UNIVERSIDADE FEDERAL DO PARANÁ

ALEXANDRE ARIAS JUNIOR

THE CAUCHY PROBLEM FOR $3-$EVOLUTION OPERATORS WITH DATA IN GEVREY AND GELFAND SHILOV TYPE SPACES

CURITIBA

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Orientador: Prof Dr. Alexandre Kirilov

Coorientador: Prof Dr. Marco Cappiello

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RESUMO

Estudamos o Problema de Cauchy para uma classe de operadores de $3-$evolução com termos de ordem inferior dependendo tanto da variável temporal quanto da espacial e assumindo valores complexos. Consideramos o problema nos ambientes funcionais Gevrey e Gevrey com decaimento exponencial no infinito (classes de funções Gelfand-Shilov), obtendo resultados de existência e boa colocação. Também estudamos a parte Friedrichs de operadores pseudodiferencias com símbolos pertencentes as classes SG, obtendo uma expansão assintótica e regularidade precisa para o mesmo. Por fim, provamos um resultado sobre invariância espectral para operadores nas classes SG.

Palavras-chaves: Equações de p-evolução. Funções Gevrey. Funções Gelfand-Shilov. Operadores pseudodiferenciais de ordem infinita. Desigualdade sharp Gårding

ABSTRACT

We study the Cauchy Problem for a class of $3-$evolution operators with complex-valued lower order terms depending both on the time and space variables. The problem is treated in the functional settings of the Gevrey and Gevrey with exponential decay at infinity (Gelfand-Shilov) functions. We achieve existence and well-posedness results. We also study the Friedrichs part for pseudodifferential operators with symbols belonging to the SG classes, obtaining a precise asymptotic expansion and regularity for the same. Finally, we prove a result of spectral invariance for operators in the SG classes.

Keywords: $p-$evolution equations. Gevrey spaces. Gelfand-Shilov spaces. Infinite order pseudodifferential operators. Sharp Gårding inequality.

Contents

Introduction

A linear partial differential operator of evolution type is given by

$$P(t,x,D_t,D_x) = D_t^m + \sum_{j=1}^{m} a_j(t,x,D_x) D_t^{m-j},$$

where $m \geq 1$ is a positive integer, $t \geq 0$ is the time variable, $x \in \mathbb{R}^n$ is the space variable and each $a_j(t,x,D)$ is a differential operator

$$a_j(t,x,D_x) = \sum_{\nu} a_{j,\nu}(t,x) D_x^{\nu},$$

where ν runs through a finite set. The equation

$$P(t,x,D_t,D_x)u(t,x) = f(t,x)$$

is said to be a linear differential equation of the evolution type. Famous examples of evolution equations are given by:

- Wave equation:
$$\partial_t^2 u(t,x) - \Delta_x u(t,x) = 0;$$

- Heat equation:
$$\partial_t u(t,x) - \Delta_x u(t,x) = 0;$$

- Schrödinger equation:
$$i\partial_t u(t,x) - \Delta_x u(t,x) = 0.$$

The Cauchy problem associated with $P(t,x,D_t,D_x)$ is expressed by

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x), \\ D_t^j u(0,x) = g_j(x), \end{cases}$$

where the Cauchy data $f(t,x)$ and $g_j(x)$, $j = 0,\dots,m-1$, are given. A survey which contains a good historical background and presents important results concerning evolution equations can be found in [37] by S. Mizohata.

Now we describe the evolution problem we are interested in. Consider

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x),\ (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = g(x),\ x \in \mathbb{R}, \end{cases} \tag{1}$$

where $P(t,x,D_t,D_x)$ is a linear differential operator of the form

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t,x)D_x^j, \tag{2}$$

where $p \in \mathbb{N}, p \geq 1$, $a_p \in C([0,T];\mathbb{R})$, $a_p(t)$ never vanishes, $a_j \in C([0,T];\mathcal{B}^\infty(\mathbb{R}_x))$ and $\mathcal{B}^\infty(\mathbb{R})$ stands for the class of the smooth complex-valued functions defined in $\mathbb{R}$ with uniformly bounded derivatives. An operator of the form (2) is known as $p-$evolution operator in the literature. Note that in the case $p = 1$ we have a strictly hyperbolic operator, for $p = 2$ we have Schrödinger type operator and for $p = 3$ the principal part is the same as in the Korteweg-De Vries operator.

Let us recall the definitions of Sobolev and Gevrey Sobolev spaces before moving on to the results. Let $s \in \mathbb{R}$, $\rho > 0$ and $\theta \geq 1$. The Sobolev space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle D_x \rangle^s u \in L^2(\mathbb{R}^n)\},$$

where $\langle D_x \rangle^s$ is the pseudodifferential operator given by the symbol $\langle \xi \rangle^s := (1 + |\xi|^2)^{\frac{s}{2}}$ and $\mathscr{S}'(\mathbb{R}^n)$ stands for the space of tempered distributions. The Gevrey Sobolev space $H^s_{\rho;\theta}(\mathbb{R}^n)$ is given in turn by

$$H^s_{\rho;\theta}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle D_x \rangle^s e^{\rho \langle D_x \rangle^{\frac{1}{\theta}}} u \in L^2(\mathbb{R}^n)\},$$

where $e^{\rho \langle D_x \rangle^{\frac{1}{\theta}}}$ is the pseudodifferential operator given by the symbol $e^{\rho \langle \xi \rangle^{\frac{1}{\theta}}}$.

The Cauchy problem (1) is said to be well-posed in $H^s(\mathbb{R})$ when for any Cauchy data $f \in C([0,T];H^s(\mathbb{R}))$ and $g \in H^s(\mathbb{R})$, there exists a unique solution $u \in C([0,T];H^s(\mathbb{R}))$. When for any data $f \in C([0,T];H^s(\mathbb{R}))$ and $g \in H^s(\mathbb{R})$, there exists a unique solution $u \in C([0,T];H^{s-\delta}(\mathbb{R}))$, for some $\delta > 0$, satisfying (1), the Cauchy problem is said to be well-posed in $H^\infty(\mathbb{R}) := \cap_s H^s(\mathbb{R})$. Note that we have a $\delta > 0$ loss of derivatives in this case. Concerning the Gevrey context, if for any data $f \in C([0,T];H^s_{\rho;\theta}(\mathbb{R}))$ and $g \in H^s_{\rho;\theta}(\mathbb{R})$ (for some $\rho > 0$)

there exists a unique solution $u \in C([0,T]; H^s_{\rho';\theta}(\mathbb{R}))$ with $0 < \rho' < \rho$, then we say that the Cauchy problem (1) is well-posed in $\mathcal{H}^\infty_\theta(\mathbb{R}) := \cup_{\rho>0} H^s_{\rho;\theta}(\mathbb{R})$.

Well-posedness for the Cauchy problem (1) has been widely investigated. Regarding necessary conditions for $H^\infty(\mathbb{R})$ well-posedness, some decay condition on the imaginary part of the subleader coefficient is required, see [28] by W. Ichinose for $p = 2$ and [4] by A. Ascanelli, C. Boiti and L. Zanghirati for general $p \geq 3$. In [19], the author M. Dreher obtained necessary conditions for well-posedness of Schrödinger equations ($p = 2$) in Gevrey type spaces.

Now we turn our attention to sufficient conditions. We begin with the Schrödinger case, that is $p = 2$. Let us consider $a_2(t)$ constant, $x \in \mathbb{R}^n$ and

$$Im\, a_1(t,x) = \mathcal{O}(|x|^{-\sigma}),\ \sigma > 0,\ \text{as } |x| \to \infty,$$

uniformly with respect to t. In this situation, K. Kajitani and A. Baba proved in [33] that we have

- $H^s(\mathbb{R}^n)$ well-posedness if $\sigma > 1$;
- $H^\infty(\mathbb{R}^n)$ well-posedness if $\sigma = 1$;
- $\mathcal{H}^\infty_\theta(\mathbb{R}^n)$ well-posedness if $\sigma < 1$ and the coefficients are Gevrey regular with index $s_0 > 1$ and $s_0 \leq \theta < \frac{1}{1-\sigma}$.

In [14] M. Cicognani and M. Reissig extended this result to degenerate Schrödinger equations, that is, when the coefficient $a_2(t)$ possibly vanishes. For $p > 2$, there are few results limited to the $H^\infty(\mathbb{R})$ setting. Namely, for $p = 3$, in [13] M. Cicognani and F. Colombini obtained $H^\infty(\mathbb{R})$ well-posedness under the following conditions:

$$|Im\, a_2(t,x)| \leq C\langle x\rangle^{-1}, \quad |Re\, a_2(t,x)| + |Im\, a_1(t,x)| \leq C\langle x\rangle^{-\frac{1}{2}},$$

for every $t \in [0,T]$ and $x \in \mathbb{R}$, where $\langle x\rangle := \sqrt{1+|x|^2}$. For general $p \geq 3$, $H^\infty(\mathbb{R})$ well-posedness is achieved by A. Ascanelli, C. Boiti and L. Zanghirati in [3] under suitable assumptions. We do not know any result concerning sufficient conditions for well-posedness in Gevrey type spaces for $p \geq 3$.

The Cauchy problem for p-evolution equations with $p \geq 3$ has a significant importance in mathematics, especially in view of their connections with semilinear models appearing in fluid dynamics. For instance, linearizations of models like Korteweg-De Vries equation and its generalizations turn out to be 3-evolution linear equations. Since we are interested in extending our study to semilinear p-evolution equations, the first step is to establish the linear theory. In Chapter 3 we prove a well-posedness result in $\mathcal{H}^\infty_\theta(\mathbb{R})$ for the Cauchy problem (1) with $p = 3$ (see Theorem 4.1), and the same phenomenon of loss of regularity appearing in [14, 33] is observed.

The Chapter 4 of the thesis analyzes the effect of the behavior of the data for $|x| \to +\infty$ on the regularity of the solution. In fact, despite the precise assumptions required on the decay at infinity of the coefficients, the aforementioned results do not give information on the behavior at the infinity $|x| \to +\infty$ for the solution. This motivates the study of the Cauchy problem (1) in a weighted functional framework and the analysis of the effect of decaying data on the regularity of the solution.

Let us introduce the weighted functional spaces we just mentioned above. Consider $m = (m_1, m_2) \in \mathbb{R}^2$, $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ and $s, \theta > 1$. The weighted Sobolev space $H^m(\mathbb{R}^n)$ is defined by

$$H^m(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle x\rangle^{m_1}\langle D_x\rangle^{m_2} u \in L^2(\mathbb{R}^n)\},$$

while the Gelfand-Shilov Sobolev space $H^m_{\rho;s,\theta}(\mathbb{R}^n)$ is given by

$$H^m_{\rho;s,\theta}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle x\rangle^{m_2}\langle D_x\rangle^{m_2} e^{\rho_2\langle x\rangle^{\frac{1}{s}}} e^{\rho_1\langle D_x\rangle^{\frac{1}{\theta}}} u \in L^2(\mathbb{R}^n)\}.$$

These Sobolev spaces measure simultaneously regularity and behavior at infinity.

Imposing the following condition on the coefficients

$$|\partial_x^\beta a_j(t,x)| \leq C_\beta \langle x\rangle^{-\frac{j}{p-1}-\beta}, \quad j = 0, \dots, p-1, \tag{3}$$

A. Ascanelli and M. Cappiello proved in [5] that for any Cauchy data $f \in C([0,T]; H^m(\mathbb{R}))$ and $g \in H^m(\mathbb{R})$ ($m = (m_1, m_2)$), there exists a unique solution $u \in C([0,T]; H^{(m_1, m_2-\delta)}(\mathbb{R}))$, for some $\delta > 0$, satisfying (1). In this case, with respect to the initial data, the solution presents a different behavior at infinity but it has the same regularity. In particular, well-posedness in the Schwartz space $\mathscr{S}(\mathbb{R})$ is achieved. We remark that the condition (3) is needed in order to work with SG pseudodifferential operators.

Concerning the Schrödinger case, that is $p = 2$, in Gelfand-Shilov setting, A. Ascanelli and M. Cappiello obtained an existence result in [6]. More precisely, let $\sigma \in (0,1)$ such that $s_0 < \frac{1}{1-\sigma}$ and assume that the coefficients satisfy

$$|\partial_x^\beta Im\, a_1(t,x)| \leq C^{|\beta|+1}\beta!^{s_0}\langle x\rangle^{-\sigma}$$

and

$$|\partial_x^\beta Re\, a_1(t,x)| + |\partial_x^\beta a_0(t,x)| \leq C^{|\beta|+1}\beta!^{s_0}\langle x\rangle^{1-\sigma-|\beta|}.$$

Moreover, let $m \in \mathbb{R}^2$, $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$ and $s, \theta > 1$ such that $\theta > s_0$, $s_0 \leq s < \frac{1}{1-\sigma}$. Then for any data $f \in C([0,T]; H^m_{\rho;s,\theta}(\mathbb{R}))$ and $g \in H^m_{\rho;s,\theta}(\mathbb{R})$, the Cauchy problem (1) admits a solution $u \in C([0,T]; H^m_{(\rho_1,-\delta);s,\theta}(\mathbb{R}))$ for every $\delta > 0$. In this case, the solution has the same Gevrey regularity with respect to the initial data, but may present an exponential growth at the infinity. We also note that this is the first time that an algebrically growth in the coefficients is allowed in this type of problem. In Chapter 4 we analyze the same problem in the case $p = 3$, cf. Theorem 4.1.

The main objective of this thesis is to study the Cauchy problem (1) in the Gevrey and Gelfand-Shilov frameworks where the degree of evolution is $p = 3$. The principal results that we have achieved are Theorem 3.1 and Theorem 4.1. The first concerns the Gevrey case and the second the Gelfand-Shilov setting.

Now we briefly describe the contents presented in this work. In Chapters 1 and 2 we collect the main notations and mathematical tools for the development of the later chapters. Our main contributions here are:

- Subsection 2.2, where we develop a calculus for a class of pseudodifferential operators with symbols of infinite order satisfying Gevrey estimates;

- Section 2.5 where we study the Friedrichs part of pseudodifferential operators with symbols in the SG and $SG_{\mu,\nu}$ classes, obtaining a precise asymptotic expansion and regularity for the same (see Theorems 2.17 and 2.19);

- Section 2.6, where we prove a result concerning the spectral invariance for SG operators satisfying Gevrey estimates.

Chapters 3 and 4 are entirely devoted to prove the main results of this work, namely Theorems 3.1 and 4.1. The principal sources of inspiration here are: [3, 5, 6, 13, 14, 33]. We would also like to point out that the efforts put into these chapters gave rise to the following two works [31] and [30].

Finally, in the last short Chapter 5 we outline some new plans for the future, based on the work presented in the thesis. The main purposes are the extension of the results obtained for the case $p = 3$ to $p-$evolution equations of arbitrary order in the linear case, the study of necessary conditions for well-posedness in Gevrey type spaces for $p \geq 3$ and lastly to consider semilinear p-evolution equations also in the Gevrey setting.

Chapter 1

Preliminaries

1.1 Basic Notations and Formulas

We start with the multi-index notation. We use $\mathbb{N}_0^n$ to represent the set of all multi-indices, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. For $\alpha, \beta \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$ we will use the following notations:

- $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ (multi-index length);
- $\beta \leq \alpha \iff \beta_j \leq \alpha_j,\ j = 1, \ldots, n$;
- $\alpha! = \alpha_1! \ldots \alpha_n!$;
- if $\beta \leq \alpha$,
$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!};$$
- $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$;
- partial derivatives are denoted by $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$, where $\partial_{x_j} = \frac{\partial}{\partial_{x_j}}$;
- $D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$, where $D_{x_j} = -i\partial_{x_j}$.

If $\xi, x \in \mathbb{R}^n$, we set
$$\xi x := \xi \cdot x = \sum_{j=0}^{n} \xi_j x_j, \quad |x|^2 = \sum_{j=0}^{n} x_j^2, \quad \langle x \rangle = \sqrt{1 + |x|^2}.$$

Observe that $\langle x \rangle$ is a smooth function and its asymptotic behavior is equivalent to $1 + |x|$, more precisely
$$\langle x \rangle \leq 1 + |x| \leq \sqrt{2} \langle x \rangle, \quad x \in \mathbb{R}^n.$$

In general one can prove that, for any $m \in \mathbb{R}$,

$$|\partial_x^\beta \langle x\rangle^m| \leq C^{|\beta|}\beta!\langle x\rangle^{m-|\beta|},$$

for some constant $C > 0$ independent of β. We also recall the *Peetre's inequality*: for any $s \in \mathbb{R}$ there is $c_s > 0$ such that

$$\langle x+\xi\rangle^s \leq c_s\langle x\rangle^{|s|}\langle \xi\rangle^s, \quad x,\xi \in \mathbb{R}^n.$$

We denote by $C^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$. For any $f \in C^\infty(\mathbb{R}^n)$ we have *Taylor's formula*: given $N \in \mathbb{N}$,

$$f(x+\xi) = \sum_{|\alpha|<N} \frac{\xi^\alpha}{\alpha!}(\partial^\alpha f)(x) + N\sum_{|\alpha|=N}\frac{\xi^\alpha}{\alpha!}\int_0^1 (1-\theta)^{N-1}(\partial^\alpha f)(x+\theta\xi)d\theta, \quad x,\xi \in \mathbb{R}^n.$$

There are two derivative formulas largely used throughout this work. The first one is the *Leibniz rule*:

$$D^\alpha(fg) = \sum_{\alpha_1+\alpha_2=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!}D^{\alpha_1}fD^{\alpha_2}g = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}D^\beta fD^{\alpha-\beta}g.$$

The second one is the *Faà di Bruno formula*: if $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{R}$, then

$$\partial_x^\alpha(f\circ g)(x) = \sum_{j=1}^{|\alpha|}\frac{f^{(j)}(g(x))}{j!}\sum_{\substack{\alpha^1+\ldots+\alpha^j=\alpha\\ |\alpha^\nu|\geq 1}}\frac{\alpha!}{\alpha^1!\ldots\alpha^j!}\prod_{\nu=1}^j \partial_x^{\alpha^\nu}g(x).$$

1.2 Standard Factorial Inequalities

Here we collect some well known formulas for factorials and binomial coefficients that we will use extensively in this work. These formulas are useful in the study of Gevrey and Gelfand-Shilov functions. We begin with the *generalized Newton formula*: if $N \in \mathbb{N}$ and $a_1, \ldots a_n$ are real numbers, then

$$(a_1+\ldots+a_n)^N = \sum_{\substack{|\alpha|=N\\ \alpha\in\mathbb{N}_0^n}}\frac{N!}{\alpha_1!\ldots\alpha_n!}\prod_{i=1}^n a_i^{\alpha_i}.$$

Particularly, when $a_1 = \ldots = a_n = 1$ it follows

$$n^N = \sum_{|\alpha|=N}\frac{N!}{\alpha_1!\ldots\alpha_n!},$$

which implies

$$|\alpha|! \leq n^{|\alpha|}\alpha!, \quad \alpha \in \mathbb{N}_0^n.$$

If $n = 2$, we obtain

$$2^N = \sum_{j+k=N} \frac{N!}{j!k!}, \quad (j+k)! \le 2^{j+k} j!k!,$$

hence, for $\alpha, \beta \in \mathbb{N}_0^n$,

$$(\alpha+\beta)! \le 2^{|\alpha+\beta|}\alpha!\beta!, \quad \alpha!\beta! \le (\alpha+\beta)!, \quad \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} = 2^{|\alpha|}.$$

A second block of inequalities follows from the Taylor expansion of the exponential function

$$e^t = \sum_{N\ge 0} \frac{t^N}{N!}, \quad t \ge 0.$$

The above formula implies

$$t^N \le N! e^t, \quad N \in \mathbb{N}_0, t \ge 0.$$

In particular, for $t = N$ we obtain

$$N^N \le N! e^N, \quad N \in \mathbb{N}_0,$$

whereas obviously,

$$N! \le N^N, \quad N \in \mathbb{N}_0.$$

We finish observing the cardinality of the two subsequent sets of multi-indices

$$\#\{\alpha \in N_0^n : |\alpha| \le m\} = \frac{(m+n)!}{m!n!},$$

$$\#\{\alpha \in \mathbb{N}_0^n : |\alpha| = m\} = \frac{(m+n-1)!}{m!(n-1)!}.$$

1.3 Some Functional Spaces and Fourier Transform

As usual $L^2(\mathbb{R}^n)$ stands for the *Hilbert* space of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^2}^2 := \int |f(x)|^2 dx \le \infty.$$

The $L^2(\mathbb{R}^n)$ inner product is given by

$$\langle f, g\rangle_{L^2} := \int f(x)\overline{g(x)}dx, \quad f, g \in L^2(\mathbb{R}^n).$$

The *Schwartz* space $\mathscr{S}(\mathbb{R}^n)$ is defined as the space of all smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)| \le \infty, \quad \alpha, \beta \in \mathbb{N}_0^n.$$

The space $\mathscr{S}(\mathbb{R}^n)$ possesses a Fréchet topology once equipped with the usual seminorms, namely,

$$\|f\|_{\ell,\mathscr{S}} := \max_{|\alpha+\beta|\le\ell} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)|, \quad \ell\in\mathbb{N}_0,\ f\in\mathscr{S}(\mathbb{R}^n).$$

The elements of its topological dual $\mathscr{S}'(\mathbb{R}^n)$ are called tempered distributions.

The *Fourier* transform is defined by

$$\mathcal{F}(f)(\xi) := \widehat{f}(\xi) = \int e^{-i\xi x} f(x)dx, \quad \xi\in\mathbb{R}^n,$$

and its inverse is given by

$$\mathcal{F}^{-1}(f)(x) := \int e^{i\xi x} f(\xi) đ\xi, \quad x\in\mathbb{R}^n,$$

where $đ\xi = (2\pi)^{-n}d\xi$ and f belongs to $\mathscr{S}(\mathbb{R}^n)$. It is well known that $\mathcal{F}$ defines an isomorphism on $\mathscr{S}(\mathbb{R}^n)$ which extends to an isomorphism on $\mathscr{S}'(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. We also point out the subsequent well known formulas

$$\int f(x)\overline{g(x)}dx = \int \mathcal{F}(\xi)\overline{\mathcal{F}(g)(\xi)}đ\xi \quad \textit{(Parseval formula)},$$

$$\|f\|_{L^2} = (2\pi)^{-\frac{n}{2}}\|\mathcal{F}(f)\|_{L^2} \quad \textit{(Plancherel formula)}.$$

1.4 Gevrey and Gelfand-Shilov Type Spaces

In this Section, we introduce the functional spaces in which many of our results will be achieved. We begin defining the uniform Gevrey classes. A detailed exposition concerning Gevrey spaces can be found in [43].

For $s>1$ and $A>0$ we denote by $G^s(\mathbb{R}^n;A)$ the Banach space of all smooth functions f such that

$$\|f\|_{G^s(\mathbb{R}^n;A)} := \sup_{\substack{x\in\mathbb{R}^n\\ \alpha\in\mathbb{N}_0}} |\partial_x^\beta f(x)|A^{-|\beta|}\beta!^{-s} < +\infty.$$

We then set the Gevrey space $G^s(\mathbb{R}^n)$ as the inductive limite of the Banach spaces $G^s(\mathbb{R}^n;A)$, that is

$$G^s(\mathbb{R}^n) = \bigcup_{A>0} G^s(\mathbb{R}^n;A).$$

Now we define the so-called Gelfand-Shilov spaces. These spaces were firstly introduced by *I.M. Gelfand* and *G.E. Shilov*, see [22]. Here we only give the basic definitions and properties. For more details we adress the reader to [12, 22, 42, 45] and Chapter 6 of [39].

For $s, \theta \geq 1$ and $A, B > 0$ we say that a smooth function f belongs to $\mathcal{S}_{s,B}^{\theta,A}(\mathbb{R}^n)$ if there is $C > 0$ such that

$$|x^\beta \partial_x^\alpha f(x)| \leq C A^{|\alpha|} B^{|\beta|} \alpha!^\theta \beta!^s,$$

for every $\alpha, \beta \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$. The norm

$$\|f\|_{\theta,s,A,B} = \sup_{\substack{x\in\mathbb{R}^n \\ \alpha,\beta\in\mathbb{N}_0^n}} |x^\beta \partial_x^\alpha f(x)| A^{-|\alpha|} B^{-|\beta|} \alpha!^{-\theta} \beta!^{-s}, \quad f \in \mathcal{S}_{s,B}^{\theta,A}(\mathbb{R}^n),$$

turns $\mathcal{S}_{s,B}^{\theta,A}(\mathbb{R}^n)$ into a Banach space. We define

$$\mathcal{S}_s^\theta(\mathbb{R}^n) = \bigcup_{A,B>0} \mathcal{S}_{s,B}^{\theta,A}(\mathbb{R}^n), \quad \Sigma_s^\theta(\mathbb{R}^n) = \bigcap_{A,B>0} \mathcal{S}_{s,B}^{\theta,A}(\mathbb{R}^n),$$

and equip them with their limit type topologies coming from the Banach spaces $\mathcal{S}_{s,B}^{\theta,A}(\mathbb{R}^n)$.

Remark 1.1. *When $\theta = s$ we simply write $\mathcal{S}_\theta$, Σ_θ instead of $\mathcal{S}_\theta^\theta, \Sigma_\theta^\theta$.*

Remark 1.2. *If $\theta + s < 1$ the spaces $\mathcal{S}_s^\theta(\mathbb{R}^n)$ are trivial. On the other hand the function $e^{-\frac{|x|^2}{2}}$ belongs to $\mathcal{S}_{\frac{1}{2}}(\mathbb{R}^n)$. Concerning the projective classes we have $\Sigma_s^\theta(\mathbb{R}^n) \neq \{0\}$ if, and only if, $\theta + s \geq \frac{1}{2}$ and $(\theta, s) \neq (\frac{1}{2}, \frac{1}{2})$.*

Remark 1.3. *We may also define, for $C, \varepsilon > 0$, the Banach space $\mathcal{S}_{s,\varepsilon}^{\theta,C}(\mathbb{R}^n)$ given by the smooth functions f such that there is $C_1 > 0$ satisfying*

$$\|f\|_{s,\theta}^{\varepsilon,C} := \sup_{\substack{x\in\mathbb{R}^n \\ \alpha\in\mathbb{N}_0^n}} C^{-|\alpha|} \alpha!^{-\theta} e^{\varepsilon|x|^{\frac{1}{s}}} |\partial_x^\alpha f(x)| < \infty,$$

and we have (with equivalent topologies)

$$\mathcal{S}_s^\theta(\mathbb{R}^n) = \bigcup_{C,\varepsilon>0} \mathcal{S}_{s,\varepsilon}^{\theta,C}(\mathbb{R}^n), \quad \Sigma_s^\theta(\mathbb{R}^n) = \bigcap_{C,\varepsilon>0} \mathcal{S}_{s,\varepsilon}^{\theta,C}(\mathbb{R}^n).$$

It is easy to see that the following inclusions are continuous (for every $\varepsilon > 0$)

$$\Sigma_s^\theta(\mathbb{R}^n) \subset \mathcal{S}_s^\theta(\mathbb{R}^n) \subset \Sigma_{s+\varepsilon}^{\theta+\varepsilon}(\mathbb{R}^n).$$

We shall denote by $(\mathcal{S}_s^\theta)'(\mathbb{R}^n)$, $(\Sigma_s^\theta)'(\mathbb{R}^n)$ the respective dual spaces. Concerning the action of the Fourier transform, we have the subsequent isomorphisms

$$\mathcal{F} : \Sigma_s^\theta(\mathbb{R}^n) \to \Sigma_\theta^s(\mathbb{R}^n), \quad \mathcal{F} : \mathcal{S}_s^\theta(\mathbb{R}^n) \to \mathcal{S}_\theta^s(\mathbb{R}^n),$$

$$\mathcal{F} : (\Sigma_s^\theta)'(\mathbb{R}^n) \to (\Sigma_\theta^s)'(\mathbb{R}^n), \quad \mathcal{F} : (\mathcal{S}_s^\theta)'(\mathbb{R}^n) \to (\mathcal{S}_\theta^s)'(\mathbb{R}^n).$$

Chapter 2

Global Pseudodifferential Operators

Let $p(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ be a linear partial differential operator with coefficients $a_\alpha \in \mathcal{B}^\infty(\mathbb{R}^n)$, where

$$\mathcal{B}^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |\partial^\beta f(x)| < \infty, \forall \beta \in \mathbb{N}_0^n\}.$$

By standard properties of the Fourier transform, the action of $p(x, D)$ in *Schwartz* spaces can be written as

$$p(x, D)u(x) = \int e^{i\xi x} p(x, \xi) \widehat{u}(\xi) đ\xi, \quad x \in \mathbb{R}^n, u \in \mathscr{S}(\mathbb{R}^n), \tag{2.1}$$

where $p(x, \xi)$ stands for the symbol of $p(x, D)$, namely $p(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha$ for every $x, \xi \in \mathbb{R}^n$. Notice that $p(x, \xi)$ satisfies an estimate of the following form: for every $\alpha, \beta \in \mathbb{N}_0^n$ there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m - |\alpha|}, \quad x, \xi \in \mathbb{R}^n. \tag{2.2}$$

The right hand side of (2.1) indeed makes sense for every smooth function $p(x, \xi)$ satisfying an estimate of the type (2.2), where $m \in \mathbb{R}$. We call symbols such functions whereas the operators defined by (2.1) are known as pseudodifferential operators. When the symbol is polynomially bounded, we say that the symbol has finite order. In the situation where it satisfies an exponential estimate, we say that the symbol has infinite order. Observe that if we are dealing with operators of infinite order, to obtain a convergent integral in (2.1) some stronger decay condition for $\widehat{u}$ is required (for instance, we can take u belonging to a suitable Gelfand-Shilov class).

We also consider operators given by symbols which are polynomially bounded with respect to x and ξ simultaneously. More precisely, symbols satisfying ($m_1, m_2 \in \mathbb{R}$)

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m_1-|\alpha|} \langle x \rangle^{m_2-|\beta|}, \quad x, \xi \in \mathbb{R}^n. \tag{2.3}$$

Pseudodifferential operators given by these symbols are known as pseudodifferential operators of SG type (cf. [44]).

Pseudodifferential operators are a fundamental mathematical tool for the development of this thesis. In this chapter we collect the basic properties in the finite order frame, see Section 2.1, and we develop some new results concerning pseudodifferential operators of inifinite order with symbols satisfying Gevrey estimates, see Sections 2.2 and 2.3. Some references for this part are: [1, 8, 9, 11, 10, 27, 35, 39, 43, 46, 47].

In Section 2.4 we recall two results which are very important for the subsequent chapters, namely the *sharp Gårding* and the *Fefferman-Phong* inequalities.

The classical strategy to prove the sharp Gårding inequality is the following: if the symbol $p(x,\xi)$ of a given operator $p(x,D)$ is such that $Re\, p(x,\xi) \geq 0$, then it is possible to decompose $p(x,D)$ as a sum of a positive definite part and a smaller order remainder term. In the approach proposed in [35], this positive part p_F is called *Friedrichs* part. Using the ideas present there, in Section 2.5 we define a SG version of p_F and obtain a precise estimate for the remainder term $p - p_F$, see Theorems 2.17 and 2.19.

To finish, Section 2.6 is devoted to the study of the so-called spectral invariance problem for SG operators given by Gevrey regular symbols.

Remark 2.1. *The term global in the title of the Chapter is due to the fact that we are dealing with symbols satisfying uniform estimates on the entire* $\mathbb{R}^{2n}$.

2.1 $S^m(\mathbb{R}^{2n})$ and $SG^m(\mathbb{R}^{2n})-$Pseudodifferential Operators

The goal of this Section is to give a brief introduction to pseudodifferential operators with symbols in $S^m(\mathbb{R}^{2n})$ and $SG^m(\mathbb{R}^{2n})$. The statements and proofs concerning the standard *Hörmander* classes S^m can be found in [35] and [46]. For the SG frame we adress the reader to [8, 16, 39, 40].

We start considering the standard Hörmander classes $S^m(\mathbb{R}^{2n})$.

Definition 2.1. *Let $m \in \mathbb{R}$. We say that $p \in S^m(\mathbb{R}^{2n})$ when $p \in C^\infty(\mathbb{R}^{2n})$ and for every $\alpha, \beta \in \mathbb{N}_0^n$ there exists $C_{\alpha,\beta} > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|}, \quad x, \xi \in \mathbb{R}^n.$$

We recall that $S^m(\mathbb{R}^{2n})$ is a *Fréchet* space endowed with the seminorms, for $\ell \in \mathbb{N}_0$,

$$|p|_{S^m,\ell} := \max_{|\alpha+\beta| \leq \ell} \sup_{x,\xi \in \mathbb{R}^n} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \langle \xi \rangle^{-m+|\alpha|}, \quad p \in S^m(\mathbb{R}^{2n}).$$

Pseudodifferential operators with symbols in $S^m(\mathbb{R}^{2n})$ are continuous from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$ and they extend to continuous maps from $\mathscr{S}'(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$. Moreover, denoting by $H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$, the *Sobolev* space

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle D \rangle^s u \in L^2(\mathbb{R}^n)\},$$

where $\langle D \rangle^s$ is the operator given by the symbol $\langle \xi \rangle^s \in S^s(\mathbb{R}^{2n})$, then an operator with symbol in $S^m(\mathbb{R}^{2n})$ extends to a bounded map from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$ for every $s \in \mathbb{R}$. Furthermore, the norm of $p(x,D)$ as an operator from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$ is bounded in terms of a finite number of seminorms of p in $S^m(\mathbb{R}^{2n})$.

The next results concern composition, adjoint and transpose for operators with symbols in $S^m(\mathbb{R}^{2n})$ classes. In order to state these results we need the definition of asymptotic expansion in $S^m(\mathbb{R}^{2n})$.

Definition 2.2. *Let $\{m_j\}_{j \in \mathbb{N}_0}$ be a sequence of real numbers such that $m_j \to -\infty$ and $m_j \geq m_{j+1}$, for every $j \in \mathbb{N}_0$. Consider moreover $p \in S^m(\mathbb{R}^{2n})$ and $p_j \in S^{m_j}(\mathbb{R}^{2n})$, for every $j \in \mathbb{N}_0$. We say that p is asymptotic to $\sum_j p_j$ in $S^m(\mathbb{R}^{2n})$, and denote $p \sim \sum_j p_j$ in $S^m(\mathbb{R}^{2n})$, when*

$$p - \sum_{j<N} p_j \in S^{m_N},$$

for every $N \in \mathbb{N}$.

Theorem 2.1. *Let $p \in S^m(\mathbb{R}^{2n})$ and $q \in S^{m'}(\mathbb{R}^{2n})$. There exists $c \in S^{m+m'}(\mathbb{R}^{2n})$ such that $c(x,D) = p(x,D) \circ q(x,D)$ and*

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) \text{ in } S^{m+m'}(\mathbb{R}^{2n}).$$

Theorem 2.2. *Let $p \in S^m(\mathbb{R}^{2n})$. Let moreover $p^*(x,D)$ and ${}^tp(x,D)$ be the L^2-adjoint and transpose of $p(x,D)$ respectively. There exist symbols $a, b \in S^m(\mathbb{R}^{2n})$ such that $a(x,D) = p^*(x,D)$, $b(x,D) = {}^tp(x,D)$ and*

$$a(x,\xi) \sim \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} \overline{\partial_\xi^\alpha D_x^\alpha p(x,\xi)}, \quad b(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} (\partial_\xi^\alpha D_x^\alpha p)(x,-\xi) \text{ in } S^m(\mathbb{R}^{2n}).$$

Now we turn our attention to SG symbol classes.

Definition 2.3. *Let $m = (m_1, m_2) \in \mathbb{R}^2$. We say that $p \in SG^m(\mathbb{R}^{2n})$ when p belongs to $C^\infty(\mathbb{R}^{2n})$ and for every $\alpha, \beta \in \mathbb{N}_0^n$ there exists $C_{\alpha,\beta} > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m_1 - |\alpha|} \langle x \rangle^{m_2 - |\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

We recall that $SG^m(\mathbb{R}^{2n})$ is a Fréchet space endowed with the seminorms, for $\ell \in \mathbb{N}_0$,

$$|p|_{SG^m,\ell} := \max_{|\alpha+\beta| \leq \ell} \sup_{x,\xi \in \mathbb{R}^n} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \langle \xi \rangle^{-m_1+|\alpha|} \langle x \rangle^{-m_2+|\beta|}, \quad p \in SG^m(\mathbb{R}^{2n}).$$

Pseudodifferential operators with symbols in $SG^m(\mathbb{R}^{2n})$ are continuous from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$ and extend to continuous maps fom $\mathscr{S}'(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$. Moreover, denoting by $H^{(s_1,s_2)}(\mathbb{R}^n)$, with $s = (s_1, s_2) \in \mathbb{R}^2$, the *weighted Sobolev* space

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle x \rangle^{s_2} \langle D_x \rangle^{s_1} u \in L^2(\mathbb{R}^n)\},$$

we know that an operator with symbol in $SG^m(\mathbb{R}^{2n})$ extends to a bounded map from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$, for every $s \in \mathbb{R}^2$.

Before stating the results about composition, adjoint and transpose for operators with symbols in $SG^m(\mathbb{R}^{2n})$ classes, we need the concept of *smoothing* operator in the SG classes. We say that $r(x,D)$ is a SG smoothing operator whenever its symbol $r(x,\xi) \in SG^m$ for every $m \in \mathbb{R}^2$. Equivalently, $r(x,D)$ is smoothing if $r(x,D) : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ continuously. Observe that $\cap_{m \in \mathbb{R}^2} SG^m(\mathbb{R}^{2n}) = \mathscr{S}(\mathbb{R}^{2n})$.

Definition 2.4. *Consider $\{m_j\}_{j \in \mathbb{N}_0} \subset \mathbb{R}^2$, $m_j = (m_{1,j}, m_{2,j})$, a sequence such that $m_{i,j} \to -\infty$ and $m_{i,j} \geq m_{i,j+1}$, for every $i = 1,2$ and $j \in \mathbb{N}_0$. Consider moreover $p \in SG^m(\mathbb{R}^{2n})$ and $p_j \in SG^{m_j}(\mathbb{R}^{2n})$, for every $j \in \mathbb{N}_0$. We say that p is asymptotic to $\sum_j p_j$ in $SG^m(\mathbb{R}^{2n})$, and denote $p \sim \sum_j p_j$ in $SG^m(\mathbb{R}^{2n})$, when*

$$p - \sum_{j<N} p_j \in SG^{m_N},$$

for every $N \in \mathbb{N}$.

Theorem 2.3. *Let $p \in SG^m(\mathbb{R}^{2n})$ and $q \in SG^{m'}(\mathbb{R}^{2n})$. There exist $c \in SG^{m+m'}(\mathbb{R}^{2n})$ and a smoothing operator $r(x,D)$ such that $p(x,D) \circ q(x,D) = c(x,D) + r(x,D)$ and*

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) \text{ in } SG^{m+m'}(\mathbb{R}^{2n}).$$

Theorem 2.4. *Let $p \in SG^m(\mathbb{R}^{2n})$. Let moreover $p^*(x,D)$ and ${}^tp(x,D)$ be the L^2-adjoint and transpose of $p(x,D)$ respectively. There exist $a,b \in SG^m(\mathbb{R}^{2n})$ and $r_1(x,D), r_2(x,D)$ smoothing operators such that $p^*(x,D) = a(x,D) + r_1(x,D)$, ${}^tp(x,D) = b(x,D) + r_2(x,D)$ and*

$$a(x,\xi) \sim \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!}\overline{\partial_\xi^\alpha D_x^\alpha p(x,\xi)}, \quad b(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(\partial_\xi^\alpha D_x^\alpha p)(x,-\xi) \text{ in } SG^m(\mathbb{R}^{2n}).$$

2.2 $S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $S^m_{\mu,\nu}(\mathbb{R}^{2n})-$Pseudodifferential Operators

In this section we develop a global calculus for pseudodifferential operators of infinite and finite order (with respect to ξ) given by symbols satisfying Gevrey estimates. The local case can be found in [25] (finite order) and in [47] (infinite order). Our proofs are strongly inspired by the Appendix A of [6] and by [9, 47].

We begin defining the symbol classes.

Definition 2.5. *Let $A > 0$, $m \in \mathbb{R}$ and $\mu, \nu > 1$. We denote by $S^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$ the Banach space of all functions $a \in C^\infty(\mathbb{R}^{2n})$ satisfying the following condition:*

$$\|a\|_A := \sup_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ x,\xi\in\mathbb{R}^n}} A^{-|\alpha+\beta|}\alpha!^{-\mu}\beta!^{-\nu}\langle\xi\rangle^{-m+|\alpha|}|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| \leq +\infty.$$

We set

$$S^m_{\mu,\nu}(\mathbb{R}^{2n}) := \bigcup_{A>0} S^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$$

endowed with the inductive limit topology of the Banach spaces $S^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$.

Definition 2.6. *Let $\mu, \nu, \theta > 1$ and $A, c > 0$. We denote by $S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n}; A, c)$ the Banach space of all functions $a \in C^\infty(\mathbb{R}^{2n})$ satisfying the following condition:*

$$\|a\|_{A,c} := \sup_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ x,\xi\in\mathbb{R}^n}} A^{-|\alpha+\beta|}\alpha!^{-\mu}\beta!^{-\nu}\langle\xi\rangle^{|\alpha|}e^{-c|\xi|^{\frac{1}{\theta}}}|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| \leq +\infty.$$

We set

$$S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n}) := \bigcup_{A,c>0} S^m_{\mu,\nu;\theta}(\mathbb{R}^{2n}; A, c)$$

endowed with the inductive limit topology of the Banach spaces $S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n}; A, c)$.

Remark 2.2. *In the situation of $\mu = \nu$ we simply write $S^m_\mu(\mathbb{R}^{2n})$ and $S^\infty_{\mu;\theta}(\mathbb{R}^{2n})$ instead of $S^m_{\mu,\mu}(\mathbb{R}^{2n})$ and $S^\infty_{\mu,\mu;\theta}(\mathbb{R}^{2n})$.*

Remark 2.3. *We have the obvious inclusion $S^m_{\mu,\nu}(\mathbb{R}^{2n}) \subset S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ for every $m \in \mathbb{R}$ and $\theta > 1$.*

2.2.1 Continuity on $\Sigma_s^\theta(\mathbb{R}^n)$

We now prove that pseudodifferential operators of the form (2.1) act continuously on Gelfand-Shilov spaces. Before going to the continuity result, we need the two following technical lemmas.

Lemma 2.1. *Let $s > 1$, $\zeta > 0$ and define*

$$m_{s,\zeta}(x) = \sum_{j=0}^{\infty} \frac{\zeta^j \langle x\rangle^{2j}}{j!^{2s}}, \quad x \in \mathbb{R}^n.$$

Then for every $\varepsilon > 0$ there is a constant $C := C_{s,\varepsilon} > 0$ such that

$$C^{-1} e^{(2s-\varepsilon)\zeta^{\frac{1}{2s}} \langle x\rangle^{\frac{1}{s}}} \leq m_{s,\zeta}(x) \leq C e^{(2s+\varepsilon)\zeta^{\frac{1}{2s}} \langle x\rangle^{\frac{1}{s}}}, \quad x \in \mathbb{R}^n.$$

Proof. See Proposition 2.4 of [29]. □

Lemma 2.2. *Let $s \geq 1$ and $\varepsilon > 0$. Then*

$$|x|^{|\alpha|} e^{-\varepsilon|x|^{\frac{1}{s}}} \leq \left(\frac{ns}{\varepsilon}\right)^{s|\alpha|} \alpha!^s, \quad x \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n.$$

Proof. Set $f(t) = t^{|\alpha|} e^{-\varepsilon t^{\frac{1}{s}}}$. It is not difficult to see that $\bar{t} = (\frac{s|\alpha|}{\varepsilon})^s$ maximizes the function $f(t)$. Hence, for every $t \in \mathbb{R}$,

$$\begin{aligned} t^{|\alpha|} e^{-\varepsilon t^{\frac{1}{s}}} &= f(t) \leq f(\bar{t}) = \left(\frac{s|\alpha|}{\varepsilon}\right)^{s|\alpha|} e^{-s|\alpha|} \\ &= \left(\frac{s}{\varepsilon}\right)^{s|\alpha|} \left(\frac{|\alpha|^{|\alpha|}}{e^{|\alpha|}}\right)^s \leq \left(\frac{ns}{\varepsilon}\right)^{s|\alpha|} \alpha!^s. \end{aligned}$$

□

Proposition 2.1. *Let $\mu, \nu, s, \theta > 1$ such that $s > \mu$ and $\theta > \nu$. Let moreover $p \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Then, the operator $p(x, D)$ is continuous from $\Sigma_s^\theta(\mathbb{R}^n)$ to $\Sigma_s^\theta(\mathbb{R}^n)$ and it extends to a linear continuous operator from $(\Sigma_s^\theta)'(\mathbb{R}^n)$ to $(\Sigma_s^\theta)'(\mathbb{R}^n)$.*

Proof. Let $F \subset \Sigma_s^\theta(\mathbb{R}^n)$ be a bounded set. Then $\mathcal{F}(F) \subset \Sigma_\theta^s(\mathbb{R}^n)$ is a bounded set, more precisely, for every $C, r > 0$ there exists $H > 0$ (independent of f) satisfying

$$|\partial_\xi^\alpha \widehat{f}(\xi)| \leq H C^\alpha \alpha!^s e^{-r|\xi|^{\frac{1}{\theta}}}, \quad x \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n,$$

for every $f \in F$.

Recalling that

$$\frac{1}{m_{s,\zeta}(x)} \sum_{j=0}^{\infty} \frac{\zeta^j}{j!^{2s}} (1 - \Delta_\xi)^j e^{i\xi x} = e^{i\xi x},$$

we have, for $\beta \in \mathbb{N}_0^n$,

$$\begin{aligned}
\partial_x^\beta p(x,D)(f)(x) &= \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \int e^{i\xi x}(i\xi)^{\beta_1}\partial_x^{\beta_2} p(x,\xi)\widehat{f}(\xi) đ\xi \\
&= \frac{1}{m_{s,\zeta}(x)} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \int \sum_{j=0}^\infty \frac{\zeta^j}{j!^{2s}}(1-\Delta_\xi)^j e^{i\xi x}(i\xi)^{\beta_1}\partial_x^{\beta_2} p(x,\xi)\widehat{f}(\xi) đ\xi \\
&= \frac{1}{m_{s,\zeta}(x)} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \int e^{i\xi x} \sum_{j=0}^\infty \frac{\zeta^j}{j!^{2s}}(1-\Delta_\xi)^j [(i\xi)^{\beta_1}\partial_x^{\beta_2} p(x,\xi)\widehat{f}(\xi)] đ\xi.
\end{aligned}$$

Now we will estimate $|(1-\Delta_\xi)^j[(i\xi)^{\beta_1}\partial_x^{\beta_2}p(x,\xi)\widehat{f}(\xi)]| =: \dagger$. First observe that, denoting $J=(j_1,\dots,j_n)$,

$$(1-\Delta_\xi)^j = \sum_{j_0+J=j} \frac{j!}{j_0!J!}(-1)^{|J|}\partial_\xi^{2J}.$$

Since $s>\mu$ and $\theta>\nu$, we have

$$\begin{aligned}
\dagger &\leq \sum_{j_0+J=j} \frac{j!}{j_0!J!} \sum_{\lambda_1+\lambda_2+\lambda_3=2J} \frac{(2J)!}{\lambda_1!\lambda_2!\lambda_3!} \frac{\beta_1!}{(\beta_1-\lambda_1)!} |\xi^{\beta_1-\lambda_1}||\partial_\xi^{\lambda_2}\partial_x^{\beta_2}p(x,\xi)||\partial_\xi^{\lambda_3}\widehat{f}(\xi)| \\
&\leq \sum_{j_0+J=j} \frac{j!}{j_0!J!} \sum_{\lambda_1+\lambda_2+\lambda_3=2J} \frac{(2J)!}{\lambda_1!\lambda_2!\lambda_3!} \frac{\beta_1!}{(\beta_1-\lambda_1)!} |\xi^{\beta_1-\lambda_1}| C_p^{|\lambda_2+\beta_2+1|}\lambda_2!^\mu\beta_2!^\nu \langle\xi\rangle^{-|\lambda_2|}e^{c_p|\xi|^{\frac{1}{\theta}}} \\
&\times HC^{|\lambda_3|}\lambda_3!^s e^{-r|\xi|^{\frac{1}{\theta}}} \\
&\leq (\tilde{C}_pH)C^{|\beta_2|}\beta_2!^\theta \sum_{j_0+J=j} \frac{j!}{j_0!J!} \sum_{\lambda_1+\lambda_2+\lambda_3=2J} \frac{(2J)!}{\lambda_1!\lambda_2!\lambda_3!} \frac{\beta_1!}{(\beta_1-\lambda_1)!} \\
&\times |\xi^{\beta_1-\lambda_1}| e^{(c_p-\frac{r}{2})|\xi|^{\frac{1}{\theta}}} C^{|\lambda_2+\lambda_3|}\lambda_2!^s\lambda_3!^s e^{-\frac{r}{2}|\xi|^{\frac{1}{\theta}}}.
\end{aligned}$$

By Lemma 2.2, for any $A_1>0$ we can choose $r(\theta,n)=r>2c_p$ such that

$$\frac{\beta_1!}{(\beta_1-\lambda_1)!}|\xi^{\beta_1-\lambda_1}|e^{(c_p-\frac{r}{2})|\xi|^{\frac{1}{\theta}}} \leq \frac{\beta_1!}{(\beta_1-\lambda_1)!}A_1^{|\beta_1-\lambda_1|}(\beta_1-\lambda_1)!^\theta \leq \tilde{H}A_1^{|\beta_1|}\beta_1!^\theta A_1^{|\lambda_1|}\lambda_1!^s.$$

Therefore, by standard factorial inequalities, for every $C,D>0$ there exists $H_{p,r,C,D} := H$ such that

$$\dagger \leq HC^{|\beta|}\beta_1!^\theta\beta_2!^\theta e^{-\frac{r}{2}|\xi|^{\frac{1}{\theta}}} j!^{2s}D^j, \quad x,\xi\in\mathbb{R}^n, j\in\mathbb{N}_0, f\in F.$$

By the above inequality and Lemma 2.1, we can take $\varepsilon>0$ satisfying $\varepsilon<2s$ and for every $\zeta>0$ we may choose $D>0$ such that $\zeta D<1$ in order to get

$$\begin{aligned}
|\partial_x^\beta p(x,D_x)(f)(x)| &\leq HC_{s,\varepsilon}e^{-(2s-\varepsilon)\zeta^{\frac{1}{2s}}\langle x\rangle^{\frac{1}{s}}} \\
&\times \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} C^{|\beta|}\beta!_1^\theta\beta_2!^\nu \int e^{-\frac{r}{2}|\xi|^{\frac{1}{\theta}}} đ\xi \sum_j (\zeta D)^j.
\end{aligned}$$

Therefore, for any $C_1 > 0$ and $r_1 > 0$ we can find a constant $H_{1,C_1,r_1} := H_1 > 0$ such that

$$|\partial_x^\beta p(x,D)(f)(x)| \leq H_1 C_1^\beta \beta!^\theta e^{-r_1|x|^{\frac{1}{s}}}, \quad x \in \mathbb{R}^n,$$

for every $f \in F$. In other words, we proved that $p(F)$ is a bounded subset of $\Sigma_s^\theta(\mathbb{R}^n)$.

In order to define the extension to the dual space we observe that, for any $u, v \in \Sigma_s^\theta(\mathbb{R}^n)$, we may write

$$\int p(x,D)u(x)v(x)dx = \int \widehat{u}(\xi)Q(v)(\xi)d\xi,$$

where Q is defined by

$$Q(v)(\xi) = \int e^{i\xi x} p(x,\xi)v(x)đx.$$

Now we shall prove that Q maps $\Sigma_s^\theta(\mathbb{R}^n)$ to $\Sigma_\theta^s(\mathbb{R}^n)$ continuously. Once more we consider $F \subset \Sigma_s^\theta(\mathbb{R}^n)$ bounded, that is, for every $C, c > 0$ we find $H_{C,c} := H > 0$ satisfying

$$|\partial_x^\beta f(x)| \leq HC^{|\beta|}\beta!^\theta e^{-c|x|^{\frac{1}{s}}}, \quad x \in \mathbb{R}^n, \beta \in \mathbb{N}_0^n, f \in F.$$

For $f \in F$ we have

$$\begin{aligned} \partial_\xi^\alpha Q(f)(\xi) &= \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} \int e^{i\xi x}(ix)^{\alpha_1}\partial_\xi^{\alpha_2} p(x,\xi)f(x)đx \\ &= \frac{1}{m_{\theta,\zeta}(\xi)} \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} \int e^{i\xi x} \sum_{j=0}^\infty \frac{\zeta^j}{j!^{2\theta}} \underbrace{(1-\Delta_x)^j[(ix)^{\alpha_1}\partial_\xi^{\alpha_2}p(x,\xi)f(x)]}_{:=g_{j,\alpha_1,\alpha_2}(x,\xi)} đx. \end{aligned}$$

Since $s > \mu$ and $\theta > \nu$, in analogous manner as before, we obtain that for every C, D there exists $h > 0$ satisfying

$$|g_{j,\alpha_1,\alpha_2}(x,\xi)| \leq hC^{|\alpha|}\alpha_1!^s\alpha_2!^s D^j j!^{2\theta} e^{c_p|\xi|^{\frac{1}{\theta}}} e^{-\tilde{c}|x|^{\frac{1}{s}}}.$$

Hence, by Lemma 2.1 and standard factorial inequalities,

$$\begin{aligned} |\partial_\xi^\alpha Q(f)(x)| &\leq \frac{1}{m_{\theta,\zeta}(\xi)} \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} hC^{|\alpha|}\alpha_1!^s\alpha_2!^s e^{c_p|\xi|^{\frac{1}{\theta}}} \int \sum_{j=0}^\infty (\zeta D)^j e^{-\tilde{c}|x|^{\frac{1}{s}}} đx \\ &\leq C_{\theta,\varepsilon} \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} hC^{|\alpha|}\alpha_1!^s\alpha_2!^s e^{(c_p-(2\theta-\varepsilon)\zeta^{\frac{1}{2\theta}})|\xi|^{\frac{1}{\theta}}} \int \sum_{j=0}^\infty (\zeta D)^j e^{-\tilde{c}|x|^{\frac{1}{s}}} đx. \end{aligned}$$

Choosing wisely the constants, we conclude that $Q(F)$ is a bounded subset of $\Sigma_\theta^s(\mathbb{R}^n)$. Finally, since the Fourier transform defines an isomorphism from $\Sigma_\theta^s(\mathbb{R}^n)$ to $\Sigma_s^\theta(\mathbb{R}^n)$ we can define the continuous extension

$$p(x,D)u(v) = u(\mathcal{F}(Qv)), \quad v \in \Sigma_s^\theta(\mathbb{R}^n), u \in (\Sigma_s^\theta)'(\mathbb{R}^n).$$

$\square$

Remark 2.4. *Using the same argument as in the proof above, it is not difficult to prove that operators with finite order symbols in $S^m_{\mu,\nu}(\mathbb{R}^2 n)$ are continuous from $\mathcal{S}^\theta_s(\mathbb{R}^n)$ into $\mathcal{S}^\theta_s(\mathbb{R}^n)$ if $s \geq \mu$ and $\theta \geq \nu$ and it extends continuously to the dual space $(\mathcal{S}^\theta_s)'(\mathbb{R}^n)$, cf. also Theorem $A.4$ in [10].*

2.2.2 Asymptotic Sums

In order to develop a symbolic calculus for the classes of pseudodifferential operators we are dealing with, we will need the following concept of asymptotic sums.

Definition 2.7. *(i) We say that the formal sum $\sum_{j=0}^{\infty} a_j$ belongs to $FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ when $a_j \in C^\infty(\mathbb{R}^{2n})$ and there are constants $H, C, c, B > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)| \leq HC^{|\alpha+\beta|+2j} \alpha!^\mu \beta!^\nu j!^{\mu+\nu-1} \langle \xi \rangle^{-|\alpha|-j} e^{c|\xi|^{\frac{1}{\theta}}},$$

for every $\alpha, \beta \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $\langle \xi \rangle \geq B(j) := Bj^{\mu+\nu-1}$.

(ii) We say that the formal sum $\sum_{j=0}^{\infty} a_j$ belongs to $FS^m_{\mu,\nu}(\mathbb{R}^{2n})$ when $a_j \in C^\infty(\mathbb{R}^{2n})$ and there are constants $H, C, B > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)| \leq HC^{|\alpha+\beta|+2j} \alpha!^\mu \beta!^\nu j!^{\mu+\nu-1} \langle \xi \rangle^{m-|\alpha|-j},$$

for every $\alpha, \beta \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $\langle \xi \rangle \geq B(j) := Bj^{\mu+\nu-1}$.

Remark 2.5. *We may consider $S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ as a subset of $FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ in the following sense, for $a \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ we set $a_0 := a$ and $a_j := 0$ ($j \geq 1$), then $a = \sum_j a_j \in FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. On the other hand if $\sum_j b_j \in FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$, then $b_0 \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Analogous considerations hold for $FS^m_{\mu,\nu}(\mathbb{R}^{2n})$ and $S^m_{\mu,\nu}(\mathbb{R}^{2n})$.*

Remark 2.6. *For every $m \in \mathbb{R}$ and $\theta > 1$ we have $FS^m_{\mu,\nu}(\mathbb{R}^{2n}) \subset FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$.*

Remark 2.7. *Let $a, b \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Define, for $j \geq 0$,*

$$c_j = \sum_{|\alpha|=j} \frac{(-1)^{|\alpha|}}{\alpha!} \overline{\partial_\xi^\alpha D_x^\alpha a}, \quad d_j = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b.$$

Then $\sum_j c_j, \sum_j d_j \in FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Analogous considerations for $S^m_{\mu,\nu}(\mathbb{R}^{2n})$.

Definition 2.8. *Let $\sum_j a_j, \sum_j b_j \in FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. We write $\sum_j a_j \sim \sum_j b_j$ if there are constants $H, C, c, B > 0$ satisfying*

$$\left| \partial_\xi^\alpha \partial_x^\beta \sum_{j<N} \{a_j - b_j\}(x,\xi) \right| \leq HC^{|\alpha+\beta|+2N} \alpha!^\mu \beta!^\nu N!^{\mu+\nu-1} \langle \xi \rangle^{-|\alpha|-N} e^{c|\xi|^{\frac{1}{\theta}}},$$

for every $\alpha, \beta \in N_0^n$, $x \in \mathbb{R}^n$, $N \in \mathbb{N}$ and $\langle \xi \rangle \geq B(N) := BN^{\mu+\nu-1}$. Analogous definition for $FS^m_{\mu,\nu}(\mathbb{R}^{2n})$.

Proposition 2.2. *Let $\sum_j a_j \in FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. There is $a \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ such that $a \sim \sum_j a_j$ in $FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Analogous result for $FS^m_{\mu,\nu}(\mathbb{R}^{2n})$.*

Proof. Let $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ satisfying $\psi(\xi) = 0$ for $\langle \xi \rangle \leq 2$, $\psi(\xi) = 1$ for $\langle \xi \rangle \geq 3$ and

$$|\partial_\xi^\alpha \psi(\xi)| \leq C_\psi^{|\alpha|+1} \alpha!^\mu, \quad \xi \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n.$$

For a large constant $R > 1$ to be chosen, we define $\psi_0(\xi) = 1$ and

$$\psi_j(\xi) = \psi\left(\frac{\xi}{R(j)}\right), \quad j \geq 1,$$

where $R(j) := Rj^{\mu+\nu-1}$ for every $j \in \mathbb{N}$. Now let us remark that

- $\langle \xi \rangle > 3R(j) \implies \langle R(j)^{-1}\xi \rangle > 3 \implies \psi_i = 1, i \leq j$;
- $\langle \xi \rangle \leq R(j) \implies \langle R(j)^{-1}\xi \rangle \leq 2 \implies \psi_i = 0, i \geq j$.

By hypothesis there are constants $H, C, c, B > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a_j| \leq HC^{|\alpha+\beta|+j} \alpha!^\mu \beta!^\nu j!^{\mu+\nu-1} \langle \xi \rangle^{-|\alpha|-j} e^{c|\xi|^{\frac{1}{\theta}}},$$

for every $\alpha, \beta \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $\langle \xi \rangle \geq Bj^{\mu+\nu-1}$. For $R > B$ we set

$$a(x, \xi) = \sum_{j=0}^\infty \psi_j(\xi) a_j(x, \xi), \quad x, \xi \in \mathbb{R}^n.$$

We will prove that $a \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $a \sim \sum_j a_j$ in $FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. We have

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| &\leq \sum_{j=0}^\infty \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} |\partial_\xi^{\alpha_1} \psi_j(\xi)| |\partial_\xi^{\alpha_2} \partial_x^\beta a_j(x, \xi)| \\
&\leq \sum_{j=0}^\infty \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} \frac{1}{R(j)^{|\alpha_1|}} C_\psi^{|\alpha_1|+1} \alpha_1!^\mu HC^{|\alpha_2+\beta|+j} \alpha_2!^\mu \beta!^\nu j!^{\mu+\nu-1} \langle \xi \rangle^{-j-|\alpha_2|} e^{c|\xi|^{\frac{1}{\theta}}}.
\end{aligned}$$

On the support of $\partial_\xi^{\alpha_1} \psi_j(\xi)$ holds $\langle \xi \rangle \geq R(j) (> B(j))$ and $\langle \xi \rangle \leq 3R(j)$ (whenever $\alpha_1 \neq 0$). Hence

$$\langle \xi \rangle^{-j} \leq R^{-j} j^{-j(\mu+\nu-1)}, \quad R(j)^{-|\alpha_1|} \leq 3^{|\alpha_1|} \langle \xi \rangle^{-|\alpha_1|},$$

and therefore, for $R > C^2$,

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq \tilde{C}^{|\alpha+\beta|+1} \alpha!^\mu \beta!^\nu \langle \xi \rangle^{-|\alpha|} e^{c|\xi|^{\frac{1}{\theta}}} \sum_{j=0}^\infty \left(\frac{C^2}{R}\right)^j$$

for every $\alpha, \beta \in \mathbb{N}_0^n$, $x, \xi \in \mathbb{R}^n$.

