

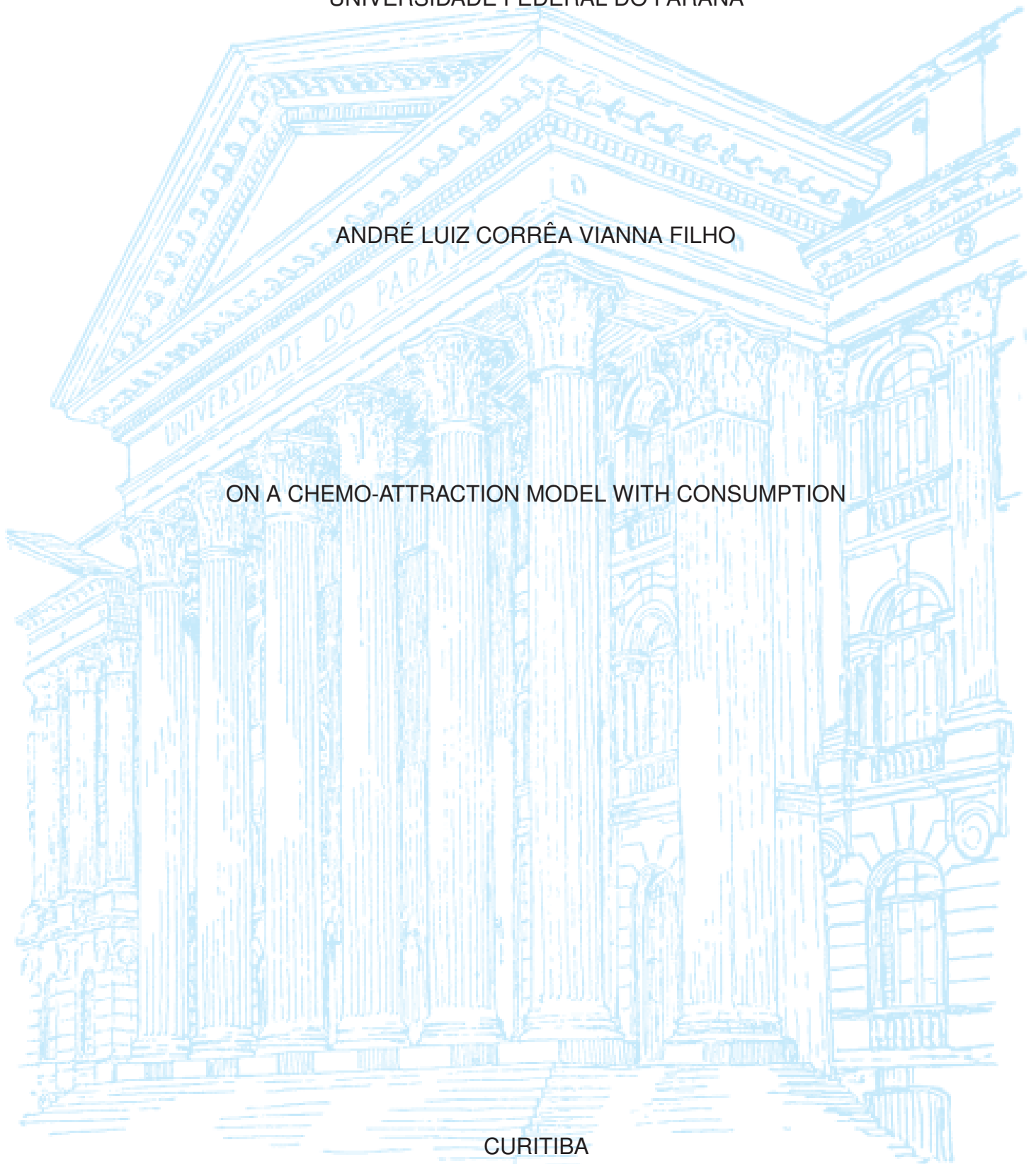
UNIVERSIDADE FEDERAL DO PARANÁ

ANDRÉ LUIZ CORRÊA VIANNA FILHO

ON A CHEMO-ATTRACTION MODEL WITH CONSUMPTION

CURITIBA

2022



ANDRÉ LUIZ CORRÊA VIANNA FILHO

ON A CHEMO-ATTRACTION MODEL WITH CONSUMPTION

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Orientador: Professor Pedro Danizete Damázio
Coorientador: Professor Francisco Manuel Guillén-González

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RESUMO

Neste trabalho, investigamos um modelo de quimioatração com consumo, que chamamos de modelo original, em domínios não-suaves e limitados de $\mathbb{R}^d$ ($d = 1, 2, 3$). Os termos não-lineares relacionados aos fenômenos de quimiotaxia e consumo introduzem muitas dificuldades na análise do modelo original, principalmente para sua simulação numérica. A fim de superar estas dificuldades, propomos e analisamos um modelo regularizado, estabelecendo resultados sobre a existência, unicidade, propriedades de regularidade e convergência para o modelo original. Após estabelecer a convergência do modelo regularizado para o modelo original, procedemos à aproximação numérica do modelo regularizado. Introduzimos três esquemas numéricos totalmente discretos baseados em Elementos Finitos. Para cada esquema, estudamos a existência de solução, a convergência para o modelo regularizado, as condições para a unicidade de solução e as taxas de erro em termos dos parâmetros de discretização.

Palavras-chaves: quimiotaxia; quimioatração; consumo; regularização, elementos finitos.

ABSTRACT

In this work we investigate a chemo-attraction with consumption model, which we call the original model, in non-smooth bounded domains of $\mathbb{R}^d$ ($d = 1, 2, 3$). The non-linear terms related to the chemotaxis and consumption phenomena introduce many difficulties to the analysis of the original model, mainly for its numerical simulation. In order to overcome these difficulties, we propose and analyze a regularized model, establishing results on existence, uniqueness, regularity's properties and convergence towards the original model. After establishing convergence of the regularized model towards the original model, we proceed to the numerical approximation of the regularized model. We introduce three fully discrete numerical schemes based on Finite Elements. For each scheme, we study the existence of solution, the convergence towards the regularized model, the conditions for uniqueness of solution and the accurate rates for the error in terms of the discretization parameters.

Key-words: chemotaxis; chemo-attraction; consumption; regularization; finite elements.

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INTRODUCTION

Chemotaxis is the movement of cells in response to the concentration gradient of a chemical substance. Cells can be attracted or repelled by the greater concentration of this chemical substance. This phenomenon is present in many physiological events, such as wound healing, immune cells migration, among others. Chemotaxis also plays an important role in undesired events, such as inflammatory diseases and cancer metastasis (MURPHY, 2001) (WANG et al., 2011). Besides, there are also studies which call the attention to the effects of chemotaxis in the migration of bacteria towards polluting substances and its usefulness in the degradation of these substances (PANDEY; JAIN, 2002) (PARALES et al., 2000). All this makes chemotaxis a matter of very practical interest.

One of the first models for chemotaxis is attributed to Evelyn Keller and Lee Segel (KELLER; SEGEL, 1970) (KELLER; SEGEL, 1971). The contribution of the former work (PATLAK, 1953) to the development of the aforementioned model is also considered. Since then, many mathematicians have studied this model and others which can be obtained from the Keller-Segel model in order to develop models for other situations involving chemotaxis as well as to obtain more realistic models, from the biological point of view (BELLOMO et al., 2015) (HORSTMANN, 2003) (HORSTMANN, 2004).

In the present work, we are interested in a model in which the cells are attracted by a chemical substance that they consume. Let Ω be a bounded domain of $\mathbf{R}^d$ ($d = 1, 2, 3$) and let Γ be its boundary. Let $u(t, x)$ and $v(t, x)$ be real functions defined on $x \in \Omega$ and $t > 0$ representing the density of cell population and the concentration of chemical substance, respectively. This model is governed by the set of equations

$$\begin{aligned}\partial_t u - \Delta u &= -\nabla \cdot (u \nabla v) \\ \partial_t v - \Delta v &= -u^s v,\end{aligned}$$

where $\nabla \cdot (u \nabla v)$ is the chemotaxis term and u^s is the consumption rate, with $s \geq 1$. We complete the system (which we will call problem (P)) with the Neumann boundary and

initial conditions:

$$\begin{aligned}
 \partial_t u - \Delta u &= -\nabla \cdot (u \nabla v) \\
 \partial_t v - \Delta v &= -u^s v \\
 \partial_\eta u \Big|_\Gamma &= \partial_\eta v \Big|_\Gamma = 0 \\
 u(0) &= u^0, \quad v(0) = v^0,
 \end{aligned} \tag{P}$$

where the initial conditions u^0 and v^0 are non-negative functions.

Theoretical properties of this model have been studied in different contexts when $s = 1$: existence of very regular and uniformly bounded solutions is proved in (TAO; WINKLER, 2012) for smooth and convex 2D domains; in (TAO, 2011), the author imposes a smallness condition on v^0 and obtain existence and uniqueness of global solution. In (WINKLER, 2012) and (JIANG et al., 2015) this model is coupled with fluids and studied in convex and non-convex domains, respectively. We remark that all these works are carried out considering smooth domains. We also observe that, in (JIANG et al., 2015), the authors extend the results of (WINKLER, 2012) to non-convex domains, but some estimates that were not time-dependent become time-dependent.

In the context of chemotaxis models, researchers are also interested in the control theory of these models. If we consider the chemotaxis model with chemoattraction and production, we have the study (RYU; YAGI, 2001). Concerning the chemorepulsion model with production, we refer to the studies (GUILLÉN-GONZÁLEZ et al., 2020c), (GUILLÉN-GONZÁLEZ et al., 2020b) and (GUILLÉN-GONZÁLEZ et al., 2020a). For the present chemo-attraction model with consumption, we have the study (LÓPEZ-RÍOS; VILLAMIZAR-ROA, 2021), on the optimal control of the chemo-attraction model coupled with fluids.

Another interesting and growing research topic on chemotaxis models is their numerical approximation. Besides the practical interest in the applications of the chemotaxis models, these models present interesting challenges to their numerical approximation. Therefore, it is expected that finding the appropriate technique to deal with these difficulties may be useful for the approximation of other PDE models (GUTIERREZ-SANTACREU; RODRÍGUEZ-GALVÁN, 2021).

Although the numerical simulation of chemotaxis models is a relevant and growing research topic, when we turn to problem (P), we still find a relatively small amount of studies on its numerical approximation. This is probably due to the difficulty that is imposed by the complex relation between the chemoattraction and consumption terms. As we can see in Section 3, the procedures on which the theoretical analysis relies are difficult to adapt in a numerical method.

To the best of our knowledge, we can cite two studies: (DUARTE-RODRÍGUEZ et al., 2021) and (GUILLÉN-GONZÁLEZ; TIERRA, 2021). In (DUARTE-RODRÍGUEZ et al., 2021) a chemotaxis-Navier-Stokes system is approached via Finite Elements

(FE) and an analysis of optimal error estimates is carried out, assuming the existence of a sufficiently regular solution. In (GUILLÉN-GONZÁLEZ; TIERRA, 2021), inspired by the treatment given to the chemo-repulsion model with linear production in (GUILLÉN-GONZÁLEZ et al., 2019), several finite element schemes are designed to approximate the system, focusing on properties such as conservation of cells, energy stability and approximate positivity rather than convergence.

In view of the relatively low number of studies on the numerical approximation of (P) and the probable reasons for it, the objective of the present study is to design an adequate background to the development of numerical approximations of the solutions of the chemotaxis models (P). As far as we know, there are not studies on the model (P) in the cases $s > 1$, neither from the theoretical nor from the numerical point of view. In this sense, this work gives the following main contributions:

- (i) generalization of the model with the consumption term $u^s v$ for $s \geq 1$ (instead of only uv);
- (ii) enlargement of the class of considered domains, maintaining the no blow-up effect in the $2D$ case and some time-independent estimates for spatial derivatives in $2D$ and $3D$ domains;
- (iii) the introduction and analysis of a regularized model (existence, uniqueness, regularity, positivity, a priori estimates and convergence towards the original model) that will be useful for designing numerical schemes that will be used in order to approximate the original model;
- (iv) the design of fully discrete numerical schemes for the approximation of the regularized model.

In order to reach such results, the adopted procedure consists in proposing the following regularized problems, which depend on a truncation parameter $m \in \mathbb{N}$,

$$\begin{aligned}
 \partial_t u_m - \Delta u_m &= -\nabla \cdot (a_m(u_m) \nabla v_m) \\
 \partial_t v_m - \Delta v_m &= -a_m(u_m)^s v_m \\
 \partial_\eta u_m|_\Gamma &= \partial_\eta v_m|_\Gamma = 0 \\
 u_m(0) &= u^0, \quad v_m(0) = v^0,
 \end{aligned} \tag{P}_m$$

where $a_m(\cdot)$ is the following truncation of the identity, from above and from below:

$$a_m(u) = \begin{cases} 0, & \text{if } u \leq 0 \\ u, & \text{if } u \in [0, m-1] \\ C^2 \text{ extension}, & \text{if } u \in (m-1, m], \\ m, & \text{if } u \geq m. \end{cases} \tag{1}$$

Remark 1. The C^2 extension can be chosen so that $0 \leq a'_m(u) \leq C$, $\forall u \in \mathbb{R}$, for some $C > 0$, and $a''_m(u)$ is uniformly bounded, for $u \geq \beta > 0$, with respect to m . ■

With the objective of enlarging the class of considered domains, we state and demonstrate our results in terms of the regularity of the Poisson-Neumann problem

$$\begin{cases} -\Delta v = f \text{ in } \Omega, \\ \partial_\eta v|_\Gamma = 0, \int_\Omega v(x) dx = 0, \end{cases}$$

in which the function f satisfies the compatibility condition

$$\int_\Omega f dx = 0,$$

(see definition 21 in page 23); and, when necessary, in terms of the following technical hypothesis:

Hypothesis (H1). For each $z \in H^2(\Omega)$ such that $\partial_\eta z|_\Gamma = 0$ there is a sequence $\{\rho_n\} \subset C^2(\overline{\Omega})$ such that $\partial_\eta \rho_n|_\Gamma = 0$ and $\rho_n \rightarrow z$ in $H^2(\Omega)$.

In order to show that the hypothesis (H1) is not too restrictive, we prove in Lemma 47 that if Γ is of class $C^{2,1}$ then hypothesis (H1) is verified.

Let us consider the average of u^0

$$u^* = \frac{1}{|\Omega|} \int_\Omega u^0(x) dx.$$

Now we highlight our main results of this PhD Thesis. Regarding existence, regularity, and uniqueness of solution, we point out the following three results, whose complete proofs are found in Chapter 2 and Sections 3.1 and 3.2, respectively.

Theorem 2. (Existence of global solution for the regularized problem (P_m)) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that the Neumann problem (1.2) has the H^2 -regularity (see definition 21 in page 23). Let $u^0 \in H^1(\Omega)$ and $v^0 \in H^2(\Omega)$ be non-negative functions. Then, for each $m \in \mathbb{N}$ the regularized problem (P_m) has a unique solution (u_m, v_m) such that:

$$u_m - u^* \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)),$$

$$\partial_t u_m \in L^2(0, \infty; L^2(\Omega)),$$

$$v_m \in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; H^2(\Omega)),$$

$$\partial_t v_m \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)).$$

If moreover the regularity $W^{2,3}$ -regularity of (1.2) is imposed and $u^0 \in H^2(\Omega)$ then

$$(u_m - u^*), v_m \in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; H^2(\Omega)),$$

$$\partial_t u_m, \Delta u_m, \partial_t v_m, \Delta v_m \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)).$$

Theorem 3. (3D. Existence of global weak solutions) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that the Neumann problem (1.2) has the H^2 -regularity (see definition 21 in page 23) and Hypothesis (H1) is satisfied. Let $u^0 \in L^{1+\varepsilon}(\Omega)$, for some $\varepsilon > 0$, if $s = 1$, and $u^0 \in L^s(\Omega)$, if $s > 1$, and $v^0 \in H^1(\Omega) \cap L^\infty(\Omega)$ be non-negative functions. Then there is a non-negative weak solution (u, v) of the original problem (P), for $s \geq 1$, obtained through a limit of non-negative solutions (u_m, v_m) of the regularized problems (P_m) as $m \rightarrow \infty$ and such that

$$\begin{cases} \int_{\Omega} u(t, x) dx = \int_{\Omega} u^0(x) dx, \text{ a.e. } t \in (0, \infty) \\ 0 \leq v(t, x) \leq \|v^0\|_{L^\infty(\Omega)}, \text{ a.e. } (t, x) \in (0, \infty) \times \Omega, \\ \begin{cases} u \in L^\infty(0, \infty; L^s(\Omega)) \cap L_{loc}^{5s/3}([0, \infty); L^{5s/3}(\Omega)), & \text{if } s \geq 1, \\ u^{s/2} \nabla v \in L^2(0, \infty; L^2(\Omega)), & \text{if } s \geq 1, \end{cases} \\ \begin{cases} \nabla u \in L^2(0, \infty; L^s(\Omega)) \cap L_{loc}^{\frac{5s}{3+s}}([0, \infty); L^{\frac{5s}{3+s}}(\Omega)), & \text{if } s \in [1, 2), \\ \nabla u \in L^2(0, \infty; L^2(\Omega)), & \text{if } s \geq 2, \end{cases} \\ \begin{cases} u \nabla v \in L^2(0, \infty; L^s(\Omega)), & \text{if } s \in [1, 2), \\ u \nabla v \in L^2(0, \infty; L^2(\Omega)), & \text{if } s \geq 2 \end{cases} \end{cases}$$

and

$$\begin{aligned} v &\in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)), \\ \nabla v &\in L^4(0, \infty; L^4(\Omega)), \end{aligned}$$

Remark 4. We remark that, from the regularities of u and v that are listed in Theorem 3, we can conclude that

$$\begin{cases} u_t \in L^2\left(0, \infty; (W^{1,s/(s-1)}(\Omega))'\right), & \text{if } s \in [1, 2), \\ u_t \in L^2\left(0, \infty; (H^1(\Omega))'\right), & \text{if } s \geq 2 \end{cases}$$

and

$$v_t \in L^2(0, \infty; L^{3/2}(\Omega)).$$

Attending to the regularity of (u, v) given so far, one has that (u, v) satisfies the u -equation of (P) in a variational sense, while the v -equation is satisfied *a.e.* in $(0, \infty) \times \Omega$. Moreover, the initial conditions have a sense because, thanks to the regularity of u, v, u_t and v_t , (u, v) is weakly continuous from $[0, \infty)$ to $L^s(\Omega) \times H^1(\Omega)$, if $s \in [1, 2]$, and $L^2(\Omega) \times H^1(\Omega)$, if $s \geq 2$ (see Chapter 3 of (TEMAM, 1977)). ■

Remark 5. Note that, for $s \in [1, 2]$, the regularity of ∇u increases as s increases. When we consider $s > 2$, the regularity of ∇u and u_t does not increase as s increases anymore. But we observe that for $s \geq 1$ the function $u^{s/2}$ has the s -independent regularity $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$. On the other hand, the regularity of the function v is independent of s . ■

Theorem 6. (2D. Existence and uniqueness of global strong solution) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain such that the Neumann problem (1.2) has the $W^{2,3}$ -regularity (see definition 21 in page 23) and hypothesis (H1) is satisfied. Let $u^0 \in H^2(\Omega)$ and $v^0 \in H^2(\Omega)$ be such that $u^0 \geq 0$ and $v^0 \geq 0$. Then there is a unique solution (u, v) for the original problem (P), for $s \geq 1$, such that

$$\begin{aligned} u, v &\in L^\infty(0, \infty; H^2(\Omega)), \\ \Delta u, \Delta v, u_t, v_t &\in L^2(0, \infty; H^1(\Omega)), \\ (u - u^*), v &\in L^2(0, \infty; W^{2,3}(\Omega)). \end{aligned}$$

In particular, u does not blow-up neither in finite nor in infinite time, that is $u \in L^\infty(0, \infty; L^\infty(\Omega))$. Consequently, there is $m_0 \in \mathbb{N}$ such that

$$(u_m, v_m) = (u, v) \text{ a.e. in } (0, \infty) \times \Omega,$$

for all $m \in [m_0, \infty)$.

We observe that in 3D domains it is not possible to state a complete result such as Theorem 6. This is due to the gap between the regularity of the solutions provided by the existence result and the regularity needed to prove a uniqueness. To the best of our knowledge, the existence or non-existence of blow up in the 3D case is an open problem.

Concerning the numerical approximation of the regularized problem, let $(0, T)$ be a time interval to be divided in $N \in \mathbb{N}$ sub-intervals (t_{n-1}, t_n) , $h > 0$ and $k = T/N$ be parameters of the space and time discretization, respectively. Let U^h and V^h be convenient function spaces of polynomial by parts and globally continuous functions to be defined later, and let u_h^0 and v_h^0 be convenient approximations of the initial conditions u^0 and v^0 , respectively. Let $\chi > 0$ be a given chemotactic constant. For $n = 1, 2, \dots, N$, we introduce the following fully discrete schemes:

Knowing $(u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h$, find $(u_h^n, v_h^n) \in U_h \times V_h$ such that

$$\begin{aligned} \left(\frac{u_h^n - u_h^{n-1}}{k}, \varphi_h \right) + (\nabla u_h^n, \nabla \varphi_h) &= \chi (a_m(u_h^n) \nabla v_h^n, \nabla \varphi_h), \quad \forall \varphi_h \in U_h, \\ \left(\frac{v_h^n - v_h^{n-1}}{k}, \phi_h \right) + (\nabla v_h^n, \nabla \phi_h) &= - (a_m(u_h^n)^s v_h^n, \phi_h), \quad \forall \phi_h \in V_h; \end{aligned} \quad (NLNS)$$

Knowing $(u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h$, find $(u_h^n, v_h^n) \in U_h \times V_h$ such that

$$\begin{aligned} \left(\frac{u_h^n - u_h^{n-1}}{k}, \varphi_h \right) + (\nabla u_h^n, \nabla \varphi_h) &= \chi (a_m(u_h^n) \nabla v_h^n, \nabla \varphi_h), \quad \forall \varphi_h \in U_h, \\ \left(\frac{v_h^n - v_h^{n-1}}{k}, \phi_h \right) + (\nabla v_h^n, \nabla \phi_h) &= - (a_m(u_h^{n-1})^s v_h^n, \phi_h), \quad \forall \phi_h \in V_h; \end{aligned} \quad (SLNS)$$

Knowing $(u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h$, find $(u_h^n, v_h^n) \in U_h \times V_h$ such that

$$\begin{aligned} \left(\frac{u_h^n - u_h^{n-1}}{k}, \varphi_h \right) + (\nabla u_h^n, \nabla \varphi_h) &= \chi(a_m(u_h^{n-1}) \nabla v_h^n, \nabla \varphi_h), \quad \forall \varphi_h \in U_h, \\ \left(\frac{u_h^n - u_h^{n-1}}{k}, \phi_h \right) + (\nabla v_h^n, \nabla \phi_h) &= -(a_m(u_h^{n-1})^s v_h^n, \phi_h), \quad \forall \phi_h \in V_h. \end{aligned} \quad (LNS)$$

At each step n of $(NLNS)$ we have to solve two non-linear algebraic equations, that's why we call it "non-linear numerical scheme" and denote it by $(NLNS)$. If we consider the scheme $(SLNS)$, at each step n , we can first solve a linear equation and then solve the non-linear one. We call this scheme "semi-linear numerical scheme" and denote it by $(SLNS)$. Finally, at each step of (LNS) , we have to solve two linear equations which can be solved separately. Then we call it "linear numerical scheme" and denote it by (LNS) .

Now we highlight our main results on the aforementioned numerical schemes. For the precise definitions and complete proof, we refer the reader to Section 1.4 and Chapter 4.

Theorem 7. (Unconditional existence) *For each time step n we have existence of solution for $(NLNS)$, $(SLNS)$ and (LNS) , independently of the parameters h and k .*

Theorem 8. (Convergence towards the regularized model) *The numerical schemes $(NLNS)$, $(SLNS)$ and (LNS) converge towards the regularized model (P_m) as $(h, k) \rightarrow 0$.*

Theorem 9. (Conditional uniqueness for the non-linear schemes) *For each time step n , we have uniqueness of solution:*

1. for (LNS) , independently of the parameters h , and k ;
2. for $(NLNS)$, if we choose h and k such that

$$\frac{k}{h^{d+2}} \leq \frac{C}{\chi^2 \|v^0\|_{L^2(\Omega)}^2} \quad \text{and} \quad \frac{k^2}{h^d} \leq \frac{C}{\chi^2 m^{2s} \|v^0\|_{L^2(\Omega)}^2};$$

3. for $(SLNS)$, if we choose h and k such that

$$\frac{k}{h^{d+2}} \leq \frac{C}{\chi^2 \|v^0\|_{L^2(\Omega)}^2}.$$

Theorem 10. (Error estimates) *Let (u_m, v_m) be the solution of (P_m) . There exists $C = C(u_m, v_m, \chi, m, T) > 0$ such that the error estimates*

$$\|u_m(t_n) - u_h^n\|_{L^2(\Omega)}^2 + \|v_m(t_n) - v_h^n\|_{L^2(\Omega)}^2 \leq C(h^2 + k^2);$$

and

$$k \sum_{n=1}^N \|u_m(t_n) - u_h^n\|_{H^1(\Omega)}^2 + k \sum_{n=1}^N \|v_m(t_n) - v_h^n\|_{H^1(\Omega)}^2 \leq C(h^2 + k^2)$$

hold:

1. for (LNS), independently of h and k ;
2. for (NLNS), if we choose k such that

$$1 - \left(\frac{2}{5} + \chi^2 C(v_m) + C(s, m) \|v_m\|_{L^\infty(0,T;L^\infty(\Omega))}^2\right)k > 0,$$

where $C(v_m) = C(\|v_m\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 + \|v_m\|_{L^\infty(0,T;W^{1,6}(\Omega))}^4)$;

3. for (SLNS), if we choose k such that

$$1 - \left(\frac{2}{5} + \chi^2 C(v_m)\right)k > 0,$$

where $C(v_m) = C(\|v_m\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 + \|v_m\|_{L^\infty(0,T;W^{1,6}(\Omega))}^4)$.

This thesis is organized as follows. In Chapter 1 we present the basic theoretical and classical results which will be used in the rest of the thesis.

Chapter 2 is devoted to discuss the regularity's properties of the solutions of regularized problem (P_m) . We begin by proving the existence of solution for the regularized problems, for each fixed $m \in \mathbb{N}$ via Galerkin's method. That is, for each fixed $m \in \mathbb{N}$, we define convenient finite dimensional spaces, with the dimension n increasing, prove the existence of approximate solutions, obtain *a priori* bounds that are independent of n and then pass to the limit as $n \rightarrow \infty$ in order to establish the existence of solution to the regularized problem (P_m) . In the sequel we prove uniqueness of solution and end the chapter by improving the regularity and by studying some properties of the solution. The additional regularity and the studied properties are useful for Chapter 3 as well as for Chapter 4, about the numerical approximation of the regularized model (P_m) .

Concerning Chapter 3, in Section 3.1 we deal with the proof of Theorem 3. This is done by passing to the limit as $m \rightarrow \infty$ in the solutions of the regularized problems, studied in Chapter 2. To do so, we need to establish *a priori* m -independent estimates for the solutions of the regularized problems. In Subsection 3.1.1 we show some formal computations in order to illustrate how these estimates are obtained. The subsequent two subsections are devoted to the obtaining of the rigorous estimates. In Subsection 3.1.2 we prove an energy inequality. The analysis must be separated according to the values of $s \geq 1$. In Subsection 3.1.3 we deal with a term which appears on the right-hand side of the energy inequality, independently of s , and whose appropriate estimation requires attention.

In Subsection 3.1.4 we obtain the m -independent estimates and pass to the limit as $m \rightarrow \infty$, finishing the proof of Theorem 3. Finally, Section 3.2 is dedicated to the proof of Theorem 6, on the two dimensional case. In order to improve the regularity of solution in 2D, we prove better m -independent estimates than the ones of Subsection 3.1.4. Then we can pass to the limit as $m \rightarrow \infty$ and prove the existence of a more regular solution, which is unique.

Finally, in Chapter 4, we deal with the numerical approximation of the regularized problems (P_m) , for each fixed $m \in \mathbb{N}$. We begin with the time discretization of (P_m) in Section 4.1. We introduce three different time discretizations and prove their convergence towards (P_m) . Then, in Section 4.2 we introduce the fully discrete numerical schemes (LNS) , $(NLNS)$ and $(SLNS)$ and discuss their properties with more details. For each numerical scheme, we study the existence of solution, the convergence towards (P_m) , the conditions for uniqueness of solution and the error estimates in terms of the discretization parameters.

CHAPTER 1

NOTATION AND PRELIMINARY RESULTS

1.1 Lebesgue, Sobolev and Bochner spaces

Let $\Omega \subset \mathbb{R}^d$ be an open bounded Lipschitz domain. Considering the Lebesgue's integral, we denote by $L^p(\Omega)$, $p \in [1, \infty)$ the spaces of p -integrable functions, with the norm

$$\|v\|_{L^p(\Omega)} = \left(\int_{\Omega} |v(x)|^p dx \right)^{1/p}, \quad \forall v \in L^p(\Omega),$$

and by $L^\infty(\Omega)$ the set of essentially bounded functions, with the norm

$$\|v\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x \in \Omega} \{|v(x)|\}, \quad \forall v \in L^\infty(\Omega).$$

The spaces $L^p(\Omega)$ are Banach spaces, for $p \in [1, \infty]$. When $p = 2$, the space $L^2(\Omega)$ is a Hilbert space and its norm is induced by the inner product

$$(u, v) = \int_{\Omega} u(x)v(x) dx, \quad \forall u, v \in L^2(\Omega).$$

We denote by $W^{l,p}(\Omega)$ the Sobolev space

$$W^{l,p}(\Omega) = \{v \in L^p(\Omega) \mid D^\alpha v \in L^p(\Omega) \text{ if } |\alpha| \leq l\},$$

which are Banach spaces with any of the equivalent norms

$$\|v\|_{W^{l,p}(\Omega)} = \left| (\|D^\alpha v\|_{L^p(\Omega)})_{\alpha \leq l} \right|_{\mathbb{R}^k}, \quad \forall v \in W^{l,p}(\Omega),$$

where $(\|D^\alpha v\|_{L^p(\Omega)})_{\alpha \leq l}$ is a vector in $\mathbb{R}^k$, for some $k \in \mathbb{N}$, and $|\cdot|_{\mathbb{R}^k}$ is a norm in $\mathbb{R}^k$. We write $H^l(\Omega)$ for $W^{l,2}(\Omega)$. The space $H^l\Omega$ is a Hilbert space with the norm

$$\|v\|_{H^l(\Omega)} = \left| (\|D^\alpha v\|_{L^2(\Omega)})_{\alpha \leq l} \right|_2, \quad \forall v \in H^l(\Omega),$$

where $|\cdot|_2$ is the Euclidian norm induced by the canonical inner product in $\mathbb{R}^k$.

If a function $v \in H^1(\Omega)$ and $\Delta v \in L^2(\Omega)$ then we can define the trace of the normal derivative of v , which will be denoted by $\partial_\eta v \Big|_\Gamma$.

There are some inequalities that are very useful when we want to estimate the integral of a product of functions or when we want to study the regularity of some functions. Here we present a generalized version of this inequality, an interpolation inequality and Young's Inequality.

Lemma 11 (Generalized Hölder's Inequality). *Let $f_1, f_2, \dots, f_k$ be functions such that $f_i \in L^{p_i}(\Omega)$, for all i , with $1 \leq p_i \leq \infty$ and $\sum_{i=1}^k \frac{1}{p_i} \leq 1$. Set*

$$f(x) = \prod_{i=1}^k f_i(x).$$

Let $1 \leq p \leq \infty$ be such that $\frac{1}{p} = \sum_{i=1}^k \frac{1}{p_i}$. Then $f \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \leq \prod_{i=1}^k \|f_i\|_{L^{p_i}(\Omega)}.$$

Proof. See the exercise 4.4 of (BREZIS, 2010). ■

Lemma 12. *Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be open, bounded and locally Lipschitz domain.*

1. Let $1 \leq p < q \leq \infty$, $\theta \in (0, 1)$ and $r \in [p, q]$, with $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. If $f \in L^p(\Omega) \cap L^q(\Omega)$ then $f \in L^r(\Omega)$ and

$$\|f\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)}^\theta \|f\|_{L^q(\Omega)}^{1-\theta}.$$

2. There exist (different) constants $\beta > 0$ such that

(i) if $d = 2$ then

$$\|v\|_{L^4(\Omega)} \leq \beta \|v\|_{L^2(\Omega)}^{1/2} \|v\|_{H^1(\Omega)}^{1/2}, \forall v \in H^1(\Omega);$$

(ii) if $d = 3$ then

$$\|v\|_{L^4(\Omega)} \leq \beta \|v\|_{L^2(\Omega)}^{1/4} \|v\|_{H^1(\Omega)}^{3/4}, \forall v \in H^1(\Omega);$$

(iii) if $d = 3$ then

$$\|v\|_{L^3(\Omega)} \leq \beta \|v\|_{L^2(\Omega)}^{1/2} \|v\|_{H^1(\Omega)}^{1/2}, \forall v \in H^1(\Omega).$$

Proof. See (BREZIS, 1984) and (TEMAM, 1977) ■

Lemma 13 (Young's Inequality). *Let A and B be non-negative real numbers and let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

If we consider that, given any $\delta > 0$,

$$AB = (p\delta)^{1/p} A \frac{B}{(p\delta)^{1/p}},$$

we can write

$$AB \leq \delta A^p + \frac{B^q}{(p\delta)^{q/pq}}.$$

Proof. We can find its proof in (BARTLE, 1995), where it is used in order to prove Hölder's inequality. ■

Lemma 14 (Poincaré Inequality). *Let Ω be a locally Lipschitz bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$, and let $v \in H^1(\Omega)$ and $v^* = \frac{1}{|\Omega|} \int_{\Omega} v(x) dx$. Then there is a constant $C = C(\Omega) > 0$ which is independent of v and such that*

$$\|v - v^*\|_{H^1(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}.$$

Proof. It can be proved by contradiction as in (EVANS, 2010). ■

Lemma 15. *Let Ω be a Lipschitz domain and let $z \in H^2(\Omega)$. Then, for each $\delta > 0$, there is a $C(\delta) > 0$ such that*

$$\|\nabla z\|_{L^2(\partial\Omega)} \leq C(\delta) \|z\|_{L^2(\Omega)} + \delta \|D^2 z\|_{L^2(\Omega)}.$$

Proof. This is Lemma 2.4 in (PASSO et al., 1998). It can be proved by contradiction as Lemma 14. ■

Now let X be a Banach space and let (a, b) be an interval. We denote by $L^p(a, b; X)$ the Bochner space

$$L^p(a, b; X) = \left\{ v : (a, b) \longrightarrow X \mid \int_a^b \|v(t)\|_X^p dt < \infty \right\}$$

with the norm

$$\|v\|_{L^p(a, b; X)} = \left(\int_a^b \|v(t)\|_X^p dt \right)^{1/p}.$$

For $p = \infty$ we define

$$L^\infty(a, b; X) = \left\{ v : (a, b) \longrightarrow X \mid \operatorname{ess\,sup}_{t \in (a, b)} \|v(t)\|_X < \infty \right\}$$

with the norm

$$\|v\|_{L^\infty(a,b;X)} = \operatorname{ess\,sup}_{t \in (a,b)} \|v(t)\|_X.$$

If $p = 2$ and X is a Hilbert space then the norm $\|\cdot\|_{L^2(a,b;X)}$ can be obtained from the inner product

$$(u, v)_{L^2(a,b;X)} = \int_a^b (u(t), v(t))_X \, dt, \quad \forall u, v \in L^2(a, b; X).$$

In particular, the spaces $L^p(0, T; L^p(\Omega))$ can be identified with the spaces $L^p(\Omega \times (0, T))$ by Fubini's theorem.

The spaces $L^p(a, b; X)$, with their respective norms $\|\cdot\|_{L^p(a,b;X)}$ are Banach spaces. If X is a Hilbert space then $L^2(a, b; X)$ is also a Hilbert space.

In order to deal with the consumption term of (P) or (P_m) the following two lemmas will be useful.

Lemma 16. *Let w_1 and w_2 be non-negative real numbers. For each $s \geq 1$ we have*

$$|w_2^s - w_1^s| \leq s |w_2 + w_1|^{s-1} |w_2 - w_1|.$$

Proof. Without loss of generality, suppose $w_1 \leq w_2$. Then

$$\begin{aligned} |w_2^s - w_1^s| &= s \left| \int_{w_1}^{w_2} r^{s-1} \, dr \right| \\ &\leq s \left| \int_{w_1}^{w_2} \sup_{r \in (w_1, w_2)} r^{s-1} \, dr \right| \\ &\leq s \left| \int_{w_1}^{w_2} |w_2 + w_1|^{s-1} \, dr \right| \\ &= s |w_2 + w_1|^{s-1} |w_2 - w_1|. \end{aligned}$$

■

Lemma 17. *Let $p \in (1, \infty)$ and let $\{w_n\}$ be a sequence of non-negative functions in $L^p(0, T; L^p(\Omega))$ such that $w_n \rightarrow w$ in $L^p(0, T; L^p(\Omega))$ as $n \rightarrow \infty$. Then, for every $r \in (1, p)$ $w_n^r \rightarrow w^r$ in $L^{p/r}(0, T; L^{p/r}(\Omega))$ as $n \rightarrow \infty$.*

Proof. Notice that w is also a non-negative function. Hence, by Lemma 16, we have

$$|w_n^r - w^r| \leq r |w_n + w|^{r-1} |w_n - w|.$$

Then

$$\|w_n^r - w^r\|_{L^{p/r}(0,T;L^{p/r}(\Omega))} = \left(\int_0^T \int_\Omega |w_n^r - w^r|^{p/r} \, dx \, dt \right)^{r/p}$$

$$\leq r \left(\int_0^T \int_{\Omega} |w_n + w|^{\frac{(r-1)p}{r}} |w_n - w|^{p/r} dx dt \right)^{r/p}.$$

Since $w_n, w \in L^p(0, T; L^p(\Omega))$, we can use Hölder's inequality in the term on the right hand side of the last inequality with the exponents $r/(r-1)$ and r , respectively. Then we obtain

$$\|w_n^r - w^r\|_{L^{p/r}(0, T; L^{p/r}(\Omega))} \leq r \left(\int_0^T \|w_n + w\|_{L^p(\Omega)}^{\frac{(r-1)p}{r}} \|w_n - w\|_{L^p(\Omega)}^{p/r} dt \right)^{r/p}$$

Again, using Hölder's inequality in the term on the right hand side of the last inequality with the exponents $r/(r-1)$ and r , respectively, we obtain

$$\begin{aligned} \|w_n^r - w^r\|_{L^{p/r}(0, T; L^{p/r}(\Omega))} &\leq r \left(\|w_n + w\|_{L^p(0, T; L^p(\Omega))}^{\frac{(r-1)p}{r}} \|w_n - w\|_{L^p(0, T; L^p(\Omega))}^{p/r} \right)^{r/p} \\ &= \|w_n + w\|_{L^p(0, T; L^p(\Omega))}^{r-1} \|w_n - w\|_{L^p(0, T; L^p(\Omega))}. \end{aligned}$$

Since $\|w_n + w\|_{L^p(0, T; L^p(\Omega))}$ is uniformly bounded with respect to $n \in \mathbb{N}$ and

$$\|w_n - w\|_{L^p(0, T; L^p(\Omega))} \rightarrow 0$$

as $n \rightarrow \infty$, we conclude that $\|w_n^r - w^r\|_{L^{p/r}(0, T; L^{p/r}(\Omega))} \rightarrow 0$ as $n \rightarrow \infty$, finishing the proof. ■

Lemma 18. *Let $p \in (1, \infty)$. Let $\{w_m\}$ be a sequence of nonnegative functions which is uniformly bounded in $L^\infty(0, \infty; L^p(\Omega))$ with respect to m . If there is $\alpha > 0$ such that*

$$\int_{\Omega} w_m(t, x) dx \geq \alpha, \text{ a.e. } t \in (0, \infty), \forall m \in \mathbb{N},$$

then there exist $\beta > 0$ and $m_0 \in \mathbb{N}$ large enough such that

$$\int_{\Omega} a_m(w_m(t, x)) dx \geq \beta, \text{ a.e. } t \in (0, \infty), \forall m \geq m_0.$$

Proof. For every $t \in (0, \infty)$ such that it makes sense, let

$$S_m(t) = \left\{ x \in \Omega \mid w_m(t, x) > m \right\}.$$

Then, for almost every $t \in (0, \infty)$, we have

$$\int_{S_m(t)} m^p dx + \int_{\Omega \setminus S_m(t)} a_m(w_m(t, x))^p dx \leq \int_{\Omega} a_m(w_m(t, x))^p dx \leq C_1(p).$$

This implies

$$|S_m(t)| = \int_{S_m(t)} dx \leq C_1(p) \left(\frac{1}{m} \right)^p. \quad (1.1)$$

We have

$$\begin{aligned}
\int_{\Omega} a_m(w_m(t, x)) \, dx &= \int_{\Omega \setminus S_m(t)} w_m(t, x) \, dx + \int_{S_m(t)} a_m(w_m(t, x)) \, dx \\
&= \int_{\Omega} w_m(t, x) \, dx - \int_{S_m(t)} w_m(t, x) - a_m(w_m(t, x)) \, dx \\
&\geq \alpha - \int_{S_m(t)} w_m(t, x) - a_m(w_m(t, x)) \, dx.
\end{aligned}$$

To finish the proof, we show that

$$\lim_{m \rightarrow \infty} \int_{S_m(t)} w_m(t, x) - a_m(w_m(t, x)) \, dx = 0,$$

uniformly with respect to $t \in (0, \infty)$. In fact, using Hölder's inequality and (1.1), we obtain

$$\begin{aligned}
\int_{S_m(t)} w_m(t, x) - a_m(w_m(t, x)) \, dx &\leq \left(\int_{S_m(t)} |w_m(t, x) - a_m(w_m(t, x))|^p \, dx \right)^{\frac{1}{p}} \left(\int_{S_m(t)} dx \right)^{\frac{p-1}{p}} \\
&\leq C_2(p) \left(\int_{S_m(t)} dx \right)^{\frac{p-1}{p}} \\
&\leq C(p) \left(\frac{1}{m} \right)^{p-1}.
\end{aligned}$$

for almost every $t \in (0, \infty)$. Therefore we have

$$\int_{\Omega} a_m(w_m(t, x)) \, dx \geq \alpha - C(p) \left(\frac{1}{m} \right)^{p-1}$$

and then we can choose m_0 large enough such that

$$\int_{\Omega} a_m(w_m(t, x)) \, dx \geq \beta = \frac{\alpha}{2},$$

for example, completing the proof. ■

The following result (Corollary 4 and Corollary 6 of (SIMON, 1986)) establishes a criterion of compactness in Bochner spaces.

Lemma 19 (Compactness in Bochner spaces). *Let X, B and Y be Banach spaces, let*

$$F = \left\{ f \in L^1(0, T; Y) \mid \partial_t f \in L^1(0, T; Y) \right\}.$$

and

$$\partial F / \partial t = \left\{ \partial_t f \mid f \in F \right\}.$$

Suppose that $X \subset B \subset Y$, with compact embedding $X \subset B$ and continuous embedding $B \subset Y$.

1. Let the set F be bounded in $L^p(0, T; X)$, with $1 \leq p < \infty$, and $\partial F/\partial t$ be bounded in $L^1(0, T; Y)$. Then F is relatively compact in $L^p(0, T; B)$.

Let F be bounded in $L^\infty(0, T; X)$ and $\partial F/\partial t$ be bounded in $L^r(0, T; Y)$, for some $r > 1$. Then F is relatively compact in $C([0, T]; B)$;

2. Let the set F be bounded in $L^q(0, T; B) \cap L^1(0, T; X)$, for $1 < q \leq \infty$, and $\partial F/\partial t$ be bounded in $L^1(0, T; Y)$. Then F is relatively compact in $L^p(0, T; B)$, for $1 \leq p < q$.

Lemma 20 (Gronwall's inequality). Let f, g and h be non-negative functions such that $g, h \in L^1(0, T)$, for some $T > 0$. Let

$$G(t) = \int_0^t g(r) dr \text{ and } H(t) = \int_0^t h(r) dr.$$

If f is such that

$$\frac{d}{dt}f(t) \leq g(t) + h(t)f(t), \text{ a.e. } t \in (0, T),$$

then

$$f(t) \leq \left(G(t) + f(0)\right)e^{H(t)}, \text{ a.e. } t \in (0, T).$$

1.2 Regularity results

We will need some information about the solution v of the Neumann problem

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ \partial_\eta v|_\Gamma = 0, & \int_\Omega v(x) dx = 0, \end{cases} \quad (1.2)$$

in which the function f satisfies the compatibility condition

$$\int_\Omega f dx = 0.$$

Definition 21. Suppose $v \in H^1(\Omega)$ is a weak solution of (1.2) with $f \in W^{n,p}(\Omega)$. If it implies that $v \in W^{n+2,p}(\Omega)$, with

$$\|v\|_{W^{n+2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)},$$

then we say that the Neumann problem (1.2) has the $W^{n+2,p}$ -regularity.

If $p = 2$ we will write H^{n+2} -regularity. ■

Remark 22. According to Grisvard (GRISVARD, 1985), if $f \in L^p(\Omega)$ and the boundary Γ is at least of $C^{1,1}$ class, then the solution v of the Neumann problem (1.2) belongs to $W^{2,p}(\Omega)$ with continuous dependence on the data (that is, problem (1.2) has the $W^{2,p}$ -regularity). The aforementioned result is also true if Ω is a polygon, that is, a polyhedron in $\mathbb{R}^2$, or if Ω is convex and $p = 2$. For more regular domains, if $f \in W^{l,p}(\Omega)$ and the boundary Γ is at least of $C^{l+1,1}$ class, then the solution v of the Neumann problem (1.2) belongs to $W^{l+2,p}(\Omega)$. This regularity, which is valid for more regular domains, will not be used in this PhD Thesis. ■

1.3 Positive and negative parts of a function

In this subsection we define the positive and negative parts of a function.

If $v \in L^p(\Omega)$, $1 \leq p \leq \infty$, we define the positive and negative parts of v as

$$v_+(x) = \max \{0, v(x)\} \text{ and } v_-(x) = -\max \{0, -v(x)\}, \text{ a.e. } x \in \Omega, \quad (1.3)$$

respectively. Note that $v = v_+ + v_-$ and $|v| = v_+ - v_-$.

Lemma 23. If $v \in H^1(\Omega)$ then $v_+, v_- \in H^1(\Omega)$ with

$$\nabla v_+(x) = \begin{cases} \nabla v, & \text{if } v(x) > 0, \\ 0, & \text{if } v(x) < 0, \end{cases}$$

and

$$\nabla v_-(x) = \begin{cases} \nabla v, & \text{if } v(x) < 0, \\ 0, & \text{if } v(x) > 0. \end{cases}$$

1.4 Preliminary Results of the Numerical Study

Here we present some results and notation that will be used in the sequel.

Lemma 24. Let X be a finite dimensional Hilbert space with scalar product $[\cdot, \cdot]$ and norm $[\cdot]$ and let P be a continuous mapping from X into itself. Suppose there is a $\beta > 0$ such that

$$[P(\xi), \xi] > 0, \text{ for } [\xi] = \beta > 0.$$

Then there exists $\xi \in X$, with $[\xi] \leq \beta$ such that $P(\xi) = 0$.

Proof. Lemma 1.4 of (TEMAM, 1977). ■

Lemma 25. (Discrete Gronwall's Inequality) Let (k_n) be a sequence of non-negative real numbers. Consider the sequence (a_n) , with $a_n \geq 0$, and a constant $c_0 \geq 0$ such that

$$a_0 \leq c_0$$

and

$$a_n \leq c_0 + \sum_{j=0}^{n-1} k_j a_j, \quad \forall n \geq 1.$$

Then, for all $n \geq 1$ we have

$$a_n \leq c_0 \exp \left(\sum_{j=1}^{n-1} k_j \right).$$

In order to prove Lemma 25, we will use the following lemma.

Lemma 26. *Let $\{k_m\}$ be a sequence of non-negative real numbers. We have*

$$1 + k_0 + \sum_{n=1}^{m-1} \left(\prod_{j=0}^{n-1} (1 + k_j) \right) k_n \leq \prod_{n=0}^{m-1} (1 + k_n)$$

Proof. We have

$$1 + k_0 + \sum_{n=1}^{m-1} \left(\prod_{j=0}^{n-1} (1 + k_j) \right) k_n = 1 + k_0 + (1 + k_0)k_1 + (1 + k_0)(1 + k_1)k_2 + \cdots + \prod_{j=0}^{m-2} (1 + k_j)k_{m-1}$$

Note that all of the terms on the right hand side of the equality are distinct and are also terms of $\prod_{n=0}^{m-1} (1 + k_n)$. That is, $\prod_{n=0}^{m-1} (1 + k_n)$ has more terms than $1 + k_0 + \sum_{n=1}^{m-1} \left(\prod_{j=0}^{n-1} (1 + k_j) \right) k_n$. Then, since $k_m \geq 0$, for all m , we prove the stated result. ■

Proof of Lemma 25. First of all, let us prove by induction that

$$a_m \leq c_0 \prod_{n=0}^{m-1} (1 + k_n), \quad m \geq 1. \quad (1.4)$$

In fact, we have $a_0 \leq c_0$ and then

$$a_1 \leq c_0 + k_0 a_0 \leq c_0 (1 + k_0).$$

That is, (1.4) is proved for $m = 1$. Now suppose that (1.4) is valid for $m \leq l$, for some $l \geq 1$. Then we have

$$\begin{aligned} a_{l+1} &\leq c_0 \sum_{n=0}^l k_n a_n \leq c_0 + c_0 k_0 + \sum_{n=1}^l c_0 k_n \left(\prod_{j=0}^{n-1} (1 + k_j) \right) = \\ &= c_0 \left(1 + k_0 \sum_{n=1}^l \left(\prod_{j=0}^{n-1} (1 + k_j) \right) k_n \right) \end{aligned}$$

and, by Lemma 26, we obtain

$$a_{l+1} \leq c_0 \prod_{n=0}^l (1 + k_n),$$

finishing the proof of (1.4) for $l + 1$. Therefore we proved (1.4) by induction.

Now, note that, for all $x \geq 0$, we have

$$1 + x \leq \exp(x).$$

Using this inequality in (1.4), we prove the desired result. ■

We are going to approximate the functions u_m and v_m in the domain $[0, T] \times \Omega$, for a finite $T > 0$, where Ω is a polyhedral domain. For this purpose, we will use spaces of functions which are polynomials by parts. In this work, these spaces will be associated to a division of the domain in N -simplexes. A d -simplex is the convex hull of $d + 1$ points in $\mathbb{R}^d$. In particular, a 2-simplex is a triangle and a 3-simplex is a tetrahedron. We will refer to these simplexes as triangles and call the division of the domain a triangulation.

Let Ω be a locally Lipschitz polyhedron. We will denote the triangulations of Ω by $\mathcal{T}_h$ and the triangles by K . The triangulations depend on a parameter $h > 0$ which is greater than the radius of the balls in which the “triangles” K are “inscribed in”.

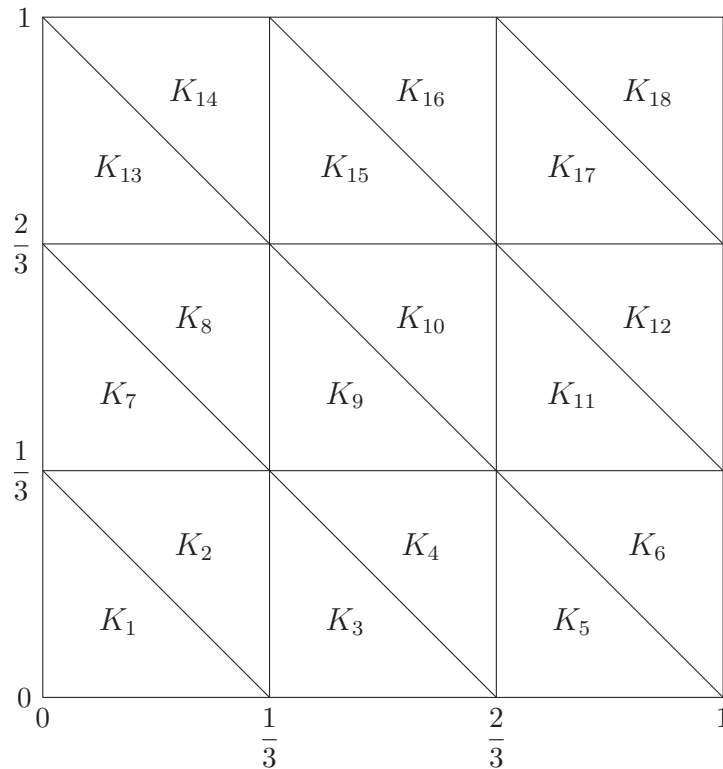


Figure 1 – Triangulation with $h = 1/3$.

