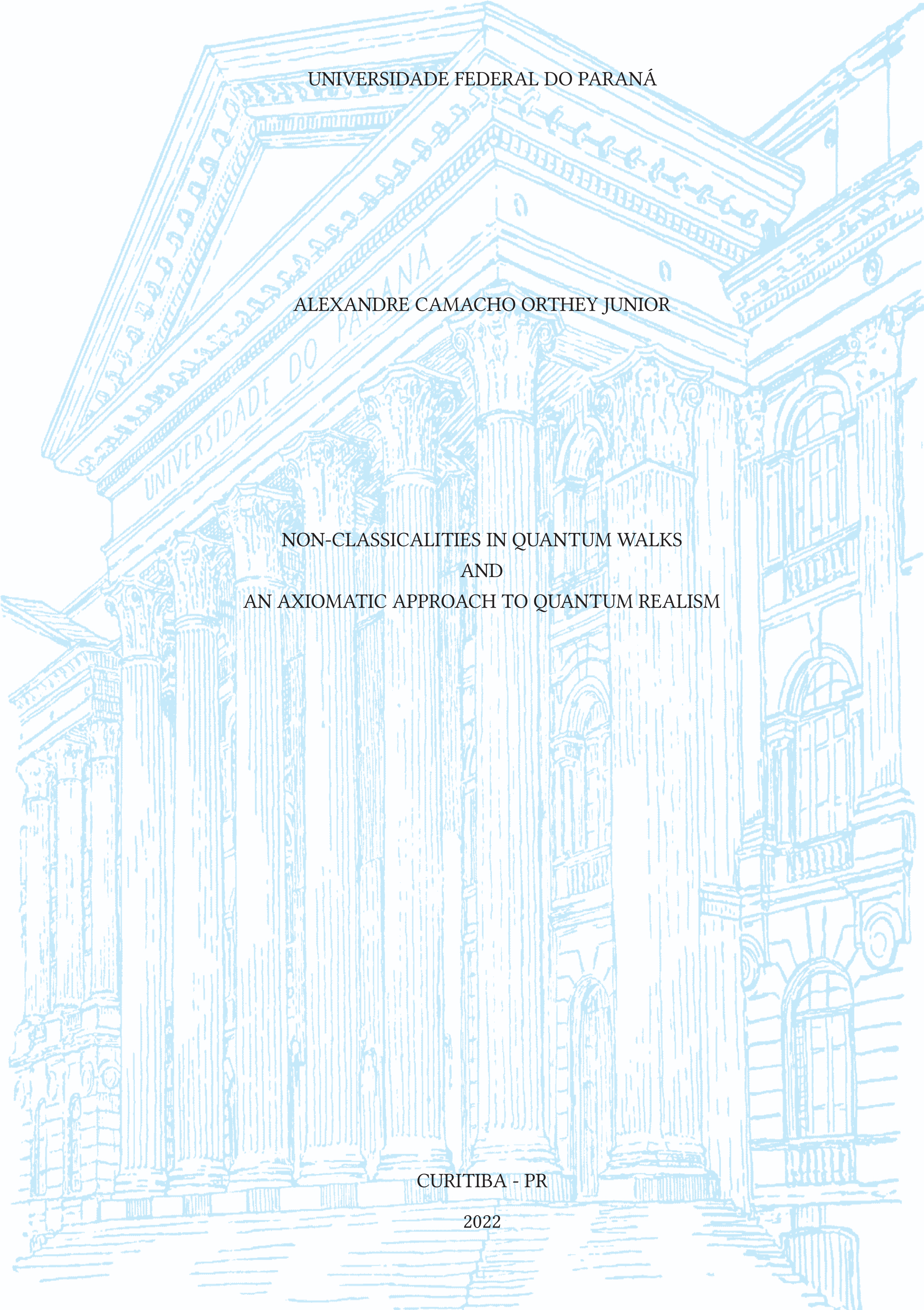

UNIVERSIDADE FEDERAL DO PARANÁ

ALEXANDRE CAMACHO ORTHEY JUNIOR

NON-CLASSICALITIES IN QUANTUM WALKS
AND
AN AXIOMATIC APPROACH TO QUANTUM REALISM

CURITIBA - PR

2022

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Tese apresentada como requisito parcial para a obtenção do título de Doutor em Ciências pelo Programa de Pós-Graduação em Física do Setor de Ciências Exatas da Universidade Federal do Paraná.

Orientador: Prof. Dr. Renato Moreira Angelo

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To all the LGBTQ+ Brazilian scientists.

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"Alexandre, you have to be quantum. The world is quantum!"
Prof. Fernando D. Sasse (personal communication, Joinville, Brazil, 2015)

RESUMO

No cerne das estranhezas da mecânica quântica estão a superposição de estados e a complementariedade de Bohr, noções conflitantes com a nossa percepção de realidade física macroscópica. Recentemente, uma hipótese de realismo foi formulada assumindo que a mecânica quântica constitui uma teoria física completa. Esta hipótese parte de uma ideia que é compartilhada também por defensores do Darwinismo Quântico: de que a codificação de informação sobre um dado observável em um grau de liberdade físico é uma condição necessária para que tal observável se torne um elemento de realidade física. Nesta tese, nós exploramos tal proposta de realismo dentro da teoria quântica em duas partes. Na Parte I nós estudamos um sistema físico conhecido como *caminhadas quânticas* e analisamos como se dá a emergência de realidade física objetiva de observáveis de spins durante a evolução de diversas não-classicalidades entre os subsistemas, a citar, não-localidade de Bell, direcionamento quântico, emaranhamento, discórdia quântica, irrealismo e não-localidade baseada em realismo. Motivados por esta análise, nós buscamos, na Parte II, nos aprofundar ainda mais no conceito de realismo dentro da mecânica quântica. Tomando a ideia de fluxo de informação do sistema para o ambiente como condição necessária para a emergência de realidade física, nós construímos uma axiomatização para o aqui chamado realismo *quântico*—em oposição ao realismo *clássico*. Nossa estratégia consiste em listar alguns princípios motivados fisicamente que sejam capazes de caracterizar o realismo quântico de maneira independente de "métrica". Introduzimos alguns critérios que definem monótonas e medidas de realidade e, em seguida, procuramos potenciais candidatos dentro de algumas teorias da informação célebres (entropias de von Neumann, Rényi e Tsallis) e também por medidas geométricas (distâncias do traço, Hilbert-Schmidt, Bures e Hellinger). Construímos explicitamente algumas classes de quantificadores entrópicos e geométricos, entre os quais que alguns satisfazem todos os axiomas propostos e, portanto, podem ser tomados como estimativas fiéis para o grau de realidade (ou definidade) de um dado observável físico. Nós esperamos que nossa estrutura possa oferecer uma base formal para futuras discussões sobre aspectos fundamentais da mecânica quântica.

Palavras-chaves: realismo. correlações quânticas. caminhadas quânticas. teoria de recursos quânticos. Darwinismo quântico.

ABSTRACT

At the heart of the strangeness of quantum mechanics are the superposition of states and Bohr's complementarity, notions that are in conflict with our perception of macroscopic physical reality. Recently, a realism hypothesis has been formulated assuming that quantum mechanics constitutes a complete physical theory. This hypothesis starts from an idea that is also shared by supporters of Quantum Darwinism: that the encoding of information about a given observable in a physical degree of freedom is a necessary condition for such an observable to become an element of physical reality. In this thesis, we explore such a proposal of realism within quantum theory into two parts. In Part I we study a physical system known as *quantum walks* and analyze how the emergence of objective physical reality of spin observables occurs during the evolution of several non-classicalities between subsystems, namely, Bell nonlocality, quantum steering, entanglement, quantum discord, irrealism, and realism-based nonlocality. Motivated by this analysis, we seek, in Part II, to get even further into the concept of realism within quantum mechanics. Taking the idea of information flow from the system to the environment as a necessary condition for the emergence of physical reality, we build an axiomatization for the here called *quantum* realism—as opposed to *classical* realism. Our strategy is to list some physically motivated principles that are capable of characterizing quantum realism in a "metric" independent way. We introduce some criteria that define monotones and measures of reality and then we look for potential candidates within some famous information theories (von Neumann, Rényi and Tsallis entropies) and also by geometric measures (trace, Hilbert- Schmidt, Bures, and Hellinger distances). We explicitly build some classes of entropic and geometric quantifiers, among which some satisfy all the proposed axioms and, therefore, can be taken as faithful estimates for the degree of reality (or definiteness) of a given physical observable. We hope that our framework can provide a formal basis for future discussions of fundamental aspects of quantum mechanics.

Key-words: realism. quantum correlations. quantum walks. quantum resource theories. quantum Darwinism.

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LIST OF ABBREVIATIONS

iff if, and only if,

CC Classical-Classical

CHSH Clauser, Horne, Shimony, and Holt

CJWR Cavalcanti, Jones, Wiseman, and Reid

CPTP Completely Positive Trace-Preserving

EPR Einstein, Podolsky, and Rosen

LF Local Friendliness

LHS Local Hidden State

LHVM Local Hidden Variable Model

LOCC Local Operations and Classical Communications

MUB Maximally Unbiased Bases

POVM Positive Operator-Valued Measure

QC Quantum-Classical

QD Quantum Darwinism

QRT Quantum Resource Theory

LIST OF SYMBOLS

▲	End of example
■	End of proof
$:=$	Equals by definition
θ, ϕ	Azimuthal and polar angles, respectively
$\mathcal{A}, \mathcal{B}, \ldots$	Partitions
$\mathfrak{B}(\mathcal{H}_{\mathcal{A},\mathcal{B},\ldots})$	Set of positive semi-definite linear operators that act over the Hilbert space $\mathcal{H}_{\mathcal{A},\mathcal{B},\ldots}$
$A, B, \ldots$	Operators in $\mathfrak{B}(\mathcal{H}_{\mathcal{A},\mathcal{B},\ldots})$
$S(\cdot)$	von Neumann entropy
$S_\alpha(\cdot)$	Rényi entropy
$S_q(\cdot)$	Tsallis entropy
$S_L(\cdot)$	Linear entropy
$S^{(q,s)}(\cdot)$	Generalized entropy
$D(\cdot\|\|\cdot)$	von Neumann relative entropy
$D_\alpha(\cdot\|\|\cdot)$	Rényi divergence
$D_q(\cdot\|\|\cdot)$	Tsallis relative entropy
$\mathfrak{I}_A(\rho)$	Irreality of A given ρ
$\mathfrak{R}_A(\rho)$	Reality of A given ρ
$\Phi_A(\cdot)$	Non-selective projective measurement of A
$H_{\mathcal{A}\|\mathcal{B}}(\cdot)$	Conditional quantum entropy of $\mathcal{A}$ given $\mathcal{B}$
$I(\rho)$	Quantum informational content of ρ
$I_{\mathcal{A}\|\mathcal{B}}(\cdot)$	Conditional quantum information of $\mathcal{A}$ given $\mathcal{B}$
$I_{\mathcal{A}:\mathcal{B}}(\cdot)$	Mutual quantum information between $\mathcal{A}$ and $\mathcal{B}$

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INTRODUCTION

When we try to understand at a deeper conceptual level the quantum mechanics predictions we usually do it through our perspective of objective physical reality. Such kind of attempt generally leads us to intriguing interpretations and conflicting ideas, although based on correct results. One important source of conflict is the superposition principle, which is far beyond the idea of *being in two places at the same time* because in order to *be* in two places such an object must really *be* there. Take a coherent quantum superposition such as $\sqrt{1-p}\,|0\rangle + e^{i\theta}\sqrt{p}\,|1\rangle$, for instance. The idea that a physical quantity (*e.g.*, spin or polarization) is an element of reality, that is, is well determined all the time independently of any observer, does not fit for such a preparation. In fact, this classical physics notion of reality is valid for a preparation like $(1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|$, which, however, is incapable of encapsulate the fundamental relative phase θ. Thus, in the absence of a conception of things that lie in an indefinite state, we appeal to our classical rationale. Consequently, when we are confronted with results from, *e.g.*, the double-slit experiment indicating that the particle interfered with itself, we assess the situation intuitively through our classical notion of objective reality and untimely conclude that the particle passed both slits at the same time (unless one is willing to accept nonlocal elements of reality, such as the Bohmian trajectories [1, 2]). Such a conclusion should at least cause a feeling of discomfort.

This thesis deals with the concept of realism and its violations in the context of quantum mechanics into two parts: the first one is more computational and focused on the analysis of a specific system (quantum walks); and the second one is more conceptual, where we will propose a set of axioms in order to better formalize the concept of *quantum realism* previously proposed in the literature. Before introducing these specific problems, we will address the issue regarding realism. In most cases the term realism is taken as a synonym for "classical reality", which may be identified with the dogma according to which all the systems exist and have well defined physical properties at every instant of time regardless the presence or action of any observer (brain-endowed systems). As we will see, quantum theory poses challenges when faced with this definition.

The discussion about elements of the physical reality in the context of quantum mechanics takes us back to the seminal work of Einstein, Podolsky, and Rosen (EPR) [3], where the authors call into question the completeness of the quantum theory. EPR proposed two conditions for the success of a physical theory: correctness and completeness. One should agree that, so far, quantum theory is correct since every one of its predictions agrees with the experiments. The idea of completeness, however, is more intricate: *"every element of physical reality must have a counterpart in the physical theory"*. Here, element of reality was defined in the following way: *"If, without disturbing the system in any way, we can predict with certainty*

(...) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity". In order to assess the completeness of quantum theory, EPR analyzed a *Gedankenexperiment* (thought experiment) based on position and momentum observables. Here, we are going to discuss Bohm's [4] version of that Gedankenexperiment, which is simpler. A pair of completely entangled spin-1/2 particles prepared in the singlet state

$$|\psi_s\rangle = \frac{|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}} \tag{1}$$

is spatially separated after its formation. Quantum theory predicts an anti-correlation of spin measurement outcomes for equal measurement directions. That means that, after the collapse of the wave function corresponding to the spin of the first particle, there must be an element of reality associated with the spin of the second particle—which is in a far site—because the observer can predict its result with certainty without disturbing it. Now, if one were to measure an orthogonal direction of the spin of the first particle, that is, an incompatible observable, the predictions would be the same as before (since the singlet is rotational invariant), which would lead to two different wave functions designated for the same reality of the far particle. Because EPR assume that no event can influence another one outside its light cone, the reality of both incompatible observables should have been established since the formation of the pair, rather than being induced by a *"spooky action at a distance"*. Since the uncertainty principle prevents simultaneous definite predictions for incompatible observables, the authors accepted the thesis that quantum theory is not complete. In other words, they claim that there are elements of reality not predicted by the theory. Therefore, there might be some lacking information in the quantum state $|\psi_s\rangle$—some *hidden variable*—that could in principle predetermine the apparently random result of a quantum measurement.

EPR's rationale was immediately confronted by Bohr [5], who argued that complementary physical quantities associated with incompatible observables cannot be elements of reality in the same experimental arrangement. Indeed, we can see that EPR use a *counterfactual definiteness* assumption: "if one *were to* measure", "the predictions *would be*", "which *would lead*"[1]. That is, EPR's claim is based on events that have not actually happened, but their measurement results are treated as defined values. Żukowsky and Brukner [6], for instance, argue that *"this is in striking disagreement with quantum formalism and the complementarity principle"*.

Three decades after the publication of EPR's paper, Bell (first in 1964 [7] and then in 1976 [8]) proved a no-go theorem claiming that any model based on local hidden variables cannot be consistent with the predictions of quantum mechanics. We will cover Bell's argument in more detail in the first chapter on fundamental concepts, but for now, it's enough to know the following. Based on a set of assumptions, Bell derived inequalities involving expected

[1] In fact, these are not phrases taken from the EPR article, but they have the same meaning stated by the authors. The reader can reach the same conclusion by assessing the article [3] right after Eq. (8) in p. 779.

values. The violation of these inequalities by any experiment necessarily implies the violation of at least one of the assumptions assumed by Bell.

One of the major problems regarding Bell's most famous result is that Bell's 1964 theorem [7] uses a different set of assumptions compared to Bell's 1976 theorem [8], nonetheless leading to the same inequality. These two sets are composed by, respectively: (i) *locality* (in the sense of *parameter independence*: Bob's inputs do not influence Alice's outcomes [9]) and *predetermination* (the existence of hidden variables that predetermine the result of a measurement [10]); and (ii) *local causality* (*"correlations between physical events in different space-time regions should be explicable in terms of physical events in the overlap of the backward light cones"* [8]). Common to both sets are also the extra hypotheses of: *no-superdeterminism* (choices in the future do not influence results in the present [11]), *absoluteness of observed events* (also called *macroreality*: every actually performed measurement has an observer-independent result [12]), and the existence of a relativistic *space-time* structure[2].

Given the undeniable success of quantum mechanics in fitting experimental data, what people do, in general, when reporting an experimental violation of a Bell inequality is to ignore the extra set of hypotheses (or assume it to be true in advance) and say for short that nature itself is incompatible with the local causality hypothesis. This phenomenon, conventionally referred to as Bell nonlocality [14], has been verified through several loophole-free tests [15–20]. Interestingly, local causality has been acknowledged as a compound assumption [6], stronger than locality but weaker than no-signaling (the impossibility of faster-than-light communication, which has never been seen violated), so that no tension whatsoever exists with relativity principles. Note that locality+predetermination do not imply local causality, being that the source of misunderstanding regarding Bell's theorems: it is possible to obtain Bell inequalities from both of them, but they are not equivalent.

Yet, others prefer to say that a violation of a Bell inequality implies that nature is in conflict with the *local realism* hypothesis, or *local realistic* theories, where *realism* could refer to the predetermination assumption (*e.g.*, the experimental articles [15–20]). However, realism in this case is a vague and not-specified term, since even Bell does not mention it in any of his seminal works[3]. Even so, some [10] choose to give up predetermination instead of locality in order to peacefully coexist with relativity, since quantum mechanics is already intrinsically non-deterministic. However, as we are going to see, giving up predetermination in order to make quantum mechanics a correct theory is not enough. It has been claimed that there is a behavior even more "quantum" than Bell nonlocality.

Very recently, an extraordinary development have bring the above debate to a new

[2] See the recent works of Wiseman and Cavalcanti [12, 13] for a very meticulous discussion on all the assumptions underlying Bell's theorem, upon which our discourse was based.

[3] Besides that, if realism in this case refers only to predetermination, why not use just *local predetermination* hypothesis instead? In the work of Gisin [21] we find a deeper discussion on the vagueness of the term realism when used in the context of Bell inequalities and why a proper definition of it still needs to be made.

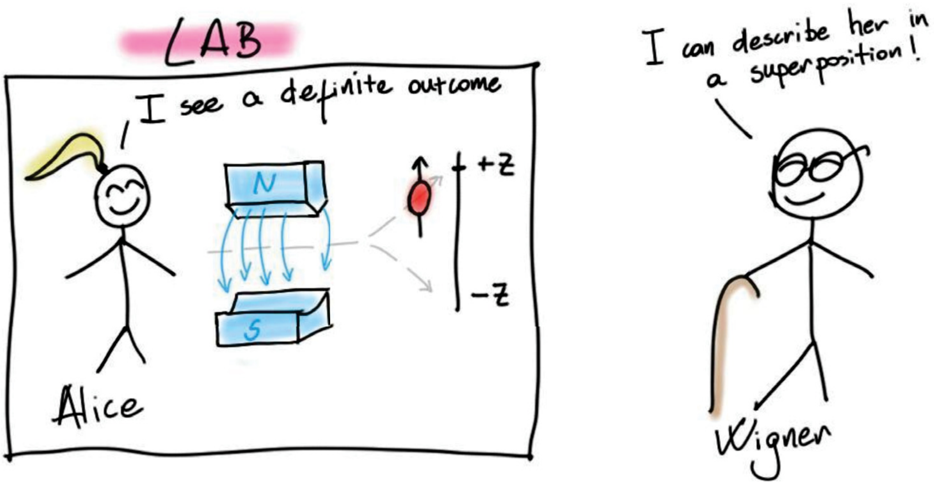

Figure 1 – The original Wigner's friend scenario (or paradox) [24]. In her laboratory, Alice performs a quantum measurement—represented here by a Stern-Gerlach experiment over a particle initially in $|\psi_0\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. After looking at the mark left by the particle on the wall in the position, say, $+z$, she registers a definite outcome and assigns to her particle the spin state $|\psi\rangle = |\uparrow\rangle$. Wigner, on the other way, lies outside the perfectly isolated lab. From his point of view, he can assign to Alice and her particle an entangled global state $|\Psi\rangle = (|\text{Alice registers } +z\rangle\,|\uparrow\rangle + |\text{Alice registers } -z\rangle\,|\downarrow\rangle)/\sqrt{2}$. There is nothing in quantum theory that forbids him to do that. Only if he opens the door lab, one of the two realities for Alice and her particle will be settled. The paradox is: from her perspective, she was never in superposition. Who is right? Actually performed measurements always have absolute results? Or the reality status depends on the observer?

level. Based on the well-known Wigner's friend scenario (see Fig. 1), Brukner [22] proposed another no-go theorem. In his approach, Alice and Bob are superobservers performing quantum operations on their respective space-like separated friends' laboratories. Charlie and Debbie (Alice's and Bob's friends, respectively) share an entangled pair of quantum particles, and by making measurements on it they establish correlations on the posterior measurements made by Alice and Bob. Inspired on that, Bong *et al.* [23] have proved and experimentally verified that *"if a superobserver can perform arbitrary quantum operations on an observer and its environment, then no physical theory can satisfy"* the entire *Local Friendliness* (LF) set of hypotheses: absoluteness of observed events, no-superdeterminism, and locality. Note that predetermination is not included here, but if we add it we obtain the conditions to the first Bell's theorem. This means that it is possible to have a scenario where we observe a violation of a Bell inequality while holding every assumption of the LF set. The opposite, however, is impossible.

Whatever perspective one may adopt in assessing the quantum phenomena, the task of combining the algebraic structure of the theory with the experienced physical reality is always an issue. In effect, it has been suggested that many of the interpretations of quantum mechanics known to date can be divided into two groups, depending on their attitude toward (the emergence of) realism [25, 26]. They are: *intrinsic realism* and *participatory realism*. The first one advocates the view that the measurement probabilities (given by $|\psi|^2$) are properties intrinsic only, and just only, to the observed system itself. In this sense, intrinsic realism can be divided into another two groups: the ψ*-ontic* view, which says that there is an objective reality and the wave function ψ represents this reality on its whole; and the ψ*-epistemic* view, which says that ψ represents partial knowledge about an underlying physical reality. Participatory realism, on the other way, endorses precisely the opposite. According to that view, the probabilities depend on not just the system itself but also on the observer's experience.

Hypothesis sets such as the LF one reinforce the above divisions over interpretations of quantum mechanics. However, the failure of the LF set can be due to any one of its three assumptions, and the experiment performed by Bong *et al.* [23] cannot tell us which one it is. Those who opt for an interpretation of quantum mechanics based on intrinsic realism should accept absoluteness of observed events at the expense of locality or no-superdeterminism, as are the cases of Bohmian mechanics [1, 2] or retrocausal models [27], respectively. Those in favor of participatory realism deny that actually observed events have results that are independent of any observer, as it is the case with the Copenhagen interpretation [28], the relational quantum mechanics [29], and QBism [30].

Let us quickly summarize what we have discussed so far. Violations of Bell inequalities are presented as evidence that nature is incompatible with local realistic models. However, there is still no consensus on what is meant by "realistic" within quantum mechanics. Each type of interpretation of quantum mechanics has a different view on this issue (and it is *really* an issue since superpositions are macroscopically counterintuitive). Furthermore, it has been argued that no-go theorems such as those proposed by Bell [7, 8], Brukner [22], and Bong *et al.* [23] provide us with the basic assumptions behind the classical behavior that are keys to choosing between different interpretations of quantum mechanics.

One of the prominent frameworks accounting for the emergence of an objective reality from the quantum substratum that is claimed to be *"free of interpretation"* is Quantum Darwinism (QD) [31]. Corroborated by recent experiments [32–34], this model claims that reality emerges when information about a quantum system gets prolifically copied into the environment. While in decoherence theory the environment is taken as an information sink, in QD the environment is structured, containing several fragments that interact with the system and correlate with it. In the environment, different observers access the same information about the system through redundant copies encoded in different fragments of the environment. When different observers can agree that they have accessed the same information (and so the access must be innocuous), the quantum state of the system is said to exist objectively. The term "Darwinism" comes, then, from the concept that only states that multiply (or procreate) their information several times in the environment can survive and, therefore, emerge from the quantum world to macroscopic classicality.

A very recent result on QD [35] suggests that there is a certain universality in the way in which the fragments of the environment acquire information about the system of interest. The more fragments that are correlated with the system (no matter how weak the correlation), the more mutual quantum information is shared between system and environment and, consequently, the more objectively real the state of the system becomes. Thus, once we accept a non-instantaneous transition to classical reality, it makes sense thinking of an intermediate state of affairs, the one prior to the definitive achievement of realism. Presumably, any gradation of "nonrealism" would be possible *a priori*. This was the intuition leading Bilobran and Angelo (BA) to introduce the so-called *irreality* (the complement of reality)—an operational quantifier intended to diagnose how far a given physical quantity is from full definiteness [36]. The criterion of realism envisaged in this approach, henceforth referred to as *BA's realism hypothesis*, does not imply full classical reality, since situations are shown to exist where the z-component of spin is an element of reality whereas the x-component is not. The basic premise employed by BA is that a measurement establishes an element of reality for the measured observable, even when the measurement outcome is not revealed. If a given state is not altered by an unrevealed measurement, then this means that this state already implied an element of reality before the measurement. Hence, the uncertainties associated with the measured observable are of subjective essence and the state is epistemic. Many developments followed from this framework, from a novel notion of nonlocality [37–39] to foundational aspects of quantum theory [40, 41] and their proof-of-principle [42, 43] to the realization that irreality is a quantum resource [44].

The abolishment of the idea that all observables are well determined all the time in a classical sense, leads to alternative and enlightening new perspectives, as Engelbert and Angelo have shown in their discussion on Hardy's paradox [41]. In their work, the authors' analysis suggests that if one gives up classical realism, then locality can be restored and the Hardy's paradox vanishes. We believe that a similar approach can lead us to a better understanding of the nonclassical behaviors characteristic of other quantum systems. Throughout this thesis, we are going to reject *classical* realism and accept that sometimes some physical quantities do not have definite values previously to the measurement—and that is not a matter of subjective ignorance. In other words, we are going to accept *quantum* realism from BA's view [36] and we are going to study it on two fronts: Parts I and II. First, we will analyze a particular system from the point of view of quantum realism in order to observe how the emergence of physical reality occurs during the evolution of several distinct non-classical features—or *non-classicalities*. The system under scrutiny in our work obeys a dynamics known as quantum walk and the following section is entirely devoted to introducing this type of dynamics and which non-classicalities we are interested in. In the second part, we will direct our attention to the quantifier of physical reality proposed by BA. Inspired by well-established quantum resource theories such as coherence and entanglement, we will propose an axiomatization for quantum realism in an attempt to further formalize its conceptual framework. There is a section devoted to this matter on the next pages.

Before starting, it is useful to spell out the meaning that shall be presumed from the term "quantum realism" throughout our work. It is not connected to the *existence* of a system, which is taken for granted from the outset, but rather to the *definiteness of a physical quantity* prior to any observer's intervention. When this scenario is realized, the corresponding quantity is said to be an *element of reality*. Unlike classical reality, quantum realism does not presume all physical quantities to be elements of reality simultaneously. "Definiteness", by its turn, does not mean "total absence of uncertainties", which would be equivalent to the condition of full predictability appearing in EPR's approach. It actually refers to the absence of quantum uncertainties for a particular physical quantity.

Without further ado, let us introduce the two topics that we are going to talk about within quantum realism.

QUANTUM STUFF IN QUANTUM WALKS

Originally introduced as quantum versions of classical random walks—with some foundational motivations and potential applications to quantum optics—, quantum walks [45] have now achieved the status of an ubiquitous tool for studies in areas like quantum computation [46–48], quantum thermodynamics [49, 50], and foundations of quantum theory [51]. Generically speaking, a quantum walk refers to the dynamics of a particle (the walker) whose motion is conditioned to some internal degree of freedom ("the coin"). Some of the usual formulations of this problem consist of confining the walker motion to a dimensionless discrete structure of space-time and modeling the internal coin with a spin-1/2 algebra. By virtue of the superposition principle, interference patterns typically develop over time, which produces a distinctive mark of quantum walks, namely, ballistic spreading [52] (see Fig. 2). Interestingly, the mathematical formalism of quantum walks is platform independent, meaning that other physical quantities can be used as internal and external degrees of freedom. In fact, it has been shown that energy levels [51] or light polarization [53] perfectly implement the notion of internal coin, whereas the walker position can be suitably emulated with time encoding [53], photonic orbital angular momentum [54], or even actual physical position [55]. References [56, 57] are excellent starting points for the study of quantum walks and Ref. [58] offers a review of physical implementations.

Another relevant feature of a quantum walk is the ability to produce quantum resources. Since information about the spin is shared with the position every time the particle takes a step, quantum correlations are created between these degrees of freedom, especially in the form of entanglement [59, 60]. For instances involving two quantum walkers [61–64], the production of nonclassical features becomes even more sophisticated. Different partitions exist and entanglement can be found between the subsystems (position-spin of one particle with position-spin of another particle) [65, 66], the spins [67], and the positions [68]. Incidentally, it is precisely the presence of interaction—and entanglement—between the walkers that makes it

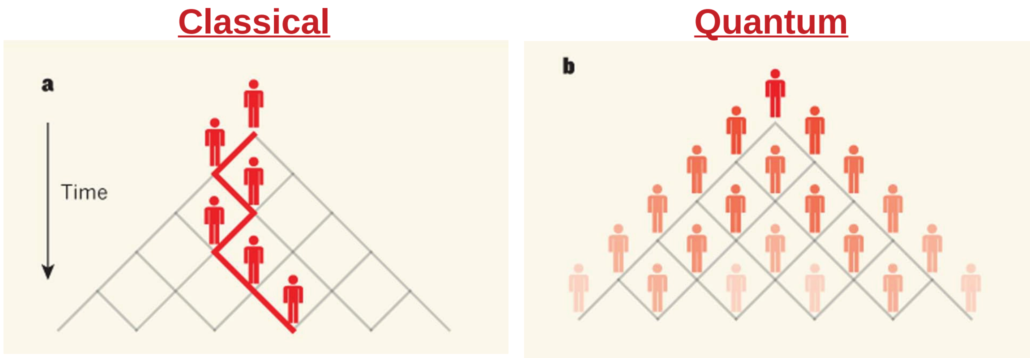

Figure 2 – While in a classical random walk (a) we can see well defined trajectories guided by, say, the result of a coin toss game, in a quantum walk (b) the superposition principle allows the walker to be in a superposition of spatial positions. Figure source: L. Sansoni, "Integrated devices for quantum information with polarization encoded qubits", Ph.D. Thesis presentation (Sapienza Università di Roma, 2012).

possible to solve, for example, a wider range of graph isomorphism problems when compared to noninteracting walkers [69]. However, to the best of our knowledge, there is no diagnosis of the presence or dynamical creation of other quantum resources during a quantum walk. Such a resource overview may lead to new perspectives for the use and generation of quantum resources in the fields where the quantum walks apply.

The first part of this thesis aims at advancing the above-delineated framework by systematically dissecting a given two-particle quantum walk with respect to its potentialities in producing several types of nonclassical features, in particular, violations of BA's realism hypothesis as well as general quantum correlations. We intend to analytically assess the behaviors (over time and asymptotically) of some well-established notions, such as entanglement and genuine multipartite entanglement [71], quantum discord [72, 73], symmetrical quantum discord [74], quantum steering [75, 76], Bell nonlocality [7, 14], realism-based nonlocality [37], and, most importantly, quantum irreality [36].

While the global state evolves unitarily, thus conserving its initial degree of purity, we can say in advance—as we are going to show in Chap. 2—that most of the aforementioned non-classicalities decrease with time between the bipartitions of the system, with some eventual occurrences of sudden deaths. On the other hand, some spin observables are shown to persist violating realism even when the walkers are arbitrarily far apart from each other and some noise is introduced in the initial two-spin state. This implies that all the involved degrees of freedom remain quantumly linked throughout the time evolution so that no individual element of reality can be claimed to exist.

The investigation proposed above can be found in Chap. 2, whereas the necessary fundamental concepts are presented in Chap. 1. In Sec. 2.1 and 2.2, we introduce a simplified model which proves crucial for our purposes. This model offers considerable analytical power for the treatment of the problem as it avoids the implementation of recursive codes to treat matrices whose dimension increases with time as $4(2t+1)^2$. In Sec. 2.3, we show how our results regarding entanglement between spin and position of one particle agrees with previous studies. In Sec. 2.4, we show that genuine fourpartite entanglement increases over time in the global state, thus "conserving" the total amount of resource furnished initially. Section 2.5 provides an exhaustive study of the dynamics of the nonclassical features associated with the two-spin state, thus regarding the spatial degrees of freedom as an external noisy channel. Concluding remarks about this part of our work are reserved to Chap. 3.

Everything in Chap. 2 are original results and were published at the end of the second year of this doctorate [A. C. Orthey and R. M. Angelo, "Nonlocality, quantum correlations, and violations of classical realism in the dynamics of two noninteracting quantum walkers", Phys. Rev. A **100**, 042110 (2019)].

DEEPER INTO QUANTUM REALISM

BA's reality condition check protocol is practical and intuitive. If an unrevealed measurement of a physical quantity does not change the state of the system, then that measurement was innocuous and it only accessed a previously defined value. In other words, there was no creation of entanglement between the system and the environment (*i.e.*, observers plus measurement apparatuses). With classical coins, this reasoning becomes even clearer. After being thrown up, a coin lands on the back of one hand and we hide it with the other. The unrevealed measurement is done as the result is already set. Raising our hand and revealing the face of the coin that was up will in no way change its result. In this sense, macroscopic objects like a coin easily satisfy the idea of classical realism: all physical quantities are well defined all the time—even before measurement—independent of observers. Notwithstanding, the interested reader might ask: if the macroscopic coin satisfies classical realism, then how can it have a well-defined result while spinning in the air? In this case, someone could take sequential photos of the coin and record its trajectory. At all times, the coin will have a single side facing up (albeit tilted) and the act of photographing does not change this fact[4]. At no time is the coin in a superposition of states.

Once we are faced with the oddities of quantum mechanics, the notion of classical realism needs to be relaxed, giving space to the aforementioned quantum realism. This is necessary mainly due to the Heisenberg uncertainty principle, which sometimes prohibits the definiteness of incompatible observables simultaneously. When we switch from base z to base x, a spin-up state $|+z\rangle$—which contains an element of reality for $S_z = \frac{\hbar}{2}\sigma_z$ because $S_z\,|+z\rangle = \frac{\hbar}{2}\,|+z\rangle$—reveals itself in superposition for the S_x observable, which configures the

[4] Of course, modeling a system that generates random numbers with a classical coin ends up encountering unavoidable difficulties when we start making deeper questions because the coin is a macroscopic object and has a side face. At some instants (comprising a measure zero set in the continuous-time interval) the coin will be vertically aligned. This does not make the measurement result undefined, after all, there is only one side of the coin that is facing up: the side face.

absence of an element of reality in this new direction.

Let us be a little more technical. Adhering to BA's criterion, the quantum state

$$\rho = (1-p)\,|a_1\rangle\,\langle a_1| + p\,|a_2\rangle\,\langle a_2|, \tag{2}$$

with $A\,|a_{1,2}\rangle = a_{1,2}\,|a_{1,2}\rangle$, is then taken as an example of scenario in which quantum realism is established for the observable A, even though mere subjective uncertainties are present when $p \in (0,1)$. To see this, we check what happens with the state when it is submitted to a nonselective measurement of $A = \sum_i a_i A_i$ (where $A_i = |a_i\rangle\,\langle a_i|$). Since $\sum_i A_i \rho A_i = \rho$, then BA's criterion of realism is satisfied, meaning that the element of reality that would presumably be installed by the measurement is already there before the measurement. In other words, in this case, reality is not dictated by the measurement and the present uncertainties only reflect subjective ignorance (ρ is an epistemic state). This is consistent with the fact that no "interference pattern" would be observed with respect to the observable A. Thus, while in BA's approach A is an element of reality for all p, in EPR's this is so only when $p = 0$ or $p = 1$ (full predictability regimes). The differences between these approaches can be further emphasized in multipartite settings. As thoroughly discussed in Ref. [39], while EPR would claim that the spin observables $S_{x,z}$ are simultaneous elements of reality for the singlet state, BA's criterion implies that these observables actually are maximally irreal (see Example 7 in Sec. 1.3.5). As another example, consider the bipartite separable state $\rho_{\text{sep}} = \sum_\lambda p_\lambda \rho_\lambda^{\mathcal{A}} \otimes \rho_\lambda^{\mathcal{B}}$. It immediately satisfies Bell's local causality hypothesis but does not imply BA realism for a vast set of observables, since $\sum_i (A_i \otimes \mathbb{1}_{\mathcal{B}})\rho_{\text{sep}}(A_i \otimes \mathbb{1}_{\mathcal{B}}) \neq \rho_{\text{sep}}$.

So is it all a matter of superposition of states? In other words, *does quantum realism equal incoherence?* The short answer is **no**. In multipartite systems, quantum correlations play a crucial role in the emergence of physical reality from the quantum world. In fact, as we are going to see in detail in Sec. 1.3.5, irreality equals coherence plus non-optimized quantum discord [36]. When subsystems of composite systems interact, they share information about their states. The more information is shared, the more the subsystems move towards realism, which is one of the main results of Dieguez and Angelo in Ref. [40]. This view is, in a way, also shared by supporters of QD, with the extra condition that the subsystem of interest must share information with several many other subsystems so that we have objective reality.

So far, so good. So, what is the problem? The first problem regarding BA's realism that we can think of here is the very origin of its quantifier. BA took an operational approach—specifically connected with non-selective collapse—and chose a particular monotone in terms of the von Neumann entropy to measure how far a quantum state is from full realism. Because it was an *ad hoc* choice, that approach does leave space for discussions about the existence of more general monotones and the possible connection with general quantum information theories and measurement theories. In fact, as it was conceived, BA's measure of physical reality carries with it features that derive from intrinsic properties related to the von Neumann entropy. In this sense, it is difficult to separate what is an attribute pertinent to quantum realism itself from those that are mathematical consequences of the chosen monotone (*e.g.*, concavity, additivity, etc.).

Must the amount of quantum realism of a composed system equal the sum of its parts? Must it increase or decrease due to quantum operations like the discard of information? And what about the realism regarding incompatible observables? Should there be an upper bound for the sum of them? Once we chose the measure proposed by BA as “the true one”, the answers to the above questions just become consequences of such a quantifier. In fact, it is actually natural in the development of physics that we accept that nature behaves in this or that way based on mathematical consequences. Experimental violations of Bell’s inequalities, for instance, were not reported until almost two decades after the theoretical prediction was published [77]. The same goes for gravitational waves detected nearly a century after the publication of Einstein’s work on general relativity [78]. The difference between these examples and BA’s measure of physical reality is that the latter was not completely modeled from physical principles. Indeed, it accuses full realism for an observable only, and just only, for states of reality regarding that observable. However, the construction of that measure lacks a more profound physical motivation.

Given the above, it seems very difficult to figure out what quantum mechanics is all about without a proper framing of the notion of realism. Moving in this direction, the second part of this thesis constitutes an attempt to formalize the idea of quantum realism and, therefore, a measure of physical reality through an axiomatization. Our axioms will be physically motivated and connected to an informational description of the measurement dynamics. Let us explore this kind of description through the following example.

The double-slit experiment with electrons[5] shows us how matter can sometimes present wave-like behavior [see Fig. 3 (a)]. In this experiment, we can interpret that elements of physical reality cannot be assigned to the position of the particle after it passes through the double-slit. In other words, we can say that its position is not real, or that the particle and its attributes do not satisfy the idea of classical realism. On the other hand, the particle can satisfy a slightly less restricted notion of realism for another of its attributes (*e.g.*, momentum), being that one of the central ideas of BA’s quantum realism. However, when the double-slit is preceded by a lightweight slit that registers the direction taken by the particle [see Fig. 3 (b)], the interference pattern disappears. Proposed by Bohr [5] and experimentally realized by Liu *et al.* [84], this experiment shows how the acquisition of information about the position of the particle by another degree of freedom precludes its wave-like behavior. The momentum of the lightweight slit and the position of the particle become entangled, which results in a loss

5 When we look at history, diffraction effects with electron beams had already been reported by Jönsson in 1961 [79]. However, it was not until 1965 that Feynman [80] argued that true evidence of the wave behavior of matter in such an experiment would only be achieved with single electron shots. It was only in the following decade that such an accomplishment was performed, this time by Merli *et al.* [81] still with an electron biprism—not an actual double-slit. In fact, an experiment with actual slits—just like Fig. 3 (a)—was reported by Batelaan and colleagues in 2013 [82, 83].

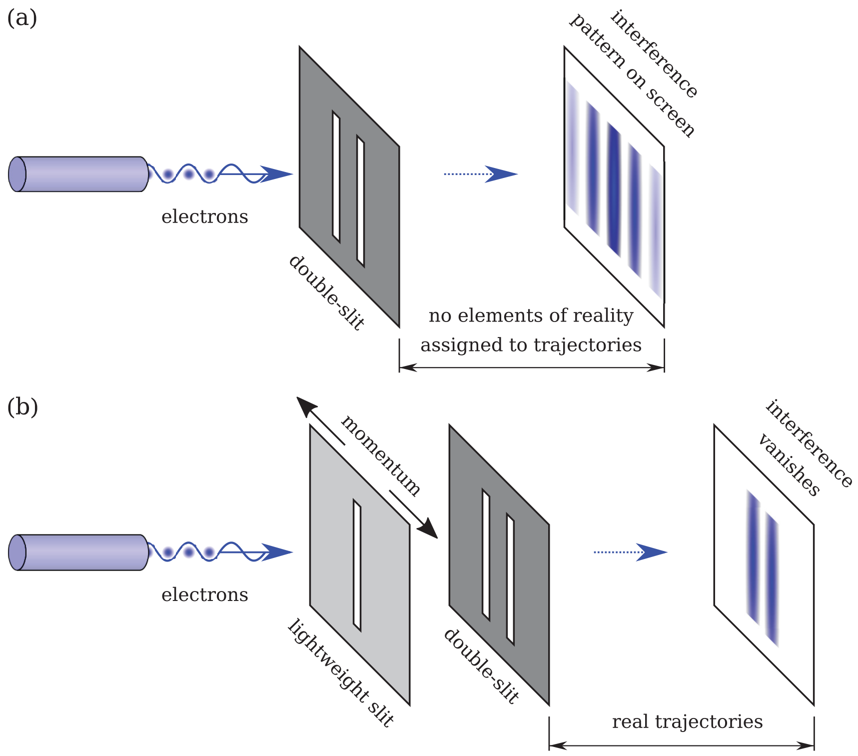

Figure 3 – (a) Depiction of a standard double-slit experiment with electrons. In the absence of information about which slit the particle passed through, it is not possible to assign an element of reality to the trajectory of the particle between the slits and the screen. In this case, electrons behave like a wave and we see an interference pattern on the screen. (b) Modified double-slit experiment, preceded by a lightweight slit. Here, the lightweight slit acts as a path informant. By conservation of momentum, the slit moves in the opposite direction to the electron's path, keeping information about which subsequent slit the electron passes through. When we look at the screen, we notice that the interference pattern disappears. In this case, we can say that there is an element of reality associated with the position of the electron since it behaved like a particle. Figure source (with modifications): https://commons.wikimedia.org/wiki/File:Double-slit.svg

of coherence for the position state of the particle. With that, we say that the trajectory of the particle became real and the process by which that happened was the flux of information from one subsystem to the other. This interpretation is supported by some recent works like Refs. [36, 42, 43, 85], especially Ref. [40]—whose arguments will be revisited in Appendix A.

Our first axiom for quantum realism will then be given by the direct equivalence between *the flow of quantum information from the measured system to the environment* and *the increase in the degree of reality of the observable that is been accessed*. What we are trying to achieve with this definition is precisely to stipulate a structure for the variation of the degree of reality in terms of informational dynamics. We hope with this to obtain a functional whose form comes from an actual physical process. However, the way in which we are going to measure the quantum information in the measurement dynamics will depend on the quantum information theory to be chosen. Since the variation in the quantum information of interest is due to a quantum operation (an unrevealed projective measurement), it is necessary to use a quantum *conditional* information measure (as opposed to the standard quantum conditional entropy). Generally speaking, conditional information measures can be constructed by any quantum divergence or distance between the state of interest and a state that contains zero conditional information, *i.e.*, states such as $\rho = \rho_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}/d_{\mathcal{B}}$ and $\rho = \mathbb{1}_{\mathcal{A}}/d_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$, where $\rho_{\mathcal{A},\mathcal{B}} = \mathrm{Tr}_{\mathcal{B},\mathcal{A}}(\rho)$ and $d_{\mathcal{A},\mathcal{B}} = \dim \mathcal{H}_{\mathcal{A},\mathcal{B}}$. Our search for quantifiers, therefore, takes place within the quantum information theories of Rényi [86–100] and Tsallis [101–106], whose scopes extend the one induced by the von Neumann entropy [107], because they provide us with well-established quantum divergences. In addition, we also extend our search within some measures of geometrical nature, namely, the trace distance, the Hilbert-Schmidt distance [108, 109], and the Bures [109, 110] and Hellinger distances [109, 111].

The axioms that follow will be inspired by the formal structure of Quantum Resource Theories (QRTs) like those of entanglement [112] and coherence [113]. We are going to revisit the concept of QRTs in Chap. 4, but for now we can say in advance that they are useful frameworks to characterize and catalog quantum resources based onto two concepts: free states (null resource states) and free operations (operations that do not create more resource). Although quantum realism cannot be considered as a quantum resource *per se*, we can think of it as the amount of quantum resource that is destroyed by a measurement. Thus, the idea of free states and free operations that constitute QRTs comes in handy. Finally, once we have specified our axioms, we can divide our reality quantifiers into two categories: *reality monotones* and *reality measures*, the former requiring a smaller set of axioms to be satisfied.

Continuing our search for the formalization of quantum realism, we will close the results of this thesis with a talk on QD. As we have already seen, QD and quantum realism share the idea that a system moves towards classicality when there is the storage of information about its state in another degree of freedom. The fundamental difference between those views lies in the fact that the QD requires a large environment for the information to be redundantly copied and, thus, be available to several observers. What is the logical implication between QD and quantum realism? Is one more fundamental than the other? Inspired in Ref. [35] we will show in Chap. 7 that the objective reality of QD necessarily implies the existence of BA reality elements—at least for qubit systems.

The second part of the thesis is organized as follows. In Chap. 4, we deeply review the concepts of quantum divergences and distances, the elements of the aforementioned quantum information theories (von Neumann, Rényi, and Tsallis), and we briefly review QRTs. In Chap. 5, we present our list of axioms for quantum realism. In Chap. 6, we explicitly build reality monotones and measures in consonance with the proposed axioms. In Chap. 7 we make the connection between QD and quantum realism. Concluding remarks on Part II are left to Chap. 8. Final comments about the whole thesis can be found in Chap. 9

PUBLICATIONS AND WORKS IN PROGRESS

This doctoral thesis resulted in the following works:

- A. C. Orthey and R. M. Angelo, “Nonlocality, quantum correlations, and violations of classical realism in the dynamics of two noninteracting quantum walkers”, Phys. Rev. A **100**, 042110 (2019);
- A. C. Orthey and R. M. Angelo, “Quantum realism: Axiomatization and quantification”, Phys. Rev. A **105**, 052218 (2022).

Other works in preparation:

- A. C. Orthey, “Geometrical quantifiers for quantum realism”;
- A. C. Orthey, “Quantum Darwinism implies Quantum Realism for qubits”.

In preparation but not related to this thesis:

- A. C. S. Costa, A. C. Orthey, R. Uola and S. Wollmann, “The effect of random measurements for the detection of quantum steering”.

Published during doctorate but not related to this thesis (results from Master dissertation):

- A. C. Orthey and E. P. M. Amorim, “Connecting velocity and entanglement in quantum walks”, Phys. Rev. A **99**, 032320 (2019);
- A. C. Orthey and E. P. Amorim, “Weak disorder enhancing the production of entanglement in quantum walks”, Braz. J. Phys. **49**, 595 (2019).

Part I

QUANTUM STUFF IN QUANTUM WALKS

1 FUNDAMENTAL CONCEPTS I

This chapter aims to revisit all the fundamental concepts needed for the text that follows. Nothing here is new except for the notation. We start with the concepts of classical and quantum entropies and their elements of information theory, move on to non-classicalities, and end with an introduction to quantum walkers.

1.1 ENTROPIES

The word *entropy* was introduced by Clausius in his 1865 work (originally in German [117], but some translated parts to English can be found in [118]) to designate the quantity S in the thermodynamic relation

$$\Delta S = S_t - S_0 = \int_0^t \frac{dQ}{T}, \tag{1.1}$$

where Q and T stand for heat and absolute temperature, respectively. In his own words, Clausius stated:

> *"I propose to name the magnitude S the entropy of the body, from the Greek word τροπή [tropē], a transformation. I have intentionally formed the word entropy so as to be as similar as possible to the word energy, since both these quantities, which are to be known by these names, are so nearly related to each other in their physical significance that a certain similarity in their names seemed to me advantageous."* R. Clausius in Ref. [118].