Now observe that for $N \geq 1$ and $\langle\xi\rangle \geq 3R(N)$ we have

$$a(x,\xi) - \sum_{j<N} a_j(x,\xi) = \underbrace{\sum_{j<N}(1-\psi_j(\xi))a_j(x,\xi)}_{=0} + \sum_{j\geq N}\psi_j(\xi)a_j(x,\xi)$$
$$= \sum_{j\geq 0}\psi_{j+N}(\xi)a_{j+N}(x,\xi).$$

Proceeding in an analogous way as before, we get (possibly enlarging R)

$$\left|\partial_\xi^\alpha\partial_x^\beta(a - \sum_{j<N} a_j)(x,\xi)\right| \leq C^{|\alpha+\beta+2N+1|}\alpha!^\mu\beta!^\nu N!^{\mu+\nu-1}\langle\xi\rangle^{-N-|\alpha|}e^{c|\xi|^{\frac{1}{\theta}}},$$

for every $\alpha, \beta \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$, $N \geq 1$ and $\langle\xi\rangle \geq 3R(N)$. □

We need the following technical lemma.

Lemma 2.3. *Let $M, B, r > 0$ and $\rho \geq 1$. Define*

$$h(\lambda) = \inf_{0\leq N\leq B\lambda^{\frac{1}{\rho}}} \frac{M^{rN}N!^r}{\lambda^{\frac{rN}{\rho}}}, \quad \lambda > 0.$$

Then there are $C, \tau > 0$ such that $h(\lambda) \leq Ce^{-\tau\lambda^{\frac{1}{\rho}}}$, for every $\lambda > 0$.

Proof. See Lemma 3.2.4 of [43]. □

The next result tells us how to define the regularizing operators for our classes.

Proposition 2.3. *Let $a, b \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $\sum_j a_j \in FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. If $a \sim \sum_j a_j \sim b$ in $FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $\theta > \mu+\nu-1$, then there are constanst $H, C, c > 0$ such that*

$$|\partial_\xi^\alpha\partial_x^\beta(a-b)(x,\xi)| \leq HC^{|\alpha+\beta|}\alpha!^\mu\beta!^\nu e^{-c|\xi|^{\frac{1}{r}}}, \quad x,\xi \in \mathbb{R}^n, \alpha,\beta \in \mathbb{N}_0^n,$$

where $r \geq \mu+\nu-1$.

Proof. By hypothesis we can find constants $C, c, B > 0$ such that

$$|\partial_\xi^\alpha\partial_x^\beta(a-b)(x,\xi)| \leq C^{|\alpha+\beta|+2N+1}\alpha!^\mu\beta!^\nu N!^{\mu+\nu-1}\langle\xi\rangle^{-|\alpha|-N}e^{c|\xi|^{\frac{1}{\theta}}}$$

for every $\alpha, \beta \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$, $N \geq 0$ and $\langle\xi\rangle \geq B(N) := BN^{\mu+\nu-1}$. Hence

$$|\partial_\xi^\alpha\partial_x^\beta(a-b)(x,\xi)| \leq C^{|\alpha+\beta|+1}\alpha!^\mu\beta!^\nu\langle\xi\rangle^{-|\alpha|}e^{c|\xi|^{\frac{1}{\theta}}} \inf_{0\leq N\leq(B^{-1}\langle\xi\rangle)^{\frac{1}{\mu+\nu-1}}} \frac{C^{2N}N!^{\mu+\nu-1}}{\langle\xi\rangle^N}.$$

By Lemma 2.3 we find $\tau > 0$ and a new constant $C > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta (a-b)(x,\xi)| \leq C^{|\alpha+\beta|+1} \alpha!^\mu \beta!^\nu \langle \xi \rangle^{-|\alpha|} e^{c|\xi|^{\frac{1}{\theta}}} e^{-\tau \langle \xi \rangle^{\frac{1}{\mu+\nu-1}}}.$$

Since $\theta > \mu+\nu-1$ we may conclude

$$|\partial_\xi^\alpha \partial_x^\beta (a-b)(x,\xi)| \leq \tilde{C}^{|\alpha+\beta|+1} \alpha!^\mu \beta!^\nu \langle \xi \rangle^{-|\alpha|} e^{-\frac{\tau}{2} \langle \xi \rangle^{\frac{1}{\mu+\nu-1}}}.$$

□

Analogously we can obtain a similar result for the classes with finite order and in this case we do not need any hypothesis over θ.

Proposition 2.4. *Let $a, b \in S^m_{\mu,\nu}(\mathbb{R}^{2n})$ and $\sum_j a_j \in FS^m_{\mu,\nu}(\mathbb{R}^{2n})$. If $a \sim \sum_j a_j \sim b$ in $FS^m_{\mu,\nu}(\mathbb{R}^{2n})$ then there are constants $H, C, c > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta (a-b)(x,\xi)| \leq HC^{|\alpha+\beta|} \alpha!^\mu \beta!^\nu e^{-r|\xi|^{\frac{1}{r}}}, \quad x, \xi \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n,$$

where $r \geq \mu+\nu-1$.

Definition 2.9. *Given $\tilde{r} > 1$, we denote by $\mathcal{K}_{\tilde{r}}$ the space of all symbols $q \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ such that for every $r \geq \tilde{r}$ there exist $C, c > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq C^{|\alpha+\beta|+1} \alpha!^r \beta!^r e^{-c|\xi|^{\frac{1}{r}}}, \quad x, \xi \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n.$$

We say that $Q = q(x,D)$ is $\tilde{r}-$regularizing whenever its symbol $q(x,\xi)$ belongs to $\mathcal{K}_{\tilde{r}}$.

2.2.3 Regularity of the Kernel

In this subsection we study the regularity and decay properties of the kernel of our operators. Let $p \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. The *Schwartz* kernel of $p(x,D)$ can be represented (formally) by

$$K_p(x,y) = \int e^{i\xi(x-y)} p(x,\xi) đ\xi, \quad x, y \in \mathbb{R}^n.$$

To be more rigorous, the Schwartz kernel of $p(x,D)$ is defined as the unique $K_p \in (\Sigma_s^\theta)'(\mathbb{R}^{2n}_{(x,y)})$ satisfying the following property

$$\int p(x,D)u(x)v(x)dx = K_p(v \otimes u), \quad u, v \in \Sigma_s^\theta(\mathbb{R}^n),$$

where $v \otimes u(x,y) := v(x)u(y)$. Before proceeding with the next result we will need the following lemma.

Lemma 2.4. *Let $\mu > 1$. For every $R > 0$ there exists a partition of the unity $\{\psi_N(\xi)\}_{N=0}^{\infty}$ satisfying*

- *supp*$(\psi_0) \subset \{\xi \in \mathbb{R}^n : \langle\xi\rangle \leq 3R\}$*;*
- *supp*$(\psi_N) \subset \{\xi : 2N^{\mu} \leq \langle\xi\rangle \leq 3(N+1)^{\mu}\}$*, for every* $N \geq 1$*;*
- *there exists* $A > 0$ *such that*

$$|\partial_{\xi}^{\alpha}\psi_N(\xi)| \leq A^{|\alpha|+1}\alpha!^{\mu}\{R\max\{1, N^{\mu}\}\}^{-|\alpha|}, \quad \xi \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n.$$

Proof. Let $\psi \in C^{\infty}(\mathbb{R}^n)$ such that $\psi(\xi) = 1$ for $\langle\xi\rangle \leq 2$, $\psi(\xi) = 0$ for $\langle\xi\rangle \geq 3$ and

$$|\partial_{\xi}^{\alpha}\psi(\xi)| \leq C^{|\alpha|+1}\alpha!^{\mu}, \quad \alpha \in \mathbb{N}_0^n, \xi \in \mathbb{R}^n.$$

Defining

$$\psi_0(\xi) = \psi\left(\frac{\xi}{R}\right), \quad \psi_N(\xi) = \psi\left(\frac{\xi}{R(N+1)^{\mu}}\right) - \psi\left(\frac{\xi}{RN^{\mu}}\right), N \in \mathbb{N},$$

we obtain the desired sequence $\{\psi_N\}_{N=0}^{\infty}$. $\square$

Proposition 2.5. *Let $p \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ where $\nu > 1$ and $\theta > \mu > 1$. Let moreover $k \in (0,1)$. Then there exist $C, c > 0$ such that*

$$|\partial_x^{\beta}\partial_y^{\gamma}K_p(x,y)| \leq C^{|\beta|+|\gamma|+1}\beta!^{\max\{\mu,\nu\}}\gamma!^{\mu}e^{-c|x-y|^{\frac{1}{\mu}}}$$

for every $\beta, \gamma \in \mathbb{N}_0^n$ and for every $x, y \in \mathbb{R}^n$ such that $|x-y| \geq k$.

Proof. We may decompose K_p in the following way

$$K_p(x,y) = \int e^{i\xi(x-y)}p(x,\xi)đ\xi = \sum_{N=0}^{\infty}\underbrace{\int e^{i\xi(x-y)}\psi_N(\xi)p(x,\xi)đ\xi}_{=:K_N(x,y)},$$

where $\{\psi_N\}_{N=0}^{\infty}$ is a partition of the unity as in Lemma 2.4. Let x, y such that $|x-y| > k$ and take $h \in \{1, \ldots, n\}$ satisfying $|x_h - y_h| \geq \frac{k}{n}$. For every $\beta, \gamma \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ we write

$$\begin{aligned}
\partial_x^{\beta}\partial_y^{\gamma}K_N(x,y) &= \sum_{\beta_1+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!}\int e^{i\xi(x-y)}(-i\xi)^{\gamma}(i\xi)^{\beta_1}\partial_x^{\beta_2}p(x,\xi)\psi_N(\xi)đ\xi \\
&= \sum_{\beta_1+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!}(-1)^N(x_h-y_h)^{-N}\int e^{i\xi(x-y)}D_{\xi_h}^N\{(-i\xi)^{\gamma}(i\xi)^{\beta_1}\partial_x^{\beta_2}p(x,\xi)\psi_N(\xi)\}đ\xi \\
&= \sum_{\beta_1+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!}(-1)^N\frac{(x_h-y_h)^{-N}}{m_{\mu,\zeta}(x-y)}
\end{aligned}$$

$$\times \int e^{i\xi(x-y)} \sum_{j=0}^{\infty} \frac{\zeta^j}{j!^{2\mu}} \underbrace{(1-\Delta_\xi)^j D^N_{\xi_h} \{(-i\xi)^\gamma (i\xi)^{\beta_1} \partial_x^{\beta_2} p(x,\xi) \psi_N(\xi)\}}_{=:h_{j,N,\beta,\gamma}(x,\xi)} đ\xi.$$

Now notice that

$$|h_{j,N,\beta,\gamma}(x,\xi)| \leq \sum_{j_0+|J|=j} \frac{j!}{j_0!J!} \sum_{\lambda_1+\lambda_2+\lambda_3=2J} \frac{(2J)!}{\lambda_1!\lambda_2!\lambda_3!} \sum_{N_1+N_2+N_3=N} \frac{N!}{N_1!N_2!N_3!} |\partial_\xi^{\lambda_1+N_1e_h} \xi^{\gamma+\beta_1}|$$
$$\times |\partial_x^{\beta_2} \partial_\xi^{\lambda_2+N_2e_h} p(x,\xi)||\partial_\xi^{\lambda_3+N_3e_h} \psi_N(\xi)|$$
$$\leq \sum_{j_0+|J|=j} \frac{j!}{j_0!J!} \sum_{\lambda_1+\lambda_2+\lambda_3=2J} \frac{(2J)!}{\lambda_1!\lambda_2!\lambda_3!} \sum_{N_1+N_2+N_3=N} \frac{N!}{N_1!N_2!N_3!} \frac{(\beta_1+\gamma)!}{(\beta_1+\gamma-\lambda_1-N_1e_h)!}$$
$$\times |\xi|^{|\beta_1+\gamma-\lambda_1-N_1e_h|} C_p^{|\beta_2+\lambda_2|+N_2+1} \beta_2!^\nu (\lambda_2+N_2e_h)!^\mu \langle\xi\rangle^{-|\lambda_2|-N_2} e^{c_p|\xi|^{\frac{1}{\theta}}}$$
$$\times A^{|\lambda_3|+N_3} (\lambda_3+N_3e_h)!^\mu R^{-|\lambda_3|-N_3} N^{-\mu(|\lambda_3|-N_3)}.$$

Since, for $N \geq 1$, $2RN^\mu \leq \langle\xi\rangle \leq 3R(N+1)^\mu$ holds true on the support of ψ_N, we obtain

$$|h_{j,N,\beta,\gamma}(x,\xi)| \leq \sum_{j_0+|J|=j} \frac{j!}{j_0!J!} \sum_{\lambda_1+\lambda_2+\lambda_3=2J} \frac{(2J)!}{\lambda_1!\lambda_2!\lambda_3!} \sum_{N_1+N_2+N_3=N} \frac{N!}{N_1!N_2!N_3!} 2^{|\beta_1+\gamma+\lambda_1|+N_1}$$
$$\times \lambda_1!N_1!\{3R(N+1)^\mu\}^{|\beta_1+\gamma|}\{2RN^\mu\}^{-|\lambda_1|-N_1} C_p^{|\beta_2+\lambda_2|+N_2+1} \beta_2!^\nu \beta_2!^\nu 2^{|\lambda_2|+N_2} \lambda_2!^\mu N_2!^\mu$$
$$\times \{2RN^\mu\}^{-|\lambda_2|-N_2} e^{c_p\{3R(N+1)^\mu\}^{\frac{1}{\theta}}} A^{|\lambda_3|+N_3} 2^{|\lambda_3|+N_3} \lambda_3!^\mu N_3!^\mu \{RN^\mu\}^{-|\lambda_3|-N_3}$$
$$\leq \tilde{C}_1^{|\beta+\gamma|+1} \beta_1!^\mu \beta_2!^\nu \gamma!^\mu C_2^j j!^{2\mu} \left(\frac{\tilde{C}_3}{R}\right)^N e^{c_p\{3R(N+1)^\mu\}^{\frac{1}{\theta}}},$$

where $C_1 = C_1(R,A,p)$, $C_2 = C_2(A,p)$ and $\tilde{C}_3 = \tilde{C}_3(A,p)$ is independent of R. Due the hypothesis $\theta > \mu$, there is $C_{p,R} > 0$ such that

$$e^{c_p\{3R(N+1)^\mu\}^{\frac{1}{\theta}}} \leq C_{p,R} e^{N+1}, \quad N \in \mathbb{N}_0,$$

hence

$$|h_{j,N,\beta,\gamma}(x,\xi)| \leq C_1^{|\beta+\gamma|+1} \beta_1!^\mu \beta_2!^\nu \gamma!^\mu C_2^j j!^{2\mu} \left(\frac{C_3}{R}\right)^N,$$

where $C_1 = C_1(p,A,R)$ and $C_3 = C_3(A,p)$ is independent of R. From the previous estimates we deduce

$$|\partial_x^\beta \partial_y^\gamma K_N(x,y)| \leq C^{|\beta+\gamma|+1} \beta!^{\max\{\mu,\nu\}} \gamma!^\mu e^{-c|x-y|^{\frac{1}{\mu}}} \sum_{j\geq 0} (\zeta C_2)^j \int_{\text{supp}\psi_N} đ\xi \left(\frac{nC_2}{kR}\right)^N.$$

Choosing $\zeta > 0$ such that $\zeta C_2 < 1$ and $R > 0$ large enough we complete the proof. □

Analogously we can obtain a version of the above propostion in the frame of finite order. We point out that none hypothesis over θ are required in this context.

Proposition 2.6. *Let $p \in S^m_{\mu,\nu}(\mathbb{R}^{2n})$ where $\mu, \nu > 1$. Let moreover $k \in (0,1)$. Then there exist $C, c > 0$ such that*

$$|\partial_x^\beta \partial_y^\gamma K_p(x,y)| \leq C^{|\beta|+|\gamma|+1} \beta!^{\max\{\mu,\nu\}} \gamma!^\mu e^{-c|x-y|^{\frac{1}{\mu}}}$$

for every $\beta, \gamma \in \mathbb{N}_0^n$ and for every $x, y \in \mathbb{R}^n$ such that $|x-y| \geq k$.

Let us remark that from $K_p(x,y)$ we can (formally) recover the symbol $p(x,\xi)$ using the next formula

$$p(x,\xi) = \int e^{-i\xi(x-y)} K_p(x,y) dy, \quad x, \xi \in \mathbb{R}^n.$$

Proposition 2.7. *Let $p \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and assume*

$$|\partial_x^\beta \partial_y^\gamma K_p(x,y)| \leq C^{|\beta+\gamma|+1} \beta!^\mu \gamma!^\mu e^{-c|x-y|^{\frac{1}{\mu}}}$$

for every $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$. Then $p(x,\xi)$ satisfies (for new constants $C, c > 0$)

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C^{|\alpha+\beta|+1} \alpha!^\mu \beta!^\mu e^{-c|\xi|^{\frac{1}{\mu}}}, \quad x, \xi \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n.$$

Proof. For $\alpha, \beta \in \mathbb{N}_0^n$ we may write

$$\begin{aligned}
\partial_\xi^\alpha \partial_x^\beta p(x,\xi) &= \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \int e^{-i\xi(x-y)} (-i\xi)^{\beta_1} (-i(x-y))^\alpha \partial_x^{\beta_2} K_p(x,y) dy \\
&= \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \frac{1}{m_{\mu,\zeta}(\xi)} \\
&\quad \times \int e^{-i\xi(x-y)} (i\xi)^{\beta_1} \sum_{j=0}^\infty \frac{\zeta^j}{j!^{2\mu}} \underbrace{(1-\Delta_y)^j \{(-i(x-y))^\alpha \partial_x^{\beta_2} K_p(x,y)\}}_{=:h_{j,\alpha,\beta_2}(x,y)} dy.
\end{aligned}$$

Now notice that

$$\begin{aligned}
&|h_{j,\alpha,\beta_2}(x,y)| \leq \sum_{j_0+|J|=j} \frac{j!}{j_0!J!} \sum_{\lambda_1+\lambda_2=2J} \frac{(2J)!}{\lambda_1!\lambda_2!} |\partial_y^{\lambda_1}(x-y)^\alpha| |\partial_x^{\beta_2} \partial_y^{\lambda_2} K_p(x,y)| \\
&\leq \sum_{j_0+|J|=j} \frac{j!}{j_0!J!} \sum_{\lambda_1+\lambda_2=2J} \frac{(2J)!}{\lambda_1!\lambda_2!} \frac{\alpha!}{(\alpha-\lambda_1)!} |(x-y)^{\alpha-\lambda_1}| e^{-\frac{c}{2}|x-y|^{\frac{1}{\mu}}} C^{|\beta_2+\lambda_2|+1} \beta_2!^\mu \lambda_2!^\mu e^{-\frac{c}{2}|x-y|^{\frac{1}{\mu}}} \\
&\leq C_1^{|\alpha+\beta|+1} \alpha!^\mu \beta!^\mu e^{-\frac{c}{2}|x-y|^{\frac{1}{\mu}}} C_2^j j!^{2\mu}.
\end{aligned}$$

On the other hand we have

$$\left| \frac{(i\xi)^{\beta_1}}{m_{\mu,\zeta}(\xi)} \right| \leq C_\zeta^{|\beta_1|+1} \beta_1!^\mu e^{-c_\zeta |\xi|^{\frac{1}{\mu}}}.$$

Choosing $\zeta > 0$ small enough we conclude the proof. $\square$

2.2.4 Global Gevrey Amplitude Classes

In order to study the product, adjoint and transpose for our classes of pseudodifferential operators, it is useful to enlarge them by considering a more general notion of symbols, namely the amplitudes.

Definition 2.10. *We say that a smooth function $a(x, y, \xi)$ belongs to the class $S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{3n})$ if there exist constants $H, C, c > 0$ satisfying*

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \xi)| \le HC^{|\alpha+\beta+\gamma|}\alpha!^\mu \beta!^\nu \gamma!^\nu \langle\xi\rangle^{-|\alpha|} e^{c|\xi|^{\frac{1}{\theta}}}, \quad x, \xi \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n. \tag{2.4}$$

Analogously we define the finite order version $S^m_{\mu,\nu}(\mathbb{R}^{3n})$ by replacing $e^{c|\xi|^{\frac{1}{\theta}}}$ with $\langle\xi\rangle^m$ in (2.4).

For a given amplitude $a(x, y, \xi)$ we associate it to the operator $A = a(x, y, D)$ defined by

$$A(f)(x) = \iint e^{i\xi(x-y)} a(x, y, \xi) f(y) dy đ\xi, \quad x \in \mathbb{R}^n, \tag{2.5}$$

where $f \in \Sigma_s^\theta(\mathbb{R}^n)$ for $\theta > \nu$ and $s > \mu$. We observe that the integral in (2.5) is not absolutely convergent in general. The precise meaning of the right hand side in (2.5) is given by the following oscillatory integral

$$A(f)(x) = \lim_{\delta\to 0} \iint e^{i\xi(x-y)} \chi(\delta\xi) a(x, y, \xi) f(y) dy đ\xi,$$

where $\chi \in \Sigma_{\min\{s,\theta\}}(\mathbb{R}^n)$ and $\chi(0) = 1$.

Proposition 2.8. *Let A be the operator defined in (2.5). If $\mu < s$ and $\nu < \theta$, then A maps $\Sigma_s^\theta(\mathbb{R}^n)$ to $\Sigma_s^\theta(\mathbb{R}^n)$ continuously.*

Proof. Let F be a bounded subset of $\Sigma_s^\theta(\mathbb{R}^n)$. Then for every $C, r > 0$ we find $H_{C,r} = H > 0$ satisfying

$$|\partial_x^\beta f(x)| \le HC^{|\beta|}\beta!^\theta e^{-r|x|^{\frac{1}{s}}}, \quad x \in \mathbb{R}^n, \beta \in \mathbb{N}_0^n, f \in F.$$

Observe that, for any $\gamma, \beta \in \mathbb{N}_0^n$,

$$\begin{aligned} x^\gamma \partial_x^\beta A(f)(x) &= \lim_{\delta\to 0} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \iint x^\gamma e^{i\xi(x-y)} (ix)^{\beta_1} \partial_x^{\beta_2} a(x, y, \xi) f(y) \chi(\delta\xi) dy đ\xi \\ &= \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \iint \{D_\xi^\gamma e^{i\xi x}\} e^{-i\xi y} (i\xi)^{\beta_1} \partial_x^{\beta_2} a(x, y, \xi) f(y) dy đ\xi \\ &= \lim_{\delta\to 0} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} (-1)^{|\gamma|} \sum_{\gamma_1+\gamma_2+\gamma_3+\gamma_4=\gamma} \frac{\gamma!}{\gamma_1!\gamma_2!\gamma_3!\gamma_4!} \end{aligned}$$

$$
\begin{aligned}
&\times \iint e^{i\xi(x-y)}(-y)^{\gamma_1} i^{|\beta_1-\gamma_2|} \frac{\beta_1!}{(\beta_1-\gamma_2)!} \xi^{\beta_1-\gamma_2} D_\xi^{\gamma_3} \partial_x^{\beta_2} a(x,y,\xi) f(y) D_\xi^{\gamma_4} \chi(\delta\xi) dy đ\xi \\
&= \lim_{\delta\to 0} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!}(-1)^{|\gamma|} \sum_{\gamma_1+\gamma_2+\gamma_3+\gamma_4=\gamma} \frac{\gamma!}{\gamma_1!\gamma_2!\gamma_3!\gamma_4!} i^{|\beta_1-\gamma_2|} \iint \frac{e^{i\xi(x-y)}}{m_{\zeta,\theta}(\xi)} \\
&\times \sum_{j\geq 0} \frac{\zeta^j}{j!^{2\theta}} \frac{\beta_1!}{(\beta_1-\gamma_2)!} \xi^{\beta_1-\gamma_2} D_\xi^{\gamma_4}\chi(\delta\xi) \underbrace{(1-\Delta_y)^j \{(-y)^{\gamma_1} D_\xi^{\gamma_3} \partial_x^{\beta_2} a(x,y,\xi) f(y)\}}_{=:\dagger} dy đ\xi.
\end{aligned}
$$

Since $s > \mu$ and $\theta > \nu$, proceeding as in the proof of Proposition 2.1, we obtain the following estimate: for every $h, C > 0$ there exist $H_{h,C} = H > 0$, $c_a > 0$ and $\tilde{r}_C = \tilde{r} > 0$ such that

$$
|\dagger| \leq HC^{\beta_2+\gamma_1+\gamma_3} \beta_2!^\theta \gamma_1!^s \gamma_3!^s h^j j!^{2\theta} e^{c_a|\xi|^{\frac{1}{\theta}}} e^{-\tilde{r}|y|^{\frac{1}{s}}}.
$$

On the other hand, using Lemma 2.1, we get that for every $C > 0$ there exist $H_C = H > 0$ and $\zeta_C = \zeta > 0$ such that

$$
\left| \frac{\beta_1!}{(\beta_1-\gamma_2)!} \xi^{\beta_1-\gamma_2} D_\xi^{\gamma_4} \chi(\delta\xi) \right| \leq HC^{\beta_1+\gamma_2+\gamma_4} \gamma_2!^s \gamma_4!^s \beta_1!^\theta e^{-\frac{2\theta-\varepsilon}{2}\zeta^{\frac{1}{2\theta}} \langle\xi\rangle^{\frac{1}{\theta}}},
$$

uniformly with respect to $\delta \in (0,1)$. Thus, standard factorial inequalities imply

$$
|x^\gamma \partial_x^\beta A(f)(x)| \tilde{H} C^{|\beta+\gamma|} \beta!^\theta \gamma!^s \sum_{j\geq 0} (h\zeta)^j.
$$

Taking $h < \zeta^{-1}$ we obtain that $A(F)$ is a bounded subset of $\Sigma_s^\theta(\mathbb{R}^n)$.

Moreover, if we ignore x^γ and ∂_x^β in the above computation, using the Lebesgue Dominated Convergence Theorem we get the following expression for $A(f)(x)$:

$$
A(f)(x) = \iint e^{i\xi(x-y)} \frac{1}{m_{\theta,\zeta}(\xi)} \sum_{j\geq 0} \frac{\zeta^j}{j!^{2\theta}} (1-\Delta_y)\{a(x,y,\xi)f(y)\} dy đ\xi.
$$

In particular, $A(f)(x)$ does not depend on the choice of χ. □

Remark 2.8. *Let $a \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{3n})$ and for each $j \in \mathbb{N}_0$ set*

$$
p_j(x,\xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} (\partial_\xi^\alpha D_y^\alpha a)(x,x,\xi), \quad x,\xi \in \mathbb{R}^n.
$$

Then $\sum_j p_j \in FS^\infty_{\mu,\nu;\theta}(\mathbb{R}^{2n})$.

Given $a \in S^\infty_{\mu,\nu;\theta}(\mathbb{R}^{3n})$, the Schwartz kernel of the corresponding $A = a(x,y,D)$ is formally given by

$$
K_a(x,y) = \int e^{i\xi(x-y)} a(x,y,\xi) đ\xi.
$$

Repeating the arguments in the proof of Proposition 2.5, we obtain the following results.

Proposition 2.9. *Let $a \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{3n})$ where $\nu > 1$ and $\theta > \mu > 1$. Let moreover $k \in (0,1)$. Then there exist $C, c > 0$ such that*

$$|\partial_x^\beta \partial_y^\gamma K_a(x,y)| \leq C^{|\beta|+|\gamma|+1} \beta!^{\max\{\mu,\nu\}} \gamma!^{\max\{\mu,\nu\}} e^{-c|x-y|^{\frac{1}{\mu}}}$$

for every $\beta, \gamma \in \mathbb{N}_0^n$ and for every $x, y \in \mathbb{R}^n$ such that $|x-y| \geq k$.

Proposition 2.10. *Let $a \in S^m_{\mu,\nu}(\mathbb{R}^{3n})$ where $\mu, \nu > 1$. Let moreover $k \in (0,1)$. Then there exist $C, c > 0$ such that*

$$|\partial_x^\beta \partial_y^\gamma K_a(x,y)| \leq C^{|\beta|+|\gamma|+1} \beta!^{\max\{\mu,\nu\}} \gamma!^{\max\{\mu,\nu\}} e^{-c|x-y|^{\frac{1}{\mu}}}$$

for every $\beta, \gamma \in \mathbb{N}_0^n$ and for every $x, y \in \mathbb{R}^n$ such that $|x-y| \geq k$.

Remark 2.9. *Let $a \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{3n})$ and assume*

$$|\partial_x^\beta \partial_y^\gamma K_a(x,y)| \leq C^{|\beta+\gamma_1|+1} \beta!^{\max\{\mu,\nu\}} \gamma!^{\max\{\mu,\nu\}} e^{-c|x-y|^{\frac{1}{\mu}}} \tag{2.6}$$

for every $x, y \in \mathbb{R}^n$ and $\beta, \gamma \in \mathbb{N}_0^n$. By the proof of Proposition 2.7, we conclude that the symbol

$$r(x,\xi) = \int e^{-i\xi(x-y)} K_a(x,y)dy, \quad x, \xi \in \mathbb{R}^n,$$

is in $\mathcal{K}_{\max\{\mu,\nu\}}$. Now let us remark that $a(x,y,D) = r(x,D)$. Indeed, for $u \in \Sigma_s^\theta$, with $\theta > \nu$ and $s > \mu$, we have

$$\begin{aligned}
r(x,D)u(x) &= \int e^{i\xi x} r(x,\xi)\widehat{u}(\xi)đ\xi \\
&= \int e^{i\xi x} \int e^{-i\xi(x-y)} K_a(x,y)dy\, \widehat{u}(\xi)đ\xi \\
&= \int K_a(x,y) \int e^{i\xi y}\widehat{u}(\xi)đ\xi dy \\
&= \int K_a(x,y)u(y)dy \\
&= a(x,y,D)u(x).
\end{aligned}$$

Summing up, if the kernel of $a(x,y,D)$ satisfies (2.6), then we may find a symbol $r \in \mathcal{K}_{\max\{\mu,\nu\}}$ such that $a(x,y,D) = r(x,D)$.

The next result concerns the relation between operators given by amplitudes and operators given by symbols.

Theorem 2.5. *Let $a \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{3n})$ where $\mu, \nu > 1$ and $\theta > \mu + \nu - 1$. There exist symbols $p \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{n})$ and $r \in \mathcal{K}_{\mu+\nu-1}$ such that $a(x, y, D) = p(x, D) + r(x, D)$ and*

$$p(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!}(\partial^{\alpha}_{\xi} D^{\alpha}_{y} a)(x, x, \xi) \text{ in } FS^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}).$$

Proof. We start considering $\chi(x, y) \in C^{\infty}(\mathbb{R}^{2n})$ satisfying

$$\chi(x, y) = \begin{cases} 1, \text{ if } |x-y| \geq \frac{1}{2} \\ 0, \text{ if } |x-y| \leq \frac{1}{4} \end{cases}, \qquad |\partial^{\beta}_{x} \partial^{\gamma}_{y} \chi(x, y)| \leq C_{\chi}^{|\beta+\gamma|+1} \gamma!^{\nu} \beta!^{\nu}.$$

We may decompose $a(x, y, \xi)$ as the following sum of elements of $S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{3n})$:

$$a(x, y, \xi) = \chi(x, y)a(x, y, \xi) + (1 - \chi(x, y))a(x, y, \xi).$$

Furthermore, since $\theta > \mu + \nu - 1 > \mu$, it follows from Proposition 2.9 and Remark 2.9 that χa defines a $\max\{\mu, \nu\}-$regularizing operator. Therefore, we may assume a supported in the set

$$\{(x, y, \xi) : |x - y| \leq 2^{-1}, \xi \in \mathbb{R}^{n}\},$$

after a possibly perturbation by a $\max\{\mu, \nu\}-$regularizing operator.

For each $j \in \mathbb{N}_0$, set

$$p_j(x, \xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!}(\partial^{\alpha}_{\xi} D^{\alpha}_{y} a)(x, x, \xi), \quad x, \xi \in \mathbb{R}^{n}.$$

Let us consider $\phi(\xi) \in C^{\infty}(\mathbb{R}^{n})$ such that $\phi(\xi) = 0$ for $\langle\xi\rangle \leq 2$, $\phi(\xi) = 1$ for $\langle\xi\rangle \geq 3$ and

$$|\partial^{\alpha}_{\xi} \phi(\xi)| \leq C_{\phi}^{|\alpha|+1} \alpha!^{\mu}, \quad \xi \in \mathbb{R}^{n}, \alpha \in \mathbb{N}^{n}_{0}.$$

Setting $\phi_0 \equiv 1$ and, for a large constant $R > 0$,

$$\phi_j(\xi) = \phi\left(\frac{\xi}{Rj^{\mu+\nu-1}}\right), \quad \xi \in \mathbb{R}^{n}, j \geq 1,$$

from Proposition 2.2 (for $R > 0$ large enough) we have

$$p(x, \xi) = \sum_{j=0}^{\infty} \phi_j(\xi) p_j(x, \xi) \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}), \quad p(x, \xi) \sim \sum_{j=0}^{\infty} p_j(x, \xi) \text{ in } FS^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}).$$

Since we can write

$$a(x, y, D) = p(x, D) + a(x, y, D) - p(x, D),$$

to conclude the proof, it only remains to prove that the kernel $K(x, y)$ of $a(x, y, D) - p(x, D)$ satisfies the following estimate

$$|\partial^{\beta}_{x} \partial^{\gamma}_{y} K(x, y)| \leq C^{|\beta+\gamma|+1} (\beta!\gamma!)^{\mu+\nu-1} e^{-c|x-y|^{\frac{1}{\mu+\nu-1}}}. \tag{2.7}$$

Observing that

$$\sum_{N=0}^{k}(\phi_N - \phi_{N+1})\sum_{j\le N} p_j = \sum_{N=0}^{k}\phi_N p_N - \phi_{k+1}\sum_{N=0}^{k} p_N, \quad \sum_{N=0}^{k}(\phi_N - \phi_{N+1})a = \phi_0 a - \phi_{k+1}a,$$

we obtain the following identity

$$\begin{aligned} a(x,y,\xi) - p(x,\xi) &= (1-\phi_0(\xi))a(x,y,\xi) \\ &\quad + \sum_{N=0}^{\infty}(\phi_N - \phi_{N+1})(\xi)\left(a(x,y,\xi) - \sum_{j\le N} p_j(x,\xi)\right) \\ &= \sum_{N=0}^{\infty}(\phi_N - \phi_{N+1})(\xi)\left(a(x,y,\xi) - \sum_{j\le N} p_j(x,\xi)\right). \end{aligned}$$

Consequently

$$K(x,y) = K_0(x,y) + \sum_{N=1}^{\infty} K_N(x,y),$$

where

$$K_N(x,y) = \int e^{i\xi(x-y)}(\phi_N - \phi_{N+1})(\xi)\left(a(x,y,\xi) - \sum_{j\le N} p_j(x,\xi)\right) đ\xi.$$

Taylor's formula gives, for $N \ge 1$,

$$a(x,y,\xi) = \sum_{|\alpha|\le N}\frac{1}{\alpha!}(y-x)^\alpha(\partial_y^\alpha a)(x,x,\xi) + R_{N+1}(x,y,\xi),$$

where

$$R_{N+1}(x,y,\xi) = (N+1)\sum_{|\alpha|=N}\frac{(y-x)^\alpha}{\alpha!}\int_0^1(\partial_y^\alpha a)(x,x+t(y-x),\xi)(1-t)^N dt.$$

Using the definition of p_j and integration by parts, we obtain that ($N \ge 1$)

$$\begin{aligned} K_N(x,y) &= \int e^{i(x-y)\xi}\sum_{1\le|\alpha|\le N}\frac{1}{\alpha!}\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_1\ge 1}}\frac{\alpha!}{\alpha_1!\alpha_2!}\partial_\xi^{\alpha_1}(\phi_N - \phi_{N+1})(\xi)(\partial_\xi^{\alpha_2}D_y^\alpha a)(x,x,\xi)đ\xi \\ &\quad + \sum_{|\alpha|=N+1}\frac{N+1}{\alpha!}\int e^{i\xi(x-y)}\sum_{\alpha_1+\alpha_2=\alpha}\partial_\xi^{\alpha_1}(\phi_N - \phi_{N+1})(\xi) \\ &\quad \times \int_0^1(1-t)^N(\partial_\xi^{\alpha_2}D_y^\alpha a)(x,y+t(y-x),\xi)dt\, đ\xi \\ &\quad = \tilde{W}_N(x,y) + W_N(x,y). \end{aligned}$$

Since we will prove absolute convergence for the series and integrals we are dealing with, we can re-arrange the terms of the sum under the integral sign. Notice also that

$$\sum_{N\ge|\alpha|}\partial_\xi^\beta(\phi_N - \phi_{N-1})(\xi) = \partial_\xi^\beta\phi_{|\alpha|}(\xi)$$

and, for an absolute convergent series,

$$\sum_{N=1}^{\infty}\sum_{1\le|\alpha|\le N} c_{N,\alpha} = \sum_{N\ge 1}\sum_{|\alpha|=1} c_{N,\alpha} + \sum_{N\ge 2}\sum_{|\alpha|=2} c_{N,\alpha} + \sum_{N\ge 3}\sum_{|\alpha|=3} c_{N,\alpha} + \dots$$
$$= \sum_{j=1}^{\infty}\sum_{N\ge j}\sum_{|\alpha|=j} c_{j,\alpha}.$$

Then we conclude that

$$K(x,y) = K_0(x,y) + \sum_{j=1}^{\infty} I_j(x,y) + \sum_{N=1}^{\infty} W_N(x,y),$$

where

$$I_j(x,y) = \sum_{|\alpha|=j}\frac{1}{\alpha!}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1\ge 1}}\frac{\alpha!}{\alpha_1!\alpha_2!}\int e^{i\xi(x-y)}\partial_\xi^{\alpha_1}\phi_j(\xi)(\partial_\xi^{\alpha_2}D_y^{\alpha}a)(x,x,\xi)đ\xi.$$

Finally, to finish the proof it is sufficient to prove that K_0, $\sum_j I_j$ and $\sum_N W_N$ satisfy an estimate of the type (2.7). Since the computations are quite similar, we only verify the estimate for $\sum_j I_j$. For $\beta,\gamma\in\mathbb{N}_0^n$,

$$\partial_x^\beta\partial_y^\gamma I_j(x,y) = \sum_{|\alpha|=j}\frac{1}{\alpha!}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1\ge 0}}\frac{\alpha!}{\alpha_1!\alpha_2!}\sum_{\beta_1+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!}$$
$$\times\int e^{i\xi(x-y)}\underbrace{(-i\xi)^\gamma(i\xi)^{\beta_1}\partial_\xi^{\alpha_1}\phi_j(\xi)\partial_x^{\beta_2}(\partial_\xi^{\alpha_2}D_y^{\alpha}a)(x,x,\xi)}_{=:h_{j,\gamma,\alpha,\beta}(x,\xi)}đ\xi.$$

We recall that on the support of ϕ_j we have the inequalities $Rj^{\mu+\nu-1}\le\langle\xi\rangle\le 3Rj^{\mu+\nu-1}$. Hence

$$|h_{j,\gamma,\alpha,\beta}(x,y)| \le |\xi|^{|\gamma+\beta_1|}C_\phi^{|\alpha_1|+1}\alpha_1!^\mu(Rj^{\mu+\nu-1})^{-|\alpha_1|}C_a^{|\alpha+\beta_2|+1}\alpha_2!^\mu\beta_2!^\nu\alpha!^\nu\langle\xi\rangle^{-|\alpha_2|}e^{c_a|\xi|^{\frac{1}{\theta}}}$$
$$\le C_{R,\phi,a}^{|\beta+\gamma|+1}\alpha_1!^\mu\alpha_2!^\mu\alpha!^\nu\beta_2!^\nu j^{(\mu+\nu-1)|\gamma+\beta_1|}C_{a,\phi}^j(Rj^{\mu+\nu-1})^{-j}e^{c_a(3R)^{\frac{1}{\theta}}j^{\frac{\mu+\nu-1}{\theta}}}.$$

Now notice that $j^{|\gamma+\beta_1|}\le C^{|\gamma+\beta_1|}\gamma!\beta!$ and, using the hypothesis $\theta>\mu+\nu-1$,

$$e^{c_a(3R)^{\frac{1}{\theta}}j^{\frac{\mu+\nu-1}{\theta}}}\le C_{a,R}\,e^j,\quad j\ge 1.$$

Hence

$$|h_{j,\gamma,\alpha,\beta}(x,y)|\le C_{R,\phi,a}^{|\beta+\gamma|+1}\alpha_1!^\mu\alpha_2!^\mu\alpha!^\nu\beta_2!^\nu\gamma!^{\mu+\nu-1}\beta_1!^{\mu+\nu-1}\left(\frac{C_{a,\phi}}{R}\right)^j j^{-j(\mu+\nu-1)},$$

where $C_{a,\phi}$ is independent of $R>0$. Therefore, choosing R large, we obtain

$$\left|\partial_x^\beta\partial_y^\gamma\sum_{j=0}^{\infty}I_j(x,y)\right|\le C^{|\beta+\gamma|+1}(\beta!\gamma!)^{\mu+\nu-1}.$$

To finish, since we are assuming $|x-y| \le 2^{-1}$ on the support of $a(x,y,\xi)$, we may write

$$\left|\partial_x^\beta \partial_y^\gamma \sum_{j=0}^{\infty} I_j(x,y)\right| \le \tilde{C}^{|\beta+\gamma|+1}(\beta!\gamma!)^{\mu+\nu-1} e^{-|x-y|^{\frac{1}{\mu+\nu-1}}}.$$

□

2.2.5 Adjoint, Transpose and Composition Formulas

In this section we shall prove that the classes of pseudodifferential operators that we are dealing with are closed under composition, taking adjoints and transposes modulo regularizing operators. We start with the adjoint and transpose formulas.

Theorem 2.6. *Let $p \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ with $\mu,\nu > 1$ and $\theta > \mu+\nu-1$. Let moreover $p^*(x,D)$ and ${}^tp(x,D)$ the L^2 adjoint and the transpose of $p(x,D)$ respectively. Then there exist symbols $q_j \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $r_j \in \mathcal{K}_{\mu+\nu-1}$, $j=1,2$, such that*

$$p^*(x,D) = q_1(x,D) + r_1(x,D), \quad q_1(x,\xi) \sim \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} \overline{\partial_\xi^\alpha D_x^\alpha p(x,\xi)} \text{ in } FS^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$$

and

$${}^tp(x,D) = q_2(x,D) + r_2(x,D), \quad q_2(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(\partial_\xi^\alpha D_x^\alpha p)(x,-\xi) \text{ in } FS^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}).$$

Proof. We define $a_1(y,\xi) = \overline{p(y,\xi)} \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{3n})$ and $a_2(y,\xi) = p(y,-\xi) \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{3n})$. Therefore $p^*(x,D) = a_1(y,D)$, ${}^tp(x,D) = a_2(y,D)$. Applying Theorem 2.5 for a_1 and a_2, we conclude the proof. □

Before going to the composition formula, we need the following lemma.

Lemma 2.5. *Let $\mu,\nu > 1$ and $\theta > \mu+\nu-1$. Let moreover $r \in \mathcal{K}_{\mu+\nu-1}$ and $p \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Then ${}^tr(x,D)$, $r^*(x,D)$ and $p(x,D)\circ r(x,D)$ are given by symbols in $\mathcal{K}_{\mu+\nu-1}$.*

Proof. We know that ${}^tr(x,D) = a_1(y,D)$, where $a_1(y,\xi) = r(y,-\xi)$. By Proposition 2.10 and the exponential decay of r, we easily conclude that

$$|\partial_x^\beta \partial_y^\gamma K_{a_1}(x,y)| \le C^{|\beta+\gamma|+1}(\beta!\gamma!)^{\mu+\nu-1} e^{-c|x-y|^{\frac{1}{\mu+\nu-1}}}. \tag{2.8}$$

Hence Remark 2.9 gives that $a_1(y,D)$ is given by a $\mu+\nu-1-$regularizing operator. In analogous way we prove the result for the adjoint.

Now let us observe that ${}^t({}^t r) = r$ and ${}^t r = q$, with $q \in \mathcal{K}_{\mu+\nu-1}$. We also notice that

$$ {}^t q(x, D)u(x) = \int e^{i\xi x} \int e^{-i\xi y} q(y, -\xi)u(y)dy\, đ\xi, $$

hence

$$ p(x, D) \circ r(x, D) = \int e^{i\xi(x-y)} p(x, \xi)q(y, -\xi)u(y)dyđ\xi. $$

In other words, the composition $p \circ r$ is an operator with amplitude $b(x, y, \xi) = p(x, \xi)q(y, -\xi)$. It is not difficult to conclude that $K_b(x, y)$ satisfies an estimate of the type (2.8). Once more making use of Remark 2.9, we get that $p \circ r$ is a regularizing operator. □

Theorem 2.7. *Let* $\mu, \nu > 1$ *and* $\theta > \mu + \nu - 1$. *Let moreover* $p, q \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. *Then there exist symbols* $s \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ *and* $r \in \mathcal{K}_{\mu+\nu-1}$ *such that*

$$ p(x, D)\circ q(x, D) = s(x, D)+r(x, D), \quad s(x, D) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial^{\alpha}_{\xi} p(x, \xi)D^{\alpha}_{x} q(x, \xi) \text{ in } FS^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}). $$

Proof. First we observe that ${}^t q(x, D) = q_1(x, D) + r_1(x, D)$, where $q_1 \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$, $r_1(x, \xi)$ belongs to $\mathcal{K}_{\mu+\nu-1}$ and

$$ q_1(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial^{\alpha}_{\xi} D^{\alpha}_{x} q(x, -\xi) \text{ in } FS^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}). \tag{2.9} $$

Now we write

$$ p(x, D) \circ q(x, D) = p(x, D) \circ {}^t q_1(x, D) + p(x, D) \circ {}^t r_1(x, D). $$

We have that $p \circ {}^t q_1$ is given by the amplitude $a(x, y, \xi) := p(x, \xi)q_1(y, -\xi)$ and $p \circ {}^t r_1$ is given by a regularizing operator. Applying Theorem 2.5 to $a(x, y, \xi)$ and using (2.9), we conclude the proof. □

Remark 2.10. *We point out that we can obtain analogous results concerning the product, transpose and adjoints of pseudodifferential operators with symbols in* $S^{m}_{\mu,\nu}(\mathbb{R}^{n})$. *Moreover, in the frame of finite order we do not need any hypothesis over* θ.

2.2.6 Gevrey Sobolev Spaces

We finish with a brief discussion about Gevrey Sobolev spaces. The main result of Chapter 3 will be achieved in these spaces.

Definition 2.11. *Let $m, \rho \in \mathbb{R}$ and $\theta > 1$. We define the Gevrey Sobolev space $H^m_{\rho;\theta}(\mathbb{R}^n)$ by*

$$H^m_{\rho;\theta}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle D\rangle^m e^{\rho\langle D\rangle^{\frac{1}{\theta}}} u \in L^2(\mathbb{R}^n)\},$$

where $e^{\rho\langle D\rangle^{\frac{1}{\theta}}}$ stands for the pseudodifferential operator with symbol $e^{\rho\langle \xi\rangle^{\frac{1}{\theta}}} \in S^{\infty}_{1,1;\theta}(\mathbb{R}^{2n})$. Gevrey Sobolev spaces are Hilbert spaces with the following inner product

$$\langle u, v\rangle_{H^m_{\rho;\theta}} = \langle \langle D\rangle^m e^{\rho\langle D\rangle^{\frac{1}{\theta}}} u, \langle D\rangle^m e^{\rho\langle D\rangle^{\frac{1}{\theta}}} v\rangle_{L^2}, \quad u, v \in H^m_{\rho;\theta}(\mathbb{R}^n).$$

Remark 2.11. *Plancherel formula gives us that $H^{m'}_{\rho';\theta}(\mathbb{R}^n) \subseteq H^m_{\rho;\theta}(\mathbb{R}^n)$ whenever $m' \geq m$ and $\rho' \geq \rho$. Being more precise, we have*

$$\|f\|_{H^m_{\rho;\theta}} \leq \|f\|_{H^{m'}_{\rho';\theta}}, \quad f \in H^{m'}_{\rho';\theta}(\mathbb{R}^n).$$

We recall that $H^\infty(\mathbb{R}^n) \subset \dot{\mathcal{B}}^\infty(\mathbb{R}^n)$, where

$$H^\infty(\mathbb{R}^n) = \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}), \quad \dot{\mathcal{B}}^\infty(\mathbb{R}^n) = \{f \in \mathcal{B}^\infty(\mathbb{R}^n) : \lim_{|x|\to\infty} \partial^\alpha_x f(x) = 0, \forall\, \alpha \in \mathbb{N}^n_0\}.$$

For any $\rho > 0$, $\theta > 1$ and $m \in \mathbb{R}$ we have $H^m_{\rho;\theta}(\mathbb{R}^n) \subset H^\infty(\mathbb{R})$. The next result shows us that $H^m_{\rho;\theta}(\mathbb{R}^n)$ is a subset of $G^\theta(\mathbb{R}^n)$ for every $m \in \mathbb{R}$, provided that $\rho > 0$.

Proposition 2.11. *Let $m \in \mathbb{R}$, $\theta > 1, \rho > 0$ and $f \in H^m_{\rho;\theta}(\mathbb{R}^n)$. Then there exist $C_1 > 0$ and $C_\theta > 0$ such that*

$$|\partial^\alpha_x f(x)| \leq C_1 \left(\frac{C_\theta}{\rho}\right)^{|\alpha|} \alpha!^\theta \|f\|_{H^m_{\rho;\theta}}, \quad x \in \mathbb{R}^n, \alpha \in \mathbb{N}^n_0.$$

Proof. We can write

$$\partial^\alpha_x f(x) = \int e^{i\xi x}\xi^\alpha \widehat{f}(\xi) đ\xi = \int e^{i\xi x}\xi^\alpha \langle\xi\rangle^{-m} e^{-\rho\langle\xi\rangle^{\frac{1}{\theta}}} \langle\xi\rangle^m e^{\rho\langle\xi\rangle^{\frac{1}{\theta}}} \widehat{f}(\xi) đ\xi.$$

Hence

$$|\partial^\alpha_x f(x)| \leq \|\xi^\alpha \langle\xi\rangle^{-m} e^{-\rho\langle\xi\rangle^{\frac{1}{\theta}}}\|_{L^2} \|f\|_{H^m_{\rho;\theta}}.$$

Now the result follows from Lemma 2.2. □

The next result concerns the action of pseudodifferential operators of finite order in Gevrey Sobolev spaces.

Theorem 2.8. *Let $p \in S^{m'}_{\mu,\nu}(\mathbb{R}^{2n})$. Let moreover $\theta > \mu + \nu - 1$ and $m, \rho \in \mathbb{R}$. Then*

$$p(x, D) : H^m_{\rho;\theta}(\mathbb{R}^n) \to H^{m-m'}_{\rho;\theta}(\mathbb{R}^n), \text{ continuously.}$$

Proof. Since $\theta > \mu + \nu - 1$, from the composition formula we have

$$e^{\rho\langle D\rangle^{\frac{1}{\theta}}} \circ p(x,D) \circ e^{-\rho\langle D\rangle^{\frac{1}{\theta}}} = a(x,D) + r(x,D),$$

where $a \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^n)$, $r \in \mathcal{K}_{\mu+\nu-1}$ and

$$a(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha e^{\rho\langle\xi\rangle^{\frac{1}{\theta}}} D_x^\alpha p(x,\xi) e^{-\rho\langle\xi\rangle^{\frac{1}{\theta}}} = \sum_\alpha a_\alpha(x,\xi) \text{ in } FS^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}).$$

Now we shall prove that $\sum_\alpha a_\alpha$ is of finite order. For this let us study more carefully the above asymptotic expansion. By Faà di Bruno formula, for $|\alpha| \geq 1$,

$$\alpha! a_\alpha(x,\xi) = D_x^\alpha p(x,\xi) \sum_{\ell=1}^{|\alpha|} \frac{1}{\ell!} \sum_{\substack{\alpha_1+\ldots+\alpha_\ell=\alpha \\ \alpha_\nu\geq 1, \nu=1,\ldots,\ell}} \frac{\alpha!}{\alpha_1!\ldots\alpha_\ell!} \prod_{\nu=1}^{\ell} \rho \partial_\xi^{\alpha_\nu} \langle\xi\rangle^{\frac{1}{\theta}}.$$

Therefore, for any $\gamma, \beta \in \mathbb{N}_0^n$,

$$\begin{aligned}
|\partial_\xi^\gamma \partial_x^\beta \alpha! a_\alpha(x,\xi)| &\leq \sum_{\gamma_1+\gamma_2=\gamma} \frac{\gamma!}{\gamma_1!\gamma_2!} |\partial_\xi^{\gamma_1} \partial_x^{\beta+\alpha} p(x,\xi)| \\
&\quad \times \sum_{\ell=1}^{|\alpha|} \frac{1}{\ell!} \sum_{\alpha_1+\ldots+\alpha_\ell=\alpha} \frac{\alpha!}{\alpha_1!\ldots\alpha_\ell!} \left| \partial_\xi^{\gamma_2} \prod_{\nu=1}^{\ell} \rho \partial_\xi^{\alpha_\nu} \langle\xi\rangle^{\frac{1}{\theta}} \right| \\
&\leq C^{|\gamma+\beta+\alpha|+1} \gamma!^\mu \beta!^\nu \alpha!^{1+\nu} \langle\xi\rangle^{m'-|\gamma|-|\alpha|} \sum_{\ell=1}^{|\alpha|} \frac{\langle\xi\rangle^{\frac{\ell}{\theta}}}{\ell!}.
\end{aligned}$$

Now consider, for $R > 0$ large, $\psi_j(\xi)$ the sequence defined in the proof of Proposition 2.2 and set

$$b(x,\xi) = \sum_{j=0}^{\infty} \psi_j(\xi) b_j(x,\xi) \in S^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n}),$$

where $b_j = \sum_{|\alpha|=j} a_\alpha$. We have $b \sim \sum_{j\geq 0} b_j$ in $FS^{\infty}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Now will prove that b belongs to $S^{m'}_{\mu,\nu}(\mathbb{R}^{2n})$. We write

$$b(x,\xi) = p(x,\xi) + \sum_{j\geq 1} \psi_j(\xi) b_j(x,\xi).$$

We infer that $\sum_{j\geq 1} \psi_j b_j \in S^{m'-(1-\frac{1}{\theta})}_{\mu,\nu}(\mathbb{R}^{2n})$. Indeed, for any $\alpha, \beta \in \mathbb{N}_0^n$

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta \sum_{j\geq 1} \psi_j(\xi) b_j(x,\xi)| &\leq \sum_{j\geq 1} \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} |\partial_\xi^{\alpha_1} \psi_j(\xi)| |\partial_\xi^{\alpha_2} \partial_x^\beta b_j(x,\xi)| \\
&= \sum_{j\geq 0} \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} |\partial_\xi^{\alpha_1} \psi_{j+1}(\xi)| |\partial_\xi^{\alpha_2} \partial_x^\beta b_{j+1}(x,\xi)| \\
&\leq \sum_{j\geq 0} \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} \frac{1}{R(j+1)^{|\alpha_1|}} C_\psi^{|\alpha_1|} \alpha_1!^\mu
\end{aligned}$$

$$\times C^{|\alpha_2+\beta|+j+1}\alpha_2!^\mu \beta!^\nu (j+1)!^\nu \langle\xi\rangle^{m'-|\alpha_2|-j-1}\sum_{\ell=1}^{j+1}\frac{\langle\xi\rangle^{\frac{\ell}{\theta}}}{\ell!}$$

$$=\sum_{j\geq 0}\sum_{\alpha_1+\alpha_2=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!}\frac{1}{R(j+1)^{|\alpha_1|}}C_\psi^{|\alpha_1|}\alpha_1!^\mu$$

$$\times C^{|\alpha_2+\beta|+j+1}\alpha_2!^\mu \beta!^\nu (j+1)!^\nu \langle\xi\rangle^{m'-(1-\frac{1}{\theta})-|\alpha_2|}\sum_{\ell=1}^{j+1}\frac{\langle\xi\rangle^{-j+\frac{\ell-1}{\theta}}}{\ell!}.$$

On the support of $\partial_\xi^{\alpha_1}\psi_{j+1}$ we have $\langle\xi\rangle \geq R(j+1)$ and $\langle\xi\rangle \leq 3R(j+1)$ (whenever $\alpha_1\neq 0$). Hence $R(j+1)^{-|\alpha_1|}\leq 3^{|\alpha_1|}\langle\xi\rangle^{-|\alpha_1|}$ and, since $\theta>\mu+\nu-1$,

$$\begin{aligned}\langle\xi\rangle^{-j+\frac{\ell-1}{\theta}} &\leq \{R(j+1)^{\mu+\nu-1}\}^{-j+\frac{\ell-1}{\theta}}\\ &= R^{-j+\frac{\ell-1}{\theta}}(j+1)^{-j(\mu+\nu-1)}(j+1)^{\frac{\mu+\nu-1}{\theta}(\ell-1)}\\ &\leq R^{-j(1-\frac{1}{\theta})}(j+1)!^{-(\mu+\nu-1)}(j+1)^{\ell-1}\\ &\leq R^{-j(1-\frac{1}{\theta})}(j+1)!^{-(\mu+\nu-1)}e^{j+1}(\ell-1)!.\end{aligned}$$

Therefore

$$|\partial_\xi^\alpha\partial_x^\beta\sum_{j\geq 1}\psi_j(\xi)b_j(x,\xi)| \leq C_1^{|\alpha+\beta+1|}\alpha!^\mu\beta!^\nu\langle\xi\rangle^{m'-(1-\frac{1}{\theta})-|\alpha|}\sum_{j\geq 0}\left(\frac{C_2}{R^{(1-\frac{1}{\theta})}}\right)^j.$$

Enlarging R if necessary, we conclude $\sum_{j\geq 1}\psi_j b_j \in S_{\mu,\nu}^{m'-(1-\frac{1}{\theta})}(\mathbb{R}^{2n})$. Hence, $b\in S_{\mu,\nu}^{m'}(\mathbb{R}^{2n})$ as we intended to prove.

Finally, since we may write

$$e^{\rho\langle D\rangle^{\frac{1}{\theta}}}\circ p(x,D)\circ e^{-\rho\langle D\rangle^{\frac{1}{\theta}}} = p(x,D)+p_\theta(x,D)+r_1(x,D),$$

where $p_\theta(x,\xi)\in S_{\mu,\nu}^{m'-(1-\theta)}(\mathbb{R}^{2n})$ and $r_1(x,\xi)\in\mathcal{K}_{\mu+\nu-1}$, we have

$$\begin{aligned}&\|p(x,D)u\|_{H^{m-m'}_{\rho;\theta}}=\| e^{\rho\langle D\rangle^{\frac{1}{\theta}}}p(x,D)u\|_{H^{m-m'}}=\| e^{\rho\langle D\rangle^{\frac{1}{\theta}}}p(x,D)e^{-\rho\langle D\rangle^{\frac{1}{\theta}}}(e^{\rho\langle D\rangle^{\frac{1}{\theta}}}u)\|_{H^{m-m'}}\\ &=\|(p+p_\theta+r_1)(x,D)(e^{\rho\langle D\rangle^{\frac{1}{\theta}}}u)\|_{H^{m-m'}}\leq C\| e^{\rho\langle D\rangle^{\frac{1}{\theta}}}u\|_{H^m}=C\|u\|_{H^m_{\rho;\theta}}.\end{aligned}$$

□

Theorem 2.9. *Let $\mu,\nu>1$ and $\theta>\mu+\nu-1$. Let moreover $r\in\mathcal{K}_{\mu+\nu-1}$. Then for every $m\in\mathbb{R}$ and $\tilde\rho\in\mathbb{R}$*

$$r(x,D): H^m_{\tilde\rho;\theta}\to\bigcap_{\rho\in\mathbb{R}}H^m_{\rho;\theta},\ \textit{continuously}.$$

Proof. Let $\rho \in \mathbb{R}$. Since $\theta > \mu + \nu - 1$ we have that $e^{\rho \langle D \rangle^{\frac{1}{\theta}}} \circ r(x, D) \circ e^{-\tilde{\rho} \langle D \rangle^{\frac{1}{\theta}}}$ defines a $(\mu + \nu - 1)-$regularizing operator. Hence

$$\begin{aligned} \| r(x,D)u \|_{H^m_{\rho;\theta}} &= \| e^{\rho \langle D \rangle^{\frac{1}{\theta}}} \circ r(x,D) \circ e^{-\tilde{\rho} \langle D \rangle^{\frac{1}{\theta}}} (e^{\tilde{\rho} \langle D \rangle^{\frac{1}{\theta}}} u) \|_{H^m} \\ &\leq C \| e^{\tilde{\rho} \langle D \rangle^{\frac{1}{\theta}}} u \|_{H^m} = C \| u \|_{H^m_{\tilde{\rho};\theta}} . \end{aligned}$$

□

2.3 $SG^{\tau,\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n})$, $SG^{\infty,\tau}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$ and $SG^{m}_{\mu,\nu}(\mathbb{R}^{2n})-$Pseudodifferential Operators

In this section, we recall some facts and results about SG pseudodifferential operators with symbols satisfying Gevrey estimates. We address the reader to [10] for the finite order case and to [6] for the infinte order framework. Now we give the definition of the symbol classes.