Let $l \in \mathbb{N}$, we define the spaces of polynomials of degree less than or equal to l by parts by

$$\mathbb{P}_l(\mathcal{T}_h(\Omega)) = \left\{ v \in C^0(\overline{\Omega}); v|_K \text{ is a polynomial of degree } \leq l, \forall K \in \mathcal{T}_h(\Omega) \right\} \quad (1.5)$$

We are going to consider only admissible triangulations according to the following definition.

Definition 27 (Admissible triangulation). A triangulation $\mathcal{T}_h(\Omega)$ of Ω is called admissible if

1. $\Omega(h) := \left(\bigcup_{K \in \mathcal{T}_h(\Omega)} K \right) \subset \overline{\Omega}$;
2. If K and K' are different triangles of $\mathcal{T}_h(\Omega)$ then $\text{int}(K) \cap \text{int}(K') = \emptyset$ and, or $K \cap K'$ is empty, or $K \cap K'$ consists of one point, or $K \cap K'$ is a common face to K and K' . ■

Let h_K be the radius of the smallest circle which encloses K and h'_K be the radius of the greatest circle which is contained in K . Considering an admissible triangulation $\mathcal{T}_h(\Omega)$, we define

$$h = \max_{K \in \mathcal{T}_h(\Omega)} h_K, \quad (1.6)$$

$$h' = \min_{K \in \mathcal{T}_h(\Omega)} h'_K, \quad (1.7)$$

$$\sigma(h) = \max_{K \in \mathcal{T}_h(\Omega)} \frac{h_K}{h'_K}. \quad (1.8)$$

We want to make $h \rightarrow 0$, but maintaining the following regularity of the triangulation.

Definition 28 (Regular family of triangulations). We say that a family $\{\mathcal{T}_h(\Omega)\}_{h \in \mathcal{H}}$ of admissible triangulations is regular if $\sigma(h)$ remains bounded as $h \rightarrow 0$, that is, there are $h^* > 0$ and $C = C(h^*) > 0$, independent of $h > 0$, such that

$$\sigma(h) \leq C < \infty, \forall h \leq h^*. \quad \blacksquare$$

Definition 29 (Affine family of triangulations). We say that a family $\{\mathcal{T}_h(\Omega)\}_{h \in \mathcal{H}}$ of admissible triangulations is an affine family of triangulations if all of its triangles are affine equivalent to a reference triangle, the triangle $\hat{K}$ with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$, for example. That is, if for each K there is a unique invertible affine mapping $F_K : \hat{K} \mapsto K$. ■

In order to use an inverse inequalities, we will also consider that the family of triangulations satisfies an inverse assumption, in the sense of (CIARLET, 2002). It is sufficient to assume that the family of triangulations is quasi-uniform, according to the next definition.

Definition 30 (Quasi-uniform triangulation). *We say that a family $\{\mathcal{T}_h(\Omega)\}_{h \in \mathcal{H}}$ of admissible triangulations is quasi-uniform if there are $h^* > 0$ and $C = C(h^*) > 0$, independent of $h > 0$, such that*

$$\frac{\max_{K \in \mathcal{T}_h(\Omega)} \text{diam}(K)}{\min_{K \in \mathcal{T}_h(\Omega)} \text{diam}(K)} \leq C, \quad \forall h \leq h^*.$$

■

These assumptions allow us to use the inverse inequality given by Theorem 3.2.6 of (CIARLET, 2002).

Lemma 31. *Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. Let z be a function of $\mathbb{P}_1(\mathcal{T}_h(\Omega))$ and assume that the family $\{\mathcal{T}_h(\Omega)\}_{h \in \mathcal{H}}$ of admissible triangulations is regular, affine and quasi-uniform. Then there is $h^* > 0$ and positive constant $C = C(h^*)$ such that, for all $h \leq h^*$, we have:*

1. $\|\nabla z\|_{L^\infty(\Omega)}^2 \leq \frac{C}{h^{d+2}} \|z\|_{L^2(\Omega)}^2;$
2. $\|z\|_{L^\infty(\Omega)}^2 \leq \frac{C}{h^d} \|z\|_{L^2(\Omega)}^2;$
3. $\|\nabla z\|_{L^\infty(\Omega)}^2 \leq \frac{C}{h^{d/3}} \|\nabla z\|_{L^6(\Omega)}^2.$

Proof. These inequalities stem from Theorem 3.2.6 of (CIARLET, 2002). ■

Let $\Pi_h : L^1(\Omega) \rightarrow \mathbb{P}_1(\mathcal{T}_h(\Omega))$ be a linear operator such that, for $n = 0, 1$ and $1 \leq p \leq \infty$, the following stability and accurate properties hold:

$$\|\Pi_h \varphi\|_{W^{n,p}(\Omega)} \leq C \|\varphi\|_{W^{n,p}(\Omega)}, \quad \forall \varphi \in W^{n,p}(\Omega) \quad (1.9)$$

and

$$\|\Pi_h \varphi - \varphi\|_{W^{n,2}(\Omega)} \leq Ch \|\varphi\|_{W^{n+1,2}(\Omega)}, \quad \forall \varphi \in W^{n+1,2}(\Omega). \quad (1.10)$$

Remark 32. We can take the operator Π_h as the Scott-Zhang operator (see (SCOTT; ZHANG, 1990) (BRENNER et al., 2008)). ■

For the time discretization we will divide the interval $[0, T]$ in N uniform intervals denoted by $I_n = (t_{n-1}, t_n)$, with $t_0 = 0$ and $t_N = T$, of length

$$k = \frac{T}{N}$$

and use the following notation. If $\{z^n\}_n$ is a sequence of functions, then

$$\delta_t z^n = \frac{z^n - z^{n-1}}{k}, \quad \forall n \geq 1. \quad (1.11)$$

We will have to deal with the product

$$(\delta_t z^n, z^n) = \int_{\Omega} \frac{(z^n - z^{n-1})}{k} z^n \, dx$$

and the following lemma is very useful.

Lemma 33. *If $z^n, z^{n-1} \in L^2(\Omega)$ then*

$$(\delta_t z^n, z^n) = \frac{1}{2k} \left(\|z^n\|_{L^2(\Omega)}^2 - \|z^{n-1}\|_{L^2(\Omega)}^2 + \|z^n - z^{n-1}\|_{L^2(\Omega)}^2 \right).$$

Proof. This identity can be obtained by straightforward calculations from the numerical identity

$$2(a - b)a = a^2 - b^2 + (a - b)^2.$$

It can be found in (TEMAM, 1977). ■

CHAPTER 2

THE REGULARIZED PROBLEM

In this chapter we define and analyze the regularized model (P_m) defined in the introduction. We recall that, in order to define the regularized version of (P) we consider the truncation of the identity (1). We remark that if $u \in H^1(\Omega)$ then $a_m(u) \in H^1(\Omega)$ and

$$\nabla a_m(u) = a'_m(u) \nabla u.$$

We recall that $0 \leq a'_m(\cdot) \leq C$, because of Remark 1. Therefore $a_m(\cdot)$ is uniformly Lipschitz, with the Lipschitz constant equal to C .

Let $u^0, v^0 \in H^2(\Omega)$ be non-negative functions. Considering the truncation (1), we denote the regularized problem (P_m) by (P_m^a) :

Find u_m and v_m such that:

$$\begin{aligned} \partial_t u_m - \Delta u_m &= -\nabla \cdot (a_m(u_m) \nabla v_m) \\ \partial_t v_m - \Delta v_m &= -a_m(u_m)^s v_m \\ \partial_\eta u_m|_\Gamma &= \partial_\eta v_m|_\Gamma = 0 \\ u_m(0) &= u^0, \quad v_m(0) = v^0, \end{aligned} \tag{P_m^a}$$

In all this chapter, we are interested in problem (P_m^a) . We study the problem (P_m^a) for each fixed $m \in \mathbb{N}$. Then we drop the m -subscript, denoting (u_m, v_m) by (u, v) . First, we are going to prove the existence of approximate solutions given by the Galerkin's method.

2.1 Existence of Solution to a Regularized Problem

In this section, we are going to use Galerkin's method in order to obtain approximate solutions for the regularized problem. In order to prove a stronger regularity, we will need the second order derivatives $a''_m(u)$. But the approximate solutions can have negative values and then the second order derivatives $a''_m(u)$ are not well defined, in general.

That is the reason why we are going to consider a slightly modified version of the truncation a_m presented in (1) during the application of the Galerkin's method. We are going to use the truncation b_m given by

$$b_m(u) = \begin{cases} -1, & \text{if } u \leq -1 \\ C^2 \text{ extension,} & \text{if } u \in (-1, 0) \\ a_m(u), & \text{if } u \geq 0, \end{cases} \quad (2.1)$$

where a_m , defined in (1), is given by

$$a_m(u) = \begin{cases} 0, & \text{if } u \leq 0 \\ u, & \text{if } u \in [0, m-1] \\ C^2 \text{ extension,} & \text{if } u \in (m-1, m], \\ m, & \text{if } u \geq m. \end{cases}$$

Remark 34. The truncation b_m can be chosen so that $0 \leq b'_m(u) \leq C$ and $|b''_m(u)| \leq C$, for all $u \in \mathbb{R}$, uniformly with respect to m and n . ■

Using the truncation $b_m(\cdot)$ we define the problem (P_m^b) :

Find u and v such that

$$\begin{aligned} \partial_t u_m - \Delta u_m &= -\nabla \cdot (b_m(u_m) \nabla v_m) \\ \partial_t v_m - \Delta v_m &= -a_m(u_m)^s v_m \\ \partial_\eta u_m|_\Gamma &= \partial_\eta v_m|_\Gamma = 0 \\ u_m(0) &= u^0, \quad v_m(0) = v^0. \end{aligned} \quad (P_m^b)$$

As it was mentioned in the beginning of this chapter, we are interested in (P_m^a) , but before dealing with (P_m^a) , in order to study the regularity of the solution of (P_m^a) , we have to study (P_m^b) . Since we are going to prove the existence of solution via Galerkin's method, we also consider a variational formulation of (P_m^b) :

$$\begin{aligned} (\partial_t u, \varphi) + (\nabla u, \nabla \varphi) &= (b_m(u) \nabla v, \nabla \varphi), \quad \forall \varphi \in H^1(\Omega), \\ (\partial_t v, \phi) + (\nabla v, \nabla \phi) &= -(a_m(u)^s v, \phi), \quad \forall \phi \in H^1(\Omega), \\ u(0) &= u^0, \quad v(0) = v^0. \end{aligned}$$

2.1.1 Approximate solutions and *a priori* estimates for (P_m^b)

In order to study the regularized problem, we use the Galerkin's method based on the set of eigenvalues $\{\lambda_j\} \subset \mathbb{R}$ and eigenfunctions $\{\varphi_j\} \subset H^2(\Omega)$ of the operator $(-\Delta + I)$ with homogeneous Neumann boundary condition. In particular we have

$$-\Delta \varphi_j = (\lambda_j - 1) \varphi_j, \quad \text{and} \quad \Delta^2 \varphi_j = (\lambda_j - 1)^2 \varphi_j, \quad j = 1, \dots, n \quad (2.2)$$

Let X^n be the finite n -dimensional space generated by the first n elements of the set $\{\varphi_j\}$, that is,

$$X^n = \{f \in H^2(\Omega) \mid f(x) = \sum_{j=1}^n \alpha_j \varphi_j(x)\}.$$

Because of (2.2), if $f \in X^n$ then $\Delta f, \Delta^2 f \in X^n$.

Then, we look for approximate solutions u_n and v_n of the form

$$u_n(t, x) = \sum_{j=1}^n g_j^n(t) \varphi_j(x)$$

$$v_n(t, x) = \sum_{j=1}^n h_j^n(t) \varphi_j(x).$$

The approximated problems are: find $g_j^n, h_j^n \in C^1(0, \infty)$ such that

$$(\partial_t u_n, \varphi_j) + (\nabla u_n, \nabla \varphi_j) = (b_m(u_n) \nabla v_n, \nabla \varphi_j), \quad (2.3)$$

$$(\partial_t v_n, \varphi_j) - (\Delta v_n, \varphi_j) = -(a_m(u_n)^s v_n, \varphi_j), \quad (2.4)$$

$$u_n(0) = u_n^0, \quad v_n(0) = v_n^0, \quad (2.5)$$

for $j = 1, \dots, n$. The functions u_n^0 and v_n^0 are orthogonal projections of u^0 and v^0 , respectively, from $H^2(\Omega)$ into X^n .

This is an autonomous non-linear system of ODEs with respect to the functions g_j^n and h_j^n , $j = 1, \dots, n$. The right hand side is continuous with respect to the variables g_j^n and h_j^n . In fact, it contains products of the variables g_j^n and h_j^n , which are continuous with respect to these variables. The term that contains the approximation $a_m(u_n)$ is also continuous because a_m is continuous. Local existence of solutions is given by the *Caratheodory's theorem*. Global existence follows from the subsequent n -independent *a priori* estimates and an extension result. These results can be found in Coddington and Levinson (CODDINGTON; LEVINSON, 1972).

The rest of this subsection is devoted to the proof of the following lemma.

Lemma 35. *The solutions of (2.3)-(2.5) satisfies the following n -independent bounds:*

1. *If the Poisson-Neumann problem (1.2) has the H^2 -regularity then*

$$u_n, v_n \text{ are bounded in } L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),$$

$$\partial_t u_n \text{ is bounded in } L^2(0, T; L^2(\Omega)),$$

$$\partial_t v_n \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega));$$

2. If the Poisson-Neumann problem (1.2) has the $W^{2,3}$ -regularity then

$$u_n, v_n \text{ are bounded in } L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,3}(\Omega)),$$

$$\partial_t u_n, \partial_t v_n \text{ are bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

■

In what follows, unless otherwise stated, we suppose that the Poisson-Neumann problem (1.2) has the H^2 -regularity. In order to obtain n -independent a priori estimates for the functions u_n and v_n , we test the v_n -equation by v_n and find

$$\frac{d}{dt} \|v_n\|_{L^2(\Omega)}^2 + 2 \|\nabla v_n\|_{L^2(\Omega)}^2 + 2 \|a_m(u_n)^{\frac{s}{2}} v_n\|_{L^2(\Omega)}^2 \leq 0.$$

Integrating in t we conclude that

$$\begin{aligned} v_n \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ a_m(u_n)^{s/2} v_n \text{ is bounded in } L^2(0, T; L^2(\Omega)). \end{aligned} \quad (2.6)$$

Then we test the u_n -equation by u_n and the v_n -equation by $-\Delta v_n$. This gives us

$$\frac{d}{dt} \|u_n\|_{L^2(\Omega)}^2 + \|\nabla u_n\|_{L^2(\Omega)}^2 \leq m^2 \|\nabla v_n\|_{L^2(\Omega)}^2.$$

$$\frac{d}{dt} \|\nabla v_n\|_{L^2(\Omega)}^2 + \|\Delta v_n\|_{L^2(\Omega)}^2 \leq m^{2s} \|v_n\|_{L^2(\Omega)}^2$$

Integrating with respect to t and using (2.6) we obtain

$$u_n \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad (2.7)$$

$$v_n \text{ is bounded in } L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)). \quad (2.8)$$

For the time derivative of v_n , we have

$$\partial_t v_n(t, x) = \sum_{j=1}^n \frac{d}{dt} h_j^n(t) \varphi_j(x)$$

and therefore $\partial_t v_n(t, \cdot) \in X^n$, for each $t \in (0, T)$. Then we can test (2.4) by $\partial_t v_n(t, \cdot)$. This gives us

$$\begin{aligned} \|\partial_t v_n(t)\|_{L^2(\Omega)}^2 &= (\Delta v_n(t), \partial_t v_n(t)) - (a_m(u_n)^s v_n, \partial_t v_n(t)) \\ &\leq \|\Delta v_n(t)\|_{L^2(\Omega)} \|\partial_t v_n(t)\|_{L^2(\Omega)} + m^s \|v_n(t)\|_{L^2(\Omega)} \|\partial_t v_n(t)\|_{L^2(\Omega)}. \end{aligned}$$

And this implies that

$$\|\partial_t v_n(t)\|_{L^2(\Omega)}^2 \leq \|\Delta v_n(t)\|_{L^2(\Omega)} + m^s \|v_n(t)\|_{L^2(\Omega)}.$$

Hence, using (2.6) and (2.8) we can conclude that

$$\partial_t v_n \text{ is bounded in } L^2(0, T; L^2(\Omega)). \quad (2.9)$$

Now, using the hypothesis that the Poisson-Neumann problem (1.2) has the H^2 -regularity, we can obtain some stronger *a priori* estimates for u_n and v_n . We can test the v_n -equation (2.4) by $\Delta^2 v_n \in X^n$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta v_n\|_{L^2(\Omega)}^2 + \|\nabla \Delta v_n\|_{L^2(\Omega)}^2 &= s \int_{\Omega} v_n a'_m(u_n) a_m(u_n)^{s-1} \nabla u_n \nabla \Delta v_n \, dx + \\ &+ \int_{\Omega} a_m(u_n)^s \nabla v_n \nabla \Delta v_n \, dx \\ &\leq s m^{s-1} \|v_n\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^2(\Omega)} \|\nabla \Delta v_n\|_{L^2(\Omega)} + \\ &+ m^s \|\nabla v_n\|_{L^2(\Omega)} \|\nabla \Delta v_n\|_{L^2(\Omega)} \\ &\leq C s^2 m^{2s-2} \|\nabla u_n\|_{L^2(\Omega)}^2 \|v_n\|_{L^\infty(\Omega)}^2 + \frac{1}{4} \|\nabla \Delta v_n\|_{L^2(\Omega)}^2 + \\ &+ C m^{2s} \|\nabla v_n\|_{L^2(\Omega)}^2 + \frac{1}{4} \|\nabla \Delta v_n\|_{L^2(\Omega)}^2 \\ &\leq C s^2 m^{2s-2} \|\nabla u_n\|_{L^2(\Omega)}^2 \|\Delta v_n\|_{L^2(\Omega)}^2 + \\ &+ C s^2 m^{2s-2} \|\nabla u_n\|_{L^2(\Omega)}^2 \|v_n\|_{L^2(\Omega)}^2 + C m^{2s} \|\nabla v_n\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla \Delta v_n\|_{L^2(\Omega)}^2. \end{aligned}$$

This gives us

$$\begin{aligned} \frac{d}{dt} \|\Delta v_n\|_{L^2(\Omega)}^2 + \|\nabla \Delta v_n\|_{L^2(\Omega)}^2 &\leq C s^2 m^{2s-2} \|\nabla u_n\|_{L^2(\Omega)}^2 \|\Delta v_n\|_{L^2(\Omega)}^2 \\ &+ C s^2 m^{2s-2} \|\nabla u_n\|_{L^2(\Omega)}^2 \|v_n\|_{L^2(\Omega)}^2 + C m^{2s} \|\nabla v_n\|_{L^2(\Omega)}^2. \end{aligned} \quad (2.10)$$

If we integrate (2.10) and use (2.7), (2.6) and (2.8) we find

$$\Delta v_n \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)). \quad (2.11)$$

Now that we have (2.11), we can also test the u_n -equation by $-\Delta u_n \in X^n$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \|\Delta u_n\|_{L^2(\Omega)}^2 &= \int_{\Omega} b_m(u_n) \Delta v_n \Delta u_n \, dx \\ &+ \int_{\Omega} b'_m(u_n) \nabla u_n \cdot \nabla v_n \Delta u_n \, dx. \end{aligned}$$

We recall that $|b_m(\cdot)| \leq m$ and $0 \leq b'_m(\cdot) \leq \tilde{C}$, for some positive constant $\tilde{C}$. Then we can use this fact and Hölder's inequality to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \|\Delta u_n\|_{L^2(\Omega)}^2 &\leq m \|\Delta v_n\|_{L^2(\Omega)} \|\Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C} \|\nabla u_n\|_{L^3(\Omega)} \|\nabla v_n\|_{L^6(\Omega)} \|\Delta u_n\|_{L^2(\Omega)}. \end{aligned}$$

Now we consider the continuous embedding $L^6(\Omega) \subset H^1(\Omega)$ and the interpolation inequality for the L^3 -norm of Lemma 12-2, in three dimensional domains. It gives us

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \|\Delta u_n\|_{L^2(\Omega)}^2 &\leq m \|\Delta v_n\|_{L^2(\Omega)} \|\Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C}_1 \|\nabla u_n\|_{L^2(\Omega)}^{1/2} \|\nabla u_n\|_{H^1(\Omega)}^{1/2} \|\nabla v_n\|_{H^1(\Omega)} \|\Delta u_n\|_{L^2(\Omega)}. \end{aligned}$$

It stems from the Poisson-Neumann problem that u_n and v_n satisfy, with Neumann homogeneous boundary conditions, that $\|\nabla u_n\|_{H^1(\Omega)} \leq \tilde{C} \|\Delta u_n\|_{L^2(\Omega)}$ and $\|\nabla v_n\|_{H^1(\Omega)} \leq \tilde{C} \|\Delta v_n\|_{L^2(\Omega)}$. Hence we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \|\Delta u_n\|_{L^2(\Omega)}^2 &\leq m \|\Delta v_n\|_{L^2(\Omega)} \|\Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C}_2 \|\nabla u_n\|_{L^2(\Omega)}^{1/2} \|\Delta v_n\|_{L^2(\Omega)} \|\Delta u_n\|_{L^2(\Omega)}^{3/2}. \end{aligned}$$

Now, using Young's inequality with the convenient weight (Lemma 13), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \|\Delta u_n\|_{L^2(\Omega)}^2 &\leq C m^2 \|\Delta v_n\|_{L^2(\Omega)}^2 + \frac{1}{4} \|\Delta u_n\|_{L^2(\Omega)}^2 + \\ &+ C \|\Delta v_n\|_{L^2(\Omega)}^4 \|\nabla u_n\|_{L^2(\Omega)}^2 + \frac{1}{4} \|\Delta u_n\|_{L^2(\Omega)}^2. \end{aligned}$$

From this we conclude that

$$\frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \|\Delta u_n\|_{L^2(\Omega)}^2 \leq C m^2 \|\Delta v_n\|_{L^2(\Omega)}^2 + C \|\Delta v_n\|_{L^2(\Omega)}^4 \|\nabla u_n\|_{L^2(\Omega)}^2. \quad (2.12)$$

Integrating (2.12) with respect to t and using (2.7) and (2.11) we obtain

$$u_n \text{ is bounded in } L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)). \quad (2.13)$$

Now we can use (2.7), (2.6), (2.8), (2.11) and (2.13) to conclude that

$$\partial_t u_n \text{ is bounded in } L^2(0, T; L^2(\Omega)) \quad (2.14)$$

In fact, we know that $\partial_t u_n(t, \cdot) \in X^n$, for each $t \in (0, T)$ because of

$$\partial_t u_n(t, x) = \sum_{j=1}^n \frac{d}{dt} g_j^n(t) \varphi_j(x),$$

Then we can test (2.3) by $\partial_t u_n$ and obtain

$$\|\partial_t u_n\|_{L^2(\Omega)}^2 = -(\nabla u_n, \nabla \partial_t u_n) + (b_m(u_n) \nabla v_n, \nabla \partial_t u_n).$$

Because of the regularity of u_n we can integrate by parts in the right hand side of the above equality. This gives us

$$\|\partial_t u_n\|_{L^2(\Omega)}^2 = (\Delta u_n, \partial_t u_n) - (b_m(u_n) \Delta v_n, \partial_t u_n) - (b'_m(u_n) \nabla u_n \cdot \nabla v_n, \partial_t u_n).$$

Then, recalling that $|b_m(u_n)| \leq m$ and $|b'_m(u_n)| \leq \tilde{C}$ for some constant positive $\tilde{C}$, we can use Hölder's inequality to conclude that

$$\begin{aligned} \|\partial_t u_n\|_{L^2(\Omega)}^2 &\leq \|\Delta u_n\|_{L^2(\Omega)} \|\partial_t u_n\|_{L^2(\Omega)} + m \|\Delta v_n\|_{L^2(\Omega)} \|\partial_t u_n\|_{L^2(\Omega)} \\ &\quad + \tilde{C} \|\nabla u_n\|_{L^3(\Omega)} \|\nabla v_n\|_{L^6(\Omega)} \|\partial_t u_n\|_{L^2(\Omega)}. \end{aligned}$$

Using the continuous embedding $L^3(\Omega) \subset L^6(\Omega) \subset H^1(\Omega)$, in three dimensional domains, we can obtain

$$\begin{aligned} \|\partial_t u_n\|_{L^2(\Omega)}^2 &\leq \|\Delta u_n\|_{L^2(\Omega)} \|\partial_t u_n\|_{L^2(\Omega)} + m \|\Delta v_n\|_{L^2(\Omega)} \|\partial_t u_n\|_{L^2(\Omega)} \\ &\quad + \tilde{C}_1 \|\nabla u_n\|_{H^1(\Omega)} \|\nabla v_n\|_{H^1(\Omega)} \|\partial_t u_n\|_{L^2(\Omega)}. \end{aligned}$$

It stems from the Poisson-Neumann problem that u_n and v_n satisfy, with Neumann homogeneous boundary conditions, that $\|\nabla u_n\|_{H^1(\Omega)} \leq \tilde{C} \|\Delta u_n\|_{L^2(\Omega)}$ and $\|\nabla v_n\|_{H^1(\Omega)} \leq \tilde{C} \|\Delta v_n\|_{L^2(\Omega)}$. Hence we have

$$\begin{aligned} \|\partial_t u_n\|_{L^2(\Omega)}^2 &\leq \|\Delta u_n\|_{L^2(\Omega)} + m \|\Delta v_n\|_{L^2(\Omega)} \\ &\quad + \tilde{C}_2 \|\Delta u_n\|_{L^2(\Omega)} \|\Delta v_n\|_{L^2(\Omega)}. \end{aligned}$$

Then we can conclude (2.14) from (2.7), (2.6), (2.8), (2.11) and (2.13).

Analogously, we notice that $\Delta \partial_t v_n(t, \cdot) \in X^n$, for each $t \in (0, T)$, because

$$\Delta \partial_t v_n(t, x) = \sum_{j=1}^n \frac{d}{dt} g_j^n(t) \Delta \varphi_j(x) = \sum_{j=1}^n \frac{d}{dt} g_j^n(t) \lambda_j \varphi_j(x).$$

Then we can test (2.4) by $\Delta \partial_t v_n$, and again by $\partial_t v_n$, in view of the new *a priori* bounds, and use (2.7), (2.6), (2.8), (2.11) and (2.13) to conclude that

$$\partial_t v_n \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)). \quad (2.15)$$

Remark 36. The *a priori* estimates obtained so far are valid for each finite $T > 0$, therefore the Galerkin solution (u_n, v_n) is defined up to infinity time. ■

Finally, we discuss the case in which we suppose that the Poisson-Neumann problem (1.2) has the $W^{2,3}$ -regularity. Because of (2.11) and the interpolation inequality for the L^3 -norm

$$\|f\|_{L^3(\Omega)} \leq C \|f\|_{L^2(\Omega)}^{1/2} \|f\|_{H^1(\Omega)}^{1/2}, \quad \forall f \in H^1(\Omega),$$

given by Lemma 12-2, we have

$$\Delta v_n \text{ is bounded in } L^4(0, T; L^3(\Omega))$$

and the $W^{2,3}$ -regularity of the Poisson-Neumann problem allows us to conclude that

$$D^2 v_n \text{ is bounded in } L^4(0, T; L^3(\Omega)). \quad (2.16)$$

Now we test the u_n -equation by $\Delta^2 u_n$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta u_n\|_{L^2(\Omega)}^2 + \|\nabla \Delta u_n\|_{L^2(\Omega)}^2 &= \int_{\Omega} b_m(u_n) \nabla \Delta v_n \cdot \nabla \Delta u_n \, dx \\ &+ \int_{\Omega} b'_m(u_n) \Delta v_n \nabla u_n \cdot \nabla \Delta u_n \, dx \\ &+ \int_{\Omega} b'_m(u_n) \nabla u_n \cdot D^2 v_n \nabla \Delta u_n \, dx \\ &+ \int_{\Omega} b'_m(u_n) \nabla v_n \cdot D^2 u_n \nabla \Delta u_n \, dx \\ &+ \int_{\Omega} b''_m(u_n) |\nabla u_n|^2 \nabla v_n \cdot \nabla \Delta u_n \, dx. \end{aligned}$$

We recall that $|b_m(u_n)| \leq m$ and $|b'_m(u_n)|, |b''_m(u_n)| \leq \tilde{C}$, for some $\tilde{C} > 0$. Then, using Hölder's inequality we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta u_n\|_{L^2(\Omega)}^2 + \|\nabla \Delta u_n\|_{L^2(\Omega)}^2 &\leq m \|\nabla \Delta v_n\|_{L^2(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C} \|\Delta v_n\|_{L^3(\Omega)} \|\nabla u_n\|_{L^6(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C} \|D^2 v_n\|_{L^3(\Omega)} \|\nabla u_n\|_{L^6(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C} \|\nabla v_n\|_{L^6(\Omega)} \|D^2 u_n\|_{L^3(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C} \|\nabla u_n\|_{L^6(\Omega)}^2 \|\nabla v_n\|_{L^6(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \end{aligned}$$

Using the $W^{2,3}$ -regularity we can write

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta u_n\|_{L^2(\Omega)}^2 + \|\nabla \Delta u_n\|_{L^2(\Omega)}^2 &\leq m \|\nabla \Delta v_n\|_{L^2(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C} \|\Delta v_n\|_{L^3(\Omega)} \|\nabla u_n\|_{L^6(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C} \|D^2 v_n\|_{L^3(\Omega)} \|\nabla u_n\|_{L^6(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C} \|\nabla v_n\|_{L^6(\Omega)} \|\Delta u_n\|_{L^3(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C} \|\nabla u_n\|_{L^6(\Omega)}^2 \|\nabla v_n\|_{L^6(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \end{aligned}$$

Now we recall the continuous embedding $L^6(\Omega) \subset H^1(\Omega)$, the interpolation inequality for the L^3 -norm (Lemma 12), in three dimensional domains, and the fact that it stems from the Poisson-Neumann problem that u_n and v_n satisfy, with Neumann homogeneous boundary conditions, that $\|\nabla u_n\|_{H^1(\Omega)} \leq \tilde{C} \|\Delta u_n\|_{L^2(\Omega)}$ and $\|\nabla v_n\|_{H^1(\Omega)} \leq \tilde{C} \|\Delta v_n\|_{L^2(\Omega)}$. It gives us

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta u_n\|_{L^2(\Omega)}^2 + \|\nabla \Delta u_n\|_{L^2(\Omega)}^2 &\leq m \|\nabla \Delta v_n\|_{L^2(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C}_1 \|\Delta v_n\|_{L^3(\Omega)} \|\Delta u_n\|_{L^2(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C}_1 \|D^2 v_n\|_{L^3(\Omega)} \|\Delta u_n\|_{L^2(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)} \\ &+ \tilde{C}_1 \|\Delta v_n\|_{L^2(\Omega)} \|\Delta u_n\|_{L^2(\Omega)}^{1/2} \|\nabla \Delta u_n\|_{L^2(\Omega)}^{3/2} \end{aligned}$$

$$+ \tilde{C}_1 \|\Delta u_n\|_{L^2(\Omega)}^2 \|\Delta v_n\|_{L^2(\Omega)} \|\nabla \Delta u_n\|_{L^2(\Omega)}.$$

Finally, we use Young's inequality with the convenient weight (Lemma 13) to obtain

$$\begin{aligned} \frac{d}{dt} \|\Delta u_n\|_{L^2(\Omega)}^2 + \|\nabla \Delta u_n\|_{L^2(\Omega)}^2 &\leq C(m, \delta) \|\nabla \Delta v_n\|_{L^2(\Omega)}^2 + \delta \|\nabla \Delta u_n\|_{L^2(\Omega)}^2 \\ &+ \left[C(\delta) \|\Delta v_n\|_{L^3(\Omega)}^2 \right] \|\Delta u_n\|_{L^2(\Omega)}^2 + \delta \|\nabla \Delta u_n\|_{L^2(\Omega)}^2 \\ &+ \left[C(\delta) \|D^2 v_n\|_{L^3(\Omega)}^2 \right] \|\Delta u_n\|_{L^2(\Omega)}^2 + \delta \|\nabla \Delta u_n\|_{L^2(\Omega)}^2 \\ &+ \left[C(\delta) \|\Delta v_n\|_{L^2(\Omega)}^2 \right] \|\Delta u_n\|_{L^2(\Omega)}^2 + \delta \|\nabla \Delta v_n\|_{L^2(\Omega)}^2 \\ &+ \left[C(\delta) \|\Delta u_n\|_{L^2(\Omega)}^2 \|\Delta v_n\|_{L^2(\Omega)}^2 \right] \|\Delta u_n\|_{L^2(\Omega)}^2 \\ &+ \delta \|\nabla \Delta u_n\|_{L^2(\Omega)}^2, \end{aligned}$$

for some constants $C(m, \delta), C(\delta) > 0$. From this inequality, (2.16) and the *a priori* n -independent bounds obtained so far we can conclude that

$$\Delta u_n \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad (2.17)$$

and then, analogously to the obtaining of (2.15), we can conclude that

$$\partial_t u_n \text{ is bounded in } L^2(0, T; H^1(\Omega)). \quad (2.18)$$

These bounds depend on m but are independent of n . They will be important to establish the existence a stronger solution of the regularized problem (P_m^a) .

2.1.2 Convergence to weak solutions of (P_m^b)

Let us consider the *a priori* n -independent estimates obtained under the hypothesis that the Poisson-Neumann problem has the H^2 -regularity. According to the compactness results in the weak and weak* topologies (chapter 3 of Brezis (BREZIS, 2010)) and compactness results in Bochner spaces (corollary 8 of Simon (SIMON, 1986)), we can conclude from (2.6), (2.7), (2.11), (2.14) and (2.9) that, up to a subsequence, for each $T > 0$,

$$\begin{aligned} u_n &\rightarrow u \text{ weakly* in } L^\infty(0, T; H^1(\Omega)), \\ u_n &\rightarrow u \text{ weakly in } L^2(0, T; H^2(\Omega)), \\ u_n &\rightarrow u \text{ strongly in } L^2(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega)), \end{aligned} \quad (2.19)$$

$$\begin{aligned} v_n &\rightarrow v \text{ weakly* in } L^\infty(0, T; H^2(\Omega)), \\ v_n &\rightarrow v \text{ strongly in } C([0, T]; H^1(\Omega)). \end{aligned} \quad (2.20)$$

Hence, using the information given by Remarks 1 and 34, Lemma 16 and by (2.19) and (2.7), we obtain

$$\begin{aligned} b_m(u_n) &\rightarrow b_m(u) \text{ strongly in } L^2(0, T; L^2(\Omega)), \\ a_m(u_n)^s &\rightarrow a_m(u)^s \text{ strongly in } L^2(0, T; L^2(\Omega)). \end{aligned} \quad (2.21)$$

We rewrite the approximate problem as its equivalent formulation

$$\left. \begin{aligned} - \int_0^T (u_n(t), \varphi_j) \xi_t(t) dt + \int_0^T (\nabla u_n(t), \nabla \varphi_j) \xi(t) dt \\ = \int_0^T (b_m(u_n(t)) \nabla v_n(t), \nabla \varphi_j) \xi(t) dt, \end{aligned} \right\} \quad (2.22)$$

$$\left. \begin{aligned} - \int_0^T (v_n(t), \varphi_j) \chi_t(t) dt + \int_0^T (\nabla v_n(t), \nabla \varphi_j) \chi(t) dt = \\ - \int_0^T (a_m(u_n(t))^s v_n(t), \varphi_j) \chi(t) dt, \end{aligned} \right\} \quad (2.23)$$

$$u_n(0) = u_n^0, \quad v_n(0) = v_n^0, \quad (2.24)$$

for $j = 1, \dots, m$ and for all $\xi, \chi \in C_c^1(0, T)$. Using the convergences (2.19), (2.20) and (2.21) it is possible to pass to the limit in (2.22) and (2.23), for each φ_j and conclude that

$$(u_t(t), \varphi_j) + (\nabla u(t), \nabla \varphi_j) = (b_m(u(t)) \nabla v(t), \nabla \varphi_j), \quad \forall j \in \mathbb{N}, \text{ a.e } t \in (0, T), \quad (2.25)$$

$$(v_t(t), \varphi_j) + (\nabla v(t), \nabla \varphi_j) = -(a_m(u(t))^s v(t), \varphi_j), \quad \forall j \in \mathbb{N}, \text{ a.e } t \in (0, T). \quad (2.26)$$

Because of linearity, we can notice that (2.25) and (3.16) are valid for any linear combination of the functions of $\{\varphi_j\}_{j \in \mathbb{N}}$. Considering u and v to be fixed, the linear forms which appear in (2.25) and (3.16) are continuous with respect to the H^1 -norm and the space of all the linear combinations of the functions of $\{\varphi_j\}_{j \in \mathbb{N}}$ is dense in $H^1(\Omega)$. Then we can extend the linear forms by density to all $\varphi \in H^1(\Omega)$. Thus we obtain

$$(u_t(t), \varphi) + (\nabla u(t), \nabla \varphi) = (b_m(u(t)) \nabla v(t), \nabla \varphi), \quad \forall \varphi \in H^1(\Omega), \text{ a.e } t \in (0, T), \quad (2.27)$$

$$(v_t(t), \varphi) + (\nabla v(t), \nabla \varphi) = -(a_m(u(t))^s v(t), \varphi), \quad \forall \varphi \in H^1(\Omega), \text{ a.e } t \in (0, T). \quad (2.28)$$

We also observe that, because of the convergence in $C([0, T]; L^2(\Omega))$ in (2.19),

$$u_n^0 = u_n(0) \rightarrow u(0) \text{ strongly in } L^2(\Omega),$$

as $n \rightarrow \infty$. But we also have

$$u_n^0 \rightarrow u^0 \text{ strongly in } L^2(\Omega),$$

as $n \rightarrow \infty$, because u_n^0 is the L^2 -projection of u^0 into X^n . Therefore we conclude that $u(0) = u^0$. Analogously, we have $v(0) = v^0$.

But the regularity of u and v , which stems from the convergences (2.19) and (2.20) and the n -uniform bounds (2.14) and (2.15), allows us to conclude that

$$\partial_t u, \Delta u, \nabla \cdot (b_m(u) \nabla v) \in L^2(0, T; L^2(\Omega)), \quad (2.29)$$

$$\partial_t v, \Delta v, a_m(u)^s v \in L^2(0, T; H^1(\Omega)) \quad (2.30)$$

and therefore the equations are satisfied in the following sense:

$$u_t(t, x) - \Delta u(t, x) = -\nabla \cdot (b_m(u(t, x)) \nabla v(t, x)), \text{ a.e. } (t, x) \in (0, T) \times \Omega, \quad (2.31)$$

$$v_t(t, x) - \Delta v(t, x) = -a_m(u(t, x))^s v(t, x), \text{ a.e. } (t, x) \in (0, T) \times \Omega, \quad (2.32)$$

with $u(0, x) = u^0(x)$ and $v(0, x) = v^0(x)$, a.e. $x \in \Omega$.

One can prove that u and v are unique by straightforward calculations.

Lemma 37. *The solution of (P_m^b) is unique.*

Proof. Suppose that (u_1, v_1) and (u_2, v_2) are such that

$$u_1, u_2 \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),$$

$$v_1, v_2 \in L^\infty(0, T; H^2(\Omega))$$

and satisfy (2.27) and (2.28). Define $w = u_2 - u_1$ and $z = v_2 - v_1$. Then w satisfies

$$\begin{aligned} (w_t(t), \varphi) + (\nabla w(t), \nabla \varphi) &= ([b_m(u_2(t)) - b_m(u_1(t))] \nabla v_2(t), \nabla \varphi) \\ &+ (b_m(u_1(t)) \nabla z, \nabla \varphi), \quad \forall \varphi \in H^1(\Omega), \text{ a.e. } t \in (0, T), \end{aligned} \quad (2.33)$$

and z satisfies

$$\begin{aligned} (z_t(t), \varphi) + (\nabla z(t), \nabla \varphi) &+ (a_m(u_1(t))^s z(t), \varphi) \\ &= -([a_m(u_2(t))^s - a_m(u_1(t))^s] v_2(t), \varphi), \quad \forall \varphi \in H^1(\Omega), \text{ a.e. } t \in (0, T). \end{aligned} \quad (2.34)$$

with $w(0) = z(0) = 0$.

Now we test (2.33) by w and (2.34) by z . Since m is fixed, $a_m(u_1)$ and $a_m(u_2)$ are bounded. And, because of Remark 34, $b_m(\cdot)$ is uniformly Lipschitz with constant 1. Then, concerning the function w , we have

$$\begin{aligned} \frac{d}{dt} \|w\|_{L^2(\Omega)}^2 + 2 \|\nabla w\|_{L^2(\Omega)}^2 &\leq \int_{\Omega} |w \nabla v_2 \nabla w| \, dx \\ &+ m \|\nabla z\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \\ &\leq \|w\|_{L^3(\Omega)} \|\nabla v_2\|_{L^6(\Omega)} \|\nabla w\|_{L^2(\Omega)} \\ &+ m \|\nabla z\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)}. \end{aligned}$$

Recalling Lemma 16, regarding the function z we find

$$\frac{d}{dt} \|z\|_{L^2(\Omega)}^2 + 2 \|\nabla z\|_{L^2(\Omega)}^2 + \|a_m(u_1)^{s/2} z\|_{L^2(\Omega)}^2$$

$$\leq m^{s-1} \|w\|_{L^2(\Omega)} \|v_2\|_{L^\infty(0,T;L^\infty(\Omega))} \|z\|_{L^2(\Omega)}.$$

Using Young's inequality we can write

$$\begin{aligned} \frac{d}{dt} \|w\|_{L^2(\Omega)}^2 + (2 - \delta) \|\nabla w\|_{L^2(\Omega)}^2 &\leq C(\delta) \|\nabla v_2\|_{L^6(\Omega)}^4 \|w\|_{L^2(\Omega)}^2 \\ &\quad + C(\delta) m^2 \|\nabla z\|_{L^2(\Omega)}^2 \end{aligned} \quad (2.35)$$

and

$$\begin{aligned} \frac{d}{dt} \|z\|_{L^2(\Omega)}^2 + (2 - \delta) \|\nabla z\|_{L^2(\Omega)}^2 \\ \leq m^{2s-2} \|v_2\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \|z\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2. \end{aligned} \quad (2.36)$$

We can choose $\delta = 1$ and from (2.35) and (2.36) we can obtain

$$\begin{aligned} \frac{d}{dt} \left[\|w\|_{L^2(\Omega)}^2 + 2Cm^2 \|z\|_{L^2(\Omega)}^2 \right] &+ \|\nabla w\|_{L^2(\Omega)}^2 + 2Cm^2 \|\nabla z\|_{L^2(\Omega)}^2 \\ &\leq \|\nabla v_2\|_{L^6(\Omega)}^4 \|w\|_{L^2(\Omega)}^2 + Cm^2 \|\nabla z\|_{L^2(\Omega)}^2 \\ &\quad + m^{2s-2} \|v_2\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \|z\|_{L^2(\Omega)}^2 + 2Cm^2 \|w\|_{L^2(\Omega)}^2. \end{aligned}$$

The term $Cm^2 \|\nabla z\|_{L^2(\Omega)}^2$ in the right hand side can be absorbed by the term

$$2Cm^2 \|\nabla z\|_{L^2(\Omega)}^2$$

in the left hand side.

Then, taking into account that $v_2 \in L^\infty(0, T; W^{1,6}(\Omega))$, using Gronwall's inequality (Lemma 20) and the initial conditions $w(0) = z(0) = 0$, we conclude that $(u_1, v_1) = (u_2, v_2)$ ■

Thus, in this subsection, for each fixed $m \in \mathbb{N}$, we have proved the existence and uniqueness of $u_m \in L_{loc}^\infty([0, \infty); H^1(\Omega)) \cap L_{loc}^2([0, \infty); H^2(\Omega))$, $v_m \in L_{loc}^\infty([0, \infty); H^2(\Omega))$, solution of (2.31)-(2.32), that is, of (P_m^b) .

2.1.3 Regularity and other properties of the regularized solutions of (P_m^a)

Continuing the analysis, in the present subsection we finally prove the existence and uniqueness of solution to the problem (P_m^a) . First we show that the solution of (P_m^a) and (P_m^b) are the same.

Lemma 38. (Pointwise estimates for (u, v)) *Let (u, v) be the solution of (P_m^b) . Therefore:*

1. *If $u^0(x) \geq 0$ a.e. $x \in \Omega$ then $u(t, x) \geq 0$ a.e. $(t, x) \in (0, \infty) \times \Omega$;*

2. If $v^0(x) \geq 0$ a.e. $x \in \Omega$ and $v^0 \in L^\infty(\Omega)$ then

$$\operatorname{ess\,inf}_{x \in \Omega} \{v^0(x)\} e^{-m^s t} \leq v(t, x) \leq \|v^0\|_{L^\infty(\Omega)} \quad \text{a.e. } (t, x) \in (0, \infty) \times \Omega;$$

In particular, if there is $\beta > 0$ such that $v^0(x) \geq \beta$ a.e. in Ω , then

$$v(t, x) \geq \beta e^{-m^s t}, \forall (t, x) \in [0, \infty) \times \Omega.$$

Proof. By testing the u -equation of (P_m^b) by $(u)_-$ and using that $|b_m(u_-)| \leq C |u_-|$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_-\|_{L^2(\Omega)}^2 + \|\nabla u_-\|_{L^2(\Omega)}^2 &= \int_{\Omega} b_m(u) \nabla v \cdot \nabla u_- \, dx \\ &= \int_{\Omega} b_m(u_-) \nabla v \cdot \nabla u_- \, dx \\ &\leq \|u_-\|_{L^3(\Omega)} \|\nabla v\|_{L^6(\Omega)} \|\nabla u_-\|_{L^2(\Omega)} \\ &\leq C_1 \|u_-\|_{L^2(\Omega)}^{1/2} \|u_-\|_{H^1(\Omega)}^{1/2} \|\nabla v\|_{L^6(\Omega)} \|\nabla u_-\|_{L^2(\Omega)} \\ &\leq C_2 \|u_-\|_{L^2(\Omega)} \|\nabla v\|_{L^6(\Omega)} \|\nabla u_-\|_{L^2(\Omega)} \\ &\quad + C_2 \|u_-\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^6(\Omega)} \|\nabla u_-\|_{L^2(\Omega)}^{3/2}. \end{aligned}$$

Hence, using Hölder's inequality and that $v \in L^\infty(0, T; W^{1,6}(\Omega))$ we can find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_-\|_{L^2(\Omega)}^2 + \|\nabla u_-\|_{L^2(\Omega)}^2 &\leq C_3 \|u_-\|_{L^2(\Omega)}^2 \\ &\quad + C_3 \|u_-\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u_-\|_{L^2(\Omega)}^2, \end{aligned}$$

and then

$$\frac{d}{dt} \|u_-\|_{L^2(\Omega)}^2 + \|\nabla u_-\|_{L^2(\Omega)}^2 \leq C \|u_-\|_{L^2(\Omega)}^2.$$

Note that $(u^0)_- = 0$, by hypothesis. Therefore, if we apply Gronwall's inequality (Lemma 20) we conclude that $u_-(t, x) = 0$, a.e. $(t, x) \in (0, T) \times \Omega$, for all $T > 0$, that is, $u(t, x) \geq 0$, a.e. $(t, x) \in (0, \infty) \times \Omega$.

In order to establish the positivity of v we define the function

$$V(t) = \operatorname{ess\,inf}_{x \in \Omega} \{v^0(x)\} e^{-m^s t}.$$

Clearly, $V(t) > 0, \forall t \in [0, \infty)$ and V is a sub-solution of the v -equation of (P_m^b) . In fact, we have

$$V_t(t) - \Delta V(t) = -m^s V(t) \leq -a_m(u(t, x))^s V(t),$$

Comparing V and v we conclude that $v(t, x) \geq V(t)$ a.e. in $(0, \infty) \times \Omega$.

The upper bound on v can be obtained an analogous argument, but now using the super-solution $V(t, x) = \|v^0\|_{L^\infty(\Omega)} e^{-m^s t}$. ■

Corollary 39. *The unique solution of (P_m^b) is the unique solution of (P_m^a) and vice versa.*

Proof. After Lemma 38, we notice that, because of the definition of a_m and b_m (see (1) and (2.1), respectively) and the positivity of u , we have $b_m(u) = a_m(u)$. Therefore the function (u, v) , solution of (P_m^b) , is also a solution of (P_m^a) .

In addition to it, considering the regularity of the solution, analogously to the proof of uniqueness of solution for (P_m^b) (Lemma (37)), we can demonstrate the uniqueness of solution to (P_m^a) . This proves that the unique solution of (P_m^a) and (P_m^b) are the same. ■

Therefore, so far, for each fixed $m \in \mathbb{N}$, we have proved the existence and uniqueness of $u \in L_{loc}^\infty([0, \infty); H^1(\Omega)) \cap L_{loc}^2([0, \infty); H^2(\Omega))$, $v \in L_{loc}^\infty([0, \infty); H^2(\Omega))$, solution of (P_m^a) . Then, from now on, we will deal only with problem (P_m^a) .

What is missing is the proof of the uniform in time regularity

$$\begin{aligned} (u_m - u^*) &\in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)), \\ v_m &\in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; H^2(\Omega)). \end{aligned} \tag{2.37}$$

Remark 40. Considering the convergences (2.19) and (2.20), we already have

$$\begin{aligned} (u_m - u^*) &\in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\ v_m &\in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^2(\Omega)), \end{aligned}$$

for each finite $T > 0$. We want to prove that this regularity is valid up to $T = \infty$, that is, (2.37). ■

Besides this regularity, we also provide other properties of the solution. More precisely, we proved the positivity in Lemma 38 and we will prove some direct m -independent estimates in Lemma 41. With Lemma 42 we begin the proof of (2.37).

Lemma 41. *Let (u, v) be the solution of (P_m^a) . We have the following estimates (independent of m):*

1. For every $t \geq 0$,

$$\|u(t)\|_{L^1(\Omega)} = \int_{\Omega} u(t, x) \, dx = \|u^0\|_{L^1(\Omega)} = u^* |\Omega|;$$

2. For every $t > 0$,

$$\|v(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla v(s)\|_{L^2(\Omega)}^2 \, ds + \int_0^t \|a_m(u(s))^{s/2} v(s)\|_{L^2(\Omega)}^2 \, ds \leq \|v^0\|_{L^2(\Omega)}^2,$$

which allows us to conclude that ∇v is bounded in $L^2(0, \infty; L^2(\Omega))$, independently of $m \in \mathbb{N}$.

Proof. To prove the first item, we integrate the u -equation of (P_m^a) and take into account that $u \geq 0$, thanks to Lemma 38. The second item can be proved by testing the v_m -equation of (P_m^a) by $2v$. ■

Lemma 42. *For each fixed $m \in \mathbb{N}$, the following regularity at infinity time to u and v holds:*

$$(u - u^*), v \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)).$$

Proof. From Lemma 41.1 we have $\int_{\Omega} (u(t) - u^*) dx = 0, \forall t \in [0, \infty)$, that is, $u - u^*$ is a null mean function. Besides, by testing the u_m -equation of (P_m^a) by $2u$ and using the estimate given in Lemma 41.2, we arrive at

$$\|u(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla u(s)\|_{L^2(\Omega)}^2 ds \leq \|u^0\|_{L^2(\Omega)}^2 + m^2 \|v^0\|_{L^2(\Omega)}^2.$$

The latter allows us to conclude that

$$\nabla u \in L^2(0, \infty; L^2(\Omega)). \quad (2.38)$$

Hence, using the Poincaré type inequality of Lemma 14 we can prove that $u - u^* \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega))$.

