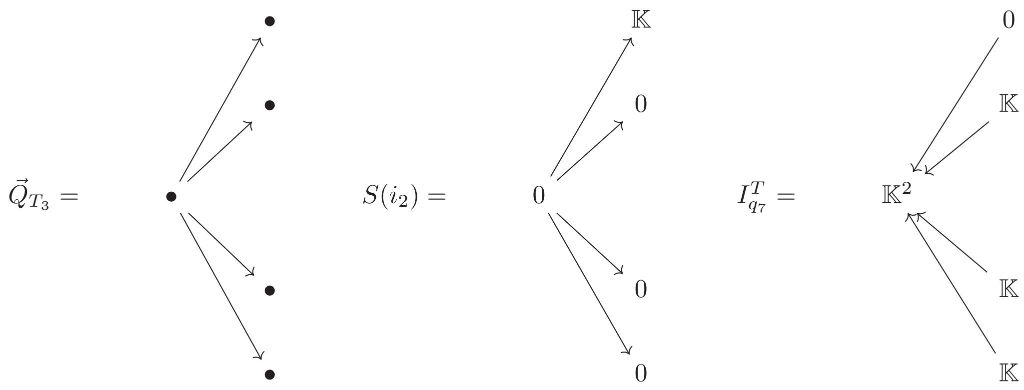

*For the vertices that do not belong to the slice $T$ we need longer slices in order to find the representations for each vertex. Consider $T_2 = s^+_{i_1}s^+_{i_2}s^+_{i_3}s^+_{i_4}s^+_{i_5}(T)$, and $T_3 = s^+_{i_1}s^+_{i_2}s^+_{i_3}s^+_{i_4}s^+_{i_5}s^+_{i_1}(T)$ then we have the representations*

$$I^T_{q_6} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_4}\Phi^-_{i_5}(S(i_1)), \quad I^T_{q_7} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_4}\Phi^-_{i_5}\Phi^-_{i_1}(S(i_2)), \quad I^T_{q_8} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_4}\Phi^-_{i_5}\Phi^-_{i_1}(S(i_3))$$
$$I^T_{q_9} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_4}\Phi^-_{i_5}\Phi^-_{i_1}(S(i_4)), \quad \text{and} \quad I^T_{q_{10}} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_4}\Phi^-_{i_5}\Phi^-_{i_1}(S(i_5)).$$

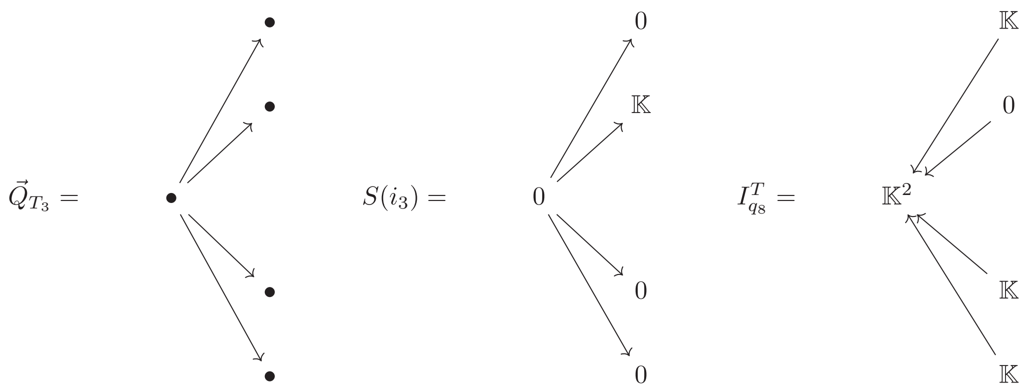
*With representation $I^T_{q_6}$ as,*



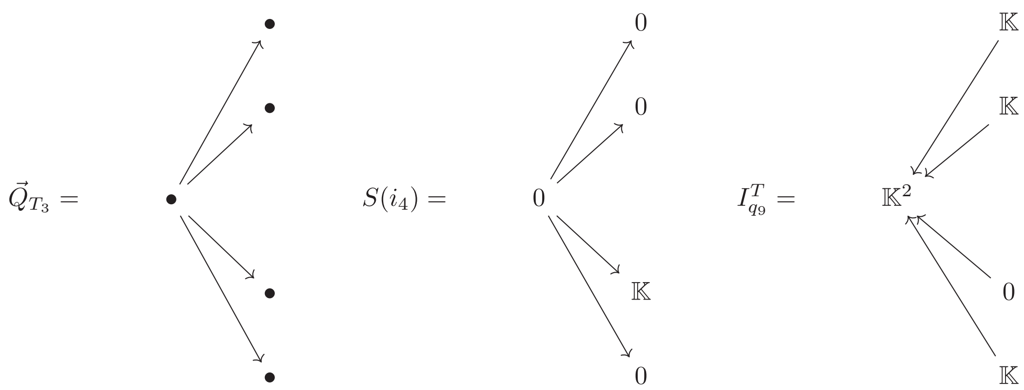
*representation* $I_{q_7}^T$ *as,*



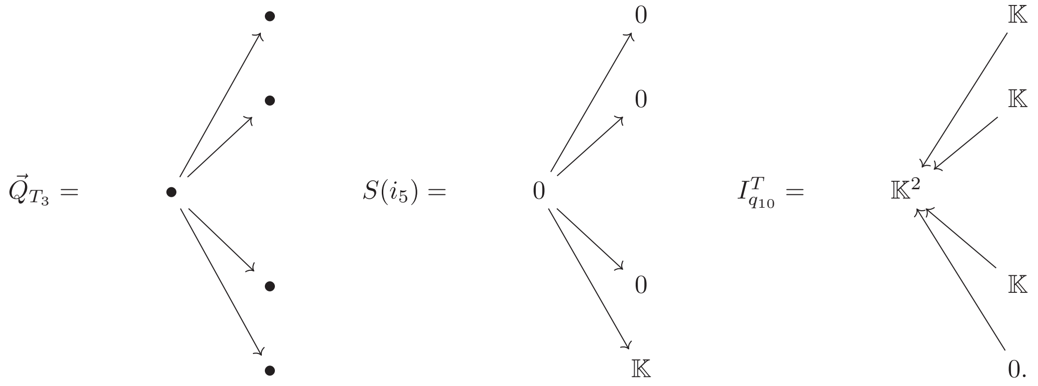

*representation* $I_{q_8}^T$ *as,*




*representation* $I_{q_9}^T$ *as,*



*and representation* $I^T_{q_{10}}$ *as,*




**Theorem 4.4.4.** *Let $Q$ be a bipartite graph and $(T, q)$ a pair with $T$ a slice and $q$ a vertex of $\mathbb{Z}Q$. Consider the representations $I^T_q$ as defined in Theorem 4.4.1.*

*(a) If $q = (i, n) \in T$ then $I^T_q = P(i)$ is the indecomposable projective representation.*

*(b) The construction is invariant under the translation $\tau$ defined by* (4.7)*, which means that $I^{\tau T}_{\tau q} = I^T_q$.*

*(c) If $q \succeq T$ then $I^T_{\tau^{-1}q} = I^{\tau T}_q = C^-(I^T_q)$, where $C^-$ is the Coxeter functor as in* (4.5)*.*

*(d) Each representation $I^T_q$ is indecomposable and preprojective, possibly zero.*

*Proof.* (a) Suppose that $q = (i, n) \in T$ and $I^T_q = \Phi^-_{i_1} \cdots \Phi^-_{i_n}(S(i))$. We saw in Theorem 2.2.11 that if $\vec{Q}' = s^+_k(\vec{Q})$ and $k \neq i$ we have $\Phi^-_k(P'(i)) = P(i)$ is the indecomposable representation of $\vec{Q}$. The way we defined the representations $I^T_q$ we have that $q$ is a sink for $T'$ and so $S(i) = P(i)^n$ is the projective indecomposable representation for $s^+_{i_{n-1}} \cdots s^+_{i_1}\vec{Q}$. So $I^T_q = \Phi^-_{i_1} \cdots \Phi^-_{i_n}(P(i)^n) = P(i)$ since all the index $i_1, \cdots, i_n$ are different from $i$. Therefore, $I^T_q$ are indecomposable projective representations.

(b) We have that $\tau T$ gives us another slice of $\mathbb{Z}Q$ and that slice give the same orientation for the graph $Q$ by Lemma 4.3.12, from the uniqueness of the construction we have that

$$I^{\tau T}_{\tau q} = I^T_q.$$

(c) By the item (2) we have that

$$I^{\tau T}_q = I^{\tau T}_{\tau\tau^{-1}q} = I^T_{\tau^{-1}q}.$$

We can write $C^- = \Phi^-_{i_1} \cdots \Phi^-_{i_\ell}$. If $T' = \tau T$ and we want to write $T'$ from $T$ by reflections, we will need to use each $i \in I$ once, so we can choose to write it by $T' = s^-_{i_1} \cdots s^-_{i_\ell}(T)$ and $T = s^+_{i_\ell} \cdots s^+_{i_1}(T')$. We have that $q \succeq T$, so by Theorem 4.4.1 we have that

$$I^{T'}_q = I^{\tau T}_q = \Phi^-_{i_1} \cdots \Phi^-_{i_\ell}(I^T_q) = C^-(I^T_q).$$

(d) If $q \succeq T$, write $q = \tau^{-n} q'$ with $q' \in T$ and $n \geq 0$ then by item (3) we have

$$I_q^T = I_{\tau^{-n}q'}^T = (C^-)^n(I_{q'}^T).$$

We have that $q' \in T$ therefore from item (1) we have that $I_{q'}^T = P(i)$ then

$$I_q^T = (C^-)^n(I_{q'}^T) = (C^-)^n(P(i)).$$

By applying $(C^+)^{n+1}$ we have

$$(C^+)^{n+1}(I_q^T) = (C^+)^{n+1}((C^-)^n(P(i))) = (C^+)(P(i)) = 0$$

and $I_q^T$ is preprojective.

□

**Corollary 4.4.5.** *Item (a) of Theorem 4.4.4 gives all the indecomposable projective representations of $Q$.*

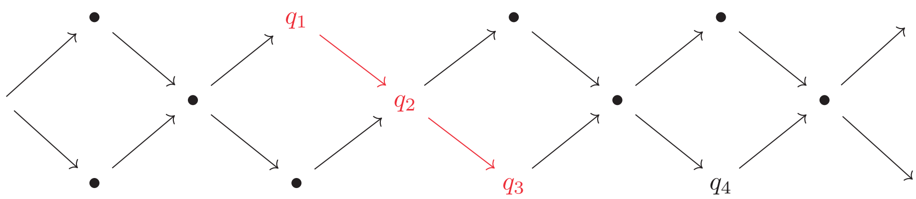
*Proof.* By Theorem 1.4.4 we know that the number of indecomposable projective representations is the number of vertices of the quiver $Q$, consider $\mid I \mid = \ell$. The number of vertices of $T$ is also $\ell$. Theorem 4.4.4 $(a)$ gives us $\ell$ indecomposable projective representation. Therefore, we have all the indecomposable projective representation. □

Let us see an example where we have representations $I_q^T = 0$ for some $q \in \mathbb{Z}Q$.

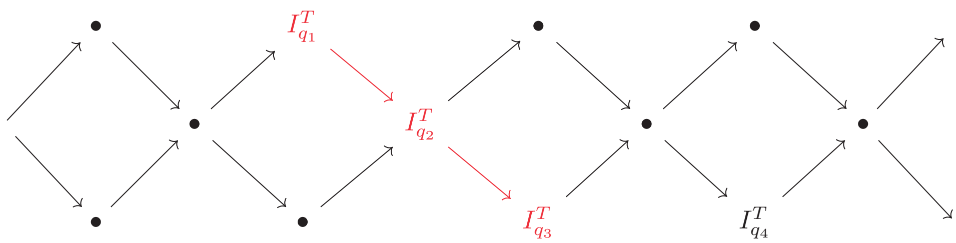
**Example 4.4.6.** *Consider the graph $A_3$*

$$i_1 \text{ ——— } i_2 \text{ ——— } i_3.$$

*We have the quiver $\mathbb{Z}Q$ with slice $T$ in red, see below the vertices that we are going to calculate the associated representation $I_q^T$,*




*Therefore we have the following representations in the quiver $\mathbb{Z}Q$,*



*We have one vertex in the slice that is also a sink, so we have the simple representation in that vertex, $I^T_{q_1} = S(i_1)$ with $S(i_1)$ relative to the quiver $\vec{Q}_T$. The quiver $\vec{Q}_T$ and the representation can be seen as follows,*

$$\vec{Q}_T = \qquad \bullet \longleftarrow \bullet \longleftarrow \bullet$$

$$S(i_1) = \qquad \mathbb{K} \longleftarrow 0 \longleftarrow 0.$$

*For the other vertices of the slice $T$ we have to use other slices, $T_1 = s^+_{i_1}(T)$, $T_2 = s^+_{i_1}s^+_{i_2}(T)$, and $T_3 = s^+_{i_1}s^+_{i_2}s^+_{i_1}s^+_{i_3}s^+_{i_2}(T)$ with representations*

$$I^T_{q_2} = \Phi^-_{i_1}(S(i_2)), \quad I^T_{q_3} = \Phi^-_{i_1}\Phi^-_{i_2}(S(i_3)), \quad \textit{and} \quad I^T_{q_4} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_1}\Phi^-_{i_3}\Phi^-_{i_2}(S(i_3)).$$

*With representation $I^T_{q_2}$ as,*

$$\vec{Q}_{T_1} = \qquad \bullet \longrightarrow \bullet \longleftarrow \bullet$$

$$S(i_2) = \qquad 0 \longrightarrow \mathbb{K} \longleftarrow 0$$

$$I^T_{q_2} = \qquad \mathbb{K} \longleftarrow \mathbb{K} \longleftarrow 0$$

*representation $I^T_{q_3}$ as,*

$$\vec{Q}_{T_2} = \qquad \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$S(i_3) = \qquad 0 \longleftarrow 0 \longrightarrow \mathbb{K}$$

$$I^T_{q_3} = \qquad \mathbb{K} \longleftarrow \mathbb{K} \longleftarrow \mathbb{K}$$

*representation $I^T_{q_4}$ as,*

$$\vec{Q}_{T_3} = \qquad \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$S(i_3) = \qquad 0 \longleftarrow 0 \longrightarrow \mathbb{K}$$

$$I^T_{q_4} = \qquad 0 \longleftarrow 0 \longleftarrow 0.$$

*Therefore, $I^T_{q_4} = 0$.*

The way we constructed the representation $I^T_q$ allow it to be zero, as we saw in Example 4.4.6. However, for the Auslander-Reiten quiver we are interested in studying the representations $I^T_q$ that are different from zero, for that we have the following definition.

**Definition 4.4.7.** *Let $Q$ be a bipartite graph and let $T \subset \mathbb{Z}Q$ be a slice. We define the preprojective Auslander-Reiten quiver of $\vec{Q}_T$ as the subquiver of $\mathbb{Z}Q$ with the set of vertices*

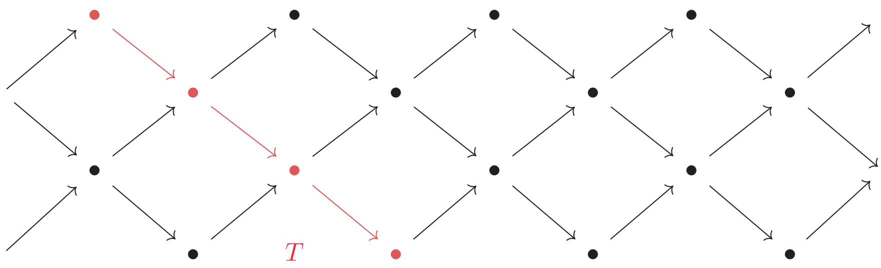
$$\Delta_T = \{q \in \mathbb{Z}Q \mid I^T_q \neq 0\} \subset \mathbb{Z}Q.$$

*If $Q$ is Dynkin, we will drop the word "preprojective" and we will say that $\Delta$ is the Auslander-Reiten quiver of $\vec{Q}_T$.*

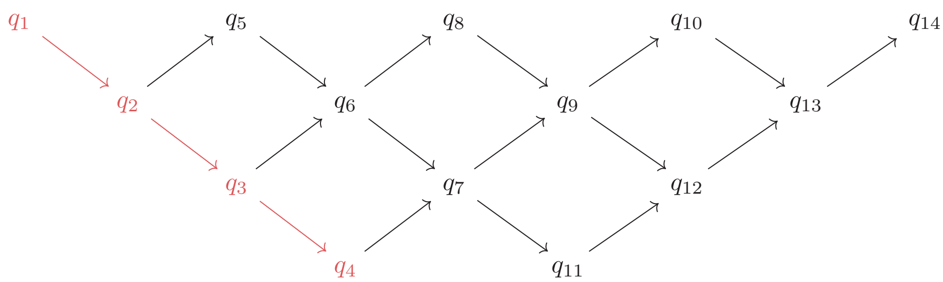
**Example 4.4.8.** *Consider the graph $A_4$,*

$$i_1 \longrightarrow\!\!\!\!\!- \; i_2 \; \text{———} \; i_3 \; \text{———} \; i_4.$$

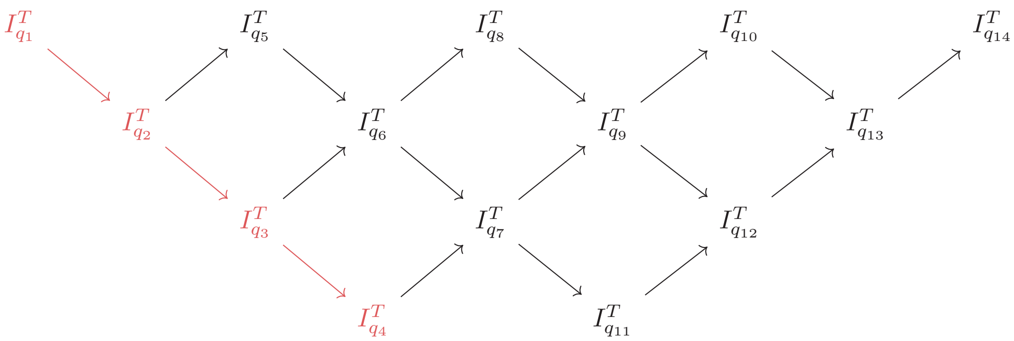
*We have the quiver $\mathbb{Z}Q$ with slice $T$ in red,*




*We want to find the Auslander-Reiten quiver for this slice, for that we need to calculate the representations $I_q^T$ with $q \in \mathbb{Z}Q$. By Theorem 4.4.1 we know that if $q$ is strictly below $T$ then $I_q^T = 0$. See below the vertices that we will calculate the representations $I_q^T$,*




*That will give us the representations,*




*As $q_1$ is a sink, we have $I_{q_1}^T = S(i_1)$ with*

$$\vec{Q}_T = \qquad \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet$$

$$S(i_1) = \qquad \mathbb{K} \longleftarrow 0 \longleftarrow 0 \longleftarrow 0.$$

*For the other representations we need some slices. Consider*

$$\begin{aligned}
T_1 =& s^+_{i_1}(T), \quad T_2 = s^+_{i_1}s^+_{i_2}(T), \quad T_3 = s^+_{i_1}s^+_{i_2}s^+_{i_3}(T), \quad T_4 = s^+_{i_1}s^+_{i_2}s^+_{i_1}s^+_{i_3}(T),\\
T_5 =& s^+_{i_1}s^+_{i_2}s^+_{i_1}s^+_{i_3}s^+_{i_2}s^+_{i_4}(T), \quad T_6 = s^+_{i_1}s^+_{i_2}s^+_{i_1}s^+_{i_3}s^+_{i_2}s^+_{i_4}s^+_{i_3}s^+_{i_1}(T), \quad T_7 = s^+_{i_1}s^+_{i_2}s^+_{i_3}s^+_{i_1}s^+_{i_2}s^+_{i_1}s^+_{i_4}s^+_{i_3}s^+_{i_2}(T),\\
T_8 =& s^+_{i_1}s^+_{i_2}s^+_{i_3}s^+_{i_4}s^+_{i_1}s^+_{i_2}s^+_{i_3}(T), \quad T_9 = s^+_{i_1}s^+_{i_2}s^+_{i_3}s^+_{i_1}s^+_{i_2}s^+_{i_1}s^+_{i_4}s^+_{i_3}s^+_{i_2}s^+_{i_4}(T),\\
T_{10} =& s^+_{i_1}s^+_{i_2}s^+_{i_1}s^+_{i_3}s^+_{i_2}s^+_{i_4}s^+_{i_3}s^+_{i_1}s^+_{i_2}s^+_{i_1}s^+_{i_4}s^+_{i_3}(T), \quad \textit{and} \quad T_{11} = s^+_{i_1}s^+_{i_2}s^+_{i_3}s^+_{i_1}s^+_{i_2}s^+_{i_4}s^+_{i_3}s^+_{i_1}s^+_{i_2}s^+_{i_4}s^+_{i_3}s^+_{i_1}s^+_{i_2}(T)
\end{aligned}$$

*with representations,*

$$\begin{aligned}
&I^T_{q_2} = \Phi^-_{i_1}(S(i_2)), \quad I^T_{q_3} = \Phi^-_{i_1}\Phi^-_{i_2}(S(i_3)), \quad I^T_{q_4} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}(S(i_4)), \quad I^T_{q_5} = \Phi^-_{i_1}\Phi^-_{i_2}(S(i_1)),\\
&I^T_{q_6} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_1}\Phi^-_{i_3}(S(i_2)), \quad I^T_{q_7} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_1}\Phi^-_{i_3}\Phi^-_{i_2}\Phi^-_{i_4}(S(i_3)), \quad I^T_{q_8} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_1}\Phi^-_{i_3}\Phi^-_{i_2}\Phi^-_{i_4}(S(i_1))\\
&I^T_{q_9} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_1}\Phi^-_{i_3}\Phi^-_{i_2}\Phi^-_{i_4}\Phi^-_{i_3}\Phi^-_{i_1}(S(i_2)), \quad I^T_{q_{10}} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_1}\Phi^-_{i_4}\Phi^-_{i_3}\Phi^-_{i_2}(S(i_1)),\\
&I^T_{q_{11}} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_4}\Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}(S(i_4)), \quad I^T_{q_{12}} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_1}\Phi^-_{i_4}\Phi^-_{i_3}\Phi^-_{i_2}\Phi^-_{i_4}(S(i_3)),\\
&I^T_{q_{13}} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_1}\Phi^-_{i_3}\Phi^-_{i_2}\Phi^-_{i_4}\Phi^-_{i_3}\Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_1}\Phi^-_{i_4}\Phi^-_{i_3}(S(i_2)),\\
&\textit{and} \quad I^T_{q_{14}} = \Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_3}\Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_4}\Phi^-_{i_3}\Phi^-_{i_1}\Phi^-_{i_2}\Phi^-_{i_4}\Phi^-_{i_3}\Phi^-_{i_1}\Phi^-_{i_2}(S(i_1)).
\end{aligned}$$

*With representation* $I^T_{q_2}$ *as,*

$$\begin{aligned}
\vec{Q}_{T_1} &= \quad \bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet\\
S(i_2) &= \quad 0 \longrightarrow \mathbb{K} \longleftarrow 0 \longleftarrow 0\\
I^T_{q_2} &= \quad \mathbb{K} \longleftarrow \mathbb{K} \longleftarrow 0 \longleftarrow 0
\end{aligned}$$