Definition 2.12. *Let $\tau \in \mathbb{R}$, $\kappa, \theta, \mu, \nu > 1$ and $A, c > 0$.*

(i) We denote by $SG^{\tau,\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}; A, c)$ the Banach space of all functions $p \in C^\infty(\mathbb{R}^{2n})$ satisfying the following condition:

$$\|p\|_{A,c} := \sup_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ x,\xi \in \mathbb{R}^n}} A^{-|\alpha+\beta|} \alpha!^{-\mu} \beta!^{-\nu} \langle \xi \rangle^{-\tau+|\alpha|} \langle x \rangle^{|\beta|} e^{-c|x|^{\frac{1}{\kappa}}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| < +\infty.$$

We set

$$SG^{\tau,\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}) = \bigcup_{A,c>0} SG^{\tau,\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}; A, c)$$

endowed with the inductive limit topology of the Banach spaces $SG^{\tau,\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}; A, c)$.

(ii) We denote by $SG^{\infty,\tau}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}; A, c)$ the Banach space of all functions $p \in C^\infty(\mathbb{R}^{2n})$ satisfying the following condition:

$$\|p\|_{A,c} := \sup_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ x,\xi \in \mathbb{R}^n}} A^{-|\alpha+\beta|} \alpha!^{-\mu} \beta!^{-\nu} \langle \xi \rangle^{|\alpha|} \langle x \rangle^{-\tau+|\beta|} e^{-c|\xi|^{\frac{1}{\kappa}}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| < +\infty.$$

We set

$$SG^{\infty,\tau}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}) = \bigcup_{A,c>0} SG^{\infty,\tau}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}; A, c)$$

endowed with the inductive limit topology of the Banach spaces $SG^{\infty,\tau}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}; A, c)$.

Definition 2.13. *Let $\mu, \nu \geq 1$, $m = (m_1, m_2) \in \mathbb{R}^2$ and $A > 0$. We denote by $SG^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$ the Banach space of all functions $p \in C^\infty(\mathbb{R}^{2n})$ satisfying the following condition:*

$$\|p\|_A := \sup_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ x,\xi\in\mathbb{R}^n}} A^{-|\alpha+\beta|}\alpha!^{-\mu}\beta!^{-\nu}\langle\xi\rangle^{-m_1+|\alpha|}\langle x\rangle^{-m_2+|\beta|}|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)| < +\infty.$$

We set

$$SG^m_{\mu,\nu}(\mathbb{R}^{2n}) = \bigcup_{A,c>0} SG^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$$

endowed with the inductive limit topology of the Banach spaces $SG^m_{\mu,\nu}(\mathbb{R}^{2n}; A)$.

Remark 2.12. *When $\mu = \nu$ we simply write $SG^m_\mu(\mathbb{R}^{2n})$, $SG^{\tau,\infty}_{\mu;\kappa}(\mathbb{R}^{2n})$, $SG^{\infty,\tau}_{\mu;\theta}(\mathbb{R}^{2n})$ instead of $SG^m_{\mu,\mu}(\mathbb{R}^{2n})$, $SG^{\tau,\infty}_{\mu,\mu;\kappa}(\mathbb{R}^{2n})$, $SG^{\infty,\tau}_{\mu,\mu;\theta}(\mathbb{R}^{2n})$.*

As usual, given a symbol $p(x,\xi)$ we shall denote by $p(x,D)$ the pseudodifferential operator defined as

$$p(x,D)u(x) = \int e^{ix\xi}p(x,\xi)\widehat{u}(\xi)đ\xi, \quad x \in \mathbb{R}^n,$$

where u belongs to some suitable function space. We have the following continuity results.

Proposition 2.12. *Let $\tau \in \mathbb{R}$, $s > \mu > 1$, $\nu > 1$ and $p \in SG^{\tau,\infty}_{\mu,\nu;s}(\mathbb{R}^{2n})$. Then for every $\theta > \nu$ and $s > \mu$ the operator $p(x,D)$ is continuous on $\Sigma^\theta_s(\mathbb{R}^n)$ and it extends to a continuous map on $(\Sigma^\theta_s)'(\mathbb{R}^n)$.*

Proposition 2.13. *Let $\tau \in \mathbb{R}$, $\theta > \nu > 1$, $\mu > 1$ and $p \in SG^{\infty,\tau}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$. Then for every $s > \mu$ and $\theta > \nu$, the operator $p(x,D)$ is continuous on $\Sigma^\theta_s(\mathbb{R}^n)$ and it extends to a continuous map on $(\Sigma^\theta_s)'(\mathbb{R}^n)$.*

The proof of Propositions 2.12 and 2.13 can be derived following the argument in the proof of [6, Proposition 2.3].

Now we define the notion of asymptotic expansion and state some fundamental results, whose proofs can be found in the Appendix A of [6]. For $t_1, t_2 \geq 0$ set

$$Q_{t_1,t_2} = \{(x,\xi) \in \mathbb{R}^{2n} : \langle x\rangle < t_1 \text{ and } \langle\xi\rangle < t_2\}$$

and $Q^e_{t_1,t_2} = \mathbb{R}^{2n} \setminus Q_{t_1,t_2}$. When $t_1 = t_2 = t$ we simply write Q_t and Q^e_t.

Definition 2.14. *We say that:*

(i) $\sum_{j\geq 0} a_j \in FSG^{\tau,\infty}_{\mu,\nu;\kappa}$ *if* $a_j \in C^\infty(\mathbb{R}^{2n})$ *and there exist* $C, c, B > 0$ *satisfying*

$$|\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)| \leq C^{|\alpha|+|\beta|+2j+1} \alpha!^\mu \beta!^\nu j!^{\mu+\nu-1} \langle\xi\rangle^{\tau-|\alpha|-j} \langle x\rangle^{-|\beta|-j} e^{c|x|^{\frac{1}{\kappa}}}$$

for $\alpha, \beta \in \mathbb{N}_0^n$, $j \geq 0$ *and* $(x,\xi) \in Q^e_{B(j)}$, *where* $B(j) := Bj^{\mu+\nu-1}$.

(ii) $\sum_{j\geq 0} a_j \in FSG^{\infty,\tau}_{\mu,\nu;\theta}$ *if* $a_j \in C^\infty(\mathbb{R}^{2n})$ *and there exist* $C, c, B > 0$ *satisfying*

$$|\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)| \leq C^{|\alpha|+|\beta|+2j+1} \alpha!^\mu \beta!^\nu j!^{\mu+\nu-1} \langle\xi\rangle^{-|\alpha|-j} e^{c|\xi|^{\frac{1}{\theta}}} \langle x\rangle^{\tau-|\beta|-j}$$

for $\alpha, \beta \in \mathbb{N}_0^n$, $j \geq 0$ *and* $(x,\xi) \in Q^e_{B(j)}$.

(iii) $\sum_{j\geq 0} a_j \in FSG^m_{\mu,\nu}$ *if* $a_j \in C^\infty(\mathbb{R}^{2n})$ *and there exist* $C, c, B > 0$ *satisfying*

$$|\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)| \leq C^{|\alpha|+|\beta|+2j+1} \alpha!^\mu \beta!^\nu j!^{\mu+\nu-1} \langle\xi\rangle^{m_1-|\alpha|-j} \langle x\rangle^{m_2-|\beta|-j}$$

for $\alpha, \beta \in \mathbb{N}_0^n$, $j \geq 0$ *and* $(x,\xi) \in Q^e_{B(j)}$.

Definition 2.15. *Given* $\sum_{j\geq 0} a_j$, $\sum_{j\geq 0} b_j$ *in* $FSG^{\tau,\infty}_{\mu,\nu;\kappa}$ *we say that* $\sum_{j\geq 0} a_j \sim \sum_{j\geq 0} b_j$ *in* $FSG^{\tau,\infty}_{\mu,\nu;\kappa}$ *if there exist* $C, c, B > 0$ *satisfying*

$$\left|\partial_\xi^\alpha \partial_x^\beta \sum_{j<N} (a_j - b_j)(x,\xi)\right| \leq C^{|\alpha|+|\beta|+2N+1} \alpha!^\mu \beta!^\nu N!^{\mu+\nu-1} \langle\xi\rangle^{\tau-|\alpha|-N} \langle x\rangle^{-|\beta|-N} e^{c|x|^{\frac{1}{\kappa}}}$$

for $\alpha, \beta \in \mathbb{N}_0^n$, $N \geq 1$ *and* $(x,\xi) \in Q^e_{B(N)}$. *Analogous definition for the classes* $FSG^{\infty,\tau}_{\mu,\nu;\theta}$, $FSG^m_{\mu,\nu}$.

Remark 2.13. *If* $\sum_{j\geq 0} a_j \in FSG^{\tau,\infty}_{\mu,\nu;\kappa}$, *then* $a_0 \in SG^{\tau,\infty}_{\mu,\nu;\kappa}$. *Given* $a \in SG^{\tau,\infty}_{\mu,\nu;\kappa}$ *and setting* $b_0 = a$, $b_j = 0$, $j \geq 1$, *we have* $a = \sum_{j\geq 0} b_j$. *Hence we can consider* $SG^{\tau,\infty}_{\mu,\nu;\kappa}$ *as a subset of* $FSG^{\tau,\infty}_{\mu,\nu;\kappa}$.

Proposition 2.14. *Given* $\sum_{j\geq 0} a_j \in FSG^{\tau,\infty}_{\mu,\nu;\kappa}$, *there exists* $a \in SG^{\tau,\infty}_{\mu,\nu;\kappa}$ *such that* $a \sim \sum_{j\geq 0} a_j$ *in* $FSG^{\tau,\infty}_{\mu,\nu;\kappa}$. *Analogous results hold for the classes* $SG^{\infty,\tau}_{\mu,\nu;\theta}$ *and* $FSG^m_{\mu,\nu}$.

Proposition 2.15. *Let* $a, b \in SG^{\tau,\infty}_{\mu,\nu;\kappa}$ *and* $\sum_{j\geq 0} a_j \in FSG^{\tau,\infty}_{\mu,\nu;\kappa}$ *such that* $a \sim \sum_{j\geq 0} a_j \sim b$ *in* $FSG^{\tau,\infty}_{\mu,\nu;\kappa}$. *If* $\kappa > \mu + \nu - 1$, *then* $a - b \in \mathcal{S}_\delta(\mathbb{R}^{2n})$ *for every* $\delta \geq \mu + \nu - 1$. *Analogous result for the classes* $FSG^{\infty,\tau}_{\mu,\nu;\theta}$ *and* $FSG^m_{\mu,\nu}$.

Concerning the symbolic calculus and the continuous mapping properties on the Gelfand-Shilov Sobolev spaces we have the following results, whose proofs can be found again in [6].

Theorem 2.10. *Let* $p \in SG^{\tau,\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n})$, $q \in SG^{\tau',\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n})$ *with* $\kappa > \mu + \nu - 1$. *Then the* L^2 *adjoint* p^* *and the composition* $p \circ q$ *have the following structure:*

- $p^*(x,D) = a(x,D) + r(x,D)$ *where* $r \in \mathcal{S}_{\mu+\nu-1}(\mathbb{R}^{2n})$, $a \in SG^{\tau,\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n})$, *and*

$$a(x,\xi) \sim \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} \overline{\partial_\xi^\alpha D_x^\alpha p(x,\xi)} \text{ in } FSG^{\tau,\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}).$$

- $p(x,D) \circ q(x,D) = b(x,D) + s(x,D)$, *where* $s \in \mathcal{S}_{\mu+\nu-1}(\mathbb{R}^{2n})$, $b \in SG^{\tau+\tau',\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n})$ *and*

$$b(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) \text{ in } FSG^{\tau+\tau',\infty}_{\mu,\nu;\kappa}(\mathbb{R}^{2n}).$$

Analogous results hold for the class $SG^{\infty,\tau}_{\mu,\nu;\theta}(\mathbb{R}^{2n})$.

We finish this section with a brief discussion concerning the Gelfand-Shilov Sobolev spaces. For details we refer once more [6].

Definition 2.16. *Let* $m = (m_1, m_2), \rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ *and* $s, \theta > 1$. *We define the Gelfand-Shilov Sobolev space* $H^m_{\rho;s,\theta}(\mathbb{R}^n)$ *by*

$$H^m_{\rho;s,\theta}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle x\rangle^{m_2} \langle D\rangle^{m_1} e^{\rho_2 \langle x\rangle^{\frac{1}{s}}} e^{\rho_1 \langle D\rangle^{\frac{1}{\theta}}} u \in L^2(\mathbb{R}^n)\},$$

where $\langle D\rangle^{m_1}$ *and* $e^{\rho_1 \langle D\rangle^{\frac{1}{\theta}}}$ *are the the pseudodifferential operators with symbols* $\langle \xi\rangle^{m_1}$ *and* $e^{\rho\langle \xi\rangle^{\frac{1}{\theta}}} \in S^\infty_{1,1;\theta}(\mathbb{R}^{2n})$, *respectively. These spaces are Hilbert spaces with the following inner product*

$$\langle u, v\rangle_{H^m_{\rho;s,\theta}} = \langle \langle x\rangle^{m_2} \langle D\rangle^{m_1} e^{\rho_2 \langle x\rangle^{\frac{1}{s}}} e^{\rho_1 \langle D\rangle^{\frac{1}{\theta}}} u, \langle x\rangle^{m_2} \langle D\rangle^{m_1} e^{\rho_2 \langle x\rangle^{\frac{1}{s}}} e^{\rho_1 \langle D\rangle^{\frac{1}{\theta}}} v\rangle_{L^2},$$

for every $u, v \in H^m_{\rho;s,\theta}(\mathbb{R}^n)$.

Remark 2.14. *We have* $H^m_{\rho;s,\theta}(\mathbb{R}^n) \subseteq H^{m'}_{\rho';s',\theta'}(\mathbb{R}^n)$ *continuously, whenever that* $s \le s'$, $\theta \le \theta'$, $m_j \ge m_j'$ *and* $\rho_j \ge \rho_j'$, $j = 1, 2$.

Remark 2.15. *The Gelfand-Shilov spaces can be expressed in terms of the Gelfand-Shilov Sobolev spaces* $H^m_{\rho;s,\theta}(\mathbb{R}^n)$ *in the following way*

$$\mathcal{S}^\theta_s(\mathbb{R}^n) = \bigcup_{\rho_1,\rho_2>0} H^m_{\rho;s,\theta}(\mathbb{R}^n), \quad \Sigma^\theta_s(\mathbb{R}^n) = \bigcap_{\rho_1,\rho_2>0} H^m_{\rho;s,\theta}(\mathbb{R}^n),$$

for every $m \in \mathbb{R}^2$.

We have the following continuity result on Gelfand-Shilov Sobolev spaces, see [6] Theorem A.18 for the proof.

Theorem 2.11. *Let* $p \in SG^{m'}_{\mu,\nu}(\mathbb{R}^{2n})$ *for some* $m' \in \mathbb{R}^2$. *Then for every* $m, \rho \in \mathbb{R}^2$ *and* s, θ *such that* $\min\{s,\theta\} > \mu+\nu-1$ *the operator* $p(x,D)$ *maps* $H^m_{\rho;s,\theta}(\mathbb{R}^n)$ *into* $H^{m-m'}_{\rho;s,\theta}(\mathbb{R}^n)$ *continuously.*

2.4 Sharp Gårding and Fefferman-Phong Inequalities

This section is devoted to recall the so-called Fefferman-Phong and sharp Gårding inequalities. Let $p(x,\xi)$ be a symbol of order m belonging to the standard Hörmander classes. Then the two following theorems hold.

Theorem 2.12 (Sharp Gårding Inequality)**.** *If* $Re\, p(x,\xi) \geq 0$*, then there exists a real constant* C *such that*

$$Re\, \langle p(x,D)u, u\rangle_{L^2(\mathbb{R}^n)} \geq -C\|u\|_{\frac{m-1}{2}}, \quad u \in \mathscr{S}(\mathbb{R}^n).$$

Proof. See Theorem 4.4 of [35]. □

Theorem 2.13 (Fefferman-Phong Inequality)**.** *If* $p(x,\xi) \geq 0$*, then there exists a real constant* C *such that*

$$Re\, \langle p(x,D)u, u\rangle_{L^2(\mathbb{R}^n)} \geq -C\|u\|_{\frac{m-2}{2}}, \quad u \in \mathscr{S}(\mathbb{R}^n).$$

Proof. See [20]. □

Remark 2.16. *We remark that the Fefferman-Phong inequality holds only for scalar symbols (cf. [41]) whereas the sharp Gårding holds more generally for matrix valued symbols (cf. [35]).*

2.5 Sharp Gårding in SG and $SG_{\mu,\nu}$ Settings

The sharp Gårding inequality for a pseudodifferential operator has been first proved by Hörmander [26] and by Lax and Nirenberg [36] for symbols in the standard Hörmander classes $S^m(\mathbb{R}^{2n})$. Later on several different proofs and extensions of this result have been provided by many authors cf. [21, 27, 35, 38]. In particular, the inequality has been extended to symbols defined in terms of a general metric, cf. [27, Theorem 18.6.7] and to matrix valued pseudo-differential operators, cf. [27, Lemma 18.6.13] and [35, Theorem 4.4 page 134].

In all the proofs of the sharp Gårding inequality the operator $p(x,D)$ is decomposed as the sum of a positive definite part and a remainder term. In the approach proposed in [35], this positive definite part p_F is called *Friedrichs* part.

Although the results in [27] are extremely general and sharp, in some applications more detailed information on the remainder term is needed. In particular, it is important to state not only the order but also the asymptotic expansion of $p - p_F$. This is needed in particular in the study of the p-evolution equations. The classical approach to study the Cauchy Problem for these equations is based on a reduction to an auxiliary problem via a suitable change of variable and on a repeated application of sharp Gårding inequality which needs at every step to understand the precise form of all the remainder terms, cf. [3]. When the coefficients $a_j(t,x)$ are uniformly bounded with respect to x, this is possible using Theorem 4.2 in [35], where the asymptotic expansion of $p_F - p$ is given in the frame of classical Hörmander classes.

The initial value problem for $p-$evolution equations can be also studied in a weighted functional setting admitting polynomially bounded coefficients (cf. [5]), which cannot be treated in the theory of standard Hörmander classes but are included in the SG framework. For this purpose we need a variant of [35, Theorem 4.2] for SG operators with a precise information on the asymptotic expansion of $p - p_F$.

Another challenging issue is to study $p-$evolution equations on Gelfand-Shilov spaces. A first step in this direction has been done in the cases $p = 2$, that is for Schrödinger-type equations, see [6], and $p = 3$, see Chapter 4 of this thesis. In both these cases, it is sufficient to apply the sharp Gårding inequality only once. To treat p-evolution equations for $p > 3$, however, we need to apply the iterative procedure described above. In addition, a precise estimate of the Gevrey regularity of the terms in the asymptotic expansion of $p - p_F$ is also needed.

In this section we, hopefully, provide appropriate tools for both the aforementioned issues. This will be achieved by defining in a suitable way the Friedrichs part of our operators and by studying in detail its asymptotic expansion and its regularity. With this purpose we will prove two separate results for the SG and $SG_{\mu,\nu}$ symbol classes. To finish, it is worth to mention that the results obtained in this Section led to the following work [32].

2.5.1 $SG_{\rho,\delta}$ Pseudodifferential Operators

In this section we recall some basic facts about $SG_{\rho,\delta}$ pseudodifferential operators which will be used in the sequel. Although for our applications we are interested to prove the main results for the SG and $SG_{\mu,\nu}$ classes, in order to prove them we need to consider more general symbols which are defined as follows.

Definition 2.17. *Given* $m = (m_1, m_2) \in \mathbb{R}^2$, $\rho = (\rho_1, \rho_2) \in (0,1]^2$, $\delta = (\delta_1, \delta_2) \in [0,1)^2$, *with* $\delta_j < \rho_j$, $j = 1, 2$, *we denote by* $SG^m_{\rho,\delta}$ *the space of all functions* $p(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ *such that*

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \langle \xi \rangle^{-m_1+\rho_1|\alpha|-\delta_1|\beta|} \langle x \rangle^{-m_2+\rho_2|\beta|-\delta_2|\alpha|} < \infty. \tag{2.10}$$

We recall that $SG^m_{\rho,\delta}$ is a Fréchet space endowed with the seminorms

$$|p|_\ell := \sup_{\substack{(x,\xi)\in\mathbb{R}^{2n} \\ |\alpha+\beta|\leq\ell}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \langle\xi\rangle^{-m_1+\rho_1|\alpha|-\delta_1|\beta|} \langle x\rangle^{-m_2+\rho_2|\beta|-\delta_2|\alpha|},$$

for $\ell \in \mathbb{N}_0$. A specific calculus for this class can be found in [8]. Pseudodifferential operators with symbols in $SG^m_{\rho,\delta}$ are linear and continuous from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$ and extend to linear and continuous maps from $\mathscr{S}'(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$. Moreover, we know that an operator with symbol in $SG^m_{\rho,\delta}$ extends to a linear and continuous map from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$ for every $s \in \mathbb{R}^2$.

Definition 2.18. *Let, for $j \in \mathbb{N}_0$, $p_j \in SG^{(m_{1,j},m_{2,j})}_{\rho,\delta}$, where $m_{1,j}$, $m_{2,j}$ are non increasing sequences and $m_{1,j} \to -\infty$, $m_{2,j} \to -\infty$, when $j \to \infty$. We say that $p \in C^\infty(\mathbb{R}^{2n})$ has the asymptotic expansion*

$$p(x,\xi) \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi)$$

if for any $N \in \mathbb{N}$ we have

$$p(x,\xi) - \sum_{j=0}^{N-1} p_j(x,\xi) \in SG^{(m_{1,N},m_{2,N})}_{\rho,\delta}.$$

Given $p_j \in SG^{(m_{1,j},m_{2,j})}_{\rho,\delta}$ as in the previous definition, we can find $p \in SG^{(m_{1,0},m_{2,0})}_{\rho,\delta}$ such that $p \sim \sum p_j$. Furthermore, if there is q such that $q \sim \sum p_j$, then

$$p - q \in SG^{-\infty} := \cap_{m\in\mathbb{R}^2} SG^m_{\rho,\delta} = \mathscr{S}(\mathbb{R}^{2n}),$$

cf. [8, Theorem 2]. The class $SG^m_{\rho,\delta}$ is closed under adjoints. Namely, given $p \in SG^m_{\rho,\delta}$ and denoting by P^* be the L^2 adjoint of $p(x,D)$, we can write $P^* = p^*(x,D) + R'$, where p^* is a symbol in $SG^m_{\rho,\delta}$ admitting the asymptotic expansion

$$p^*(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \alpha!^{-1} \partial_\xi^\alpha D_x^\alpha \overline{p(x,\xi)}$$

and $R' : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. The class $SG^\infty_{\rho,\delta} := \cup_{m\in\mathbb{R}^2} SG^m_{\rho,\delta}$ possesses algebra properties with respect to composition. Namely, given $p \in SG^m_{\rho,\delta}$ and $q \in SG^{m'}_{\rho,\delta}$, then there exists a symbol $s \in SG^{m+m'}_{\rho,\delta}$ such that $p(x,D)q(x,D) = s(x,D) + R$ where R is a smoothing operator $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. Moreover,

$$s(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \alpha!^{-1} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi),$$

cf. [8, Theorem 3].

We now consider Gevrey regular symbols.

Definition 2.19. *Fixed* $C > 0$ *we denote by* $SG^m_{\rho,\delta;(\mu,\nu)}(\mathbb{R}^{2n}; C)$ *the space of all smooth functions* $p(x,\xi)$ *such that*

$$|p|_C := \sup_{\alpha,\beta\in\mathbb{N}_0^n} C^{-|\alpha|-|\beta|}\alpha!^{-\mu}\beta!^{-\nu} \times \sup_{x,\xi\in R^n} \langle\xi\rangle^{-m_1+\rho_1|\alpha|-\delta_1|\beta|}\langle x\rangle^{-m_2+\rho_2|\beta|-\delta_2|\alpha|}|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)| < +\infty.$$

We set $SG^m_{\rho,\delta;(\mu,\nu)}(\mathbb{R}^{2n}) = \bigcup_{C>0} SG^m_{\rho,\delta;(\mu,\nu)}(\mathbb{R}^{2n}; C)$.

Equipping $SG^m_{\rho,\delta;(\mu,\nu)}(\mathbb{R}^{2n}; C)$ with the norm $|\cdot|_C$ we obtain a Banach space and we can endow $SG^m_{\rho,\delta;(\mu,\nu)}(\mathbb{R}^{2n})$ with the topology of inductive limit of Banach spaces. A complete calculus for operators with symbols in this class can be found in [10]. Here we recall only the main results. Since $SG^m_{\rho,\delta;(\mu,\nu)} \subset SG^m_{\rho,\delta}$, the previous mapping properties on the Schwartz and weighted Sobolev spaces hold true for operators with symbols in $SG^m_{\rho,\delta;(\mu,\nu)}$. By the way the, natural functional setting for these operators is given by the Gelfand-Shilov spaces of type $\mathcal{S}$. For every $\mu' \geq \mu/(1-\delta_2), \nu' \geq \nu/(1-\delta_1)$, an operator with symbol in $SG^m_{\rho,\delta;(\mu,\nu)}$ is linear and continuous from $\mathcal{S}^{\nu'}_{\mu'}(\mathbb{R}^n)$ to itself and extends to a linear continuous map from the dual space $(\mathcal{S}^{\nu'}_{\mu'})'(\mathbb{R}^n)$ into itself, see [10, Theorem A.4].

The notion of asymptotic expansion for symbols in $SG^m_{\rho,\delta;(\mu,\nu)}$ can be defined in terms of formal sums, cf. [10]. Here, to obtain our results we need to use a refined notion. All the next results can be obtained in the same way as in [10] without changing the arguments in the proofs.

For $t_1, t_2 \geq 0$ set

$$Q_{t_1,t_2} = \{(x,\xi) \in \mathbb{R}^{2n} : \langle x\rangle < t_1 \text{ and } \langle\xi\rangle < t_2\}$$

and $Q^e_{t_1,t_2} = \mathbb{R}^{2n} \setminus Q_{t_1,t_2}$. When $t_1 = t_2 = t$ we simply write Q_t and Q^e_t.

Definition 2.20. *Let* $\bar\sigma_j = (k_j, \ell_j)$ *be a sequence such that* $k_0 = \ell_0 = 0$*,* k_j, ℓ_j *are strictly increasing,* $k_{j+N} \geq k_j + k_N$*,* $\ell_{j+N} \geq \ell_j + \ell_N$*, for* $j, N \in \mathbb{N}_0$*, and* $k_j \geq \Lambda_1 j$*,* $\ell_j \geq \Lambda_2 j$ *for* $j \geq 1$*, for some* $\Lambda_1, \Lambda_2 > 0$*. We say that* $\sum\limits_{j\geq 0} p_j \in F_{\bar\sigma}SG^m_{\rho,\delta;(\mu,\nu)}$ *if* $p_j \in C^\infty(\mathbb{R}^{2n})$ *and there are* $C, c, B > 0$ *satisfying*

$$|\partial_\xi^\alpha\partial_x^\beta p_j(x,\xi)| \leq C^{|\alpha|+|\beta|+2j+1}\alpha!^\mu\beta!^\nu j!^{\mu+\nu-1}\langle\xi\rangle^{m_1-\rho_1|\alpha|+\delta_1|\beta|-k_j}\langle x\rangle^{m_2-\rho_2|\beta|+\delta_2|\alpha|-\ell_j}$$

for $\alpha, \beta \in \mathbb{N}_0^n$*,* $j \geq 0$ *and* $(x,\xi) \in Q^e_{B_2(j),B_1(j)}$*, where* $B_i(j) = (Bj^{\mu+\nu-1})^{\frac{1}{\Lambda_i}}$*,* $i = 1, 2$.

Definition 2.21. *Given* $\sum_{j\geq 0} p_j, \sum_{j\geq 0} q_j \in F_{\bar{\sigma}}SG^m_{\mu,\nu}$, *we say that* $\sum_{j\geq 0} p_j \sim \sum_{j\geq 0} q_j$ *in* $F_{\bar{\sigma}}SG^m_{\rho,\delta;(\mu,\nu)}$ *if there are* $C, c, B > 0$ *satisfying*

$$\left|\partial_\xi^\alpha \partial_x^\beta \sum_{j<N}(p_j - q_j)(x,\xi)\right| \leq C^{|\alpha|+|\beta|+2N+1}\alpha!^\mu \beta!^\nu N!^{\mu+\nu-1}\langle\xi\rangle^{m_1-\rho_1|\alpha|+\delta_1|\beta|-k_N}\langle x\rangle^{m_2-\rho_2|\beta|+\delta_2|\alpha|-\ell_N}$$

for $\alpha, \beta \in \mathbb{N}_0^n$, $N \geq 1$ *and* $(x,\xi) \in Q^e_{B_2(N),B_1(N)}$.

Remark 2.17. *If* $k_j = (\rho_1 - \delta_1)j, \ell_j = (\rho_2 - \delta_2)j$ *and* $\Lambda_i = \rho_i - \delta_i, i = 1, 2$, *we simply write* $FSG^m_{\rho,\delta;(\mu,\nu)}$ *and we recover the usual definitions presented in [10]. If moreover* $\rho = (1, 1), \delta = (0, 0)$, *we use the notation* $FSG^m_{(\mu,\nu)}$.

Remark 2.18. *If* $\sum_{j\geq 0} p_j \in F_{\bar{\sigma}}SG^m_{\rho,\delta;(\mu,\nu)}$, *then* $p_0 \in SG^m_{\rho,\delta;(\mu,\nu)}$. *Given* $p \in SG^m_{\rho,\delta;(\mu,\nu)}$ *and setting* $p_0 = p$, $p_j = 0$, $j \geq 1$, *we have* $p = \sum_{j\geq 0} p_j$. *Hence we can consider* $SG^m_{\rho,\delta;(\mu,\nu)}$ *as a subset of* $F_{\bar{\sigma}}SG^m_{\rho,\delta;(\mu,\nu)}$.

Proposition 2.16. *Given* $\sum_{j\geq 0} p_j \in F_{\bar{\sigma}}SG^m_{\rho,\delta;(\mu,\nu)}$, *there exists* $p \in SG^m_{\rho,\delta;(\mu,\nu)}$ *such that* $p \sim \sum_{j\geq 0} p_j$ *in* $F_{\bar{\sigma}}SG^m_{\rho,\delta;(\mu,\nu)}$.

Proposition 2.17. *Let* $p \in SG^{(0,0)}_{\rho,\delta;(\mu,\nu)}$ *such that* $p \sim 0$ *in* $F_{\bar{\sigma}}SG^{(0,0)}_{\rho,\delta;(\mu,\nu)}$. *Then* $p \in \mathcal{S}_r(\mathbb{R}^{2n})$ *for* $r \geq \max\{\frac{1}{\tilde{\Lambda}}(\mu+\nu-1), \mu+\nu-1\}$, *where* $\tilde{\Lambda} = \min\{\Lambda_1, \Lambda_2\}$.

Proposition 2.18. *Let* $p \in SG^m_{\rho,\delta;(\mu,\nu)}$ *and let* P^* *be the* L^2 *adjoint of* $p(x,D)$. *Then there exists a symbol* $p^* \in SG^m_{\rho,\delta;(\mu,\nu)}$ *such that* $P^* = p^*(x,D) + R$, *where* R *is a* $\mathcal{S}_r$*-regularizing operator for any* $r \geq \frac{\mu+\nu-1}{\min\{\varrho_1-\delta_1,\varrho_2-\delta_2\}}$. *Moreover*

$$p^*(x,\xi) \sim \sum_{j\geq 0}\sum_{|\alpha|=j} \alpha!^{-1}\partial_\xi^\alpha D_x^\alpha \overline{p(x,\xi)} \quad \text{in} \quad FSG^m_{\rho,\delta;(\mu,\nu)}.$$

Proposition 2.19. *Let* $p \in SG^m_{\rho,\delta;(\mu,\nu)}$, $q \in SG^{m'}_{\rho,\delta;(\mu,\nu)}$. *Then, there exists a symbol* $s \in SG^{m+m'}_{\rho,\delta;(\mu,\nu)}$ *such that* $p(x,D)q(x,D) = s(x,D) + R'$ *where* R' *is a* $\mathcal{S}_r$*-regularizing operator for any* $r \geq \frac{\mu+\nu-1}{\min\{\varrho_1-\delta_1,\varrho_2-\delta_2\}}$. *Moreover*

$$s(x,\xi) \sim \sum_{j\geq 0}\sum_{|\alpha|=j} \alpha!^{-1}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) \quad \text{in} \quad FSG^{m+m'}_{\rho,\delta;(\mu,\nu)}.$$

2.5.2 Oscillatory Integrals and Operators with Double Symbols

In order to define the Friedrichs part of an operator it is necessary to extend the notion of pseudodifferential operator as in [35] by considering more general symbols called *double symbols*. We start considering some amplitude classes.

Definition 2.22 (Amplitudes)**.** *For $m \in \mathbb{R}^2$ and $\delta \in [0,1)^2$ we define $\mathcal{A}^m_\delta(\mathbb{R}^{2n})$ as the space of all smooth functions $a(\eta, y)$ such that*

$$|\partial_\eta^\alpha \partial_y^\beta a(\eta, y)| \leq C_{\alpha,\beta} \langle \eta \rangle^{m_1+\delta_1|\beta|} \langle y \rangle^{m_2+\delta_2|\alpha|}, \quad \eta, y \in \mathbb{R}^n.$$

For $\ell \in \mathbb{N}_0$ and $a \in \mathcal{A}^m_\delta(\mathbb{R}^{2n})$, the seminorms

$$|a|_\ell = \max_{|\alpha+\beta|\leq \ell} \sup_{\eta,y\in\mathbb{R}^n} \{|\partial_\eta^\alpha \partial_y^\beta a(\eta, y)| \langle \eta \rangle^{-(m_1+\delta_1|\beta|)} \langle y \rangle^{-(m_2+\delta_2|\alpha|)}\},$$

turn $\mathcal{A}^m_\delta(\mathbb{R}^{2n})$ into a Fréchet space.

Remark 2.19. *In [35, Chapter 1, Section 6], the special case $\mathcal{A}^{(m,\tau)}_{(\delta,0)}(\mathbb{R}^n)$, where $m \in \mathbb{R}$, $\tau > 0$ and $\delta \in [0,1)$, is treated.*

Definition 2.23 (Oscillatory Integral)**.** *For $a \in \mathcal{A}^m_{\delta,\tau}$ we define*

$$\begin{aligned} Os - [e^{-i\eta y} a(\eta, y)] &= Os - \iint e^{-i\eta y} a(\eta, y) dy đ\eta \\ &:= \lim_{\varepsilon \to 0} \iint e^{-i\eta y} \chi_\varepsilon(\eta, y) a(\eta, y) dy đ\eta, \end{aligned}$$

where $\chi_\varepsilon(\eta, y) = \chi(\varepsilon\eta, \varepsilon y)$ and χ is a Schwartz function on $\mathbb{R}^{2n}$ such that $\chi(0,0) = 1$.

Theorem 2.14. *Let $a \in \mathcal{A}^m_\delta(\mathbb{R}^{2n})$. If $\ell, \ell' \in \mathbb{N}_0$ satisfy*

$$-2\ell(1-\delta_1) + m_1 < -n, \quad -2\ell'(1-\delta_2) + m_2 < -n,$$

then $|\langle y \rangle^{-2\ell'} \langle D_\eta \rangle^{2\ell'} \{\langle \eta \rangle^{-2\ell} \langle D_y \rangle^{2\ell} a(\eta, y)\}|$ belongs to $L^1(\mathbb{R}^{2n})$ and we have

$$Os - [e^{-i\eta y} a(\eta, y)] = \iint e^{-i\eta y} \langle y \rangle^{-2\ell'} \langle D_\eta \rangle^{2\ell'} \{\langle \eta \rangle^{-2\ell} \langle D_y \rangle^{2\ell} a(\eta, y)\} dy đ\eta.$$

Furthermore, there exists $C_{\ell,\ell'} > 0$ independent of $a \in \mathcal{A}^m_{\delta,\tau}(\mathbb{R}^{2n})$ such that

$$|Os - [e^{-i\eta y} a(\eta, y)]| \leq C_{\ell,\ell'} |a|_{2(\ell+\ell')}. \tag{2.11}$$

Proof. Integration by parts gives

$$Os - [e^{-i\eta y} a] = \lim_{\varepsilon \to 0} \iint e^{-i\eta y} \langle y \rangle^{-2\ell'} \langle D_\eta \rangle^{2\ell'} \{\langle \eta \rangle^{-2\ell} \langle D_y \rangle^{2\ell} (a\chi_\varepsilon)\} dy đ\eta,$$

where $\chi_\varepsilon(\eta, y) = \chi(\varepsilon\eta, \varepsilon y)$. Since

$$\langle D_\eta \rangle^{2\ell'} = \sum_{\ell_0'+|L'|=\ell'} \frac{\ell'!}{\ell_0'! L'!} D_\eta^{2L'}, \quad \langle D_y \rangle^{2\ell} = \sum_{\ell_0+|L|=\ell} \frac{\ell!}{\ell_0! L!} D_y^{2L},$$

where $L' = (\ell'_1, \ldots, \ell'_n)$ and $L = (\ell_1, \ldots, \ell_n)$, we have

$$
\begin{aligned}
\langle D_\eta \rangle^{2\ell'} \{\langle \eta \rangle^{-2\ell} \langle D_y \rangle^{2\ell} (\chi_\varepsilon a)\} &= \sum_{\ell'_0 + |L'| = \ell'} \frac{\ell'!}{\ell'_0! L'!} D_\eta^{2L'} \{\langle \eta \rangle^{-2\ell} \langle D_y \rangle^{2\ell} (\chi_\varepsilon a)\} \\
&= \sum_{\ell'_0 + |L'| = \ell'} \frac{\ell'!}{\ell'_0! L'!} \sum_{\alpha_1 + \alpha_2 = 2L'} \frac{(2L')!}{\alpha_1! \alpha_2!} D_\eta^{\alpha_1} \langle \eta \rangle^{-2\ell} \cdot D_\eta^{\alpha_2} \langle D_y \rangle^{2\ell} (\chi_\varepsilon a) \\
&= \sum_{\ell'_0 + |L'| = \ell'} \frac{\ell'!}{\ell'_0! L'!} \sum_{\alpha_1 + \alpha_2 = 2L'} \frac{(2L')!}{\alpha_1! \alpha_2!} D_\eta^{\alpha_1} \langle \eta \rangle^{-2\ell} \\
&\times \sum_{\ell_0 + |L| = \ell} \frac{\ell!}{\ell_0! L!} D_\eta^{\alpha_2} D_y^{2L} (\chi_\varepsilon a) \\
&= \sum_{\ell'_0 + |L'| = \ell'} \frac{\ell'!}{\ell'_0! L'!} \sum_{\alpha_1 + \alpha_2 = 2L'} \frac{(2L')!}{\alpha_1! \alpha_2!} D_\eta^{\alpha_1} \langle \eta \rangle^{-2\ell} \sum_{\ell_0 + |L| = \ell} \frac{\ell!}{\ell_0! L!} \\
&\times \sum_{\alpha'_1 + \alpha'_2 = \alpha_2} \sum_{\beta_1 + \beta_2 = 2L} \frac{\alpha_2!}{\alpha'_1! \alpha'_2!} \frac{(2L)!}{\beta_1! \beta_2!} D_\eta^{\alpha'_1} D_y^{\beta_1} \chi_\varepsilon D_\eta^{\alpha'_2} D_y^{\beta_2} a.
\end{aligned}
$$

Hence we obtain the following estimates, for ε in $[0, 1]$,

$$
\begin{aligned}
&|\langle D_\eta \rangle^{2\ell'} \{\langle \eta \rangle^{-2\ell} \langle D_y \rangle^{2\ell} (\chi_\varepsilon a)\}| \leq \sum_{\ell'_0 + |L'| = \ell'} \frac{\ell'!}{\ell'_0! L'!} \sum_{\alpha_1 + \alpha_2 = 2L'} \frac{(2L')!}{\alpha_1! \alpha_2!} \\
&\times C_0^{|\alpha_1| + 1} \alpha_1! \langle \eta \rangle^{-2\ell - |\alpha_1|} \sum_{\ell_0 + |L| = \ell} \frac{\ell!}{\ell_0! L!} \sum_{\alpha'_1 + \alpha'_2 = \alpha_2} \sum_{\beta_1 + \beta_2 = 2L} \frac{\alpha_2!}{\alpha'_1! \alpha'_2!} \frac{(2L)!}{\beta_1! \beta_2!} \\
&\times \varepsilon^{|\alpha'_1 + \beta_1|} C_\chi^{|\alpha'_1| + |\beta_1| + 1} \alpha'_1!^\mu \beta_1!^\nu |a|_{|\alpha'_2 + \beta_2|} \langle \eta \rangle^{m_1 + \delta_1 |\beta_2|} \langle y \rangle^{m_2 + \delta_2 |\alpha'_2|} \\
&\leq C_{\ell, \ell'} |a|_{2(\ell + \ell')} \langle \eta \rangle^{m - 2\ell(1 - \delta_1)} \langle y \rangle^{m_2 + 2\ell' \delta_2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle y \rangle^{-2\ell'} |\langle D_\eta \rangle^{2\ell'} \{\langle \eta \rangle^{-2\ell} \langle D_y \rangle^{2\ell} (\chi_\varepsilon a)\}| \leq \\
&\qquad \leq C_{\ell, \ell'} |a|_{2(\ell + \ell')} \langle \eta \rangle^{m_1 - 2\ell(1 - \delta_1)} \langle y \rangle^{m_2 - 2\ell'(1 - \delta_2)}.
\end{aligned}
$$

Finally, by Lemma 6.3 at page 47 of [35] and Lebesgue dominated convergence theorem we obtain

$$
Os - [e^{-i\eta y} a] = \iint e^{-i\eta y} \langle y \rangle^{-2\ell'} \langle D_\eta \rangle^{2\ell'} \{\langle \eta \rangle^{-2\ell} \langle D_y \rangle^{2\ell} a\} dy \bar{d}\eta,
$$

$$
|Os - [e^{-i\eta y} a]| \leq C_{\ell, \ell'} |a|_{2(\ell + \ell')} \iint \langle \eta \rangle^{m_1 - 2\ell(1 - \delta_1)} \langle y \rangle^{m_2 - 2\ell'(1 - \delta_2)} dy \bar{d}\eta.
$$

□

Following the ideas in the proofs of Theorems 6.7 and 6.8 of [35, Chapter 1, Section 6] one can obtain the following result.

Proposition 2.20. *Let $a \in \mathcal{A}^m_\delta(\mathbb{R}^{2n})$, $\alpha, \beta \in \mathbb{N}_0$ and $\eta_0, y_0 \in \mathbb{R}^n$. Then*

(i) $Os - [e^{-i\eta y} y^\alpha a] = Os - [(-D_\eta)^\alpha e^{-i\eta y} a] = Os - [e^{-i\eta y} D^\alpha_\eta a]$;

(ii) $Os - [e^{-i\eta y} \eta^\beta a] = Os - [(-D_y)^\beta e^{-i\eta y} a] = Os - [e^{-i\eta y} D^\beta_y a]$;

(iii) $Os - [e^{-i\eta y} a(\eta, y)] = Os - [e^{-i(\eta-\eta_0)(y-y_0)} a(\eta - \eta_0, y - y_0)]$.

Now we define the double symbol classes.

Definition 2.24. *Let $m = (m_1, m_2), m' = (m'_1, m'_2) \in \mathbb{R}^2$ and $\rho = (\rho_1, \rho_2)$, $\delta = (\delta_1, \delta_2)$ such that $0 \le \delta_j < \rho_j \le 1, j = 1, 2$. We denote by $SG^{m,m'}_{\rho,\delta}(\mathbb{R}^{4n})$ the space of all functions $p \in C^\infty(\mathbb{R}^{4n})$ such that for any $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^n_0$ there is $C^{\alpha',\beta'}_{\alpha,\beta} > 0$ for which*

$$\begin{aligned} |p^{\alpha,\alpha'}_{\beta,\beta'}(x, \xi, x', \xi')| \le & C^{\alpha',\beta'}_{\alpha,\beta} \langle \xi \rangle^{m_1 - \rho_1|\alpha|} \langle \xi' \rangle^{m'_1 - \rho_1|\alpha'|} \langle \xi; \xi' \rangle^{\delta_1|\beta+\beta'|} \\ & \times \langle x \rangle^{m_2 - \rho_2|\beta|} \langle x' \rangle^{m'_2 - \rho_2|\beta'|} \langle x; x' \rangle^{\delta_2|\alpha+\alpha'|} \end{aligned} \tag{2.12}$$

for every $x, x', \xi, \xi' \in \mathbb{R}^n$, where $p^{\alpha,\alpha'}_{\beta,\beta'} = \partial^\alpha_\xi \partial^{\alpha'}_{\xi'} D^\beta_x D^{\beta'}_{x'} p$ and $\langle z; z' \rangle = \sqrt{1 + |z|^2 + |z'|^2}$ for every $z, z' \in \mathbb{R}^n$.

Denoting by $|p|^{m,m'}_{\alpha,\alpha',\beta,\beta'}$ the supremum over $x, \xi, x', \xi' \in \mathbb{R}^n$ of

$$\begin{aligned} & |p^{\alpha,\alpha'}_{\beta,\beta'}(x, \xi, x', \xi')| \langle \xi \rangle^{-m_1 + \rho_1|\alpha|} \langle \xi' \rangle^{-m'_1 + \rho_1|\alpha'|} \langle \xi; \xi' \rangle^{-\delta_1|\beta+\beta'|} \\ & \qquad \times \langle x \rangle^{-m_2 + \rho_2|\beta|} \langle x' \rangle^{-m'_2 + \rho_2|\beta'|} \langle x; x' \rangle^{-\delta_2|\alpha+\alpha'|}, \end{aligned}$$

the space $SG^{m,m'}_{\rho,\delta}$ is a Fréchet space whose topology is defined by the family of seminorms

$$|p|^{m,m'}_\ell := \sup_{|\alpha+\beta+\alpha'+\beta'| \le \ell} |p|^{m,m'}_{\alpha,\alpha',\beta,\beta'}.$$

Definition 2.25. *Let $m = (m_1, m_2), m' = (m'_1, m'_2) \in \mathbb{R}^2$, $\rho = (\rho_1, \rho_2)$, $\delta = (\delta_1, \delta_2)$ such that $0 \le \delta_j < \rho_j \le 1, j = 1, 2$, and let $\mu, \nu \ge 1$. We denote by $SG^{m,m'}_{\rho,\delta;(\mu,\nu)}(\mathbb{R}^{4n})$ the space of all functions $p \in C^\infty(\mathbb{R}^{4n})$ such that for some $C > 0$:*

$$\begin{aligned} |p^{\alpha,\alpha'}_{\beta,\beta'}(x, \xi, x', \xi')| \le & C^{|\alpha+\beta+\alpha'+\beta'|} (\alpha!\alpha'!)^\mu (\beta!\beta'!)^\nu \langle \xi \rangle^{m_1 - \rho_1|\alpha|} \langle \xi' \rangle^{m'_1 - \rho_1|\alpha'|} \langle \xi; \xi' \rangle^{\delta_1|\beta+\beta'|} \\ & \times \langle x \rangle^{m_2 - \rho_2|\beta|} \langle x' \rangle^{m'_2 - \rho_2|\beta'|} \langle x; x' \rangle^{\delta_2|\alpha+\alpha'|}. \end{aligned} \tag{2.13}$$

For $C > 0$ the space $SG^{m,m'}_{\rho,\delta;(\mu,\nu)}(\mathbb{R}^{4n}; C)$ of all smooth functions $p(x, \xi, x', \xi')$ such that (2.13) holds for a fixed $C > 0$, is a Banach space with norm

$$|p|^{m,m'}_C := \sup_{\alpha,\alpha',\beta,\beta' \in \mathbb{N}^n_0} C^{-|\alpha+\alpha'+\beta+\beta'|} (\alpha!\alpha'!)^{-\mu} (\beta!\beta'!)^{-\nu} |p|^{m,m'}_{\alpha,\alpha',\beta,\beta'}.$$

After that we define $SG^{m,m'}_{\rho,\delta;(\mu,\nu)} = \bigcup_{C>0} SG^{m,m'}_{\rho,\delta;(\mu,\nu)}(\mathbb{R}^{4n}; C)$ as an inductive limit of Banach spaces.

Definition 2.26. *For $p \in SG^{m,m'}_{\rho,\delta}$ we define*

$$p(x, D_x, x', D_{x'})u(x) := \int e^{i\xi(x-x')} e^{i\xi'(x'-x'')} p(x, \xi, x', \xi') u(x'') dx'' đ\xi' dx' đ\xi$$
$$= \int e^{i\xi(x-x')} e^{i\xi' x'} p(x, \xi, x', \xi') \widehat{u}(\xi') đ\xi' dx' đ\xi,$$

for every $u \in \mathscr{S}(\mathbb{R}^n)$.

Lemma 2.6. *Let $p \in SG^{m,m'}_{\rho,(0,\delta_2)}(\mathbb{R}^{4n})$. For any multi-indices $\alpha, \alpha', \beta, \beta'$ set $q = p^{\alpha,\alpha'}_{\beta,\beta'}$ and for $\theta \in [-1, 1]$ define*

$$q_\theta(x, \xi) = Os - \iint e^{-i\eta y} q(x, \xi + \theta\eta, x + y, \xi) dy đ\eta, \quad x, \xi \in \mathbb{R}^n.$$

Then $\{q_\theta\}_{|\theta|\leq 1}$ is bounded in $SG^{\tau}_{\rho,(0,\delta_2)}(\mathbb{R}^{2n})$, where $\tau = (\tau_1, \tau_2)$, $\tau_1 = m_1 + m_1' - \rho_1|\alpha + \alpha'|$, $\tau_2 = m_2 + m_2' - \rho_2|\beta + \beta'| + \delta_2|\alpha + \alpha'|$. Furthermore, for any $\ell \in \mathbb{N}_0$ there are $\ell' := \ell'(\ell) \in \mathbb{N}_0$ and $C_{\ell,\ell'} > 0$ independent of θ such that

$$|q_\theta|^{\tau}_{\ell} \leq C_{\ell,\ell'} |p|^{m,m'}_{\ell'}.$$

Proof. First notice that $q(x, \xi + \theta\eta, x + y, \xi) \in \mathcal{A}^{(m_1, m_2' + \delta_2|\alpha+\alpha'|)}_{(0,\delta_2)}(\mathbb{R}^{2n}_{\eta,y})$, therefore $q_\theta(x, \xi)$ is well defined for any fixed ξ, x, θ. Given $\gamma, \mu \in \mathbb{N}_0^n$ we may write, omitting $(x, \xi + \theta\eta, x + y, \xi)$ in the notation,

$$\partial_x^\gamma \partial_\xi^\mu q = \sum_{\substack{\mu' \leq \mu \\ \gamma' \leq \gamma}} \frac{\mu!}{\mu'!(\mu - \mu')!} \frac{\gamma!}{\gamma'!(\gamma - \gamma')!} p^{(\alpha+\mu', \alpha'+\mu-\mu')}_{(\beta+\gamma', \beta'+\gamma-\gamma')}. \tag{2.14}$$

To prove that $\{q_\theta\}_{|\theta|\leq 1}$ is bounded in $SG^{\tau}_{\rho,(0,\delta_2)}(\mathbb{R}^{2n})$ it is sufficient to show that

$$|q_\theta(x, \xi)| \leq C_1 |p|^{m,m'}_{\ell_1} \langle \xi \rangle^{\tau_1} \langle x \rangle^{\tau_2} \tag{2.15}$$

for some $C_1 > 0$ and $\ell_1 \in \mathbb{N}_0$ depending on $\alpha, \beta, \alpha', \beta'$. Indeed, if (2.15) holds, then we can estimate the derivatives of q_θ as follows:

$$|\partial_x^\gamma \partial_\xi^\mu q_\theta(x, \xi)| = \left| Os - \iint e^{-iy\eta} \partial_x^\gamma \partial_\xi^\mu q(x, \xi + \theta\eta, x + y, \xi) dy đ\eta \right|$$
$$\leq \sum_{\substack{\gamma' \leq \gamma \\ \mu' \leq \mu}} \binom{\mu}{\mu'} \binom{\gamma}{\gamma'} \left| Os - \iint e^{-iy\eta} p^{(\alpha+\mu', \alpha'+\mu-\mu')}_{(\beta+\gamma', \beta'+\gamma-\gamma')}(x, \xi + \theta\eta, x + y, \xi) dy đ\eta \right|$$
$$\leq \sum_{\substack{\gamma' \leq \gamma \\ \mu' \leq \mu}} \binom{\mu}{\mu'} \binom{\gamma}{\gamma'} C_1(\alpha, \alpha', \beta, \beta', \gamma, \mu) |p|^{m,m'}_{\ell_1}$$
$$\times \langle \xi \rangle^{m_1 + m_1' - \rho_1|\alpha+\mu'+\alpha'+\mu-\mu'|} \langle x \rangle^{m_2 + m_2' - \rho_2|\beta+\gamma'+\beta'+\gamma-\gamma'| + \delta_2|\alpha+\mu'+\alpha'+\mu-\mu'|}$$

$$\leq C|p|_{\ell_1}^{m,m'}\langle\xi\rangle^{m_1+m_1'-\rho_1|\alpha+\alpha'+\mu|}\langle x\rangle^{m_2+m_2'-\rho_2|\beta+\beta'+\gamma|+\delta_2|\alpha+\alpha'+\mu|}$$

where C and ℓ_1 depend of $\alpha,\alpha',\beta,\beta',\gamma,\mu$ and does not depend of θ.

Now we will show that (2.15) holds true. Observe that

$$e^{-i\eta y}=(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell}(1-\langle x\rangle^{2\delta_2}\Delta_y)^{\ell}e^{-i\eta y},$$

therefore

$$q_\theta(x,\xi)=Os-\iint e^{-iy\eta}r_\theta(x,\xi;\eta,y)dy đ\eta,$$

where

$$r_\theta(x,\xi;\eta,y)=(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell}(1-\langle x\rangle^{2\delta_2}\Delta_y)^{\ell}q(x,\xi+\theta\eta,x+y,\xi).$$

If we take ℓ satisfying $2\ell>|m_1|+n$, then r_θ is integrable with respect to η. Now set a cutoff function $\chi(y)$ such that $\chi(y)=1$ for $|y|\leq 4^{-1}\langle x\rangle$ and $\chi(y)=0$ for $|y|\geq 2^{-1}\langle x\rangle$. Then we can write

$$Os-\iint e^{-iy\eta}r_\theta(x,\xi;\eta,y)dy đ\eta=I_1+I_2+I_3,$$

where

$$I_1=\int_{\mathbb{R}^n_\eta}\int_{|y|\leq 4^{-1}\langle x\rangle^{\delta_2}}e^{-iy\eta}r_\theta(x,\xi;\eta,y)\chi(y)dy đ\eta,$$

$$I_2=\int_{\mathbb{R}^n_\eta}\int_{4^{-1}\langle x\rangle^{\delta_2}\leq|y|\leq 2^{-1}\langle x\rangle}e^{-iy\eta}r_\theta(x,\xi;\eta,y)\chi(y)dy đ\eta,$$

$$I_3=Os-[e^{-iy\eta}r_\theta(x,\xi;\eta,y)(1-\chi(y))].$$

Let us obtain a useful inequality when $|y|\leq 2^{-1}\langle x\rangle$. Since

$$|\langle x+y\rangle-\langle x\rangle|\leq\int_0^1\left|\frac{d}{dt}\langle x+ty\rangle\right|dt\leq\int_0^1|y|\frac{|x+ty|}{\langle x+ty\rangle}dt\leq|y|,$$

then, for $|y|\leq 2^{-1}\langle x\rangle$, we have

$$\frac{1}{2}\langle x\rangle\leq\langle x+y\rangle\leq\frac{3}{2}\langle x\rangle,\quad \langle x;x+y\rangle\leq\langle x\rangle+|x+y|\leq\frac{5}{2}\langle x\rangle.$$

Now we shall proceed to estimate I_1,I_2,I_3. We begin with I_1. With aid of Petree's inequality and using the fact that $\rho_2>\delta_2$ we obtain, for $|y|\leq 4^{-1}\langle x\rangle^{\delta_2}$ and $|\theta|\leq 1$,

$$|r_\theta(x,\xi;\eta,y)|\leq(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell}\sum_{\ell_0+|L|=\ell}\frac{\ell!}{\ell_0!L!}\langle x\rangle^{2\delta_2|L|}|p_{(\beta,\beta'+2L)}^{(\alpha,\alpha')}|$$

$$\leq(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell}\langle x\rangle^{2\delta_2|L|}\sum_{\ell_0+|L|=\ell}\frac{\ell!}{\ell_0!L!}|p|_{|\alpha+\alpha'+\beta+\beta'|+2\ell}^{m,m'}$$

$$\times \langle \xi+\theta\eta\rangle^{m_1-\rho_1|\alpha|}\langle\xi\rangle^{m_1'-\rho_1|\alpha'|}\langle x\rangle^{m_2-\rho_2|\beta|}\langle x+y\rangle^{m_2'-\rho_2|\beta'+2L|}\langle x;x+y\rangle^{\delta_2|\alpha+\alpha'|}$$
$$\leq C^k|p|_k^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2}(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell}\langle\eta\rangle^{|m_1|+\rho_1|\alpha|},$$

where $k=|\alpha+\beta+\alpha'+\beta'|+2\ell$. Therefore, for $2\ell>|m_1|+\rho_1|\alpha|+n$,

$$\begin{aligned}|I_1| &= \left|\int_{\mathbb{R}^n_\eta}\int_{|y|\leq 4^{-1}\langle x\rangle^{\delta_2}} e^{-iy\eta}r_\theta(x,\xi;\eta,y)\chi(y)dy\bar{d}\eta\right|\\
&\leq C^k|p|_k^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2}\int(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell}\langle\eta\rangle^{|m_1|+\rho_1|\alpha|}\bar{d}\eta\int_{|y|\leq\frac{\langle x\rangle^{\delta_2}}{4}}dy\\
&\leq C^k|p|_k^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2}\int\langle\eta\rangle^{-2\ell}\langle\langle x\rangle^{-\delta_2}\eta\rangle^{|m_1|+\rho_1|\alpha|}\bar{d}\eta\int_{|y|\leq\frac{\langle x\rangle^{\delta_2}}{4}}\langle x\rangle^{-\delta_2 n}dy\\
&\leq C^k|p|_k^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2}\int\langle\eta\rangle^{-2\ell+|m_1|+\rho_1|\alpha|}\bar{d}\eta\prod_{j=1}^n\int_{|y_j|\leq\frac{\langle x\rangle^{\delta_2}}{4}}\langle x\rangle^{-\delta_2 n}dy_j\\
&\leq C^k\int\langle\eta\rangle^{-n}\bar{d}\eta|p|_k^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2}.\end{aligned}$$

To estimate I_2 and I_3 it is useful to study $|\Delta_\eta^{\ell_1}r_\theta|$. We have

$$\begin{aligned}|\Delta_\eta^{\ell_1}r_\theta| &\leq \sum_{|Q|=\ell_1}\frac{\ell_1!}{Q!}\sum_{Q_1+Q_2=2Q}\frac{(2Q)!}{Q_1!Q_2!}|\partial_\eta^{Q_1}(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell}|\cdot|(1-\langle x\rangle^{2\delta_2}\Delta_y)^\ell\partial_\eta^{Q_2}q|\\
&\leq \sum_{|Q|=\ell_1}\frac{\ell_1!}{Q!}\sum_{Q_1+Q_2=2Q}\frac{(2Q)!}{Q_1!Q_2!}C^{|Q_1|+\ell+1}\langle x\rangle^{\delta_2|Q_1|}Q_1!(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell-|Q_1|}|\theta|^{|Q_2|}\\
&\quad\times|(1-\langle x\rangle^{2\delta_2}\Delta_y)^\ell p_{(\beta,\beta')}^{(\alpha+Q_2,\alpha')}|.\end{aligned}$$

Noticing that

$$\begin{aligned}|(1-\langle x\rangle^{2\delta_2}\Delta_y)^\ell p_{(\beta,\beta')}^{(\alpha+Q_2,\alpha')}| &\leq \sum_{\ell_0+|L|=\ell}\frac{\ell!}{\ell_0!L!}\langle x\rangle^{2\delta_2|L|}|p_{(\beta,\beta'+2L)}^{(\alpha+Q_2,\alpha')}|\\
&\leq \sum_{\ell_0+|L|=\ell}\frac{\ell!}{\ell_0!L!}\langle x\rangle^{2\delta_2|L|}|p|_{\tilde{k}}^{m,m'}\langle\xi+\theta\eta\rangle^{m_1-\rho_1|\alpha+Q_2|}\langle\xi\rangle^{m_1'-\rho_1|\alpha'|}\\
&\quad\times\langle x\rangle^{m_2-\rho_2|\beta|}\langle x+y\rangle^{m_2'-\rho_2|\beta'+2L|}\langle x;x+y\rangle^{\delta_2|\alpha+\alpha'+Q_2|},\end{aligned}$$

where $\tilde{k}=|\alpha+\alpha'+\beta+\beta'|+2(\ell_1+\ell)$, we obtain

$$\begin{aligned}|\Delta_\eta^{\ell_1}r_\theta| &\leq \sum_{|Q|=\ell_1}\frac{\ell_1!}{Q!}\sum_{Q_1+Q_2=2Q}\frac{(2Q)!}{Q_1!Q_2!}C^{|Q_1|+\ell+1}\langle x\rangle^{\delta_2|Q_1|}Q_1!\\
&\quad\times(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell-|Q_1|}\sum_{\ell_0+|L|=\ell}\frac{\ell!}{\ell_0!L!}\langle x\rangle^{2\delta_2|L|}|p|_{\tilde{k}}^{m,m'}\langle\xi+\theta\eta\rangle^{m_1-\rho_1|\alpha+Q_2|}\\
&\quad\times\langle\xi\rangle^{m_1'-\rho_1|\alpha'|}\langle x\rangle^{m_2-\rho_2|\beta|}\langle x+y\rangle^{m_2'-\rho_2|\beta'+2L|}\langle x;x+y\rangle^{\delta_2|\alpha+\alpha'+Q_2|}.\end{aligned}$$

Now we proceed with the estimate for I_2. If $|y| \leq 2^{-1}\langle x\rangle$ we get

$$|\Delta_\eta^\ell r_\theta| \leq C^{\tilde{k}+1}|p|_{\tilde{k}}^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2+2\delta_2\ell}(1+\langle x\rangle^{2\delta_2}|\eta|^2)^{-\ell}\langle\eta\rangle^{|m_1|+\rho_1|\alpha|},$$

therefore, using integration by parts and assuming $2\ell > |m_1| + \rho_1|\alpha| + 2n$,

$$|I_2| \leq C^{\tilde{k}+1}|p|_{\tilde{k}}^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2+\delta_2(2\ell-n)}\int_{\frac{\langle x\rangle^{\delta_2}}{4}\leq|y|\leq\frac{\langle x\rangle}{2}}|y|^{-2\ell}dy.$$

For $|y| \geq 4^{-1}\langle x\rangle^{\delta_2}$ we may write

$$|y|^{-2\ell} \leq 2^{2\ell}\prod_{j=1}^{n}(|y_j| + \frac{\langle x\rangle^{\delta_2}}{4})^{-\frac{2\ell}{n}},$$

and then

$$\int_{\frac{\langle x\rangle^{\delta_2}}{4}\leq|y|\leq\frac{\langle x\rangle}{2}}|y|^{-2\ell}dy \leq 2^{2\ell}\prod_{j=1}^{n}\int(|y_j| + \frac{\langle x\rangle^{\delta_2}}{4})^{-\frac{2\ell}{n}}dy_j \leq C^\ell\langle x\rangle^{\delta_2(n-2\ell)}.$$

After that

$$|I_2| \leq \tilde{C}^{\tilde{k}+1}|p|_{\tilde{k}}^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2}.$$

Finally, we take care of I_3. If $|y| \geq 4^{-1}\langle x\rangle$ we have $\langle x+y\rangle \leq 5|y|$ and $\langle x;x+y\rangle \leq 9|y|$. Hence, for $|y| \geq 4^{-1}\langle x\rangle$, we may write

$$|\Delta_\eta^{\ell_1} r_\theta| \leq C^{\tilde{k}+1}|p|_{\tilde{k}}^{m,m'}\langle\xi\rangle^{\tau_1}\langle\eta\rangle^{|m_1|+\rho_1|\alpha|}\langle x\rangle^{m_2-\rho_2|\beta|}|y|^{|m_2'|+\delta_2|\alpha+\alpha'|+2\delta_2(\ell+\ell_1)}$$

and therefore, choosing $\ell,\ell_1 \in \mathbb{N}_0$ satisfying $2\ell > |m_1| + \rho_1|\alpha| + 2n$ and $2\ell_1(1-\delta_2) \geq |m_2'| + \delta_2|\alpha+\alpha'| + 2\delta_2\ell + 2n$,

$$\begin{aligned}|I_3| \leq\ & C^{\tilde{k}+1}|p|_{\tilde{k}}^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{m_2-\rho_2|\beta|-\delta_2 n}\int\langle\eta\rangle^{|m_1|+\rho_1|\alpha|-2\ell}đ\eta \\ &\times\int_{|y|\geq 4^{-1}\langle x\rangle}|y|^{|m_2'|+\delta_2|\alpha+\alpha'|+2\delta_2\ell-2\ell_1(1-\delta_2)}dy.\end{aligned}$$

Setting $r = 2\ell_1(1-\delta_2) - |m_2'| - \delta_2|\alpha+\alpha'| - 2\delta_2\ell$, we obtain

$$|I_3| \leq C^{\tilde{k}+1}|p|_{\tilde{k}}^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{m_2-\rho_2|\beta|}\int\langle\eta\rangle^{-2n}đ\eta\langle x\rangle^{n(1-\delta_2)-r}.$$

Choosing ℓ_1 such that $r > -m_2 + \rho_2|\beta'| - \delta_2|\alpha+\alpha'| + n(1-\delta_2)$, we get

$$|I_3| \leq C^{\tilde{k}+1}|p|_{\tilde{k}}^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2}\int\langle\eta\rangle^{-2n}đ\eta.$$

Gathering all the previous computations and choosing $\ell,\ell_1 \in \mathbb{N}_0$ satisfying

$$2\ell \geq |m_1| + \rho_1|\alpha| + 2n,$$

$$2\ell_1(1-\delta_2) \geq 2|m_2'| + \rho_2|\beta'| + \delta_2|\alpha+\alpha'| + 2\delta_2\ell + 2n,$$

we have

$$|q_\theta(x,\xi)| \leq C^{\tilde{k}}|p|_{\tilde{k}}^{m,m'}\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2},$$

where $\tilde{k} = |\alpha+\beta+\alpha'+\beta'| + 2(\ell+\ell_1)$. This concludes the proof. $\square$

Remark 2.20. *Let $a \in C^\infty(\mathbb{R}^n)$ such that $|\partial_x^\beta a(x)| \leq C_\beta \langle x\rangle^{m_2}$, for $\beta \in \mathbb{N}_0^n$. For each fixed x, we can look at $a(x+\cdot)$ as an amplitude in $\mathcal{A}_{(0,0)}^{(0,|m_2|)}(\mathbb{R}^{2n})$. Moreover (cf. Section 6, Chapter 1 of [35])*

$$Os - [e^{-i\eta y}a(x+y)] = Os - [e^{-i\eta(y-x)}a(y)] = a(x).$$

Theorem 2.15. *Let $p(x,\xi,x',\xi') \in SG_{\rho,(0,\delta_2)}^{m,m'}$ and set*

$$p_L(x,\xi) = Os - \iint e^{-i\eta y}p(x,\xi+\eta,x+y,\xi)dy đ\eta, \quad x,\xi \in \mathbb{R}^n.$$

Then $p_L \in SG_{\rho,(0,\delta_2)}^{m+m'}$, $p(x,D_x,x',D_{x'}) = p_L(x,D_x)$ and

$$p_L(x,\xi) \sim \sum_{j\in\mathbb{N}_0}\sum_{|\alpha|=j}\frac{1}{\alpha!}(\partial_\xi^\alpha D_{x'}^\alpha p)(x,\xi,x,\xi)$$

Furthermore, given $\ell \in \mathbb{N}_0$ there is $\ell_0 := \ell_0(\ell) \in \mathbb{N}_0$ such that

$$|p_L|_\ell^{m+m'} \leq C_{\ell,\ell_0}|p|_{\ell_0}^{(m,m')}.$$

Proof. First we notice that repeating the ideas in the proof of [35, Lemma 2.3, page 65] we can conclude that $p_L = p$ as operators.