Next, we use this fact and take v as a test function in the v -equation

$$v_t - \Delta v + a_m(u^*)^s v = -(a_m(u)^s - a_m(u^*)^s)v.$$

We obtain

$$\begin{aligned} \frac{d}{dt} \|v(t)\|_{L^2(\Omega)}^2 + 2 \|\nabla v(t)\|_{L^2(\Omega)}^2 + a_m(u^*) \|v(t)\|_{L^2(\Omega)}^2 = \\ - \int_{\Omega} (a_m(u(t, x))^s - a_m(u^*)^s) v(t, x)^2 dx. \end{aligned} \quad (2.39)$$

Then, recalling that $|v_m(t, x)| \leq \|v^0\|_{L^\infty(\Omega)}$ (Lemma 38) the right hand side of the above inequality can be estimated by

$$\left| \int_{\Omega} (a_m(u)^s - a_m(u^*)^s) v^2 dx \right| \leq \|v^0\|_{L^\infty(\Omega)} \int_{\Omega} |a_m(u)^s - a_m(u^*)^s| v dx.$$

Noting that

$$|a_m(u)^s - a_m(u^*)^s| \leq s |a_m(u) + a_m(u^*)|^{s-1} |a_m(u) - a_m(u^*)|,$$

because of Lemma 16, we obtain

$$\left| \int_{\Omega} (a_m(u)^s - a_m(u^*)^s) v^2 dx \right| \leq \|v^0\|_{L^\infty(\Omega)} s \int_{\Omega} |a_m(u) + a_m(u^*)|^{s-1} |a_m(u) - a_m(u^*)| v dx.$$

Now we recall that m is fixed and therefore $a_m(r) \leq m$, for all $r \in \mathbb{R}$. Then we have

$$\left| \int_{\Omega} (a_m(u)^s - a_m(u^*)^s) v^2 dx \right| \leq \|v^0\|_{L^\infty(\Omega)} (2m)^{s-1} s \int_{\Omega} |a_m(u) - a_m(u^*)| v dx.$$

Hence, using Hölder's inequality we obtain

$$\left| \int_{\Omega} (a_m(u)^s - a_m(u^*)^s) v^2 dx \right| \leq (2m)^{s-1} s \|v^0\|_{L^\infty(\Omega)} \|u - u^*\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.$$

Finally, using Young's inequality (Lemma 13) with the convenient weight, we obtain

$$\left| \int_{\Omega} (a_m(u)^s - a_m(u^*)^s) v^2 dx \right| \leq C(u^*, v^0, s) m^{s-1} \|u - u^*\|_{L^2(\Omega)}^2 + \frac{a_m(u^*)^s}{2} \|v\|_{L^2(\Omega)}^2,$$

where $C(u^*, v^0, s) > 0$ is a constant (independent of t and x). Now, considering also the terms of the left hand side of (2.39), one has

$$\frac{d}{dt} \|v(t)\|_{L^2(\Omega)}^2 + 2 \|\nabla v(t)\|_{L^2(\Omega)}^2 + a_m(u^*) \|v(t)\|_{L^2(\Omega)}^2 \leq C(u^*, v^0, s) m^{s-1} \|u(t) - u^*\|_{L^2(\Omega)}^2.$$

Note that $a_m(u^*)$ is a fixed positive real number if $u^0 \neq 0$ and $\|u(t) - u^*\|_{L^2(\Omega)}^2 \in L^1(0, \infty)$. Hence we can conclude that $v \in L^2(0, \infty; L^2(\Omega))$ and, together with Lemma 41.2, we finally conclude that $v \in L^2(0, \infty; H^1(\Omega))$. ■

Using the regularity of Lemma 42 and recalling that $a_m(u)$, $a'_m(u)$ and v belong to $L^\infty(0, \infty; L^\infty(\Omega))$, we can test the v_m -equation of problem (P_m^a) by $-\Delta v$, proceed as we did in order to obtain (2.8). Using the regularity up to infinity time obtained so far we can conclude that

$$v \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)), \quad (2.40)$$

Next, we would like to test the v_m -equation of (P_m^a) by $\Delta^2 v$. But we don't have information about $\Delta^2 v$. In order to justify the calculations, we recall the regularity (2.30) of v . Because of this regularity, we can take the gradient of the v_m -equation of (P_m^a) and then test the resulting equation by $\nabla \Delta v$. This gives us

$$(\nabla v_t, \nabla \Delta v) + \|\nabla \Delta v\|_{L^2(\Omega)}^2 = -(a_m(u)^s \nabla v, \nabla \Delta v) - (v a'_m(u) a_m(u)^{s-1} \nabla u, \nabla \Delta v)$$

and then

$$\frac{1}{2} \frac{d}{dt} \|\Delta v\|_{L^2(\Omega)}^2 + \|\nabla \Delta v\|_{L^2(\Omega)}^2 = -(a_m(u)^s \nabla v, \nabla \Delta v) - (v a'_m(u) a_m(u)^{s-1} \nabla u, \nabla \Delta v).$$

Therefore, proceeding as we did in order to obtain (2.11), if we use the regularity up to infinity time obtained so far we can conclude that actually

$$v \in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; H^2(\Omega)), \quad (2.41)$$

$$\Delta v \in L^2(0, \infty; H^1(\Omega)). \quad (2.42)$$

Note that (2.41) is the regularity of the function v that appears in (2.37), which we want to prove. Then, in order to finish the proof of (2.37) we must prove the regularity of u_m . To do so, we test the u_m -equation of problem (P_m^a) by $-\Delta u$. Using (2.42), we can proceed as we did in order to obtain (2.13) and conclude that

$$(u - u^*) \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)).$$

This concludes the proof of (2.37).

We remind that the regularity (2.37) is proved under the assumption that the Poisson-Neumann problem (1.2) has the H^2 -regularity. In order to finish the chapter, we discuss the regularity of u and v in case we suppose that the Neumann problem (1.2) has the $W^{2,3}$ -regularity.

In this case, because of the *a priori* estimates (2.17), (2.18), (2.11) and (2.15) for u_n and v_n , we can conclude that u and v have the regularity

$$\begin{aligned} u - u^* &\in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,3}(\Omega)) \\ u_t, \Delta u &\in L^2(0, T; H^1(\Omega)), \end{aligned}$$

$$\begin{aligned} v &\in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,3}(\Omega)), \\ v_t, \Delta v &\in L^2(0, T; H^1(\Omega)). \end{aligned}$$

But now that we have (2.37), in order to obtain the regularity up to infinity time, we would like to test the u_m -equation of (P_m^a) by $\Delta^2 u$. Again, we don't have information about $\Delta^2 u$. Then we have to proceed as we did above in order to obtain (2.41) and (2.42). Proceeding like this we can conclude that

$$\begin{aligned} u - u^* &\in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; W^{2,3}(\Omega)) \\ u_t, \Delta u &\in L^2(0, \infty; H^1(\Omega)), \end{aligned} \tag{2.43}$$

$$\begin{aligned} v &\in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; W^{2,3}(\Omega)), \\ v_t, \Delta v &\in L^2(0, \infty; H^1(\Omega)). \end{aligned} \tag{2.44}$$

CHAPTER 3

THE ORIGINAL PROBLEM

In this Chapter we will obtain m -independent estimates to (u, v) , the solution of problem (P_m^a) . Here, this problem will be denoted by (P_m) :

$$\begin{aligned} \partial_t u_m - \Delta u_m &= -\nabla \cdot (a_m(u_m) \nabla v_m) \\ \partial_t v_m - \Delta v_m &= -a_m(u_m)^s v_m \\ \partial_\eta u_m \Big|_\Gamma &= \partial_\eta v_m \Big|_\Gamma = 0 \\ u_m(0) &= u^0, \quad v_m(0) = v^0, \end{aligned} \tag{P_m}$$

in order to pass to the limit as $m \rightarrow \infty$ and prove the existence of solution to the original problem (P). We also study the regularity of the solution of (P) obtained in the existence result.

To do that, we assume that problem (1.2) has the H^2 -regularity, Hypothesis (H1) holds and the initial conditions u^0 and v^0 are non-negative. This makes sense because of the physical meaning of these functions.

For each $m \in \mathbb{N}$, we will consider (u_m, v_m) to be the solution of (P_m) with initial data $u_m^0 \in H^1(\Omega)$ and $v_m^0 \in H^2(\Omega)$, with u_m^0 and v_m^0 being approximations of u^0 and v^0 such that $u_m^0, v_m^0 \geq 0$,

$$u_m^0 \rightarrow u^0 \text{ strongly in } L^p(\Omega), \text{ as } m \rightarrow \infty, \tag{3.1}$$

for $p = 1 + \varepsilon$, for some $\varepsilon > 0$, if $s = 1$, and $p = s$, if $s > 1$, and

$$v_m^0 \rightarrow v^0 \text{ strongly in } H^1(\Omega) \cap L^\infty(\Omega), \text{ as } m \rightarrow \infty. \tag{3.2}$$

We begin by the 3D case proving Theorem 3 in Section 3.1. Section 3.2 is devoted to the proof of Theorem 6.

3.1 Proof of Theorem 3

In this Section we are going to prove Theorem 3. In order to do it, we will obtain m -independent *a priori* estimates for u_m and v_m and then use compactness results to pass to the limit as $m \rightarrow \infty$ in problem (P_m) , thus proving the existence of solution to (P).

We remark that, from Subsection 2.1.3, we already have some m -independent *a priori* estimates for u_m and v_m from which we can conclude that

$$v_m \text{ is bounded in } L^\infty(0, \infty; L^\infty(\Omega)), \quad (3.3)$$

$$u_m \text{ is bounded in } L^\infty(0, \infty; L^1(\Omega)) \quad (3.4)$$

and

$$\nabla v_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)), \quad (3.5)$$

from Lemmas 38 and 41. These m -uniform bounds will be used through this Section in the process of obtaining m -independent *a priori* estimates, but they are not sufficient to pass to the limit as $m \rightarrow \infty$ in all the terms of (P_m) .

Therefore, in the present Section we are going to obtain stronger m -independent *a priori* estimates for u_m and v_m . In Subsection 3.1.1, we perform some formal computations in order to illustrate how these stronger estimates will be obtained. In Subsection 3.1.2, we will prove three intermediate inequalities, considering $s = 1$, $s \in (1, 2)$ and $s \geq 2$, respectively, which will be useful in the process of obtaining the aforementioned m -independent *a priori* estimates. In Subsection 3.1.3 we give special attention to a term that must be adequately estimated and finish the Subsection with some m -uniform *a priori* estimates. Finally, in Subsection 3.1.4, we obtain the sufficient m -uniform bounds for u_m and v_m from the m -uniform estimates obtained in Subsection 3.1.3 and pass to the limit as $m \rightarrow \infty$, proving the existence of solution for (P) in 3D bounded domains. Subsection 3.1.4 is divided in the cases $s \in [1, 2)$ and $s \geq 2$.

3.1.1 Formal computations

The basic idea is that the effects of the consumption and chemotaxis terms cancel. First of all, we present some formal calculations to illustrate how it works. Suppose (u, v) is a regular enough solution to the original problem (P). In order to obtain estimates to u and v we consider a function $g(u)$ such that $g''(u) = u^{s-2}$; formally, supposing u, v are strictly positive, we can use

$$g'(u) = \begin{cases} \frac{u^{s-1}}{(s-1)} & , \text{ if } s > 1, \\ \ln(u) & , \text{ if } s = 1. \end{cases} \quad (3.6)$$

as a test function in the u -equation of (P), obtaining

$$\frac{d}{dt} \int_{\Omega} g(u) dx + \int_{\Omega} g''(u) |\nabla u|^2 dx = \int_{\Omega} u g''(u) \nabla v \cdot \nabla u$$

and, since $u g''(u) = u^{s-1}$, we have

$$\frac{d}{dt} \int_{\Omega} g(u) dx + \int_{\Omega} u^{s-2} |\nabla u|^2 dx = \int_{\Omega} u^{s-1} \nabla v \cdot \nabla u. \quad (3.7)$$

On the other hand, we can test the v -equation of (P) by

$$-\Delta \ln(v) = -\nabla \cdot \left(\frac{\nabla v}{v} \right) = -\frac{1}{v} \Delta v + \frac{1}{v^2} |\nabla v|^2.$$

Then we obtain

$$-\int_{\Omega} v_t \Delta \ln(v) dx = -\int_{\Omega} \Delta v \Delta \ln(v) dx + \int_{\Omega} u^s v \Delta \ln(v) dx.$$

For the term which involves the time derivative of v we use

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v} dx &= \int_{\Omega} \left(\frac{\nabla v}{\sqrt{v}} \right)_t \cdot \frac{\nabla v}{\sqrt{v}} dx \\ &= -\frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} v_t dx + \int_{\Omega} (\nabla v)_t \frac{\nabla v}{v} dx \\ &= -\frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} v_t dx - \int_{\Omega} v_t \nabla \cdot \frac{\nabla v}{v} dx \\ &= -\frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} v_t dx - \int_{\Omega} v_t \Delta(\ln(v)) dx \end{aligned}$$

in order to write

$$-\int_{\Omega} v_t \Delta(\ln(v)) dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} v_t dx. \quad (3.8)$$

In the last term on the right hand side of (3.8) we use $v_t = \Delta v - u^s v$. This leads us to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v} dx + \int_{\Omega} \frac{|\nabla v|^2}{v} dx + \frac{1}{2} \int_{\Omega} u^s \frac{|\nabla v|^2}{v} dx \\ = \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} \Delta v dx - s \int_{\Omega} u^{s-1} \nabla u \cdot \nabla v dx. \end{aligned} \quad (3.9)$$

Now, note that the last term in the right hand side of (3.9) is equal to $-s$ times the term that appears in the right hand side of (3.7). Hence, if we add (3.9) to s times (3.7), these two terms cancel each other and we obtain the time differential equation

$$\begin{aligned} \frac{d}{dt} \left[s \int_{\Omega} g(u) dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v} dx \right] + s \int_{\Omega} u^{s-2} |\nabla u|^2 dx + \\ + \int_{\Omega} u^s \frac{|\nabla v|^2}{v} dx = - \int_{\Omega} \frac{|\Delta v|^2}{v} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} \Delta v dx. \end{aligned} \quad (3.10)$$

Since all the terms in the left hand side of (3.10) are positive, we only have to estimate the term

$$-\int_{\Omega} \frac{|\Delta v|^2}{v} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} \Delta v dx \quad (3.11)$$

in the right hand side (see Lemma 54 below).

In the sequel, we consider the regularized problem (P_m) and its solution (u_m, v_m) instead of the original problem (P) and (u, v) . In this case, we have to deal with the presence of the truncation $a_m(\cdot)$ in the chemotaxis and consumption terms and with the fact that u_m is non-negative, but not necessarily strictly positive (Lemma 38).

We will separate the cases $s = 1$, $s \in (1, 2)$ and $s \geq 2$, due to the differences that the cases present. First we will obtain an inequality for the case $s = 1$ in Lemma 44, then we will obtain the analogous inequality for the cases $s \in [1, 2)$ and $s \geq 2$ in Lemmas 45 and 46, respectively. In any case, the term

$$-\int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx,$$

which will be the rigorous version of the term (3.11), will remain on the right hand side of the inequality. This term will be treated in the Subsection 3.1.3.

Remark 43. Note that if $s = 1$ then g' , given by (3.6), and g'' have a singularity at zero. If $s \in (1, 2)$, then only g'' is singular at zero and, if $s \geq 2$, then neither g' nor g'' have any singularity. ■

3.1.2 An intermediate inequality

In this subsection, we are going to obtain an inequality which the solutions of the regularized problems satisfy, for each $m \in \mathbb{N}$. Due to the difference of the test functions involved in this process (see Remark 43), we will separate the analysis in three cases: $s = 1$, $s \in (1, 2)$ and $s \geq 2$. For each case we will obtain the corresponding inequality, which we call intermediate inequality. These intermediate inequalities are the first step in the process of obtaining m -independent estimates for the solutions of the regularized problems (P_m) .

We denote by (u, v) the solution of the regularized problem instead of (u_m, v_m) . By the time we have established the m -independent estimates we go back to the notation (u, v) for the solution of the original problem (P) and (u_m, v_m) for the solution of the regularized problem (P_m) .

Lemma 44 (Intermediate inequality for $s = 1$). *The solution (u, v) of the problem (P_m) satisfies, for $\alpha \in (0, 1/4)$,*

$$\frac{d}{dt} \left[\int_{\Omega} g_m(u) dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} dx \right]$$

$$\begin{aligned}
& +2 \int_{\Omega} |\nabla[a_m(u) + 1]^{1/2}|^2 dx + \frac{1}{2} \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{(v + \alpha)} dx \\
& \leq - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx + 2 \int_{\Omega} |\nabla v|^2 dx,
\end{aligned} \tag{3.12}$$

where $g_m(r)$ is a non-negative primitive of $\ln(a_m(r) + 1)$.

Proof. In order to prove (3.12) we will use the cancellation effect mentioned in Subsection 3.1.1. Since now we are dealing with the regularized problem (instead of the original problem), we must pay attention to two technical difficulties that arise: the presence of the truncation $a_m(\cdot)$ in the chemotaxis and consumption terms and the fact that u and v are non-negative, but not strictly positive. For $s = 1$, this means that instead of using $g'(u) = \ln(u)$ as a test function in the u -equation of (P_m) , we must use $g'(a_m(u) + 1) = \ln(a_m(u) + 1)$, in order to preserve the cancellation effect, avoid divisions by zero and invalid values for the argument of $\ln(\cdot)$; and instead of using $-\Delta \ln(v) = -\nabla \cdot \left(\frac{\nabla v}{v} \right)$ as a test function in the v -equation of (P_m) , we must use

$$-\Delta \ln(v + \alpha) = -\nabla \cdot \left(\frac{\nabla v}{v + \alpha} \right) = -\frac{1}{v + \alpha} \Delta v + \frac{1}{(v + \alpha)^2} |\nabla v|^2,$$

where $\alpha > 0$ is a real number to be chosen later, in order to avoid divisions by zero. With this in mind, we proceed to the proof.

We begin by using $\varphi = \ln(a_m(u) + 1)$ in the u -equation of problem (P_m) to obtain

$$\frac{d}{dt} \int_{\Omega} g_m(u) dx + \int_{\Omega} \frac{a'_m(u)}{a_m(u) + 1} |\nabla u|^2 dx = \left(\frac{a_m(u)}{a_m(u) + 1} \nabla v, \nabla a_m(u) \right),$$

where

$$g_m(u) = \int_0^u \ln(a_m(r) + 1) dr$$

is a primitive of $\ln(a_m(r) + 1)$. Since $0 \leq a'_m(u) \leq 1$, by Remark 1, we have $(a'_m(u))^2 \leq a'_m(u)$, we can write

$$\begin{aligned}
\int_{\Omega} \frac{a'_m(u)}{a_m(u) + 1} |\nabla u|^2 dx & \geq \int_{\Omega} \frac{(a'_m(u))^2}{a_m(u) + 1} |\nabla u|^2 dx \\
& = \int_{\Omega} \frac{|\nabla a_m(u)|^2}{a_m(u) + 1} dx \\
& = \int_{\Omega} \left| \frac{\nabla a_m(u)}{\sqrt{a_m(u) + 1}} \right|^2 dx \\
& = 4 \int_{\Omega} |\nabla[a_m(u) + 1]^{1/2}|^2 dx.
\end{aligned}$$

Hence we have

$$\frac{d}{dt} \int_{\Omega} g_m(u) dx + 4 \int_{\Omega} |\nabla[a_m(u) + 1]^{1/2}|^2 dx = \left(\frac{a_m(u) + 1 - 1}{a_m(u) + 1} \nabla v, \nabla a_m(u) \right)$$

$$\begin{aligned}
&= (\nabla v, \nabla a_m(u)) - \left(\nabla v, \frac{\nabla a_m(u)}{a_m(u) + 1} \right) \\
&\leq (\nabla v, \nabla a_m(u)) + 2 \|\nabla v\|_{L^2(\Omega)} \|\nabla[a_m(u) + 1]^{1/2}\|_{L^2(\Omega)},
\end{aligned}$$

because $a_m(u) + 1 \geq 1$ implies $\sqrt{a_m(u) + 1} \leq a_m(u) + 1$. Then we obtain

$$\frac{d}{dt} \int_{\Omega} g_m(u) dx + 3 \int_{\Omega} |\nabla[a_m(u) + 1]^{1/2}|^2 dx \leq (\nabla v, \nabla a_m(u)) + \|\nabla v\|_{L^2(\Omega)}^2 \quad (3.13)$$

In order to deal with the v -equation of (P_m) , first we test the v -equation of (P_m) by $-\Delta(\ln(v + \alpha))$. This gives us

$$-\int_{\Omega} v_t \Delta(\ln(v + \alpha)) dx = -\int_{\Omega} \Delta v \Delta(\ln(v + \alpha)) dx + \int_{\Omega} a_m(u) v \Delta(\ln(v + \alpha)) dx.$$

Then we use a rigorous version of (3.8),

$$-\int_{\Omega} v_t \Delta(\ln(v + \alpha)) dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} v_t dx,$$

in order to write

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} dx &= -\frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} v_t dx \\
&\quad - \int_{\Omega} \Delta v \Delta(\ln(v + \alpha)) dx + \int_{\Omega} a_m(u) v \Delta(\ln(v + \alpha)) dx.
\end{aligned}$$

Now, for the first term in the right hand side of the above identity we recall that $(v)_t = \Delta v - a_m(u)v$; for the second term in the right hand side we recall that

$$-\Delta \ln(v + \alpha) = -\nabla \cdot \left(\frac{\nabla v}{v + \alpha} \right) = -\frac{1}{v + \alpha} \Delta v + \frac{1}{(v + \alpha)^2} |\nabla v|^2;$$

and for the last term of the right hand side we integrate by parts. It follows that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} dx &= -\frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx + \frac{1}{2} \int_{\Omega} a_m(u) v \frac{|\nabla v|^2}{(v + \alpha)^2} dx \\
&\quad - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx \\
&\quad + \int_{\Omega} a_m(u) v \Delta(\ln(v + \alpha)) dx \\
&= \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx + \frac{1}{2} \int_{\Omega} a_m(u) \frac{v}{(v + \alpha)} \frac{|\nabla v|^2}{(v + \alpha)} dx \\
&\quad - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx - \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{v + \alpha} dx \\
&\quad - \int_{\Omega} \frac{v}{v + \alpha} \nabla a_m(u) \cdot \nabla v dx \\
&\leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx - \frac{1}{2} \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{(v + \alpha)} dx
\end{aligned}$$

$$\begin{aligned}
& - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx - \int_{\Omega} \frac{v + \alpha}{v + \alpha} \nabla a_m(u) \cdot \nabla v dx \\
& + \int_{\Omega} \frac{\alpha}{v + \alpha} \nabla a_m(u) \cdot \nabla v dx \\
& \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx - \frac{1}{2} \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{(v + \alpha)} dx \\
& - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx - \int_{\Omega} \nabla a_m(u) \cdot \nabla v dx \\
& + \int_{\Omega} \frac{\alpha}{\sqrt{v + \alpha}} \frac{\nabla a_m(u)}{\sqrt{1 + a_m(u)}} \sqrt{1 + a_m(u)} \frac{\nabla v}{\sqrt{v + \alpha}} dx \\
& \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx - \frac{1}{2} \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{(v + \alpha)} dx \\
& - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx - \int_{\Omega} \nabla a_m(u) \cdot \nabla v dx \\
& + 2 \int_{\Omega} \sqrt{\alpha} |\nabla[a_m(u) + 1]^{1/2}| \left| \sqrt{a_m(u) + 1} \right| \frac{|\nabla v|}{\sqrt{v + \alpha}} dx,
\end{aligned}$$

where, in the last inequality, we used $\sqrt{\frac{\alpha}{v + \alpha}} \leq 1$. If we add the above inequality to the inequality (3.13), then the terms $\int_{\Omega} \nabla a_m(u) \cdot \nabla v dx$ cancel and we obtain

$$\begin{aligned}
& \frac{d}{dt} \left[\int_{\Omega} g_m(u) dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} dx \right] + 3 \int_{\Omega} |\nabla[a_m(u) + 1]^{1/2}|^2 dx + \\
& + \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{(v + \alpha)} dx \leq \\
& \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx + \int_{\Omega} |\nabla v|^2 dx \\
& + 2 \int_{\Omega} \sqrt{\alpha} |\nabla[a_m(u) + 1]^{1/2}| \left| \sqrt{a_m(u) + 1} \right| \frac{|\nabla v|}{\sqrt{v + \alpha}} dx.
\end{aligned}$$

We can deal with the last term in the right hand side of the inequality using Holder's and Young's inequality,

$$\begin{aligned}
& 2 \int_{\Omega} \sqrt{\alpha} |\nabla[a_m(u) + 1]^{1/2}| \left| \sqrt{a_m(u) + 1} \right| \frac{|\nabla v|}{\sqrt{v + \alpha}} dx \\
& \leq 2\sqrt{\alpha} \left(\int_{\Omega} (a_m(u) + 1) \frac{|\nabla v|^2}{v + \alpha} dx \right)^{1/2} \|\nabla[a_m(u) + 1]^{1/2}\|_{L^2(\Omega)} \\
& \leq \alpha \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{v + \alpha} dx + \|\nabla[a_m(u) + 1]^{1/2}\|_{L^2(\Omega)}^2 + \alpha \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{v + \alpha} dx \\
& \leq \alpha \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{v + \alpha} dx + \|\nabla[a_m(u) + 1]^{1/2}\|_{L^2(\Omega)}^2 + \int_{\Omega} |\nabla v|^2 dx.
\end{aligned}$$

Therefore, we can choose $\alpha < 1/4$ and finally obtain

$$\frac{d}{dt} \left[\int_{\Omega} g_m(u) dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} dx \right] + 2 \int_{\Omega} |\nabla[a_m(u) + 1]^{1/2}|^2 dx +$$

$$\begin{aligned}
& + \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{4} \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{(v + \alpha)} dx \leq \\
& \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx + 2 \int_{\Omega} |\nabla v|^2 dx.
\end{aligned}$$

■

Now we prove the inequality considering $s \in (1, 2)$.

Lemma 45 (Intermediate inequality for $s \in (1, 2)$). *The solution (u, v) of the problem (P_m) satisfies, for $\alpha \in (0, 1/4s)$,*

$$\begin{aligned}
& \frac{d}{dt} \left[s \int_{\Omega} g_m(u) dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} dx \right] + \frac{1}{4} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{(v + \alpha)} dx \\
& \leq - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx,
\end{aligned} \tag{3.14}$$

where $g_m(r)$ is a non-negative primitive of

$$g'_m(r) = \frac{a_m(r)^{s-1}}{(s-1)}.$$

Proof. In order to prove (3.14) we will use the cancellation effect between the chemotaxis and consumption terms mentioned in the beginning of this Section. Analogously to the case $s = 1$ in order to preserve the cancellation effect and avoid divisions by zero, for $s \in (1, 2)$, instead of using $g'(u)$ as a test function in the u -equation of (P_m) , we should consider the sequence $\{1/j\}_{j \in \mathbb{N}}$ and use

$$g'_{m,j}(u) = \frac{(a_m(u) + 1/j)^{s-1}}{(s-1)},$$

and instead of using $-\Delta \ln(v) = -\nabla \cdot \left(\frac{\nabla v}{v} \right)$ as a test function in the v -equation of (P_m) , we must use

$$-\Delta \ln(v + \alpha) = -\nabla \cdot \left(\frac{\nabla v}{v + \alpha} \right) = -\frac{1}{v + \alpha} \Delta v + \frac{1}{(v + \alpha)^2} |\nabla v|^2,$$

where $\alpha > 0$ is a real number to be chosen later. Due to the complexity of the procedures that are involved in the demonstration of this Lemma, we divide the proof in four main steps:

1. first we obtain an inequality from the u -equation of problem (P_m) ;
2. then we obtain an inequality from the v -equation of problem (P_m) ;
3. in the sequel, from these two inequalities we obtain (3.16);

4. finally, we pass to the limit as in (3.16) as $j \rightarrow \infty$, which leads us to (3.14) and concludes the proof.

STEP 1: In the first step we deal with the u -equation of (P_m) .

We begin by using $\varphi = g'_{m,j}(u) = (a_m(u) + 1/j)^{s-1}/(s-1)$ in the u -equation of (P_m) to obtain

$$\frac{d}{dt} \int_{\Omega} g_{m,j}(u) \, dx + \int_{\Omega} \frac{a'_m(u)}{(a_m(u) + 1/j)^{2-s}} |\nabla u|^2 \, dx = \left(\frac{a_m(u)}{(a_m(u) + 1/j)^{2-s}} \nabla v, \nabla a_m(u) \right),$$

where

$$g_{m,j}(r) = \int_0^r \frac{(a_m(\theta) + 1/j)^{s-1}}{(s-1)} \, d\theta$$

is a primitive of $(a_m(r) + 1/j)^{s-1}/(s-1)$. Since $0 \leq a'_m(u) \leq 1$, by Remark 1, we have $(a'_m(u))^2 \leq a'_m(u)$, we can write

$$\begin{aligned} \int_{\Omega} \frac{a'_m(u)}{(a_m(u) + 1/j)^{2-s}} |\nabla u|^2 \, dx &\geq \int_{\Omega} \frac{(a'_m(u))^2}{(a_m(u) + 1/j)^{2-s}} |\nabla u_m|^2 \, dx \\ &= \int_{\Omega} \frac{|\nabla a_m(u)|^2}{(a_m(u) + 1/j)^{2-s}} \, dx \\ &= \int_{\Omega} \left| \frac{\nabla a_m(u)}{(a_m(u) + 1/j)^{1-s/2}} \right|^2 \, dx \\ &= \frac{4}{s^2} \int_{\Omega} |\nabla [a_m(u) + 1]^{s/2}|^2 \, dx. \end{aligned}$$

and hence we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} g_{m,j}(u) \, dx + \frac{4}{s^2} \int_{\Omega} |\nabla [a_m(u) + 1/j]^{s/2}|^2 \, dx \\ &= \left(\frac{a_m(u) + 1/j - 1/j}{(a_m(u) + 1/j)^{2-s}} \nabla v, \nabla a_m(u) \right) \\ &= ((a_m(u) + 1/j)^{s-1} \nabla v, \nabla a_m(u)) \\ &\quad - \left(\nabla v, \left(\frac{1/j}{(a_m(u) + 1/j)} \right)^{1-s/2} \frac{(1/j)^{s/2} \nabla a_m(u)}{(a_m(u) + 1/j)^{1-s/2}} \right) \\ &\leq ((a_m(u) + 1/j)^{s-1} \nabla v, \nabla a_m(u)) \\ &\quad + \frac{2}{s} (1/j)^{s/2} \|\nabla v\|_{L^2(\Omega)} \|\nabla [a_m(u) + 1/j]^{s/2}\|_{L^2(\Omega)}, \end{aligned}$$

because $a_m(u) + 1/j \geq 1/j$. Then, using Young's inequality of Lemma 13, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} g_{m,j}(u) \, dx + \frac{3}{s^2} \int_{\Omega} |\nabla [a_m(u) + 1/j]^{s/2}|^2 \, dx \\ \leq ((a_m(u) + 1/j)^{s-1} \nabla v, \nabla a_m(u)) + (1/j)^s \|\nabla v\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.15}$$

STEP 2: In the second step, we deal with the v -equation of (P_m) .

In order to deal with the v -equation of (P_m) , first we test the v -equation of (P_m) by $-\Delta(\ln(v + \alpha))$. This gives us

$$-\int_{\Omega} v_t \Delta(\ln(v + \alpha)) \, dx = -\int_{\Omega} \Delta v \Delta(\ln(v + \alpha)) \, dx + \int_{\Omega} a_m(u)^s v \Delta(\ln(v + \alpha)) \, dx.$$

Then we use a rigorous version of (3.8),

$$-\int_{\Omega} v_t \Delta(\ln(v + \alpha)) \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} v_t \, dx,$$

in order to write

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} \, dx &= -\frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} v_t \, dx \\ &\quad - \int_{\Omega} \Delta v \Delta(\ln(v + \alpha)) \, dx + \int_{\Omega} a_m(u)^s v \Delta(\ln(v + \alpha)) \, dx. \end{aligned}$$

Now, for the first term in the right hand side of the above identity we recall that $(v)_t = \Delta v - a_m(u)^s v$; for the second term in the right hand side we recall that

$$-\Delta \ln(v + \alpha) = -\nabla \cdot \left(\frac{\nabla v}{v + \alpha} \right) = -\frac{1}{v + \alpha} \Delta v + \frac{1}{(v + \alpha)^2} |\nabla v|^2;$$

and for the last term of the right hand side we integrate by parts. It follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} \, dx &= -\frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v \, dx + \frac{1}{2} \int_{\Omega} a_m(u)^s v \frac{|\nabla v|^2}{(v + \alpha)^2} \, dx - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} \, dx \\ &\quad + \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v \, dx + \int_{\Omega} a_m(u)^s v \Delta(\ln(v + \alpha)) \, dx \\ &= \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v \, dx + \frac{1}{2} \int_{\Omega} a_m(u)^s \frac{v}{(v + \alpha)} \frac{|\nabla v|^2}{(v + \alpha)} \, dx \\ &\quad - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} \, dx - \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{v + \alpha} \, dx \\ &\quad - s \int_{\Omega} \frac{v}{v + \alpha} a_m(u)^{s-1} \nabla a_m(u) \cdot \nabla v \, dx \\ &\leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v \, dx - \frac{1}{2} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{(v + \alpha)} \, dx \\ &\quad - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} \, dx - s \int_{\Omega} \frac{v + \alpha}{v + \alpha} a_m(u)^{s-1} \nabla a_m(u) \cdot \nabla v \, dx \\ &\quad + s \int_{\Omega} \frac{\alpha}{v + \alpha} a_m(u)^{s-1} \nabla a_m(u) \cdot \nabla v \, dx \\ &\leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v \, dx - \frac{1}{2} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{(v + \alpha)} \, dx \\ &\quad - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} \, dx - s \int_{\Omega} a_m(u)^{s-1} \nabla a_m(u) \cdot \nabla v \, dx \\ &\quad + \alpha s \int_{\Omega} (a_m(u) + 1/j)^{s-1} |\nabla a_m(u)| \frac{|\nabla v|}{v + \alpha} \, dx \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v+\alpha)^2} \Delta v \, dx - \frac{1}{2} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{(v+\alpha)} \, dx \\
&\quad - \int_{\Omega} \frac{|\Delta v|^2}{v+\alpha} \, dx - s \int_{\Omega} a_m(u)^{s-1} \nabla a_m(u) \cdot \nabla v \, dx \\
&\quad + 2 \int_{\Omega} \sqrt{\alpha} |\nabla[a_m(u) + 1/j]^{s/2}| |(a_m(u) + 1/j)^{s/2}| \frac{|\nabla v|}{\sqrt{v+\alpha}} \, dx,
\end{aligned}$$

where, in the last inequality, we used that $\frac{\alpha}{v+\alpha} \leq \sqrt{\frac{\alpha}{v+\alpha}} \leq 1$.

STEP 3: In the third step, we use the inequalities which we obtained from the u -equation and from the v -equation of (P_m) in steps 1 and 2, respectively, in order to obtain a third inequality.

If we add the above inequality to s times the inequality (3.15), then we obtain

$$\begin{aligned}
&\frac{d}{dt} \left[s \int_{\Omega} g_{m,j}(u) dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v+\alpha} \, dx \right] + \frac{3}{s} \int_{\Omega} |\nabla[a_m(u) + 1/j]^{s/2}|^2 \, dx \\
&\quad + \int_{\Omega} \frac{|\Delta v|^2}{v+\alpha} \, dx + \frac{1}{2} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{(v+\alpha)} \, dx \\
&\leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v+\alpha)^2} \Delta v \, dx + (1/j)^s \int_{\Omega} |\nabla v|^2 \, dx \\
&\quad + 2 \int_{\Omega} \sqrt{\alpha} |\nabla[a_m(u) + 1/j]^{s/2}| |(a_m(u) + 1/j)^{s/2}| \frac{|\nabla v|}{\sqrt{v+\alpha}} \, dx \\
&\quad + s \int_{\Omega} \left[(a_m(u) + 1/j)^{s-1} - a_m(u)^{s-1} \right] \nabla a_m(u) \cdot \nabla v \, dx.
\end{aligned}$$

Next, we deal with the third term in the right hand side of the previous inequality using Holder's and Young's inequality,

$$\begin{aligned}
&2 \int_{\Omega} \sqrt{\alpha} |\nabla[a_m(u) + 1/j]^{s/2}| |(a_m(u) + 1/j)^{s/2}| \frac{|\nabla v|}{\sqrt{v+\alpha}} \, dx \\
&\leq 2\sqrt{\alpha} \left[\int_{\Omega} (a_m(u) + 1/j)^s \frac{|\nabla v|^2}{v+\alpha} \, dx \right]^{1/2} \|\nabla[a_m(u) + 1/j]^{s/2}\|_{L^2(\Omega)} \\
&\leq \alpha s \int_{\Omega} (a_m(u) + 1/j)^s \frac{|\nabla v|^2}{v+\alpha} \, dx + \frac{1}{s} \|\nabla[a_m(u) + 1/j]^{s/2}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Therefore, we can choose $\alpha \leq 1/(4s)$ and finally obtain

$$\begin{aligned}
&\frac{d}{dt} \left[s \int_{\Omega} g_{m,j}(u) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v+\alpha} \, dx \right] + \frac{2}{s} \int_{\Omega} |\nabla[a_m(u) + 1/j]^{s/2}|^2 \, dx \\
&\quad + \int_{\Omega} \frac{|\Delta v|^2}{v+\alpha} \, dx + \frac{1}{2} \int_{\Omega} \left[a_m(u)^s - \frac{1}{2}(a_m(u) + 1/j)^s \right] \frac{|\nabla v|^2}{(v+\alpha)} \, dx \\
&\leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v+\alpha)^2} \Delta v \, dx + (1/j)^s \int_{\Omega} |\nabla v|^2 \, dx \\
&\quad + s \int_{\Omega} \left[a_m(u)^{s-1} - (a_m(u) + 1/j)^{s-1} \right] \nabla a_m(u) \cdot \nabla v \, dx.
\end{aligned}$$

In order to avoid problems with divisions by zero in the term $\frac{2}{s} \int_{\Omega} |\nabla[a_m(u) + 1/j]^{s/2}|^2 dx$ as we take the limit as $j \rightarrow \infty$, we use the fact that this term is non-negative and write

$$\begin{aligned} & \frac{d}{dt} \left[s \int_{\Omega} g_{m,j}(u) dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} dx \right] \\ & + \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} \left[a_m(u)^s - \frac{1}{2} (a_m(u) + 1/j)^s \right] \frac{|\nabla v|^2}{(v + \alpha)} dx \\ & \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v dx + (1/j)^s \int_{\Omega} |\nabla v|^2 dx \\ & + s \int_{\Omega} \left[a_m(u)^{s-1} - (a_m(u) + 1/j)^{s-1} \right] \nabla a_m(u) \cdot \nabla v dx, \end{aligned} \quad (3.16)$$

STEP 4: In the last step, we pass to the limit as $j \rightarrow \infty$.

Now we show that, passing to the limit as $j \rightarrow \infty$, we recover the cancellation of the chemotaxis and consumption terms. To deal with the passage to the limit as $j \rightarrow \infty$, we remind that $m \in \mathbb{N}$ is fixed and that the functions u and v have the regularity

$$(u - u^*), v \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)) \quad \text{and} \quad u_t, v_t \in L^2(0, \infty; L^2(\Omega)).$$

This means that there is a zero measure set $\mathcal{N} \subset (0, \infty)$ such that if we choose a fixed $t^* \in (0, \infty) \setminus \mathcal{N}$ then we have

$$u_t(t^*, \cdot), u(t^*, \cdot), v(t^*, \cdot), \nabla u(t^*, \cdot), \nabla v(t^*, \cdot), \Delta v(t^*, \cdot) \in L^2(\Omega)$$

and

$$a_m(u(t^*, \cdot)) \in L^\infty(\Omega).$$

Therefore each integral of the inequality (3.16) is well defined and (3.16) is satisfied for each $t^* \in (0, \infty) - \mathcal{N}$.

We want to take to the limit as $j \rightarrow \infty$ in (3.16). We are going to do it term by term. Let $t^* \in (0, \infty) - \mathcal{N}$ and let us first consider the term

$$\frac{d}{dt} \int_{\Omega} g_{m,j}(u(t^*, x)) dx = \int_{\Omega} u_t(t^*, x) \frac{(a_m(u(t^*, x)) + 1/j)^{s-1}}{s-1} dx. \quad (3.17)$$

Since $t^* \in (0, \infty) - \mathcal{N}$, we have

$$u_t(t^*, \cdot), u(t^*, \cdot) \in L^2(\Omega),$$

$$a_m(u(t^*, \cdot)) \in L^\infty(\Omega)$$

and then the integral (3.17) is well defined.

Next, we define the functions $f, F, f_j \in L^1(\Omega)$, for all $j \in \mathbb{N}$, by

$$f_j(x) = u_t(t^*, x) \frac{(a_m(u(t^*, x)) + 1/j)^{s-1}}{s-1}, \quad f(x) = u_t(t^*, x) \frac{a_m(u(t^*, x))^{s-1}}{s-1}$$

$$\text{and} \quad F(x) = |f_1(x)| = |u_t(t^*, x)| \frac{(a_m(u(t^*, x)) + 1)^{s-1}}{s-1}.$$

Then, for almost every $x \in \Omega$, $f_j(x) \rightarrow f(x)$ and $|f_j(x)| \leq F(x)$. Therefore, using the Dominated Convergence Theorem we can conclude that

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{d}{dt} \int_{\Omega} g_{m,j}(u(t^*, x)) \, dx &= \lim_{j \rightarrow \infty} \int_{\Omega} u_t(t^*, x) \frac{(a_m(u(t^*, x)) + 1/j)^{s-1}}{s-1} \, dx \\ &= \int_{\Omega} u_t(t^*, x) \frac{a_m(u(t^*, x))^{s-1}}{s-1} \, dx = \frac{d}{dt} \int_{\Omega} g_m(u(t^*, x)) \, dx. \end{aligned}$$

We can follow this reasoning and take the limit as $j \rightarrow \infty$ in the other terms of the (3.16), still considering the fixed time $t^* \in (0, \infty) - \mathcal{N}$. Using the Dominated Convergence Theorem again we conclude that

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{1}{2} \int_{\Omega} \left[a_m(u(t^*, x))^s - \frac{1}{2} (a_m(u(t^*, x)) + 1/j)^s \right] \frac{|\nabla v(t^*, x)|^2}{(v(t^*, x) + \alpha)} \, dx \\ = \frac{1}{4} \int_{\Omega} a_m(u(t^*, x))^s \frac{|\nabla v(t^*, x)|^2}{(v(t^*, x) + \alpha)} \, dx, \\ \lim_{j \rightarrow \infty} (1/j)^s \int_{\Omega} |\nabla v(t^*, x)|^2 \, dx = 0 \end{aligned}$$

and

$$\lim_{j \rightarrow \infty} s \int_{\Omega} \left[a_m(u(t^*, x))^{s-1} - (a_m(u(t^*, x)) + 1/j)^{s-1} \right] \nabla a_m(u(t^*, x)) \cdot \nabla v(t^*, x) \, dx = 0.$$

Then, since the limit preserves inequalities, after we take the limit as $j \rightarrow \infty$ in (3.16), we obtain

$$\begin{aligned} \frac{d}{dt} \left[s \int_{\Omega} g_m(u(t^*, x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v(t^*, x)|^2}{v(t^*, x) + \alpha} \, dx \right] + \int_{\Omega} \frac{|\Delta v(t^*, x)|^2}{v(t^*, x) + \alpha} \, dx \\ + \frac{1}{4} \int_{\Omega} a_m(u(t^*, x))^s \frac{|\nabla v(t^*, x)|^2}{(v(t^*, x) + \alpha)} \, dx \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v(t^*, x)|^2}{(v(t^*, x) + \alpha)^2} \Delta v(t^*, x) \, dx \end{aligned}$$

Since it is valid for any $t^* \in (0, \infty) - \mathcal{N}$, the inequality is valid for almost every $t \in (0, \infty)$. Therefore (3.14) holds. $\blacksquare$

Lemma 46 (Intermediate inequality for $s \geq 2$). *The solution (u, v) of the problem (P_m) satisfies, for $\alpha \in (0, 1/4s)$,*

$$\begin{aligned} & \frac{d}{dt} \left[s \int_{\Omega} g_m(u) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} \, dx \right] \\ & + \frac{3}{s} \int_{\Omega} |\nabla a_m(u)^{s/2}|^2 \, dx + \frac{1}{4} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{(v + \alpha)} \, dx \\ & \leq - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v \, dx, \end{aligned} \tag{3.18}$$

where $g_m(r)$ is a non-negative primitive of

$$g'_m(r) = \frac{a_m(r)^{s-1}}{(s-1)}.$$

Proof. Similarly to what happens in the cases $s = 1$ and $s \in (1, 2)$, for $s \geq 2$, instead of using $g'(u)$ as a test function in the u -equation of (P_m) , we should use

$$g'_m(u) = \frac{(a_m(u))^{s-1}}{(s-1)},$$

and, again instead of using $-\Delta \ln(v) = -\nabla \cdot \left(\frac{\nabla v}{v} \right)$ as a test function in the v -equation of (P_m) , we must use

$$-\Delta \ln(v + \alpha) = -\nabla \cdot \left(\frac{\nabla v}{v + \alpha} \right) = -\frac{1}{v + \alpha} \Delta v + \frac{1}{(v + \alpha)^2} |\nabla v|^2,$$

where $\alpha > 0$ is a real number to be chosen later. We begin by using $\varphi = g'_m(u) = (a_m(u))^{s-1}/(s-1)$ in the u -equation of (P_m) to obtain

$$\frac{d}{dt} \int_{\Omega} g_m(u) \, dx + \int_{\Omega} (a_m(u))^{s-2} a'_m(u) |\nabla u|^2 \, dx = (a_m(u) (a_m(u))^{s-2} \nabla v, \nabla a_m(u)),$$

where

$$g_m(r) = \int_0^r \frac{(a_m(\theta))^{s-1}}{(s-1)} \, d\theta$$

is a primitive of $(a_m(r))^{s-1}/(s-1)$. Since $0 \leq a'_m(u) \leq 1$, by Remark 1, we have $(a'_m(u))^2 \leq a'_m(u)$, we can write

$$\begin{aligned} \int_{\Omega} a'_m(u) (a_m(u))^{s-2} |\nabla u|^2 \, dx &\geq \int_{\Omega} (a'_m(u))^2 (a_m(u))^{s-2} |\nabla u|^2 \, dx \\ &= \int_{\Omega} (a_m(u))^{s-2} |\nabla a_m(u)|^2 \, dx \\ &= \int_{\Omega} |(a_m(u))^{s/2-1} \nabla a_m(u)|^2 \, dx \\ &= \frac{4}{s^2} \int_{\Omega} |\nabla (a_m(u))^{s/2}|^2 \, dx. \end{aligned}$$

Then we obtain

$$\frac{d}{dt} \int_{\Omega} g_m(u) \, dx + \frac{4}{s^2} \int_{\Omega} |\nabla [a_m(u)]^{s/2}|^2 \, dx \leq (a_m(u)^{s-1} \nabla v, \nabla a_m(u)). \quad (3.19)$$

In order to deal with the v -equation of (P_m) , first we test the v -equation of (P_m) by $-\Delta(\ln(v + \alpha))$. This gives us

$$-\int_{\Omega} v_t \Delta(\ln(v + \alpha)) \, dx = -\int_{\Omega} \Delta v \Delta(\ln(v + \alpha)) \, dx + \int_{\Omega} a_m(u) v \Delta(\ln(v + \alpha)) \, dx.$$

Then we use a rigorous version of (3.8),

$$-\int_{\Omega} v_t \Delta(\ln(v + \alpha)) \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} v_t \, dx,$$

in order to write

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} \, dx &= -\frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} v_t \, dx \\ &\quad - \int_{\Omega} \Delta v \Delta(\ln(v + \alpha)) \, dx + \int_{\Omega} a_m(u) v \Delta(\ln(v + \alpha)) \, dx. \end{aligned}$$

Now, for the first term in the right hand side of the above identity we recall that $(v)_t = \Delta v - a_m(u)^s v$; for the second term in the right hand side we recall that

$$-\Delta \ln(v + \alpha) = -\nabla \cdot \left(\frac{\nabla v}{v + \alpha} \right) = -\frac{1}{v + \alpha} \Delta v + \frac{1}{(v + \alpha)^2} |\nabla v|^2;$$

and for the last term of the right hand side we integrate by parts. We can follow great part of the calculations that are made for the v -equation in Lemma 44 and in Step 2 of Lemma 45 and obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} \, dx &\leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v \, dx - \frac{1}{2} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{(v + \alpha)} \, dx \\ &\quad - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} \, dx - s \int_{\Omega} a_m(u)^{s-1} \nabla a_m(u) \cdot \nabla v \, dx \\ &\quad + 2 \int_{\Omega} \sqrt{\alpha} |\nabla[a_m(u)]^{s/2}| |a_m(u)^{s/2}| \frac{|\nabla v|}{\sqrt{v + \alpha}} \, dx, \end{aligned} \tag{3.20}$$

If we add the above inequality to s times the inequality (3.19), then the term

$$s \int_{\Omega} a_m(u)^{s-1} \nabla a_m(u) \cdot \nabla v \, dx,$$

which appears in s times (3.19) cancels with the term

$$-s \int_{\Omega} a_m(u)^{s-1} \nabla a_m(u) \cdot \nabla v \, dx,$$

which comes from (3.20) and we obtain

$$\begin{aligned} \frac{d}{dt} \left[s \int_{\Omega} g_m(u) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v + \alpha} \, dx \right] &+ \frac{4}{s} \int_{\Omega} |\nabla[a_m(u)]^{s/2}|^2 \, dx \\ &+ \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} \, dx + \frac{1}{2} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{(v + \alpha)} \, dx \\ &\leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v + \alpha)^2} \Delta v \, dx \\ &\quad + 2 \int_{\Omega} \sqrt{\alpha} |\nabla[a_m(u)]^{s/2}| |a_m(u)^{s/2}| \frac{|\nabla v|}{\sqrt{v + \alpha}} \, dx. \end{aligned}$$

Next, we deal with the second term in the right hand side of the previous inequality using Holder's and Young's inequality,

$$\begin{aligned}
& 2 \int_{\Omega} \sqrt{\alpha} |\nabla[a_m(u)]^{s/2}| |a_m(u)^{s/2}| \frac{|\nabla v|}{\sqrt{v+\alpha}} dx \\
& \leq 2\sqrt{\alpha} \left[\int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{v+\alpha} dx \right]^{1/2} \|\nabla[a_m(u)]^{s/2}\|_{L^2(\Omega)} \\
& \leq \alpha s \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{v+\alpha} dx + \frac{1}{s} \|\nabla[a_m(u)]^{s/2}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Therefore, we can choose $\alpha \leq 1/(4s)$ and finally obtain

$$\begin{aligned}
\frac{d}{dt} \left[s \int_{\Omega} g_m(u) dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v+\alpha} dx \right] + \frac{3}{s} \int_{\Omega} |\nabla[a_m(u)]^{s/2}|^2 dx \\
+ \int_{\Omega} \frac{|\Delta v|^2}{v+\alpha} dx + \frac{1}{4} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{(v+\alpha)} dx \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v+\alpha)^2} \Delta v dx
\end{aligned}$$

where

$$g_m(r) = \int_0^r \frac{(a_m(\theta))^{s-1}}{(s-1)} d\theta$$

is a primitive of $(a_m(r))^{s-1}/(s-1)$. Therefore (3.18) holds. ■

3.1.3 The treatment of the term $-\int_{\Omega} \frac{|\Delta v|^2}{v+\alpha} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v+\alpha)^2} \Delta v dx$ of the intermediate inequality

In the last subsection we separated the analysis according to the values of s , obtaining (3.12) if $s = 1$, (3.14) if $s \in (1, 2)$ and (3.18) if $s \geq 2$. In any case, the term

$$-\int_{\Omega} \frac{|\Delta v|^2}{v+\alpha} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{(v+\alpha)^2} \Delta v dx \quad (3.21)$$

appears in the right hand side of the inequality and its adequate estimation is not straightforward.

In this subsection we present and prove the necessary results and use them to adequately estimate (3.21). At the end of the subsection we also prove an energy inequality that will allow us to obtain m -independent estimates for the function v that are valid up to infinity time in the next subsection.

At this stage, we will need hypothesis (H1) (see page 11). First, we show that, as we mentioned in the introduction, hypothesis (H1) is valid if the Poisson-Neumann problem has the $W^{3,3}$ -regularity (see definition 21 in page 23). According to (GRISVARD, 1985), this is true if Γ is at least $C^{2,1}$, for example. The validity of Hypothesis (H1) in other domains of practical interest, such as polyhedra and polygons, is an open question.

Lemma 47. Suppose that the Poisson-Neumann problem has the $W^{3,p}$ -regularity, for any $p > 3$, and let $z \in H^2(\Omega)$ such that $\partial_\eta z|_\Gamma = 0$. Then there is a sequence $\{\rho_n\} \subset C^2(\overline{\Omega})$, with $\partial_\eta \rho_n|_\Gamma = 0$, which converges to z in $H^2(\Omega)$.

Proof. For any fixed $z \in H^2(\Omega)$ such that $\partial_\eta z|_\Gamma = 0$, define $f = -\Delta z + z$. Note that $f \in L^2(\Omega)$ and $z \in H^2(\Omega)$ is the solution of

$$\begin{cases} -\Delta z + z = f \\ \partial_\eta z|_\Gamma = 0. \end{cases} \quad (3.22)$$

Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of mollifiers of f , that is, $f_n \in C_c^\infty(\Omega)$ and $f_n \rightarrow f$ in $L^2(\Omega)$ as $n \rightarrow \infty$. Then, for each fixed $n \in \mathbb{N}$, consider the following problem: Find $\rho_n : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} -\Delta \rho_n + \rho_n = f_n \\ \partial_\eta \rho_n|_\Gamma = 0. \end{cases} \quad (3.23)$$

Considering the hypothesis that the Poisson-Neumann problem has the $W^{3,3}$ -regularity, we can conclude that, for each $n \in \mathbb{N}$, there is one, and only one, function $\rho_n \in W^{3,3}(\Omega) \subset C^2(\overline{\Omega})$ such that $\partial_\eta \rho_n|_\Gamma = 0$ which solves problem (3.23).