Some years later, Boltzmann [119] obtained a statistical formulation for Clausius' entropy, resulting in the relation

$$\frac{\Delta S}{\Delta(\ln \Omega)} = \text{const.}, \tag{1.2}$$

where Ω stands for the number of all possible microstates available to the system. However, it was Planck [120] who presented the formula

$$S = k_B \ln \Omega \tag{1.3}$$

with the so called Boltzmann constant k_B and no arbitrary additive constant S_0. Here the meaning of the word entropy becomes less nebulous. Since Ω is associated with the probability of finding the system in a given macrostate, entropy can be considered as a measure of system disorder. The more microstates are accessible to the system, the lower its predictability and, consequently, the greater the degree of the system disorder.

Another way to conceive entropy is through the idea of how much uncertainty—in the sense of ignorance—there is in the state of a physical system. The more microstates are accessible to the state, the larger is the uncertainty that we as observers have about it. Entropy measures this degree of uncertainty. Suppose there is only one possibility of microstate for a given system, *i.e.*, $\Omega = 1$. Therefore, $S = 0$, which means that the uncertainty intrinsic to the system is null. Now, if the system changes over time without external interference, its microstates go from a configuration of less to a higher probability. Thus, the uncertainty rises as well as entropy.

A closer relation between entropy and the concept of information emerged at the very beginning of information theory, with Shannon [121]. Although the concepts of information entropy of Shannon's work and thermodynamic entropy of Clausius's work are analogous, the former was not inspired by the latter. However, in conversation with Tribus and McIrvine in 1961 (as the authors report in Ref. [122]), Shannon reveals the origin of his choice by the term *entropy*:

> *"My greatest concern was what to call it. I thought of calling it 'information', but the word was overly used, so I decided to call it 'uncertainty'. When I discussed it with John von Neumann, he had a better idea. von Neumann told me, 'You should call it entropy, for two reasons: In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, nobody knows what entropy really is, so in a debate you will always have the advantage.' "* C. Shannon in Ref. [122].

1.1.1 Classical distributions and the Shannon entropy

Shannon defined the *information entropy* of a random variable X by

$$H(X) := -\sum_x p_x \log p_x, \tag{1.4}$$

where p_x is the probability of x to occur. In other words, H is the average of the information content $h(x) = -\log(p_x)$, associated to the variable X, that is, the amount of information you gain by knowing X. Alternatively, we can also think about $H(X)$ as the uncertainty about X before we learn its value. The entropy is maximum when all events are equally probable and zero when only one variable has the chance of occurring, since $\sum_x p_x = 1$.

The concept of Shannon entropy can be extended to more than one variable. Let Y be another random variable with probability p_y for the outcome y. The joint probability $p_{x,y}$ associated to the pair of outcomes (x, y) give us the *joint entropy*

$$H(X, Y) := -\sum_{xy} p_{x,y} \log p_{x,y}. \tag{1.5}$$

If the events are independent, *i.e.*, $p_{x,y} = p_x p_y$, then the entropy becomes additive, that is, $H(X,Y) = H(X) + H(Y)$. However, if $p_{x,y}$ can not be decomposed as a product, then when we know something about Y our ignorance about X must decrease. This translates what is known as the entropy of X conditional on knowing Y,

$$H(X|Y) := \sum_y p_y H(X|y), \tag{1.6}$$

where

$$H(X|y) := -\sum_x p_{x|y} \log p_{x|y} \tag{1.7}$$

is the entropy of X conditioned to the outcome y, with $p_{x|y} = p_{x,y}/p_y$. A closer look on (1.6) leads us to write

$$\begin{aligned} H(X|Y) &= -\sum_{xy} p_y \frac{p_{x,y}}{p_y} \log \frac{p_{x,y}}{p_y} && (1.8)\\ &= -\sum_{xy} p_{x,y} \log p_{x,y} + \sum_{xy} p_{x,y} \log p_y && (1.9)\\ &= H(X,Y) - H(Y), && (1.10) \end{aligned}$$

where $p_y = \sum_x p_{x,y}$.

In order to understand how close to each other are two probability distributions p_x and q_x of the same variable X, one can use the Shannon *relative entropy*

$$H(p_x||q_x) := \sum_x p_x \log \frac{p_x}{q_x} = -H(X) - \sum_x p_x \log q_x \geqslant 0, \tag{1.11}$$

where the last equality holds *iff* $p_x = q_x$.

Furthermore, Shannon entropy satisfies the following properties, whose proofs can be found in [107]:

Theorem 1 (Basic properties of Shannon entropy)**.**

1. ***Non-negativity:*** $H(X) \geqslant 0$*. Equality holds iff* $X = \{x_1\}$*;*
2. ***Symmetry:*** $H(X,Y) = H(Y,X)$*;*
3. $H(X,Y) \geqslant H(X)$*. Equality holds iff* Y *is a function of* X*;*
4. ***Subadditivity:*** $H(X,Y) \leqslant H(X) + H(Y)$*. Equality holds iff* X *and* Y *are independent from each other;*
5. ***Upper bound:*** $H(X) \leqslant \log d$*, where* $d = \dim X$*.*

Together with the above properties, comes the concept of *mutual information*: the information that X and Y have in common,

$$I(X:Y) \coloneqq H(X) + H(Y) - H(X,Y) \geqslant 0, \tag{1.12}$$

which is non-negative because of the subadditivity property. Since we are summing up the information available of both sets, what they have in common in counted twice, that's why we subtract $H(X,Y)$. Another way to define mutual information is through the expression

$$J(X:Y) \coloneqq H(X) - H(X|Y), \tag{1.13}$$

which gives us $I(X:Y) = J(X:Y)$ thanks to (1.10). Yet, mutual information in the form of J makes direct reference to the measurement process, that is, equation (1.13) interprets mutual information as the information we can get about X after measuring Y. On the other hand, the quantity I does not have such a requirement, and the fact that both expressions are equal is due to something very intrinsic to classical physics: it's always possible to get information without disturbing the system. This certainly is not the case for quantum physics, where the quantum versions of I and J are not equal most of the time, as we will see in the following sections.

1.1.2 Density operators and the von Neumann entropy

At the very beginning of the development of quantum theory, von Neumann was already trying to find a statistical description for quantum systems, giving rise to the density operator ρ in his paper from 1927 [123]. The following revision is based on the textbook of Nielsen and Chuang [107] and the lecture notes of Landi [124]. The density operator describes an ensemble of quantum systems that can be in different (normalized) quantum states $|\psi_i\rangle$ with probabilities p_i (such that $\sum_i p_i = 1$) as

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{1.14}$$

the so called *mixed states*, since $|\psi_i\rangle\langle\psi_i|$ are the pure ones. The mixed state ρ is a positive semi-definite linear operator (*i.e.*, it has only non-negative eigenvalues) that acts over the Hilbert space $\mathcal{H}$ where the quantum states $|\psi_i\rangle$ live in. Since $|\psi_i\rangle \in \mathcal{H}$, we say that $\rho \in \mathfrak{B}(\mathcal{H})$. The state ρ is also Hermitian ($\rho^\dagger = \rho$) and it has unit trace, that is

$$\operatorname{Tr}\rho = \sum_i p_i \operatorname{Tr}\left(|\psi_i\rangle\langle\psi_i|\right) = \sum_i p_i \langle\psi_i|\psi_i\rangle = \sum_i p_i = 1, \tag{1.15}$$

where $\operatorname{Tr} |\vartheta\rangle\langle\varphi| = \langle\varphi|\vartheta\rangle$ for arbitrary $|\vartheta\rangle$ and $|\varphi\rangle$ quantum states. More generally, if A is an observable, that is, an Hermitian operator acting on $\mathcal{H}$, it admits a spectral decomposition $A = \sum_a \lambda_a |a\rangle\langle a|$ with $\lambda_a \in \mathbb{R}$ eigenvalues and $|a\rangle \in \mathcal{H}$ eigenstates. The operators $|a\rangle\langle a| \eqqcolon A_a$ are called projectors, since $A_a^2 = A_a$. That said, a real function $f:\mathbb{R}\to\mathbb{R}$ can be applied to A as

$$f(A) = \sum_a f(\lambda_a) \left|a\right\rangle \left\langle a\right|. \tag{1.16}$$

Trace function

The trace function is formally defined as

$$\operatorname{Tr} A := \sum_k \langle k| A |k\rangle , \tag{1.17}$$

where $\{|k\rangle\}$ is any orthonormal basis for $\mathcal{H}$. From the completeness relation $\sum_j |j\rangle \langle j| = \mathbb{1}$, where $\{|i\rangle\}$ is another orthonormal basis for $\mathcal{H}$, it is easy to see that the trace is independent of the basis:

$$\sum_k \langle k| A |k\rangle = \sum_{kj} \langle k|j\rangle \langle j| A |k\rangle = \sum_{kj} \langle j| A |k\rangle \langle k|j\rangle = \sum_j \langle j| A |j\rangle . \tag{1.18}$$

A useful choice of basis to calculate the trace of A is that one that diagonalizes it, *i.e.*, $\{|a\rangle\}$. Therefore

$$\operatorname{Tr} A = \sum_{a'a} \lambda_{a'} \langle a|a'\rangle \langle a'|a\rangle = \sum_a \lambda_a, \tag{1.19}$$

since $\langle a|a'\rangle = \delta_{aa'}$. It is clear now that the trace is a property of the operator and not of the basis we choose. Because of that, the trace function plays a fundamental role in calculating the expected value of observables in quantum mechanics. For pure states, the expected value of an observable A for the state $|\psi\rangle$ is given by $\langle A\rangle_\psi = \langle\psi| A |\psi\rangle$. It is possible to extend this concept to mixed states ρ by

$$\langle A\rangle_\rho := \sum_i p_i \langle\psi_i| A |\psi_i\rangle \tag{1.20}$$

$$= \sum_{ik} p_i \langle\psi_i|k\rangle \langle k| A |\psi_i\rangle \tag{1.21}$$

$$= \sum_{ik} \langle k| A p_i |\psi_i\rangle \langle\psi_i|k\rangle \tag{1.22}$$

$$= \operatorname{Tr}(A\rho). \tag{1.23}$$

Time evolution and projective measurements

Let us remember that the unitary time evolution of a quantum state in the Schrödinger picture is given by

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle , \tag{1.24}$$

where H is the Hamiltonian. Therefore, for density operators, we have

$$\rho(t) = e^{-\frac{iHt}{\hbar}} \rho(0) e^{\frac{iHt}{\hbar}} . \tag{1.25}$$

By applying the Leibniz rule for derivatives in the above equation we obtain the Liouville-von Neumann equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \rho(t) = [H, \rho(t)] , \tag{1.26}$$

which is just the Schrödinger's equation in terms of density operators. A projective measurement, however, cannot be described by (1.26). Actually, it constitutes a postulate of the quantum theory—something that is put forward *ad hoc*. A projective measurement by some observable A written in its spectral form $A = \sum_a \lambda_a |a\rangle \langle a|$ of a pure state $|\psi\rangle = |a\rangle$ always returns the same eigenstate $|a\rangle$ with outcome λ_a,

$$A|\psi\rangle = \sum_{a'} \lambda_{a'} |a'\rangle \langle a'|a\rangle = \sum_{a'} \lambda_{a'} \delta_{aa'} |a'\rangle = \lambda_a |a\rangle . \tag{1.27}$$

When $|\psi\rangle$ is any other thing but an eigenstate of A, the projective measurement will result the outcome λ_a with probability $p_a = |\langle a|\psi\rangle|^2$—the well-known *Born rule* [125]. After the measurement of A, the system will randomly collapse to the normalized state

$$|\psi\rangle \to |\psi_a\rangle = \frac{A_a |\psi\rangle}{\sqrt{p_a}}. \tag{1.28}$$

This is called *the postulate of reduction*, and it can be extended to mixtures of quantum states as

$$\rho \to \rho_a = \frac{A_a \rho A_a}{\mathrm{Tr}\,(A_a \rho)}, \tag{1.29}$$

where $p_a = \mathrm{Tr}\,(A_a\rho)$ is the probability of resulting the outcome λ_a by measuring A in ρ. Note that here we use the term "randomly collapse" only in a mathematical sense without assigning any specific physical interpretation for the centennial dilemma regarding the measurement problem. Regardless of the interpretation of quantum theory that we choose for the nature of the quantum state (which goes beyond the debate ψ-ontic/ψ-epistemic [25]), there is a mathematical discontinuity from the point of view of the observer in the description of the observed system during the measurement procedure. Some questions about this subject will be addressed in Part II. For now, let us stay with the basics of the density operator algebra.

As a matter of fact, there is a way to describe a projective measurement as a unitary evolution. In order to do that, we need to introduce *partial traces* and *quantum channels*.

Partial traces

Suppose now that ρ describes a statistical mixture of bipartite states, *i.e.*, $\rho \in \mathfrak{B}(\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$. It is possible to make reference to only one part of the Hilbert space by means of the partial trace operation:

$$\rho_{\mathcal{A}} := \mathrm{Tr}_{\mathcal{B}}(\rho) \qquad \text{and} \qquad \rho_{\mathcal{B}} := \mathrm{Tr}_{\mathcal{A}}(\rho). \tag{1.30}$$

The partial trace operation over, say, $\mathcal{A}$, is given by

$$\mathrm{Tr}_{\mathcal{A}}(\rho) := \sum_i \langle i| \rho |i\rangle , \tag{1.31}$$

where $\{|i\rangle\}$ is any orthonormal basis for $\mathcal{H}_{\mathcal{A}}$. See the following examples:

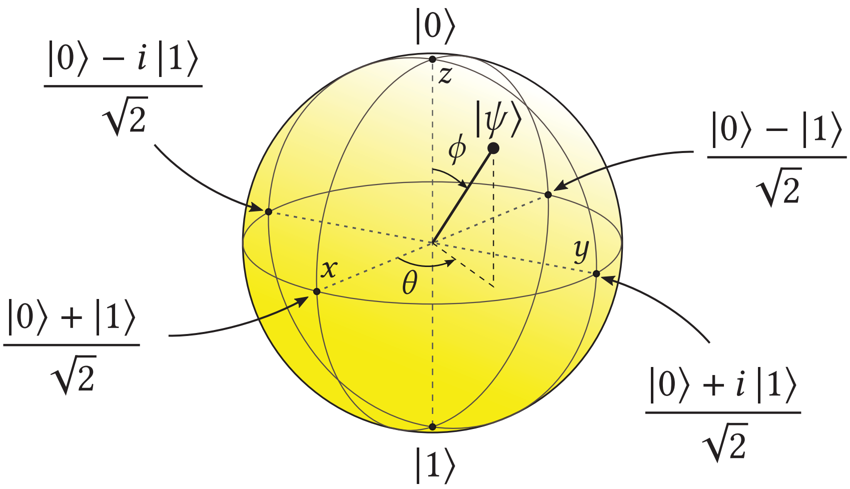

Figure 4 – The Bloch sphere. The qubit $|\psi\rangle$ given by (1.36) have a correspondent point on the surface of the Bloch sphere. Here, θ and ϕ are the azimuthal and polar angles, respectively.

Example 1. A generic bipartite mixed state in $\mathfrak{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$ can be written as

$$\rho = \sum_{ijkl} p_{ijkl} |\alpha_i\rangle \langle\alpha_j| \otimes |\beta_k\rangle \langle\beta_l| \,. \tag{1.32}$$

The partial trace over $\mathcal{A}$ can be computed by

$$\rho_{\mathcal{B}} = \sum_{aijkl} p_{ijkl} \langle\alpha_a|\alpha_i\rangle \langle\alpha_j|\alpha_a\rangle \otimes |\beta_k\rangle \langle\beta_l| \tag{1.33}$$

$$= \sum_{aijkl} p_{ijkl}\delta_{ai}\delta_{ja} |\beta_k\rangle \langle\beta_l| \tag{1.34}$$

$$= \sum_{akl} p_{aakl} |\beta_k\rangle \langle\beta_l| \tag{1.35}$$

▲

Example 2. The *qubit* is the quantum analogous of the classical bit of information because it can be in a quantum superposition between $|0\rangle$ and $|1\rangle$ as $|\psi\rangle = a|0\rangle + b|1\rangle$, where the complex coefficients satisfy $|a|^2 + |b|^2 = 1$. A clever way to represent a qubit is through a spherical parametrization

$$|\psi\rangle = \cos\frac{\phi}{2}|0\rangle + \sin\frac{\phi}{2} e^{i\theta}|1\rangle \,, \tag{1.36}$$

where $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$. Note that we could have another complex phase multiplying $|0\rangle$, but this would lead to an irrelevant global phase. Having said that, a qubit can be geometrically represented in what is called *the Bloch sphere* (see Fig. 4). A general mixed qubit state, however, can be written with two more parameters:

$$\rho = \lambda_{0,0}|0\rangle\langle 0| + \lambda_{0,1}|0\rangle\langle 1| + \lambda_{1,0}|1\rangle\langle 0| + \lambda_{1,1}|1\rangle\langle 1| \,. \tag{1.37}$$

If we choose the canonical matrix representation given by

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle 0| \equiv \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \langle 1| \equiv \begin{pmatrix} 0 & 1 \end{pmatrix}, \tag{1.38}$$

such that

$$|0\rangle\langle 0| \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |0\rangle\langle 1| \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad |1\rangle\langle 0| \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad |1\rangle\langle 1| \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.39}$$

then we can write

$$\rho \equiv \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} \\ \lambda_{1,0} & \lambda_{1,1} \end{pmatrix}. \tag{1.40}$$

Of course there are some constrains between the $\lambda_{i,j}$ parameters because we must have $\mathrm{Tr}\,\rho = 1$ and $\rho^\dagger = \rho$ (therefore, only three parameters are needed actually). A pure two-qubit state written with four parameters, will have the form

$$|\Psi\rangle = a\,|0,0\rangle + b\,|0,1\rangle + c\,|1,0\rangle + d\,|1,1\rangle\,, \tag{1.41}$$

where

$$|0,0\rangle \equiv |0\rangle \otimes |0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{with} \quad |0,0\rangle\langle 0,0| \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.42}$$

and so on. Note that the tensor product for quantum states in the Hilbert spaces becomes the Kronecker product of matrices in the matrix representation. Therefore, a mixed two-qubit state ϱ will be represented as a 4x4 matrix

$$\varrho = \sum_{ijkl=0}^{1} \lambda_{ijkl}\,|i,j\rangle\langle k,l| \equiv \begin{pmatrix} \lambda_{0000} & \lambda_{0001} & \lambda_{0010} & \lambda_{0011} \\ \lambda_{0100} & \lambda_{0101} & \lambda_{0110} & \lambda_{0111} \\ \lambda_{1000} & \lambda_{1001} & \lambda_{1010} & \lambda_{1011} \\ \lambda_{1100} & \lambda_{1101} & \lambda_{1110} & \lambda_{1111} \end{pmatrix}, \tag{1.43}$$

where $|i,j\rangle\langle k,l| \equiv |i\rangle\langle k| \otimes |j\rangle\langle l|$. The partial traces over the first and the second qubit are, respectively,

$$\varrho_{\mathcal{B}} = \sum_{ajl=0}^{1} \lambda_{ajal}\,|j\rangle\langle l| \equiv \begin{pmatrix} \lambda_{0000} + \lambda_{1010} & \lambda_{0001} + \lambda_{1011} \\ \lambda_{0100} + \lambda_{1110} & \lambda_{0101} + \lambda_{1111} \end{pmatrix}, \tag{1.44}$$

$$\varrho_{\mathcal{A}} = \sum_{bik=0}^{1} \lambda_{ibkb}\,|i\rangle\langle k| \equiv \begin{pmatrix} \lambda_{0000} + \lambda_{0101} & \lambda_{0010} + \lambda_{0111} \\ \lambda_{1000} + \lambda_{1101} & \lambda_{1010} + \lambda_{1111} \end{pmatrix}. \tag{1.45}$$

▲

Every trace can be decomposed in partial traces

$$\mathrm{Tr}\,\rho = \mathrm{Tr}_{\mathcal{A}}\mathrm{Tr}_{\mathcal{B}}\rho = \mathrm{Tr}_{\mathcal{B}}\mathrm{Tr}_{\mathcal{A}}\rho. \tag{1.46}$$

With that in mind, a closely related fact from the above is the following. When dealing with logarithms of product states, we have

$$\ln(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) = (\ln \rho_{\mathcal{A}}) \otimes \mathbb{1}_{\mathcal{B}} + \mathbb{1}_{\mathcal{A}} \otimes (\ln \rho_{\mathcal{B}}) \tag{1.47}$$

because

$$\ln(\rho_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}) = \sum_{ij} \ln(\lambda_i^{\mathcal{A}}) |a_i\rangle \langle a_i| \otimes |b_j\rangle \langle b_j| , \tag{1.48}$$

$$= \sum_{i} \ln(\lambda_i^{\mathcal{A}}) |a_i\rangle \langle a_i| \otimes \sum_j |b_j\rangle \langle b_j| , \tag{1.49}$$

$$= (\ln \rho_{\mathcal{A}}) \otimes \mathbb{1}_{\mathcal{B}}, \tag{1.50}$$

where $\lambda_i^{\mathcal{A}}$ and $|a_i\rangle$ are the eigenvalues and eigenvectors of $\rho_{\mathcal{A}}$ and similarly for $\mathcal{B}$. Note that, usually, people just write $\ln(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) = \ln(\rho_{\mathcal{A}}) + \ln(\rho_{\mathcal{B}})$ for shortness. Suppose we want to calculate the quantity, say, $\mathrm{Tr}\,[\sigma \ln(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}})]$ where $\sigma \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$ (which appears in relative entropies as we are going to see further). We can do

$$\mathrm{Tr}\,[\sigma \ln(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}})] = \mathrm{Tr}\,(\sigma \ln \rho_{\mathcal{A}}) + \mathrm{Tr}\,(\sigma \ln \rho_{\mathcal{B}}), \tag{1.51}$$

which from (1.46) reads

$$\mathrm{Tr}\,[\sigma \ln(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}})] = \mathrm{Tr}\,_{\mathcal{A}}\mathrm{Tr}\,_{\mathcal{B}}(\sigma \ln \rho_{\mathcal{A}}) + \mathrm{Tr}\,_{\mathcal{B}}\mathrm{Tr}\,_{\mathcal{A}}(\sigma \ln \rho_{\mathcal{B}}), \tag{1.52}$$

$$= \mathrm{Tr}\,_{\mathcal{A}}(\sigma_{\mathcal{A}} \ln \rho_{\mathcal{A}}) + \mathrm{Tr}\,_{\mathcal{B}}(\sigma_{\mathcal{B}} \ln \rho_{\mathcal{B}}). \tag{1.53}$$

The last partial traces, then, can be understood as usual traces since they act only over their own spaces.

Quantum channels

A quantum channel (or quantum operation) is a completely positive trace preserving (CPTP) map. A map Λ is completely positive *iff* it can be written in the operator sum representation

$$\Lambda(\rho) = \sum_i K_i \rho K_i^{\dagger}, \tag{1.54}$$

where K_i are known as *Kraus operators*. A map Λ will be trace preserving *iff*

$$\sum_i K_i^{\dagger} K_i = \mathbb{1}. \tag{1.55}$$

In addition, a quantum channel Λ will be called a *bistochastic map iff* it is also *unital*, that is, $\Lambda(\mathbb{1}/d) = \mathbb{1}/d$. An excellent revision about these special types of maps can be found in Chap. 10 of Ref. [126].

Example 3. An unrevealed projective measurement is an example of a bistochastic map. Let $A = \sum_a a A_a$ be an observable with $A_a = |a\rangle \langle a|$ projectors satisfying $\sum_a A_a = \mathbb{1}$. With this, we can define the map Φ_A such that

$$\Phi_A(\rho) = \sum_a A_a \rho A_a, \tag{1.56}$$

which is immediately CPTP, since $A_a^\dagger = A_a$ and $A_a A_{a'} = \delta_{aa'} A_a$. In addition, this implies that $\Phi_A(\mathbb{1}/d) = \mathbb{1}/d$. Physically, what $\Phi_A(\rho)$ represents is an unrevealed, or non-selective, projective measurement. The experiment was realized and for all practical purposes the quantum state collapsed, but the experimentalist—for some reason—do not know the result (maybe she has not looked at the results on the lab computer yet). However, she can make a statistical prediction of what she will found after the measurement of A if she looks at the screen, which is exactly $\Phi_A(\rho)$. ▲

Stinespring's dilation Theorem

With partial traces and quantum channels in mind, we can introduce the Stinespring's dilation theorem[1]:

Theorem 2 (Stinespring's dilation)**.** *Let $\Lambda : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ be a CPTP map between states on a finite-dimensional Hilbert space $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{H}_\mathcal{E}$ and a unitary operation U on $\mathcal{H} \otimes \mathcal{H}_\mathcal{E}$ such that*

$$\Lambda(\rho) = \mathrm{Tr}_\mathcal{E}\left[U\left(\rho \otimes |e_0\rangle\langle e_0|\right)U^\dagger\right], \qquad \forall \rho \in \mathfrak{B}(\mathcal{H}). \tag{1.57}$$

The $\mathcal{H}_\mathcal{E}$ space is usually called auxiliary or ancillary space.

Note that Λ can be any CPTP map, which includes projective measurements like (1.56). Therefore, for any projective measurement, there is always a *bigger* Hilbert space in which that measurement can be seen as a unitary evolution that entangles (couples) the state of the system that is been measured with an environment[2]. Some call this informally "the church of the bigger Hilbert space".

Generalized measurements

Before ending our introduction to density states and quantum measurements and move to entropies, it is important to define *generalized measurements.* A generalized measurement is defined from a set of Kraus operators $\{M_i\}$ satisfying

$$\sum_i M_i^\dagger M_i = \mathbb{1}. \tag{1.58}$$

If the outcome of the measurement is i, then the state after the measurement and respective probability of obtaining it are

$$\rho \to \rho_i = \frac{M_i \rho M_i^\dagger}{p_i}, \qquad \text{and} \qquad p_i = \mathrm{Tr}\left(M_i \rho M_i^\dagger\right). \tag{1.59}$$

[1] The formulation presented here—in terms of CPTP maps—was taken from Sec. 8.2 of Ref. [107]. For the original version, see Ref. [127].

[2] Which can be an auxiliary space, a measurement apparatus, the laboratory, and/or the experimentalist. It will depend on how good we are and how much time we have available to write and solve all the initial quantum states and Schrödinger equations for each one of the particles that compose the system+environment representation.

Note that if $M_i = M_i^\dagger$ and $M_i^2 = M_i$, then $M_i^\dagger M_i = M_i$, which is specifically the case of a projective measurement [Eq. (1.29)]. Now, by the cyclic permutation of the trace, we can write $p_i = \mathrm{Tr}\,(M_i^\dagger M_i \rho)$ and define the positive operators

$$E_i = M_i^\dagger M_i, \tag{1.60}$$

which are called *effects* and they satisfy $\sum_i E_i = \mathbb{1}$, $E_i^\dagger = E_i$, and $E_i \geqslant 0$. Therefore, all the probabilities regarding an specific measurement can be obtained by these effects through $p_i = \mathrm{Tr}\,(E_i \rho)$, but the point is: the set of measurement operators $\{M_i\}$ is not uniquely defined. If we are only interested in the probabilities, there is some freedom of choice regarding the set of measurement operators. The state of the system after the measurement, however, can be quite different depending on the set $\{M_i\}$. These effects E_i define what is called a Positive Operator-Valued Measure (POVM).

Example 4. An example of POVM is the *monitoring* map [40]

$$\mathcal{M}_A^\epsilon(\rho) := (1-\epsilon)\rho + \epsilon\Phi_A(\rho), \tag{1.61}$$

which interpolates between weak and strong (projective) measurements by the intensity parameter $\epsilon \in [0,1]$. One can verify that the effects E_i of $\mathcal{M}_A^\epsilon$ are given by $E_0 = \sqrt{1-\epsilon}\mathbb{1}$ and $E_{i\neq 0} = \sqrt{\epsilon}A_i$. Indeed,

$$\sum_{i=0}^{d_\mathcal{A}} E_i \rho E_i^\dagger = (1-\epsilon)\rho + \epsilon \sum_{i=1}^{d_\mathcal{A}} A_i \rho A_i = \mathcal{M}_A^\epsilon(\rho), \tag{1.62}$$

where $d_\mathcal{A} = \dim \mathcal{H}_\mathcal{A}$. ▲

von Neumann entropy

In the same year that von Neumann introduced his density operator for quantum states, he also presented an expression for the entropy of a quantum state ρ as

$$S(\rho) := -\mathrm{Tr}\,(\rho \ln \rho) = -\sum_i \lambda_i \ln \lambda_i, \tag{1.63}$$

where λ_i are the eigenvalues of ρ [128]. Years later, von Neumann compiled his results in a book—originally in German [129] with a posterior translated version to English [130]—which became a reference to the subject. From (1.63) we can see that for pure states $\rho = |\psi\rangle\langle\psi|$ we have $S(\rho) = 0$, which means no ignorance about the state. For maximally mixed states $\rho = \mathbb{1}/d$ we have $S(\rho) = \ln d$, where $d = \dim \mathcal{H}$, which corresponds to complete ignorance about the mixture.

Some important properties of the von Neumann entropy are listed bellow with respective proofs in [107]:

Theorem 3 (Basic properties of the von Neumann entropy).

1. ***Non-negativity:*** *$S(\rho) \geqslant 0$. Equality holds iff ρ is pure;*

2. ***Invariance under unitary transformations:*** *If $UU^\dagger = \mathbb{1}$, then*

$$S(U\rho U^\dagger) = S(\rho). \tag{1.64}$$

3. ***Subadditivity:*** *For $\rho \in \mathfrak{B}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})$ we have*

$$S(\rho) \leqslant S(\rho_\mathcal{A}) + S(\rho_\mathcal{B}), \tag{1.65}$$

where the equality holds iff $\rho = \rho_\mathcal{A} \otimes \rho_\mathcal{B}$;

4. ***Strong subadditivity:*** *For $\rho \in \mathfrak{B}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B} \otimes \mathcal{H}_\mathcal{C})$ we have*

$$S(\rho) + S(\rho_\mathcal{B}) \leqslant S(\rho_{\mathcal{A}\mathcal{B}}) + S(\rho_{\mathcal{B}\mathcal{C}}), \tag{1.66}$$

where the equality holds iff $\rho = \rho_{\mathcal{A}\mathcal{B}} \otimes \rho_\mathcal{C}$;

5. ***Joint entropy theorem:*** *Suppose p_i are probabilities, $|i\rangle$ are orthogonal states for a system in $\mathcal{H}_\mathcal{A}$ and ρ_i is any set of density operators that acts over $\mathcal{H}_\mathcal{B}$. Then*

$$S\left(\sum_i p_i \,|i\rangle \langle i| \otimes \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i); \tag{1.67}$$

6. ***Concavity:*** *If $\rho = \sum_i p_i \rho_i$, then*

$$S\left(\sum_i p_i \rho_i\right) \geqslant \sum_i p_i S(\rho_i); \tag{1.68}$$

7. ***Upper bound:*** *If $\rho \in \mathfrak{B}(\mathcal{H})$, then*

$$S(\rho) \leqslant \ln \dim \mathcal{H}. \tag{1.69}$$

As for the Shannon relative entropy, von Neumann entropy also allows for a quantifier for the proximity of two quantum statistical descriptions ρ and σ

$$D(\rho||\sigma) := \mathrm{Tr}\,(\rho \ln \rho - \rho \ln \sigma). \tag{1.70}$$

The non-negativity of the relative entropy is not a straightforward result. As a matter of fact, it constitutes a theorem [107]:

Theorem 4 (Klein's Inequality)**.** *The quantum relative entropy is non-negative,*

$$D(\rho||\sigma) \geqslant 0, \tag{1.71}$$

where equality holds iff $\rho = \sigma$.

Actually, Klein's inequality has a generalized version which can be quite useful:

Theorem 5 (Generalized Klein's Inequality)**.** *Suppose $f(\cdot) : \mathbb{R}_+^* \to \mathbb{R}$ is a convex function, inducing a natural function $f(\cdot)$ on Hermitian operators. If A and B are Hermitian operators, than*

$$\mathrm{Tr}\,\left[f(A) - f(B) - (A - B)f'(B)\right] \geqslant 0. \tag{1.72}$$

Moreover, if f is strictly convex, then equality holds iff $A = B$.

1.2 ELEMENTS OF QUANTUM INFORMATION THEORY

On the quantum context, one form of conditional entropy of part $\mathcal{A}$ given that we have knowledge about part $\mathcal{B}$ is defined as

$$H_{\mathcal{A}|\mathcal{B}}(\rho) \coloneqq S(\rho) - S(\rho_{\mathcal{B}}). \tag{1.73}$$

Unlike Shannon conditional entropy, von Neumann conditional entropy can be negative. This means that sometimes our ignorance about some part of the state can be greater than that of the whole state. In situations like this, the informational content can be stored in the correlations between the subsystems, rather than the subsystems themselves.

Example 5. Let's take the singlet state (1) and define $\uparrow = 0$ and $\downarrow = 1$. By following the recipe (1.43), the singlet state can be represented by

$$\rho_s = |\psi_s\rangle\langle\psi_s| \equiv \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.74}$$

Immediately from (1.44), one has $\rho_{\mathcal{B}} = \mathbb{1}_2/2$. From the formula (1.63), we obtain $S(\rho_{\mathcal{B}}) = \ln 2$. Therefore, $H_{\mathcal{A}|\mathcal{B}}(\rho_s) = -\ln 2$, since $S(\rho_s) = 0$. ▲

Another interesting fact is that it is possible to use definition (1.73) to rewrite the strong subadditivity property (1.66) as

$$H_{\mathcal{A}|\mathcal{B}C}(\rho) \leqslant H_{\mathcal{A}|\mathcal{B}}(\rho_{\mathcal{A}\mathcal{B}}), \tag{1.75}$$

which means that the discard of information about one subsystem (in this case the partial trace over C) increases the conditional entropy.

Now, if we take the same path of Shannon entropy to define an analogous mutual information of a quantum state ρ regarding a bipartite space $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, an equivalence like that of (1.12) and (1.13) is not always true. Thus, let us first define the *quantum mutual information* of ρ as

$$I_{\mathcal{A}:\mathcal{B}}(\rho) \coloneqq S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}) - S(\rho). \tag{1.76}$$

It is straightforward to see that the quantum mutual information of a product state $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$ is zero, since $S(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) = S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}})$. Now, a definition of quantum mutual information inspired by (1.13) must take into account the measurement procedure in one part of the system, let's say $\mathcal{H}_{\mathcal{B}}$, by some observable $B = \sum_b bB_b$ with projectors $B_b = |b\rangle\langle b|$ satisfying $\sum_b |b\rangle\langle b| = \mathbb{1}$. Having said that, the quantum mutual information can also be defined as

$$J(\rho|\mathcal{B}) := S(\rho_{\mathcal{A}}) - S(\rho_{\mathcal{A}}|\mathcal{B}), \tag{1.77}$$

where

$$S(\rho_{\mathcal{A}}|\mathcal{B}) := \sum_b p_b S(\rho_{\mathcal{A}|b}) \tag{1.78}$$

is the conditional entropy of $\rho_{\mathcal{A}}$ given that a measurement was made in part $\mathcal{B}$, with $\rho_{\mathcal{A}|b} = \mathrm{Tr}_{\mathcal{B}}(B_b \rho B_b)/p_b$ and probability $p_b = \mathrm{Tr}\,(B_b \rho)$. The difference between (1.76) and (1.77) will give rise to the quantum discord, a quantum correlation to be defined in Sec. 1.3.4.

Another useful quantity is the available information regarding a state ρ in a space $\mathcal{H}$ of dimension d,

$$I(\rho) := \ln d - S(\rho). \tag{1.79}$$

This quantity is maximum for pure states and zero for maximally mixed ones. Now, if we consider a bipartite state $\rho \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$, where $\dim(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}) = d_{\mathcal{A}} d_{\mathcal{B}}$, one can obtain the relation

$$I(\rho) = I(\rho_{\mathcal{A}}) + I(\rho_{\mathcal{B}}) + I_{\mathcal{A}:\mathcal{B}}(\rho). \tag{1.80}$$

Indeed, by applying (1.79) on (1.80) one has

$$\begin{aligned} I(\rho_{\mathcal{A}}) + I(\rho_{\mathcal{B}}) + I_{\mathcal{A}:\mathcal{B}}(\rho) &= \ln d_{\mathcal{A}} - S(\rho_{\mathcal{A}}) + \ln d_{\mathcal{B}} - S(\rho_{\mathcal{B}}) + S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}) - S(\rho), && (1.81)\\ &= \ln\,(d_{\mathcal{A}} d_{\mathcal{B}}) - S(\rho), && (1.82)\\ &= I(\rho). && (1.83) \end{aligned}$$

The relation (1.80) tells us that the total available information about the state is composed of local and global terms, that is, it depends not only on the information regarding each part of the system but also on interactions between them. Even more interesting is the fact that (1.80) implies the conservation of information in a closed system (which follows from the unitary invariance of von Neumann entropy). The left side of (1.80) is constant while the right side admits transmutations between local information $[I(\rho_{\mathcal{A}})$ and $I(\rho_{\mathcal{B}})]$ and nonlocal information $[I_{\mathcal{A}:\mathcal{B}}(\rho)]$ [131].

1.3 NONCLASSICAL FEATURES

This section is dedicated to present each one of the nonclassical features that were mentioned in the introduction.

1.3.1 Bell nonlocality

Nearly three decades after EPR discussed the supposed incompleteness of quantum mechanics as a physical theory, Bell [7] proved that if we try to complete quantum mechanics with local hidden variables, we cannot reproduce every one of quantum mechanics predictions. A simplified version of his argument is as follows (the interested reader can take a look in Ref. [132] for a more profound discussion). Suppose two scientists, Alice and Bob, share a bipartite state $\rho \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$. The joint probability $p(a, b|A, B)$ of outcomes a and b for observables A and B made by Alice and Bob on parties $\mathcal{A}$ and $\mathcal{B}$, respectively, cannot, in general, be written as a product of independent probability distributions, that is,

$$p(a, b|A, B) \neq p(a|A)p(b|B). \tag{1.84}$$

Such an assertion does not indicate, *a priori*, any "spooky action at a distance" since the correlations could have their origins on local interactions at the preparation of the state. However, even if the state was already defined since its preparation, quantum mechanics does not provide such information because measurement results are random. Let us complete the quantum mechanical description with a hidden variable λ, possibly following a distribution $p(\lambda)$ satisfying $\int p(\lambda)\mathrm{d}\lambda = 1$, that could restore factorizability, that is,

$$p(a, b|A, B) = \int \mathrm{d}\lambda\, p(\lambda)p(a|A, \lambda)p(b|B, \lambda), \tag{1.85}$$

thus satisfying EPR criticism. The above expression is usually called local causality hypothesis, or also, *Bell locality hypothesis*. It means that all available statistical information about the outcomes a and b comes from (i) the local choice of observables A and B and (ii) some hidden variable λ that we could not have access for any reason. This hidden variable supposedly defines the state before the measurement, and together with its local aspect, prevents any explanations based on some action at a distance that could provide the correlations between results made at space-like separated sites. Through this Local Hidden Variable Model (LHVM), the expected value of a measurement can be obtained by

$$\langle A \otimes B \rangle = \iint \mathrm{d}a\mathrm{d}b\, p(a, b|A, B)ab, \tag{1.86}$$

$$= \iint \mathrm{d}a\mathrm{d}b \int \mathrm{d}\lambda\, p(\lambda)p(a|A, \lambda)p(b|B, \lambda)ab. \tag{1.87}$$

Considering a situation where two measurements can be done over each party, namely A, A', B, and B', with outcomes $a, a', b, b' \in \{-1, 1\}$, one can derive, after some algebra, the Clauser-Horne-Shimony-Holt (CHSH) inequality [133]

$$\mathbb{B} := \big| \langle A \otimes B \rangle + \langle A' \otimes B \rangle + \langle A \otimes B' \rangle - \langle A' \otimes B' \rangle \big| \leq 2. \tag{1.88}$$

Contexts, *i.e.*, states and measurement choices, which violate the above inequality necessarily violate the Bell locality hypothesis (1.85), as in the following example.

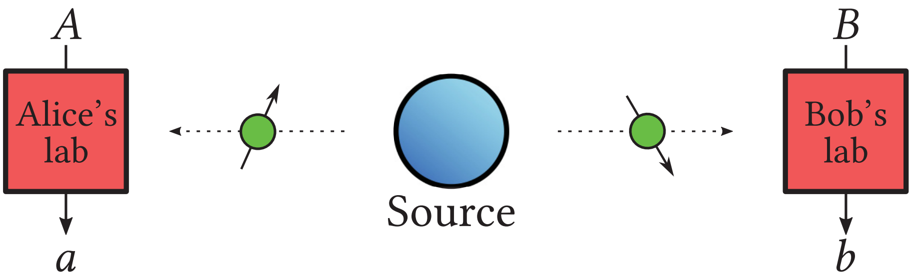

Figure 5 – Depiction of a Bell experiment. Alice and Bob receive each one a particle in their space-like separated labs of a pair that came out from a single source. Alice measures the observable A and registers the outcome a. Bob measures the observable B and registers the outcome b. The experiment is repeated several times and the joint probability $P(a, b|A, B)$ is registered.

Example 6. If we use the standard quantum mechanical description for the expected value, *i.e.*,

$$\langle A \otimes B \rangle_\rho = \mathrm{Tr}\ [(A \otimes B)\, \rho]\,, \tag{1.89}$$

we can calculate the average results that an experimentalist will obtain in a laboratory if one measures the spin observables A and B in the singlet state (1.74). Let us specify how to do that calculation. Any spin-1/2 observable A can be written as the inner product of a three-dimensional unitary vector $\hat{e} = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$ with the *Pauli vector*

$$\vec{\sigma} := (\sigma_x, \sigma_y, \sigma_z) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \tag{1.90}$$

which is a "vector" constituted of the three Pauli matrices. The vector $\hat{e}$ designates the direction of measurement and the Pauli matrices are associated to the canonical basis (1.38). For instance, if we measure the spin of the particle in the $\hat{z} = (0, 0, 1)$ direction, we obtain either the result $+\hbar/2$ or $-\hbar/2$ and the particle collapses to either the state $|0\rangle$ or $|1\rangle$, respectively (see the Bloch sphere in Fig. 4). In terms of projectors, we have

$$\left(\frac{\hbar}{2} |0\rangle \langle 0| - \frac{\hbar}{2} |1\rangle \langle 1| \right) |\psi_s\rangle \equiv \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |\psi_s\rangle = \frac{\hbar}{2} \sigma_z |\psi_s\rangle\,, \tag{1.91}$$

where $\hbar\sigma_z/2 =: S_z$ is the spin observable in the z direction. For now on we can suppress the constant $\hbar/2$. Analogously, any spin observable can be written in the above form as $\hat{e} \cdot \vec{\sigma}$. Now, let us choose clever directions to measure the singlet state (1.74) and to see a violation of the CHSH inequality. If we take orthogonal measurement directions $A = \hat{e}_1 \cdot \vec{\sigma}$ and $A' = \hat{e}_2 \cdot \vec{\sigma}$, and also orthogonal measurement directions $B = -(\hat{e}_1 + \hat{e}_2) \cdot \vec{\sigma}/\sqrt{2}$ and $B' = (-\hat{e}_1 + \hat{e}_2) \cdot \vec{\sigma}/\sqrt{2}$ in (1.88), we obtain $\mathbb{B} = 2\sqrt{2}$, which is a clearly violation of the CHSH inequality. Therefore, for this choice of observables, the state (1.74) is said to be *Bell nonlocal*, or that it violates the local causality hypothesis. ▲

Since $p(\lambda)$ is completely generic, the previous counterexample is sufficient to conclude that there is no LHVM that describes all the predictions of quantum mechanics.

Instead of simply diagnosing the presence of Bell nonlocality, we can quantify it. To this end, we can adopt a usual strategy according to which one takes the maximal violation of the inequality (1.88) as a quantifier for the degree of Bell nonlocality of the state. Here we follow the approach put forward in Refs. [134, 135]. We start with Luo's result [136], which ensures that every two-qubit state can be written, up to local unitary operations, as

$$\zeta = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \vec{a}\cdot\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{b}\cdot\vec{\sigma} + \sum_{i=1}^{3} c_i\,\sigma_i\otimes\sigma_i\right). \tag{1.92}$$

where $\{\vec{a},\vec{b},\vec{c}\}\in\mathbb{R}^3$ and $(\sigma_1,\sigma_2,\sigma_3) = (\sigma_x,\sigma_y,\sigma_z)$. For this state, the Bell-nonlocality quantifier proposed in Ref. [134] can be expressed in the form

$$\mathcal{B}(\zeta) = \max\left\{0, \frac{\sqrt{\vec{c}\cdot\vec{c} - c_{\min}^2} - 1}{\sqrt{2}-1}\right\}, \tag{1.93}$$

where $c_{\min} = \min\{|c_1|,|c_2|,|c_3|\}$. Although we are not showing how the authors deduce the above equation, we can provide some insights. When decomposed in form (1.92), vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ carry different aspects regarding the state due to the operators they multiply. Vector $\vec{c}$ is the only one that carries information regarding non-local aspects. That is why Eq. (1.93) (specifically for this inequality) only depends on vector $\vec{c}$.

1.3.2 Quantum steering

Quantum steering (sometimes called EPR steering), or just steering, signalizes the capability of an observer to steer the state of a system in a remote site via local measurements [75]. Thus, the *nonsteerability hypothesis* is defined by constraining Bob's statistical description in (1.85) as coming from a quantum state,

$$p(a,b|A,B) = \int \mathrm{d}\lambda\, p(\lambda)p(a|A,\lambda)p_q(b|B,\lambda), \tag{1.94}$$

where

$$p_q(b|B,\lambda) = \mathrm{Tr}\left(B_b\rho_\lambda^{\mathcal{B}}\right) \tag{1.95}$$

and B_b is a projector, although POVMs can be used instead (see Ref. [76] for a review on quantum steering). If it is possible to find a state $\rho_\lambda^{\mathcal{B}}$ to reproduce the probability $p(a,b|A,B)$ in (1.94), then the state shared by Alice and Bob is said to be *nonsteerable* or, equivalently, it is consistent with a local hidden state (LHS) model, that is, Bob does not need to believe that his state can be steered by Alice's choice of measurement basis. In the absence of a quantum description $\rho_\lambda^{\mathcal{B}}$, the shared state is called *steerable*. However, it is hard to find an LHS model in order to identify if the state is unsteerable. Therefore, similarly to Bell nonlocality, it is easier to discover if the state is steerable by searching for violations of inequalities, like the ones from the work of Cavalcanti, Jones, Wiseman, and Reid (CJWR) [137],

$$\frac{1}{\sqrt{n}}\left|\sum_{i=1}^{n}\langle A_i\otimes B_i\rangle\right| \leqslant 1, \tag{1.96}$$

where n is the number of measurements made in each site of a partite state. In scenarios where two measurements are performed per site on a two-qubit system, steering becomes identical to Bell nonlocality, as demonstrated by Costa and Angelo in Ref. [134] (specifically for the CJWR inequality). On the other hand, quantum steering and Bell nonlocality become distinguishable when at least three measurements are allowed per site, in which case the following quantum steering quantifier can be derived for the general two-qubit state (1.92) as the maximal violation of (1.96):

$$\mathcal{S}(\zeta) = \max\left\{0, \frac{\sqrt{\vec{c}\cdot\vec{c}} - 1}{\sqrt{3} - 1}\right\}. \tag{1.97}$$

Note that the nonsteerablility hypothesis (1.94) is more restrict than the Bell locality (1.85), which means that if a state is Bell nonlocal, then it is steerable. The contrary implication is not always true.

1.3.3 Entanglement

Entanglement is related to the degree of inseparability of a quantum state [71]. By restricting both statistical descriptions for Alice's and Bob's parties in (1.85), one obtains the *separability hypothesis*

$$p(a,b|A,B) = \int \mathrm{d}\lambda\, p(\lambda) p_q(a|A,\lambda) p_q(b|B,\lambda) \tag{1.98}$$

where

$$p_q(a|A,\lambda) = \mathrm{Tr}\left(A_a \rho_\lambda^{\mathcal{A}}\right), \tag{1.99a}$$

$$p_q(b|B,\lambda) = \mathrm{Tr}\left(B_b \rho_\lambda^{\mathcal{B}}\right), \tag{1.99b}$$

which implies that

$$p(a,b|A,B) = \int \mathrm{d}\lambda\, p(\lambda)\, \mathrm{Tr}\left[(A_a \otimes B_b)\left(\rho_\lambda^{\mathcal{A}} \otimes \rho_\lambda^{\mathcal{B}}\right)\right], \tag{1.100}$$

$$= \mathrm{Tr}\,(A_a \otimes B_b \rho), \tag{1.101}$$

with

$$\rho = \int \mathrm{d}\lambda\, p(\lambda) \rho_\lambda^{\mathcal{A}} \otimes \rho_\lambda^{\mathcal{B}}. \tag{1.102}$$

In other words, if a bipartite state ρ can be written as a sum of product states, then it is called *separable*, otherwise, it is called *entangled*. In order to quantify the degree of separability, an *entanglement measure* must satisfy a list of properties such as continuity, subadditivity, convexity, no increasing under local operations and classical communications (LOCC), and others [138, 139]. For bipartite pure quantum states $|\psi\rangle$, the entanglement can be measured by

$$E_S(|\psi\rangle) := S(\rho_{\mathcal{A}}) = S(\rho_{\mathcal{B}}), \tag{1.103}$$

where S is the von Neumann entropy and $\rho_{\mathcal{A},\mathcal{B}} = \mathrm{Tr}_{\mathcal{B},\mathcal{A}}(|\psi\rangle\langle\psi|)$. In addition, the *linear entropy* $S_L(\sigma) = 1 - \mathrm{Tr}\,\sigma^2$ can also be used as an alternative to the von Neumann one. In the case of a mixed state, the quantification of entanglement gets trickier in higher dimensions. For a generic two-qubit state ρ, entanglement can be computed by means of the concurrence [140]:

$$E_C(\rho) := \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}, \tag{1.104}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the eigenvalues of the operator $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ and ρ^* is the complex conjugate of ρ.