Applying Lemma 2.6 for $\alpha = \alpha' = \beta' = \beta = 0$, we obtain that $p_L \in SG_{\rho,(0,\delta_2)}^{m+m'}$.

Now by Taylor formula we may write

$$\begin{aligned} p(x,\xi+\eta,x+y,\xi) &= \sum_{|\alpha|<N}\frac{\eta^\alpha}{\alpha!}(\partial_\xi^\alpha p)(x,\xi,x+y,\xi) + \\ &+ N\sum_{|\gamma|=N}\frac{\eta^\gamma}{\gamma!}\int_0^1(1-\theta)^{N-1}(\partial_\xi^\gamma p)(x,\xi+\theta\eta,x+y,\xi)d\theta. \end{aligned}$$

Integration by parts and Remark 2.20 give

$$\begin{aligned} Os - [e^{-i\eta y}\eta^\alpha(\partial_\xi^\alpha p)(x,\xi,x+y,\xi)] &= Os - [e^{-i\eta y}D_y^\alpha(\partial_\xi^\alpha p)(x,\xi,x+y,\xi)] \\ &= (\partial_\xi^\alpha D_{x'}^\alpha p)(x,\xi,x,\xi) \end{aligned}$$

and

$$Os-[e^{-i\eta y}\eta^\gamma \int_0^1 (1-\theta)^{N-1}(\partial_\xi^\gamma p)(x,\xi+\theta\eta,x+y,\xi)d\theta] =$$
$$Os-[e^{-i\eta y}\int_0^1 (1-\theta)^{N-1}(\partial_\xi^\gamma D_{x'}^\gamma p)(x,\xi+\theta\eta,x+y,\xi)d\theta].$$

Hence

$$p_L(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!}(\partial_\xi^\alpha D_{x'}^\alpha p)(x,\xi,x,\xi) + r_N(x,\xi),$$

and Lemma 2.6 implies $r_N \in SG^{m+m'-N(\rho-(0,\delta_2))}_{\rho,(0,\delta_2)}$. □

In order to obtain the same kind of result for the classes $SG^{m,m'}_{\rho,(0,\delta_2);(\mu,\nu)}$, we need an analogous of Lemma 2.6 with a precise estimate of the Gevrey regularity.

Lemma 2.7. *Let* $p \in SG^{m,m'}_{\rho,(0,\delta_2);(\mu,\nu)}(\mathbb{R}^{4n};A)$ *for some* $A>0$. *For any multi-indices* $\alpha,\alpha',\beta,\beta'$ *set* $q = p^{\alpha,\alpha'}_{\beta,\beta'}$ *and, for* $\theta\in[-1,1]$ *consider* q_θ *as in Lemma 2.6. Then*

$$|\partial_\xi^\sigma \partial_x^\gamma q_\theta(x,\xi)| \leq |p|_A^{m,m'}(CA^r)^k(\alpha!\alpha'!\sigma!)^{\tilde{\mu}}(\beta!\beta'!\gamma!)^{\tilde{\nu}}\langle x\rangle^{\tau_2-\rho_2|\gamma|+\delta_2|\sigma|}\langle\xi\rangle^{\tau_1-\rho_1|\sigma|} \tag{2.16}$$

where $k=|\alpha+\beta+\alpha'+\beta'+\sigma+\gamma|$, $\tilde{\mu} = (1+\frac{\delta_2\rho_1+\delta_2}{1-\delta_2})\mu+\rho_1\nu$, $\tilde{\nu} = \frac{\rho_2}{1-\delta_2}\mu+\nu$, τ_1 *and* τ_2 *are as in Lemma 2.6 and* C, r *are positive constants depending only on* ρ,δ,m,m',μ,ν *and* n.

Proof. Following the ideas presented in the proof of Lemma 2.6 and using standard factorial inequalities we obtain

$$|q_\theta(x,\xi)| \leq |p|_A^{m,m'}(CA)^{\tilde{k}}\ell!^{2\nu}\ell_1!^{2\mu}(\alpha!\alpha'!)^\mu(\beta!\beta'!)^\nu\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2},$$

where $C>0$ depends only of μ,ν,n,m_1, $\tilde{k} = |\alpha+\beta+\alpha'+\beta'|+2(\ell+\ell_1)$ and ℓ,ℓ_1 are positive integers satisfying $2\ell \geq |m_1|+\rho_1|\alpha|+2n$, and

$$2\ell_1(1-\delta_2) \geq 2|m_2'|+\rho_2|\beta'|+\delta_2|\alpha+\alpha'|+2\delta_2\ell+2n.$$

In particular if we choose

$$\ell = \left\lfloor \frac{|m_1|}{2}+\frac{\rho_1}{2}|\alpha|\right\rfloor + n+1,$$
$$\ell_1 = \left\lfloor \frac{1}{1-\delta_2}\left(|m_2'|+\delta_2\ell+\frac{\rho_2}{2}|\beta'|+\frac{\delta_2}{2}|\alpha+\alpha'|\right)\right\rfloor + n+1,$$

where $\lfloor\cdot\rfloor$ stands for the floor function, then we obtain

$$|q_\theta(x,\xi)| \leq |p|_A^{m,m'}(CA^r)^{|\alpha+\beta+\alpha'+\beta'|}\alpha!^{\tilde{\mu}}\alpha'!^{\mu(1+\frac{\delta_2}{1-\delta_2})}\beta!^\nu\beta'!^{\tilde{\nu}}(\beta!\beta'!)^\nu\langle\xi\rangle^{\tau_1}\langle x\rangle^{\tau_2}.$$

From the last estimate and (2.14) we get (2.16). □

As a consequence of Lemma 2.7 we have the following result.

Theorem 2.16. *Let* $p \in SG^{m,m'}_{\rho,(0,\delta_2);(\mu,\nu)}(\mathbb{R}^{4n})$*. Then* p_L *belongs to* $SG^{m+m'}_{\rho,(0,\delta_2);(\tilde{\mu},\tilde{\nu})}$ *and*

$$p_L(x,\xi) \sim \sum_{j\in\mathbb{N}_0^n} p_j(x,\xi) \quad in \quad FSG^{m+m'}_{\rho,(0,\delta_2);(\tilde{\mu},\tilde{\nu})},$$

where

$$p_j(x,\xi) = \sum_{|\alpha|=j} \alpha!^{-1}(\partial_\xi^\alpha D_{x'}^\alpha p)(x,\xi,x,\xi)$$

and $\tilde{\mu}$ *and* $\tilde{\nu}$ *are as in Lemma 2.7.*

Theorem 2.16 states that p_L has a lower Gevrey regularity than p since $\tilde{\mu} > \mu$ and $\tilde{\nu} > \nu$. However we observe that if $p \in SG^{m,m'}_{\rho,(0,\delta_2);(\mu,\nu)}$, then $\sum_{j\in\mathbb{N}_0} p_j \in FSG^{m+m'}_{\rho,(0,\delta_2);(\mu,\nu)}$. So by Proposition 2.16 there exists $q \in SG^{m+m'}_{\rho,(0,\delta_2);(\mu,\nu)}$ such that $q \sim \sum p_j$ in $FSG^{m+m'}_{\rho,(0,\delta_2);(\mu,\nu)}$. On the other hand we have $p_L \sim \sum p_j$ in $FSG^{m+m'}_{\rho,(0,\delta_2);(\tilde{\mu},\tilde{\nu})}$. Hence $p_L - q \sim 0$ in $FSG^{m+m'}_{\rho,(0,\delta_2);(\tilde{\mu},\tilde{\nu})}$ which implies that $p_L = q + r$, where r belongs to the Gelfand-Shilov space $\mathcal{S}_{\tilde{\mu}+\tilde{\nu}-1}(\mathbb{R}^{2n})$. This means that we can write p_L as the sum of a symbol with the same orders and regularity as p plus a remainder term which has a lower Gevrey regularity but with orders small as we want.

2.5.3 The Friedrichs Part

Fix $q \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$ supported on $Q = \{\sigma \in \mathbb{R}^n : |\sigma| \le 1\}$, such that q is even, $\int q(\sigma)^2 d\sigma = 1$ and $|\partial_\sigma^\alpha q(\sigma)| \le C_q^{|\alpha|+1}\alpha!^s$, where $1 < s \le \min\{\mu,\nu\}$.

Lemma 2.8. *For* $\tau, \tau' \in (0,1)$ *set* $F : \mathbb{R}^{3n} \to \mathbb{R}$ *given by*

$$F(x,\xi,\zeta) = q(\langle x\rangle^{\tau'}\langle\xi\rangle^{-\tau}(\zeta-\xi))\langle\xi\rangle^{-\frac{\tau n}{2}}\langle x\rangle^{\frac{\tau' n}{2}}, \tag{2.17}$$

for $x,\xi,\zeta \in \mathbb{R}^n$*. Then, for any* $\alpha,\beta \in \mathbb{N}_0^n$*, we have*

$$\begin{aligned}\partial_\xi^\alpha\partial_x^\beta F(x,\xi,\zeta) &= \langle x\rangle^{\frac{\tau' n}{2}}\langle\xi\rangle^{-\frac{\tau n}{2}} \sum_{\substack{|\gamma|\le|\alpha| \\ \gamma_1\le\gamma}} \sum_{|\delta|\le|\beta|} \psi_{\alpha\gamma\gamma_1}(\xi)\phi_{\beta\delta\gamma\gamma_1}(x) \\ &\quad \times(\langle x\rangle^{\tau'}\langle\xi\rangle^{-\tau}(\zeta-\xi))^{\gamma_1+\delta}(\partial^{\gamma+\delta}q)(\langle x\rangle^{\tau'}\langle\xi\rangle^{-\tau}(\zeta-\xi)),\end{aligned}$$

where $\psi_{\alpha\gamma\gamma_1}$ *and* $\phi_{\beta\delta\gamma\gamma_1}$ *satisfy the following estimates:*

$$|\partial_\xi^\mu\psi_{\alpha\gamma\gamma_1}(\xi)| \le C_{\alpha\mu}\langle\xi\rangle^{-|\alpha|+(1-\tau)|\gamma-\gamma_1|-|\mu|}, \tag{2.18}$$

$$|\partial_x^\nu\phi_{\beta\delta\gamma\gamma_1}(x)| \le C_{\beta\nu}\langle x\rangle^{-|\beta|+\tau'|\gamma-\gamma_1|-|\nu|}, \tag{2.19}$$

for every $\mu,\nu \in \mathbb{N}_0^n$*.*

The lemma can be proved by induction on $|\alpha+\beta|$ following the same argument as in the proof of [35, Lemma 4.1 page 129]. Observing that $|\gamma-\gamma_1| \le |\gamma| \le |\alpha|$ we have $\psi_{\alpha\gamma\gamma_1}(\xi)\phi_{\beta\delta\gamma\gamma_1}(x) \in SG^{(-\tau|\alpha|,-|\beta|+\tau'|\alpha|)}(\mathbb{R}^{2n})$. Finally we remark that for $\alpha=\beta=0$ we have $\psi_{\alpha\gamma\gamma_1} \equiv \phi_{\beta\delta\gamma\gamma_1} \equiv 1$.

Definition 2.27. *Let $p \in SG^m$. Let moreover $F(x,\xi,\zeta)$ be defined by (2.17) with $\tau=\tau'=\frac{1}{2}$. We set the Friedrichs part of p by*

$$p_F(\xi,x',\xi') = \int F(x',\xi,\zeta)p(x',\zeta)F(x',\xi',\zeta)\,d\zeta, \quad x',\xi,\xi' \in \mathbb{R}^n.$$

The following properties can be easily achieved (cf. Theorem 4.3 of [35]).

Proposition 2.21. *Let $p \in SG^m$ and let p_F be its Friedrichs part. For $u,v \in \mathscr{S}(\mathbb{R}^n)$, the following conditions hold:*

(i) If $p(x,\xi)$ is real, then $\langle p_F u, v\rangle_{L^2} = \langle u, p_F v\rangle_{L^2}$;

(ii) If $p(x,\xi) \ge 0$, then $\langle p_F u, u\rangle_{L^2} \ge 0$;

(iii) If $p(x,\xi)$ is purely imaginary, then $\langle p_F u, v\rangle_{L^2} = -\langle u, p_F v\rangle_{L^2}$;

(iv) If Re $p(x,\xi) \ge 0$, then Re $\langle p_F u, u\rangle_{L^2} \ge 0$.

Theorem 2.17. *Let $p \in SG^m(\mathbb{R}^{2n})$ and let p_F be its Friedrichs part. Then $p_{F,L} \in SG^m(\mathbb{R}^{2n})$ and $p_{F,L} - p \in SG^{m-(1,1)}(\mathbb{R}^{2n})$. Moreover*

$$p_{F,L}(x,\xi) - p(x,\xi) \sim \sum_{|\beta|=1} q_{0,\beta}(x,\xi) + \sum_{|\alpha+\beta|\ge 2} q_{\alpha,\beta}(x,\xi),$$

where, for $|\beta|=1$,

$$\begin{aligned} q_{0,\beta}(x,\xi) = &\sum_{\beta_1+\beta_2+\beta_3=\beta} D_x^{\beta_3} p(x,\xi) \sum_{|\gamma|\le|\beta|}\sum_{|\delta|\le|\beta_1|} \psi_{\beta\gamma\gamma}(\xi)\phi_{\beta_1\delta\gamma\gamma}(x) \\ &\times \sum_{|\delta'|\le|\beta_2|} \phi_{\beta_2\delta'00}(x) \int \sigma^{\gamma+\delta+\delta'}(\partial^{\gamma+\delta}q)(\sigma)(\partial^{\delta'}q)(\sigma)d\sigma, \end{aligned}$$

with $\psi_{\beta\gamma\gamma}\phi_{\beta_1\delta\gamma\gamma} \in SG^{(-|\beta|,-|\beta_1|)}(\mathbb{R}^{2n})$, $\phi_{\beta_2\delta'00} \in SG^{(0,-|\beta_2|)}(\mathbb{R}^{2n})$ and, for $|\alpha+\beta| \ge 2$:

$$\begin{aligned} q_{\alpha,\beta}(x,\xi) = &\sum_{\beta_1+\beta_2+\beta_3=\beta} \frac{(\langle\xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}})^{|\alpha|}}{\alpha!\beta_1!\beta_2!\beta_3!} \\ &\times \sum_{\substack{|\gamma|\le|\beta| \\ \gamma_1\le\gamma}} \sum_{|\delta|\le|\beta_1|} \psi_{\beta\gamma\gamma_1}(\xi)\phi_{\beta_1\delta\gamma\gamma_1}(x) \sum_{|\delta'|\le|\beta_2|} \phi_{\beta_2\delta'00}(x) \end{aligned}$$

$$\times \int_Q \sigma^{\alpha+\gamma_1+\delta_1+\delta'_1}(\partial^{\gamma+\delta}q)(\sigma)(\partial^{\delta'}q)(\sigma)\,d\sigma \cdot \partial_\xi^\alpha D_x^{\beta_3} p(x,\xi)$$

with $\psi_{\beta\gamma\gamma_1}\phi_{\beta_1\delta\gamma\gamma_1} \in SG^{(-\frac{1}{2}|\beta|,-|\beta_1|+\frac{1}{2}|\beta|)}(\mathbb{R}^{2n})$, $\phi_{\beta_2\delta'00} \in SG^{(0,-|\beta_2|)}(\mathbb{R}^{2n})$.

We need the following technical lemma whose proof follows by a simple compactness argument.

Lemma 2.9. *For* $\tau \in (0,1)$ *there is* $C > 0$ *such that*

$$C^{-1}\langle\xi\rangle \le \langle \xi + \zeta\langle\xi\rangle^\tau\rangle \le C\langle\xi\rangle,$$

for every $\xi \in \mathbb{R}^n$ *and* $|\zeta| \le 1$.

Proof of Theorem 2.17. From Leibniz formula and Cauchy-Schwartz inequality we get

$$|\partial_\xi^\alpha \partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} p_F(\xi,x',\xi')| \le \sum_{\beta'_1+\beta'_2+\beta'_3=\beta'} \frac{\beta'!}{\beta'_1!\beta'_2!\beta'_3!} \left[\int |\partial_\xi^\alpha \partial_{x'}^{\beta'_1} F(x',\xi,\zeta)|^2\,d\zeta\right]^{\frac{1}{2}}$$
$$\times \left[\int |\partial_{x'}^{\beta'_2} p(x',\zeta)\partial_{\xi'}^{\alpha'}\partial_{x'}^{\beta'_3} F(x',\xi',\zeta)|^2\,d\zeta\right]^{\frac{1}{2}}.$$

Now by a change of variables we obtain

$$|\partial_\xi^\alpha \partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} p_F(\xi,x',\xi')| \le \langle\xi\rangle^{\frac{n}{4}}\langle x'\rangle^{-\frac{n}{2}}\langle\xi'\rangle^{\frac{n}{4}} \sum_{\beta'_1+\beta'_2+\beta'_3=\beta'} \frac{\beta'!}{\beta'_1!\beta'_2!\beta'_3!}$$
$$\times \left[\int_Q |(\partial_\xi^\alpha \partial_{x'}^{\beta'_1} F)(x',\xi,\langle x'\rangle^{-\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}}\sigma+\xi)|^2\,d\sigma\right]^{\frac{1}{2}}$$
$$\times \left[\int_Q |(\partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'_2} F)(x',\xi',\langle x'\rangle^{-\frac{1}{2}}\langle\xi'\rangle^{\frac{1}{2}}\sigma+\xi')(\partial_{x'}^{\beta'_3} p)(x',\langle x'\rangle^{-\frac{1}{2}}\langle\xi'\rangle^{\frac{1}{2}}\sigma+\xi')|^2 d\sigma\right]^{\frac{1}{2}}.$$

Applying Lemma 2.8 we obtain

$$|\partial_\xi^\alpha \partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} p_F(\xi,x',\xi')| \le \sum_{\beta'_1+\beta'_2+\beta'_3=\beta'} \frac{\beta'!}{\beta'_1!\beta'_2!\beta'_3!}$$
$$\times \left(\int \sum_{\substack{|\gamma|\le|\alpha| \\ \gamma_1\le\gamma}} \sum_{|\delta|\le|\beta'_1|} \left|\psi_{\alpha\gamma\gamma_1}(\xi)\phi_{\beta'_1\delta\gamma\gamma_1}(x')\sigma^{\gamma_1+\delta}(\partial^{\gamma+\delta}q)(\sigma)\right|^2\,d\sigma\right)^{\frac{1}{2}}$$
$$\times \left(\int \sum_{\substack{|\gamma|\le|\alpha'| \\ \gamma_1\le\gamma \\ |\delta|\le|\beta'_2|}} \left|\psi_{\alpha'\gamma\gamma_1}(\xi')\phi_{\beta'_2\delta\gamma\gamma_1}(x')\sigma^{\gamma_1+\delta}(\partial^{\gamma+\delta}q)(\sigma)\right|^2 \left|\partial_{x'}^{\beta'_3} p(x',\langle x'\rangle^{-\frac{1}{2}}\langle\xi'\rangle^{\frac{1}{2}}\sigma+\xi')\right|^2\,d\sigma\right)^{\frac{1}{2}}.$$

We now observe that by Lemma 2.9

$$\left|\partial_{x'}^{\beta_3'} p(x', \langle x'\rangle^{-\frac{1}{2}}\langle \xi'\rangle^{\frac{1}{2}}\sigma + \xi')\right| \leq C_{\beta_3'}\langle \xi'\rangle^{m_1}\langle x'\rangle^{m_2-|\beta_3'|}.$$

Since $\psi_{\alpha\gamma\gamma_1}\phi_{\beta_1'\delta\gamma\gamma_1} \in SG^{(-\frac{|\alpha|}{2}, -|\beta_1'|+\frac{|\alpha|}{2})}$ and $\psi_{\alpha'\gamma\gamma_1}\phi_{\beta_2'\delta\gamma\gamma_1} \in SG^{(-\frac{|\alpha'|}{2}, -|\beta_2'|+\frac{|\alpha'|}{2})}$ we obtain

$$|\partial_\xi^\alpha \partial_{\xi'}^{\alpha'}\partial_{x'}^{\beta'} p_F(\xi, x', \xi')| \leq C_{\alpha\alpha'\beta'}\langle \xi\rangle^{-\frac{|\alpha|}{2}}\langle \xi'\rangle^{m_1-\frac{|\alpha'|}{2}}\langle x'\rangle^{m_2-|\beta'|+\frac{|\alpha+\alpha'|}{2}},$$

that is $p_F \in SG^{(0,0),(m_1,m_2)}_{(1/2,1),(0,1/2)}$. Then, by Theorem 2.15 $p_{F,L} \in SG^{m}_{(1/2,1),(0,1/2)}$ and

$$p_{F,L}(x,\xi) \sim \sum_\beta \frac{1}{\beta!}(\partial_\xi^\beta D_{x'}^\beta p_F)(\xi, x, \xi) = \sum_\beta \tilde{p}_\beta(x,\xi),$$

which implies that $p_{F,L} - \sum_{|\beta|<N} \tilde{p}_\beta \in SG^{m-\frac{N}{2}(1,1)}_{(1/2,1),(0,1/2)}$ for every $N \in \mathbb{N}$. To improve this result let us study more carefully the above asymptotic expansion. Note that

$$\begin{aligned}
\tilde{p}_\beta(x,\xi) &= \sum_{\beta_1+\beta_2+\beta_3=\beta} \frac{1}{\beta_1!\beta_2!\beta_3!}\int \partial_\xi^\beta D_x^{\beta_1} F(x,\xi,\zeta) D_x^{\beta_3} p(x,\zeta) D_x^{\beta_2} F(x,\xi,\zeta)d\zeta \\
&= \sum_{\beta_1+\beta_2+\beta_3=\beta} \frac{1}{\beta_1!\beta_2!\beta_3!} \sum_{\substack{|\gamma|\leq|\beta| \\ \gamma_1\leq\gamma}} \sum_{|\delta|\leq|\beta_1|} \psi_{\beta\gamma\gamma_1}(\xi)\phi_{\beta_1\delta\gamma\gamma_1}(x) \sum_{|\delta'|\leq|\beta_2|} \psi_{\beta_2\delta'00}(x) \\
&\times \int_Q (D_x^{\beta_3}p)(x, \langle \xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}}\sigma + \xi)\sigma^{\gamma_1+\delta+\delta'}(\partial^{\gamma+\delta}q)(\sigma)(\partial^{\delta'}q)(\sigma)d\sigma.
\end{aligned}$$

By Taylor formula we can write

$$\begin{aligned}
D_x^{\beta_3} p(x, \langle \xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}}\sigma + \xi) &= \sum_{|\alpha|<N} \frac{(\langle \xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}})^{|\alpha|}\sigma^\alpha}{\alpha!}\partial_\xi^\alpha D_x^{\beta_3} p(x,\xi) \\
&+ N\sum_{|\alpha|=N} \frac{\langle \xi\rangle^{\frac{N}{2}}\langle x\rangle^{-\frac{N}{2}}\sigma^\alpha}{\alpha!}\int_0^1 (1-\theta)^{N-1}(\partial_\xi^\alpha D_x^{\beta_3}p)(x, \theta\langle \xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}}\sigma + \xi)d\theta.
\end{aligned}$$

Then we get

$$\begin{aligned}
\tilde{p}_\beta(x,\xi) = &\sum_{\beta_1+\beta_2+\beta_3=\beta} \sum_{|\alpha|<N} \frac{(\langle \xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}})^{|\alpha|}}{\alpha!\beta_1!\beta_2!\beta_3!} \\
&\times \sum_{\substack{|\gamma|\leq|\beta| \\ \gamma_1\leq\gamma}} \sum_{|\delta|\leq|\beta_1|} \psi_{\beta\gamma\gamma_1}(\xi)\phi_{\beta_1\delta\gamma\gamma_1}(x) \sum_{|\delta'|\leq|\beta_2|} \phi_{\beta_2\delta'00}(x) \\
&\times \int_Q \sigma^{\alpha+\gamma_1+\delta+\delta'}\partial^{\gamma+\delta}q(\sigma)\partial^{\delta'}q(\sigma)\, d\sigma \cdot \partial_\xi^\alpha D_x^{\beta_3}p(x,\xi) + r_{\beta,N}(x,\xi)
\end{aligned}$$

where

$$r_{\beta,N}(x,\xi) = \sum_{\beta_1+\beta_2+\beta_3=\beta} \sum_{|\alpha|=N} \frac{N}{\alpha!} \cdot \frac{(\langle \xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}})^N}{\beta_1!\beta_2!\beta_3!}$$

$$\times \sum_{\substack{|\gamma|\leq|\beta| \\ \gamma_1\leq\gamma}} \sum_{|\delta|\leq|\beta_1|} \psi_{\beta\gamma\gamma_1}(\xi)\phi_{\beta_1\delta\gamma\gamma_1}(x) \sum_{|\delta'|\leq|\beta_2|} \phi_{\beta_2\delta'00}(x)$$

$$\times \int_Q \sigma^{\gamma_1+\delta+\delta'+\alpha}\partial^{\gamma+\delta}q(\sigma)\partial^{\delta'}q(\sigma) \int_0^1 (1-\theta)^{N-1}(\partial_\xi^\alpha D_x^{\beta_3}p)(x, \theta\langle\xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}}\sigma+\xi)\, d\theta d\sigma. \tag{2.20}$$

Using Lemma 2.9 we get that $r_{\beta,N}$ belongs to $SG^{(m_1-\frac{1}{2}(N+|\beta|), m_2-\frac{1}{2}(|\beta|+N))}$, whereas

$$q_{\alpha,\beta}(x,\xi) = \sum_{\beta_1+\beta_2+\beta_3=\beta} \frac{(\langle\xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}})^{|\alpha|}}{\alpha!\beta_1!\beta_2!\beta_3!}$$
$$\times \sum_{\substack{|\gamma|\leq|\beta| \\ \gamma_1\leq\gamma}} \sum_{|\delta|\leq|\beta_1|} \psi_{\beta\gamma\gamma_1}(\xi)\phi_{\beta_1\delta\gamma\gamma_1}(x) \sum_{|\delta'|\leq|\beta_2|} \phi_{\beta_2\delta'00}(x)$$
$$\times \int_Q \sigma^{\alpha+\gamma_1+\delta_1+\delta'_1}\partial^{\gamma+\delta}q(\sigma)\partial^{\delta'}q(\sigma)\, d\sigma \cdot \partial_\xi^\alpha D_x^{\beta_3}p(x,\xi)$$

belongs to $SG^{(m_1-\frac{1}{2}(|\alpha|+|\beta|), m_2-\frac{1}{2}(|\beta|+|\alpha|))}$. Hence

$$\sum_{|\alpha+\beta|=j} q_{\alpha,\beta} \in SG^{m-\frac{1}{2}(j,j)}.$$

Then we can find a symbol $t(x,\xi)$ such that

$$t(x,\xi) \sim \sum_{j\in\mathbb{N}_0} \sum_{|\alpha+\beta|=j} q_{\alpha,\beta}(x,\xi).$$

Since, for every $N \in \mathbb{N}$,

$$\tilde{p}_\beta(x,\xi) - \sum_{|\alpha|<N} q_{\alpha,\beta}(x,\xi) \in SG^{m-\frac{1}{2}(N+|\beta|, N+|\beta|)}_{(\frac{1}{2},1),(0,\frac{1}{2})}(\mathbb{R}^{2n}),$$

we obtain that $p_{F,L} - t \in \mathscr{S}(\mathbb{R}^{2n})$, and therefore

$$p_{F,L}(x,\xi) \sim \sum_{j\in\mathbb{N}_0} \sum_{|\alpha+\beta|=j} q_{\alpha,\beta}(x,\xi).$$

To finish the proof, let us analyze more carefully the functions $q_{\alpha,\beta}$ when $|\alpha+\beta| \leq 1$. First we notice that if $\alpha = \beta = 0$ we have $q_{0,0}(x,\xi) = p(x,\xi)$. If $|\alpha| = 1$ and $\beta = 0$,

$$q_{\alpha,0}(x,\xi) = \frac{\langle\xi\rangle^{\frac{|\alpha|}{2}}\langle x\rangle^{-\frac{|\alpha|}{2}}}{\alpha!} \int \sigma^\alpha q(\sigma)^2 d\sigma = 0,$$

because $\sigma^\alpha q^2(\sigma)$ is an odd function. In the case $|\alpha| = 0$ and $|\beta| = 1$ we have

$$q_{0,\beta}(x,\xi) = \sum_{\beta_1+\beta_2+\beta_3=\beta} D_x^{\beta_3}p(x,\xi) \sum_{\substack{|\gamma|\leq|\beta| \\ |\gamma_1|\leq|\gamma|}} \sum_{|\delta|\leq|\beta_1|} \psi_{\beta\gamma\gamma_1}(\xi)\phi_{\beta_1\delta\gamma\gamma_1}(x)$$

$$\times \sum_{|\delta'|\leq|\beta_2|} \phi_{\beta_2\delta' 00}(x) \int \zeta^{\gamma_1+\delta+\delta'}(\partial^{\gamma+\delta}q)(\zeta)(\partial^{\delta'}q)(\zeta)d\zeta.$$

If $|\gamma_1| < |\gamma|$ in the above formula, we have $\gamma_1 = 0$ and $|\gamma| = 1$ and, since q is even,

$$\int \zeta^{\delta+\delta'}(\partial^{e_j+\delta}q)(\zeta)(\partial^{\delta'}q)(\zeta)d\zeta = 0, \ j = 1, \ldots, n.$$

Therefore

$$\begin{aligned} q_{0,\beta}(x,\xi) = & \sum_{\beta_1+\beta_2+\beta_3=\beta} D_x^{\beta_3}p(x,\xi) \sum_{|\gamma|\leq|\beta|} \sum_{|\delta|\leq|\beta_1|} \psi_{\beta\gamma\gamma}(\xi)\phi_{\beta_1\delta\gamma\gamma}(x) \\ & \times \sum_{|\delta'|\leq|\beta_2|} \phi_{\beta_2\delta' 00}(x) \int \zeta^{\gamma+\delta+\delta'}(\partial^{\gamma+\delta}q)(\zeta)(\partial^{\delta'}q)(\zeta)d\zeta, \end{aligned}$$

and by Lemma 2.8 $q_{0,\beta} \in SG^{m-(1,1)}(\mathbb{R}^{2n})$. Hence

$$p_{F,L}(x,\xi) - p(x,\xi) \sim \sum_{|\beta|=1} q_{0,\beta}(x,\xi) + \sum_{|\alpha+\beta|\geq 2} q_{\alpha,\beta}(x,\xi)$$

and in particular that $p_{F,L} - p \in SG^{m-(1,1)}(\mathbb{R}^{2n})$. □

Proposition 2.21 and Theorem 2.17 imply the well known sharp Gårding inequality.

Theorem 2.18. *Let* $p \in SG^m(\mathbb{R}^{2n})$*. If Re* $p(x,\xi) \geq 0$*, then*

$$\mathit{Re}\,(p(x,D)u,u)_{L^2} \geq -C\|u\|^2_{H^{\frac{1}{2}(m-(1,1))}} \qquad u \in \mathscr{S}(\mathbb{R}^n),$$

for some positive constant C*.*

Proof. Setting $q = p - p_{F,L} \in SG^{m-(1,1)}(\mathbb{R}^{2n})$ and recalling that p_F and $p_{F,L}$ define the same operator, we may write, by (iv) of Proposition 2.21:

$$\mathit{Re}\,(p(x,D)u,u)_{L^2} = \mathit{Re}\,(q(x,D)u,u)_{L^2} + \mathit{Re}\,(p_F u,u)_{L^2} \geq \mathit{Re}\,(q(x,D)u,u)_{L^2}.$$

Now observe that for any $s = (s_1,s_2) \in \mathbb{R}^2$

$$\begin{aligned} |(q(x,D)u,u)_{L^2}| &= |(\langle x\rangle^{s_2}\langle D_x\rangle^{s_1}q(x,D)u, \langle x\rangle^{-s_2}\langle D_x\rangle^{-s_1}u)_{L^2}| \\ &\leq \|q(x,D)u\|_{H^s}\|u\|_{H^{-s}} \leq C\|u\|_{H^{s+m-(1,1)}}\|u\|_{H^{-s}}. \end{aligned}$$

Choosing $s = \frac{1}{2}[(1,1) - m]$ we conclude that

$$\mathit{Re}\,(p(x,D)u,u)_{L^2} \geq -C\|u\|^2_{H^{\frac{1}{2}(m-(1,1))}}.$$

□

In order to study the Friedrichs part of symbols satisfying Gevrey estimates, we will need the following version of the Faà di Bruno formula. Given smooth functions $g : \mathbb{R}^n \to \mathbb{R}^p$, $g = (g_1, \ldots, g_p), f : \mathbb{R}^p \to \mathbb{R}$ and $\gamma \in \mathbb{N}_0^n - \{0\}$ we have

$$\partial^\gamma (f \circ g)(x) = \sum \frac{\gamma!}{k_1! \ldots k_\ell!} (\partial^{k_1+\cdots+k_n} f)(g(x)) \prod_{j=1}^{\ell} \prod_{i=1}^{p} \left[\frac{1}{\delta_j!} \partial^{\delta_j} g_i(x) \right]^{k_{ji}}, \tag{2.21}$$

where the sum is taken over all $\ell \in \mathbb{N}$, all sets $\{\delta_1, \ldots, \delta_\ell\}$ of ℓ distinct elements of $\mathbb{N}_0^n - \{0\}$ and all $(k_1, \ldots, k_\ell) \in (\mathbb{N}_0^p - \{0\})^\ell$, such that

$$\gamma = \sum_{s=1}^{\ell} |k_s| \delta_s.$$

It is possible to show that there is a constant $C > 0$ such that

$$\sum \frac{(k_1 + \ldots + k_\ell)!}{k_1! \ldots k_\ell!} \leq C^{|\gamma|+1}, \quad \gamma \in \mathbb{N} - \{0\},$$

and $|k_1 + \ldots + k_\ell|! \leq |\gamma|!$, where the summation and $k_1, \ldots, k_\ell$ are as in(2.21). For a proof of these assertions we refer to Proposition 4.3 (pg 9), Corollary 4.5 (pg 11) and Lemma 4.8 (pg 12) of [7].

Let $p \in SG^m_{(\mu,\nu)}$. We already know that $p_F \in SG^{(0,0),(m_1,m_2)}_{(\frac{1}{2},1),(0,\frac{1}{2})}$. Now we want to obtain a precise information about the Gevrey regularity of p_F. By Faà di Bruno formula,

$$\begin{aligned}
\partial_x^\beta \partial_\xi^\alpha q(\langle \xi \rangle^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2}} (\zeta - \xi)) &= \partial_x^\beta \sum_{\ell, k_1, \ldots, k_\ell} \frac{\alpha!}{k_1! \ldots k_\ell!} \\
&\times (\partial^{k_1+\ldots+k_\ell} q)(\langle \xi \rangle^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2}} (\zeta - \xi)) \prod_{j=1}^{\ell} \langle x \rangle^{\frac{|k_j|}{2}} \prod_{i=1}^{n} \left[\frac{1}{\delta_j!} \partial_\xi^{\delta_j} \{ \langle \xi \rangle^{-\frac{1}{2}} (\zeta_i - \xi_i) \} \right]^{k_{ji}} \\
&= \sum_{\ell, k_1, \ldots, k_\ell} \frac{\alpha!}{k_1! \ldots k_\ell!} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1! \beta_2!} \partial_x^{\beta_1} (\partial^{k_1+\ldots+k_\ell} q)(\langle \xi \rangle^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2}} (\zeta - \xi)) \\
&\times \partial_x^{\beta_2} \prod_{j=1}^{\ell} \langle x \rangle^{\frac{|k_j|}{2}} \prod_{i=1}^{n} \left[\frac{1}{\delta_j!} \partial_\xi^{\delta_j} \{ \langle \xi \rangle^{-\frac{1}{2}} (\zeta_i - \xi_i) \} \right]^{k_{ji}} \\
&= \sum_{\ell, k_1, \ldots, k_\ell} \frac{\alpha!}{k_1! \ldots k_\ell!} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1! \beta_2!} \sum_{\ell', k'_1, \ldots, k'_{\ell'}} \frac{\beta_1!}{k'_1! \ldots k'_{\ell'}!} \\
&\times (\partial^{(k_1+\ldots+k_\ell)+(k'_1+\ldots+k'_{\ell'})} q)(\langle \xi \rangle^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2}} (\zeta - \xi)) \\
&\times \prod_{j'=1}^{\ell'} \prod_{i'=1}^{n} \left[\frac{1}{\delta'_{j'}!} \partial_x^{\delta'_j} \langle \xi \rangle^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2}} (\zeta_{i'} - \xi_{i'}) \right]^{k'_{j'i'}} \\
&\times \sum_{\sigma_1+\ldots+\sigma_\ell=\beta_2} \frac{\beta_2!}{\sigma_1! \ldots \sigma_\ell!} \prod_{j=1}^{\ell} \partial_x^{\sigma_j} \langle x \rangle^{\frac{|k_j|}{2}} \prod_{i=1}^{n} \left[\frac{1}{\delta_j!} \partial_\xi^{\delta_j} \{ \langle \xi \rangle^{-\frac{1}{2}} (\zeta_i - \xi_i) \} \right]^{k_{ji}},
\end{aligned}$$

hence

$$\partial_x^\beta \partial_\xi^\alpha q(\langle\xi\rangle^{-\frac{1}{2}}\langle x\rangle^{\frac{1}{2}}(\zeta-\xi)) = \sum_{\ell,k_1,\ldots,k_\ell} \frac{\alpha!}{k_1!\ldots k_\ell!} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!}$$
$$\times \sum_{\ell',k'_1,\ldots,k'_{\ell'}} \frac{\beta_1!}{k'_1!\ldots k'_{\ell'}!} (\partial^{(k_1+\ldots+k_\ell)+(k'_1+\ldots+k'_{\ell'})} q)(\langle\xi\rangle^{-\frac{1}{2}}\langle x\rangle^{\frac{1}{2}}(\zeta-\xi))$$
$$\times \prod_{j'=1}^{\ell'}\prod_{i'=1}^{n} \left[\frac{1}{\delta'_{j'}!}\partial_x^{\delta'_{j'}}\langle x\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}}\right]^{k'_{j'i'}} \left[\langle\xi\rangle^{-\frac{1}{2}}\langle x\rangle^{\frac{1}{2}}(\zeta_{i'}-\xi_{i'})\right]^{k'_{j'i'}}$$
$$\times \sum_{\sigma_1+\ldots+\sigma_\ell=\beta_2} \frac{\beta_2!}{\sigma_1!\ldots\sigma_\ell!} \prod_{j=1}^{\ell} \partial_x^{\sigma_j}\langle x\rangle^{\frac{1}{2}|k_j|} \prod_{i=1}^{n} \left[\frac{1}{\delta_j!}\{\partial_\xi^{\delta_j}\langle\xi\rangle^{-\frac{1}{2}}(\zeta_i-\xi_i) - \delta_{ji}\partial_\xi^{\delta_j-e_i}\langle\xi\rangle^{-\frac{1}{2}}\}\right]^{k_{ji}}.$$

Noticing that $\langle x\rangle^{\frac{1}{2}}\langle\xi\rangle^{-\frac{1}{2}}|\zeta-\xi|\le 1$ on the support of q, we have

$$|\partial_x^\beta\partial_\xi^\alpha q(\langle\xi\rangle^{-\frac{1}{2}}\langle x\rangle^{\frac{1}{2}}(\zeta-\xi))| \le \tilde{C}_{q,s}^{|\alpha+\beta|+1}(\alpha!\beta!)^s\langle x\rangle^{\frac{|\alpha|}{2}-|\beta|}\langle\xi\rangle^{-\frac{|\alpha|}{2}}.$$

We now apply the above inequality to estimate the derivatives of F. We have

$$\begin{aligned}|\partial_\xi^\alpha\partial_x^\beta F(x,\xi,\zeta)| &\le \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_1=\beta}} \frac{\alpha!\beta!}{\alpha_1!\beta_1!\alpha_2!\beta_2!}\partial_\xi^{\alpha_1}\langle\xi\rangle^{-\frac{n}{4}}\partial_x^{\beta_1}\langle x\rangle^{\frac{n}{4}}|\partial_\xi^{\alpha_2}\partial_x^{\beta_2}q(\langle\xi\rangle^{-\frac{1}{2}}\langle x\rangle^{\frac{1}{2}}(\zeta-\xi))|\\ &\le C_{q,s}^{|\alpha+\beta|+1}(\alpha!\beta!)^s\langle\xi\rangle^{-\frac{n}{4}-\frac{|\alpha|}{2}}\langle x\rangle^{\frac{n}{4}+\frac{|\alpha|}{2}-|\beta|}. \qquad (2.22)\end{aligned}$$

Finally we proceed with the estimates for p_F. Denoting

$$Q_{x,\xi} = \{\zeta\in\mathbb{R}^n\colon \langle x\rangle^{\frac{1}{2}}\langle\xi\rangle^{-\frac{1}{2}}|\zeta-\xi|<1\}, \quad x,\xi\in\mathbb{R}^n,$$

we obtain

$$|\partial_\xi^\alpha\partial_{\xi'}^{\alpha'}\partial_{x'}^{\beta'}p_F(\xi,x',\xi')| \le \sum_{\beta_1+\beta_2+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!\beta_3!}\left[\int_{Q_{x,\xi}}|\partial_\xi^\alpha\partial_x^{\beta'_1}F(\xi,x',\zeta)|^2d\zeta\right]^{\frac{1}{2}}$$
$$\times\left[\int_{Q_{x',\xi'}}|\partial_x^{\beta'_3}p(x',\zeta)\partial_\xi^{\alpha'}\partial_x^{\beta'_2}F(x',\xi',\zeta)|^2d\zeta\right]^{\frac{1}{2}}$$
$$\le \sum_{\beta_1+\beta_2+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!\beta_3!}C_{q,s}^{|\alpha+\alpha'+\beta'_1+\beta'_2|+2}(\alpha!\alpha'!\beta'_1!\beta'_2!)^s\langle\xi\rangle^{-\frac{|\alpha|}{2}}\langle\xi'\rangle^{-\frac{|\alpha'|}{2}}$$
$$\times\langle x\rangle^{\frac{1}{2}|\alpha+\alpha'|-|\beta'_1+\beta'_2|}\left[\int_{Q_{x',\xi}}\langle\xi\rangle^{-\frac{n}{2}}\langle x'\rangle^{\frac{n}{2}}d\zeta\right]^{\frac{1}{2}}\cdot\left[\int_{Q_{x',\xi'}}\langle\xi'\rangle^{-\frac{n}{2}}\langle x'\rangle^{\frac{n}{2}}|\partial_x^{\beta'_3}p(x',\zeta)|^2d\zeta\right]^{\frac{1}{2}}$$
$$\le \sum_{\beta_1+\beta_2+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!\beta_3!}C_{q,s}^{|\alpha+\alpha'+\beta'_1+\beta'_2|+2}(\alpha!\alpha'!\beta'_1!\beta'_2!)^s\langle\xi\rangle^{-\frac{|\alpha|}{2}}\langle\xi'\rangle^{-\frac{|\alpha'|}{2}}$$
$$\times\langle x\rangle^{\frac{|\alpha+\alpha'|}{2}-|\beta'_1+\beta'_2|}\left[\int_{|\zeta|<1}d\zeta\right]^{\frac{1}{2}}\left[\int_{|\zeta|<1}|\partial_x^{\beta'_3}p(x',\langle x'\rangle^{-\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}}\zeta+\xi')|^2d\zeta\right]^{\frac{1}{2}}.$$

Using Lemma 2.9 and recalling that $s \leq \min\{\mu, \nu\}$,

$$|\partial_\xi^\alpha \partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} p_F| \leq C^{|\alpha+\alpha'+\beta'|+1}(\alpha!\alpha'!)^\mu \beta'!^\nu \langle x\rangle^{m_2-|\beta'|+\frac{1}{2}|\alpha+\alpha'|}\langle\xi\rangle^{-\frac{|\alpha|}{2}}\langle\xi'\rangle^{m_1-\frac{|\alpha'|}{2}},$$

which means $p_F \in SG^{(0,0),(m_1,m_2)}_{(\frac{1}{2},1),(0,\frac{1}{2});(\mu,\nu)}(\mathbb{R}^{4n})$.

Now we discuss the asymptotic expansion of p_F, when $p \in SG^m_{(\mu,\nu)}$. In the following we will use the notation of the proof of Theorem 2.17. We have

$$p_{F,L}(x,\xi) \sim \sum_\beta \tilde{p}_\beta(x,\xi) \text{ in } FSG^m_{(\frac{1}{2},1),(0,\frac{1}{2});(\tilde{\mu},\tilde{\nu})},$$

and by Lemma 2.7 and Taylor formula we may write

$$\left|\partial_\xi^\theta \partial_x^\sigma (p_{F,L} - \sum_{|\beta|<N} \tilde{p}_\beta)(x,\xi)\right| \leq C^{|\theta+\sigma|+2N+1}\theta!^{\tilde{\mu}}\sigma!^{\tilde{\nu}}N!^{\tilde{\mu}+\tilde{\nu}-1}\langle\xi\rangle^{m_1-\frac{|\theta|}{2}-\frac{N}{2}}\langle x\rangle^{m_2-|\sigma|+\frac{|\theta|}{2}-\frac{N}{2}} \tag{2.23}$$

for every $\theta, \sigma \in \mathbb{N}_0^n$, $x, \xi \in \mathbb{R}^n$ and $N \in \mathbb{N}$, where $\tilde{\mu} = \frac{3}{2}\mu + \frac{1}{2}\nu$ and $\tilde{\nu} = \nu + 2\mu$. We also have, for every $\beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}$,

$$\tilde{p}_\beta(x,\xi) - \sum_{|\alpha|<N} q_{\alpha,\beta}(x,\xi) = r_{\beta,N}(x,\xi).$$

where $r_{\beta,N}$ is given as in (2.20).

Performing the change of variables $\sigma = (\zeta - \xi)\langle x\rangle^{\frac{1}{2}}\langle\xi\rangle^{-\frac{1}{2}}$, we obtain

$$\begin{aligned} r_{\beta,N}(x,\xi) = &\sum_{\beta_1+\beta_2+\beta_3=\beta} \sum_{|\alpha|=N} \frac{N}{\alpha!} \cdot \frac{(\langle\xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}})^N}{\beta_1!\beta_2!\beta_3!} \\ &\times \sum_{\substack{|\gamma|\leq|\beta| \\ \gamma_1\leq\gamma}} \sum_{|\delta|\leq|\beta_1|} \psi_{\beta\gamma\gamma_1}(\xi)\phi_{\beta_1\delta\gamma\gamma_1}(x) \sum_{|\delta'|\leq|\beta_2|} \phi_{\beta_2\delta'00}(x) \\ &\times \int_{Q_{x,\xi}} ((\zeta-\xi)\langle x\rangle^{\frac{1}{2}}\langle\xi\rangle^{-\frac{1}{2}})^{\gamma_1+\delta+\delta'+\alpha}\partial^{\gamma+\delta}q((\zeta-\xi)\langle x\rangle^{\frac{1}{2}}\langle\xi\rangle^{-\frac{1}{2}})\partial^{\delta'}q((\zeta-\xi)\langle x\rangle^{\frac{1}{2}}\langle\xi\rangle^{-\frac{1}{2}}) \\ &\cdot \int_0^1 (1-\theta)^{N-1}(\partial_\xi^\alpha D_x^{\beta_3}p)(x,\theta\zeta+(1-\theta)\xi)\,d\theta\langle x\rangle^{\frac{n}{2}}\langle\xi\rangle^{-\frac{n}{2}}d\zeta. \end{aligned}$$

By Lemma 2.8 we get

$$\begin{aligned} r_{\beta,N}(x,\xi) = &\sum_{\beta_1+\beta_2+\beta_3=\beta} \sum_{|\alpha|=N} \frac{N}{\alpha!} \cdot \frac{(\langle\xi\rangle^{\frac{1}{2}}\langle x\rangle^{-\frac{1}{2}})^N}{\beta_1!\beta_2!\beta_3!} \int_{Q_{x,\xi}} (\partial_\xi^\beta \partial_x^{\beta_1} F)(x,\xi,\zeta)(\partial_x^{\beta_2}F)(x,\xi,\zeta) \\ &\times \int_0^1 (1-\theta)^{N-1}(\partial_\xi^\alpha D_x^{\beta_3}p)(x,\theta\zeta+(1-\theta)\xi)\,d\theta d\zeta. \end{aligned}$$

Now, there exists $K > 0$ such that

$$K^{-1}\langle\xi\rangle \leq \langle\theta\zeta+(1-\theta)\xi\rangle \leq K\langle\xi\rangle, \quad |\theta| < 1, \zeta \in Q_{x,\xi}, x, \xi \in \mathbb{R}^n.$$

Then using (2.22), since $s < \min\{\mu, \nu\}$, we obtain

$$\begin{aligned}
|\partial_\xi^\gamma \partial_x^\delta r_{\beta,N}(x,\xi)| &\le C^{|\gamma+\delta|+2(N+|\beta|)+1} \gamma!^\mu \delta!^\nu \beta!^{s+\nu-1} N!^{\mu-1} \\
&\quad \times \langle \xi \rangle^{m_1-|\gamma|-\frac{N+|\beta|}{2}} \langle x \rangle^{m_2-|\delta|-\frac{N+|\beta|}{2}} \underbrace{\int_{Q_{x,\xi}} \langle \xi \rangle^{-\frac{n}{2}} \langle x \rangle^{\frac{n}{2}} d\sigma}_{=\int_{|\sigma|\le 1} d\sigma} \\
&\le C^{|\gamma+\delta|+2(N+|\beta|)+1} \gamma!^\mu \delta!^\nu \beta!^{s+\nu-1} N!^{\mu-1} \langle \xi \rangle^{m_1-|\gamma|-\frac{N+|\beta|}{2}} \langle x \rangle^{m_2-|\delta|-\frac{N+|\beta|}{2}},
\end{aligned} \tag{2.24}$$

for every $\gamma, \delta \in \mathbb{N}_0^n$, $x, \xi \in \mathbb{R}^n$ and $N \in \mathbb{N}$. Now by (2.23) and (2.24), we get

$$p_{F,L}(x,\xi) \sim \sum_{j \in \mathbb{N}_0} \sum_{|\alpha+\beta|=j} q_{\alpha,\beta}(x,\xi) \text{ in } FSG^m_{(\frac{1}{2},1),(0,\frac{1}{2});(\tilde{\mu},\tilde{\nu})}.$$

To improve the above asymptotic expansion, note that for $j \ge 2$

$$|\partial_\xi^\gamma \partial_x^\delta \sum_{|\alpha+\beta|=j} q_{\alpha,\beta}(x,\xi)| \le C^{|\gamma+\delta|+2j+1} \gamma!^\mu \delta!^\nu j!^{\mu+\nu-1} \langle x \rangle^{m_2-|\delta|-\frac{j}{2}} \langle \xi \rangle^{m_1-|\gamma|-\frac{j}{2}},$$

and

$$|\partial_\xi^\gamma \partial_x^\delta \sum_{|\alpha+\beta|=1} q_{\alpha,\beta}(x,\xi)| \le C^{|\theta+\sigma|+2j+1} \gamma!^\mu \delta!^\nu j!^{\mu+\nu-1} \langle x \rangle^{m_2-|\delta|-1} \langle \xi \rangle^{m_1-|\gamma|-1},$$

for every $\gamma, \delta \in \mathbb{N}_0^n$, $x, \xi \in \mathbb{R}^n$, hence

$$\sum_{j \in \mathbb{N}_0} \sum_{|\alpha+\beta|=j} q_{\alpha,\beta}(x,\xi) \in F_{(k_j,\ell_j)} SG^m_{(\mu,\nu)},$$

where $k_0 = \ell_0 = 0$, $k_1 = \ell_1 = 1$, $k_j = \ell_j = \frac{j}{2}$. Then there exists $q \in SG^m_{(\mu,\nu)}(\mathbb{R}^{2n})$ such that

$$q(x,\xi) \sim \sum_{\alpha,\beta} q_{\alpha,\beta}(x,\xi) \;\text{ in } F_{(k_j,\ell_j)} SG^m_{(\mu,\nu)}.$$

Repeating the argument at the end of Subection 2.5.2 we can write $p_{F,L}(x,\xi) = q(x,\xi) + r(x,\xi)$, where r belongs to the Gelfand-Shilov space $\mathcal{S}_{\tilde{\mu}+\tilde{\nu}-1}(\mathbb{R}^{2n})$. Summing up we obtain the following result.

Theorem 2.19. *Let $p \in SG^m_{(\mu,\nu)}$ and p_F be its Friedrichs part. Then we can write $p_{F,L} = q + r$, with $r \in \mathcal{S}_{\tilde{\mu}+\tilde{\nu}-1}(\mathbb{R}^{2n})$ and*

$$q(x,\xi) \sim p(x,\xi) + \sum_{|\beta|=1} q_{0,\beta}(x,\xi) + \sum_{|\alpha+\beta|\ge 2} q_{\alpha,\beta}(x,\xi) \;\text{ in } F_{(k_j,\ell_j)} SG^m_{(\mu,\nu)}$$

where $k_0 = \ell_0 = 0$, $k_1 = \ell_1 = 1$, $k_j = \ell_j = \frac{j}{2}$. Moreover, the symbols $q_{0,\beta} \in SG^{m-(1,1)}_{(\mu,\nu)}(\mathbb{R}^{2n})$ and $q_{\alpha,\beta} \in SG^{m-\frac{|\alpha+\beta|}{2}(1,1)}_{(\mu,\nu)}(\mathbb{R}^{2n})$ are the same as in Theorem 2.17.

2.6 Spectral Invariance for $SG^m_{\mu,\nu}(\mathbb{R}^{2n})-$Pseudodifferential Operators

Let $p \in SG^{0,0}(\mathbb{R}^{2n})$, then we know that $p(x,D)$ extends to a bounded operator on $L^2(\mathbb{R}^n)$. Suppose that $p(x,D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bijective. The question is to determine whether or not the inverse $\{p(x,D)\}^{-1}$ is also an SG operator of order $(0,0)$. This is known as the spectral invariance problem and it has an affirmative answer, see [17].

Now assume that p is also Gevrey regular, that is $p \in SG^{0,0}_{\mu,\nu}(\mathbb{R}^{2n})$, and $p(x,D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bijective. By the initial discussion we know that p^{-1} is an $SG^{0,0}$ pseudodifferential operator, but what can we say about its Gevrey regularity? Following the ideas presented in [17], we will prove that the symbol of p^{-1} satisfies Gevrey estimates, whenever so does the symbol $p(x,\xi)$.

We start recalling the definition of Fredholm operators.

Definition 2.28. *Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. We say that T is Fredholm whenever the following conditions are satisfied*

(i) the range $R(T)$ of T is a closed subspace of Y;

(ii) the kernel $N(T)$ of T is finite dimensional;

(iii) the kernel $N(T^*)$ of the adjoint T^* is finite dimensional.

The index of a Fredholm operator T is defined by $i(T) = dimN(T) - dimN(T^*)$.

Theorems 2.20, 2.21, 2.22 here below can be found in Chapters 20 and 21 of [46].