Since the functions z and ρ_n solve the problems (3.22) and (3.23), respectively, the functions $z - \rho_n$ solve the problem

$$\begin{cases} -\Delta(z - \rho_n) + (z - \rho_n) = (f - f_n) \\ \partial_\eta(z - \rho_n)|_\Gamma = 0. \end{cases} \quad (3.24)$$

Then, using $(z - \rho_n)$ and $-\Delta(z - \rho_n)$ as test functions in (3.24) we can conclude that

$$\|z - \rho_n\|_{L^2(\Omega)}, \|\nabla(z - \rho_n)\|_{L^2(\Omega)}, \|\Delta(z - \rho_n)\|_{L^2(\Omega)} \leq \|f - f_n\|_{L^2(\Omega)},$$

which implies that

$$\|z - \rho_n\|_{H^2(\Omega)} \leq C \|f - f_n\|_{L^2(\Omega)}.$$

Since $f_n \rightarrow f$ in $L^2(\Omega)$ as $n \rightarrow \infty$, the latter implies that $\rho_n \rightarrow z$ in $H^2(\Omega)$ as $n \rightarrow \infty$, finishing the proof. $\blacksquare$

Next, we present some technical results which are needed to obtain estimates for the solution v of (P_m) .

Lemma 48. Let $h \in W^{2,\infty}(\mathbb{R})$ and let $z : \Omega \rightarrow \mathbb{R}$ be a $C^2(\overline{\Omega})$ function such that $\partial_\eta z|_\Gamma = 0$. Then

$$\int_\Omega h'(z) |\nabla z|^2 \Delta z \, dx = -\frac{1}{3} \int_\Omega h''(z) |\nabla z|^4 \, dx + \frac{2}{3} \int_\Omega h(z) |D^2 z|^2 \, dx +$$

$$-\frac{2}{3} \int_{\Omega} h(z) |\Delta z|^2 dx - \frac{1}{3} \int_{\Gamma} h(z) \nabla |\nabla z|^2 \cdot \eta d\Gamma.$$

Proof. See the appendix of (PASSO et al., 1998) for a proof based on successive integration by parts. ■

Lemma 49. Let $z \in C^2(\overline{\Omega})$ is a strictly positive function such that $\partial_{\eta} z|_{\Gamma} = 0$. Then there holds

$$-\int_{\Omega} \frac{|\Delta z|^2}{z} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla z|^2}{z^2} \Delta z dx = -\int_{\Omega} z |D^2 \ln(z)|^2 dx + \frac{1}{2} \int_{\Gamma} \frac{1}{z} \nabla |\nabla z|^2 \cdot \eta d\Gamma$$

Proof. Since z is strictly positive in $\overline{\Omega}$ we choose $h \in W^{2,\infty}(\mathbb{R})$ to be a function such that $h(z) = -1/z$. Then we use Lemma 48 in order to write

$$\begin{aligned} -\int_{\Omega} \frac{|\Delta z|^2}{z} dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla z|^2}{z^2} \Delta z dx &= -\int_{\Omega} \frac{|D^2 z|^2}{z} dx - \int_{\Omega} \frac{|\nabla z|^2}{z^2} \Delta z dx \\ &\quad - \frac{1}{2} \int_{\Omega} \frac{|\nabla z|^4}{z^3} dx + \frac{1}{2} \int_{\Gamma} \frac{1}{z} \nabla |\nabla z|^2 \cdot \eta d\Gamma. \end{aligned}$$

Then we prove the desired identity noting that

$$-\int_{\Omega} \frac{|D^2 z|^2}{z} dx - \int_{\Omega} \frac{|\nabla z|^2}{z^2} \Delta z dx - \frac{1}{2} \int_{\Omega} \frac{|\nabla z|^4}{z^3} dx = -\int_{\Omega} z |D^2 \ln(z)|^2 dx.$$

This last equation can be found in the Lemma 3.3 of (WINKLER, 2012). ■

In the next Lemma we look at what kind of estimates we can obtain from the term $\int_{\Omega} z |D^2 \ln(z)|^2 dx$.

Lemma 50. Assume hypothesis (H1) holds and let $\alpha \in \mathbb{R}$ be a positive real number. Let $z \in H^2(\Omega)$ be a positive function such that $z \geq \alpha$ in Ω and $\partial_{\eta} z|_{\Gamma} = 0$. Then there is a constant $C > 0$, which is independent of $\alpha > 0$, such that

$$\int_{\Omega} \frac{|\nabla z|^4}{z^3} dx \leq (2 + \sqrt{3})^2 \int_{\Omega} z |D^2 \ln(z)|^2 dx, \quad (3.25)$$

and

$$\int_{\Omega} \frac{|\Delta z|^2}{z} dx \leq C \int_{\Omega} z |D^2 \ln(z)|^2 dx. \quad (3.26)$$

Proof. The result above is valid if $z \in C^2(\overline{\Omega})$ according to Lemma 3.3 of (WINKLER, 2012). In order to extend it to $z \in H^2(\Omega)$ we use hypothesis (H1). In this case, there is a sequence $\{z_j\}$ such that $z_j \in C^2(\overline{\Omega})$, $\partial_{\eta} z_j|_{\Gamma} = 0$ and $z_j \rightarrow z$ in $H^2(\Omega)$. The functions z_j are not necessarily positive, but, since, in $3\mathbb{D}$, convergence in $H^2(\Omega)$ implies convergence in $L^\infty(\Omega)$ and $z \geq \alpha > 0$ in Ω , this implies that we can choose $j_0 \in \mathbb{N}$ large enough so that $z_j \geq \alpha/2$ in Ω for $j \geq j_0$. Then we can use Lemma 3.3 of (WINKLER, 2012) to obtain

$$\int_{\Omega} \frac{|\nabla z_j|^4}{z_j^3} dx \leq (2 + \sqrt{3})^2 \int_{\Omega} z_j |D^2 \ln(z_j)|^2 dx,$$

for $j \geq j_0$. Now we pass to the limit as $j \rightarrow \infty$ and this finishes the proof of (3.25).

In order to prove (3.26) we note that $\frac{\Delta z}{\sqrt{z}} = \sqrt{z} \Delta \ln(z) + \frac{|\nabla z|^2}{z^{3/2}}$, from which we have

$$\int_{\Omega} \frac{|\Delta z|^2}{z} dx \leq 2 \int_{\Omega} z |\Delta \ln(z)|^2 dx + 2 \int_{\Omega} \frac{|\nabla z|^4}{z^3} dx.$$

Then we can use $|\Delta \ln(z)|^2 \leq 3 |D^2 \ln(z)|^2$ and (3.25) in order to conclude (3.26). ■

The next two results will allow us to estimate the boundary integral.

Lemma 51. *Let $h \in W^{2,\infty}(\mathbb{R})$ and let $z : \Omega \rightarrow \mathbb{R}$ be a $C^2(\overline{\Omega})$ function and such that $\partial_{\eta} z|_{\Gamma} = 0$ and let $\Gamma = \bigcup_{i=1}^m \Gamma_i$, each Γ_i defined through a parametrization of one variable of $\mathbb{R}^3$ by the other two. Then there is $C > 0$ such that, for all i , one has*

$$\left| \int_{\Gamma_i} h(z) \nabla |\nabla z|^2 \cdot \eta d\Gamma_i \right| \leq C \int_{\Gamma_i} |h(z)| |\nabla z|^2 d\Gamma_i.$$

Proof. See (GUILLÉN-GONZÁLEZ; RODRÍGUEZ-BELLIDO, 2015). ■

Lemma 52. *Let Ω be a Lipschitz domain and let $z \in H^2(\Omega)$. Then, for each $\delta > 0$, there is a $C(\delta) > 0$*

$$\|\nabla z\|_{L^2(\Gamma)} \leq C(\delta) \|\nabla z\|_{L^2(\Omega)} + \delta \|\Delta z\|_{L^2(\Omega)}.$$

Proof. Applying Lemma 15 to the zero average function $z - \frac{1}{|\Omega|} \int_{\Omega} z dx$ we can conclude that for each $\delta_1 > 0$ that we choose there is a $C(\delta_1) > 0$ such that

$$\|\nabla z\|_{L^2(\Gamma)} \leq C(\delta_1) \|\nabla z\|_{L^2(\Omega)} + \delta_1 \|D^2 z\|_{L^2(\Omega)}.$$

Since

$$z - \frac{1}{|\Omega|} \int_{\Omega} z dx \in H^2(\Omega),$$

with zero mean and $\partial_{\eta} z|_{\Gamma} = 0$, it follows that $z - \frac{1}{|\Omega|} \int_{\Omega} z dx$ is the unique solution of the Poisson-Neumann problem (1.2) with $f = \Delta z \in L^2(\Omega)$.

Then, from the H^2 -regularity of (1.2) we conclude that there is $C > 0$ such that

$$\|\nabla z\|_{L^2(\Gamma)} \leq C(\delta_1) \|\nabla z\|_{L^2(\Omega)} + C\delta_1 \|\Delta z\|_{L^2(\Omega)}.$$

Since it is valid for any $\delta_1 > 0$, we can write $\delta := C\delta_1$ and obtain the desired result. ■

Remark 53. For the constant $C(\delta)$ of Lemma 52 we possibly have

$$\lim_{\delta \rightarrow 0} C(\delta) = +\infty,$$

because the constant $C(\delta_1)$ obtained from Lemma 15 may also explode as $\delta_1 \rightarrow \infty$. ■

Now we are in position to obtain better estimates for the solution v .

Lemma 54. *Let v be the solution of the v -equation of (P_m) . Then there exist constants $C, C(\alpha) > 0$, such that, for a.e. $t \in [0, \infty)$, it holds*

$$\begin{aligned} & - \int_{\Omega} \frac{|\Delta v(t)|^2}{v(t) + \alpha} dx + \frac{1}{2} \int_{\Omega} \frac{1}{(v(t) + \alpha)^2} |\nabla v(t)|^2 \Delta v(t) dx \\ & \leq -C \left(\int_{\Omega} \frac{|\nabla v(t)|^4}{(v(t) + \alpha)^3} dx + \int_{\Omega} \frac{|\Delta v(t)|^2}{v(t) + \alpha} dx \right) + C(\alpha) \|\nabla v(t)\|_{L^2(\Omega)}^2. \end{aligned}$$

Proof. We will apply some of the previous lemmas to the function $z = z(t) = v(t) + \alpha$. Note that, since $0 \leq v(t) \leq \|v^0\|_{L^\infty(\Omega)}$, we have $\alpha \leq z(t) \leq \|v^0\|_{L^\infty(\Omega)} + \alpha$. Now we divide the proof in three main steps.

STEP 1:

First of all, we are going to obtain the inequality

$$\begin{aligned} & - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} \frac{1}{(v + \alpha)^2} |\nabla v|^2 \Delta v dx \leq - \int_{\Omega} (v + \alpha) |D^2 \ln(v + \alpha)|^2 dx \\ & \quad + \frac{\delta}{\alpha} \|\Delta v\|_{L^2(\Omega)}^2 + C(\delta) \|\nabla v\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.27)$$

In fact, from hypothesis (H1), we have the existence of a sequence $\{z_j\}$ such that $z_j \in C^2(\bar{\Omega})$, the trace of the normal derivative of z_j is zero and $\|z(t) - z_j\|_{H^2(\Omega)} \rightarrow 0$. Analogously to Lemma 50, we can choose the sequence $\{z_j\}$ to be such that

$$\frac{\alpha}{2} \leq z_j(t, x) \leq 2(\|v^0\|_{L^\infty(\Omega)} + \alpha) \text{ in } \Omega, \quad \forall j \geq j_0. \quad (3.28)$$

From Lemma 49, we have

$$- \int_{\Omega} \frac{|\Delta z_j|^2}{z_j} dx + \frac{1}{2} \int_{\Omega} \frac{1}{z_j^2} |\nabla z_j|^2 \Delta z_j dx = - \int_{\Omega} z_j |D^2 \ln(z_j)|^2 dx + \frac{1}{2} \int_{\Gamma} \frac{1}{z_j} \nabla |\nabla z_j|^2 \cdot n d\Gamma.$$

Now we apply Lemma 51 and Lemma 52 (in this order) to obtain

$$\begin{aligned} & - \int_{\Omega} \frac{|\Delta z_j|^2}{z_j} dx + \frac{1}{2} \int_{\Omega} \frac{1}{z_j^2} |\nabla z_j|^2 \Delta z_j dx \leq - \int_{\Omega} z_j |D^2 \ln(z_j)|^2 dx + C \int_{\Gamma} \frac{1}{z_j} |\nabla z_j|^2 d\Gamma \\ & \leq - \int_{\Omega} z_j |D^2 \ln(z_j)|^2 dx + \frac{C}{\alpha} \|\nabla z_j\|_{L^2(\Gamma)}^2 \\ & \leq - \int_{\Omega} z_j |D^2 \ln(z_j)|^2 dx + \frac{\delta}{\alpha} \|\Delta z_j\|_{L^2(\Omega)}^2 \\ & \quad + \frac{C(\delta)}{\alpha} \|\nabla z_j\|_{L^2(\Omega)}^2, \end{aligned}$$

where $\delta > 0$ is supposed to be a sufficiently small number to be chosen later.

Now, recalling that $z = v + \alpha$, we use the fact that $z_j \rightarrow z$ in $H^2(\Omega)$ in order to pass to the limit and obtain the inequality (3.27).

STEP 2:

Next, we apply Lemma 50, with $z = v + \alpha$, to (3.27) in order to obtain

$$\begin{aligned} - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} \frac{1}{(v + \alpha)^2} |\nabla v|^2 \Delta v dx &\leq -\tilde{C} \left(\int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \int_{\Omega} \frac{|\nabla v|^4}{(v + \alpha)^3} dx \right) \\ &\quad + \frac{\delta}{\alpha} \|\Delta v\|_{L^2(\Omega)}^2 + C(\delta) \|\nabla v\|_{L^2(\Omega)}^2 \end{aligned}$$

and, since

$$\|\Delta v\|_{L^2(\Omega)}^2 = \int_{\Omega} (v + \alpha) \frac{|\Delta v|^2}{v + \alpha} dx \leq (\|v^0\|_{L^\infty(\Omega)} + \alpha) \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx,$$

we obtain

$$\begin{aligned} - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} \frac{1}{(v + \alpha)^2} |\nabla v|^2 \Delta v dx &\leq -\tilde{C} \left(\int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \int_{\Omega} \frac{|\nabla v|^4}{(v + \alpha)^3} dx \right) \\ &\quad + \frac{\delta}{\alpha} (\|v^0\|_{L^\infty(\Omega)} + \alpha) \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx \\ &\quad + C(\delta) \|\nabla v\|_{L^2(\Omega)}^2. \end{aligned}$$

STEP 3:

Then, choosing $\delta = \delta(\alpha) > 0$ small enough such that

$$-C = -\tilde{C} + \frac{2\delta(\alpha)}{\alpha} (\|v^0\|_{L^\infty(\Omega)} + \alpha) < 0 \quad (3.29)$$

and denoting $C(\delta(\alpha))$ by $C(\alpha)$, we can write

$$\begin{aligned} - \int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \frac{1}{2} \int_{\Omega} \frac{1}{(v + \alpha)^2} |\nabla v|^2 \Delta v dx &\leq -C \left(\int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \int_{\Omega} \frac{|\nabla v|^4}{(v + \alpha)^3} dx \right) \\ &\quad + C(\alpha) \|\nabla v\|_{L^2(\Omega)}^2, \end{aligned}$$

finishing the proof. ■

Remark 55. In the inequality proved in Lemma 54, both constants, C and $C(\alpha)$, depend on α . But, because of the application of Lemma 52, we possibly have

$$\lim_{\alpha \rightarrow 0} C(\alpha) = +\infty.$$

■

Using Lemma 54 in the inequalities (3.12), (3.14) and (3.18) we obtain the following energy inequalities.

Lemma 56 (Energy inequalities). 1. If $s = 1$ the solution (u, v) of (P_m) satisfies, for a.e. $t \in (0, \infty)$,

$$\begin{aligned} & \frac{d}{dt} \left[\int_{\Omega} \left[g_m(u) + \frac{1}{2} \frac{|\nabla v|^2}{v + \alpha} \right] dx \right] + 2 \int_{\Omega} |\nabla [a_m(u + 1)]^{1/2}| dx \\ & + \frac{1}{2} \int_{\Omega} a_m(u) \frac{|\nabla v|^2}{v + \alpha} dx + C \left(\int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \int_{\Omega} \frac{|\nabla v|^4}{(v + \alpha)^3} dx \right) \\ & \leq [C(\alpha) + 2] \|\nabla v\|_{L^2(\Omega)}^2, \end{aligned} \quad (3.30)$$

where g_m is a non-negative primitive of g'_m defined in Lemma 44;

2. If $s \in (1, 2)$ the solution (u, v) of (P_m) satisfies, for a.e. $t \in (0, \infty)$,

$$\begin{aligned} & \frac{d}{dt} \left[s \int_{\Omega} \left[g_m(u) + \frac{1}{2} \frac{|\nabla v|^2}{v + \alpha} \right] dx \right] + \frac{1}{2} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{v + \alpha} dx \\ & + C \left(\int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \int_{\Omega} \frac{|\nabla v|^4}{(v + \alpha)^3} dx \right) \leq C(\alpha) \|\nabla v\|_{L^2(\Omega)}^2, \end{aligned} \quad (3.31)$$

where g_m is a non-negative primitive of g'_m defined in Lemma 45;

3. If $s \geq 2$ the solution (u, v) of (P_m) satisfies, for a.e. $t \in (0, \infty)$,

$$\begin{aligned} & \frac{d}{dt} \left[s \int_{\Omega} \left[g_m(u) + \frac{1}{2} \frac{|\nabla v|^2}{v + \alpha} \right] dx \right] + \frac{3}{s} \int_{\Omega} |\nabla [a_m(u)]^{s/2}| dx \\ & + \frac{1}{2} \int_{\Omega} a_m(u)^s \frac{|\nabla v|^2}{v + \alpha} dx + C \left(\int_{\Omega} \frac{|\Delta v|^2}{v + \alpha} dx + \int_{\Omega} \frac{|\nabla v|^4}{(v + \alpha)^3} dx \right) \\ & \leq C(\alpha) \|\nabla v\|_{L^2(\Omega)}^2, \end{aligned} \quad (3.32)$$

where g_m is a non-negative primitive of g'_m defined in Lemma 46;

Proof. If we apply Lemma 54 to the terms on the right hand side of (3.12), (3.14) and (3.18), respectively, we obtain the desired result. ■

To finish the subsection, we give three inequalities which stem from (3.30), (3.31) and (3.32), for the cases $s = 1$, $s \in (1, 2)$ and $s \geq 2$, respectively. We integrate each one of these inequalities with respect to t in order to obtain m -independent estimates.

Case $s = 1$:

Beginning with the case $s = 1$, if we integrate (3.30), for each $T \in (0, \infty)$ we obtain that

$$\begin{aligned}
& \int_{\Omega} g_m(u(T, x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v(T, x)|^2}{v(T, x) + \alpha} \, dx + 2 \int_0^T \int_{\Omega} |\nabla[a_m(u(t, x)) + 1]^{1/2}| \, dx \, dt \\
& \quad + \frac{1}{2} \int_0^T \int_{\Omega} a_m(u(t, x)) \frac{|\nabla v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt \\
& \quad + C \left(\int_0^T \int_{\Omega} \frac{|\Delta v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt + \int_0^T \int_{\Omega} \frac{|\nabla v(t, x)|^4}{(v(t, x) + \alpha)^3} \, dx \, dt \right) \leq \\
& \leq [C(\alpha) + 2] \int_0^T \|\nabla v(t, x)\|_{L^2(\Omega)}^2 \, dt + \int_{\Omega} g_m(u^0(x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v^0(x)|^2}{v^0(x) + \alpha} \, dx.
\end{aligned} \tag{3.33}$$

But we remind that, because of Lemma 41, we have

$$\int_0^T \|\nabla v\|_{L^2(\Omega)}^2 \, dt \leq \|v^0\|_{L^2(\Omega)}^2, \quad \forall T > 0. \tag{3.34}$$

We also remind that, in this case, that is, when $s = 1$, $g_m(u^0)$ is defined in Lemma 44 as

$$\int_0^{u^0} \ln(a_m(\theta) + 1) \, d\theta.$$

And since $a_m(\theta) \leq \theta$, we have

$$g_m(u^0) = \int_0^{u^0} \ln(a_m(\theta) + 1) \, d\theta \leq \int_0^{u^0} \ln(\theta + 1) \, d\theta \leq \int_0^{u^0} (\theta + 1) \, d\theta = \frac{1}{2} |u^0|^2 + u^0.$$

Therefore $g_m(u^0)$ can be estimated independently of $m \in \mathbb{N}$ by

$$\int_{\Omega} g_m(u^0) \, dx \leq \|u^0\|_{L^2(\Omega)}^2 + |\Omega| u^*. \tag{3.35}$$

Then, applying (3.34) and (3.35) to (3.33) we conclude that, for the case $s = 1$, we have

$$\begin{aligned}
& \int_{\Omega} g_m(u(T, x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v(T, x)|^2}{v(T, x) + \alpha} \, dx + 2 \int_0^T \int_{\Omega} |\nabla[a_m(u(t, x)) + 1]^{1/2}| \, dx \, dt \\
& \quad + \frac{1}{2} \int_0^T \int_{\Omega} a_m(u(t, x)) \frac{|\nabla v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt \\
& \quad + C \left(\int_0^T \int_{\Omega} \frac{|\Delta v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt + \int_0^T \int_{\Omega} \frac{|\nabla v(t, x)|^4}{(v(t, x) + \alpha)^3} \, dx \, dt \right) \\
& \leq C(\alpha, u^0, v^0),
\end{aligned} \tag{3.36}$$

where the constant $C(\alpha, u^0, v^0)$ is given by

$$C(\alpha, u^0, v^0) = [C(\alpha) + 2] \|v^0\|_{L^2(\Omega)}^2 + \|u^0\|_{L^2(\Omega)}^2 + |\Omega| u^* + \frac{1}{2} \int_{\Omega} \frac{|\nabla v^0(x)|^2}{v^0(x) + \alpha} \, dx$$

Case $s > 1$:

Now, considering the case $s \in (1, 2)$, if we integrate (3.31) we find

$$\begin{aligned}
& s \int_{\Omega} g_m(u(T, x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v(T, x)|^2}{v(T, x) + \alpha} \, dx \\
& + \frac{1}{2} \int_0^T \int_{\Omega} a_m(u(t, x))^s \frac{|\nabla v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt \\
& + C \left(\int_0^T \int_{\Omega} \frac{|\Delta v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt + \int_0^T \int_{\Omega} \frac{|\nabla v(t, x)|^4}{(v(t, x) + \alpha)^3} \, dx \, dt \right) \\
& \leq C(\alpha) \int_0^T \|\nabla v(t, x)\|_{L^2(\Omega)}^2 \, dt + s \int_{\Omega} g_m(u^0(x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v^0(x)|^2}{v^0(x) + \alpha} \, dx \\
& \leq C(\alpha) \|v^0\|_{L^2(\Omega)}^2 + s \int_{\Omega} g_m(u^0(x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v^0(x)|^2}{v^0(x) + \alpha} \, dx.
\end{aligned} \tag{3.37}$$

And, finally, considering the case $s \geq 2$, if we integrate (3.32), we find

$$\begin{aligned}
& s \int_{\Omega} g_m(u(T, x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v(T, x)|^2}{v(T, x) + \alpha} \, dx + \frac{3}{s} \int_0^T \int_{\Omega} \nabla[a_m(u(t, x))]^{s/2} \, dx \, dt \\
& + \frac{1}{2} \int_0^T \int_{\Omega} a_m(u(t, x))^s \frac{|\nabla v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt \\
& + C \left(\int_0^T \int_{\Omega} \frac{|\Delta v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt + \int_0^T \int_{\Omega} \frac{|\nabla v(t, x)|^4}{(v(t, x) + \alpha)^3} \, dx \, dt \right) \\
& \leq C(\alpha) \int_0^T \|\nabla v(t, x)\|_{L^2(\Omega)}^2 \, dt + s \int_{\Omega} g_m(u^0(x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v^0(x)|^2}{v^0(x) + \alpha} \, dx \\
& \leq C(\alpha) \|v^0\|_{L^2(\Omega)}^2 + s \int_{\Omega} g_m(u^0(x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v^0(x)|^2}{v^0(x) + \alpha} \, dx.
\end{aligned} \tag{3.38}$$

Now we recall that, either for the case $s \in (1, 2)$, in Lemma 45, or for the case $s \geq 2$, in Lemma 46, the function $g_m(u^0)$ is defined as

$$g_m(u^0) = \int_0^{u^0} \frac{(a_m(\theta))^{s-1}}{s-1} \, d\theta.$$

Again, if we remind that $a_m(\theta) \leq \theta$ then we can write

$$g_m(u^0) = \int_0^{u^0} \frac{(a_m(\theta))^{s-1}}{s-1} \, d\theta \leq \int_0^{u^0} \frac{\theta^{s-1}}{s-1} \, d\theta = \frac{(u^0)^s}{s(s-1)}. \tag{3.39}$$

Then, considering (3.34) and (3.39), from (3.37) and (3.38), for the case $s \in (1, 2)$, we obtain

$$\begin{aligned}
& s \int_{\Omega} g_m(u(T, x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v(T)|^2}{v(T, x) + \alpha} \, dx + \frac{1}{2} \int_0^T \int_{\Omega} a_m(u(t, x))^s \frac{|\nabla v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt \\
& + C \left(\int_0^T \int_{\Omega} \frac{|\Delta v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt + \int_0^T \int_{\Omega} \frac{|\nabla v(t, x)|^4}{(v(t, x) + \alpha)^3} \, dx \, dt \right) \leq \\
& \leq C(\alpha, s, u^0, v^0)
\end{aligned} \tag{3.40}$$

and, for the case $s \geq 2$,

$$\begin{aligned}
& s \int_{\Omega} g_m(u(T, x)) \, dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v(T, x)|^2}{v(T, x) + \alpha} \, dx + \frac{3}{s} \int_0^T \int_{\Omega} \nabla[a_m(u(t, x))]^{s/2} \, dx \, dt \\
& + \frac{1}{2} \int_0^T \int_{\Omega} a_m(u(t, x))^s \frac{|\nabla v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt \\
& + C \left(\int_0^T \int_{\Omega} \frac{|\Delta v(t, x)|^2}{v(t, x) + \alpha} \, dx \, dt + \int_0^T \int_{\Omega} \frac{|\nabla v(t, x)|^4}{(v + \alpha)^3} \, dx \, dt \right) \leq \\
& \leq C(\alpha, s, u^0, v^0),
\end{aligned} \tag{3.41}$$

where the constant $C(\alpha, s, u^0, v^0)$ is defined by

$$C(\alpha, s, u^0, v^0) = \|v^0\|_{L^2(\Omega)}^2 + \frac{\|u^0\|_{L^s(\Omega)}^s}{s(s-1)} + \frac{1}{2} \int_{\Omega} \frac{|\nabla v^0|^2}{v^0 + \alpha} \, dx.$$

3.1.4 m -independent estimates and passage to the limit as $m \rightarrow \infty$

Now we use again the notation (u_m, v_m) for the solution of the regularized problem (P_m) and (u, v) for the solution of the original problem (P) . We will use (3.36), (3.40) and (3.41) to obtain m -independent estimates for (u_m, v_m) that will allow us to pass to the limit as $m \rightarrow \infty$ and prove the existence of solution to the original problem as a limit of solutions of the regularized problems. First we will obtain some m -independent estimates for v_m that can be extracted either from (3.40), (3.40) or (3.41). Next we will need to separate the analysis in two cases: $s \in [1, 2)$ and $s \geq 2$.

Either from (3.36), (3.40) or (3.41), for every $T \in (0, \infty)$, we have

$$\int_{\Omega} \frac{|\nabla v_m(T, x)|^2}{v_m(T, x) + \alpha} \, dx \leq C(\alpha, s, u^0, v^0). \tag{3.42}$$

By Lemma 38, we have

$$0 \leq v_m(t, x) + \alpha \leq \|v^0\|_{L^\infty(\Omega)} + \alpha, \text{ a.e. } (t, x) \in (0, \infty) \times \Omega, \tag{3.43}$$

and hence

$$|\nabla v_m(T, x)|^2 = (v_m(T, x) + \alpha) \frac{|\nabla v_m(T, x)|^2}{(v_m(T, x) + \alpha)} \leq \left(\|v^0\|_{L^\infty(\Omega)} + \alpha \right) \frac{|\nabla v_m(T, x)|^2}{(v_m(T, x) + \alpha)}$$

Then, using the m -independent estimate (3.42) we can conclude that

$$\begin{aligned}
\|\nabla v_m(T)\|_{L^2(\Omega)}^2 &= \int_{\Omega} |\nabla v_m(T, x)|^2 \, dx \\
&\leq \left(\|v^0\|_{L^\infty(\Omega)} + \alpha \right) \int_{\Omega} \frac{|\nabla v_m(T, x)|^2}{(v_m(T, x) + \alpha)} \, dx \\
&\leq \left(\|v^0\|_{L^\infty(\Omega)} + \alpha \right) C(\alpha, s, u^0, v^0),
\end{aligned}$$

that is,

$$\nabla v_m \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)). \quad (3.44)$$

Analogously, either from (3.36), (3.40) or (3.41) we can obtain

$$\begin{aligned} \int_0^T \int_\Omega a_m(u_m(t, x))^s \frac{|\nabla v_m(t, x)|^2}{v_m(t, x) + \alpha} dx dt &\leq [C(\alpha) + 2] \|v^0\|_{L^2(\Omega)}^2 \\ &+ \int_\Omega g_m(u^0) dx + \frac{1}{2} \int_\Omega \frac{|\nabla v^0|^2}{v^0 + \alpha} dx. \end{aligned} \quad (3.45)$$

Considering again (3.43) we can write

$$\begin{aligned} a_m(u_m(t, x))^s |\nabla v_m(t, x)|^2 &= (v_m(t, x) + \alpha) a_m(u_m(t, x))^s \frac{|\nabla v_m(t, x)|^2}{v_m(t, x) + \alpha} \\ &\leq \left(\|v^0\|_{L^\infty(\Omega)} + \alpha \right) a_m(u_m(t, x))^s \frac{|\nabla v_m(t, x)|^2}{v_m(t, x) + \alpha}. \end{aligned}$$

Then, using the m -independent estimate (3.45) we can conclude that

$$a_m(u_m)^{s/2} \nabla v_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \quad (3.46)$$

Continuing with this procedure, either from (3.36), (3.40) or (3.41) we can obtain

$$\nabla v_m \text{ is bounded in } L^4(0, \infty; L^4(\Omega)) \quad (3.47)$$

and

$$\Delta v_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \quad (3.48)$$

In particular, since $\frac{\partial}{\partial \eta} v_m \Big|_{\partial \Omega} = 0$, considering the auxiliary function

$$v_m(t, x) - \frac{1}{|\Omega|} \int_\Omega v_m(t, x) dx,$$

one can conclude that

$$\|\nabla v_m(t, \cdot)\|_{H^1(\Omega)} \leq C \|\Delta v_m(t, \cdot)\|_{L^2(\Omega)}.$$

Therefore, from (3.48) we can also conclude that

$$\nabla v_m \text{ is bounded in } L^2(0, \infty; H^1(\Omega)). \quad (3.49)$$

Using the obtained results until this point we analyze the existence of solution in 3D, first for $s \in [1, 2)$ and then for $s \geq 2$.

3.1.4.1 m -independent estimates and passage to the limit for $s \in [1, 2)$

Let

$$\forall r > 0, \quad g'(r) = \begin{cases} \ln(r) & \text{if } s = 1, \\ r^{s-1}/(s-1) & \text{if } s \in (1, 2) \end{cases}$$

and let

$$g(r) = \int_0^r g'(\theta) d\theta = \begin{cases} r \ln(r) - r & \text{if } s = 1, \\ r^s/s(s-1) & \text{if } s \in (1, 2). \end{cases}$$

Notice that $g''(r) = r^{s-2}$, $\forall r > 0$, in all cases.

We test the u_m -equation of (P_m) by $g'(u_m + 1)$ and obtain

$$\begin{aligned} \int_{\Omega} \partial_t u_m \, g'(u_m + 1) \, dx + \int_{\Omega} (u_m + 1)^{s-2} |\nabla[u_m + 1]|^2 \, dx \\ = \int_{\Omega} a_m(u_m) \nabla v_m (u_m + 1)^{s-2} \nabla u_m \, dx \end{aligned} \quad (3.50)$$

Now we recall that

$$(u_m + 1)^{s-2} |\nabla[u_m + 1]|^2 = |(u_m + 1)^{(s/2-1)} \nabla[u_m + 1]|^2 = \frac{4}{s^2} |\nabla[u_m + 1]^{s/2}|^2.$$

Then we can use it in (3.50) and write

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} g(u_m + 1) \, dx + \frac{4}{s^2} \int_{\Omega} |\nabla[u_m + 1]^{s/2}|^2 \, dx \\ = \int_{\Omega} a_m(u_m) \nabla v_m \cdot (u_m + 1)^{s-2} \nabla u_m \, dx \\ = \int_{\Omega} a_m(u_m) (u_m + 1)^{s/2-1} \nabla v_m (u_m + 1)^{s/2-1} \cdot \nabla u_m \, dx \\ = \frac{2}{s} \int_{\Omega} \frac{a_m(u_m)}{(u_m + 1)^{1-s/2}} \nabla v_m \cdot \nabla[u_m + 1]^{s/2} \, dx \\ = \frac{2}{s} \int_{\Omega} \frac{a_m(u_m)^{1-s/2}}{(u_m + 1)^{1-s/2}} a_m(u_m)^{s/2} \nabla v_m \cdot \nabla[u_m + 1]^{s/2} \, dx \\ \leq \frac{2}{s} \left(\int_{\Omega} a_m(u_m)^s |\nabla v_m|^2 \, dx \right)^{1/2} \left(\int_{\Omega} |\nabla[u_m + 1]^{s/2}|^2 \, dx \right)^{1/2} \end{aligned}$$

and thus we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} g(u_m + 1) \, dx + \frac{2}{s^2} \int_{\Omega} |\nabla[u_m + 1]^{s/2}|^2 \, dx \\ \leq \frac{1}{2} \int_{\Omega} a_m(u_m)^s |\nabla v_m|^2 \, dx. \end{aligned}$$

Integrating with respect to t from 0 to T , for any fixed $T \in (0, \infty)$, we obtain

$$\begin{aligned} \int_{\Omega} g(u_m(T) + 1) \, dx + \frac{2}{s^2} \int_0^T \int_{\Omega} |\nabla[u_m + 1]^{s/2}|^2 \, dx \, dt \\ \leq \frac{1}{2} \int_0^T \int_{\Omega} a_m(u_m)^s |\nabla v_m|^2 \, dx \, dt + \int_{\Omega} g(u^0 + 1) \, dx. \end{aligned}$$

Then, because of the definition of g and (3.46) we conclude that

$$[u_m + 1]^{s/2} \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)) \quad (3.51)$$

and

$$\nabla[u_m + 1]^{s/2} \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \quad (3.52)$$

Remark 57. When $s = 1$, (3.51) is equivalent to

$$u_m \text{ is bounded in } L^\infty(0, \infty; L^1(\Omega)).$$

The latter, and thus (3.51) for $s = 1$, stems from Lemma 41-1. For $s = 1$ we also have

$$(u_m + 1)\ln(u_m + 1) \text{ is bounded in } L^\infty(0, \infty; L^1(\Omega)),$$

because of the definition of g . ■

Since

$$\nabla u_m = \nabla(u_m + 1) = \nabla((u_m + 1)^{s/2})^{2/s} = \frac{2}{s}(u_m + 1)^{1-s/2} \nabla(u_m + 1)^{s/2}, \quad (3.53)$$

taking into account that we are considering $s \in [1, 2)$, we can use (3.51) to obtain

$$(u_m + 1)^{1-s/2} \text{ is bounded in } L^\infty(0, \infty; L^{2s/(2-s)}(\Omega))$$

and then (3.52) to conclude that

$$\nabla u_m \text{ is bounded in } L^2(0, \infty; L^s(\Omega)).$$

Hence, using the Poincaré inequality for zero mean functions (Lemma 14),

$$u_m - u^* \text{ is bounded in } L^\infty(0, \infty; L^s(\Omega)) \cap L^2(0, \infty; W^{1,s}(\Omega)). \quad (3.54)$$

Considering the chemotaxis term of the u_m -equation of (P_m) , we can write $a_m(u_m)\nabla v_m$ as

$$a_m(u_m)\nabla v_m = a_m(u_m)^{1-s/2} a_m(u_m)^{s/2} \nabla v_m.$$

Then, we have $a_m(u_m)^{1-s/2}$ bounded in $L^\infty(0, \infty; L^{2s/(2-s)}(\Omega))$, because of (3.54), and $a_m(u_m)^{s/2} \nabla v_m$ bounded in $L^2(0, \infty; L^2(\Omega))$, because of (3.48), and hence we can conclude that

$$a_m(u_m)\nabla v_m \text{ is bounded in } L^2(0, \infty; L^s(\Omega)). \quad (3.55)$$

Then, if we consider the u_m -equation of (P_m) , from (3.54) and (3.55) we conclude that

$$\partial_t u_m \text{ is bounded in } L^2\left(0, \infty; (W^{1,s/(s-1)}(\Omega))'\right). \quad (3.56)$$

Now we turn to the v_m -equation, rewritten as

$$\partial_t v_m - \Delta v_m + a_m(u^*)^s v_m = -(a_m(u_m)^s - a_m(u^*)^s) v_m. \quad (3.57)$$

Analyzing the term on the right hand side of (3.57), we have

$$a_m(u_m)^s - a_m(u^*)^s \text{ is bounded in } L^2(0, \infty; L^{3/2}(\Omega)). \quad (3.58)$$

In fact, using Lemma 16, we obtain

$$|a_m(u_m)^s - a_m(u^*)^s| \leq s |a_m(u_m) + a_m(u^*)|^{s-1} |u_m - u^*|.$$

Then, considering the m -uniform bound (3.54) and the Sobolev embedding $L^{3s/(3-s)}(\Omega) \subset W^{1,s}(\Omega)$ we obtain (3.58).

With this information, now we can test (3.57) by v_m , obtaining

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_m\|_{L^2(\Omega)}^2 + \|\nabla v_m\|_{L^2(\Omega)}^2 + \frac{a_m(u^*)^s}{2} \|v_m\|_{L^2(\Omega)}^2 &\leq C \int_{\Omega} |a_m(u_m)^s - a_m(u^*)^s| v_m^2 dx \\ &\leq C \|v_m^0\|_{L^\infty(\Omega)} \|a_m(u_m)^s - a_m(u^*)^s\|_{L^{3/2}(\Omega)} \|v_m\|_{L^3(\Omega)} \\ &\leq C(\delta) \|a_m(u_m)^s - a_m(u^*)^s\|_{L^{3/2}(\Omega)}^2 + \delta \|v_m\|_{L^2(\Omega)}^2 + \delta \|\nabla v_m\|_{L^2(\Omega)}^2. \end{aligned}$$

Note that $a_m(u^*) = \min\{u^*, m\}$. Hence if $u_0 \not\equiv 0$ then there is $k > 0$ such that $a_m(u^*) \geq k > 0$, for all $m \in \mathbb{N}$. Hence, choosing $\delta > 0$ small enough, we can conclude that there is $\beta > 0$ such that

$$\frac{1}{2} \frac{d}{dt} \|v_m\|_{L^2(\Omega)}^2 + \beta \|\nabla v_m\|_{L^2(\Omega)}^2 + \beta \|v_m\|_{L^2(\Omega)}^2 \leq C \|v_m^0\|_{L^\infty(\Omega)}^2 \|a_m(u_m)^s - a_m(u^*)^s\|_{L^{3/2}(\Omega)}^2.$$

Therefore, integrating with respect to t and using (3.58) we obtain

$$v_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \quad (3.59)$$

With the m -uniform bounds obtained so far we can obtain a m -uniform bound for $\partial_t v_m$ in $L^2(0, \infty; L^{3/2}(\Omega))$. In fact, going back to (3.57), reminding that v_m is uniformly bounded in $L^\infty(0, \infty; L^\infty(\Omega))$ with respect to m and considering (3.59) and (3.58), we conclude that

$$\partial_t v_m \text{ is bounded in } L^2(0, \infty; L^{3/2}(\Omega)). \quad (3.60)$$

Now we are going to obtain m -uniform bounds for $\{u_m\}$ which are necessary in order to pass to the limit as $m \rightarrow \infty$ in the terms of the equations of (P_m) . Because of (3.54) and (3.52), we have that

$$(u_m + 1)^{s/2} \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \text{ for every finite } T > 0.$$

Using the Sobolev inequality $H^1(\Omega) \subset L^6(\Omega)$ and interpolation inequalities we obtain

$$(u_m)^{s/2} \text{ is bounded in } L^{10/3}(0, T; L^{10/3}(\Omega)),$$

which is equivalent to

$$u_m \text{ is bounded in } L^{5s/3}(0, T; L^{5s/3}(\Omega)). \quad (3.61)$$

Because of (3.53) (remind that $s \in [1, 2)$), we also have

$$u_m \text{ is bounded in } L^{5s/(3+s)}(0, T; W^{1,5s/(3+s)}(\Omega)). \quad (3.62)$$

We observe that $W^{1,5s/(3+s)}(\Omega) \subset L^q(\Omega)$, with continuous embedding for $q \in [1, 15s/(9-2s)]$ and compact embedding for $q \in [1, 15s/(9-2s))$. Then, since $s \in [1, 2)$, we have $\frac{5s}{3} < \frac{15s}{9-2s}$ and therefore the embedding $W^{1,5s/(3+s)}(\Omega) \subset L^{5s/3}(\Omega)$ is compact. Note also that $q = \frac{5s}{3} \geq \frac{5}{3} > 1$.

Now we can use Lemma 19 with

$$X = W^{1,5s/(3+s)}(\Omega), \quad B = L^{5s/3}(\Omega), \quad Y = (H^3(\Omega))'$$

and $q = 5s/3$, to conclude that there is a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) and a limit function u such that, for each $T > 0$,

$$u_m \longrightarrow u \text{ weakly in } L^{5s/(3+s)}(0, T; W^{1,5s/(3+s)}(\Omega)),$$

and

$$u_m \longrightarrow u \text{ strongly in } L^p(0, T; L^p(\Omega)), \quad \forall p \in [1, 5s/3), \quad (3.63)$$

Using the Dominated Convergence Theorem we can conclude from (3.63) that

$$a_m(u_m) \rightarrow u \text{ strongly in } L^p(0, T; L^p(\Omega)), \quad \forall p \in (1, 5s/3). \quad (3.64)$$

It stems from the convergence (3.64) and Lemma 17 that

$$(a_m(u_m))^s \rightarrow u^s \text{ strongly in } L^p(0, T; L^p(\Omega)), \quad \forall p \in (1, 5/3). \quad (3.65)$$

The convergence of v_m is better. For each fixed $T > 0$, there is a subsequence of $\{v_m\}$ (still denoted by $\{v_m\}$) and a limit function v such that

$$\begin{aligned} v_m &\rightarrow v \text{ weakly}^* \text{ in } L^\infty(0, \infty; L^\infty(\Omega)), \\ v_m &\rightarrow v \text{ weakly in } L^2(0, \infty; H^2(\Omega)), \\ \nabla v_m &\rightarrow \nabla v \text{ weakly in } L^4(0, \infty; L^4(\Omega)), \\ v_m &\rightarrow v \text{ strongly in } L^2(0, T; H^1(\Omega)), \\ \text{and } \partial_t v_m &\rightarrow \partial_t v \text{ weakly in } L^2(0, \infty; L^{3/2}(\Omega)). \end{aligned} \quad (3.66)$$

Now we are going to use the weak and strong convergences obtained so far to pass to the limit as $m \rightarrow \infty$ in the equations of problem (P_m) . We are going to identify the limits of the non-linear terms related to chemotaxis and consumption,

$$a_m(u_m)\nabla v_m \text{ and } a_m(u_m)^s v_m,$$

respectively, with

$$u \nabla v \text{ and } u^s v.$$

In fact, considering the chemotaxis term, we have to deal with the difference

$$\int_0^T \int_{\Omega} [a_m(u_m) \nabla v_m - u \nabla v] \cdot \nabla \varphi \, dx \, \psi(t) \, dt, \quad (3.67)$$

for any $\varphi \in C_c^\infty(\Omega)$ and $\psi \in C_c^\infty(0, T)$. We can write it as

$$\begin{aligned} & \int_0^T \int_{\Omega} [a_m(u_m) \nabla v_m - u \nabla v] \cdot \nabla \varphi \, dx \, \psi(t) \, dt \\ &= \int_0^T \int_{\Omega} [a_m(u_m) - u] \nabla v_m \nabla \varphi \, dx \, \psi(t) \, dt \\ & \quad + \int_0^T \int_{\Omega} u [\nabla v_m - \nabla v] \nabla \varphi \, dx \, \psi(t) \, dt. \end{aligned}$$

Then, using the convergences (3.66), (3.64) and the m -uniform bound (3.47) we can conclude that (3.67) goes to zero as $m \rightarrow \infty$.

Considering the consumption term, we have to deal with the difference

$$\int_0^T \int_{\Omega} [a_m(u_m)^s v_m - u^s v] \varphi \, dx \, \psi(t) \, dt, \quad (3.68)$$

for any $\varphi \in C^\infty(\Omega)$ and $\psi \in C_c^\infty(0, \infty)$. We can write it as

$$\begin{aligned} & \left| \int_0^T \int_{\Omega} [a_m(u_m)^s v_m - u^s v] \varphi \, dx \, \psi(t) \, dt \right| \leq \\ & \leq C \int_0^T \int_{\Omega} |a_m(u_m)^s - u^s| v_m |\varphi| \, dx \, |\psi(t)| \, dt \\ & \quad + C \int_0^T \int_{\Omega} u^s |v_m - v| |\varphi| \, dx \, |\psi(t)| \, dt. \end{aligned}$$

Then, using the convergences (3.65), (3.66) and the m -uniform upper bound on v_m given by Lemma 38 we can conclude that (3.68) goes to zero as $m \rightarrow \infty$.

This concludes the identification of the limits of $a_m(u_m) \nabla v_m$ and $a_m(u_m)^s v_m$ with $u \nabla v$ and $u^s v$, respectively. The passage to the limit in the linear terms is simpler and is omitted here.

Then, we can conclude that the limit functions u and v of (3.63) and (3.66) satisfy the equations of the original problem (P) in the sense that

$$(u_t(t), \varphi) + (\nabla u(t), \nabla \varphi) = (u(t) \nabla v(t), \nabla \varphi), \text{ a.e } t \in (0, T),$$

$$(v_t(t), \varphi) + (\nabla v(t), \nabla \varphi) = -((u(t))^s v(t), \varphi), \text{ a.e } t \in (0, T).$$

for each $\varphi \in C^\infty(\Omega)$.

In order to finish the proof of Theorem 3 we must obtain the regularity (up to infinite time) which is claimed for u .

From (3.51) and (3.52) there exists a subsequence of $\{(u_m + 1)^{s/2}\}$, still denoted by $\{(u_m + 1)^{s/2}\}$, and a limit function w such that

$$\begin{aligned} (u_m + 1)^{s/2} &\longrightarrow \varphi \text{ weakly* in } L^\infty(0, \infty; L^2(\Omega)) \\ \nabla(u_m + 1)^{s/2} &\longrightarrow \nabla\varphi \text{ weakly in } L^2(0, \infty; L^2(\Omega)). \end{aligned} \quad (3.69)$$

Then, using the strong convergence (3.63), the continuity of the function $u_m \mapsto f(u_m) = (u_m + 1)^{s/2}$ and the Dominated Convergence Theorem, we prove that $\varphi = (u + 1)^{s/2}$.

Analogously, because of (3.48) we can conclude that, up to a subsequence, there is a limit function y such that

$$a_m(u_m)^{s/2} \nabla v_m \longrightarrow \phi \text{ weakly in } L^2(0, \infty; L^2(\Omega)). \quad (3.70)$$

And using the convergences (3.65) and (3.66) we can conclude that $\phi = u^{s/2} \nabla v$.

Therefore we have proved the global in time regularity

$$\begin{aligned} (u + 1)^{s/2} &\in L^\infty(0, \infty; L^2(\Omega)), \\ \nabla(u + 1)^{s/2} &\in L^2(0, \infty; L^2(\Omega)), \\ u^{s/2} \nabla v &\in L^2(0, \infty; L^2(\Omega)). \end{aligned} \quad (3.71)$$

Considering (3.71) and proceeding as in the obtaining of (3.54) and (3.55) we conclude the global in time regularity

$$\begin{aligned} u &\in L^\infty(0, \infty; L^s(\Omega)), \\ \nabla u, u \nabla v &\in L^2(0, \infty; L^s(\Omega)), \end{aligned}$$

finishing the proof of Theorem 3 in the case $s \in [1, 2)$.