1.3.4 Quantum discord

Introduced by Olliver and Zurek [72], and independently by Henderson and Vedral [73], *quantum discord* was conceived as the minimum deviation between two different ways of quantifying mutual information for quantum states, namely, (1.12) and (1.13):

$$\mathcal{D}_{\mathcal{B}}(\rho) := \min_B \left[I_{\mathcal{A}:\mathcal{B}}(\rho) - J(\rho|\mathcal{B})\right], \tag{1.105}$$

$$= \min_B \left[\sum_b p_b S(\rho_{\mathcal{A}|b}) + S(\rho_{\mathcal{B}}) - S(\rho)\right]. \tag{1.106}$$

Note that, in general, $\mathcal{D}_{\mathcal{B}}(\rho) \neq \mathcal{D}_{\mathcal{A}}(\rho)$. Later on, Rulli and Sarandy [74] further explored the idea that quantum discord can also be viewed as the sensitivity of mutual information to minimally disturbing projective measurements conducted locally. We are going to show how to do that. Consider a CPTP unital map Φ_B as a local unrevealed measurement procedure such that

$$\Phi_B(\rho) = \sum_b (\mathbb{1}_{\mathcal{A}} \otimes B_b)\rho(\mathbb{1}_{\mathcal{A}} \otimes B_b) = \sum_b \rho_{\mathcal{A}|b} \otimes p_b |b\rangle\langle b|, \tag{1.107}$$

by an observable $B = \sum_b bB_b$ with projectors $B_b = |b\rangle\langle b|$ satisfying $\sum_b B_b = \mathbb{1}_{\mathcal{B}}$, where

$$\rho_{\mathcal{A}|b} = \frac{\mathrm{Tr}_{\mathcal{B}}(B_b\rho B_b)}{p_b} \qquad \text{and} \qquad p_b = \mathrm{Tr}\,(B_b\rho). \tag{1.108}$$

By applying the joint entropy theorem [see Eq. (1.67)] on $\Phi_B(\rho)$, one has

$$S(\Phi_B(\rho)) = H(p_b) + \sum_b p_b S(\rho_{\mathcal{A}|b}). \tag{1.109}$$

In addition,

$$H(p_b) = -\sum_b p_b \ln p_b = S\left(\sum_b p_b |b\rangle\langle b|\right) = S(\mathrm{Tr}_{\mathcal{A}}(\Phi_B(\rho))) = S(\Phi_B(\rho_{\mathcal{B}})). \tag{1.110}$$

Now, we substitute (1.110) in (1.109) and then in (1.106) to obtain

$$\mathcal{D}_{\mathcal{B}}(\rho) = \min_B \left[S(\Phi_B(\rho)) - S(\Phi_B(\rho_{\mathcal{B}})) + S(\rho_{\mathcal{B}}) - S(\rho)\right]. \tag{1.111}$$

Since $\Phi_B(\rho_{\mathcal{A}}) = \rho_{\mathcal{A}}$, one can add zero in the above equation to arrive at

$$\mathcal{D}_{\mathcal{B}}(\rho) = \min_B \left[S(\Phi_B(\rho)) - S(\Phi_B(\rho_{\mathcal{B}})) - S(\Phi_B(\rho_{\mathcal{A}})) + S(\rho_{\mathcal{B}}) + S(\rho_{\mathcal{A}}) - S(\rho)\right], \tag{1.112}$$

which clearly reduces to

$$\mathcal{D}_{\mathcal{B}}(\rho) \coloneqq \min_B \left[I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\Phi_B(\rho))\right]. \tag{1.113}$$

The quantum discord of a state $\rho \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$ relative to $\mathcal{B}$ is null when ρ possesses only classical correlation between each party, *i.e.*, only for quantum-classical (QC) states,

$$\rho_{\mathrm{QC}} = \sum_i \rho_{\mathcal{A}|i} \otimes p_i \left|i\right\rangle \left\langle i\right|, \tag{1.114}$$

and classical-classical (CC) states,

$$\rho_{\mathrm{CC}} = \sum_i p_i \left|i\right\rangle \left\langle i\right| \otimes \left|i\right\rangle \left\langle i\right|. \tag{1.115}$$

1.3.5 Quantum irreality

In Ref. [36], BA introduced a quantifier of *irreality*—a measure that indicates by how much the hypothesis of realism is violated. Their operational criterion of physical reality is as follows. Suppose we have a source that gives us infinitely many copies of a bipartite state. These copies are sent to a tomography procedure that furnishes the description $\rho \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$. Now, suppose that a secret agent starts to intercept each one of those copies and, before the tomography procedure, the agent measures always the same observable $A = \sum_i a_i A_i$, where $A_i = \left|a_i\right\rangle \left\langle a_i\right| \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}})$ are projectors such that $A_i A_j = \delta_{ij} A_i$ and $\sum_i A_i = \mathbb{1}_{\mathcal{A}}$, but the result is kept secret (see Fig. 6). Without having access to the measurement outcomes, the best description we can have now for the state is

$$\Phi_A(\rho) = \sum_i (A_i \otimes \mathbb{1}_{\mathcal{B}})\rho(A_i \otimes \mathbb{1}_{\mathcal{B}}) = \sum_i p_i A_i \otimes \rho_{\mathcal{B}|i}, \tag{1.116}$$

where $\rho_{\mathcal{B}|i} = \mathrm{Tr}_{\mathcal{A}}(A_i \rho A_i)/p_i$ is the state of the untouched part of the system conditioned to the part $\mathcal{A}$ outcome and $p_i = \mathrm{Tr}\,(A_i \rho)$ is the probability of the outcome a_i. Now, we can choose to interpret that the observable A is now an *element of reality* for $\Phi_A(\rho)$, since new measurements of the same observable will not disturb the system, i.e., $\Phi_A(\Phi_A(\rho)) = \Phi_A(\rho)$. The act of measuring A again will just reveal a pre-existing reality. Thus, the authors in Ref. [36] define that

Definition 1 (Element of reality)**.** The observable A is real for the state ρ *iff*

$$\Phi_A(\rho) = \rho. \tag{1.117}$$

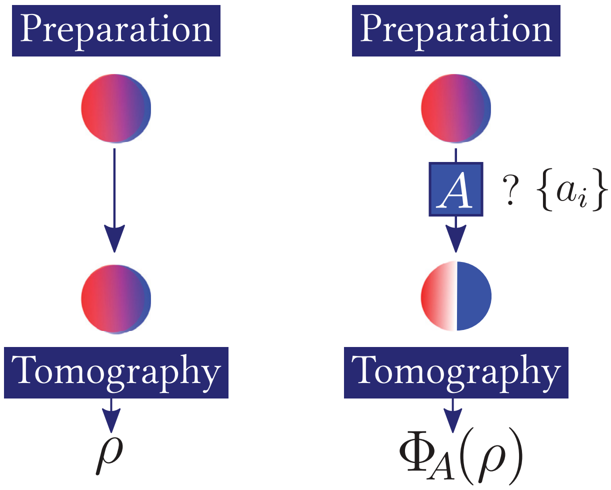

Figure 6 – Illustration of the operational criterion of BA's reality. (left) A given source provides infinitely many copies of a state (Preparation) to be sent to a tomography procedure in order to obtain the description ρ. (right) Each copy is intercepted and always the same observable A is measured, but the result a_i is kept secret. The unrevealed measured state $\Phi_A(\rho)$ is obtained by tomography and the descriptions are compared. A is real for the preparation ρ *iff* $\Phi_A(\rho) = \rho$ (illustration inspired on the work of Savi [141]).

A first proposal to quantify the violation of the above criterion was then introduced as the *irreality* of A given ρ by[3]

$$\mathfrak{I}_A(\rho) \coloneqq S(\Phi_A(\rho)) - S(\rho). \tag{1.118}$$

This measure is non-negative, since von Neumann entropy does not decrease under projective measurements, and it vanishes only, and just only, for states of A-reality (1.117). To prove that, we need the following lemma [142]:

Lemma 1. *For any function f, any quantum states ρ and σ, and any observable A, we have*

$$\mathrm{Tr}\left[\rho\, f(\Phi_A(\sigma))\right] = \mathrm{Tr}\left[\Phi_A(\rho) f(\Phi_A(\sigma))\right]. \tag{1.119}$$

The reader can find the proof in Appendix B. By the above lemma and the Klein's inequality (1.71), we immediately reach

$$\mathfrak{I}_A(\rho) = D(\rho||\Phi_A(\rho)) \geqslant 0, \tag{1.120}$$

where the last equality holds *iff* $\Phi_A(\rho) = \rho$. Interestingly, for any ρ on $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, one shows that

$$\mathfrak{I}_A(\rho) = \mathfrak{I}_A(\rho_\mathcal{A}) + D_A(\rho), \tag{1.121}$$

where $D_A(\rho) = I(\rho) - I(\Phi_A(\rho))$ is the measurement-dependent discord (note that $\mathcal{D}_\mathcal{A}(\rho) = \min_A D_A(\rho)$). This decomposition reveals that irreality actually captures both (i) information about local coherence and (ii) correlation changes induced by local measurements.

[3] In Part II we are going to revisit this concept through its complement, the *reality*, and discuss it in a deeper way with an axiomatic approach. Indeed, in Chap. 6 we are going to generalize this quantifier with the Rényi and Tsallis quantum information theories and, in addition, we are going to propose alternative geometric-based quantifiers for the irreality.

It is important to note differences between EPR's and BA's elements of reality. Although both of them agree that the eigenstates of an observable are completely real for that observable, they disagree for correlated states and maximally mixed states. See the following examples.

Example 7. While EPR say that every observable is real for the singlet state $\rho_s = |\psi_s\rangle\langle\psi_s|$ because the state is already defined before the measurement (and quantum mechanics is "supposedly incomplete", and that is why we need hidden variables to describe the "apparently" random measurement results), BA's criterion claims full irreality. Indeed, one can write any 2-dimensional observable

$$A = \sum_i a_i A_i = A_+ - A_- \tag{1.122}$$

in terms of its projectors

$$A_\pm = \frac{1}{2}\left(\mathbb{1}_{\mathcal{A}} \pm \hat{v}\cdot\vec{\sigma}\right), \tag{1.123}$$

where the unitary vector $\hat{v} \in \mathbb{R}^3$ gives the measurement direction, to obtain

$$\Phi_A(\rho_s) = \frac{1}{2}\left[\rho + (\hat{v}\cdot\vec{\sigma})\,\rho\,(\hat{v}\cdot\vec{\sigma})\right]. \tag{1.124}$$

A direct calculation using any software of symbolic computation gives $S(\Phi_A(\rho_s)) = \ln 2$ for any $\hat{v}$. Since $S(\rho_s) = 0$, one has $\mathfrak{I}_A(\rho_s) = \ln 2$. ▲

Example 8. Now, imagine that we ignore the far particle in the previous example, which mathematically implies that we are tracing out the $\mathcal{B}$ part of the system: $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho_s) = \mathbb{1}_{\mathcal{A}}/2$. Therefore, $\mathfrak{I}_A(\rho_{\mathcal{A}}) = 0$ for any A. This makes sense when we accept the following argument. A maximally mixed state represents a fair classical coin that, in turn, lies always in a real state of affairs. For instance, imagine a classical coin toss game. The coin lands on the back of one hand and then we cover it with the other. The most skeptical of physicists (and perhaps here we should leave out our beloved QBist friends [30]) would say that the state of the coin is already real, even if we cannot predict its outcome. The simple act of raising the hand will not interfere with the definiteness of the physical quantity associated with the face of the coin that is already turned up. We can say that the observable $A =$ *turned-up-face* of the coin is already an element of reality, *i.e.*, $\mathfrak{I}_A(\mathbb{1}_{\mathcal{A}}/2) = 0$. ▲

After all, is A real or not? It depends. Locally, the element of reality exists, but globally, correlations with the whole prevent the prevalence of physical reality. Of course, the classical coin analogy for the state $\rho_{\mathcal{A}} = \mathbb{1}_{\mathcal{A}}/2$ is just that: an analogy. The nature of the electron spin is still unclear and, therefore, subject to interpretations based on the experimental results.

1.3.6 Realism-based nonlocality

One of the implications of the irreality measure (1.118) is the realism-based nonlocality, a notion of nonlocal behavior proposed initially by BA [36] and further developed by

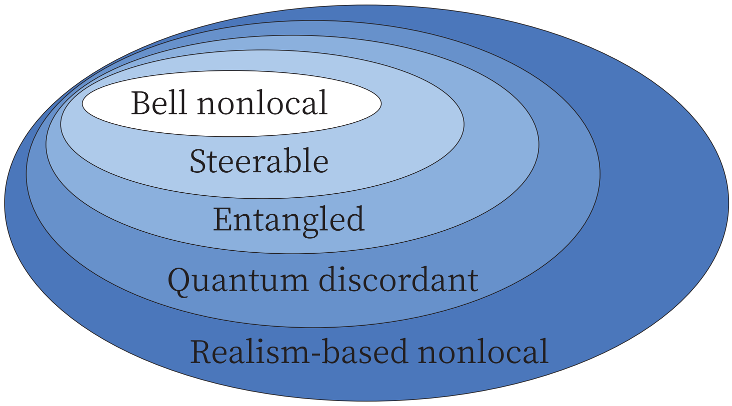

Figure 7 – Venn diagram representing the hierarchy of contextual independent nonclassical aspects: Bell nonlocality, Steering, Entanglement, Quantum discord, and Realism-based nonlocality. Each ellipse represents a set of states that contains the highlighted nonclassical feature.

Gomes and Angelo [37, 143]. The *BA locality hypothesis* is stated as

$$\mathfrak{I}_A(\rho) = \mathfrak{I}_A(\Phi_B(\rho)), \tag{1.125}$$

that is, the degree of irreality of an observable A acting at $\mathcal{A}$ should not change due to projective non-selective measurements made by B in a far site $\mathcal{B}$. Violations of (1.125) can be quantified by

$$\eta_{AB}(\rho) := \mathfrak{I}_A(\rho) - \mathfrak{I}_A(\Phi_B(\rho)), \tag{1.126}$$

$$= S(\Phi_A(\rho)) + S(\Phi_B(\rho)) - S(\Phi_{AB}(\rho)) - S(\rho), \tag{1.127}$$

giving rise to the *contextual realism-based nonlocality*. As the name suggests, this measure depends on the context (A, B, ρ). It is always non-negative and vanishes for product states $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$ or when ρ is a state of reality for A or B[4]. Then, the (context independent) *realism-based nonlocality* $\mathcal{N}$, that is, a nonlocal aspect inherent to the state, can be defined by taking the maximization over all available observables,

$$\mathcal{N}(\rho) := \max_{A,B} \eta_{AB}(\rho). \tag{1.128}$$

This measure is always non-negative and it is null for product states $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$. In the case of pure states, the realism-based nonlocality reduces to entanglement. When compared to the nonclassical features discussed previously, realism-based nonlocality has shown to be the most ubiquitous one [143]. Even for nondiscordant states like (1.115), the measure $\mathcal{N}$ can be nonzero. Let us take the contextual realism-based nonlocality for ρ_{CC},

$$\eta_{AB}(\rho_{CC}) = \eta_{AB}\left(\sum_i p_i A_i' \otimes B_i'\right), \tag{1.129}$$

$$= H(\{p_i\}) + \sum_i S\left[\Phi_A(A_i') \otimes \Phi_B(B_i')\right] - S\left[\sum_i p_i \Phi_A(A_i') \otimes \Phi_B(B_i')\right]. \tag{1.130}$$

[4] The sufficiency for this statement is yet unknown.

The quantity $\mathcal{N}(\rho_{CC})$ will be nonzero if at least one context (A, B, ρ_{CC}) provides $\eta_{AB}(\rho_{CC}) > 0$. In fact, for observables A and B that are maximally incompatible with $A' = \sum_i a_i' A_i'$ and $B' = \sum_i b_i' B_i'$, respectively, one has $\Phi_A(A_i') = \mathbb{1}_{\mathcal{A}}/d_{\mathcal{A}}$ and $\Phi_B(B_i') = \mathbb{1}_{\mathcal{B}}/d_{\mathcal{B}}$, and therefore $\eta_{AB}(\rho_{CC}) = H(\{p_i\})$, which is positive. The hierarchy satisfied by the contextual independent nonclassical aspects presented so far is depicted in Fig. 7.

1.4 QUANTUM WALKS

Quantum walks are the quantum version of classical random walks[5]. In the classical version, a walker takes discrete steps depending on the result of a coin toss game (or any other random event with dichotomous results). In the quantum analog, the walker is a particle with a spin-like internal degree of freedom whose step to the right or left is conditioned by its internal degree. One possible description of the system can be formulated, *e.g.*, by setting that spin up goes right and spin down goes left. If the particle is in a superposition between spin up and down, then it makes a step as a superposition between left and right. The random aspect inherent to the quantum walk evolution is due to a rotation in the spin of the particle before each step. Thus, because of the superposition principle, some aspects emerge from the quantum walk making it dramatically distinct from its classical version (check out Fig. 2 again in the Introduction).

1.4.1 Formal structure

The state of a one-dimensional quantum walker belongs to a Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_X$, where $\mathcal{H}_S$, spanned by $\{|\uparrow\rangle, |\downarrow\rangle\}$, refers to a spin-1/2 space state ($\hbar = 1$) and $\mathcal{H}_X$, spanned by a discrete basis $\{|x\rangle : x \in \mathbb{Z}\}$, denotes the space state associated with the dimensionless discrete position X. Let

$$|\psi_0\rangle = \left(\cos\frac{\alpha}{2}|\uparrow\rangle + \sin\frac{\alpha}{2}|\downarrow\rangle\right) \otimes \sum_{x=-\infty}^{\infty} f(x)|x\rangle \tag{1.131}$$

be the initial state such that $\alpha \in [0, \pi]$ and f is the initial probability amplitude for the walker position. The single-step unitary evolution is determined by the operator [45]

$$U = D\,(C \otimes \mathbb{1}_X), \tag{1.132}$$

where D is the conditional displacement operator,

$$D = \sum_x \Big(|\uparrow\rangle\langle\uparrow| \otimes |x+1\rangle\langle x| + |\downarrow\rangle\langle\downarrow| \otimes |x-1\rangle\langle x| \Big), \tag{1.133}$$

and C is the so-called quantum coin, a SU(2) matrix which here is chosen to be the standard unbiased Hadamard coin:

$$C \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1.134}$$

[5] For a very gentle introduction see Sec. 1.1 in Ref. [144] in Portuguese or Ref. [56] in English.

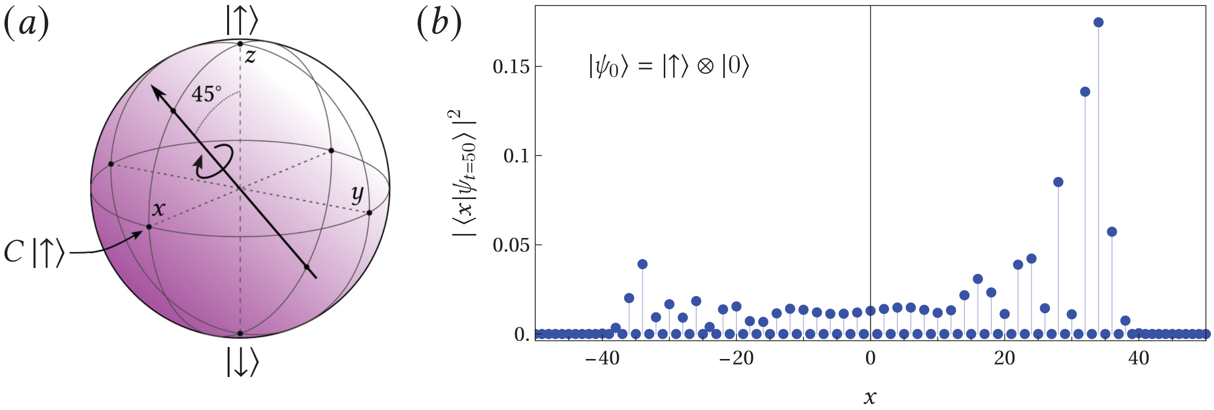

Figure 8 – (a) Depiction of the rotation in the spin state $|{\uparrow}\rangle$ due to the action of the Hadamard coin. (b) Numerical probability distribution $|\langle x|\psi_t\rangle|^2$ of a quantum walk under the action of a Hadamard coin with initial state $|\psi_0\rangle = |{\uparrow}\rangle \otimes |0\rangle$ at $t = 50$. Note that, if t is even (odd), the particle can only be found in a even (odd) position.

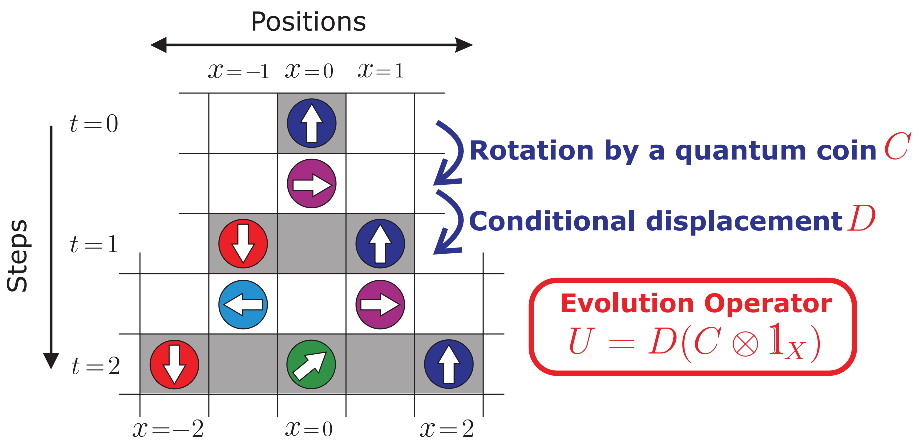

Figure 9 – Two time steps of a quantum walk starting with spin state $|{\uparrow}\rangle$ at position state $|x = 0\rangle$.

When represented in the Bloch sphere, a qubit under the action of the Hadamard coin is rotated by 180°around the axis $(\hat{x}+\hat{z})/\sqrt{2}$, as in Fig. 8 ($a$). See also Fig. 9 for a depiction of the quantum walk evolution.

Example 9. Consider a quantum walker in the initial state $|\psi_{t=0}\rangle = |{\uparrow}\rangle \otimes |0\rangle$, which corresponds to $\alpha = 0$ and $f(x) = \delta_{x,0}$ in (1.131). Now, let us use the canonical basis

$$|{\uparrow}\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |{\downarrow}\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.135}$$

We can calculate the state of the walker after one time-step by

$$|\psi_{t=1}\rangle = U\,|\psi_{t=0}\rangle\,, \tag{1.136}$$

$$= D(C \otimes \mathbb{1}_X)(|{\uparrow}\rangle \otimes |0\rangle), \tag{1.137}$$

$$= D\left(C\,|{\uparrow}\rangle \otimes \mathbb{1}_X\,|0\rangle\right), \tag{1.138}$$

$$= D\left(\frac{|{\uparrow}\rangle + |{\downarrow}\rangle}{\sqrt{2}} \otimes |0\rangle\right), \tag{1.139}$$

$$= \left(\sum_x |\uparrow\rangle \langle\uparrow| \otimes |x+1\rangle \langle x| + |\downarrow\rangle \langle\downarrow| \otimes |x-1\rangle \langle x| \right) \left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \otimes |0\rangle \right), \tag{1.140}$$

$$= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |1\rangle + |\downarrow\rangle \otimes |-1\rangle \right). \tag{1.141}$$

As we can see, the position and the spin degrees of freedom can become entangled during the time evolution of the quantum walk. This means that some level of information regarding the spin state of the walker becomes encoded in the position state. After the second step, we also start to see quantum interference in the time evolution:

$$|\psi_{t=2}\rangle = U\,|\psi_{t=1}\rangle, \tag{1.142}$$

$$= D(C \otimes \mathbb{1}_X) \frac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |1\rangle + |\downarrow\rangle \otimes |-1\rangle \right), \tag{1.143}$$

$$= \frac{1}{2} |\uparrow\rangle \otimes |2\rangle + \frac{1}{2} \left(|\uparrow\rangle + |\downarrow\rangle \right) \otimes |0\rangle - \frac{1}{2} |\downarrow\rangle \otimes |-2\rangle \tag{1.144}$$

At the third time-step, we start to see a shift in the probability $p_x = |\langle x|\psi_t\rangle|^2$ towards the right due to the initial spin state that we have chosen:

$$|\psi_{t=3}\rangle = \frac{1}{2\sqrt{2}} |\uparrow\rangle \otimes |3\rangle + \left(\frac{|\uparrow\rangle}{\sqrt{2}} + \frac{|\downarrow\rangle}{2\sqrt{2}} \right) \otimes |1\rangle - \frac{1}{2\sqrt{2}} |\uparrow\rangle \otimes |-1\rangle + \frac{1}{2\sqrt{2}} |\downarrow\rangle \otimes |-3\rangle. \tag{1.145}$$

In Fig. 8 (b) we show a numerical evaluation of $|\langle x|\psi_{t=50}\rangle|^2$. ▲

Remark 1. The class of spin states indicated in (1.131) represents a circle in the xz plane of the Bloch sphere, that is, states with no phase difference between $|\uparrow\rangle$ and $|\downarrow\rangle$. Some studies have shown that, when employed along with the Hadamard coin, such class of states is sufficiently general, in the sense that they can yield every possible production rate of spin-position entanglement [145], as well as every possible dispersion [115]. Still, there is some lack of generality because only one combination of features—entanglement and dispersion—can be simulated through this approach [115].

Now, let us explore a broader class of initial position distributions $f(x)$.

1.4.2 The Gaussian states

The walker state after t steps can be written as

$$|\psi_t\rangle = U^t\,|\psi_0\rangle = \sum_x \Big[a_t(\alpha, x)\,|\uparrow\rangle + b_t(\alpha, x)\,|\downarrow\rangle \Big] \otimes |x\rangle, \tag{1.146}$$

with normalization condition

$$\sum_x \left(|a_t(\alpha, x)|^2 + |b_t(\alpha, x)|^2 \right) = 1 \tag{1.147}$$

and a dimensionless time $t \in \mathbb{N}$. If the initial distribution $|f(x)|^2$ is sharply localized, the spin amplitudes $a_t(\alpha, x)$ and $b_t(\alpha, x)$ evolve according to a highly oscillatory pattern, a well-known characteristic of local states [see Fig. 8 (b)]. Fourier analysis combined with the stationary phase method [146] define a largely applied scheme to achieve analytical results, such as those reported for long-time dispersion rates [147–149] and asymptotic entanglement [60, 115, 145, 150, 151]. This approach, however, is not appropriate for our purposes because we are interested in looking at the whole dynamics of nonclassical features quantifiers. To this end, we adopt throughout this thesis a model according to which the initial distribution is given by the Gaussian function

$$f(x) = \frac{1}{\sqrt{K}} \exp\left(-\frac{x^2}{4\sigma_0^2}\right), \tag{1.148}$$

where σ_0 is the dimensionless dispersion and

$$K = \sum_x \exp\left(-\frac{x^2}{2\sigma_0^2}\right) \tag{1.149}$$

is the normalization constant. This choice is rather convenient, for it is known that, whenever σ_0 is sufficiently large, such state preserves not only its Gaussianity over time [152] but also the interesting properties of ballistic spreading and entanglement creation. Figure 10 gives a comparison of the probability distributions $|\langle x|\,\psi_t\rangle|^2$ at $t = 100$ for quantum walkers initially prepared in a local ($\sigma_0 = 0.2$) and in a broad ($\sigma_0 = 5$) Gaussian state for different initial spin states ($\alpha = 0, \pi/4$, and $3\pi/4$). The region wherein the walker is likely to be found increases with time as $2t + 2\sigma_0$ and the analytical treatment of the problem for long times remains unfeasible even for Gaussian states.

1.4.3 Entanglement in quantum walks

Since information about the spin is being encoded at the positions the quantum walker is passing through, we have entanglement being produced during the time evolution of the system. Because the initial state (1.131) is pure and the time evolution operator (1.132) is unitary, the state $|\psi_t\rangle$ remains pure. Therefore, the entanglement between the degrees of freedom in $\mathcal{H}_S$ and $\mathcal{H}_X$ can be calculated using (1.103). Having said that, we just need to obtain the reduced density matrix relative to the spins by tracing out the positions

$$\rho_S = \mathrm{Tr}_X\left(|\psi_t\rangle\langle\psi_t|\right) = \begin{pmatrix} \sum_x |a_t(\alpha,x)|^2 & \sum_x a_t(\alpha,x)b_t^*(\alpha,x) \\ \sum_x a_t^*(\alpha,x)b_t(\alpha,x) & \sum_x |b_t(\alpha,x)|^2 \end{pmatrix} =: \begin{pmatrix} \Theta & \Gamma \\ \Gamma^* & 1-\Theta \end{pmatrix}. \tag{1.150}$$

Let us choose the von Neumann entropy (1.63) in order to calculate the entanglement in $|\psi_t\rangle$. To do that, we need the eigenvalues of ρ_S, which are

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \sqrt{(1-2\Theta)^2 + 4|\Gamma|^2}\right). \tag{1.151}$$

Therefore, the entanglement is given by

$$E(|\psi_t\rangle) = S(\rho_S) = -\lambda_+ \ln\lambda_+ - \lambda_- \ln\lambda_-. \tag{1.152}$$

In Fig. 11, we present a numerical evaluation of the time evolution of the above equation for

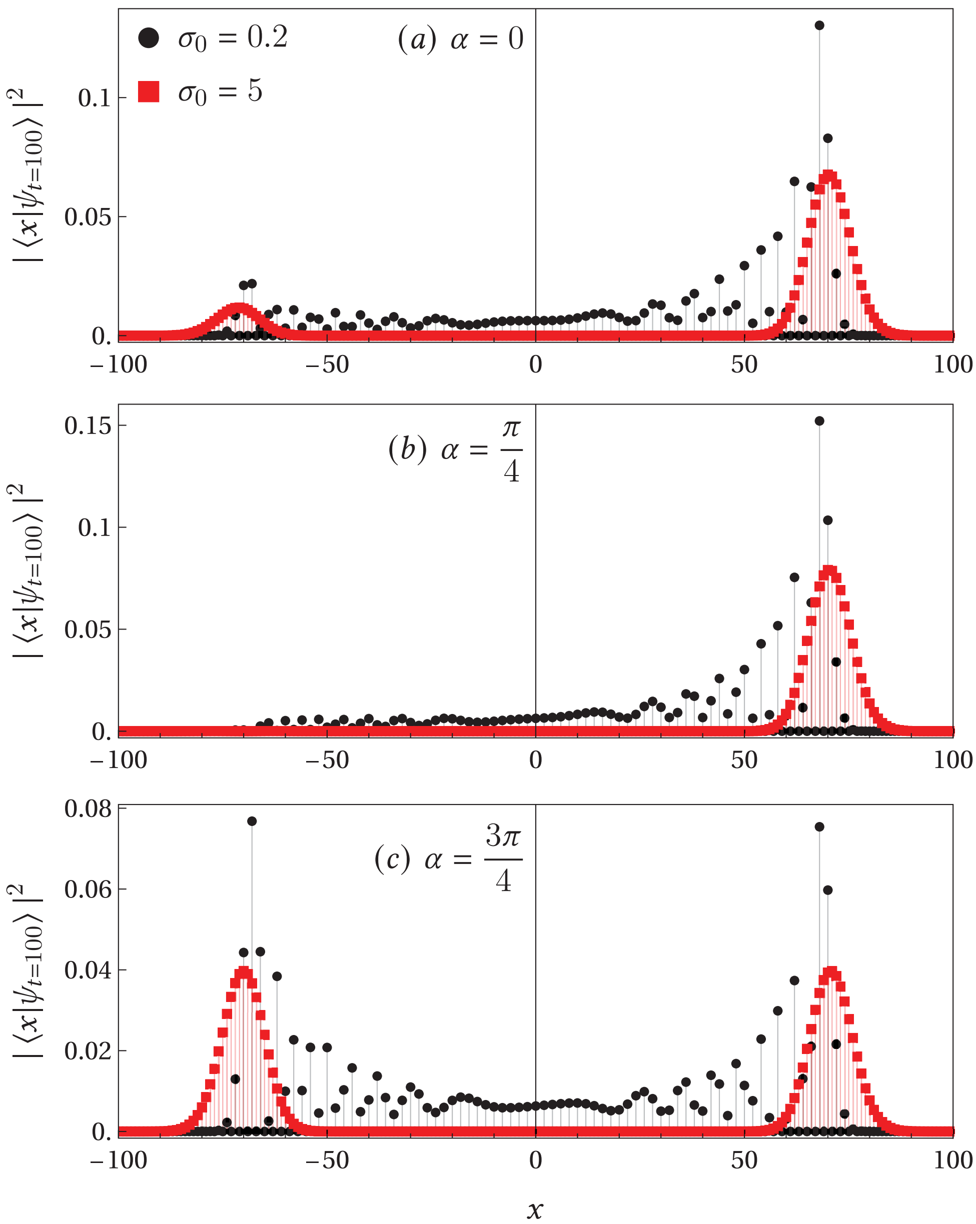

Figure 10 – Numerical probability distributions $|\langle x|\psi_t\rangle|^2$ of quantum walks with (a) $\alpha = 0$, (b) $\alpha = \pi/4$, and (c) $\alpha = 3\pi/4$ [see Eq. (1.131)] for local ($\sigma_0 = 0.2$, black circles) and broad ($\sigma_0 = 5$, red squares) Gaussian states, at $t = 100$, as a function of the dimensionless position x. The greater the initial dispersion, the more effective the maintenance of the Gaussian shape over time.

the same six different initial configurations of Fig. 10. In Fig. 11 we can see some fundamental characteristics about the production of entanglement in quantum walks. The first one is that the entanglement reaches an asymptotic limit over time, regardless of the distance traveled by the walker. This asymptotic limit depends both on the quantum coin used in the temporal evolution and on the characteristics of the walker's initial state. The more spatially centered the walker's state, the greater our uncertainty regarding the moment, which leads to high oscillations both in the probability distribution and in the production of entanglement. In the case of wide Gaussian states, the uncertainty at the moment is smaller, which leads to a

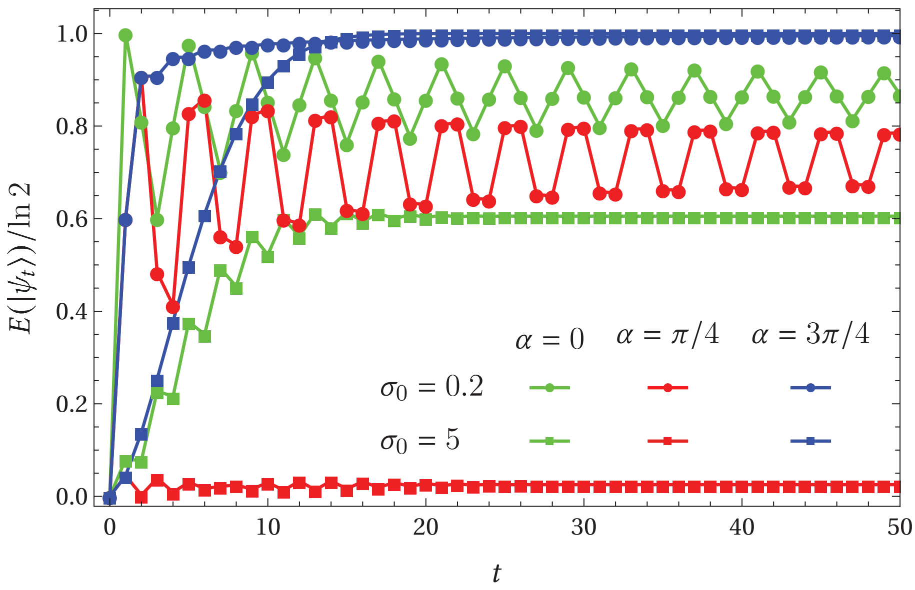

Figure 11 – Numerical simulation of the entanglement [see Eq. (1.152)] as a function of time of a quantum walker for six different initial conditions: $\sigma_0 = 0.2$ (circles) and $\sigma_0 = 5$ (squares), and $\alpha = 0$ (green), $\alpha = \pi/4$ (red), and $\alpha = 3\pi/4$ (blue).

better-defined trajectory and the asymptotic entanglement limit is reached more quickly. From Fig. 8 (a), we can see that the spin state given by $\alpha = 3\pi/4$ is orthogonal to the eigenstates of the Hadamard coin [which rotates around the $(\hat{x} + \hat{z})/\sqrt{2}$ axis]. Thus, this state is maximally affected by the transformation caused by C and, therefore, leads to the maximum correlation between spin and position. The exact opposite occurs with spin states with $\alpha = \pi/4$ direction.

Another way to produce maximally entangled states in quantum walks is through the addition of dynamical disorder in the time evolution (1.132) by randomly chosen a different quantum coin C at each time-step. That fact was numerically and analytically proved by Vieira *et al.* in Refs. [153, 154]. The optimal rate of disorder is actually very low and it depends on σ_0, as showed in Ref. [116]. Nonetheless, for the purposes of this thesis, we will not study dynamically disordered quantum walks.

2 IRREALITY, QUANTUM CORRELATIONS, AND NONLOCALITY AMONG QUANTUM WALKERS

This chapter is dedicated to presenting the discussion and related results of our research into two quantum walker systems published in Physical Review A [A. C. Orthey and R. M. Angelo, "Nonlocality, quantum correlations, and violations of classical realism in the dynamics of two noninteracting quantum walkers", Phys. Rev. A **100**, 042110 (2019)]. Following the discussion made in the Introduction, we are going to explore the dynamical evolution of several non-classical features in a system composed of two non-interacting quantum walkers. First, we are going to present our simplified Gaussian model in order to analytically assess the system.

2.1 SIMPLIFIED GAUSSIAN MODEL FOR QUANTUM WALKS

Let us introduce the fundamental ingredients of our simplified Gaussian model for quantum walks. First, we employ the approximation

$$K = \sum_{x=-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma_0^2}\right) = \vartheta_3\left(0, e^{-1/2\sigma_0^2}\right) \cong \sqrt{2\pi\sigma_0^2}, \tag{2.1}$$

where

$$\vartheta_3(z,q) = \sum_{x=-\infty}^{+\infty} q^{x^2} e^{2xiz} \tag{2.2}$$

is the *Jacobi theta function*. This approximation fails only for $\sigma_0 < 1$, a domain that will henceforth be out of scope. Second, for the description of the long-time Gaussian distributions illustrated in Fig. 10, we propose the ansatz

$$a_t(\alpha,x) = q_a^+(\alpha)\, g_t^+(x) + q_a^-(\alpha)\, g_t^-(x), \tag{2.3a}$$

$$b_t(\alpha,x) = q_b^+(\alpha)\, g_t^+(x) + q_b^-(\alpha)\, g_t^-(x), \tag{2.3b}$$

where

$$g_t^\pm(x) = \frac{(\pm 1)^t}{\left(2\pi\sigma_0^2\right)^{1/4}} \exp\left[-\frac{\left(x \mp t/\sqrt{2}\right)^2}{4\sigma_0^2}\right] \tag{2.3c}$$

and

$$q_u^\pm(\alpha) = \frac{1}{4}\left[\mathfrak{c}_u^\pm \cos\left(\frac{\alpha}{2}\right) + \mathfrak{s}_u^\pm \sin\left(\frac{\alpha}{2}\right)\right], \tag{2.3d}$$

with $\mathfrak{c}_u^\pm$ and $\mathfrak{s}_u^\pm$ $(u = a, b)$ being the coefficients that will adjust our model to the exact numerical results. The above formulas were derived with basis on preliminary numerical studies. Note that the amplitudes $g_t^\pm(x)$ move with speed $1/\sqrt{2}$ (a hallmark of the Hadamard walk). The oscillatory form proposed for $q_u^\pm(\alpha)$ is naturally induced by the structure of the initial state

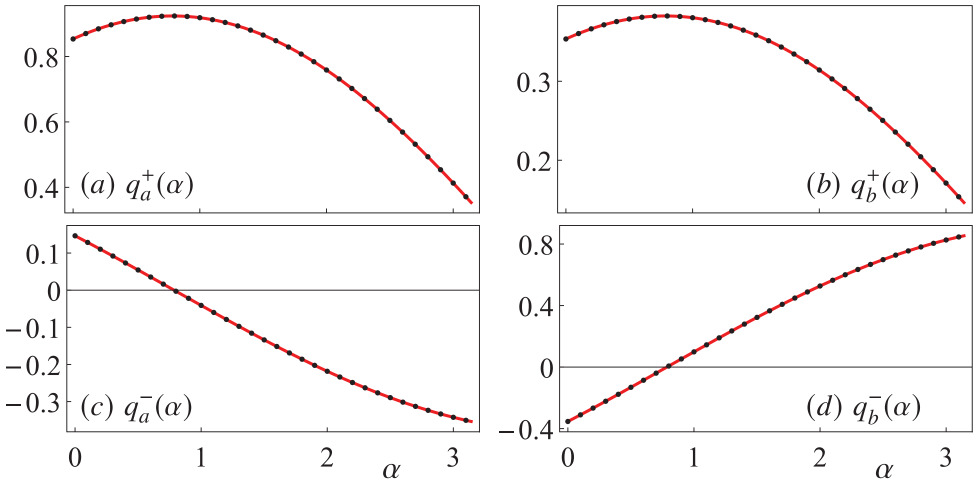

Figure 12 – Comparison between analytical model $q^{\pm}_{a,b}$ (red line) and numerical data (black points) as a function of α at $t = 100$ and $\sigma_0 = 10$.

(1.131). After an extensive numerical analysis (see Fig. 12), involving many different values of α, σ_0, and t, we have found[1]

$$\mathfrak{c}_a^{\pm} \cong 2 \pm \sqrt{2}, \qquad \mathfrak{s}_a^{\pm} \cong \pm\sqrt{2}, \tag{2.4a}$$

$$\mathfrak{c}_b^{\pm} \cong \pm\sqrt{2}, \qquad \mathfrak{s}_b^{\pm} \cong 2 \mp \sqrt{2}. \tag{2.4b}$$

Equations (1.131)-(2.4) define our quantum-walk simplified model. The quality of this model was tested via evaluation of the fidelity $|\langle\psi_t^{\text{sim}}|\psi_t\rangle|^2$ of the state $|\psi_t\rangle$, computed with our simplified model, with respect to the state $|\psi_t^{\text{sim}}\rangle$, derived via numerical simulation. For sufficiently broad states ($\sigma_0 \gtrsim 5$) and several values of $\{\sigma_0, \alpha, t\}$ the fidelity was never less than 99.8%. Also noteworthy is the fact that in Eq. (2.3c) we have tacitly assumed that the dispersion of each Gaussian maintains its initial value σ_0, which proved to be a rather good approximation whenever $\sigma_0 \gg 1$. For future convenience, we note that for $\alpha = 0$ and $\alpha = \pi$, which imply the initial states

$$|\uparrow\rangle \otimes \sum_x f(x)\,|x\rangle =: |\psi_0^{\uparrow}\rangle\,, \tag{2.5a}$$

$$|\downarrow\rangle \otimes \sum_x f(x)\,|x\rangle =: |\psi_0^{\downarrow}\rangle\,, \tag{2.5b}$$

the above model leads to the respective solutions:

$$\sum_x \Big[a_t(0,x)\,|\uparrow\rangle + b_t(0,x)\,|\downarrow\rangle\Big] \otimes |x\rangle =: |\psi_t^{\uparrow}\rangle\,, \tag{2.6a}$$

$$\sum_x \Big[a_t(\pi,x)\,|\uparrow\rangle + b_t(\pi,x)\,|\downarrow\rangle\Big] \otimes |x\rangle =: |\psi_t^{\downarrow}\rangle\,. \tag{2.6b}$$

[1] Two years after the publication of our results in Ref. [38], Vieira *et al.* [155] have solved this problem analytically.

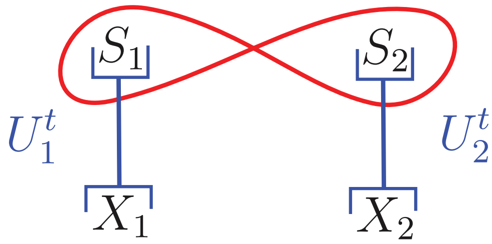

Figure 13 – Depiction of the dynamics of the temporal evolution between the four parts of the Hilbert space that contain the state of the two quantum walkers: two initially entangled qubits S_1 and S_2 that are posteriorly coupled with the position degrees of freedom X_1 and X_2 through the discrete time unitary evolutions U_1^t and U_2^t.

2.2 TWO WALKERS

Consider now two walkers, named 1 and 2, whose state vector lies in the Hilbert space $\mathcal{H} = \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} \otimes \mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}$. In our model, we consider a scenario where the spins $S_{1,2}$ are initially prepared in a maximally entangled state (the singlet state) while the positions $X_{1,2}$ of the walkers are described by Gaussian distributions centered at the origins of their (distinct) coordinate systems. The joint state reads

$$|\Psi_0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \otimes \sum_{x_1,x_2} f(x_1) f(x_2) \, |x_1, x_2\rangle \, , \tag{2.7}$$

with f given by Eq. (1.148). Moreover, we assume that the walkers do not interact with each other and with the external universe. Concretely, we can conceive an instance such that, after getting their spins maximally correlated, the particles are put to walk in distinct laboratories, which can be arbitrarily separated in space. Each walker is governed by its own unitary dynamics and the eventual emergence of any nonclassical feature between initially independent degrees of freedom (S_1 and X_2, for example) must be accomplished thanks to the only quantum resource encoded in the joint state, namely, two-qubit entanglement. The reader can found a depiction of our two quantum walker system in Fig. 13.

To obtain the time-evolved state vector, we should first realize that the state (2.7) can be spanned in terms of the kets (2.5) as

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}} \left(|\psi_0^{\uparrow}\rangle \, |\psi_0^{\downarrow}\rangle - |\psi_0^{\downarrow}\rangle \, |\psi_0^{\uparrow}\rangle \right) . \tag{2.8}$$

Linearity immediately allows us to use the solutions (2.6) to write

$$|\Psi_t\rangle = \frac{1}{\sqrt{2}} \left(|\psi_t^{\uparrow}\rangle \, |\psi_t^{\downarrow}\rangle - |\psi_t^{\downarrow}\rangle \, |\psi_t^{\uparrow}\rangle \right) , \tag{2.9}$$

which can be more explicitly written as

$$|\Psi_t\rangle = \sum_{x_1,x_2} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{2t^2 + 4x_{cm}^2 + x_r^2}{8\sigma_0^2} \right) |s_t(x_r)\rangle \otimes |x_1, x_2\rangle \, , \tag{2.10}$$

where we have introduced, for the sake of notational simplicity, $x_{cm} = (x_1 + x_2)/2$ (the center of mass position), $x_r = x_2 - x_1$ (the relative position), the non-normalized state

$$|s_t(x_r)\rangle = \sinh\left(\frac{t x_r}{2\sqrt{2}\,\sigma_0^2}\right)|\beta_{23}\rangle + \cosh\left(\frac{t x_r}{2\sqrt{2}\,\sigma_0^2}\right)|B_4\rangle\,, \tag{2.11}$$

and the Bell basis

$$\begin{aligned} |B_1\rangle &= \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}, \qquad |B_2\rangle = \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}, \\ |B_3\rangle &= \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}, \qquad |B_4\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}, \end{aligned} \tag{2.12}$$

which allowed us to write $|\beta_{23}\rangle \equiv (|B_2\rangle - |B_3\rangle)/\sqrt{2}$.

It is clear from the result (2.10) that none of the original degrees of freedom $\{S_1, S_2, X_1, X_2\}$ factorizes for $t > 0$. On the other hand, the two-spin state (2.11) depends only on the relative coordinate, so that the state associated with the center of mass must factorize. This can be proved as follows. Let us replace laboratory positions $\{x_1, x_2\}$ with center of mass and relative coordinates $\{x_{cm}, x_r\}$ through the usual map [156, 157]

$$|x_1\rangle \otimes |x_2\rangle \mapsto |x_2 - x_1\rangle_r \otimes \left|\frac{x_1 + x_2}{2}\right\rangle_{cm}, \tag{2.13}$$

which links every state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with a counterpart in $\mathcal{H}_{cm} \otimes \mathcal{H}_r$ (assuming walkers with equal masses). Using this map and changing dummy variables in summations, we rewrite the state (2.10) in an explicitly separable form, $|\Psi_t\rangle = |\Theta\rangle_{cm} \otimes |\Phi_t\rangle_r$, where

$$|\Theta\rangle_{cm} = \sum_{x_{cm}} \frac{1}{(\pi\sigma_0^2)^{1/4}} \exp\left(-\frac{x_{cm}^2}{2\sigma_0^2}\right)|x_{cm}\rangle\,, \tag{2.14a}$$

$$|\Phi_t\rangle_r = \sum_{x_r} \frac{1}{(4\pi\sigma_0^2)^{1/4}} \exp\left(-\frac{2t^2 + x_r^2}{8\sigma_0^2}\right)|s_t(x_r)\rangle \otimes |x_r\rangle\,. \tag{2.14b}$$

This completes the proof. An interesting observation can now be made for the spins. By separating the summation for x_r in parcels with $x_r < 0$, $x_r = 0$, and $x_r > 0$, we can compute the asymptotic state

$$|\Phi_\infty\rangle = \frac{1}{\sqrt{2}}\left(|\phi_\infty^+\rangle \otimes |S_+\rangle - |\phi_\infty^-\rangle \otimes |S_-\rangle\right), \tag{2.15}$$

where

$$|\phi_t^\pm\rangle = \sum_{x_r > 0} \frac{1}{(4\pi\sigma_0^2)^{1/4}} \exp\left(-\frac{(t - x_r/\sqrt{2})^2}{4\sigma_0^2}\right)|\pm x_r\rangle\,, \tag{2.16a}$$

$$|S_\pm\rangle = \frac{|\beta_{23}\rangle \pm |B_4\rangle}{\sqrt{2}}, \tag{2.16b}$$

and $\phi_\infty^\pm = \lim_{t\to\infty} |\phi_t^\pm\rangle$. We see, therefore, that by measuring the sign of the relative coordinate, one makes the two-spin state collapse to either $|S_+\rangle$ or $|S_-\rangle$, which constitute peculiar coherent superpositions of Bell states.