Theorem 2.20. *[Atkinson] Let X, Y separable Hilbert spaces. Then a bounded operator $A : X \to Y$ is Fredholm if and only if there exist $B : Y \to X$ bounded, $K_1 : X \to Y$ and $K_2 : Y \to X$ compact operators such that*

$$BA = I_X - K_1, \qquad AB = I_Y - K_2.$$

Theorem 2.21. *Let X, Y, Z be separable Hilbert spaces and let $A : X \to Y$, $B : Y \to Z$ be Fredholm operators. Then*

(i) $B \circ A : X \to Z$ is Fredholm and $i(BA) = i(B) + i(A)$;

(ii) $Y = N(A^t) \oplus R(A)$.

Remark 2.21. *Let X be a Hilbert space and $K : X \to X$ a compact operator. Then $I - K$ is Fredholm and $i(I - K) = 0$.*

Theorem 2.22. *Let $p \in SG^{m_1,m_2}(\mathbb{R}^{2n})$ such that $p(x, D) : H^{s_1+m_1,s_2+m_2}(\mathbb{R}^n) \to H^{m_1,m_2}(\mathbb{R}^n)$ is Fredholm for some $s_1, s_2 \in \mathbb{R}$. Then p is SG-elliptic, that is there exist $C, R > 0$ such that*

$$|p(x,\xi)| \geq C\langle\xi\rangle^{m_1}\langle x\rangle^{m_2} \qquad \text{for} \quad (x,\xi) \in Q^e_R,$$

where $Q^e_R = \{(x,\xi) \in \mathbb{R}^{2n} : \langle\xi\rangle \geq R \text{ or } \langle x\rangle \geq R\}$.

Theorem 2.23. *Let $p \in SG^{m_1,m_2}_{\mu,\nu}(\mathbb{R}^{2n})$ be SG-elliptic. Then there is $q \in SG^{-m_1,-m_2}_{\mu,\nu}(\mathbb{R}^{2n})$ such that*

$$p(x, D) \circ q(x, D) = I + r_1(x, D), \qquad q(x, D) \circ p(x, D) = I + r_2(x, D),$$

where $r_1, r_2 \in \mathcal{S}_{\mu+\nu-1}(\mathbb{R}^{2n})$.

Proof. See Theorem 6.3.16 of [39]. □

In order to prove the main result of this section, we need the following technical lemma.

Lemma 2.10. *Let $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be bounded operator such that A and A^* map $L^2(\mathbb{R}^n)$ into $\Sigma_r(\mathbb{R}^n)$ continuously. Then the Schwartz kernel of A belongs to $\Sigma_r(\mathbb{R}^{2n})$.*

Proof. Since $\Sigma_r(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is a nuclear Fréchet space, (cf. [18]), by Propositions 2.1.7 and 2.1.8 of [24], we have that A is defined by a kernel $H(x, y)$ and we have the following representations

$$H(x,y) = \sum_{j\in\mathbb{N}_0} a_j f_j(x) g_j(y) = \sum_{j\in\mathbb{N}_0} \tilde{a}_j \tilde{f}_j(x)\tilde{g}_j(y),$$

where $a_j, \tilde{a}_j \in \mathbb{C}$, $\tilde{f}_j(x), g_j(y) \in \Sigma_r(\mathbb{R}^n)$, $f_j(x), \tilde{g}_j(y) \in L^2(\mathbb{R}^n)$, $\sum_j |a_j| < \infty$, $\sum_j |\tilde{a}_j| < \infty$, $\tilde{f}_j(x), g_j(y)$ converge to zero in $\Sigma_r(\mathbb{R}^n)$ and $f_j(x), \tilde{g}_j(y)$ converge to zero in $L^2(\mathbb{R}^n)$.

We now use the following characterization: $H(x, y) \in \Sigma_r(\mathbb{R}^{2n})$ if and only if

$$\sup_{\alpha,\beta\in\mathbb{N}^n_0} \left\| \frac{x^\alpha y^\beta H(x,y)}{C^{|\alpha|+|\beta|}\alpha!^r\beta!^r} \right\|_{L^2} < \infty \quad \text{and} \quad \sup_{\alpha,\beta\in\mathbb{N}^n_0} \left\| \frac{\xi^\alpha \eta^\beta \widehat{H}(\xi,\eta)}{C^{|\alpha|+|\beta|}\alpha!^r\beta!^r} \right\|_{L^2} < \infty$$

for every $C > 0$, and prove that both the latter conditions hold. Note that

$$\|y^\beta H(x,y)\|^2_{L^2} = \iint \left| \sum_{j\in\mathbb{N}_0} a_j f_j(x) y^\beta g_j(y) \right|^2 dx\, dy$$

$$
\begin{aligned}
&= \iint \left| \sum_{j\in\mathbb{N}_0} a_j^{\frac{1}{2}} f_j(x) a_j^{\frac{1}{2}} y^\beta g_j(y) \right|^2 dx\, dy \\
&\le \int \sum_{j\in\mathbb{N}_0} |a_j^{\frac{1}{2}} f_j(x)|^2 dx \int \sum_{j\in\mathbb{N}_0} |a_j^{\frac{1}{2}} y^\beta g_j(y)|^2 dy \\
&= \sum_{j\in\mathbb{N}_0} |a_j| \|f_j\|_{L^2}^2 \sum_{j\in\mathbb{N}_0} |a_j| \|y^\beta g_j\|_{L^2}^2.
\end{aligned}
$$

Since g_j converges to zero in $\Sigma_r(\mathbb{R}^n)$, we have

$$
\|y^\beta g_j(y)\|_{L^2} = \left\| \frac{y^\beta g_j(y)}{C^{|\beta|}\beta!^r} \right\|_{L^2} C^{|\beta|}\beta!^r \le C^{|\beta|}\beta!^r \sup_{j\in\mathbb{N}_0} \left\| \frac{y^\beta g_j(y)}{C^{|\beta|}\beta!^r} \right\|_{L^2},
$$

for every $C > 0$, and therefore

$$
\|y^\beta H(x,y)\|_{L^2} \le \left(\sum_{j\in\mathbb{N}_0} |a_j| \right)^2 \sup_{j\in\mathbb{N}_0} \|f_j\|_{L^2} \sup_{j\in\mathbb{N}_0} \left\| \frac{y^\beta g_j(y)}{C^{|\beta|}\beta!^r} \right\|_{L^2} C^{|\beta|}\beta!^r.
$$

Hence

$$
\sup_{\beta\in\mathbb{N}_0^n} \left\| \frac{y^\beta H(x,y)}{C^{|\beta|}\beta!^r} \right\|_{L^2} < \infty \iff \sup_{N\in\mathbb{N}_0} \left\| \frac{\langle y\rangle^N H(x,y)}{C^N N!^r} \right\|_{L^2} < \infty,
$$

for every $C > 0$. Using the representation $\sum \tilde{a}_j \tilde{f}_j(x)\tilde{g}_j(y)$, analogously we obtain

$$
\sup_{\alpha\in\mathbb{N}_0^n} \left\| \frac{x^\alpha H(x,y)}{C^{|\alpha|}\alpha!^r} \right\|_{L^2} < \infty \iff \sup_{N\in\mathbb{N}_0} \left\| \frac{\langle x\rangle^N H(x,y)}{C^N N!^r} \right\|_{L^2} < \infty,
$$

for every $C > 0$.

Now note that, for every $N \in \mathbb{N}_0$, $x, y \in \mathbb{R}^n$,

$$
\langle (x,y)\rangle^N = (\langle x\rangle^2 + |y|^2)^{\frac{N}{2}} \le (\langle x\rangle + \langle y\rangle)^N \le 2^{N-1}(\langle x\rangle^N + \langle y\rangle^N).
$$

Therefore, for every $C > 0$,

$$
\left\| \frac{\langle x,y\rangle^N H(x,y)}{C^N N!^r} \right\|_{L^2} \le \left\| \frac{\langle x\rangle^N H(x,y)}{C_1^N N!^r} \right\|_{L^2} + \left\| \frac{\langle y\rangle^N H(x,y)}{C_1^N N!^r} \right\|_{L^2},
$$

where $C_1 = (2^{-1}C)^N$. Hence, for every $C > 0$,

$$
\sup_{N\in\mathbb{N}_0} \left\| \frac{\langle x,y\rangle^N H(x,y)}{C^N N!^r} \right\|_{L^2} < \infty.
$$

Since the Fourier transform defines an isomorphism on $L^2(\mathbb{R}^{2n})$ and on $\Sigma_r(\mathbb{R}^{2n})$, we have

$$
\widehat{H}(\xi,\eta) = \sum_{j\in\mathbb{N}_0} a_j \widehat{f}_j(\xi)\widehat{g}_j(\eta) = \sum_{j\in\mathbb{N}_0} \tilde{a}_j \widehat{\tilde{f}}_j(\xi)\widehat{\tilde{g}}_j(\eta),
$$

where $a_j, \tilde{a}_j \in \mathbb{C}$, $\widehat{\tilde{f}}_j(\xi), \widehat{g}_j(\eta) \in \Sigma_r(\mathbb{R}^n)$, $\widehat{f}_j(\xi), \widehat{\tilde{g}}_j(\eta) \in L^2(\mathbb{R}^n)$, $\sum_j |a_j| < \infty$, $\sum_j |\tilde{a}_j| < \infty$, $\widehat{\tilde{f}}_j(\xi), \widehat{g}_j(\eta)$ converge to zero in $\Sigma_r(\mathbb{R}^n)$ and $\widehat{f}_j(\xi), \widehat{\tilde{g}}_j(\eta)$ converge to zero in $L^2(\mathbb{R}^n)$. In an analogous manner as before we get, for every $C > 0$,

$$\sup_{N \in \mathbb{N}_0} \left\| \frac{\langle \xi, \eta \rangle^N \widehat{H}(\xi, \eta)}{C^N N!^r} \right\|_{L^2} < \infty.$$

Hence $H(x, y) \in \Sigma_r(\mathbb{R}^{2n})$. $\square$

Theorem 2.24. *Let $p \in SG^{0,0}_{\mu,\nu}(\mathbb{R}^{2n})$ such that $p(x, D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bijective. Then $\{p(x, D)\}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a pseudodifferential operator given by a symbol $\tilde{p} = q + \tilde{k}$ where $q \in SG^{0,0}_{\mu,\nu}(\mathbb{R}^{2n})$ and $\tilde{k} \in \Sigma_r(\mathbb{R}^{2n})$ for every $r > \mu + \nu - 1$.*

Proof. Since $p(x, D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bijective, then $p(x, D)$ is Fredholm and

$$i(p(x, D)) = dim N(p(x, D)) - dim N(p^t(x, D)) = 0.$$

Therefore by Theorem 2.22 $p(x, \xi)$ is SG-elliptic and by Theorem 2.23 there is $q \in SG^{0,0}_{\mu,\nu}(\mathbb{R}^{2n})$ such that

$$q(x, D) \circ p(x, D) = I + r(x, D), \qquad p(x, D) \circ q(x, D) = I + s(x, D),$$

for some $r, s \in \mathcal{S}_{\mu+\nu-1}(\mathbb{R}^{2n})$. In particular $r(x; D), s(x, D)$ are compact operators on $L^2(\mathbb{R}^n)$. By Theorem 2.20 $q(x, D)$ is a Fredholm operator and we have

$$i(q(x, D)) = i(q(x, D)) + i(p(x, D)) = i(q(x, D) \circ p(x, D)) = i(I + r(x, D)) = 0.$$

Note that $N(q(x, D))$ and $N(q^t(x, D))$ are subspaces of $\mathcal{S}_{\mu+\nu-1}(\mathbb{R}^n)$. Indeed, take $f \in N(q)$ and $g \in N(q^t)$, then

$$0 = p(x, D) \circ q(x, D)f = (I + s(x, D))f \implies f = -s(x, D)f,$$

$$0 = p^t(x, D) \circ q^t(x, D)g = (q(x, D) \circ p(x, D))^t g = (I + r(x, D))^t g \implies g = -r^t(x, D)g.$$

Since $L^2(\mathbb{R}^n)$ is a separable Hilbert space and $N(q(x, D))$ is closed, we have the following decompositions

$$L^2 = N(q) \oplus N(q)^\perp, \qquad L^2 = N(q^t) \oplus R_{L^2}(q),$$

where $R_{L^2}(q)$ denotes the range of $q(x, D)$ as an operator on $L^2(\mathbb{R}^n)$.

Consider the projection $\pi : L^2 \to N(q)$ of L^2 onto $N(q)$ with null space $N(q)^\perp$, an linear isomorphism $F : N(q) \to N(q^t)$ and $i : N(q^t) \to L^2$ the inclusion. Set $Q = i \circ F \circ \pi$. Then $Q : L^2 \to L^2$ is bounded and its image is contained in $N(q^t) \subset \mathcal{S}_{\mu+\nu-1}$. It is not difficult to see that $Q^* = \tilde{i} \circ F^* \circ \pi_{N(q^t)}$, where $\tilde{i}$ is the inclusion of $N(q)$ into L^2 and $\pi_{N(q^t)}$ is the orthogonal projection of L^2 onto $N(q)$. Since $\mathcal{S}_{\mu+\nu-1} \subset \Sigma_r$, then by Lemma 2.10, Q is given by a kernel in Σ_r.

We will now show that $q + Q$ is a bijective parametrix of p. Indeed, let $u = u_1 + u_2$ in $N(q) \oplus N(q^t)$ such that $(q + Q)u = 0$. Then $0 = qu_2 + (i \circ F)u_1 \in R_{L^2}(q) \oplus N(q^t)$. Hence $qu_2 = 0$ and $i \circ Fu_1 = 0$ which implies that $u = 0$. In order to prove that Q is surjective, consider $f = f_1 + f_2 \in R_{L^2}(q) \oplus N(q^t)$. There exist $u_1 \in L^2$ and $u_2 \in N(q)$ such that $qu_1 = f_1$ and $Fu_2 = f_2$. Now write $u_1 = v_1 + v_2 \in N(q) \oplus N(q)^\perp$. Then $q(u_1) = q(v_2)$ and therefore $(q + Q)(v_2 + u_2) = f_1 + f_2 = f$. Finally notice that

$$p(x, D) \circ (q(x, D) + Q) = I + s + p(x, D) \circ Q = I + s'(x, D),$$

$$(q(x, D) + Q) \circ p(x, D) = I + r + Q \circ p(x, D) = I + r'(x, D),$$

where $r', s' \in \Sigma_r(\mathbb{R}^{2n})$.

Now set $\tilde{q} = q + Q$. Therefore $\tilde{q} \circ p = I + r' : L^2 \to L^2$ is bijective. Define $k = -(I + r')^{-1} \circ r'$. Then $(I + r')(I + k) = I$ and $k = -r' - r'k$. Observe that

$$k^t = -\{r'\}^t - k^t\{r'\}^t = -\{r'\}^t + \{r'\}^t\{(I + r')^{-1}\}^t\{r'\}^t.$$

Hence k, k^t map L^2 into Σ_r and by Lemma 2.10 we have that k is given by a kernel in $\Sigma_r(\mathbb{R}^{2n})$.

To finish the proof, it is enough to notice that

$$p^{-1} \circ \tilde{q}^{-1} = (\tilde{q} \circ p)^{-1} = (I + r')^{-1} = I + k \implies p^{-1} = (I + k) \circ \tilde{q}.$$

$\square$

Chapter 3

Cauchy Problem for $3-$Evolution Operators With Data in Gevrey Type Spaces

3.1 Introduction and Main Result

Let us consider for $(t,x) \in [0,T] \times \mathbb{R}$ the following class of $3-$ evolution pseudodifferential operators

$$P(t,x,D_t,D_x) = D_t + a_3(t,D_x) + a_2(t,x,D_x) + a_1(t,x,D_x) + a_0(t,x,D_x), \tag{3.1}$$

where $a_3(t,\xi)$ has order 3, $a_3(t,\xi)$ is real-valued and $a_j(t,x,D)$ has order j (with respect to ξ), $j = 0,1,2.$ We are interested in the Cauchy Problem associated with the operator P with data in suitable Gevrey classes. The main result of this chapter reads as follows.

Theorem 3.1. *Let $s_0 > 1$ and $\sigma \in (\frac{1}{2}, 1)$ such that $s_0 < \frac{1}{2(1-\sigma)}$. Let $P(t,x,D_t,D_x)$ be an operator like in (3.1) satisfying the following assumptions:*

(i) $a_3(t,\xi) \in C([0,T], S^3_1(\mathbb{R}^2))$ *and there exist* $C_{a_3}, R_{a_3} > 0$ *such that*

$$|\partial_\xi a_3(t,\xi)| \geq C_{a_3}\xi^2, \quad t \in [0,T],\ |\xi| \geq R_{a_3};$$

(ii) $a_2(t,x,\xi) \in C([0,T]; S^2_{1,s_0}(\mathbb{R}^2))$ *and there is* $C_{a_2} > 0$ *such that*

$$|\partial_\xi^\alpha \partial_x^\beta a_2(t,x,\xi)| \leq C_{a_2}^{\alpha+\beta+1}\alpha!\beta!^{s_0}\langle\xi\rangle^{2-\alpha}\langle x\rangle^{-\sigma}, \quad t \in [0,T],\ x,\xi \in \mathbb{R},\ \alpha,\beta \in \mathbb{N}_0;$$

(iii) $a_1(t,x,\xi) \in C([0,T]; S^1_{1,s_0}(\mathbb{R}))$ *and there exists* $C_{a_1} > 0$ *such that*

$$|Im\, a_1(t,x,\xi)| \leq C_{a_1} \langle \xi \rangle \langle x \rangle^{-\frac{\sigma}{2}}, \quad t \in [0,T],\ x, \xi \in \mathbb{R};$$

(iv) $a_0(t,x,\xi) \in C([0,T], S^0_{1,s_0}(\mathbb{R}^2))$.

Let $\theta > 1$ *such that* $s_0 \leq \theta < \frac{1}{2(1-\sigma)}$ *and let* $f \in C([0,T]; H^m_{\rho;\theta}(\mathbb{R}))$ *and* $g \in H^m_{\rho;\theta}(\mathbb{R})$*, where* $m, \rho \in \mathbb{R}$ *and* $\rho > 0$*. Then the Cauchy problem*

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = g(x), \quad x \in \mathbb{R}, \end{cases} \tag{3.2}$$

admits a unique solution $u \in C([0,T]; H^m_{\tilde{\rho};\theta}(\mathbb{R}))$ *for some* $\tilde{\rho} < \rho$*; moreover the solution satisfies the following energy estimate*

$$\|u(t)\|^2_{H^m_{\tilde{\rho};\theta}} \leq C \left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau)\|^2_{H^m_{\rho;\theta}} d\tau \right), \tag{3.3}$$

for all $t \in [0,T]$ *and for some* $C > 0$.

Remark 3.1. *Theorem 3.1 implies that the Cauchy Problem (3.2) is well-posed in the class*

$$\mathcal{H}^\infty_\theta(\mathbb{R}) = \bigcup_{\rho > 0} H^m_{\rho;s}(\mathbb{R}).$$

Note that $\mathcal{H}^\infty_\theta(\mathbb{R}) \subset G^\theta(\mathbb{R})$ *(cf. Proposition 2.11).*

Remark 3.2. *We notice that the solution* u *exhibits a loss of regularity with respect to the initial data in the sense that it belongs to* $H^m_{\tilde{\rho};\theta}$ *for some* $\tilde{\rho} < \rho$*. Moreover, the decay rate* σ *of the coefficients imposes restrictions on the values of* θ *for which the Cauchy problem (3.2) is well-posed. Such phenomena are typical of this type of problems and they appear also in the papers [14, 33] .*

Remark 3.3. *Let us make some comments on the decay assumptions (iii) and (iv) in Theorem 3.1. In the Schrödinger case (*$p = 2$*), we know from [33] that the decay condition* $\langle x \rangle^{-\sigma}$ *on the imaginary part of the subleader coefficient,* $\sigma \in (0,1)$*, leads to Gevrey well-posedness for* $s_0 \leq \theta < 1/(1-\sigma)$*. Here we prove that, for the* $3-$*evolution case, the decay condition* $a_2(t,x) \sim \langle x \rangle^{-\sigma}$*,* $\sigma \in (1/2, 1)$*, together with a weaker decay assumption on the (lower order) term Im*$a_1(t,x) \sim \langle x \rangle^{-\sigma/2}$ *is sufficient to get Gevrey well-posedness for* $\theta \in [s_0, \{2(1-\sigma)\}^{-1})$*. Comparing these results, we point out that here we need to assume that also Re*$a_2 \sim \langle x \rangle^{-\sigma}$ *in* *order to control the term appearing in* (3.25)*. It is not clear at this moment if this assumption is only technical or it is a necessary condition to obtain our result. Another natural question arising is the following: what happens for* $\sigma \in (0, 1/2]$*? At the moment we cannot prove Gevrey well-posedness for small* σ*, nor provide a counterexample.*

Remark 3.4. *In this remark, we describe a model class of linear differential operators that fits in the hypothesis of Theorem 3.1.*

Let $s_0 > 1$ *and* $\sigma \in (\frac{1}{2}, 1)$ *such that* $s_0 < \frac{1}{2(1-\sigma)}$*. Now consider the operator*

$$P(t, x, D_t, D_x) = D_t + a_3(t)D_x^3 + a_2(t, x)D_x^2 + a_1(t, x)D_x^1 + a_0(t, x), \tag{3.4}$$

where the coefficients satisfy

(i) $a_3(t)$ *is a continuous real-valued function which never vanishes;*

(ii) $a_2(t, x) \in C([0, T]; G^{s_0}(\mathbb{R}))$ *and there exists* $C_{a_2} > 0$ *such that*

$$|\partial_x^\beta a_2(t, x)| \leq C_{a_2}^{\beta+1} \beta!^{s_0} \langle x \rangle^{-\sigma};$$

(iii) $a_1(t, x) \in C([0, T]; G^{s_0}(\mathbb{R}^2))$ *and there exists* C_{a_1} *such that*

$$|Im\, a_1(t, x)| \leq C_{a_1} \langle x \rangle^{-\frac{\sigma}{2}};$$

(iv) $a_0(t, x) \in C([0, T]; G^{s_0}(\mathbb{R}))$.

Then P *is under the hypothesis of Theorem 3.1.*

3.2 Strategy of the Proof

To prove Theorem 3.1 we need to perform a suitable change of variable. In fact, if we set

$$iP = \partial_t + ia_3(t, D) + \sum_{j=0}^{2} ia_j(t, x, D) = \partial_t + ia_3(t, D) + A(t, x, D),$$

since $a_3(t, \xi)$ is real-valued, we have

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_{L^2}^2 &= 2Re\, \langle \partial_t u(t), u(t) \rangle_{L^2} \\
&= 2Re\, \langle iPu(t), u(t) \rangle_{L^2} - 2Re\, \langle ia_3(t, D)u(t), u(t) \rangle_{L^2} - 2Re\, \langle Au(t), u(t) \rangle_{L^2} \\
&\leq \|Pu(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 - \langle (A + A^*)u(t), u(t) \rangle_{L^2}.
\end{aligned}$$

However, $A + A^*$ is an operator of order 2, so we cannot derive an energy inequality in L^2 from the estimate above. The idea is then to conjugate the operator iP by a suitable invertible pseudodifferential operator $e^\Lambda(t, x, D)$ with a symbol of the form $e^{\Lambda(t,x,\xi)}$ in order to obtain

$$(iP)_\Lambda := e^\Lambda \circ (iP) \circ \{e^\Lambda\}^{-1} = \partial_t + ia_3(t, D) + \{A_{2,\Lambda} + A_{1,\Lambda} + A_{\frac{1}{\theta},\Lambda} + r_{0,\Lambda}\}(t, x, D),$$

where $A_{j,\Lambda}$ has symbol $a_{j,\Lambda}(t, x, \xi)$ of order j but with $Re\, a_{j,\Lambda} \geq 0$, for $j = \frac{1}{\theta}, 1, 2$, and $r_{0,\Lambda}(t, x, D)$ has symbol $r_{0,\Lambda}(t, x, D)$ of order zero. In this way, applying Fefferman-Phong inequality to $A_{2,\Lambda}$ (see [20]) and sharp Gårding inequality to $A_{1,\Lambda}$ and $A_{\frac{1}{\theta},\Lambda}$ (see Theorem 1.7.15 of [39]), we obtain the estimate from below

$$Re\, \langle\{A_{2,\Lambda} + A_{1,\Lambda} + A_{\frac{1}{\theta},\Lambda}\}v(t), v(t)\rangle_{L^2} \geq -c\|v(t)\|^2_{L^2},$$

and therefore for the solution v of the Cauchy problem associated to the operator P_Λ we get

$$\frac{d}{dt}\|v(t)\|^2_{L^2} \leq C(\|(iP)_\Lambda v(t)\|^2_{L^2} + \|v(t)\|^2_{L^2}).$$

Gronwall inequality then gives the desired energy estimate for the conjugated operator $(iP)_\Lambda$. By standard arguments in the energy method, we then obtain that the Cauchy problem associated with P_Λ is well-posed in any Sobolev space $H^m(\mathbb{R})$.

Finally, we turn back to our original Cauchy problem (3.2). The problem (3.2) is in fact equivalent to

$$\begin{cases} P_\Lambda(t, D_t, x, D_x)v(t, x) = e^\Lambda(t, x, D_x)f(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}, \\ v(0, x) = e^\Lambda(0, x, D_x)g(x), \quad x \in \mathbb{R}, \end{cases} \tag{3.5}$$

in the sense that if u solves (3.2) then $v = e^\Lambda u$ solves (3.5), and if v solves (3.5) then $u = \{e^\Lambda\}^{-1}v$ solves (3.2). In this step, the continuous mapping properties of e^Λ and $\{e^\Lambda\}^{-1}$ will play an important role.

The operator e^Λ will be a pseudodifferential operator (of infinite order) with symbol $e^{\Lambda(t,x,\xi)}$, and the function $\Lambda(t, x, \xi)$ will be of the form

$$\Lambda(t, x, \xi) = k(t)\langle\xi\rangle_h^{\frac{1}{\theta}} + \lambda_2(x, \xi) + \lambda_1(x, \xi) = k(t)\langle\xi\rangle_h^{\frac{1}{\theta}} + \tilde{\Lambda}(x, \xi), \quad t \in [0, T],\ x, \xi \in \mathbb{R},$$

where $\tilde{\Lambda} = \lambda_2 + \lambda_1 \in S_\mu^{2(1-\sigma)}(\mathbb{R}^2)$, $k \in C^1([0, T]; \mathbb{R})$ is a positive non-increasing function to be chosen later on and $\langle\xi\rangle_h := \sqrt{h^2 + \xi^2}$ with $h \geq 1$ a large parameter. The replacement of $\langle\xi\rangle$ with $\langle\xi\rangle_h$ is useful in several parts of the proof (for instance, to obtain the invertibility of the operator $e^{\Lambda}(t,x,D)$) and does not change the symbol classes (cf. Remark 3.5). Now we briefly explain the main role of each part of the change of variables. The transformation with λ_2 will change the terms of order 2 into the sum of a positive operator of the same order plus a remainder of order 1; the transformation with λ_1 will not change the terms of order 2, but it will turn the terms of order 1 into the sum of a positive operator plus a remainder of order less than $1/\theta$. Finally, the transformation with k will correct this remainder term. We also observe that since $2(1-\sigma) < 1/\theta$ the leading part is $k(t)\langle\xi\rangle_h^{\frac{1}{\theta}}$, hence the inverse of $e^{\Lambda}(t,x,D)$ possesses regularizing properties with respect to the spaces $H^m_{\rho;\theta}$, because $k(t)$ has positive sign.

The precise definitions of λ_2 and λ_1 will be given in Section 3.3. Since Λ admits an algebraic growth in the ξ variable, then $e^{\Lambda(t,x,\xi)}$ presents an exponential growth. This is one of the reasons for which we need pseudodifferential operators of infinite order.

Remark 3.5. *It is not difficult to conclude that*

$$|\partial_\xi^\alpha \langle\xi\rangle_h^{m_1}| \leq C_{m_1}^{|\alpha|}\alpha!\langle\xi\rangle_h^{m_1-|\alpha|}, \quad \xi\in\mathbb{R}^n, \alpha\in\mathbb{N}_0^n,$$

where C_{m_1} *is a positive constant independent of* h.

Now let us consider the class of symbols $S^m_{h,\mu,\nu}(\mathbb{R}^{2n})$ *obtained replacing the weight* $\langle\xi\rangle$ *by* $\langle\xi\rangle_h$, *that is,* $p\in S^m_{h,\mu,\nu}$ *if there exists* $C>0$ *such that*

$$|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)| \leq C^{|\alpha+\beta|+1}\alpha!^\mu\beta!^\nu\langle\xi\rangle_h^{m-|\alpha|}, \quad \alpha,\beta\in\mathbb{N}_0^n, x,\xi\in\mathbb{R}^n.$$

Since $h\geq 1$, *we have*

$$\langle\xi\rangle \leq \langle\xi\rangle_h \leq h\langle\xi\rangle, \quad \xi\in\mathbb{R}^n.$$

Hence we may conclude that $S^m_{\mu,\nu}(\mathbb{R}^{2n}) = S^m_{h,\mu,\nu}(\mathbb{R}^{2n})$ *with equivalent topologies. Analogous considerations for all the other classes of symbols treated in Chapter 2.*

We also remark that $\langle\xi\rangle$ *and* $\langle\xi\rangle_h$ *are asymptotically equivalent in the sense*

$$\lim_{|\xi|\to\infty}\frac{\langle\xi\rangle}{\langle\xi\rangle_h} = \lim_{|\xi|\to\infty}\frac{\langle\xi\rangle_h}{\langle\xi\rangle} = 1.$$

Hence, for any $\varepsilon>0$, *there exists* $R_{h,\varepsilon}>0$ *such that*

$$\langle\xi\rangle_h \leq (1+\varepsilon)\langle\xi\rangle, \quad |\xi|\geq R_{h,\varepsilon}.$$

3.3 Definition and Properties of $\lambda_2(x,\xi)$ and $\lambda_1(x,\xi)$

For $M_2, M_1>0$ to be chosen, we define

$$\lambda_2(x,\xi) = M_2 w\left(\frac{\xi}{h}\right)\int_0^x \langle y\rangle^{-\sigma}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right)dy, \quad (x,\xi)\in\mathbb{R}^2, \tag{3.6}$$

$$\lambda_1(x,\xi) = M_1 w\left(\frac{\xi}{h}\right)\langle\xi\rangle_h^{-1}\int_0^x \langle y\rangle^{-\frac{\sigma}{2}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right)dy, \quad (x,\xi)\in\mathbb{R}^2, \tag{3.7}$$

where

$$w(\xi) = \begin{cases} 0, & |\xi|\le 1, \\ -\mathrm{sgn}(\partial_\xi a_3(t,\xi)), & |\xi| > R_{a_3}, \end{cases} \qquad \psi(y) = \begin{cases} 1, & |y|\le \frac{1}{2}, \\ 0, & |y|\ge 1, \end{cases}$$

$|\partial_\xi^\alpha w(\xi)| \le C_w^{\alpha+1}\alpha!^\mu$, $|\partial_y^\beta \psi(y)| \le C_\psi^{\beta+1}\beta!^\mu$, for some $\mu > 1$ which we can take arbitrarly close to 1. Note that $\langle y\rangle \le \langle\xi\rangle_h^2$ on the support of $\psi(\langle y\rangle\langle\xi\rangle_h^{-2})$ and $\sqrt{2}h \le \langle\xi\rangle_h \le \langle R_{a_3}\rangle h$ on the support of $w'(h^{-1}\xi)$. Note also that thanks to the assumption (i) in Theorem 3.1, the function w is well defined.

We have the following result for $\lambda_2(x,\xi)$.

Lemma 3.1. *Let $\lambda_2(x,\xi)$ be defined in (3.6). Then*

(i) $|\partial_\xi^\alpha \lambda_2(x,\xi)| \le C_{\lambda_2}^{\alpha+1}\alpha!^\mu\langle\xi\rangle_h^{2(1-\sigma)-\alpha}$, *for* $\alpha\ge 0$;

(ii) $|\partial_\xi^\alpha \lambda_2(x,\xi)| \le C_{\lambda_2}^{\alpha+1}\alpha!^\mu\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{1-\sigma}$, *for* $\alpha\ge 0$;

(iii) $|\partial_\xi^\alpha\partial_x^\beta \lambda_2(x,\xi)| \le C_{\lambda_2}^{\alpha+\beta+1}\alpha!^\mu\beta!^\mu\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{-\sigma-(\beta-1)}$, *for* $\alpha\ge 0, \beta\ge 1$,

where $C>0$ is independent of $h>1$. In particular, $\lambda_2(x,\xi)\in S_\mu^{2(1-\sigma)}(\mathbb{R}^2)\cap SG_\mu^{0,1-\sigma}(\mathbb{R}^2)$.

Proof. We denote by $\chi_\xi(x)$ the characteristic function of the set $\{x\in\mathbb{R} : \langle x\rangle \le \langle\xi\rangle_h^2\}$. Now note that

$$|\lambda_2(x,\xi)| \le M_2\int_0^{|x|}\langle y\rangle^{-\sigma}\chi_\xi(y)dy = \int_0^{\min\{|x|,\langle\xi\rangle_h^2\}}\langle y\rangle^{-\sigma}dy,$$

hence

$$|\lambda_2(x,\xi)| \le \frac{M_2}{1-\sigma}\min\{\langle\xi\rangle_h^{2(1-\sigma)}, \langle x\rangle^{1-\sigma}\}.$$

For $\alpha\ge 1$ we have

$$\begin{aligned}
|\partial_\xi^\alpha\lambda_2(x,\xi)| &\le M_2\sum_{\alpha_1+\alpha_2=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!}\left|w^{(\alpha_1)}\left(\frac{\xi}{h}\right)\right| h^{-\alpha_1}\left|\int_0^x \chi_\xi(y)\langle y\rangle^{-\sigma}\partial_\xi^{\alpha_2}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right)\right| \\
&\le M_2\sum_{\alpha_1+\alpha_2=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!}\left|w^{(\alpha_1)}\left(\frac{\xi}{h}\right)\right| h^{-\alpha_1}\int_0^{|x|}\chi_\xi(y)\langle y\rangle^{-\sigma} \\
&\times\sum_{j=1}^{\alpha_2}\frac{\left|\psi^{(j)}\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right)\right|}{j!}\sum_{\gamma_1+\ldots+\gamma_j=\alpha_2}\frac{\alpha_2!}{\gamma_1!\ldots\gamma_j!}\prod_{\nu=1}^{j}\langle y\rangle|\partial_\xi^{\gamma_\nu}\langle\xi\rangle_h^{-2}|dy
\end{aligned}$$

$$
\begin{aligned}
&\leq M_2 \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} C_w^{\alpha_1+1} \alpha_1!^\mu \langle \xi \rangle_h^{-\alpha_1} \langle R_{a_3} \rangle^{\alpha_1} \\
&\times \tilde{C}_\psi^{\alpha_2+1} \alpha_2!^\mu \langle \xi \rangle_h^{-\alpha_2} \int_0^{|x|} \chi_\xi(y) \langle y \rangle^{-\sigma} dy \\
&\leq M_2 C_{w,\psi,R_{a_3}}^{\alpha+1} \alpha!^\mu \langle \xi \rangle_h^{-\alpha} \int_0^{|x|} \chi_\xi(y) \langle y \rangle^{-\sigma} dy \\
&\leq \frac{M_2}{1-\sigma} C_{w,\psi,R_{a_3}}^{\alpha+1} \alpha!^\mu \langle \xi \rangle_h^{-\alpha} \min\{\langle \xi \rangle_h^{2(1-\sigma)}, \langle x \rangle^{1-\sigma}\}.
\end{aligned}
$$

For $\alpha \geq 0$ and $\beta \geq 1$ we have

$$
\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta \lambda_2(x,\xi)| &\leq M_2 \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta-1}} \frac{\alpha!}{\alpha_2!} \frac{(\beta-1)!}{\beta_1!\beta_2!} \left| w^{(\alpha_1)}\left(\frac{\xi}{h}\right) \right| |\partial_x^{\beta_1} \langle x \rangle^{-\sigma}| \left| \partial_\xi^{\alpha_2} \partial_x^{\beta_2} \psi \left(\frac{\langle x \rangle}{\langle \xi \rangle_h^2} \right) \right| \\
&\leq M_2 \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta-1}} \frac{\alpha!}{\alpha_2!} \frac{(\beta-1)!}{\beta_1!\beta_2!} C_w^{\alpha_1+1} \alpha_1!^\mu \langle R_{a_3} \rangle^{\alpha_1} \langle \xi \rangle_h^{-\alpha_1} C^{\beta_1} \beta_1! \langle x \rangle^{-\sigma-\beta_1} \\
&\times \chi_\xi(x) \sum_{j=1}^{\alpha_2+\beta_2} \frac{\left| \psi^{(j)}\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^2} \right) \right|}{j!} \sum_{\substack{\gamma_1+\ldots+\gamma_j=\alpha_2 \\ \lambda_1+\ldots+\lambda_j=\beta_2}} \frac{\alpha_2!\beta_2!}{\gamma_1!\lambda_1!\ldots\gamma_j!\lambda_j!} \prod_{\nu=1}^{j} |\partial_x^{\lambda_\nu} \langle x \rangle| |\partial_\xi^{\gamma_\nu} \langle \xi \rangle_h^{-2}| \\
&\leq M_2 \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta-1}} \frac{\alpha!}{\alpha_2!} \frac{(\beta-1)!}{\beta_1!\beta_2!} C_w^{\alpha_1+1} \alpha_1!^\mu \langle R_{a_3} \rangle^{\alpha_1} \langle \xi \rangle_h^{-\alpha_1} C^{\beta_1} \beta_1! \langle x \rangle^{-\sigma-\beta_1} \\
&\times \tilde{C}_\psi^{\alpha_2+\beta_2+1} \alpha_2!^\mu \beta_2!^\mu \langle x \rangle^{-\beta_2} \langle \xi \rangle_h^{-\alpha_2} \\
&\leq M_2 C_{\psi,w,R_{a_3}}^{\alpha+\beta+1} \alpha!^\mu (\beta-1)!^\mu \langle x \rangle^{-\sigma-(\beta-1)} \langle \xi \rangle_h^{-\alpha}.
\end{aligned}
$$

□

For $\lambda_1(x,\xi)$ we have:

Lemma 3.2. *Let $\lambda_1(x,\xi)$ be defined in (3.7). Then*

(i) $|\partial_\xi^\alpha \lambda_1(x,\xi)| \leq C_{\lambda_1}^{\alpha+1} \alpha!^\mu \langle \xi \rangle_h^{1-\sigma-\alpha}$, *for* $\alpha \geq 0$;

(ii) $|\partial_\xi^\alpha \lambda_1(x,\xi)| \leq C_{\lambda_1}^{\alpha+1} \alpha!^\mu \langle \xi \rangle_h^{-1-\alpha} \langle x \rangle^{1-\frac{\sigma}{2}}$, *for* $\alpha \geq 0$;

(iii) $|\partial_\xi^\alpha \lambda_1(x,\xi)| \leq C_{\lambda_1}^{\alpha+1} \alpha!^\mu \langle \xi \rangle_h^{-\alpha} \langle x \rangle^{1-\sigma}$, *for* $\alpha \geq 0$;

(iv) $|\partial_\xi^\alpha \partial_x^\beta \lambda_1(x,\xi)| \leq C_{\lambda_1}^{\alpha+\beta+1} \alpha!^\mu \beta!^\mu \langle \xi \rangle_h^{-1-\alpha} \langle x \rangle^{-\frac{\sigma}{2}-(\beta-1)}$, *for* $\alpha \geq 0, \beta \geq 1$;

(v) $|\partial_\xi^\alpha \partial_x^\beta \lambda_1(x,\xi)| \leq C_{\lambda_1}^{\alpha+\beta+1} \alpha!^\mu \beta!^\mu \langle \xi \rangle_h^{-\alpha} \langle x \rangle^{-\sigma-(\beta-1)}$, *for* $\alpha \geq 0, \beta \geq 1$,

where $C > 0$ is independent of $h \geq 1$. Particularly $\lambda_1 \in S_\mu^{1-\sigma}(\mathbb{R}^2) \cap SG_\mu^{0,1-\sigma}(\mathbb{R}^2)$.

Proof. The demonstration of this result is quite similar to the proof of Lemma 3.1. For this reason, we will omit it. $\square$

Since we need to consider the symbols $e^{\lambda_2(x,\xi)}$ and $\lambda_1(x,\xi)$, the two next results will play an important role.

Proposition 3.1. *Let* $s \geq 1$. *Then:*

(i) $\lambda(x,\xi) \in S_\mu^{\frac{1}{s}}(\mathbb{R}^2)$ *implies* $e^{\lambda(x,\xi)} \in S_{\mu;s}^{\infty}(\mathbb{R}^2)$*;*

(ii) $\lambda(x,\xi) \in SG_\mu^{0,\frac{1}{s}}(\mathbb{R}^2)$ *implies* $e^{\lambda(x,\xi)} \in SG_{\mu;s}^{0,\infty}(\mathbb{R}^2)$*;*

(iii) $\lambda(x,\xi) \in SG_\mu^{\frac{1}{s},0}(\mathbb{R}^2)$ *implies* $e^{\lambda(x,\xi)} \in SG_{\mu;s}^{\infty,0}(\mathbb{R}^2)$.

Proof. Considering that the proofs of (i), (ii) and (ii) are very similar, we only focus in (i). For $|\alpha+\beta| \geq 1$, Faà di Bruno formula gives

$$\partial_{(x,\xi)}^{(\beta,\alpha)} e^{\lambda(x,\xi)} = \sum_{j=1}^{|\alpha+\beta|} \frac{e^{\lambda(x,\xi)}}{j!} \sum_{\substack{\alpha_1+\ldots+\alpha_j=\alpha \\ \beta_1+\ldots+\beta_j=\beta}} \frac{\alpha!\beta!}{\alpha_1!\beta_1!\ldots\alpha_j!\beta_j!} \prod_{\nu=1}^{j} \partial_\xi^{\alpha_\nu}\partial_x^{\beta_\nu}\lambda(x,\xi).$$

Thus, from standard factorial inequalities we get

$$|\partial_\xi^\alpha \partial_x^\beta e^{\lambda(x,\xi)}| \leq C^{|\alpha+\beta|+1}\alpha!^\mu\beta!^\mu\langle\xi\rangle^{-|\alpha|}e^{\lambda(x,\xi)}\sum_{j=1}^{|\alpha+\beta|}\frac{\langle\xi\rangle^{\frac{j}{s}}}{j!}.$$

Hence,

$$|\partial_\xi^\alpha \partial_x^\beta e^{\lambda(x,\xi)}| \leq \tilde{C}^{|\alpha+\beta|+1}\alpha!^\mu\beta!^\mu\langle\xi\rangle^{-|\alpha|}e^{c|x|^{\frac{1}{s}}}.$$

$\square$

Proposition 3.2. *Let* $\rho, m \in \mathbb{R}$ *and let* $s,\mu > 1$ *with* $s > 2\mu - 1$. *Let* $\lambda \in S_\mu^{\frac{1}{\kappa}}(\mathbb{R}^2)$. *Then:*

(i) if $\kappa > s$, *then the operator* $e^{\lambda(x,D)}$ *is continuous from* $H^m_{\rho;s}(\mathbb{R})$ *into* $H^m_{\rho-\delta;s}(\mathbb{R})$ *for every* $\delta > 0$*;*

(ii) if $\kappa = s$, *then the operator* $e^{\lambda(x,D)}$ *is continuous from* $H^m_{\rho;s}(\mathbb{R})$ *into* $H^m_{\rho-\delta;s}(\mathbb{R})$ *for every*

$$\delta > C(\lambda) := \sup\{\lambda(x,\xi)/\langle\xi\rangle^{1/s} : (x,\xi) \in \mathbb{R}^2\}.$$

Proof. Consider $\phi(\xi) \in G^{\mu}(\mathbb{R})$ a cut-off function such that, for a large positive constant K, $\phi(\xi) = 1$ for $|\xi| < K/2$, $\phi(\xi) = 0$ for $|\xi| > K$ and $0 \leq \phi(\xi) \leq 1$ for every $\xi \in \mathbb{R}$. We split the symbol $e^{\lambda(x,\xi)}$ as

$$e^{\lambda(x,\xi)} = \phi(\xi)e^{\lambda(x,\xi)} + (1 - \phi(\xi))e^{\lambda(x,\xi)} = a_1(x, \xi) + a_2(x, \xi). \tag{3.8}$$

Since ϕ has compact support, we have in particular $a_1(x, \xi) \in S^0_{\mu}$. On the other hand, given any $\delta > 0$ and choosing K large enough, since $\kappa > s$ we may write $|\lambda(x, \xi)|\langle\xi\rangle^{-1/s} < \delta$ on the support of $a_2(x, \xi)$. Hence we obtain

$$a_2(x, \xi) = e^{\delta\langle\xi\rangle^{1/s}}(1 - \phi(\xi))e^{\lambda(x,\xi)-\delta\langle\xi\rangle^{1/s}}$$

with $(1-\phi(\xi))e^{\lambda(x,\xi)-\delta\langle\xi\rangle^{1/s}}$ of order 0 because $\lambda(x, \xi)-\delta\langle\xi\rangle^{1/s} < 0$ on the support of $(1-\phi(\xi))$. Thus, (4.8) becomes

$$e^{\lambda(x,\xi)} = a_1(x, \xi) + e^{\delta\langle\xi\rangle^{1/s}}\tilde{a}_2(x, \xi),$$

with a_1 and $\tilde{a}_2$ of order 0. Since by Theorem 2.8 the operators $a_1(x, D)$ and $\tilde{a}_2(x, D)$ map continuously $H^m_{\rho;s}$ into itself, then we obtain (i). The proof of (ii) follows a similar argument. □

3.4 Invertibility of $e^{\tilde{\Lambda}(x,D)}$, $\tilde{\Lambda}(x, \xi) = (\lambda_2 + \lambda_1)(x, \xi)$

In this section we construct the inverse of the operator $e^{\tilde{\Lambda}}(x, D)$ where

$$\tilde{\Lambda}(x, \xi) = \lambda_2(x, \xi) + \lambda_1(x, \xi), \quad x, \xi \in \mathbb{R}^n,$$

and we prove that the inverse acts continuously on Gevrey Sobolev spaces. By Lemmas 3.1 and 3.2 we have $\tilde{\Lambda}(x, \xi) \in SG^{0,2(1-\sigma)}_{\mu}(\mathbb{R}^2) \cap S^{2(1-\sigma)}_{\mu}(\mathbb{R}^2)$. Therefore $e^{\tilde{\Lambda}(x,\xi)} \in SG^{0,\infty}_{\mu;\frac{1}{1-\sigma}}(\mathbb{R}^2) \cap S^{\infty}_{\mu;\frac{1}{2(1-\sigma)}}(\mathbb{R}^2)$, see Proposition 3.1. To construct the inverse of $e^{\tilde{\Lambda}}(x, D)$ we need to use the L^2 adjoint of $e^{-\tilde{\Lambda}}(x, D)$ (cf. [6, 33, 34]), defined as the operator given by the amplitude $e^{-\tilde{\Lambda}(y,\xi)}$, that is,

$$\{e^{-\tilde{\Lambda}}(x, D)\}^*u(x) = \iint e^{i(x-y)\xi}e^{-\tilde{\Lambda}(y,\xi)}u(y)\, dy đ\xi, \quad x \in \mathbb{R}. \tag{3.9}$$

Assuming $\mu > 1$ such that $1/(1-\sigma) > 2\mu - 1$, by results from calculus, we may write

$$\{e^{-\tilde{\Lambda}}(x, D)\}^* = a_1(x, D) + r_1(x, D),$$

where $a_1 \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha e^{-\tilde{\Lambda}}$ in $FSG^{0,\infty}_{\mu;1/(1-\sigma)}(\mathbb{R}^2)$, $r_1 \in \mathcal{S}_{2\mu-1}(\mathbb{R}^2)$, and

$$e^{\tilde{\Lambda}}(x,D)\circ\{e^{-\tilde{\Lambda}}(x,D)\}^* = e^{\tilde{\Lambda}}\circ a_1(x,D) + e^{\tilde{\Lambda}}\circ r_1(x,D) = a_2(x,D) + r_2(x,D) + e^{\tilde{\Lambda}}\circ r_1(x,D),$$

where

$$a_2 \sim \sum_{\alpha,\beta}\frac{1}{\alpha!\beta!}\partial_\xi^\alpha e^{\tilde{\Lambda}}\partial_\xi^\beta D_x^{\alpha+\beta}e^{-\tilde{\Lambda}} = \sum_\gamma \frac{1}{\gamma!}\partial_\xi^\gamma(e^{\tilde{\Lambda}}D_x^\gamma e^{-\tilde{\Lambda}}) \text{ in } FSG^{0,\infty}_{\mu;1/(1-\sigma)}(\mathbb{R}^2)$$

and $r_2 \in \mathcal{S}_{2\mu-1}(\mathbb{R}^2)$. Therefore

$$e^{\tilde{\Lambda}}(x,D)\circ\{e^{-\tilde{\Lambda}}(x,D)\}^* = a(x,D) + r(x,D),$$

where $a \sim \sum_\gamma \frac{1}{\gamma!}\partial_\xi^\gamma(e^{\tilde{\Lambda}}D_x^\gamma e^{-\tilde{\Lambda}})$ in $FSG^{0,\infty}_{\mu;1/(1-\sigma)}(\mathbb{R}^2)$ and $r \in \mathcal{S}_{2\mu-1}(\mathbb{R}^2)$.

Now let us study more carefully the asymptotic expansion

$$\sum_{\gamma\geq 0}\frac{1}{\gamma!}\partial_\xi^\gamma(e^{\tilde{\Lambda}}D_x^\gamma e^{-\tilde{\Lambda}}) = \sum_{\gamma\geq 0} r_{1,\gamma}.$$

Note that

$$e^{\tilde{\Lambda}(x,\xi)}D_x^\gamma e^{-\tilde{\Lambda}(x,\xi)} = \sum_{j=1}^{\gamma}\frac{(-1)^\gamma}{j!}\sum_{\gamma_1+\cdots+\gamma_j=\gamma}\frac{\gamma!}{\gamma_1!\ldots\gamma_j!}\prod_{\ell=1}^{j}D_x^{\gamma_\ell}\tilde{\Lambda}(x,\xi),$$

hence, for $\alpha,\beta\geq 0$,

$$\begin{aligned}
|\partial_\xi^\alpha\partial_x^\beta r_{1,\gamma}| &\leq \frac{1}{\gamma!}\sum_{j=1}^{\gamma}\frac{1}{j!}\sum_{\gamma_1+\cdots+\gamma_j=\gamma}\frac{\gamma!}{\gamma_1!\ldots\gamma_j!}\sum_{\alpha_1+\cdots+\alpha_j=\alpha+\gamma}\ \sum_{\beta_1+\cdots+\beta_j=\beta}\frac{(\alpha+\gamma)!}{\alpha_1!\ldots\alpha_j!}\frac{\beta!}{\beta_1!\ldots\beta_j!}\\
&\quad\times\prod_{\ell=1}^{j}|\partial_\xi^{\alpha_\ell}\partial_x^{\beta_\ell+\gamma_\ell}\tilde{\Lambda}(x,\xi)|\\
&\leq \frac{1}{\gamma!}\sum_{j=1}^{\gamma}\frac{1}{j!}\sum_{\gamma_1+\cdots+\gamma_j=\gamma}\frac{\gamma!}{\gamma_1!\ldots\gamma_j!}\sum_{\alpha_1+\cdots+\alpha_j=\alpha+\gamma}\ \sum_{\beta_1+\cdots+\beta_j=\beta}\frac{(\alpha+\gamma)!}{\alpha_1!\ldots\alpha_j!}\frac{\beta!}{\beta_1!\ldots\beta_j!}\\
&\quad\times\prod_{\ell=1}^{j}C_{\tilde{\Lambda}}^{\alpha_\ell+\beta_\ell+\gamma_\ell+1}\alpha_\ell!^\mu(\beta_\ell+\gamma_\ell)!^\mu\langle\xi\rangle_h^{-\alpha_\ell}\langle x\rangle^{1-\sigma-\beta_\ell-\gamma_\ell}\\
&\leq C^{\alpha+\beta+2\gamma+1}\alpha!^\mu\beta!^\mu\gamma!^{2\mu-1}\langle\xi\rangle_h^{-\alpha-\gamma}\sum_{j=1}^{\gamma}\frac{\langle x\rangle^{(1-\sigma)j-\beta-\gamma}}{j!}.
\end{aligned}$$

We shall consider the following sets

$$Q_{t_1,t_2;h} = \{(x,\xi)\in\mathbb{R}^2 : \langle x\rangle < t_1 \text{ and } \langle\xi\rangle_h < t_2\}$$

and $Q^e_{t_1,t_2;h} = \mathbb{R}^2 - Q_{t_1,t_2;h}$. When $t_1 = t_2 = t$ we simply write $Q_{t;h}$ and $Q^e_{t;h}$.

Let $\psi(x,\xi) \in C^\infty(\mathbb{R}^2)$ such that $\psi \equiv 0$ on $Q_{2;h}$, $\psi \equiv 1$ on $Q^e_{3;h}$, $0 \le \psi \le 1$ and

$$|\partial_\xi^\alpha \partial_x^\beta \psi(x,\xi)| \le C_\psi^{\alpha+\beta+1}\alpha!^\mu \beta!^\mu,$$

for every $x, \xi \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{N}_0$. Now set $\psi_0 \equiv 1$ and, for $j \ge 1$,

$$\psi_j(x,\xi) := \psi\left(\frac{x}{R(j)}, \frac{\xi}{R(j)}\right),$$

where $R(j) = Rj^{2\mu-1}$ and $R > 0$ is a large constant. Let us recall that

- $(x,\xi) \in Q^e_{3R(j)} \implies \left(\frac{x}{R(j)}, \frac{\xi}{R(j)}\right) \in Q^e_3 \implies \psi_i(x,\xi) = 1$, for $i \le j$;
- $(x,\xi) \in Q_{R(j)} \implies \left(\frac{x}{R(j)}, \frac{\xi}{R(j)}\right) \in Q_2 \implies \psi_i(x,\xi) = 0$, for $i \ge j$.

Defining $b(x,\xi) = \sum_{j\ge 0} \psi_j(x,\xi) r_{1,j}(x,\xi)$, following the same ideas in the proof of Proposition 2.14, we have that $b(x,\xi) \in SG^{0,\infty}_{\mu;\frac{1}{1-\sigma}}(\mathbb{R}^2)$ and

$$b(x,\xi) \sim \sum_{j\ge 0} r_{1,j}(x,\xi) \text{ in } FSG^{0,\infty}_{\mu;\frac{1}{1-\sigma}}(\mathbb{R}^2).$$

We will show that $b(x,\xi) \in SG^{0,0}_\mu(\mathbb{R}^{2n})$. Indeed, first we write

$$b(x,\xi) = 1 + \sum_{j\ge 1} \psi_j(x,\xi) r_{1,j}(x,\xi) = 1 + \sum_{j\ge 0} \psi_{j+1}(x,\xi) r_{1,j+1}(x,\xi).$$

On the support of $\partial_\xi^{\alpha_1}\partial_x^{\beta_1}\psi_{j+1}$ we have

$$\langle x\rangle \le 3R(j+1) \quad \text{and} \quad \langle \xi\rangle_h \le 3R(j+1),$$

whenever $\alpha_1 + \beta_1 \ge 1$. Hence

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta \sum_{j\ge 0} \psi_{j+1} r_{1,j+1}| &\le \sum_{j\ge 0} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\frac{\beta!}{\beta_1!\beta_2!} |\partial_\xi^{\alpha_1}\partial_x^{\beta_1}\psi_{j+1}(x,\xi)||\partial_\xi^{\alpha_2}\partial_x^{\beta_2} r_{1,j+1}(x,\xi)| \\
&\le \sum_{j\ge 0} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\frac{\beta!}{\beta_1!\beta_2!}\frac{1}{R(j+1)^{(\alpha_1+\beta_1)}} C_\psi^{\alpha_1+\beta_1+1}\alpha_1!^\mu \beta_1!^\mu \\
&\times C^{\alpha_2+\beta_2+2(j+1)+1}\alpha_2!^\mu\beta_2!^\mu (j+1)!^{2\mu-1}\langle\xi\rangle_h^{-\alpha_2-(j+1)}\langle x\rangle^{-\beta_1-(j+1)} \sum_{\ell=1}^{j+1} \frac{\langle x\rangle^{(1-\sigma)\ell}}{\ell!} \\
&\le \sum_{j\ge 0} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \frac{\alpha!}{\alpha_1!\alpha_2!}\frac{\beta!}{\beta_1!\beta_2!}\frac{1}{R(j+1)^{(\alpha_1+\beta_1)}} C_\psi^{\alpha_1+\beta_1+1}\alpha_1!^\mu \beta_1!^\mu \\
&\times C^{\alpha_2+\beta_2+2(j+1)+1}\alpha_2!^\mu\beta_2!^\mu (j+1)!^{2\mu-1}\langle\xi\rangle_h^{-\alpha_2-(j+1)}\langle x\rangle^{-\sigma-\beta_1} \sum_{\ell=1}^{j+1} \frac{\langle x\rangle^{(1-\sigma)(\ell-1)-j}}{\ell!}
\end{aligned}$$

$$\leq \tilde{C}^{\alpha+\beta+1}(\alpha!\beta!)^{\mu}\langle\xi\rangle_h^{-1-\alpha}\langle x\rangle^{-\sigma-\beta}\sum_{j\geq 0} C^{2j}(j+1)!^{2\mu-1}\langle\xi\rangle_h^{-j}\sum_{\ell=0}^{j+1}\frac{\langle x\rangle^{(1-\sigma)(\ell-1)-j}}{\ell!}.$$

We also have that

$$\langle x\rangle \geq R(j+1) \quad \text{or} \quad \langle\xi\rangle_h \geq R(j+1)$$

holds true on the support of $\partial_\xi^{\alpha_1}\partial_x^{\beta_1}\psi_{j+1}$. If $\langle\xi\rangle_h \geq R(j+1)$, then

$$\langle\xi\rangle_h^{-j} \leq R^{-j}(j+1)^{-j(2\mu-1)} \leq R^{-j}(j+1)!^{-(2\mu-1)}.$$

On the other hand, since we are assuming $\mu > 1$ such that $2\mu - 1 < \frac{1}{1-\sigma}$, if $\langle x\rangle \leq R(j+1)$ we obtain

$$\begin{aligned}\langle x\rangle^{(1-\sigma)(\ell-1)-j} &\leq R^{(1-\sigma)(\ell-1)-j}\{(j+1)^{2\mu-1}\}^{(1-\sigma)(\ell-1)-j}\\ &\leq R^{-\sigma j}(j+1)^{\ell-1-j(2\mu-1)}\\ &= R^{-\sigma j}(j+1)^{\ell-1}(j+1)!^{-(2\mu-1)}.\end{aligned}$$

Enlarging $R > 0$ if necessary, we can infer that $\sum_{j\geq 1} r_{1,j} \in SG_{\mu}^{-1,-\sigma}(\mathbb{R}^2)$.

In analogous way it is possible to prove that $\sum_{j\geq k} r_{1,j} \in SG_{\mu}^{-k,-\sigma k}(\mathbb{R}^2)$. Hence, we may conclude

$$b(x,\xi) - \sum_{j<k} r_{1,j}(x,\xi) \in SG_{\mu}^{-k,-\sigma k}(\mathbb{R}^2), \quad k \in \mathbb{N},$$

that is, $b \sim \sum_j r_{1,j}$ in $SG_{\mu}^{0,0}(\mathbb{R}^2)$.

Since $a \sim \sum r_{1,j}$ in $FSG^{0,\infty}_{\mu;1/(1-\sigma)}(\mathbb{R}^2)$, $b \sim \sum r_{1,j}$ in $FSG^{0,\infty}_{\mu;1/(1-\sigma)}(\mathbb{R}^2)$ we have $a - b \in \mathcal{S}_{2\mu-1}(\mathbb{R}^2)$. Hence we may write

$$e^{\tilde{\Lambda}} \circ \{e^{-\tilde{\Lambda}}\}^* = I + \tilde{r}(x,D) + \bar{r}(x,D) = I + r(x,D),$$

where $\tilde{r} \in SG_{\mu}^{-1,-\sigma}(\mathbb{R}^2)$, $\tilde{r} \sim \sum_{\gamma\geq 1} r_{1,\gamma}(x,\xi)$ in $SG_{\mu}^{-1,-\sigma}(\mathbb{R}^2)$ and $\bar{r} \in \mathcal{S}_{2\mu-1}(\mathbb{R}^2)$. In particular $r \in SG_{2\mu-1}^{-1,-\sigma}(\mathbb{R}^2)$, therefore we have

$$\begin{aligned}|\partial_\xi^\alpha\partial_x^\beta r(x,\xi)| &\leq C_{\alpha,\beta}\langle\xi\rangle_h^{-1-\alpha}\langle x\rangle^{-\sigma-\beta}\\ &\leq C_{\alpha,\beta}h^{-1}\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{-\sigma-\beta}.\end{aligned}$$

This implies that the $(0,0)-$seminorms of $r(x,\xi)$ are bounded by h^{-1}. Choosing h large enough, we obtain that $I + r(x,D)$ is invertible on $L^2(\mathbb{R})$ and its inverse $(I + r(x,D))^{-1}$ is given by the Neumann series $\sum_{j\geq 0}(-r(x,D))^j$. The composition $\{e^{-\tilde{\Lambda}}\}^* \circ \sum_j(-r(x,D))^j$ is then a right inverse on L^2 for $e^{\tilde{\Lambda}}(x,D)$.