3.1.4.2 m - independent estimates and passage to the limit for $s \geq 2$

If we consider the functions u_m , the procedure for the case in which $s \geq 2$ is slightly different. First we remind that, from (3.41), we have

$$\nabla a_m(u_m)^{s/2} \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \quad (3.72)$$

We also remind that we defined $g_m(r) = a_m(r)^{s-1}/(s-1)$, for $s \geq 2$. Then we have

$$a_m(r)^s = s \int_0^r a'_m(\theta) a_m(\theta)^{s-1} d\theta \leq s \int_0^r a_m(\theta)^{s-1} d\theta = s(s-1)g_m(r).$$

Therefore it also stems from (3.41) that

$$a_m(u_m)^{s/2} \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)). \quad (3.73)$$

From (3.73) and (3.72) we also conclude that

$$a_m(u_m)^{s/2} \text{ is bounded in } L^{10/3}(0, T; L^{10/3}(\Omega)),$$

that is,

$$a_m(u_m) \text{ is bounded in } L^{5s/3}(0, T; L^{5s/3}(\Omega)). \quad (3.74)$$

For each fixed $m \in \mathbb{N}$, consider the zero measure set $\mathcal{N} \subset (0, \infty)$ such that

$$u_m(t^*, \cdot), v_m(t^*, \cdot) \in H^1(\Omega), \quad \forall t^* \in (0, \infty) - \mathcal{N}.$$

Then, for each fixed $t^* \in (0, \infty) - \mathcal{N}$, let us consider the sets

$$\{0 \leq u_m \leq 1\} = \left\{ x \in \Omega \mid 0 \leq u_m(t^*, x) \leq 1 \right\}$$

and

$$\{u_m \geq 1\} = \left\{ x \in \Omega \mid u_m(t^*, x) \geq 1 \right\}.$$

Now note that

$$\begin{aligned} \int_{\Omega} a_m(u_m(t^*, x))^2 |\nabla v_m(t^*, x)|^2 dx &= \int_{\{0 \leq u_m \leq 1\}} a_m(u_m(t^*, x))^2 |\nabla v_m(t^*, x)|^2 dx \\ &\quad + \int_{\{u_m \geq 1\}} a_m(u_m(t^*, x))^2 |\nabla v_m(t^*, x)|^2 dx. \end{aligned}$$

Hence, since $s \geq 2$, we have

$$\begin{aligned} \int_{\Omega} a_m(u_m(t^*, x))^2 |\nabla v_m(t^*, x)|^2 dx &\leq \int_{\{0 \leq u_m \leq 1\}} |\nabla v_m(t^*, x)|^2 dx \\ &\quad + \int_{\{u_m \geq 1\}} a_m(u_m(t^*, x))^s |\nabla v_m(t^*, x)|^2 dx \\ &\leq \int_{\Omega} |\nabla v_m(t^*, x)|^2 dx \\ &\quad + \int_{\Omega} a_m(u_m(t^*, x))^s |\nabla v_m(t^*, x)|^2 dx. \end{aligned}$$

The last inequality is valid for all $t^* \in (0, \infty) - \mathcal{N}$, then if we integrate in the variable t we obtain

$$\begin{aligned} \int_0^\infty \int_{\Omega} a_m(u_m(t, x))^2 |\nabla v_m(t, x)|^2 dx dt &\leq \int_0^\infty \int_{\Omega} |\nabla v_m(t, x)|^2 dx dt \\ &\quad + \int_0^\infty \int_{\Omega} a_m(u_m(t, x))^s |\nabla v_m(t, x)|^2 dx dt. \end{aligned}$$

Therefore by Lemma 41-2 and (3.46) we can conclude that

$$a_m(u_m) \nabla v_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \quad (3.75)$$

Now we test the u_m -equation of problem (P_m) by u_m . This gives us

$$\frac{1}{2} \frac{d}{dt} \|u_m\|_{L^2(\Omega)}^2 + \|\nabla u_m\|_{L^2(\Omega)}^2 = \int_{\Omega} a_m(u_m) \nabla v_m \cdot \nabla u_m dx$$

$$\begin{aligned}
&\leq \left(\int_{\Omega} a_m(u_m)^2 |\nabla v_m|^2 dx \right)^{1/2} \|\nabla u_m\|_{L^2(\Omega)} \\
&\leq \frac{1}{2} \int_{\Omega} a_m(u_m)^2 |\nabla v_m|^2 dx + \frac{1}{2} \|\nabla u_m\|_{L^2(\Omega)}^2,
\end{aligned}$$

and hence we have

$$\frac{d}{dt} \|u_m\|_{L^2(\Omega)}^2 + \|\nabla u_m\|_{L^2(\Omega)}^2 \leq \int_{\Omega} a_m(u_m)^2 |\nabla v_m|^2 dx.$$

Integrating with respect to t from 0 to T , for any fixed $T \in (0, \infty)$, we obtain

$$\begin{aligned}
\|u_m(T)\|_{L^2(\Omega)}^2 + \int_0^T \|\nabla u_m(t)\|_{L^2(\Omega)}^2 dt &\leq \int_0^T \int_{\Omega} a_m(u_m(t, x))^2 |\nabla v_m(t, x)|^2 dx dt \\
&\quad + \|u^0\|_{L^2(\Omega)}^2.
\end{aligned}$$

Therefore, we conclude from (3.75) that

$$u_m \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)) \quad (3.76)$$

and

$$\nabla u_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \quad (3.77)$$

If we consider the u_m -equation of (P_m) , we have

$$(\partial_t u_m, \varphi) + (\nabla u_m, \nabla \varphi) = (a_m(u_m) \nabla v_m, \nabla \varphi),$$

for sufficiently regular φ . Because of (3.77) and (3.75) we conclude that

$$\partial_t u_m \text{ is bounded in } L^2(0, \infty; (H^1(\Omega))'). \quad (3.78)$$

Let $(a_m(u_m)^s)^* = \frac{1}{|\Omega|} \int_{\Omega} a_m(u_m)^s dx$, from (3.73) and (3.72), we can also conclude that

$$\nabla a_m(u_m)^s \text{ is bounded in } L^2(0, \infty; L^1(\Omega)).$$

In view of Lemma 14, the latter implies

$$a_m(u_m)^s - (a_m(u_m)^s)^* \text{ is bounded in } L^2(0, \infty; W^{1,1}(\Omega)) \quad (3.79)$$

and, in particular, by the Sobolev embedding, we have

$$a_m(u_m)^s - (a_m(u_m)^s)^* \text{ is bounded in } L^2(0, \infty; L^{3/2}(\Omega)). \quad (3.80)$$

Now we consider the v_m -equation of (P_m) written as

$$\partial_t v_m - \Delta v_m + (a_m(u_m)^s)^* v_m = -(a_m(u_m)^s - (a_m(u_m)^s)^*) v_m. \quad (3.81)$$

Testing (3.81) by v_m and using Holder's inequality we can obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_m\|_{L^2(\Omega)}^2 + \|\nabla v_m\|_{L^2(\Omega)}^2 + (a_m(u_m)^s)^* \|v_m\|_{L^2(\Omega)}^2 \\ \leq C \|v_m^0\|_{L^\infty(\Omega)} \|a_m(u_m)^s - (a_m(u_m)^s)^*\|_{L^{3/2}(\Omega)} \|v_m\|_{L^3(\Omega)} \\ \leq C \|a_m(u_m)^s - (a_m(u_m)^s)^*\|_{L^{3/2}(\Omega)}^2 + \delta \|v_m\|_{L^2(\Omega)}^2 + \delta \|\nabla v_m\|_{L^2(\Omega)}^2. \end{aligned}$$

In order to bound $(a_m(u_m)^s)^*$ from below, we will apply Lemma (18). Indeed,

$$(a_m(u_m)^s)^* \geq C \left(\int_{\Omega} a_m(u_m) dx \right)^s$$

and applying Lemma 18 (with $w_m = u_m$, $p = 2$, and using (3.76)) we conclude that there exist $\beta > 0$ and m_0 large enough such that $(a_m(u_m)^s)^* \geq \beta > 0$, a.e. $t \in (, \infty)$, for all $m \geq m_0$. Therefore

$$\frac{1}{2} \frac{d}{dt} \|v_m\|_{L^2(\Omega)}^2 + (1 - \delta) \|\nabla v_m\|_{L^2(\Omega)}^2 + (\beta - \delta) \|v_m\|_{L^2(\Omega)}^2 \leq C \|a_m(u_m)^s - (a_m(u_m)^s)^*\|_{L^{3/2}(\Omega)}^2.$$

Now, choosing δ small enough, integrating the last inequality with respect to t and using (3.80) we obtain

$$v_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \quad (3.82)$$

With the m -independent *a priori* bounds obtained so far we can also give an m -independent *a priori* bound for $\partial_t v_m$. In fact, if we consider again the equation (3.81), then the m -independent estimate in the L^∞ -norm for v_m given by Lemma 38-2 and the m -independent *a priori* bounds (3.82), (3.80) and (3.48) allow us to conclude that

$$\partial_t v_m \text{ is bounded in } L^2(0, \infty; L^{3/2}(\Omega)). \quad (3.83)$$

Now, using (3.76), (3.77) and (3.78) we can conclude that there is a function u such that

$$\begin{aligned} u - u^* &\in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)), \\ \partial_t u &\in L^2(0, \infty; (H^1(\Omega))') \end{aligned}$$

and

$$\begin{aligned} u_m &\longrightarrow u \text{ weakly}^* \text{ in } L^\infty(0, \infty; L^2(\Omega)), \\ \nabla u_m &\longrightarrow \nabla u \text{ weakly in } L^2(0, \infty; L^2(\Omega)), \\ \partial_t u_m &\longrightarrow u \text{ weakly in } L^2\left(0, \infty; (H^1(\Omega))'\right), \\ u_m &\longrightarrow u \text{ strongly in } L^2(0, T; L^2(\Omega)), \forall T > 0. \end{aligned} \quad (3.84)$$

Using the Dominated Convergence Theorem we can also prove that

$$a_m(u_m) \longrightarrow u \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (3.85)$$

The latter implies that $\{a_m(u_m)\}$ is a Cauchy sequence in $L^2(0, T; L^2(\Omega))$. Since $a_m(u_m)$ is bounded in $L^{5s/3}(0, T; L^{5s/3}(\Omega))$, it is also true that $\{a_m(u_m)\}$ is a Cauchy

sequence in $L^p(0, T; L^p(\Omega))$, for all $p \in [2, 5s/3]$. Therefore, for each $p \in (2, 5s/3)$, $a_m(u_m)$ converges to a certain limit w_p . But, since we are considering a finite measure domain Ω , the convergence for p also implies the convergence for every $q \in (1, p)$. Then (3.85) implies that $w_p = u$ and we conclude that

$$a_m(u_m) \longrightarrow u \text{ strongly in } L^p(0, T; L^p(\Omega)), \forall p \in (1, 5s/3). \quad (3.86)$$

From the convergence (3.86) and Lemma 17 we also conclude that

$$a_m(u_m)^s \longrightarrow u^s \text{ strongly in } L^p(0, T; L^p(\Omega)), \forall p \in (1, 5/3). \quad (3.87)$$

Remark 58. From (3.86) we can conclude that $u \in L^{5s/3}(0, T; L^{5s/3}(\Omega))$. ■

Remark 59. From the m -uniform bound (3.73) we can conclude that there is a subsequence of $\{a_m(u_m)\}$ which converges weakly* to some limit in $L^\infty(0, \infty; L^s(\Omega))$. From this and the convergence (3.86) we can conclude that, up to a subsequence

$$a_m(u_m) \rightarrow u \text{ weakly* in } L^\infty(0, \infty; L^s(\Omega)).$$

Therefore we can conclude that

$$u \in L^\infty(0, \infty; L^s(\Omega)).$$

■

For $s \geq 2$, if we consider the functions v_m , we have m -independent estimates for the same norms that we had for $s \in [1, 2)$. Then we can also conclude that there is a subsequence of $\{v_m\}$ (still denoted by $\{v_m\}$) and a function v such that

$$v \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)),$$

$$v_t \in L^2(0, \infty; L^{3/2}(\Omega))$$

and

$$\begin{aligned} v_m &\rightarrow v \text{ weakly* in } L^\infty(0, \infty; L^\infty(\Omega)), \\ v_m &\rightarrow v \text{ weakly in } L^2(0, \infty; H^2(\Omega)), \\ \nabla v_m &\rightarrow \nabla v \text{ weakly in } L^4(0, \infty; L^4(\Omega)), \\ v_m &\rightarrow v \text{ strongly in } L^2(0, T; H^1(\Omega)), \\ \text{and } \partial_t v_m &\rightarrow \partial_t v \text{ weakly in } L^2(0, \infty; L^{3/2}(\Omega)). \end{aligned} \quad (3.88)$$

Now we do as in the previous subsection, in which we dealt with the case $s \in [1, 2)$, and look at the non-linear terms related to the chemotaxis and the consumption.

Again, if we consider the chemotaxis term, we have to deal with the difference

$$\int_0^T \int_\Omega [a_m(u_m) \nabla v_m - u \nabla v] \cdot \nabla \varphi \, dx \, \psi(t) \, dt, \quad (3.89)$$

for any $\varphi \in C_c^\infty(\Omega)$ and $\psi \in C_c^\infty(0, T)$. Now it is convenient to write it as

$$\begin{aligned} & \int_0^T \int_\Omega [a_m(u_m) \nabla v_m - u \nabla v] \cdot \nabla \varphi \, dx \, \psi(t) \, dt \\ &= \int_0^T \int_\Omega [a_m(u_m) - u] \nabla v_m \nabla \varphi \, dx \, \psi(t) \, dt \\ & \quad + \int_0^T \int_\Omega u [\nabla v_m - \nabla v] \nabla \varphi \, dx \, \psi(t) \, dt. \end{aligned}$$

Then, using the convergences (3.86), (3.66) and the m -uniform bound (3.47) we can conclude that (3.89) goes to zero as $m \rightarrow \infty$.

Considering the consumption term, we have to deal with the difference

$$\int_0^T \int_\Omega [a_m(u_m)^s v_m - u^s v] \varphi \, dx \, \psi(t) \, dt, \quad (3.90)$$

for any $\psi \in C_c^\infty(0, \infty)$. Summing and subtracting the term $u^s v_m$ inside the brackets, we can write it as

$$\begin{aligned} \int_0^T \int_\Omega [a_m(u_m)^s v_m - u^s v] \varphi \, dx \, \psi(t) \, dt &= \int_0^T \int_\Omega [a_m(u_m)^s - u^s] v_m \varphi \, dx \, \psi(t) \, dt \\ & \quad + \int_0^T \int_\Omega u^s [v - v_m] \varphi \, dx \, \psi(t) \, dt. \end{aligned}$$

Then, using the convergences (3.87), (3.88) and the m -uniform upper bound on v_m given by Lemma 38 we can conclude that (3.90) goes to zero as $m \rightarrow \infty$.

This concludes the identification of the limits of $a_m(u_m) \nabla v_m$ and $a_m(u_m)^s v_m$ with $u \nabla v$ and $u^s v$, respectively. The passage to the limit as $m \rightarrow \infty$ in the linear terms is simpler and will be omitted here.

Then, we can conclude that the limit functions u and v of (3.84) and (3.88) satisfy the equations of the original problem (P) in the sense that

$$(u_t(t), \varphi) + (\nabla u(t), \nabla \varphi) = (u(t) \nabla v(t), \nabla \varphi), \quad a.e \, t \in (0, T),$$

$$(v_t(t), \varphi) + (\nabla v(t), \nabla \varphi) = -((u(t))^s v(t), \varphi), \quad a.e \, t \in (0, T).$$

for each $\varphi \in C^\infty(\Omega)$.

But u , v , u_t and v_t have the following regularity, which stems from the m -uniform bounds obtained in this subsection:

$$u - u^* \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)) \cap L_{loc}^{5s/3}(0, \infty; L^{5s/3}(\Omega)), \quad (3.91)$$

$$u_t \in L^2(0, \infty; (H^1(\Omega))'),$$

$$v \in L^\infty(0, \infty; L^\infty(\Omega)) \cap L^2(0, \infty; H^2(\Omega))$$

$$\nabla v \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)), \quad (3.92)$$

$$v_t \in L^2(0, \infty; L^{3/2}(\Omega)).$$

and the equations are actually satisfied in the following sense:

$$\begin{aligned} (u_t(t), \varphi) + (\nabla u(t), \nabla \varphi) &= (u(t) \nabla v(t), \nabla \varphi), \quad a.e \ t \in (0, T), \ \forall \varphi \in H^1(\Omega), \\ v_t(t, x) - \Delta v(t, x) &= -(u(t, x))^s v(t, x), \quad a.e \ (t, x) \in (0, T) \times \Omega. \end{aligned}$$

This finishes the proof of existence of solution to the original problem (P) as a limit of solutions of the regularized problems (P_m) for $s \geq 2$.

Next, in order to finish Section 3.1, we present some remarks.

Remark 60. We can't prove uniqueness of solution in the class of solutions (u, v) which have the regularity obtained in the existence result. ■

Remark 61. (Time independent estimates) Since any norm is weakly lower semi-continuous, the limit functions u and v satisfy the same time independent estimates that we established for the functions u_m and v_m . ■

Remark 62. (No blow-up hypothesis) Suppose that, for $s \geq 1$, the solution (u, v) has the regularity obtained in the existence result (one can consider the less regular case, $s = 1$) and, in addition, $u \in L^\infty(0, T; L^\infty(\Omega))$. Then we can prove that there is $n_0 \in \mathbb{N}$ such that $(u, v) = (u_m, v_m)$, for all $m \geq n_0$.

In fact, due to (3.52) we conclude that

$$\nabla[u + 1]^{1/2} \in L^2(0, T; L^2(\Omega)),$$

and since

$$u \in L^\infty(0, T; L^\infty(\Omega)),$$

it implies that

$$\nabla u \in L^2(0, T; L^2(\Omega)).$$

Now we remind that the v -equation is satisfied in the sense that

$$v_t(t, x) - \Delta v(t, x) = -u(t, x)^s v(t, x), \quad a.e. \text{ in } (0, T) \times \Omega,$$

with $u^s v \in L^\infty(0, T; L^\infty(\Omega))$. Therefore we can conclude that

$$\begin{aligned} u &\in L^\infty(0, T; L^\infty(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ v &\in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^2(\Omega)), \end{aligned} \tag{3.93}$$

and (u, v) is a solution of (P) in the sense that

$$\begin{aligned} (u_t(t), \varphi) + (\nabla u(t), \nabla \varphi) &= (u(t) \nabla v(t), \nabla \varphi), \quad \forall \varphi \in H^1(\Omega), \quad a.e \text{ in } (0, T), \\ v_t(t, x) - \Delta v(t, x) &= -(u(t, x))^s v(t, x), \quad a.e \text{ in } (0, T) \times \Omega. \end{aligned}$$

Since $u \in L^\infty(0, T; L^\infty(\Omega))$, there is $n_0 \in \mathbb{N}$ such that $a_m(u(t, x)) = u(t, x)$, for all $m \geq n_0$, and therefore (u, v) is also a solution of the regularized problem (P_m) . The regularity of (u, v) is enough to prove that $(u, v) = (u_m, v_m)$, for all $m \geq n_0$. ■

3.2 Existence and Uniqueness in 2D

As we could see until this point, the requirements that hypothesis (H1) is satisfied and that the Neumann problem (1.2) has the H^2 -regularity are sufficient in order to show the existence of solution to the original problem (P) as a limit of solutions of the regularized problems (P_m) .

In the sequel, we show that, in the two dimensional case, we can improve the results on the regularity of the solution. We will also prove the uniqueness of solution. The key point is the inequality of Lemma 12-2 for 2 dimensional domains.

3.2.1 Uniqueness in 2D

Theorem 63. *In the two dimensional case, for $s \in [1, 2)$, we have uniqueness of solution in the class of functions (u, v) such that*

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad (3.94)$$

and

$$v \in L^\infty(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)). \quad (3.95)$$

For $s \geq 2$, we have uniqueness of solution in the class of functions (u, v) such that (3.94), (3.95) and

$$u \in L^{\frac{4s}{2-\varepsilon}}(0, T; L^{2s}(\Omega)), \quad (3.96)$$

$$u \in L^{4s-4}(0, T; L^{4s-4}(\Omega)), \quad (3.97)$$

Remark 64. The regularities (3.94) and (3.95) imply

$$u_t \in L^2(0, T; (H^1(\Omega))'). \quad (3.98)$$

■

Now we prove Theorem 63.

Proof of Theorem 63. Suppose (u_1, v_1) and (u_2, v_2) are two solutions of the original problem (P) with the regularity mentioned in the beginning of this paragraph. Define $u = u_2 - u_1$ and $v = v_2 - v_1$. Then u satisfies

$$\begin{aligned} (u_t(t), \varphi) + (\nabla u(t), \nabla \varphi) &= (u(t) \nabla v_2(t), \nabla \varphi) \\ &+ (u_1(t) \nabla v, \nabla \varphi), \quad \forall \varphi \in H^1(\Omega), \text{ a.e } t \in (0, T), \end{aligned} \quad (3.99)$$

and v satisfies

$$\begin{aligned} v_t(t) - \Delta v(t) &= -[(u_2(t))^s - (u_1(t))^s]v_2(t) \\ &\quad - (u_1(t))^s v(t), \text{ a.e } t \in (0, T), \end{aligned} \quad (3.100)$$

with $u(0) = v(0) = 0$. Note that we can conclude from (3.99) that u is a zero mean function.

Case $s \in [1, 2)$:

Now we test (3.99) by u . We obtain

$$\begin{aligned} \frac{d}{dt} \|u\|_{L^2(\Omega)}^2 + 2 \|\nabla u\|_{L^2(\Omega)}^2 &\leq \|u\|_{L^4(\Omega)} \|\nabla v_2\|_{L^4(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\ &\quad + \|u_1\|_{L^4(\Omega)} \|\nabla v\|_{L^4(\Omega)} \|\nabla u\|_{L^2(\Omega)}. \end{aligned}$$

Then, using the interpolation inequality of Lemma 12-2, for 2D domains, and the Poincaré inequality for zero mean functions of Lemma 14, we have

$$\begin{aligned} \frac{d}{dt} \|u\|_{L^2(\Omega)}^2 + 2 \|\nabla u\|_{L^2(\Omega)}^2 &\leq \|u\|_{L^4(\Omega)} \|\nabla v_2\|_{L^4(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\ &\quad + \|u_1\|_{L^4(\Omega)} \|\nabla v\|_{L^4(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\ &\leq C \|u\|_{L^2(\Omega)}^{1/2} \|\nabla v_2\|_{L^4(\Omega)} \|\nabla u\|_{L^2(\Omega)}^{3/2} \\ &\quad + C \|u_1\|_{L^4(\Omega)} \|\nabla v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{H^1(\Omega)}^{1/2} \|\nabla u\|_{L^2(\Omega)} \end{aligned}$$

Using Young's inequality we can obtain, for each $\delta > 0$,

$$\begin{aligned} \frac{d}{dt} \|u\|_{L^2(\Omega)}^2 + 2 \|\nabla u\|_{L^2(\Omega)}^2 &\leq C(\delta) \|\nabla v_2\|_{L^4(\Omega)}^4 \|u\|_{L^2(\Omega)}^2 + C(\delta) \|u_1\|_{L^4(\Omega)}^4 \|\nabla v\|_{L^2(\Omega)}^2 \\ &\quad + \delta \|\nabla u\|_{L^2(\Omega)}^2 + \delta \|v\|_{H^2(\Omega)}^2. \end{aligned}$$

Taking $\delta = 1/2$ we find

$$\begin{aligned} \frac{d}{dt} \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 &\leq C \|\nabla v_2\|_{L^4(\Omega)}^4 \|u\|_{L^2(\Omega)}^2 \\ &\quad + C(\|u_1\|_{L^4(\Omega)}^4 + 1) \|\nabla v\|_{L^2(\Omega)}^2 \\ &\quad + C \|v\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\Delta v\|_{L^2(\Omega)}^2 \\ &\leq C \|\nabla v_2\|_{L^4(\Omega)}^4 \|u\|_{L^2(\Omega)}^2 \\ &\quad + C(\|u_1\|_{L^4(\Omega)}^4 + 1) \|v\|_{H^1(\Omega)}^2 \\ &\quad + \frac{1}{2} \|\Delta v\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.101)$$

Next we test (3.100) by v . We obtain

$$\frac{d}{dt} \|v\|_{L^2(\Omega)}^2 + 2 \|\nabla v\|_{L^2(\Omega)}^2 + \left\| u_1^{s/2} v \right\|_{L^2(\Omega)}^2 \leq \int_{\Omega} |u_2^s - u_1^s| |v_2| |v|.$$

Using

$$|u_2^s - u_1^s| \leq s |u_2 + u_1|^{s-1} |u_2 - u_1| ,$$

from Lemma 16 we have

$$\begin{aligned} \frac{d}{dt} \|v\|_{L^2(\Omega)}^2 + 2 \|\nabla v\|_{L^2(\Omega)}^2 + \left\| u_1^{s/2} v \right\|_{L^2(\Omega)}^2 \\ \leq s \int_{\Omega} |u_2 + u_1|^{s-1} |u| |v_2| |v| \\ \leq s \|v_2\|_{L^\infty(\Omega)} \|u_2 + u_1\|_{L^{2s}(\Omega)}^{s-1} \|u\|_{L^2(\Omega)} \|v\|_{L^{2s}(\Omega)} \\ \leq \tilde{C} \|u_2 + u_1\|_{L^{2s}(\Omega)}^{s-1} \|u\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} . \end{aligned}$$

If we apply Young's inequality we obtain

$$\begin{aligned} \frac{d}{dt} \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \left\| u_1^{s/2} v \right\|_{L^2(\Omega)}^2 \\ \leq C \|u_2 + u_1\|_{L^{2s}(\Omega)}^{2s-2} \|u\|_{L^2(\Omega)}^2 + C \|v\|_{H^1(\Omega)}^2 . \end{aligned} \quad (3.102)$$

Finally we test (3.100) by $-\Delta v$. It gives us

$$\begin{aligned} \frac{d}{dt} \|\nabla v\|_{L^2(\Omega)}^2 + 2 \|\Delta v\|_{L^2(\Omega)}^2 \\ = -2 \int_{\Omega} \left([(u_2)^s - (u_1)^s] v_2 + (u_1)^s v \right) \Delta v \, dx \\ \leq \tilde{C} \|[(u_2)^s - (u_1)^s] v_2\|_{L^2(\Omega)}^2 + \tilde{C} \|(u_1)^s v\|_{L^2(\Omega)}^2 \\ + \frac{1}{2} \|\Delta v\|_{L^2(\Omega)}^2 . \end{aligned}$$

We must estimate the terms

$$\tilde{C} \|[(u_2)^s - (u_1)^s] v_2\|_{L^2(\Omega)}^2$$

and

$$\tilde{C} \|(u_1)^s v\|_{L^2(\Omega)}^2 .$$

For the first of these two terms we can use

$$|u_2^s - u_1^s| \leq s |u_2 + u_1|^{s-1} |u_2 - u_1| ,$$

from Lemma 16 and find

$$\begin{aligned} \tilde{C} \|[(u_2)^s - (u_1)^s] v_2\|_{L^2(\Omega)}^2 &= \tilde{C} \int_{\Omega} |(u_2)^s - (u_1)^s|^2 |v_2|^2 \, dx \\ &\leq \tilde{C} s \|v_2\|_{L^\infty(\Omega)} \int_{\Omega} |u_2 + u_1|^{2s-2} |u_2 - u_1|^2 \, dx \end{aligned}$$

$$\begin{aligned}
&\leq \tilde{C}_1 \|u_2 + u_1\|_{L^4(\Omega)}^{2s-2} \|u\|_{L^{8/(6-2s)}(\Omega)}^2 \\
&\leq \tilde{C}_2 \|u_2 + u_1\|_{L^4(\Omega)}^{2s-2} \|u\|_{L^4(\Omega)}^2 \\
&\leq \bar{C} \|u_2 + u_1\|_{L^4(\Omega)}^{2s-2} \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\
&\leq C \|u_2 + u_1\|_{L^4(\Omega)}^{4s-4} \|u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2.
\end{aligned}$$

For the second term we have

$$\begin{aligned}
\tilde{C} \|(u_1)^s v\|_{L^2(\Omega)}^2 &= \int_{\Omega} |u_1^s|^2 |v|^2 \, dx \\
&\leq \tilde{C} \|u_1\|_{L^4(\Omega)}^{2s} \|v\|_{L^{4/(2-s)}(\Omega)}^2 \\
&\leq \tilde{C}_1 \|u_1\|_{L^4(\Omega)}^{2s} \|v\|_{H^1(\Omega)}^2.
\end{aligned}$$

Using the estimates of these two terms we obtain

$$\begin{aligned}
\frac{d}{dt} \|\nabla v\|_{L^2(\Omega)}^2 + \|\Delta v\|_{L^2(\Omega)}^2 &\leq C \|u_2 + u_1\|_{L^4(\Omega)}^{4s-4} \|u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 \\
&\quad + C \|u_1\|_{L^4(\Omega)}^{2s} \|v\|_{H^1(\Omega)}^2.
\end{aligned} \tag{3.103}$$

Now, taking into account that

$$\|\nabla v\|_{L^2(\Omega)}^2 \leq \|v\|_{H^1(\Omega)}^2,$$

if we sum up (3.101), (3.102) and (3.103), we obtain

$$\begin{aligned}
\frac{d}{dt} \left(\|u\|_{L^2(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2 \right) &+ \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\Delta v\|_{L^2(\Omega)}^2 \\
&\leq C \|\nabla v_2\|_{L^4(\Omega)}^4 \|u\|_{L^2(\Omega)}^2 \\
&\quad + C(\|u_1\|_{L^4(\Omega)}^4 + 1) \|v\|_{H^1(\Omega)}^2 \\
&\quad + C \|u_2 + u_1\|_{L^{2s}(\Omega)}^{2s-2} \|u\|_{L^2(\Omega)}^2 + C \|v\|_{H^1(\Omega)}^2 \\
&\quad + C \|u_2 + u_1\|_{L^4(\Omega)}^{4s-4} \|u\|_{L^2(\Omega)}^2 \\
&\quad + C \|u_1\|_{L^4(\Omega)}^{2s} \|v\|_{H^1(\Omega)}^2.
\end{aligned} \tag{3.104}$$

Finally, recalling the regularity hypotheses, in particular,

$$u_1, u_2, \nabla v_1, \nabla v_2 \in L^4(0, T; L^4(\Omega)),$$

recalling that $u(0) = v(0) = 0$ and applying Gronwall's inequality (Lemma 20) to (3.104) we conclude that $w = v = 0$, that is, $u_1 = u_2$ and $v_1 = v_2$.

Case $s \geq 2$:

For $s \geq 2$, we start by testing (3.99) by u and we also obtain

$$\begin{aligned} \frac{d}{dt} \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 &\leq C \|\nabla v_2\|_{L^4(\Omega)}^4 \|u\|_{L^2(\Omega)}^2 \\ &\quad + C(\|u_1\|_{L^4(\Omega)}^4 + 1) \|v\|_{H^1(\Omega)}^2 \\ &\quad + \frac{1}{2} \|\Delta v\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.105)$$

Next we test (3.100) by v . We obtain

$$\frac{d}{dt} \|v\|_{L^2(\Omega)}^2 + 2 \|\nabla v\|_{L^2(\Omega)}^2 + \left\| u_1^{s/2} v \right\|_{L^2(\Omega)}^2 \leq \int_{\Omega} |u_2^s - u_1^s| |v_2| |v|.$$

Using

$$|u_2^s - u_1^s| \leq s |u_2 + u_1|^{s-1} |u_2 - u_1|,$$

from Lemma 16 we have

$$\begin{aligned} \frac{d}{dt} \|v\|_{L^2(\Omega)}^2 + 2 \|\nabla v\|_{L^2(\Omega)}^2 + \left\| u_1^{s/2} v \right\|_{L^2(\Omega)}^2 \\ \leq s \int_{\Omega} |u_2 + u_1|^{s-1} |u| |v_2| |v| \\ \leq s \|v_2\|_{L^\infty(\Omega)} \|u_2 + u_1\|_{L^{4s-4}(\Omega)}^{s-1} \|u\|_{L^2(\Omega)} \|v\|_{L^4(\Omega)} \\ \leq \tilde{C} \|u_2 + u_1\|_{L^{4s-4}(\Omega)}^{s-1} \|u\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned}$$

If we apply Young's inequality we obtain

$$\begin{aligned} \frac{d}{dt} \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \left\| u_1^{s/2} v \right\|_{L^2(\Omega)}^2 \\ \leq C \|u_2 + u_1\|_{L^{4s-4}(\Omega)}^{2s-2} \|u\|_{L^2(\Omega)}^2 + C \|v\|_{H^1(\Omega)}^2. \end{aligned} \quad (3.106)$$

Finally we test (3.100) by $-\Delta v$. It gives us

$$\begin{aligned} \frac{d}{dt} \|\nabla v\|_{L^2(\Omega)}^2 + 2 \|\Delta v\|_{L^2(\Omega)}^2 \\ = -2 \int_{\Omega} \left([(u_2)^s - (u_1)^s] v_2 + (u_1)^s v \right) \Delta v \, dx \\ \leq \tilde{C} \|[(u_2)^s - (u_1)^s] v_2\|_{L^2(\Omega)}^2 + \tilde{C} \|(u_1)^s v\|_{L^2(\Omega)}^2 \\ + \frac{1}{2} \|\Delta v\|_{L^2(\Omega)}^2. \end{aligned}$$

We must estimate the terms

$$\tilde{C} \|[(u_2)^s - (u_1)^s] v_2\|_{L^2(\Omega)}^2$$

and

$$\tilde{C} \|(u_1)^s v\|_{L^2(\Omega)}^2.$$

For the first of these two terms we can use

$$|u_2^s - u_1^s| \leq s |u_2 + u_1|^{s-1} |u_2 - u_1|,$$

from Lemma 16 and find

$$\begin{aligned} \tilde{C} \|[(u_2)^s - (u_1)^s] v\|_{L^2(\Omega)}^2 &= \tilde{C} \int_{\Omega} |(u_2)^s - (u_1)^s|^2 |v|^2 dx \\ &\leq \tilde{C} s \|v\|_{L^\infty(\Omega)}^2 \int_{\Omega} |u_2 + u_1|^{2s-2} |u_2 - u_1|^2 dx \\ &\leq \tilde{C} s \|v^0\|_{L^\infty(\Omega)}^2 \|u_2 + u_1\|_{L^{4s-4}(\Omega)}^{2s-2} \|u\|_{L^4(\Omega)}^2 \\ &\leq \bar{C} \|u_2 + u_1\|_{L^{4s-4}(\Omega)}^{2s-2} \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\ &\leq C \|u_2 + u_1\|_{L^{4s-4}(\Omega)}^{4s-4} \|u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2. \end{aligned}$$

For the second term we have

$$\begin{aligned} \tilde{C} \|(u_1)^s v\|_{L^2(\Omega)}^2 &= \int_{\Omega} |u_1^s|^2 |v|^2 dx \\ &\leq \tilde{C} \|u_1\|_{L^{2s}(\Omega)}^{2s} \|v\|_{L^\infty(\Omega)}^2 \\ &\leq \tilde{C}_1 \|u_1\|_{L^{2s}(\Omega)}^{2s} \|v\|_{W^{1,2+\varepsilon}(\Omega)}^2 \\ &\leq \tilde{C}_2 \|u_1\|_{L^{2s}(\Omega)}^{2s} \|v\|_{H^1(\Omega)}^{2-\varepsilon} \|v\|_{H^2(\Omega)}^\varepsilon \\ &\leq \tilde{C}_3 \|u_1\|_{L^{2s}(\Omega)}^{\frac{4s}{2-\varepsilon}} \|v\|_{H^1(\Omega)}^2 + \delta \|v\|_{H^2(\Omega)}^2 \\ &\leq \tilde{C}_4 (\|u_1\|_{L^{2s}(\Omega)}^{\frac{4s}{2-\varepsilon}} + 1) \|v\|_{H^1(\Omega)}^2 + \delta' \|\Delta v\|_{L^2(\Omega)}^2. \end{aligned}$$

Using the estimates of these two terms we obtain

$$\begin{aligned} \frac{d}{dt} \|\nabla v\|_{L^2(\Omega)}^2 + \|\Delta v\|_{L^2(\Omega)}^2 &\leq C \|u_2 + u_1\|_{L^{4s-4}(\Omega)}^{4s-4} \|u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 \\ &\quad + C (\|u_1\|_{L^{2s}(\Omega)}^{\frac{4s}{2-\varepsilon}} + 1) \|v\|_{H^1(\Omega)}^2 + \delta' \|\Delta v\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.107}$$

Now, choosing a small enough positive δ' and taking into account that

$$\|\nabla v\|_{L^2(\Omega)}^2 \leq \|v\|_{H^1(\Omega)}^2,$$

if we sum up (3.105), (3.106) and (3.107), we obtain

$$\begin{aligned}
& \frac{d}{dt} \left(\|u\|_{L^2(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2 \right) + \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\Delta v\|_{L^2(\Omega)}^2 \\
& \leq C \|\nabla v_2\|_{L^4(\Omega)}^4 \|u\|_{L^2(\Omega)}^2 \\
& \quad + C(\|u_1\|_{L^4(\Omega)}^4 + 1) \|v\|_{H^1(\Omega)}^2 \\
& \quad + C \|u_2 + u_1\|_{L^{4s-4}(\Omega)}^{2s-2} \|u\|_{L^2(\Omega)}^2 + C \|v\|_{H^1(\Omega)}^2 \\
& \quad + C \|u_2 + u_1\|_{L^{4s-4}(\Omega)}^{4s-4} \|u\|_{L^2(\Omega)}^2 \\
& \quad + C(\|u_1\|_{L^{2s}(\Omega)}^{\frac{4s}{2-\varepsilon}} + 1) \|v\|_{H^1(\Omega)}^2.
\end{aligned} \tag{3.108}$$

Finally, recalling the regularity hypotheses, in particular,

$$u_1, u_2 \in L^{\frac{4s}{2-\varepsilon}}(0, T; L^{2s}(\Omega)) \cap L^{4s-4}(0, T; L^{4s-4}(\Omega))$$

and

$$\nabla v_1, \nabla v_2 \in L^4(0, T; L^4(\Omega)),$$

recalling that $u(0) = v(0) = 0$ and applying Gronwall's inequality (Lemma 20) to (3.108) we conclude that $w = v = 0$, that is, $u_1 = u_2$ and $v_1 = v_2$. $\blacksquare$

3.2.2 Proof of Theorem 6

In the two dimensional case, we can study the regularity of the solution (u, v) of the original problem for all $s \geq 2$ at the same time. Since these solutions can be obtained as a limit of the regularized solutions, (u_m, v_m) , as $m \rightarrow \infty$, we will go back to the part where we obtained m -independent estimates for (u_m, v_m) , the solution of (P_m) , now considering the initial data $(u^0, v^0) \in H^2(\Omega) \times H^2(\Omega)$. We are going to find better m -independent estimates for u_m and v_m and conclude that the solution provided by the existence result is in fact much more regular.

For the two dimensional case, we can use u_m^p , for $1 \leq p < \infty$, as a test function in the u_m -equation to obtain

$$\begin{aligned}
& \frac{1}{p+1} \frac{d}{dt} \int_{\Omega} u_m^{p+1}(x) \, dx + p \int_{\Omega} u_m^{p-1}(x) |\nabla u_m(x)|^2 \, dx \\
& = p \int_{\Omega} a_m(u_m(x)) \nabla v(x) u_m^{p-1}(x) \nabla u_m(x) \, dx \\
& \leq p \int_{\Omega} u_m^p(x) |\nabla v(x)| |\nabla u_m(x)| \, dx \\
& \leq p \int_{\Omega} u_m^{p/2+1/2}(x) |\nabla v(x)| u_m^{p/2-1/2}(x) |\nabla u_m(x)| \, dx
\end{aligned}$$

$$\begin{aligned}
&\leq p \|u_m^{p/2+1/2}(x)\|_{L^4(\Omega)} \|\nabla v_m\|_{L^4(\Omega)} \left(\int_{\Omega} u_m^{p-1}(x) |\nabla u_m(x)|^2 dx \right)^{1/2} \\
&\leq Cp \|u_m^{p/2+1/2}\|_{L^2(\Omega)}^{1/2} \|u_m^{p/2+1/2}\|_{H^1(\Omega)}^{1/2} \|\nabla v_m\|_{L^4(\Omega)} \left(\int_{\Omega} u_m^{p-1}(x) |\nabla u_m(x)|^2 dx \right)^{3/4} \\
&\leq C(p) \|u_m^{p/2+1/2}\|_{L^2(\Omega)} \|\nabla v_m\|_{L^4(\Omega)} \left(\int_{\Omega} u_m^{p-1}(x) |\nabla u_m(x)|^2 dx \right)^{1/2} \\
&\quad + C(p) \|u_m^{p/2+1/2}\|_{L^2(\Omega)}^{1/2} \|\nabla v_m\|_{L^4(\Omega)} \left(\int_{\Omega} u_m^{p-1}(x) |\nabla u_m(x)|^2 dx \right)^{3/4},
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\frac{1}{p+1} \frac{d}{dt} \int_{\Omega} u_m^{p+1}(x) dx + \frac{p}{2} \int_{\Omega} u_m^{p-1}(x) |\nabla u_m(x)|^2 dx &\leq C(p) \|\nabla v_m\|_{L^4(\Omega)}^2 \int_{\Omega} u_m^{p+1}(x) dx \\
&\quad + C(p) \|\nabla v_m\|_{L^4(\Omega)}^4 \int_{\Omega} u_m^{p+1}(x) dx.
\end{aligned}$$

By (3.47) and (3.49) we have

$$\|\nabla v_m\|_{L^4(\Omega)}^2, \|\nabla v_m\|_{L^4(\Omega)}^4 \in L^1(0, \infty).$$

Then, Gronwall's inequality (Lemma 20) leads us to

$$u_m \text{ is bounded in } L^\infty(0, \infty; L^{p+1}(\Omega)), \text{ for } 1 \leq p < \infty, \quad (3.109)$$

$$u_m^{p-1} \nabla u_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)), \text{ for } 1 \leq p < \infty. \quad (3.110)$$

Remark 65. The m -independent bounds in (3.109) and (3.110) depend exponentially on p . ■

Then we recall from (2.37) that

$$\partial_t v_m, \Delta v_m \in L^2(0, \infty; H^1(\Omega)).$$

Hence,

$$\begin{aligned}
\nabla(\partial_t v_m) - \nabla \Delta v_m &= -s a_m(u_m)^{s-1} \nabla a_m(u_m) v_m \\
&\quad - a_m(u_m)^s \nabla v_m, \text{ a.e. } (t, x) \in (0, \infty) \times \Omega.
\end{aligned} \quad (3.111)$$

Since all the terms in this equation are in $L^2(0, \infty; L^2(\Omega))$, we can take the inner product with $-\nabla \Delta v_m$ and integrate over Ω . Using integration by parts, Hölder's and Young's inequalities and the m -uniform upper bound on v_m given by Lemma 38, we obtain

$$\begin{aligned}
\frac{d}{dt} \|\Delta v_m\|_{L^2(\Omega)}^2 + \|\nabla \Delta v_m\|_{L^2(\Omega)}^2 &\leq \frac{\|v^0\|_{L^\infty(\Omega)}^2}{2} \|a_m(u_m)^{s-1} \nabla a_m(u_m)\|_{L^2(\Omega)}^2 \\
&\quad + C \|a_m(u_m)\|_{L^4(\Omega)}^2 \|\nabla v_m\|_{L^4(\Omega)}^2.
\end{aligned} \quad (3.112)$$

Then, integrating (3.112) with respect to t and using (3.110) with $p = s$, (3.109) with $p = 3$ and (3.47) we conclude that

$$v_m \text{ is bounded in } L^\infty(0, \infty; H^2(\Omega)), \quad (3.113)$$

$$\Delta v_m \text{ is bounded in } L^2(0, \infty; H^1(\Omega)).$$

Now, analogously to the obtaining of (3.112), we can also test (3.111) by $\nabla \partial_t v_m$. Then, using the m -independent estimate in the L^∞ -norm for v_m given by Lemma 38-2, (3.109) with $p = 3$, (3.110) $p = s$ and (3.113) we also conclude that

$$\partial_t v_m \text{ is bounded in } L^2(0, \infty; H^1(\Omega)). \quad (3.114)$$

Then, because of the regularity of the solutions (u_m, v_m) , we can use $-\Delta u_m$ as a test function in the u -equation of (P_m) . We obtain

$$\begin{aligned} \frac{d}{dt} \|\nabla u_m\|_{L^2(\Omega)}^2 + 2 \|\Delta u_m\|_{L^2(\Omega)}^2 &= \int_{\Omega} a_m(u_m) \Delta v_m \Delta u_m \, dx \\ &\quad + \int_{\Omega} \nabla a_m(u_m) \cdot \nabla v_m \Delta u_m \, dx \\ &\leq \|a_m(u_m)\|_{L^4(\Omega)} \|\Delta v_m\|_{L^4(\Omega)} \|\Delta u_m\|_{L^2(\Omega)} \\ &\quad + \|\nabla a_m(u_m)\|_{L^4(\Omega)} \|\nabla v_m\|_{L^4(\Omega)} \|\Delta u_m\|_{L^2(\Omega)} \\ &\leq \|u_m\|_{L^4(\Omega)} \|\Delta v_m\|_{L^4(\Omega)} \|\Delta u_m\|_{L^2(\Omega)} \\ &\quad + \|\nabla u_m\|_{L^4(\Omega)} \|\nabla v_m\|_{L^4(\Omega)} \|\Delta u_m\|_{L^2(\Omega)} \\ &\leq \|u_m\|_{L^4(\Omega)} \|\Delta v_m\|_{L^4(\Omega)} \|\Delta u_m\|_{L^2(\Omega)} \\ &\quad + \tilde{C} \|\nabla u_m\|_{L^2(\Omega)}^{1/2} \|\nabla u_m\|_{H^1(\Omega)}^{1/2} \|\nabla v_m\|_{L^4(\Omega)} \|\Delta u_m\|_{L^2(\Omega)} \\ &\leq \|u_m\|_{L^4(\Omega)} \|\Delta v_m\|_{L^4(\Omega)} \|\Delta u_m\|_{L^2(\Omega)} \\ &\quad + \tilde{C} \|\nabla u_m\|_{L^2(\Omega)}^{1/2} \|\nabla v_m\|_{L^4(\Omega)} \|\Delta u_m\|_{L^2(\Omega)}^{3/2}. \end{aligned}$$

Using Young's inequality (Lemma 13) we get

$$\begin{aligned} \frac{d}{dt} \|\nabla u_m\|_{L^2(\Omega)}^2 + 2 \|\Delta u_m\|_{L^2(\Omega)}^2 &\leq \frac{1}{2} \|u_m\|_{L^4(\Omega)}^2 \|\Delta v_m\|_{L^4(\Omega)}^2 \\ &\quad + C \|\nabla v_m\|_{L^4(\Omega)}^4 \|\nabla u_m\|_{L^2(\Omega)}^2 \\ &\quad + \|\Delta u_m\|_{L^2(\Omega)}^2. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \frac{d}{dt} \|\nabla u_m\|_{L^2(\Omega)}^2 + \|\Delta u_m\|_{L^2(\Omega)}^2 &\leq \frac{1}{2} \|u_m\|_{L^4(\Omega)}^2 \|\Delta v_m\|_{L^4(\Omega)}^2 \\ &\quad + C \|\nabla v_m\|_{L^4(\Omega)}^4 \|\nabla u_m\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.115)$$

Therefore, using (3.113) and Gronwall's inequality (Lemma 20) in (3.115) we conclude that

$$\nabla u_m \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)). \quad (3.116)$$

$$\Delta u_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)). \quad (3.117)$$

Considering the u_m -equation of (P_m) ,

$$\partial_t u_m - \Delta u_m = -a_m(u_m) \Delta v_m - \nabla a_m(u_m) \nabla v_m,$$

(3.109), (3.113), (3.116) and (3.117) we obtain

$$\partial_t u_m \text{ is bounded in } L^2(0, \infty; L^2(\Omega)), \quad (3.118)$$

We can use (3.109), (3.116), (3.117), (3.113), (3.118) and (3.114) and compactness results in the weak*, weak and strong topologies and the uniqueness for functions which satisfy (3.96), (3.97), (3.94) and (3.95) to conclude that there are functions u and v such that

$$\begin{aligned} u &\in L^\infty(0, \infty; H^1(\Omega)), \\ \nabla u, \Delta u &\in L^2(0, \infty; L^2(\Omega)) \end{aligned}$$

and

$$\begin{aligned} v &\in L^\infty(0, \infty; H^2(\Omega)), \\ \nabla v, \Delta v &\in L^2(0, \infty; H^1(\Omega)) \end{aligned}$$

which satisfy

$$u_t(t, x) - \Delta u(t, x) = \nabla \cdot (u(t, x) \nabla v(t, x)), \text{ a.e. } (t, x) \in (0, \infty) \times \Omega, \quad (3.119)$$

$$v_t(t, x) - \Delta v(t, x) = -u(t, x) v(t, x), \text{ a.e. } (t, x) \in (0, \infty) \times \Omega. \quad (3.120)$$

Now the homogeneous Neumann boundary condition is satisfied by u and v in the sense of $H^{-1/2}(\partial\Omega)$.

Finally, if now we suppose that Ω has the $W^{2,3}$ -regularity then we can prove better a m -independent estimates for u_m . The procedure is analogous to the one that was used in order to obtain the bounds (2.17), in the Subsection 2.1.1. We would like to test the u_m -equation of (P_m) by $\Delta^2 u_m$, but we don't have information about $\Delta^2 u_m$.

Therefore we are going to do like we did in order to obtain (3.113): we took the gradient of the v_m -equation of (P_m) , which gave us (3.111), and then tested (3.111) by $\nabla \Delta v_m$. Then, first we are going take the gradient of the u_m -equation of (P_m) and then we are going to test the resulting equation by $\nabla \Delta u_m$.

Before doing this, we recall that, in case the Poisson-Neumann problem (1.2) has the $W^{2,3}$ -regularity, the functions u_m and v_m have the regularity (2.43) and (2.44), respectively. Hence, if we take the gradient in the u_m -equation of (P_m) we obtain

$$\begin{aligned} \nabla(\partial_t u_m) - \nabla \Delta u_m &= -a_m(u_m) \nabla \Delta v_m - \nabla a_m(u_m) \Delta v_m \\ &\quad - D^2 v_m \nabla a_m(u_m) - D^2 a_m(u_m) \nabla v_m. \end{aligned} \quad (3.121)$$

Now we test (3.121) by $\nabla \Delta u_m$. Using the m -uniform bounds obtained so far we can conclude that

$$\Delta u_m \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)). \quad (3.122)$$

And if we look at (3.121) again we can also conclude that

$$\partial_t u_m \text{ is bounded in } L^2(0, \infty; H^1(\Omega)). \quad (3.123)$$

Therefore we summarize our findings for the 2D case:

- If Ω satisfies the hypothesis (H1) and has the H^2 -regularity then

$$\begin{aligned} u &\in L^\infty(0, \infty; H^1(\Omega)), \\ \Delta u, u_t &\in L^2(0, \infty; L^2(\Omega)), \\ u - u^* &\in L^2(0, \infty; H^2(\Omega)). \end{aligned} \quad (3.124)$$

and

$$\begin{aligned} v &\in L^\infty(0, \infty; H^2(\Omega)), \\ \Delta v, v_t &\in L^2(0, \infty; H^1(\Omega)), \\ v &\in L^2(0, \infty; H^2(\Omega)). \end{aligned} \quad (3.125)$$

- If Ω satisfies the hypothesis (H1) and has the $W^{2,3}$ -regularity then

$$\begin{aligned} u &\in L^\infty(0, \infty; H^2(\Omega)), \\ \Delta u, u_t &\in L^2(0, \infty; H^1(\Omega)), \\ u - u^* &\in L^2(0, \infty; W^{2,3}(\Omega)). \end{aligned} \quad (3.126)$$

and

$$\begin{aligned} v &\in L^\infty(0, \infty; H^2(\Omega)), \\ \Delta v, v_t &\in L^2(0, \infty; H^1(\Omega)), \\ v &\in L^2(0, \infty; W^{2,3}(\Omega)). \end{aligned} \quad (3.127)$$

Remark 66. Note that if Ω is such that hypothesis (H1) is satisfied and the Neumann problem (1.2) has the $W^{2,3}$ -regularity then the function u belongs to $L^\infty(0, \infty; L^\infty(\Omega))$, that is, the density of population is bounded and therefore does not blow up in infinite time. ■

CHAPTER 4

NUMERICAL SCHEMES

Until this point, we proposed the regularized problems (P_m) , we studied their solutions (u_m, v_m) and proved that, up to a subsequence, $\{(u_m, v_m)\}_{m \in \mathbb{N}}$ converges to a solution of the original problem (P) as $m \rightarrow \infty$. We showed that this solution (u, v) is unique and more regular in the two dimensional case.

Besides, in the two dimensional case, we showed that we do not need to take $m \rightarrow \infty$. In Section 3.2, we showed that we have

$$(u, v) = (u_{m_0}, v_{m_0}),$$

for a sufficiently large $m_0 \in \mathbb{N}$ (and for any $m > m_0$, too).

Then, in this chapter we propose three fully discrete numerical schemes to approximate the solution of the regularized problem (P_m) . We deal with a slightly modified version of problem (P_m) , with a constant $\chi > 0$ in the chemotaxis term. The problem which we will call problem (P_m) from now on is

$$\begin{aligned} \partial_t u_m - \Delta u_m &= -\chi \nabla \cdot (a_m(u_m) \nabla v_m) \\ \partial_t v_m - \Delta v_m &= -a_m(u_m)^s v_m \\ \frac{\partial u_m}{\partial n} \Big|_{\partial \Omega} &= \frac{\partial v_m}{\partial n} \Big|_{\partial \Omega} = 0 \\ u_m(0) &= u_0, \quad v_m(0) = v_0. \end{aligned}$$

This constant $\chi > 0$ in the chemotaxis term of problem (P) appears in some articles, such as (WINKLER, 2012) and (TAO, 2011), for example. The presence of this constant does not make difference for the theoretical analysis carried on so far, but there is evidence that it makes difference for the numerical approximation of the problem.

Chapter 4 is divided as follows. In Section 4.1 we introduce three time discretizations for (P_m) and prove their convergence. In Section 4.2 we propose three fully discrete numerical schemes, we study the existence and the uniqueness of solution, we prove their convergence towards (P_m) and we study the error estimates.

4.1 Time Discretization

In this section, we introduce the time discretization of the regularized model (P_m) based on the following variational formulation of the regularized problem (P_m) :

$$\begin{aligned} (\partial_t u_m, \varphi) + (\nabla u_m, \nabla \varphi) &= \chi(a_m(u_m) \nabla v_m, \nabla \varphi), \quad \forall \varphi \in H^1(\Omega), \\ (\partial_t v_m, \phi) + (\nabla v_m, \nabla \phi) &= -(a_m(u_m)^s v_m, \phi), \quad \forall \phi \in H^1(\Omega), \\ u_m(0) &= u^0, \quad v_m(0) = v^0. \end{aligned} \quad (P_m^V)$$

Since it is clear that we are talking about the solutions of the regularized problem, from this point on, we write u and v instead of u_m and v_m .

Let $T > 0$ be a finite real number and suppose we want to approximate the solution of the regularized model in the time interval $[0, T]$. Now we divide $[0, T]$ in N sub-intervals and define $k = T/N$ and $t_n = nk$, for $n = 0, 1, 2, \dots, N$.

Considering u^0 and v^0 to be the initial conditions of problem (P_m^V) , we define three different time discretizations for the problem (P_m^V) :

$$\begin{aligned} \text{Knowing } u^{n-1}, v^{n-1} \in L^2(\Omega), \text{ find } u^n, v^n \in H^1(\Omega) \text{ such that} \\ (\delta_t u^n, \varphi) + (\nabla u^n, \nabla \varphi) &= \chi(a_m(u^n) \nabla v^n, \nabla \varphi), \quad \forall \varphi \in H^1(\Omega), \\ (\delta_t v^n, \phi) + (\nabla v^n, \nabla \phi) &= -(a_m(u^n)^s v^n, \phi), \quad \forall \phi \in H^1(\Omega); \end{aligned} \quad (NLTD)$$

$$\begin{aligned} \text{Knowing } u^{n-1}, v^{n-1} \in L^2(\Omega), \text{ find } u^n, v^n \in H^1(\Omega) \text{ such that} \\ (\delta_t u^n, \varphi) + (\nabla u^n, \nabla \varphi) &= \chi(a_m(u^n) \nabla v^n, \nabla \varphi), \quad \forall \varphi \in H^1(\Omega), \\ (\delta_t v^n, \phi) + (\nabla v^n, \nabla \phi) &= -(a_m(u^{n-1})^s v^n, \phi), \quad \forall \phi \in H^1(\Omega); \end{aligned} \quad (SLTD)$$

$$\begin{aligned} \text{Knowing } u^{n-1}, v^{n-1} \in L^2(\Omega), \text{ find } u^n, v^n \in H^1(\Omega) \text{ such that} \\ (\delta_t u^n, \varphi) + (\nabla u^n, \nabla \varphi) &= \chi(a_m(u^{n-1}) \nabla v^n, \nabla \varphi), \quad \forall \varphi \in H^1(\Omega), \\ (\delta_t v^n, \phi) + (\nabla v^n, \nabla \phi) &= -(a_m(u^{n-1})^s v^n, \phi), \quad \forall \phi \in H^1(\Omega). \end{aligned} \quad (LTD)$$

At each step of the first scheme we have to solve two non-linear differential equations, that's why we call it "non-linear time discretization" and denote it by $(NLTD)$. If we consider the second scheme, we have to solve, at each step, a non-linear differential equation and a linear one, that's why we call it "semi-linear time discretization" and denote it by $(SLTD)$. Finally, at each step of the third scheme, we have to solve two linear differential equations. Then we call it "linear time discretization" and denote it by (LTD) .