*representation* $I^T_{q_3}$ *as,*

$$\begin{aligned}
\vec{Q}_{T_2} &= \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet\\
S(i_3) &= \quad 0 \longleftarrow 0 \longrightarrow \mathbb{K} \longleftarrow 0\\
I^T_{q_3} &= \quad \mathbb{K} \longleftarrow \mathbb{K} \longleftarrow \mathbb{K} \longleftarrow 0
\end{aligned}$$

*representation* $I^T_{q_4}$ *as,*

$$\begin{aligned}
\vec{Q}_{T_3} &= \quad \bullet \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet\\
S(i_4) &= \quad 0 \longleftarrow 0 \longleftarrow 0 \longrightarrow \mathbb{K}\\
I^T_{q_4} &= \quad \mathbb{K} \longleftarrow \mathbb{K} \longleftarrow \mathbb{K} \longleftarrow \mathbb{K}
\end{aligned}$$

*representation* $I^T_{q_5}$ *as,*

$$\vec{Q}_{T_2} = \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

$$S(i_1) = \quad \mathbb{K} \longleftarrow 0 \longrightarrow 0 \longleftarrow 0$$

$$I^T_{q_5} = \quad 0 \longleftarrow \mathbb{K} \longleftarrow 0 \longleftarrow 0$$

*representation* $I^T_{q_6}$ *as,*

$$\vec{Q}_{T_4} = \quad \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$S(i_2) = \quad 0 \longrightarrow \mathbb{K} \longleftarrow 0 \longrightarrow 0$$

$$I^T_{q_6} = \quad 0 \longleftarrow \mathbb{K} \longleftarrow \mathbb{K} \longleftarrow 0$$

*representation* $I^T_{q_7}$ *as,*

$$\vec{Q}_{T_5} = \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

$$S(i_3) = \quad 0 \longleftarrow 0 \longrightarrow \mathbb{K} \longleftarrow 0$$

$$I^T_{q_7} = \quad 0 \longleftarrow \mathbb{K} \longleftarrow \mathbb{K} \longleftarrow \mathbb{K}$$

*representation* $I^T_{q_8}$ *as,*

$$\vec{Q}_{T_5} = \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

$$S(i_1) = \quad \mathbb{K} \longleftarrow 0 \longrightarrow 0 \longleftarrow 0$$

$$I^T_{q_8} = \quad 0 \longleftarrow 0 \longleftarrow \mathbb{K} \longleftarrow 0$$

*representation* $I^T_{q_9}$ *as,*

$$\vec{Q}_{T_6} = \quad \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$S(i_2) = \quad 0 \longrightarrow \mathbb{K} \longleftarrow 0 \longrightarrow 0$$

$$I^T_{q_9} = \quad 0 \longleftarrow 0 \longleftarrow \mathbb{K} \longleftarrow \mathbb{K}$$

*representation* $I^T_{q_{10}}$ *as,*

$$\vec{Q}_{T_7} = \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$S(i_1) = \quad \mathbb{K} \longleftarrow 0 \longrightarrow 0 \longrightarrow 0$$

$$I^T_{q_{10}} = \quad 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \mathbb{K}$$

*representation $I^T_{q_{11}}$ as,*

$$\vec{Q}_{T_8} = \qquad \bullet \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$S(i_4) = \qquad 0 \longleftarrow 0 \longleftarrow 0 \longrightarrow \mathbb{K}$$

$$I^T_{q_{11}} = \qquad 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0$$

*representation $I^T_{q_{12}}$ as,*

$$\vec{Q}_{T_9} = \qquad \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

$$S(i_3) = \qquad 0 \longleftarrow 0 \longrightarrow \mathbb{K} \longleftarrow 0$$

$$I^T_{q_{12}} = \qquad 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0$$

*representation $I^T_{q_{13}}$ as,*

$$\vec{Q}_{T_{10}} = \qquad \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$S(i_2) = \qquad 0 \longrightarrow \mathbb{K} \longleftarrow 0 \longrightarrow 0$$

$$I^T_{q_{13}} = \qquad 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0$$

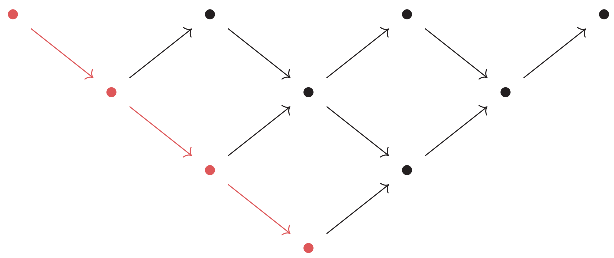
*and representation $I^T_{q_{14}}$ as,*

$$\vec{Q}_{T_{11}} = \qquad \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

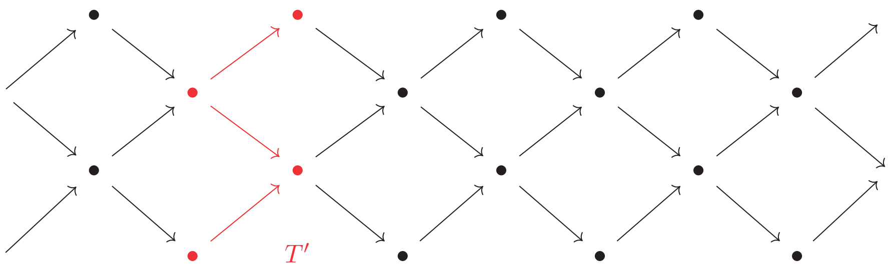
$$S(i_1) = \qquad \mathbb{K} \longleftarrow 0 \longrightarrow 0 \longrightarrow 0$$

$$I^T_{q_{14}} = \qquad 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0.$$

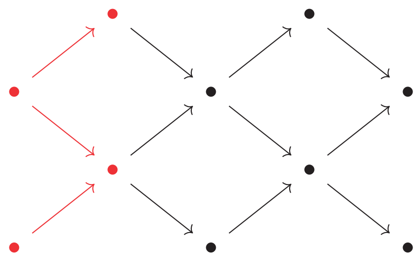
*Therefore, the Auslander-Reiten quiver for the slice $T$ is,*



*If we choose another slice $T'$ in red in the quiver $\mathbb{Z}Q$ as*



*then we can follow the same steps as before and find the following Auslander-Reiten quiver for the slice $T'$ as*



**Theorem 4.4.9.** *Let $Q$ be a bipartite graph and let $T$ be a slice of $\mathbb{Z}Q$. Then the map*

$$\phi : \Delta_T \to Ind(\vec{Q}_T)$$
$$q \mapsto I_q^T$$

*is a bijection between $\Delta_T$ as defined in Definition 4.4.7 and the set $Ind(\vec{Q}_T)$ of isomorphism classes of nonzero preprojective indecomposable representations of $\vec{Q}_T$.*

*Proof.* By Theorem 4.4.4 (4) we know that each $I_q^T$ is a preprojective indecomposable representation, so $I_q^T \in \mathrm{Ind}(\vec{Q}_T)$.

On the other hand, if $I$ is a preprojective indecomposable representation then by Theorem 4.2.5

$$I = (C^-)^n(P(i))$$

for some $n \geq 0$. For $q = (i, h_i) \in T$ we have by Theorem 4.4.4 item (1) that $I_q^T = P(i)$. Then

$$I = (C^-)^n(P(i)) = (C^-)^n(I_q^T) = I_{\tau^{-n}q}^T$$

therefore each preprojective indecomposable representation can be written as $I_q^T$ for some $q$. It follows that $\phi$ is surjective.

Now it remains to show that two representations of the form $(C^-)^n(P(i))$ are not isomorphic. Indeed, if

$$I = (C^-)^{n_1}(P(i_1)) = (C^-)^{n_2}(P(i_2)).$$

We have that $n_1$ is taken to be the maximum such that $(C^+)^n(I) \neq 0$, then $n_1 = n_2$. So $P(i_1) = P(i_2)$, which only occurs if $i_1 = i_2$. Which means that the function $\phi$ is injective, and hence bijective. □

This theorem relates the combinatorics of slices of the path algebra $\mathbb{Z}Q$ of a bipartite graph $Q$ to the representation theory of the quiver $\vec{Q}_T$ obtained by restricting $Q$ to $T$. It shows that the set of isomorphism classes of nonzero preprojective indecomposable representations of $\vec{Q}_T$ is in bijection with the set of elements of the root lattice $\Delta_T$ of $T$.

**Theorem 4.4.10.** *Let $Q$ be a Dynkin graph, and let $T, \Delta_T$ be as in Definition 4.4.7. Let $w_0 \in W$ be as in Lemma A.11.1 and write $w_0 = s_{i_r} \cdots s_{i_1}$ to be a reduced expression for $w_0$ adapted to orientation $\Omega_T$, also let $T' = s^+_{i_r} \cdots s^+_{i_1} T$. Then*

$$\Delta_T = \{q \in \mathbb{Z}Q \mid T \preceq q \prec T'\}.$$

*Proof.* Denote by

$$\Delta'_T = \{q \in \mathbb{Z}Q \mid T \preceq q \prec T'\}.$$

Each $q \in \Delta'_T$ is such that $I^T_q \neq 0$ then $q \in \Delta_T$ and $\Delta'_T \subseteq \Delta_T$. Therefore, $\{I^T_q; q \in \Delta'_T\}$ are all the indecomposable representations $I_1, \cdots, I_r$ with,

$$\begin{aligned} I_1 =& S(i_1), & \dim\,(I_1) = \gamma_1 = \alpha_{i_1}, \\ I_2 =& \Phi^-_{i_1}(S(i_2)), & \dim\,(I_2) = \gamma_2 = s_{i_1}(\alpha_{i_2}), \\ &\vdots & \\ I_r =& \Phi^-_{i_1} \cdots \Phi^-_{i_{r-1}}(S(i_r)), & \dim\,(I_r) = \gamma_r = s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r}). \end{aligned} \tag{4.9}$$

As in (3.6).

Then the function,

$$\phi : \Delta'_T \to \mathrm{Ind}(\vec{Q}_T)$$
$$q \mapsto I^T_q$$

is injective, since $\Delta'_T$ is a subset of $\Delta_T$. By Theorem 4.4.9 we have that $\Delta_T$ is in bijection with $\mathrm{Ind}(\vec{Q}_T)$. The set $\Delta'_T$ has $\ell + r - \ell = r$ elements and the set $\mathrm{Ind}(\vec{Q}_T)$ has $r$ elements as we can see in (4.9). Therefore, the function is a bijection and the only possibility for this to happen is if $\Delta'_T = \Delta_T$. □

**Corollary 4.4.11.** *The slice $T' = s^+_{i_r} \cdots s^+_{i_1} T$ used in the proof of Theorem 4.4.10, does not depend on the choice of the reduced expression $w_0 = s_{i_r} \cdots s_{i_1}$.*

*Proof.* If $w_0 = s_{i_r} \cdots s_{i_1} = s_{j_r} \cdots s_{j_1}$ then we have that $T' = s_{i_r} \cdots s_{i_1}(T) = s_{j_r} \cdots s_{j_1}(T)$ but by Theorem 4.3.19 the sequence is defined uniquely up to interchanging $s^+_i s^+_j \leftrightarrow s^+_j s^+_i$ for $i, j$ not connected in $Q$ □

By Proposition 3.2.1, the nonzero indecomposable representations are in bijection with positive roots in the Dynkin case, so we have a bijection

$$\Delta_T \simeq \mathsf{R}_+.$$

Therefore, by Example 4.4.8 we have that the positive root system $\mathsf{R}_+$ associated with the graph $A_4$ has 10 elements.

**Theorem 4.4.12.** *Let $Q$ be a Dynkin graph, $T \subset \mathbb{Z}Q$ a slice, and $\Delta_T$ the Auslander-Reiten quiver. Define a partial order $\preceq$ on $\Delta_T$, and thus on Ind$(\vec{Q}_T)$, by*

$$q \preceq q' \iff \text{there is a path in } \Delta_T \text{ from } q \text{ to } q'.$$

*(a) For any choice of reduced expression $w_0 = s_{i_r} \cdots s_{i_1}$ adapted to $\vec{Q}_T$, the order on $\mathsf{R}_+$ defined by (3.6) is compatible with the partial order $\preceq$, which means that if $I_k \preceq I_n$ then $k \leq n$.*

*(b) Any linear order on $\mathsf{R}_+$ compatible with $\preceq$ can be obtained from some reduced expression for $w_0$ adapted to $\vec{Q}_T$.*

*(c) We have that*

$$\begin{aligned} Hom_{\vec{Q}_T}(I_q, I_{q'}) \neq 0 \quad &\Rightarrow \quad q \preceq q', \\ Ext^1_{\vec{Q}_T}(I_q, I_{q'}) \neq 0 \quad &\Rightarrow \quad q \prec q'. \end{aligned}$$

*Proof.* (a) Consider the sequence of slices

$$\begin{aligned} T_1 &= T, \\ T_2 &= s^+_{i_1}(T), \\ &\vdots \\ T_r &= s^+_{i_{r-1}} \cdots s^+_{i_1}(T) \\ T_{r+1} &= s^+_{i_r} \cdots s^+_{i_1}(T) = T'. \end{aligned}$$

Since every $T_i$ differs from $T_{i-1}$ in a single place, there exists only one $q_i \in \mathbb{Z}Q$ such that $q_i \in T_i$ and $q_i \succ T_{i-1}$ for $i = 2, \cdots, r$ and for $i = 1$ we have that $q_1 \in T_1$ and $q_1 \prec T_2$.

The Auslander-Reiten quiver has the set of vertices:

$$\Delta_T = \{q \in \mathbb{Z}Q; T \preceq q \prec T'\} = \{q_1, \cdots, q_r\}.$$

By Theorem 4.4.1 item (2) and Theorem 4.4.10 we have that

$$I^T_{q_1} = S(i_1) = I_1.$$

Since $w_0$ is adapted to $\vec{Q}_T$ then $i_1$ is a sink for $\vec{Q}_T$. As $T_k = s^+_{i_{k-1}} \cdots s^+_{i_1}(T)$ and $q_k \succeq T_k$ then

$$\begin{aligned} I^T_{q_2} &= \Phi^-_{i_1}(I^{T_2}_{q_2}) = \Phi^-_{i_1}(S(i_2)) = I_2 \\ &\vdots \\ I^T_{q_r} &= \Phi^-_{i_1} \cdots \Phi^-_{i_{r-1}}(S(i_r)) = I_r, \end{aligned}$$

so we have all the indecomposable representations of a Dynkin quiver.

Now it is enough to show that if $q_k \preceq q_n$ then $k \leq n$. If $q_k \preceq q_n$ then there exists a path in $\Delta_T$ from $q_k$ to $q_n$, and by Proposition 4.3.18 we have that $p \succeq T$ if and only if there exists a path in $\mathbb{Z}Q$ from some vertex of $T$ to $p$. Since $q_k \in T_k$ we have that $q_n \succeq T_k$. If $q_n \in T_k$ then $n = k$ and if $q_n \succ T_k$ then $n > k$, so $k \leq n$.

(b) If we have a total order in $\mathsf{R}_+$ we have that $\mathsf{R}_+ = \{\alpha_1, \cdots, \alpha_r\}$ and if $\alpha_i \preceq \alpha_j$ then $i \leq j$. If the order is compatible with $\preceq$ from $\Delta_T$ this means that there is a total order in $\Delta_T$. So we have a sequence of slices $T_0 \preceq T_1 \preceq \cdots \preceq T_r = T'$ where each $T_k$ is obtained from $T_{k-1}$ by applying one reflection $s^+_{i_k}$ and that defines a reduced expression for $w_0$ adapted to $\vec{Q}_T$,

$$w_0 = s_{i_r} \cdots s_{i_1}.$$

(c) Note that $q \preceq q'$ if and only if for every total order $\leq$ compatible with $\preceq$ we have that $q \leq q'$. By Theorem 3.3.3

$$\mathrm{Hom}_{\vec{Q}}(I_a, I_b) = 0 \quad \text{for } a > b,$$
$$\mathrm{Ext}^1_{\vec{Q}}(I_a, I_b) = 0 \quad \text{for } a \leq b.$$

Therefore,

$$\mathrm{Hom}_{\vec{Q}}(I_q, I_{q'}) \neq 0 \quad \text{for } q \leq q' \Rightarrow q \preceq q',$$
$$\mathrm{Ext}^1_{\vec{Q}}(I_q, I_{q'}) \neq 0 \quad \text{for } q > q' \Rightarrow q \succ q'.$$

□

This theorem is an important result in the study of the Auslander-Reiten quivers of Dynkin quivers. It provides a partial order on the indecomposable representations of the preprojective algebra that is compatible with the Auslander-Reiten structure. The fact that any linear order compatible with this partial order can be obtained from a reduced expression for $w_0$ adapted to $\vec{Q}_T$ is a useful tool for constructing Auslander-Reiten quivers and studying their properties. The implications of the last part of the theorem for Hom and Ext groups provide important information about the structure of the quiver and the relationships between its representations. Overall, this theorem plays a key role in understanding the representation theory of Dynkin quivers and their preprojective algebras.

# Appendices

# Appendix A

# ROOT SYSTEM

For the reader's convenience we include all definitions and results related to the theory of finite root systems as an appendix for this dissertation. This study will be made from an axiomatic point of view to avoid the Lie theory approach. We follow [12] in this chapter.

We will use four axioms to define root systems and from that we can prove all the properties we need along this dissertation. We will start this appendix studying the definition and some properties of reflections, since in the axioms we will use this definition.

## A.1 Reflections in a Euclidean space

Fix a euclidean space $\mathsf{E}$, which is a finite dimensional vector space over $\mathbb{R}$ endowed with an inner product $(\ ,\ )$. And set $\ell = \dim(\mathsf{E})$.

Geometrically, an orthogonal reflection on the euclidean space $\mathsf{E}$ is an invertible linear operator on $\mathsf{E}$ that leaves a hyperplane point wise fixed and sends any vector orthogonal to that hyperplane into its negative. The reader can see that in Figure A.1.

A reflection is an orthogonal operator, which is an operator that preserves the inner product on $\mathsf{E}$, using symbols this means that $(s(\alpha), s(\beta)) = (\alpha, \beta)$. Any nonzero vector $\alpha$ determines a reflection $s_\alpha$, with respect to the hyperplane

$$P_\alpha = \{\beta \in \mathsf{E}; (\beta, \alpha) = 0\}. \tag{A.1}$$

Naturally nonzero vectors proportional to $\alpha$ yield the same reflection. Given a vector $\alpha \in \mathsf{E} - \{0\}$ we have the reflection

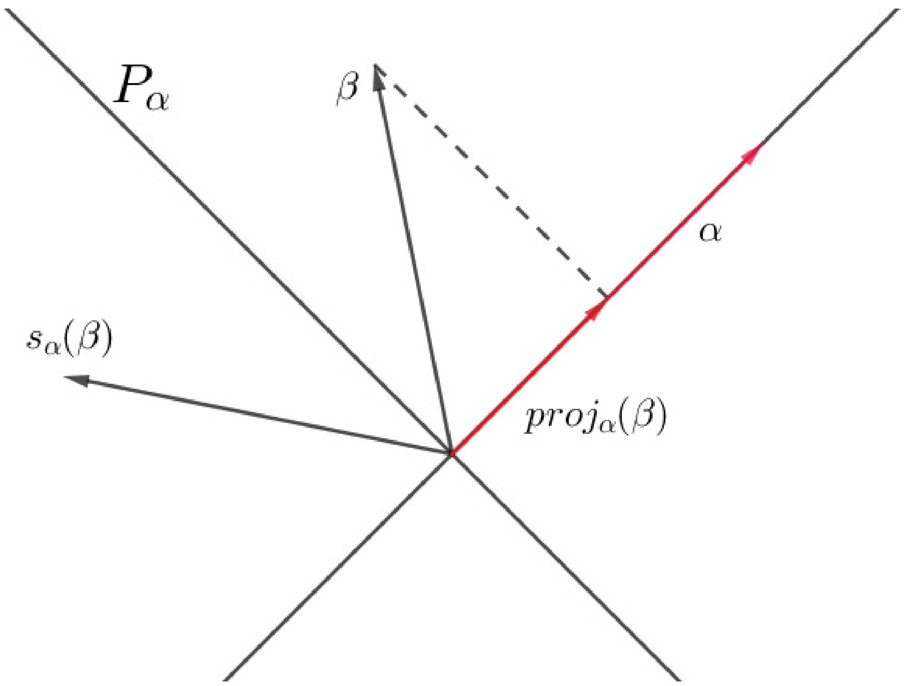
$$s_\alpha : \mathsf{E} \to \mathsf{E} \ \text{ with } \ s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

For ease of notation we define

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)},$$

which will be frequently used through this appendix.

Figure A.1: Reflection of $\beta$




Notice that $\langle \beta, \alpha \rangle$ is linear only in the first coordinate, indeed for $\alpha, \beta, \gamma \in \mathsf{E}$ and $t \in \mathbb{R}$ we have

$$\langle \beta + \gamma, \alpha \rangle = \frac{2(\beta + \gamma, \alpha)}{(\alpha, \alpha)} = \frac{2\left((\beta, \alpha) + (\gamma, \alpha)\right)}{(\alpha, \alpha)} = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} + \frac{2(\gamma, \alpha)}{(\alpha, \alpha)} = \langle \beta, \alpha \rangle + \langle \gamma, \alpha \rangle$$
$$\langle t\beta, \alpha \rangle = \frac{2(t\beta, \alpha)}{(\alpha, \alpha)} = t\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = t\,\langle \beta, \alpha \rangle$$
$$\langle \beta, t\alpha \rangle = \frac{2(\beta, t\alpha)}{(t\alpha, t\alpha)} = \frac{2t(\beta, \alpha)}{t^2(\alpha, \alpha)} = \frac{1}{t}\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \neq t\,\langle \beta, \alpha \rangle\,.$$

Further, for each $\beta \in \mathsf{E}$ we have

$$s_\alpha s_\alpha(\beta) = s_\alpha(\beta - \langle \beta, \alpha \rangle \alpha) = s_\alpha(\beta) - \langle \beta, \alpha \rangle s_\alpha(\alpha) = \beta - \langle \beta, \alpha \rangle \alpha + \langle \beta, \alpha \rangle \alpha = \beta,$$

so $s_\alpha$ is an involution and $s_\alpha = s_\alpha^{-1}$.

**Lemma A.1.1.** *Let* $\mathsf{R}$ *be a finite set which spans* $\mathsf{E}$*. Suppose all reflections* $s_\alpha$ *with* $\alpha \in \mathsf{R}$ *leave* $\mathsf{R}$ *invariant, which means that* $s_\alpha(\mathsf{R}) = \mathsf{R}$*. If* $s \in GL(\mathsf{E})$ *and* $P$ *is a hyperplane of* $\mathsf{E}$ *are such that,*

*(i)* $s(\mathsf{R}) = \mathsf{R}$*;*

*(ii)* $s(\beta) = \beta \;\forall\; \beta \in P$*;*

*(iii)* $s(\alpha) = -\alpha$ *for some* $\alpha \in \mathsf{R}$*.*

*Then* $s = s_\alpha$ *and* $P = P_\alpha$*.*

*Proof.* Let $\tau = ss_\alpha = ss_\alpha^{-1}$. Then $\tau(\mathsf{R}) = \mathsf{R}$ and

$$\tau(\alpha) = ss_\alpha(\alpha) = s(-\alpha) = -s(\alpha) = \alpha.$$

Since $\tau$ is the composition of linear transformations then $\tau(\beta) = \beta$ for all $\beta \in \mathbb{R}\alpha$ so $\tau$ act as the identity on the subspace $\mathbb{R}\alpha$.