By Theorem 2.24 we have

$$(I + r(x,D))^{-1} = q(x,D) + k(x,D),$$

where $q \in SG^{0,0}_{2\mu-1}(\mathbb{R}^2)$, $k \in \Sigma_\delta(\mathbb{R}^2)$ for every $\delta > 2(2\mu-1) - 1 = 4\mu - 3$. Choosing $\mu > 1$ close enough to 1, we have that δ can be chosen arbitrarily close to 1. Hence, by Theorem 2.8, for every fixed $\theta > 1$, we can find $\mu > 1$ such that

$$(I + r(x,D))^{-1} : H^{m'}_{\rho';\theta} \to H^{m'}_{\rho';\theta}$$

is continuous for every $m', \rho' \in \mathbb{R}$. Analogously one can show the existence of a left inverse of e^{Λ} with the same properties. Summing up, we have obtained the following result.

Lemma 3.3. *Let $\theta > 1$ and take $\mu > 1$ such that $\theta > 4\mu - 3$. There is $h_0 \geq 1$ such that for every $h \geq h_0$, the operator $e^{\tilde{\Lambda}}(x,D)$ is invertible on $L^2(\mathbb{R})$ and its inverse is given by*

$$\{e^{\tilde{\Lambda}}(x,D)\}^{-1} = \{e^{-\tilde{\Lambda}(x,D)}\}^* \circ (I + r(x,D))^{-1} = \{e^{-\tilde{\Lambda}(x,D)}\}^* \circ \sum_{j\geq 0} (-r(x,D))^j,$$

where $r \in SG^{-1,-\sigma}_{2\mu-1}(\mathbb{R}^2)$ and $r \sim \sum_{\gamma\geq 1} \frac{1}{\gamma!}\partial_\xi^\gamma (e^{\tilde{\Lambda}} D_x^\gamma e^{-\tilde{\Lambda}})$ in $SG^{-1,-\sigma}_{2\mu-1}(\mathbb{R}^2)$. Moreover, the symbol of $(I + r(x,D))^{-1}$ belongs to $SG^{(0,0)}_\delta$ for every $\delta > 4\mu - 3$ and it maps continuously $H^{m'}_{\rho';\theta}$ into itself for any $\rho', m' \in \mathbb{R}$.

We conclude this section writing $\{e^{\tilde{\Lambda}}(x,D)\}^{-1}$ in a more precise way. From the asymptotic expansion of the symbol $r(x,\xi)$ we have

$$\{e^{\tilde{\Lambda}}(x,D)\}^{-1} = \{e^{-\tilde{\Lambda}}(x,D)\}^* \circ (I - r(x,D) + (r(x,D))^2 + q_{-3}(x,D)),$$

where q_{-3} denotes an operator with symbol in $SG^{-3,-3\sigma}_\delta(\mathbb{R}^2)$ for every $\delta > 4\mu - 3$. Now note that

$$r = i\partial_\xi\partial_x\tilde{\Lambda} + \frac{1}{2}\partial_\xi^2(\partial_x^2\tilde{\Lambda} - [\partial_x\tilde{\Lambda}]^2) + q_{-3} = q_{-1} + q_{-2} + q_{-3}$$

and

$$\begin{aligned}
(r(x,D))^2 &= (q_{-1} + q_{-2} + q_{-3})(x,D) \circ (q_{-1} + q_{-2} + q_{-3})(x,D) \\
&= q_{-1}(x,D) \circ q_{-1}(x,D) + q_{-3}(x,D) \\
&= \mathrm{op}\left\{-[\partial_\xi\partial_x\tilde{\Lambda}]^2 + q_{-3}\right\}
\end{aligned}$$

for a new element q_{-3} in the same space. We finally obtain:

$$\{e^{\tilde{\Lambda}}\}^{-1} = \{e^{-\tilde{\Lambda}}\}^* \circ \left[I + \mathrm{op}\left(-i\partial_\xi\partial_x\tilde{\Lambda} - \frac{1}{2}\partial_\xi^2(\partial_x^2\tilde{\Lambda} - [\partial_x\tilde{\Lambda}]^2) - [\partial_\xi\partial_x\tilde{\Lambda}]^2 + q_{-3}\right)\right], \quad (3.10)$$

where $q_{-3} \in SG^{-3,-3\sigma}_\delta(\mathbb{R}^2)$. Since we deal with operators of order not exceeding 3, in the next sections we are going to use frequently formula (3.10) for the inverse of $e^{\tilde{\Lambda}}(x,D)$.

3.5 Conjugation of iP

In this section we will perform the conjugation of iP by the operator $e^{\Lambda}(t,x,D)$, where

$$\Lambda(t,x,\xi) = k(t)\langle\xi\rangle_h^{\frac{1}{s}} + \tilde{\Lambda}(x,\xi) = k(t)\langle\xi\rangle_h^{\frac{1}{s}} + \lambda_2(x,\xi) + \lambda_1(x,\xi)$$

and $k \in C^1([0,T];\mathbb{R})$ is a non-increasing function (to be chosen later on) such that $k(T) > 0$. Since the inverse of $e^{\tilde{\Lambda}}$ is given by

$$\{e^{-\tilde{\Lambda}}(x,D)\}^* \circ \sum_{j\geq 0}(-r(x,D))^j$$

we shall need to work with products of the type

$$e^{\tilde{\Lambda}}(x,D) \circ p(x,D) \circ \{e^{-\tilde{\Lambda}}(x,D)\}^*,$$

where $p(x,D)$ is given by a symbol of finite order. The next result shows us how to compute these types of conjugations.

Theorem 3.2. *Let p be a symbol satisfying*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_A A^{|\alpha+\beta|}\alpha!^k \beta!^k \langle\xi\rangle_h^{m-|\alpha|},$$

and let Λ satisfying

$$|\partial_\xi^\alpha \Lambda(x,\xi)| \leq \rho_0 A^{|\alpha|}\alpha!^k \langle\xi\rangle_h^{\frac{1}{k}-|\alpha|}, \quad \alpha \in \mathbb{N}_0,$$

and

$$|\partial_\xi^\alpha \partial_x^\beta \Lambda(x,\xi)| \leq \rho_0 A^{|\alpha|}\alpha!^k \beta!^k \langle\xi\rangle_h^{-|\alpha|}, \quad \alpha \in \mathbb{N}_0, \beta > 0. \tag{3.11}$$

Then there exist $\delta > 0$ and $h_0 = h_0(A) \geq 1$ such that if $\rho_0 \leq \delta A^{-\frac{1}{k}}$ and $h \geq h_0$, then

$$e^{\Lambda}(x,D)p(x,D)\{e^{-\Lambda}(x,D)\}^* = op\left(\sum_{|\alpha+\beta|<N} \frac{1}{\alpha!\beta!}\partial_\xi^\alpha\{\partial_\xi^\beta e^{\Lambda(x,\xi)} D_x^\beta p(x,\xi) D_x^\alpha e^{-\Lambda(x,\xi)}\}\right)$$
$$+ r_N(x,D) + r_\infty(x,D),$$

where

$$|\partial_\xi^\alpha \partial_x^\beta r_N(x,\xi)| \leq C_{\rho_0,A,k}(C_k A)^{|\alpha+\beta|+2N}\alpha!^k \beta!^k N!^{2k-1}\langle\xi\rangle_h^{m-(1-\frac{1}{k})N-|\alpha|},$$
$$|\partial_\xi^\alpha \partial_x^\beta r_\infty(x,\xi)| \leq C_{\rho_0,A,k}(C_k A)^{|\alpha+\beta|+2N}\alpha!^k \beta!^k N!^{2k-1} e^{-c_k A^{-\frac{1}{k}}\langle\xi\rangle_h^{\frac{1}{k}}}.$$

Remark 3.6. *Theorem 3.2 is totally inspired by Section 6 of the first part of [34]. Our approach follows the same ideas developed there, but with some small modifications in the results and proofs. Namely, we work on a Taylor expansion that was not performed in [34].*

3.5.1 Proof of Theorem 3.2

This section contains the proof of Theorem 3.2. Let us introduce the main ingredients of the proof. First of all, we need to extend Gevrey regular symbols to the complex domain.

Given a symbol $p \in S^m_\kappa(\mathbb{R}^{2n}; A)$ and a cutoff function $\chi \in C^\infty_c(\mathbb{R})$ such that

$$|\partial_t^j \chi(t)| \leq C^{j+1} j!^{k'}, \quad \chi(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| \geq 2, \end{cases}$$

where $1 < k' < k$. For $x, \xi \in \mathbb{R}^n$ and $|y| \leq B_1 \langle \xi \rangle_h^{\frac{1}{k}-1}$, $|\eta| \leq B_1 \langle \xi \rangle_h^{\frac{1}{k}}$ (where $B_1 > 0$) we define an almost analytic extension of $p(x, \xi)$ by

$$p(x+iy, \xi+i\eta) = \sum_{\delta, \gamma \in \mathbb{N}_0^n} \frac{1}{\delta! \gamma!} \partial_\xi^\gamma \partial_x^\delta p(x, \xi)(iy)^\delta (i\eta)^\gamma \chi(b_{|\delta|} \langle \xi \rangle_h^{\frac{1}{k}-1}) \chi(b_{|\gamma|} \langle \xi \rangle_h^{\frac{1}{k}-1}), \tag{3.12}$$

where the sequence $\{b_j\}_{j \in \mathbb{N}_0}$ is given by $b_0 = 1$ and $b_j = Bj!^{\frac{k-1}{j}}$ with $B = DAB_1$ and D is a large positive constant. Formula (3.12) is the Taylor series of $p(x+iy, \xi+i\eta)$ centered at (x, ξ) multiplied by some cutoff functions. These cutoff functions will ensure the convergence of the series. Namely, we have the following result which is proved in [34, Proposition 5.2].

Proposition 3.3. *Let $x, \xi, y, \eta \in \mathbb{R}^n$ such that $|y| \leq B_1 \langle \xi \rangle_h^{\frac{1}{k}-1}$ and $|\eta| \leq B_1 \langle \xi \rangle_h^{\frac{1}{k}}$. Let moreover $p(x+iy, \xi+i\eta)$ the almost analytic extension defined by (3.12). Then there are positive constants C_A, C_k, c_k such that*

$$|\partial_\xi^\alpha \partial_\eta^\rho \partial_x^\beta \partial_y^\lambda p(x+iy, \xi+i\eta)| \leq C_A (C_k A)^{|\alpha+\beta+\rho+\lambda|} \alpha!^k \beta!^k \rho!^k \lambda!^k \langle \xi \rangle_h^{m-|\alpha+\rho|}, \tag{3.13}$$

$$|\overline{\partial}_j \partial_\xi^\alpha \partial_\eta^\rho \partial_x^\beta \partial_y^\lambda p(x+iy, \xi+i\eta)| \leq C_A (C_k A)^{|\alpha+\beta+\rho+\lambda|} \alpha!^k \beta!^k \rho!^k \lambda!^k \langle \xi \rangle_h^{m-|\alpha+\rho|} e^{-c_k B^{-\frac{1}{k-1}} \langle \xi \rangle_h^{\frac{1}{k}}}, \tag{3.14}$$

where $\overline{\partial}_j$ stands for $2^{-1}(\partial_{x_j} + i\partial_{y_j})$ or $2^{-1}(\partial_{\xi_j} + i\partial_{\eta_j})$, $j = 1, 2, \ldots, n$.

For the proofs of the next results we recall that

$$Os - \iint e^{-iy\eta} e^{ix\eta} a(y) dy đ\eta = Os - \iint e^{-i\eta y} a(y+x) dy đ\eta = a(x), \quad x \in \mathbb{R}^n, \tag{3.15}$$

where $a \in \mathcal{B}^\infty(\mathbb{R}^n)$, see the example in the end of Section 6, Chapter 1 of [35]. Theorem 3.2 is a direct consequence of the two following propositions.

Proposition 3.4. *Under the same assumptions of Theorem 3.2, there exists* $\delta > 0$ *such that if* $\rho_0 \leq \delta A^{-\frac{1}{k}}$*, then*

$$e^{\Lambda}(x, D) \circ p(x, D) = op(e^{\Lambda(x,\xi)} s_N(x, \xi)) + q_N(x, D) + r_\infty(x, D),$$

where

$$s_N(x, \xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} e^{-\Lambda(x,\xi)} \{\partial_\xi^\alpha e^{\Lambda(x,\xi)}\} D_x^\alpha p(x, \xi), \tag{3.16}$$

$$|\partial_\xi^\alpha \partial_x^\beta q_N(x, \xi)| \leq C_{\rho_0,A,k} (C_k A)^{|\alpha+\beta|+2N} \alpha!^k \beta!^k N!^{2k-1} \langle \xi \rangle_h^{m-(1-\frac{1}{k})N-|\alpha|},$$

$$|\partial_\xi^\alpha \partial_x^\beta r_\infty(x, \xi)| \leq C_{\rho_0,A,k} (C_k A)^{|\alpha+\beta|+2N} \alpha!^k \beta!^k N!^{2k-1} e^{-c_k A^{-\frac{1}{k}} \langle \xi \rangle_h^{\frac{1}{k}}}.$$

Proof. Arguing as in the proof of [34, Theorem 6.9], we can write the symbol $s(x, \xi)$ of $e^{\Lambda}(x, D) \circ p(x, D)$ as the following oscillatory integral

$$s(x, \xi) = Os - \iint e^{-i\eta y} e^{\Lambda(x,\xi+\eta)} p(x + y, \xi) dy đ\eta.$$

Applying Taylor's formula and using (3.15) we obtain

$$s(x, \xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_\xi^\alpha e^{\Lambda(x,\xi)} D_x^\alpha p(x, \xi) + r_N(x, \xi),$$

where

$$r_N(x, \xi) = \sum_{|\alpha|=N} \frac{1}{\alpha!} Os - \iint e^{-iy\eta} D_x^\alpha p(x + y, \xi) \int_0^1 (1 - \theta)^{N-1} \partial_\xi^\alpha e^{\Lambda(x,\xi+\theta\eta)} d\theta dy đ\eta.$$

Therefore

$$e^{\Lambda}(x, D) \circ p(x, D) = \mathrm{op}(e^{\Lambda(x,\xi)} s_N(x, \xi)) + r_N(x, D),$$

where s_N is given by (3.16). Take now $\chi(t) \in C_c^\infty(\mathbb{R})$ such that

$$|\partial_t^j \chi(t)| \leq C^{j+1} j!^{k'} \ (1 < k' < k), \quad \chi(t) = \begin{cases} 1, & |t| \leq \frac{1}{4} \\ 0, & |t| \geq \frac{1}{2} \end{cases},$$

and set $\chi(\xi, \eta) = \chi(\langle \eta \rangle \langle \xi \rangle_h^{-1})$, $\xi, \eta \in \mathbb{R}^n$. Note that

$$\frac{1}{2} \langle \xi \rangle_h \leq \langle \xi + \theta\eta \rangle_h \leq \frac{3}{2} \langle \xi \rangle_h$$

for every $\xi, \eta \in$ supp $\chi(\xi, \eta)$ and $|\theta| \leq 1$. We can split the operator $r_N(x, D)$ as

$$r_N(x, D)u(x) = \sum_{|\alpha|=N} \frac{1}{\alpha!} \int e^{i\xi x + \Lambda(x,\xi)} Os - \iint e^{-i\eta y} \int_0^1 (1 - \theta)^{N-1} e^{\Lambda(x,\xi+\theta\eta)-\Lambda(x,\xi)}$$

$$\times\, w^{\alpha}(\Lambda; x, \xi+\theta\eta)d\theta\,\{D_x^{\alpha}p\}(x+y,\xi)dy đ\eta\, \widehat{u}(\xi)đ\xi$$
$$= \int e^{i\xi x+\Lambda(x,\xi)} r_N'(x,\xi)\widehat{u}(\xi)đ\xi + \int e^{ix\xi} r_N''(x,\xi)\widehat{u}(\xi)đ\xi,$$

where for $\alpha, \beta \in \mathbb{N}_0^n$ we set $w^{\alpha}_{\beta}(\Lambda; x, \xi) = e^{-\Lambda(x,\xi)}\partial_\xi^\alpha D_x^\beta e^{\Lambda(x,\xi)}$,

$$r_N'(x,\xi) = \lim_{\varepsilon\to 0} \sum_{|\alpha|=N} \frac{1}{\alpha!} \iint e^{-i\eta y} \int_0^1 (1-\theta)^{N-1} e^{\Lambda(x,\xi+\theta\eta)-\Lambda(x,\xi)} w^{\alpha}(\Lambda; x, \xi+\theta\eta)d\theta$$
$$\times \{D_x^\alpha p\}(x+y,\xi)\chi(\xi,\eta)\chi_\varepsilon(y,\eta)dyđ\eta,$$

$$r_N''(x,\xi) = \lim_{\varepsilon\to 0} \sum_{|\alpha|=N} \frac{1}{\alpha!} \iint e^{-i\eta y} \int_0^1 (1-\theta)^{N-1} \partial_\xi^\alpha e^{\Lambda(x,\xi+\theta\eta)} d\theta$$
$$\times \{D_x^\alpha p\}(x+y,\xi)(1-\chi)(\xi,\eta)\chi_\varepsilon(y,\eta)dyđ\eta,$$

and $\chi_\varepsilon(y,\eta) = \chi(\varepsilon y)\chi(\varepsilon\eta)$, $\chi \in \Sigma_{k'}(\mathbb{R}^n)$, $\chi(0) = 1$.

Now we work on $r_N'(x,\xi)$. Notice first that

$$\Lambda(x,\xi+\theta\eta) - \Lambda(x,\xi) = \eta\cdot\theta \int_0^1 \nabla_\xi \Lambda(x, \xi+\tilde\theta\theta\eta)d\tilde\theta = \eta\cdot\Lambda_\xi(x,\xi,\eta,\theta),$$

then we obtain

$$r_N'(x,\xi) = \lim_{\varepsilon\to 0} \sum_{|\alpha|=N} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N-1} \iint e^{-i\eta(y+i\Lambda_\xi(x,\xi,\eta,\theta))} w^{\alpha}(\Lambda; x, \xi+\theta\eta)$$
$$\times \{D_x^\alpha p\}(x+y,\xi)\chi(\xi,\eta)\chi_\varepsilon(y,\eta)dyđ\eta d\theta.$$

Observe that

$$|\partial_\xi^\alpha \partial_\eta^\gamma \partial_x^\beta \Lambda_\xi(x,\xi,\eta,\theta)| \le C_k\rho_0 A (C_k A)^{|\alpha+\gamma+\beta|}\alpha!^k\gamma!^k\beta!^k \langle\xi\rangle_h^{\frac{1}{k}-1-|\alpha+\gamma|},$$

for every $\xi, \eta \in$ supp $\chi(\xi,\eta)$ and $|\theta| \le 1$. For $x, \xi \in \mathbb{R}^n$ and $|y| \le C_k\delta A^{\frac{k-1}{k}}\langle\xi\rangle_h^{\frac{1}{k}-1}$, consider now the almost analytic extension

$$p(x+iy,\xi) = \sum_\delta \frac{1}{\delta!}\partial_x^\delta p(x,\xi)(iy)^\delta \chi(b_{|\delta|}\langle\xi\rangle_h^{1-\frac{1}{k}}),$$

where $b_j = Bj!^{\frac{k-1}{j}}$, $B = DA^{\frac{2k-1}{k}}$ and D is a large constant. Choosing $D = A^{-1}\tilde D$ with $\tilde D$ large enough, in view of (3.14), we obtain

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\lambda \overline{\partial}_j p(x+iy,\xi)| \le C_{A,k}(C_kA)^{|\alpha+\beta+\lambda|}\{\alpha!\lambda!\beta!\}^k \langle\xi\rangle_h^{m-|\alpha|} e^{-c_k A^{-\frac{1}{k}}\langle\xi\rangle_h^{\frac{1}{k}}}, \tag{3.17}$$

where $\overline{\partial}_j$ stands for $2^{-1}(\partial_{x_j} + i\partial_{y_j})$. In this way we may write $r'_N(x,\xi)$ as

$$r'_N(x,\xi) = \lim_{\varepsilon\to 0} \sum_{|\alpha|=N} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N-1} \int_{\mathbb{R}^n_\eta} \int_{\gamma(x,\xi,\eta,\theta)} e^{-i\eta z} w^\alpha(\Lambda; x, \xi+\theta\eta)$$

$$\times \{D_x^\alpha p\}(x+z-i\Lambda_\xi(x,\xi,\eta,\theta),\xi)\chi(\xi,\eta)\chi_\varepsilon(z-i\Lambda_\xi(x,\xi,\eta,\theta),\eta)dz\, đ\eta\, d\theta,$$

where

$$\gamma(x,\xi,\eta,\theta) = \{y + i\Lambda_\xi(x,\xi,\eta,\theta) \colon y \in \mathbb{R}^n\}.$$

Stokes formula implies

$$r'_N(x,\xi) = \lim_{\varepsilon\to 0} \sum_{|\alpha|=N} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N-1} \iint_{\mathbb{R}^{2n}_{(y,\eta)}} e^{-i\eta y} w^\alpha(\Lambda; x, \xi+\theta\eta)$$

$$\times \{D_x^\alpha p\}(x+y-i\Lambda_\xi(x,\xi,\eta,\theta),\xi)\chi(\xi,\eta)\chi_\varepsilon(y-i\Lambda_\xi(x,\xi,\eta,\theta),\eta)dy\, đ\eta\, d\theta$$

$$+ \lim_{\varepsilon\to 0} \sum_{|\alpha|=N} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N-1} \int_{\mathbb{R}^n_\eta} \sum_{j=1}^n \iint_{\Gamma(x,\xi,\eta,\theta)} e^{-i\eta z} w^\alpha(\Lambda; x, \xi+\theta\eta)$$

$$\times \overline{\partial}_{z_j}\{(D_x^\alpha p)(x+z-i\Lambda_\xi(x,\xi,\eta,\theta),\xi)\chi_\varepsilon(z-i\Lambda_\xi(x,\xi,\eta,\theta),\eta)\}\chi(\xi,\eta)d\overline{z}_j \wedge dz\, đ\eta\, d\theta$$

$$= r'_{1,N}(x,\xi) + r'_{2,N}(x,\xi),$$

where

$$\Gamma(x,\xi,\eta,\theta) = \{y + it\Lambda_\xi(x,\xi,\eta,\theta) \colon y \in \mathbb{R}^n,\, t \in [0,1]\}.$$

To finish the proof, it only remains to estimate $r'_{1,N}(x,\xi)$, $r'_{2,N}(x,\xi)$ and $r''_N(x,\xi)$. We begin with $r'_{1,N}(x,\xi)$: setting

$$a_{\varepsilon,\theta}(x,\xi;y,\eta) = w^\alpha(\Lambda; x, \xi+\theta\eta)\{D_x^\alpha p\}(x+y-i\Lambda_\xi(x,\xi,\eta,\theta),\xi)\chi(\xi,\eta)\chi_\varepsilon(y-i\Lambda_\xi(x,\xi,\eta,\theta),\eta),$$

we have, for any $\ell_1, \ell_2 \in \mathbb{N}_0$,

$$\partial_\xi^{\alpha'}\partial_x^{\beta'} r'_{1,N}(x,\xi) = \lim_{\varepsilon\to 0} \sum_{|\alpha|=N} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N-1} \iint e^{-iy\eta}$$

$$\times \langle y\rangle^{-2\ell_2}\langle D_\eta\rangle^{2\ell_2}\{\langle\eta\rangle^{-2\ell_1}\langle D_y\rangle^{2\ell_1}\partial_\xi^{\alpha'}\partial_x^{\beta'} a_{\varepsilon,\theta}(x,\xi;y,\eta)\}dy đ\eta\, d\theta.$$

Standard computations give

$$|\langle D_\eta\rangle^{2\ell_2}\{\langle\eta\rangle^{-2\ell_1}\langle D_y\rangle^{2\ell_1}\partial_\xi^{\alpha'}\partial_x^{\beta'} a_{\varepsilon,\theta}(x,\xi;y,\eta)\}| \leq C_{\rho_0,A,k}(C_kA)^{|\alpha'+\beta'|+2(|\alpha|+\ell_1+\ell_2)}\alpha'!^k\beta'!^k$$

$$\times (\alpha!\ell_1!\ell_2!)^{2k}\langle\xi\rangle_h^{m-|\alpha'|-|\alpha|(1-\frac{1}{k})}\langle\eta\rangle^{-2\ell_1},$$

for every $x, y \in \mathbb{R}^n$, $\xi, \eta \in$ supp $\chi(\xi,\eta)$ and $\theta, \varepsilon \in (0,1)$. Therefore

$$|\partial_\xi^{\alpha'}\partial_x^{\beta'} r'_{1,N}(x,\xi)| \leq C_{\rho_0,A,k}(C_kA)^{|\alpha'+\beta'|+2N}\alpha'!^k\beta'!^k N!^{2k-1}\langle\xi\rangle_h^{m-|\alpha'|-N(1-\frac{1}{k})}$$

$$\times (C_kA)^{+2(\ell_1+\ell_2)}(\ell_1!\ell_2!)^{2k} \iint \langle y\rangle^{-2\ell_2}\langle\eta\rangle^{-2\ell_1}dy đ\eta.$$

Hence, choosing $2\ell_1$ and $2\ell_2$ larger than n, we get

$$|\partial_\xi^{\alpha'}\partial_x^{\beta'}r'_{1,N}(x,\xi)| \le C_{\rho_0,A,k,n}(C_kA)^{|\alpha'+\beta'|+2N}\alpha'!^k\beta'!^kN!^{2k-1}\langle\xi\rangle_h^{m-|\alpha'|-N(1-\frac{1}{k})}.$$

Now we consider $r'_{2,N}(x,\xi)$. Taking coordinates (y,t) in $\Gamma(x,\xi,\eta,\theta)$, we get $z = y + it\Lambda_\xi$ and $d_{\bar{z}_j}\wedge d_z = 2i\Lambda_{\xi_j}dtdy$, hence we may write (omitting the argument (x,ξ,η,θ) in Λ_ξ)

$$\begin{aligned}
r'_{2,N} &= \lim_{\varepsilon\to 0}\sum_{|\alpha|=N}\frac{1}{\alpha!}\int_0^1(1-\theta)^{N-1}\int_{\mathbb{R}^n_\eta}\chi(\xi,\eta)\sum_{j=1}^n\int_0^1\int_{\mathbb{R}^n_y}e^{-i\eta(y+it\Lambda_\xi)}w^\alpha(\Lambda;x,\xi+\theta\eta)\\
&\quad\times\overline{\partial}_{z_j}\{\{D_x^\alpha p\}(x+y+i(t-1)\Lambda_\xi,\xi)\chi_\varepsilon(y+i(t-1)\Lambda_\xi,\eta)\}2i\Lambda_{\xi_j}dydt\,đ\eta\,d\theta\\
&= \lim_{\varepsilon\to 0}\sum_{|\alpha|=N}\frac{1}{\alpha!}\sum_{j=1}^n\int_0^1\int_0^1(1-\theta)^{N-1}e^{-t\Lambda(x,\xi)}\iint_{\mathbb{R}^{2n}}e^{-iy\eta}e^{t\Lambda(x,\xi+\theta\eta)}w^\alpha(\Lambda;x,\xi+\theta\eta)\\
&\quad\times\chi(\xi,\eta)2i\Lambda_{\xi_j}\overline{\partial}_{z_j}\{\{D_x^\alpha p\}(x+y+i(t-1)\Lambda_\xi,\xi)\chi_\varepsilon(y+i(t-1)\Lambda_\xi,\eta)\}dyđ\eta\,dtd\theta.
\end{aligned}$$

Setting

$$\begin{aligned}
b_{j,\varepsilon,\theta}(x,\xi;y,\eta,t) &= e^{(1-t)\Lambda(x,\xi)}e^{t\Lambda(x,\xi+\theta\eta)}w^\alpha(\Lambda;x,\xi+\theta\eta)\chi(\xi,\eta)\\
&\quad\times 2i\Lambda_{\xi_j}\overline{\partial}_{z_j}\{\{D_x^\alpha p\}(x+y+i(t-1)\Lambda_\xi,\xi)\chi_\varepsilon(y+i(t-1)\Lambda_\xi,\eta)\},
\end{aligned}$$

we have for any $\ell_1,\ell_2\in\mathbb{N}_0$,

$$\begin{aligned}
\partial_\xi^{\alpha'}\partial_x^{\beta'}\{e^{\Lambda(x,\xi)}r'_{2,N}(x,\xi)\} &= \lim_{\varepsilon\to 0}\sum_{|\alpha|=N}\frac{1}{\alpha!}\sum_{j=1}^n\int_0^1\int_0^1(1-\theta)^{N-1}\iint e^{-iy\eta}\\
&\quad\langle y\rangle^{-2\ell_2}\langle D_\eta\rangle^{2\ell_2}\{\langle\eta\rangle^{-2\ell_1}\langle D_y\rangle^{2\ell_1}\partial_\xi^{\alpha'}\partial_x^{\beta'}b_{j,\varepsilon,\theta}(x,\xi;y,\eta,t)\}dyđ\eta\,dtd\theta.
\end{aligned}$$

Since $b_{j,\varepsilon,\theta}$ contains $\overline{\partial}_{z_j}$ we obtain the following estimate

$$\begin{aligned}
|\langle D_\eta\rangle^{2\ell_2}\{\langle\eta\rangle^{-2\ell_1}\langle D_y\rangle^{2\ell_1}\partial_\xi^{\alpha'}\partial_x^{\beta'}b_{j,\varepsilon,\theta}(x,\xi;y,\eta,t)\}| &\le C_{\rho_0,A,k}(C_kA)^{|\alpha'+\beta'|+2(|\alpha|+\ell_1+\ell_2)}\alpha'!^k\beta'!^k\\
&\quad\times\{\alpha!\ell_1!\ell_2!\}^{2k}e^{5\rho_0\langle\xi\rangle_h^{\frac{1}{k}}}\langle\xi\rangle_h^m e^{-c_kA^{-\frac{1}{k}}\langle\xi\rangle_h^{-\frac{1}{k}}}\langle\eta\rangle^{-2\ell_1},
\end{aligned}$$

for every $x,y\in\mathbb{R}^n$, $x,\xi\in$ supp $\chi(\xi,\eta)$, $t,\theta,\varepsilon\in(0,1)$ and $j=1,\ldots,n$. Hence

$$\begin{aligned}
|\partial_\xi^{\alpha'}\partial_x^{\beta'}\{e^{\Lambda(x,\xi)}r'_{2,N}(x,\xi)\}| &\le C_{\rho_0,A,k,n}(C_kA)^{|\alpha'+\beta'|+2(N+\ell_1+\ell_2)}\alpha'!^k\beta'!^kN!^{2k-1}\\
&\quad\times\{\ell_1!\ell_2!\}^{2k}e^{5\rho_0\langle\xi\rangle_h^{\frac{1}{k}}}\langle\xi\rangle_h^m e^{-c_kA^{-\frac{1}{k}}\langle\xi\rangle_h^{-\frac{1}{k}}}\iint\langle y\rangle^{-2\ell_2}\langle\eta\rangle^{-2\ell_1}dyđ\eta.
\end{aligned}$$

Recalling that $\rho_0 \leq \delta A^{-\frac{1}{k}}$ and choosing ℓ_1, ℓ_2 larger than $n/2$,

$$|\partial_\xi^{\alpha'}\partial_x^{\beta'}\{e^{\Lambda(x,\xi)}r'_{2,N}(x,\xi)\}| \leq C_{\rho_0,A,k,n,m}(C_kA)^{|\alpha'+\beta'|+2N}\alpha'!^k\beta'!^kN!^{2k-1}e^{(5\delta-\frac{c_k}{2})A^{-\frac{1}{k}}\langle\xi\rangle_h^{\frac{1}{k}}}.$$

To finish we estimate $r''_N(x,\xi)$. We recall that $\langle\eta\rangle \geq 4^{-1}\langle\xi\rangle_h$ on the support of the function $(1-\chi)(\xi,\eta)$. Integrating by parts we get

$$\begin{aligned} r''_N(x,\xi) = \lim_{\varepsilon\to 0}\sum_{|\alpha|=N}\frac{1}{\alpha!}\iint e^{-i\eta y}\int_0^1(1-\theta)^{N-1}\langle y\rangle^{-2\ell_2} \\ \times \underbrace{\langle D_\eta\rangle^{2\ell_2}\left(\langle\eta\rangle^{-2\ell_1}\langle D_y\rangle^{2\ell_1}\left[\partial_\xi^\alpha e^{\Lambda(x,\xi+\theta\eta)}\{D_x^\alpha p\}(x+y,\xi)(1-\chi)(\xi,\eta)\chi_\varepsilon(y,\eta)\right]\right)}_{=:c_{\varepsilon,\theta,\ell_1,\ell_2}(x,\xi;y,\eta)} \\ \times d\theta\, dy đ\eta. \end{aligned}$$

Noticing that

$$\begin{aligned} |\partial_\xi^\alpha\partial_x^\beta\partial_\eta^\gamma e^{\Lambda(x,\xi+\theta\eta)}| &\leq \sum_{j=1}^{|\alpha+\beta+\gamma|}\frac{e^{\Lambda(x,\xi+\theta\eta)}}{j!}\sum_{(\beta_1,\alpha_1,\gamma_1)+\cdots+(\beta_j,\alpha_j,\gamma_j)=(\alpha,\beta,\gamma)}\frac{\beta!\alpha!\gamma!}{\beta_1!\alpha_1!\gamma_1!\ldots\beta_j!\alpha_j!\gamma_j!} \\ &\quad\times\prod_{\ell=1}^j\{\partial_\xi^{\alpha_\ell+\gamma_\ell}\partial_x^{\beta_\ell}\Lambda\}(x,\xi+\theta\eta) \\ &\leq (C_kA)^{|\beta+\alpha+\gamma|}\{\alpha!\beta!\gamma!\}^ke^{\Lambda(x,\xi+\theta\eta)}\sum_{j=1}^{|\alpha+\beta+\gamma|}\frac{\rho_0^j\langle\xi+\theta\eta\rangle_h^{\frac{j}{k}}}{j!} \\ &\leq (C_kA)^{|\beta+\alpha+\gamma|}\{\alpha!\beta!\gamma!\}^ke^{2\rho_0\langle\xi+\theta\eta\rangle_h^{\frac{1}{k}}} \\ &\leq (C_kA)^{|\beta+\alpha+\gamma|}\{\alpha!\beta!\gamma!\}^ke^{2\rho_0(\langle\xi\rangle_h^{\frac{1}{k}}+\langle\eta\rangle^{\frac{1}{k}})}, \end{aligned}$$

we easily get

$$\begin{aligned} |\partial_\xi^{\alpha'}\partial_x^{\beta'}c_{\varepsilon,\theta,\ell_1,\ell_2}(x,\xi;y,\eta)| \leq C_{A,k}(C_kA)^{|\alpha'+\beta'|+2(|\alpha|+\ell_1+\ell_2)}\alpha'!^k\beta'!^k\{\alpha!\ell_1!\ell_2!\}^{2k} \\ \times\langle\xi\rangle_h^me^{2\rho_0(\langle\xi\rangle_h^{\frac{1}{k}}+\langle\eta\rangle^{\frac{1}{k}})}\langle\eta\rangle^{-2\ell_1}, \end{aligned}$$

hence

$$\begin{aligned} |\partial_\xi^{\alpha'}\partial_x^{\beta'}r''_N(x,\xi)| \leq C_{A,k}(C_kA)^{|\alpha'+\beta'|+2N}\alpha'!^k\beta'!^kN!^{2k-1}\langle\xi\rangle_h^me^{2\rho_0\langle\xi\rangle_h^{\frac{1}{k}}} \\ \times\iint(C_kA)^{2(\ell_1+\ell_2)}\langle y\rangle^{-2\ell_2}\langle\eta\rangle^{-2\ell_1}e^{2\rho_0\langle\eta\rangle^{\frac{1}{k}}}dy đ\eta. \end{aligned}$$

Taking ℓ_2 larger than $n/2$ and noticing that the above inequality holds for every ℓ_1 we obtain

$$|\partial_\xi^{\alpha'}\partial_x^{\beta'}r''_N(x,\xi)| \leq C_{A,k,n}(C_kA)^{|\alpha'+\beta'|+2N}\alpha'!^k\beta'!^kN!^{2k-1}\langle\xi\rangle_h^me^{2\rho_0\langle\xi\rangle_h^{\frac{1}{k}}}$$

$$\times \int e^{-2c_k A^{-\frac{1}{k}}\langle\eta\rangle^{\frac{1}{k}}} e^{2\rho_0\langle\eta\rangle^{\frac{1}{k}}}\, đ\eta.$$

Since $\langle\eta\rangle \geq 4^{-1}\langle\xi\rangle_h$ on the support of $(1-\chi)(\xi,\eta)$ and we are assuming $\rho \leq \delta A^{-\frac{1}{k}}$, we conclude that

$$|\partial_\xi^{\alpha'}\partial_x^{\beta'} r''_N(x,\xi)| \leq C_{A,k,n}(C_kA)^{|\alpha'+\beta'|+2N}\alpha'!^k\beta'!^kN!^{2k-1}\langle\xi\rangle_h^m e^{(2\delta-4^{-\frac{1}{k}}c_k)A^{-\frac{1}{k}}\langle\xi\rangle_h^{\frac{1}{k}}}$$
$$\times \int e^{(2\delta-c_k)A^{-\frac{1}{k}}\langle\eta\rangle^{\frac{1}{k}}}\, đ\eta.$$

Therefore, possibly shrinking $\delta > 0$, we get

$$|\partial_\xi^{\alpha'}\partial_x^{\beta'} r''_N(x,\xi)| \leq C_{A,k,n,m}(C_kA)^{|\alpha'+\beta'|+2N}\alpha'!^k\beta'!^kN!^{2k-1}e^{-\tilde{\delta}A^{-\frac{1}{k}}\langle\xi\rangle_h^{\frac{1}{k}}}.$$

□

Proposition 3.5. *Under the assumptions of Theorem 3.2, there exist* $\delta > 0$ *and* $h_0(A) \geq 1$ *such that if* $h \geq h_0$ *and* $\rho_0 \leq \delta A^{-\frac{1}{k}}$ *we may write the product* $op(pe^{\Lambda}) \circ \{e^{-\Lambda}(x,D)\}^*$ *as follows*

$$op(pe^{\Lambda}) \circ \{e^{-\Lambda}(x,D)\}^* = s_{N'}(x,D) + q_{N'}(x,D) + r_\infty(x,D),$$

where

$$s_{N'}(x,\xi) = \sum_{|\alpha|<N'} \frac{1}{\alpha!}\partial_\xi^\alpha\{e^{\Lambda(x,\xi)}p(x,\xi)D_x^\alpha e^{-\Lambda(x,\xi)}\}, \tag{3.18}$$

$$|\partial_\xi^\alpha\partial_x^\beta q_{N'}(x,\xi)| \leq C_{\rho_0,A,k}(C_kA)^{|\alpha+\beta|+2N}\alpha!^k\beta!^kN'!^{2k-1}\langle\xi\rangle_h^{m-(1-\frac{1}{k})N'-|\alpha|}, \tag{3.19}$$

$$|\partial_\xi^\alpha\partial_x^\beta r_\infty(x,\xi)| \leq C_{\rho_0,A,k}(C_kA)^{|\alpha+\beta|+2N}\alpha!^k\beta!^kN'!^{2k-1}e^{-c_kA^{-\frac{1}{k}}\langle\xi\rangle_h^{\frac{1}{k}}}. \tag{3.20}$$

Proof. We have

$$\{e^{-\Lambda}\}^*u(x) = \iint e^{i\xi(x-y)}e^{-\Lambda(y,\xi)}u(y)dyđ\xi$$
$$= \int e^{i\xi x}\int e^{-i\xi y}e^{-\Lambda(y,\xi)}u(y)dy\, đ\xi,$$

which implies

$$\mathrm{op}(pe^{\Lambda}) \circ \{e^{-\Lambda}\}^*u(x) = \iint e^{i\xi(x-y)}e^{\Lambda(x,\xi)-\Lambda(y,\xi)}p(x,\xi)u(y)dyđ\xi.$$

In this way the symbol $\sigma(x,\xi)$ of the composition $\mathrm{op}(pe^{\Lambda}) \circ \{e^{-\Lambda}\}^*$ is given by

$$\sigma(x,\xi) = Os - \iint e^{-i\eta y}e^{\Lambda(x,\xi+\eta)-\Lambda(x+y,\xi+\eta)}p(x,\xi+\eta)dyđ\eta.$$

By Taylor's formula and (3.15) we obtain

$$\begin{aligned}\sigma(x,\xi) &= \sum_{|\alpha'|<N'} \frac{1}{\alpha!}\partial_\xi^\alpha\{e^{\Lambda(x,\xi)}p(x,\xi)D_x^\alpha e^{-\Lambda(x,\xi)}\} \\ &+ N' \sum_{|\alpha|=N'} \frac{1}{\alpha!}\int_0^1 (1-\theta)^{N'-1} \\ &\times Os - \iint e^{-iy\eta}\partial_\xi^\alpha\{e^{\Lambda(x,\xi+\theta\eta)}p(x,\xi+\theta\eta)D_x^\alpha e^{-\Lambda(x+y,\xi+\theta\eta)}\}dy đ\eta\, d\theta \\ &= s_{N'}(x,\xi) + r_{N'}(x,\xi).\end{aligned}$$

By Leibniz formula we have

$$\begin{aligned}\partial_\xi^\alpha\{e^{\Lambda(x,\xi+\theta\eta)}p(x,\xi+\theta\eta)D_x^\alpha e^{-\Lambda(x+y,\xi+\theta\eta)}\} &= \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} \\ &\times \partial_\xi^{\alpha_1}e^{\Lambda(x,\xi+\theta\eta)}\partial_\xi^{\alpha_2}p(x,\xi+\theta\eta)\partial_\xi^{\alpha_3}D_x^\alpha e^{-\Lambda(x+y,\xi+\theta\eta)} \\ &= e^{\Lambda(x,\xi+\theta\eta)-\Lambda(x+y,\xi+\theta\eta)}f_\alpha(x,y+x,\xi+\theta\eta),\end{aligned}$$

where

$$f_\alpha(x,y,\xi) := \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} w^{\alpha_1}(\Lambda;x,\xi)\partial_\xi^{\alpha_2}p(x,\xi)w_\alpha^{\alpha_3}(-\Lambda;y,\xi).$$

Performing the change of variables $y \mapsto y-x$ and $\eta \mapsto \theta^{-1}\{\eta-\xi\}$ we now obtain

$$\begin{aligned}r_{N'}(x,\xi) &= N' \sum_{|\alpha|=N'} \frac{1}{\alpha!}\int_0^1 (1-\theta)^{N'-1} \\ &\times Os - \iint e^{-i(y-x)(\frac{\eta-\xi}{\theta})}e^{\Lambda(x,\eta)-\Lambda(y,\eta)}f_\alpha(x,y,\eta)\theta^{-n}dy đ\eta\, d\theta.\end{aligned}$$

Writing

$$\Lambda(y,\eta)-\Lambda(x,\eta) = (y-x)\cdot \underbrace{\int_0^1 \nabla_x\Lambda(x+s(y-x),\eta)ds}_{=:\Lambda_x(x,y,\eta)},$$

we get

$$\begin{aligned}r_{N'}(x,\xi) &= \lim_{\varepsilon\to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!}\int_0^1 (1-\theta)^{N'-1}\int e^{i(y-x)\frac{\xi}{\theta}}\int e^{-i(\frac{y-x}{\theta})(\eta-i\theta\Lambda_x(x,y,\eta))} \\ &\times f_\alpha(x,y,\eta)\chi_\varepsilon(y,\eta)\theta^{-n}đ\eta\, dy\, d\theta.\end{aligned}$$

Let $\gamma(x,y) = \{\zeta\in\mathbb{C}^n : \exists\,\eta\in\mathbb{R}^n, \zeta = \eta - i\theta\Lambda_x(x,y,\eta)\}$ and

$$\Gamma(x,y) = \{\zeta : \exists\,\eta\in\mathbb{R}^n, \exists t\in[0,1], \zeta = \eta - it\theta\Lambda_x(x,y,\eta)\}.$$

It follows from [34, Lemma 5.4] (page 23) that there exists $h_0(A) \geq 1$ such that if $h \geq h_0(A)$, then we can find a smooth function $\varXi(x, y, \zeta)$ (with x and y as parameters) satisfying

$$\begin{cases} \zeta = \varXi(x, y; \zeta) - i\theta\Lambda_x(x, y, \varXi(x, y; \zeta)), \ \forall \zeta \in \Gamma(x, y), \\ \varXi(x, y; \eta + i\theta\Lambda_x(x, y, \eta)) = \eta, \ \forall \eta \in \mathbb{R}^n \end{cases}$$

and the following estimates

$$|\partial_x^\beta \partial_y^\delta \partial_{Re\,\zeta}^\alpha \partial_{Im\,\zeta}^\lambda \varXi(x, y, ; \zeta)| \leq C_{\rho_0, A}(CA^2)^{|\alpha+\lambda+\beta+\delta|}\{\alpha!\beta!\lambda!\delta!\}^k \langle Re\,\zeta\rangle_h^{\frac{1}{k}-|\alpha+\lambda|}, \tag{3.21}$$

$$|\overline{\partial}_{\zeta_j} \partial_x^\beta \partial_y^\delta \partial_{Re\,\zeta}^\alpha \partial_{Im\,\zeta}^\lambda \varXi(x, y, ; \zeta)| \leq C_{\rho_0, A}(CA^2)^{|\alpha+\lambda+\beta+\delta|}\{\alpha!\beta!\lambda!\delta!\}^k \langle Re\,\zeta\rangle_h^{\frac{1}{k}-|\alpha+\lambda|} e^{-c_k A^{-\frac{1}{k}} \langle Re\,\zeta\rangle_h^{\frac{1}{k}}}. \tag{3.22}$$

Hence we may write

$$\begin{aligned} r_{N'}(x, \xi) = \lim_{\varepsilon \to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N'-1} \int e^{i(y-x)\frac{\xi}{\theta}} \\ \times \int_{\gamma(x,y)} e^{-i(\frac{y-x}{\theta})\zeta} \underbrace{J(x, y; \zeta) f_\alpha(x, y, \varXi(x, y; \zeta))\chi_\varepsilon(y, \varXi(x, y; \zeta))}_{=:H_\varepsilon(x,y,\zeta)} \theta^{-n} đ\zeta\, dy\, d\theta, \end{aligned}$$

where $J(x, y, \zeta) = Det\{\frac{\partial \varXi}{\partial \zeta}(x, y, \zeta)\}$. Stokes formula implies

$$\begin{aligned} r_{N'}(x, \xi) &= \lim_{\varepsilon \to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N'-1} \int e^{i(y-x)\frac{\xi}{\theta}} \int_{\mathbb{R}^n_\eta} e^{-i(\frac{y-x}{\theta})\eta} H_\varepsilon(x, y, \eta)\theta^{-n} đ\eta\, dy\, d\theta \\ &+ \lim_{\varepsilon \to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N'-1} \int e^{i(y-x)\frac{\xi}{\theta}} \\ &\times \sum_{j=1}^n \iint_{\Gamma(x,y)} e^{-i(\frac{y-x}{\theta})\zeta} \overline{\partial}_{\zeta_j} H_\varepsilon(x, y, \zeta)\theta^{-n} d\bar{\zeta}_j \wedge đ\zeta\, dy\, d\theta. \end{aligned}$$

Take $\chi(t) \in C_c^\infty(\mathbb{R})$ such that

$$|\partial_t^j \chi(t)| \leq C^{j+1} j!^{k'} \ (1 < k' < k), \quad \chi(t) = \begin{cases} 1, & |t| \leq \frac{1}{4}, \\ 0, & |t| \geq \frac{1}{2}, \end{cases}$$

and set $\chi(\xi, \eta) = \chi(\langle \eta \rangle \langle \xi \rangle_h^{-1})$, $\xi, \eta \in \mathbb{R}^n$. Then

$$\begin{aligned} r_{N'}(x, \xi) &= \lim_{\varepsilon \to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N'-1} \iint e^{-iy\eta} H_\varepsilon(x, x+y, \xi+\theta\eta)\chi(\xi, \eta) đ\eta\, dy\, d\theta \\ &+ \lim_{\varepsilon \to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N'-1} \iint e^{-iy\eta} H_\varepsilon(x, x+y, \xi+\theta\eta)(1-\chi)(\xi, \eta) đ\eta\, dy\, d\theta \end{aligned}$$

$$
\begin{aligned}
&+\lim_{\varepsilon\to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!}\int_0^1 (1-\theta)^{N'-1}\int e^{i(y-x)\frac{\xi}{\theta}} \\
&\times \sum_{j=1}^{n} \iint_{\Gamma(x,y)} e^{-i(\frac{y-x}{\theta})\zeta}\overline{\partial}_{\zeta_j} H_\varepsilon(x,y,\zeta)\theta^{-n} d\bar{\zeta}_j \wedge đ\zeta\, dy\, d\theta \\
&= r_{1,N'}(x,\xi) + r_{2,N'}(x,\xi) + r_{3,N'}(x,\xi).
\end{aligned}
$$

Now we estimate the three symbols above. Since the following computations are quite similar to the ones made in Proposition 3.4, we only explain the main ideas.

We start with $r_{1,N'}(x,\xi)$. We observe that since we are assuming (3.11), we have

$$
\begin{aligned}
|\partial_\xi^{\alpha'}\partial_x^{\beta'}\partial_\eta^{\rho}\partial_y^{\lambda} f_\alpha(x,x+y,\xi+\theta\eta)| &\leq C_{k,A,\rho_0}(C_k A)^{|\alpha'+\beta'+\rho+\lambda|+2|\alpha|} \\
&\times \{\alpha'!\beta'!\rho!\lambda!\}^k \alpha!^{2k-1}\langle\xi\rangle^{m-|\alpha|(1-\frac{1}{k})-|\alpha'+\rho|},
\end{aligned}
$$

for every $(\xi,\eta)\in$ supp$\{\chi(\xi,\eta)\}$, and henceforth (in view of (3.21))

$$
\begin{aligned}
|\partial_\xi^{\alpha'}\partial_x^{\beta'}\partial_\eta^{\rho}\partial_y^{\lambda} H_\varepsilon(x,x+y,\xi+\theta\eta)| &\leq C_{k,A,\rho_0}(C_k A)^{|\alpha'+\beta'+\rho+\lambda|+2|\alpha|} \\
&\times \{\alpha'!\beta'!\rho!\lambda!\}^k \alpha!^{2k-1}\langle\xi\rangle^{m-|\alpha|(1-\frac{1}{k})-|\alpha'+\rho|},
\end{aligned}
$$

so, integrating by parts, we can obtain estimate (3.19) for $r_{1,N'}(x,\xi)$.

To deal with $r_{2,N'}(x,\xi)$, notice that thanks to (3.11) we obtain

$$
|\partial_\xi^{\alpha'}\partial_x^{\beta'}\partial_\eta^{\rho}\partial_y^{\lambda} f_\alpha(x,x+y,\xi+\theta\eta)| \leq C_{k,A,\rho_0}(C_k A)^{|\alpha'+\beta'+\rho+\lambda|+2|\alpha|}\{\alpha'!\beta'!\rho!\lambda!\}^k \alpha!^{2k-1}\langle\xi+\theta\eta\rangle^{m}
$$

and therefore

$$
|\partial_\xi^{\alpha'}\partial_x^{\beta'}\partial_\eta^{\rho}\partial_y^{\lambda} H_\varepsilon(x,x+y,\xi+\theta\eta)| \leq C_{k,A,\rho_0}(C_k A)^{|\alpha'+\beta'+\rho+\lambda|+2|\alpha|}\{\alpha'!\beta'!\rho!\lambda!\}^k \alpha!^{2k-1}\langle\xi+\theta\eta\rangle^{m}.
$$

Integrating by parts as in the proof of Proposition 3.4 and using that $4\langle\eta\rangle \geq \langle\xi\rangle_h$ on the support of $(1-\chi)(\xi,\eta)$, we can obtain an exponential decay like in (3.20) also for $r_{2,N'}(x,\xi)$.

Finally, we explain how to treat $r_{3,N'}(x,\xi)$. Consider coordinates

$$
\varphi(x,y;t,\eta) = \eta - it\theta\Lambda_x(x,y,\eta)(=\zeta),
$$

for every $t\in(0,1)$ and $\eta\in\mathbb{R}^n$, in $\Gamma(x,y)$, then $d\bar{\zeta}_j\wedge đ\zeta = J_j(x,y,t,\eta)dtđ\eta$, where

$$
J_j(x,y,t,\eta) = Det\left\{\frac{\partial}{\partial(t,\eta)}(\eta_j + it\theta\Lambda_{x_j}(x,y,\eta), \eta - it\theta\Lambda_x(x,y,\eta))\right\}.
$$

Hence

$$
r_{2,N'}(x,\xi) = \lim_{\varepsilon\to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!}\sum_{j=1}^{n}\int_0^1 (1-\theta)^{N'-1}\int e^{i(y-x)\frac{\xi}{\theta}} \int_0^1\int_{\mathbb{R}^n_\eta} e^{-i(\frac{y-x}{\theta})(\eta - it\theta\Lambda_x(x,y,\eta))}
$$

$$\times J_j(x,y,t,\eta)\overline{\partial}_{\zeta_j} H_\varepsilon(x,y,\eta - it\theta\Lambda_x(x,y,\eta))\theta^{-n} dt đ\eta\, dy\, d\theta$$

$$= \lim_{\varepsilon\to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!} \sum_{j=1}^{n} \int_0^1 (1-\theta)^{N'-1} \int_0^1 \iint e^{-iy\eta} e^{-ty\Lambda_x(x,y+x,\xi+\theta\eta)}$$

$$\times J_j(x,y+x,t,\xi+\theta\eta)\overline{\partial}_{\zeta_j} H_\varepsilon(x,y+x,\xi+\theta\eta - ty\theta\Lambda_x(x,y+x,\xi+\theta\eta)) đ\eta dy\, dt\, d\theta.$$

Now we split $r_{2,N'}(x,\xi)$ as

$$r_{2,N'}(x,\xi) = \lim_{\varepsilon\to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!} \sum_{j=1}^{n} \int_0^1 (1-\theta)^{N'-1} \int_0^1 \iint e^{-iy\eta} e^{t(\Lambda(x,\xi+\theta\eta)-\Lambda(x+y,\xi+\theta\eta))} \chi(\xi,\eta)$$

$$\times J_j(x,y+x,t,\xi+\theta\eta)\overline{\partial}_{\zeta_j} H_\varepsilon(x,y+x,\xi+\theta\eta - it\theta\Lambda_x(x,y+x,\xi+\theta\eta)) đ\eta dy\, dt\, d\theta$$

$$+ \lim_{\varepsilon\to 0} N' \sum_{|\alpha|=N'} \frac{1}{\alpha!} \sum_{j=1}^{n} \int_0^1 (1-\theta)^{N'-1} \int_0^1 \iint e^{-iy\eta} e^{t(\Lambda(x,\xi+\theta\eta)-\Lambda(x+y,\xi+\theta\eta))} (1-\chi)(\xi,\eta)$$

$$\times J_j(x,y+x,t,\xi+\theta\eta)\overline{\partial}_{\zeta_j} H_\varepsilon(x,y+x,\xi+\theta\eta - it\theta\Lambda_x(x,y+x,\xi+\theta\eta)) đ\eta dy\, dt\, d\theta$$

$$= r'_{2,N'}(x,\xi) + r''_{2,N'}(x,\xi).$$

Using that $\langle \xi+\theta\eta\rangle_h$ is comparable with $\langle\xi\rangle_h$ on the support of $\chi(\xi,\eta)$, the exponential decay coming from $\overline{\partial}_{\zeta_j} H_\varepsilon$ and the hypothesis $\rho_0 \le \delta A^{-\frac{1}{k}}$, we obtain an estimate like in (3.20) for $r'_{2,N'}$. Integrating by parts and taking into account the fact that $4\langle\eta\rangle \ge \langle\xi\rangle_h$ on the support of $(1-\chi)(\xi,\eta)$, we obtain that also $r''_{2,N'}$ satisfies the estimate (3.20). □

Remark 3.7. *Shrinking $\delta > 0$ if necessary, we may conclude that*

$$r_\infty(x,D) \circ \{e^{-\Lambda}(x,D)\}^* = \tilde r_\infty(x,D),$$

where $\tilde r_\infty(x,D)$ is still a regularizing operator, that is, it satisfies an estimate like in (3.20).

3.5.2 Conjugation of iP by $e^{\tilde\Lambda}$

Before making the conjugation, let us make some remarks. By Lemmas 3.1 and 3.2 we get

$$|\partial_\xi^\alpha \partial_x^\beta \tilde\Lambda(x,\xi)| \le \begin{cases} C_{\tilde\Lambda}^{|\alpha+\beta|+1} \alpha!^\mu \beta!^\mu \langle\xi\rangle_h^{2(1-\sigma)-\alpha}, \\ C_{\tilde\Lambda}^{|\alpha+\beta|+1} \alpha!^\mu \beta!^\mu \langle\xi\rangle_h^{-\alpha}, \text{ if } \beta > 0, \end{cases}$$

where $C_{\tilde\Lambda}$ is a constant depending only on $M_2, M_1, C_w, C_\psi, \mu, \sigma$. Moreover, since we are assuming $2(1-\sigma) < \frac{1}{\theta}$ we also get

$$|\partial_\xi^\alpha \partial_x^\beta \tilde\Lambda(x,\xi)| \le C_{\tilde\Lambda}^{|\alpha+\beta|+1} \alpha!^\mu \beta!^\mu \langle\xi\rangle_h^{2(1-\sigma)-\alpha}$$

$$
\begin{aligned}
&= C_{\tilde\Lambda}^{|\alpha+\beta|+1}\alpha!^{\mu}\beta!^{\mu}\langle\xi\rangle_h^{\frac{1}{\theta}-\alpha}\langle\xi\rangle_h^{2(1-\sigma)-\frac{1}{\theta}} \\
&\leq C_{\tilde\Lambda}h^{\frac{1}{\theta}-2(1-\sigma)}C_{\tilde\Lambda}^{|\alpha+\beta|}\alpha!^{\mu}\beta!^{\mu}\langle\xi\rangle_h^{\frac{1}{\theta}-\alpha},
\end{aligned}
$$

therefore

$$\rho_0(\tilde\Lambda) := h^{\frac{1}{\theta}-2(1-\sigma)}C_{\tilde\Lambda}$$

can be assumed as small as we want, provided that $h > h_0(M_2, M_1, C_{\tilde\Lambda}, \theta, \sigma)$. Hence we may use Theorem 3.2 to compute $e^{\tilde\Lambda}(x,D)\circ(iP)\circ\{e^{\tilde\Lambda}(x,D)\}^{-1}$.

First we note that $e^{\tilde\Lambda}(x,D)\circ\partial_t\circ\{e^{\tilde\Lambda}(x,D)\}^{-1} = \partial_t$, because $\tilde\Lambda(x,\xi) = \lambda_2(x,\xi) + \lambda_1(x,\xi)$ is independent of t.

- Conjugation of $ia_3(t,D)$: Since a_3 does no depend on x we may use Proposition 3.5 to compute (omitting (t,x,D) in the notation)

$$e^{\tilde\Lambda}\circ ia_3\circ\{e^{-\tilde\Lambda}\}^* = ia_3 + \partial_\xi\{ia_3D_x(-\tilde\Lambda)\} + \frac{1}{2}\partial_\xi^2\{ia_3[D_x^2(-\tilde\Lambda) + (D_x\tilde\Lambda)^2]\} + q_3 + r_\infty.$$

Since $x-$derivatives kill the $\xi-$growth given by the integrals of $\tilde\Lambda$, we can conclude that q_3 has order zero. Composing with the Neumman series we get from (3.10)

$$
\begin{aligned}
e^{\tilde\Lambda}(ia_3)\{e^{\tilde\Lambda}\}^{-1} &= \text{op}\left(ia_3 - \partial_\xi(a_3\partial_x\tilde\Lambda) + \frac{i}{2}\partial_\xi^2[a_3(\partial_x^2\tilde\Lambda - (\partial_x\tilde\Lambda)^2)] + q_3 + r_\infty\right) \\
&\quad\circ\text{op}\left(1 - i\partial_\xi\partial_x\tilde\Lambda - \frac{1}{2}\partial_\xi^2(\partial_x^2\tilde\Lambda - [\partial_x\tilde\Lambda]^2) - [\partial_\xi\partial_x\tilde\Lambda]^2 + q_{-3}\right) \\
&= ia_3 - \partial_\xi(a_3\partial_x\tilde\Lambda) + \frac{i}{2}\partial_\xi^2\{a_3(\partial_x^2\tilde\Lambda - \{\partial_x\tilde\Lambda\}^2)\} + a_3\partial_\xi\partial_x\tilde\Lambda - i\partial_\xi a_3\partial_\xi\partial_x^2\tilde\Lambda \\
&\quad + i\partial_\xi(a_3\partial_x\tilde\Lambda)\partial_\xi\partial_x\tilde\Lambda - \frac{i}{2}a_3\{\partial_\xi^2(\partial_x^2\tilde\Lambda + [\partial_x\tilde\Lambda]^2) + 2[\partial_\xi\partial_x\tilde\Lambda]^2\} + r_0 + r \\
&= ia_3 - \partial_\xi a_3\partial_x\tilde\Lambda + \frac{i}{2}\partial_\xi^2\{a_3[\partial_x^2\tilde\Lambda - (\partial_x\tilde\Lambda)^2]\} - i\partial_\xi a_3\partial_\xi\partial_x^2\tilde\Lambda \\
&\quad + i\partial_\xi(a_3\partial_x\tilde\Lambda)\partial_\xi\partial_x\tilde\Lambda - \frac{i}{2}a_3\{\partial_\xi^2(\partial_x^2\tilde\Lambda + [\partial_x\tilde\Lambda]^2) + 2(\partial_\xi\partial_x\tilde\Lambda)^2\} + r_0 + r,
\end{aligned}
$$

where $r_0 \in C([0,T]; S_\delta^0(\mathbb{R}^2))$ and r is a new regularizing term. Writing $\tilde\Lambda = \lambda_2 + \lambda_1$ and observing that $D_x\lambda_1$ has order -1 we get

$$
\begin{aligned}
e^{\tilde\Lambda}(ia_3)\{e^{\tilde\Lambda}\}^{-1} &= ia_3 - \partial_\xi a_3\partial_x\lambda_2 - \partial_\xi a_3\partial_x\lambda_1 + \frac{i}{2}\partial_\xi^2\{a_3(\partial_x^2\lambda_2 - \{\partial_x\lambda_2\}^2)\} - i\partial_\xi a_3\partial_\xi\partial_x^2\lambda_2 \\
&\quad + i\partial_\xi(a_3\partial_x\lambda_2)\partial_\xi\partial_x\lambda_2 - \frac{i}{2}a_3\{\partial_\xi^2(\partial_x^2\lambda_2 + [\partial_x\lambda_2]^2) + 2[\partial_\xi\partial_x\lambda_2]^2\} + r_0 + r,
\end{aligned}
$$

for a new zero order term r_0. Setting

$$\begin{aligned} d_1(t,x,\xi) = &\frac{1}{2}\partial_\xi^2\{a_3(\partial_x^2\lambda_2 - \{\partial_x\lambda_2\}^2)\} - \partial_\xi a_3 \partial_\xi \partial_x^2 \lambda_2 \\ &+ \partial_\xi(a_3\partial_x\lambda_2)\partial_\xi\partial_x\lambda_2 - \frac{1}{2}a_3\{\partial_\xi^2(\partial_x^2\lambda_2 + [\partial_x\lambda_2]^2) + 2[\partial_\xi\partial_x\lambda_2]^2\} \end{aligned}$$

we may write

$$e^{\tilde{\Lambda}}(ia_3)\{e^{\tilde{\Lambda}}\}^{-1} = ia_3 - \partial_\xi a_3 \partial_x \lambda_2 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 + r_0 + r,$$

where d_1 is a real-valued symbol of order 1 which does not depend on λ_1. Moreover, we have the following estimate

$$|\partial_\xi^\alpha \partial_x^\beta d_1(t,x,\xi)| \leq C_{\lambda_2}^{\alpha+\beta+1}\alpha!^\mu \beta!^\mu \langle\xi\rangle^{1-\alpha};$$

where C_{λ_2} is a constant independent of λ_1.