All three time discretizations can be studied, for instance, via Galerkin's method and, for each time step size, $k = T/N$, we can prove the existence and uniqueness of

solution. But note that the time discretizations have different features. In fact, in $(NLTD)$, we must solve the non-linear equations on u^n and v^n simultaneously. On the other hand, in $(SLTD)$, we can first solve a linear equation on v^n and then a non-linear equation on u^n . In (LTD) , we can also solve a linear equation on v^n first and then an equation on u^n , that is also linear in this case.

Another difference arises if we consider the positivity of the solutions u^n and v^n . If we consider u^0 and v^0 to be non-negative functions, we can prove that the functions v^n obtained from $(NLTD)$, $(SLTD)$ and (LTD) , for $n = 1, 2, \dots, N$, are also non-negative. Besides, we can also prove that $v^n(x) \leq \|v^{n-1}\|_{L^\infty(\Omega)}$ a.e. $x \in \Omega$ and therefore we have

$$0 \leq v^n(x) \leq \|v^{n-1}\|_{L^\infty(\Omega)}, \text{ a.e. } x \in \Omega.$$

On the other hand, we can only prove that the functions u^n given by $(NLTD)$ and $(SLTD)$ are non-negative. Although the functions u_n given by (LTD) might assume negative values, the chemotaxis and consumption speed are always non-negative, because we use the truncation $a_m(u^{n-1})$ instead of u^{n-1} .

We next prove the convergence of the time discretizations towards the regularized model as the time step size k goes to zero. To this end, we prove the following energy inequalities.

Lemma 67 (A priori estimates). *If Ω is a locally Lipschitz domain then the solutions u^n and v^n of $(NLTD)$, $(SLTD)$ and (LTD) satisfy*

$$\|v^n\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \|v^j - v^{j-1}\|_{L^2(\Omega)}^2 + 2 \sum_{j=1}^n k \|\nabla v^j\|_{L^2(\Omega)}^2 \leq \|v^0\|_{L^2(\Omega)}^2, \quad (4.1)$$

$$\|u^n\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \|u^j - u^{j-1}\|_{L^2(\Omega)}^2 + \sum_{j=1}^n k \|\nabla u^j\|_{L^2(\Omega)}^2 \leq \|u^0\|_{L^2(\Omega)}^2 + \frac{m^2}{2} \|v^0\|_{L^2(\Omega)}^2, \quad (4.2)$$

for all $n \geq 1$.

Proof. First we use $\phi = 2kv^n$ in v^n -equation of the respective time discretization, $(NLTD)$, $(SLTD)$ or (LTD) . In this part, instead of the index n , use the index j . Considering any time discretization, we obtain

$$(\delta_t v^j, 2kv^j) + 2k \|\nabla v^j\|_{L^2(\Omega)}^2 \leq 0.$$

Then we use Lemma 33 in order to obtain

$$\|v^j\|_{L^2(\Omega)}^2 + \|v^j - v^{j-1}\|^2 + 2k \|\nabla v^j\|_{L^2(\Omega)}^2 \leq \|v^{j-1}\|_{L^2(\Omega)}^2$$

Summing this inequality from $j = 1$ to n we have (4.1).

Now, considering again the index j instead of n , we use $\varphi = 2ku^j$ as a test function in the u^j -equation of the respective time discretization. After using Lemma 33, Hölder's inequality, manipulating the resulting equation, summing from $j = 1$ to n and using the fact that, owing to (4.1),

$$2k \sum_{j=1}^n \|\nabla v^j\|_{L^2(\Omega)}^2 \leq \|v^0\|_{L^2(\Omega)}^2,$$

we obtain

$$\begin{aligned} \|u^n\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \|u^j - u^{j-1}\|_{L^2(\Omega)}^2 &+ \sum_{j=1}^n k \|\nabla u^j\|_{L^2(\Omega)}^2 \\ &\leq \|u^0\|_{L^2(\Omega)}^2 + m^2 k \sum_{j=1}^n \|\nabla v^j\|_{L^2(\Omega)}^2 \\ &\leq \|u^0\|_{L^2(\Omega)}^2 + \frac{m^2}{2} \|v^0\|_{L^2(\Omega)}^2. \end{aligned}$$

This finishes the proof of (4.2). ■

Next, we prove the convergence of the time discretizations towards the regularized model. In order to do it, considering $n = 1, 2, \dots, N$, we first define the piecewise continuous functions u_k^r, u_k^l and the locally linear and globally continuous function u_k by

$$u_k^r(t, x) = u^n(x), \text{ if } t \in [t_{n-1}, t_n), \quad (4.3)$$

$$u_k^l(t, x) = u^{n-1}(x), \text{ if } t \in [t_{n-1}, t_n), \quad (4.4)$$

$$u_k(t, x) = \frac{(t_n - t)u^{n-1}(x) + (t - t_{n-1})u^n(x)}{k}, \text{ if } t \in [t_{n-1}, t_n). \quad (4.5)$$

Analogously, we define the functions v_k^r, v_k^l and v_k .

Remark 68. We index the functions $u_k^r, u_k^l, u_k, v_k^r, v_k^l$ and v_k by the sub-index k . We remark that $k = T/N$ is the time step size and, in order to attain better and better approximations, we take smaller and smaller time step sizes. Therefore, we want to study the behavior of the sequences $\{u_k^r\}, \{u_k^l\}, \{u_k\}, \{v_k^r\}, \{v_k^l\}$ and $\{v_k\}$ as $k \rightarrow 0$. But we remind that $k \rightarrow 0$ within the bounded and discrete set of possible values $\{T/N\}_{N \in \mathbb{N}}$, as the number of sub-intervals, N , goes to infinity. ■

Now we can state the result on the convergence of the time discretizations (*LTD*), (*SLTD*) and (*NLTD*).

Theorem 69. *Let Ω be a locally Lipschitz domain and let (u, v) be the weak solution of the regularized problem (P_m) , that is, a solution of (P_m^V) . Consider the pairs (u^n, v^n) , $n = 0, 1, \dots, N$, given by one of the time discretizations (*LTD*), (*SLTD*) and (*NLTD*)*

and let $u_k^r, u_k^l, u_k, v_k^r, v_k^l$ and v_k be the respective associated functions, defined as in (4.3), (4.4) and (4.5). Then, as $k \rightarrow 0$, the sequences $\{u_k^r\}_k, \{u_k^l\}_k$ and $\{u_k\}_k$ converge to the function u and the sequences $\{v_k^r\}_k, \{v_k^l\}_k$ and $\{v_k\}_k$ converge to the function v weakly* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$.

Remark 70. We remark that the whole sequences converge as the time step size k goes to zero because of the uniqueness of the limit. ■

Proof. We demonstrate this Theorem for the case in which the functions (u^n, v^n) are given by (SLTD). The proofs of the convergences of (NLTD) and (LTD) are analogous.

Using the functions $u_k^r, u_k^l, u_k, v_k^r, v_k^l$ and v_k , we can rewrite the equations of (SLTD) as

$$\begin{aligned} (\partial_t u_k, \varphi) + (\nabla u_k^r, \nabla \varphi) &= \chi(a_m(u_k^r) \nabla v_k^r, \nabla \varphi), \quad \forall \varphi \in H^1(\Omega) \\ (\partial_t v_k, \phi) + (\nabla v_k^r, \nabla \phi) &= -(a_m(u_k^l)^s v_k^r, \phi), \quad \forall \phi \in H^1(\Omega). \end{aligned} \quad (4.6)$$

Taking Lemma 67 into account, we can obtain the following k -independent bounds for $u_k^r, u_k^l, u_k, v_k^r, v_k^l$ and v_k :

$$\begin{aligned} u_k^r, u_k^l, u_k, v_k^r, v_k^l, v_k &\text{ are bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ \partial_t u_k, \partial_t v_k &\text{ are bounded in } L^2(0, T; H^{-1}(\Omega)). \end{aligned} \quad (4.7)$$

Besides, using the estimate

$$\sum_{j=1}^n \|u^j - u^{j-1}\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \|v^j - v^{j-1}\|_{L^2(\Omega)}^2 \leq C(m, u^0, v^0),$$

given in (4.1) and (4.2), we can also prove that, as $k \rightarrow 0$,

$$\|u_k^r - u_k^l\|_{L^2(0, T; L^2(\Omega))}, \|u_k^r - u_k\|_{L^2(0, T; L^2(\Omega))} \longrightarrow 0 \quad (4.8)$$

and

$$\|v_k^r - v_k^l\|_{L^2(0, T; L^2(\Omega))}, \|v_k^r - v_k\|_{L^2(0, T; L^2(\Omega))} \longrightarrow 0. \quad (4.9)$$

Now consider subsequences of the sequences $\{u_k^r\}_k, \{u_k^l\}_k, \{u_k\}_k, \{v_k^r\}_k, \{v_k^l\}_k$ and $\{v_k\}_k$, still denoted by the index k . Then, using:

1. the k -independent bounds (4.7),
2. the weak convergence and compactness results in Bochner spaces
3. and the convergences (4.8) and (4.9);

we can conclude that there are functions $\bar{u}, \bar{v} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ such that (up to a subsequence)

$$\begin{aligned} u_k^r, u_k^l, u_k &\longrightarrow \bar{u} \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)) \\ u_k^r, u_k^l, u_k &\longrightarrow \bar{u} \text{ weakly in } L^2(0, T; H^1(\Omega)) \\ u_k^r, u_k^l, u_k &\longrightarrow \bar{u} \text{ strongly in } L^2(0, T; L^2(\Omega)), \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} v_k^r, v_k^l, v_k &\longrightarrow \bar{v} \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \\ v_k^r, v_k^l, v_k &\longrightarrow \bar{v} \text{ weakly in } L^2(0, T; H^1(\Omega)), \\ v_k^r, v_k^l, v_k &\longrightarrow \bar{v} \text{ strongly in } L^2(0, T; L^2(\Omega)). \end{aligned} \quad (4.11)$$

Using these convergences we can pass to the limit in (4.6) as $k \rightarrow 0$ and conclude that $(\bar{u}, \bar{v})$ is a solution of (P_m^V) . By the uniqueness of solution of the problem (P_m^V) , we conclude that $(\bar{u}, \bar{v}) = (u, v)$.

Thus we proved that every subsequence of each sequence $\{u_k^r\}_k, \{u_k^l\}_k$ and $\{u_k\}_k$ has a subsequence which converges to the same limit, u , weakly* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$. For each sequence, this implies the convergences of the whole sequence. Analogously, we can conclude the convergence of each sequence $\{v_k^r\}_k, \{v_k^l\}_k$ and $\{v_k\}_k$ to v weakly* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$, finishing the proof. ■

4.2 Fully Discrete Numerical Schemes

Based on the time discretizations (*NLTD*), (*SLTD*) and (*LTD*), we propose fully discrete schemes that we use to approximate the functions u and v . For each resulting scheme, we study the existence, uniqueness and stability of the solutions, as well as the convergence towards the regularized model and the error estimates.

We begin with the linear numerical scheme. Although the analysis of this scheme is not simple, it imposes fewer difficulties than the analyses of the non-linear and the semi-linear schemes, which are carried out in the sequel.

4.2.1 Linear numerical scheme

Based on the linear time discretization (*LTD*), we can propose a linear fully discrete numerical scheme for the approximation of the regularized problem.

Consider the spaces $U_h = V_h = \mathbb{P}_1(\mathcal{T}_h(\Omega))$. Let $u_h^0 = \Pi_h(u^0)$ and $v_h^0 = \Pi_h(v^0)$. The fully discrete scheme that we propose in this subsection is:

Knowing $(u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h$, find $(u_h^n, v_h^n) \in U_h \times V_h$ such that

$$(\delta_t u_h^n, \varphi_h) + (\nabla u_h^n, \nabla \varphi_h) = \chi(a_m(u_h^{n-1}) \nabla v_h^n, \nabla \varphi_h), \quad \forall \varphi_h \in U_h, \quad (LNS)$$

$$(\delta_t v_h^n, \phi_h) + (\nabla v_h^n, \nabla \phi_h) = - (a_m(u_h^{n-1})^s v_h^n, \phi_h), \quad \forall \phi_h \in V_h.$$

Now that we defined the linear numerical scheme (LNS), we prove the existence and uniqueness of the discrete solutions.

Theorem 71. *Given $u_h^{n-1} \in U_h$ and $v_h^{n-1} \in V_h$, there exists a unique solution (u_h^n, v_h^n) to (LNS).*

Proof. We remark that the linear numerical scheme (LNS) has a unique solution and is easy to solve. In fact, similarly to the time discretization (LTD), in the linear numerical scheme (LNS), both equations are linear and we can first solve the v_h^n -equation and then the u_h^n -equation. Now the problem is posed in finite dimensional spaces and we can also note that the matrices of the linear systems generated by the v_h^n -equation and the u_h^n -equation are positive definite and therefore we can use the Conjugate Gradient method in order to solve these linear systems.

Taking it into account, if we first look at the v_h^n -equation, we see that, given $u_h^{n-1}, v_h^{n-1} \in U_h \times V_h$, it has a unique solution $v^n \in V_h$. Then we use this unique v^n and $u_h^{n-1}, v_h^{n-1} \in U_h \times V_h$ to solve the u_h^n -equation, which also has a unique solution $u_h^n \in U_h$. ■

We also study some properties of the discrete solutions u_h^n and v_h^n .

Remark 72. (On the positivity of v_h^n). Using the mass lumping technique and the acute angle restriction on the triangulation, we can prove that if $v_h^{n-1} \geq 0$ then $v_h^n \geq 0$. We refer to (GUTIERREZ-SANTACREU; RODRÍGUEZ-GALVÁN, 2021) for the mass lumping technique. For simplicity, we continue the analysis without using mass lumping. ■

Theorem 73. *The fully discrete scheme is conservative with respect to the population of cells, that is,*

$$\int_{\Omega} u_h^n dx = \int_{\Omega} u_h^0 dx, \quad \text{for all } n = 1, 2, \dots, N. \quad (4.12)$$

For the chemical density we have:

$$1. \text{ if } v_h^n \geq 0 \text{ then } \int_{\Omega} v_h^n dx \leq \int_{\Omega} v_h^{n-1} dx \text{ for all } n = 1, 2, \dots, N;$$

2. if $\int_{\Omega} u_h^0 dx > 0$ then

$$\int_{\Omega} |v_h^n|^2 dx < \int_{\Omega} |v_h^{n-1}|^2 dx, \text{ for all } n = 1, 2, \dots, N.$$

Proof. The conservation of the population of cells is obtained directly by testing $\varphi_h = 1$ in the u_h^n -equation of (LNS) . Concerning the chemical density, in order to prove the first item, we use $\phi_h = 1$ in the v_h^n -equation of (LNS) and the hypothesis on the positivity of v_h^n . For the second item, we have to use $\phi_h = 2kv_h^n$ as a test function in the v_h^n -equation of (LNS) . We obtain

$$2 \int_{\Omega} \delta_t v_h^n v_h^n dx + 2k \int_{\Omega} |\nabla v_h^n|^2 dx + 2k \int_{\Omega} a_m(u_h^{n-1})^s |v_h^n|^2 dx = 0$$

and, if we apply Lemma 33 to the first term in the left hand side, we find

$$\begin{aligned} \int_{\Omega} |v_h^n|^2 dx + \int_{\Omega} |v_h^n - v_h^{n-1}|^2 dx + 2k \int_{\Omega} |\nabla v_h^n|^2 dx \\ + 2k \int_{\Omega} a_m(u_h^{n-1})^s |v_h^n|^2 dx = \int_{\Omega} |v_h^{n-1}|^2 dx. \end{aligned}$$

From the latter we conclude that

$$\int_{\Omega} |v_h^n|^2 dx \leq \int_{\Omega} |v_h^{n-1}|^2 dx, \text{ for all } n = 1, 2, \dots, N.$$

Now we suppose that

$$\int_{\Omega} |v_h^n|^2 dx = \int_{\Omega} |v_h^{n-1}|^2 dx, \text{ for some } n.$$

Then we conclude that

$$\int_{\Omega} |\nabla v_h^n|^2 dx = 0$$

and

$$\int_{\Omega} a_m(u_h^{n-1})^s |v_h^n|^2 dx = 0.$$

Hence v_h^n is constant in Ω and

$$\int_{\Omega} a_m(u_h^{n-1})^s dx = 0.$$

The latter implies that $(u_h^{n-1})_+ = 0$ in Ω . Therefore, because of (4.12) we have

$$\int_{\Omega} u_h^0 dx = \int_{\Omega} u_h^{n-1} dx = \int_{\Omega} (u_h^{n-1})_- dx \leq 0.$$

This proves the contrapositive of the desired result. ■

After studying some properties of the discrete solutions u_h^n and v_h^n , we prove the convergence towards the weak solution of the regularized model. First we prove a stability result.

Lemma 74. *If Ω is a polyhedron then we have the following stability results for the approximations u_h^n and v_h^n :*

$$\begin{aligned} \|v_h^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \|v_h^i - v_h^{i-1}\|_{L^2(\Omega)}^2 + 2 \sum_{i=1}^n k \|\nabla v_h^i\|_{L^2(\Omega)}^2 &\leq \|v_h^0\|_{L^2(\Omega)}^2, \\ \|u_h^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \|u_h^i - u_h^{i-1}\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla u_h^i\|_{L^2(\Omega)}^2 \\ &\leq \|u_h^0\|_{L^2(\Omega)}^2 + \frac{m^2}{2} \|v_h^0\|_{L^2(\Omega)}^2. \end{aligned}$$

Proof. The proof is analogous to the proof of Lemma 67. First we use $\phi_h = v_h^n$ in the v_h^n -equation and then we use $\varphi_h = u_h^n$ as a test function in the u_h^n -equation. Lemma 33 will be useful again. ■

Remark 75. The stability of v_h^n and ∇v_h^n in the norm of $L^2(\Omega)$ does not depend on m . On the other hand, the stability of u_h^n and ∇u_h^n depends on m . ■

Similarly to the study of the convergence of the time discretizations (*NLTD*), (*SLTD*) and (*LTD*), we prove the convergence of the linear fully discrete numerical scheme (*LNS*) towards the regularized problem. Considering $n = 1, 2, \dots, N$, we define the functions u_{hk}^r , u_{hk}^l and u_{hk} by

$$u_{hk}^r(t, x) = u_h^n(x), \text{ if } t \in [t_{n-1}, t_n), \quad (4.13)$$

$$u_{hk}^l(t, x) = u_h^{n-1}(x), \text{ if } t \in [t_{n-1}, t_n), \quad (4.14)$$

$$u_{hk}(t, x) = \frac{(t_n - t)u_h^{n-1}(x) + (t - t_{n-1})u_h^n(x)}{k}, \text{ if } t \in [t_{n-1}, t_n). \quad (4.15)$$

Analogously, we define the functions v_{hk}^r , v_{hk}^l and v_{hk} .

Now, taking Remark 68 into account, we can state the result on the convergence of the linear fully discrete scheme (*LNS*).

Theorem 76. *Let Ω be a polyhedron and let (u, v) be the weak solution of the regularized problem (P_m) , that is, a solution of (P_m^V) . Consider the pairs (u_h^n, v_h^n) , $n = 0, 1, \dots, N$, given by the linear fully discrete scheme (*LNS*) and let u_{hk}^r , u_{hk}^l , u_{hk} , v_{hk}^r , v_{hk}^l and v_{hk} be the respective associated functions, defined as in (4.13), (4.14) and (4.15). Then, as $(h, k) \rightarrow (0, 0)$, the sequences $\{u_{hk}^r\}_{(h,k)}$, $\{u_{hk}^l\}_{(h,k)}$ and $\{u_{hk}\}_{(h,k)}$ converge to the*

function u and the sequences $\{v_{hk}^r\}_{(h,k)}$, $\{v_{hk}^l\}_{(h,k)}$ and $\{v_{hk}\}_{(h,k)}$ converge to the function v weakly* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$.

Proof. Using the functions u_{hk}^r , u_{hk}^l , u_{hk} , v_{hk}^r , v_{hk}^l and v_{hk} , we can rewrite the equations of (LNS) as

$$\begin{aligned} (\partial_t u_{hk}, \varphi_h) + (\nabla u_{hk}^r, \nabla \varphi_h) &= \chi(a_m(u_{hk}^r) \nabla v_{hk}^r, \nabla \varphi_h) \\ (\partial_t v_{hk}, \phi_h) + (\nabla v_{hk}^r, \nabla \phi_h) &= - (a_m(u_{hk}^l)^s v_{hk}^r, \phi_h). \end{aligned} \quad (4.16)$$

Taking Lemma 67 into account, we can obtain the following h, k -independent bounds for u_{hk}^r , u_{hk}^l , u_{hk} , v_{hk}^r , v_{hk}^l and v_{hk} :

$$u_{hk}^r, u_{hk}^l, u_{hk}, v_{hk}^r, v_{hk}^l, v_{hk} \text{ are bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \quad (4.17)$$

Besides, we can also prove that, as $k \rightarrow 0$,

$$\|u_{hk}^r - u_{hk}^l\|_{L^2(0, T; L^2(\Omega))}, \|u_{hk}^r - u_{hk}\|_{L^2(0, T; L^2(\Omega))} \longrightarrow 0 \quad (4.18)$$

and

$$\|v_{hk}^r - v_{hk}^l\|_{L^2(0, T; L^2(\Omega))}, \|v_{hk}^r - v_{hk}\|_{L^2(0, T; L^2(\Omega))} \longrightarrow 0. \quad (4.19)$$

In order to bound the time derivative $\partial_t u_{hk}$ we remark that, for each $t \in (0, T)$, we have $\partial_t u_{hk}(t, \cdot) \in U_h$. Let $P_{L^2}^h(\varphi) \in U_h$ be the L^2 -projection of φ into U^h , we also recall the stability of the L^2 -projection in $H^1(\Omega)$, that is,

$$\|P_{L^2}^h(\varphi)\|_{H^1(\Omega)} \leq \|\varphi\|_{H^1(\Omega)}, \quad \forall \varphi \in H^1(\Omega).$$

Therefore, for any $\varphi \in H^1(\Omega)$ we have

$$(\partial_t u_{hk}, \varphi) = (\partial_t u_{hk}, P_{L^2}^h(\varphi)).$$

Then, for each $\varphi \in H^1(\Omega)$, we can test the u -equation of (4.16) by $P_{L^2}^h \varphi \in U^h$, obtaining

$$\begin{aligned} (\partial_t u_{hk}, \varphi) &= - (\nabla u_{hk}^r, \nabla P_{L^2}^h(\varphi)) + (a_m(u_{hk}^l) \nabla v_{hk}^r, P_{L^2}^h(\varphi)) \\ &\leq \|\nabla u_{hk}^r\|_{L^2(\Omega)} \|\nabla P_{L^2}^h(\varphi)\|_{L^2(\Omega)} + m \|\nabla v_{hk}^r\|_{L^2(\Omega)} \|\nabla P_{L^2}^h(\varphi)\|_{L^2(\Omega)} \\ &\leq C \|\nabla u_{hk}^r\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)} + Cm \|\nabla v_{hk}^r\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)}. \end{aligned}$$

Then we can conclude that

$$\frac{|(\partial_t u_{hk}, \varphi)|}{\|\varphi\|_{H^1(\Omega)}} \leq C \|\nabla u_{hk}^r\|_{L^2(\Omega)} + Cm \|\nabla v_{hk}^r\|_{L^2(\Omega)}, \quad \forall \varphi \in H^1(\Omega).$$

Analogously, we can prove that

$$\frac{|(\partial_t v_{hk}, \phi)|}{\|\phi\|_{H^1(\Omega)}} \leq C \|\nabla v_{hk}^r\|_{L^2(\Omega)} + Cm^s \|v_{hk}^r\|_{L^2(\Omega)}, \quad \forall \phi \in H^1(\Omega).$$

Therefore, using the h, k -independent bounds in (4.17) we can conclude that

$$\partial_t u_{hk}, \partial_t v_{hk} \text{ are bounded in } L^2(0, T; H^{-1}(\Omega)). \quad (4.20)$$

Now consider subsequences of the sequences $\{u_{hk}^r\}_{h,k}$, $\{u_{hk}^l\}_{h,k}$, $\{u_{hk}\}_{h,k}$, $\{v_{hk}^r\}_{h,k}$, $\{v_{hk}^l\}_{h,k}$ and $\{v_{hk}\}_{h,k}$, still denoted by the index k . Then, using:

1. the h, k -independent bounds, (4.17) and (4.20),
2. the weak convergence and compactness results in Bochner spaces,
3. and the convergences (4.18) and (4.19),

we can conclude that there are functions $\bar{u}, \bar{v} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ such that (up to a subsequence)

$$\begin{aligned} u_{hk}^r, u_{hk}^l, u_{hk} &\longrightarrow \bar{u} \text{ weakly* in } L^\infty(0, T; L^2(\Omega)) \\ u_{hk}^r, u_{hk}^l, u_{hk} &\longrightarrow \bar{u} \text{ weakly in } L^2(0, T; H^1(\Omega)) \\ u_{hk}^r, u_{hk}^l, u_{hk} &\longrightarrow \bar{u} \text{ strongly in } L^2(0, T; L^2(\Omega)), \end{aligned} \quad (4.21)$$

and

$$\begin{aligned} v_{hk}^r, v_{hk}^l, v_{hk} &\longrightarrow \bar{v} \text{ weakly* in } L^\infty(0, T; L^2(\Omega)), \\ v_{hk}^r, v_{hk}^l, v_{hk} &\longrightarrow \bar{v} \text{ weakly in } L^2(0, T; H^1(\Omega)), \\ v_{hk}^r, v_{hk}^l, v_{hk} &\longrightarrow \bar{v} \text{ strongly in } L^2(0, T; L^2(\Omega)). \end{aligned} \quad (4.22)$$

Using these convergences we can pass to the limit in (4.16) as $h, k \rightarrow 0$ and conclude that $(\bar{u}, \bar{v})$ is a solution of (P_m^V) . In fact, for each $\varphi, \phi \in H^1(\Omega)$, we can use (4.16) and the operator Π_h , defined in Section 1.4, to write

$$\begin{aligned} (\partial_t u_{hk}, \Pi_h \varphi) + (\nabla u_{hk}^r, \nabla \Pi_h \varphi) &= \chi(a_m(u_{hk}^r) \nabla v_{hk}^r, \nabla \Pi_h \varphi) \\ (\partial_t v_{hk}, \Pi_h \phi) + (\nabla v_{hk}^r, \nabla \Pi_h \phi) &= - (a_m(u_{hk}^l)^s v_{hk}^r, \Pi_h \phi). \end{aligned} \quad (4.23)$$

Then, in addition to the convergences (4.21) and (4.22), if we take into account that

$$\Pi_h \varphi \rightarrow \varphi \text{ and } \Pi_h \phi \rightarrow \phi \text{ strongly in } H^1(\Omega), \quad (4.24)$$

as $h \rightarrow 0$, we can pass to the limit as $h, k \rightarrow 0$ and conclude that $(\bar{u}, \bar{v})$ is a solution of (P_m^V) . By the uniqueness of solution of the problem (P_m^V) , we conclude that $(\bar{u}, \bar{v}) = (u, v)$.

Thus we proved that every subsequence of each sequence $\{u_{hk}^r\}_{h,k}$, $\{u_{hk}^l\}_{h,k}$ and $\{u_{hk}\}_{h,k}$ has a subsequence which converges to the same limit, u , weakly* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$. For each sequence, this implies the convergences of the whole sequence. Analogously, we can

conclude the convergence of each sequence $\{v_{hk}^r\}_{h,k}$, $\{v_{hk}^l\}_{h,k}$ and $\{v_{hk}\}_{h,k}$ to v weakly* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$, finishing the proof. ■

Once proved the convergence of the fully discrete scheme (LNS) towards the regularized problem, we turn to the study of the error estimates.

4.2.1.1 Error estimates for the linear fully discrete numerical scheme

In this subsection we want to estimate the total errors $u_h^n - u(t_n)$ and $v_h^n - v(t_n)$. In order to estimate these errors, we will use the operator Π_h defined in Section 1.4, as well as its properties

$$\|\Pi_h \varphi\|_{W^{n,p}(\Omega)} \leq C \|\varphi\|_{W^{n,p}(\Omega)}, \quad \forall \varphi \in W^{n,p}(\Omega) \quad (4.25)$$

and

$$\|\Pi_h \varphi - \varphi\|_{W^{n,2}(\Omega)} \leq Ch \|\varphi\|_{W^{n+1,2}(\Omega)}, \quad \forall \varphi \in W^{n+1,2}(\Omega), \quad (4.26)$$

for $n = 0, 1$ and $1 \leq p \leq \infty$. We will use this operator in order to write the total error $u_h^n - u(t_n)$ as

$$u_h^n - u(t_n) = u_h^n - \Pi_h(u(t_n)) + \Pi_h(u(t_n)) - u(t_n).$$

This way we separate the error in two components: the interpolation error $\Pi_h(u(t_n)) - u(t_n)$, for which property (4.26) provides us with an estimate in terms of the space discretization parameter h ; and the fully discrete error $u_h^n - \Pi_h(u(t_n))$ which belongs to the finite dimensional space U_h . We can do the same with the total error $v_h^n - v(t_n)$.

Therefore, in order to estimate $u_h^n - u(t_n)$, we just have to estimate the fully discrete errors

$$e_u^n = u_h^n - \Pi_h u(t_n) \quad \text{and} \quad e_v^n = v_h^n - \Pi_h v(t_n),$$

for the linear fully discrete scheme (LNS).

We remind that, for simplicity, when needed, we will denote the interval (t_{n-1}, t_n) by I_n .

Theorem 77. *Suppose that the solution (u, v) of the regularized problem (P_m^V) has the regularity*

$$u, v \in L^2(0, T; H^2(\Omega)),$$

$$v \in L^\infty(0, T; W^{1,6}(\Omega)),$$

$$u_t, v_t \in L^2(0, T; H^1(\Omega)).$$

Then there exist positive constants $C(u, v, \chi, m, T)$ (the constants may be different for each estimate) such that

$$\|e_u^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla e_u^i\|_{L^2(\Omega)}^2 \leq C(u, v, \chi, m, T)(h^2 + k^2)$$

and

$$\|e_v^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla e_v^i\|_{L^2(\Omega)}^2 \leq C(u, v, \chi, m, T)(h^2 + k^2).$$

Proof. Let us begin with the u -equation of (P_m^V) . We integrate this equation from t_{n-1} to t_n and subtract k times the u_h^n -equation of (LNS) :

$$\begin{aligned} k\delta_t(u(t_n) - u_h^n, \varphi_h) + \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla u_h^n, \nabla \varphi_h) dt \\ = \chi \int_{t_{n-1}}^{t_n} (a_m(u(t))\nabla v(t) - a_m(u_h^{n-1})\nabla v_h^n, \nabla \varphi_h) dt. \end{aligned}$$

Now we sum and subtract the convenient terms involving the operator Π_h in order to make the error e_u^n appear. Using this procedure, some terms, which we call consistency errors, also arise. We denote these consistency errors by the letter $\mathcal{E}$, with the appropriate indexes. For the first term we have

$$\begin{aligned} k\delta_t(u(t_n) - u_h^n, \varphi_h) &= k\delta_t(u(t_n) - \Pi_h u(t_n), \varphi_h) + k\delta_t(\Pi_h u(t_n) - u_h^n, \varphi_h) \\ &= \int_{t_{n-1}}^{t_n} (u_t(t) - \Pi_h u_t(t), \varphi_h) dt + k\delta_t(\Pi_h u(t_n) - u_h^n, \varphi_h) \\ &= -\mathcal{E}_{u,1}^n(\varphi_h) + k\delta_t(e_u^n, \varphi_h). \end{aligned}$$

For the second term,

$$\begin{aligned} \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla u_h^n, \nabla \varphi_h) dt &= \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla \Pi_h u(t), \nabla \varphi_h) dt \\ + \int_{t_{n-1}}^{t_n} (\nabla \Pi_h u(t) - \nabla u_h^n, \nabla \varphi_h) dt &+ \int_{t_{n-1}}^{t_n} (\nabla \Pi_h u(t_n) - \nabla u_h^n, \nabla \varphi_h) dt \\ &= -\mathcal{E}_{u,a,1}^n(\varphi_h) - \mathcal{E}_{u,a,2}^n(\varphi_h) + k(\nabla e_u^n, \nabla \varphi_h). \end{aligned}$$

For the term in the right hand side we have,

$$\begin{aligned} \chi \left(\int_{t_{n-1}}^{t_n} (a_m(u(t))\nabla v(t) - a_m(u_h^{n-1})\nabla v_h^n, \nabla \varphi_h) dt \right. \\ = \chi \left(\int_{t_{n-1}}^{t_n} (a_m(u(t))[\nabla v(t) - \nabla \Pi_h v(t)], \nabla \varphi_h) dt \right. \\ \left. + \int_{t_{n-1}}^{t_n} (a_m(u(t))\nabla \Pi_h[v(t) - v(t_n)], \nabla \varphi_h) dt \right) \end{aligned}$$

$$\begin{aligned}
& + \int_{t_{n-1}}^{t_n} ([a_m(u(t)) - a_m(\Pi_h u(t))] \nabla \Pi_h v(t_n), \nabla \varphi_h) dt \\
& + \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t)) - a_m(\Pi_h u(t_{n-1}))] \nabla \Pi_h v(t_n), \nabla \varphi_h) dt \\
& + \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1})) - a_m(u_h^{n-1})] \nabla \Pi_h v(t_n), \nabla \varphi_h) dt \\
& + \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1}) [\nabla \Pi_h v(t_n) - \nabla v_h^n], \nabla \varphi_h) dt \\
& = \mathcal{E}_{u,c,1}^n(\varphi_h) + \mathcal{E}_{u,c,2}^n(\varphi_h) + \mathcal{E}_{u,c,3}^n(\varphi_h) + \mathcal{E}_{u,c,4}^n(\varphi_h) \\
& + \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1})) - a_m(u_h^{n-1})] \nabla \Pi_h v(t_n), \nabla \varphi_h) dt \\
& + \chi \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1}) \nabla e_v^n, \nabla \varphi_h) dt.
\end{aligned}$$

Now we turn to the v -equation of (P_m^V) . Analogously to what we have done with the u -equation, we integrate the v -equation of (P_m^V) from t_{n-1} to t_n and subtract k times the v_h^n -equation of (LNS) :

$$\begin{aligned}
& k\delta_t(v(t_n) - v_h^n, \phi_h) + \int_{t_{n-1}}^{t_n} (\nabla v(t) - \nabla v_h^n, \nabla \phi_h) dt \\
& = \int_{t_{n-1}}^{t_n} (a_m(u(t))^s v(t) - a_m(u_h^{n-1})^s v_h^n, \phi_h) dt.
\end{aligned}$$

For the first term we have

$$\begin{aligned}
k\delta_t(v(t_n) - v_h^n, \phi_h) & = k\delta_t(v(t_n) - \Pi_h v(t_n), \phi_h) + k\delta_t(\Pi_h v(t_n) - v_h^n, \phi_h) \\
& = \int_{t_{n-1}}^{t_n} (v_t(t) - \Pi_h v_t(t), \phi_h) dt + k\delta_t(\Pi_h v(t_n) - v_h^n, \phi_h) \\
& = -\mathcal{E}_{v,1}^n(\phi_h) + k\delta_t(e_v^n, \phi_h).
\end{aligned}$$

For the second term,

$$\begin{aligned}
& \int_{t_{n-1}}^{t_n} (\nabla v(t) - \nabla v_h^n, \nabla \phi_h) dt = \int_{t_{n-1}}^{t_n} (\nabla v(t) - \nabla \Pi_h v(t), \nabla \phi_h) dt \\
& + \int_{t_{n-1}}^{t_n} (\nabla \Pi_h(v(t) - v(t_n)), \nabla \phi_h) dt + \int_{t_{n-1}}^{t_n} (\nabla \Pi_h v(t_n) - \nabla v_h^n, \nabla \phi_h) dt \\
& = -\mathcal{E}_{v,a,1}^n(\phi_h) - \mathcal{E}_{v,a,2}^n(\phi_h) + k(\nabla e_v^n, \nabla \phi_h).
\end{aligned}$$

For the term in the right hand side we have,

$$\begin{aligned}
& - \int_{t_{n-1}}^{t_n} (a_m(u(t))^s v(t) - a_m(u_h^{n-1})^s v_h^n, \phi_h) dt \\
& = - \int_{t_{n-1}}^{t_n} (a_m(u(t))^s [v(t) - \Pi_h v(t)], \phi_h) dt
\end{aligned}$$

$$\begin{aligned}
& - \int_{t_{n-1}}^{t_n} (a_m(u(t))^s \Pi_h[v(t) - v(t_n)], \phi_h) \, dt \\
& - \int_{t_{n-1}}^{t_n} ([a_m(u(t))^s - a_m(\Pi_h u(t))^s] \Pi_h v(t_n), \phi_h) \, dt \\
& - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t))^s - a_m(\Pi_h u(t_{n-1}))^s] \Pi_h v(t_n), \phi_h) \, dt \\
& - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1}))^s - a_m(u_h^{n-1})^s] \Pi_h v(t_n), \phi_h) \, dt \\
& - \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1})^s [\Pi_h v(t_n) - v_h^n], \phi_h) \, dt \\
& = \mathcal{E}_{v,c,1}^n(\phi_h) + \mathcal{E}_{v,c,2}^n(\phi_h) + \mathcal{E}_{v,c,3}^n(\phi_h) + \mathcal{E}_{v,c,4}^n(\phi_h) \\
& - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1}))^s - a_m(u_h^{n-1})^s] \Pi_h v(t_n), \phi_h) \, dt \\
& - \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1})^s e_v^n, \phi_h) \, dt.
\end{aligned}$$

Let us define the total consistency errors:

$$\mathcal{E}_u^n(\cdot) = \mathcal{E}_{u,1}^n(\cdot) + \mathcal{E}_{u,a,1}^n(\cdot) + \mathcal{E}_{u,a,2}^n(\cdot) + \mathcal{E}_{u,c,1}^n(\cdot) + \mathcal{E}_{u,c,2}^n(\cdot) + \mathcal{E}_{u,c,3}^n(\cdot) + \mathcal{E}_{u,c,4}^n(\cdot)$$

and

$$\mathcal{E}_v^n(\cdot) = \mathcal{E}_{v,1}^n(\cdot) + \mathcal{E}_{v,a,1}^n(\cdot) + \mathcal{E}_{v,a,2}^n(\cdot) + \mathcal{E}_{v,c,1}^n(\cdot) + \mathcal{E}_{v,c,2}^n(\cdot) + \mathcal{E}_{v,c,3}^n(\cdot) + \mathcal{E}_{v,c,4}^n(\cdot).$$

Then the fully discrete errors e_u^n and e_v^n satisfy the following equalities for all $\varphi_h \in U_h$ and $\phi_h \in V_h$,

$$\begin{aligned}
k\delta_t(e_u^n, \varphi_h) + k(\nabla e_u^n, \nabla \varphi_h) &= \mathcal{E}_u^n(\varphi_h) + \chi \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1}) \nabla e_v^n, \nabla \varphi_h) \, dt \\
&+ \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1})) - a_m(u_h^{n-1})] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt,
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
k\delta_t(e_v^n, \varphi_h) + k(\nabla e_v^n, \nabla \varphi_h) &+ \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1})^s e_v^n, \phi_h) \, dt = \mathcal{E}_v^n(\phi_h) \\
&- \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1}))^s - a_m(u_h^{n-1})^s] \Pi_h v(t_n), \phi_h) \, dt.
\end{aligned} \tag{4.28}$$

Then let us estimate $\mathcal{E}_u^n(\varphi_h)$ and $\mathcal{E}_v^n(\phi_h)$. We begin by $\mathcal{E}_{u,1}^n(\varphi_h)$. We have

$$\begin{aligned}
\mathcal{E}_{u,1}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (\Pi_h u_t(t) - u_t(t), \varphi_h) \, dt \\
&\leq \int_{t_{n-1}}^{t_n} \|\Pi_h u_t(t) - u_t(t)\|_{L^2(\Omega)} \|\varphi_h\|_{L^2(\Omega)} \, dt \\
&\leq Ch \int_{t_{n-1}}^{t_n} \|u_t(t)\|_{H^1(\Omega)} \, dt \|\varphi_h\|_{L^2(\Omega)}
\end{aligned}$$

$$\leq Chk^{1/2} \|u_t\|_{L^2(I_n; H^1(\Omega))} \|\varphi_h\|_{L^2(\Omega)}.$$

Now we estimate $\mathcal{E}_{u,a,1}^n(\varphi_h)$ and $\mathcal{E}_{u,a,2}^n(\varphi_h)$. We have

$$\begin{aligned} \mathcal{E}_{u,a,1}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla \Pi_h u(t), \nabla \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|\nabla \Pi_h u(t) - \nabla u(t)\|_{L^2(\Omega)} \, dt \, \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} \|u\|_{L^2(I_n; H^2(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_{u,a,2}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (\nabla \Pi_h(u(t) - u(t_n)), \nabla \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|\nabla \Pi_h(u(t) - u(t_n))\|_{L^2(\Omega)} \, dt \, \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq \int_{t_{n-1}}^{t_n} \int_t^{t_n} \|u_t(s)\|_{H^1(\Omega)} \, ds \, dt \, \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Ck^{3/2} \|u_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)}. \end{aligned}$$

Estimating $\mathcal{E}_{u,c,1}^n(\varphi_h)$, $\mathcal{E}_{u,c,2}^n(\varphi_h)$, $\mathcal{E}_{u,c,3}^n(\varphi_h)$ and $\mathcal{E}_{u,c,4}^n(\varphi_h)$ we obtain

$$\begin{aligned} \mathcal{E}_{u,c,1}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))[\nabla v(t) - \nabla \Pi_h v(t)], \nabla \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|a_m(u(t))\|_{L^\infty(\Omega)} \|\nabla v(t) - \nabla \Pi_h v(t)\|_{L^2(\Omega)} \, dt \, \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} m \|v\|_{L^2(I_n; H^2(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)}, \end{aligned}$$

$$\begin{aligned} \mathcal{E}_{u,c,2}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))\nabla \Pi_h[v(t) - v(t_n)], \nabla \varphi_h) \, dt \\ &\leq Ck^{3/2} \|a_m(u(t))\|_{L^\infty(I_n; L^\infty(\Omega))} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Cmk^{3/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)}, \end{aligned}$$

$$\begin{aligned} \mathcal{E}_{u,c,3}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} ([a_m(u(t)) - a_m(\Pi_h u(t))]\nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt \\ &\leq Ck^{1/2} \|a_m(u(t)) - a_m(\Pi_h u(t))\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)}. \end{aligned}$$

$$\mathcal{E}_{u,c,4}^n(\varphi_h) = \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t)) - a_m(\Pi_h u(t_{n-1}))]\nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt$$

$$\leq Ck^{3/2} \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)} .$$

Similarly, we have

$$\mathcal{E}_{v,1}^n(\phi_h) \leq Chk^{1/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} ,$$

$$\mathcal{E}_{v,a,1}^n(\phi_h) \leq Chk^{1/2} \|v\|_{L^2(I_n; H^2(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)}$$

and

$$\mathcal{E}_{v,a,2}^n(\phi_h) \leq Ck^{3/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)} .$$

Estimating $\mathcal{E}_{v,c,1}^n(\phi_h)$, $\mathcal{E}_{v,c,2}^n(\phi_h)$, $\mathcal{E}_{v,c,3}^n(\phi_h)$ and $\mathcal{E}_{v,c,4}^n(\phi_h)$ we obtain

$$\begin{aligned} \mathcal{E}_{v,c,1}^n(\phi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))^s [v(t) - \Pi_h v(t)], \phi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|a_m(u(t))^s\|_{L^\infty(\Omega)} \|v(t) - \Pi_h v(t)\|_{L^2(\Omega)} \, dt \|\phi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} \|a_m(u)^s\|_{L^\infty(I_n; L^\infty(\Omega))} \|v\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} \\ &\leq Cm^s h k^{1/2} \|v\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} , \end{aligned}$$

$$\begin{aligned} \mathcal{E}_{v,c,2}^n(\phi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))^s \Pi_h [v(t) - v(t_n)], \phi_h) \, dt \\ &\leq Ck^{3/2} \|a_m(u)^s\|_{L^\infty(I_n; L^\infty(\Omega))} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} \\ &\leq Cm^s k^{3/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} , \end{aligned}$$

$$\begin{aligned} \mathcal{E}_{v,c,3}^n(\phi_h) &= \int_{t_{n-1}}^{t_n} ([a_m(u(t))^s - a_m(\Pi_h u(t))^s] \Pi_h v(t_n), \phi_h) \, dt \\ &\leq Ck^{1/2} \|[a_m(u) + a_m(\Pi_h u)]^{s-1}\|_{L^\infty(I_n; L^\infty(\Omega))} \|a_m(u) - a_m(\Pi_h u)\|_{L^2(I_n; L^3(\Omega))} \\ &\quad \cdot \|v(t_n)\|_{L^6(\Omega)} \|\phi_h\|_{L^2(\Omega)} \\ &\leq Cm^{s-1} h k^{1/2} \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v\|_{L^\infty(I_n; L^6(\Omega))} \|\phi_h\|_{L^2(\Omega)} . \end{aligned}$$

$$\mathcal{E}_{v,c,4}^n(\phi_h) = \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t))^s - a_m(\Pi_h u(t_{n-1}))^s] \Pi_h v(t_n), \phi_h) \, dt$$

$$\leq C m^{s-1} k^{3/2} \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v\|_{L^\infty(I_n; L^6(\Omega))} \|\phi_h\|_{L^2(\Omega)}.$$

Then,

$$\begin{aligned} \mathcal{E}_u^n(\varphi_h) &\leq C h k^{1/2} \left(\|u_t\|_{L^2(I_n; L^2(\Omega))} \|\varphi_h\|_{L^2(\Omega)} + \left[\|u\|_{L^2(I_n; H^2(\Omega))} \right. \right. \\ &\quad \left. \left. + m \|v\|_{L^2(I_n; H^2(\Omega))} \right. \right. \\ &\quad \left. \left. + \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \right] \|\nabla \varphi_h\|_{L^2(\Omega)} \right) \\ &+ C k^{3/2} \left(\|u_t\|_{L^2(I_n; H^1(\Omega))} + m \|v_t\|_{L^2(I_n; H^1(\Omega))} \right. \\ &\quad \left. + \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \right) \|\nabla \varphi_h\|_{L^2(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_v^n(\phi_h) &\leq C h k^{1/2} \left(\|v_t\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} + \|v\|_{L^2(I_n; H^2(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)} \right. \\ &\quad \left. + \left[m^s \|v\|_{L^2(I_n; H^1(\Omega))} \right. \right. \\ &\quad \left. \left. + m^{s-1} \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v\|_{L^\infty(I_n; L^6(\Omega))} \right] \|\phi_h\|_{L^2(\Omega)} \right) \\ &+ C k^{3/2} \left(\|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)} + \left[m^s \|v_t\|_{L^2(I_n; H^1(\Omega))} \right. \right. \\ &\quad \left. \left. + \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v\|_{L^\infty(I_n; L^6(\Omega))} \right] \|\phi_h\|_{L^2(\Omega)} \right). \end{aligned}$$

Using Young's inequality we have

$$\begin{aligned} \mathcal{E}_u^n(\varphi_h) &\leq \tilde{C}(\delta) h^2 \left[\|u\|_{L^2(I_n; H^2(\Omega))}^2 + m^2 \|v\|_{L^2(I_n; H^2(\Omega))}^2 \right. \\ &\quad \left. + \|u\|_{L^2(I_n; W^{1,3}(\Omega))}^2 \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))}^2 \right] + \frac{\delta}{2} k \|\nabla \varphi_h\|_{L^2(\Omega)}^2 \\ &\quad + \tilde{C}(\delta) h^2 \|v_t\|_{L^2(I_n; L^2(\Omega))}^2 + \frac{\delta}{2} k \|\varphi_h\|_{L^2(\Omega)}^2 \\ &\quad + \tilde{C}(\delta) k^2 \left[\|u_t\|_{L^2(I_n; H^1(\Omega))}^2 + m^2 \|v_t\|_{L^2(I_n; H^1(\Omega))}^2 \right. \\ &\quad \left. + \|u_t\|_{L^2(I_n; L^3(\Omega))}^2 \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))}^2 \right] + \delta k \|\nabla \varphi_h\|_{L^2(\Omega)}^2, \end{aligned}$$

and we can write

$$\mathcal{E}_u^n(\varphi_h) \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|\varphi_h\|_{L^2(\Omega)}^2 + 2\delta k \|\nabla \varphi_h\|_{L^2(\Omega)}^2, \quad (4.29)$$

where, for each n , $\tilde{C}(I_n, \delta, m, u, v) > 0$ is a constant which depends on the norms of u, v, u_t and v_t in the interval I_n . Note that, because of the regularity of u and v in the whole interval $(0, T)$, there is a constant $C(\delta, u, v, m) > 0$ such that

$$\sum_{n=0}^N \tilde{C}(I_n, \delta, m, u, v) \leq C(\delta, u, v, m). \quad (4.30)$$

Analogously, we can obtain

$$\mathcal{E}_v^n(\phi_h) \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|\phi_h\|_{L^2(\Omega)}^2 + \delta k \|\nabla \phi_h\|_{L^2(\Omega)}^2. \quad (4.31)$$

Then, if we test (4.27) by e_u^n , (4.28) by e_v^n and use the estimates (4.29) and (4.31) we obtain

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 - \|e_u^{n-1}\|_{L^2(\Omega)}^2 + \|e_u^n - e_u^{n-1}\|_{L^2(\Omega)}^2 \\
& + k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) \\
& + \delta k \|e_u^n\|_{L^2(\Omega)}^2 + 2\delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 + \chi \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1}) \nabla e_v^n, \nabla e_u^n) dt \\
& + \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1})) - a_m(u_h^{n-1})] \nabla \Pi_h v(t_n), \nabla e_u^n) dt,
\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}
& \|e_v^n\|_{L^2(\Omega)}^2 - \|e_v^{n-1}\|_{L^2(\Omega)}^2 + \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + k \|a_m(u_h^{n-1})^{s/2} e_v^n\|_{L^2(\Omega)}^2 \\
& \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|e_v^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_v^n\|_{L^2(\Omega)}^2 \\
& - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1}))^s - a_m(u_h^{n-1})^s] \Pi_h v(t_n), e_v^n) dt.
\end{aligned} \tag{4.33}$$

We still have to estimate some terms of (4.32) and (4.33). We have

$$\chi \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1}) \nabla e_v^n, \nabla e_u^n) dt \leq m^2 \chi^2 C(\delta) k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2, \tag{4.34}$$

$$\begin{aligned}
& \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1})) - a_m(u_h^{n-1})] \nabla \Pi_h v(t_n), \nabla e_u^n) dt \\
& \leq k \chi \|e_u^{n-1}\|_{L^3(\Omega)} \|v(t_n)\|_{W^{1,6}(\Omega)} \|\nabla e_u^n\|_{L^2(\Omega)} \\
& \leq C(\delta) k \chi^2 \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 \|e_u^{n-1}\|_{L^3(\Omega)}^2 + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\
& \leq C(\delta) k \chi^2 \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 \|e_u^{n-1}\|_{L^2(\Omega)} \|e_u^{n-1}\|_{H^1(\Omega)} + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\
& \leq C(\delta) k \chi^2 \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 \|e_u^{n-1}\|_{L^2(\Omega)}^2 \\
& + C(\delta) k \chi^2 \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 \|e_u^{n-1}\|_{L^2(\Omega)} \|\nabla e_u^{n-1}\|_{L^2(\Omega)} + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\
& \leq \left(C(\delta) \chi^2 \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 + C(\delta) \chi^4 \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^4 \right) k \|e_u^{n-1}\|_{L^2(\Omega)}^2 \\
& + \delta k \|\nabla e_u^{n-1}\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
& \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1}))^s - a_m(u_h^{n-1})^s] \Pi_h v(t_n), e_v^n) dt \\
& \leq C m^{s-1} k \|e_u^{n-1}\|_{L^2(\Omega)} \|\Pi_h(v(t_n))\|_{L^\infty(\Omega)} \|e_v^n\|_{L^2(\Omega)} \\
& \leq C(\delta) m^{2(s-1)} k \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \|e_u^{n-1}\|_{L^2(\Omega)}^2 + k \delta \|e_v^n\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.36}$$

Using it in (4.32) and (4.33), we have

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 - \|e_u^{n-1}\|_{L^2(\Omega)}^2 + \|e_u^n - e_u^{n-1}\|_{L^2(\Omega)}^2 + k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\
& \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|e_u^n\|_{L^2(\Omega)}^2 + 4\delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\
& + m^2 C(\delta, \chi) k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + C(\delta, \chi, \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}) k \|e_u^{n-1}\|_{L^2(\Omega)}^2 \\
& + \delta k \|\nabla e_u^{n-1}\|_{L^2(\Omega)}^2
\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}
& \|e_v^n\|_{L^2(\Omega)}^2 - \|e_v^{n-1}\|_{L^2(\Omega)}^2 + \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + k \|a_m(u_h^{n-1})^{s/2} e_v^n\|_{L^2(\Omega)}^2 \\
& \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + 2\delta k \|e_v^n\|_{L^2(\Omega)}^2 \\
& + \delta k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + C(\delta, v) m^{2(s-1)} k \|e_u^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.38}$$

Let $C_0 = 2m^2 C(\delta, \chi)$ and note that

$$\begin{aligned}
\|e_v^n\|_{L^2(\Omega)}^2 &= \|e_v^n - e_v^{n-1} + e_v^{n-1}\|_{L^2(\Omega)}^2 \leq \left[\|e_v^n - e_v^{n-1}\|_{L^2(\Omega)} + \|e_v^{n-1}\|_{L^2(\Omega)} \right]^2 \\
&\leq 2 \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 + 2 \|e_v^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Then we can sum (4.37) and C_0 times (4.38) to obtain

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 - \|e_u^{n-1}\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 - C_0 \|e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 2\delta k) \|e_u^n - e_u^{n-1}\|_{L^2(\Omega)}^2 + (1 - 2\delta k) C_0 \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 4\delta) k \|\nabla e_u^n\|_{L^2(\Omega)}^2 + k \left(\frac{C_0}{2} - \delta C_0 \right) \|\nabla e_v^n\|_{L^2(\Omega)}^2 \\
& \leq (1 + C_0) \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) \\
& + (1 + C(\delta, \chi, v) + C(\delta, v) m^{2(s-1)} C_0) k \|e_u^{n-1}\|_{L^2(\Omega)}^2 \\
& + \delta k \|\nabla e_u^{n-1}\|_{L^2(\Omega)}^2 + 2\delta k C_0 \|e_v^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.39}$$

If we sum (4.39) from 1 to n , we find

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 \\
& + (1 - 2\delta k) \sum_{j=1}^n \|e_u^j - e_u^{j-1}\|_{L^2(\Omega)}^2 + (1 - 2\delta k) C_0 \sum_{j=1}^n \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 4\delta) k \sum_{j=1}^n \|\nabla e_u^j\|_{L^2(\Omega)}^2 + k \left(\frac{C_0}{2} - \delta C_0 \right) \sum_{j=1}^n \|\nabla e_v^j\|_{L^2(\Omega)}^2 \\
& \leq (1 + C_0) C(\delta, u, v, m) (h^2 + k^2) \\
& + \sum_{j=0}^{n-1} (1 + C(\delta, \chi, v) + C(\delta, v) m^{2(s-1)} C_0) k \|e_u^j\|_{L^2(\Omega)}^2 \\
& + \delta k \sum_{j=0}^{n-1} \|\nabla e_u^j\|_{L^2(\Omega)}^2 + \sum_{j=0}^{n-1} 2\delta k C_0 \|e_v^j\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.40}$$

We can simplify the term

$$\delta k \sum_{j=0}^{n-1} \|\nabla e_u^j\|_{L^2(\Omega)}^2$$

in the right hand side of (4.40) with the term

$$(1 - 4\delta) k \sum_{j=1}^n \|\nabla e_u^j\|_{L^2(\Omega)}^2,$$

which is in the left hand side of (4.40). Recalling that $u_h^0 = \Pi_h u^0$, that is, $e_u^0 = 0$, we have

$$\begin{aligned}
(1 - 4\delta) k \sum_{j=1}^n \|\nabla e_u^j\|_{L^2(\Omega)}^2 - \delta k \sum_{j=0}^{n-1} \|\nabla e_u^j\|_{L^2(\Omega)}^2 &= (1 - 4\delta) \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\
&+ (1 - 5\delta) k \sum_{j=1}^{n-1} \|\nabla e_u^j\|_{L^2(\Omega)}^2 \geq (1 - 5\delta) k \sum_{j=1}^n \|\nabla e_u^j\|_{L^2(\Omega)}^2.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 \\
& + (1 - 2\delta k) \sum_{j=1}^n \|e_u^j - e_u^{j-1}\|_{L^2(\Omega)}^2 + (1 - 2\delta k) C_0 \sum_{j=1}^n \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 5\delta) k \sum_{j=1}^n \|\nabla e_u^j\|_{L^2(\Omega)}^2 + k \left(\frac{C_0}{2} - \delta C_0 \right) \sum_{j=1}^n \|\nabla e_v^j\|_{L^2(\Omega)}^2 \\
& \leq (1 + C_0) C(\delta, u, v, m) (h^2 + k^2) \\
& + \sum_{j=0}^{n-1} (1 + C(\delta, \chi, v) + C(\delta, v) m^{2(s-1)} C_0) k \|e_u^j\|_{L^2(\Omega)}^2 \\
& + \sum_{j=0}^{n-1} 2\delta k C_0 \|e_v^j\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.41}$$

Now we can choose $\delta > 0$ small enough and apply the discrete Gronwall's inequality (Lemma 25) in order to obtain (we recall that $u_h^0 = \Pi_h u^0$ and $v_h^0 = \Pi_h v^0$)

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 \\
& + C \sum_{i=1}^n \|e_u^i - e_u^{i-1}\|_{L^2(\Omega)}^2 + CC_0 \sum_{i=1}^n \|e_v^i - e_v^{i-1}\|_{L^2(\Omega)}^2 \\
& + C \sum_{i=1}^n k \|\nabla e_u^i\|_{L^2(\Omega)}^2 + CC_0 \sum_{i=1}^n k \|\nabla e_v^i\|_{L^2(\Omega)}^2 \\
& \leq \tilde{C}(u, v, m, T)(h^2 + k^2),
\end{aligned} \tag{4.42}$$

as we wanted to prove. ■

Remark 78. Note that $\tilde{C}(u, v, m, T)$ depends exponentially on m , due to the discrete Gronwall's inequality (Lemma 25). ■

4.2.2 Non-linear numerical scheme

Consider the spaces $U_h = V_h = \mathbb{P}_1(\mathcal{T}_h(\Omega))$. Let $u_h^0 = \Pi_h u^0$ and $v_h^0 = \Pi_h v^0$. The fully discrete scheme that we propose in this subsection is:

Knowing $(u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h$, find $(u_h^n, v_h^n) \in U_h \times V_h$ such that

$$(\delta_t u_h^n, \varphi_h) + (\nabla u_h^n, \nabla \varphi_h) = \chi(a_m(u_h^n) \nabla v_h^n, \nabla \varphi_h), \quad \forall \varphi_h \in U_h, \tag{NLNS}$$

$$(\delta_t v_h^n, \phi_h) + (\nabla v_h^n, \nabla \phi_h) = -(a_m(u_h^n)^s v_h^n, \phi_h), \quad \forall \phi_h \in V_h.$$

Now that we defined the non-linear numerical scheme (NLNS), we study the existence and uniqueness of the discrete solutions.