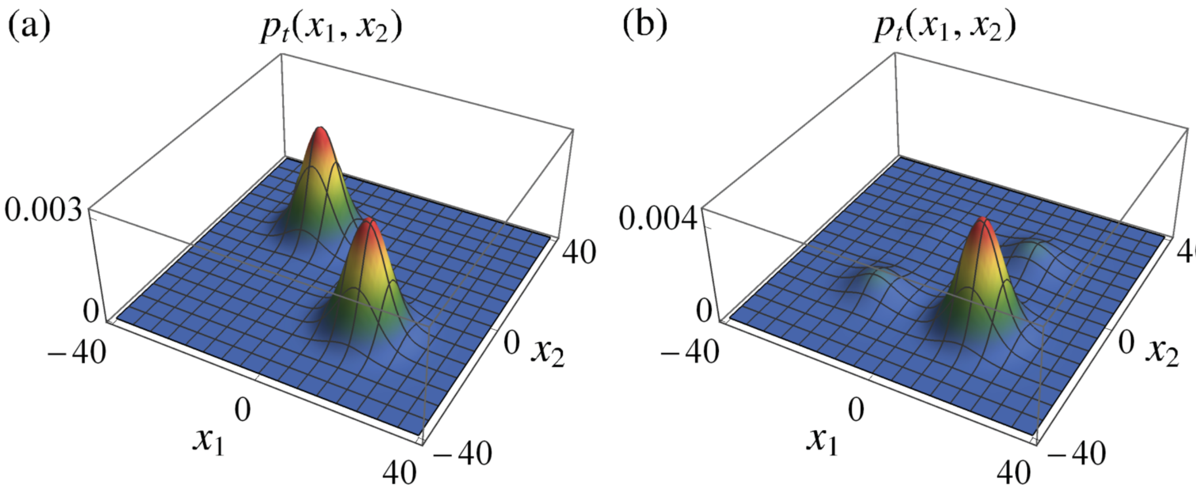

Figure 14 – Probability distribution $p_t(x_1, x_2)$ at $t = 20$ of two quantum walkers with initial states (a) $|\Psi_0\rangle$ [see Eq. (2.7)] and (b) $|\psi_0^{\uparrow}\rangle\, |\psi_0^{\downarrow}\rangle$ [see Eqs. (2.5)], both with $\sigma_0 = 5$. A correlated two-spin state is more effective in producing strong spatial anticorrelations.

Now we show that correlations develop between walkers' positions. The probability

$$p_t(x_1, x_2) = \mathrm{Tr}\ (\Omega_t\, |x_1, x_2\rangle\, \langle x_1, x_2|) \tag{2.17}$$

of finding them at the respective locations (x_1, x_2), at time t, given the state $\Omega_t = |\Psi_t\rangle\, \langle \Psi_t|$, results in

$$p_t(x_1, x_2) = \frac{1}{4\pi\sigma_0^2} \exp\left(-\frac{x_{cm}^2}{\sigma_0^2}\right) \left\{ \exp\left[-\frac{\left(t - x_r/\sqrt{2}\right)^2}{2\sigma_0^2}\right] + \exp\left[-\frac{\left(t + x_r/\sqrt{2}\right)^2}{2\sigma_0^2}\right] \right\}, \tag{2.18}$$

whose maximum value occurs for $x_{cm} = 0$ and $x_r = \pm t\sqrt{2}$, that is, $x_1 = -x_2 = \pm t/\sqrt{2}$. This implies a notorious spatial anticorrelation for walkers' positions. Such effect is not observed, for instance, when the joint state is given by $|\psi_t^{\uparrow}\rangle\, |\psi_t^{\downarrow}\rangle$ [see Eqs. (2.6)]—a scenario where the walkers start in a fully uncorrelated state and evolve without any interaction. It is immediate to conclude, therefore, that it is the presence of the initial correlations between the spins that induces the development of spatial correlations (similar results have been reported for local states [61]). Figure 14 illustrates this result. While the walkers are more likely to be found at the anticorrelated locations $x_1 = -x_2 = \pm 20/\sqrt{2}$ at the instant $t = 20$ when the spins start in the singlet state [Fig. 14(a)], such strong correlation does not appear when the spins are prepared in $|\uparrow\downarrow\rangle$ [Fig. 14(b)]. Even though the spacetime is modeled as discrete, numerical simulations are throughout presented with continuous variables, which render the results easier to appreciate.

The reliability of our Gaussian model was again checked via direct comparisons with simulations. To this end we evaluated the fidelity $|\langle \Psi_t^{\mathrm{sim}}|\Psi_t\rangle|^2$ between the state $|\Psi_t^{\mathrm{sim}}\rangle$, computed via numerical simulations, and the state $|\Psi_t\rangle$, derived with our Gaussian model. Some typical results are presented in Tab. 1. We see that the Gaussian model is fairly good for sufficiently broad states ($\sigma_0 \gtrsim 3$) and performs better for small times, since in this regime

σ_0	1	2	3	5	10
$t = 50$	0.3284	0.9098	0.9802	0.9947	0.9987
$t = 100$	0.1709	0.7973	0.9641	0.9939	0.9987

Table 1 – Fidelity $|\langle\Psi_t^{\text{sim}}|\Psi_t\rangle|^2$ of the state $|\Psi_t\rangle$, derived through our simplified Gaussian model, with the numerical simulation $|\Psi_t^{\text{sim}}\rangle$, for two noninteracting quantum walkers, at times $t = 50$ and 100, for different values of σ_0 and the initial state (2.7).

the spreading of the wave packets (not implemented in our model) is less significant. Similar behaviors for the fidelity were observed for generic spin states, thus indicating the broad adequacy of our model.

In possession of solution (2.10), we are ready to conduct a thorough study of several nonclassical features that develop over time in the two-body quantum walk under scrutiny. Basically, we divide the results in three parts. In the first, we show briefly how our model agrees with the results of other authors regarding the entanglement between spin and the position of quantum walkers. In the second, we show that genuine fourpartite entanglement is monotonically generated during the walk. In the third, we consider a Bell scenario where the spatial degrees of freedom constitute noisy channels for the spins and then investigate the time evolution of several nonclassical features quantifiers.

2.3 ENTANGLEMENT BETWEEN SPIN AND POSITION

Since the global state (1.146) is pure, the entanglement $E_{SX}(|\psi_t\rangle)$ between the spin S and the walker position X can be computed via the linear entropy $S_L(\sigma) = 1 - \text{Tr}\,\sigma^2$ of the reduced state

$$\rho_S = \text{Tr}_X\left(|\psi_t\rangle\langle\psi_t|\right) \equiv \begin{pmatrix} \sum_x a_t^2 & \sum_x a_t b_t \\ \sum_x a_t b_t & \sum_x b_t^2 \end{pmatrix}. \tag{2.19}$$

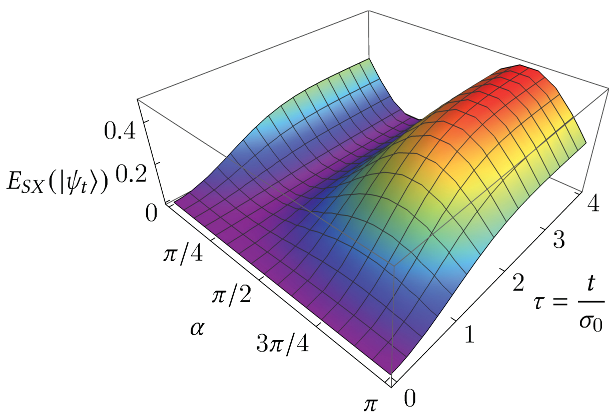

Figure 15 – Entanglement between spin and position given by (2.20) as a function of the scaled time $\tau = t/\sigma_0$ and the angle α of the initial state (1.131).

Simple calculations then show that

$$E_{SX}(|\psi_t\rangle) := S_L(\rho_S) = \frac{1-\sin 2\alpha}{4}\,(1-\mathcal{E}_t)\,, \tag{2.20}$$

where

$$\mathcal{E}_t = \exp\left(\frac{-t^2}{2\sigma_0^2}\right). \tag{2.21}$$

We see that maximum (minimum) entanglement production will be attained during the walk when $\alpha = 3\pi/4$ ($\alpha = \pi/4$), in agreement with previous works [60]. The factor $\mathcal{E}_t$, which will be ubiquitous in our model, controls the production of entanglement in a way such that the sharper the initial distribution the faster the entanglement production. See Fig. 15.

2.4 GENUINE FOURPARTITE ENTANGLEMENT

We have seen above that the initial entanglement between the spins induces the development of (presumably quantum) spatial correlations, after all, the walkers do not interact with each other. Naturally, one may ask whether entanglement can also be created among some other degrees of freedom, as for instance between $X_{1(2)}$ and $S_{2(1)}$, or even among all degrees of freedom in an inextricable way. Now we show that the latter type of entanglement does indeed take place.

Our analysis is based on the measure of genuine multipartite entanglement (GME) introduced by Ma *et al.* in Ref. [158]. This quantifier is very convenient to our purposes because it assumes a simple computational form for multipartite pure states. Given a pure state $|\Phi\rangle \in \bigotimes_{i=1}^{n}\mathcal{H}_i$, the authors defined the GME-concurrence of $|\Phi\rangle$ as

$$C_{\text{GME}}\left(|\Phi\rangle\right) := \min_{\gamma_i\in\gamma}\sqrt{2\,S_L(\rho_{\gamma_i})}, \tag{2.22}$$

where $S_L(\rho)$ is the linear entropy of ρ and $\gamma = \{\gamma_i\}$ is the set of all possible parts defining the bipartitions of the state. According to this definition, GME will be present only if the state is nonseparable in every bipartition, that is, if the reduced states ρ_{γ_i} of $|\Phi\rangle$ are all nonpure. In our system, two examples of parts γ_i are X_1 (for the bipartition $X_1|X_2S_1S_2$) and X_2S_1 (for $X_2S_1|X_1S_2$), for which one finds the respective reduced states $\rho_{X_1} = \text{Tr}_{X_2S_1S_2}\Omega_t$ and $\rho_{X_2S_1} = \text{Tr}_{X_1S_2}\Omega_t$, with $\Omega_t = |\Psi_t\rangle\langle\Psi_t|$. Using the state (2.10) we computed all possible reduced states[2] ρ_{γ_i}, whose schematics can be found in Fig. 16 . For instance, for the two-spin state we found

$$\rho_S = \frac{1-\mathcal{E}_t}{2}\,|\beta_{23}\rangle\langle\beta_{23}| + \frac{1+\mathcal{E}_t}{2}\,|B_4\rangle\langle B_4|\,, \tag{2.23}$$

with $S = S_1S_2$. Analytical expressions were then obtained for the respective linear entropies, the results being

$$S_L(\rho_{S_j}) = \frac{1}{2}, \qquad\qquad S_L(\rho_{X_j}) = \frac{1}{2}\,(1-\mathcal{E}_t)\,, \tag{2.24a}$$

[2] For all practical purposes, the summations over positions, which emerge in the partial trace, can be safely substituted by integrals. This has been checked for Gaussian states with $\sigma_0 = 5$, in which case the difference between a discrete sum over a closed two-dimensional box of width $200 + 2t$ and an integral over the whole $\mathbb{R}^2$ was never greater than 10^{-15}.

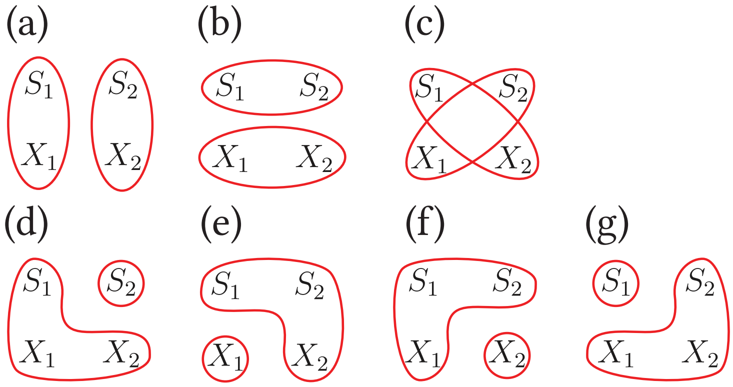

Figure 16 – Schematics of all seven possible bipartitions of the Hilbert space $\mathcal{H} = \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} \otimes \mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}$

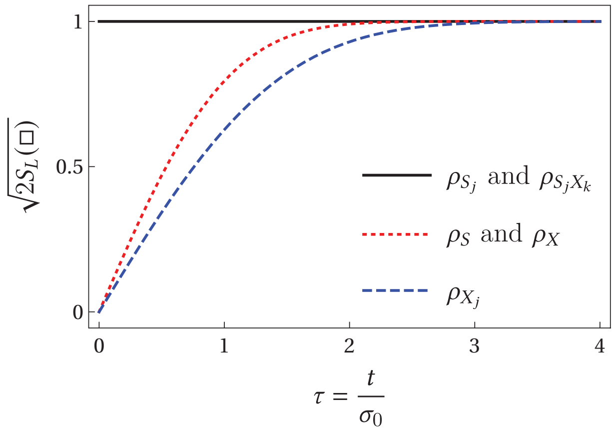

Figure 17 – Dynamical evolution of the entanglement for each bipartite division given by Eqs. (2.24) as a function of the scaled time $\tau = t/\sigma_0$. The GME of the global state $|\Psi_t\rangle$ is given, therefore, by the blue dashed line. In Fig. 16, we have the following correspondence: black line (a, c, d, and g), red dotted line (b) and blue dashed line (e and f).

$$S_L(\rho_{S_jX_k}) = \frac{1}{2}, \qquad S_L(\rho_S) = S_L(\rho_X) = \frac{1}{2}\left(1 - \mathcal{E}_t^2\right). \tag{2.24b}$$

with $j, k \in \{1, 2\}$ and $X = X_1X_2$. From these relations and the definition (2.22), one finds

$$C_{\text{GME}}(|\Psi_t\rangle) = \sqrt{1 - \mathcal{E}_t}, \tag{2.25}$$

which is a monotonically increasing function of time and can also be written as monotonic functions of the bipartite entanglement quantifiers $S_L(\rho_{X_j})$ and $S_L(\rho_X)$. These results indicate that even though each walker evolves independently, the presence of entanglement between the spins at the beginning of the walk allows the global state of the walkers to develop genuine fourpartite entanglement over time. Later on, this interpretation will be corroborated by further evidences. Finally, note that all bipartitions will be equally entangled as $t \longrightarrow \infty$, as we can see in Fig. 17.

2.5 NONCLASSICAL ASPECTS DYNAMICS BETWEEN SPINS

In this section, we confine our attention to the spins only. This leads us to special Bell scenarios where information about the spins of the particles are encoded, via quantum

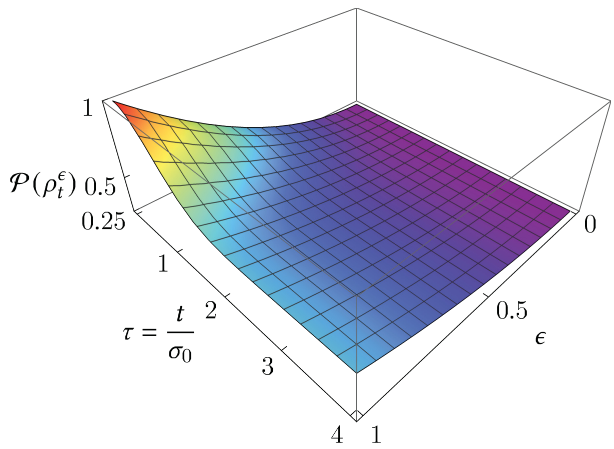

Figure 18 – Purity of the reduced density state ρ_t^ϵ as a function of the scaled time $\tau = t/\sigma_0$ and the noise-related factor ϵ. Note that we always have $\mathcal{P}(\rho_t^\epsilon) \geqslant 0.25$.

correlations, on spatial degrees of freedom—a mechanism that tends to degrade the resources present in the two-spin state. As a materialization of such scenarios, we can envisage instances similar to those recently proposed for witnessing aspects of quantum gravity [159, 160], where the spin value defines the path to be taken by the particle (as also happens in a Stern-Gerlach experiment) and then each specific path couples with the gravitation source in a particular manner. In this framework, the spatial degrees of freedom are expected to play the role of a noisy channel, whose effect over the two-spin state varies during the motion of the walkers. We now investigate how several nonclassical aspects present in the two-state spin varies with time under the aforementioned noisy channel.

To give more generality to our study, we consider that the spins are initially prepared in the Werner state

$$\rho_\epsilon^W = (1-\epsilon)\frac{\mathbb{1}}{4} + \epsilon \left|B_4\right\rangle \left\langle B_4\right|, \tag{2.26}$$

with $\epsilon \in [0,1]$. This formulation considers a white noise of amplitude $1-\epsilon$ over the singlet state $|B_4\rangle$. Assuming Gaussian amplitudes for the positions, as in Eq. (2.7), the initial state of the two-walker model becomes $\rho_0 = \rho_\epsilon^W \otimes |\varphi_1, \varphi_2\rangle \langle \varphi_1, \varphi_2|$, where $|\varphi_i\rangle = \sum_{x_i} f(x_i) |x_i\rangle$. Applying the time evolution operator $U_1^t U_2^t$ [see Eq. (1.132)] and tracing over the positions yield (see Appendix C.1 for the details)

$$\rho_t^\epsilon = (1-\epsilon)\frac{\mathbb{1}}{4} + \epsilon\, \rho_S, \tag{2.27}$$

with ρ_S being the time-dependent density operator (2.23). From now on, we restore the time dependence in the notation. The purity of the two-spin state ρ_t^ϵ reads

$$\mathcal{P}(\rho_t^\epsilon) = \mathrm{Tr}\left[(\rho_t^\epsilon)^2\right] = \frac{1}{4}\left[1 + \epsilon^2\left(1 + 2\mathcal{E}_t^2\right)\right], \tag{2.28}$$

which monotonically decreases with time (and with the noise amplitude $1-\epsilon$) towards the asymptotic value $(1+\epsilon^2)/4$. This shows that the spatial variables indeed get more correlated with the spins as the walk takes place. Moreover, the decoherence and the whole dynamics of the two-spin state is controlled by the decay factor $\mathcal{E}_t$ which, by its turn, is determined by the initial dispersion σ_0 of the Gaussian amplitudes. The broader the spatial distributions, the larger the time scale within which the two-spin state keeps its coherence. Accordingly, a completely delocalized walker ($\sigma_0 \to \infty$) will never have its position correlated with its spin during the walk. This is reasonable since, in this case, it is difficult to defend that, being everywhere, the walker really walks. See Fig. 18.

2.5.1 Bell nonlocality

Let us now apply the formalism introduced in Sec. 1.3.1 to investigate the presence of Bell nonlocality in the spin state ρ_t^ϵ. As a first step, it is instructive to look at the nonlocality induced by ρ_t^ϵ when there is no white noise ($\epsilon = 1$ and $\rho_t^{\epsilon=1} = \rho_S$). It can be directly demonstrated by taking $A_\pm$ and $B_\pm$ in the form $\hat{v}_i \cdot \vec{\sigma}$, with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ being the vector composed of Pauli matrices and $\hat{v}_i \in \mathbb{R}^3$ unit vectors. By letting $\hat{v}_i$ assume orthogonal directions $\hat{e}_1$ and $\hat{e}_2$, for particle 1, and $-(\hat{e}_1 + \hat{e}_2)/\sqrt{2}$ and $(-\hat{e}_1 + \hat{e}_2)/\sqrt{2}$, for particle 2, one shows that

$$\mathbb{B}_t = \frac{1 + 3\mathcal{E}_t}{\sqrt{2}}, \tag{2.29}$$

which implies a CHSH-inequality (1.88) violation for

$$t < \sigma_0 \sqrt{2 \ln \left(\frac{3}{2\sqrt{2} - 1} \right)} \cong 0.995\, \sigma_0. \tag{2.30}$$

This means that for $t > \sigma_0$ Bell nonlocality will no longer be detected with those specific measurement directions. This can be explained as follows. For long times, each walker's spatial distribution gets sufficiently correlated with its spin. This effect is illustrated in Fig. 19, where the probability distributions

$$p_t(x_1) = \mathrm{Tr}\,(\Omega_t \left|x_1\right\rangle \left\langle x_1\right|) \tag{2.31}$$

and

$$p_t^\mu(x_1) = \mathrm{Tr}\,(\Omega_t \left|x_1\right\rangle \left\langle x_1\right| \otimes \left|\mu\right\rangle \left\langle \mu\right|), \qquad \mu = \uparrow, \downarrow, \tag{2.32}$$

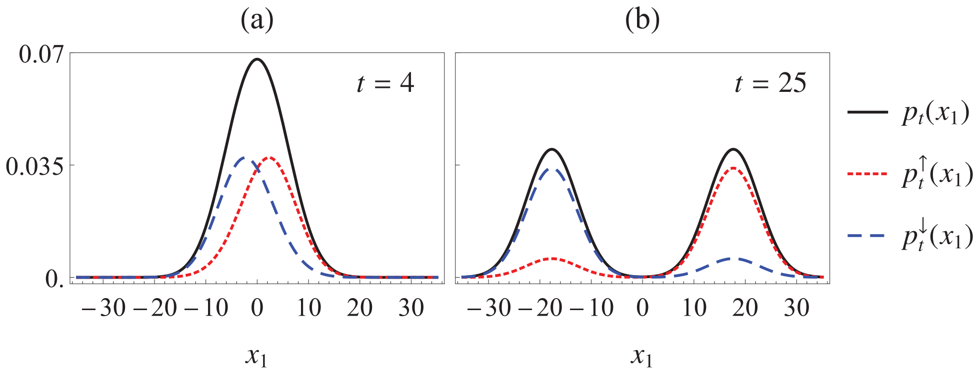

Figure 19 – Probability distributions $p_t(x_1)$ (black line), $p_t^\uparrow(x_1)$ (red dotted line), and $p_t^\downarrow(x_1)$ (blue dashed line) of finding particle 1 at position x_1, at position x_1 with spin up, and at position x_1 with spin down, respectively, at instants (a) $t = 4$ and (b) $t = 25$. The initial dispersion is $\sigma_0 = 5$.

for the particle 1 (similarly for particle 2) are shown at two different instants: (a) just before the Bell-nonlocality sudden death and (b) long after this. As a consequence of the correlations generated between spin and position (of each walker), the power of the noisy channel on the two-spin state increases and the nonlocal correlations degrade.

Let us now quantify how much the state ρ_t^ϵ is Bell nonlocal. Adapted to the form (1.92), the state (2.27) is such that $\vec{a} = \vec{b} = \vec{0}$ and $\vec{c} = (-\epsilon, -\epsilon\,\mathcal{E}_t, -\epsilon\,\mathcal{E}_t)$, from which we find

$$\mathcal{B}(\rho_t^\epsilon) = \left(1 + \sqrt{2}\right) \max\left\{0, \epsilon\sqrt{1 + \mathcal{E}_t^2} - 1\right\}. \tag{2.33}$$

It follows that Bell nonlocality will be present only for

$$t < \sigma_0 \sqrt{\ln\left(\frac{\epsilon^2}{1 - \epsilon^2}\right)} \equiv t_{\mathcal{B}}. \tag{2.34}$$

This shows that for any state ρ_t^ϵ with $\epsilon \in (1/\sqrt{2}, 1)$ there will be a finite critical time $t_{\mathcal{B}}$ after which the two-spin state will become Bell local. Such "sudden-death time" is substantially postponed as the white noise becomes very small ($\epsilon \longrightarrow 1$), in which case Bell nonlocality, as measured by $\mathcal{B}(\rho_t^\epsilon)$, will vanish only asymptotically [see later Fig. 23(a)]. For high levels of white noise ($\epsilon \leq 1/\sqrt{2}$), Bell nonlocality never manifests itself.

2.5.2 Quantum steering

For the state under scrutiny here, the measure (1.97) introduced in Sec. 1.3.2 reduces to

$$\mathcal{S}(\rho_t^\epsilon) = \left(\frac{1 + \sqrt{3}}{2}\right) \max\left\{0, \epsilon\sqrt{1 + 2\mathcal{E}_t^2} - 1\right\}, \tag{2.35}$$

which indicates the existence of steering as long as

$$t < \sigma_0 \sqrt{\ln\left(\frac{2\epsilon^2}{1 - \epsilon^2}\right)} \equiv t_{\mathcal{S}}. \tag{2.36}$$

It follows that a sudden-death time $t_{\mathcal{S}}$ will exist for quantum steering whenever $\epsilon \in (1/\sqrt{3}, 1)$ and that $t_{\mathcal{S}}$ can be made arbitrarily large for reduced amounts of white noise (again, see later Fig. 23(a) for an illustration of this behavior). For $\epsilon \leq 1/\sqrt{3}$, ρ_t^ϵ is nonsteerable[3].

2.5.3 Entanglement

A straightforward calculation from (1.104) gives

$$E(\rho_t^\epsilon) = \frac{1}{2} \max\left\{0, \epsilon(1 + 2\mathcal{E}_t) - 1\right\}, \tag{2.37}$$

[3] A method based on LHS and semi-definite program has been developed that predicts the presence of steering for $\epsilon > 1/2$ [161], which agrees with the results of Ref. [75]. The method employed here is based on linear steering inequalities (see Ref. [134] and references therein) and, although less powerful, allows for analytical analysis and is in full agreement with a recently introduced geometrical quantifier [162].

which predicts entanglement for

$$t < \sigma_0 \sqrt{2\ln\left(\frac{2\epsilon}{1-\epsilon}\right)} \equiv t_E. \tag{2.38}$$

A well defined instant t_E will exist for entanglement sudden death if $\epsilon \in (1/3, 1)$. While entanglement will vanish only asymptotically for reduced values of white noise ($\epsilon \longrightarrow 1$), as it will be shown in Fig. 23(a), it will not occur if $\epsilon \leq 1/3$.

Two points are now worth noticing. First, in the regime of no white noise ($\epsilon = 1$), we have $E(\rho_t^{\epsilon=1}) = \mathcal{E}_t$, which attaches an interesting interpretation to the damping factor. Moreover, we can revisit Sec. 2.4 and write the complementarity relation

$$C^2_{\text{GME}}(|\Psi_t\rangle) + E(\rho_t^{\epsilon=1}) = 1, \tag{2.39}$$

which explicitly shows that fourpartite entanglement develops over time at the expense of the two-spin entanglement. Second, in the domain $\epsilon \in (1/\sqrt{2}, 1)$, wherein the sudden-death times are all well defined, one has

$$t_{\mathcal{B}} < t_{\mathcal{S}} < t_E, \tag{2.40}$$

which corroborates the current knowledge according to which Bell nonlocality is the most fragile quantum resource, whereas entanglement is the least one [75, 134, 135]. The reader can find a visual confirmation of the relation (2.40) in Fig. 20.

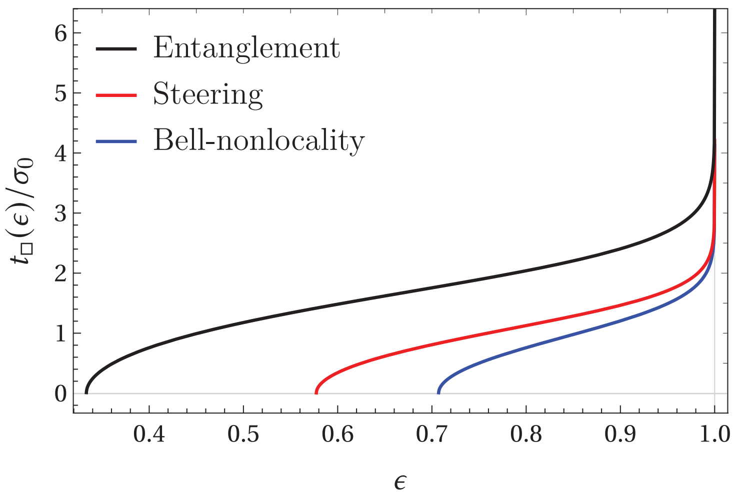

Figure 20 – Scaled sudden death-times $t_\square(\epsilon)/\sigma_0$ for the state ρ_t^ϵ as a function of ϵ for (from top to bottom) entanglement (black), steering (red), and Bell nonlocality (blue). Note that when $\epsilon \longrightarrow 1$, all sudden death-times go to infinity.

2.5.4 Quantum discord

To compute quantum discord in our model through (1.113), we consider the observable $B = \hat{v}_2 \cdot \vec{\sigma}$, with unit vector $\hat{v}_2(\theta_2, \phi_2) = (\cos\theta_2 \sin\phi_2, \sin\theta_2 \sin\phi_2, \cos\phi_2)$ and projectors $B_\pm = (\mathbb{1} \pm \hat{v}_2 \cdot \vec{\sigma})/2$. Direct calculations produce

$$S\left(\text{Tr}_{S_{1,2}}\rho_t^\epsilon\right) = S\left(\Phi_B\left(\text{Tr}_{S_1}\rho_t^\epsilon\right)\right) = \ln 2 \tag{2.41}$$

and the formal result

$$\mathcal{D}_{S_2}(\rho_t^\epsilon) = \min_B \left[S\left(\Phi_B(\rho_t^\epsilon)\right) - S\left(\rho_t^\epsilon\right)\right]. \tag{2.42}$$

Some more algebra gives

$$S(\rho_t^\epsilon) = \left(\frac{1-\epsilon}{2}\right)\ln 2 + \left(\frac{1+\epsilon}{2}\right) H\left(\frac{1}{2} + \frac{\epsilon\,\mathcal{E}_t}{1+\epsilon}\right) + H\left(\frac{1+\epsilon}{2}\right), \tag{2.43}$$

with the Shannon entropy

$$H(u) = -u\ln u - (1-u)\ln(1-u). \tag{2.44}$$

Numerical analyses revealed that $(\theta_2, \phi_2) = (0, \frac{\pi}{4})$ define the optimal observable B for all times, with which we have been able to compute $\min_B S(\Phi_B(\rho_t^\epsilon))$ and then obtain

$$\mathcal{D}_{S_2}(\rho_t^\epsilon) = \frac{1+\epsilon}{2}\left[\ln 2 - H\left(\frac{1}{2} + \frac{\epsilon\,\mathcal{E}_t}{1+\epsilon}\right)\right]. \tag{2.45}$$

In contrast with what we have for Bell nonlocality, quantum steering, and entanglement, there is no domain of ϵ for which the quantum discord of ρ_t^ϵ suddenly vanishes. In fact, regardless of the white-noise level, quantum discord vanishes only asymptotically (see Fig. 23). By symmetry, one can straightforwardly conclude that $\mathcal{D}_{S_1}(\rho_t^\epsilon) = \mathcal{D}_{S_2}(\rho_t^\epsilon)$.

We can also quantify the sensitivity of total correlations to unread measurements conducted separately in both sites. This information is captured by the so-called symmetrical quantum discord [74], which for a state ρ is formally written as

$$\mathcal{D}(\rho) := \min_{A,B}\left[I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\Phi_{AB}(\rho))\right], \tag{2.46}$$

where

$$\Phi_{AB}(\rho) = \sum_{a,b} (A_a \otimes B_b)\,\rho\,(A_a \otimes B_b), \tag{2.47}$$

for observables $A = \sum_a aA_a$ and $B = \sum_b bB_b$ acting on $\mathcal{H}_{S_1}$ and $\mathcal{H}_{S_2}$, respectively. Also in this case we have been able to analytically conduct all the calculations and prove that $\mathcal{D}(\rho_t^\epsilon) = \mathcal{D}_{S_{1,2}}(\rho_t^\epsilon)$. Hence, hereafter we make no distinction between quantum discord and its symmetrical counterpart.

2.5.5 Irreality of spin observables

Now we discuss aspects of quantum irreality introduced in Sec. 1.3.5. For the state under scrutiny, because $\varrho_{S_1} = \mathrm{Tr}_{S_2}\rho_t^\epsilon = \mathbb{1}/2$ it follows that $\Phi_A(\varrho_{S_1}) = \varrho_{S_1}$ for all A on $\mathcal{H}_{S_1}$, which implies that $\mathfrak{I}_A(\varrho_{S_1}) = 0$. As a consequence, $\mathfrak{I}_A(\rho_t^\epsilon) = D_{S_1}(\rho_t^\epsilon)$ and, therefore,

$$\mathfrak{I}_A(\rho_t^\epsilon) \geq \mathcal{D}(\rho_t^\epsilon). \tag{2.48}$$

This relation is important because it establishes a lower bound for the irreality of all observables A on $\mathcal{H}_{S_1}$. Since for $\epsilon > 0$ quantum discord vanishes only asymptotically, then it is guaranteed

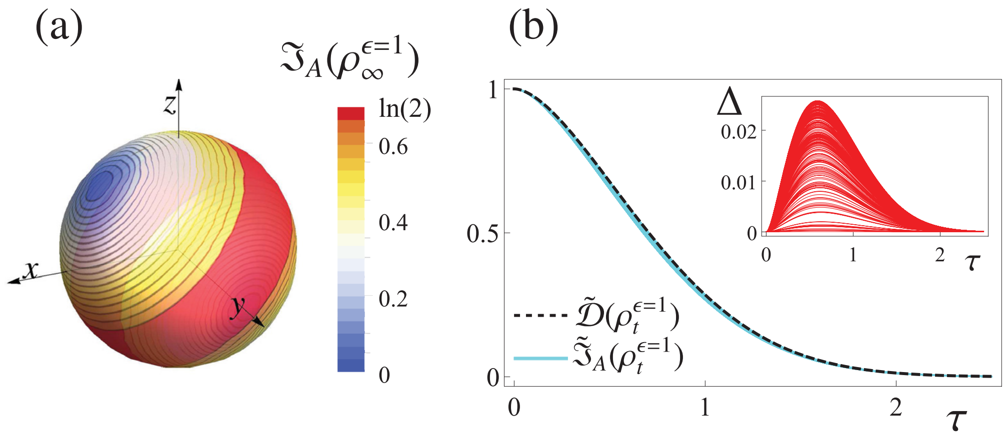

Figure 21 – (a) Contour plot for the asymptotic irreality $\mathfrak{I}_A(\rho_\infty^{\epsilon=1})$ of an observable $A = \hat{v}_1 \cdot \vec{\sigma}$ on $\mathcal{H}_{S_1}$, in spherical coordinates, for the no-noise regime ($\epsilon = 1$). The color scale goes from 0 (blue) to ln 2 (red). (b) Scaled irreality $\tilde{\mathfrak{I}}_A(\rho_t^{\epsilon=1})$ (cyan curves) as a function of the scaled time $\tau = t/\sigma_0$ for 200 randomly chosen measurement directions $\hat{v}_1$. The scaled discord $\tilde{\mathcal{D}}(\rho_t^{\epsilon=1})$ (black dashed curve) defines a tight upper bound [see Eq. (2.50)]. In the inset, the difference $\Delta = \tilde{\mathcal{D}}(\rho_t^{\epsilon=1}) - \tilde{\mathfrak{I}}_A(\rho_t^{\epsilon=1})$ is plotted for each of the 200 measurement directions, showing results never greater than 0.03.

that no element of reality will exist at short times. On the other hand, for the regime of maximum white noise ($\epsilon = 0$), every observable will always be an element of reality, since $\rho_t^{\epsilon=0} = \mathbb{1}/4$ and then $\mathfrak{I}_A(\rho_t^{\epsilon=0}) = \mathcal{D}(\rho_t^{\epsilon=0}) = 0$.

It is interesting to look also at the no-noise regime. Introducing the unit vector $\hat{v}_1(\theta_1, \phi_1)$ to define a generic observable $A = \hat{v}_1 \cdot \vec{\sigma}$ for the spin S_1, we find a lengthy and nonenlightening analytical function for $\mathfrak{I}_A(\rho_t^{\epsilon})$ (omitted). For $\epsilon = 1$, though, an interesting universal behavior is found. Since in this case the initial state (the singlet) is rotationally invariant, any observable is maximally irreal at $t = 0$. As the Gaussian packets start to split themselves and get correlated with the spins, irreality becomes direction dependent and typically decays with time, eventually reaching the asymptotic value

$$\mathfrak{I}_A(\rho_\infty^{\epsilon=1}) = H\left(\frac{1 + v_{\theta_1\phi_1}}{2}\right), \tag{2.49}$$

where $v_{\theta\phi} = (\cos\phi + \cos\theta \sin\phi)/\sqrt{2}$. A panoramic view of the asymptotic irreality is presented in Fig. 21(a). First of all, it is seen that $\mathfrak{I}_A(\rho_\infty^{\epsilon=1}) = 0$ only for two particular observables, namely, $\pm(\sigma_x + \sigma_z)/\sqrt{2}$ (center of the blue circle and its antipode), which are directly related to the quantum coin (1.134) that we have adopted for the walk [see Fig. 8 (a)]. This happens because the position of each walker correlates with its respective coin, thus establishing its reality. For any other observable, we have $\mathfrak{I}_A(\rho_\infty^{\epsilon=1}) > 0$, which reveals a broad scenario of quantum irrealism. In particular, since the Shannon entropy $H(u)$ reaches its maximum for $u = 1/2$, there is a continuous set of observables, defined by $v_{\theta_1\phi_1} = 0$, for which the asymptotic irreality reaches the maximum value ln 2. This set corresponds to the center of the red strip. We have checked that, in fact, these observables remain maximally irreal for every instant of time.

Interestingly, we also found that the way irreality gets to the asymptote (2.49) is nearly direction-independent (as long as we exclude the aforementioned maximal-irreality set). After some numerical incursions, we have been able to show that

$$\tilde{\mathfrak{I}}_A(\rho_t^{\epsilon=1}) := \frac{\mathfrak{I}_A(\rho_t^{\epsilon=1}) - \mathfrak{I}_A(\rho_\infty^{\epsilon=1})}{\mathfrak{I}_A(\rho_0^{\epsilon=1}) - \mathfrak{I}_A(\rho_\infty^{\epsilon=1})} \lessapprox \frac{\mathcal{D}(\rho_t^{\epsilon=1})}{\mathcal{D}(\rho_0^{\epsilon=1})} =: \tilde{\mathcal{D}}(\rho_t^{\epsilon=1}), \tag{2.50}$$

with $\mathfrak{I}_A(\rho_0^{\epsilon=1}) = \mathcal{D}(\rho_0^{\epsilon=1}) = \ln 2$. The cyan curves presented in Fig. 21(b) illustrate the behavior of the scaled irreality $\tilde{\mathfrak{I}}_A(\rho_t^{\epsilon=1})$ for 200 randomly chosen directions $\hat{v}_1(\theta_1, \phi_1)$ as a function of the scaled time $\tau = t/\sigma_0$. Clearly, the curves do not significantly deviate from each other and are all upper bounded by the scaled discord $\tilde{\mathcal{D}}(\rho_t^{\epsilon=1})$ (black dashed line). As shown in the inset, $0 \leq \tilde{\mathcal{D}}(\rho_t^{\epsilon=1}) - \tilde{\mathfrak{I}}_A(\rho_t^{\epsilon=1}) < 0.03$. Hence, to a pretty good accuracy we can state that the scaled irreality is determined by the scaled discord, which is observable independent. It follows, therefore, that there is an approximate class of universality for the irreality behavior, which is likely to emerge from the fact that the initial state of the spins is the rotationally invariant singlet.

2.5.6 Realism-based nonlocality

It is clear from the above that, in contrast with all the other types of nonclassical aspects studied so far, irreality can be preserved during the quantum walk. Presumably, a similar behavior can exist for the realism-based nonlocality introduced in Sec. 1.3.6. Through (1.127), we can compute the contextual realism-based nonlocality $\eta_{AB}(\rho_t^{\epsilon})$ for the context defined by generic observables $A = \hat{v}_1 \cdot \vec{\sigma}$ and $B = \hat{v}_2 \cdot \vec{\sigma}$. For the maximum-noise scenario, we directly obtain $\eta_{AB}(\rho_t^{\epsilon=0}) = 0$, since for the state $\rho_t^{\epsilon=0} = \mathbb{1}/4$ all observables are elements of reality.

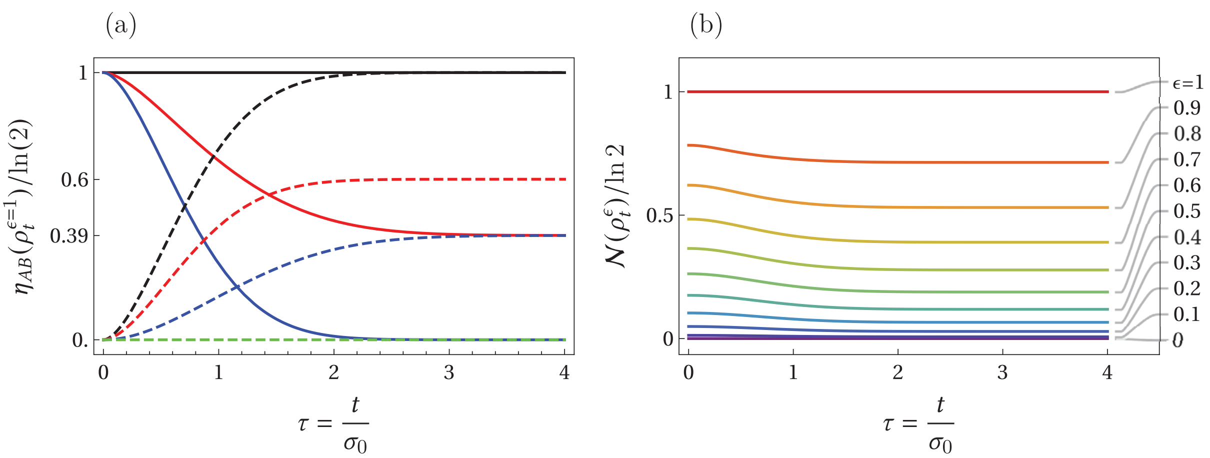

Figure 22 – (a) Behavior of the normalized contextual realism-based nonlocality $\eta_{AB}(\rho_t^{\epsilon=1})/\ln 2$, in the no-noise regime ($\epsilon = 1$), as a function of the scaled time $\tau = t/\sigma_0$, for the following pairs of observables (contexts) in continuous lines from top to bottom: $A = B = \sigma_y$ (black line), $A = B = \sigma_z$ (red line), $A = B = (\sigma_x + \sigma_z)/\sqrt{2}$ (blue line); and in dashed lines from top to bottom:$A = (\sigma_x - \sigma_z)/\sqrt{2}$ and $B = \sigma_y$ (black line), $A = \sigma_x$ and $B = \sigma_y$ (red line), $A = \sigma_z$ and $B = \sigma_x$ (blue line), $A = (\sigma_x + \sigma_z)/\sqrt{2}$ and $B = \sigma_y$ (green line). (b) Behavior of $\mathcal{N}(\rho_t^{\epsilon})$ given by (2.52) as a function of the scaled time τ for several values of ϵ.

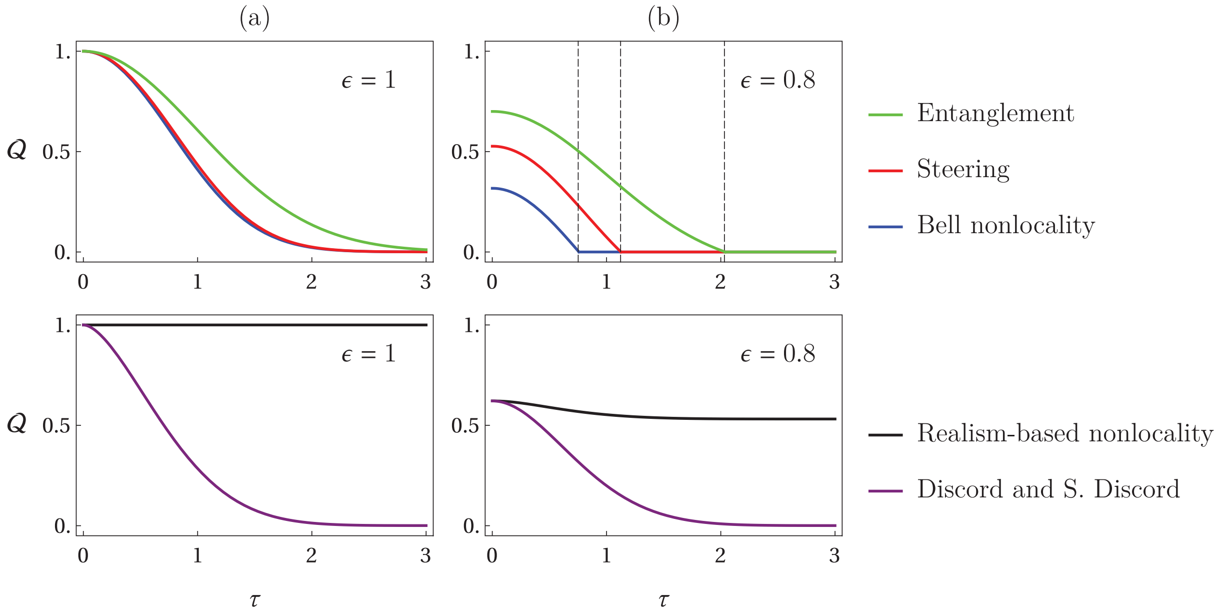

Figure 23 – All the (observable independent) nonclassical aspects quantifiers Q computed in this work for the two-spin state ρ_t^ϵ as a function of the scaled times $\tau = t/\sigma_0$ for (a) $\epsilon = 1.0$ (left panels) and (b) $\epsilon = 0.8$ (right panels), where Q assumes in upper panels: Bell nonlocality $\mathcal{B}$ (blue bottom line), quantum steering $\mathcal{S}$ (red middle line), entanglement E (green top line); and in bottom panels: normalized (symmetrical) quantum discord $\mathcal{D}/\ln 2$ (black bottom line), and normalized realism-based nonlocality $\mathcal{N}/\ln 2$ (purple top line). The vertical dashed lines in the upper right panel refer to the sudden-death times given by Eqs. (2.34), (2.36), and (2.38). nonclassical aspects typically decreases with both time and the amount $1-\epsilon$ of noise, but realism-based nonlocality survives.

On the other hand, in the other extreme ($\epsilon = 1$), all sorts of behaviors can be found for the contextual realism-based nonlocality, as is illustrated in Fig. 22(a). For the asymptotic values of the contextual realism-based nonlocality we have found

$$\eta_{AB}(\rho_\infty^{\epsilon=1}) = H\left(\frac{1+\nu_{\theta_1\phi_1}}{2}\right) + H\left(\frac{1+\nu_{\theta_2\phi_2}}{2}\right) - H\left(\frac{1+\nu_{\theta_1\phi_1}\nu_{\theta_2\phi_2}}{2}\right). \tag{2.51}$$

Therefore, there exists an infinite set of observables, defined by $(\nu_{\theta_1\phi_1}, \nu_{\theta_2\phi_2}) = (0,0)$, for which the contextual realism-based nonlocality will asymptotically reach its maximum value $\ln 2$.

It is also interesting to look at the realism-based nonlocality (1.128). Even though the maximization over $\{A, B\}$ implies a hard mathematical problem in general, numerical and analytical incursions on $\eta_{AB}(\rho_t^\epsilon)$ give us the clues for the accomplishment of such task. For instance, we see from Fig. 22(a), that the choice $A = B = \sigma_y$ is optimal. In fact, we have verified that parallel direction measurements $\hat{v}(\theta,\phi)$ satisfying $\nu_{\theta\phi} = 0$, that is, observables in the circle represented by the red strip in Fig. 21(a), provide the maximization. We then find

$$\mathcal{N}(\rho_t^\epsilon) = \mathcal{D}(\rho_t^\epsilon) + H\left(\frac{1+\epsilon\,\mathcal{E}_t}{2}\right) - H\left(\frac{1+\epsilon}{2}\right), \tag{2.52}$$

with the symmetrical quantum discord $\mathcal{D}(\rho_t^\epsilon)$ being given by Eq. (2.45). Realism-based nonlocality is similar to quantum discord in that they never experiment sudden death. On the other hand, while the latter vanishes asymptotically, the former behaves as (see Figs. 22(b) and 23)

$$\mathcal{N}(\rho_\infty^\epsilon) = \ln 2 - H\left(\frac{1+\epsilon}{2}\right), \tag{2.53}$$

which vanishes as $t \to \infty$ only in the maximum noise regime ($\epsilon = 0$). In fact, it directly follows from Eq. (2.52) that $\mathcal{N}(\rho_t^\epsilon) \geq \mathcal{D}(\rho_t^\epsilon)$. Therefore, in flagrant contrast with the other nonclassical aspects, $\mathcal{N}(\rho_t^\epsilon)$ manifests itself as the most resilient one, which is in full agreement with previously conducted studies [143].

2.6 TAKEAWAY MESSAGE

- The simplified Gaussian model is good (fidelity 0.99) to describe large walker packages ($\sigma_0 \geq 5$). Check again Tab. 1;
- Two non-interacting quantum walkers develop multipartite entanglement if their spins are initially entangled;
- The positions steal coherence from the spins as the walkers move;
- Correlations between the spins decrease with time and all of them, except realism-based nonlocality, go to zero (see Fig. 23).
- Spins asymptotically become real in the directions given by the coin eigenvectors [see Fig. 21 (a)].