- Conjugation of $ia_2(t,x,D)$: for $N \in \mathbb{N}$ such that $2 - N(1-\frac{1}{\theta}) \leq 0$, Theorem 3.2 gives

$$\begin{aligned} e^{\tilde{\Lambda}} \circ ia_2(t,x,D) \circ \{e^{-\tilde{\Lambda}}\}^* = ia_2(t,x,D) + \text{op}\underbrace{\left(\sum_{1\leq\alpha+\beta<N} \frac{1}{\alpha!\beta!}\partial_\xi^\alpha\{\partial_\xi^\beta e^{\tilde{\Lambda}} D_x^\beta(ia_2) D_x^\alpha e^{-\tilde{\Lambda}}\}\right)}_{=:(ia_2)_N} \\ + \tilde{r}_0(t,x,D) + \tilde{r}(t,x,D), \end{aligned}$$

where $\tilde{r}_0$ has order zero and $\tilde{r}$ is a $\theta-$regularizing term. By the hypothesis on a_2, we obtain

$$|\partial_\xi^\alpha \partial_x^\beta (ia_2)_N(t,x,\xi)| \leq C_{a_2,\tilde{\Lambda}}^{\alpha+\beta+1}\alpha!^\mu \beta!^{s_0}\langle\xi\rangle^{2-[2\sigma-1]-\alpha}\langle x\rangle^{-\sigma}.$$

Composing with the Neumann series and using the fact that $\partial_x\lambda_1$ has order -1 we get

$$\begin{aligned} e^{\tilde{\Lambda}} \circ ia_2 \circ \{e^{\tilde{\Lambda}}\}^{-1} &= (ia_2 + (ia_2)_N + \tilde{r}_0 + \tilde{R}) \circ (I - i\partial_\xi\partial_x\lambda_2 + q_{-2}) \\ &= ia_2 + (ia_2)_N + a_2 \circ \partial_\xi\partial_x\lambda_2 - i(ia_2)_N \circ \partial_\xi\partial_x\lambda_2 + r_0 + r \\ &= ia_2 + \underbrace{(ia_2)_N - i(ia_2)_N\partial_\xi\partial_x\lambda_2}_{=:(ia_2)_{\tilde{\Lambda}}} + a_2\partial_\xi\partial_x\lambda_2 + r_0 + r, \end{aligned}$$

where r_0 has order zero, r is a $\theta-$regularizing term and $(a_2)_{\tilde{\Lambda}}$ satisfies

$$|\partial_\xi^\alpha \partial_x^\beta (ia_2)_{\tilde{\Lambda}}(t,x,\xi)| \leq C_{a_2,\tilde{\Lambda}}^{\alpha+\beta+1}\alpha!^\mu \beta!^{s_0}\langle\xi\rangle^{2-[2\sigma-1]-\alpha}\langle x\rangle^{-\sigma},$$

in particular

$$|(ia_2)_{\tilde{\Lambda}}(t,x,\xi)| \leq C_{a_2,\tilde{\Lambda}}\langle\xi\rangle_h^{2-[2\sigma-1]}\langle x\rangle^{-\sigma}. \tag{3.23}$$

- Conjugation of $ia_1(t,x,D)$:

$$\begin{aligned} e^{\tilde{\Lambda}} \circ (ia_1)(t,x,D) \circ \{e^{\tilde{\Lambda}}\}^{-1} &= (ia_1 + (ia_1)_{\tilde{\Lambda}} + r_1)(t,x,D) \sum_{j\geq 0}(-r)^j \\ &= ia_1(t,x,D) + (ia_1)_{\tilde{\Lambda}}(t,x,D) + r_0(t,x,D) + r(t,x,D), \end{aligned}$$

where r_0 has order zero, r is a $\theta-$regularizing operator and

$$(ia_1)_{\tilde{\Lambda}} \sim \sum_{|\alpha+\beta|\geq 1} \frac{1}{\alpha!\beta!}\partial_\xi^\alpha\{\partial_\xi^\beta e^{\tilde{\Lambda}} D_x^\beta (ia_1) D_x^\alpha e^{-\tilde{\Lambda}}\} \text{ in } S^{2(1-\sigma)}_{\mu,s_0}. \tag{3.24}$$

- Conjugation of $ia_0(t,x,D)$: $e^{\tilde{\Lambda}} \circ (ia_0)(t,x,D) \circ \{e^{\tilde{\Lambda}}\}^{-1} = r_0(t,x,D) + r(t,x,D)$, where r_0 has order zero and r is a $\theta-$regularizing term.

Gathering all the previous computations we may write (omitting (t,x,D) in the notation)

$$\begin{aligned} e^{\tilde{\Lambda}}(iP)\{e^{\tilde{\Lambda}}\}^{-1} &= \partial_t + ia_3 - \partial_\xi a_3 \partial_x \lambda_2 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 \\ &+ ia_2 + (ia_2)_{\tilde{\Lambda}} + a_2 \partial_\xi \partial_x \lambda_2 + ia_1 + (ia_1)_{\tilde{\Lambda}} + r_0 + r, \end{aligned}$$

where $d_1 \in S^1_{1,s_0}$, d_1 is real-valued, d_1 does not depend of λ_1, $(ia_2)_{\tilde{\Lambda}}$ satisfies (3.23), $(ia_1)_{\tilde{\Lambda}}$ satisfies (3.24), $r_0 \in C([0,T]; S^0_{\delta,s_0}(\mathbb{R}^2))$ and r is a $\theta-$regularizing operator.

3.5.3 Conjugation of $e^{\tilde{\Lambda}}(iP)\{e^{\tilde{\Lambda}}\}^{-1}$ by $e^{k(t)\langle D\rangle_h^{\frac{1}{\theta}}}$

Let us recall that the function $k(t)$ satisfies $k \in C^1([0,T];\mathbb{R})$, $k'(t) \leq 0$ and $k(t) > 0$ for every $t \in [0,T]$. We shall use the following lemma (which is a special case of Theorem 3.2).

Lemma 3.4. *Let* $a \in C([0,T], S^m_{\mu,\theta}(\mathbb{R}^2; A))$*, where* $1 < \mu \leq \theta$*. There exists* $\delta > 0$ *such that if* $k(t) \leq \delta A^{-\frac{1}{\theta}}$ *then*

$$e^{k(t)\langle D\rangle_h^{\frac{1}{\theta}}} \circ a(t,x,D) \circ e^{-k(t)\langle D\rangle_h^{\frac{1}{\theta}}} = a(t,x,D) + b(t,x,D) + r_\infty(t,x,D),$$

where $b \sim \sum_{j\geq 1} \frac{1}{j!}\partial_\xi^j e^{k(t)\langle \xi\rangle_h^{\frac{1}{\theta}}} D_x^j a\, e^{-k(t)\langle \xi\rangle_h^{\frac{1}{\theta}}}$ *in* $S^{m-(1-\frac{1}{\theta})}_{\mu,\theta}(\mathbb{R}^2)$ *and* $r_\infty(t,x,D)$ *is a* $\theta-$*regularizing operator.*

- Conjugation of ∂_t: $e^{k(t)\langle D\rangle_h^{\frac{1}{\theta}}}\, \partial_t\, e^{-k(t)\langle D\rangle_h^{\frac{1}{\theta}}} = \partial_t - k'(t)\langle D\rangle_h^{\frac{1}{\theta}}$.

- Conjugation of $ia_3(t,D)$: since a_3 does not depend of x, we simply have

$$e^{k(t)\langle D\rangle_h^{\frac{1}{\theta}}} \circ ia_3(t,D) \circ e^{-k(t)\langle D\rangle_h^{\frac{1}{\theta}}} = ia_3(t,D).$$

- Conjugation of op$\{ia_2 - \partial_\xi a_3 \partial_x \lambda_2\}$:

$$e^{k(t)\langle D\rangle_h^{\frac{1}{\theta}}} \circ (ia_2 - \partial_\xi a_3 \partial_x \lambda_2)(t,x,D) \circ e^{-k(t)\langle D\rangle_h^{\frac{1}{\theta}}} = ia_2(t,x,D) \\ - \mathrm{op}(\partial_\xi a_3 \partial_x \lambda_2) + (b_{2,k} + r_0 + r)(t,x,D)$$

where r_0 has order zero, r is a $\theta-$regularizing term and $b_{2,k}(t,x,\xi) \in C([0,T]; S^{1+\frac{1}{\theta}}_{\mu,s_0}(\mathbb{R}^2))$,

$$|b_{2,k}(t,x,\xi)| \leq \max\{1, k(t)\} C_{s,\lambda_2} \langle \xi \rangle_h^{1+\frac{1}{\theta}} \langle x \rangle^{-\sigma}, \quad x, \xi \in \mathbb{R}^n. \tag{3.25}$$

- Conjugation of $(ia_2)_{\tilde{\Lambda}}(t,x,D)$:

$$e^{k(t)\langle D\rangle_h^{\frac{1}{\theta}}} \circ (ia_2)_{\tilde{\Lambda}}(t,x,D) \circ e^{-k(t)\langle D\rangle_h^{\frac{1}{\theta}}} = \{(ia_2)_{k,\tilde{\Lambda}} + r_0 + r\}(t,x,D),$$

where r_0 has order zero, r is a $\theta-$regularizing term and $(ia_2)_{k,\tilde{\Lambda}} \in C([0,T]; S^{2-[2\sigma-1]}_{\mu,s_0})$,

$$|\partial_\xi^\alpha \partial_x^\beta (ia_2)_{k,\tilde{\Lambda}}(t,x,\xi)| \leq (\max\{k(t),1\} C_{a_2,\tilde{\Lambda}})^{\alpha+\beta+1} \alpha!^\mu \beta!^{s_0} \langle \xi \rangle^{2-[2\sigma-1]-\alpha} \langle x \rangle^{-\sigma}.$$

In particular

$$|(ia_2)_{k,\tilde{\Lambda}}(t,x,\xi)| \leq \max\{k(t),1\} C_{a_2,\tilde{\Lambda}} \langle \xi \rangle_h^{2-[2\sigma-1]} \langle x \rangle^{-\sigma}. \tag{3.26}$$

- Conjugation of op$\{ia_1 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 + a_2 \partial_\xi \partial_x \lambda_2\}$: we have (omitting (t,x,D) in the notation)

$$e^{k(t)\langle D\rangle_h^{\frac{1}{\theta}}} \circ (ia_1 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 + a_2 \partial_\xi \partial_x \lambda_2) \circ e^{-k(t)\langle D\rangle_h^{\frac{1}{\theta}}} \\ = ia_1 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 + a_2 \partial_\xi \partial_x \lambda_2 + b_{1,k} + r_0 + r,$$

where r_0 has order zero, r is a $\theta-$regularizing term and $b_{1,k}(t,x,\xi) \in C([0,T]; S^{\frac{1}{\theta}}_{\mu,s_0})$,

$$|b_{1,k}(t,x,\xi)| \leq k(t) C_{\tilde{\Lambda}} \langle \xi \rangle_h^{\frac{1}{\theta}}, \quad x \in \mathbb{R}, \xi \in \mathbb{R}. \tag{3.27}$$

- Conjugation of $(ia_1)_{\tilde{\Lambda}}(t,x,D)$:

$$e^{k(t)\langle D\rangle_h^{\frac{1}{\theta}}} \circ (ia_1)_{\tilde{\Lambda}}(t,x,D) \circ e^{-k(t)\langle D\rangle_h^{\frac{1}{\theta}}} = \{(ia_1)_{k,\tilde{\Lambda}} + r_0 + r\}(t,x,D),$$

where r_0 has order zero, r is a $\theta-$regularizing term and $(ia_1)_{k,\tilde{\Lambda}} \in C([0,T]; S^{2(1-\sigma)}_{\mu,s_0})$,

$$|(ia_1)_{k,\tilde{\Lambda}}(t,x,\xi)| \leq C_{\tilde{\Lambda}} \langle \xi \rangle_h^{2(1-\sigma)}, \quad x, \xi \in \mathbb{R}. \tag{3.28}$$

Finally, gathering all the previous computations we obtain the following expression for the conjugated opeartor

$$\begin{aligned} e^{\Lambda} \circ (iP) \circ \{e^{\Lambda}\}^{-1} = {} & \partial_t + ia_3(t,D) \\ & + \mathrm{op}(ia_2 - \partial_\xi a_3 \partial_x \lambda_2 + b_{2,k} + (ia_2)_{k,\tilde\Lambda}) \\ & + \mathrm{op}(ia_1 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 + a_2 \partial_\xi \partial_x \lambda_2) \\ & + \mathrm{op}(-k'(t)\langle \xi \rangle_h^{\frac{1}{\theta}} + b_{1,k} + (ia_1)_{k,\tilde\Lambda}) + (r_0 + r)(t,x,D) \end{aligned}$$

where $b_{2,k}$ satisfies (3.25), $(ia_2)_{k,\tilde\Lambda}$ satisfies (3.26), $b_{1,k}$ satisfies (3.27), $(ia_1)_{k,\tilde\Lambda}$ satisfies (3.28), r_0 has order zero and r is a $\theta-$regularizing term.

3.6 Estimates from Below for the Real Parts

In this Section, we will derive some estimates from below for the real parts of the lower order terms of $(iP)_\Lambda$ and we use them to achieve a well-posedness result for the Cauchy problem (3.5). We start noticing that for $|\xi| > hR_{a_3}$ we have

$$\begin{aligned} -\partial_\xi a_3 \partial_x \lambda_2 &= |\partial_\xi a_3| M_2 \langle x \rangle^{-\sigma} \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^2}\right) \\ &= |\partial_\xi a_3| M_2 \langle x \rangle^{-\sigma} - |\partial_\xi a_3| M_2 \langle x \rangle^{-\sigma} \left[1 - \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^2}\right)\right], \end{aligned}$$

$$\begin{aligned} -\partial_\xi a_3 \partial_x \lambda_1 &= |\partial_\xi a_3| M_1 \langle \xi \rangle_h^{-1} \langle x \rangle^{-\frac{\sigma}{2}} \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^2}\right) \\ &= |\partial_\xi a_3| M_1 \langle \xi \rangle_h^{-1} \langle x \rangle^{-\frac{\sigma}{2}} - |\partial_\xi a_3| M_1 \langle \xi \rangle_h^{-1} \langle x \rangle^{-\frac{\sigma}{2}} \left[1 - \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^2}\right)\right]. \end{aligned}$$

We also observe that

$$-|\partial_\xi a_3| M_2 \langle x \rangle^{-\sigma} \left[1 - \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^2}\right)\right] \geq -2^\sigma C_{a_3} M_2 \langle \xi \rangle_h^{2(1-\sigma)},$$

$$-|\partial_\xi a_3| M_1 \langle \xi \rangle_h^{-1} \langle x \rangle^{-\frac{\sigma}{2}} \left[1 - \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^2}\right)\right] \geq -2^\sigma C_{a_3} M_1 \langle \xi \rangle_h^{1-\sigma},$$

because $\langle x \rangle \geq \frac{1}{2}\langle \xi \rangle_h^2$ on the support of $(1-\psi)(\langle x \rangle \langle \xi \rangle_h^{-2})$.

In this way we may write ($|\xi| > R_{a_3} h$)

$$e^{\Lambda} \circ (iP) \circ \{e^{\Lambda}\}^{-1} = \partial_t + ia_3(t,D) + \tilde a_2(t,x,D) + \tilde a_1(t,x,D) + \tilde a_\theta(t,x,D) + r_0(t,x,D),$$

where r_0 is an operator of order 0 and

$$Re\, \tilde a_2 = -Im\, a_2 + |\partial_\xi a_3| M_2 \langle x \rangle^{-\sigma} + Re\, b_{2,k} + Re\, (ia_2)_{k,\tilde\Lambda},$$

$$Im\,\tilde a_2 = Re\,a_2 + Im\,b_{2,k} + Im\,(a_2)_{k,\tilde\Lambda},$$

$$Re\,\tilde a_1 = -Im\,a_1 + |\partial_\xi a_3| M_1 \langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}} + Re\,a_2 \partial_\xi\partial_x\lambda_2,$$

$$\begin{aligned} Re\,\tilde a_\theta &= -k'(t)\langle\xi\rangle_h^{\frac{1}{\theta}} + Re\,b_{1,k} + Re\,(ia_1)_{k,\tilde\Lambda} \\ &\quad - |\partial_\xi a_3| M_2\langle x\rangle^{-\sigma}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\right] - |\partial_\xi a_3| M_1 \langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\right]. \end{aligned}$$

Now we decompose $iIm\,\tilde a_2$ into its Hermitian and anti-Hermitian part:

$$i\mathrm{Im}\tilde a_2 = \frac{i\mathrm{Im}\tilde a_2 + (i\mathrm{Im}\tilde a_2)^*}{2} + \frac{i\mathrm{Im}\tilde a_2 - (i\mathrm{Im}\tilde a_2)^*}{2} = H_{Im\,\tilde a_2} + A_{Im\,\tilde a_2};$$

we have that $2Re\,\langle A_{Im\,\tilde a_2}u, u\rangle = 0$, while $H_{Im\,\tilde a_2}$ has symbol

$$\sum_{\alpha\geq 1}\frac{i}{2\alpha!}\partial_\xi^\alpha D_x^\alpha \mathrm{Im}\tilde a_2(t,x,\xi) = \underbrace{\sum_{\alpha\geq 1}\frac{i}{2\alpha!}\partial_\xi^\alpha D_x^\alpha Re\,a_2}_{=:c(t,x,\xi)} + \underbrace{\sum_{\alpha\geq 1}\frac{i}{2\alpha!}\partial_\xi^\alpha D_x^\alpha\{Im\,b_{2,k} + Im\,(a_2)_{k,\tilde\Lambda}\}}_{=:e(t,x,\xi)}.$$

The hypothesis on a_2 implies

$$|c(t,x,\xi)| \leq C_c\langle\xi\rangle\langle x\rangle^{-\sigma},$$

with C_c depending only on Rea_2, whereas from (3.25), (3.26) and using that $2(1-\sigma)\leq\frac{1}{\theta}$ we obtain

$$|e(t,x,\xi)| \leq C_{e,k,\tilde\Lambda}\langle\xi\rangle^{\frac{1}{\theta}}\langle x\rangle^{-\sigma},$$

with $C_{e,k.\tilde\Lambda}$ depending on $k(t)$ and $\tilde\Lambda$.

We are ready to obtain the desired estimates from below. Using the above decomposition we get

$$\begin{aligned} e^\Lambda\circ(iP)\circ\{e^\Lambda\}^{-1} &= \partial_t + ia_3(t,D) + Re\,\tilde a_2(t,x,D) + A_{Im\,\tilde a_2}(t,x,D) \\ &\quad + (\tilde a_1 + c + e)(t,x,D) + \tilde a_\theta(t,x,D) + r_0(t,x,D). \end{aligned}$$

Note that $\langle\xi\rangle_h^2 \leq 2\xi^2$ provided that $|\xi| > R_{a_3}h$. Estimating the terms of order 2 we get

$$\begin{aligned} Re\,\tilde a_2 &\geq M_2\frac{C_{a_3}}{2}\langle\xi\rangle_h^2\langle x\rangle^{-\sigma} - C_{a_2}\langle\xi\rangle_h^2\langle x\rangle^{-\sigma}\langle x\rangle^{-\sigma} \\ &\quad - \max\{1,k(t)\}C_{\lambda_2}\langle\xi\rangle_h^{1+\frac{1}{\theta}} - \max\{1,k(t)\}C_{\tilde\Lambda}\langle\xi\rangle_h^{2-[2\sigma-1]}\langle x\rangle^{-\sigma} \\ &\geq \left(M_2\frac{C_{a_3}}{2} - C_{a_2} - \max\{1,k(t)\}C_{\lambda_2}h^{-(1-\frac{1}{\theta})} - \max\{1,k(t)\}C_{\tilde\Lambda}h^{-(2\sigma-1)}\right)\langle\xi\rangle_h^2\langle x\rangle^{-\sigma}, \end{aligned} \tag{3.29}$$

For the terms of order 1 we obtain

$$Re\,(\tilde a_1 + c + e) \geq M_1\frac{C_{a_3}}{2}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} - C_{a_1}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} - C_{a_2,\lambda_2}\langle\xi\rangle_h\langle x\rangle^{-2\sigma} \tag{3.30}$$

$$
\begin{aligned}
&- C_c\langle\xi\rangle_h\langle x\rangle^{-\sigma} - C_{e,k,\tilde{\Lambda}}\langle\xi\rangle_h^{\frac{1}{\theta}}\langle x\rangle^{-\sigma} \\
&\geq \left(M_1\frac{C_{a_3}}{2} - C_{a_1} - C_{a_2,\lambda_2} - C_c - C_{e,k,\tilde{\Lambda}}h^{-(1-\frac{1}{\theta})}\right)\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}}.
\end{aligned}
$$

Finally, for the terms of order $\frac{1}{\theta}$ we have

$$
\begin{aligned}
Re\,\tilde{a}_\theta &\geq -k'(t)\langle\xi\rangle_h^{\frac{1}{\theta}} - k(t)C_{\tilde{\Lambda}}\langle\xi\rangle_h^{\frac{1}{\theta}} - C_{\tilde{\Lambda}}\langle\xi\rangle_h^{2(1-\sigma)} - 2^\sigma C_{a_3}M_2\langle\xi\rangle_h^{2(1-\sigma)} - 2^\sigma C_{a_3}M_1\langle\xi\rangle_h^{1-\sigma} \\
&\geq -(k'(t) + C_1k(t) + C_2)\langle\xi\rangle_h^{\frac{1}{\theta}},
\end{aligned}
$$

where $C_2 = \tilde{C}_2h^{-[\frac{1}{\theta}-2(1-\sigma)]}$ and $C_1, \tilde{C}_2$ depend on $\tilde{\Lambda}$ but not on h. Setting

$$
k(t) = e^{-C_1t}k(0) - \frac{1-e^{-C_1t}}{C_1}C_2, \quad t \in [0,T],
$$

we obtain $k'(t) \leq 0$ and $k'(t) + C_1k(t) + C_2 = 0$. Note that for any choice of $k(0) > 0$, we can choose h large enough in order to obtain $k(t) > 0$ on $[0,T]$.

From the previous estimates from below we obtain the following proposition.

Proposition 3.6. *For any sufficiently small choice of* $k(0) > 0$ *there exist* $M_2, M_1 > 0$ *and a large parameter* $h_0 = h_0(k(0), M_2, M_1, T, \theta, \sigma) > 0$ *such that for every* $h \geq h_0$ *the Cauchy problem* (3.5) *is well-posed in Sobolev spaces* $H^m(\mathbb{R})$*. More precisely, for any* $\tilde{g} \in H^m(\mathbb{R})$ *and* $\tilde{f} \in C([0,T]; H^m(\mathbb{R}))$*, there is a unique solution* $v \in C([0,T]; H^m(\mathbb{R})) \cap C^1([0,T]; H^{m-3}(\mathbb{R}))$ *such that the following energy estimate holds*

$$
\|v(t)\|^2_{H^m} \leq C\left(\|\tilde{g}\|^2_{H^m} + \int_0^t \|\tilde{f}(\tau)\|^2_{H^m}d\tau\right), \quad t \in [0,T].
$$

Proof. Let $k(0) > 0$. Take $M_2 > 0$ such that

$$
M_2\frac{C_{a_3}}{2} - C_{a_2} > 0, \tag{3.31}
$$

and after that set $M_1 > 0$ in such way that

$$
M_1\frac{C_{a_3}}{2} - C_{a_1} - C_{a_2,\lambda_2} - C_c > 0. \tag{3.32}
$$

Finally, making the parameter h_0 large enough, we obtain $k(T) > 0$ and

$$
M_2\frac{C_{a_3}}{2} - C_{a_2} - \max\{1, k(t)\}C_{\lambda_2}h^{-(1-\frac{1}{\theta})} - \max\{1, k(t)\}C_{\tilde{\Lambda}}h^{-(2\sigma-1)} \geq 0, \tag{3.33}
$$

$$
M_1\frac{C_{a_3}}{2} - C_{a_1} - C_{a_2,\lambda_2} - C_c - C_{e,k,\tilde{\Lambda}}h^{-(1-\frac{1}{\theta})} > 0. \tag{3.34}
$$

With these choices $Re\,\tilde{a}_2(t,x,\xi), Re\,(\tilde{a}_1+c+e)(t,x,\xi), Re\,\tilde{a}_\theta(t,x,\xi)$ are non negative for large $|\xi|$. Applying the Fefferman-Phong inequality to Re$\tilde{a}_2$ we have

$$Re\langle Re\,\tilde{a}_2(t,x,D)v,v\rangle_{L^2} \geq -C\|v\|^2_{L^2}, \quad v \in \mathscr{S}(\mathbb{R}).$$

By the sharp Gårding inequality we also obtain that

$$Re\langle(\tilde{a}_1+c+e)(t,x,D)v,v\rangle_{L^2} \geq -C\|v\|^2_{L^2}, \quad v \in \mathscr{S}(\mathbb{R})$$

and

$$Re\langle\tilde{a}_\theta(t,x,D)v,v\rangle_{L^2} \geq -C\|v\|^2_{L^2}, \quad v \in \mathscr{S}(\mathbb{R}).$$

As a consequence we get the energy estimate

$$\frac{d}{dt}\|v(t)\|^2_{L^2} \leq C'(\|v(t)\|^2_{L^2} + \|(iP)_\Lambda v(t)\|^2_{L^2}),$$

which gives us the well-posedness on $H^m(\mathbb{R})$ for the Cauchy problem (3.5). □

3.7 Proof of Theorem 3.1

Finally we are ready to prove Theorem 3.1. With this purpose, take initial data satisfying $f \in C([0,T], H^m_{\rho;\theta}(\mathbb{R})), g \in H^m_{\rho;\theta}(\mathbb{R})$, for some $m \in \mathbb{R}$ and $\rho > 0$. Now choose $k(0) < \rho$ and M_2, M_1 large enough so that Proposition 3.6 holds true. We have

$$e^\Lambda(t,x,D)f \in C([0,T]; H^m_{\rho-(k(0)+\delta);\theta}(\mathbb{R})), \quad e^\Lambda(0,x,D)g \in H^m_{\rho-(k(0)+\delta);\theta}(\mathbb{R})$$

for every $\delta > 0$. Since $k(0) < \rho$ and $k(t)$ is non-increasing, we may conclude $e^\Lambda f \in C([0,T]; H^m(\mathbb{R}))$, $e^\Lambda g \in H^m(\mathbb{R})$. Proposition 3.6 gives $h_0 > 0$ large such that for $h \geq h_0$ the Cauchy Problem associated with P_Λ is well-posed in Sobolev spaces. Namely, there exists a unique $v \in C([0,T]; H^m)$ satisfying

$$\begin{cases} P_\Lambda v(t,x) = e^\Lambda(t,x,D)f(t,x), & (t,x) \in [0,T]\times\mathbb{R}, \\ v(0,t) = e^\Lambda(0,x,D)g(x), & x \in \mathbb{R}^n, \end{cases}$$

and

$$\|v(t)\|^2_{H^m} \leq C\left(\|e^\Lambda g\|^2_{H^m} + \int_0^t \|e^\Lambda f(\tau)\|^2_{H^m} d\tau\right), \quad t \in [0,T]. \tag{3.35}$$

Setting $u = \{e^\Lambda\}^{-1}v$ we obtain a solution for our original problem, that is

$$\begin{cases} Pu(t,x) = f(t,x), & (t,x) \in [0,T]\times\mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}. \end{cases}$$

Now let us study which space the solution $u(t,x)$ belongs to. We have

$$u = \{e^{\Lambda}\}^{-1}v = \{e^{-\tilde{\Lambda}}\}^{*} \underbrace{\sum_{j}(-r)^{j}}_{\text{order zero}} e^{-k(t)\langle D\rangle_h^{\frac{1}{\theta}}} v,$$

where $v \in H^m$. Noticing that $k(T) > 0$ and k is non-increasing, we achieve

$$e^{-k(t)\langle D\rangle_h^{\frac{1}{\theta}}} v = e^{-k(T)\langle D\rangle_h^{\frac{1}{\theta}}} \underbrace{e^{(k(T)-k(t))\langle D\rangle_h^{\frac{1}{\theta}}}}_{\text{order zero}} v \in H^m_{k(T);\theta}(\mathbb{R}).$$

Hence $\{e^{\Lambda}\}^{-1}v \in H^m_{k(T)-\delta;\theta}$ for every $\delta > 0$. Moreover, from (3.35) we obtain that u satisfies the following energy estimate

$$\begin{aligned}
\|u(t)\|^2_{H^m_{k(T)-\delta;\theta}} &= \|\{e^{\Lambda}(t)\}^{-1}v(t)\|^2_{H^m_{k(T)-\delta;\theta}} \leq C_1\|v(t)\|^2_{H^m} \\
&\leq C_2\left(\|e^{\Lambda}(0)g\|^2_{H^m} + \int_0^t \|e^{\Lambda}(\tau)f(\tau)\|^2_{H^m}d\tau\right) \\
&\leq C_3\left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau)\|^2_{H^m_{\rho;\theta}}d\tau\right), \quad t \in [0,T].
\end{aligned}$$

Summing up, given $f \in C([0,T], H^m_{\rho;\theta}(\mathbb{R})), g \in H^m_{\rho;\theta}(\mathbb{R})$ for some $m \in \mathbb{R}$ and $\rho > 0$, we find a solution $u \in C([0,T]; H^m_{\rho';\theta}(\mathbb{R}))$ ($\rho' < \rho$) for the Cauchy problem associated with the operator P and initial data f, g.

Now it only remains to prove the uniqueness of the solution. To this aim, assume $u_1, u_2 \in C([0,T]; H^m_{\rho';\theta}(\mathbb{R}))$ such that

$$\begin{cases} Pu_j = f, \\ u_j(0) = g. \end{cases}$$

For a new choice of $k(0) < \rho'$ and applying once more Proposition 3.6, we may find new parameters $M_2, M_1 > 0$ and $h_0 > 0$ such that the Cauch Problem associated with

$$P_{\Lambda'} = e^{\Lambda'} \circ P \circ \{e^{\Lambda'}\}^{-1}$$

is well-posed in H^m, where $\Lambda'(t,x,\xi)$ represents the symbol corresponding to the transformation associated with the new parameters $k(0), M_2, M_1, h_0$. Since $e^{\Lambda'}f, e^{\Lambda'}g, e^{\Lambda'}u_j \in H^m$ and u_j, $j = 1, 2$, satisfy

$$\begin{cases} P_{\Lambda'}e^{\Lambda'}u_j = e^{\Lambda'}f \\ e^{\Lambda'}u_j(0) = e^{\Lambda'}g, \end{cases}$$

we must have $e^{\Lambda'}u_1 = e^{\Lambda'}u_2$ and therefore $u_1 = u_2$. This concludes the proof. $\square$

Chapter 4

Cauchy Problem for $3-$Evolution Operators With Data in Gelfand-Shilov Type Spaces

4.1 Introduction and Main Result

Let us consider for $(t,x) \in [0,T] \times \mathbb{R}$ the $3-$evolution operator

$$P(t,x,D_t,D_x) = D_t + a_3(t,D_x) + a_2(t,x,D_x) + a_1(t,x,D_x) + a_0(t,x,D_x). \tag{4.1}$$

As in the previous chapter, we assume that $a_3(t,\xi)$ has order 3, $a_3(t,\xi)$ is a real-valued and $a_j(t,x,\xi)$ has order j (with respect to ξ), $j = 0,1,2$. We are interested in the Cauchy Problem associated with the operator P with data in some suitable Gelfand-Shilov class. The main result of this chapter reads as follows.

Theorem 4.1. *Consider $s_0 > 1$ and $\sigma \in (0,1)$ such that $s_0 < \frac{1}{1-\sigma}$. Let $P(t,x,D_t,D_x)$ be an operator like in (4.1) satisfying the following assumptions:*

(i) $a_3(t,\xi) \in C([0,T], S^3_1(\mathbb{R}^2))$, *$a_3(t,\xi)$ is real-valued and there exist $C_{a_3}, R_{a_3} > 0$ such that*

$$|\partial_\xi a_3(t,\xi)| \geq C_{a_3}|\xi|^2, \quad t \in [0,T], |\xi| > R_{a_3};$$

(ii) *Re* $a_2(t,x,\xi) \in C([0,T], SG^{2,0}_{1,s_0}(\mathbb{R}^2))$, *Im* $a_2(t,x,\xi) \in C([0,T], SG^{2,-\sigma}_{1,s_0}(\mathbb{R}^2))$;

(iii) *Re* $a_1(t,x,\xi) \in C([0,T], SG^{1,1-\sigma}_{1,s_0}(\mathbb{R}^2))$, *Im* $a_1(t,x,\xi) \in C([0,T], SG^{1,-\frac{\sigma}{2}}(\mathbb{R}^2))$;

(iv) $a_0(t,x,\xi) \in C([0,T], SG^{0,1-\sigma}_{1,s_0}(\mathbb{R}^2))$.

Let $s,\theta > 1$ *such that* $s_0 \leq s < \frac{1}{1-\sigma}$ *and* $\theta > s_0$ *and let* $f \in C([0,T]; H^m_{\rho;s,\theta}(\mathbb{R}))$ *and* $g \in H^m_{\rho;s,\theta}(\mathbb{R})$*, where* $m,\rho \in \mathbb{R}^2$ *with* $\rho_2 > 0$*. Then the Cauchy problem*

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{4.2}$$

admits a solution $u \in C([0,T]; H^m_{(\rho_1,-\tilde{\delta});s,\theta}(\mathbb{R}))$ *for every* $\tilde{\delta} > 0$*; moreover the solution satisfies the following energy estimate*

$$\|u(t)\|^2_{H^m_{(\rho_1,-\tilde{\delta});s,\theta}} \leq C\left(\|g\|^2_{H^m_{\rho;s,\theta}} + \int_0^t \|f(\tau)\|^2_{H^m_{\rho;s,\theta}} d\tau\right), \tag{4.3}$$

for all $t \in [0,T]$ *and for some* $C > 0$.

Remark 4.1. *Observe that the solution obtained in Theorem 4.1 has the same Gevrey regularity as the initial data but may lose the decay exhibited at* $t = 0$ *and admit an exponential growth for* $|x| \to \infty$ *when* $t > 0$*. Moreover, the loss* $\rho_2 + \tilde{\delta}$ *for an arbitrary* $\tilde{\delta} > 0$ *in the behavior at infinity is independent of* θ*,* s *and* ρ_1*. We also observe that the result holds for every* $\theta > s_0$ *without any upper bound on the regularity index* θ*, that is, there is no relation between the rate of decay of the data and the Gevrey regularity of the solution. Both these phenomena had been already observed in the case* $p = 2$*, see [6].*

Remark 4.2. *Let us describe a model class of linear differential operators that fits Theorem 4.1. For this purpose, we need to define a class of functions, namely, the uniform Gevrey functions which are polynomially bounded. Being more precise, let* $m \in \mathbb{R}$*,* $s_0 > 1$ *and* $A > 0$*. We set the Banach space* $G^{s_0,m}(\mathbb{R}, A)$ *of all smooth functions* $f(x)$ *such that*

$$\sup_{\alpha \in \mathbb{N}_0, x \in \mathbb{R}} |\partial_x^\beta f(x)| A^{-|\beta|} \beta!^{-s_0} \langle x \rangle^{-m+|\beta|} < \infty.$$

After that, we take the inductive limit of the Banach spaces $G^{s_0,m}(\mathbb{R}, A)$*, more precisely*

$$G^{s_0,m}(\mathbb{R}) = \bigcup_{A>0} G^{s_0,m}(\mathbb{R}, A).$$

Now consider the linear partial differential operator

$$P(t,x,D_t,D_x) = D_t + a_3(t)D_x^3 + a_2(t,x)D_x^2 + a_1(t,x)D_x^1 + a_0(t,x), \tag{4.4}$$

where the coefficients satisfy

(i) $a_3(t)$ *is a continuous real-valued function which never vanishes;*

(ii) *Re* $a_2 \in C([0,T], G^{s_0,0}(\mathbb{R}))$*, Im* $a_2 \in C([0,T], G^{s_0,-\sigma}(\mathbb{R}))$*;*

(iii) *Re* $a_1 \in C([0,T], G^{s_0,1-\sigma}(\mathbb{R}))$*, Im* $a_1 \in C([0,T], G^{s_0,-\frac{\sigma}{2}}(\mathbb{R}))$*;*

(iv) $a_0 \in C([0,T], G^{s_0,1-\sigma}(\mathbb{R}))$.

Then P is under the hypothesis of Theorem 3.1. Notice that the coefficients $a_j(t,x)$ are continuous with respect to t and Gevrey regular of index s_0 with respect to x. Moreover, $Im\, a_2(t,x)$ and $Im\, a_1(t,x)$ behave like $\langle x\rangle^{-\sigma}$ and $\langle x\rangle^{-\frac{\sigma}{2}}$ respectively, meanwhile $Re\, a_2(t,x)$ is uniformly bounded and $Re\, a_1(t,x), a_0(t,x)$ may admit a polynomial growth $\langle x\rangle^{1-\sigma}$.

Example 4.1. *For $t \in [0,T]$ and $x \in \mathbb{R}$ consider the following initial value problem*

$$\begin{cases} D_t u + D_x^3 u + a_2(t,x)D_x^2 u + a_1(t,x)D_x u + a_0(t,x)u = 0, \\ u(0,x) = e^{-\langle x\rangle^{1-\sigma}}, \end{cases} \tag{4.5}$$

where

$$a_2(t,x) = i(t-1)(1-\sigma)x\langle x\rangle^{-\sigma-1},$$

$$a_1(t,x) = 2(t-1)(1-\sigma)[\langle x\rangle^{-\sigma-1} - (\sigma+1)x^2\langle x\rangle^{-\sigma-3}],$$

$$a_0(t,x) = i\langle x\rangle^{1-\sigma} + i(t-1)(1-\sigma^2)[3x\langle x\rangle^{-\sigma-3} - (\sigma+3)x^3\langle x\rangle^{-\sigma-5}].$$

Notice that the coefficients a_j are analytic and satisfy the decay conditions of Theorem 4.1. Moreover the initial datum belongs to $\mathscr{S}(\mathbb{R})$ since $\sigma \in (0,1)$. It is easy to verify that the problem (4.5) admits the solution

$$u(t,x) = e^{(t-1)\langle x\rangle^{1-\sigma}} \notin C([0,T], \mathscr{S}(\mathbb{R})),$$

if $T \geq 1$. Analogously, $u \notin C([0,T], H^\infty(\mathbb{R}))$. More precisely, we notice that the solution has the same regularity as the initial data but it grows exponentially for $|x| \to \infty$ when $t \geq 1$.

This example shows us that the solution may present an exponential growth (for large values of t) even if the initial data decays exponentially, at least in the critical case $s_0 = \frac{1}{1-\sigma}$.

4.2 Strategy of the Proof

Here we briefly outline the strategy of the proof of Theorem 4.1. The idea is quite similar to that one applied in Theorem 3.1. The main difference is that now the change of variables will concentrate the loss of ”regularity” in the Sobolev indices which measure the behavior at infinity.

Once more we have

$$iP = \partial_t + ia_3(t,D) + \sum_{i=0}^{2} ia_i(t,x,D) = \partial_t + ia_3(t,D) + A(t,x,D)$$

and noticing that $a_3(t,\xi)$ is real we obtain

$$\frac{d}{dt}\|u(t)\|^2_{L^2} \leq \|Pu(t)\|^2_{L^2} + \|u(t)\|^2_{L^2} - \langle (A+A^*)u(t), u(t)\rangle_{L^2}.$$

Since $(A+A)^*(t) \in SG^{2,1-\sigma}(\mathbb{R}^{2n})$ we cannot derive an energy inequality in a straightforward way. Again, the idea is to conjugate the operator iP by a suitable pseudodifferential operator $e^{\Lambda}(t,x,D)$ in order to get

$$(iP)_\Lambda := e^{\Lambda}(iP)\{e^{\Lambda}\}^{-1} = \partial_t + ia_3(t,D) + \{a_{2,\Lambda} + a_{1,\Lambda} + a_{\sigma,\Lambda} + r_{0,\Lambda}\}(t,x,D),$$

where $a_{2,\Lambda} \in SG^{2,0}(\mathbb{R}^2)$, $a_{1,\Lambda} \in SG^{1,1-\sigma}(\mathbb{R}^2)$, $a_{\sigma,\Lambda} \in SG^{0,\sigma}(\mathbb{R}^2)$ and $r_0 \in SG^{0,0}(\mathbb{R}^2)$, but with $Re\, a_{j,\Lambda} \geq 0$, for $j = \sigma, 1, 2$. In this way, with the aid of Fefferman-Phong and sharp Gårding inequalities, we obtain that the Cauchy problem associated with P_Λ is well-posed in the weighted Sobolev spaces $H^m(\mathbb{R})$, $m \in \mathbb{R}^2$.

The operator e^{Λ} will be a pseudodifferential operator with symbol $e^{\Lambda(t,x,\xi)}$, and the function $\Lambda(t,x,\xi)$ will be of the form

$$\Lambda(t,x,\xi) = k(t)\langle x\rangle_h^{1-\sigma} + \lambda_2(x,\xi) + \lambda_1(x,\xi), \quad t \in [0,T],\ x,\xi \in \mathbb{R},$$

where $h > 1$ is a large parameter, $\lambda_1, \lambda_2 \in SG_\mu^{0,1-\sigma}(\mathbb{R}^2)$ and $k(t)$ is a $C^1([0,T];\mathbb{R})$ non increasing function to be chosen later on. The transformation with λ_2 will change the terms of $\xi-$order 2 into the sum of a positive operator plus a remainder of $\xi-$order 1; the transformation with λ_1 will not change the terms of order 2, but it will turn the terms of order 1 into the sum of a positive operator plus a remainder of $\xi-$order 0 and $x-$order less than $1-\sigma$. Finally, the transformation with $k(t)$ will correct these remainder terms.

4.3 Definition and Properties of $\lambda_2(x,\xi)$ and $\lambda_1(x,\xi)$

Let M_2 and M_1 be positive constants to be chosen later on. We define

$$\lambda_2(x,\xi) = M_2 w\left(\frac{\xi}{h}\right)\int_0^x \langle y\rangle^{-\sigma} dy, \quad (x,\xi) \in \mathbb{R}^2, \tag{4.6}$$

$$\lambda_1(x,\xi) = M_1 w\left(\frac{\xi}{h}\right)\langle\xi\rangle_h^{-1}\int_0^x \langle y\rangle^{-\frac{\sigma}{2}}\psi\left(\frac{\langle y\rangle^\sigma}{\langle\xi\rangle_h^2}\right)dy, \quad (x,\xi)\in\mathbb{R}^2, \tag{4.7}$$

where

$$w(\xi) = \begin{cases} 0, & |\xi|\leq 1, \\ -\mathrm{sgn}(\partial_\xi a_3(t,\xi)), & |\xi| > R_{a_3}, \end{cases} \qquad \psi(y) = \begin{cases} 1, & |y|\leq \frac{1}{2}, \\ 0, & |y|\geq 1, \end{cases}$$

$|\partial^\alpha w(\xi)| \leq C_w^{\alpha+1}\alpha!^\mu$, $|\partial^\beta\psi(y)|\leq C_\psi^{\beta+1}\beta!^\mu$ for some $\mu > 1$ which we can take arbitrarily close to 1.

Lemma 4.1. *Let* $\lambda_2(x,\xi)$ *as in (4.6). Then*

(i) $|\lambda_2(x,\xi)| \leq \frac{M_2}{1-\sigma}\langle x\rangle^{1-\sigma}$;

(ii) $|\partial_x^\beta\lambda_2(x,\xi)| \leq M_2 C^\beta \beta!\langle x\rangle^{1-\sigma-\beta}$, *for* $\beta\geq 1$;

(iii) $|\partial_\xi^\alpha\partial_x^\beta\lambda_2(x,\xi)| \leq M_2 C^{\alpha+\beta+1}\alpha!^\mu\beta!\chi_{E_{h,R_{a_3}}}(\xi)\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{1-\sigma-\beta}$, *for* $\alpha\geq 1, \beta\geq 0$,

where $E_{h,R_{a_3}} = \{\xi\in\mathbb{R}\colon h\leq\xi\leq R_{a_3}h\}$ *and the constant* $C>0$ *is independent of* h. *In particular* $\lambda_2(x,\xi)\in SG_\mu^{0,1-\sigma}(\mathbb{R}^2)$.

Proof. First note that

$$|\lambda_2(x,\xi)| = M_2\left|w\left(\frac{\xi}{h}\right)\right|\int_0^{|x|}\langle y\rangle^{-\sigma}dy \leq M_2\int_0^{\langle x\rangle} y^{-\sigma}dy = \frac{M_2}{1-\sigma}\langle x\rangle^{1-\sigma}.$$

For $\beta\geq 1$

$$|\partial_x^\beta\lambda_2(x,\xi)| \leq M_2\left|w\left(\frac{\xi}{h}\right)\right||\partial_x^{\beta-1}\langle x\rangle^{-\sigma}| \leq M_2 C^{\beta-1}(\beta-1)!\langle x\rangle^{1-\sigma-\beta}.$$

For $\alpha\geq 1$

$$\begin{aligned}|\partial_\xi^\alpha\lambda_2(x,\xi)| &\leq M_2 h^{-\alpha}\left|w^{(\alpha)}\left(\frac{\xi}{h}\right)\right|\int_0^{\langle x\rangle} y^{-\sigma}dy \\ &\leq \frac{M_2}{1-\sigma}C_w^{\alpha+1}\langle R_{a_3}\rangle^\alpha\alpha!^\mu\chi_{E_{h,R_{a_3}}}(\xi)\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{1-\sigma}.\end{aligned}$$

Finally, for $\alpha,\beta\geq 1$

$$\begin{aligned}|\partial_\xi^\alpha\partial_x^\beta\lambda_2(x,\xi)| &\leq M_2 h^{-\alpha}\left|w^{(\alpha)}\left(\frac{\xi}{h}\right)\right|\partial_x^{\beta-1}\langle x\rangle^{-\sigma} \\ &\leq M_2 C_w^{\alpha+1}\langle R_{a_3}\rangle^\alpha C^{\beta-1}\alpha!^\mu(\beta-1)!\chi_{E_{h,R_{a_3}}}(\xi)\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{1-\sigma-\beta}.\end{aligned}$$

□

For the function $\lambda_1(x,\xi)$ we can prove the following estimates.

Lemma 4.2. *Let* $\lambda_1(x,\xi)$ *as in (4.7). Then for all* $\alpha, \beta \geq 0$

(i) $|\partial_\xi^\alpha \partial_x^\beta \lambda_1(x,\xi)| \leq M_1 C^{\alpha+\beta+1}(\alpha!\beta!)^\mu \langle\xi\rangle_h^{-1-\alpha}\langle x\rangle^{1-\frac{\sigma}{2}-\beta}$;

(ii) $|\partial_\xi^\alpha \partial_x^\beta \lambda_1(x,\xi)| \leq M_1 C^{\alpha+\beta+1}(\alpha!\beta!)^\mu \langle\xi\rangle_h^{-\alpha}\langle x\rangle^{1-\sigma-\beta}$,

where the constant $C > 0$ *is independent of* h. *In particular* $\lambda_1(x,\xi) \in SG_\mu^{0,1-\sigma}(\mathbb{R}^2)$.

Proof. Denote by $\chi_\xi(x)$ the characteristic function of the set $\{x \in \mathbb{R}\colon \langle x\rangle^\sigma \leq \langle\xi\rangle_h^2\}$. For $\alpha = \beta = 0$ we have

$$|\lambda_1(x,\xi)| \leq M_1 \left| w\left(\frac{\xi}{h}\right)\right| \langle\xi\rangle_h^{-1} \int_0^{\langle x\rangle} y^{-\frac{\sigma}{2}} dy \leq \frac{2}{2-\sigma} M_1 \langle\xi\rangle_h^{-1}\langle x\rangle^{1-\frac{\sigma}{2}},$$

and

$$|\lambda_1(x,\xi)| \leq M_1 \left| w\left(\frac{\xi}{h}\right)\right| \int_0^{\langle x\rangle} \langle\xi\rangle_h^{-1}\langle y\rangle^{-\frac{\sigma}{2}}\chi_\xi(y)dy \leq \frac{M_1}{1-\sigma}\langle x\rangle^{1-\sigma}.$$

For $\alpha \geq 1$, with the aid of Faà di Bruno formula, we have

$$\begin{aligned}
|\partial_\xi^\alpha \lambda_1(x,\xi)| &\leq M_1 \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} h^{-\alpha_1}\left| w^{(\alpha_1)}\left(\frac{\xi}{h}\right)\right| \partial_\xi^{\alpha_2}\langle\xi\rangle_h^{-1} \\
&\quad\times \left| \int_0^x \langle y\rangle^{-\frac{\sigma}{2}} \partial_\xi^{\alpha_3}\psi\left(\frac{\langle y\rangle^\sigma}{\langle\xi\rangle_h^2}\right) dy\right| \\
&\leq M_1 \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} C_w^{\alpha_1+1}\langle R_{a_3}\rangle^{\alpha_3}\alpha_1!^\mu \langle\xi\rangle_h^{-\alpha_1} C^{\alpha_2}\alpha_2!\langle\xi\rangle_h^{-1-\alpha_2} \\
&\quad\times \int_0^{\langle x\rangle} \langle y\rangle^{-\frac{\sigma}{2}}\chi_\xi(y) \sum_{j=1}^{\alpha_3} \frac{\left|\psi^{(j)}\left(\frac{\langle y\rangle^{1-\frac{1}{s}}}{\langle\xi\rangle_h^2}\right)\right|}{j!} \sum_{\gamma_1+\dots+\gamma_j=\alpha_3} \frac{\alpha_3!}{\gamma_1!\dots\gamma_j!} \prod_{\ell=1}^j \partial_\xi^{\gamma_\ell}\langle\xi\rangle_h^{-2}\langle y\rangle^\sigma dy \\
&\leq M_1 \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} C_w^{\alpha_1+1}\langle R_{a_3}\rangle^{\alpha_3}\alpha_1!^\mu \langle\xi\rangle_h^{-\alpha_1} C^{\alpha_2}\alpha_2!\langle\xi\rangle_h^{-1-\alpha_2} \\
&\quad\times \int_0^{\langle x\rangle} \langle y\rangle^{-\frac{\sigma}{2}}\chi_\xi(y) \sum_{j=1}^{\alpha_3} C_\psi^{j+1} j!^{\mu-1} \sum_{\gamma_1+\dots+\gamma_j=\alpha_3} \frac{\alpha_3!}{\gamma_1!\dots\gamma_j!} \prod_{\ell=1}^j C^{\gamma_\ell+1}\gamma_\ell!\langle\xi\rangle_h^{-\gamma_\ell} dy \\
&\leq M_1 C_{\{w,\psi,\sigma,R_{a_3}\}}^{\alpha+1}\alpha!^\mu\langle\xi\rangle_h^{-1-\alpha}\langle y\rangle^{1-\frac{\sigma}{2}},
\end{aligned}$$

and

$$\begin{aligned}
|\partial_\xi^\alpha \lambda_1(x,\xi)| &\leq M_1 \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} h^{-\alpha_1}\left| w^{(\alpha_1)}\left(\frac{\xi}{h}\right)\right| \partial_\xi^{\alpha_2}\langle\xi\rangle_h^{-1} \\
&\quad\times \left| \int_0^x \langle y\rangle^{-\frac{\sigma}{2}} \partial_\xi^{\alpha_3}\psi\left(\frac{\langle y\rangle^\sigma}{\langle\xi\rangle_h^2}\right) dy\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq M_1 \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} C_w^{\alpha_1+1}\langle R_{a_3}\rangle^{\alpha_3}\alpha_1!^{\mu}\langle\xi\rangle_h^{-\alpha_1}C^{\alpha_2}\alpha_2!\langle\xi\rangle_h^{-\alpha_2} \\
&\times \int_0^{\langle x\rangle} \langle\xi\rangle_h^{-1}\langle y\rangle^{-\frac{\sigma}{2}}\chi_\xi(y)\sum_{j=1}^{\alpha_3}\frac{\left|\psi^{(j)}\left(\frac{\langle y\rangle^\sigma}{\langle\xi\rangle_h^2}\right)\right|}{j!}\sum_{\gamma_1+\cdots+\gamma_j=\alpha_3}\frac{\alpha_3!}{\gamma_1!\ldots\gamma_j!}\prod_{\ell=1}^{j}\partial_\xi^{\gamma_\ell}\langle\xi\rangle_h^{-2}\langle y\rangle^\sigma dy \\
&\leq M_1 \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!} C_w^{\alpha_1+1}\langle R_{a_3}\rangle^{\alpha_3}\alpha_1!^{\mu}\langle\xi\rangle_h^{-\alpha_1}C^{\alpha_2}\alpha_2!\langle\xi\rangle_h^{-\alpha_2} \\
&\times \int_0^{\langle x\rangle}\langle y\rangle^{-\sigma}\chi_\xi(y)\sum_{j=1}^{\alpha_3}C_\psi^{j+1}j!^{\mu-1}\sum_{\gamma_1+\cdots+\gamma_j=\alpha_3}\frac{\alpha_3!}{\gamma_1!\ldots\gamma_j!}\prod_{\ell=1}^{j}C^{\gamma_\ell+1}\gamma_\ell!\langle\xi\rangle_h^{-\gamma_\ell}dy \\
&\leq M_1 C_{\{w,\psi,\sigma,R_{a_3}\}}^{\alpha+1}\alpha!^{\mu}\langle\xi\rangle_h^{-\alpha}\langle y\rangle^{1-\sigma}.
\end{aligned}
$$

For $\beta \geq 1$ we have

$$
\begin{aligned}
|\partial_x^\beta\lambda_1(x,\xi)| &\leq M_1\langle\xi\rangle_h^{-1}\chi_\xi(x)\sum_{\beta_1+\beta_2=\beta-1}\frac{(\beta-1)!}{\beta_1!\beta_2!}\partial_x^{\beta_1}\langle x\rangle^{-\frac{\sigma}{2}}\sum_{j=1}^{\beta_2}\frac{\left|\psi^{(j)}\left(\frac{\langle y\rangle^\sigma}{\langle\xi\rangle_h^2}\right)\right|}{j!} \\
&\times\sum_{\delta_1+\cdots+\delta_j=\beta_2}\frac{\beta_2!}{\delta_1!\ldots\delta_j!}\prod_{\ell=1}^{j}\langle\xi\rangle_h^{-2}\partial_x^{\delta_\ell}\langle x\rangle^\sigma \\
&\leq M_1\langle\xi\rangle_h^{-1}\chi_\xi(x)\sum_{\beta_1+\beta_2=\beta-1}\frac{(\beta-1)!}{\beta_1!\beta_2!}C^{\beta_1+1}\beta_1!^{\mu}\langle x\rangle^{-\frac{\sigma}{2}-\beta_1}\sum_{j=1}^{\beta_2}C_\psi^{j+1}j!^{\mu-1} \\
&\times\sum_{\delta_1+\cdots+\delta_j=\beta_2}\frac{\beta_2!}{\delta_1!\ldots\delta_j!}\prod_{\ell=1}^{j}C^{\delta_\ell+1}\delta_\ell!\langle x\rangle^{-\delta_\ell} \\
&\leq M_1C_\psi^{\alpha+\beta+1}(\beta-1)!^{\mu}\langle\xi\rangle_h^{-1}\chi_\xi(x)\langle x\rangle^{1-\sigma-\beta} \\
&\leq M_1C_\psi^{\alpha+\beta+1}(\beta-1)!^{\mu}\langle x\rangle^{1-\sigma-\beta}
\end{aligned}
$$

Finally, for $\alpha, \beta \geq 1$ we have

$$
\begin{aligned}
|\partial_\xi^\alpha\partial_x^\beta\lambda_1(x,\xi)| &\leq M_1\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}h^{-\alpha_1}\left|w^{(\alpha_1)}\left(\frac{\xi}{h}\right)\right|\partial_\xi^{\alpha_2}\langle\xi\rangle_h^{-1}\sum_{\beta_1+\beta_2=\beta-1}\frac{(\beta-1)!}{\beta_1!\beta_2!} \\
&\times\partial_x^{\beta_1}\langle x\rangle^{\frac{\sigma}{2}}\left|\partial_\xi^{\alpha_3}\partial_x^{\beta_2}\psi\left(\frac{\langle x\rangle^\sigma}{\langle\xi\rangle_h^2}\right)\right| \\
&\leq M_1\chi_\xi(x)\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}h^{-\alpha_1}\left|w^{(\alpha_1)}\left(\frac{\xi}{h}\right)\right|\partial_\xi^{\alpha_2}\langle\xi\rangle_h^{-1}\sum_{\beta_1+\beta_2=\beta-1}\frac{(\beta-1)!}{\beta_1!\beta_2!}\partial_x^{\beta_1}\langle x\rangle^{-\frac{\sigma}{2}} \\
&\times\sum_{j=1}^{\alpha_3+\beta_2}\frac{\left|\psi^{(j)}\left(\frac{\langle x\rangle^\sigma}{\langle\xi\rangle_h^2}\right)\right|}{j!}\sum_{\gamma_1+\cdots+\gamma_j=\alpha_3}\sum_{\delta_1+\cdots+\delta_j=\beta_2}\frac{\alpha_3!}{\gamma_1!\ldots\gamma_j!}\frac{\beta_2!}{\delta_1!\ldots\delta_j!}\prod_{\ell=1}^{j}\partial_\xi^{\gamma_\ell}\langle\xi\rangle_h^{-2}\partial_x^{\delta\ell}\langle x\rangle^\sigma \\
&\leq M_1\chi_\xi(x)\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}C_w^{\alpha_1+1}\alpha_1!^{\mu}\langle R_{a_3}\rangle^{\alpha_1}\langle\xi\rangle_h^{-\alpha_1}C^{\alpha_2+1}\alpha_2!\langle\xi\rangle_h^{-1-\alpha_2}
\end{aligned}
$$

$$\times \sum_{\beta_1+\beta_2=\beta-1} \frac{(\beta-1)!}{\beta_1!\beta_2!} C^{\beta_1+1}\beta_1!\langle x\rangle^{-\frac{1}{2}(1-\frac{1}{s})-\beta_1} \sum_{j=1}^{\alpha_3+\beta_2} C_\psi^{j+1} j!^{\mu-1}$$
$$\times \sum_{\gamma_1+\cdots+\gamma_j=\alpha_3} \sum_{\delta_1+\cdots+\delta_j=\beta_2} \frac{\alpha_3!}{\gamma_1!\ldots\gamma_j!} \frac{\beta_2!}{\delta_1!\ldots\delta_j!} \prod_{\ell=1}^{j} C^{\gamma_\ell+1}\gamma_\ell!\langle\xi\rangle_h^{-2-\gamma_\ell} C^{\delta_\ell+1}\delta_\ell!\langle x\rangle^{\sigma-\delta_\ell}$$
$$\le M_1\chi_\xi(x) C_{\{w,\psi,R_{a_3}\}}^{\alpha+\beta+1}\alpha!^\mu(\beta-1)!^\mu\langle\xi\rangle_h^{-1-\alpha}\langle x\rangle^{1-\frac{\sigma}{2}-\beta}$$
$$\le M_1 C_{\{w,\psi,R_{a_3}\}}^{\alpha+\beta+1}\alpha!^\mu(\beta-1)!^\mu\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{1-\sigma-\beta}.$$

□

We end this section with the following continuity result.