Theorem 79 (Unconditional existence). *Given $u_h^{n-1} \in U_h$ and $v_h^{n-1} \in V_h$, there exists a solution of (NLNS).*

Proof. Showing existence of solution to (NLNS) is equivalent to showing the existence of solution to

Knowing $(u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h$, find $(u_h^n, v_h^n) \in U_h \times V_h$ such that

$$(\delta_t u_h^n, \varphi_h) + m^2 \chi^2 (\delta_t v_h^n, \phi_h) + (\nabla u_h^n, \nabla \varphi_h) + m^2 \chi^2 (\nabla v_h^n, \nabla \phi_h) \tag{4.43}$$

$$-\chi(a_m(u_h^n) \nabla v_h^n, \nabla \varphi_h) + m^2 \chi^2 (a_m(u_h^n)^s v_h^n, \phi_h) = 0, \quad \forall \varphi_h \in U_h, \quad \forall \phi_h \in V_h.$$

Therefore we will prove that the solution (4.43) is unique using Lemma 24. We choose $X_h = U_h \times V_h$, the inner product $[\cdot, \cdot]$ is the inner product of $H^1(\Omega) \times H^1(\Omega)$ and

$P_h : X_h \rightarrow X_h$ is given by the relation

$$\begin{aligned} [P_h((u_h, v_h)), (\varphi_h, \phi_h)] &= (u_h, \varphi_h) + m^2 \chi^2 (v_h, \phi_h) + k (\nabla u_h, \nabla \varphi_h) \\ &+ m^2 \chi^2 k (\nabla v_h, \nabla \phi_h) - k \chi (a_m(u_h) \nabla v_h, \nabla \varphi_h) + m^2 \chi^2 k (a_m(u_h)^s v_h, \phi_h) \\ &- (u_h^{n-1}, \varphi_h) - m^2 \chi^2 (v_h^{n-1}, \phi_h), \end{aligned}$$

for all $(u_h, v_h), (\varphi_h, \phi_h) \in X_h$. The mapping P_h is continuous and

$$\begin{aligned} [P_h((u_h, v_h)), (u_h, v_h)] &\geq \|u_h\|_{L^2(\Omega)}^2 + m^2 \chi^2 \|v_h\|_{L^2(\Omega)}^2 + k \|\nabla u_h\|_{L^2(\Omega)}^2 \\ &+ m^2 \chi^2 k \|\nabla v_h\|_{L^2(\Omega)}^2 + m^2 \chi^2 k \|a_m(u_h)^{\frac{s}{2}} v_h\|_{L^2(\Omega)}^2 \\ &- km \chi \|\nabla v_h\|_{L^2(\Omega)} \|\nabla u_h\|_{L^2(\Omega)} - \|u_h^{n-1}\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)} \\ &- m^2 \chi^2 \|v_h^{n-1}\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \\ &\geq \frac{1}{2} \|u_h\|_{L^2(\Omega)}^2 + \frac{m^2 \chi^2}{2} \|v_h\|_{L^2(\Omega)}^2 + \frac{k}{2} \|\nabla u_h\|_{L^2(\Omega)}^2 \\ &+ \frac{m^2 \chi^2}{2} k \|\nabla v_h\|_{L^2(\Omega)}^2 - \|u_h^{n-1}\|_{L^2(\Omega)}^2 - m^2 \chi^2 \|v_h^{n-1}\|_{L^2(\Omega)}^2 \\ &\geq C(k, \chi, m) \|(u_h, v_h)\|_{H^1(\Omega) \times H^1(\Omega)}^2 \\ &- \|u_h^{n-1}\|_{L^2(\Omega)}^2 - m^2 \chi^2 \|v_h^{n-1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Then, for $\beta > 0$ large enough and $[(u_h, v_h)] = \beta$ we have $[P_h((u_h, v_h)), (u_h, v_h)] > 0$. Therefore, by Lemma 24, we conclude that there exists a pair $(u_h, v_h) \in X_h$ which satisfies (4.43) and thus (NLNS). ■

Lemma 80. For any solution (u_h^n, v_h^n) (NLNS), the properties given by Lemma 73 and the stability results of Lemma 74 hold.

Proof. The proofs are analogous to the proofs of lemmas 73 and 74. ■

Theorem 81. Let Ω be a polyhedron and let (u, v) be the weak solution of the regularized problem (P_m) , that is, a solution of (P_m^V) . For each $n \in \{1, \dots, N\}$ let the pair (u_h^n, v_h^n) be a solution given by the existence result for the non-linear fully discrete scheme (NLNS), Theorem 79. Considering the aforementioned pairs (u_h^n, v_h^n) , let $u_{hk}^r, u_{hk}^l, u_{hk}, v_{hk}^r, v_{hk}^l$ and v_{hk} be the respective associated functions, defined as in (4.13), (4.14) and (4.15). Then, as $(h, k) \rightarrow (0, 0)$, the sequences $\{u_{hk}^r\}_{(h,k)}$, $\{u_{hk}^l\}_{(h,k)}$ and $\{u_{hk}\}_{(h,k)}$ converge to the function u and the sequences $\{v_{hk}^r\}_{(h,k)}$, $\{v_{hk}^l\}_{(h,k)}$ and $\{v_{hk}\}_{(h,k)}$ converge to the function v weakly* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$.

Proof. The demonstration of this result is analogous to the demonstration of the Theorem 76. ■

Once proved the convergence of the fully discrete scheme (*NLNS*) towards the regularized problem, we address the questions on the uniqueness of solution and error estimates.

Theorem 82 (Conditional uniqueness). *If we choose h and k such that*

$$\frac{k}{h^{d+2}} \leq \frac{C}{\chi^2 \|v^0\|_{L^2(\Omega)}^2} \text{ and } \frac{k^2}{h^d} \leq \frac{C}{\chi^2 m^{2s} \|v^0\|_{L^2(\Omega)}^2},$$

*then the solution of the scheme (*NLNS*) is unique.*

Proof. Suppose that there are two solutions of (*NLNS*), $(u_{h,1}, v_{h,1})$ and $(u_{h,2}, v_{h,2})$. We remind that, since $(u_{h,1}, v_{h,1}), (u_{h,2}, v_{h,2}) \in U_h \times V_h$, we have $u_{h,1}, v_{h,1}, u_{h,2}, v_{h,2} \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$. Let $w_h = u_{h,1} - u_{h,2}$ and $z_h = v_{h,1} - v_{h,2}$. We prove that $w_h = z_h = 0$. If $(u_{h,1}, v_{h,1})$ and $(u_{h,2}, v_{h,2})$ are solutions of (*NLNS*) then we have

$$\begin{aligned} (w_h, \varphi_h) + k(\nabla w_h, \nabla \varphi_h) &= k\chi(a_m(u_{h,1})\nabla z_h, \nabla \varphi_h) \\ &\quad + k\chi((a_m(u_{h,1}) - a_m(u_{h,2}))\nabla v_{h,2}, \nabla \varphi_h), \\ (z_h, \phi_h) + k(\nabla z_h, \nabla \phi_h) &= -k(a_m(u_{h,1})^s z_h, \phi_h) \\ &\quad - k((a_m(u_{h,1})^s - a_m(u_{h,2})^s)v_{h,2}, \phi_h), \end{aligned}$$

for all $(\varphi_h, \phi_h) \in X_h$.

Now we take $(\varphi_h, \phi_h) = (w_h, z_h)$ and obtain

$$\begin{aligned} \|w_h\|_{L^2(\Omega)}^2 + k\|\nabla w_h\|_{L^2(\Omega)}^2 &= k\chi(a_m(u_{h,1})\nabla z_h, \nabla w_h) \\ &\quad + k\chi((a_m(u_{h,1}) - a_m(u_{h,2}))\nabla v_{h,2}, \nabla w_h), \\ \|z_h\|_{L^2(\Omega)}^2 + k\|\nabla z_h\|_{L^2(\Omega)}^2 &+ k\|a_m(u_{h,1})^{s/2} z_h\|_{L^2(\Omega)}^2 \\ &= -k((a_m(u_{h,1})^s - a_m(u_{h,2})^s)v_{h,2}, z_h), \end{aligned} \tag{4.44}$$

Using Hölder's and Young's inequalities in the first equation of (4.44), we can obtain

$$\begin{aligned} \|w_h\|_{L^2(\Omega)}^2 + k\|\nabla w_h\|_{L^2(\Omega)}^2 &\leq \frac{k\chi^2 m^2}{2} \|\nabla z_h\|_{L^2(\Omega)}^2 + \frac{k}{2} \|\nabla w_h\|_{L^2(\Omega)}^2 \\ &\quad + \frac{k\chi^2}{2} \|\nabla v_{h,2}\|_{L^\infty(\Omega)}^2 \|w_h\|_{L^2(\Omega)}^2 + \frac{k}{2} \|\nabla w_h\|_{L^2(\Omega)}^2, \end{aligned}$$

Then, using the inverse inequality (Lemma 31-1)

$$\|\nabla v_{h,2}\|_{L^\infty(\Omega)}^2 \leq \frac{\hat{C}}{h^{d+2}} \|v_{h,2}\|_{L^2(\Omega)}^2$$

and the estimate of $\|v_{h,2}\|_{L^2(\Omega)}^2$ given by Lemma 80, we conclude that

$$(1 - k\chi^2 \frac{\hat{C} \|v^0\|_{L^2(\Omega)}^2}{2h^{d+2}}) \|w_h\|_{L^2(\Omega)}^2 \leq \frac{k\chi^2 m^2}{2} \|\nabla z_h\|_{L^2(\Omega)}^2. \tag{4.45}$$

In order to deal with the term on the right hand side of (4.45) we sum (4.45) to $\frac{\chi^2 m^2}{2}$ times the second equation of (4.44). Then we obtain

$$\begin{aligned} & (1 - k \frac{\hat{C} \chi^2 \|v^0\|_{L^2(\Omega)}^2}{h^{d+2}}) \|w_h\|_{L^2(\Omega)}^2 + \frac{\chi^2 m^2}{2} \|z_h\|_{L^2(\Omega)}^2 \\ & + \frac{\chi^2 m^2}{2} k \|a_m(u_{h,1})^{s/2} z_h\|_{L^2(\Omega)}^2 \leq -\frac{\chi^2 m^2}{2} k ((a_m(u_{h,1})^s - a_m(u_{h,2})^s) v_{h,2}, z_h). \end{aligned} \quad (4.46)$$

Now, using Hölder's and Young's inequalities we can estimate the term in the right hand side by

$$\begin{aligned} \frac{\chi^2 m^2}{2} k ((a_m(u_{h,1})^s - a_m(u_{h,2})^s) v_{h,2}, z_h) & \leq \frac{\chi^2 m^{s+1}}{2} k \|w_h\|_{L^2(\Omega)} \|v_{h,2}\|_{L^\infty(\Omega)} \|z_h\|_{L^2(\Omega)} \\ & \leq \frac{1}{4} \|w_h\|_{L^2(\Omega)}^2 + \frac{\chi^4 m^{2s+2}}{4} k^2 \|v_{h,2}\|_{L^\infty(\Omega)}^2 \|z_h\|_{L^2(\Omega)}^2. \end{aligned}$$

Using the inverse inequality (Lemma 31-2)

$$\|v_{h,2}\|_{L^\infty(\Omega)}^2 \leq \frac{\hat{C}}{h^d} \|v_{h,2}\|_{L^2(\Omega)}^2$$

and the estimate of $\|v_{h,2}\|_{L^2(\Omega)}^2$ given by Lemma 80, we find

$$\begin{aligned} & \frac{\chi^2 m^2}{2} k ((a_m(u_{h,1})^s - a_m(u_{h,2})^s) v_{h,2}, z_h) \\ & \leq \frac{1}{4} \|w_h\|_{L^2(\Omega)}^2 + k^2 \frac{\hat{C} \chi^2 m^{2s} \|v^0\|_{L^2(\Omega)}^2}{h^d} \frac{\chi^2 m^2}{4} \|z_h\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.47)$$

Using (4.47) in (4.46) we obtain

$$\begin{aligned} & \frac{1}{2} \left(\frac{3}{2} - k \frac{\hat{C} \chi^2 \|v^0\|_{L^2(\Omega)}^2}{h^{d+2}} \right) \|w_h\|_{L^2(\Omega)}^2 + \frac{1}{2} \left(1 - k^2 \frac{\hat{C} \chi^2 m^{2s} \|v^0\|_{L^2(\Omega)}^2}{2h^d} \right) \chi^2 m^2 \|z_h\|_{L^2(\Omega)}^2 \\ & + \frac{\chi^2 m^2}{2} k \|a_m(u_{h,1})^{s/2} z_h\|_{L^2(\Omega)}^2 \leq 0. \end{aligned}$$

Then, with χ and m fixed, if we choose k and h such that

$$\left(\frac{3}{2} - k \frac{\hat{C} \chi^2 \|v^0\|_{L^2(\Omega)}^2}{h^{d+2}} \right) > 0 \text{ and } \left(1 - k^2 \frac{\hat{C} \chi^2 m^{2s} \|v^0\|_{L^2(\Omega)}^2}{2h^d} \right) > 0,$$

we conclude that $w_h = z_h = 0$. ■

4.2.2.1 Error estimates for the fully discrete non-linear scheme

Now we study the error estimates. We want to estimate the fully discrete errors

$$e_u^n = u_h^n - \Pi_h u(t_n) \quad \text{and} \quad e_v^n = v_h^n - \Pi_h v(t_n).$$

We remind that, for simplicity, when needed, we will denote the interval (t_{n-1}, t_n) by I_n .

Theorem 83. Suppose that the solution (u, v) of the regularized problem (P_m^V) has the regularity

$$\begin{aligned} u, v &\in L^2(0, T; H^2(\Omega)), \\ v &\in L^\infty(0, T; W^{1,6}(\Omega)), \\ u_t, v_t &\in L^2(0, T; H^1(\Omega)). \end{aligned}$$

If we choose k such that

$$k < \frac{1}{\left(\frac{2}{5} + \chi^2 C(v) + C(s, m) \|v\|_{L^\infty(0, T; L^\infty(\Omega))}^2\right)},$$

where $C(v) = C(\|v\|_{L^\infty(0, T; W^{1,6}(\Omega))}^2 + \|v\|_{L^\infty(0, T; W^{1,6}(\Omega))}^4)$. Then there exists a positive constant $C(u, v, \chi, m, T)$ such that

$$\|e_u^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla e_u^i\|_{L^2(\Omega)}^2 \leq C(u, v, \chi, m, T)(h^2 + k^2)$$

and

$$\|e_v^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla e_v^i\|_{L^2(\Omega)}^2 \leq C(u, v, \chi, m, T)(h^2 + k^2).$$

Proof. Many terms of the consistency errors will be the same as in the linear case but, for completeness, we will show all the computations.

Integrate the u -equation of (P_m^V) from t_{n-1} to t_n and subtract k times the u_h^n -equation of $(NLNS)$:

$$\begin{aligned} k\delta_t(u(t_n) - u_h^n, \varphi_h) + \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla u_h^n, \nabla \varphi_h) dt \\ = \chi \int_{t_{n-1}}^{t_n} (a_m(u(t)) \nabla v(t) - a_m(u_h^n) \nabla v_h^n, \nabla \varphi_h) dt. \end{aligned}$$

Sum and subtract the convenient terms involving the operator Π_h . For the first term we have

$$\begin{aligned} k\delta_t(u(t_n) - u_h^n, \varphi_h) &= k\delta_t(u(t_n) - \Pi_h u(t_n), \varphi_h) + k\delta_t(\Pi_h u(t_n) - u_h^n, \varphi_h) \\ &= \int_{t_{n-1}}^{t_n} (u_t(t) - \Pi_h u_t(t), \varphi_h) dt + k\delta_t(\Pi_h u(t_n) - u_h^n, \varphi_h) \\ &= -\mathcal{E}_{u,1}^n(\varphi_h) + k\delta_t(e_u^n, \varphi_h). \end{aligned}$$

For the second term,

$$\int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla u_h^n, \nabla \varphi_h) dt = \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla \Pi_h u(t), \nabla \varphi_h) dt$$

$$\begin{aligned}
& + \int_{t_{n-1}}^{t_n} (\nabla \Pi_h(u(t) - u(t_n)), \nabla \varphi_h) \, dt + \int_{t_{n-1}}^{t_n} (\nabla \Pi_h u(t_n) - \nabla u_h^n, \nabla \varphi_h) \, dt \\
& = -\mathcal{E}_{u,a,1}^n(\varphi_h) - \mathcal{E}_{u,a,2}^n(\varphi_h) + k(\nabla e_u^n, \nabla \varphi_h).
\end{aligned}$$

For the term in the right hand side we have,

$$\begin{aligned}
& \chi \left(\int_{t_{n-1}}^{t_n} (a_m(u(t)) \nabla v(t) - a_m(u_h^n) \nabla v_h^n), \nabla \varphi_h \right) \, dt \\
& = \chi \left(\int_{t_{n-1}}^{t_n} (a_m(u(t)) [\nabla v(t) - \nabla \Pi_h v(t)], \nabla \varphi_h) \, dt \right. \\
& \quad + \int_{t_{n-1}}^{t_n} (a_m(u(t)) \nabla \Pi_h [v(t) - v(t_n)], \nabla \varphi_h) \, dt \\
& \quad + \int_{t_{n-1}}^{t_n} ([a_m(u(t)) - a_m(\Pi_h u(t))] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt \\
& \quad + \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t)) - a_m(\Pi_h u(t_n))] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt \\
& \quad + \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt \\
& \quad \left. + \int_{t_{n-1}}^{t_n} (a_m(u_h^n) [\nabla \Pi_h v(t_n) - \nabla v_h^n], \nabla \varphi_h) \, dt \right) \\
& = \mathcal{E}_{u,c,1}^n(\varphi_h) + \mathcal{E}_{u,c,2}^n(\varphi_h) + \mathcal{E}_{u,c,3}^n(\varphi_h) + \mathcal{E}_{u,c,4}^n(\varphi_h) \\
& + \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt \\
& \quad + \chi \int_{t_{n-1}}^{t_n} (a_m(u_h^n) \nabla e_v^n, \nabla \varphi_h) \, dt.
\end{aligned}$$

Now integrate v -equation of (P_m^V) from t_{n-1} to t_n and subtract k times the v_h^n -equation of (LNS) :

$$\begin{aligned}
& k\delta_t(v(t_n) - v_h^n, \phi_h) + \int_{t_{n-1}}^{t_n} (\nabla v(t) - \nabla v_h^n, \nabla \phi_h) \, dt \\
& = \int_{t_{n-1}}^{t_n} (a_m(u(t))^s v(t) - a_m(u_h^n)^s v_h^n, \phi_h) \, dt.
\end{aligned}$$

For the first term we have

$$\begin{aligned}
& k\delta_t(v(t_n) - v_h^n, \phi_h) = k\delta_t(v(t_n) - \Pi_h v(t_n), \phi_h) + k\delta_t(\Pi_h v(t_n) - v_h^n, \phi_h) \\
& = \int_{t_{n-1}}^{t_n} (v_t(t) - \Pi_h v_t(t), \phi_h) \, dt + k\delta_t(\Pi_h v(t_n) - v_h^n, \phi_h) \\
& = -\mathcal{E}_{v,1}^n(\phi_h) + k\delta_t(e_v^n, \phi_h).
\end{aligned}$$

For the second term,

$$\int_{t_{n-1}}^{t_n} (\nabla v(t) - \nabla v_h^n, \nabla \phi_h) \, dt = \int_{t_{n-1}}^{t_n} (\nabla v(t) - \nabla \Pi_h v(t), \nabla \phi_h) \, dt$$

$$\begin{aligned}
& + \int_{t_{n-1}}^{t_n} (\nabla \Pi_h(v(t) - v(t_n)), \nabla \phi_h) \, dt + \int_{t_{n-1}}^{t_n} (\nabla \Pi_h v(t_n) - \nabla v_h^n, \nabla \phi_h) \, dt \\
& = -\mathcal{E}_{v,a,1}^n(\phi_h) - \mathcal{E}_{v,a,2}^n(\phi_h) + k(\nabla e_v^n, \nabla \phi_h).
\end{aligned}$$

For the term in the right hand side we have,

$$\begin{aligned}
& - \int_{t_{n-1}}^{t_n} (a_m(u(t))^s v(t) - a_m(u_h^n)^s v_h^n, \phi_h) \, dt \\
& = - \int_{t_{n-1}}^{t_n} (a_m(u(t))^s [v(t) - \Pi_h v(t)], \phi_h) \, dt \\
& \quad - \int_{t_{n-1}}^{t_n} (a_m(u(t))^s \Pi_h [v(t) - v(t_n)], \phi_h) \, dt \\
& \quad - \int_{t_{n-1}}^{t_n} ([a_m(u(t))^s - a_m(\Pi_h u(t))^s] \Pi_h v(t_n), \phi_h) \, dt \\
& \quad - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t))^s - a_m(\Pi_h u(t_n))^s] \Pi_h v(t_n), \phi_h) \, dt \\
& \quad - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n))^s - a_m(u_h^n)^s] \Pi_h v(t_n), \phi_h) \, dt \\
& \quad - \int_{t_{n-1}}^{t_n} (a_m(u_h^n)^s [\Pi_h v(t_n) - v_h^n], \phi_h) \, dt \\
& = \mathcal{E}_{v,c,1}^n(\phi_h) + \mathcal{E}_{v,c,2}^n(\phi_h) + \mathcal{E}_{v,c,3}^n(\phi_h) + \mathcal{E}_{v,c,4}^n(\phi_h) \\
& \quad - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n))^s - a_m(u_h^n)^s] \Pi_h v(t_n), \phi_h) \, dt \\
& \quad - \int_{t_{n-1}}^{t_n} (a_m(u_h^n)^s e_v^n, \phi_h) \, dt.
\end{aligned}$$

Let

$$\mathcal{E}_u^n(\cdot) = \mathcal{E}_{u,1}^n(\cdot) + \mathcal{E}_{u,a,1}^n(\cdot) + \mathcal{E}_{u,a,2}^n(\cdot) + \mathcal{E}_{u,c,1}^n(\cdot) + \mathcal{E}_{u,c,2}^n(\cdot) + \mathcal{E}_{u,c,3}^n(\cdot) + \mathcal{E}_{u,c,4}^n(\cdot)$$

and

$$\mathcal{E}_v^n(\cdot) = \mathcal{E}_{v,1}^n(\cdot) + \mathcal{E}_{v,a,1}^n(\cdot) + \mathcal{E}_{v,a,2}^n(\cdot) + \mathcal{E}_{v,c,1}^n(\cdot) + \mathcal{E}_{v,c,2}^n(\cdot) + \mathcal{E}_{v,c,3}^n(\cdot) + \mathcal{E}_{v,c,4}^n(\cdot).$$

Then the fully discrete errors e_u^n and e_v^n satisfy the equations

$$\begin{aligned}
k\delta_t(e_u^n, \varphi_h) + k(\nabla e_u^n, \nabla \varphi_h) &= \mathcal{E}_u^n(\varphi_h) + \chi \int_{t_{n-1}}^{t_n} (a_m(u_h^n) \nabla e_v^n, \nabla \varphi_h) \, dt \\
&+ \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt,
\end{aligned} \tag{4.48}$$

$$\begin{aligned}
k\delta_t(e_v^n, \varphi_h) + k(\nabla e_v^n, \nabla \varphi_h) &+ \int_{t_{n-1}}^{t_n} (a_m(u_h^n)^s e_v^n, \phi_h) \, dt = \mathcal{E}_v^n(\phi_h) \\
&- \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n))^s - a_m(u_h^n)^s] \Pi_h v(t_n), \phi_h) \, dt.
\end{aligned} \tag{4.49}$$

Then let us estimate $\mathcal{E}_u^n(\varphi_h)$ and $\mathcal{E}_v^n(\phi_h)$. We begin by $\mathcal{E}_{u,1}^n(\varphi_h)$. We have

$$\begin{aligned}\mathcal{E}_{u,1}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (\Pi_h u_t(t) - u_t(t), \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|\Pi_h u_t(t) - u_t(t)\|_{L^2(\Omega)} \|\varphi_h\|_{L^2(\Omega)} \, dt \\ &\leq Ch \int_{t_{n-1}}^{t_n} \|u_t(t)\|_{H^1(\Omega)} \, dt \|\varphi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} \|u_t\|_{L^2(I_n; H^1(\Omega))} \|\varphi_h\|_{L^2(\Omega)}.\end{aligned}$$

Now we estimate $\mathcal{E}_{u,a,1}^n(\varphi_h)$ and $\mathcal{E}_{u,a,2}^n(\varphi_h)$. We have

$$\begin{aligned}\mathcal{E}_{u,a,1}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla \Pi_h u(t), \nabla \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|\nabla \Pi_h u(t) - \nabla u(t)\|_{L^2(\Omega)} \, dt \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} \|u\|_{L^2(I_n; H^2(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_{u,a,2}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (\nabla \Pi_h(u(t) - u(t_n)), \nabla \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|\nabla \Pi_h(u(t) - u(t_n))\|_{L^2(\Omega)} \, dt \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq \int_{t_{n-1}}^{t_n} \int_t^{t_n} \|u_t(s)\|_{H^1(\Omega)} \, ds \, dt \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Ck^{3/2} \|u_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)}.\end{aligned}$$

Estimating $\mathcal{E}_{u,c,1}^n(\varphi_h)$, $\mathcal{E}_{u,c,2}^n(\varphi_h)$, $\mathcal{E}_{u,c,3}^n(\varphi_h)$ and $\mathcal{E}_{u,c,4}^n(\varphi_h)$ we obtain

$$\begin{aligned}\mathcal{E}_{u,c,1}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))[\nabla v(t) - \nabla \Pi_h v(t)], \nabla \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|a_m(u(t))\|_{L^\infty(\Omega)} \|\nabla v(t) - \nabla \Pi_h v(t)\|_{L^2(\Omega)} \, dt \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} m \|v\|_{L^2(I_n; H^2(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{u,c,2}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))\nabla \Pi_h[v(t) - v(t_n)], \nabla \varphi_h) \, dt \\ &\leq Ck^{3/2} \|a_m(u(t))\|_{L^\infty(I_n; L^\infty(\Omega))} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Cmk^{3/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{u,c,3}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} ([a_m(u(t)) - a_m(\Pi_h u(t))] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt \\
&\leq Ck^{1/2} \|a_m(u(t)) - a_m(\Pi_h u(t))\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \|\nabla \varphi_h\|_{L^2(\Omega)} \\
&\leq Chk^{1/2} \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \|\nabla \varphi_h\|_{L^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{u,c,4}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t)) - a_m(\Pi_h u(t_n))] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt \\
&\leq Ck^{3/2} \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \|\nabla \varphi_h\|_{L^2(\Omega)} .
\end{aligned}$$

Similarly, we have

$$\mathcal{E}_{v,1}^n(\phi_h) \leq Chk^{1/2} \|v_t(t)\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} ,$$

$$\mathcal{E}_{v,a,1}^n(\phi_h) \leq Chk^{1/2} \|v\|_{L^2(I_n; H^2(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)}$$

and

$$\mathcal{E}_{v,a,2}^n(\phi_h) \leq Ck^{3/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)} .$$

Estimating $\mathcal{E}_{v,c,1}^n(\phi_h)$, $\mathcal{E}_{v,c,2}^n(\phi_h)$, $\mathcal{E}_{v,c,3}^n(\phi_h)$ and $\mathcal{E}_{v,c,4}^n(\phi_h)$ we obtain

$$\begin{aligned}
\mathcal{E}_{v,c,1}^n(\phi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))^s [v(t) - \Pi_h v(t)], \phi_h) \, dt \\
&\leq \int_{t_{n-1}}^{t_n} \|a_m(u(t))^s\|_{L^\infty(\Omega)} \|v(t) - \Pi_h v(t)\|_{L^2(\Omega)} \, dt \|\phi_h\|_{L^2(\Omega)} \\
&\leq Chk^{1/2} \|a_m(u)^s\|_{L^\infty(I_n; L^\infty(\Omega))} \|v\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} \\
&\leq Cm^s h k^{1/2} \|v\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} ,
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{v,c,2}^n(\phi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))^s \Pi_h [v(t) - v(t_n)], \phi_h) \, dt \\
&\leq Ck^{3/2} \|a_m(u)^s\|_{L^\infty(I_n; L^\infty(\Omega))} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} \\
&\leq Cm^s k^{3/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} ,
\end{aligned}$$

$$\mathcal{E}_{v,c,3}^n(\phi_h) = \int_{t_{n-1}}^{t_n} ([a_m(u(t))^s - a_m(\Pi_h u(t))^s] \Pi_h v(t_n), \phi_h) \, dt$$

$$\begin{aligned}
&\leq Ck^{1/2} \left\| [a_m(u) + a_m(\Pi_h u)]^{s-1} \right\|_{L^\infty(I_n; L^\infty(\Omega))} \times \\
&\quad \times \|a_m(u) - a_m(\Pi_h u)\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \|\phi_h\|_{L^2(\Omega)} \\
&\leq Cm^{s-1} h k^{1/2} \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \|\phi_h\|_{L^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{v,c,4}^n(\phi_h) &= \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t))^s - a_m(\Pi_h u(t_n))^s] \Pi_h v(t_n), \phi_h) \, dt \\
&\leq Cm^{s-1} k^{3/2} \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \|\phi_h\|_{L^2(\Omega)}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{E}_u^n(\varphi_h) &\leq Chk^{1/2} \left(\|u_t(t)\|_{L^2(I_n; L^2(\Omega))} \|\varphi_h\|_{L^2(\Omega)} + \left[\|u\|_{L^2(I_n; H^2(\Omega))} \right. \right. \\
&\quad \left. \left. + m \|v\|_{L^2(I_n; H^2(\Omega))} \right. \right. \\
&\quad \left. \left. + \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \right] \|\nabla \varphi_h\|_{L^2(\Omega)} \right) \\
&\quad + Ck^{3/2} \left(\|u_t\|_{L^2(I_n; H^1(\Omega))} + m \|v_t\|_{L^2(I_n; H^1(\Omega))} \right. \\
&\quad \left. + \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \right) \|\nabla \varphi_h\|_{L^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_v^n(\phi_h) &\leq Chk^{1/2} \left(\|v_t(t)\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} + \|v\|_{L^2(I_n; H^2(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)} \right. \\
&\quad \left. + \left[m^s \|v\|_{L^2(I_n; H^1(\Omega))} \right. \right. \\
&\quad \left. \left. + \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \right] \|\phi_h\|_{L^2(\Omega)} \right) \\
&\quad + Ck^{3/2} \left(\|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)} + \left[m^s \|v_t\|_{L^2(I_n; H^1(\Omega))} \right. \right. \\
&\quad \left. \left. + \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \right] \|\phi_h\|_{L^2(\Omega)} \right).
\end{aligned}$$

Using Young's inequality we have

$$\begin{aligned}
\mathcal{E}_u^n(\varphi_h) &\leq \tilde{C}(\delta) h^2 \left[\|u\|_{L^2(I_n; H^2(\Omega))}^2 + m^2 \|v\|_{L^2(I_n; H^2(\Omega))}^2 \right. \\
&\quad \left. + \|u\|_{L^2(I_n; W^{1,3}(\Omega))}^2 \|v(t_n)\|_{W^{1,6}(\Omega)}^2 \right] + \frac{\delta}{2} k \|\nabla \varphi_h\|_{L^2(\Omega)}^2 \\
&\quad + \tilde{C}(\delta) h^2 \|v_t(t)\|_{L^2(I_n; L^2(\Omega))}^2 + \frac{\delta}{2} k \|\varphi_h\|_{L^2(\Omega)}^2 \\
&\quad + \tilde{C}(\delta) k^2 \left[\|u_t\|_{L^2(I_n; H^1(\Omega))}^2 + m^2 \|v_t\|_{L^2(I_n; H^1(\Omega))}^2 \right. \\
&\quad \left. + \|u_t\|_{L^2(I_n; L^3(\Omega))}^2 \|v(t_n)\|_{W^{1,6}(\Omega)}^2 \right] + \delta k \|\nabla \varphi_h\|_{L^2(\Omega)}^2,
\end{aligned}$$

and we can write

$$\mathcal{E}_u^n(\varphi_h) \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|\varphi_h\|_{L^2(\Omega)}^2 + 2\delta k \|\nabla \varphi_h\|_{L^2(\Omega)}^2, \quad (4.50)$$

where $\tilde{C}(I_n, \delta, m, u, v) > 0$ is a constant which depends on the norms of u, v, u_t and v_t in the intervals I_n . Note that there is a constant $C(\delta, u, v, m) > 0$ such that

$$\sum_{n=0}^N \tilde{C}(I_n, \delta, m, u, v) \leq C(\delta, u, v, m). \quad (4.51)$$

Analogously, we can obtain

$$\mathcal{E}_v^n(\phi_h) \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|\phi\|_{L^2(\Omega)}^2 + \delta k \|\nabla \phi_h\|_{L^2(\Omega)}^2. \quad (4.52)$$

Then, if we use e_u^n as a test function in (4.48), e_v^n as a test function in (4.49) and the estimates (4.50) and (4.52) we obtain

$$\begin{aligned} & \|e_u^n\|_{L^2(\Omega)}^2 - \|e_u^{n-1}\|_{L^2(\Omega)}^2 + \|e_u^n - e_u^{n-1}\|_{L^2(\Omega)}^2 \\ & + k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) \\ & + \delta k \|e_u^n\|_{L^2(\Omega)}^2 + 2\delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 + \chi \int_{t_{n-1}}^{t_n} (a_m(u_h^n) \nabla e_v^n, \nabla e_u^n) dt \\ & + \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla e_u^n) dt, \end{aligned} \quad (4.53)$$

and

$$\begin{aligned} & \|e_v^n\|_{L^2(\Omega)}^2 - \|e_v^{n-1}\|_{L^2(\Omega)}^2 + \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\ & + k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + k \|a_m(u_h^{n-1})^{s/2} e_v^n\|_{L^2(\Omega)}^2 \\ & \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|e_v^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_v^n\|_{L^2(\Omega)}^2 \\ & - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n))^s - a_m(u_h^n)^s] \Pi_h v(t_n), e_v^n) dt. \end{aligned} \quad (4.54)$$

We still have to estimate some terms of (4.53) and (4.54):

$$\chi \int_{t_{n-1}}^{t_n} (a_m(u_h^n) \nabla e_v^n, \nabla e_u^n) dt \leq m^2 \chi^2 C(\delta) k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2, \quad (4.55)$$

$$\begin{aligned} & \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla e_u^n) dt \\ & \leq k \chi \|e_u^n\|_{L^3(\Omega)} \|\nabla \Pi_h v(t_n)\|_{L^6(\Omega)} \|\nabla e_u^n\|_{L^2(\Omega)} \\ & \leq k \chi C_1 \|e_u^n\|_{L^2(\Omega)}^{1/2} \|e_u^n\|_{H^1(\Omega)}^{1/2} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla e_u^n\|_{L^2(\Omega)} \\ & \leq k \chi C_2 \|e_u^n\|_{L^2(\Omega)}^{1/2} \left(\|e_u^n\|_{L^2(\Omega)}^{1/2} + \|\nabla e_u^n\|_{L^2(\Omega)}^{1/2} \right) \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla e_u^n\|_{L^2(\Omega)} \\ & = k \chi C_2 \|e_u^n\|_{L^2(\Omega)} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla e_u^n\|_{L^2(\Omega)} \end{aligned}$$

$$+k \chi C_2 \|e_u^n\|_{L^2(\Omega)}^{1/2} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla e_u^n\|_{L^2(\Omega)}^{3/2}.$$

Using Young's inequality we can conclude that

$$\begin{aligned} & \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla e_u^n) dt \\ & \leq \chi^2 C k \left(\|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 + \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^4 \right) \|e_u^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\ & = \chi^2 C(v) k \|e_u^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2, \end{aligned} \quad (4.56)$$

where $C(v) = C(\|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 + \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^4)$. The last term that we have to estimate in (4.54) is

$$\begin{aligned} & \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n))^s - a_m(u_h^n)^s] \Pi_h v(t_n), e_v^n) dt \\ & \leq s m^{s-1} k \|e_u^n\|_{L^2(\Omega)} \|\Pi_h(v(t_n))\|_{L^\infty(\Omega)} \|e_v^n\|_{L^2(\Omega)} \\ & \leq C(\delta, s) m^{2(s-1)} k \|v(t_n)\|_{L^\infty(\Omega)}^2 \|e_u^n\|_{L^2(\Omega)}^2 + k \delta \|e_v^n\|_{L^2(\Omega)}^2 \\ & \leq C(\delta, s) m^{2(s-1)} k \|v^0\|_{L^\infty(\Omega)}^2 \|e_u^n\|_{L^2(\Omega)}^2 + k \delta \|e_v^n\|_{L^2(\Omega)}^2. \end{aligned}$$

Using it in (4.53) and (4.54), we have

$$\begin{aligned} & \|e_u^n\|_{L^2(\Omega)}^2 - \|e_u^{n-1}\|_{L^2(\Omega)}^2 + \|e_u^n - e_u^{n-1}\|_{L^2(\Omega)}^2 + k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\ & \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|e_u^n\|_{L^2(\Omega)}^2 + 4\delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\ & \quad + m^2 C(\delta, \chi) k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + \chi^2 C(v) k \|e_u^n\|_{L^2(\Omega)}^2, \end{aligned} \quad (4.57)$$

and

$$\begin{aligned} & \|e_v^n\|_{L^2(\Omega)}^2 - \|e_v^{n-1}\|_{L^2(\Omega)}^2 + \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\ & \quad + k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + k \|a_m(u_h^{n-1})^{s/2} e_v^n\|_{L^2(\Omega)}^2 \\ & \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + 2\delta k \|e_v^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_v^n\|_{L^2(\Omega)}^2 \\ & \quad + C(\delta, s) m^{2(s-1)} \|v^0\|_{L^\infty(\Omega)}^2 \|e_u^n\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.58)$$

Let $C_0 = 2m^2 C(\delta, \chi)$ and note that

$$\begin{aligned} \|e_v^n\|_{L^2(\Omega)}^2 &= \|e_v^n - e_v^{n-1} + e_v^{n-1}\|_{L^2(\Omega)}^2 \leq \left[\|e_v^n - e_v^{n-1}\|_{L^2(\Omega)} + \|e_v^{n-1}\|_{L^2(\Omega)} \right]^2 \\ &\leq 2 \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 + 2 \|e_v^{n-1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Then we can obtain

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 - \|e_u^{n-1}\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 - C_0 \|e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + \left(1 - (2\delta + \chi^2 C(v) + C(\delta, s)m^{2(s-1)} \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^2)k\right) \|e_u^n - e_u^{n-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 2\delta k)C_0 \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 4\delta)k \|\nabla e_u^n\|_{L^2(\Omega)}^2 + k\left(\frac{C_0}{2} - \delta C_0\right) \|\nabla e_v^n\|_{L^2(\Omega)}^2 \\
& \leq (1 + C_0)\tilde{C}(I_n, \delta)(h^2 + k^2) + 2\delta k C_0 \|e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + (2\delta + \chi^2 C(v) + C(\delta, s)m^{2(s-1)} \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^2)k \|e_u^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.59}$$

If we sum (4.59) from 1 to n , we find

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 + \left(1 - (2\delta + \chi^2 C(v) \right. \\
& \left. + C(\delta, s)m^{2(s-1)} \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^2)k\right) \sum_{j=1}^n \|e_u^j - e_u^{j-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 2\delta k)C_0 \sum_{j=1}^n \|e_v^j - e_v^{j-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 4\delta)k \sum_{j=1}^n \|\nabla e_u^j\|_{L^2(\Omega)}^2 + k\left(\frac{C_0}{2} - \delta C_0\right) \sum_{j=1}^n \|\nabla e_v^j\|_{L^2(\Omega)}^2 \\
& \leq (1 + C_0)C(\delta, u, v, m)(h^2 + k^2) + \sum_{j=0}^{n-1} 2\delta k C_0 \|e_v^j\|_{L^2(\Omega)}^2 \\
& + \sum_{j=0}^{n-1} (2\delta + \chi^2 C(v) + C(\delta, s)m^{2(s-1)} \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^2)k \|e_u^j\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.60}$$

Now, we realize that, in order to keep the non-negative terms in the left hand side of (4.60), we must choose δ , k and h such that

$$1 - 4\delta > 0, \quad 1 - 2\delta k > 0$$

and

$$1 - (2\delta + \chi^2 C(v) + C(\delta, s)m^{2(s-1)} \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^2)k > 0.$$

It suggests that we must choose $\delta < 1/4$, let it be $\delta = 1/5$. Then we must choose h and k such that

$$1 - \left(\frac{2}{5} + \chi^2 C(v) + C(\delta, s)m^{2(s-1)} \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^2\right)k > 0.$$

After that, we can apply the discrete Gronwall's inequality (Lemma 25) in order to obtain (we recall that $u_h^0 = \Pi_h u^0$ and $v_h^0 = \Pi_h v^0$)

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 \\
& + C \sum_{i=1}^n \|e_u^i - e_u^{i-1}\|_{L^2(\Omega)}^2 + CC_0 \sum_{i=1}^n \|e_v^i - e_v^{i-1}\|_{L^2(\Omega)}^2 \\
& + C \sum_{i=1}^n k \|\nabla e_u^i\|_{L^2(\Omega)}^2 + CC_0 \sum_{i=1}^n k \|\nabla e_v^i\|_{L^2(\Omega)}^2 \\
& \leq \tilde{C}(u, v, \chi, m, T)(h^2 + k^2),
\end{aligned} \tag{4.61}$$

■

4.2.3 Semi-linear numerical scheme

Consider the spaces $U_h = V_h = \mathbb{P}_1(\mathcal{T}_h(\Omega))$. Let $u_h^0 = \Pi_h u^0$ and $v_h^0 = \Pi_h v^0$. The fully discrete scheme that we propose in this subsection is:

Knowing $(u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h$, find $(u_h^n, v_h^n) \in U_h \times V_h$ such that

$$(\delta_t u_h^n, \varphi_h) + (\nabla u_h^n, \nabla \varphi_h) = \chi (a_m(u_h^n) \nabla v_h^n, \nabla \varphi_h), \quad \forall \varphi_h \in U_h, \tag{SLNS}$$

$$(\delta_t v_h^n, \phi_h) + (\nabla v_h^n, \nabla \phi_h) = - (a_m(u_h^{n-1}) s v_h^n, \phi_h), \quad \forall \phi_h \in V_h.$$

Now that we defined the semi-linear numerical scheme (NLNS), we study the existence and uniqueness of the discrete solutions.

Theorem 84 (Unconditional existence). *Given $u_h^{n-1} \in U_h$ and $v_h^{n-1} \in V_h$, there exists a solution of (SLNS).*

Proof. Given u_h^{n-1} and v_h^{n-1} we obtain a unique v_h^n which satisfies the v_h^n -equation of (SLNS). Then we must prove that, given u_h^{n-1} and v_h^n , there is a u_h^n which satisfies the u_h^n -equation of (SLNS). In order to do it, we will use Lemma 24. We choose $X_h = U_h$, the inner product $[\cdot, \cdot]$ is the inner product of $H^1(\Omega)$ and $P_h : X_h \rightarrow X_h$ is given by the relation

$$\begin{aligned}
[P_h(u_h), \varphi_h] &= (u_h, \varphi_h) + k (\nabla u_h, \nabla \varphi_h) \\
&\quad - k \chi (a_m(u_h) \nabla v_h^n, \nabla \varphi_h) - (u_h^{n-1}, \varphi_h),
\end{aligned}$$

for all $u_h, \varphi_h \in X_h$. The mapping P_h is continuous and

$$\begin{aligned}
[P_h(u_h), u_h] &\geq \|u_h\|_{L^2(\Omega)}^2 + k \|\nabla u_h\|_{L^2(\Omega)}^2 - km \chi \|\nabla v_h^n\|_{L^2(\Omega)} \|\nabla u_h\|_{L^2(\Omega)} \\
&\quad - \|u_h^{n-1}\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2} \|u_h\|_{L^2(\Omega)}^2 + \frac{k}{2} \|\nabla u_h\|_{L^2(\Omega)}^2 - \frac{k(m\chi)^2}{2} \|\nabla v_h^n\|_{L^2(\Omega)}^2 - \frac{1}{2} \|u_h^{n-1}\|_{L^2(\Omega)}^2 \\
&\geq C(k) \|u_h\|_{H^1(\Omega)}^2 - \frac{k(m\chi)^2}{2} \|\nabla v_h^n\|_{L^2(\Omega)}^2 - \frac{1}{2} \|u_h^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Then, for $\beta > 0$ large enough and $u_h \in X_h$ such that $\|u_h\|_{H^1(\Omega)} = \beta$ we have

$$[P_h(u_h), u_h] > 0.$$

Therefore, by Lemma 24, we conclude that there exists $u_h \in X_h$ which satisfies the u_h^n -equation of (SLNS). ■

Lemma 85. *For any solution (u_h^n, v_h^n) of (SLNS), the properties given by Lemma 73 and the stability results of Lemma 74 hold.*

Proof. The proofs are analogous to the proofs of Lemmas 73 and 74. ■

Theorem 86. *Let Ω be a polyhedron and let (u, v) be the weak solution of the regularized problem (P_m) , that is, a solution of (P_m^V) . For each $n \in \{1, \dots, N\}$ let the pair (u_h^n, v_h^n) be a solution given by the existence result for the non-linear fully discrete scheme (SLNS), Theorem 84. Considering the aforementioned pairs (u_h^n, v_h^n) , let $u_{hk}^r, u_{hk}^l, u_{hk}, v_{hk}^r, v_{hk}^l$ and v_{hk} be the respective associated functions, defined as in (4.13), (4.14) and (4.15). Then, as $(h, k) \rightarrow (0, 0)$, the sequences $\{u_{hk}^r\}_{(h,k)}$, $\{u_{hk}^l\}_{(h,k)}$ and $\{u_{hk}\}_{(h,k)}$ converge to the function u and the sequences $\{v_{hk}^r\}_{(h,k)}$, $\{v_{hk}^l\}_{(h,k)}$ and $\{v_{hk}\}_{(h,k)}$ converge to the function v weakly* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$.*

Proof. The demonstration of this result is analogous to the demonstration of the Theorems 76 and 86. ■

Once proved the convergence of the fully discrete scheme (SLNS) towards the regularized problem, we address the questions on the uniqueness of solution and error estimates.