Therefore, $\tau$ induces a liner map

$$\tau' : \frac{\mathsf{E}}{\mathbb{R}\alpha} \to \frac{\mathsf{E}}{\mathbb{R}\alpha} \text{ defined by } \tau'(\overline{\beta}) = \overline{\tau(\beta)},$$

and moreover $\tau' = \text{id}$. In fact, since $\mathsf{E} = P \oplus \mathbb{R}\alpha$, given any $\beta \in \mathsf{E}$ and writing $\beta = \gamma + b\alpha$, for some $\gamma \in P$ and $b \in \mathbb{R}$, we have

$$\begin{aligned}\tau(\beta) = \tau(\gamma + b\alpha) = \tau(\gamma) + b\tau(\alpha) &= ss_\alpha(\gamma) + b\alpha = s(\gamma - \langle \gamma, \alpha \rangle \alpha) + b\alpha \\ &= s(\gamma) - \langle \gamma, \alpha \rangle s(\alpha) + b\alpha = \gamma + \langle \gamma, \alpha \rangle \alpha + b\alpha = \gamma + (\langle \gamma, \alpha \rangle + b)\alpha.\end{aligned}$$

Since $\overline{\beta} = \overline{\gamma}$ and $\overline{\tau(\beta)} = \overline{\gamma + (\langle p, \alpha \rangle + b)\alpha} = \overline{\gamma}$, then

$$\tau'(\overline{\beta}) = \overline{\tau(\beta)} = \overline{\gamma} = \overline{\beta}.$$

It follows that $\tau$ is equivalent to

$$\begin{bmatrix} 1 & A \\ 0 & \text{Id}_{\ell-1} \end{bmatrix},$$

for some $1 \times (\ell - 1)$-matrix $A$ and $\text{Id}_{\ell-1}$ denotes the identity square matrix of size $\ell - 1$. Then all eigenvalues of $\tau$ are 1, and the minimal polynomial $m_\tau(t)$ of $\tau$ divides $(t-1)^\ell$.

On the other hand, since $\mathsf{R}$ is finite and $\tau(\mathsf{R}) = \mathsf{R}$ it follows that $\tau|_{\mathsf{R}}$ is an element of the symmetric group $S_{\mathsf{R}}$. In particular, there exists $k$ such that $\tau^k|_{\mathsf{R}} = \text{Id}_{\mathsf{R}}$. Since $\mathsf{R}$ spans $\mathsf{E}$ we have that $\tau^k = \text{Id}_{\mathsf{E}}$ and hence $m_\tau(t)$ divides $t^k - 1$. Combined with the previous step, this shows that $m_\tau(t)$ divides $t - 1 = \gcd(t^k - 1, (t-1)^\ell)$ which implies $\tau = 1$ and thus $s = s_\alpha$. □

## A.2 Root systems

Root systems are a fundamental concept in the theory of Lie algebras, with wide-ranging applications in mathematics and physics. A root system is a set of vectors in a Euclidean space that satisfy certain axioms, which reflect the symmetry of the set. Root systems have an elegant mathematical structure that encodes information about the underlying Lie algebra, making them a powerful tool for studying Lie theory. In this section, we will provide a detailed introduction to root systems, starting with their definition and basic properties.

**Definition A.2.1.** *A subset $\mathsf{R}$ of the euclidean space $\mathsf{E}$ is called a root system in $\mathsf{E}$ if the following axioms are satisfied,*

*(R1) $\mathsf{R}$ is finite, spans $\mathsf{E}$ and does not contain $0$.*

*(R2) If $\alpha \in \mathsf{R}$, the only multiples of $\alpha$ in $\mathsf{R}$ are $\pm\alpha$.*

*(R3) If $\alpha \in \mathsf{R}$, the reflection $s_\alpha$ leaves $\mathsf{R}$ invariant.*

*(R4) If $\alpha, \beta \in \mathsf{R}$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.*

Notice that if we replace the given inner product on $\mathsf{E}$ by a positive multiple then the axioms would not be affected. Since only ratios of inner products occur. We define the rank of the root system $\mathsf{R}$ to be $\ell = \dim(\mathsf{E})$.

Let $\mathsf{R}$ be a root system in $\mathsf{E}$. Denote by $\mathscr{W}$ the subgroup of $GL(\mathsf{E})$ generated by the reflections $s_\alpha$, with $\alpha \in \mathsf{R}$. By $(R3)$ we have that $\mathscr{W}$ permutes the set $\mathsf{R}$, which by $(R1)$ is finite and spans $\mathsf{E}$. This allows us to identify $\mathscr{W}$ with a subgroup of the symmetric group on $\mathsf{R}$, in particular this implies that $\mathscr{W}$ is finite. The group $\mathscr{W}$ is called the Weyl group of $\mathsf{R}$ and plays an extremely important role in the root system theory. The following lemma shows how certain automorphisms of $\mathsf{E}$ act on $\mathscr{W}$ by conjugation.

**Lemma A.2.2.** *Let $\mathsf{R}$ be a root system in $\mathsf{E}$ with Weyl group $\mathscr{W}$. If $s \in GL(\mathsf{E})$ leaves $\mathsf{R}$ invariant, then $ss_\alpha s^{-1} = s_{s(\alpha)}$ for all $\alpha \in \mathsf{R}$. Moreover, for each $\alpha, \beta \in \mathsf{R}$ we have $\langle \beta, \alpha \rangle = \langle s(\beta), s(\alpha) \rangle$.*

*Proof.* As $s$ and $s_\alpha$ leave $\mathsf{R}$ invariant so does $ss_\alpha s^{-1}$. We can write $ss_\alpha s^{-1}(s(\beta)) = ss_\alpha(\beta)$ and that gives us the equality

$$ss_\alpha(\beta) = s(s_\alpha(\beta)) = s(\beta - \langle \beta, \alpha \rangle \alpha) = s(\beta) - \langle \beta, \alpha \rangle s(\alpha).$$

If $\gamma \in s(P_\alpha)$ then $\gamma = s(\delta)$ for some $\delta \in P_\alpha$ and hence

$$ss_\alpha s^{-1}(\gamma) = ss_\alpha s^{-1}(s(\delta)) = s(\delta) - \langle \delta, \alpha \rangle s(\alpha) = s(\delta) = \gamma.$$

Therefore, $ss_\alpha s^{-1}$ fixes point wise the hyperplane $s(P_\alpha)$. Further

$$ss_\alpha s^{-1}(s(\alpha)) = s(\alpha) - \langle \alpha, \alpha \rangle s(\alpha) = s(\alpha) - 2s(\alpha) = -s(\alpha),$$

and hence $ss_\alpha s^{-1}$ sends $s(\alpha)$ to $-s(\alpha)$. By Lemma A.1.1 we have that $ss_\alpha s^{-1} = s_{s(\alpha)}$ and the first claim follows.

For the second claim, using the previous one we have, for each $\beta \in R$,

$$s_{s(\alpha)}(s(\beta)) = s(\beta) - \langle s(\beta), s(\alpha) \rangle s(\alpha) = s(\beta) - \langle \beta, \alpha \rangle s(\alpha) = s(s_\alpha(\beta)) = ss_\alpha s^{-1}(s(\beta)).$$

Thus,

$$\begin{aligned} s(\beta) - \langle s(\beta), s(\alpha) \rangle s(\alpha) &= s(\beta) - \langle \beta, \alpha \rangle s(\alpha) \\ - \langle s(\beta), s(\alpha) \rangle s(\alpha) &= - \langle \beta, \alpha \rangle s(\alpha) \\ \langle s(\beta), s(\alpha) \rangle s(\alpha) &= \langle \beta, \alpha \rangle s(\alpha) \\ \langle s(\beta), s(\alpha) \rangle &= \langle \beta, \alpha \rangle \end{aligned}$$

and that gives us the second assertion of the lemma. □

Consider two root systems $\mathsf{R}$ and $\mathsf{R}'$ defined in euclidean spaces $\mathsf{E}$ and $\mathsf{E}'$, respectively. There is a natural way to define an isomorphism between these root systems.

**Definition A.2.3.** *Consider the root system $\mathsf{R}$ of $\mathsf{E}$ and the root system $\mathsf{R}'$ of $\mathsf{E}'$. We say that the two root systems are isomorphic if there exists a vector space isomorphism $\phi : \mathsf{E} \to \mathsf{E}'$ that sends $\mathsf{R}$ to $\mathsf{R}'$ and such that $\langle \phi(\beta), \phi(\alpha) \rangle = \langle \beta, \alpha \rangle$ for each pair of roots $\alpha, \beta \in \mathsf{R}$.*

From the Definition A.2.3 it follows that,

$$\begin{aligned} s_{\phi(\alpha)}(\phi(\beta)) &= \phi(\beta) - \langle \phi(\beta), \phi(\alpha) \rangle \, \phi(\alpha) = \phi(\beta) - \langle \beta, \alpha \rangle \, \phi(\alpha) \\ &= \phi(\beta - \langle \beta, \alpha \rangle \, \alpha) = \phi(s_\alpha(\beta)). \end{aligned} \tag{A.2}$$

For the root system $\mathsf{R}$ we have the associated Weyl group $\mathscr{W}$ and for the root system $\mathsf{R}'$ we have the associated Weyl group $\mathscr{W}'$. By (A.2) we have that an isomorphism of root systems $\phi$ as in Definition A.2.3 induces a natural isomorphism of Weyl groups,

$$\begin{aligned} \varphi : \mathscr{W} &\to \mathscr{W}' \\ s &\mapsto \phi \circ s \circ \phi^{-1}. \end{aligned}$$

In view of Lemma A.2.2, an automorphism of $\mathsf{R}$ induces an automorphism of $\mathsf{E}$. In particular, we can regard $\mathscr{W}$ as a subgroup of $\mathrm{Aut}(\mathsf{E})$.

## A.3 Examples

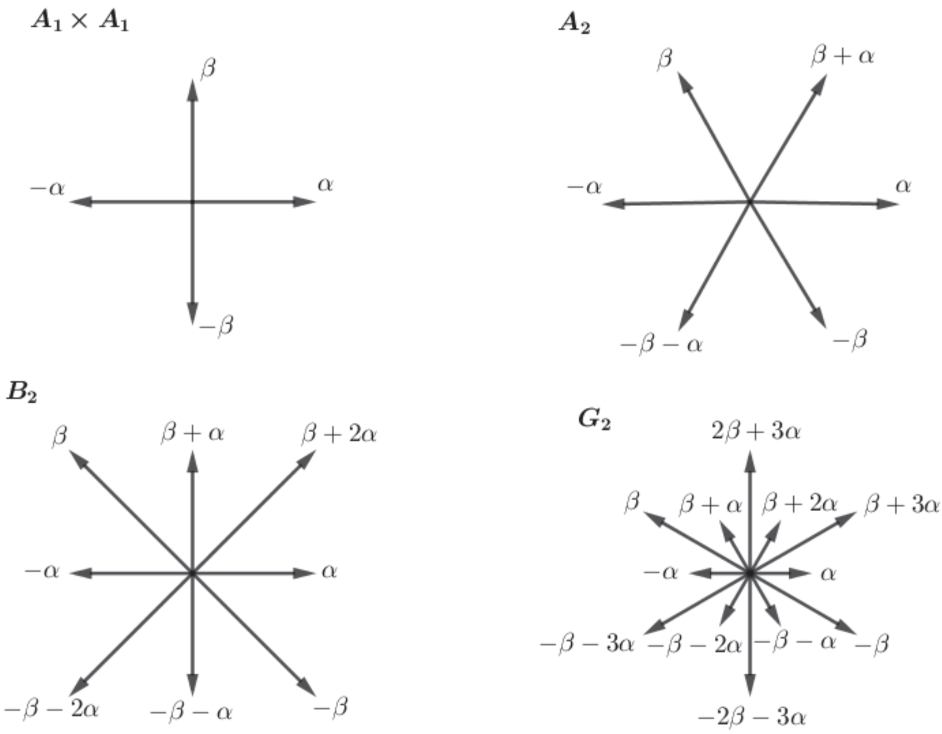
When $\ell \leq 2$ we can describe $\mathsf{R}$ simply drawing a picture. In view of $(R2)$, there is only one possibility in case $\ell = 1$ as it can be seen in the Figure A.2.

Figure A.2: Rank 1 Example


It is easy to check that this actually is a root system, with Weyl group of order 2. This root system is said to be of $A_1$ type.

For rank $\ell = 2$ we have more possibilities, four of which are depicted in Figure A.3. These possibilities turn out to be the only ones, any other possibility is isomorphic to one of these four.

Figure A.3: Rank 2 Examples




## A.4 Pairs of roots

Remember that in Definition A.2.1 the item $(R4)$ states that for all $\alpha, \beta \in \mathsf{R}$ we have that $\langle \beta, \alpha \rangle \in \mathbb{Z}$. Further, the cosine of the angle $\theta$ between vectors $\alpha, \beta \in \mathsf{E}$ is given by

$$||\alpha|| ||\beta|| \cos\theta = (\alpha, \beta).$$

That is going to limit the possible angles occurring between pairs of roots.

Consider the equalities

$$\langle \beta, \alpha \rangle = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = 2\frac{||\beta|| ||\alpha|| \cos\theta}{||\alpha|| ||\alpha||} = 2\frac{||\beta||}{||\alpha||}\cos\theta \tag{A.3}$$

and

$$\langle \alpha, \beta \rangle = 2\frac{(\alpha, \beta)}{(\beta, \beta)} = 2\frac{||\alpha|| ||\beta|| \cos\theta}{||\beta|| ||\beta||} = 2\frac{||\alpha||}{||\beta||}\cos\theta. \tag{A.4}$$

Multiplying (A.4) and (A.3) gives us,

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \left(2\frac{||\alpha||}{||\beta||}\cos\theta\right)\left(2\frac{||\beta||}{||\alpha||}\cos\theta\right) = 4\cos^2\theta. \tag{A.5}$$

Therefore, $0 \leq \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \leq 4$ and moreover $\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle$ must have the same sign. Assuming $\alpha \neq \pm\beta$ and $||\beta|| \geq ||\alpha||$ the only possibilities are the ones portrayed in Table A.1.

The reader can check in the Figure A.3 that the angles and relative lengths we have in the Table A.1 are the ones we get there.

Table A.1: Possibilities of angles

| $\langle\alpha,\beta\rangle$ | $\langle\beta,\alpha\rangle$ | $\theta$ | $\|\|\beta\|\|^2/\|\|\alpha\|\|^2$ |
|---|---|---|---|
| 0 | 0 | $\pi/2$ | undetermined |
| 1 | 1 | $\pi/3$ | 1 |
| $-1$ | $-1$ | $2\pi/3$ | 1 |
| 1 | 2 | $\pi/4$ | 2 |
| $-1$ | $-2$ | $3\pi/4$ | 2 |
| 1 | 3 | $\pi/6$ | 3 |
| $-1$ | $-3$ | $5\pi/6$ | 3 |

**Lemma A.4.1.** *Let $\alpha, \beta$ be non-proportional roots. If $(\alpha,\beta) > 0$, which means that the angle between $\alpha$ and $\beta$ is strictly acute, then $\alpha-\beta$ is a root. Similarly, if $(\alpha,\beta) < 0$ then $\alpha+\beta$ is a root.*

*Proof.* The second assertion follows from the first. It is enough to apply the first assertion to $-\beta$ instead of $\beta$. We have $(\alpha,-\beta) = -(\alpha,\beta) > 0$ and then $\alpha-(-\beta) = \alpha+\beta$ is a root.

To prove the first assertion note that $(\alpha,\beta) > 0$ if and only if $\langle\alpha,\beta\rangle > 0$. Inspecting Table A.1 it follows that either $\langle\alpha,\beta\rangle = 1$ or $\langle\beta,\alpha\rangle = 1$. If $\langle\alpha,\beta\rangle = 1$ we have $s_\beta(\alpha) = \alpha-\beta \in \mathsf{R}$, by $(R3)$. Similarly, if $\langle\beta,\alpha\rangle = 1$ then $s_\alpha(\beta) = \beta-\alpha \in \mathsf{R}$, by $(R3)$ and $s_{\beta-\alpha}(\beta-\alpha) = \alpha-\beta \in \mathsf{R}$, also by $(R3)$. In both cases $\alpha-\beta$ is a root. □

As an application, consider a pair of non–proportional roots $\alpha, \beta$. Let us look at all roots of the form $\beta+i\alpha$, where $i \in \mathbb{Z}$, the $\alpha$-string through $\beta$. Let $r, q \in \mathbb{Z}^+$ be the largest integers for which $\beta - r\alpha \in \mathsf{R}$ and $\beta+q\alpha \in \mathsf{R}$. We shall prove that the $\alpha$-string is unbroken, i.e., for all $i$ between $-r$ and $q$ we have $\beta+i\alpha \in \mathsf{R}$. In fact, if the $\alpha$-string breaks we have that there exists $p, m$ with $-r \le p < m \le q$ such that

$$\beta+p\alpha \in \mathsf{R}, \quad \beta+(p+1)\alpha \notin \mathsf{R}, \quad \beta+(m-1)\alpha \notin \mathsf{R}, \quad \beta+m\alpha \in \mathsf{R}.$$

By Lemma A.4.1 we have that $(\alpha, \beta+p\alpha) \ge 0$, $(\alpha,\beta+s\alpha) \le 0$, since otherwise $\beta+(p+1)\alpha$ a and $\beta+(m-1)\alpha$ would be roots. Then we have

$$(\alpha,\beta+p\alpha) = (\alpha,\beta)+p(\alpha,\alpha) \ge 0, \quad (\alpha,\beta+s\alpha) = (\alpha,\beta)+s(\alpha,\alpha) \le 0.$$

Hence, $p(\alpha,\alpha) \ge s(\alpha,\alpha)$ which implies $p \ge m$, contradicting the fact that $p < m$. We conclude that the $\alpha$-string through $\beta$ is unbroken from $\beta - r\alpha$ to $\beta+q\alpha$.

Notice that for $-r \le i \le q$ such that $\beta+i\alpha \in \mathsf{R}$ we have,

$$s_\alpha(\beta+i\alpha) = s_\alpha(\beta)+is_\alpha(\alpha) = \beta - \langle\beta,\alpha\rangle\,\alpha - i\alpha = \beta+(-\langle\beta,\alpha\rangle - i)\alpha \in \mathsf{R}.$$

Which means that $s_\alpha$ just add a multiple of $\alpha$ to $\beta$. We have that $-r \le (-\langle\beta,\alpha\rangle - i) \le q$, since otherwise $(-\langle\beta,\alpha\rangle - i)$ would contradict the maximality of $r$ or $q$. Then this $\alpha$-string is invariant under $s_\alpha$.

Let $f$ be the function such as

$$f : [-r, q] \to [-r, q]$$
$$i \mapsto -i - \langle \beta, \alpha \rangle$$

where $[-r, q]$ is the interval of integer numbers between $-r$ e $q$. Then $s_\alpha(\beta + i\alpha) = \beta + f(i)\alpha$. Note that $f$ is injective, in fact

$$f(i_1) = f(i_2) \ \Rightarrow \ -i_1 - \langle \beta, \alpha \rangle = -i_2 - \langle \beta, \alpha \rangle \ \Rightarrow \ i_1 = i_2.$$

Therefore, $f$ is bijective. Besides that, $f$ is decreasing, indeed

$$i_1 \geq i_2 \ \Rightarrow \ -i_1 - \langle \beta, \alpha \rangle \leq -i_2 - \langle \beta, \alpha \rangle \ \Rightarrow \ f(i_1) \leq f(i_2).$$

Then geometrically $s_\alpha$ just reverses the string, since $s_\alpha(\beta + i\alpha) = \beta + f(i)\alpha$. In particular, $f(q) = -r$ so $s_\alpha(\beta + q\alpha) = \beta - r\alpha$. But we also can write $s_\alpha(\beta + q\alpha) = \beta - \langle \beta, \alpha \rangle \alpha - q\alpha$, and then,

$$\beta - \langle \beta, \alpha \rangle \alpha - q\alpha = \beta - r\alpha,$$

and finally we obtain

$$r - q = \langle \beta, \alpha \rangle .$$

We have that an $\alpha$-string through $\beta$ has $q + r + 1$ elements and by Table A.1,

$$\langle \beta + q\alpha, \alpha \rangle \leq 3 \qquad \text{and} \qquad \langle \beta - r\alpha, \alpha \rangle \geq -3.$$

Then

$$\langle \beta, \alpha \rangle + 2q \leq 3 \qquad \text{and} \qquad -\langle \beta, \alpha \rangle + 2r \leq 3.$$

By adding the two equations we have

$$2q + 2r \leq 6 \qquad \Rightarrow \qquad q + r + 1 \leq 4.$$

In particular, we have proved that root strings are of length at most 4.

## A.5 Basis and Weyl chambers

The concept of root systems in Lie algebras is closely related to the notion of a basis, and the Weyl group is an important mathematical object that arises naturally in the study of root systems. A Weyl chamber is a subset of the Euclidean space that is bounded by hyperplanes associated with the roots of the root system, and the chambers are used to classify the elements of the Weyl group. In this section, we will explore the concept of Weyl chambers and their relationship with the basis of a root system.

**Definition A.5.1.** *A subset $\Delta$ of $\mathsf{R}$ is called a basis of the root system $\mathsf{R}$ if it satisfies two properties.*

*(B1) The subset $\Delta$ is a basis of* $\mathsf{E}$*.*

*(B2) Each root $\beta$ can be written as $\beta = \sum k_\alpha \alpha$ with $\alpha \in \Delta$ and integral coefficients $k_\alpha$ all non–negative or all non–positive.*

The roots $\alpha \in \Delta$ are called simple roots. Since $\Delta$ is a basis for $\mathsf{E}$ then $\mathrm{Card}(\Delta) = \ell$ and the expression for $\beta$ in item $(B2)$ of Definition A.5.1 is unique. With this we can define the height of a root relative to the basis $\Delta$ by

$$\mathrm{ht}(\beta) = \sum_{\alpha \in \Delta} k_\alpha.$$

If every $k_\alpha \geq 0$ we say that $\beta$ is positive and write $\beta \succ 0$, if every $k_\alpha \leq 0$ we say that $\beta$ is negative and write $\beta \prec 0$. We denote the collection of all positive roots relative to $\Delta$ by $\mathsf{R}_+$ and the negative ones by $\mathsf{R}_-$, we can write this sets as

$$\mathsf{R}_+ = \{\beta \in \Delta; \beta \succ 0\} \quad \text{and} \quad \mathsf{R}_- = \{\beta \in \Delta; \beta \prec 0\}.$$

Clearly $\mathsf{R}_- = -\mathsf{R}_+$. Indeed, if $\beta \in \mathsf{R}_-$ then

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \text{ with } k_\alpha \leq 0.$$

We have that $-k_\alpha \geq 0$, so

$$\beta = -\sum_{\alpha \in \Delta} (-k_\alpha)\alpha \in -\mathsf{R}_+.$$

If $\beta \in -\mathsf{R}_+$ then

$$\beta = -\sum_{\alpha \in \Delta} k_\alpha \alpha \text{ with } k_\alpha \geq 0.$$

We have that $-k_\alpha \leq 0$, so

$$\beta = \sum_{\alpha \in \Delta} (-k_\alpha)\alpha \in \mathsf{R}_-.$$

Consider $\alpha$ and $\beta$ to be positive roots. If $\alpha + \beta$ is a root then $\alpha + \beta$ is also a positive root. Actually, $\succ$ defines a partial order on $\mathsf{E}$. For $\lambda, \mu \in \mathsf{E}$, we define that $\lambda \prec \mu$ if and only if $\mu - \lambda$ is a sum of positive roots or $\lambda = \mu$.

The next important theorem is about the existence of a basis for every root system. In the examples of rank 1 and 2 the roots labelled as $\alpha$ and $\beta$ form a basis in each case. Notice that the angle between $\alpha$ and $\beta$ is obtuse, which means that $(\alpha, \beta) \leq 0$. As we will see in the next lemma, that is no coincidence.