3 CONCLUDING REMARKS ON PART I

Quantum walk studies often demand numerical simulations, which do not always allow for the access of some refined physical aspects. In this first part of our work, by introducing a Gaussian model—which proved to be quite accurate for delocalized walkers ($\sigma_0 \gg 1$)—we were able to conduct a profound analysis of the nonclassical feature dynamics in a two-walker system. Previous studies [115, 145, 152, 163–167] allow us to optimistically speculate upon the applicability of our model even to scenarios involving more localized states.

Starting with a single quantum walker, we derived an analytical expression for the entanglement between spin and position [Eq. (2.20)]. This result reveals that the production of entanglement is regulated by a parameter that controls the initial coherence of the spin state. For the problem of two noninteracting walkers, with spins prepared in the singlet state, we showed that genuine fourpartite entanglement is created throughout the walk, monotonically increasing with time [Eq. (2.25)], at the expense of two-spin correlations [Eq. (2.39)]. This reveals a scenario where the total amount of resource is conserved. Also, we found that by measuring the sign of the relative coordinate, the spins can be prepared in superpositions of Bell states.

With respect to nonclassical aspects between the spins, our results are graphically summarized in Fig. 23, for two noise regimes, where the quantifiers are separated into two rows, according to their susceptibility to sudden death. The panels in the upper row show the behaviors over time of Bell nonlocality [Eq. (2.33)], quantum steering [Eq. (2.35)], and entanglement [Eq. (2.37)], while in the lower row simulations are presented for (symmetrical) quantum discord [Eq. (2.45)] and realism-based nonlocality [Eq. (2.52)]. Besides showing a clear chronology of deaths, which is formally stated in the relations (2.40), our findings corroborate the view according to which there is a strict hierarchy [143] among the quantifiers, in such a way that the existence of Bell nonlocality implies steering, which implies entanglement, which implies quantum discord, which then implies realism-based nonlocality, while the converse sequence of implications is false. Moreover, it is clear that realism-based nonlocality is the only type of nonclassical feature that survives upon the noisy channels considered. From such aspect, an urgent demand arises aiming at characterizing the potential of this nonclassical feature as a useful quantum resource.

Finally, from a foundational viewpoint, lessons can be learned with respect to (ir)realism. According to the relation (2.48), since quantum discord vanishes only asymptotically, generic spin variables cannot be elements of reality. In fact, given the presence of fourpartite quantum correlations, the positions cannot be either. There are only two specific spin observables that asymptotically behaves as elements of reality [check again the center of the blue circles in Fig. 21(a)], and these are closely related with the quantum coin C [Eq.

(1.134)]. From Fig. 8 (a), we can see that these elements of reality are in fact the eigenvectors of C, since the coin produces rotations around the axis that crosses the blue circles. The real reason for this relationship still eludes our understanding, since the dynamics of the quantum walk ceases to be trivial after a few steps. For now, we can at least predict that other quantum coins (*e.g.*, the Fourier coin [56]) would produce other elements of reality in directions that follow the same logic.

Altogether, our findings reinforce the potential of quantum walks as a rich arena for studies involving information-theoretic and foundational issues, such as the interconversion of bipartite to multipartite entanglement and the dynamics of further nonclassical aspects, from nonlocality to (multipartite) quantum correlations and violations of classical realism.

Part II

DEEPER INTO QUANTUM REALISM

4 FUNDAMENTAL CONCEPTS II

This chapter aims at revisiting fundamental concepts that will be necessary for the next ones. Unlike Chap. 1, this chapter contains some new elements that will be presented in Sec. 4.3 with proper indication. Everything else remains a bibliographic review, with some adaptations to the notation. The goal here is to introduce some mathematical tools to distinguish quantum states and then show how to represent quantum informational content and conditional information with them. Conditional information constitute the basic element from which we will axiomatize quantum realism. At the end of this chapter, we will also make a brief introduction to quantum resource theories since our axiomatization of quantum realism is inspired on them. Before all, let us expand our notion of entropy.

4.1 UNIFIED ENTROPY

Beyond the Shannon classical entropy (introduced in Sec. 1.1.1), there are other propositions to quantify the degree of uncertainty regarding a system, like those proposed by Rényi [168] and Tsallis [101]. Quantum versions of them are also defined and they are contemplated, together with von Neumann entropy, by the so called *unified entropy* [169–171]

$$S_{(q,s)}(\rho) := \frac{(\mathrm{Tr}\,\rho^q)^s - 1}{s(1-q)}, \tag{4.1}$$

where $q > 0$, $q \neq 1$, and $s \neq 0$. For $q \longrightarrow 1$ and any s we have von Neumann entropy

$$\lim_{q\to 1} S_{(q,s)}(\rho) \equiv S(\rho) = -\mathrm{Tr}\,(\rho \ln \rho). \tag{4.2}$$

In the case of $s \longrightarrow 0$, one can obtain the *quantum Rényi entropies*

$$\lim_{s\to 0} S_{(q,s)}(\rho) \equiv S_q^R(\rho) = \frac{\ln \mathrm{Tr}\,\rho^q}{1-q}. \tag{4.3}$$

Furthermore, *quantum Tsallis entropies* are reached for $s = 1$,

$$S_{(q,1)}(\rho) \equiv S_q^T(\rho) = \frac{\mathrm{Tr}\,\rho^q - 1}{1-q}. \tag{4.4}$$

It is also possible, by setting $q = 2$ and $s = 1$, to obtain *linear entropy*

$$S_{(2,1)}(\rho) \equiv S_L(\rho) = 1 - \mathrm{Tr}\,\rho^2. \tag{4.5}$$

One can see by direct calculation that the limiting case of $q \longrightarrow 1$ for both Rényi and Tsallis entropies gives us back von Neumann entropy. In fact,

$$\lim_{q\to 1} S_q^R(\rho) \overset{\text{L'H}}{=} \lim_{q\to 1} \frac{\frac{\mathrm{d}}{\mathrm{d}q} \ln \mathrm{Tr}\,\rho^q}{\frac{\mathrm{d}}{\mathrm{d}q}(1-q)} \tag{4.6}$$

$$= \lim_{q\to 1} \frac{\frac{\mathrm{d}}{\mathrm{d}q} \ln \sum_i \lambda_i^q}{\frac{\mathrm{d}}{\mathrm{d}q}(1-q)} \tag{4.7}$$

$$= \lim_{q\to 1} \frac{\frac{1}{\sum_i \lambda_i^q} \sum_i \lambda_i^q \ln \lambda_i}{-1} \tag{4.8}$$

$$= \lim_{q\to 1} \frac{\frac{1}{\mathrm{Tr}\,\rho^q} \mathrm{Tr}\,\rho^q \ln \rho}{-1} \tag{4.9}$$

$$= -\frac{\mathrm{Tr}\,\rho \ln \rho}{\mathrm{Tr}\,\rho} \tag{4.10}$$

$$= -\mathrm{Tr}\,\rho \ln \rho, \tag{4.11}$$

where L'H indicates L'Hôpital's rule, $\{\lambda_i\}$ are the eigenvalues of ρ, and $\mathrm{Tr}\,\rho = 1$. Analogously, one has

$$\lim_{q\to 1} S_q^T(\rho) \overset{\text{L'H}}{=} \lim_{q\to 1} \frac{\frac{\mathrm{d}}{\mathrm{d}q}(\mathrm{Tr}\,\rho^q - 1)}{\frac{\mathrm{d}}{\mathrm{d}q}(1-q)} = \lim_{q\to 1} \frac{\mathrm{Tr}\,\rho^q \ln \rho}{-1} = -\mathrm{Tr}\,\rho \ln \rho. \tag{4.12}$$

Not all special cases of unified entropy satisfy additivity for product states, as one can see by the relation

$$S_{(q,s)}(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) = S_{(q,s)}(\rho_{\mathcal{A}}) + S_{(q,s)}(\rho_{\mathcal{B}}) + (1-q)sS_{(q,s)}(\rho_{\mathcal{A}})S_{(q,s)}(\rho_{\mathcal{B}}), \tag{4.13}$$

where additivity is recovered for $q \longrightarrow 1$ or $s \longrightarrow 0$. However, all (q,s)-entropies satisfy Schur concavity

$$\sigma \prec \rho \quad \Rightarrow \quad S_{(q,s)}(\rho) \leqslant S_{(q,s)}(\sigma), \tag{4.14}$$

where $\prec$ means the symbol for majorization, *i.e.*, if $\{r_i\}$ and $\{s_i\}$ are the decreasingly organized eigenvalues of ρ and σ, respectively, then

$$\sigma \prec \rho \quad \Leftrightarrow \quad \sum_{i=1}^{n} s_i \leqslant \sum_{i=1}^{n} r_i, \tag{4.15}$$

where $n \leqslant N = \max\{\mathrm{rank}(\rho), \mathrm{rank}(\sigma)\}$. If $\mathrm{rank}(\rho) \leqslant \mathrm{rank}(\sigma)$ we just complete the vector r_i with zeros. Now, suppose σ is a pure state and ρ is a general mixed state. Then, by (4.14), one can prove the non-negativity of the unified entropy

$$\sum_{i=1}^{n} r_i \leqslant \sum_{i=1}^{n} s_i = 1 \quad \Rightarrow \quad 0 \leqslant S_{(q,s)}(\rho). \tag{4.16}$$

Similarly, by comparing a maximally mixed state $\rho^* = \mathbb{1}/N$ with a general one, one can found the upper bound of unified entropy

$$\sum_{i=1}^{n} \frac{\mathbb{1}}{N} \leqslant \sum_{i=1}^{n} r_i \quad \Rightarrow \quad S_{(q,s)}(\rho) \leqslant S_{(q,s)}\left(\frac{1}{N}\right) = \frac{\left(\sum_i^N \frac{1}{N^q}\right)^s - 1}{(1-q)s} = \frac{N^{(1-q)s} - 1}{(1-q)s}. \tag{4.17}$$

Under a bistochastic map Λ, unified entropy never decreases, *i.e.*,

$$S_{(q,s)}(\Lambda(\rho)) \geqslant S_{(q,s)}(\rho). \tag{4.18}$$

Moreover, equality holds *iff* $\Lambda(\rho) = U\rho U^{\dagger}$.

Concavity is another property satisfied by unified entropy only for a set of parameters q and s (see Proposition 3 in Ref. [169]). If ρ and σ are density operators and $\lambda \in [0,1]$, then

$$S_{(q,r)}(\lambda\rho + (1-\lambda)\sigma) \geqslant \lambda S_{(q,s)}(\rho) + (1-\lambda)S_{(q,s)}(\rho), \tag{4.19}$$

for all q and s satisfying either $0 \leqslant q \leqslant 1$ and $qs \leqslant 1$ or $q \geqslant 1$ and $qs \geqslant 1$.

4.2 DISTINGUISHING QUANTUM STATES

Essentially, there are two ways of distinguishing one quantum state from another: *distances* (or *metrics*) and *divergences* (or *relative entropies*). The former is of geometric nature an it requires to be symmetric, while the latter is an entropic measure and it does not require symmetry in its entries.

4.2.1 Norms and Distances

A *norm* on a complex vector space $\mathcal{X}$ is a real-valued function whose value at $x \in \mathcal{X}$ is denoted by $\|x\|$ and must satisfy the following [172]:

Definition 2 (Norm)**.** Every norm $\|x\|$ must be/satisfy:

1. **Non-negative:** $\|x\| \geqslant 0$;
2. **Positive definite:** $\|x\| = 0 \quad \textit{iff} \quad x = \mathbf{0}$;
3. **Absolutely homogeneous**: $\|\alpha x\| = |\alpha|\|x\|, \ \forall\alpha \in \mathbb{C}$;
4. **Triangle inequality:** $\|x+y\| \leqslant \|x\| + \|y\|$.

If the triangle inequality is not satisfied, then we have a *seminorm.* One norm of particular interest is the Schatten p-norm [108] of an operator X

$$\|X\|_p = \left[\operatorname{Tr}\left(|X|^p\right)\right]^{\frac{1}{p}}, \tag{4.20}$$

where p is a real number such that $p \geqslant 1$ and $|X| := \sqrt{X^{\dagger}X}$. The Schatten p-norms of density operators $\rho \in \mathfrak{B}(\mathcal{H})$ (*i.e.* $\rho \geqslant 0$, $\operatorname{Tr}\rho = 1$, and $\rho^{\dagger} = \rho$) satisfy the following [108]:

Theorem 6 (Basic properties of Schatten p-norms)**.**

1. ***Unitary invariance:*** *If U and V are unitary operators, then*

$$\|U\rho V^{\dagger}\|_p = \|\rho\|_p; \tag{4.21}$$

2. ***Multiplicativity under tensor products:***

$$\|\rho \otimes \sigma\|_p = \|\rho\|_p \|\sigma\|_p; \tag{4.22}$$

3. ***Holder's inequality:*** *Take* $p, q, r \in [1, \infty)$ *such that* $1/p + 1/q = 1/r$, *then*

$$\|\rho\sigma\|_r \leqslant \|\rho\|_p \|\sigma\|_q; \tag{4.23}$$

4. ***Sub-mutiplicativity:***

$$\|\rho\sigma\|_p \leqslant \|\rho\|_p \|\sigma\|_p; \tag{4.24}$$

5. ***Monotonicity:*** *If* $1 \leqslant p \leqslant p' \leqslant \infty$, *then*

$$1 \geqslant \|\rho\|_1 \geqslant \|\rho\|_p \geqslant \|\rho\|_{p'} \geqslant \|\rho\|_\infty, \tag{4.25}$$

where equality holds for pure states.

6. ***Pinching inequality:*** *If* C *is a CPTP unital map, then*

$$\|C(\rho)\|_p \leqslant \|\rho\|_p, \tag{4.26}$$

where equality holds if ρ *is a maximally mixed state. Also, equality holds if* $p = 1$. *In the particular case that* C *is a non-selective measurement (e.g.,* $C = \Phi_A$*), then equality hols iff* $\rho = C(\rho)$ *(see Theorem 5.2 in Ref. [173]).*

7. ***Contractivity:*** *If* $p = 1$ *and* Λ *is a CPTP map, then*

$$\|\Lambda(\rho)\|_1 \leqslant \|\rho\|_1. \tag{4.27}$$

8. ***Audenaert's inequality for*** p***-norms:*** *For any bipartite state* $\rho \in \mathfrak{B}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})$, *the inequality*

$$\|\rho\|_p + 1 \geqslant \|\rho_\mathcal{A}\|_p + \|\rho_\mathcal{B}\|_p \tag{4.28}$$

holds for $p > 1$ *[174].*

Some cases of p-norms receive special names. The reader can find a selection of them in Table 2.

p	Name	$\|X\|_p$	Alternative computation
1	Trace norm	$\mathrm{Tr}\,\|X\|$	$\sum_i \|\lambda_i\|$
2	Hilbert-Schimidt norm	$\sqrt{\mathrm{Tr}\,X^2}$	$\sqrt{\sum_{ij} \|X_{ij}\|^2}$
$+\infty$	Operator norm	$\max_i\{\|\lambda_i\|\}$	—

Table 2 – Three different special cases of Schatten p-norms. For the sake of clarity, λ_i and X_{ij} are the eigenvalues and matrix elements of X, respectively. Note that, $|X| := \sqrt{X^\dagger X}$.

Let us now define the concept of metric [172]:

Definition 3 (Metric)**.** A *metric* (or *distance function*, or just *distance*) is a function $d : M \times M \mapsto \mathbb{R}$ that maps each par $x, y \in M$ onto a real number $d(x, y)$ satisfying

1. **Positive definiteness:** $d(x, y) \geqslant 0$, where the equality holds *iff* $x = y$;
2. **Symmetry:** $d(x, y) = d(y, x)$;
3. **Triangle inequality:** $d(x, z) \leqslant d(x, y) + d(y, z)$.

Positive definiteness is essential to obtain proper distinguishing measures. In addition, other properties are necessary for such measures to have a physical meaning. In the context of density states $\rho, \sigma \in \mathfrak{B}(\mathcal{H})$, a metric is said to be *contractive* under a CPTP map Λ *iff*

$$d(\Lambda(\rho), \Lambda(\sigma)) \leqslant d(\rho, \sigma). \tag{4.29}$$

In particular, contractive metrics are unitarily invariant, that is,

$$d(U\rho U^\dagger, U\sigma U^\dagger) = d(\rho, \sigma). \tag{4.30}$$

Having said that, we can use norms to find proper metrics. Note that it is not strictly necessary to use norms when defining a metric since other functionals can be used instead. Without further ado, the *trace distance* d_{Tr} is given by

$$d_{\mathrm{Tr}}(\rho, \sigma) := \|\sigma - \rho\|_1 = \mathrm{Tr}\,|\sigma - \rho|. \tag{4.31}$$

Similarly, the *Hilbert-Schmidt distance* d_{HS} is defined as

$$d_{\mathrm{HS}}(\rho, \sigma) := \|\sigma - \rho\|_2 = \sqrt{\mathrm{Tr}\,|\sigma - \rho|^2}. \tag{4.32}$$

As matter of fact, the trace and the Hilbert-Schmidt distances are special cases of the L_p*-distances* d_p (also called *Schatten* p*-distances*) [108, 109]

$$d_p(\rho, \sigma) := \|\sigma - \rho\|_p = \left(\mathrm{Tr}\,|\sigma - \rho|^p\right)^{\frac{1}{p}}, \tag{4.33}$$

for all $p \geqslant 1$. Although all the L_p-distances are unitarily invariant because they inherit this property from the Schatten p-norms, the only L_p-distance that is contractive under CPTP maps is the trace distance ($p = 1$) [175]. However, all L_p-distances are jointly convex

$$d_p\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \leqslant \sum_i p_i d_p(\rho_i, \sigma_i). \tag{4.34}$$

Note that, any power $p > 1$ of d_p is also jointly convex. In addition, we have the *Bures distance* d_{Bu} [109, 110]

$$d_{\mathrm{Bu}}(\rho, \sigma) := \left[2 - 2\sqrt{F(\rho, \sigma)}\right]^{\frac{1}{2}}, \tag{4.35}$$

where $F(\rho, \sigma)$ is the *Uhlmann fidelity* [176]

$$F(\rho,\sigma) := \|\sqrt{\sigma}\sqrt{\rho}\|_1^2 = \left[\mathrm{Tr}\,\left(\sqrt{\rho}\sigma\sqrt{\rho}\right)^{\frac{1}{2}}\right]^2 . \tag{4.36}$$

Note that, the Uhlmann fidelity is symmetric. Moreover, if $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$, than $F(\rho,\sigma) = |\langle\psi|\phi\rangle|^2$. Last but not least, we can cite the *quantum Hellinger distance* d_{He} [109, 111]

$$d_{\mathrm{He}}(\rho,\sigma) := \|\sqrt{\sigma} - \sqrt{\rho}\|_2 = \left(2 - 2\mathrm{Tr}\,\sqrt{\sigma}\sqrt{\rho}\right)^{\frac{1}{2}} . \tag{4.37}$$

It is interesting to note that if ρ and σ commute, then $d_{\mathrm{Bu}}(\rho,\sigma) = d_{\mathrm{He}}(\rho,\sigma)$. Both the Bures and the Hellinger distances are contractive under CPTP maps. The joint convexity, however, is only satisfied by the square of them, *i.e.*,

$$d_{\mathrm{Bu}}^2\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) \leqslant \sum_i p_i d_{\mathrm{Bu}}^2(\rho_i,\sigma_i), \tag{4.38}$$

with an identical expression for d_{He} [109]. The reader can find a summary of properties about all these metrics in Tab. 3.

	d_{Tr}	d_{HS}	d_{HS}^2	d_p	d_p^p	d_{Bu}	d_{Bu}^2	d_{He}	d_{He}^2
Continuity	✔	✔	✔	✔	✔	✔	✔	✔	✔
Positive definiteness	✔	✔	✔	✔	✔	✔	✔	✔	✔
Unitary invariance	✔	✔	✔	✔	✔	✔	✔	✔	✔
Joint convexity	✔	✔	✔	✔	✔	✘	✔	✘	✔
Contractivity	✔	✘	✘	✘	✘	✔	✔	✔	✔

Table 3 – Summary of properties that are satisfied by the trace distance d_{Tr}, the Hilbert-Schmidt distance d_{HS}, the L_p-distances d_p with finite $p > 1$ and $p \neq 2$, the Bures distance d_{Bu}, the squared Bures distance d_{Bu}^2, the Hellinger distance d_{He}, and the squared Hellinger distance d_{He}^2 between any pair of density states.

4.2.2 Quantum Divergences

Quantum divergences (or relative entropies) are measures of the distinctiveness of positive operators. These measures are known for their usefulness and versatility in defining several quantum information concepts, in particular, the one that will be shown to be of key relevance in the next chapter, namely, the quantum conditional information. We now review three divergence measures, namely, the von Neumann relative entropy, the Rényi divergences, and the Tsallis relative entropies.

4.2.2.1 von Neumann relative entropy

The von Neumann relative entropy, also known as the Umegaki relative entropy [177], is one of the most used divergences in quantum information theory. It is defined as

$$D(\rho||\sigma) := \frac{\mathrm{Tr}\ [\rho(\ln\rho - \ln\sigma)]}{\mathrm{Tr}\,\rho}, \tag{4.39}$$

where $\rho > 0$, $\sigma \geqslant 0$ and $\ker\sigma \subseteq \ker\rho$, where ker stands for kernel of the operator[1]. The factor $\mathrm{Tr}\,\rho$ ensures that $D(\lambda\rho||\lambda\sigma) = D(\rho||\sigma)$ for all $\lambda > 0 \in \mathbb{R}$. $D(\rho||\sigma)$ is a continuous functional satisfying (whenever $\rho \geqslant \sigma$) the positive definiteness property:

$$D(\rho||\sigma) \geqslant 0,\ \text{with equality holding iff}\ \rho = \sigma. \tag{4.40}$$

The von Neumann relative entropy also satisfies the following properties: (i) unitary invariance,

$$D\left(U\rho U^\dagger||U\sigma U^\dagger\right) = D(\rho||\sigma), \tag{4.41}$$

for any unitary U; (ii) additivity,

$$D\left(\bigotimes_i \rho_i \middle\| \bigotimes_i \sigma_i\right) = \sum_i D(\rho_i||\sigma_i); \tag{4.42}$$

(iii) joint convexity,

$$D\left(\sum_i p_i\rho_i \middle\| \sum_i p_i\sigma_i\right) \leqslant \sum_i p_i D(\rho_i||\sigma_i); \tag{4.43}$$

and (iv) data processing inequality (DPI),

$$D\left(\Lambda(\rho)||\Lambda(\sigma)\right) \leqslant D(\rho||\sigma), \tag{4.44}$$

also known as contractivity or monotonicity under quantum channels Λ.

4.2.2.2 Rényi divergences

Constituting a generalization of the von Neumann relative entropy, the Rényi divergences [86] are defined as

$$D_\alpha(\rho||\sigma) := \frac{1}{\alpha-1}\ln\frac{\mathrm{Tr}\ \left(\rho^\alpha\sigma^{1-\alpha}\right)}{\mathrm{Tr}\,\rho}, \tag{4.45}$$

for $\alpha \in (0,1) \cup (1,+\infty)$ and the same conditions of quantity (4.39). Here, we also have $D_\alpha(\lambda\rho||\lambda\sigma) = D_\alpha(\rho||\sigma)$, for any positive real λ. Equation (4.45) is said a generalization of Eq. (4.39) because $D_{\alpha\to1}(\rho||\sigma) = D(\rho||\sigma)$. Another relative entropy comprised by the Rényi divergences is the min-relative entropy $D_{\min}(\rho||\sigma) := \lim_{\alpha\to0} D_\alpha(\rho||\sigma) = -\ln[\mathrm{Tr}\,(\rho^0\sigma)/\mathrm{Tr}\,\rho]$ where ρ^0 is the projection onto the support of ρ, as defined by Datta [88] (see Table 4 for a

[1] $|\psi\rangle \in \ker\rho$ *iff* $\rho\,|\psi\rangle = 0$.

summary of properties and Appendix C.2.1 for more details about the min-relative entropy). Some properties that are satisfied by the von Neumann relative entropy encounter, however, some restrictions in the Rényi generalization: joint convexity is valid only for $\alpha \in (0,1)$ and DPI only for $\alpha \in (0,1) \cup (1,2]$ (see Ref. [90] and references therein). The rest of the properties remain intact. A variant of definition (4.45) is the so-called sandwiched Rényi divergence, which was independently proposed by Müller-Lennert *et al.* [93] and Wilde *et al.* [94] as

$$\widetilde{D}_\alpha(\rho||\sigma) := \frac{1}{\alpha-1}\ln\left\{\frac{1}{\operatorname{Tr}\rho}\operatorname{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha}\right]\right\}, \tag{4.46}$$

with the same conditions of quantity (4.45). Besides reducing to the von Neumann relative entropy as $\alpha \to 1$, the divergence (4.46) reproduces other famous relative entropies, such as the collisional relative entropy ($\alpha = 2$) [87] and the max-relative entropy $D_{\max}(\rho||\sigma) := \lim_{\alpha\to+\infty}\widetilde{D}_\alpha(\rho||\sigma)$ [88] (see Table 4 for a summary of properties and Appendix C.2.1 for more details about the collisional and the max-relative entropies). The sandwiched Rényi divergence satisfies the same properties as its counterpart (4.45) but for different ranges of parameters: joint convexity is valid only for $\alpha \in [1/2,1)$ while DPI holds for $\alpha \in [1/2,1) \cup (1,+\infty)$ [91, 92]. It was proved for $\alpha \in (0,1)$ [95] and $\alpha > 1$ [94] that the inequality $\widetilde{D}_\alpha(\rho,\sigma) \leqslant D_\alpha(\rho,\sigma)$ is always true, where the equality holds iff $[\rho,\sigma] = 0$. The necessity of this statement was noted in Ref. [98].

The issue concerning the commutativity of operators raised the discussion about the use of the divergence (4.46) instead of (4.45). However, as pointed out by Gupta and Wilde [97], $D_\alpha(\rho||\sigma)$ "is perfectly well defined" when ρ and σ do not commute and, in fact, this divergence has proven to be useful for discrimination tasks in some contexts when $\alpha \in (0,1)$, including the limiting case $\alpha \to 0$ (see Ref. [97] and references therein). The problem with definition (4.45) is that it does not satisfy DPI for $\alpha \in (2,+\infty)$, a large range that is in fact covered by the sandwiched version, including its limiting case ($\alpha \to +\infty$) known as max-relative entropy [87]. Since divergences are fundamental tools for one to distinguish a quantum state from another, it is expected that after the action of a quantum channel the states become less distinguishable and, therefore, DPI is an essential property for quantum information. Nonetheless, to obtain reality quantifiers it will be sufficient for us to focus on the original version of the Rényi divergence, since all the results will directly have a counterpart for the sandwiched version.

4.2.2.3 Tsallis relative entropies

To close this section on quantum divergences, let us revisit the Tsallis relative entropies, originally proposed by Abe [103]. Here we adopt the form

$$D_q(\rho||\sigma) := \frac{\operatorname{Tr}\left[\rho^q\left(\ln_q\rho - \ln_q\sigma\right)\right]}{\operatorname{Tr}\rho} = \frac{\operatorname{Tr}\left(\rho - \rho^q\sigma^{1-q}\right)}{(1-q)\operatorname{Tr}\rho}, \tag{4.47}$$

where $q \in (0,1)$ and $\ln_q(x) := (x^{1-q}-1)/(1-q)$. As pointed out by Rastegin [106], the definition (4.47) can be extended to $q > 1$ if $\ker\sigma \subseteq \ker\rho$. The normalization guarantees that

$D_q(\lambda\rho||\lambda\sigma) = D_q(\rho||\sigma)$ for any $\lambda > 0 \in \mathbb{R}$. When $q \longrightarrow 1$, we regain the von Neumann relative entropy. The Tsallis relative entropies and the Rényi divergences share several properties. $D_q(\rho||\sigma)$ is a continuous and positive definite functional in ρ and σ for $q \in (0,1) \cup (1,+\infty)$. In addition, the Tsallis relative entropies satisfy unitary invariance for $q \in (0,1) \cup (1,+\infty)$, and both joint convexity and DPI for $\alpha \in (0,1) \cup (1,2]$ [104, 106]. Most importantly, they are pseudo-additive, that is

$$D_q(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}||\sigma_{\mathcal{A}} \otimes \sigma_{\mathcal{B}}) = D_q(\rho_{\mathcal{A}}||\sigma_{\mathcal{A}}) + D_q(\rho_{\mathcal{B}}||\sigma_{\mathcal{B}}) + (q-1)D_q(\rho_{\mathcal{A}}||\sigma_{\mathcal{A}})D_q(\rho_{\mathcal{B}}||\sigma_{\mathcal{B}}). \quad (4.48)$$

We refer the reader to Table 4 for a summary of properties that are satisfied by each divergence presented in this section.

	D	D_α	$D_{\min}$	$\widetilde{D}_\alpha$	$D_{\max}$	D_q
Continuity	✔	✔	✘	✔	✘	✔
Positive def.	✔	✔	✘	✔	✔	✔
Unitary inv.	✔	✔	✔	✔	✔	✔
Additivity	✔	✔	✔	✔	✔	✘
J. convex.	✔	$\alpha \in (0,1)$	✔	$\alpha \in [1/2, 1)$	✘	$q \in (0,2] - \{1\}$
DPI	✔	$\alpha \in (0,2] - \{1\}$	✔	$\alpha \in [1/2, +\infty) - \{1\}$	✔	$q \in (0,2] - \{1\}$

Table 4 – Summary of properties satisfied by the von Neumann relative entropy D, the Rényi divergence D_α, the min-relative entropy $D_{\min} := D_{\alpha\to 0}$, the sandwiched Rényi divergence $\widetilde{D}_\alpha$, the max-relative entropy $D_{\max} := \widetilde{D}_{\alpha\to+\infty}$, and the Tsallis relative entropy D_q, for any pair $\{\rho, \sigma\}$ of density operators.

4.3 ELEMENTS OF QUANTUM INFORMATION THEORY WITH QUANTUM DIVERGENCES

We review now the representation of quantum informational content and conditional quantum information by means of the above introduced divergences. Note that, the Rényi conditional information measures [Eq. (4.63)], as well as the Tsallis informational content and conditional information [Eqs. (4.65) and (4.69)] constitute original proposals.

4.3.1 von Neumann information theory

The largest divergence implied by Eq. (4.39) emerges when one considers a generic pure state, $\psi = |\psi\rangle\langle\psi|$, and the maximally mixed one, $\mathbb{1}/d$, with $d = \dim\mathcal{H}$. We have $D(\psi||\mathbb{1}/d) = S(\mathbb{1}/d) = \ln d$ (sometimes referred to as the normalization condition $D(\mathbb{1}||\mathbb{1}/d) = S(\mathbb{1}/d)$ [93]), where

$$S(\rho) := -\frac{\mathrm{Tr}\,(\rho\ln\rho)}{\mathrm{Tr}\,\rho} \quad (4.49)$$

is the von Neumann entropy of ρ. The quantum informational content $I(\rho)$ of a quantum state ρ is a concept complementary to ignorance, that is, $I(\rho)+S(\rho)=I^{\max}=S^{\max}$ with $S^{\max}=S(\mathbb{1}/d)=\ln d=I(\psi)=I^{\max}$ (meaning that the entropy of a maximally mixed state $\mathbb{1}/d$ equals the informational content of a pure state ψ). In terms of the relative entropy, information can be defined as

$$I(\rho):=D(\rho||\mathbb{1}/d)=\ln d-S(\rho), \tag{4.50}$$

where we have used the fact that $\operatorname{Tr}\rho\ln\mathbb{1}/d=\operatorname{Tr}\rho\ln(1/d)\mathbb{1}=\ln(1/d)$. Since pure states (resp. maximally mixed states) have maximum (resp. minimum) informational content, $I(\rho)$ is itself a direct measure of purity. One can make a further interpretation of $I(\rho)$ referring back to the map (1.116). Consider the pairs $\{A,A'\}$ and $\{B,B'\}$ of noncommuting operators acting on $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$, respectively, and forming maximally unbiased bases (MUB). One has

$$\Phi_{AA'}(\rho)\equiv\Phi_A\Phi_{A'}(\rho)=\Phi_{A'}\Phi_A(\rho)=\frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}}\otimes\rho_{\mathcal{B}}, \tag{4.51}$$

where $\rho_{\mathcal{B}}=\operatorname{Tr}_{\mathcal{A}}(\rho)$, and similarly for $\{B,B'\}$. For the whole context $\mathbb{C}=\{A,A',B,B'\}$, we can write $\Phi_{\mathbb{C}}(\rho)=\frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}}\otimes\frac{\mathbb{1}_{\mathcal{B}}}{d_{\mathcal{B}}}=\mathbb{1}/d$, with $d=d_{\mathcal{A}}d_{\mathcal{B}}$. This is a state of null irreality—or full classical reality—, since for any observable X one has $\Phi_X(\mathbb{1}/d)=\mathbb{1}/d$, that is, a nonselective measurement of X cannot change the established state of affairs. Therefore, we can rewrite Eq. (4.50) in the form $I(\rho)=D(\rho||\Phi_{\mathbb{C}}(\rho))$, which allows us to interpret the informational content as the divergence of ρ with respect to its classical counterpart $\Phi_{\mathbb{C}}(\rho)$.

Equation (4.39) can also be used to define the quantum conditional entropy of a quantum state ρ,

$$H_{\mathcal{A}|\mathcal{B}}(\rho):=-D(\rho||\mathbb{1}_{\mathcal{A}}\otimes\rho_{\mathcal{B}}). \tag{4.52}$$

It can be checked by means of (1.53) that this formula yields the usual relation $H_{\mathcal{A}|\mathcal{B}}=S(\rho)-S(\rho_{\mathcal{B}})$ from Sec. 1.2. The conditional entropy can alternatively be defined through an optimization process over the subspace $\mathcal{B}$, since $\inf_{\sigma_{\mathcal{B}}}D(\rho||\mathbb{1}_{\mathcal{A}}\otimes\sigma_{\mathcal{B}})=D(\rho||\mathbb{1}_{\mathcal{A}}\otimes\rho_{\mathcal{B}})$. Here, inf stands for the infimum[2]. By its turn, the conditional information of ρ can also be defined through the *information-ignorance complementarity*, that is,

$$I_{\mathcal{A}|\mathcal{B}}(\rho)+H_{\mathcal{A}|\mathcal{B}}(\rho)=H^{\max}_{\mathcal{A}|\mathcal{B}}=H_{\mathcal{A}|\mathcal{B}}\left(\frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}}\otimes\rho_{\mathcal{B}}\right)=\ln d_{\mathcal{A}}, \tag{4.53}$$

a relation that will be taken as a fundamental premise in all information theories throughout this thesis. We then write

$$I_{\mathcal{A}|\mathcal{B}}(\rho):=\ln d_{\mathcal{A}}-H_{\mathcal{A}|\mathcal{B}}(\rho)=D\left(\rho\,\middle\|\,\frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}}\otimes\rho_{\mathcal{B}}\right). \tag{4.54}$$

Because both the entries in the above divergence are normalized density operators, one has $0\leqslant I_{\mathcal{A}|\mathcal{B}}(\rho)\leqslant\ln d$. Also, the conditional information can be decomposed as

$$I_{\mathcal{A}|\mathcal{B}}(\rho)=I(\rho_{\mathcal{A}})+I_{\mathcal{A}:\mathcal{B}}(\rho), \tag{4.55}$$

[2] The infimum of a set I is the greatest of the lower bounds of I. For example, if $I=\{1,2,3,4\}$, than $\inf I=1$. Now, if $I=\{x\in\mathbb{R}\mid 0<x<1\}$, than $\inf I=0$. The infimum of a set does not necessarily belongs to the set itself. The same logic applies for the sup, *i.e.*, the supremum.

where $I(\rho_{\mathcal{A}}) = D\left(\rho_{\mathcal{A}}||\mathbb{1}_{\mathcal{A}}/d_{\mathcal{A}}\right)$ is the informational content of part $\mathcal{A}$ and

$$I_{\mathcal{A}:\mathcal{B}}(\rho) = D\left(\rho||\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}\right) \tag{4.56}$$

is the mutual information (1.76)—now written as a divergence. Since the mutual information is a measure of total correlations between the parts, $I_{\mathcal{A}|\mathcal{B}}$ can be said to be composed of "local" and "global" information.

4.3.2 Rényi quantum information theory

The generalization for the Rényi scenario from the von Neumann information theory is straightforward. By using Eq. (4.45) one checks the validity of the normalization condition, $D_\alpha(\psi||\mathbb{1}/d) = \ln d = S_\alpha(\mathbb{1}/d)$, where

$$S_\alpha(\rho) := -\frac{1}{\alpha - 1}\ln\frac{\operatorname{Tr}\rho^\alpha}{\operatorname{Tr}\rho}, \tag{4.57}$$

is the quantum Rényi entropy of ρ. The Rényi informational content of ρ can be defined as

$$I_\alpha(\rho) := D_\alpha(\rho||\mathbb{1}/d) = \ln d - S_\alpha(\rho), \tag{4.58}$$

which reproduces Eq. (4.50) as $\alpha \to 1$. See Appendix C.2.2 for more details. Since ρ commutes with $\mathbb{1}/d$, the original and the sandwiched Rényi divergences result in the same informational content. Here as well, we can interpret the informational content as the amount by which ρ diverges from a full reality state, that is, $I_\alpha(\rho) = D_\alpha\left(\rho||\Phi_{\mathbb{C}}(\rho)\right)$.

It is usual to define the Rényi conditional entropy in at least two different ways [99]:

$$H^{\alpha\downarrow}_{\mathcal{A}|\mathcal{B}}(\rho) := -D_\alpha(\rho||\mathbb{1}_{\mathcal{A}} \otimes \rho_{\mathcal{B}}), \tag{4.59a}$$

$$H^{\alpha\uparrow}_{\mathcal{A}|\mathcal{B}}(\rho) := -\inf_{\sigma_{\mathcal{B}}} D_\alpha(\rho||\mathbb{1}_{\mathcal{A}} \otimes \sigma_{\mathcal{B}}), \tag{4.59b}$$

with $\sigma_{\mathcal{B}} \in \mathfrak{B}(\mathcal{H}_{\mathcal{B}})$. The arrows are used to express the relation $H^{\alpha\uparrow}_{\mathcal{A}|\mathcal{B}} \geqslant H^{\alpha\downarrow}_{\mathcal{A}|\mathcal{B}}$. It is noteworthy that, unlike its von Neumann counterpart (4.52), the Rényi conditional entropy cannot be expanded as $S_\alpha(\rho) - S_\alpha(\rho_{\mathcal{B}})$. Moreover, as emphasized by Tomamichel *et al.* [96], proposals along these lines lead to conceptual problems, such as the invalidation of DPI. From the complementarity relation

$$I^{\alpha\uparrow,\downarrow}_{\mathcal{A}|\mathcal{B}}(\rho) + H^{\alpha\uparrow,\downarrow}_{\mathcal{A}|\mathcal{B}}(\rho) = [H^{\alpha}_{A|B}]_{\max} = H^{\alpha\uparrow,\downarrow}_{\mathcal{A}|\mathcal{B}}\left(\frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}} \otimes \rho_{\mathcal{B}}\right) = \ln d_{\mathcal{A}}, \tag{4.60}$$

we propose the Rényi conditional information measures

$$I^{\alpha\uparrow}_{\mathcal{A}|\mathcal{B}}(\rho) := \ln d_{\mathcal{A}} - H^{\alpha\downarrow}_{\mathcal{A}|\mathcal{B}}(\rho), \tag{4.61a}$$

$$I^{\alpha\downarrow}_{\mathcal{A}|\mathcal{B}}(\rho) := \ln d_{\mathcal{A}} - H^{\alpha\uparrow}_{\mathcal{A}|\mathcal{B}}(\rho), \tag{4.61b}$$

with arrows justified by the relation $I^{\alpha\downarrow}_{\mathcal{A}|\mathcal{B}}(\rho) \leqslant I^{\alpha\uparrow}_{\mathcal{A}|\mathcal{B}}(\rho)$. For any quantum channel $\Lambda_{\mathcal{B}\to\mathcal{B}'}$, both measures satisfy DPI, that is,

$$I^{\alpha\uparrow,\downarrow}_{\mathcal{A}|\mathcal{B}'}(\Lambda(\rho)) \leqslant I^{\alpha\uparrow,\downarrow}_{\mathcal{A}|\mathcal{B}}(\rho) \tag{4.62}$$

for $\alpha \in (0,1)\cup(1,2]$ (including the limiting case $\alpha \to 0$) and $\alpha \in [1/2,1)\cup(1,+\infty)$, respectively [96]. Both conditional information measures are convex under mixing for $\alpha \in (0,1)$ and $\alpha \in [1/2,1)$, respectively [99]. Finally, by use of Eq. (4.45) we have

$$I^{\alpha\uparrow}_{\mathcal{A}|\mathcal{B}}(\rho) = D_\alpha\left(\rho \,\middle\|\, \frac{\mathbb{1}_\mathcal{A}}{d_\mathcal{A}} \otimes \rho_\mathcal{B}\right), \tag{4.63a}$$

$$I^{\alpha\downarrow}_{\mathcal{A}|\mathcal{B}}(\rho) = \inf_{\sigma_\mathcal{B}} D_\alpha\left(\rho \,\middle\|\, \frac{\mathbb{1}_\mathcal{A}}{d_\mathcal{A}} \otimes \sigma_\mathcal{B}\right). \tag{4.63b}$$

The reader can find more details about these relations in Appendix C.2.2.

4.3.3 Tsallis quantum information theory

From Eq. (4.47) we find $D_q(\psi||\mathbb{1}/d) = d^{q-1}S_q(\mathbb{1}/d)$, where

$$S_q(\rho) := -\mathrm{Tr}\,\left(\rho^q \ln_q \rho\right) = -\frac{\mathrm{Tr}\,(\rho - \rho^q)}{(1-q)\,\mathrm{Tr}\,\rho} \tag{4.64}$$

is the Tsallis entropy of ρ [101, 102] and $S_q(\mathbb{1}/d) = \ln_q d$. Note that, differently from the structure found for the previous information theories, here the normalization relation is such that $D_q(\psi||\mathbb{1}/d) \neq S_q(\mathbb{1}/d)$. This suggests that it may be convenient to "correct" either D_q or S_q by means of a scaling factor like d^{1-q} or d^{q-1}. To preserve the fundamental status of the information-ignorance complementarity, we then define the Tsallis informational content as

$$I_q(\rho) := d^{1-q}D_q(\rho||\mathbb{1}/d) = \ln_q d - S_q(\rho). \tag{4.65}$$

Similarly to what can be found in Ref. [105], let us define the Tsallis conditional entropy as

$$H^q_{\mathcal{A}|\mathcal{B}}(\rho) := -D_q(\rho||\mathbb{1}_\mathcal{A} \otimes \rho_\mathcal{B}). \tag{4.66}$$

One may wonder whether $-\inf_{\sigma_\mathcal{B}} D_q(\rho||\mathbb{1}_\mathcal{A} \otimes \sigma_\mathcal{B})$ would be an admissible formulation as well. Although we believe there is no reason why this proposal should be ruled out *a priori*, we are not aware of any study supporting it. Following previous rationales, we now look for a conditional information satisfying the information-ignorance relation

$$I^q_{\mathcal{A}|\mathcal{B}}(\rho) + H^q_{\mathcal{A}|\mathcal{B}}(\rho) = [H^q_{\mathcal{A}|\mathcal{B}}]_{\max} = H^q_{\mathcal{A}|\mathcal{B}}\left(\frac{\mathbb{1}_\mathcal{A}}{d_\mathcal{A}} \otimes \rho_\mathcal{B}\right) = \ln_q d_\mathcal{A}. \tag{4.67}$$

We then find

$$I^q_{\mathcal{A}|\mathcal{B}}(\rho) := \ln_q d_\mathcal{A} - H^q_{\mathcal{A}|\mathcal{B}}(\rho), \tag{4.68}$$

Using the above formulas, one shows that

$$I^q_{\mathcal{A}|\mathcal{B}}(\rho) = d_\mathcal{A}^{1-q}\, D_q\left(\rho \,\middle\|\, \frac{\mathbb{1}_\mathcal{A}}{d_\mathcal{A}} \otimes \rho_\mathcal{B}\right), \tag{4.69}$$

which correctly reduces to the result (4.54) as $q \to 1$. Up to the scaling factor $d_{\mathcal{A}}^{1-q}$, proved necessary in the present scenario, we note by Eqs. (4.54), (4.63a), and (4.69) that it is possible to maintain a unified picture for the definition of the conditional informational of ρ in terms of its divergence with respect to its full reality counterpart, $\Phi_{AA'}(\rho) = \frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}} \otimes \rho_{\mathcal{B}}$.

4.4 QUANTUM RESOURCE THEORIES

QRTs [178] comprehend a useful way to classify and study non-classicalities based on the tasks that they are necessary to. A QRT is characterized through its (i) free states, (ii) resource states, and (iii) free operations. Free states are those that do not contain that quantum resource, unlike the resource states. Free operations constitute those operations that do not create more quantum resource. Hence, free operations applied to free states produce only free states. For instance, coherence [113] and entanglement [112] of quantum states are the most known quantum resources. Just recently, irreality was argued to be a QRT too [44]. Let us revisit entanglement and irreality under the framework of QRTs:

Example 10. An entangled state is a bipartite state that can not be written as a separable state, *i.e.*, $\rho = \sum_i p_i \rho_i^{\mathcal{A}} \otimes \rho_i^{\mathcal{B}}$ [see Eq. (1.102)]. Therefore, separable states have null entanglement, which configure them as the free states of the QRT of entanglement. The resource states, in this case, are all the entangled ones. Now, the free operations are those operations that do not create more entanglement, namely, the LOCCs. It is not possible, for instance, to entangle two particles by applying operations on just one of them (a local operation) nor by transmitting information from one lab to the other through a phone call (a classical communication). Some tasks in which entanglement is a necessary resource are quantum cryptography [179] and quantum teleportation [180]. ▲

Example 11. When it comes to the irreality $\mathfrak{I}_A$, it is mandatory that we mention which observable is under scrutiny. Hence, one can immediately see that A-reality states, *i.e.*, states such that $\rho = \Phi_A(\rho)$, are the free states of the QRT of $\mathfrak{I}_A$. Consequently, every state that violates BA's realism hypothesis is a resource state. The free operations of irreality must be those that do not create more indefiniteness, namely, weak and projective measurements of A. ▲

Note that, a resource theory does not specify how to quantify its own resource. Some authors argue that a mathematical function must satisfy a set of properties based on physical arguments in order to proper quantify a given quantum resource [178]. Depending on the properties that are satisfied by a given quantifier, it can be identified as a *monotone* or as a *measure*, albeit that classification varies from one QRT to the other. Entanglement, for instance, can be quantified through several different ways. For pure quantum states, the linear entropy (4.5), the von Neumann entropy (1.63), and, more generally, the Rényi entropy (4.3) of the reduced state $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho)$ of a bipartite state $\rho \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$ are considered proper *entanglement measures*. *Entanglement monotones*, on the other way, satisfy a smaller list of properties. Any pair of parameters (q, s) that results in a concave unified entropy [see Eq.

(4.19)] gives us an entanglement monotone [181]

$$E^{(q,s)}(|\psi\rangle) := S_{(q,s)}\left(\mathrm{Tr}_{\mathcal{B}}(|\psi\rangle\langle\psi|)\right) = S_{(q,s)}\left(\mathrm{Tr}_{\mathcal{A}}(|\psi\rangle\langle\psi|)\right), \tag{4.70}$$

for q and s satisfying either $0 \leqslant q \leqslant 1$ and $rs \leqslant 1$ or $r \geqslant 1$ and $rs \geqslant 1$, which is a broader class of entanglement quantifiers than the entanglement measures.

Let the continuous non-negative functional $\rho \mapsto Q(\rho)$ be a quantum resource quantifier, $\mathcal{F}_Q$ its set of free states, and Λ_Q its free operations. Some of the aforementioned elementary properties can be stated as [178]:

1. *Vanishing for free states:*
$$Q(\rho) = 0 \quad \textit{iff} \quad \rho \in \mathcal{F}_Q; \tag{4.71}$$

2. *Monotonicity:*
$$Q(\rho) \geqslant Q\left(\Lambda_Q(\rho)\right); \tag{4.72}$$

3. *Convexity:*
$$Q\left(\sum_i p_i\rho_i\right) \leqslant \sum_i p_i Q(\rho_i); \tag{4.73}$$

4. *Subadditivity:*
$$Q(\rho \otimes \sigma) \leqslant Q(\rho) + Q(\sigma); \tag{4.74}$$

We should mention that, the first two are absolutely essential since they deal with the basics of a QRT. The last two, on the other side, represent convenient mathematical properties without which the meaning of a quantum resource would be harmed. This is due to the fact that mixing or adding states moves the system towards classicality. Other properties that are even less essential are *additivity* [when the equality holds in (4.74)] and *flag additivity*. The latter will be addressed in Sec. 5.3

5 AXIOMS FOR QUANTUM REALISM

This chapter and the next one aim at presenting the main results of Part II. These results were published in Ref. [A. C. Orthey and R. M. Angelo, "Quantum realism: Axiomatization and quantification", Phys. Rev. A **105**, 052218 (2022)]. Here, we are going to present the axioms for quantum realism which are fundamentally based on the flux of conditional information and the QRTs.

5.1 REALITY AND INFORMATION

As we have seen in the last chapter, quantum resource theories have shown to be a powerful framework to characterize a given quantum effect [178]. Incidentally, quantum realism cannot be thought of as a quantum resource because reality abounds for free in the classical regime. On the other hand, quantum realism is complementary to quantum irrealism (as quantified by irreality, which is believed to be a quantum resource [44]). With this inversion in mind, we seek inspiration in the formal structure of quantum resource theories to guide our axiomatization of quantum realism.