Theorem 87 (Conditional uniqueness). *Given (u_h^{n-1}, v_h^{n-1}) , let (u_h^n, v_h^n) be a solution of (SLNS). The function v_h^n is uniquely determined independently of the values of the discretization parameters h and k . Besides, if we choose h and k such that*

$$\frac{k}{h^{(d+2)/2}} < \frac{C}{\chi^2 \|v^0\|_{L^2(\Omega)}^2},$$

then the function u_h^n is also uniquely determined and therefore (u_h^n, v_h^n) is the unique solution of (SLNS).

Proof. Suppose that there are two pair of solution of $(SLNS)$, $(u_{h,1}, v_{h,1})$ and $(u_{h,2}, v_{h,2})$. We remind that, since $(u_{h,1}, v_{h,1}), (u_{h,2}, v_{h,2}) \in X_h$ then $u_{h,1}, v_{h,1}, u_{h,2}, v_{h,2} \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$. Let $w_h = u_{h,1} - u_{h,2}$ and $z_h = v_{h,1} - v_{h,2}$, if $(u_{h,1}, v_{h,1})$ and $(u_{h,2}, v_{h,2})$ are solutions of $(SLNS)$ then we have

$$(z_h, \phi_h) + k(\nabla z_h, \nabla \phi_h) + k(a_m(u^{n-1_h})^s z_h, \phi_h) = 0, \quad \forall \phi_h \in V_h.$$

Using $\phi = z_h$ as a test function in this equation we can conclude that $z_h = 0$, that is, $v_{h,1} = v_{h,2}$. Let us write $v_{h,1} = v_{h,2} = v_h^n$.

Then w_h satisfies the following equation:

$$(w_h, \varphi_h) + k(\nabla w_h, \nabla \varphi_h) = k\chi((a_m(u_{h,1}) - a_m(u_{h,2}))\nabla v_h^n, \nabla \varphi_h)$$

for all $\varphi_h \in U_h$. If we take $\varphi = w_h$ and use Holder's and Young's inequalities we can obtain

$$(1 - \frac{k\chi^2}{4} \|\nabla v_h^n\|_{L^\infty(\Omega)}^2) \|w_h\|_{L^2(\Omega)}^2 \leq 0$$

Then, using the inverse inequality (Lemma 31-1)

$$\|\nabla v_h^n\|_{L^\infty(\Omega)}^2 \leq \frac{\hat{C}}{h^{d+2}} \|v_h^n\|_{L^2(\Omega)}^2$$

and the estimate of $\|v_h^n\|_{L^2(\Omega)}^2$ given by Lemma 80, we conclude that there is a constant $C > 0$ such that

$$(1 - \frac{k\chi^2 C \|v^0\|_{L^2(\Omega)}^2}{4h^{(d+2)/2}}) \|w_h\|_{L^2(\Omega)}^2 \leq 0$$

Therefore, if we choose h and k such that

$$\frac{k\chi^2 C \|v^0\|_{L^2(\Omega)}^2}{4h^{(d+2)/2}} < 1,$$

we can conclude that $w_h = 0$. ■

Once proved the convergence of the fully discrete scheme $(SLNS)$ towards the regularized problem, we turn to the study of the error estimates.

4.2.3.1 Error estimates for the fully discrete semi-linear numerical scheme

Now we study the error estimates. We want to estimate the fully discrete errors

$$e_u^n = u_h^n - \Pi_h u(t_n) \quad \text{and} \quad e_v^n = v_h^n - \Pi_h v(t_n).$$

We remind that, for simplicity, when needed, we will denote the interval (t_{n-1}, t_n) by I_n .

Theorem 88. Suppose that the solution (u, v) of the regularized problem (P_m^V) has the regularity

$$\begin{aligned} u, v &\in L^2(0, T; H^2(\Omega)), \\ v &\in L^\infty(0, T; W^{1,6}(\Omega)), \\ u_t, v_t &\in L^2(0, T; H^1(\Omega)). \end{aligned}$$

If we choose k such that

$$k < \frac{1}{\left(\frac{2}{5} + \chi^2 C(v)\right)},$$

where $C(v) = C(\|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^2 + \|v\|_{L^\infty(0,T;W^{1,6}(\Omega))}^4)$. Then there exists a positive constant $C(u, v, \chi, m, T)$ such that

$$\|e_u^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla e_u^i\|_{L^2(\Omega)}^2 \leq C(u, v, \chi, m, T)(h^2 + k^2)$$

and

$$\|e_v^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla e_v^i\|_{L^2(\Omega)}^2 \leq C(u, v, \chi, m, T)(h^2 + k^2).$$

Proof. Integrate the u -equation of (P_m^V) from t_{n-1} to t_n and subtract k times the u_h^n -equation of $(SLNS)$:

$$\begin{aligned} k\delta_t(u(t_n) - u_h^n, \varphi_h) &+ \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla u_h^n, \nabla \varphi_h) dt \\ &+ \chi \int_{t_{n-1}}^{t_n} (a_m(u(t))\nabla v(t) - a_m(u_h^n)\nabla v_h^n, \nabla \varphi_h) dt. \end{aligned}$$

Sum and subtract the convenient terms involving the operator Π_h . For the first term we have

$$\begin{aligned} k\delta_t(u(t_n) - u_h^n, \varphi_h) &= k\delta_t(u(t_n) - \Pi_h u(t_n), \varphi_h) + k\delta_t(\Pi_h u(t_n) - u_h^n, \varphi_h) \\ &= \int_{t_{n-1}}^{t_n} (u_t(t) - \Pi_h u_t(t), \varphi_h) dt + k\delta_t(\Pi_h u(t_n) - u_h^n, \varphi_h) \\ &= -\mathcal{E}_{u,1}^n(\varphi_h) + k\delta_t(e_u^n, \varphi_h). \end{aligned}$$

For the second term,

$$\begin{aligned} \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla u_h^n, \nabla \varphi_h) dt &= \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla \Pi_h u(t), \nabla \varphi_h) dt \\ &+ \int_{t_{n-1}}^{t_n} (\nabla \Pi_h u(t) - \nabla u_h^n, \nabla \varphi_h) dt \\ &= -\mathcal{E}_{u,a,1}^n(\varphi_h) - \mathcal{E}_{u,a,2}^n(\varphi_h) + k(\nabla e_u^n, \nabla \varphi_h). \end{aligned}$$

For the term in the right hand side we have,

$$\begin{aligned}
& \chi \left(\int_{t_{n-1}}^{t_n} (a_m(u(t)) \nabla v(t) - a_m(u_h^n) \nabla v_h^n), \nabla \varphi_h \right) dt \\
&= \chi \left(\int_{t_{n-1}}^{t_n} (a_m(u(t)) [\nabla v(t) - \nabla \Pi_h v(t)], \nabla \varphi_h) dt \right. \\
&\quad \left. + \int_{t_{n-1}}^{t_n} (a_m(u(t)) \nabla \Pi_h [v(t) - v(t_n)], \nabla \varphi_h) dt \right. \\
&\quad \left. + \int_{t_{n-1}}^{t_n} ([a_m(u(t)) - a_m(\Pi_h u(t))] \nabla \Pi_h v(t_n), \nabla \varphi_h) dt \right. \\
&\quad \left. + \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t)) - a_m(\Pi_h u(t_n))] \nabla \Pi_h v(t_n), \nabla \varphi_h) dt \right. \\
&\quad \left. + \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla \varphi_h) dt \right. \\
&\quad \left. + \int_{t_{n-1}}^{t_n} (a_m(u_h^n) [\nabla \Pi_h v(t_n) - \nabla v_h^n], \nabla \varphi_h) dt \right) \\
&= \mathcal{E}_{u,c,1}^n(\varphi_h) + \mathcal{E}_{u,c,2}^n(\varphi_h) + \mathcal{E}_{u,c,3}^n(\varphi_h) + \mathcal{E}_{u,c,4}^n(\varphi_h) \\
&+ \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla \varphi_h) dt \\
&\quad + \chi \int_{t_{n-1}}^{t_n} (a_m(u_h^n) \nabla e_v^n, \nabla \varphi_h) dt.
\end{aligned}$$

Now we turn to the v -equation of (P_m^V) . Analogously to what we have done with the u -equation, we integrate the v -equation of (P_m^V) from t_{n-1} to t_n and subtract k times the v_h^n -equation of $(SLNS)$:

$$\begin{aligned}
& k \delta_t (v(t_n) - v_h^n, \phi_h) + \int_{t_{n-1}}^{t_n} (\nabla v(t) - \nabla v_h^n, \nabla \phi_h) dt \\
&= \int_{t_{n-1}}^{t_n} (a_m(u(t))^s v(t) - a_m(u_h^{n-1})^s v_h^n, \phi_h) dt.
\end{aligned}$$

For the first term we have

$$\begin{aligned}
& k \delta_t (v(t_n) - v_h^n, \phi_h) = k \delta_t (v(t_n) - \Pi_h v(t_n), \phi_h) + k \delta_t (\Pi_h v(t_n) - v_h^n, \phi_h) \\
&= \int_{t_{n-1}}^{t_n} (v_t(t) - \Pi_h v_t(t), \phi_h) dt + k \delta_t (\Pi_h v(t_n) - v_h^n, \phi_h) \\
&= -\mathcal{E}_{v,1}^n(\phi_h) + k \delta_t (e_v^n, \phi_h).
\end{aligned}$$

For the second term,

$$\begin{aligned}
& \int_{t_{n-1}}^{t_n} (\nabla v(t) - \nabla v_h^n, \nabla \phi_h) dt = \int_{t_{n-1}}^{t_n} (\nabla v(t) - \nabla \Pi_h v(t), \nabla \phi_h) dt \\
&+ \int_{t_{n-1}}^{t_n} (\nabla \Pi_h (v(t) - v(t_n)), \nabla \phi_h) dt + \int_{t_{n-1}}^{t_n} (\nabla \Pi_h v(t_n) - \nabla v_h^n, \nabla \phi_h) dt
\end{aligned}$$

$$= -\mathcal{E}_{v,a,1}^n(\phi_h) - \mathcal{E}_{v,a,2}^n(\phi_h) + k(\nabla e_v^n, \nabla \phi_h).$$

For the term in the right hand side we have,

$$\begin{aligned} & - \int_{t_{n-1}}^{t_n} (a_m(u(t))^s v(t) - a_m(u_h^{n-1})^s v_h^n], \phi_h) dt \\ &= - \int_{t_{n-1}}^{t_n} (a_m(u(t))^s [v(t) - \Pi_h v(t)], \phi_h) dt \\ & \quad - \int_{t_{n-1}}^{t_n} (a_m(u(t))^s \Pi_h [v(t) - v(t_n)], \phi_h) dt \\ & \quad - \int_{t_{n-1}}^{t_n} ([a_m(u(t))^s - a_m(\Pi_h u(t))^s] \Pi_h v(t_n), \phi_h) dt \\ & \quad - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t))^s - a_m(\Pi_h u(t_{n-1}))^s] \Pi_h v(t_n), \phi_h) dt \\ & \quad - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1}))^s - a_m(u_h^{n-1})^s] \Pi_h v(t_n), \phi_h) dt \\ & \quad - \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1})^s [\Pi_h v(t_n) - v_h^n], \phi_h) dt \\ &= \mathcal{E}_{v,c,1}^n(\phi_h) + \mathcal{E}_{v,c,2}^n(\phi_h) + \mathcal{E}_{v,c,3}^n(\phi_h) + \mathcal{E}_{v,c,4}^n(\phi_h) \\ & \quad - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1}))^s - a_m(u_h^{n-1})^s] \Pi_h v(t_n), \phi_h) dt \\ & \quad - \int_{t_{n-1}}^{t_n} (a_m(u_h^{n-1})^s e_v^n, \phi_h) dt. \end{aligned}$$

Let

$$\mathcal{E}_u^n(\cdot) = \mathcal{E}_{u,1}^n(\cdot) + \mathcal{E}_{u,a,1}^n(\cdot) + \mathcal{E}_{u,a,2}^n(\cdot) + \mathcal{E}_{u,c,1}^n(\cdot) + \mathcal{E}_{u,c,2}^n(\cdot) + \mathcal{E}_{u,c,3}^n(\cdot) + \mathcal{E}_{u,c,4}^n(\cdot)$$

and

$$\mathcal{E}_v^n(\cdot) = \mathcal{E}_{v,1}^n(\cdot) + \mathcal{E}_{v,a,1}^n(\cdot) + \mathcal{E}_{v,a,2}^n(\cdot) + \mathcal{E}_{v,c,1}^n(\cdot) + \mathcal{E}_{v,c,2}^n(\cdot) + \mathcal{E}_{v,c,3}^n(\cdot) + \mathcal{E}_{v,c,4}^n(\cdot).$$

Then the fully discrete errors e_u^n and e_v^n satisfy the equations

$$\begin{aligned} k\delta_t(e_u^n, \varphi_h) + k(\nabla e_u^n, \nabla \varphi_h) &= \mathcal{E}_u^n(\varphi_h) + \chi \int_{t_{n-1}}^{t_n} (a_m(u_h^n) \nabla e_v^n, \nabla \varphi_h) dt \\ & \quad + \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla \varphi_h) dt, \end{aligned} \quad (4.62)$$

$$\begin{aligned} k\delta_t(e_v^n, \varphi_h) + k(\nabla e_v^n, \nabla \varphi_h) &+ \int_{t_{n-1}}^{t_n} (a_m(u_h^n)^s e_v^n, \phi_h) dt = \mathcal{E}_v^n(\phi_h) \\ & - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n))^s - a_m(u_h^n)^s] \Pi_h v(t_n), \phi_h) dt. \end{aligned} \quad (4.63)$$

Then let us estimate $\mathcal{E}_u^n(\varphi_h)$ and $\mathcal{E}_v^n(\phi_h)$. We begin by $\mathcal{E}_{u,1}^n(\varphi_h)$. We have

$$\begin{aligned}\mathcal{E}_{u,1}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (\Pi_h u_t(t) - u_t(t), \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|\Pi_h u_t(t) - u_t(t)\|_{L^2(\Omega)} \|\varphi_h\|_{L^2(\Omega)} \, dt \\ &\leq Ch \int_{t_{n-1}}^{t_n} \|u_t(t)\|_{H^1(\Omega)} \, dt \|\varphi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} \|u_t\|_{L^2(I_n; H^1(\Omega))} \|\varphi_h\|_{L^2(\Omega)}.\end{aligned}$$

Now we estimate $\mathcal{E}_{u,a,1}^n(\varphi_h)$ and $\mathcal{E}_{u,a,2}^n(\varphi_h)$. We have

$$\begin{aligned}\mathcal{E}_{u,a,1}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (\nabla u(t) - \nabla \Pi_h u(t), \nabla \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|\nabla \Pi_h u(t) - \nabla u(t)\|_{L^2(\Omega)} \, dt \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} \|u\|_{L^2(I_n; H^2(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_{u,a,2}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (\nabla \Pi_h(u(t) - u(t_n)), \nabla \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|\nabla \Pi_h(u(t) - u(t_n))\|_{L^2(\Omega)} \, dt \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq \int_{t_{n-1}}^{t_n} \int_t^{t_n} \|u_t(s)\|_{H^1(\Omega)} \, ds \, dt \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Ck^{3/2} \|u_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)}.\end{aligned}$$

Estimating $\mathcal{E}_{u,c,1}^n(\varphi_h)$, $\mathcal{E}_{u,c,2}^n(\varphi_h)$, $\mathcal{E}_{u,c,3}^n(\varphi_h)$ and $\mathcal{E}_{u,c,4}^n(\varphi_h)$ we obtain

$$\begin{aligned}\mathcal{E}_{u,c,1}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))[\nabla v(t) - \nabla \Pi_h v(t)], \nabla \varphi_h) \, dt \\ &\leq \int_{t_{n-1}}^{t_n} \|a_m(u(t))\|_{L^\infty(\Omega)} \|\nabla v(t) - \nabla \Pi_h v(t)\|_{L^2(\Omega)} \, dt \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Chk^{1/2} m \|v\|_{L^2(I_n; H^2(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{u,c,2}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))\nabla \Pi_h[v(t) - v(t_n)], \nabla \varphi_h) \, dt \\ &\leq Ck^{3/2} \|a_m(u(t))\|_{L^\infty(I_n; L^\infty(\Omega))} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)} \\ &\leq Cmk^{3/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \varphi_h\|_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{u,c,3}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} ([a_m(u(t)) - a_m(\Pi_h u(t))] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt \\
&\leq Ck^{1/2} \|a_m(u(t)) - a_m(\Pi_h u(t))\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \|\nabla \varphi_h\|_{L^2(\Omega)} \\
&\leq Chk^{1/2} \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \|\nabla \varphi_h\|_{L^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{u,c,4}^n(\varphi_h) &= \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t)) - a_m(\Pi_h u(t_n))] \nabla \Pi_h v(t_n), \nabla \varphi_h) \, dt \\
&\leq Ck^{3/2} \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \|\nabla \varphi_h\|_{L^2(\Omega)} .
\end{aligned}$$

Similarly, we have

$$\mathcal{E}_{v,1}^n(\phi_h) \leq Chk^{1/2} \|v_t(t)\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} ,$$

$$\mathcal{E}_{v,a,1}^n(\phi_h) \leq Chk^{1/2} \|v\|_{L^2(I_n; H^2(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)}$$

and

$$\mathcal{E}_{v,a,2}^n(\phi_h) \leq Ck^{3/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)} .$$

Estimating $\mathcal{E}_{v,c,1}^n(\phi_h)$, $\mathcal{E}_{v,c,2}^n(\phi_h)$, $\mathcal{E}_{v,c,3}^n(\phi_h)$ and $\mathcal{E}_{v,c,4}^n(\phi_h)$ we obtain

$$\begin{aligned}
\mathcal{E}_{v,c,1}^n(\phi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))^s [v(t) - \Pi_h v(t)], \phi_h) \, dt \\
&\leq \int_{t_{n-1}}^{t_n} \|a_m(u(t))^s\|_{L^\infty(\Omega)} \|v(t) - \Pi_h v(t)\|_{L^2(\Omega)} \, dt \|\phi_h\|_{L^2(\Omega)} \\
&\leq Chk^{1/2} \|a_m(u)^s\|_{L^\infty(I_n; L^\infty(\Omega))} \|v\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} \\
&\leq Cm^s h k^{1/2} \|v\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{v,c,2}^n(\phi_h) &= \int_{t_{n-1}}^{t_n} (a_m(u(t))^s \Pi_h [v(t) - v(t_n)], \phi_h) \, dt \\
&\leq Ck^{3/2} \|a_m(u)^s\|_{L^\infty(I_n; L^\infty(\Omega))} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} \\
&\leq Cm^s k^{3/2} \|v_t\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)}
\end{aligned}$$

$$\mathcal{E}_{v,c,3}^n(\phi_h) = \int_{t_{n-1}}^{t_n} ([a_m(u(t))^s - a_m(\Pi_h u(t))^s] \Pi_h v(t_n), \phi_h) \, dt$$

$$\begin{aligned}
&\leq Ck^{1/2} \left\| [a_m(u) + a_m(\Pi_h u)]^{s-1} \right\|_{L^\infty(I_n; L^\infty(\Omega))} \times \\
&\quad \times \|a_m(u) - a_m(\Pi_h u)\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \|\phi_h\|_{L^2(\Omega)} \\
&\leq Cm^{s-1} h k^{1/2} \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \|\phi_h\|_{L^2(\Omega)}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{v,c,4}^n(\phi_h) &= \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t))^s - a_m(\Pi_h u(t_{n-1}))^s] \Pi_h v(t_n), \phi_h) dt \\
&\leq Cm^{s-1} k^{3/2} \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \|\phi_h\|_{L^2(\Omega)}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{E}_u^n(\varphi_h) &\leq Chk^{1/2} \left(\|u_t(t)\|_{L^2(I_n; L^2(\Omega))} \|\varphi_h\|_{L^2(\Omega)} + \left[\|u\|_{L^2(I_n; H^2(\Omega))} \right. \right. \\
&\quad \left. \left. + m \|v\|_{L^2(I_n; H^2(\Omega))} \right. \right. \\
&\quad \left. \left. + \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \right] \|\nabla \varphi_h\|_{L^2(\Omega)} \right) \\
&\quad + Ck^{3/2} \left(\|u_t\|_{L^2(I_n; H^1(\Omega))} + m \|v_t\|_{L^2(I_n; H^1(\Omega))} \right. \\
&\quad \left. + \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{W^{1,6}(\Omega)} \right) \|\nabla \varphi_h\|_{L^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_v^n(\phi_h) &\leq Chk^{1/2} \left(\|v_t(t)\|_{L^2(I_n; H^1(\Omega))} \|\phi_h\|_{L^2(\Omega)} + \|v\|_{L^2(I_n; H^2(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)} \right. \\
&\quad \left. + \left[m^s \|v\|_{L^2(I_n; H^1(\Omega))} \right. \right. \\
&\quad \left. \left. + \|u\|_{L^2(I_n; W^{1,3}(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \right] \|\phi_h\|_{L^2(\Omega)} \right) \\
&\quad + Ck^{3/2} \left(\|v_t\|_{L^2(I_n; H^1(\Omega))} \|\nabla \phi_h\|_{L^2(\Omega)} + \left[m^s \|v_t\|_{L^2(I_n; H^1(\Omega))} \right. \right. \\
&\quad \left. \left. + \|u_t\|_{L^2(I_n; L^3(\Omega))} \|v(t_n)\|_{L^6(\Omega)} \right] \|\phi_h\|_{L^2(\Omega)} \right).
\end{aligned}$$

Using Young's inequality we have

$$\begin{aligned}
\mathcal{E}_u^n(\varphi_h) &\leq \tilde{C}(\delta) h^2 \left[\|u\|_{L^2(I_n; H^2(\Omega))}^2 + m^2 \|v\|_{L^2(I_n; H^2(\Omega))}^2 \right. \\
&\quad \left. + \|u\|_{L^2(I_n; W^{1,3}(\Omega))}^2 \|v(t_n)\|_{W^{1,6}(\Omega)}^2 \right] + \frac{\delta}{2} k \|\nabla \varphi_h\|_{L^2(\Omega)}^2 \\
&\quad + \tilde{C}(\delta) h^2 \|v_t(t)\|_{L^2(I_n; L^2(\Omega))}^2 + \frac{\delta}{2} k \|\varphi_h\|_{L^2(\Omega)}^2 \\
&\quad + \tilde{C}(\delta) k^2 \left[\|u_t\|_{L^2(I_n; H^1(\Omega))}^2 + m^2 \|v_t\|_{L^2(I_n; H^1(\Omega))}^2 \right. \\
&\quad \left. + \|u_t\|_{L^2(I_n; L^3(\Omega))}^2 \|v(t_n)\|_{W^{1,6}(\Omega)}^2 \right] + \delta k \|\nabla \varphi_h\|_{L^2(\Omega)}^2,
\end{aligned}$$

and we can write

$$\mathcal{E}_u^n(\varphi_h) \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|\varphi_h\|_{L^2(\Omega)}^2 + 2\delta k \|\nabla \varphi_h\|_{L^2(\Omega)}^2, \quad (4.64)$$

where $\tilde{C}(I_n, \delta, m, u, v) > 0$ is a constant which depends on the norms of u, v, u_t and v_t in the intervals I_n . Note that there is a constant $C(\delta, u, v, m) > 0$ such that

$$\sum_{n=0}^N \tilde{C}(I_n, \delta, m, u, v) \leq C(\delta, u, v, m). \quad (4.65)$$

Analogously, we can obtain

$$\mathcal{E}_v^n(\phi_h) \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|\phi_h\|_{L^2(\Omega)}^2 + \delta k \|\nabla \phi_h\|_{L^2(\Omega)}^2. \quad (4.66)$$

Then, if we use e_u^n as a test function in (4.62), e_v^n as a test function in (4.63) and the estimates (4.64) and (4.66) we obtain

$$\begin{aligned} & \|e_u^n\|_{L^2(\Omega)}^2 - \|e_u^{n-1}\|_{L^2(\Omega)}^2 + \|e_u^n - e_u^{n-1}\|_{L^2(\Omega)}^2 + \\ & k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) \\ & + \delta k \|e_u^n\|_{L^2(\Omega)}^2 + 2\delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 + \chi \int_{t_{n-1}}^{t_n} (a_m(u_h^n) \nabla e_v^n, \nabla e_u^n) dt \\ & + \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla e_u^n) dt, \end{aligned} \quad (4.67)$$

and

$$\begin{aligned} & \|e_v^n\|_{L^2(\Omega)}^2 - \|e_v^{n-1}\|_{L^2(\Omega)}^2 + \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\ & + k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + k \|a_m(u_h^{n-1})^{s/2} e_v^n\|_{L^2(\Omega)}^2 \\ & \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + \delta k \|e_v^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_v^n\|_{L^2(\Omega)}^2 \\ & - \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1}))^s - a_m(u_h^{n-1})^s] \Pi_h v(t_n), e_v^n) dt. \end{aligned} \quad (4.68)$$

We still have to estimate some terms of (4.67) and (4.68):

$$\chi \int_{t_{n-1}}^{t_n} (a_m(u_h^n) \nabla e_v^n, \nabla e_u^n) dt \leq m^2 \chi^2 C(\delta) k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2, \quad (4.69)$$

$$\begin{aligned} & \chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla e_u^n) dt \\ & \leq k \chi \|e_u^n\|_{L^3(\Omega)} \|\nabla \Pi_h v(t_n)\|_{L^6(\Omega)} \|\nabla e_u^n\|_{L^2(\Omega)} \\ & \leq k \chi C_1 \|e_u^n\|_{L^2(\Omega)}^{1/2} \|e_u^n\|_{H^1(\Omega)}^{1/2} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla e_u^n\|_{L^2(\Omega)} \\ & \leq k \chi C_2 \|e_u^n\|_{L^2(\Omega)}^{1/2} \left(\|e_u^n\|_{L^2(\Omega)}^{1/2} + \|\nabla e_u^n\|_{L^2(\Omega)}^{1/2} \right) \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla e_u^n\|_{L^2(\Omega)} \end{aligned}$$

$$\begin{aligned}
&= k \chi C_2 \|e_u^n\|_{L^2(\Omega)} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla e_u^n\|_{L^2(\Omega)} \\
&+ k \chi C_2 \|e_u^n\|_{L^2(\Omega)}^{1/2} \|v\|_{L^\infty(I_n; W^{1,6}(\Omega))} \|\nabla e_u^n\|_{L^2(\Omega)}^{3/2}.
\end{aligned}$$

Using Young's inequality we can conclude that

$$\begin{aligned}
&\chi \int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_n)) - a_m(u_h^n)] \nabla \Pi_h v(t_n), \nabla e_u^n) dt \\
&\leq \chi^2 C k \left(\|v\|_{L^\infty(0,T; W^{1,6}(\Omega))}^2 + \|v\|_{L^\infty(0,T; W^{1,6}(\Omega))}^4 \right) \|e_u^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \\
&= \chi^2 C(v) k \|e_u^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2,
\end{aligned} \tag{4.70}$$

where $C(v) = C(\|v\|_{L^\infty(0,T; W^{1,6}(\Omega))}^2 + \|v\|_{L^\infty(0,T; W^{1,6}(\Omega))}^4)$.

The last term that we have to estimate in (4.68) is

$$\begin{aligned}
&\int_{t_{n-1}}^{t_n} ([a_m(\Pi_h u(t_{n-1}))^s - a_m(u_h^{n-1})^s] \Pi_h v(t_n), e_v^n) dt \\
&\leq C m^{s-1} k \|e_u^{n-1}\|_{L^2(\Omega)} \|\Pi_h(v(t_n))\|_{L^\infty(\Omega)} \|e_v^n\|_{L^2(\Omega)} \\
&\leq C(\delta, s) m^{2(s-1)} k \|v\|_{L^\infty(0,T; L^\infty(\Omega))}^2 \|e_u^{n-1}\|_{L^2(\Omega)}^2 + k \delta \|e_v^n\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.71}$$

Using it in (4.67) and (4.68), we have

$$\begin{aligned}
&\|e_u^n\|_{L^2(\Omega)}^2 - \|e_u^{n-1}\|_{L^2(\Omega)}^2 + \|e_u^n - e_u^{n-1}\|_{L^2(\Omega)}^2 \\
&+ k \|\nabla e_u^n\|_{L^2(\Omega)}^2 \leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) \\
&+ \delta k \|e_u^n\|_{L^2(\Omega)}^2 + 4\delta k \|\nabla e_u^n\|_{L^2(\Omega)}^2 + m^2 C(\delta, \chi) k \|\nabla e_v^n\|_{L^2(\Omega)}^2 \\
&+ C(v) \chi^2 k \|e_u^n\|_{L^2(\Omega)}^2,
\end{aligned} \tag{4.72}$$

and

$$\begin{aligned}
&\|e_v^n\|_{L^2(\Omega)}^2 - \|e_v^{n-1}\|_{L^2(\Omega)}^2 + \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\
&+ k \|\nabla e_v^n\|_{L^2(\Omega)}^2 + k \|a_m(u_h^{n-1})^{s/2} e_v^n\|_{L^2(\Omega)}^2 \\
&\leq \tilde{C}(I_n, \delta, m, u, v)(h^2 + k^2) + 2\delta k \|e_v^n\|_{L^2(\Omega)}^2 + \delta k \|\nabla e_v^n\|_{L^2(\Omega)}^2 \\
&+ C(\delta, s) m^{2(s-1)} k \|v\|_{L^\infty(0,T; L^\infty(\Omega))}^2 \|e_u^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.73}$$

Let $C_0 = 2m^2 C(\delta, \chi)$ and note that

$$\begin{aligned}
\|e_v^n\|_{L^2(\Omega)}^2 &= \|e_v^n - e_v^{n-1} + e_v^{n-1}\|_{L^2(\Omega)}^2 \leq \left[\|e_v^n - e_v^{n-1}\|_{L^2(\Omega)} + \|e_v^{n-1}\|_{L^2(\Omega)} \right]^2 \\
&\leq 2 \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 + 2 \|e_v^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Then we can obtain

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 - \|e_u^{n-1}\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 - C_0 \|e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + \left(1 - (2\delta + C(v)\chi^2)k\right) \|e_u^n - e_u^{n-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 2\delta k)C_0 \|e_v^n - e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 4\delta)k \|\nabla e_u^n\|_{L^2(\Omega)}^2 + k\left(\frac{C_0}{2} - \delta C_0\right) \|\nabla e_v^n\|_{L^2(\Omega)}^2 \\
& \leq (1 + C_0)\tilde{C}(I_n, \delta)(h^2 + k^2) + 2\delta k C_0 \|e_v^{n-1}\|_{L^2(\Omega)}^2 \\
& + (2\delta + C(v)\chi^2 + C(\delta, s)m^{2(s-1)}s^2 \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^2)k \|e_u^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.74}$$

If we sum (4.74) from 1 to n , we find

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 \\
& + \left(1 - (2\delta + C(v)\chi^2)k\right) \sum_{j=1}^n \|e_u^j - e_u^{j-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 2\delta k)C_0 \sum_{j=1}^n \|e_v^j - e_v^{j-1}\|_{L^2(\Omega)}^2 \\
& + (1 - 4\delta)k \sum_{j=1}^n \|\nabla e_u^j\|_{L^2(\Omega)}^2 + k\left(\frac{C_0}{2} - \delta C_0\right) \sum_{j=1}^n \|\nabla e_v^j\|_{L^2(\Omega)}^2 \\
& \leq (1 + C_0)C(\delta, u, v, m)(h^2 + k^2) + \sum_{j=0}^{n-1} 2\delta k C_0 \|e_v^j\|_{L^2(\Omega)}^2 \\
& + \sum_{j=0}^{n-1} (2\delta + C(v)\chi^2 + C(\delta, s)m^{2(s-1)}s^2 \|v\|_{L^\infty(0,T;L^\infty(\Omega))}^2)k \|e_u^j\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.75}$$

Now, we realize that, in order to keep the non-negative terms in the left hand side of (4.75), we must choose δ and k such that

$$1 - 4\delta > 0, \quad 1 - 2\delta k > 0$$

and

$$1 - (2\delta + C(v)\chi^2)k > 0,$$

It suggests that we must choose $\delta < 1/4$, let it be $\delta = 1/5$. Then we must choose k such that

$$1 - \left(\frac{2}{5} + C(v)\chi^2\right)k > 0,$$

After that, we can apply the discrete Gronwall's inequality (Lemma 25) in order to obtain

$$\begin{aligned}
& \|e_u^n\|_{L^2(\Omega)}^2 + C_0 \|e_v^n\|_{L^2(\Omega)}^2 \\
& + C \sum_{i=1}^n \|e_u^i - e_u^{i-1}\|_{L^2(\Omega)}^2 + CC_0 \sum_{i=1}^n \|e_v^i - e_v^{i-1}\|_{L^2(\Omega)}^2 \\
& + C \sum_{i=1}^n k \|\nabla e_u^i\|_{L^2(\Omega)}^2 + CC_0 \sum_{i=1}^n k \|\nabla e_v^i\|_{L^2(\Omega)}^2 \\
& \leq \tilde{C}(u, v, \chi, m, T)(h^2 + k^2),
\end{aligned} \tag{4.76}$$

■

CONCLUSION

Chemotaxis plays an important role in many biological processes, such as the movement of the immune cells, the migration of bacteria and cancer metastasis, for example. There are also some studies which call the attention to the effects of chemotaxis in the migration of bacteria towards polluting substances and its usefulness in the degradation of these substances. All this makes chemotaxis a matter of very practical interest.

One of the first models for chemotaxis is attributed to Keller and Segel (KELLER; SEGEL, 1970) (KELLER; SEGEL, 1971). The contribution of (PATLAK, 1953) to the development of the aforementioned model is also considered. Since then, many mathematicians studied this and other related models which were obtained from the Keler-Segel model with the aim of arriving at models for other situations related to chemotaxis as well as more biologically realistic models.

In this work, we were interested in the chemoattraction model with consumption

$$\begin{aligned}
 \partial_t u - \Delta u &= -\nabla \cdot (u \nabla v) \\
 \partial_t v - \Delta v &= -u^s v \\
 \partial_\eta u \Big|_\Gamma &= \partial_\eta v \Big|_\Gamma = 0 \\
 u(0) &= u^0, \quad v(0) = v^0,
 \end{aligned} \tag{P}$$

with $s \geq 1$. As pointed out in the introduction, on one hand, the numerical approximation of chemotaxis models is an important and growing research topic. On the other hand, concerning the chemoattraction model (P), most of the studies on (P) are dedicated to its theoretical analysis, dealing only with $s = 1$, and, to the best of our knowledge, we can cite two studies, (DUARTE-RODRÍGUEZ et al., 2021) and (GUILLÉN-GONZÁLEZ; TIERRA, 2021), on the numerical approximation of (P), also for $s = 1$.

Due to the relevance of the discussion on the numerical approximation of chemotaxis models and the relatively low number of studies on the numerical approximation of the chemoattraction model (P), we identified a research opportunity related to the numerical approximation of (P). However, the non-linear terms related to chemotaxis and consumption introduce many difficulties to the analysis of the original model, mainly for

its numerical simulation. In order to deal with these difficulties, our approach consisted in proposing a sequence of regularized models

$$\begin{aligned}
 \partial_t u_m - \Delta u_m &= -\nabla \cdot (a_m(u_m) \nabla v_m) \\
 \partial_t v_m - \Delta v_m &= -a_m(u_m)^s v_m \\
 \partial_\eta u_m \Big|_\Gamma &= \partial_\eta v_m \Big|_\Gamma = 0 \\
 u_m(0) &= u^0, \quad v_m(0) = v^0,
 \end{aligned} \tag{P}_m$$

depending on a parameter $m \in \mathbb{N}$, proving convergence towards the original model (P) as $m \rightarrow \infty$ and, finally, proposing and analyzing discrete numerical schemes with the aim of approximating the regularized models, for each fixed $m \in \mathbb{N}$.

With this approach we managed to give the following contributions:

- (i) generalization of the model with the consumption term $u^s v$ for $s \geq 1$ (instead of only uv);
- (ii) enlargement of the class of considered domains, maintaining the no blow-up effect in the $2D$ case and some time-independent estimates for spatial derivatives in $2D$ and $3D$ domains;
- (iii) the introduction and analysis of a regularized model (existence, uniqueness, regularity, positivity, a priori estimates and convergence towards the original model) that are be useful for designing numerical schemes that are used in order to approximate the original model;
- (iv) the design of fully discrete numerical schemes for the approximation of the regularized model.

In the sequel, we summarize the main results obtained in this PhD. thesis starting from the introduction and analysis of the regularized model, going through the analysis of the original model and arriving at the design and analysis of the numerical methods.

Theorem. (Existence of global solution for the regularized problem (P_m)) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that the Neumann problem (1.2) has the H^2 -regularity (see definition 21 in page 23). Let $u^0 \in H^1(\Omega)$ and $v^0 \in H^2(\Omega)$ be non-negative functions. Then, for each $m \in \mathbb{N}$ the regularized problem (P_m) has a unique solution (u_m, v_m) such that:*

$$u_m - u^* \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)),$$

$$\partial_t u_m \in L^2(0, \infty; L^2(\Omega)),$$

$$v_m \in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; H^2(\Omega)),$$

$$\partial_t v_m \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)).$$

If moreover the regularity $W^{2,3}$ -regularity of (1.2) is imposed and $u^0 \in H^2(\Omega)$ then

$$(u_m - u^*), v_m \in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; H^2(\Omega)),$$

$$\partial_t u_m, \Delta u_m, \partial_t v_m, \Delta v_m \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)).$$

Proof. Proved in Chapter 2. ■

Theorem. (3D. Existence of global weak solutions for (P)) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that the Neumann problem (1.2) has the H^2 -regularity (see definition 21 in page 23) and Hypothesis (H1) is satisfied. Let $u^0 \in L^{1+\varepsilon}(\Omega)$, for some $\varepsilon > 0$, if $s = 1$, and $u^0 \in L^s(\Omega)$, if $s > 1$, and $v^0 \in H^1(\Omega) \cap L^\infty(\Omega)$ be non-negative functions. Then there is a non-negative weak solution (u, v) of the original problem (P), for $s \geq 1$, obtained through a limit of non-negative solutions (u_m, v_m) of the regularized problems (P_m) as $m \rightarrow \infty$ and such that

$$\begin{cases} \int_{\Omega} u(t, x) dx = \int_{\Omega} u^0(x) dx, \text{ a.e. } t \in (0, \infty) \\ 0 \leq v(t, x) \leq \|v^0\|_{L^\infty(\Omega)}, \text{ a.e. } (t, x) \in (0, \infty) \times \Omega, \end{cases}$$

$$\begin{cases} u \in L^\infty(0, \infty; L^s(\Omega)) \cap L_{loc}^{5s/3}([0, \infty); L^{5s/3}(\Omega)), \text{ if } s \geq 1, \\ u^{s/2} \nabla v \in L^2(0, \infty; L^2(\Omega)), \text{ if } s \geq 1, \end{cases}$$

$$\begin{cases} \nabla u \in L^2(0, \infty; L^s(\Omega)) \cap L_{loc}^{\frac{5s}{3+s}}([0, \infty); L^{\frac{5s}{3+s}}(\Omega)), \text{ if } s \in [1, 2), \\ \nabla u \in L^2(0, \infty; L^2(\Omega)), \text{ if } s \geq 2, \end{cases}$$

$$\begin{cases} u \nabla v \in L^2(0, \infty; L^s(\Omega)), \text{ if } s \in [1, 2), \\ u \nabla v \in L^2(0, \infty; L^2(\Omega)), \text{ if } s \geq 2 \end{cases}$$

and

$$v \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)),$$

$$\nabla v \in L^4(0, \infty; L^4(\Omega)),$$

Proof. Proved in Section 3.1. ■

Theorem. (2D. Existence and uniqueness of global strong solution to (P)) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain such that the Neumann problem (1.2) has the $W^{2,3}$ -regularity (see definition 21 in page 23) and hypothesis (H1) is satisfied. Let $u^0 \in H^2(\Omega)$ and $v^0 \in H^2(\Omega)$ be such that $u^0 \geq 0$ and $v^0 \geq 0$. Then there is a unique solution (u, v) for the original problem (P), for $s \geq 1$, such that

$$u, v \in L^\infty(0, \infty; H^2(\Omega)),$$

$$\Delta u, \Delta v, u_t, v_t \in L^2(0, \infty; H^1(\Omega)),$$

$$(u - u^*), v \in L^2(0, \infty; W^{2,3}(\Omega)).$$

In particular, u does not blow-up neither in finite nor in infinite time, that is $u \in L^\infty(0, \infty; L^\infty(\Omega))$. Consequently, there is $m_0 \in \mathbb{N}$ such that

$$(u_m, v_m) = (u, v) \text{ a.e. in } (0, \infty) \times \Omega,$$

for all $m \in [m_0, \infty)$.

Proof. Proved in Section 3.2. ■

After these results on (P_m) and (P) , we introduced and analyzed the discrete numerical schemes based on the finite element method in order to approximate the regularized models, for each $m \in \mathbb{N}$, on a finite time interval $(0, T)$. Consider the spaces $U_h = V_h = \mathbb{P}_1(\mathcal{T}_h(\Omega))$ and let $u_h^0 = \Pi_h(u^0)$ and $v_h^0 = \Pi_h(v^0)$, where $\Pi_h : L^1(\Omega) \rightarrow \mathbb{P}_1(\mathcal{T}_h(\Omega))$ is the Scott-Zhang operator of (SCOTT; ZHANG, 1990). The fully discrete schemes are:

$$\left. \begin{aligned} &\text{Knowing } (u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h, \text{ find } (u_h^n, v_h^n) \in U_h \times V_h \text{ such that} \\ &(\delta_t u_h^n, \varphi_h) + (\nabla u_h^n, \nabla \varphi_h) = \chi(a_m(u_h^{n-1}) \nabla v_h^n, \nabla \varphi_h), \quad \forall \varphi_h \in U_h, \\ &(\delta_t v_h^n, \phi_h) + (\nabla v_h^n, \nabla \phi_h) = -(a_m(u_h^{n-1})^s v_h^n, \phi_h), \quad \forall \phi_h \in V_h, \end{aligned} \right\} \quad (LNS)$$

$$\left. \begin{aligned} &\text{Knowing } (u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h, \text{ find } (u_h^n, v_h^n) \in U_h \times V_h \text{ such that} \\ &(\delta_t u_h^n, \varphi_h) + (\nabla u_h^n, \nabla \varphi_h) = \chi(a_m(u_h^n) \nabla v_h^n, \nabla \varphi_h), \quad \forall \varphi_h \in U_h, \\ &(\delta_t v_h^n, \phi_h) + (\nabla v_h^n, \nabla \phi_h) = -(a_m(u_h^n)^s v_h^n, \phi_h), \quad \forall \phi_h \in V_h \end{aligned} \right\} \quad (NLNS)$$

and

$$\left. \begin{aligned} &\text{Knowing } (u_h^{n-1}, v_h^{n-1}) \in U_h \times V_h, \text{ find } (u_h^n, v_h^n) \in U_h \times V_h \text{ such that} \\ &(\delta_t u_h^n, \varphi_h) + (\nabla u_h^n, \nabla \varphi_h) = \chi(a_m(u_h^n) \nabla v_h^n, \nabla \varphi_h), \quad \forall \varphi_h \in U_h, \\ &(\delta_t v_h^n, \phi_h) + (\nabla v_h^n, \nabla \phi_h) = -(a_m(u_h^{n-1})^s v_h^n, \phi_h), \quad \forall \phi_h \in V_h. \end{aligned} \right\} \quad (SLNS)$$

We remark that the discrete scheme (LNS) is a linear numerical scheme and, at each time step, we can solve two decoupled linear systems of equations with positive definite matrices. At each time step of $(NLNS)$, the non-linear numerical scheme, we must solve two non-linear systems of equations simultaneously. Finally, at each time step of $(SLNS)$, the semi-linear numerical scheme, we must solve a system of linear equations and a system of non-linear equations which can be solved separately. We have the following results.

Theorem. (On the existence and uniqueness of solution for the numerical schemes) For each time step n we have:

1. existence and uniqueness of solution (u_h^n, v_h^n) to (LNS) independently of the choices of the time and space discretization parameters k and h ;
2. existence of solution (u_h^n, v_h^n) to (NLNS) independently of the choices of the time and space discretization parameters k and h and uniqueness of solution if k and h are chosen so that

$$\frac{k}{h^{d+2}} \leq \frac{C}{\chi^2 \|v^0\|_{L^2(\Omega)}^2} \text{ and } \frac{k^2}{h^d} \leq \frac{C}{\chi^2 m^{2s} \|v^0\|_{L^2(\Omega)}^2};$$

3. existence of solution (u_h^n, v_h^n) to (SLNS) independently of the choices of the time and space discretization parameters k and h and uniqueness of solution if k and h are chosen so that

$$\frac{k}{h^{(d+2)/2}} < \frac{C}{\chi^2 \|v^0\|_{L^2(\Omega)}^2}.$$

Proof. Proved in Theorems 71, 79, 82, 84 and 87. ■

Lemma. (On the a priori estimates) Consider a set of pairs (u_h^n, v_h^n) , $n = 0, 1, \dots, N$, given either by (LNS), (NLNS) or (SLNS). Independently of the choices of the time and space discretization parameters k and h we have the a priori estimates (independent of k , h and n)

$$\begin{aligned} \|v_h^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \|v_h^i - v_h^{i-1}\|_{L^2(\Omega)}^2 + 2 \sum_{i=1}^n k \|\nabla v_h^i\|_{L^2(\Omega)}^2 &\leq \|v_h^0\|_{L^2(\Omega)}^2, \\ \|u_h^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \|u_h^i - u_h^{i-1}\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla u_h^i\|_{L^2(\Omega)}^2 &\leq \|u_h^0\|_{L^2(\Omega)}^2 + \frac{m^2}{2} \|v_h^0\|_{L^2(\Omega)}^2. \end{aligned}$$

Proof. Proved in Lemmas 74, 80 and 85. ■

Theorem. (On the convergence of the numerical schemes) Let Ω be a polyhedron and let (u_m, v_m) be the weak solution of the regularized problem (P_m) , for a fixed $m \in \mathbb{N}$. Consider the pairs (u_h^n, v_h^n) , $n = 0, 1, \dots, N$, all given either by (LNS), (NLNS) or (SLNS), and let $u_{hk}^r, u_{hk}^l, u_{hk}, v_{hk}^r, v_{hk}^l$ and v_{hk} be the respective associated functions, defined as in (4.13), (4.14) and (4.15). Then, as $(h, k) \rightarrow (0, 0)$, the sequences $\{u_{hk}^r\}_{(h,k)}$, $\{u_{hk}^l\}_{(h,k)}$ and $\{u_{hk}\}_{(h,k)}$ converge to the function u_m and the sequences $\{v_{hk}^r\}_{(h,k)}$, $\{v_{hk}^l\}_{(h,k)}$ and $\{v_{hk}\}_{(h,k)}$ converge to the function v_m weakly* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$.

Proof. Proved in Theorems 76, 81 and 86. ■

Theorem. (On the error estimates) Let Ω be a polyhedron and let (u_m, v_m) be the weak solution of the regularized problem (P_m) , for a fixed $m \in \mathbb{N}$. Consider the pairs (u_h^n, v_h^n) , $n = 0, 1, \dots, N$, all given either by (LNS) , $(NLNS)$ or $(SLNS)$, and let us define the discrete errors $e_u^n = u_h^n - \Pi_h u_m(t_n)$ and $e_v^n = v_h^n - \Pi_h v_m(t_n)$. Assume that (u_m, v_m) has the regularity

$$\begin{aligned} u_m, v_m &\in L^2(0, T; H^2(\Omega)), \\ v_m &\in L^\infty(0, T; W^{1,6}(\Omega)), \\ \partial_t u_m, \partial_t v_m &\in L^2(0, T; H^1(\Omega)). \end{aligned}$$

Then there exist positive constants $C(u_m, v_m, \chi, m, T)$ (the constants may be different for each estimate) such that

$$\|e_u^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla e_u^i\|_{L^2(\Omega)}^2 \leq C(u_m, v_m, \chi, m, T)(h^2 + k^2)$$

and

$$\|e_v^n\|_{L^2(\Omega)}^2 + \sum_{i=1}^n k \|\nabla e_v^i\|_{L^2(\Omega)}^2 \leq C(u_m, v_m, \chi, m, T)(h^2 + k^2)$$

1. independently of k and h in case (u_h^n, v_h^n) , $n = 0, 1, \dots, N$, are given either by (LNS) ;
2. if we choose k such that

$$k < \frac{1}{\left(\frac{2}{5} + \chi^2 C(v_m) + C(s, m) \|v_m\|_{L^\infty(0, T; L^\infty(\Omega))}^2\right)},$$

in case (u_h^n, v_h^n) , $n = 0, 1, \dots, N$, are given either by (LNS)

3. if we choose k such that

$$k < \frac{1}{\left(\frac{2}{5} + \chi^2 C(v_m)\right)},$$

in case (u_h^n, v_h^n) , $n = 0, 1, \dots, N$, are given either by $(SLNS)$.

Proof. Proved in Theorems 77, 83 and 88. ■

If we stick to the theoretical results obtained on the numerical schemes until now, we will tend to judge the linear scheme (LNS) as more advantageous. In fact, in comparison with the non-linear schemes, it is easier to solve and even its theoretical analysis is simpler, because none of the results regarding the linear scheme, including the ones on the uniqueness of solution and error estimates, depend on the choice of the discretization parameters k and h .

But a disadvantage of the linear scheme (LNS) appears when we turn to the study of the positivity of the fully discrete cell density u_h^n . In (GUILLÉN-GONZÁLEZ et al., 2018), a study on a chemorepulsion model, the results obtained by the authors through the numerical simulations give an evidence that the conservation of the positivity of the fully discrete cell density might contribute to avoiding spurious oscillations of the numerical solutions, hence being a desirable feature to a numerical scheme. Numerical simulations performed by us indicate that conservation of the positivity of the discrete cell density is also important in the context of the P problem. And it is not clear how this feature can be obtained for the linear scheme.

On the other hand, since the non-linear schemes ($NLNS$) and ($SLNS$) are obtained from the time discretizations ($NLTD$) and ($SLTD$), which preserve the positivity of the time discrete cell density, we expect that it helps us to study the conservation of the positivity of the fully discrete cell density u_h^n . This is also indicated by numerical simulations performed by us. Besides, there are also some studies on related chemotaxis models which give us insights on how it can be done when we consider a scheme where the equation for u_h^n is chosen to be non-linear, such as (GUILLÉN-GONZÁLEZ; TIERRA, 2021), (GUTIERREZ-SANTACREU; RODRÍGUEZ-GALVÁN, 2021) and (GUILLÉN-GONZÁLEZ et al., 2018), for example.

Therefore, further studies are needed to decide which numerical scheme is more appropriate to approximate the chemoattraction model with consumption (P).

For the future, first of all, we intend to submit the results on problems (P) and (P_m) for publication. The subsequent step will be performing numerical simulations with the proposed numerical schemes (LNS), ($NLNS$) and ($SLNS$) in order to compare their performance and to verify the theoretical results obtained so far.

As it was mentioned before, numerical simulations performed by us already indicate that the conservation of the positivity of the fully discrete cell density u_h^n may be an important feature to the numerical schemes in the context of problem (P) and that the non-linear numerical schemes ($NLNS$) and ($SLNS$) are better suited to the conservation of the positivity of u_h^n than the linear scheme (LNS). Therefore, we also intend to introduce and study numerical schemes which preserve the positivity of u_h^n . To this end, besides the cited articles, (GUILLÉN-GONZÁLEZ; TIERRA, 2021), (GUTIERREZ-SANTACREU; RODRÍGUEZ-GALVÁN, 2021) and (GUILLÉN-GONZÁLEZ et al., 2018), we may also consider other techniques such as upwind finite element method and discontinuous Galerkin methods (ACOSTA-SOBA et al., 2021).

Finally, with the experience on the study of problem (P) obtained so far, another interesting and possible research topic is the control of (P). As it was mentioned in the introduction, the control of chemotaxis models is a relevant research topic and, in the context of problem (P), a starting point would be the introduction of a bilinear control

in the concentration of chemical substance equation, as it was done in (GUILLÉN-GONZÁLEZ et al., 2020c), for a chemorepulsion model with production.

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