**Lemma A.5.2.** *Let $\Delta$ be a basis for* $\mathsf{R}$ *and $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$. We have that $(\alpha, \beta) \leq 0$ and $\alpha - \beta$ is not a root.*

*Proof.* Otherwise, $(\alpha, \beta) > 0$. Since $\alpha \neq \beta$, by assumption, and $\alpha \neq -\beta$ since $-\beta \notin \Delta$. Lemma A.4.1 then says that $\alpha - \beta$ is a root, but this violates item $(B2)$ in Definition A.5.1. □

**Theorem A.5.3.** *Any root system* $\mathsf{R}$ *has a basis.*

The proof of Theorem $A.5.3$ is constructive, which means that we will actually show how to construct a basis for any root system. In order to do that we will need the following two lemmas and some definitions.

**Lemma A.5.4.** *The union of finitely many hyperplanes $P_\alpha$ with $\alpha \in \mathsf{R}$ cannot exhaust $\mathsf{E}$.*

*Proof.* Since $\mathsf{R}$ is a finite set, we can let $r$ be the number of elements in $\mathsf{R}$, so we can write $\mathsf{R} = \{\alpha_1, \cdots, \alpha_r\}$. We want to show that

$$\mathsf{E} \neq \bigcup_{i=1}^{r} P_{\alpha_i}.$$

We proceed by induction in $r$, which clearly starts for $r = 1$. As $P_{\alpha_1}$ is a hyperplane we have that $\dim(P_{\alpha_1}) = \dim(E) - 1$, then obviously $\mathsf{E} \neq P_{\alpha_1}$.

Now suppose that

$$\mathsf{E} = \bigcup_{i=1}^{r+1} P_{\alpha_i}.$$

By the induction hypothesis there exists an element $x \in \mathsf{E}$ such that

$$x \notin \bigcup_{i=1}^{r} P_{\alpha_i} \quad \Longrightarrow \quad x \in P_{\alpha_{r+1}}.$$

On the other hand, let $y \in \mathsf{E}$ be such that $y \notin P_{\alpha_{r+1}}$, then

$$y \in \bigcup_{i=1}^{r} P_{\alpha_i}.$$

Consider the set $V = \{x + \lambda y; \lambda \in \mathbb{R}\}$. The set $V$ is infinite, since $\mathbb{R}$ is infinite. There exists an integer $i$ such that $1 \leq i \leq r$ and distinct elements $\lambda, \mu \in \mathbb{R}$, such that $x + \lambda y, x + \mu y \in P_{\alpha_i}$. Therefore,

$$(x + \lambda y) - (x + \mu y) = (\lambda - \mu) y \in P_{\alpha_i}.$$

So $y \in P_{\alpha_i}$ which means that $x \in P_{\alpha_i}$. Which contradicts the choice of $x$. □

**Lemma A.5.5.** *The intersection of positive open half-spaces associated with any basis $B = \{\gamma_1, \cdots, \gamma_\ell\}$ of $\mathsf{E}$ is non-void.*

*Proof.* An open half-space associated with $\gamma_i$ is the set $\Omega_i = \{\beta \in \mathsf{E}; (\beta, \gamma_i) > 0\}$. The intersection of the open half-spaces associated with the basis $B$ is

$$\Omega = \bigcap_{i=1}^{\ell} \Omega_i.$$

Define

$$U_i := \operatorname{span}(\gamma_1, \cdots, \gamma_{i-1}, \gamma_{i+1}, \cdots, \gamma_\ell) \quad \text{and} \quad \delta_i = \operatorname{proj}_{U_i^\perp}(\gamma_i).$$

We claim that $(\delta_i, \gamma_i) > 0$. In fact, let $\{u_1, \cdots, u_\ell\}$ be an orthonormal basis of $\mathsf{E}$. Then

$$\operatorname{proj}_{U_i^\perp}(\gamma_i) = \sum_{i-1}^{\ell} (u_i, \gamma_i) u_i$$

and

$$(\delta_i, \gamma_i) = \left(\mathrm{proj}_{U_i^\perp}(\gamma_i), \gamma_i\right) = \left(\sum_{i-1}^{\ell}(u_i, \gamma_i)u_i, \gamma_i\right) = \sum_{i-1}^{\ell}(u_i, \gamma_i)(u_i, \gamma_i) = \sum_{i-1}^{\ell}(u_i, \gamma_i)^2 \geq 0.$$

If $(\delta_i, \gamma_i) = 0$ then $\gamma_i \in U_i$ which is impossible by the definition of $U_i$, so $(\delta_i, \gamma_i) > 0$ which proves the claim. Note further that $(\delta_j, \gamma_i) = 0$, for $j \neq i$, since $\gamma_i \in U_j$.

Finally, set $\delta = \sum \delta_j$. Then $\delta \in \Omega$ since

$$(\delta, \gamma_i) = \left(\sum_{i-1}^{\ell} \delta_j, \gamma_i\right) = \sum_{i-1}^{\ell}(\delta_j, \gamma_i) = (\delta_i, \gamma_i) > 0.$$

□

Given $\gamma \in \mathsf{E}$ define $\mathsf{R}_+(\gamma) = \{\alpha \in \mathsf{R}; (\gamma, \alpha) > 0\}$ to be the set of roots lying on the "positive" side of the hyperplane orthogonal to $\gamma$.

**Definition A.5.6.** *We say that $\gamma \in \mathsf{E}$ is regular if*

$$\gamma \in \mathsf{E} - \bigcup_{\alpha \in \mathsf{R}} P_\alpha$$

*and singular otherwise.*

Note that Lemma A.5.4 implies the existence of regular elements. If $\gamma$ is regular it is clear that $\mathsf{R} = \mathsf{R}_+(\gamma) \cup -\mathsf{R}_+(\gamma)$.

**Definition A.5.7.** *An element $\alpha \in \mathsf{R}_+(\gamma)$ is said to be decomposable if $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in \mathsf{R}_+(\gamma)$ and indecomposable otherwise.*

The following result guarantees the existence of a basis for a root system, and it is actually the proof of Theorem A.5.3.

**Theorem A.5.8.** *Let $\gamma \in \mathsf{E}$ be regular. Then the set $\Delta(\gamma)$ of all indecomposable roots in $\mathsf{R}_+(\gamma)$ is a basis of $\mathsf{R}$, and every basis is obtainable is this manner.*

*Proof.* The proof is proceeded in steps.

(1) Each root in $\mathsf{R}_+(\gamma)$ is a non-negative $\mathbb{Z}$-linear combination of $\Delta(\gamma)$. Otherwise, some $\alpha \in \mathsf{R}_+(\gamma)$ cannot be so written, choose $\alpha$ so that $(\gamma, \alpha)$ is as small as possible. Obviously $\alpha$ itself cannot be in $\Delta(\gamma)$, since then $\alpha$ would be a $\mathbb{Z}$-linear combination of $\Delta(\gamma)$. So $\alpha$ is decomposable, which means $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \mathsf{R}_+(\gamma)$, whence

$$(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2).$$

But each of the $(\gamma, \beta_i)$ is positive since $\beta_i \in \mathsf{R}_+(\gamma)$ and $(\gamma, \beta_i) < (\gamma, \alpha)$ for $i = 1, 2$. Therefore, to avoid contradicting the minimality of $(\gamma, \alpha)$ we have that $\beta_1$ and $\beta_2$ must each be a nonnegative $\mathbb{Z}$-linear combination of $\Delta(\gamma)$, whence $\alpha$ also is. This contradiction proves the original assertion.

(2) If $\alpha, \beta \in \Delta(\gamma)$ then $(\alpha, \beta) \leq 0$ unless $\alpha = \beta$. Otherwise, $\alpha - \beta$ is a root and so either

$$\alpha - \beta \in \mathsf{R}_+(\gamma) \text{ or } \beta - \alpha \in \mathsf{R}_+(\gamma).$$

In the first case, $\alpha = \beta + (\alpha - \beta)$, which says that $\alpha$ is decomposable. In the second case, $\beta = \alpha + (\beta - \alpha)$ is decomposable. This contradicts the assumption that $\alpha, \beta$ are indecomposable.

(3) The set $\Delta(\gamma)$ is linearly independent. Suppose $\sum r_\alpha \alpha = 0$, where $\alpha \in \Delta(\gamma)$ and $r_\alpha \in \mathbb{R}$. Separating the indices $\alpha$ for which $r_\alpha > 0$ from those for which $r_\alpha < 0$, we can rewrite this as

$$\sum u_\alpha \alpha = \sum t_\beta \beta \text{ with } u_\alpha, t_\beta > 0.$$

Note that the sets of $\alpha$'s and $\beta$'s are disjoint. Consider $\epsilon = \sum u_\alpha \alpha = \sum t_\beta \beta$. Then

$$(\epsilon, \epsilon) = \sum u_\alpha t_\beta (\alpha, \beta) \leq 0$$

by step (2). We know that $(\epsilon, \epsilon) \geq 0$ and this forces $(\epsilon, \epsilon) = 0$ and then $\epsilon = 0$. Therefore,

$$0 = (\gamma, \epsilon) = \sum u_\alpha (\gamma, \alpha),$$

but $(\gamma, \alpha) > 0 \ \forall \ \alpha \in \Delta(\gamma)$. As the sum is zero then each $u_\alpha = 0$. Similarly, we can get to all $t_\beta = 0$. Which is a contradiction since we started with $u_\alpha, t_\beta > 0$. There is not any $r_\alpha > 0$ or $r_\alpha < 0$ then $r_\alpha = 0$ for all $\alpha \in \Delta(\gamma)$. Whence the set is linearly independent.

(4) Finally, we can show that $\Delta(\gamma)$ is a basis of $\mathsf{R}$. Since $\mathsf{R} = \mathsf{R}_+(\gamma) \cup -\mathsf{R}_+(\gamma)$, the requirement $(B2)$ of Definition A.5.1 is satisfied thanks to step (1). Also, from step (1) it follows that $\Delta(\gamma)$ spans $\mathsf{E}$, which combined with step (3) yields $(B1)$ of Definition A.5.1.

(5) Each basis $\Delta$ of $\mathsf{R}$ has the form $\Delta(\gamma)$ for some regular $\gamma \in \mathsf{E}$. Given $\Delta$ let $\gamma \in \mathsf{E}$ so that $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$. This is possible since the intersection of positive open half-spaces associated with any basis of $\mathsf{E}$ is non-void by Lemma A.5.5. In view of $(B2)$ of Definition A.5.1, $\gamma$ is regular and if $\beta \in \mathsf{R}_+$ then $\beta = \sum k_\alpha \alpha$ with $\alpha \in \Delta$ and $k_\alpha > 0$. Then we can write

$$(\gamma, \beta) = (\gamma, \sum k_\alpha \alpha) = \sum k_\alpha (\gamma, \alpha) > 0,$$

so $\beta \in \mathsf{R}_+(\gamma)$ and

$$\mathsf{R}_+ \subset \mathsf{R}_+(\gamma). \tag{A.6}$$

The same occurs for $\mathsf{R}_- \subset -\mathsf{R}_+(\gamma)$. We know we can write

$$\mathsf{R} = \mathsf{R}_+ \cup -\mathsf{R}_+ = \mathsf{R}_+(\gamma) \cup -\mathsf{R}_+(\gamma).$$

By (A.6) we have that $\mathsf{R}_+ = \mathsf{R}_+(\gamma)$ and $-\mathsf{R}_+ = -\mathsf{R}_+(\gamma)$. As $\mathsf{R}_+ = \mathsf{R}_+(\gamma)$ we have that the set $\Delta$ consists of indecomposable elements, which means that $\Delta \subset \Delta(\gamma)$. As we have $\text{Card}(\Delta) = \text{Card}(\Delta(\gamma)) = \ell$ then $\Delta = \Delta(\gamma)$.

□

**Remark A.5.9.** *The argument given in item* (3) *of Theorem A.5.8 actually shows that any set of vectors lying strictly on one side of a hyperplane in* $\mathsf{E}$ *and forming pairwise obtuse angles must be linearly independent.*

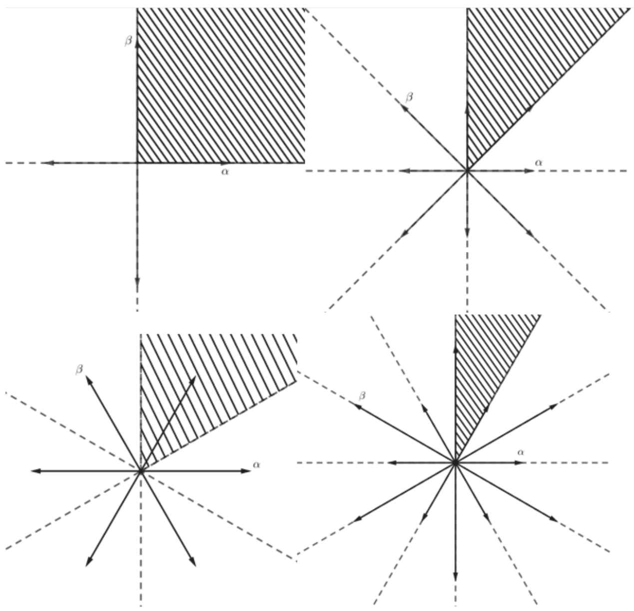
The hyperplanes $P_\alpha$ with $\alpha \in \mathsf{R}$, partition $\mathsf{E}$ into finitely many regions. The connected components of

$$\mathsf{E} - \bigcup_{\alpha} P_\alpha$$

are called the open Weyl chambers of $\mathsf{E}$ and denoted by $\mathscr{C}$. Each regular $\gamma \in \mathsf{E}$ belongs to precisely one Weyl chamber, denoted by $\mathscr{C}(\gamma)$. Let $\gamma' \in E$ be also regular then we have that $\mathscr{C}(\gamma) = \mathscr{C}(\gamma')$ if and only if $\gamma$ and $\gamma'$ lie on the same side of each hyperplane $P_\alpha$ with $\alpha \in \mathbb{R}$. More specifically, $\mathsf{R}_+(\gamma) = \mathsf{R}_+(\gamma')$, or equivalently $\Delta(\gamma) = \Delta(\gamma')$. This shows that Weyl chambers are in natural 1–1 correspondence with basis of $\mathsf{R}$.

If $\Delta = \Delta(\gamma)$ then we write $\mathscr{C}(\Delta) = \mathscr{C}(\gamma)$, and we say that this is the fundamental Weyl chamber of $\Delta$. We have that $\mathscr{C}(\Delta)$ is the open convex set consisting of all $\gamma \in \mathsf{E}$ which satisfy $(\gamma, \alpha) > 0$ with $\alpha \in \Delta$. In rank 2 the Weyl chambers are drawn as seen in Figure A.4. The shaded chamber is the one fundamental relative to the basis $\{\alpha, \beta\}$.

Figure A.4: Weyl chamber for rank 2




The Weyl group sends one Weyl chamber onto another, explicitly, for $s \in \mathscr{W}$ and $\gamma \in \mathsf{E}$ regular we have

$$s(\mathscr{C}(\gamma)) = \mathscr{C}(s\gamma).$$

On the other hand, $\mathscr{W}$ permutes basis $s \in \mathscr{W}$ sends $\Delta$ to $s(\Delta)$, which is again a basis. These two actions of $\mathscr{W}$ are in fact compatible with the above correspondence between Weyl chambers and basis. We have $s(\Delta(\gamma)) = \Delta(s\gamma)$, since $(s\gamma, s\alpha) = (\gamma, \alpha)$.

**Definition A.5.10.** *Given a root system* $\mathsf{R}$ *we define the root lattice*

$$L = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \simeq \mathbb{Z}^I.$$

*We also define the positive and negative root lattice as*

$$L_+ = Z_+^I \text{ and } L_- = Z_-^I. \tag{A.7}$$

## A.6 Lemmas on simple roots

Let $\Delta$ be a fixed basis of a root system $\mathsf{R}$. We prove here several very useful lemmas about the behavior of simple roots.

**Lemma A.6.1.** *If* $\alpha \in \mathsf{R}_+$ *and* $\alpha \notin \Delta$ *then* $\alpha - \beta$ *is a positive root for some* $\beta \in \Delta$.

*Proof.* If $(\alpha, \beta) \leq 0$ for all $\beta \in \Delta$ then $\Delta \cup \{\alpha\}$ is a linearly independent set by Theorem A.5.8 item (3) and this is an absurd, since $\Delta$ is already a basis of $\mathsf{E}$. So $(\alpha, \beta) > 0$ for some $\beta \in \Delta$ and then $\alpha - \beta \in \mathsf{R}$ by Lemma A.4.1, which applies since $\beta$ cannot be proportional to $\alpha$. Consider

$$\alpha = \sum_{\gamma \in \Delta} k_\gamma \gamma \quad \text{with all} \quad k_\gamma \geq 0.$$

Then some $k_\gamma > 0$ for $\gamma \neq \beta$ since $\alpha \neq \beta$. Subtracting $\beta$ from $\alpha$ yields a $\mathbb{Z}$-linear combination of simple roots with at least one positive coefficient, the one from $\gamma \neq \beta$ continues positive. This forces all coefficients to be positive, thanks to the uniqueness of expression in $(B2)$ of Definition A.5.1. □

**Corollary A.6.2.** *Each* $\beta \in \mathsf{R}_+$ *can be written in the form* $\alpha_1 + \cdots + \alpha_k$, *with* $\alpha_i \in \Delta$ *not necessarily distinct, in such a way that each partial sum* $\alpha_1 \cdots + \alpha_i$ *is a root.*

*Proof.* This proof will be done by induction on the height of $\beta$. If $\mathrm{ht}(\beta) = 1$ there is nothing to prove.

Consider $\delta \in \mathsf{R}_+$ with $\mathrm{ht}(\delta) > 1$ and write $\delta = \sum k_\alpha \alpha$ with $\alpha \in \Delta$ and $k_\alpha \geq 0$. By Lemma A.6.1 we can find $\alpha_r \in \Delta$ such that $\delta - \alpha_r \in \mathsf{R}_+$ and $\mathrm{ht}(\delta - \alpha_r) = \mathrm{ht}(\delta) - 1$. Then $\delta - \alpha_r = \alpha_1 + \cdots + \alpha_k$ with $\alpha_i \in \Delta$ not necessarily distinct, in such a way that each partial sum $\alpha_1 \cdots + \alpha_i$ is a root. So $\delta = \alpha_1 + \cdots + \alpha_k + \alpha_r \in \mathsf{R}_+$. □

**Lemma A.6.3.** *Let* $\alpha$ *be a simple root. Then* $s_\alpha$ *permutes the positive roots other than* $\alpha$.

*Proof.* Let $\beta \in \mathsf{R}_+ - \{\alpha\}$ then $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$ com $k_\gamma \in \mathbb{Z}^+$. It is clear that $\beta$ is not proportional to $\alpha$, since we took $\beta \neq \alpha$ and $-\alpha \notin \mathsf{R}_+$. Therefore, $k_\gamma \neq 0$ for some $\gamma \neq \alpha$. We have that

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha.$$

So the coefficient of $\gamma$ in $s_\alpha(\beta)$ is still $k_\gamma$, since the only coefficient that changes is the one from $\alpha$. In other words, $s_\alpha(\beta)$ has at least one positive coefficient, relative to $\Delta$, forcing it to be positive. Moreover, $s_\alpha(\beta) \neq \alpha$ since $\alpha$ is the image of $-\alpha$. □

**Corollary A.6.4.** *Set*

$$\delta = \frac{1}{2}\sum_{\beta \succ 0} \beta.$$

*Then* $s_\alpha(\delta) = \delta - \alpha$ *for all* $\alpha \in \Delta$.

*Proof.* Write $\delta = \delta' + \frac{1}{2}\alpha$. We have that $s_\alpha(\delta') = \delta'$ since $s_\alpha$ is a bijection and

$$s_\alpha\left(\frac{1}{2}\alpha\right) = -\frac{1}{2}\alpha.$$

Then

$$s_\alpha(\delta) = s_\alpha\left(\delta' + \frac{1}{2}\alpha\right) = s_\alpha(\delta') + \frac{1}{2}s_\alpha(\alpha) = \delta' - \frac{1}{2}\alpha = \delta' + \frac{1}{2}\alpha - \alpha = \delta - \alpha.$$

□

**Lemma A.6.5.** *Let* $\alpha_1, \cdots, \alpha_t \in \Delta$, *not necessarily distinct, and write* $s_i = s_{\alpha_i}$. *If* $s_1 \cdots s_{t-1}(\alpha_t) \in \mathsf{R}_-$ *then* $s_1 \cdots s_t = s_1 \cdots s_{r-1}s_{r+1} \cdots s_{t-1}$, *for some index* $1 \leq r < t$.

*Proof.* Write $\beta_i = s_{i+1} \cdots s_{t-1}(\alpha_t)$ with $0 \leq i \leq t-2$, and $\beta_{t-1} = \alpha_t$. Since $\beta_0 = s_1 \cdots s_{t-1}(\alpha_t) \prec 0$ and $\beta_{t-1} \succ 0$ we can find the smallest index $r$ for which $\beta_r \succ 0$. Then $s_r(\beta_r) = s_r s_{r+1} \cdots s_{t-1}(\alpha_t) = \beta_{r-1} \prec 0$ and $\beta_r = \alpha_r$ by Lemma A.6.1. In general if $s \in \mathscr{W}$ that implies $s_{s(\alpha)} = s s_\alpha s^{-1}$ by Lemma A.2.2. So in particular,

$$s_r = s_{\alpha_r} = s_{\beta_r} = s_{s_{r+1}\cdots s_{t-1}(\alpha_t)} = (s_{r+1} \cdots s_{t-1})s_t(s_{t-1} \cdots s_{r+1}).$$

Multiplying both sides by $(s_1 \cdots s_{r-1})$ on the left side and by $(s_{r+1} \cdots s_t)$ on the right side we have,

$$\begin{aligned}(s_1 \cdots s_{r-1})s_r(s_{r+1} \cdots s_t) &= (s_1 \cdots s_{r-1})(s_{r+1} \cdots s_{t-1})s_t(s_{t-1} \cdots s_{r+1})(s_{r+1} \cdots s_t) \\ s_1 \cdots s_t &= s_1 \cdots s_{r-1}s_{r+1} \cdots s_{t-1}.\end{aligned}$$

□

**Corollary A.6.6.** *If* $s = s_1 \cdots s_t$ *is an expression for* $s \in \mathscr{W}$ *in terms of reflections corresponding to simple roots, with* $t$ *as small as possible, then* $s(\alpha_t) \prec 0$.

*Proof.* We have that

$$s(\alpha_t) = s_1 \cdots s_t(\alpha_t) = s_1 \cdots s_{t-1}(-\alpha_t) = -s_1 \cdots s_{t-1}(\alpha_t).$$

If $s_1 \cdots s_{t-1}(\alpha_t) \prec 0$ then by Lemma A.6.5 we could write

$$s_1 \cdots s_t = s_1 \cdots s_{r-1}s_{r+1} \cdots s_{t-1}.$$

Which contradicts the minimality of $t$. Therefore, $s_1 \cdots s_{t-1}(\alpha_t) \succ 0$ and $s(\alpha_t) = -s_1 \cdots s_{t-1}(\alpha_t) \prec 0$. □

## A.7 The Weyl group

The Weyl group is a fundamental mathematical object that arises naturally in the study of Lie algebras and Lie groups. It is a group of symmetries that preserves the root system structure and plays a central role in the classification of semisimple Lie algebras. In this section, we will explore the structure and properties of the Weyl group in detail. We will examine the relationship between the Weyl group and the root system, including how the Weyl group acts on the roots and how it generates all possible bases of the root system by permutating the basis of $\mathsf{R}$ or, equivalently, the Weyl chambers.