We start by grounding our intuition on some empirical facts. After passing through a wall with two slits, an electron has its paths described as a quantum superposition and an interference pattern is observed in the detection system (check out Fig. 3 again in the Introduction). During the flight, quantum mechanics does not ascribe a well defined position for the electron, so that its position is not an element of reality and the electron is said to behave like a wave. On the other hand, when the two slits are preceded with a very lightweight floating slit, the interference pattern disappears [5, 84] (see the double-slit quantum eraser [182] for a similar phenomenology). In this case, the entanglement created between the electron and the floating slit allows for the former to be described by a statistical mixture. It then follows that trajectory-based models are admissible so that the electron position can be claimed to be an element of reality. In other words, particle-like elements of reality emerge in this experiment because a given degree of freedom—the momentum of the lightweight slit—encodes which-way information about the electron [85]. In this case, trajectory-based models are admissible so that the electron position can be claimed to be an element of reality. These are expected to be the results of the experiment even in the absence of a huge environment, like a thermal bath.

Now, even though the supporters of quantum Darwinism would eventually claim that the conditions for the emergence of an objective reality are not met during the electron flight—for the information about the electron path has not an environment to be recorded in—we believe they would agree that the motional degree of freedom of the first slit is able to acquire information about the electron path, thus suppressing its wave-like properties. This is exactly the same viewpoint adopted by supporters of BA's realism [40, 42]. We then take this common perspective as our fundamental premise regarding the dynamical emergence of quantum realism:

The reality status of a physical observable can only increase when information about it is stored in another physical degree of freedom.

The above dictum translates the *information-reality complementarity principle* introduced by Dieguez and Angelo [40]. In Appendix A, we revisit the authors' argument using the notation to be introduced in the next section.

5.2 AXIOMS FOR REALITY MONOTONES

To formalize these ideas, we consider the functional $\rho \mapsto \mathfrak{R}_A(\rho)$, hereafter named *the reality of the observable* $A \in \mathfrak{B}(\mathcal{H}_\mathcal{A})$ *given the state* $\rho \in \mathfrak{B}(\mathcal{H}_\mathcal{S} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})$, where $\mathfrak{B}(\mathcal{H})$ is the set of positive semidefinite Hermitian operators acting on the Hilbert space $\mathcal{H}$. Let us now consider some generic dynamics involving the interest system $\mathcal{S}$ and an ancillary system $\mathcal{E}$ generically referred to as environment. Assume that the state of the composite system at an arbitrary instant of time t is given by $\upsilon_t \in \mathfrak{B}(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E})$, so that $\rho_t = \mathrm{Tr}_\mathcal{E}(\upsilon_t)$ denotes the reduced state of $\mathcal{S}$ with initial condition $\rho_{t=0} = \rho$. An alteration in the reality degree of the observable A in the time interval $[t_1, t_2]$ is here denoted as

$$\Delta\mathfrak{R}_A(t_2, t_1) := \mathfrak{R}_A(\rho_{t_2}) - \mathfrak{R}_A(\rho_{t_1}). \tag{5.1}$$

Let us also introduce another functional, $\upsilon_t \mapsto I_{\mathcal{E}|\mathcal{S}}(\upsilon_t)$, aiming at denoting how certain informational content associated with the environment is conditioned to some configuration of the system. Variations of this information with time are then described as

$$\Delta I_{\mathcal{E}|\mathcal{S}}(t_2, t_1) := I_{\mathcal{E}|\mathcal{S}}(\upsilon_{t_2}) - I_{\mathcal{E}|\mathcal{S}}(\upsilon_{t_1}). \tag{5.2}$$

We are now ready to state our main postulate.

Axiom 1 (Reality and information flow)**.** *The degree of reality of an observable A is altered in the time interval $[t_1, t_2]$ only when an amount $\Delta I_{\mathcal{E}|\mathcal{S}}(t_2, t_1)$ of information about this observable is shared with the environment, that is,*

$$\Delta\mathfrak{R}_A(t_2, t_1) \equiv \Delta I_{\mathcal{E}|\mathcal{S}}(t_2, t_1). \tag{5.3}$$

The specific mathematical structures of $\mathfrak{R}_A$ and $I_{\mathcal{E}|\mathcal{S}}$ and the sense in which information leaks into the environment will be opportunely specified for each information theory we consider. By now, the crux is realizing that this axiom implements the fundamental premise of quantum Darwinism and BA's realism, namely, that reality varies with time only through a physical process involving interactions, the establishment of correlations, and some form of information exchange. Also, the relation (5.3) attaches an informational profile to the quantifier $\mathfrak{R}_A$. Although this choice is somewhat *ad hoc* (after all, one could use, for instance, norm-based "metrics") it is very convenient for the establishment of conceptual bridges with well-known information theoretic quantities.

Our second axiom aims at making explicit reference to measurements, another fundamental process through which an element of reality emerges. In a sense, this axiom is related to the first one in that a measurement can be viewed as a process whereby information about an observable is shared with an apparatus. On the other hand, a measurement is a very special instance involving, at the last stage, updating of information in the observer's mind, a physical system whose informational dynamics is often excluded from the theoretical description. For this reason, the quantum state collapse is generally used as an effective description for the measurement process. Let us consider a nonselective measurement of a nondegenerate discrete-spectrum observable A as described by the map Φ_A [see Eq. (1.116)]. In BA's approach, the relation $\Phi_A(\rho) = \rho$ is taken as an operational criterion of realism, since measuring A and not revealing the outcomes (operations implied by Φ_A) do not change the state of affairs, thus implying that ρ is already a state for which A is an element of reality. In this circumstance, ρ is termed an A-reality state. We can also consider a monitoring of A [40], a generalized version of the unrevealed projective measurement (1.116) that is able to interpolate weak and strong measurements through the strength parameter $\epsilon \in [0,1]$. Formally, the monitoring of A is written as

$$\mathcal{M}_A^{\epsilon}(\rho) := (1-\epsilon)\,\rho + \epsilon\,\Phi_A(\rho). \tag{5.4}$$

Implementing a POVM with effects $\{\sqrt{1-\epsilon}\mathbb{1}, \sqrt{\epsilon}A_i\}$ (see Example 4), this map is expected to increase the reality of A whenever $\epsilon > 0$. The second axiom then follows.

Axiom 2 (Reality and measurements)**.** *The reality $\mathfrak{R}_A(\rho)$ is a non-negative real number bounded from above by $\mathfrak{R}_{\mathcal{A}}^{\max}$. It is maximum iff ρ is an A-reality state and never decreases upon generalized measurements of A, that is,*

$$0 \leqslant \mathfrak{R}_A(\rho) \leqslant \mathfrak{R}_A\left(\mathcal{M}_A^{\epsilon}(\rho)\right) \leqslant \mathfrak{R}_A\left(\Phi_A(\rho)\right) := \mathfrak{R}_{\mathcal{A}}^{\max}, \tag{5.5}$$

where the second and third equalities hold iff $\Phi_A(\rho) = \rho$.

Given the informational nature of the reality quantifier $\mathfrak{R}_A$ and because the maximum amount of information a given Hilbert space can codify is bounded by its dimension, the upper bound $\mathfrak{R}_{\mathcal{A}}^{\max}$ is expected to depend on $d_{\mathcal{A}} = \dim(\mathcal{H}_{\mathcal{A}})$. Note that while Axiom 1 specifies the measure unity by which reality is quantified, Axiom 2 establishes a numerical scale. The intrinsic relation between these axioms can be appreciated in terms of the dynamics imposed on the initial state $\rho \otimes |e_0\rangle\langle e_0|$ by a given unitary operator U_t^{ϵ} acting on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$. By use of the Theorem 2 (Stinespring dilation), we have

$$\rho_t = \mathrm{Tr}_{\mathcal{E}}\left[U_t^{\epsilon}\left(\rho \otimes |e_0\rangle\langle e_0|\right)U_t^{\epsilon\dagger}\right] = \mathcal{M}_A^{\epsilon}(\rho), \tag{5.6}$$

where $\rho \in \mathfrak{B}(\mathcal{H}_\mathcal{S})$ and $U^\epsilon_{t=0} = \mathbb{1}_{\mathcal{S}\mathcal{E}}$. The action of $\mathcal{M}^\epsilon_A$ generally changes the purity degree of ρ, so that some correlations with the environment and corresponding alterations in $I_{\mathcal{E}|\mathcal{S}}$ are expected to occur, in agreement with the prescription (5.3).

To state our third axiom, we appeal to the intuition that the reality status of a physical quantity should not decrease upon discard or addition of uncorrelated degrees of freedom. On the other hand, we cannot exclude the possibility of increasing the realism of a quantity when the discarded system is correlated because in this case the system state undergoes an effective decoherence process, and hence, is shifted toward classical reality.

Axiom 3 (Role of other parts)**.** ***(a)*** *Discarding a part of the system does not diminishes reality, that is,*

$$\mathfrak{R}_A\left(\mathrm{Tr}_\mathcal{X}(\rho)\right) \geqslant \mathfrak{R}_A(\rho), \tag{5.7a}$$

for $\mathcal{H}_\mathcal{X} \subseteq \mathcal{H}_\mathcal{B}$, where the equality applies when the discarded part is uncorrelated. Also, ***(b)*** *adding a fully uncorrelated system $\mathcal{Z}$ can by no means change the elements of reality of the system $\mathcal{S}$, that is*

$$\mathfrak{R}_A(\rho \otimes \Omega) = \mathfrak{R}_A(\rho), \tag{5.7b}$$

where $\Omega \in \mathfrak{B}(\mathcal{H}_\mathcal{Z})$.

From a mathematical viewpoint, we can recognize by Axioms 2 and 3 a set of maps, henceforth called *realistic operations*, that do not diminish the reality of an observable. Formally,

Definition 4. A *realistic operation* is a map $\rho \mapsto \Gamma(\rho)$ such that $\mathfrak{R}_A\left(\Gamma(\rho)\right) \geqslant \mathfrak{R}_A(\rho)$.

For the above axioms, we have identified a particular set of realistic operations, that is, $\Gamma \in \{\mathcal{M}^\epsilon_A, \mathrm{Tr}_\mathcal{X}, \otimes\,\Omega\}$.

With our fourth axiom we make a clear departure from classical reality. The point consists of implementing the intuition according to which, for a generic preparation ρ, noncommuting observables, as for instance three orthogonal spin components, cannot be simultaneous elements of reality. In other words, quantum realism is expected to be upper bounded.

Axiom 4 (Uncertainty relation)**.** *Two observables X and Y acting on $\mathcal{H}_\mathcal{A}$ cannot be simultaneous elements of reality in general, that is,*

$$\mathfrak{R}_X(\rho) + \mathfrak{R}_Y(\rho) \leqslant 2\mathfrak{R}^{\max}_\mathcal{A}. \tag{5.8}$$

The equality is expected to hold only in "classical-like" circumstances, such as $\rho = (\mathbb{1}/d_\mathcal{A}) \otimes \rho_\mathcal{B}$ or $[X, Y] = 0$. The above statement links quantum realism to Bohr's complementarity principle. Interestingly, a recent experiment conducted in a nuclear magnetic resonance platform [43] has been reported corroborating the validity of the uncertainty relation (5.8) within the information theory induced by the von Neumann entropy.

Let us consider now a collection of quantum states $\rho_i \in \mathcal{H}_S$ with associated probabilities p_i and realities $\Re_A(\rho_i)$. We do not expect the simple combination of these individual members to generate an ensemble with a lower reality status. In fact, mixing typically is an action toward classicality, so that reality is expected to be a concave functional. The fifth axiom is then stated as follows.

Axiom 5 (Mixing). *The reality of a mixture $\{p_i, \rho_i\}$ of density operators ρ_i with respective weights p_i can never decrease the installed mean reality, that is,*

$$\Re_A\left(\sum_i p_i\rho_i\right) \geqslant \sum_i p_i \Re_A(\rho_i). \tag{5.9}$$

So far, we have presented the properties that we consider sufficient to define a meaningful *reality monotone*, in the sense that, upon the processes described above, reality never decreases[1]. That is, the typical move is toward classical reality, not the opposite. Although this set of axioms is rather constraining, we shall see in the next chapter that it can be satisfied by a number of quantifiers supported not only by the standard von Neumann information theory but also by the Rényi and the Tsallis ones. This justifies the following definition.

Definition 5. A functional $\rho \mapsto \Re_A(\rho)$ satisfying Axioms 1-5 is called a *reality monotone.*

5.3 AXIOMS FOR REALITY MEASURES

In what follows we introduce two supplementary properties that can arguably be viewed as natural requirements for a reality measure.

Axiom 6 (Additivity). *The reality is an additive quantity over n independent systems each one prepared in a state ρ_i, that is,*

$$\Re_A\left(\bigotimes_{i=1}^{n} \rho_i\right) = \sum_{i=1}^{n} \Re_A(\rho_i), \tag{5.10}$$

where A, on the left-hand side, acts on each one of the n systems.

In particular, this means that given n independent (eventually far apart) systems prepared in the same state ρ, the total amount of reality of an observable A that acts on each ρ is nothing but the direct sum $n\Re_A(\rho)$.

Axiom 7 (Flagging). *The mean reality of an ensemble $\{p_i, \rho_i\}$ does not change under flagging, that is,*

$$\Re_A\left(\sum_i p_i\rho_i \otimes |x_i\rangle\langle x_i|\right) = \sum_i p_i \Re_A(\rho_i). \tag{5.11}$$

[1] With respect to Axiom 1, we are of course envisaging dynamics whereby correlations typically build up so that $\Delta I_{\mathcal{E}|\mathcal{S}}(t_2, t_1) \geqslant 0$. This is particularly true when the environment $\mathcal{E}$ is a genuine reservoir, like a thermal bath.

The flagging property [183] has recently been discussed within the context of quantum resource theories. Suppose one identifies with a flag $|x_i\rangle \in \mathcal{H}_{\mathcal{X}}$ each one of the states $\rho_i \in \mathcal{H}_{\mathcal{S}}$ of our collection. The above axiom reflects the fact that merely labelling each element of the ensemble with a flag basis $\{|x_i\rangle\}$ should not increase the mean reality. In other words, the insertion of classical correlations with respect to the flag is innocuous on average.

With the above axioms we have set the grounds to define what we propose to be a significant reality quantifier.

Definition 6. A functional $\rho \mapsto \mathfrak{R}_A(\rho)$ satisfying Axioms 1-7 is called a *reality measure.*

5.4 TAKEAWAY MESSAGE

- We have proposed physically motivated axioms for reality measures and monotones;
- Reality measures require more axioms than reality monotones, but the latter still carries minimum physical meaning;
- If we forget the result of a measurement, the reality status of such observable must not change. That is why non-selective measurements are used to define BA's realism condition, *i.e.*, $\Phi_A(\rho) = \rho$;
- The main axiom connects the reality increase with the gain of conditional information regarding the system by the environment. QD, by the way, requires the extra condition of a very big environment.
- It is not yet clear whether our list of axioms is strictly necessary or sufficient. What we present here is a kind of catalog of the properties that are and are not satisfied by this or that quantifier.

6 CASE STUDIES

Now that we have a list of axioms for quantum realism at hand, let us explore some celebrated quantum information theories in search of quantum reality quantifiers. We start with those induced by the von Neumann, Rényi, and Tsallis entropies. Then, we will propose geometric measures of quantum conditional information through the L_p, Bures, and Hellinger distances. We will see that some reality monotones can also be obtained from them.

6.1 ENTROPIC QUANTIFIERS

Let us start with the measures of conditional information $I_{\mathcal{E}|\mathcal{S}}$ that are induced by quantum divergences, namely, von Neumann relative entropy, Réniy divergences, and Tsallis relative entropies.

6.1.1 von Neumann reality measure

Through the relation $\mathfrak{R}_A(\rho_t) - \mathfrak{R}_A(\rho) = I_{\mathcal{E}|\mathcal{S}}(\upsilon_t) - I_{\mathcal{E}|\mathcal{S}}(\upsilon_0)$, Axiom 1 links the emergence of realism in the system $\mathcal{S}$ with the acquisition of information by the environment $\mathcal{E}$. Our strategy here consists of starting with the uncorrelated state $\upsilon_0 = \rho \otimes |e_0\rangle\langle e_0|$ and searching for a dynamics that yields maximum reality for A. That is, we want to find a reduced state $\rho_t = \mathrm{Tr}_{\mathcal{E}}(\upsilon_t) = \Phi_A(\rho)$ such that $\mathfrak{R}(\rho_t) = \mathfrak{R}^{\max}_{\mathcal{A}}$ as per Axiom 2, so that we can construct the A-reality measure $\mathfrak{R}_A(\rho) = \mathfrak{R}^{\max}_{\mathcal{A}} - \Delta I_{\mathcal{E}|\mathcal{S}}(t,0)$. From the additivity [Eq. (4.42)] of the von Neumann conditional information [Eq. (4.54)] we find

$$I_{\mathcal{E}|\mathcal{S}}(\upsilon_0) = D\left(\rho \otimes |e_0\rangle\langle e_0| \,\middle\|\, \rho \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}\right) = \ln d_{\mathcal{E}}. \tag{6.1}$$

Because there are no correlations in the initial state, the informational content of the environment is not conditioned to the system.

Now we consider a dynamics induced by a unitary operator U_t satisfying Eq. (5.6) with $\epsilon = 1$. It is worth noting that, by using the state $\upsilon_t = U_t\left(\rho \otimes |e_0\rangle\langle e_0|\right)U_t^\dagger$ in (4.55), we can compute the variation of $I_{\mathcal{E}|\mathcal{S}}(\upsilon_t) = I\left(\mathrm{Tr}_{\mathcal{S}}\upsilon_t\right) + I_{\mathcal{E}:\mathcal{S}}(\upsilon_t)$ in the interval $[0,t]$ and then return to Axiom 1 to better specify the notion of "information flow". The condition for the A-reality increase, $\Delta I_{\mathcal{E}|\mathcal{S}}(t,0) > 0$, will be satisfied when $I_{\mathcal{E}:\mathcal{S}}(\upsilon_t) > S\left(\mathrm{Tr}_{\mathcal{S}}(\upsilon_t)\right)$, which means that the share of information (correlations) between system and environment has to be sufficiently large for the emergence of reality. An alternative way of appreciating the role of the information flow for the emergence of realism is by writing $I(\upsilon_t) = I\big(\mathrm{Tr}_{\mathcal{E}}\upsilon_t\big) + I_{\mathcal{E}|\mathcal{S}}(\upsilon_t)$ and then noticing that $I(\upsilon_t)$ is conserved in any unitary dynamics. It readily follows that $\Delta I_{\mathcal{E}|\mathcal{S}} = I\big(\mathrm{Tr}_{\mathcal{E}}\upsilon_0\big) - I\big(\mathrm{Tr}_{\mathcal{E}}\upsilon_t\big) \equiv -\Delta I_{\mathcal{S}}$. By Axiom 1 we then have $\Delta\mathfrak{R}_A = -\Delta I_{\mathcal{S}}$, which shows that the A-reality increases whenever information "flows out of the system". For a more detailed discussion, see Appendix A. Without further ado, let us get back to the divergence. Since $\mathrm{Tr}_{\mathcal{E}}(v_t) = \Phi_A(\rho)$, we have $I_{\mathcal{E}|\mathcal{S}}(v_t) = D\left(U_t v_0 U_t^\dagger || \Phi_A(\rho) \otimes \mathbb{1}_{\mathcal{E}}/d_{\mathcal{E}}\right)$. To evaluate this quantity, let us use the following result (whose proof can be found in Appendix B):

Theorem 7. *Let the unitary evolution U_t be defined by the Stinespring dilation theorem* (5.6) *with $\epsilon = 1$. It follows that U_t commutes with $\Phi_A(\rho) \otimes \mathbb{1}_{\mathcal{E}}/d_{\mathcal{E}}$, that is,*

$$U_t \left(\Phi_A(\rho) \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}\right) U_t^\dagger = \Phi_A(\rho) \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}. \tag{6.2}$$

The above theorem means that $\Phi_A(\rho) \otimes \mathbb{1}_{\mathcal{E}}/d_{\mathcal{E}}$ does not evolve under the action of U_t. As a corollary we have $\Phi_A(\Phi_A(\rho)) = \Phi_A(\rho)$. This shows that once a state of reality is established by the conjugation of a unitary evolution and a discard, then repeating this operation is innocuous, as maximum reality cannot be further enhanced. That means that we can freely apply U_t onto $\Phi_A(\rho) \otimes \mathbb{1}_{\mathcal{E}}/d_{\mathcal{E}}$ and then use the unitary invariance of the von Neumann relative entropy to obtain $I_{\mathcal{E}|\mathcal{S}}(v_t) = D(v_0||\Phi_A(\rho) \otimes \mathbb{1}_{\mathcal{E}}/d_{\mathcal{E}})$. Using additivity again, we get

$$I_{\mathcal{E}|\mathcal{S}}(v_t) = \ln d_{\mathcal{E}} + D(\rho||\Phi_A(\rho)). \tag{6.3}$$

From Eqs. (6.1) and (6.3) we have $\Delta I_{\mathcal{E}|\mathcal{S}}(t,0) = D(\rho||\Phi_A(\rho))$ and hence $\mathfrak{R}_A(\rho) = \mathfrak{R}_{\mathcal{A}}^{\max} - D(\rho||\Phi_A(\rho))$. It is always possible to say that $D(\rho||\sigma) \leqslant \ln d_{\mathcal{A}} d_{\mathcal{B}}$, where $\rho, \sigma \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$ and $d_{\mathcal{A},\mathcal{B}} = \dim \mathcal{H}_{\mathcal{A},\mathcal{B}}$. However, Φ_A only acts over $\mathcal{H}_{\mathcal{A}}$, which means that the divergence caused by it must be at most $\ln d_{\mathcal{A}}$. In fact, this is stated by the following result (see Appendix B):

Lemma 2. *Given the reality state $\Phi_A(\rho) = \sum_i p_i A_i \otimes \rho_{\mathcal{B}|i}$, it holds that $D(\rho||\Phi_A(\rho)) \leqslant S(\Phi_A(\rho_{\mathcal{A}})) \leqslant \ln d_{\mathcal{A}}$.*

Thus, we can set $\mathfrak{R}_{\mathcal{A}}^{\max} = \ln d_{\mathcal{A}}$. This yields the reality quantifier

$$\mathfrak{R}_A(\rho) = \ln d_{\mathcal{A}} - D(\rho||\Phi_A(\rho)), \tag{6.4}$$

which is such that $\mathfrak{R}_A(\rho) \geqslant 0$ and $\mathfrak{R}_A(\Phi_A(\rho)) = \ln d_{\mathcal{A}}$, as required by Axiom 2.

A particularly interesting property of the reality quantifier (6.4) is that it allows us to formally state a complementarity relation. To see this, we can employ Lemmas 1 and 2 to demonstrate that $D(\rho||\Phi_A(\rho)) = S(\Phi_A(\rho)) - S(\rho) =: \mathfrak{I}_A(\rho)$, where $\mathfrak{I}_A(\rho)$ is the irreality (indefinite reality) of the observable A given the state ρ, as originally proposed by BA [36] and revisited in Sec. 1.3.5. We then have

$$\mathfrak{R}_A(\rho) + \mathfrak{I}_A(\rho) = \ln d_{\mathcal{A}}. \tag{6.5}$$

It becomes clear now the duality between irreality—a quantum resource *per se* [44]—and reality, which can thus be viewed as the amount of quantum resource that is destroyed when an observable is measured[1].

[1] Maybe we could say that $\mathfrak{R}_A$ is a *classical resource*.

Now we show that the quantifier (6.4) does satisfy Definition 6, which characterizes it as a reality measure. Axiom 1 was of course satisfied by construction. From DPI, we can check that any quantum channel Λ that commutes with Φ_A for every ρ, that is, $\Lambda(\Phi_A(\rho)) = \Phi_A(\Lambda(\rho))$, will never decrease the A reality. This includes monitoring maps $\mathcal{M}_A^\epsilon$ of any intensity and the discarding of parts of the system that do not include $\mathcal{A}$. The Axioms 2 and 3(a) are therefore satisfied. Along with the fact that $\Phi_A(\rho \otimes \Omega) = \Phi_A(\rho) \otimes \Omega$, additivity [Eq. (4.42)] guarantees that the quantifier (6.4) satisfies the Axiom 3(b). Therefore, we have $\mathfrak{R}_A(\Gamma(\rho)) \geqslant \mathfrak{R}_A(\rho)$, confirming that realistic operations Γ cannot make realism decrease.

That Axiom 4 is respected follows from (see Appendix B):

Lemma 3. *Consider generic observables $X, Y \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}})$ and the von Neumann reality quantifier (6.4). It follows that $\mathfrak{R}_X(\rho) + \mathfrak{R}_Y(\rho) \leqslant 2 \ln d_{\mathcal{A}}$, with equality iff $\rho = \Phi_X(\rho) = \Phi_Y(\rho)$.*

At this point, it is opportune to remark how quantum correlations influence the realism uncertainty relation (Axiom 4). From Eq. (1.121) we can see that irreality $\mathfrak{I}_A$ depends on the nonoptimized quantum discord D_A associated to the observable A. Since $D_A(\rho) \geqslant \min_A D_A(\rho) \equiv \mathcal{D}_{\mathcal{A}}(\rho)$, where $\mathcal{D}_{\mathcal{A}}$ stands for the one-sided quantum discord, one can conclude that $D_X(\rho) + D_Y(\rho) \geqslant 2\mathcal{D}_{\mathcal{A}}(\rho)$, with equality holding, for instance, for product states. Combining $\mathfrak{I}_A(\rho_{\mathcal{A}}) = D\left(\rho_{\mathcal{A}} || \Phi_A(\rho_{\mathcal{A}})\right) \geqslant 0$ with Eq. (6.5), we can verify that $\mathfrak{R}_A(\rho) \leqslant \ln d_{\mathcal{A}} - \mathcal{D}_{\mathcal{A}}(\rho)$ and

$$\mathfrak{R}_X(\rho) + \mathfrak{R}_Y(\rho) \leqslant 2\left[\ln d_{\mathcal{A}} - \mathcal{D}_{\mathcal{A}}(\rho)\right]. \tag{6.6}$$

This shows that quantum correlations, as measured by quantum discord (entanglement for pure states) forbid X and Y to be simultaneous elements of reality. Accordingly, using a nuclear magnetic resonance platform and associating X and Y with wave- and particle-like observables, researchers have recently reported on an experiment where an entangled quantum system behaves neither as a wave nor as particle [43].

The validity of Axiom 5 (mixing) comes immediately from joint convexity [Eq. (4.43)]. With respect to Axiom 6 (additivity), we should first note that

$$\mathfrak{R}_A(\rho^{\otimes n}) := \ln d_{\mathcal{A}}^n - D(\rho^{\otimes n} || \Phi_A(\rho)^{\otimes n}), \tag{6.7}$$

that is, A is presumed to act over each copy of ρ. It then follows from the identity (4.42) that $\mathfrak{R}_A(\rho^{\otimes n}) = n\mathfrak{R}_A(\rho)$.

Last but not least, to verify the validity of Axiom 7 (flagging), we start with $\mathfrak{R}_A(\rho_f) = \ln d_{\mathcal{A}} - S(\Phi_A(\rho_f)) + S(\rho_f)$ (see the proof of Lemma 2, Appendix B), with the flagged state $\rho_f = \sum_i p_i \rho_i \otimes |x_i\rangle\langle x_i|$. The joint entropy theorem yields $S(\rho_f) = H(\{p_i\}) + \sum_i p_i S(\rho_i)$, where $H(\{p_i\}) = -\sum_i p_i \ln p_i$ is the Shannon entropy of the distribution p_i [107]. Direct calculations gives $\mathfrak{R}_A(\rho_f) = \sum_i p_i \mathfrak{R}_A(\rho_i)$ with $\mathfrak{R}_A(\rho_i) = \ln d_{\mathcal{A}} - S(\Phi_A(\rho_i)) + S(\rho_i)$, which proves the point. With all that, it becomes established that the quantifier (6.4) does indeed satisfy Definition 6 and can hereafter be called a reality measure.

Referring back to Axiom 2, it is worth discussing how the reality measure (6.4) changes upon monitoring maps [Eq. (5.4)]. First, because the reality measure respects Axiom 5 (mixing), which ultimately is a statement of concavity, one can readily show that $\mathfrak{R}_A(\mathcal{M}_A^\epsilon(\rho)) \geqslant (1-\epsilon)\,\mathfrak{R}_A(\rho) + \epsilon\,\mathfrak{R}_A\,(\Phi_A(\rho))$. Then, by use of Eq. (6.5) we arrive at

$$\mathfrak{R}_A\left(\mathcal{M}_A^\epsilon(\rho)\right) - \mathfrak{R}_A(\rho) \geqslant \epsilon\,\mathfrak{I}_A(\rho). \tag{6.8}$$

This shows that a monitoring of A always increases the A reality as long as there is a nonzero amount of A irreality [40]. Second and more surprising, it turns out that the monitoring of an observable $Y \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}})$ never diminishes the reality of another observable $X \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}})$. That is stated by the following lemma:

Lemma 4. *Consider observables $X, Y \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}})$ and the von Neumann reality quantifier* (6.4)*. If $\Phi_{XY} = \Phi_{YX}$, then the monitoring of Y never decreases the reality of X, that is,*

$$\Delta := \mathfrak{R}_X(\mathcal{M}_Y^\epsilon(\rho)) - \mathfrak{R}_X(\rho) \geqslant 0, \qquad \forall \epsilon \in [0,1]. \tag{6.9}$$

This is one of the main results of Ref. [40], but the reader can find a simpler alternative proof of it based on DPI and mixing in Appendix B. Notice that inequality 6.9 generalizes Axiom 2.

Finally, it is worth noticing that the so-called *local irreality*, $\mathfrak{I}_A(\rho_{\mathcal{A}}) = S\big(\Phi_A(\rho_{\mathcal{A}})\big) - S(\rho_{\mathcal{A}})$, which relates to irreality through the formula $\mathfrak{I}_A(\rho) = \mathfrak{I}_A(\rho_{\mathcal{A}}) + D_A(\rho)$ [see Eq.(1.121)], is nothing but the measure known as *relative entropy of coherence* [184], which has been acknowledged as a quantum resource [113]. This shows that quantum irrealism is induced by both types of "quantumness", namely, quantum coherence and quantum correlations. In particular, in the absence of correlations, one has $\mathfrak{I}_A(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) = \mathfrak{I}_A(\rho_{\mathcal{A}})$, showing that coherence is sufficient to preclude classical reality. Within the coherence theory of multipartite settings, the irreality $\mathfrak{I}_A(\rho)$ turns out to be equivalent to the concept known as *quantum-incoherent relative entropy* [185]. These connections between quantum irrealism and quantum coherence measures just reinforce that $\mathfrak{I}_A(\rho)$ is a sensible quantifier of the former concept, for quantum superposition (coherence) is the fundamental mechanism responsible for the departure of the natural behavior from classical reality.

6.1.2 Rényi reality monotones

We now derive a reality quantifier based on the non-optimized conditional information (4.63a). Because this quantity and the von Neumann relative entropy share properties such as positive definiteness, unitary invariance, and additivity, we can rigidly follow the steps of the precedent section, which amounts to use Theorem 7 and Axiom 1, to directly propose the Rényi reality quantifier

$$\mathfrak{R}_A^{\alpha\downarrow}(\rho) = \ln d_{\mathcal{A}} - D_\alpha\left(\rho || \Phi_A(\rho)\right), \tag{6.10}$$

for $\alpha \in (0,1)\cup(1,+\infty)$. Since $\lim_{\alpha\to 1}\mathfrak{R}_A^{\alpha\downarrow}(\rho) = \mathfrak{R}_A(\rho)$ for any ρ and A, we have here an evident generalization of (6.4) within the Rényi quantum information theory. Inspired by the results of the previous section, we have chosen $\mathfrak{R}_{\mathcal{A}}^{\max} = \ln d_{\mathcal{A}}$ to make the quantity (6.10) always non-negative (in particular, for $\alpha \to 1$).

As we show now, the quantifier (6.10) is a reality monotone only in the restricted range $\alpha \in (0,1)$. Axioms 2 and 3(a) are satisfied whenever DPI is valid, in this case, for $\alpha \in (0,1)\cup(1,2]$. Axioms 3(b) and 4 are validated by additivity and positive definiteness, respectively. Axiom 5, however, only holds when D_α is jointly convex, that is, for $\alpha \in (0,1)$. This significantly restricts the domain wherein $\mathfrak{R}_A^\alpha$ can be termed a reality monotone. Although additivity guarantees the Axiom 6 to be respected by the monotone (6.10), there is no answer yet as to whether or not D_α satisfies flagging. Only in the affirmative case we could regard $\mathfrak{R}_A^\alpha$ as a reality measure for $\alpha \in (0,1)$.

Since the Rényi divergence is a monotonically increasing real function of α, for all $\alpha > 0$ and fixed density operators [90], the reality measure (6.10) is a monotonically decreasing real function of its parameter, meaning that

$$\mathfrak{R}_A^{\alpha\downarrow}(\rho) \geqslant \mathfrak{R}_A^{\beta\downarrow}(\rho) \tag{6.11}$$

for real non-negative numbers $\alpha \leqslant \beta$. This entails that if $\mathfrak{R}_A^{\beta\downarrow}(\rho) = \ln d_{\mathcal{A}}$ for some $\beta \geqslant 0$, meaning that $\rho = \Phi_A(\rho)$, then $\mathfrak{R}_A^{\alpha\downarrow}(\rho) = \ln d_{\mathcal{A}}$ for every $\alpha \leqslant \beta$. If, in addition, $\beta \to 1$, then all Rényi reality monotones will numerically reach the maximum $\ln d_{\mathcal{A}}$. Therefore, although Rényi reality monotones with different parameters α disagree in value when applied to non-real observables (those for which $\mathfrak{R}_A^{\alpha\downarrow}(\rho) < \ln d_{\mathcal{A}}$), they do always agree about states of reality (see Example 13 and respective Fig. 25 below).

One of the consequences of the positive definiteness property—which does not hold when we use the min-relative entropy—is that $\mathfrak{R}_A^{\alpha\downarrow}(\rho) = 0$ if and only if $\rho = \sum_i p_i A_i \otimes \rho_{\mathcal{B}|i} = \Phi_A(\rho)$, which is a classical-quantum state with zero one-sided quantum discord. This means that the lack of quantum correlations is a condition necessary for the occurrence of at least one element of reality. On the other hand, classical reality manifests itself for the preparation $\rho = (\mathbb{1}_{\mathcal{A}}/d_{\mathcal{A}}) \otimes \rho_{\mathcal{B}}$, since in this case we have $\mathfrak{R}_A^{\alpha\downarrow}(\rho) = \ln d_{\mathcal{A}}$ for any A.

Next, we present some case studies.

Example 12. Let $\rho_\epsilon \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$ be the Werner state

$$\rho_\epsilon = (1-\epsilon)\frac{\mathbb{1}}{4} + \epsilon\,\psi_s, \tag{6.12}$$

where $\epsilon \in [0,1]$, $\psi_s = |\psi_s\rangle\langle\psi_s|$, and $|\psi_s\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is the singlet state. To assess the reality degree of the spin observable $A = \hat{u}\cdot\vec{\sigma}$ acting on $\mathcal{H}_{\mathcal{A}}$, with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ being the Pauli vector, we take the projectors $A_\pm = (\mathbb{1}_{\mathcal{A}} \pm \hat{u}\cdot\vec{\sigma})/2$ with $\hat{u} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ and then compute the A-reality state $\Phi_A(\rho_\epsilon) = A_+\rho_\epsilon A_+ + A_-\rho_\epsilon A_-$. Because ρ_ϵ is rotationally

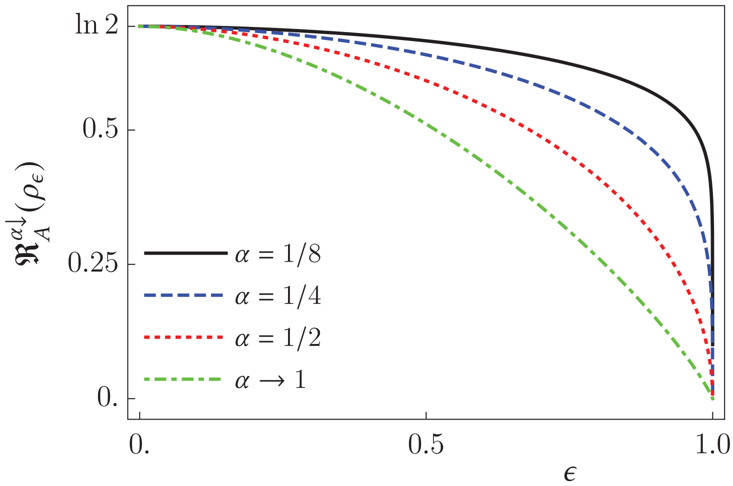

Figure 24 – Reality measure (dash-dotted green line) and Rényi monotones $\mathfrak{R}_A^{\alpha\downarrow}(\rho_\epsilon)$ for any spin observable A of the first qubit of a Werner state [Eq. (6.12)] as a function of the purity parameter ϵ (as introduced in Example 12) for: $\alpha = 1/8$ (solid black line), $\alpha = 1/4$ (dashed blue line), $\alpha = 1/2$ (dotted red line), and $\alpha \to 1$ (dash-dotted green line).

invariant, it commutes with $\Phi_A(\rho_\epsilon)$ and we have $\widetilde{D}_\alpha(\rho_\epsilon||\Phi_A(\rho_\epsilon)) = D_\alpha(\rho_\epsilon||\Phi_A(\rho_\epsilon))$ for $\alpha \in (0,1) \cup (1,+\infty)$. Therefore, both the original and the sandwiched Rényi divergences can be used within the range $\alpha \in (0,1)$ to provide a reality monotone for the Werner state. By direct calculation of $\mathfrak{R}_A^{\alpha\downarrow}(\rho_\epsilon) = \ln 2 - D_\alpha(\rho_\epsilon||\Phi_A(\rho_\epsilon))$ we find

$$\mathfrak{R}_A^{\alpha\downarrow}(\rho_\epsilon) = \ln 2 - \ln\left[\frac{(1-\epsilon)^\alpha + (1+3\epsilon)^\alpha}{4(1+\epsilon)^{\alpha-1}} + \frac{1-\epsilon}{2}\right]^{\frac{1}{\alpha-1}}. \tag{6.13}$$

Note that the special cases

$$\lim_{\alpha\to 0} \mathfrak{R}_A^{\alpha\downarrow}(\rho_\epsilon) = \begin{cases} \ln 2 & \text{if } \epsilon \in [0,1)\,, \\ 0 & \text{if } \epsilon = 1, \end{cases} \tag{6.14a}$$

$$\lim_{\alpha\to +\infty} \mathfrak{R}_A^{\alpha\downarrow}(\rho_\epsilon) = \ln 2 - \ln\left(\frac{1+3\epsilon}{1+\epsilon}\right), \tag{6.14b}$$

do not constitute reality monotones, since they do not satisfy Axioms 2 and 5, respectively. See Appendix C.2.1 for the technical details on how to calculate the Rényi divergences when $\alpha \to 0$ and $+\infty$. As we can see in Fig. 24, the monotonicity property (6.11) is verified. Also, since the Rényi monotone is concave (due to the mixing axiom) and $\mathfrak{R}_A^{\alpha\downarrow}(\psi_s) = 0$, then $\mathfrak{R}_A^{\alpha\downarrow}(\rho_\epsilon) \geqslant (1-\epsilon)\ln 2$. This result manifests itself in Fig. 24 through the concavity of the curves, a feature that is not respected by the convex function (6.14b). ▲

Example 13. Let us consider now the one-parameter two-qubit state $\rho_\mu \in \mathfrak{B}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})$ defined as

$$\rho_\mu = \frac{\mathbb{1}}{4} + \frac{\mu}{4}\left(\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y\right) + \frac{2\mu - 1}{4}\sigma_z \otimes \sigma_z, \tag{6.15}$$

where $\mu \in [0,1]$. This state is such that $\rho_{\mu=1} = |\varphi\rangle\langle\varphi|$, with $|\varphi\rangle \equiv (|00\rangle + |11\rangle)/\sqrt{2}$, and $\rho_{\mu=0} = (|01\rangle\langle 01| + |10\rangle\langle 10|)/2$. Unlike the previous example, for the observable $A = \hat{u}\cdot\vec{\sigma}$ it

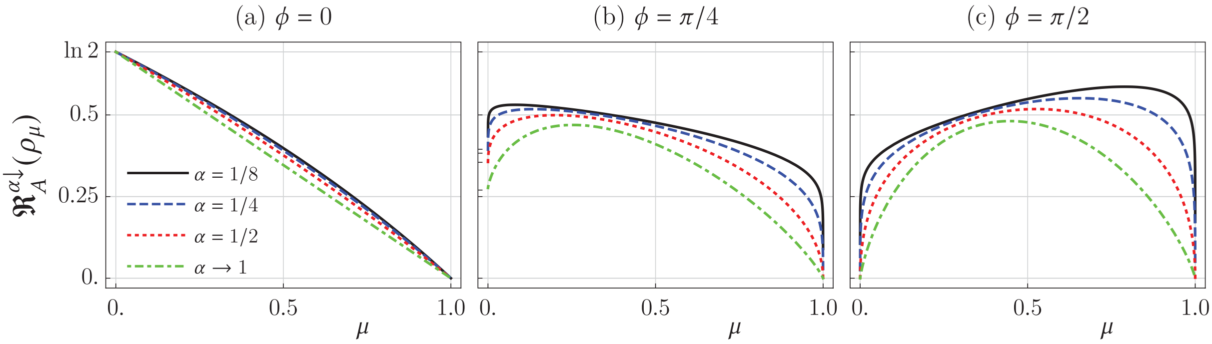

Figure 25 – Reality monotones $\mathfrak{R}_A^{\alpha\downarrow}(\rho_\mu)$ of the spin observable $A = \hat{u}\cdot\vec{\sigma}$, where $\hat{u} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\theta)$, regarding the first qubit of the ρ_μ state (6.15) as a function of μ as introduced in Example 13 for any θ and (a) $\phi = 0$, (b) $\phi = \pi/4$, and (c) $\phi = \pi/2$ and for (from top to bottom): $\alpha = 1/8$ (solid black line), $\alpha = 1/4$ (dashed blue line), $\alpha = 1/2$ (dotted red line), and $\alpha \to 1$ (dash-dotted green line).

follows that ρ_μ does not always commute with $\Phi_A(\rho_\mu)$. Indeed, our calculations show that $\mathfrak{R}_A^{\alpha\downarrow}(\rho_\mu)$ (whose lengthy and non-enlightening expression will be omitted) depends on the polar angle ϕ, as shown in Fig. 25. Again, the monotonicity relation (6.11) makes itself clear. Also, numerical simulations show that in this case a reality monotone based on $\widetilde{D}_\alpha$ for $\alpha \in [1/2, 1)$ behaves very similarly to what is presented in Fig. 25. ▲

Now we turn our attention to the optimized version (4.63b) of the Rényi conditional information. Here the derivation of a reality monotone becomes subtler because the optimization process does not keep a unique and straightforward connection with the self-contained dynamical scenario prescribed by Axiom 1. Still, some potential candidates can be proposed. Let us consider again the initial state v_0. Plugged into Eq. (4.63b), it yields

$$I^{\alpha\downarrow}_{\mathcal{E}|\mathcal{S}}(v_0) = \inf_{\sigma_\mathcal{S}} D_\alpha\left(\rho\otimes|e_0\rangle\langle e_0|\,\middle\|\,\sigma_\mathcal{S}\otimes\frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right) = \ln d_\mathcal{E}, \tag{6.16}$$

where we have used additivity and $\inf_{\sigma_\mathcal{S}} D_\alpha(\rho||\sigma_\mathcal{S}) = 0$. By applying U_t, we find

$$I^{\alpha\downarrow}_{\mathcal{E}|\mathcal{S}}(v_t) = \inf_{\sigma_\mathcal{S}} D_\alpha\left(v_t\,\middle\|\,\sigma_\mathcal{S}\otimes\frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right). \tag{6.17}$$

As before, we assume that U_t is such that $\rho_t = \mathrm{Tr}_\mathcal{E}(v_t) = \Phi_A(\rho)$. With that, we obtain $\mathfrak{R}_A^{\alpha\uparrow}(\rho_t) = \mathfrak{R}_\mathcal{A}^{\max} = \ln d_\mathcal{A}$ and, via Axiom 1,

$$\mathfrak{R}_A^{\alpha\uparrow}(\rho) = \ln d - \inf_{\sigma_\mathcal{S}} D_\alpha\left(v_t\,\middle\|\,\sigma_\mathcal{S}\otimes\frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), \tag{6.18}$$

where $v_t = U_t\left(\rho\otimes|e_0\rangle\langle e_0|\right)U_t^\dagger$, $d = d_\mathcal{A}d_\mathcal{E}$, and $d_\mathcal{E} = d_\mathcal{A}$. That this quantifier is indeed a reality monotone for $\alpha\in(0,1)$ (when $\mathfrak{R}_A^{\alpha\downarrow}$ is too) is demonstrated by Proposition 1 in Appendix B. A disadvantage of $\mathfrak{R}_A^{\alpha\uparrow}$ in comparison with $\mathfrak{R}_A^{\alpha\downarrow}$ is the presence of the environment state $|e_0\rangle$ and the observable-dependent unitary operator U_t, whose formal structure is provided in the proof of Theorem 7 (Appendix B). What is more, the optimization process may be impracticable,

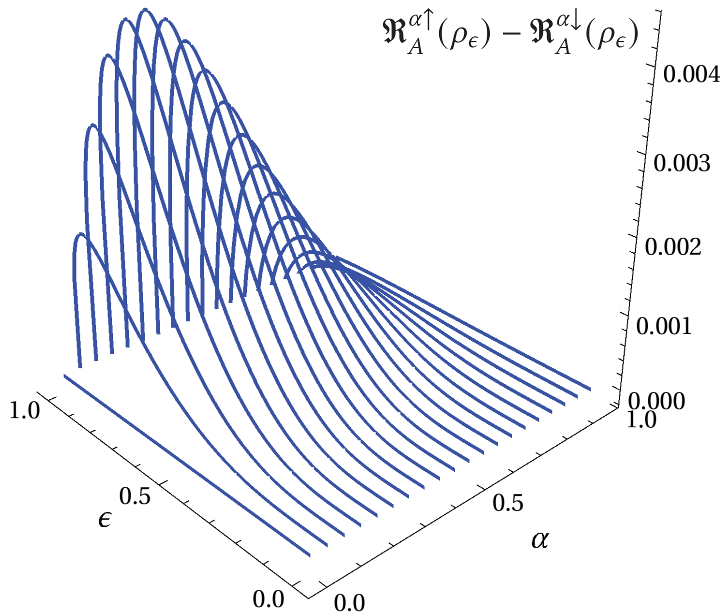

Figure 26 – Differences $\mathfrak{R}_A^{\uparrow\alpha}(\rho_\epsilon) - \mathfrak{R}_A^{\downarrow\alpha}(\rho_\epsilon)$ for the Werner state (6.12) as a function of the purity parameter $\epsilon \in (0,1)$ and $\alpha \in (0,1)$. The maximum difference, ~ 0.0044, is reached when $(\alpha, \epsilon) \sim (0.24, 0.89)$.

specially if the sandwiched Rényi divergence $\widetilde{D}_\alpha$ is used instead of D_α. Interestingly, however, by use of the quantum Sibson identity (see supplemental material of Ref. [186]), we obtain the closed form

$$\inf_{\sigma_{\mathcal{B}}} D_\alpha(\rho||\mathbb{1}_{\mathcal{A}} \otimes \sigma_{\mathcal{B}}) = \frac{\alpha}{\alpha - 1} \ln \operatorname{Tr}_{\mathcal{B}} \left\{ \left[\operatorname{Tr}_{\mathcal{A}} (\rho^\alpha)\right]^{1/\alpha} \right\}, \tag{6.19}$$

which allows us to simplify Eq. (6.18) as

$$\mathfrak{R}_A^{\alpha\uparrow}(\rho) = \ln d_{\mathcal{A}} - \frac{\alpha}{\alpha - 1} \ln \operatorname{Tr}_{\mathcal{A}} \left\{ \left[\operatorname{Tr}_{\mathcal{E}} \left(v_t^\alpha\right)\right]^{1/\alpha} \right\}, \tag{6.20}$$

for $\alpha \in (0,1)$. An example is opportune.