**Theorem A.7.1.** *Let $\Delta$ be a basis for the root system $\mathsf{R}$.*

*(a) If $\gamma \in \mathsf{E}$ and $\gamma$ is regular, then there exists $s \in \mathscr{W}$ such that $(s(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$. So $\mathscr{W}$ acts transitively on Weyl chambers.*

*(b) If $\Delta'$ is another basis for $\mathsf{R}$, then $s(\Delta') = \Delta$ for some $s \in \mathscr{W}$. So $\mathscr{W}$ acts transitively on basis.*

*(c) If $\alpha \in \mathsf{R}$ is any root, there exists $s \in \mathscr{W}$ such that $s(\alpha) \in \Delta$.*

*(d) The Weyl group $\mathscr{W}$ is generated by the reflections $s_\alpha$ with $\alpha \in \Delta$.*

*(e) If $s(\Delta) = \Delta$ and $s \in \mathscr{W}$, then $s = 1$. So $\mathscr{W}$ acts simply transitively on basis.*

*Proof.* Let $\mathscr{W}'$ be the subgroup of $\mathscr{W}$ generated by all $s_\alpha$ with $\alpha \in \Delta$. We shall prove $(a)$ to $(c)$ for $\mathscr{W}'$, then deduce that $\mathscr{W}' = \mathscr{W}$.

(a) Write
$$\delta = \frac{1}{2} \sum_{\alpha \succ 0} \alpha,$$
and choose $s \in \mathscr{W}'$ for which $(s(\gamma), \delta)$ is maximal in the set $\{(\omega(\gamma), \delta); \omega \in \mathscr{W}\}$. If $\alpha$ is simple, then $s_\alpha s$ is also in $\mathscr{W}'$. The choice of $s$ implies that
$$(s(\gamma), \delta) \geq (s_\alpha s(\gamma), \delta) = (s(\gamma), s_\alpha(\delta)) = (s(\gamma), \delta - \alpha) = (s(\gamma), \delta) - (s(\gamma), \alpha)$$
where the equality is true by Corollary A.6.4. This forces $(s(\gamma), \alpha) \geq 0$ for all $\alpha \in \Delta$. Since $\gamma$ is regular, we cannot have $(s(\gamma), \alpha) = 0$ for any $\alpha \in \mathsf{R}$, since then $\gamma$ would be orthogonal to $s^{-1}\alpha \in \mathsf{R}$. So all the inequalities are strict and $(s(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$. Therefore, $s(\gamma)$ lies in the fundamental Weyl chamber $\mathscr{C}(\Delta)$ and $s$ sends $\mathscr{C}(\gamma)$ to $\mathscr{C}(\Delta)$ as desired.

(b) Since $\mathscr{W}'$ permutes the Weyl chambers, by item $(a)$, it also permutes the basis of $\mathsf{R}$, transitively.

(c) In view of item $(b)$, it suffices to prove that each root belongs to at least one basis. Since the only roots proportional to $\alpha$ are $\pm\alpha$, the hyperplanes $P_\beta$ with $\beta \neq \pm\alpha$ are distinct from $P_\alpha$. Then there exists $\gamma \in P_\alpha$ such that $\gamma \notin P_\beta$ for all $\beta \neq \pm\alpha$. Choose $\gamma'$ close enough to $\gamma$ so that $(\gamma', \alpha) = \epsilon > 0$ while

$|(\gamma', \beta)| > \epsilon$ for all $\beta \neq \pm\alpha$. Then $\alpha \in \mathsf{R}_+(\gamma')$. If $\alpha \notin \Delta(\gamma')$ then $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \mathsf{R}_+(\gamma')$. And then $(\gamma', \beta_i) > \epsilon$, so

$$(\gamma', \alpha) = (\gamma', \beta_1) + (\gamma', \beta_2) > 2\epsilon,$$

which is a contradiction since $(\gamma', \alpha) = \epsilon$. Therefore, $\alpha \in \Delta(\gamma')$.

(d) To prove that $\mathscr{W}' = \mathscr{W}$, it is enough to show that each reflection $s_\alpha$ with $\alpha \in \mathsf{R}$ is in $\mathscr{W}'$. Given any $\alpha \in \mathsf{R}$ there exists $s \in \mathscr{W}'$ such that $\beta = s(\alpha) \in \Delta$ by item $(c)$. Then $s_\beta = s_{s(\alpha)} = s s_\alpha s^{-1}$, so $s_\alpha = s^{-1} s_\beta s \in \mathscr{W}'$.

(e) Let $s(\Delta) = \Delta$ and $s \neq 1$. If $s$ is written minimally as a product of one or more simple reflections, which is possible thanks to item $(d)$, then the Corollary A.6.6 gives $s(\alpha_t) \prec 0$. As $\alpha_t \in \Delta$ and $s(\Delta) = \Delta$ then $s(\alpha_t) \succ 0$ which is a contradiction.

□

We defined $\mathscr{W}$ to be the group generated by reflections $s_\alpha$ for all $\alpha \in \mathsf{R}$. The Theorem A.7.1 showed that in fact $\mathscr{W}$ is generated by the reflections $s_\alpha$ with $\alpha \in \Delta$. Recall that we can write $s = s_{\alpha_1} \cdots s_{\alpha_t}$ with $\alpha_i \in \Delta$, when $t$ is minimal we say that the expression is reduced. Then we can define the length of a reflection $s \in \mathscr{W}$ relative to $\Delta$ as $l(s) = t$. Define

$$N(s) = \{\alpha \in \mathsf{R}_+; s(\alpha) \prec 0\} \quad \text{and} \quad n(s) = \mid N(s) \mid . \tag{A.8}$$

**Lemma A.7.2.** *For all $s \in \mathscr{W}$ we have that $l(s) = n(s)$ for $n(s)$ as in* (A.8).

*Proof.* We proceed by induction on $l(s)$.

If $l(s) = 0$ then $s = 1$ and $n(s) = 0$, since for every positive root $\alpha$ we have that $s(\alpha) = \alpha$.

Assume the lemma is true for all $\tau \in \mathscr{W}$ with $l(\tau) < l(s)$. Write the reduced expression of $s$ as $s = s_{\alpha_1} \cdots s_{\alpha_t}$ and set $\alpha = \alpha_t$. By Corollary A.6.6 we have that $s(\alpha) \prec 0$. Then Lemma A.6.3 implies that $n(ss_\alpha) = n(s) - 1$ since $s_\alpha$ sends all positive roots different from $\alpha$ to a positive root. On the other hand, since $\alpha = \alpha_t$, we have

$$ss_\alpha = s_1 \cdots s_t s_\alpha = s_1 \cdots s_t s_t = s_1 \cdots s_{t-1}.$$

Then $l(ss_\alpha) = l(s) - 1 < l(s)$. By induction hypothesis we have $l(ss_\alpha) = n(ss_\alpha)$. Combining these statements, we obtain

$$l(ss_\alpha) = l(s) - 1 = n(s) - 1 = n(ss_\alpha).$$

Hence, $l(s) = n(s)$. □

**Lemma A.7.3.** *Let $\beta \in \mathscr{C}(\Delta)$. If $s(\beta) = \beta$ for some $s \in \mathscr{W}$ then $s = 1$.*

*Proof.* If $\beta \in \mathscr{C}(\Delta)$ then for every $\alpha \in \Delta$ we have $(\beta, \alpha) > 0$. Since $s(\beta) = \beta$ we have

$$0 < (\beta, \alpha) = (s(\beta), s(\alpha)) = (\beta, s(\alpha)) \ \forall\ \alpha \in \Delta.$$

If $s \neq 1$ we can write $s$ in terms of reflections corresponding to simple roots $s = s_1 \cdots s_t$ with $t$ minimal. As $\alpha_t \in \Delta$, we have that $(\beta, s(\alpha_t)) > 0$ and by Corollary A.6.6 we know that $s(\alpha_t) \prec 0$. This means that $s(\alpha_t) = \sum_{\alpha \in \Delta} k_\alpha \alpha$ with every $k_\alpha \leq 0$. Then

$$0 < (\beta, s(\alpha_t)) = \left(\beta, \sum_{\alpha \in \Delta} k_\alpha \alpha\right) = \sum_{\alpha \in \Delta} k_\alpha (\beta, \alpha).$$

By hypothesis $(\beta, \alpha) > 0$ so each term $k_\alpha(\beta, \alpha) \leq 0$ which is a contradiction, since this would mean that

$$\sum_{\alpha \in \Delta} k_\alpha (\beta, \alpha) \leq 0.$$

Therefore, $s = 1$. □

We saw in item $(e)$ of Theorem A.7.1 that $\mathscr{W}$ acts simply transitively on Weyl chambers. Let us get a better understanding of that. The next lemma shows that the closure $\overline{\mathscr{C}(\Delta)}$ of the fundamental Weyl chamber relative to $\Delta$ is a fundamental domain for the action of $\mathscr{W}$ on $\mathsf{E}$. Which means that each vector in $\mathsf{E}$ is $\mathscr{W}$ conjugate to precisely one point of this set.

**Lemma A.7.4.** *Let $\lambda, \mu \in \overline{\mathscr{C}(\Delta)}$. If $s(\lambda) = \mu$ for some $s \in \mathscr{W}$ then $s$ is a product of simple reflections which fix $\lambda$. In particular $\lambda = \mu$.*

*Proof.* We proceed by induction on $l(s)$. If $l(s) = 0$ then $s = 1$ and $s(\lambda) = \lambda = \mu$ so $s = 1$ is written as simple reflections that fixes $\lambda$.

Let $l(s) > 0$. By Lemma A.7.2 $n(s) > 0$ and $s$ must send some positive root to a negative root, then $s$ cannot send all simple roots to positive roots. Say $s(\alpha) \prec 0$ for $\alpha \in \Delta$. Since $\lambda, \mu \in \overline{\mathscr{C}(\Delta)}$ we have $(\mu, \alpha) \geq 0$ and $(\lambda, \alpha) \geq 0$. If we write

$$s(\alpha) = \sum_{\beta \in \Delta} k_\beta \beta \quad \text{with} \quad k_\beta \leq 0.$$

Then,

$$(\mu, s(\alpha)) = (\mu, \sum k_\beta \beta) = \sum k_\beta (\mu, \beta) \leq 0,$$

since $k_\beta \leq 0$ and $(\mu, \beta) \geq 0 \;\forall\; \beta \in \Delta$. Now

$$0 \geq (\mu, s(\alpha)) = (s^{-1}(\mu), \alpha) = (\lambda, \alpha) \geq 0,$$

and this forces $(\lambda, \alpha) = 0$. Therefore, $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha = \lambda$ and hence $(ss_\alpha)(\lambda) = \mu$. By Lemma A.6.3 and Lemma A.7.3, $l(ss_\alpha) = l(s) - 1$. Then by the induction hypothesis $ss_\alpha = s_{\alpha_1} \cdots s_{\alpha_{t-1}}$ is such that $s_{\alpha_i}$ fixes $\lambda$, so $s = s_{\alpha_1} \cdots s_{\alpha_{t-1}} s_\alpha$ and $s_{\alpha_i}, s_\alpha$ fixes $\lambda$. □

## A.8 Irreducible root systems

**Definition A.8.1.** *The root system $\mathsf{R}$ is called reducible if we can write*

$$\mathsf{R} = \mathsf{R}_1 \cup \mathsf{R}_2$$

*with $(\mathsf{R}_1, \mathsf{R}_2) = 0$ and $\mathsf{R}_1 \neq 0, \mathsf{R}_2 \neq 0$. We say that $\mathsf{R}$ is irreducible otherwise.*

We can do a similar definition for the basis of a root system.

**Definition A.8.2.** *The basis* $\Delta$ *for a root system* $\mathsf{R}$ *is called reducible if we can write*

$$\Delta = \Delta_1 \cup \Delta_2$$

*with* $(\Delta_1, \Delta_2) = 0$ *and* $\Delta_1 \neq 0, \Delta_2 \neq 0$. *We say that* $\Delta$ *is irreducible otherwise.*

In the examples we gave in section A.3 we can see that $A_1, A_2, B_2, G_2$ are irreducible, while $A_1 \times A_1$ is reducible.

**Proposition A.8.3.** *Let* $\Delta$ *be the basis for the root system* $\mathsf{R}$. *Then* $\mathsf{R}$ *is irreducible if and only if* $\Delta$ *is irreducible.*

*Proof.* Suppose that $\mathsf{R}$ is irreducible and $\Delta = \Delta_1 \cup \Delta_2$ with $(\Delta_1, \Delta_2) = 0$.

For each root $\alpha \in \mathsf{R}$ there exists $s \in \mathscr{W}$ such that $s(\alpha) \in \Delta$ by Theorem A.7.1 item $(c)$. Let $\mathsf{R}_i$ be the set of roots conjugate to a root in $\Delta_i$, for $i = 1, 2$. Then $\mathsf{R} = \mathsf{R}_1 \cup \mathsf{R}_2$. Recall that $\mathscr{W}$ is generated by simple reflections $s_\alpha$ with $\alpha \in \Delta$.

If $\alpha \in \Delta_i$ and $\beta \in \Delta_j$, with $i \neq j$, then $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta$, since $(\beta, \alpha) = 0$. Hence, each root of $\mathsf{R}_i$ is obtained from $\Delta_i$ by adding or subtracting elements of $\Delta_i$. Therefore, $(\mathsf{R}_1, \mathsf{R}_2) = 0$ and that is an absurd since $\mathsf{R}$ is irreducible.

Suppose that $\Delta$ is irreducible and $\mathsf{R} = \mathsf{R}_1 \cup \mathsf{R}_2$ with $(\mathsf{R}_1, \mathsf{R}_2) = 0$.

If $\Delta \not\subset \mathsf{R}_1$ and $\Delta \not\subset \mathsf{R}_2$ then $\Delta = \Delta_1 \cup \Delta_2$ with $\Delta_1 = \Delta \cap \mathsf{R}_1$ and $\Delta_2 = \Delta \cap \mathsf{R}_2$. Therefore, $(\Delta_1, \Delta_2) = 0$ and that contradicts the irreducibility of $\Delta$.

If $\Delta \subset \mathsf{R}_1$ then $(\Delta, \mathsf{R}_2) = 0$. Recall that $\Delta$ generates $\mathsf{E}$, so we have that $(\mathsf{E}, \mathsf{R}_2) = 0$, which is an absurd since $\mathsf{R}$ is a finite set. If $\Delta \subset \mathsf{R}_2$ the same contradiction happens. $\square$

**Lemma A.8.4.** *Let* $\mathsf{R}$ *be irreducible. Relative to the partial order* $\prec$, *there is a unique maximal root* $\theta$. *In particular,* $\alpha \neq \theta$ *implies that* $ht(\alpha) < ht(\theta)$, *and* $(\theta, \alpha) \geq 0$ *for all* $\alpha \in \Delta$. *If*

$$\theta = \sum_{\alpha \in \Delta} k_\alpha \alpha$$

*then all* $k_\alpha > 0$ *for every* $\alpha \in \Delta$.

*Proof.* Let

$$\theta = \sum_{\alpha \in \Delta} k_\alpha \alpha \in \mathsf{R}$$

be maximal with respect to $\prec$. We have that $\theta \succ 0$, indeed if $\theta \prec 0$ then $-\theta \in \mathsf{R}_+$ and $-\theta \succ \theta$.

So for $\theta = \sum k_\alpha \alpha$ it follows that $k_\alpha \geq 0$. Then we can write

$$\Delta_1 = \{\alpha \in \Delta; k_\alpha > 0\} \text{ and } \Delta_2 = \{\alpha \in \Delta; k_\alpha = 0\}.$$

Note that $\Delta = \Delta_1 \cup \Delta_2$ is a disjoint union. Suppose $\Delta_2 \neq 0$ and let $\gamma \in \Delta_2$. Then by Lemma A.5.2 $(\gamma, \alpha) \leq 0$ for all $\alpha \in \Delta - \{\gamma\}$. So

$$(\gamma, \theta) = \Big(\gamma, \sum k_\alpha \alpha\Big) = \sum k_\alpha (\gamma, \alpha) = \sum_{\alpha \neq \gamma} k_\alpha (\gamma, \alpha) \leq 0$$

since $k_\gamma = 0$ and $k_\alpha \geq 0$ for all $\alpha \neq \gamma$, then $(\gamma, \theta) \leq 0$. Since $\mathsf{R}$ is irreducible, there exists $\alpha \in \Delta_1$ such that $(\gamma, \alpha) < 0$, whence $(\gamma, \theta) < 0$. Then by Lemma A.4.1 $\theta + \gamma$ is a root, contradicting the maximality of $\theta$. Therefore, $\Delta_2$ is empty and all $k_\alpha > 0$.

This argument also shows that

$$(\alpha, \theta) \geq 0 \ \forall \ \alpha \in \Delta.$$

Since otherwise we would have $\alpha + \theta$ a root, which contradicts the maximality of $\theta$.

Now let $\beta$ be another maximal root. The preceding argument applies to $\beta$ as well, so $\beta$ involves at least one $\alpha \in \Delta$, with positive coefficient, for which $(\alpha, \theta) > 0$. It follows that $(\beta, \theta) > 0$, and either $\theta = \beta$ or $\theta - \beta$ is a root. If $\theta - \beta$ is a root, then either $\theta - \beta \in \mathsf{R}_-$ and $\theta \prec \beta$, which contradicts the maximality of $\theta$ or else $\theta - \beta \in \mathsf{R}_+$ and $\beta \prec \theta$, which contradicts the maximality of $\beta$. Therefore, $\theta$ is unique. □

**Lemma A.8.5.** *Let* $\mathsf{R}$ *be irreducible. Then* $\mathscr{W}$ *acts irreducibly on* $\mathsf{E}$*. In particular, the* $\mathscr{W}$*-orbit of a root* $\alpha$ *spans* $\mathsf{E}$*.*

*Proof.* Given a root $\alpha \in \mathsf{R}$, the $\mathscr{W}-$orbit of the root is the set

$$\mathscr{W}(\alpha) = \{s(\alpha); s \in \mathscr{W}\}.$$

The span of $\mathscr{W}(\alpha)$ is a nonzero $\mathscr{W}$-invariant subspace of $\mathsf{E}$. Indeed, for all $\beta \in \mathscr{W}(\alpha)$ we have that $\beta = s(\alpha)$ for some $s \in \mathscr{W}$. Then for all $s' \in \mathscr{W}$ we have that $s'(\beta) = s'(s(\alpha)) = s's(\alpha) \in \mathscr{W}(\alpha)$ since $s's \in \mathscr{W}$, so the second statement follows from the first.

As to the first, let $\mathsf{E}'$ be a nonzero subspace of $\mathsf{E}$ invariant under $\mathscr{W}$. If $\mathsf{E}''$ is the orthogonal complement of $\mathsf{E}'$ then for all $\beta \in \mathsf{E}''$ we have $(\beta, \mathsf{E}') = 0$. For every $s \in \mathscr{W}$ we have $s(\mathsf{E}') \subset \mathsf{E}'$ since $\mathsf{E}'$ is $\mathscr{W}$-invariant and then $s(\mathsf{E}') = \mathsf{E}'$ since $s$ is an isomorphism. Therefore,

$$(s(\beta), s(\mathsf{E}')) = (\beta, \mathsf{E}') = 0 \quad \text{and} \quad (s(\beta), s(\mathsf{E}')) = (s(\beta), \mathsf{E}').$$

So $(s(\beta), \mathsf{E}') = 0$ and $s(\beta) \in \mathsf{E}''$, which means that $\mathsf{E}''$ is $\mathscr{W}$-invariant. We can write $\mathsf{E} = \mathsf{E}' \oplus \mathsf{E}''$.

Suppose that $\alpha \in \mathsf{R}$ and $\mathsf{E}' \not\subset P_\alpha$, then $\exists \ \lambda \in \mathsf{E}' - P_\alpha$. So

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \, \alpha \in \mathsf{E}'$$

since $\mathsf{E}'$ is $\mathscr{W}$-invariant. As $\lambda \notin P_\alpha$ then $\langle \lambda, \alpha \rangle \neq 0$ and $s_\alpha(\lambda) - \lambda = -\langle \lambda, \alpha \rangle \, \alpha \in \mathsf{E}'$ which implies that $\alpha \in \mathsf{E}'$. This proves that for $\alpha \in \mathsf{R}$ either $\alpha \in \mathsf{E}'$ or else $\mathsf{E}' \subset P_\alpha$.

Thus, $\alpha \notin \mathsf{E}'$ implies $\mathsf{E}' \subset P_\alpha$ and then $(\alpha, \mathsf{E}') = 0$ so $\alpha \in \mathsf{E}''$. So each root lies in one subspace or the other. We can write $\mathsf{R} = \mathsf{R}_1 \cup \mathsf{R}_2$ with $(\mathsf{R}_1, \mathsf{R}_2) = 0$, $\mathsf{R}_1 = \mathsf{R} \cap \mathsf{E}'$ and $\mathsf{R}_2 = \mathsf{R} \cap \mathsf{E}''$ which is a partition of $\mathsf{R}$ into orthogonal subsets, forcing one or the other to be empty. Since we took $\mathsf{E}'$ to be nonzero then $\mathsf{E}'' = \emptyset$. Recall that $\mathsf{R}$ spans $\mathsf{E}$, so we conclude that $\mathsf{E}' = \mathsf{E}$. □

**Lemma A.8.6.** *Let* $\mathsf{R}$ *be irreducible. Then at most two root lengths occur in* $\mathsf{R}$ *and all roots of a given length are conjugate under* $\mathscr{W}$*.*

*Proof.* If $\alpha, \beta$ are two arbitrary roots and $(\beta, s(\alpha)) = 0$ for all $s(\alpha)$ with $s \in \mathscr{W}$, then $\beta \in \mathscr{W}(\alpha)^{\perp}$. By Lemma A.8.5 we have that $\mathscr{W}(\alpha)$ spans $\mathsf{E}$ so $\beta = 0$. Hence, there exists $s \in \mathscr{W}$ such that

$$(\beta, s(\alpha)) \neq 0. \tag{A.9}$$

Suppose that $(\alpha, \beta) \neq 0$, we know from Table A.1 that the possible ratios of squared root lengths of $\alpha, \beta$ are $1, 2, 3, 1/2, 1/3$. Let $\alpha, \beta, \gamma$ be three roots with different lengths. If $||\alpha|| = 2||\beta||$ and $||\gamma|| = 1/3||\beta||$ then $||\gamma|| = \frac{1}{6}||\alpha||$ and that is not a possible ratio. If we do other combinations we can get the ratios $6, 2/3, 3/2$ and those are not possible ratios as well. Therefore, at most two root lengths occur in $\mathsf{R}$.

Now let $\alpha, \beta$ have equal length with $(\alpha, \beta) \neq 0$, which we can do by (A.9). We may assume them to be non-orthogonal and distinct, otherwise we're done. This forces $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \pm 1$, by Table A.1. Replacing $\beta$, if needed, by $-\beta = s_\beta(\beta)$, we may assume that $\langle \alpha, \beta \rangle = 1$. Therefore,

$$(s_\alpha s_\beta s_\alpha)(\beta) = s_\alpha s_\beta(\beta - \alpha) = s_\alpha(-\beta - \alpha + \beta) = \alpha.$$

And we got one from the other only applying elements of $\mathscr{W}$. □

In case $\mathsf{R}$ is irreducible with two distinct root lengths, we call one of then long root and the other one short root. If all roots are of equal length, it is conventional to say that they are all long.

**Lemma A.8.7.** *Let* $\mathsf{R}$ *be irreducible, with two distinct root lengths. Then the maximal root* $\theta$ *of Lemma A.8.4 is long.*

*Proof.* Let $\alpha \in \mathsf{R}$ be arbitrary. It suffices to show that $(\beta, \beta) \geq (\alpha, \alpha)$ since then $||\beta|| \geq ||\alpha||$. As $(s(\alpha), s(\alpha)) = (\alpha, \alpha)$ for all $s \in \mathscr{W}$ we may replace $\alpha$ by a $\mathscr{W}$-conjugate. By Theorem A.7.1 item $(c)$ we have that there exists $s_1 \in \mathscr{W}$ such that $s_1(\alpha) \in \Delta$. By Theorem A.7.1 item $(a)$ we have that there exists $s_2 \in \mathscr{W}$ such that $(s_2(\gamma), s_1(\alpha)) > 0$, therefore

$$(s_2(\gamma), s_1(\alpha)) = (s_2(\gamma), s_2 s_2 s_1(\alpha)) = (\gamma, s_2 s_1(\alpha)) > 0$$

so $s(\alpha) = s_2 s_1(\alpha) \in \overline{\mathscr{C}(\Delta)}$. Hence, we can replace $\alpha$ by a $\mathscr{W}$-conjugate lying in the closure of the fundamental Weyl chamber relative to $\Delta$.

By Lemma A.8.4 we have that $(\theta, \alpha) \geq 0$ for all $\alpha \in \Delta$, then $\theta \in \overline{\mathscr{C}(\Delta)}$. Since $\theta$ is maximal then $\theta - \alpha \succ 0$ and $\theta - \alpha$ is a nonnegative sum of positive roots. Therefore, we have that $(\gamma, \theta - \alpha) \geq 0$ for any $\gamma \in \overline{\mathscr{C}(\Delta)}$. This fact, applied to the cases $\gamma = \theta$ and $\gamma = \alpha$ yields,

$$(\theta, \theta - \alpha) \geq 0 \Rightarrow (\theta, \theta) - (\theta, \alpha) \geq 0 \Rightarrow (\theta, \theta) \geq (\alpha, \theta)$$
$$(\alpha, \theta - \alpha) \geq 0 \Rightarrow (\alpha, \theta) - (\alpha, \alpha) \geq 0 \Rightarrow (\alpha, \theta) \geq (\alpha, \alpha)$$

then $(\theta, \theta) \geq (\alpha, \theta) \geq (\alpha, \alpha)$. □

## A.9 Cartan matrix of a root system

In this section, we will explore the concept of the Cartan matrix in detail. We will begin by defining the Cartan matrix and discussing its basic properties, such as its symmetry and its relationship with the root system. Let $\mathsf{R}$ be a root system of rank $\ell$ with basis $\Delta$ and associated Weyl group $\mathscr{W}$.

**Definition A.9.1.** *Fix an ordering for* $\Delta$ *as* $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$. *Consider the matrix* $C = (c_{ij})$ *defined by* $c_{ij} = \langle \alpha_i, \alpha_j \rangle$.

The matrix $C$ is called the Cartan matrix of $\mathsf{R}$ and the entries of $C$ are called Cartan integers.

**Example A.9.2.** *For the root systems of rank* 2*, the Cartan matrices are*

$$A_1 \times A_1 : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \; A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}; \; B_2 : \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}; \; G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The Cartan matrix is independent of the choice of $\Delta$, and that is thanks to the fact that $\mathscr{W}$ acts transitively on the collection of basis, and that is the important point for us. Moreover, the Cartan matrix is non-singular, since $\Delta$ is a basis of $\mathsf{E}$. And the Cartan matrix characterize the root system $\mathsf{R}$ completely.

**Proposition A.9.3.** *Let* $\mathsf{R}' \subset \mathsf{E}'$ *be another root system, with basis* $\Delta' = \{\alpha'_1, \cdots, \alpha'_\ell\}$. *If* $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ *for* $1 \leq i, j \leq \ell$, *then the bijection* $\alpha_i \mapsto \alpha'_i$ *extends, uniquely, to an isomorphism* $\phi : \mathsf{E} \to \mathsf{E}'$ *mapping* $\mathsf{R}$ *onto* $\mathsf{R}'$ *and satisfying* $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$ *for all* $\alpha, \beta \in \mathsf{R}$. *Therefore, the Cartan matrix of* $\mathsf{R}$ *determines* $\mathsf{R}$ *up to isomorphism.*

*Proof.* Since $\Delta$ is a basis of $\mathsf{E}$ and $\Delta'$ is a basis of $\mathsf{E}'$, there is a unique vector space isomorphism $\phi : \mathsf{E} \to \mathsf{E}'$ that sends $\alpha_i$ to $\alpha'_i$ with $1 \leq i \leq \ell$. If $\alpha, \beta \in \Delta$, the hypothesis insures that

$$s_{\phi(\alpha)}(\phi(\beta)) = s_{\alpha'}(\beta') = \beta' - \langle \beta', \alpha' \rangle \alpha' = \phi(\beta) - \langle \beta, \alpha \rangle \phi(\alpha) = \phi(\beta - \langle \beta, \alpha \rangle \alpha) = \phi(s_\alpha(\beta)).$$

In other words, the following diagram commutes for each $\alpha \in \Delta$,

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{\ \phi\ } & \mathsf{E}' \\ {\scriptstyle s_\alpha}\downarrow & & \downarrow{\scriptstyle s_{\phi(\alpha)}} \\ \mathsf{E} & \xrightarrow[\ \phi\ ]{} & \mathsf{E}' \end{array}$$

The respective Weyl groups $\mathscr{W}$ and $\mathscr{W}'$ are generated by simple reflections. It follows that the map $s \mapsto \phi \circ s \circ \phi^{-1}$ is an isomorphism of $\mathscr{W}$ onto $\mathscr{W}'$, sending $s_\alpha$ to $s_{\phi(\alpha)}$ where $\alpha \in \Delta$. Each $\beta \in \mathsf{R}$ is conjugate under $\mathscr{W}$ to a simple root, say $\beta = s(\alpha)$ where $\alpha \in \Delta$. This forces

$$\phi(\beta) = \phi(s(\alpha)) = \phi(s(\phi^{-1}\phi(\alpha))) = (\phi \circ s \circ \phi^{-1})(\phi(\alpha)) \in \mathsf{R}'.$$

It follows that $\phi$ maps $\mathsf{R}$ to $\mathsf{R}'$.

We showed that $s_{\phi(\alpha)}(\phi(\beta)) = \phi(s_\alpha(\beta))$ for all $\alpha, \beta \in \Delta$ in (A.2). By linearity, we know that this follows for every $\beta \in \mathsf{E}$, in particular to all $\beta \in \mathsf{R}$. For $\alpha \in \mathsf{R}$ we know by Theorem A.7.1 $(c)$ that there exists $w \in \mathscr{W}$ such that $w(\alpha_i) = \alpha$ for some $\alpha_i \in \Delta$. By Lemma A.2.2 we have that $s_\alpha = s_{w(\alpha_i)} = w \circ s_{\alpha_i} \circ w^{-1}$ and hence

$$\phi \circ s_\alpha \circ \phi^{-1} = \phi \circ w \circ s_{\alpha_i} \circ w^{-1} \circ \phi^{-1} = \underbrace{\phi \circ w \circ \phi^{-1}}_{=:w'} \circ \phi \circ s_{\alpha_i} \circ \phi^{-1} \circ \phi \circ w^{-1} \circ \phi^{-1}$$

$$= w' \circ s_{\phi(\alpha_i)} \circ w'^{-1} = s_{w'(\phi(\alpha_i))} = s_{\phi(w(\alpha_i))} = s_{\phi(\alpha)}.$$

For all $\alpha, \beta \in \mathsf{R}$ we have that

$$s_{\phi(\alpha)}(\phi(\beta)) = \phi(\beta) - \langle \phi(\beta), \phi(\alpha) \rangle \, \phi(\alpha)$$
$$\phi(s_\alpha(\beta)) = \phi(\beta) - \langle \beta, \alpha \rangle \, \phi(\alpha).$$

Which gives us $\langle \phi(\beta), \phi(\alpha) \rangle = \langle \beta, \alpha \rangle$. □

This proposition shows that if you have the Cartan integers you can recover the root system $\mathsf{R}$. In fact, we can devise a practical algorithm for writing all roots, or just the positive roots. Probably the best approach is to consider root strings. Start with the roots of height one, which are the simple roots. For any pair $\alpha_i \neq \alpha_j$, the integer $r$ for the $\alpha_j$-string through $\alpha_i$ is 0, which means that $\alpha_i - \alpha_j$ is not a root, thanks to Lemma A.5.2. So the integer $q$ equals $-\langle \alpha_i, \alpha_j \rangle$.

From this we can write down all roots $\alpha$ of height 2, hence all integers $\langle \alpha, \alpha_j \rangle$. For each root $\alpha$ of height 2 the integer $r$ for the $\alpha_j$-string through $\alpha$ can be determined easily. The root $\alpha_j$ can be subtracted at most once, since if we subtract twice we will have $\alpha - 2\alpha_j$, which is not a root. Then $q$ is found, since we know $r - q = \langle \alpha, \alpha_j \rangle$. The Corollary A.6.2 assures us that all positive roots are eventually obtained if we repeat this process enough times. The reader can see this process in the next example.

**Example A.9.4.** *Let us see we can find a root system given the Cartan matrix. If we have the Cartan matrix for $B_2$*

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

*We know the basis $\Delta$ has two elements. If we say that $\Delta = \{\alpha_1, \alpha_2\}$ then from the Cartan matrix we know that*

$$\langle \alpha_1, \alpha_1 \rangle = 2 \qquad \langle \alpha_1, \alpha_2 \rangle = -2$$
$$\langle \alpha_2, \alpha_1 \rangle = -1 \qquad \langle \alpha_2, \alpha_2 \rangle = 2.$$

*We will calculate the $\alpha_i$-string through $\alpha$ for $i = 1, 2$ and $\alpha \in \mathsf{R}$, for the strings through $\alpha_1$ and $\alpha_2$ we know that $\alpha_1 - \alpha_2 \notin \mathsf{R}$*

- *$\alpha_1$-string through $\alpha_2$*

  *$r = 0$ and $-q = \langle \alpha_2, \alpha_1 \rangle = -1$ then $q = 1$, so the string is*

$$\alpha_2, \alpha_2 + \alpha_1 \in \mathsf{R}_+.$$

- *$\alpha_2$-string through $\alpha_1$*

  *$r = 0$ and $-q = \langle \alpha_1, \alpha_2 \rangle = -2$ then $q = 2$, so the string is*

$$\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2 \in \mathsf{R}_+.$$

- *$\alpha_1$-string through $\alpha_1 + 2\alpha_2$*

  *$\langle \alpha_1 + 2\alpha_2, \alpha_1 \rangle = \langle \alpha_1, \alpha_1 \rangle + 2 \langle \alpha_2, \alpha_1 \rangle = 2 - 2 = 0$ since $\langle , \rangle$ is linear in the first variable, and $\alpha_1 + 2\alpha_2 - \alpha_1 = 2\alpha_2 \notin \mathsf{R}$, so $r = 0$ and then $-q = \langle \alpha_1 + 2\alpha_2, \alpha_1 \rangle = 0$ and $q = 0$, so the string is*

$$\alpha_1 + 2\alpha_2 \in \mathsf{R}_+.$$

*The root $\alpha_1 + \alpha_2$ is in both $\alpha_1$ and $\alpha_2$ strings, so we don't need to calculate the $\alpha_1$-string through $\alpha_1 + \alpha_2$ and $\alpha_2$-string through $\alpha_1 + \alpha_2$. The root $\alpha_1 + 2\alpha_2$ is in the $\alpha_2$- string, so we don't need to calculate $\alpha_2$-string through $\alpha_1 + 2\alpha_2$.*

*So, the root system is $\mathsf{R} = \mathsf{R}_+ \cup \mathsf{R}_-$ with $\mathsf{R}_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$.*

We will see a more difficult example.

**Example A.9.5.** *Let us find the root system for the Cartan matrix $D_4$*

$$\begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}.$$

*We know that the basis $\Delta$ have four elements. If we say that $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\langle \alpha_i, \alpha_j \rangle = c_{ij}$ we have that*

- *$\alpha_j$-string through $\alpha_i$ for $\{(i,j)\} = \{(1,2); (2,1); (1,3); (3,1); (2,3); (3,2)\}$*

  *$r = 0$ and $-q = \langle \alpha_i, \alpha_j \rangle = 0$ then $q = 0$, so the string is*

$$\alpha_i \in \mathsf{R}_+.$$

- *$\alpha_1$-string through $\alpha_4$*

  *$r = 0$ and $-q = \langle \alpha_4, \alpha_1 \rangle = -1$ then $q = 1$, so the string is*

$$\alpha_4, \alpha_1 + \alpha_4 \in \mathsf{R}_+.$$

- *$\alpha_2$-string through $\alpha_4$*

  *$r = 0$ and $-q = \langle \alpha_4, \alpha_2 \rangle = -1$ then $q = 1$, so the string is*

$$\alpha_4, \alpha_2 + \alpha_4 \in \mathsf{R}_+.$$

- *$\alpha_3$-string through $\alpha_4$*

  *$r = 0$ and $-q = \langle \alpha_4, \alpha_3 \rangle = -1$ then $q = 1$, so the string is*

  $$\alpha_4, \alpha_3 + \alpha_4 \in \mathsf{R}_+.$$

- *$\alpha_1$-string through $\alpha_2 + \alpha_4$*

  *$r = 0$ and $-q = \langle \alpha_2 + \alpha_4, \alpha_1 \rangle = 0 - 1$ then $q = 1$, so the string is*

  $$\alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4 \in \mathsf{R}_+.$$

- *$\alpha_3$-string through $\alpha_2 + \alpha_4$*

  *$r = 0$ and $-q = \langle \alpha_2 + \alpha_4, \alpha_3 \rangle = 0 - 1$ then $q = 1$, so the string is*

  $$\alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4 \in \mathsf{R}_+.$$

- *$\alpha_1$-string through $\alpha_3 + \alpha_4$*

  *$r = 0$ and $-q = \langle \alpha_3 + \alpha_4, \alpha_1 \rangle = 0 - 1$ then $q = 1$, so the string is*

  $$\alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 \in \mathsf{R}_+.$$

- *$\alpha_1$-string through $\alpha_2 + \alpha_3 + \alpha_4$*

  *$r = 0$ and $-q = \langle \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 \rangle = 0 + 0 - 1$ then $q = 1$, so the string is*

  $$\alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in \mathsf{R}_+.$$

- *$\alpha_4$-string through $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$*

  *$r = 0$ since $\alpha_1 + \alpha_2 + \alpha_3$ is not a root and $-q = \langle \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_4 \rangle = -1 - 1 - 1 + 2$ then $q = 1$, so the string is*

  $$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 \in \mathsf{R}_+.$$

*So, the root system is $\mathsf{R} = \mathsf{R}_+ \cup \mathsf{R}_-$ with $\mathsf{R}_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_4, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4\}$.*

## A.10 Coxeter graphs and Dynkin diagrams

Coxeter graphs and Dynkin diagrams are powerful tools for understanding the structure of root systems. A Coxeter graph is a graph that encodes the symmetry properties of a root system, while a Dynkin diagram is a variation of the Coxeter graph that provides additional information about the root system. In this section, we will explore these concepts.

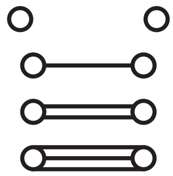
If $\alpha, \beta$ are distinct positive roots by Table A.1 we know that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2\}$ or 3. Define the Coxeter graph of $\mathsf{R}$ to be a graph having $\ell$ vertices, the $i$th joined to the $j$th with $i \neq j$ by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges.

$A_1 \times A_1$

$A_2$

$B_2$

$G_2$

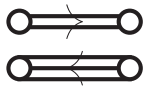


The Coxeter graph determines the numbers $\langle \alpha_i, \alpha_j \rangle$ in case all roots have equal length, since then $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle$. In case more than one root length occurs, like $B_2$ or $G_2$, the graph fails to tell us which of the pair of vertices should correspond to a short simple root, and which correspond to a long simple root.

Whenever a double or triple edge occurs in the Coxeter graph of $\mathsf{R}$, we add an arrow pointing towards the shorter one of the two roots. This additional information allows us to recover the Cartan integers. We call the resulting figure the Dynkin diagram of $\mathsf{R}$. For example

$B_2$

$G_2$



The next diagram gives us another example,

which turns out to be associated Dynkin diagram with the root system $F_4$, it is easy to recover the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

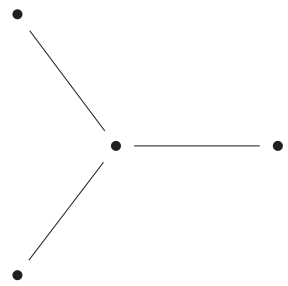
**Example A.10.1.** *Let us recover the Cartan matrix given the Dynkin diagram. If we have the diagram for* $B_2$

$B_2$

*We know that the basis $\Delta$ have two elements. If we say that the first is $\alpha_1$ and the second is $\alpha_2$, then from the diagram we know that $||\alpha_1|| > ||\alpha_2||$, $\langle \alpha_1, \alpha_2 \rangle \langle \alpha_2, \alpha_1 \rangle = 2$, $\langle \alpha_1, \alpha_2 \rangle \leq 0, \langle \alpha_2, \alpha_1 \rangle \leq 0$ and $\langle \alpha_1, \alpha_2 \rangle > \langle \alpha_2, \alpha_1 \rangle$. Then we get $\langle \alpha_1, \alpha_2 \rangle = -2$ and $\langle \alpha_2, \alpha_1 \rangle = -1$. So we have the Cartan matrix,*

$$C_1 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

**Example A.10.2.** *For the diagram $D_4$*

$D_4 =$ 

*we have the Cartan matrix*

$$\begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}.$$

By Proposition A.8.3 we have that a root system $\mathsf{R}$ is irreducible if and only if $\mathsf{R}$ or $\Delta$ cannot be partitioned into two proper and orthogonal subsets. Then it is clear that $\mathsf{R}$ is irreducible if and only if its Coxeter graph if connected. We can say in general that there will be a number of connected components in the Coxeter graph.

**Lemma A.10.3.** *Let $\mathsf{E}'$ be a subspace of $\mathsf{E}$. If a reflection $s_\alpha$ leaves $\mathsf{E}'$ invariant then either $\alpha \in \mathsf{E}'$ or else $\mathsf{E}' \subset P_\alpha$.*

*Proof.* We know that $P_\alpha = \{\beta \in \mathsf{E}; (\beta, \alpha) = 0\}$ by (A.1). Suppose that $\mathsf{E}' \not\subset P_\alpha$. Then there exists some $\beta \in \mathsf{E}'$ such that $(\beta, \alpha) \neq 0$. By hypothesis, we have that $s_\alpha(\beta) \in \mathsf{E}'$, which means that

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \, \alpha \in \mathsf{E}' \quad \text{with} \quad \langle \beta, \alpha \rangle \neq 0.$$

As $\beta \in \mathsf{E}'$ we have that $\alpha \in \mathsf{E}'$. ◻

**Proposition A.10.4.** *The root system $\mathsf{R}$ decomposes, uniquely, as the union of irreducible root systems $\mathsf{R}_i$, in subspaces $\mathsf{E}_i$ of $\mathsf{E}$, such that $\mathsf{E} = \mathsf{E}_1 \oplus \cdots \oplus \mathsf{E}_t$, the sum is the orthogonal direct sum.*

*Proof.* Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_t$ be the partition of $\Delta$ into mutually orthogonal subsets, which means that $(\Delta_i, \Delta_j) = 0 \ \forall \ i \neq j$. If $\mathsf{E}_i = \operatorname{span}\{\Delta_i\}$ then $\mathsf{E} = \mathsf{E}_1 \oplus \cdots \oplus \mathsf{E}_t$ as an orthogonal direct sum. Now let $\mathsf{R}_i = \mathsf{E}_i \cap \mathsf{R}$. We have that $\mathsf{R}_i$ is a root system in $\mathsf{E}_i$, since all the axioms of Definition A.2.1 are satisfied as seen below,

- $\mathsf{R}_i$ is finite, spans $\mathsf{E}_i$ and does not contain 0;
- if $\alpha \in \mathsf{R}_i$ then the only multiples of $\alpha$ in $\mathsf{R}_i$ are $\pm\alpha$ since those are the only ones in $\mathsf{R}$;
- if $\alpha \in \mathsf{R}_i$ then $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \, \alpha$ for all $\beta \in \mathsf{R}_i$ and that is just a $\mathbb{Z}$-linear combination of elements from $\mathsf{R}_i$ and that is in $\mathsf{R}$ and in $\mathsf{E}_i$, therefore they are in $\mathsf{R}_i$ so $s_\alpha$ leaves $\mathsf{R}_i$ invariant;

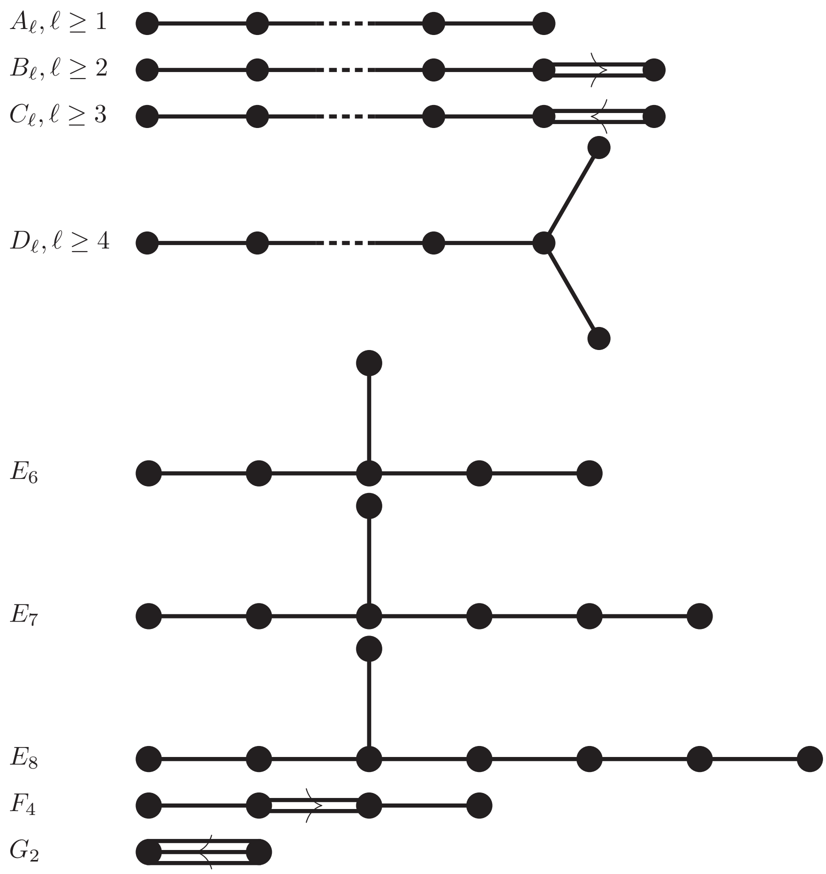
- if $\alpha, \beta \in \mathsf{R}_i$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$ since that is the case in $\mathsf{R}$.

The Weyl group of $\mathsf{E}_i$ is the restriction to $\mathsf{E}_i$ of the subgroup of $\mathscr{W}$ generated by all $s_\alpha$ with $\alpha \in \Delta_i$.