Example 14. Computing the monotone (6.20) for the Werner state (6.12) yields the result $\mathfrak{R}_A^{\alpha\uparrow}(\rho_\epsilon) = \ln 2 - \frac{\alpha}{\alpha-1} \ln \chi$, where

$$\chi = \frac{1-\epsilon}{2} + \left[\frac{(1+\epsilon)^\alpha + (1+3\epsilon)^\alpha}{2^{\alpha+1}}\right]^{1/\alpha} \tag{6.21}$$

and $\alpha \in (0,1)$. Figure 26 illustrates the slight differences between the monotones (6.20) and (6.10), which always respect $\mathfrak{R}_A^{\alpha\uparrow}(\rho_\epsilon) \geqslant \mathfrak{R}_A^{\alpha\downarrow}(\rho_\epsilon)$, as expected. ▲

Roughly speaking, the divergences in Eq. (6.18) evaluate the "minimum distance" (according to an "entropic metric") between the time evolved state v_t and the product $\sigma_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{E}}/d_{\mathcal{E}}$. One might argue, however, that it would be more reasonable to run the optimization, at every instant of time, within a set more closely related with the reduced state $\operatorname{Tr}_{\mathcal{E}}(v_t)$, which is strictly confined to the dynamics imposed by U_t. Adhering to this rationale, we start over by proposing the following adaptation in the optimized conditional information (4.63b):

$$I^{\alpha\downarrow}_{\mathcal{E}|\mathcal{S}}(U_t \upsilon_0 U_t^\dagger) = \inf_{\sigma_\mathcal{S}} D_\alpha \left(U_t \upsilon_0 U_t^\dagger \middle\| \mathrm{Tr}_\mathcal{E} \left(U_t \eta_0 U_t^\dagger \right) \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}} \right). \tag{6.22}$$

where $\eta_0 = \sigma_\mathcal{S} \otimes |e_0\rangle\langle e_0|$ and $\sigma_\mathcal{S} \in \mathfrak{B}(\mathcal{H}_\mathcal{S})$. For $t = 0$ we see that no significant change is implied to the original definition. By use of additivity and $\upsilon_0 = \rho \otimes |e_0\rangle\langle e_0|$, we easily obtain $I^{\alpha\downarrow}_{\mathcal{E}|\mathcal{S}}(\upsilon_0) = \ln d_\mathcal{E}$. Nevertheless, for $t > 0$ the optimization runs only over the initial state $\sigma_\mathcal{S}$. This preserves the dynamics imposed by U_t and avoids any artificial freedom that would otherwise be tested throughout the minimization process. Noticing that $\mathrm{Tr}_\mathcal{E}\left(U_t \eta_0 U_t^\dagger\right) = \Phi_A(\sigma_\mathcal{S})$, we can employ Theorem 7, unitary invariance, and additivity to show that $I^{\alpha\downarrow}_{\mathcal{E}|\mathcal{S}}(\upsilon_t) = \ln d_\mathcal{E} + \inf_{\sigma_\mathcal{S}} D_\alpha\left(\rho || \Phi_A(\sigma_\mathcal{S})\right)$. Employing Axiom 1 with $\bar{\mathfrak{R}}^\alpha_A(\rho_t) = \ln d_\mathcal{A}$ gives

$$\bar{\mathfrak{R}}^\alpha_A(\rho) = \ln d_\mathcal{A} - \inf_{\sigma_\mathcal{S}} D_\alpha\left(\rho || \Phi_A(\sigma_\mathcal{S})\right). \tag{6.23}$$

This is exactly the result we would obtain by restricting the optimization in Eq. (6.18) to the set of A-reality states, that is, the one constituted by states satisfying $\sigma_\mathcal{S} = \Phi_A(\sigma_\mathcal{S})$.

From the discussion conducted up until now, one can conclude that

$$\mathfrak{R}^{\alpha\downarrow}_A(\rho) \leqslant \bar{\mathfrak{R}}^\alpha_A(\rho) \leqslant \mathfrak{R}^{\alpha\uparrow}_A(\rho). \tag{6.24}$$

This is not to say, however, that we have mathematical evidence that $\bar{\mathfrak{R}}^\alpha_A$ and $\mathfrak{R}^{\alpha\downarrow}_A$ are distinct quantities. On the contrary, we do have evidence that they are equal when $\alpha \to 1$. To show this, we use Lemma 1 to obtain $D\left(\rho || \Phi_A(\sigma_\mathcal{S})\right) = D\left(\rho || \Phi_A(\rho)\right) + D\left(\Phi_A(\rho) || \Phi_A(\sigma_\mathcal{S})\right)$, which gives $\inf_{\sigma_\mathcal{S}} D\left(\rho || \Phi_A(\sigma_\mathcal{S})\right) = D\left(\rho || \Phi_A(\rho)\right)$. This readily implies that, $\bar{\mathfrak{R}}^\alpha_A = \mathfrak{R}^{\alpha\downarrow}_A$ as $\alpha \to 1$. Although we expect for the definitive solution to this problem, we can safely announce the Rényi reality monotone $\mathfrak{R}^\alpha_A(\rho)$ in the form

$$\mathfrak{R}^\alpha_A \in \{\mathfrak{R}^{\alpha\downarrow}_A, \bar{\mathfrak{R}}^\alpha_A, \mathfrak{R}^{\alpha\uparrow}_A\} \tag{6.25}$$

for $\alpha \in (0,1)$, with their respective formulas (6.10), (6.23), and (6.18), and $\mathfrak{R}^{\alpha\downarrow}_A$ thus being a lower bound for the Rényi A-reality.

Remark 2. Very similar arguments can be made toward the establishment of reality quantifiers such as $\mathfrak{R}^{\min}_A$, $\widetilde{\mathfrak{R}}_A$, and $\mathfrak{R}^{\max}_A$. Operationally, they can be directly obtained through the replacement of D_α in Eq. (6.10) by the respective divergences $D_{\min} := \lim_{\alpha\to 0} D_\alpha$, $\widetilde{D}_\alpha$ (the sandwiched Rényi divergence), and $D_{\max} := \lim_{\alpha\to+\infty} \widetilde{D}_\alpha$. The reader can find in Table 4 a summary of the properties that are satisfied by these divergences and, in Table 5, the axioms respected by the corresponding reality quantifiers. It turns out, though, that only $\widetilde{\mathfrak{R}}_A$ works as a reality monotone for some values of α.

6.1.3 Tsallis reality monotones

Unlike the Rényi divergence (4.45), the Tsallis relative entropy (4.47) is not additive on its entries. Yet, we show now that it is still possible to construct a reality monotone in this information theory. Aiming at accounting for Axiom 1, we employ Theorem 7, unitary invariance, and definitions (4.47) and (4.69) to demonstrate that $I^q_{\mathcal{E}|\mathcal{S}}(v_t) - I^q_{\mathcal{E}|\mathcal{S}}(v_0) = D_q(\rho||\Phi_A(\rho))$, where $v_0 = \rho \otimes |e_0\rangle\langle e_0|$. Now, we note that for $\rho = |\psi\rangle\langle\psi| \otimes \mathbb{1}_{\mathcal{B}}/d_{\mathcal{B}}$ and $\Phi_A(\psi) = \mathbb{1}/d_{\mathcal{A}}$ we find $D_q(\rho||\Phi_A(\rho)) = d_{\mathcal{A}}^{q-1} S_q(\mathbb{1}_{\mathcal{A}}/d_{\mathcal{A}})$, which also manifests the normalization issue. Selecting a unitary evolution such that $\mathfrak{R}^q_A(\rho_t) = \mathfrak{R}^q_A(\Phi_A(\rho)) = \ln_q d_{\mathcal{A}}$, we then propose the quantifier

$$\mathfrak{R}^q_A(\rho) = \ln_q d_{\mathcal{A}} - d_{\mathcal{A}}^{1-q} D_q(\rho||\Phi_A(\rho)), \tag{6.26}$$

which reduces to its von Neumann counterpart (6.4) as $q \to 1$. Table 4, along with the fact that $D_q(\rho \otimes \Omega||\sigma \otimes \Omega) = D_q(\rho||\sigma)$, shows that likewise the Rényi divergences, D_q satisfies all properties necessary for one to validate $\mathfrak{R}^q_A$ as a reality monotone in the domain $q \in (0, 2]$. On the other hand, even if flagging comes to eventually be proved for the Tsallis reality monotone, the lack of additivity already guarantees that $\mathfrak{R}^q_A$ will never be classified as a reality measure.

Check out Table 5 (next landscape-oriented page) for a summary of the axioms held by the Tsallis reality monotone (6.26) with respect to the parameter q. Next, we present a brief case study.

Example 15. Let us take again the Werner state (6.12). A lengthy but direct calculation of (6.26) yields

$$\mathfrak{R}^q_A(\rho_\epsilon) = \ln_q 2 - \frac{(1-\epsilon)^q - 2(1+\epsilon)^q + (1+3\epsilon)^q}{4(q-1)\,[2(1+\epsilon)]^{q-1}}, \tag{6.27}$$

for $q \in (0, 1) \cup (1, 2]$. As in Example 12, due to the rotational invariance of the singlet state it follows that $\mathfrak{R}^q_A(\rho_\epsilon)$ actually is observable independent. See Fig. 27 for numerical illustrations of the above formula. It can be checked that $\partial_q \mathfrak{R}^q_A(\rho_\epsilon) \leqslant 0$, meaning that $\mathfrak{R}^q_A(\rho_\epsilon) \geqslant \mathfrak{R}^p_A(\rho_\epsilon)$ for $q \leqslant p$. ▲

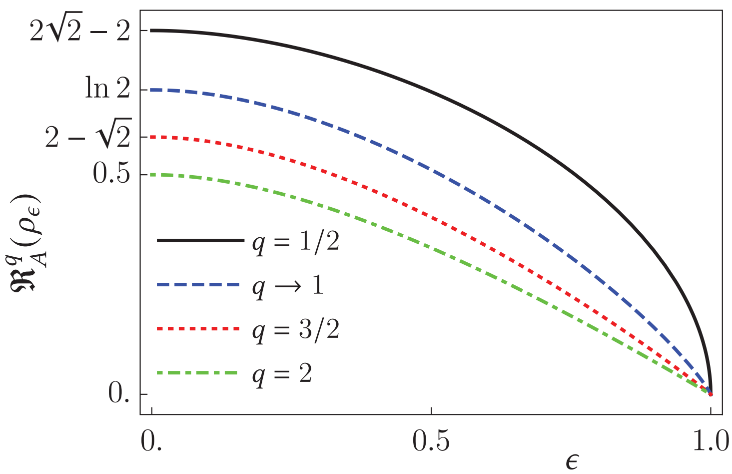

Figure 27 – Tsallis reality monotone $\mathfrak{R}^q_A(\rho_\epsilon)$ for any spin observable A of the first qubit of a Werner state (6.12) as a function of the purity parameter ϵ (as introduced in Example 12) for: $q = 1/2$ (solid black line), $q \to 1$ (dashed blue line), $q = 3/2$ (dotted red line) and $q = 2$ (dash-dotted green line).

	$\mathfrak{R}_A$	$\mathfrak{R}_A^{\alpha}$	$\mathfrak{R}_A^{\min}$	$\widetilde{\mathfrak{R}}_A^{\alpha}$	$\mathfrak{R}_A^{\max}$	$\mathfrak{R}_A^{q}$
Axiom 1 (information flow)	✔	✔	✔	✔	✔	✔
Axiom 2 (measurements)	✔	✔	✘	✔	✔	✔
Axiom 3(a) (part discard)	✔	$\alpha \in (0,1) \cup (1,2]$	✔	$\alpha \in [1/2,1) \cup (1,+\infty)$	✔	$q \in (0,1) \cup (1,2]$
Axiom 3(b) (uncorrelated part)	✔	✔	✔	✔	✔	✔
Axiom 4 (uncertainty relation)	✔	✔	✘	✔	✔	✔
Axiom 5 (mixing)	✔	$\alpha \in (0,1)$	✔	$\alpha \in [1/2,1)$	✘	$q \in (0,1) \cup (1,2]$
Axiom 6 (additivity)	✔	✔	✔	✔	✔	✘
Axiom 7 (flagging)	✔	?	?	?	✘	?

Table 5 – Summary of the axioms satisfied by the A-reality quantifiers $\mathfrak{R}_A$ [Eq. (6.4)], $\mathfrak{R}_A^{\alpha}$ [Eq. (6.25)], $\mathfrak{R}_A^{\min}$, $\widetilde{\mathfrak{R}}_A^{\alpha}$, $\mathfrak{R}_A^{\max}$ (see Remark 2), and $\mathfrak{R}_A^{q}$ [Eq. (6.26)] built out of their corresponding divergences D, D_α, $D_{\min}$, $\widetilde{D}_\alpha$, $D_{\max}$, and D_q, whose properties are listed in Table 4. For some specific parameter domains, our approach legitimates several reality monotones, namely, $\mathfrak{R}_A^{\alpha}$, $\widetilde{\mathfrak{R}}_A^{\alpha}$, and $\mathfrak{R}_A^{q}$, while only $\mathfrak{R}_A$ can be validated (up to now) as a reality measure.

6.2 GEOMETRIC QUANTIFIERS

Let us define a geometric measure of conditional information by

$$I^{\square}_{\mathcal{A}|\mathcal{B}}(\rho) := d^{n}_{\square}\left(\rho, \frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}} \otimes \rho_{\mathcal{B}}\right), \tag{6.28}$$

where $d_{\square}$ is any of the distances presented in Sec. 4.2.1 and n is some power conveniently chosen. This definition is clearly borrowed from the notion of conditional information with divergences we presented in Sec. 4.3 and inspired by the work of Roga *et al.* [111]. Immediately from the positive definiteness property of distances, we have that

$$I^{\square}_{\mathcal{A}|\mathcal{B}}(\rho) = 0 \qquad \textit{iff} \qquad \rho = \frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}} \otimes \rho_{\mathcal{B}}. \tag{6.29}$$

In addition to that, we can expect meaningful measures of conditional information from the definition (6.28) once we use distances $d^{n}_{\square}$ that respect the contractivity relation (4.29), which is nothing more than a DPI in the geometrical context. This is the case for the trace distance and the square of the Bures and the Hellinger distances.

In order to follow Axiom 1, we must obtain the conditional information of the global states $v_0 = \rho \otimes |e_0\rangle\langle e_0|$ and $v_t = U_t(v_0)U_t^{\dagger}$ by

$$I^{\square}_{\mathcal{E}|\mathcal{S}}(v_0) = d^{n}_{\square}\left(\rho \otimes |e_0\rangle\langle e_0|, \rho \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}\right), \tag{6.30a}$$

$$I^{\square}_{\mathcal{E}|\mathcal{S}}(v_t) = d^{n}_{\square}\left(\rho \otimes |e_0\rangle\langle e_0|, \Phi_A(\rho) \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}\right), \tag{6.30b}$$

where we have used Theorem 7 in the last equation. Geometric reality quantifiers can be obtained using the following recipe:

$$\mathfrak{R}^{\square}_{A}(\rho) = \mathfrak{R}^{\max}_{A}(\square) - \Delta I^{\square}_{\mathcal{E}|\mathcal{S}}(v_0, v_t), \tag{6.31}$$

where $\Delta I^{\square}_{\mathcal{E}|\mathcal{S}}(v_0, v_t) = I^{\square}_{\mathcal{E}|\mathcal{S}}(v_t) - I^{\square}_{\mathcal{E}|\mathcal{S}}(v_0)$ is the change in the geometric conditional informational content. After some algebra (see Appendix C.4), we reach the following four geometric reality quantifiers:

$$\mathfrak{R}^{\mathrm{Tr}}_{A}(\rho) = \mathfrak{R}^{\max}_{\mathcal{A}}(\mathrm{Tr}) - \left[d_{\mathrm{Tr}}\left(\rho, \frac{\Phi_A(\rho)}{d_{\mathcal{E}}}\right) - \frac{d_{\mathcal{E}} - 1}{d_{\mathcal{E}}}\right]; \tag{6.32}$$

$$\mathfrak{R}^{\mathrm{HS}}_{A}(\rho) = \mathfrak{R}^{\max}_{\mathcal{A}}(\mathrm{HS}) - \frac{1}{d_{\mathcal{E}}} d^{2}_{\mathrm{HS}}(\rho, \Phi_A(\rho)); \tag{6.33}$$

$$\mathfrak{R}^{\mathrm{Bu}}_{A}(\rho) = \mathfrak{R}^{\max}_{\mathcal{A}}(\mathrm{Bu}) - \frac{1}{\sqrt{d_{\mathcal{E}}}} d^{2}_{\mathrm{Bu}}(\rho, \Phi_A(\rho)); \tag{6.34}$$

$$\mathfrak{R}^{\mathrm{He}}_{A}(\rho) = \mathfrak{R}^{\max}_{\mathcal{A}}(\mathrm{He}) - \frac{1}{\sqrt{d_{\mathcal{E}}}} d^{2}_{\mathrm{He}}(\rho, \Phi_A(\rho)). \tag{6.35}$$

Note that $d_{\mathcal{E}} = d_{\mathcal{A}}$. We have chosen $n = 1$ for the reality quantifier based on the trace distance and $n = 2$ for the other cases. The max value $\mathfrak{R}^{\max}_{\mathcal{A}}(\square)$ can be obtained by making ρ a maximally entangled pure state in each one of the four terms $\Delta I^{\square}_{\mathcal{E}|\mathcal{S}}$. Note that, by Theorem 7, $d_{\mathcal{E}} = d_{\mathcal{A}}$. An example can be opportune:

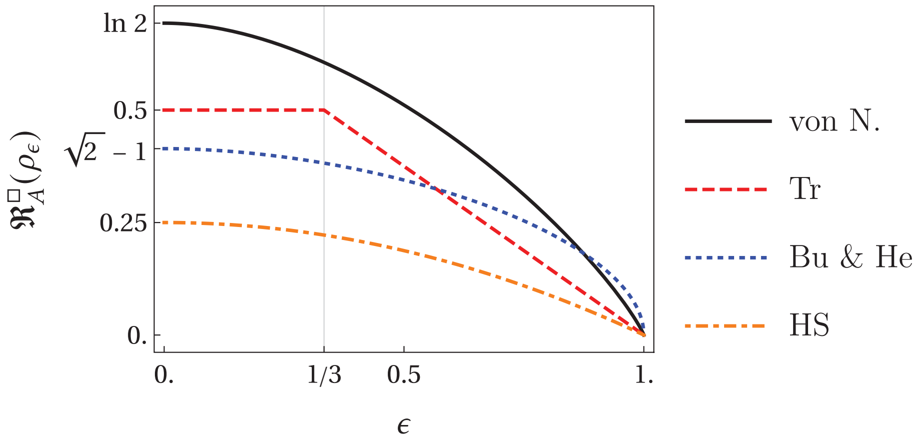

Figure 28 – Reality degree $\mathfrak{R}_A^{\square}$ of any spin-1/2 observable regarding the first particle of a Werner state ρ_ϵ from Example 16 as a function of ϵ for $\square$ = von Neumann (black line), Tr (red dashed line), Bu and He (blue dotted line), and HS (orange dash-dotted line). $\mathfrak{R}_A^{\max}(\square)$ is highlighted in the vertical axe.

Example 16. Let us see how the reality of an spin-1/2 observable $A = \hat{v} \cdot \vec{\sigma}$ regarding the first particle of the Werner state $\rho_\epsilon = (1-\epsilon)\mathbb{1}_4/4 + \epsilon \left|\psi_s\right\rangle \left\langle\psi_s\right|$, where $\left|\psi_s\right\rangle$ is the singlet state (1.74), increases with the loss of coherence within the aforementioned quantifiers. The calculations were made with the help of a software of symbolic computation and the nonenlightening expressions will be omitted, but the results can be visualized in Fig. 28. As expected for the Werner state, the reality degree does not depend on the direction of measurement $\hat{v}$. Notably, $\mathfrak{R}_A^{\mathrm{Tr}}$ violates Axiom 2 for $\epsilon \in [0, 1/3]$, which makes the trace distance unsuitable for reality quantifiers. It is also important to note that since ρ_ϵ commutes with $\Phi_A(\rho_\epsilon)$, then we have $\mathfrak{R}_A^{\mathrm{Bu}}(\rho_\epsilon) = \mathfrak{R}_A^{\mathrm{He}}(\rho_\epsilon)$. ▲

We can also propose to measure the reality degree of an observable by means of the L_p-distances (4.33). Such a quantifier could generalize the trace and the Hilbert-Schmidt realities. However, the expression we have obtained for it does not provide any further insight [see Eq. (C.55) in Appendix C.4]. Nonetheless, we present in Tab. 6 which axioms we know that are fulfilled by $\mathfrak{R}_A^{p}$.

Let us now investigate which of the seven axioms that we proposed in the last chapter are satisfied by each one of the four geometric reality quantifiers in Eqs. (6.32)-(6.35).

6.2.1 Trace reality

The Example 16 alone is sufficient to eliminate $\mathfrak{R}_A^{\mathrm{Tr}}$ as a candidate to a reality monotone since Axiom 2 is violated, that is, maximum reality is achieved by other states than $\Phi_A(\rho)$. Although the trace distance satisfies the positive definiteness property, it emerges in Eq. (6.32) with one of its entries non-normalized. Because of that, if we chose to maintain the validity of Axiom 1, we lose Axioms 2 and 4. Nonetheless, the trace distance is invariant under the addition of an uncorrelated system and it is also jointly convex, therefore satisfying

Axioms 3(b) and 5. Besides, none of the L_p-distances is additive, so Axiom 6 is not valid. For the same reason, Yu *et al.* [187] argued that the trace distance can not be used to provide a proper coherence measure. Since von Neumann reality can be interpreted as just the complementary of coherence when $\rho \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}})$, one can argue that the trace reality should do the same. However, as we have just seen, the trace distance cannot provide adequate quantifiers of reality nor coherence. Finally, the effectiveness of the flagging axiom remains unknown.

6.2.2 Hilbert-Schmidt reality

Although the Hilbert-Schmidt distance is not contractive under CPTP maps in general, it becomes contractive under CPTP *unital* maps, which is the case for projective measurements Φ_A, monitorings $\mathcal{M}_A^\epsilon$, and partial traces [see the pinching inequality (4.26)]. Also, the square of d_{HS} remains contractive because $x \mapsto x^2$ is an injective monotonically increasing function for $x \geqslant 0$. Therefore, the Axiom 3(a) of the realistic operations is valid. It is easy to see that Axioms 2, 4, and 5 are also valid given the positive definiteness and the joint convexity properties of the L_p-distances. The irrelevance of the uncorrelated [Axiom 3(B)], however, is not fulfilled because

$$d^2_{\mathrm{HS}}\left(\rho \otimes \Omega, \Phi_A(\rho) \otimes \Omega\right) = \|\rho \otimes \Omega - \Phi_A(\rho) \otimes \Omega\|_2^2, \tag{6.36}$$

$$= \|\rho - \Phi_A(\rho)\|_2^2 \|\Omega\|_2^2, \tag{6.37}$$

$$> d^2_{\mathrm{HS}}\left(\rho, \Phi_A(\rho)\right), \tag{6.38}$$

if Ω is not pure. This is, in fact, a common issue regarding quantum correlations derived from the Hilbert-Schmidt distance, as argued by Piani [188]. Having said that, the quantifier $\mathfrak{R}_A^{\mathrm{HS}}$ does not meet the minimum requirements to be considered a reality monotone.

6.2.3 Bures and Hellinger reality monotones

When we choose to keep as true the information-reality complementarity principle, the Bures and Hellinger distances are the only ones—of the ones we've seen so far—that provide us with at least a reality monotone. Since additivity is far from been reached (mainly because of the absence of the logarithm), a reality measure can not be obtained from these distances. An easy way to see that is by realizing that

$$d^2_{\mathrm{He}}\left(\rho, \sigma\right) = D_{q=\frac{1}{2}}(\rho||\sigma), \tag{6.39}$$

that is, the square of the Hellinger distance (4.37) is equal to the Tsallis relative entropy (4.47) when $q = 1/2$. As the latter is not additive, the former is not either.

The positive definiteness, the contractivity, and the joint convexity of the Bures and the Hellinger distances are sufficient features for us to verify that both $\mathfrak{R}_A^{\mathrm{He}}$ and $\mathfrak{R}_A^{\mathrm{Bu}}$ are indeed reality monotones, satisfying Axioms 1-5. In addition to that, we can also see that

$$d^2_{\mathrm{Bu}}\left(\rho, \sigma\right) = 2 - 2\exp\left(-\frac{1}{2}\widetilde{D}_{\alpha=\frac{1}{2}}(\rho||\sigma)\right), \tag{6.40}$$

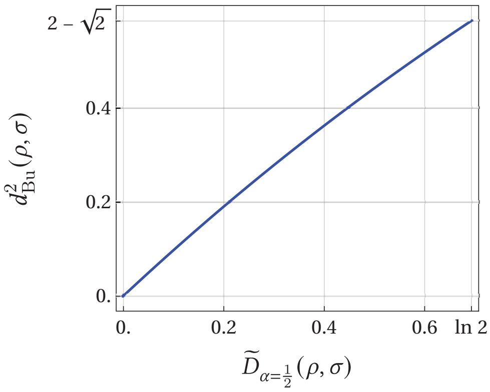

Figure 29 – Injective increasing relation between d^2_{Bu} and $\widetilde{D}_{\alpha=\frac{1}{2}}$ of any pair of quantum states given by (6.40). Identical behavior for (6.41).

$$d^2_{\mathrm{He}}(\rho,\sigma) = 2 - 2\exp\left(-\frac{1}{2}D_{\alpha=\frac{1}{2}}(\rho||\sigma)\right), \tag{6.41}$$

which means that the Bures and the Hellinger reality monotones can be written as injective increasing functions of special cases of the Rényi divergences (4.45) and (4.46), precisely when $\alpha = 1/2$. This reaffirms our argument that the Bures and the Hellinger distances provide proper reality monotones. See Fig. 29 to verify the almost linear behavior of Eqs. (6.40) and (6.41).

The main difference between the Bures and the Hellinger distances is that the latter is sensitive to the noncommutativity of states. In fact, the role of this feature in the measurability of the reality of observables is not yet clear to us—which suggests a path of research in line with the work of Martins *et al.* [189]. Nevertheless, the reader can get an idea of how close these quantifiers are to each other by the following example.

Example 17. Let us take again the state ρ_μ from Example 13. As before, $\mathfrak{R}^{\mathrm{Bu}}_A(\rho_\mu)$ and $\mathfrak{R}^{\mathrm{He}}_A(\rho_\mu)$ are also dependent only on μ and ϕ. The slightly differences between these two reality monotones due to the noncommutativity between ρ_μ and $\Phi_A(\rho_\mu)$ can be visualised in Fig. 30. ▲

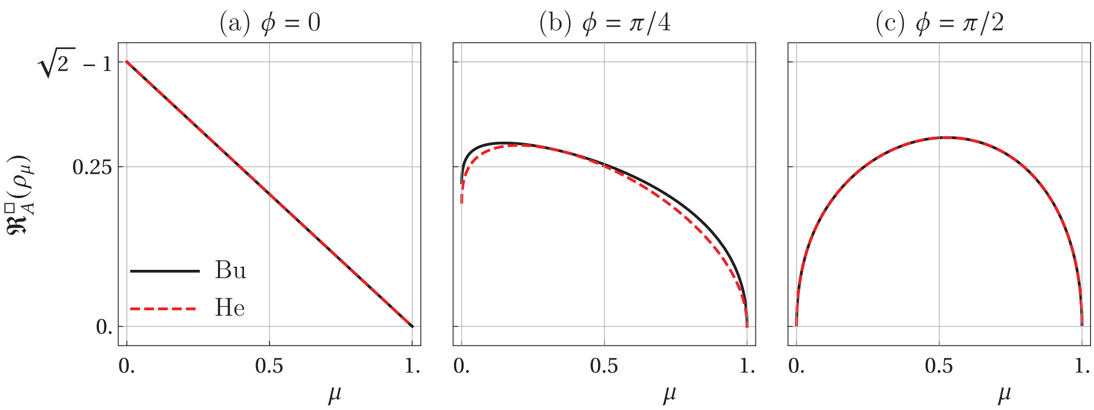

Figure 30 – Reality monotones $\mathfrak{R}^{\mathrm{Bu}}_A$ (black line) and $\mathfrak{R}^{\mathrm{He}}_A$ (red dashed line) of a spin-1/2 observable $A = \hat{v}\cdot\vec{\sigma}$, where $\hat{v} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$, regarding the first qubit of the state ρ_μ for (a) $\phi = 0$, (b) $\phi = \pi/4$, and (c) $\phi = \pi/2$.

	$\mathfrak{R}_A^{\mathrm{Tr}}$	$\mathfrak{R}_A^{\mathrm{HS}}$	$\mathfrak{R}_A^{p}$	$\mathfrak{R}_A^{\mathrm{Bu}}$	$\mathfrak{R}_A^{\mathrm{He}}$
Axiom 1 (information flow)	✔	✔	✔	✔	✔
Axiom 2 (measurements)	✘	✔	?	✔	✔
Axiom 3(a) (part discard)	✔	✔	?	✔	✔
Axiom 3(b) (uncorrelated part)	✔	✘	✘	✔	✔
Axiom 4 (uncertainty relation)	✘	✔	?	✔	✔
Axiom 5 (mixing)	✔	✔	✔	✔	✔
Axiom 6 (additivity)	✘	✘	✘	✘	✘
Axiom 7 (flagging)	?	?	?	?	?

Table 6 – Summary of the axioms satisfied by the A-reality quantifiers $\mathfrak{R}_A^{\mathrm{Tr}}$ [Eq. (6.32)], $\mathfrak{R}_A^{\mathrm{HS}}$ [Eq. (6.33)], $\mathfrak{R}_A^{p}$, $\mathfrak{R}_A^{\mathrm{Bu}}$ [Eq. (6.34)], and $\mathfrak{R}_A^{\mathrm{He}}$ [Eq. (6.35)] built out of their corresponding distance measures d_{Tr}, d_{HS}^2, d_p^p (for finite $p > 1$ and $p \neq 2$), d_{Bu}^2, and d_{He}^2, whose properties are listed in Table 3. Our approach legitimates just two geometric reality monotones, namely, $\mathfrak{R}_A^{\mathrm{Bu}}$ and $\mathfrak{R}_A^{\mathrm{He}}$.

In short, the takeaway message is as follows. In our search for genuinely geometric quantifiers for the violation of the BA's realism hypothesis, we ended up encountering precisely the monotones related to the Rényi and the Tsallis divergences that are symmetrical in their inputs ($q = \alpha = 1/2$), that is, divergences that are also genuine metrics. It is possible that these conclusions are directly related to the informational nature of entropy, which in turn is strictly necessary for us to maintain the validity of Axiom 1. The reader can find a summary of our findings in Tab. 6.

6.3 TAKEAWAY MESSAGE

- Our axiomatic approach to quantum realism recovers BA's original reality measure from von Neumann entropy;
- Reality is not equivalent to incoherence because

$$\mathfrak{R}_A(\rho) = \underbrace{\ln d_{\mathcal{A}} - \mathfrak{I}_A(\rho_{\mathcal{A}})}_{\text{incoherence}} - \underbrace{D_A(\rho)}_{\text{correlations}}. \tag{6.42}$$

- Rényi and Tsallis entropies provide reality monotones only for restricted parameter ranges;
- Within the explored distances, only those that are also quantum divergences can provide reality monotones, namely, Bures and Hellinger distances.

7 A TALK ON QUANTUM DARWINISM

This is our final chapter of results. Here, we show a logical implication between the objective reality of QD and the BA's realism hypothesis for qubit systems. This result is not yet submitted for publication.

As we saw in the introduction, QD [31] is a framework used to describe the emergence of objective physical reality from the quantum world. According to that view, the system of interest interacts with different fragments of the environment and, in this way, encodes partial information about itself in them. The pointer state that is most reproduced in the environment is what emerges as the objective physical reality once it is capable of being accessed by several observers without them interfering with the system. Inspired by a recent paper [35], we are going to show that this process logically implies BA's realism hypothesis—at least for qubits.

First, let us model our system (see Fig. 31). Let $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$, such that $\mathcal{H}_\mathcal{S} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ and $\mathcal{H}_\mathcal{E} = \bigotimes_{k=1}^{N} \mathcal{H}_{\mathcal{E}_k}$, with $\mathcal{H}_\mathcal{A}$ containing 1 qubit and $\mathcal{H}_\mathcal{E}$ containing N qubits. Note that the fragment $\mathcal{F}_m$ of the environment $\mathcal{E}$ is constituted by $\mathcal{H}_{\mathcal{F}_m} = \bigotimes_{k=1}^{m} \mathcal{H}_{\mathcal{E}_k}$, with $m \leqslant N$. As in the proof of Theorem 7, let U be the unitary operator

$$U \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{pmatrix} \tag{7.1}$$

henceforth called *c-maybe gate*, where $s^2 + c^2 = 1$. If $s = 0$, than U is the standard c-not gate. It is possible to write

$$U = \sum_{i=0}^{1} P_i \otimes T_i \tag{7.2}$$

where $P_i = A_i \otimes \mathbb{1}_\mathcal{B}$, $T_0 = \mathbb{1}_{\mathcal{E}_1}$, and $T_1 = \sigma_{s,c} \equiv \begin{pmatrix} s & c \\ -c & s \end{pmatrix}$. In particular, if $s = 0$, than $\sigma_{s,c}$ becomes the Pauli matrix σ_x. Thus,

$$U = P_0 \otimes \mathbb{1}_{\mathcal{E}_1} + P_1 \otimes \sigma_{s,c}. \tag{7.3}$$

Observe that $\sigma_{s,c} |0\rangle = s |0\rangle + c |1\rangle$ e $(\sigma_{s,c})^2 = \mathbb{1}_{\mathcal{E}_1}$. Therefore, we can write the following useful term

$$\langle 0| T_i T_j |0\rangle = s + \delta_{ij}(1 - s). \tag{7.4}$$

Now, let us see what happens when the system interact with just 1 qubit from the environment. Let us take $\rho_{\mathcal{S}\mathcal{E}_1}(0) = \rho \otimes |0^1\rangle \langle 0^1|$ in a 1 + 1-qubit space. With that,

$$\rho_{\mathcal{S}\mathcal{E}_1}(t) = U \rho_{\mathcal{S}\mathcal{E}_1}(0) U^\dagger = \sum_{ij} P_i \rho P_j \otimes T_i |0^1\rangle \langle 0^1| T_j. \tag{7.5}$$

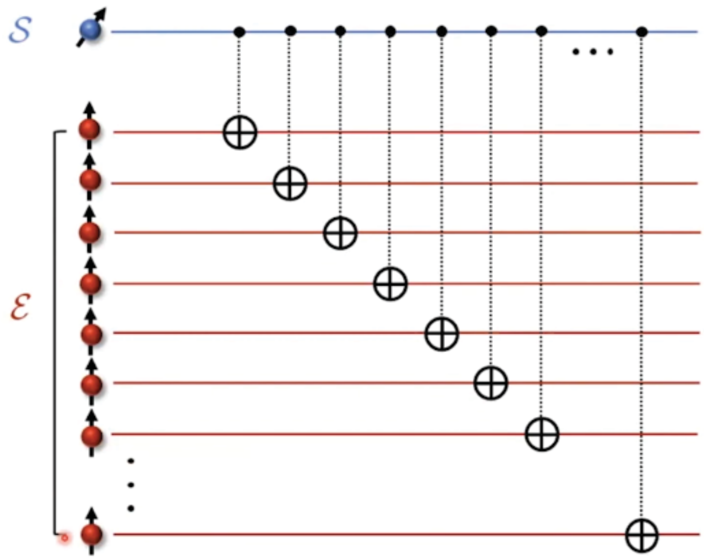

Figure 31 – A system $\mathcal{S}$ composed by just 1 qubit interacts with m qubits in the environment $\mathcal{E}$ through a c-maybe gate U. Figure source: A. Touil, *Eavesdropping on the Decohering Environment: Quantum Darwinism, Amplification, and the Origin of Objective Classical Reality*, Online workshop: Quantum Boundaries 2021, https://youtu.be/pNwBGwYF0Go (visited on 10/14/2021).

By tracing out the fragment $\mathcal{E}_1$ of the environment (which turns out to be the whole environment), we obtain

$$\begin{aligned} \mathrm{Tr}_{\mathcal{E}_1}\left[\rho_{\mathcal{S}\mathcal{E}_1}(t)\right] &= \sum_{ij} P_i\rho P_j \langle 0|\, T_i T_j \,|0\rangle , && (7.6)\\ &= \sum_{ij} P_i\rho P_j[s+\delta_{ij}(1-s)], && (7.7)\\ &= s\sum_{ij} P_i\rho P_j + (1-s)\sum_i P_i\rho P_i, && (7.8)\\ &= s\rho + (1-s)\Phi_A(\rho), && (7.9)\\ &= \mathcal{M}_A^{\epsilon=1-s}(\rho). && (7.10) \end{aligned}$$

As we can see here, the c-maybe operator written in the eigenbasis of A is precisely the unitary evolution that results in a monitoring of A with intensity $\epsilon = 1 - s$ after we trace out the environment.

Let us remember that the premise behind QD is the redundancy of information about the system that is codified in several degrees of freedom of a structured environment. Let $\rho_{\mathcal{S}\mathcal{F}_m}(0)$ be the initial state of the system $\mathcal{S}$ plus the fragment $\mathcal{F}_m$ of the environment such that

$$\rho_{\mathcal{S}\mathcal{F}_m}(0) = \rho \otimes \bigotimes_{k=1}^{m} |0^k\rangle \langle 0^k| \,. \tag{7.11}$$

Now, let $U^{(k)}$ be the c-maybe gate that acts over the pair $\mathcal{S}\mathcal{E}_k$. Therefore, we obtain (see the calculations in Appendix C.5)

$$\rho_{S\mathcal{F}_m}(t) = U^{(m)}U^{(m-1)}\cdots U^{(2)}U^{(1)}\left[\rho \bigotimes_{k=1}^{m} |0^k\rangle\langle 0^k|\right] U^{(1)}U^{(2)}\cdots U^{(m-1)}U^{(m)}, \tag{7.12}$$

$$= \sum_{ij} P_i\rho P_j \bigotimes_{k=1}^{m} T_i |0^k\rangle\langle 0^k| T_j. \tag{7.13}$$

The above equation is the entangled state that represents the situation found at the end of the experiment depicted in Fig. 31. By tracing out the whole fragment $\mathcal{F}_m$, we obtain

$$\mathrm{Tr}_{\mathcal{F}_m}\left(\rho_{S\mathcal{F}_m}\right) = \sum_{ij} P_i\rho P_j \left[s + \delta_{ij}(1-s)\right]^m, \tag{7.14}$$

$$= \sum_{i} P_i\rho P_i + s^m \sum_{i\neq j} P_i\rho P_j, \tag{7.15}$$

$$= \Phi_A(\rho) + s^m \sum_{i} P_i\rho P_i - s^m \sum_{i} P_i\rho P_i + s^m \sum_{i\neq j} P_i\rho P_j, \tag{7.16}$$

$$= (1-s^m)\Phi_A(\rho) + s^m\rho, \tag{7.17}$$

where we have used the useful term (7.4). By summoning Eq. (18) from Ref. [40], that is,

$$\left[\mathcal{M}_A^{\epsilon}\right]^m(\rho) = (1-\epsilon)^n\rho + [1-(1-\epsilon)^n]\Phi_A(\rho), \tag{7.18}$$

we have that

$$\mathrm{Tr}_{\mathcal{F}_m}\left(\rho_{S\mathcal{F}_m}\right) = \left[\mathcal{M}_A^{\epsilon=1-s}\right]^m(\rho). \tag{7.19}$$

The above equation is really a significant and even intuitive result. Each weak interaction with some part of the environment produces one more monitoring over the system. This process clearly increases the degree of reality, since monitorings are realistic operations (see Definition 4).

A closer look on (7.18) also give us that $\left[\mathcal{M}_A^{\epsilon}\right]^m(\rho) = \mathcal{M}_A^{1-(1-\epsilon)^m}(\rho)$ [40]. Having said that, when several observers access the information about ρ through a huge fragment $\mathcal{F}_m$ of the environment, *i.e.*, $m \longrightarrow +\infty$, we obtain

$$\lim_{m\to+\infty}\left[\mathcal{M}_A^{\epsilon=1-s}\right]^m(\rho) = \Phi_A(\rho), \tag{7.20}$$

since $0 \leqslant \epsilon \leqslant 1$. This configures BA's realism hypothesis from a Darwinist perspective, that is, the objective reality achieved through the redundant codification of information about the state of the system in different fragments of the environment implies the full reality of the observable A given ρ—at least for $1+m$ qubits. Moreover, no matter how weak is the interaction between system and environment, if the system finds enough parts of the environment to redundantly codify its information, it can always become as real as it wants.

The theory of QD does not determine, *a priori*, the size of the environment nor for how many observers the information about the system should be available. Here, we use 'infinite' observers as a mathematical tool, a synonym for 'many'. Depending on the quality of the c-maybe gate (parameter $1 - s = \epsilon$), few fragments may be necessary for the state to approach $\Phi_A(\rho)$ in such a way that the experimental apparatus is incapable to discern it from $\left[\mathcal{M}_A^{\epsilon}\right]^m(\rho)$. However, if we want the system to present objective reality to more observers, more fragments must be needed so that, in the limit, we must stick to Eq. (7.20).

Unquestionably, there is an urgent need to make a clearer connection between BA's reality hypothesis and the objective reality of QD. Proposals such as Spectrum Broadcast Structures and Strong Quantum Darwinism [191] seem to be promising starting points of investigation since more specifications are presented in these frameworks.

7.1 TAKEAWAY MESSAGE

- The emergence of objective reality according to Quantum Darwinism that results from the interaction, albeit weak, of the system of interest with infinite fragments of the environment implies the reality hypothesis of BA for systems of qubits, *i.e.*, it results in $\Phi_A(\rho) = \rho$.

8 CONCLUDING REMARKS ON PART II

Inspired by a significant amount of theoretical and experimental works regarding the emergence of classicality from the quantum substratum [31, 35–44, 192–194], here we have proposed an axiomatization for the concept of quantum realism. This notion is different from classical reality in a very fundamental manner, namely, non-commuting observables cannot be simultaneous elements of reality in general (Axiom 4).

Our core premise, implemented via Axiom 1, is the one permeating the aforementioned literature: an observable A emerges as an element of the physical reality only when another degree of freedom encodes information about it. By its turn, Axiom 2 highlights the role of measurements to quantum realism. In full consonance with Axiom 1—for measurements can be viewed as dynamical processes through which an apparatus get to know about the measured observable—the second axiom can yet be used, along with the Stinespring theorem [Eq. (5.6)], as a necessary criterion for one to decide when a measurement is concluded. The rationale is that we do not expect a measurement to have been finished before the establishment of reality, that is, before the instant t at which $\rho_t = \Phi_A(\rho)$ and hence $\mathfrak{R}_A(\rho_t) = \mathfrak{R}_{\mathcal{A}}^{\max}$. The role of large environments in this respect then consists of ensuring the irreversibility of the measurement. Axioms 3 and 5 complete the list of reasonable assumptions for a functional $\mathfrak{R}_A(\rho)$ to be named a reality monotone, while Axioms 6 and 7 are additional conditions legitimating a reality measure. However debatable our list of axioms may be, it furnishes an intuitive "metric independent" characterization of quantum realism, thus framing the concept in a formal structure. Moreover, as we have explicitly demonstrated (see Tables 5 and 6 for an overview), sensible reality monotones and a reality measure can be built by use of information theoretic quantities associated with the von Neumann, Rényi, and Tsallis entropies, as well as, the Bures and Hellinger distances. Regarding the latter, we have found that they are also quantum divergences and therefore carry informational properties within. This is probably the reason why the Bures and Hellinger distances provide reality monotones, unlike the trace and Hilber-Schmidt distances.

At least two technical questions are left open for future research with respect to the reality monotones and measures. The first one concerns the completion of the last line of Table 5, as well as, the last line and third column of Table 6. Indeed, the concept of flagging has been introduced only recently and formal results in this regard are still lacking for the Rényi and Tsallis divergences and the explored distances. Second, the computational advantage of using geometric measures such as the Bures and Hellinger distances in calculating reality still lacks evidence. Yet, the paths in this line were open.

With regard to resource theories, although some evidences have been put forward suggesting that the A-irreality, $\mathfrak{I}_A(\rho) = \ln d_{\mathcal{A}} - \mathfrak{R}_A(\rho)$, can be viewed as a quantum resource

[44], it would be interesting to have at hand a concrete information task wherein this concept configures a clear advantage in relation to contexts involving the A-reality state $\Phi_A(\rho)$.

Another relevant contribution of this thesis is the fact that we started a conversation between Quantum Realism and Quantum Darwinism. The presence of an element of reality for a given observable according to BA is guaranteed whenever the system of interest is correlated with infinite different fragments of the environment—no matter the strength of the interaction. It remains unclear, however, whether the opposite implication is true. A cursory analysis points to a negative answer. As a counterexample, we can go back to the lightweight double-slit experiment and verify that the particle contains an element of physical reality for its position even though it is correlated only with a single system, in this case, the lightweight slit. Therefore, we have $\Phi_A(\rho) = \rho$ without the redundancy of information required by QD.

9 FINAL COMMENTS

This thesis explores a notion of physical reality in the context of quantum mechanics in several ways. Here, we use the term "Quantum" Realism as opposed to "Classical" Realism. Our intention with this is to emphasize that, in quantum systems, the notion that "all physical quantities are well defined all the time independent of observers" falls apart. Of course, if we follow an interpretation of quantum mechanics that assumes that there are well-defined trajectories all the time for an electron around a nucleus (as is the case with Bohmian mechanics), then our nomenclature would need to be reformulated. Still, BA's equation given by $\Phi_A(\rho) = \rho$ may or may not be violated depending on the system under scrutiny—and that does not depend on the interpretation of quantum mechanics chosen by the reader.

In Part I, when we explored quantum walkers, we got in touch with the notion that quantum correlations prevent local elements of reality. This is clear at the beginning of Sec. 2.5.5 by the relation $\mathfrak{I}_A(\rho_t^\epsilon) = D_{S_1}(\rho_t^\epsilon)$. As locally the spins of the walkers are given by completely incoherent states (*i.e.*, $\mathrm{Tr}_{S_2}\rho_t^\epsilon = \mathrm{Tr}_{S_1}\rho_t^\epsilon = \mathbb{1}/2$), the irreality of the pair of spins is exclusively due to the context (non-optimized) quantum discord between them. Furthermore, another important dynamics stands out: while the entanglement between the spins decreases, the multipartite entanglement of the global state increases (Fig. 17). This leads to the idea of a flow of correlations [Eq. (2.39)]—and therefore information—which is followed by the emergence of some elements of reality associated with the spins of the walkers over time [Fig. 21 (a)].

Inspired by the investigation made in Part I, we delve into the notion of quantum realism in Part II through an axiomatization whose fundamental concept is the idea described above: the physical reality of an observable grows when there is a flow of information from the system to the environment. In the Wigner's friend scenario, for instance, the Alice+particle bipartite system does not contain an element of reality associated with the particle from Wigner's point of view, since he did not perform any measurement, that is, there was no flow of quantum conditional information. On the other hand, Alice prescribes an element of reality for her particle from the moment her measurement is performed. Also, if Wigner discards the information regarding Alice's state, the particle starts to have an element of reality (since it was completely entangled with Alice). From here, new questions arise about this quantum notion of realism. Since the unrevealed quantum measurement is essentially a partial trace on the environment, one can wonder if the partial trace performed by Wigner is a physical process. When is the quantum measurement actually performed? Our analysis points in the following direction: the measurement is only finished when an element of reality emerges. However, here a new question arises: is it possible for the element of reality to exist without an actual measurement having been previously performed? In other words, is it possible to find

$\rho = \Phi_A(\rho)$ *in nature*? BA's realism hypothesis is operational and therefore does not reveal by what mechanisms reality arises. Our Axiom 1, on the other hand, only determines how the degree of reality of an observable varies. Whether it is possible to find a state given by $\Phi_A(\rho)$ without the observable A having been previously measured or not is a question of profound importance. We believe that the negative answer to this question can rule out interpretations of quantum mechanics based on the notion of intrinsic realism in favor of those based on participatory realism, where the observer plays a crucial role in the emergence of physical reality and, possibly, the measurement results. In this way, the idea that *nothing is real until it is measured* would be accepted. Speculation or not, isn't that how physics develops?

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Appendix

APPENDIX A – REVISITING THE INFORMATION-REALITY COMPLEMENTARITY PRINCIPLE

Let us revisit the main argument on which Part II is constructed. Based on BA's realism [36], Dieguez and Angelo [40] showed the existence of a complementarity principle between reality and information by means of a map that unifies non-selective weak and projective measurements,

$$\mathcal{M}_A^\epsilon(\rho) \coloneqq (1-\epsilon)\rho + \epsilon\Phi_A(\rho), \tag{A.1}$$

which they called *monitoring*. First, they introduce the concept of *reality* $\mathfrak{R}_A$ as complementary to irreality $\mathfrak{I}_A$, just as information and ignorance (entropy),

$$\mathfrak{R}_A(\rho) + \mathfrak{I}_A(\rho) = \ln d_{\mathcal{A}}. \tag{A.2}$$

Under a monitoring $\mathcal{M}_A^\epsilon$ of intensity ϵ, the reality $\mathfrak{R}$ of the observable A given the state ρ changes by

$$\begin{aligned} \Delta\mathfrak{R}_A(\mathcal{M}_A^\epsilon(\rho), \rho) &\coloneqq \mathfrak{R}_A(\mathcal{M}_A^\epsilon(\rho)) - \mathfrak{R}_A(\rho), &\text{(A.3)}\\ &= \mathfrak{I}_A(\rho) - \mathfrak{I}_A(\mathcal{M}_A^\epsilon(\rho)), &\text{(A.4)}\\ &= S(\mathcal{M}_A^\epsilon(\rho)) - S(\rho), &\text{(A.5)} \end{aligned}$$

where the hierarchy $\mathcal{M}_A^\epsilon\Phi_A = \Phi_A\mathcal{M}_A^\epsilon = \Phi_A$ was applied to obtain (A.5) from (A.4). Since von Neumann entropy is concave [see Eq. (1.68)], one has

$$\begin{aligned} \Delta\mathfrak{R}_A(\mathcal{M}_A^\epsilon(\rho), \rho) &= S\left[(1-\epsilon)\rho + \epsilon\Phi_A(\rho)\right] - S(\rho) &\text{(A.6)}\\ &\geqslant (1-\epsilon)S(\rho) + \epsilon S(\Phi_A(\rho)) - S(\rho) &\text{(A.7)}\\ &= \epsilon\left[S(\Phi_A(\rho)) - S(\rho)\right], &\text{(A.8)} \end{aligned}$$

which leads to

$$\Delta\mathfrak{R}_A(\Phi_A(\rho), \rho) \geqslant \epsilon\mathfrak{I}_A(\rho). \tag{A.9}$$

Through the above result, the authors claim that a monitoring of intensity ϵ will always increase the reality—or, equivalently, destroy the irreality—of an observable by never less than the original amount of irreality times the intensity ϵ. When $\epsilon \to 1$, *i.e.*, a projective measurement, one has $\Delta\mathfrak{R}_A(\mathcal{M}_A^\epsilon(\rho), \rho) = \mathfrak{I}_A(\rho)$.