Now if $\alpha \notin \Delta_i$ then $s_\alpha = \beta - \langle \beta, \alpha \rangle \alpha = \beta$ for all $\beta \in \mathsf{E}_i$ since $\langle \beta, \alpha \rangle = 0$, so $s_\alpha$ acts trivially on $\mathsf{E}_i$ and $\mathsf{E}_i$ is $\mathscr{W}$-invariant. The argument that either $\alpha \in \mathsf{E}_i$ or $\mathsf{E}_i \subset P_\alpha$ given in Lemma A.10.3 shows that each root lies in one of the $\mathsf{E}_i$. Then $\mathsf{R} = \mathsf{R}_1 \cup \cdots \cup \mathsf{R}_t$. □

The following theorem establishes the classification of connected Dynkin diagram. In particular, we obtain a classification of irreducible root systems.

**Theorem A.10.5.** *If* $\mathsf{R}$ *is an irreducible root system of rank* $\ell$*, its Dynkin diagram is one of the following, with* $\ell$ *vertices in each case:*




## A.11 Longest element and the Coxeter element

In this section we will introduce two important elements of the Weyl group, the longest element and the Coxeter element. After that we will give an equation that links the two elements. We will use the length in Lemma A.7.2 to study the longest element.

**Lemma A.11.1.** *There exists a unique element* $w_0 \in \mathscr{W}$ *sending* $\mathsf{R}_+$ *to* $\mathsf{R}_-$*, relative to* $\Delta$*. Any reduced expression for* $w_0$ *must involve all* $s_\alpha$ *with* $\alpha \in \Delta$*. And* $l(w_0) = \mid \mathsf{R}_+ \mid$.

*Proof.* Note that $-\Delta = \{-\alpha; \alpha \in \Delta\}$ is also a basis for $\mathsf{R}$. Since $\mathscr{W}$ acts transitively on basis of $\mathsf{R}$, there exists $w_0 \in \mathscr{W}$ such that $s(\Delta) = -\Delta$. Then $w_0$ necessarily takes positive roots of $\mathsf{R}$ to negative roots of $\mathsf{R}$, relative to $\Delta$. If $\tau \in \mathscr{W}$ also has this property, then $w_0\tau$ takes a positive basis to another positive basis. By definition, two basis can be positive with respect to $\Delta$ only if they are equal. By Theorem A.7.1 we have that $w_0\tau = 1$. As $w_0$ has order two then $w_0 = \tau$. Therefore, $w_0$ is unique.

Let $w_0 = s_{\alpha_1} \cdots s_{\alpha_t}$ be a reduced expression for $w_0$ with $\alpha_i \in \Delta$. Suppose $\beta \in \Delta$ is not in this expression. Since $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ it is clear that $s_{\alpha_s} \cdots s_{\alpha_t}$ cannot take $\beta$ to another simple root. Since each $s_{\alpha_i}$ permutes $\mathsf{R}_+ - \{\alpha_i\}$ then $w_0(\beta) \notin \mathsf{R}_-$, which is a contradiction. Hence, a reduced expression for $w_0$ must involve all $s_\alpha$ with $\alpha \in \Delta$.

As $w_0(\mathsf{R}_+) = \mathsf{R}_-$ we have that $n(w_0) = |\mathsf{R}_+|$ and $l(w_0) = n(w_0)$, so we have $l(w_0) = |\mathsf{R}_+|$. □

**Definition A.11.2.** *An element $C \in \mathscr{W}$ is called a Coxeter element if for some ordering of simple roots $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ and $s_{\alpha_i} = s_i$, we have*

$$C = s_\ell \cdots s_1.$$

Of course, we have more than one Coxeter element. We can show that they are all conjugate in $\mathscr{W}$. If

$$\Delta_1 = \{\alpha_1, \cdots, \alpha_\ell\} \quad \text{and} \quad \Delta_2 = \{\beta_1, \cdots, \beta_\ell\}$$

are two orderings for the simple roots with

$$C_1 = s_{\alpha_\ell} \cdots s_{\alpha_1} \quad \text{and} \quad C_2 = s_{\beta_\ell} \cdots s_{\beta_1}$$

the two Coxeter elements associated to $\Delta_1$ and $\Delta_2$, respectively. Then there exists $s = C_2C_1^{-1} \in \mathscr{W}$ such that

$$sC_1 = C_2C_1^{-1}C_1 = C_2.$$

**Definition A.11.3.** *The order of a Coxeter element is called the Coxeter number of the root system, which we will denote by $h$. In other words $C^h = 1$.*

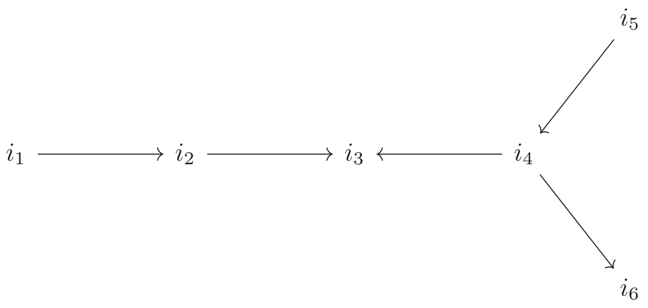
By Lemma A.11.1 we have that $\mathscr{W}$ contains a unique element $w_0$ with the longest length, which means that $l(w_0) = |R^+|$. The next theorem will give us a relation between this $w_0$ and the Coxeter element. For this we need to choose a partition for the index set $I$, we will split that set as $I = I_0 \cup I_1$. This partition of $I$ will be made so that for every edge in $\Omega$ one of the endpoints will be in $I_0$ and the other in $I_1$. This partition is called the bipartite partition.

We will define $c_0, c_1 \in \mathscr{W}$ by

$$c_0 = \prod_{i \in I_0} s_i \quad \text{and} \quad c_1 = \prod_{i \in I_1} s_i. \tag{A.10}$$

Since $(\alpha_i, \alpha_j) = 0$ for all $i, j \in I_0$ or for all $i, j \in I_1$, then all $s_i$ with $i \in I_0$ commute, and similarly for $s_i$ with $i \in I_1$, this implies that $c_0$ and $c_1$ are well-defined. By Definition A.11.2, the product $C = c_1c_0$ is a Coxeter element. Let us see in an example how this bipartite partition works.

**Example A.11.4.** *Let $\vec{Q}$ be the quiver given by*




*A bipartite partition of this quiver can be done as,*

$$I_0 = \{i_1, i_3, i_5, i_6\} \text{ and } I_1 = \{i_2, i_4\}.$$

*We can write $c_0$ and $c_1$ as,*

$$c_0 = s_{i_1} s_{i_3} s_{i_5} s_{i_6} = s_{i_6} s_{i_5} s_{i_1} s_{i_3} \text{ and } c_1 = s_{i_2} s_{i_4} = s_{i_4} s_{i_2}.$$

The next theorem is based on section 3.17 of [13].

**Theorem A.11.5.** *If $c_0, c_1$ are as defined above in* (A.10) *and $w_0 \in \mathscr{W}$ is the longest element as in Lemma A.11.1, then*

$$\begin{aligned} w_0 &= \cdots c_0 c_1 c_0 \quad (h \text{ factors}) \\ &= \cdots c_1 c_0 c_1 \quad (h \text{ factors}) \end{aligned}$$

*are reduced expressions for $w_0$, here $h$ is the Coxeter number as in Definition A.11.3.*

*Proof.* As the order of factors in $c_0, c_1$ are irrelevant then both have order two. We can write

$$C = c_1 c_0 = s_{k_1} \cdots s_{k_t} s_{k_{t+1}} \cdots s_{k_\ell} \quad \text{with} \quad k_1, \cdots, k_t, k_{t+1}, \cdots k_\ell \in I.$$

So $c_1 = s_{k_1} \cdots s_{k_t}$ and $c_0 = s_{k_{t+1}} \cdots s_{k_\ell}$. Let us write $I_0 = \{k_{t+1}, \cdots, k_\ell\}$ and $I_1 = \{k_1, \cdots, k_t\}$, which makes $I = I_0 \cup I_1$. For ease of notation write $J_0 = \{t+1, \cdots, \ell\}$, $J_1 = \{1, \cdots, t\}$ and $J = J_0 \cup J_1$.

We can write a basis $B$ for the space $\mathbb{R}^\ell$ as $B = \{\alpha_{k_1}, \cdots, \alpha_{k_\ell}\}$ and the dual basis of $B$ is $B^\vee = \{\omega_{k_1}, \cdots, \omega_{k_\ell}\}$. Which means that

$$(\alpha_{k_i}, \omega_{k_j}) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Consider the sets

$$Y = \text{span}\{\omega_{k_{t+1}}, \cdots, \omega_{k_\ell}\} \text{ and } Z = \text{span}\{\omega_{k_1}, \cdots, \omega_{k_t}\}$$

and consider the orthogonal sets of $Y$ and $Z$

$$Y^\perp = \text{span}\{\alpha_{k_1}, \cdots, \alpha_{k_t}\} \text{ and } Z^\perp = \text{span}\{\alpha_{k_{t+1}}, \cdots, \alpha_{k_\ell}\}.$$

Let $H_{k_i}$ be the hyperplane orthogonal to $\alpha_{k_i}$, which means that $H_{k_i} = \{x \in V; (x, \alpha_{k_i}) = 0\}$. Also consider

$$Y' = H_{k_1} \cap \cdots \cap H_{k_t} \text{ and } Z' = H_{k_{t+1}} \cap \cdots \cap H_{k_\ell}.$$

We know that $Y' \cap Z' = \{0\}$, since otherwise it would exist a vector $v \in V$ such that $v \in V^\perp$ and 0 is the only vector with this property. Also, $(w_{k_i}, \alpha_{k_j}) = 0$ for all $i \in J_0$ and $j \in J_1$. Then we have that $Y \subset Y'$ and analogously $Z \subset Z'$. To conclude this argument we have that $V = Y \oplus Z$ so $Y = Y'$ and $Z = Z'$. Therefore, we proved that

$$Y = \text{span}\{\omega_{k_{t+1}}, \cdots, \omega_{k_\ell}\} = H_{k_1} \cap \cdots \cap H_{k_t} \text{ and } Z = \text{span}\{\omega_{k_1}, \cdots, \omega_{k_t}\} = H_{k_{t+1}} \cap \cdots \cap H_{k_\ell}.$$

For $y \in Y$ we know that $(y, \alpha_{k_i}) = 0$ for $i \in J_1$. So $c_1(y) = y$ and $c_1$ fixes $Y$ point by point. Also, we have that $c_1(\alpha_{k_i}) = -\alpha_{k_i}$ for all $i \in J_1$, since $(\alpha_{k_i}, \alpha_{k_j}) = 0$ for all $i, j \in J_1$. So $c_1$ acts as $-1$ in $Y^\perp$. Analogously we have that $c_0$ fixes $Z$ point by point and acts as $-1$ in $Z^\perp$.

Let

$$A : \mathbb{R}^\ell \to \mathbb{R}^\ell$$
$$\omega_{k_i} \mapsto \alpha_{k_i}.$$

Since

$$\alpha_{k_j} = \sum_{i \in J} a_{ij} \omega_{k_i},$$

we can write $A = (a_{ij})$ with $a_{ij} = (\alpha_{k_i}, \alpha_{k_j})$ the matrix relative to the dual basis of the operator $A$.

We know from linear algebra that if a matrix is indecomposable, positive defined, symmetric, with non-positive entries in the diagonal, then $A$ has an eigenvalue $d$ associated to an eigenvector $(d_1, \cdots, d_\ell) \in \mathbb{R}^\ell$ with $d_i > 0$ for all $i \in J$. This means that

$$A(d_1, \cdots, d_\ell) = d(d_1, \cdots, d_\ell).$$

Consider

$$\lambda = \sum_{j \in J_1} d_j \omega_{k_j} \in Z \text{ and } \mu = \sum_{i \in J_0} d_i \omega_{k_i} \in Y$$

and write $L = \lambda\mathbb{R}$, $M = \mu\mathbb{R}$ and $P = \text{span}\{L, M\}$. We can write $A(d_1, \cdots, d_\ell)$ in two ways,

$$A(d_1, \cdots, d_\ell) = A\left(\sum_{i \in J} d_i \omega_{k_i}\right) = \sum_{i \in J} d_i A(\omega_{k_i}) = \sum_{i \in J} d_i \alpha_{k_i}$$
$$A(d_1, \cdots, d_\ell) = d(d_1, \cdots, d_\ell) = \sum_{i \in J} d d_i \omega_{k_i}$$

so,

$$\sum_{i \in J} d_i \alpha_{k_i} = \sum_{i \in J} d d_i \omega_{k_i}.$$

Making the inner product in both sides with $\alpha_{k_j}$ for $j \in J_1$ we have

$$\begin{aligned}
\left(\sum_{i\in J} d_i\alpha_{k_i}, \alpha_{k_j}\right) &= \left(\sum_{i\in J} dd_i\omega_{k_i}, \alpha_{k_j}\right) \\
\sum_{i\in J} d_i\left(\alpha_{k_i}, \alpha_{k_j}\right) &= \sum_{i\in J} dd_i\left(\omega_{k_i}, \alpha_{k_j}\right) \\
2d_j + \sum_{i\in J_0} d_i(\alpha_{k_i}, \alpha_{k_j}) &= dd_j \\
\sum_{i\in J_0} d_i a_{ij} &= (d-2)d_j
\end{aligned}$$

since $(\alpha_{k_j}, \alpha_{k_j}) = 2$ and $(\alpha_{k_i}, \alpha_{k_j}) = 0$ for all $i \in J_1$ and $i \neq j$.

Joining this information we have,

$$\begin{aligned}
(d-2)\lambda = (d-2)\sum_{j\in J_1} d_j\omega_{k_j} &= \sum_{j\in J_1}(d-2)d_j\omega_{k_j} = \sum_{j\in J_1}\left(\sum_{i\in J_0} d_i a_{ij}\right)\omega_{k_j} \\
&= \sum_{i\in J_0} d_i\left(\sum_{j\in J_1} a_{ij}\omega_{k_j}\right) = \sum_{i\in J_0} d_i\left(\sum_{j\in J_1}(\alpha_{k_i}, \alpha_{k_j})\omega_{k_j}\right) \\
&= \sum_{i\in J_0} d_i\left(-2\omega_{k_i} + \sum_{t\in J} a_{ti}\omega_{k_t}\right) = \sum_{i\in J_0} d_i\left(-2\omega_{k_i} + \alpha_{k_i}\right) \\
&= -2\sum_{i\in J_0} d_i\omega_{k_i} + \sum_{i\in J_0} d_i\alpha_{k_i} = -2\mu + v
\end{aligned}$$

where

$$(d-2)\lambda + 2\mu = v = \sum_{i\in J_0} d_i\alpha_{k_i} \in Z^{\perp}.$$

Since $v$ is orthogonal to $Z$, so it is $(d-2)\lambda + 2\mu$.

As $v \in Z^{\perp}$ then $c_0(v) = -v$ and $\lambda \in Z$ then $c_0(\lambda) = \lambda$ so

$$-(d-2)\lambda - 2\mu = -v = c_0(v) = c_0((d-2)\lambda + 2\mu) = (d-2)\lambda + 2c_0(\mu)$$

therefore,

$$\begin{aligned}
2c_0(\mu) &= -2(d-2)\lambda - 2\mu \\
c_0(\mu) &= -(d-2)\lambda - \mu.
\end{aligned}$$

The elements that belong to $P$ are of the form $a\lambda + b\mu$, so

$$c_0(a\lambda + b\mu) = ac_0(\lambda) + bc_0(\mu) = a\lambda + b(-(d-2)\lambda - \mu) = (a - b(d-2))\lambda - b\mu \in P$$

therefore $c_0(P) = P$. Analogously we can see that $c_1(P) = P$ so $C(P) = c_1c_0(P) = P$.

As $\lambda \in Z$ and $\mu \in Y$ we have that $L = \lambda\mathbb{R} \subset Z$ and $M = \mu\mathbb{R} \subset Y$ then $c_0$ fixes $L$ point by point and $c_1$ fixes $M$ point by point. So $L^{\perp} \cap P \subset Z^{\perp}$ and $c_0$ acts as $-1$ in $L^{\perp} \cap P$ so $c_0$ is a reflection in $P$ with fixed line $L$. Analogously $c_1$ is a reflection in $P$ with fixed line $M$. Therefore, $C$ acts in $P$ as a rotation.

The fundamental Weyl chamber is,

$$\mathscr{C} = \left\{ \sum \beta_i \omega_i;\ \beta_i > 0 \right\},$$

and it is closure is

$$\overline{\mathscr{C}} = \left\{ \sum \beta_i \omega_i;\ \beta_i \geq 0 \right\}.$$

We have that $\lambda, \mu \in \overline{\mathscr{C}}$ and

$$P \cap \mathscr{C} = \{a\lambda + b\mu;\ a, b > 0\}$$

then we have that $\lambda + \mu \in P \cap \mathscr{C}$ so $P \cap \mathscr{C} \neq \emptyset$.

If $C^x$ fixes $P$ point by point then it fixes some element of $\mathscr{C}$ then by Lemma A.7.3 we have that $C^x = 1$, but $h$ is the minimum such that $C^h = 1$ so $C$ has order $h$ in $P$ and acts as a rotation through $\frac{2\pi}{h}$.

If $h$ é even then $w_0 = C^{\frac{h}{2}}$ and since $C$ is a rotation through $\frac{2\pi}{h}$ then $C^{\frac{h}{2}}$ is a rotation through $\pi$, and it takes points from $P \cap \mathscr{C}$ to points of $P \cap w_0\mathscr{C}$.

Since $\overline{\mathscr{C}}$ is a fundamental domain to the action of $\mathscr{W}$, while points of $\mathscr{C}$ have trivial stabilizers in $\mathscr{W}$ then $w_0 = C^{\frac{h}{2}}$.

If $h$ is odd, looking case by case the numbers of Coxeter that have an odd $h$ is only in the $A_n$ type when $n$ is even and $h = n+1$. For the $S_{n+1}$ group in type $A_n$ with $n$ even we can see directly that a modification of the procedure above is enough to produce a reduced decomposition, $w_0$ is the product of $C^{\frac{h}{2}}$ times the first factor $c_1$ of $C$. □

Note that if $h$ is even then we can write $w_0 = C^{\frac{h}{2}}$ for $C = c_1c_0$, but if $h$ is odd then we have that

$$l(w_0) = l(\cdots c_1c_0c_1) = \cdots + l(c_1) + l(c_0) + l(c_1)$$

we can do that since $c_0, c_1$ are reduced expressions. If we write $h = 2k+1$, and we know that $C = c_1c_0 = c_0c_1$ so

$$l(w_0) = kl(c_0) + (k+1)l(c_1) = kl(c_1) + (k+1)l(c_0)$$

that implies that $l(c_0) = l(c_1)$ therefore

$$l(w_0) = kl(c_0) + (k+1)l(c_1) = kl(c_0) + (k+1)l(c_0) = (2k+1)l(c_0) = hl(c_0),$$

and we get that

$$l(c_0) = l(c_1) = \frac{l(w_0)}{h}.$$

**Corollary A.11.6.** *If $h$ is the order of the Coxeter element and $|I|$ is the rank of the root system* $\mathsf{R}$*, then*

$$|I|h = |\mathsf{R}|.$$

*Proof.* We have that $|\mathsf{R}| = 2|\mathsf{R}_+|$ and $l(w_0) = |\mathsf{R}_+|$.

If $h$ is even we have that by Theorem A.11.5

$$|\mathsf{R}| = 2|\mathsf{R}_+| = 2l(w_0) = 2l\left(C^{\frac{h}{2}}\right) = 2\left(\frac{h}{2}l(c_0) + l(c_1)\right) = |I|h.$$

If $h$ is odd we have that by Theorem A.11.5

$$|\mathsf{R}| = 2|\mathsf{R}_+| = 2l(w_0) = 2l(c_1)h = (l(c_1) + l(c_1))h = (l(c_0) + l(c_1))h = |I|h.$$

□

## A.12 Infinity root system

In the previous sections we have studied root system of finite type. However, for a second part of this dissertation we need some information coming from infinity root system. Similarly, as in the finite case, the infinity root systems are related to the weight theory of Kac-Moody algebra - certain generalizations of the simple finite dimensional Lie algebras. In this section we define and state only the necessary results which are needed in this work. Furthermore, we approach the theory of infinity root system in a way to avoid going through the Kac-Moody algebra theory, since this is not in the scope of this work. We follow [18] in this section. For a deeper interest and its connection with Kac-Moody algebra we suggest [14].

**Definition A.12.1.** *A matrix $A = (a_{ij})_{1\leq i,j\leq \ell}$, $a_{i,j} \in \mathbb{K}$ is said to be a generalized Cartan matrix if $A$ satisfies the following axioms, for all $1 \leq i, j \leq \ell$:*

*(i)* $a_{ij} \in \mathbb{Z}$

*(ii)* $a_{ii} = 2$

*(iii)* $a_{ij} \leq 0$ *if* $i \neq j$

*(iv)* $a_{ij} = 0$ *if and only if* $a_{ji} = 0$.

**Definition A.12.2.** *A realization of $A$ (over a field $\mathbb{K}$ of characteristic $0$) consists of a finite dimensional vector space $V$ and a pair of sets $\Delta = \{\alpha_1, \cdots, \alpha_\ell\} \subset V$ and $\Delta^\vee = \{\alpha_1^\vee, \cdots, \alpha_\ell^\vee\} \subset V^*$, such that*

*(i) The set $\Delta$ and $\Delta^\vee$ are linearly independent;*

*(ii)* $\langle \alpha_i, \alpha_j^\vee \rangle = a_{i,j}$,

*where $\langle\ ,\ \rangle$ denotes the natural paring $V \times V^* \to \mathbb{K}$.*

Given a realization $\mathcal{R} = (V, \Delta, \Delta^\vee)$ of $A$ we can define the Weyl group $W$ as the subgroup of $GL(V)$ generated by simple reflections $s_i : V \to V$ given by

$$s_i(v) = v - \langle v, \alpha_i^\vee \rangle \alpha_i, \quad v \in V.$$

The real root system of $\mathcal{R}$ is defined by $\mathsf{R}^{re} = W\Delta$. Define

$$Q = \bigoplus_{j=1}^{\ell} \mathbb{Z}\alpha_j, \quad Q^\vee = \bigoplus_{j=1}^{\ell} \mathbb{Z}\alpha_j^\vee.$$

Given $\beta \in Q$ and $\alpha \in R$ we have $s_\alpha(\beta) = \beta - u\alpha$ for some $u \in \mathbb{Z}$. Then the root string between $\beta$ and $s_\alpha(\beta)$ is defined as

$$[\beta, s_\alpha(\beta)] = \begin{cases} \{\beta, \beta+\alpha, \cdots, \beta+u\alpha\}, & u \geq 0 \\ \{\beta, \beta-\alpha, \cdots, \beta+u\alpha\}, & u < 0. \end{cases}$$

A subset $\Delta$ of $Q$ is said to have the root string property relative to $\mathsf{R}^{re}$ if

(i) $\mathsf{R}^{re} \subset \Delta$,

(ii) whenever $\beta \in \Delta$ and $\alpha \in \mathsf{R}^{re}$ we have $[\beta, s_\alpha(\beta)] \subset \Delta$.

**Proposition A.12.3.** *There is a unique minimal subset* $\mathsf{R}$ *of* $V$ *with root string property relative to* $\mathsf{R}^{re}$.

The set $\mathsf{R}$ above is called the root string closure of $\mathsf{R}^{re}$.

**Definition A.12.4.** *The elements of* $\mathsf{R}\backslash\mathsf{R}^{re}$ *are called imaginary roots and hence we denote by* $\mathsf{R}^{im} = \mathsf{R}\backslash\mathsf{R}^{re}$.

It is now clear that we can define a Coxeter element, similarly, as we did in Definition A.11.2 for the finite case.

**Definition A.12.5.** *Given* $\{i_1, \cdots, i_\ell\} = \{1, \cdots, \ell\}$ *we can define a Coxeter element* $C \in W$ *by*

$$C = s_{i_1} \cdots c_{i_\ell}.$$

**Definition A.12.6.** *If we write* $s \in W$ *as* $s = s_{\alpha_1} \cdots s_{\alpha_t}$ *with* $\alpha_i \in \Delta$ *and* $t$ *minimal we say that the expression of* $s$ *is reduced.*

We define the length of a reflection $s \in W$ relative to $\Delta$ as $l(s) = t$. Define

$$N(s) = \{\alpha \in \mathsf{R}^{re}_+; s(\alpha) \in \mathsf{R}^{re}_-\} \quad \text{and} \quad n(s) = \mid N(s) \mid . \tag{A.11}$$

**Lemma A.12.7.** *For all* $s \in W$ *we have that* $l(s) = n(s)$ *for* $n(s)$ *as in* (A.11).

**Lemma A.12.8.** *Let* $\alpha_1, \cdots, \alpha_t \in \Delta$, *not necessarily distinct, and write* $s_i = s_{\alpha_i}$. *If* $s_1 \cdots s_{t-1}(\alpha_t) \in \mathsf{R}^{re}_-$ *then* $s_1 \cdots s_t = s_1 \cdots s_{r-1}s_{r+1} \cdots s_{t-1}$, *for some index* $1 \leq r < t$.

**Corollary A.12.9.** *If* $s = s_1 \cdots s_t$ *is an expression for* $s \in W$ *in terms of reflections corresponding to simple roots, with* $t$ *as small as possible, then* $s(\alpha_t) \in \mathsf{R}^{re}_-$.

# Appendix B

# ORBITS

In mathematics, an orbit is a set of points related by the action of a group. Specifically, given a group $G$ acting on a set $X$, the orbit of an element $x \in X$ under the action of $G$ is the set of all elements in $X$ that can be obtained by applying some element of $G$ to $x$. Orbits are a fundamental concept in many areas of mathematics, including group theory, algebraic geometry, and representation theory. In representation theory, orbits play a crucial role in the study of group actions on vector spaces and modules, and in particular in the classification of irreducible representations.

Throughout this chapter, $\mathbb{K}$ is an algebraically closed field, all the words related to topology are referring to the Zariski topology. Notation $\overline{X}$ stands for the closure of $X$ in Zariski topology. We also fix a quiver $\vec{Q}$ with set of vertices $I$ and set of oriented edges $\Omega$. To start we need to set some notation on representation and how they translate to the orbits. We follow [16] in this appendix.

## B.1 Representation space

Representation space refers to the collection of all possible representations of a given algebraic structure. In mathematics, the study of representation spaces is a fundamental tool used to understand the underlying algebraic objects.

Let $\vec{Q} = (I, \Omega)$ be a quiver and $V = (V_i, x_h)_{i \in I, h \in \Omega} \in \mathrm{Rep}(\vec{Q})$. Denote by

$$v = \dim(V) \text{ and } v_i = \dim(V_i),\ i \in I.$$

We identify $V_i \simeq \mathbb{K}^{v_i}$. If we have an arrow $h : i \to j$ then the linear transformation can be described as $x_h : \mathbb{K}^{v_i} \to \mathbb{K}^{v_j}$, which means that

$$x_h \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{v_i}, \mathbb{K}^{v_j}) = \mathrm{Mat}_{v_j \times v_i}(\mathbb{K}).$$

Or equivalently, the linear transformation can be described by a vector $x$ in the space

$$R(\vec{Q}, v) = \bigoplus_{h \in \Omega} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{v_{s(h)}}, \mathbb{K}^{v_{t(h)}}).$$

Conversely, every $x \in R(\vec{Q}, v)$ defines a representation

$$V^x = (\{\mathbb{K}^{v_i}\}, \{x_h\}).$$

The space $R(\vec{Q}, v)$ is called the representation space. We will omit $\vec{Q}$ from the notation, when there is no ambiguity, and write just $R(v)$. Or we may also write

$$R(V) = \bigoplus_{h \in \Omega} \mathrm{Hom}_{\mathbb{K}}(V_{s(h)}, V_{t(h)})$$

for an $I$-graded vector space $V = \bigoplus V_i$. Obviously, $R(V) \simeq R(\dim(V))$.

Consider now the group

$$GL(v) = \prod_{i \in I} GL(v_i, \mathbb{K})$$

which acts in $R(v)$ by conjugation. Note that if $g \in GL(v)$ then $g = (g_i)_{i \in I}$ and if $x = (x_h)_{h \in \Omega} \in R(v)$ then $gx = ((gx)_{h \in \Omega}) \in R(v)$ where

$$(gx)_h = g_{t(h)} x_h g_{s(h)}^{-1}.$$

If we have an edge $h : i \to j$ then $(gx)_h = g_j x_h g_i^{-1}$ and the orbit of $x \in R(v)$ is

$$\mathbb{O}_x = \{gx; g \in GL(v)\}.$$

We will denote the orbit of $x \in R(v)$ by $\mathbb{O}_x$. We will also use the notation $\mathbb{O}_V$ for the orbit corresponding to a representation $V$, that is an abuse of language.

**Theorem B.1.1.** *Two elements $x, x' \in R(v)$ define isomorphic representations of $\vec{Q}$ if and only if they are in the same $GL(v)$ orbit.*

*Proof.* If $V_x \simeq V_{x'}$ then there exists $g : V_x \to V_{x'}$ that is an isomorphism. So each $g_i$ is invertible and $g_i \in GL(v_i, \mathbb{K})$ and then $g \in GL(v)$ therefore the diagram commutes,

$$\begin{array}{ccc} \mathbb{K}^{v_i} & \xrightarrow{x_h} & \mathbb{K}^{v_j} \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle g_j} \\ \mathbb{K}^{v_i} & \xrightarrow[x'_h]{} & \mathbb{K}^{v_j} \end{array}$$

which means that $x'_h g_i = g_j x_h$. Since $g_i$ is invertible there exists $g_i^{-1}$, so

$$x'_h = g_j x_h g_i^{-1} = (gx)_h$$

then $x' = gx$ and $x' \in \mathbb{O}_x$. Therefore, $\mathbb{O}_x = \mathbb{O}_{x'}$.

Conversely, if $x, x' \in R(v)$ are in the same orbit then $gx = x'$ which means $(gx)_h = x'_h$ for all $h \in \Omega$. So $x'_h = g_j x_h x_i^{-1}$ for every $h : i \to j$, then the diagram commutes,

$$\begin{array}{ccc} \mathbb{K}^{v_i} & \xrightarrow{x_h} & \mathbb{K}^{v_j} \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle g_j} \\ \mathbb{K}^{v_i} & \xrightarrow[x'_h]{} & \mathbb{K}^{v_j} \end{array}$$

Therefore $g : V_x \to V_{x'}$ is a morphism of representations with each $g_i \in GL(v_i, \mathbb{K})$. So $g_i$ is an isomorphism and $g$ is an isomorphism between representations, which means $V_x \simeq V_{x'}$. $\square$

In other words, we have a bijection

$$\text{(isomorphism classes of representations of graded dimension } v) \iff (\text{GL}(v)\text{-orbits in R}(v))$$

This result is particularly useful in the study of the classification of representations of quivers, as it allows us to reduce the problem of classifying isomorphism classes of representations to the problem of classifying orbits under the action of $GL(v)$.

**Example B.1.2.** *Let $\vec{Q}$ be the quiver given by*

$$\bullet \longrightarrow \bullet$$

*We want to find all the orbits related with the quiver $\vec{Q}$ with dimension $v = (1, 1)$. In this quiver we have only one edge in $\Omega$, so the representation space is given by*

$$R(v) = \bigoplus_{h \in \Omega} Hom\,(\mathbb{K}^{v_{s(h)}}, \mathbb{K}^{v_{t(h)}}) = Hom(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}$$

*and the group $GL(v)$ is*

$$GL(v) = \mathbb{K}^\times \times \mathbb{K}^\times,$$

*where $\mathbb{K}^\times$ are the nonzero elements of the field.*

*Let $x \in R(v)$ and $g \in GL(v)$. Then we have that $x \in \mathbb{K}$ and $g = (\alpha, \beta)$ with $\alpha, \beta \in \mathbb{K}^\times$. So*

$$gx = (\alpha, \beta)x = \beta x \alpha^{-1}.$$

*For $x = 0$ we have $\mathbb{O}_0 = \{0\}$. For $x = 1$ we have $gx = \beta\alpha^{-1}$ and by varying the $\alpha$ and $\beta$ we have all the elements of the field except the zero one, therefore $\mathbb{O}_1 = \mathbb{K}^\times$.*

*So*

$$R(v) = \mathbb{K} = \mathbb{O}_0 \cup \mathbb{O}_1.$$

*And we have two representations, up to an isomorphism, with dimension $v = (1, 1)$,*

$$\mathbb{O}_0 \simeq \mathbb{K} \xrightarrow{\;0\;} \mathbb{K}$$

$$\mathbb{O}_1 \simeq \mathbb{K} \xrightarrow{\;1\;} \mathbb{K}$$

## B.2 Properties of orbits

The group $GL(v)$ is a linear algebraic group acting on a finite dimensional vector space. In the next proposition we state the results of the general theory of orbits and algebraic groups that we will need in this chapter. For proofs and further details we suggest [21].

**Proposition B.2.1.**

*(1) Each orbit is a nonsingular algebraic variety.*

*(2) The closure of an orbit is a union of orbits in Zariski topology and for $\mathbb{K} = \mathbb{C}$ it is also in the analytic topology. Moreover, $\overline{\mathbb{O}} - \mathbb{O}$ is a union of orbits of smaller dimensions.*

*(3) For any $x \in R(v)$, the stabilizer subgroup*

$$G_x = \{g \in GL(v); g(x) = x\}$$

*is a closed algebraic subgroup in $GL(v)$, and we have natural isomorphism*

$$\begin{aligned} \mathbb{O}_x &= GL(v)/G_x, \\ T_x\mathbb{O}_x &= T_1 GL(v)/T_1(G_x), \end{aligned} \tag{B.1}$$

*where $T_1$ stands for the tangent space at identity. In particular,*

$$dim(\mathbb{O}_x) = dim(GL(v)) - dim(G_x).$$

*(4) There exists at most one orbit of dimension equal to $dim(R(v))$. If it exists, it is open and dense in $R(v)$ in Zariski topology. For $\mathbb{K} = \mathbb{C}$ it is also open and dense in the analytic topology.*

*(5) If $R(v)$ has a finite number of orbits then there exists an orbit $\mathbb{O}_x$ with $dim(\mathbb{O}_x) = dim(R(v))$.*

The properties listed are important for understanding the structure of representation spaces, which is a fundamental concept in representation theory. The fact that each orbit is a nonsingular algebraic variety and that the closure of an orbit is a union of orbits in the Zariski topology provides a way to analyze the structure of the space by studying its orbits. The concept of stabilizer subgroup is also useful in analyzing the orbits and understanding their dimension. The fact that there exists at most one orbit of dimension equal to the dimension of the representation space is also important, as it characterizes the structure of the space in terms of its orbits. Now we can see some properties that are specific to quiver representation.

**Theorem B.2.2.** *For every $x \in R(v)$, the stabilizer $G_x$ is connected in Zariski topology. For $\mathbb{K} = \mathbb{C}$ it is also connected in the analytic topology.*

*Proof.* By definition,

$$G_x = \{g \in GL(v);\ g_j x_h g_i^{-1} = x_h \text{ for any edge } h : i \to j\} = \mathrm{Aut}_{\vec{Q}}(V^x),$$

where $V^x$ is the representation of $\vec{Q}$ corresponding to $x \in R(v)$. We have that $\mathrm{Aut}_{\vec{Q}}(V)$ is the subset in $\mathrm{End}_{\vec{Q}}(V)$ consisting of operators $A$ satisfying $\det(A) \neq 0$. Thus, $\mathrm{Aut}_{\vec{Q}}(V)$ can be written as

$$\mathrm{Aut}_{\vec{Q}}(V) = \mathrm{End}_{\vec{Q}}(V) - \{A \in \mathrm{End}_{\vec{Q}}(V); \det(A) = 0\} = L - X,$$

where $L$ is a vector space and $X$ is an algebraic subvariety of codimension 1 closed in $L$. If $A \subset L - X$ is open then $A$ is open in $L$ since $A = (L - X) \cap B$ with $B$ open in $L$ and $X$ is closed in $L$. So $L - X$ is open in $L$ and $A$ is the intersection of two open subsets, which means that $A$ is open in $L$. Thus, any two nonempty open subsets intersect, and it is Zariski connected.

For $\mathbb{K} = \mathbb{C}$, connectedness in the analytic topology follows from the observation that if $m, m' \in L - X$, then the complex line through $m, m'$ intersects $X$ at finitely many points and thus $m, m'$ can be connected by a path in $L - X$. □

The importance of this theorem lies in its implications for the structure of the orbit $\mathbb{O}_x$. In particular, since $G_x$ is a closed subgroup of $GL(v)$, the orbit $\mathbb{O}_x$ is a homogeneous space $GL(v)/G_x$, which inherits the structure of a manifold from that of $GL(v)$ and $G_x$. The connectedness of $G_x$ in both the Zariski and analytic topologies ensures that $\mathbb{O}_x$ is connected as well, which is a key property for understanding the geometry and topology of the space of representations $R(v)$.

**Lemma B.2.3.** *For any $V, W \in Rep(\vec{Q})$, we have an exact sequence*

$$0 \longrightarrow Hom_{\vec{Q}}(V, W) \longrightarrow \bigoplus_i Hom_{\mathbb{K}}(V_i, W_i)$$

$$\longrightarrow \bigoplus_h Hom_{\mathbb{K}}(V_{s(h)}, W_{t(h)}) \longrightarrow Ext^1_{\vec{Q}}(V, W) \longrightarrow 0$$

*Proof.* We have the projective resolution for the representation $V$

$$0 \longrightarrow \bigoplus_{h\to\Omega} P(t(h)) \otimes \mathbb{K}h \otimes V_{s(h)} \xrightarrow{d_1} \bigoplus_{i\in I} P(i) \otimes V_i \xrightarrow{d_0} V \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_{\vec{Q}}(-, W)$ to the resolution we have

$$0 \longrightarrow \mathrm{Hom}_{\vec{Q}}(V, W) \longrightarrow \mathrm{Hom}_{\vec{Q}}\left(\bigoplus_{i\in I} P(i) \otimes V_i, W\right)$$

$$\longrightarrow \mathrm{Hom}_{\vec{Q}}\left(\bigoplus_{h\to\Omega} P(t(h)) \otimes \mathbb{K}h \otimes V_{s(h)}, W\right) \longrightarrow \mathrm{Ext}^1_{\vec{Q}}(V, W)$$

$$\longrightarrow \mathrm{Ext}^1_{\vec{Q}}\left(\bigoplus_{i\in I} P(i) \otimes V_i, W\right).$$

To prove that

$$\mathrm{Hom}_{\vec{Q}}(\bigoplus_{i\in I} P(i) \otimes V_i, W) \simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{K}}(V_i, W_i)$$

recall that

$$\mathrm{Hom}_A(L, \mathrm{Hom}_B(M, N)) \simeq \mathrm{Hom}_B(M \otimes_A L, N) \tag{B.2}$$

this can be found in [1, p. 126]. So,

$$\mathrm{Hom}_{\vec{Q}}\left(\bigoplus_{i\in I} P(i) \otimes V_i, W\right) \simeq \mathrm{Hom}_{\mathbb{K}}(V_i, \mathrm{Hom}_{\vec{Q}}(P(i), W)) \simeq \mathrm{Hom}_{\mathbb{K}}(V_i, W_i),$$

where the second equivalence is true by Theorem 1.4.3.

Using isomorphism (B.2) we have

$$\begin{aligned}\mathrm{Hom}_{\vec{Q}}\left(\bigoplus_{h\to\Omega} P(t(h))\otimes \mathbb{K}h\otimes V_{s(h)}, W\right) &\simeq \mathrm{Hom}_{\mathbb{K}}\left(\mathbb{K}h\otimes V_{s(h)}, \mathrm{Hom}_{\vec{Q}}(P(t(h)), W)\right)\\ &\simeq \mathrm{Hom}_{\mathbb{K}}\left(\mathbb{K}h\otimes V_{s(h)}, W_{t(h)}\right)\\ &\simeq \mathrm{Hom}_{\mathbb{K}}\left(V_{s(h)}, W_{t(h)}\right).\end{aligned}$$

Where the second equivalence is true by Theorem 1.4.3. The third equivalence is true since

$$\mathbb{K}h\otimes V_{s(h)} \simeq \mathbb{K}\otimes V_{s(h)} \simeq V_{s(h)}$$

by [1, p. 123].

Finally, to show that

$$\mathrm{Ext}^1_{\vec{Q}}\left(\bigoplus_{i\in I} P(i)\otimes V_i, W\right) = 0$$

we just need to show that $P(i)\otimes V_i$ is projective. Which means that $\mathrm{Hom}_{\vec{Q}}(P(i)\otimes_{\mathbb{K}} V_i, -)$ is exact. By (B.2) we have that

$$\mathrm{Hom}_{\vec{Q}}(P(i)\otimes V_i, -) \simeq \mathrm{Hom}_{\mathbb{K}}(V_i, \mathrm{Hom}_{\vec{Q}}(P(i), -)).$$

As $P(i)$ is projective then $\mathrm{Hom}_{\vec{Q}}(P(i), -)$ is exact and $\mathrm{Hom}_{\mathbb{K}}(V_i, -)$ is exact since $V_i$ is a $\mathbb{K}$ vector space with finite dimension. Therefore, $\mathrm{Hom}_{\vec{Q}}(P(i)\otimes_{\mathbb{K}} V_i, -)$ is exact. Since $P(i)\otimes V_i$ is projective then $\bigoplus_{i\in I} P(i)\otimes V_i$ is also projective and

$$\mathrm{Ext}^1_{\vec{Q}}\left(\bigoplus_{i\in I} P(i)\otimes V_i, W\right) = 0.$$

□

**Theorem B.2.4.** *Let $x\in R(v)$ and let $V^x\in Rep(\vec{Q})$ be the corresponding representation.*

*(1) We have that*

$$T_x\mathbb{O}_x \simeq \frac{End(v)}{End_{\vec{Q}}(V^x)} \text{ where } End(v) = \bigoplus_{i\in I} End(\mathbb{K}^{v_i}).$$

*(2) Let*

$$N_x\mathbb{O}_x = \frac{T_xR(v)}{T_x\mathbb{O}_x}$$

*be the normal space for the orbit $\mathbb{O}_x$. Then $N_x\mathbb{O}_x = Ext^1_{\vec{Q}}(V^x, V^x)$.*

*Proof.* (1) From (B.1) in Proposition B.2.1 item (3) we have that

$$\mathbb{O}_x = \frac{GL(v)}{G_x} \quad\text{and}\quad T_x\mathbb{O}_x = \frac{T_1GL(v)}{T_1G_x}.$$

So, $T_1GL(v) = \mathrm{End}(v)$ and $G_x = \mathrm{Aut}_{\vec{Q}}(V^x)$ then $T_1G_x = \mathrm{End}_{\vec{Q}}(V^x)$ therefore

$$T_x\mathbb{O}_x \simeq \frac{\mathrm{End}(v)}{\mathrm{End}_{\vec{Q}}(V^x)}.$$

(2) Using the exact sequence from Lemma B.2.3 with $V = W = V^x$ we have

$$0 \longrightarrow Hom_{\vec{Q}}(V^x, V^x) \longrightarrow \bigoplus_i Hom_{\mathbb{K}}((V^x)_i, (V^x)_i)$$

$$\longrightarrow \bigoplus_h Hom_{\mathbb{K}}((V^x)_{s(h)}, (V^x)_{t(h)}) \longrightarrow Ext^1_{\vec{Q}}(V^x, V^x) \longrightarrow 0.$$

Which leads us to the exact sequence

$$0 \longrightarrow \mathrm{End}_{\vec{Q}}(V^x) \xrightarrow{\alpha} \mathrm{End}(v) \xrightarrow{\beta} R(v) \longrightarrow \mathrm{Ext}^1_{\vec{Q}}(V^x, V^x) \longrightarrow 0.$$

So we have two short exact sequences

$$0 \longrightarrow \mathrm{End}_{\vec{Q}}(V^x) \xrightarrow{\alpha} \mathrm{End}(v) \xrightarrow{\beta} \mathrm{Im}(\beta) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}(\beta) \xrightarrow{\gamma} R(v) \longrightarrow \mathrm{Ext}^1_{\vec{Q}}(V^x, V^x) \longrightarrow 0.$$

From the first short exact sequence we have

$$\mathrm{Im}(\beta) \simeq \mathrm{Coker}(\alpha) \simeq \frac{\mathrm{End}(v)}{\mathrm{End}_{\vec{Q}}(V^x)} \simeq T_x\mathbb{O}_x.$$

Using that in the second sequence we have

$$0 \longrightarrow T_x\mathbb{O}_x \xrightarrow{\gamma} R(v) \longrightarrow \mathrm{Ext}^1_{\vec{Q}}(V^x, V^x) \longrightarrow 0.$$

Therefore we have that

$$\mathrm{Ext}^1_{\vec{Q}}(V^x, V^x) \simeq \mathrm{Coker}(\gamma) \simeq \frac{R(v)}{T_x\mathbb{O}_x} \simeq \frac{T_xR(v)}{T_x\mathbb{O}_x} = N_x\mathbb{O}_x.$$

□

**Corollary B.2.5.** *The orbit $\mathbb{O}_x$ is open in $R(v)$ if and only if $Ext^1_{\vec{Q}}(V^x, V^x) = 0$.*

*Proof.* If $\mathbb{O}_x$ is open in $R(v)$ then we have that $\overline{\mathbb{O}_x} = R(v)$ and $\dim(\overline{\mathbb{O}_x}) = \dim(R(v))$, since $R(v)$ is irreducible then every open set of $R(v)$ is dense in $R(v)$. We have that $\mathbb{O}_x$ is locally closed then $\dim(\mathbb{O}_x) = \dim\left(\overline{\mathbb{O}_x}\right)$. We also have that $\dim(\mathbb{O}_x) = \dim(T_x\mathbb{O}_x)$. Therefore, $\dim(T_x\mathbb{O}_x) = \dim(R(v))$ and by Theorem B.2.4

$$\mathrm{Ext}^1_{\vec{Q}}(V^x, V^x) = \mathrm{R(v)}T_x\mathbb{O}_x = 0.$$

Conversely, consider that $\mathrm{Ext}^1_{\vec{Q}}(V^x, V^x) = 0$. We saw in Theorem B.2.4 that

$$\mathrm{Ext}^1_{\vec{Q}}(V^x, V^x) \simeq \frac{R(v)}{T_x\mathbb{O}_x} = N_x\mathbb{O}_x = 0.$$

Then

$$\dim(R(v)) = \dim(T_x\mathbb{O}_x) = \dim(\mathbb{O}_x).$$

Therefore, $\mathbb{O}_x$ is open in $R(v)$ by Proposition B.2.1 item (4). □

This theorem is a fundamental result in the study of representation spaces and plays an important role in the theory of stability conditions and moduli spaces. It allows us to relate geometric properties of the orbit to algebraic properties of the Ext groups, and vice versa, which helps us to better understand the structure of representation spaces and their moduli spaces.

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