The authors' argument for the complementarity connection between reality and information is as follows. First, let us invoke Theorem 2 (Stinespring dilation), that allows us to describe the measurement from the perspective of a super-observer, that is, in a system+environment description. Since $\mathcal{M}_A^\epsilon$ is a CPTP map, one has

$$\mathcal{M}_A^\epsilon(\rho) = \mathrm{Tr}_{\mathcal{E}}\left[U_t^\epsilon\left(\rho \otimes |e_0\rangle\langle e_0|\right)U_t^{\epsilon\dagger}\right] = \rho_t, \tag{A.10}$$

where U_t^ϵ is a unitary time evolution that entangles the degrees of freedom in $\mathcal{H}_\mathcal{S} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ with an auxiliary space $\mathcal{H}_\mathcal{E}$ initially prepared as $|e_0\rangle\langle e_0|$. We can assume that the environment is in a pure state because otherwise, we can always assume that there is a bigger Hilbert space where this supposed mixed state results from a partial trace over a pure state. Let us call $\upsilon_0 = \rho \otimes |e_0\rangle\langle e_0|$ and $\upsilon_t = U_t^\epsilon \upsilon_0 U_t^{\epsilon\dagger}$ the initial and final global states, respectively, and $\sigma_t = \mathrm{Tr}_\mathcal{S}(\upsilon_t)$ the final state of the environment. The mutual information (1.76) shared by $\mathcal{S}$ and $\mathcal{E}$ in υ_t is given by

$$I_{\mathcal{S}:\mathcal{E}}(\upsilon_t) = S(\rho_t) + S(\sigma_t) - S(\upsilon_t). \tag{A.11}$$

Since the evolution is unitary, one has $S(\upsilon_t) = S(\upsilon_0)$. Thus, by denoting $\Delta S_\mathcal{S} := S(\rho_t) - S(\rho)$ and $\Delta S_\mathcal{E} := S(\sigma_t) - S(|e_0\rangle)$, one can obtain the change in mutual information as

$$\Delta I_{\mathcal{S}:\mathcal{E}}(\upsilon_t, \upsilon_0) := I_{\mathcal{S}:\mathcal{E}}(\upsilon_t) - I_{\mathcal{S}:\mathcal{E}}(\upsilon_0) = \Delta S_\mathcal{S} + \Delta S_\mathcal{E}. \tag{A.12}$$

From (1.79), one has

$$\Delta I_\mathcal{E} := I(\sigma_t) - I(|e_0\rangle) = -\Delta S_\mathcal{E}, \tag{A.13}$$

and from (A.10), one obtain

$$\Delta S_\mathcal{S} = S(\mathcal{M}_A^\epsilon(\rho)) - S(\rho) = \Delta \mathfrak{R}_A(\mathcal{M}_A^\epsilon(\rho), \rho). \tag{A.14}$$

Back to (A.12), one has

$$\Delta I_{\mathcal{S}:\mathcal{E}}(\upsilon_t, \upsilon_0) + \Delta I_\mathcal{E} = \Delta \mathfrak{R}_A(\mathcal{M}_A^\epsilon(\rho), \rho). \tag{A.15}$$

From (1.80), one has

$$\begin{aligned} &\Delta I_{\mathcal{S}:\mathcal{E}}(\upsilon_t, \upsilon_0) + \Delta I_\mathcal{E} \\ &= S(\rho_t) + S(\sigma_t) - S(\upsilon_t) - [S(\rho) + S(|e_0\rangle) - S(\upsilon_0)] - S(\sigma_t) + S(|e_0\rangle), && \text{(A.16)} \\ &= S(\rho_t) - S(\rho), && \text{(A.17)} \\ &= I(\rho) - I(\rho_t), && \text{(A.18)} \\ &=: -\Delta I_\mathcal{S}, && \text{(A.19)} \end{aligned}$$

where the invariance of total available information on closed systems was applied. Therefore, one can obtain

$$\Delta I_\mathcal{S} + \Delta \mathfrak{R}_A(\mathcal{M}_A^\epsilon(\rho), \rho) = 0. \tag{A.20}$$

The above relation tells us that variations in A's reality directly implies in variations on the information associated to the system $\mathcal{S}$. Dieguez and Angelo also show that, in the case of pure states, one has [see Eq. (4.70)]

$$\Delta I_\mathcal{S} = -S(\rho_t) = -E(\upsilon_t) \tag{A.21}$$

and, therefore, $\Delta \mathfrak{R}_A(\mathcal{M}_A^\epsilon(\rho), \rho) = E(\upsilon_t)$. In other words, the entanglement created between the system $\mathcal{S}$ and the environment $\mathcal{E}$ during the monitoring increases the reality of the observable A and vice versa.

APPENDIX B – PROOFS

The results presented in this section refers to states such that $\rho \in \mathfrak{B}(\mathcal{H}_{\mathcal{S}})$, with $\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, and the unrevealed measurements map (1.116).

Theorem 7. *Let the unitary evolution U_t be defined by the Stinespring dilation theorem (5.6) with $\epsilon = 1$. It follows that U_t commutes with $\Phi_A(\rho) \otimes \mathbb{1}_{\mathcal{E}}/d_{\mathcal{E}}$, that is,*

$$U_t\left(\Phi_A(\rho) \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}\right)U_t^\dagger = \Phi_A(\rho) \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}. \tag{B.1}$$

Proof. Take the joint state $\upsilon_0 = \rho \otimes |e_0\rangle\langle e_0| \in \mathfrak{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$, with $d_{\mathcal{E}} = \dim \mathcal{H}_{\mathcal{E}} = \dim \mathcal{H}_{\mathcal{A}}$. Write the unitary operator

$$U_t = \sum_{k=0}^{d_{\mathcal{E}}-1} P_k \otimes T_k, \tag{B.2}$$

where $P_k = A_k \otimes \mathbb{1}_{\mathcal{B}}$ is a subspace projector and T_k is a unitary operator satisfying $T_k T_k^\dagger = T_k^\dagger T_k = \mathbb{1}_{\mathcal{E}}$ and $\langle e_0|T_j^\dagger T_i|e_0\rangle = \delta_{ij}$. An example of this structure is provided by the shift operator $T_k|e_i\rangle = |e_{i+k}\rangle$ in the cyclic space with the boundary condition $T_1|e_{d_{\mathcal{E}}-1}\rangle = |e_{d_{\mathcal{E}}}\rangle = |e_0\rangle$. Its matrix representation is given by a power of the generalized Pauli operator σ_x,

$$T_k \doteq \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^k. \tag{B.3}$$

Notice that $T_i T_j = T_{i+j}$, which renders $U_t U_t = \sum_i P_i \otimes T_{2i}$. One has $\upsilon_t = U_t \upsilon_0 U_t^\dagger = \sum_{i,j} P_i \rho P_j \otimes T_i|e_0\rangle\langle e_0|T_j^\dagger$, which correctly reproduces the Stinespring relation

$$\mathrm{Tr}_{\mathcal{E}}(\upsilon_t) = \sum_{ij} P_i \rho P_j \langle e_0|T_j^\dagger T_i|e_0\rangle = \Phi_A(\rho), \tag{B.4}$$

The unitary operator (B.2) is unique up to a unitary operation over the environment. Finally, by direct application of U_t we have

$$U\left(\Phi_A(\rho) \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}\right)U^\dagger = \sum_{kij} P_i P_k \rho P_k P_j \otimes T_i \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}} T_j^\dagger, \tag{B.5}$$

$$= \sum_k P_k \rho P_k \otimes \frac{T_k T_k^\dagger}{d_{\mathcal{E}}}, \tag{B.6}$$

$$= \Phi_A(\rho) \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}, \tag{B.7}$$

which yields the desired result since P_i is a projector and T_i is unitary. ∎

Lemma 1. *For any function f, any quantum states ρ and σ, and any observable A, we have*

$$\mathrm{Tr}\,[\rho\, f(\Phi_A(\sigma))] = \mathrm{Tr}\,[\Phi_A(\rho) f(\Phi_A(\sigma))]. \tag{B.8}$$

Proof. Let us abbreviate $A_a \otimes \mathbb{1}_{\mathcal{B}}$ by just A_a. Having said that, we do

$$\mathrm{Tr}\ [\rho f(\Phi_A(\sigma))] = \mathrm{Tr} \sum_a \rho f(p_a A_a \otimes \sigma_{\mathcal{B}|a}), \tag{B.9}$$

$$= \mathrm{Tr} \sum_a \rho [A_a \otimes f(p_a \sigma_{\mathcal{B}|a})], \tag{B.10}$$

By the complementarity relation $\sum_{a'} A_{a'} = \mathbb{1}_{\mathcal{A}}$, the projector's property $A_{a'}^2 = A_{a'}$, and the cyclic permutation of the trace we have

$$\mathrm{Tr}\ [\rho f(\Phi_A(\sigma))] = \mathrm{Tr} \sum_{aa'} A_{a'}^2 \rho [A_a \otimes f(p_a \sigma_{\mathcal{B}|a})], \tag{B.11}$$

$$= \mathrm{Tr} \sum_{aa'} A_{a'} \rho [A_a \otimes f(p_a \sigma_{\mathcal{B}|a})] A_{a'}, \tag{B.12}$$

$$= \mathrm{Tr} \sum_{aa'} A_{a'} \rho A_{a'} [A_a \otimes f(p_a \sigma_{\mathcal{B}|a})], \tag{B.13}$$

$$= \mathrm{Tr} \sum_{a'} A_{a'} \rho A_{a'} \sum_a [A_a \otimes f(p_a \sigma_{\mathcal{B}|a})], \tag{B.14}$$

$$= \mathrm{Tr}\ [\Phi_A(\rho) f(\Phi_A(\sigma))]\,. \tag{B.15}$$

■

Lemma 2. *Given the reality state $\Phi_A(\rho) = \sum_i p_i A_i \otimes \rho_{\mathcal{B}|i}$, it holds that $D\,(\rho||\Phi_A(\rho)) \leqslant S\,(\Phi_A(\rho_{\mathcal{A}})) \leqslant \ln d_{\mathcal{A}}$.*

Proof. From Lemma 1 one can straightforwardly show that $D\,(\rho||\Phi_A(\rho)) = S(\Phi_A(\rho)) - S(\rho)$. Since $S(\rho) \geqslant \sum_i p_i S(\rho_{\mathcal{B}|i})$ (see Lemma 2 of Ref. [195]), we can employ the joint entropy theorem (1.67) to reach $S\,(\Phi_A(\rho)) = H(\{p_i\}) + \sum_i p_i S(\rho_{\mathcal{B}|i})$, with $H(\{p_i\})$ being the Shannon entropy of the distribution p_i, to finally obtain

$$D\,(\rho||\Phi_A(\rho)) \leqslant H(\{p_i\}) = S\,(\Phi_A(\rho_{\mathcal{A}})) \leqslant \ln d_{\mathcal{A}}. \tag{B.16}$$

■

Lemma 3. *Consider generic observables $X, Y \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}})$ and the von Neumann reality quantifier (6.4). It follows that $\mathfrak{R}_X(\rho) + \mathfrak{R}_Y(\rho) \leqslant 2 \ln d_{\mathcal{A}}$, with equality iff $\rho = \Phi_X(\rho) = \Phi_Y(\rho)$.*

Proof. By Eqs. (4.40) and (6.4) we see that the main claim is readily satisfied and that the equality holds *iff* $\rho = \Phi_X(\rho) = \Phi_Y(\rho)$, meaning that ρ must be a state of simultaneous reality for X and Y. This will certainly be the case when $[X, Y] = 0$, for X and Y will share the same set of eigenstates so that $\Phi_X = \Phi_Y$, but also for $\rho = (\mathbb{1}_{\mathcal{A}}/d_{\mathcal{A}}) \otimes \rho_{\mathcal{B}}$, as can be checked by direct calculation. ■

Note that this proof is valid also for a reality quantifier that is based on any divergence measure that respects the positive definiteness property [Eq. (4.40)].

Lemma 4. *Consider observables $X, Y \in \mathfrak{B}(\mathcal{H}_{\mathcal{A}})$ and the von Neumann reality quantifier (6.4). If $\Phi_{XY} = \Phi_{YX}$, then the monitoring of Y never decreases the reality of X, that is,*

$$\Delta := \mathfrak{R}_X(\mathcal{M}_Y^{\epsilon}(\rho)) - \mathfrak{R}_X(\rho) \geqslant 0, \qquad \forall \epsilon \in [0,1]. \tag{B.17}$$

Proof. In light of Axiom 5 (mixing) and definition (5.4), we find $\Delta \geqslant \epsilon\,[\mathfrak{R}_X(\Phi_Y(\rho)) - \mathfrak{R}_X(\rho)]$, which can be explicitly expressed as $\Delta \geqslant \epsilon\,[D\,(\rho||\Phi_X(\rho)) - D\,(\Phi_Y(\rho)||\Phi_{XY}(\rho))]$, where $\Phi_{XY}(\rho) \equiv \Phi_X\Phi_Y(\rho)$. Using the hypothesis and DPI we can write

$$D\,(\Phi_Y(\rho)||\Phi_{XY}(\rho)) = D\,(\Phi_Y(\rho)||\Phi_{YX}(\rho)) \leqslant D\,(\rho||\Phi_X(\rho)), \tag{B.18}$$

which proves that $\Delta \geqslant 0\ \forall \epsilon \in [0,1]$, as desired. It is worth noticing that, apart from the trivial scenario where $[X, Y] = 0$, the hypothesis is true also when X and Y are maximally noncommuting, that is, when their eigenstates form MUB satisfying $|\langle x_i|y_j\rangle| = 1/\sqrt{d_{\mathcal{A}}}$. ■

Lemma 5. $\|\rho\|_2^2 - \|\Phi_A(\rho)\|_2^2 = \|\rho - \Phi_A(\rho)\|_2^2$.

Proof.

$$\begin{aligned}
\|\rho - \Phi_A(\rho)\|_2^2 &= \mathrm{Tr}\,\left(\rho - \Phi_A(\rho)\right)^2, && \text{(B.19)}\\
&= \mathrm{Tr}\,\rho^2 - \mathrm{Tr}\,\rho\Phi_A(\rho) - \mathrm{Tr}\,\Phi_A(\rho)\rho + \mathrm{Tr}\,\Phi_A(\rho)^2, && \text{(B.20)}\\
&= \mathrm{Tr}\,\rho^2 - 2\mathrm{Tr}\,\rho\Phi_A(\rho) + \mathrm{Tr}\,\Phi_A(\rho)^2, && \text{(B.21)}\\
&= \mathrm{Tr}\,\rho^2 - 2\mathrm{Tr}\,\rho\Phi_A(\rho) + \mathrm{Tr}\,\rho\Phi_A(\rho), \qquad \text{(Lemma 1)} && \text{(B.22)}\\
&= \mathrm{Tr}\,\rho^2 - \mathrm{Tr}\,\rho\Phi_A(\rho), && \text{(B.23)}\\
&= \mathrm{Tr}\,\rho^2 - \mathrm{Tr}\,\Phi_A(\rho)^2, \qquad \text{(Lemma 1)} && \text{(B.24)}\\
&= \|\rho\|_2^2 - \|\Phi_A(\rho)\|_2^2. && \text{(B.25)}
\end{aligned}$$

■

Proposition 1. *$\rho \mapsto \mathfrak{R}_A^{\alpha\uparrow}(\rho)$ is a reality monotone.*

Proof. By construction, $\mathfrak{R}_A^{\alpha\uparrow}$ is in harmony with Axiom 1. We now prove that $\mathfrak{R}_A^{\alpha\uparrow}(\rho) = \mathfrak{R}^{\max}$ *iff* $\rho = \Phi_A(\rho)$, as per Axiom 2. The claim is true *iff* $D_\alpha\,(v_t||\bar{\sigma}_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{E}}/d_{\mathcal{E}}) = \ln d_{\mathcal{E}}$, where $\bar{\sigma}_{\mathcal{S}}$ is the solution for the minimization and $v_t = U_t\,(\rho \otimes |e_0\rangle\langle e_0|)\,U_t^\dagger$. Choosing $\bar{\sigma}_{\mathcal{S}} = \Phi_A(\bar{\sigma}_{\mathcal{S}})$ (an A-reality state), one can apply Theorem 7 and unitary invariance to obtain $D_\alpha\,(\rho||\Phi_A(\bar{\sigma}_{\mathcal{S}})) = 0$, which can be reached *iff* $\rho = \Phi_{\mathcal{A}}(\bar{\sigma}_{\mathcal{S}})$, meaning that ρ is an A-reality state satisfying $\Phi_A(\rho) = \rho$. Axioms 2 and 3(a) are satisfied directly from DPI and the fact that U_t^{ϵ} in Eq. (5.6) commutes with $\mathcal{M}_A^{\epsilon}$ and $\mathrm{Tr}_{\mathcal{X}}$ for $\mathcal{H}_{\mathcal{X}} \subseteq \mathcal{H}_{\mathcal{B}}$. Provided the optimization is made over $\sigma_{\mathcal{S}} \otimes \Omega \in \mathfrak{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\Omega})$, Axiom 3(b) is trivially satisfied via the additivity property. Also, because Axiom 2 applies, we can use the arguments employed for the proof of Lemma 3 to show that Axiom 4 is also true for $\mathfrak{R}_A^{\alpha\uparrow}$. Finally, the validity of Axiom 5 is immediately verified by the convexity of the conditional information (4.63b). Although additivity guarantees the agreement with Axiom 6, the flagging property has not yet been demonstrated for the quantity (4.63b), which precludes $\mathfrak{R}_A^{\alpha\uparrow}$ to be promoted to the status of a reality measure for $\alpha \in (0, 1)$. ■

APPENDIX C – CALCULATIONS

C.1 TWO QUANTUM WALKERS WITH NOISE

The initial state of the two quantum walkers is given by:

$$\rho_0 = \rho_\epsilon^W \otimes |\varphi_1, \varphi_2\rangle \langle\varphi_1, \varphi_2| , \tag{C.1}$$

$$= \left[(1-\epsilon)\frac{\mathbb{1}}{4} + \epsilon |B_4\rangle \langle B_4|\right] \otimes |\varphi_1, \varphi_2\rangle \langle\varphi_1, \varphi_2| , \tag{C.2}$$

$$= (1-\epsilon)\frac{\mathbb{1}}{4} \otimes |\varphi_1, \varphi_2\rangle \langle\varphi_1, \varphi_2| + \epsilon |B_4\rangle \langle B_4| \otimes |\varphi_1, \varphi_2\rangle \langle\varphi_1, \varphi_2| . \tag{C.3}$$

Let us remember that $U_1 = D_1(C_1 \otimes \mathbb{1}_{X_1})$ and $U_2 = D_2(C_2 \otimes \mathbb{1}_{X_2})$, with $\mathbb{1}_{X_1} \otimes \mathbb{1}_{X_2} = \mathbb{1}_X$, act over different spaces:

$$U_1^t U_2^t \rho_0 \left(U_1^t U_2^t\right)^\dagger$$

$$= (1-\epsilon) U_1^t U_2^t \left(\frac{\mathbb{1}}{4} \otimes |\varphi_1, \varphi_2\rangle \langle\varphi_1, \varphi_2|\right) \left(U_1^t U_2^t\right)^\dagger$$

$$+ \epsilon U_1^t U_2^t \left(|B_4\rangle \langle B_4| \otimes |\varphi_1, \varphi_2\rangle \langle\varphi_1, \varphi_2|\right) \left(U_1^t U_2^t\right)^\dagger , \tag{C.4}$$

$$= (1-\epsilon) U_1^{t-1} U_2^{t-1} D_1 D_2 \left(\frac{C_1 C_2 \mathbb{1} C_1 C_2}{4} \otimes \mathbb{1}_X |\varphi_1, \varphi_2\rangle \langle\varphi_1, \varphi_2| \mathbb{1}_X\right) D_1^\dagger D_2^\dagger \left(U_1^{t-1} U_2^{t-1}\right)^\dagger$$

$$+ \epsilon |\Psi_t\rangle \langle\Psi_t| , \tag{C.5}$$

$$= (1-\epsilon) U_1^{t-1} U_2^{t-1} D_1 D_2 \left(\frac{\mathbb{1}}{4} \otimes |\varphi_1, \varphi_2\rangle \langle\varphi_1, \varphi_2|\right) D_1^\dagger D_2^\dagger \left(U_1^{t-1} U_2^{t-1}\right)^\dagger + \epsilon |\Psi_t\rangle \langle\Psi_t| . \tag{C.6}$$

Now, observe that

$$D_1 \left(\frac{\mathbb{1}}{2} \otimes |\varphi_1\rangle \langle\varphi_1|\right) D_1^\dagger = \sum_{x_1 x_1'} \frac{1}{2} |\uparrow\rangle \langle\uparrow| \otimes |x_1 + 1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1' + 1|$$

$$+ \frac{1}{2} |\downarrow\rangle \langle\downarrow| \otimes |x_1 - 1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1' - 1| , \tag{C.7}$$

$$= \sum_{x_1 x_1'} \frac{1}{2} |\uparrow\rangle \langle\uparrow| \otimes \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle |x_1 + 1\rangle \langle x_1' + 1|$$

$$+ \frac{1}{2} |\downarrow\rangle \langle\downarrow| \otimes \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle |x_1 - 1\rangle \langle x_1' - 1| , \tag{C.8}$$

with identical expression for particle 2. If we trace out the positions, we obtain

$$\mathrm{Tr}_{X_1} U_1 \left(\frac{\mathbb{1}}{2} \otimes |\varphi_1\rangle \langle\varphi_1|\right) (U_1)^\dagger$$

$$= \sum_{x_1 x_1'} \frac{1}{2} |\uparrow\rangle \langle\uparrow| \otimes \delta_{x_1 x_1'} \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle + \frac{1}{2} |\downarrow\rangle \langle\downarrow| \otimes \delta_{x_1 x_1'} \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle , \tag{C.9}$$

$$= \sum_{x_1} \frac{\mathbb{1}}{2} | \langle x_1|\varphi_1\rangle |^2, \tag{C.10}$$

$$= \frac{\mathbb{1}}{2}. \tag{C.11}$$

If we apply the unitary evolution twice, we reach

$$\begin{aligned}
&[D_1(C_1 \otimes \mathbb{1}_{X_1})]D_1 \left(\frac{\mathbb{1}}{2} \otimes |\varphi_1\rangle \langle\varphi_1|\right) D_1^\dagger [D_1(C_1 \otimes \mathbb{1}_{X_1})]^\dagger \\
&= D_1 \left[\sum_{x_1 x_1'} \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes |x_1+1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'+1| \right.\\
&\left. +\frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes |x_1-1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'-1| \right] D_1^\dagger,
\end{aligned} \tag{C.12}$$

$$\begin{aligned}
&= \sum_{x_1 x_1' x_1'' x_1'''} \frac{|\uparrow\rangle\langle\uparrow|}{4} \otimes |x_1''+1\rangle \langle x_1''|x_1+1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'+1|x_1'''\rangle \langle x_1'''+1| \\
&+ \frac{|\uparrow\rangle\langle\uparrow|}{4} \otimes |x_1''+1\rangle \langle x_1''|x_1-1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'-1|x_1'''\rangle \langle x_1'''+1| \\
&+ \frac{|\downarrow\rangle\langle\downarrow|}{4} \otimes |x_1''-1\rangle \langle x_1''|x_1+1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'+1|x_1'''\rangle \langle x_1'''-1| \\
&+ \frac{|\downarrow\rangle\langle\downarrow|}{4} \otimes |x_1''-1\rangle \langle x_1''|x_1-1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'-1|x_1'''\rangle \langle x_1'''-1|,
\end{aligned} \tag{C.13}$$

$$\begin{aligned}
&= \sum_{x_1 x_1'} \frac{|\uparrow\rangle\langle\uparrow|}{4} \otimes |x_1+2\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'+2| \\
&+ \frac{|\uparrow\rangle\langle\uparrow|}{4} \otimes |x_1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'| \\
&+ \frac{|\downarrow\rangle\langle\downarrow|}{4} \otimes |x_1\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'| \\
&+ \frac{|\downarrow\rangle\langle\downarrow|}{4} \otimes |x_1-2\rangle \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \langle x_1'-2|,
\end{aligned} \tag{C.14}$$

$$\begin{aligned}
&= \sum_{x_1 x_1'} \frac{|\uparrow\rangle\langle\uparrow|}{2} \otimes \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \frac{1}{2}\left(|x_1+2\rangle \langle x_1'+2| + |x_1\rangle\langle x_1'|\right) \\
&+ \frac{|\downarrow\rangle\langle\downarrow|}{2} \otimes \langle x_1|\varphi_1\rangle \langle\varphi_1|x_1'\rangle \frac{1}{2}\left(|x_1-2\rangle \langle x_1'-2| + |x_1\rangle\langle x_1'|\right),
\end{aligned} \tag{C.15}$$

which again results in $\mathbb{1}/2$ after we trace over the positions. For t steps, this behavior remains unchanged. Therefore, from (C.6), we have

$$\mathrm{Tr}_{X_1X_2}\left[U_1^t U_2^t \rho_0 \left(U_1^t U_2^t\right)^\dagger\right] = (1-\epsilon)\frac{\mathbb{1}}{4} + \epsilon\,\rho_S =: \rho_t^\epsilon. \tag{C.16}$$

C.2 RÉNYI DIVERGENCES

C.2.1 Special cases

The *min-relative entropy*, as defined by Datta [88], is the limit of the original Rényi divergence as $\alpha \to 0$. Operationally, it comes as follows. Consider the spectral decompositions $\rho = \sum_i r_i |\lambda_i\rangle\langle\lambda_i|$ and $\sigma = \sum_j s_j |\nu_j\rangle\langle\nu_j|$. Since $f(\rho) = \sum_i f(r_i)|\lambda_i\rangle\langle\lambda_i|$, then we have $\rho^0 =$

$\sum_{i:\,r_i>0} |\lambda_i\rangle\langle\lambda_i|$, which is called the *projection onto the support of* ρ. Thus, the min-relative entropy is given by $D_0(\rho||\sigma) = -\ln \mathrm{Tr}\,\rho^0\sigma$. Explicitly, we have

$$D_{\alpha\to 0}(\rho||\sigma) = -\ln \sum_j \sum_{i:\,r_i>0} s_j |\langle\lambda_i|v_j\rangle|^2. \tag{C.17}$$

Note that, the min-relative entropy does not satisfy continuity nor positive definiteness in ρ and σ, a feature that prevents it to satisfy Axiom 2. One special case of the sandwiched Rényi divergence is the *collision relative entropy*, which was introduced in Ref. [87] in its conditional form as a generalization of the classical conditional collision entropy to the quantum theory. It is obtained when we choose $\alpha = 2$ in (4.46):

$$\widetilde{D}_2(\rho||\sigma) = \ln \mathrm{Tr}\left[\left(\sigma^{-\frac{1}{4}}\rho\sigma^{-\frac{1}{4}}\right)^2\right]. \tag{C.18}$$

Another special case of $\widetilde{D}_\alpha$ is obtained as $\alpha \to +\infty$, which is called *max-relative entropy* [88]:

$$\widetilde{D}_{\alpha\to+\infty}(\rho||\sigma) = \ln \left\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right\|_\infty. \tag{C.19}$$

Here, the *operator norm* $\|\varrho\|_\infty$ is given by the maximum eigenvalue of a density state ϱ.

C.2.2 Extra steps

Let us check the Eq. (4.58). From definition (4.45), we can directly calculate

$$I_\alpha(\rho) = D_\alpha(\rho||\mathbb{1}/d), \tag{C.20}$$
$$= \frac{1}{\alpha-1}\ln\frac{\mathrm{Tr}\,\left(\rho^\alpha(\mathbb{1}/d)^{1-\alpha}\right)}{\mathrm{Tr}\,\rho}, \tag{C.21}$$
$$= \frac{1}{\alpha-1}\ln\frac{(1/d)^{1-\alpha}\mathrm{Tr}\,\left(\rho^\alpha\mathbb{1}\right)}{\mathrm{Tr}\,\rho}, \tag{C.22}$$
$$= \frac{1}{\alpha-1}\ln\frac{\mathrm{Tr}\,\left(\rho^\alpha\right)}{\mathrm{Tr}\,\rho} + \frac{1}{\alpha-1}\ln d^{\alpha-1}, \tag{C.23}$$
$$= \ln d - S_\alpha(\rho). \tag{C.24}$$

Now, let us check the Eq. (4.63a). From (4.45), we have

$$I^{\alpha\uparrow}_{\mathcal{A}|\mathcal{B}}(\rho) = D_\alpha\left(\rho\,\middle\|\,\frac{\mathbb{1}_\mathcal{A}}{d_\mathcal{A}}\otimes\rho_\mathcal{B}\right), \tag{C.25}$$
$$= \frac{1}{\alpha-1}\ln\frac{\mathrm{Tr}\,\left[\rho^\alpha(\mathbb{1}_\mathcal{A}/d_\mathcal{A}\otimes\rho_\mathcal{B})^{1-\alpha}\right]}{\mathrm{Tr}\,\rho}, \tag{C.26}$$
$$= \frac{1}{\alpha-1}\ln\frac{(1/d_\mathcal{A})^{1-\alpha}\mathrm{Tr}\,\left[\rho^\alpha(\mathbb{1}_\mathcal{A}\otimes\rho_\mathcal{B})^{1-\alpha}\right]}{\mathrm{Tr}\,\rho}, \tag{C.27}$$
$$= \ln d_\mathcal{A} + D_\alpha(\rho||\mathbb{1}_\mathcal{A}\otimes\rho_\mathcal{B}), \tag{C.28}$$
$$= \ln d_\mathcal{A} - H^{\alpha\downarrow}_{\mathcal{A}|\mathcal{B}}(\rho). \tag{C.29}$$

C.3 TSALLIS RELATIVE ENTROPIES

C.3.1 Monotonicity on q

It is known that the Rényi divergences are monotonically increasing functions on their parameter alpha [93]. Although there is no such result in the literature regarding Tsallis relative entropies, we can conjecture its validity based on the following. Just for now, let us use superscripts T and R to better distinguish between the Tsallis and the Rényi divergences. It is easy to show that

$$D_q^T(\rho||\sigma) = \frac{1}{1-q}\left[1 - e^{(q-1)D_q^R(\rho||\sigma)}\right]. \tag{C.30}$$

The derivative of the above equation with respect to the parameter q can be written as

$$\frac{\mathrm{d}}{\mathrm{d}q}D_q^T(\rho||\sigma) = \frac{1+e^{(q-1)D_q^R(\rho||\sigma)}}{(q-1)^2}\left[-1+(q-1)D_q^R(\rho||\sigma)+(q-1)^2\frac{\mathrm{d}}{\mathrm{d}q}D_q^R(\rho||\sigma)\right]. \tag{C.31}$$

We know that $0 \leqslant D_q^R(\rho||\sigma) \leqslant \ln 2$ and also that the derivative of D_q^R must always be non-negative (see Theorem 7 in Ref. [93]), but the non-negativity of (C.31) remains an open question. However, although we have no analytical proof that (C.31) is non-negative for all $q > 0$, our numerical incursions given the aforementioned constraints point to a positive answer which indicates that in fact, $q \mapsto D_q^T(\rho||\sigma)$ is a monotonically increasing function for fixed states.

C.4 GEOMETRIC CONDITIONAL INFORMATION

For $v_0 = \rho \otimes \mathbb{1}_\mathcal{E}/d_\mathcal{E}$, the Eq. (6.28) reads:

$$\begin{aligned} I^\square_{\mathcal{E}|\mathcal{S}}(v_0) &= d^n_\square\left(v_0, \mathrm{Tr}_\mathcal{E} v_0 \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), &\text{(C.32)}\\ &= d^n_\square\left(\rho\otimes|e_0\rangle\langle e_0|, \rho\otimes\frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right). &\text{(C.33)}\end{aligned}$$

After the unitary evolution, we have:

$$\begin{aligned} I^\square_{\mathcal{E}|\mathcal{S}}(v_t) &= d^n_\square\left(v_t, \mathrm{Tr}_\mathcal{E} v_t \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), &\text{(C.34)}\\ &= d^n_\square\left(U_t\left(\rho\otimes|e_0\rangle\langle e_0|\right)U_t^\dagger, \Phi_A(\rho)\otimes\frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), &\text{(C.35)}\\ &= d^n_\square\left(U_t\left(\rho\otimes|e_0\rangle\langle e_0|\right)U_t^\dagger, U_t\left(\Phi_A(\rho)\otimes\frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right)U_t^\dagger\right), \quad \text{(Theorem 7)} &\text{(C.36)}\\ &= d^n_\square\left(\rho\otimes|e_0\rangle\langle e_0|, \Phi_A(\rho)\otimes\frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right). &\text{(C.37)}\end{aligned}$$

Now, let us evaluate the above expressions for the following specific cases:

C.4.1 L_p-distances

$$I^p_{\mathcal{E}|\mathcal{S}}(v_0) = d^p_p\left(\rho \otimes |e_0\rangle\langle e_0|, \rho \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), \tag{C.38}$$

$$= \left\|\rho \otimes |e_0\rangle\langle e_0| - \rho \otimes \tfrac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right\|^p_p, \tag{C.39}$$

$$= \|\rho\|^p_p \left\||e_0\rangle\langle e_0| - \tfrac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right\|^p_p, \tag{C.40}$$

$$= \|\rho\|^p_p \left\|\textstyle\sum_{k=0}^{d_\mathcal{E}-1}\left(\delta_{k,0} - \tfrac{1}{d_\mathcal{E}}\right)|e_k\rangle\langle e_k|\right\|^p_p, \tag{C.41}$$

$$= \|\rho\|^p_p \left\{\mathrm{Tr}\left|\sum_{k=0}^{d_\mathcal{E}-1}\left(\delta_{k,0} - \frac{1}{d_\mathcal{E}}\right)|e_k\rangle\langle e_k|\right|^p\right\}, \tag{C.42}$$

$$= \|\rho\|^p_p \left\{\sum_{k=0}^{d_\mathcal{E}-1}\left|\delta_{k,0} - \frac{1}{d_\mathcal{E}}\right|^p \mathrm{Tr}\,|e_k\rangle\langle e_k|\right\}, \tag{C.43}$$

$$= \|\rho\|^p_p \left\{\sum_{k=0}^{d_\mathcal{E}-1}\left|\delta_{k,0} - \frac{1}{d_\mathcal{E}}\right|^p\right\}, \tag{C.44}$$

$$= \|\rho\|^p_p \left\{\left(1 - \frac{1}{d_\mathcal{E}}\right)^p + \frac{d_\mathcal{E} - 1}{d^p_\mathcal{E}}\right\}, \tag{C.45}$$

$$= \|\rho\|^p_p \frac{1}{d^p_\mathcal{E}}\left[(d_\mathcal{E} - 1)^p + d_\mathcal{E} - 1\right]. \tag{C.46}$$

$$I^p_{\mathcal{E}|\mathcal{S}}(v_t) = d^p_p\left(\rho \otimes |e_0\rangle\langle e_0|, \Phi_A(\rho) \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), \tag{C.47}$$

$$= \left\|\rho \otimes |e_0\rangle\langle e_0| - \Phi_A(\rho) \otimes \tfrac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right\|^p_p, \tag{C.48}$$

$$= \left\|\textstyle\sum_{k=0}^{d_\mathcal{E}-1}\left(\delta_{k,0}\rho - \dfrac{\Phi_A(\rho)}{d_\mathcal{E}}\right) \otimes |e_k\rangle\langle e_k|\right\|^p_p, \tag{C.49}$$

$$= \mathrm{Tr}\left|\sum_{k=0}^{d_\mathcal{E}-1}\left(\delta_{k,0}\rho - \frac{\Phi_A(\rho)}{d_\mathcal{E}}\right) \otimes |e_k\rangle\langle e_k|\right|^p, \tag{C.50}$$

$$= \sum_{k=0}^{d_\mathcal{E}-1}\mathrm{Tr}\left|\delta_{k,0}\rho - \frac{\Phi_A(\rho)}{d_\mathcal{E}}\right|^p \mathrm{Tr}\,|e_k\rangle\langle e_k|, \tag{C.51}$$

$$= \sum_{k=0}^{d_\mathcal{E}-1}\mathrm{Tr}\left|\delta_{k,0}\rho - \frac{\Phi_A(\rho)}{d_\mathcal{E}}\right|^p, \tag{C.52}$$

$$= \mathrm{Tr}\left|\rho - \frac{\Phi_A(\rho)}{d_\mathcal{E}}\right|^p + (d_\mathcal{E} - 1)\,\mathrm{Tr}\left|\frac{\Phi_A(\rho)}{d_\mathcal{E}}\right|^p, \tag{C.53}$$

$$= d^p_p\left(\rho, \frac{\Phi_A(\rho)}{d_\mathcal{E}}\right) + \frac{d_\mathcal{E} - 1}{d^p_\mathcal{E}}\|\Phi_A(\rho)\|^p_p \tag{C.54}$$

$$\Delta I^p_{\mathcal{E}|\mathcal{S}}(v_t, v_0) = d^p_p\left(\rho, \frac{\Phi_A(\rho)}{d_\mathcal{E}}\right) - \frac{d_\mathcal{E} - 1}{d^p_\mathcal{E}}\left(\|\Phi_A(\rho)\|^p_p - \|\rho\|^p_p\right) - \|\rho\|^p_p\left(\frac{d_\mathcal{E} - 1}{d_\mathcal{E}}\right)^p \tag{C.55}$$

Trace distance ($p = 1$):

$$I^{\mathrm{Tr}}_{\mathcal{E}|\mathcal{S}}\left(v_0\right) = 2\left(\frac{d_{\mathcal{E}} - 1}{d_{\mathcal{E}}}\right). \tag{C.56}$$

$$I^{\mathrm{Tr}}_{\mathcal{E}|\mathcal{S}}\left(v_t\right) = d_{\mathrm{Tr}}\left(\rho, \frac{\Phi_A(\rho)}{d_{\mathcal{E}}}\right) + \frac{d_{\mathcal{E}} - 1}{d_{\mathcal{E}}}. \tag{C.57}$$

$$\Delta I^{\mathrm{Tr}}_{\mathcal{E}|\mathcal{S}}\left(v_t, v_0\right) = d_{\mathrm{Tr}}\left(\rho, \frac{\Phi_A(\rho)}{d_{\mathcal{E}}}\right) - \frac{d_{\mathcal{E}} - 1}{d_{\mathcal{E}}}. \tag{C.58}$$

Hilbert-Schmidt distance ($p = 2$):

$$I^{\mathrm{HS}}_{\mathcal{E}|\mathcal{S}}\left(v_0\right) = \frac{d_{\mathcal{E}} - 1}{d_{\mathcal{E}}}\|\rho\|_2^2. \tag{C.59}$$

$$\begin{aligned}
I^{\mathrm{HS}}_{\mathcal{E}|\mathcal{S}}\left(v_t\right) &= \mathrm{Tr}\left(\rho - \frac{\Phi_A(\rho)}{d_{\mathcal{E}}}\right)^2 + \frac{d_{\mathcal{E}} - 1}{d_{\mathcal{E}}^2}\mathrm{Tr}\,\Phi_A(\rho)^2, && \text{(C.60)}\\
&= \mathrm{Tr}\,\rho^2 - \frac{2}{d_{\mathcal{E}}}\mathrm{Tr}\,\rho\Phi_A(\rho) + \frac{1}{d_{\mathcal{E}}^2}\mathrm{Tr}\,\Phi_A(\rho)^2 + \frac{d_{\mathcal{E}} - 1}{d_{\mathcal{E}}^2}\mathrm{Tr}\,\Phi_A(\rho)^2, && \text{(C.61)}\\
&= \mathrm{Tr}\,\rho^2 - \frac{2}{d_{\mathcal{E}}}\mathrm{Tr}\,\Phi_A(\rho)^2 + \frac{1}{d_{\mathcal{E}}}\mathrm{Tr}\,\Phi_A(\rho)^2, \qquad \text{(Lemma 1)} && \text{(C.62)}\\
&= \|\rho\|_2^2 - \frac{1}{d_{\mathcal{E}}}\|\Phi_A(\rho)\|_2^2. && \text{(C.63)}
\end{aligned}$$

$$\begin{aligned}
\Delta I^{\mathrm{HS}}_{\mathcal{E}|\mathcal{S}}\left(v_t, v_0\right) &= \frac{1}{d_{\mathcal{E}}}\left(\|\rho\|_2^2 - \|\Phi_A(\rho)\|_2^2\right), && \text{(C.64)}\\
&= \frac{1}{d_{\mathcal{E}}}\|\rho - \Phi_A(\rho)\|_2^2, \qquad \text{(Lemma 5)} && \text{(C.65)}\\
&= \frac{1}{d_{\mathcal{E}}}d^2_{\mathrm{HS}}\left(\rho, \Phi_A(\rho)\right). && \text{(C.66)}
\end{aligned}$$

C.4.2 Bures distance

For the Bures distance, first we need the Uhlmann fidelity:

$$\begin{aligned}
F\left(\rho \otimes |e_0\rangle\langle e_0|, \rho \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}\right) &= \left\|\sqrt{\rho \otimes |e_0\rangle\langle e_0|}\sqrt{\rho \otimes \tfrac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}}\right\|_1^2, && \text{(C.67)}\\
&= \left\|\left(\sqrt{\rho} \otimes |e_0\rangle\langle e_0|\right)\left(\sqrt{\rho} \otimes \tfrac{\mathbb{1}_{\mathcal{E}}}{\sqrt{d_{\mathcal{E}}}}\right)\right\|_1^2, && \text{(C.68)}\\
&= \left\|\rho \otimes \tfrac{|e_0\rangle\langle e_0|}{\sqrt{d_{\mathcal{E}}}}\right\|_1^2, && \text{(C.69)}\\
&= \frac{1}{d_{\mathcal{E}}}\|\rho\|_1^2\|\,|e_0\rangle\langle e_0|\,\|_1^2, && \text{(C.70)}\\
&= \frac{1}{d_{\mathcal{E}}}. && \text{(C.71)}
\end{aligned}$$

$$\begin{aligned}
F\left(\rho \otimes |e_0\rangle\langle e_0|, \Phi_A(\rho) \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}\right) &= \left\|\sqrt{\rho \otimes |e_0\rangle\langle e_0|}\sqrt{\Phi_A(\rho) \otimes \tfrac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}}\right\|_1^2, && \text{(C.72)}\\
&= \left\|\left(\sqrt{\rho} \otimes |e_0\rangle\langle e_0|\right)\left(\sqrt{\Phi_A(\rho)} \otimes \tfrac{\mathbb{1}_{\mathcal{E}}}{\sqrt{d_{\mathcal{E}}}}\right)\right\|_1^2, && \text{(C.73)}
\end{aligned}$$

$$= \left\| \sqrt{\rho}\sqrt{\Phi_A(\rho)} \otimes \frac{|e_0\rangle\langle e_0|}{\sqrt{d_\mathcal{E}}} \right\|_1^2, \tag{C.74}$$

$$= \frac{1}{d_\mathcal{E}} \| \sqrt{\rho}\sqrt{\Phi_A(\rho)} \|_1^2 \| \, |e_0\rangle \langle e_0| \, \|_1^2, \tag{C.75}$$

$$= \frac{1}{d_\mathcal{E}} F(\rho, \Phi_A(\rho)). \tag{C.76}$$

Now, we can calculate:

$$I^{\mathrm{Bu}}_{\mathcal{E}|\mathcal{S}}(v_0) = d^2_{\mathrm{Bu}}\left(\rho \otimes |e_0\rangle \langle e_0| \, , \rho \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), \tag{C.77}$$

$$= 2 - 2\sqrt{F\left(\rho \otimes |e_0\rangle \langle e_0| \, , \rho \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right)}, \tag{C.78}$$

$$= 2 - \frac{2}{\sqrt{d_\mathcal{E}}}. \tag{C.79}$$

$$I^{\mathrm{Bu}}_{\mathcal{E}|\mathcal{S}}(v_t) = d^2_{\mathrm{Bu}}\left(\rho \otimes |e_0\rangle \langle e_0| \, , \Phi_A(\rho) \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), \tag{C.80}$$

$$= 2 - 2\sqrt{F\left(\rho \otimes |e_0\rangle \langle e_0| \, , \Phi_A(\rho) \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right)}, \tag{C.81}$$

$$= 2 - 2\sqrt{\frac{1}{d_\mathcal{E}} F\left(\rho, \Phi_A(\rho)\right)}. \tag{C.82}$$

$$\Delta I^{\mathrm{Bu}}_{\mathcal{E}|\mathcal{S}}(v_t, v_0) = \frac{1}{\sqrt{d_\mathcal{E}}}\left(2 - 2\sqrt{F(\rho, \Phi_A(\rho))}\right), \tag{C.83}$$

$$= \frac{1}{\sqrt{d_\mathcal{E}}} d^2_{\mathrm{Bu}}\left(\rho, \Phi_A(\rho)\right). \tag{C.84}$$

C.4.3 Hellinger distance

$$I^{\mathrm{He}}_{\mathcal{E}|\mathcal{S}}(v_0) = d^2_{\mathrm{He}}\left(\rho \otimes |e_0\rangle \langle e_0| \, , \rho \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), \tag{C.85}$$

$$= 2 - 2\mathrm{Tr}\left(\sqrt{\rho \otimes |e_0\rangle \langle e_0|}\sqrt{\rho \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}}\right), \tag{C.86}$$

$$= 2 - 2\mathrm{Tr}\left(\rho \otimes \frac{|e_0\rangle \langle e_0|}{\sqrt{d_\mathcal{E}}}\right), \tag{C.87}$$

$$= 2 - \frac{2}{\sqrt{d_\mathcal{E}}}\mathrm{Tr}\,(\rho)\mathrm{Tr}\,(|e_0\rangle \langle e_0|), \tag{C.88}$$

$$= 2 - \frac{2}{\sqrt{d_\mathcal{E}}}. \tag{C.89}$$

$$I^{\mathrm{He}}_{\mathcal{E}|\mathcal{S}}(v_t) = d^2_{\mathrm{He}}\left(\rho \otimes |e_0\rangle \langle e_0| \, , \Phi_A(\rho) \otimes \frac{\mathbb{1}_\mathcal{E}}{d_\mathcal{E}}\right), \tag{C.90}$$

$$= 2 - 2\mathrm{Tr}\left(\sqrt{\rho \otimes |e_0\rangle\langle e_0|}\sqrt{\Phi_A(\rho) \otimes \frac{\mathbb{1}_{\mathcal{E}}}{d_{\mathcal{E}}}}\right), \tag{C.91}$$

$$= 2 - 2\mathrm{Tr}\left(\sqrt{\rho}\sqrt{\Phi_A(\rho)} \otimes \frac{|e_0\rangle\langle e_0|}{\sqrt{d_{\mathcal{E}}}}\right), \tag{C.92}$$

$$= 2 - \frac{2}{\sqrt{d_{\mathcal{E}}}}\mathrm{Tr}\left(\sqrt{\rho}\sqrt{\Phi_A(\rho)}\right)\mathrm{Tr}\left(|e_0\rangle\langle e_0|\right), \tag{C.93}$$

$$= 2 - \frac{2}{\sqrt{d_{\mathcal{E}}}}\mathrm{Tr}\left(\sqrt{\rho}\sqrt{\Phi_A(\rho)}\right). \tag{C.94}$$

$$\Delta I^{\mathrm{He}}_{\mathcal{E}|\mathcal{S}}\left(v_t, v_0\right) = \frac{1}{\sqrt{d_{\mathcal{E}}}}\left[2 - 2\mathrm{Tr}\left(\sqrt{\rho}\sqrt{\Phi_A(\rho)}\right)\right], \tag{C.95}$$

$$= \frac{1}{\sqrt{d_{\mathcal{E}}}} d^2_{\mathrm{He}}\left(\rho, \Phi_A(\rho)\right). \tag{C.96}$$

C.5 EVOLUTION OF $1 + m$ QUBITS

$$\begin{aligned}
\rho_{\mathcal{S}\mathcal{F}_m}(t) &= U^{(m)}U^{(m-1)}\cdots U^{(2)}U^{(1)}\left[\rho \otimes \bigotimes_{k=1}^{m} |0^k\rangle\langle 0^k|\right] U^{(1)}U^{(2)}\cdots U^{(m-1)}U^{(m)} && \text{(C.97)}\\
&= U^{(m)}U^{(m-1)}\cdots U^{(2)}\left[\sum_{ij} P_i\rho P_j \otimes \left(T_i|0^1\rangle\langle 0^1|T_j\right) \otimes \bigotimes_{k=2}^{m} |0^k\rangle\langle 0^k|\right] U^{(2)}\cdots U^{(m-1)}U^{(m)} && \text{(C.98)}\\
&= U^{(m)}U^{(m-1)}\cdots\left[\sum_{iji'j'} P_{i'}P_i\rho P_j P_{j'} \otimes \left(T_i|0^1\rangle\langle 0^1|T_j\right) \otimes \left(T_{i'}|0^2\rangle\langle 0^2|T_{j'}\right) \otimes \bigotimes_{k=3}^{m} |0^k\rangle\langle 0^k|\right]\cdots U^{(m-1)}U^{(m)} && \text{(C.99)}\\
&= U^{(m)}U^{(m-1)}\cdots\left[\sum_{ij} P_i\rho P_j \otimes \left(T_i|0^1\rangle\langle 0^1|T_j\right) \otimes \left(T_i|0^2\rangle\langle 0^2|T_j\right) \otimes \bigotimes_{k=3}^{m} |0^k\rangle\langle 0^k|\right]\cdots U^{(m-1)}U^{(m)} && \text{(C.100)}\\
&= \sum_{ij} P_i\rho P_j \otimes \bigotimes_{k=1}^{m} T_i|0^k\rangle\langle 0^k|T_j && \text{(C.101)}
\end{aligned}$$