Proposition 4.1. *Let* $\rho, m \in \mathbb{R}^2$ *and* $s, \kappa, \theta, \mu > 1$ *with* $\min\{s, \theta\} > 2\mu - 1$. *Let moreover* $\lambda(x,\xi) \in SG_\mu^{0,\frac{1}{\kappa}}(\mathbb{R}^{2n})$. *Then:*

(i) if $\kappa > s$, *then the operator* $e^{\lambda(x,D)}$ *is continuous from* $H^m_{\rho;s,\theta}(\mathbb{R}^n)$ *into* $H^m_{\rho-\delta e_2;s,\theta}(\mathbb{R}^n)$ *for every* $\delta > 0$, *where* $e_2 = (0,1)$*;*

(ii) if $\kappa = s$, *then the operator* $e^{\lambda(x,D)}$ *is continuous from* $H^m_{\rho;s,\theta}(\mathbb{R}^n)$ *into* $H^m_{\rho-\delta e_2;s,\theta}(\mathbb{R}^n)$ *for every* $\delta > C(\lambda) := \sup\{\lambda(x,\xi)/\langle x\rangle^{1/s} : (x,\xi) \in \mathbb{R}^{2n}\}$.

Proof. Consider $\phi(x) \in G^\mu(\mathbb{R}^n)$ a cut-off function such that, for a large positive constant K, $\phi(x) = 1$ for $|x| < K/2$, $\phi(x) = 0$ for $|x| > K$ and $0 \le \phi(x) \le 1$ for every $x \in \mathbb{R}^n$. We split the symbol $e^{\lambda(x,\xi)}$ as

$$e^{\lambda(x,\xi)} = \phi(x)e^{\lambda(x,\xi)} + (1-\phi(x))e^{\lambda(x,\xi)} = a_1(x,\xi) + a_2(x,\xi). \tag{4.8}$$

Since ϕ has compact support and λ has order zero with respect to ξ, we have $a_1(x,\xi) \in SG_\mu^{0,0}$. On the other hand, given any $\delta > 0$ and choosing K large enough, since $\kappa > s$ we may write $|\lambda(x,\xi)|\langle x\rangle^{-1/s} < \delta$ on the support of $a_2(x,\xi)$. Hence we obtain

$$a_2(x,\xi) = e^{\delta\langle x\rangle^{1/s}}(1-\phi(x))e^{\lambda(x,\xi)-\delta\langle x\rangle^{1/s}}$$

with $(1-\phi(x))e^{\lambda(x,\xi)-\delta\langle x\rangle^{1/s}}$ of order $(0,0)$ because $\lambda(x,\xi) - \delta\langle x\rangle^{1/s} < 0$ on the support of $(1-\phi(x))$. Thus, (4.8) becomes

$$e^{\lambda(x,\xi)} = a_1(x,\xi) + e^{\delta\langle x\rangle^{1/s}}\tilde{a}_2(x,\xi),$$

a_1 and $\tilde{a}_2$ of order $(0,0)$. Since by Theorem 2.11 the operators $a_1(x,D)$ and $\tilde{a}_2(x,D)$ map continuously $H^m_{\rho;s,\theta}$ into itself, then we obtain (i). The proof of (ii) follows a similar argument and can be found in [6, Theorem 2.4]. □

4.4 Invertibility of $e^{\tilde{\Lambda}}(x, D)$, $\tilde{\Lambda}(x,\xi) = (\lambda_2 + \lambda_1)(x,\xi)$

Since $e^{\tilde{\Lambda}(x,\xi)} \in SG^{0,\infty}_{\mu;\frac{1}{1-\sigma}}(\mathbb{R}^2)$, using the calculus in the $SG^{0,\infty}_{\mu;\frac{1}{1-\sigma}}(\mathbb{R}^2)$ setting and assuming $\mu > 1$ arbitrarily close to 1, we can use what we have already done in Section 3.4 of Chapter 3. Being more precise, we have the following result.

Lemma 4.3. *For $h > 0$ large enough, the operator $e^{\tilde{\Lambda}}(x, D)$ is invertible on $L^2(\mathbb{R})$ and its inverse is given by*

$$\{e^{\tilde{\Lambda}}(x,D)\}^{-1} = \{e^{-\tilde{\Lambda}(x,D)}\}^* \circ (I + r(x,D))^{-1} = \{e^{-\tilde{\Lambda}(x,D)}\}^* \circ \sum_{j\geq 0}(-r(x,D))^j,$$

where $r \in SG^{-1,-\sigma}_{2\mu-1}(\mathbb{R}^2)$ and $r \sim \sum_{\gamma\geq 1}\frac{1}{\gamma!}\partial_\xi^\gamma(e^{\tilde{\Lambda}}D_x^\gamma e^{-\tilde{\Lambda}})$ in $SG^{-1,-\sigma}_{2\mu-1}(\mathbb{R}^2)$. Moreover, the symbol of $(I + r(x,D))^{-1}$ belongs to $SG^{(0,0)}_{\delta}(\mathbb{R}^2)$ for every $\delta > 4\mu - 3$ and it maps continuously $H^{m'}_{\rho';s}(\mathbb{R})$ into itself for any $\rho', m' \in \mathbb{R}$ and $s > 1$ (provided that $\mu > 1$ is such that $s > 8\mu - 7$). Furthermore, we can write the inverse in the following way

$$\{e^{\tilde{\Lambda}}\}^{-1} = \{e^{-\tilde{\Lambda}}\}^* \circ \left[I + op\left(-i\partial_\xi\partial_x\tilde{\Lambda} - \frac{1}{2}\partial_\xi^2(\partial_x^2\tilde{\Lambda} - [\partial_x\tilde{\Lambda}]^2) - [\partial_\xi\partial_x\tilde{\Lambda}]^2 + q_{-3}\right)\right], \tag{4.9}$$

where $q_{-3} \in SG^{-3,-3\sigma}_{\delta}(\mathbb{R}^2)$.

4.5 Conjugation of iP

In this section, we will perform the conjugation of iP by the operator $e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}} \circ e^{\Lambda(t,x,D)}$ and its inverse, where $\Lambda(t,x,\xi) = k(t)\langle x\rangle_h^{1-\sigma} + \tilde{\Lambda}(x,\xi)$ and $k \in C^1([0,T];\mathbb{R})$ is a non increasing function such that $k(T) \geq 0$.

More precisely, we will compute

$$e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}} \circ e^{k(t)\langle x\rangle_h^{1-\sigma}} \circ e^{\tilde{\Lambda}}(x,D) \circ (iP) \circ \{e^{\tilde{\Lambda}}(x,D)\}^{-1} \circ e^{-k(t)\langle x\rangle_h^{1-\sigma}} \circ e^{-\rho_1\langle D\rangle^{\frac{1}{\theta}}},$$

where $\rho_1 \in \mathbb{R}$ and $P(t,x,D_t,D_x)$ is given by 4.1. As we discussed before, the role of this conjugation is to make positive the lower order terms of the conjugated operator.

Since the adjoint $\{e^{-\tilde{\Lambda}}\}^*$ appears in the inverse $\{e^{\tilde{\Lambda}}\}^{-1}$, we need the following lemma.

Lemma 4.4. *Let $\tilde{\Lambda} \in SG^{0,1-\sigma}_{\mu}(\mathbb{R}^2)$ and $a \in SG^{m_1,m_2}_{1,s_0}(\mathbb{R}^2)$, with $\mu > 1$ such that $1/(1-\sigma) > \mu + s_0 - 1$ and $s_0 > \mu$. Then, for any $M \in \mathbb{N}$,*

$$e^{\tilde{\Lambda}}(x,D) \circ a(x,D) \circ \{e^{-\tilde{\Lambda}}(x,D)\}^* = a(x,D)$$

$$+ \, op\left(\sum_{1\leq\alpha+\beta<M} \frac{1}{\alpha!\beta!}\partial_\xi^\alpha\{\partial_\xi^\beta e^{\tilde{\Lambda}} D_x^\beta a D_x^\alpha e^{-\tilde{\Lambda}}\} + q_M \right) + r(x,D),$$

where $q_M \in SG^{m_1-M,m_2-M\sigma}_{\mu,s_0}(\mathbb{R}^2)$ *and* $r \in \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2)$.

Proof. Since $\tilde{\Lambda} \in SG^{0,1-\sigma}_{\mu}(\mathbb{R}^2)$ and $a \in SG^{m_1,m_2}_{1,s_0}(\mathbb{R}^2)$, we have $e^{\pm\tilde{\Lambda}}, a \in SG^{0,\infty}_{\mu,s_0;\frac{1}{1-\sigma}}(\mathbb{R}^2)$. Therefore, by results from calculus, we obtain

$$\{e^{-\tilde{\Lambda}}(x,D)\}^* = a_1(x,D) + r_1(x,D) \quad \text{and} \quad e^{\tilde{\Lambda}}(x,D)\circ a(x,D) = a_2(x,D) + r_2(x,D),$$

where $a_1 \in SG^{0,\infty}_{\mu,s_0;\frac{1}{1-\sigma}}(\mathbb{R}^2)$, $a_2 \in SG^{m_1,\infty}_{\mu,s_0;\frac{1}{1-\sigma}}(\mathbb{R}^2)$, $r_1, r_2 \in \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2)$ and

$$a_1 \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha e^{-\tilde{\Lambda}} \text{ in } FSG^{0,\infty}_{\mu,s_0;\frac{1}{1-\sigma}}(\mathbb{R}^2),$$

$$a_2 \sim \sum_\beta \frac{1}{\beta!}\partial_\xi^\beta e^{\tilde{\Lambda}} D_x^\beta a \text{ in } FSG^{m_1,\infty}_{\mu,s_0;\frac{1}{1-\sigma}}(\mathbb{R}^2).$$

Hence

$$\begin{aligned} e^{\tilde{\Lambda}}(x,D)\circ a(x,D)\circ\{e^{-\tilde{\Lambda}}(x,D)\}^* &= a_2(x,D)\circ a_1(x,D) + a_2(x,D)\circ r_1(x,D) \\ &+ r_2(x,D)\circ a_1(x,D) + r_2(x,D)\circ r_1(x,D) \\ &= a_3(x,D) + r_3(x,D) + a_2(x,D)\circ r_1(x,D) \\ &+ r_2(x,D)\circ a_1(x,D) + r_2(x,D)\circ r_1(x,D), \end{aligned}$$

with $a_2(x,D)\circ a_1(x,D) = a_3(x,D) + r_3(x,D)$, where $a_3 \in SG^{m_1,\infty}_{\mu,s_0;\frac{1}{1-\sigma}}(\mathbb{R}^2)$, $r_3 \in \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2)$ and

$$\begin{aligned} a_3 &\sim \sum_{\gamma,\alpha,\beta} \frac{1}{\alpha!\beta!\gamma!}\,\partial_\xi^\gamma\{\partial_\xi^\beta e^{\tilde{\Lambda}} D_x^\beta a\}\partial_\xi^\alpha D_x^{\alpha+\gamma} e^{-\tilde{\Lambda}} \\ &= \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!}\,\partial_\xi^\alpha\{\partial_\xi^\beta e^{\tilde{\Lambda}} D_x^\beta a D_x^\alpha e^{-\tilde{\Lambda}}\} \text{ in } FSG^{m_1,\infty}_{\mu,s_0;\frac{1}{1-\sigma}}. \end{aligned}$$

Thus

$$e^{\tilde{\Lambda}}(x,D)\circ a(x,D)\circ\{e^{-\tilde{\Lambda}}(x,D)\}^* = a_3(x,D) + r(x,D),$$

for some $r \in \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2)$.

Now let us study the asymptotic expansion of a_3. For $\alpha,\beta \in \mathbb{N}_0$ we have

$$\partial_\xi^\beta e^{\tilde{\Lambda}}\partial_x^\beta a \partial_x^\alpha e^{-\tilde{\Lambda}} = \partial_x^\beta a \sum_{h=1}^{\beta}\frac{1}{h!}\sum_{\beta_1+\cdots+\beta_h=\beta}\frac{\beta!}{\beta_1!\ldots\beta_h!}\prod_{\ell=1}^{h}\partial_\xi^{\beta_\ell}\tilde{\Lambda}$$

$$\times \sum_{k=1}^{\alpha} \frac{1}{k!} \sum_{\alpha_1+\cdots+\alpha_k=\alpha} \frac{\alpha!}{\alpha_1!\ldots\alpha_k!} \prod_{\ell=1}^{k} \partial_x^{\alpha_\ell}(-\tilde\Lambda).$$

Therefore, by Faà di Bruno formula, for $\gamma, \delta \in \mathbb{N}_0$, we have

$$\begin{aligned}
\partial_\xi^{\gamma+\alpha}\partial_x^{\delta}\{\partial_\xi^\beta e^{\tilde\Lambda}\partial_x^\beta a \partial_x^\alpha e^{-\tilde\Lambda}\} = &\sum_{\gamma_1+\gamma_2+\gamma_3=\gamma+\alpha}\ \sum_{\delta_1+\delta_2+\delta_3=\delta} \frac{(\gamma+\alpha)!}{\gamma_1!\gamma_2!\gamma_3!}\frac{\delta!}{\delta_1!\delta_2!\delta_3!}\ \partial_\xi^{\gamma_1}\partial_x^{\beta+\delta_1}a \\
&\times \partial_\xi^{\gamma_2}\partial_x^{\delta_2}\left(\sum_{h=1}^{\beta}\frac{1}{h!}\sum_{\beta_1+\cdots+\beta_h=\beta}\frac{\beta!}{\beta_1!\ldots\beta_h!}\prod_{\ell=1}^{h}\partial_\xi^{\beta_\ell}\tilde\Lambda\right) \\
&\times \partial_\xi^{\gamma_3}\partial_x^{\delta_3}\left(\sum_{k=1}^{\alpha}\frac{1}{k!}\sum_{\alpha_1+\cdots+\alpha_k=\alpha}\frac{\alpha!}{\alpha_1!\ldots\alpha_k!}\prod_{\ell=1}^{k}\partial_x^{\alpha_\ell}(-\tilde\Lambda)\right) \\
&= \sum_{\gamma_1+\gamma_2+\gamma_3=\gamma+\alpha}\ \sum_{\delta_1+\delta_2+\delta_3=\delta}\frac{(\gamma+\alpha)!}{\gamma_1!\gamma_2!\gamma_3!}\frac{\delta!}{\delta_1!\delta_2!\delta_3!}\ \partial_\xi^{\gamma_1}\partial_x^{\beta+\delta_1}a \\
&\times \sum_{h=1}^{\beta}\frac{1}{h!}\sum_{\beta_1+\cdots+\beta_h=\beta}\frac{\beta!}{\beta_1!\ldots\beta_h!}\sum_{\theta_1+\cdots+\theta_h=\gamma_2}\ \sum_{\sigma_1+\cdots+\sigma_h=\delta_2}\frac{\gamma_2!}{\theta_1!\ldots\theta_h!}\frac{\delta_2!}{\sigma_1!\ldots\sigma_h!} \\
&\times \prod_{\ell=1}^{h}\partial_\xi^{\theta_\ell+\beta_\ell}\partial_x^{\sigma_\ell}\tilde\Lambda \\
&\times \sum_{k=1}^{\alpha}\frac{1}{k!}\sum_{\alpha_1+\cdots+\alpha_k=\alpha}\frac{\alpha!}{\alpha_1!\ldots\alpha_k!}\sum_{\theta_1+\cdots+\theta_k=\gamma_3}\ \sum_{\sigma_1+\cdots+\sigma_k=\delta_3}\frac{\gamma_3!}{\theta_1!\ldots\theta_k!}\frac{\delta_3!}{\sigma_1!\ldots\sigma_k!} \\
&\times \prod_{\ell=1}^{k}\partial_\xi^{\theta_\ell}\partial_x^{\alpha_\ell+\sigma_\ell}(-\tilde\Lambda),
\end{aligned}$$

hence

$$\begin{aligned}
|\partial_\xi^{\gamma+\alpha}\partial_x^{\delta}(\partial_\xi^\beta e^{\tilde\Lambda}D_x^\beta a D_x^\alpha e^{-\tilde\Lambda})| \le &\sum_{\substack{\gamma_1+\gamma_2+\gamma_3=\gamma+\alpha\\ \delta_1+\delta_2+\delta_3=\delta}}\frac{(\gamma+\alpha)!}{\gamma_1!\gamma_2!\gamma_3!}\frac{\delta!}{\delta_1!\delta_2!\delta_3!}C_a^{\gamma_1+\beta+\delta_1+1}\gamma_1!^{\mu}(\beta+\gamma_1)!^{s_0} \\
&\times \langle\xi\rangle^{m_1-\gamma_1}\langle x\rangle^{m_2-\beta-\delta_1} \\
&\times \sum_{h=1}^{\beta}\frac{1}{h!}\sum_{\beta_1+\cdots+\beta_h=\beta}\frac{\beta!}{\beta_1!\ldots\beta_h!}\sum_{\theta_1+\cdots+\theta_h=\gamma_2}\ \sum_{\sigma_1+\cdots+\sigma_h=\delta_2}\frac{\gamma_2!}{\theta_1!\ldots\theta_h!}\frac{\delta_2!}{\sigma_1!\ldots\sigma_h!} \\
&\times \prod_{\ell=1}^{h}C_{\tilde\Lambda}^{\theta_\ell+\beta_\ell+\sigma_\ell+1}(\theta_\ell+\beta_\ell)!^{\mu}\sigma_\ell!^{\mu}\langle\xi\rangle^{-\theta_\ell-\beta_\ell}\langle x\rangle^{1-\sigma-\sigma_\ell} \\
&\times \sum_{k=1}^{\alpha}\frac{1}{k!}\sum_{\alpha_1+\cdots+\alpha_k=\alpha}\frac{\alpha!}{\alpha_1!\ldots\alpha_k!}\sum_{\theta_1+\cdots+\theta_k=\gamma_3}\ \sum_{\sigma_1+\cdots+\sigma_k=\delta_3}\frac{\gamma_3!}{\theta_1!\ldots\theta_k!}\frac{\delta_3!}{\sigma_1!\ldots\sigma_k!} \\
&\times \prod_{\ell=1}^{k}C_{\tilde\Lambda}^{\theta_\ell+\beta_\ell+\sigma_\ell+1}(\theta_\ell)!^{\mu}(\alpha_\ell+\sigma_\ell)!^{\mu}\langle\xi\rangle^{-\theta_\ell}\langle x\rangle^{1-\sigma-\alpha_\ell-\sigma_\ell} \\
&\le C_1^{\gamma+\delta+2(\alpha+\beta)+1}\gamma!^{\mu}\delta!^{s_0}(\alpha+\beta)!^{\mu+s_0}\langle\xi\rangle^{m_1-\gamma-(\alpha+\beta)}\langle x\rangle^{m_2-\delta-(\alpha+\beta)} \\
&\times \sum_{k=1}^{\alpha}\frac{\langle x\rangle^{k(1-\sigma)}}{k!}\sum_{h=1}^{\beta}\frac{\langle x\rangle^{h(1-\sigma)}}{h!}
\end{aligned}$$

$$\leq C_1^{\gamma+\delta+2(\alpha+\beta)+1}\gamma!^{\mu}\delta!^{s_0}(\alpha+\beta)!^{\mu+s_0}\langle\xi\rangle^{m_1-\gamma-(\alpha+\beta)}\langle x\rangle^{m_2-\delta-(\alpha+\beta)}$$
$$\times\, 2^{\alpha+\beta}\sum_{k=1}^{\alpha+\beta}\frac{\langle x\rangle^{k(1-\sigma)}}{k!}.$$

Writing

$$\sum_{j\geq 0}\sum_{\alpha+\beta=j}\frac{1}{\alpha!\beta!}\partial_\xi^\alpha\{\partial_\xi^\beta e^{\tilde\Lambda}D_x^\beta a D_x^\alpha e^{-\tilde\Lambda}\}=\sum_{j\geq 0}r_j,$$

the above estimate implies

$$|\partial_\xi^\alpha\partial_x^\beta r_j(x,\xi)|\leq C^{\alpha+\beta+2j+1}\alpha!^{\mu}\beta!^{s_0}(j)!^{\mu+s_0-1}\langle\xi\rangle^{m_1-\alpha-j}\langle x\rangle^{m_2-\beta-j}\sum_{k=1}^{j}\frac{\langle x\rangle^{k(1-\sigma)}}{k!},$$

for every $j\geq 0$, $\alpha,\beta\in\mathbb{N}_0$ and $x,\xi\in\mathbb{R}$.

Let $\psi(x,\xi)\in C^\infty(\mathbb{R}^2)$ such that $\psi\equiv 0$ on Q_2, $\psi\equiv 1$ on Q_3^e, $0\leq\psi\leq 1$ and

$$|\partial_\xi^\alpha\partial_x^\beta\psi(x,\xi)|\leq C^{\alpha+\beta+1}\alpha!^{\mu}\beta!^{s_0},$$

for every $x,\xi\in\mathbb{R}$ and $\alpha,\beta\in\mathbb{N}_0$. Now set $\psi_0\equiv 1$ and, for $j\geq 1$,

$$\psi_j(x,\xi):=\psi\left(\frac{x}{R(j)},\frac{\xi}{R(j)}\right),$$

where $R(j)=Rj^{s_0+\mu-1}$, for a large constant $R>0$.

Setting $b(x,\xi)=\sum_{j\geq 0}\psi_j(x,\xi)r_j(x,\xi)$ we have $b\in SG_{\mu,s_0}^{m_1,\infty}(\mathbb{R}^2)$ and

$$b(x,\xi)\sim\sum_{j\geq 0}r_j(x,\xi)\ \text{ in }\ FSG_{\mu,s_0}^{m_1,\infty}(\mathbb{R}^2).$$

Now we will prove that $b\in SG_{\mu,s_0}^{m_1,m_2}(\mathbb{R}^2)$. Indeed, first we note

$$b(x,\xi)=a(x,\xi)+\sum_{j\geq 0}\psi_{j+1}(x,\xi)r_{j+1}(x,\xi).$$

For any $\alpha,\beta\in\mathbb{N}_0$ we get

$$|\partial_\xi^\alpha\partial_x^\beta\sum_{j\geq 0}\{\psi_{j+1}r_{j+1}\}(x,\xi)|\leq\sum_{j\geq 0}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}}\frac{\alpha!}{\alpha_1!\alpha_2!}\frac{\beta!}{\beta_1!\beta_2!}|\partial_\xi^{\alpha_1}\partial_x^{\beta_1}\psi_{j+1}(x,\xi)||\partial_\xi^{\alpha_2}\partial_x^{\beta_2}r_{j+1}(x,\xi)|$$
$$\leq\sum_{j\geq 0}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}}\frac{\alpha!}{\alpha_1!\alpha_2!}\frac{\beta!}{\beta_1!\beta_2!}C_\psi^{\alpha_1+\beta_1+1}\alpha_1!^{\mu}\beta_1!^{s_0}R(j+1)^{-\alpha_1-\beta_1}$$
$$\times\, C^{\alpha_2+\beta_2+2(j+1)+1}\alpha_2!^{\mu}\beta_2!^{s_0}\langle\xi\rangle^{m_1-\alpha_2-j}\langle x\rangle^{m_2-\beta_2-j}\sum_{k=1}^{j}\frac{\langle x\rangle^{k(1-\sigma)}}{k!}.$$

On the support of $\partial_\xi^{\alpha_1}\partial_x^{\beta_1}\psi_{j+1}$ we have

$$\langle x\rangle\leq 3R(j+1)\quad\text{and}\quad\langle\xi\rangle\leq 3R(j+1)$$

whenever that $\alpha_1 + \beta_1 \neq 0$. Hence

$$|\partial_\xi^\alpha \partial_x^\beta \sum_{j\geq 0} \{\psi_{j+1} r_{j+1}\}(x,\xi)| \leq \tilde{C}^{\alpha+\beta+1} \alpha!^\mu \beta!^{s_0} \langle \xi \rangle^{m_1-\alpha-1} \langle x \rangle^{m_2-\beta-\sigma} \langle x \rangle^{\sigma-1}$$
$$\times \sum_{j\geq 0} C_1^{2j} j!^{\mu+s_0-1} \langle \xi \rangle^{-j} \langle x \rangle^{-j} \sum_{k=1}^{j} \frac{\langle x \rangle^{k(1-\sigma)}}{k!}.$$

We also have

$$\langle x \rangle \geq R(j+1) \quad \text{and} \quad \langle \xi \rangle \geq R(j+1)$$

on the support of $\partial_\xi^{\alpha_1} \partial_x^{\beta_1} \psi_{j+1}$. If $\langle \xi \rangle \geq R(j+1)$ we get

$$\langle \xi \rangle^{-j} \leq R^{-j} (j+1)!^{-j(\mu+s_0-1)} \leq R^{-j} j!^{-(\mu+s_0-1)}.$$

On the other hande, since $\mu + s_0 - 1 < \frac{1}{1-\sigma}$, if $\langle x \rangle \geq R(j+1)$ then

$$\begin{aligned}
\langle x \rangle^{(k-1)(1-\sigma)-j} &\leq R^{(k-1)(1-\sigma)-j} \{(j+1)^{\mu+s_0-1}\}^{(k-1)(1-\sigma)-j} \\
&\leq R^{-\sigma j} (j+1)^{(k-1)-j(\mu+s_0-1)} \\
&\leq R^{-\sigma j} e^{j+1} (k-1)! (j+1)!^{-(\mu+s_0-1)}.
\end{aligned}$$

Hence, enlarging $R > 0$ if necessary, we infer that $\sum_{j\geq 1} \psi_j r_j \in SG_{\mu,s_0}^{m_1-1,m_2-\sigma}(\mathbb{R}^2)$. Analogously one can get

$$\sum_{j\geq k} \psi_j(x,\xi) r_j(x,\xi) \in SG_{\mu,s_0}^{m_1-k,m_2-\sigma k}(\mathbb{R}^2), \quad k \in \mathbb{N}_0.$$

Therefore

$$b(x,\xi) - \sum_{j<k} r_j(x,\xi) \in SG_{\mu,s_0}^{m_1-k,m_2-\sigma k}, \quad k \in \mathbb{N},$$

which finalizes the proof. □

4.5.1 Conjugation of iP by $e^{\tilde{\Lambda}}$

We notice again that $e^{\tilde{\Lambda}} \partial_t \{e^{\tilde{\Lambda}}\}^{-1} = \partial_t$ since $\tilde{\Lambda}(x,\xi) = \lambda_2(x,\xi) + \lambda_1(x,\xi)$ does not depend on t.

- Conjugation of $ia_3(t,D)$: Since a_3 does not depend of x, applying Lemma 4.4, we have

$$e^{\tilde{\Lambda}}(x,D)(ia_3)(t,D)\{e^{-\tilde{\Lambda}}(x,D)\}^* = ia_3(t,D) + s(t,x,D) + r_3(t,x,D),$$

$s \sim \sum_{j\geq 1} \frac{1}{j!}\partial_\xi^j\{e^{\tilde{\Lambda}} ia_3 D_x^j e^{-\tilde{\Lambda}}\}$ in $SG^{2,-\sigma}_{\mu}(\mathbb{R}^2)$ and $r_3 \in C([0,T], \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2))$. Hence, using (4.9), we can write (omitting (t,x,D) in the notation)

$$
\begin{aligned}
e^{\tilde{\Lambda}}(ia_3)\{e^{\tilde{\Lambda}}\}^{-1} &= \left(ia_3 - \partial_\xi(a_3\partial_x\tilde{\Lambda}) + \frac{i}{2}\partial_\xi^2[a_3(\partial_x^2\tilde{\Lambda} - (\partial_x\tilde{\Lambda})^2)] + a_3^{(0)} + r_3\right) \\
&\quad \circ \left[I - i\partial_\xi\partial_x\tilde{\Lambda} - \frac{1}{2}\partial_\xi^2(\partial_x^2\tilde{\Lambda} - [\partial_x\tilde{\Lambda}]^2) - [\partial_\xi\partial_x\tilde{\Lambda}]^2 + q_{-3}\right] \\
&= ia_3 - \partial_\xi(a_3\partial_x\tilde{\Lambda}) + \frac{i}{2}\partial_\xi^2\{a_3(\partial_x^2\tilde{\Lambda} - \{\partial_x\tilde{\Lambda}\}^2)\} + a_3\partial_\xi\partial_x\tilde{\Lambda} - i\partial_\xi a_3\partial_\xi\partial_x^2\tilde{\Lambda} \\
&\quad + i\partial_\xi(a_3\partial_x\tilde{\Lambda})\partial_\xi\partial_x\tilde{\Lambda} - \frac{i}{2}a_3\{\partial_\xi^2(\partial_x^2\tilde{\Lambda} + [\partial_x\tilde{\Lambda}]^2) + 2[\partial_\xi\partial_x\tilde{\Lambda}]^2\} + r_0 + \bar{r} \\
&= ia_3 - \partial_\xi a_3\partial_x\tilde{\Lambda} + \frac{i}{2}\partial_\xi^2\{a_3[\partial_x^2\tilde{\Lambda} - (\partial_x\tilde{\Lambda})^2]\} - i\partial_\xi a_3\partial_\xi\partial_x^2\tilde{\Lambda} \\
&\quad + i\partial_\xi(a_3\partial_x\tilde{\Lambda})\partial_\xi\partial_x\tilde{\Lambda} - \frac{i}{2}a_3\{\partial_\xi^2(\partial_x^2\tilde{\Lambda} + [\partial_x\tilde{\Lambda}]^2) + 2(\partial_\xi\partial_x\tilde{\Lambda})^2\} + r_0 + \bar{r} \\
&= ia_3 - \partial_\xi a_3\partial_x\lambda_2 - \partial_\xi a_3\partial_x\lambda_1 + \frac{i}{2}\partial_\xi^2\{a_3(\partial_x^2\lambda_2 - \{\partial_x\lambda_2\}^2)\} - i\partial_\xi a_3\partial_\xi\partial_x^2\lambda_2 \\
&\quad + i\partial_\xi(a_3\partial_x\lambda_2)\partial_\xi\partial_x\lambda_2 - \frac{i}{2}a_3\{\partial_\xi^2(\partial_x^2\lambda_2 + [\partial_x\lambda_2]^2) + 2[\partial_\xi\partial_x\lambda_2]^2\} + r_0 + \bar{r},
\end{aligned}
$$

where $a_3^{(0)} \in C([0,T]; SG^{0,0}_{\mu})$, $r_0 \in C([0,T]; SG^{0,0}_{\delta}(\mathbb{R}^2))$ and, since we may assume $2\delta - 1 < \mu + s_0 - 1$, $\bar{r} \in C([0,T]; \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2))$.

- Conjugation of $ia_2(t,x,D)$: by Lemma 4.4 and (4.9) we get (again omitting (t,x,D))

$$
\begin{aligned}
e^{\tilde{\Lambda}}(x,D)(ia_2)\{e^{\tilde{\Lambda}}(x,D)\}^{-1} &= \left(ia_2 - \partial_\xi\{a_2\partial_x\tilde{\Lambda}\} + \partial_\xi\tilde{\Lambda}\partial_x a_2 + a_2^{(0)} + r_2\right) \\
&\quad \circ \left[I - i\partial_\xi\partial_x\tilde{\Lambda} - \frac{1}{2}\partial_\xi^2(\partial_x^2\tilde{\Lambda} - [\partial_x\tilde{\Lambda}]^2) - [\partial_\xi\partial_x\tilde{\Lambda}]^2 + q_{-3}\right] \\
&= ia_2 - \partial_\xi\{a_2\partial_x\tilde{\Lambda}\} + \partial_\xi\tilde{\Lambda}\partial_x a_2 + a_2\partial_\xi\partial_x\tilde{\Lambda} + r_0 + \bar{r} \\
&= ia_2 - \partial_\xi a_2\partial_x\tilde{\Lambda} + \partial_\xi\tilde{\Lambda}\partial_x a_2 + r_0 + \bar{r} \\
&= ia_2 - \partial_\xi a_2\partial_x\lambda_2 + \partial_\xi\lambda_2\partial_x a_2 + r_0 + \bar{r},
\end{aligned}
$$

where $a_2^{(0)} \in C([0,T]; SG^{0,0}_{\mu,s_0})$, $r_0 \in C([0,T]; SG^{0,0}_{\delta,s_0}(\mathbb{R}^2))$ and $\bar{r} \in C([0,T]; \mathcal{S}_{\delta+s_0-1}(\mathbb{R}^2))$.

- Conjugation of $ia_1(t,x,D)$:

$$
\begin{aligned}
e^{\tilde{\Lambda}}(x,D)(ia_1(t,x,D))\{e^{\tilde{\Lambda}}(x,D)\}^{-1} &= \mathrm{op}(ia_1 + a_1^{(0)} + r_1) \circ \sum_{j\geq 0}(-r)^j \\
&= ia_1(t,x,D) + \tilde{r}_0(t,x,D) + \tilde{r}(t,x,D),
\end{aligned}
$$

where $a_1^{(0)} \in C([0,T]; SG^{0,1-2\sigma}_{\mu,s_0}(\mathbb{R}^2))$, $\tilde{r_0} \in C([0,T]; SG^{0,1-2\sigma}_{\delta,s_0}(\mathbb{R}^2))$ and $\tilde{r} \in C([0,T]; \mathcal{S}_{\delta+s_0-1}(\mathbb{R}^2))$.

- Conjugation of $ia_0(t,x,D)$:

$$e^{\tilde{\Lambda}}(x,D)(ia_0(t,x,D))\{e^{\tilde{\Lambda}}(x,D)\}^{-1} = \text{op}(ia_0 + a_0^{(0)} + r_0)\sum_{j\geq 0}(-r)^j$$
$$= ia_0(t,x,D) + \tilde{\tilde{r}}_0(t,x,D) + \tilde{r}_1(t,x,D),$$

where $a_0^{(0)} \in C([0,T]; SG^{-1,1-2\sigma}_{\mu,s_0})$, $\tilde{\tilde{r}}_0 \in C([0,T]; SG^{-1,1-2\sigma}_{\delta}(\mathbb{R}^2))$ and
$\tilde{r}_1 \in C([0,T]; \mathcal{S}_{\delta+s_0-1}(\mathbb{R}^2))$.

Summing up we obtain

$$\begin{aligned} e^{\tilde{\Lambda}}(x,D)(iP)\{e^{\tilde{\Lambda}}(x,D)\}^{-1} &= \partial_t + ia_3(t,D) + ia_2(t,x,D) - \text{op}(\partial_\xi a_3 \partial_x \lambda_2) \\ &\quad + ia_1(t,x,D) - \text{op}(\partial_\xi a_3 \partial_x \lambda_1 + \partial_\xi a_2 \partial_x \lambda_2 - \partial_\xi \lambda_2 \partial_x a_2 - ib_1) \\ &\quad + ia_0(t,x,D) + r_\sigma(t,x,D) + \bar{r}(t,x,D), \end{aligned} \tag{4.10}$$

where

$$b_1 \in C([0,T]; SG^{1,-2\sigma}_{\mu,s_0}(\mathbb{R}^2)),\ b_1(t,x,\xi) \in \mathbb{R},\ b_1 \text{ does not depend on } \lambda_1, \tag{4.11}$$

and

$$r_\sigma \in C([0,T]; SG^{0,1-2\sigma}_{\delta,s_0}(\mathbb{R}^2)),\ \bar{r} \in C([0,T]; \mathcal{S}_{\delta+s_0-1}(\mathbb{R}^2)). \tag{4.12}$$

4.5.2 Conjugation of $e^{\tilde{\Lambda}}(iP)\{e^{\tilde{\Lambda}}\}^{-1}$ by $e^{k(t)\langle x\rangle_h^{1-\sigma}}$

Let us recall that we are assuming that $k \in C^1([0,T];\mathbb{R})$, $k'(t) \leq 0$ and $k(t) \geq 0$ for every $t \in [0,T]$. Following the same ideas of Lemma 4.4 one can prove the following result.

Lemma 4.5. *Let $a \in C([0,T]; SG^{m_1,m_2}_{\mu,s_0}(\mathbb{R}^2))$, where $1 < \mu < s_0$ and $\mu + s_0 - 1 < \frac{1}{1-\sigma}$. Then*

$$e^{k(t)\langle x\rangle_h^{1-\sigma}}\, a(t,x,D)\, e^{-k(t)\langle x\rangle_h^{1-\sigma}} = a(t,x,D) + b(t,x,D) + \bar{r}(t,x,D),$$

where $b \sim \sum_{j\geq 1} \frac{1}{j!} e^{k(t)\langle x\rangle_h^{1-\sigma}}\, \partial_\xi^j a\, D_x^j e^{-k(t)\langle x\rangle_h^{1-\sigma}}$ in $SG^{m_1-1,m_2-\sigma}_{\mu,s_0}(\mathbb{R}^2)$ and
$\bar{r} \in C([0,T], \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2))$.

Now we perform the conjugation by $e^{k(t)\langle x\rangle_h^{1-\sigma}}$ of the operator $e^{\tilde{\Lambda}}(iP)\{e^{\tilde{\Lambda}}\}^{-1}$ in (4.10) with the aid of Lemma 4.5.

- Conjugation of ∂_t: $e^{k(t)\langle x\rangle_h^{1-\sigma}}\, \partial_t\, e^{-k(t)\langle x\rangle_h^{1-\sigma}} = \partial_t - k'(t)\langle x\rangle_h^{1-\sigma}$.

- Conjugation of $ia_3(t,D)$:

$$\begin{aligned} e^{k(t)\langle x\rangle_h^{1-\sigma}}\, ia_3(t,D)\, e^{-k(t)\langle x\rangle_h^{1-\sigma}} &= ia_3(t,D) + \text{op}(-k(t)\partial_\xi a_3 \partial_x \langle x\rangle_h^{1-\sigma}) \\ &+ \text{op}\left(\frac{i}{2}\partial_\xi^2 a_3\{k(t)\partial_x^2\langle x\rangle_h^{1-\sigma} - k^2(t)[\partial_x\langle x\rangle_h^{1-\sigma}]^2\}\right) \\ &+ a_3^{(0)}(t,x,D) + r_3(t,x,D), \end{aligned}$$

where $a_3^{(0)} \in C([0,T]; SG^{0,-3\sigma}_{\mu,s_0}(\mathbb{R}^2))$ and $r_3 \in C([0,T]; \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2))$.

- Conjugation of $\text{op}(ia_2 - \partial_\xi a_3 \partial_x \lambda_2)$:

$$\begin{aligned} e^{k(t)\langle x\rangle_h^{1-\sigma}}\, \text{op}(ia_2 - \partial_\xi a_3\partial_x\lambda_2)\, e^{-k(t)\langle x\rangle_h^{1-\sigma}} &= ia_2(t,x,D) + \text{op}(-\partial_\xi a_3\partial_x\lambda_2) \\ + \text{op}(-k(t)\partial_\xi a_2\partial_x\langle x\rangle_h^{1-\sigma} &- ik(t)\partial_\xi\{\partial_\xi a_3\partial_x\lambda_2\}\partial_x\langle x\rangle_h^{1-\sigma}) \\ + a_2^{(0)}(t,x,D) + r_2(t,x,D), & \end{aligned}$$

where $a_2^{(0)} \in C([0,T]; SG^{0,-2\sigma}_{\mu,s_0}(\mathbb{R}^2))$ and $r_2 \in C([0,T]; \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2))$.

- Conjugation of $i(a_1+a_0)(t,x,D)$: We have

$$e^{k(t)\langle x\rangle_h^{1-\sigma}} i(a_1+a_0)(t,x,D) e^{-k(t)\langle x\rangle_h^{1-\sigma}} = (ia_1 + ia_0 + a_{1,0} + r_1)(t,x,D),$$

where

$$a_{1,0} \sim \sum_{j\geq 1} e^{k(t)\langle x\rangle_h^{1-\sigma}} \frac{1}{j!}\partial_\xi^j i(a_1+a_0)(t,x,\xi) D_x^j e^{-k(t)\langle x\rangle_h^{1-\sigma}} \text{ in } SG^{0,1-2\sigma}_{\mu,s_0}(\mathbb{R}^2)$$

and $r_1 \in C([0,T]; \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2))$. It is not difficult to verify that the following estimate holds

$$|a_{1,0}(t,x,\xi)| \leq \max\{1,k(t)\} C_T \langle x\rangle_h^{1-2\sigma}, \tag{4.13}$$

where C_T depends on a_1 and does not depend on $k(t)$.

- Conjugation of $\text{op}(-\partial_\xi a_3\partial_x\lambda_1 - \partial_\xi a_2\partial_x\lambda_2 + \partial_\xi\lambda_2\partial_x a_2 + ib_1)$: taking into account (i) of Lemma 4.2 we obtain

$$\begin{aligned} & e^{k(t)\langle x\rangle_h^{1-\sigma}} \text{op}(-\partial_\xi a_3\partial_x\lambda_1 - \partial_\xi a_2\partial_x\lambda_2 + \partial_\xi\lambda_2\partial_x a_2 + ib_1) e^{-k(t)\langle x\rangle_h^{1-\sigma}} \\ &\quad = +\text{op}(-\partial_\xi a_3\partial_x\lambda_1 - \partial_\xi a_2\partial_x\lambda_2 + \partial_\xi\lambda_2\partial_x a_2 + ib_1) + (r_0 + \bar{r})(t,x,D), \end{aligned}$$

where $r_0 \in C([0,T]; SG^{0,0}_{\mu,s_0}(\mathbb{R}^2))$ and $\bar{r} \in C([0,T]; \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2))$.

- Conjugation of $r_\sigma(t,x,D)$: $e^{k(t)\langle x\rangle_h^{1-\sigma}}\, r_\sigma(t,x,D)\, e^{-k(t)\langle x\rangle_h^{1-\sigma}} = r_{\sigma,1}(t,x,D) + \bar{r}(t,x,D)$, where $\bar{r} \in C([0,T];\mathcal{S}_{\delta+s_0-1}(\mathbb{R}^2))$, $r_{\sigma,1}$ belongs to $C([0,T]; SG^{0,1-2\sigma}_{\delta,s_0}(\mathbb{R}^2))$ and the following estimate holds

$$|r_{\sigma,1}(t,x,\xi)| \leq C_{T,\tilde{\Lambda}} \langle x\rangle_h^{1-2\sigma}, \tag{4.14}$$

where $C_{T,\tilde{\Lambda}}$ does not depend of $k(t)$.

Gathering all the previous computations we may write

$$\begin{aligned} e^{k(t)\langle x\rangle_h^{1-\sigma}} e^{\Lambda}(iP)\{e^{\Lambda}\}^{-1} e^{-k(t)\langle x\rangle_h^{1-\sigma}} &= \partial_t + ia_3(t,D) \\ &+ \mathrm{op}(-\partial_\xi a_3 \partial_x \lambda_2 + ia_2 - k(t)\partial_\xi a_3 \partial_x \langle x\rangle_h^{1-\sigma}) \\ &+ \mathrm{op}(-\partial_\xi a_3\partial_x\lambda_1 + ia_1 - \partial_\xi a_2 \partial_x \lambda_2 + \partial_\xi \lambda_2 \partial_x a_2 - k(t)\partial_\xi a_2 \partial_x \langle x\rangle_h^{1-\sigma}) \\ &+ \mathrm{op}(ib_1 + ic_1 + ia_0 - k'(t)\langle x\rangle_h^{1-\sigma} + a_{1,0} + r_{\sigma,1}) + (r_0+\bar{r})(t,x,D), \end{aligned} \tag{4.15}$$

where b_1 satisfies (4.11),

$$c_1 \in C([0,T]; SG^{1,-2\sigma}_{\mu,s_0}(\mathbb{R}^2)),\ c_1(t,x,\xi) \in \mathbb{R},\ c_1 \text{ does not depend on } \lambda_1 \tag{4.16}$$

(but c_1 depends of $\lambda_2, k(t)$), $a_{1,0}$ as in (4.13), $r_{\sigma,1}$ as in (4.14), and for some operators

$$r_0 \in C([0,T]; SG^{0,0}_{\delta,s_0}(\mathbb{R}^2)),\ \bar{r} \in C([0,T]; \mathcal{S}_{\delta+s_0-1}(\mathbb{R}^2)).$$

4.5.3 Conjugation of $e^{k(t)\langle x\rangle_h^{1-\sigma}} e^{\tilde{\Lambda}}(iP)\{e^{\tilde{\Lambda}}\}^{-1} e^{-k(t)\langle x\rangle_h^{1-\sigma}}$ by $e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}}$

Since we are considering $\theta > s_0$ and $\mu > 1$ arbitrarily close to 1, we may assume that all the previous smoothing remainder terms have symbols in $\Sigma_\theta(\mathbb{R}^2)$. In this subsection we shall use the following lemma.

Lemma 4.6. *Let $a \in SG^{m_1,m_2}_{\mu,s_0}$, where $1 < \mu < s_0$ and $\mu + s_0 - 1 < \theta$. Then*

$$e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}}\, a(x,D)\, e^{-\rho_1\langle D\rangle^{\frac{1}{\theta}}} = a(x,D) + b(x,D) + r(x,D),$$

where $b \sim \sum_{j\geq 1} \frac{1}{j!}\partial_\xi^j e^{\rho_1\langle\xi\rangle^{\frac{1}{\theta}}} D_x^j a\, e^{-\rho_1\langle\xi\rangle^{\frac{1}{\theta}}}$ in $SG^{m_1-(1-\frac{1}{\theta}),m_2-1}_{\mu,s_0}(\mathbb{R}^2)$ and $r \in \mathcal{S}_{\mu+s_0-1}(\mathbb{R}^2)$.

Let us now conjugate by $e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}}$ the operator $e^{k(t)\langle x\rangle_h^{1-\sigma}} e^{\Lambda}(iP)\{e^{\Lambda}\}^{-1} e^{-k(t)\langle x\rangle_h^{1-\sigma}}$ in (4.15). First of all we observe that since a_3 does not depend of x, we simply have

$$e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}}\, ia_3(t,D)\, e^{-\rho_1\langle D\rangle^{\frac{1}{\theta}}} = ia_3(t,D).$$

- Conjugation of op $\left(-\partial_\xi a_3 \partial_x \lambda_2 + i a_2 - k(t)\partial_\xi a_3 \partial_x \langle x\rangle_h^{1-\sigma}\right)$:

$$
\begin{aligned}
&e^{\rho_1 \langle D\rangle^{\frac{1}{\theta}}} \operatorname{op}(-\partial_\xi a_3 \partial_x \lambda_2 + i a_2 - k(t)\partial_\xi a_3 \partial_x \langle x\rangle_h^{1-\sigma})\, e^{-\rho_1 \langle D\rangle^{\frac{1}{\theta}}} \\
&\qquad = \operatorname{op}(-\partial_\xi a_3 \partial_x \lambda_2 + i a_2 - k(t)\partial_\xi a_3 \partial_x \langle x\rangle_h^{1-\sigma}) + (a_{2,\rho_1} + \bar{r})(t,x,D),
\end{aligned}
$$

where $a_{2,\rho_1} \in C([0,T], SG^{1+\frac{1}{\theta},-1}_{\mu,s_0}(\mathbb{R}^2))$, $\bar{r} \in C([0,T], \Sigma_\theta(\mathbb{R}^2))$ and the following estimate holds

$$
|\partial_\xi^\alpha \partial_x^\beta a_{2,\rho_1}(t,x,\xi)| \leq \max\{1,k(t)\}|\rho_1| C^{\alpha+\beta+1}_{\lambda_2,T} \alpha!^\mu \beta!^{s_0} \langle \xi\rangle^{1+\frac{1}{\theta}-\alpha}\langle x\rangle^{-\sigma-\beta}. \quad (4.17)
$$

In particular

$$
|a_{2,\rho_1}(t,x,\xi)| \leq \max\{1,k(t)\}|\rho_1| C_{\lambda_2,T}\langle \xi\rangle_h^{1+\frac{1}{\theta}}\langle x\rangle^{-\sigma}.
$$

- Conjugation of

$$
\operatorname{op}(-\partial_\xi a_3 \partial_x \lambda_1 + i a_1 - \partial_\xi a_2 \partial_x \lambda_2 + \partial_\xi \lambda_2 \partial_x a_2 - k(t)\partial_\xi a_2 \partial_x \langle x\rangle_h^{1-\sigma} + i b_1 + i c_1):
$$

the conjugation of this term is given by

$$
\begin{aligned}
&\operatorname{op}(-\partial_\xi a_3 \partial_x \lambda_1 + i a_1 - \partial_\xi a_2 \partial_x \lambda_2 + \partial_\xi \lambda_2 \partial_x a_2 - k(t)\partial_\xi a_2 \partial_x \langle x\rangle_h^{1-\sigma} + i b_1 + i c_1) \\
&\qquad + a_{1,\rho_1}(t,x,D) + \bar{r}(t,x,D),
\end{aligned}
$$

where $\bar{r} \in C([0,T], \Sigma_\theta(\mathbb{R}^2))$ and a_{1,ρ_1} satisfies the following estimate

$$
|\partial_\xi^\alpha \partial_x^\beta a_{1,\rho_1}(t,x,\xi)| \leq \max\{1,k(t)\}|\rho_1| C^{\alpha+\beta+1}_{\tilde{\Lambda},T} \alpha!^\mu \beta!^{s_0} \langle \xi\rangle^{\frac{1}{\theta}-\alpha}\langle x\rangle^{-\sigma-\beta}. \quad (4.18)
$$

Particularly

$$
|a_{1,\rho_1}(t,x,\xi)| \leq \max\{1,k(t)\}|\rho_1| C_{\tilde{\Lambda},T}\langle \xi\rangle_h^{\frac{1}{\theta}}\langle x\rangle^{-\sigma}.
$$

- Conjugation of op$(i a_0 - k'(t)\langle x\rangle_h^{1-\sigma} + a_{1,0} + r_{\sigma,1})$:

$$
\begin{aligned}
&e^{\rho_1 \langle D\rangle^{\frac{1}{\theta}}} \operatorname{op}(i a_0 - k'(t)\langle x\rangle_h^{1-\sigma} + a_{1,0} + r_{\sigma,1}) e^{-\rho_1 \langle D\rangle^{\frac{1}{\theta}}} \\
&\qquad = \operatorname{op}(i a_0 - k'(t)\langle x\rangle_h^{1-\sigma} + a_{1,0} + r_{\sigma,1}) + r_0(t,x,D) + \bar{r}(t,x,D),
\end{aligned}
$$

where $r_0 \in C([0,T]; SG^{(0,0)}_{\delta,s_0}(\mathbb{R}^2))$ and $\bar{r} \in C([0,T]; \Sigma_\theta(\mathbb{R}^2))$.

Gathering all the previous computations we may write

$$
\begin{aligned}
&e^{\rho_1 \langle D\rangle^{\frac{1}{\theta}}} e^{k(t)\langle x\rangle_h^{1-\sigma}} e^{\tilde{\Lambda}}(iP)\{e^{\rho_1 \langle D\rangle^{\frac{1}{\theta}}} e^{k(t)\langle x\rangle_h^{1-\sigma}} e^{\tilde{\Lambda}}\}^{-1} = \partial_t + i a_3(t,D) \\
&\qquad + \operatorname{op}\left(-\partial_\xi a_3 \partial_x \lambda_2 + i a_2 - k(t)\partial_\xi a_3 \partial_x \langle x\rangle_h^{1-\sigma} + a_{2,\rho_1} - \partial_\xi a_3 \partial_x \lambda_1\right)
\end{aligned}
$$

$$+ \operatorname{op}\left(ia_1 - \partial_\xi a_2 \partial_x \lambda_2 + \partial_\xi \lambda_2 \partial_x a_2 - k(t)\partial_\xi a_2 \partial_x \langle x\rangle_h^{1-\sigma} + ib_1 + ic_1 + a_{1,\rho_1}\right)$$
$$+ \operatorname{op}(ia_0 - k'(t)\langle x\rangle_h^{1-\sigma} + a_{1,0} + r_{\sigma,1}) + r_0(t,x,D) + \bar{r}(t,x,D), \tag{4.19}$$

where a_{2,ρ_1} as in (4.17), b_1 as in (4.11), c_1 as in (4.16), a_{1,ρ_1} as in (4.18), $a_{1,0}$ as in (4.13), $r_{\sigma,1}$ as in (4.14), and for some operators

$$r_0 \in C([0,T]; SG^{(0,0)}_{\delta,s_0}(\mathbb{R}^2)), \quad \bar{r} \in C([0,T]; \Sigma_\theta(\mathbb{R}^2)).$$

4.6 Estimates from Below for the Real Parts

In this section we will choose M_2, M_1 and $k(t)$ in order to apply Fefferman-Phong and sharp Gårding inequalities to get the desired energy estimate for the conjugated problem. By the computations of the previous section we have

$$e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}} e^{k(t)\langle x\rangle_h^{1-\sigma}} e^{\tilde{\Lambda}}(x,D)(iP)\{e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}} e^{k(t)\langle x\rangle_h^{1-\sigma}} e^{\tilde{\Lambda}}(x,D)\}^{-1}$$
$$= \partial_t + ia_3(t,D) + \sum_{j=0}^{2} \tilde{a}_j(t,x,D) + r_0(t,x,D) + \bar{r}(t,x,D),$$

where $\tilde{a}_2, \tilde{a}_1$ represent the part with $\xi-$order $2,1$ respectively and $\tilde{a}_0$ represents the part with $\xi-$order 0, but with a positive order (less than or equal to $1-\sigma$) with respect to x. Now note that

$$\begin{aligned}
Re\,\tilde{a}_2 &= -\partial_\xi a_3 \partial_x \lambda_2 - Im\, a_2 - k(t)\partial_\xi a_3 \partial_x \langle x\rangle_h^{1-\sigma} + Re\, a_{2,\rho_1},\\
Im\,\tilde{a}_2 &= Re\, a_2 + Im\, a_{2,\rho_1},\\
Re\,\tilde{a}_1 &= -\partial_\xi a_3 \partial_x \lambda_1 - Im\, a_1 - \partial_\xi Re\, a_2 \partial_x \lambda_2 + \partial_\xi \lambda_2 \partial_x Re\, a_2\\
&\quad - k(t)\partial_\xi Re\, a_2 \partial_x \langle x\rangle_h^{1-\sigma} + Re\, a_{1,\rho_1},\\
Re\,\tilde{a}_0 &= -Im a_0 - k'(t)\langle x\rangle_h^{1-\sigma} + Re\, a_{1,0} + Re\, r_{\sigma,1}.
\end{aligned}$$

Since the Fefferman-Phong inequality holds true only for scalar symbols, we need to decompose $Im\,\tilde{a}_2$ into its Hermitian and anti-Hermitian part:

$$iIm\tilde{a}_2 = \frac{iIm\tilde{a}_2 + (iIm\tilde{a}_2)^*}{2} + \frac{iIm\tilde{a}_2 - (iIm\tilde{a}_2)^*}{2} = t_1 + t_2,$$

where $2Re\langle t_2(t,x,D)u,u\rangle = 0$ and $t_1(t,x,\xi) = -\sum_{\alpha\geq 1} \frac{i}{2\alpha!}\partial_\xi^\alpha D_x^\alpha Im\tilde{a}_2(t,x,\xi)$. Observe that, using (4.17),

$$|t_1(t,x,\xi)| \leq C_{a_2}\langle\xi\rangle\langle x\rangle^{-1} + \max\{1,k(t)\}|\rho_1| C_{\lambda_2}\langle\xi\rangle^{\frac{1}{\theta}}\langle x\rangle^{-\sigma-1}$$

$$\leq \{C_{a_2} + h^{-(1-\frac{1}{\theta})}\max\{1, k(0)\}|\rho_1|C_{\lambda_2}\}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}}. \tag{4.20}$$

In this way we may write

$$\begin{aligned} &e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}}e^{k(t)\langle x\rangle_h^{1-\sigma}}e^{\tilde{\Lambda}}(x,D)(iP)\{e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}}e^{k(t)\langle x\rangle_h^{1-\sigma}}e^{\tilde{\Lambda}}(x,D)\}^{-1} \\ &\quad = \partial_t + ia_3(t,D) + (Re\,\tilde{a}_2 + t_2 + t_1 + \tilde{a}_1 + \tilde{a}_0)(t,x,D) + \tilde{r}_0(t,x,D), \end{aligned} \tag{4.21}$$

where $\tilde{r}_0$ has symbol of order $(0,0)$.

Now we are ready to choose M_2, M_1 and $k(t)$. The function $k(t)$ will be of the form $k(t) = K(T-t)$, $t \in [0,T]$, for a positive constant K to be chosen. In the following computations we shall consider $|\xi| > hR_{a_3}$. Observe that $2|\xi|^2 \geq \langle\xi\rangle_h^2$ whenever $|\xi| \geq h > 1$. For $Re\,\tilde{a}_2$ we have:

$$\begin{aligned} Re\,\tilde{a}_2 &= M_2|\partial_\xi a_3|\langle x\rangle^{-\sigma} - Im\,a_2 - k(t)\partial_\xi a_3\partial_x\langle x\rangle_h^{1-\sigma} + Re\,a_{2,\rho_1} \\ &\geq M_2C_{a_3}|\xi|^2\langle x\rangle^{-\sigma} - C_{a_2}\langle\xi\rangle_h^2\langle x\rangle^{-\sigma} \\ &- \tilde{C}_{a_3}k(0)(1-\sigma)\langle\xi\rangle_h^2\langle x\rangle_h^{-\sigma} - \max\{1,k(0)\}C_{\lambda_2,\rho_1}\langle\xi\rangle_h^{1+\frac{1}{\theta}}\langle x\rangle^{-\sigma} \\ &\geq (M_2\frac{C_{a_3}}{2} - C_{a_2} - \tilde{C}_{a_3}k(0)(1-\sigma) - \max\{1,k(0)\}C_{\lambda_2,\rho_1}\langle\xi\rangle_h^{-(1-\frac{1}{\theta})})\langle\xi\rangle_h^2\langle x\rangle^{-\sigma} \\ &\geq (M_2\frac{C_{a_3}}{2} - C_{a_2} - \tilde{C}_{a_3}k(0)(1-\sigma) - \max\{1,k(0)\}C_{\lambda_2,\rho_1}h^{-(1-\frac{1}{\theta})})\langle\xi\rangle_h^2\langle x\rangle^{-\sigma} \\ &= (M_2\frac{C_{a_3}}{2} - C_{a_2} - \tilde{C}_{a_3}KT(1-\sigma) - \max\{1,KT\}C_{\lambda_2,\rho_1}h^{-(1-\frac{1}{\theta})})\langle\xi\rangle_h^2\langle x\rangle^{-\sigma} \end{aligned}$$

For $Re\,\tilde{a}_1$, we have:

$$\begin{aligned} Re\,\tilde{a}_1 &= M_1|\partial_\xi a_3|\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}}\psi\left(\frac{\langle x\rangle^\sigma}{\langle\xi\rangle_h^2}\right) - Im\,a_1 - \partial_\xi Re\,a_2\partial_x\lambda_2 + \partial_\xi\lambda_2\partial_x Re\,a_2 \\ &- k(t)\partial_\xi Re\,a_2\partial_x\langle x\rangle_h^{1-\sigma} + Re\,a_{1,\rho_1} \\ &\geq M_1C_{a_3}|\xi|^2\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}}\psi\left(\frac{\langle x\rangle^\sigma}{\langle\xi\rangle_h^2}\right) - C_{a_1}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} - \tilde{C}_{a_2,\lambda_2}\langle\xi\rangle_h\langle x\rangle^{-\sigma} \\ &- Ck(0)(1-\sigma)\langle\xi\rangle_h\langle x\rangle_h^{-\sigma} - \max\{1,k(0)\}C_{\tilde{\Lambda},\rho_1}\langle\xi\rangle_h^{\frac{1}{\theta}}\langle x\rangle_h^{-\sigma} \\ &\geq M_1\frac{C_{a_3}}{2}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}}\psi\left(\frac{\langle x\rangle^\sigma}{\langle\xi\rangle_h^2}\right) - C_{a_1}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} - \tilde{C}_{a_2,\lambda_2}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} \\ &- Ck(0)(1-\sigma)\langle\xi\rangle_h\langle x\rangle_h^{-\frac{\sigma}{2}}\langle x\rangle^{-\frac{\sigma}{2}} - \max\{1,k(0)\}C_{\tilde{\Lambda},\rho_1}\langle\xi\rangle_h^{-(1-\frac{1}{\theta})}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} \\ &= (M_1\frac{C_{a_3}}{2} - C_{a_1} - \tilde{C}_{a_2,\lambda_2} - CKT(1-\sigma)\langle x\rangle_h^{-\frac{\sigma}{2}})\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} \\ &- \max\{1,KT\}C_{\tilde{\Lambda},\rho_1}\langle\xi\rangle_h^{-(1-\frac{1}{\theta})}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} - M_1\frac{C_{a_3}}{2}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}}\left(1-\psi\left(\frac{\langle x\rangle^\sigma}{\langle\xi\rangle_h^2}\right)\right). \end{aligned}$$

Since $\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} \leq \sqrt{2}$ on the support of $1-\psi\left(\frac{\langle x\rangle^\sigma}{\langle\xi\rangle_h^2}\right)$, we may conclude

$$Re\,\tilde{a}_1 \geq (M_1\frac{C_{a_3}}{2} - C_{a_1} - \tilde{C}_{a_2,\lambda_2} - CKT(1-\sigma)h^{-\frac{\sigma}{2}})\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}}$$

$$- \max\{1, KT\}C_{\tilde{\Lambda},\rho_1}h^{-(1-\frac{1}{\theta})}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} - M_1\frac{C_{a_3}}{2}\sqrt{2}.$$

Taking (4.20) into account we obtain

$$\begin{aligned} Re\,(\tilde{a}_1 + t_1) \geq (M_1\frac{C_{a_3}}{2} - C_{a_2} - C_{a_1} - \tilde{C}_{a_2,\lambda_2} - CKT(1-\sigma)h^{-\frac{\sigma}{2}})\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} \\ - \max\{1, KT\}(C_{\tilde{\Lambda},\rho_1} + |\rho_1|C_{\lambda_2})h^{-(1-\frac{1}{\theta})}\langle\xi\rangle_h\langle x\rangle^{-\frac{\sigma}{2}} - M_1\frac{C_{a_3}}{2}\sqrt{2}. \end{aligned}$$

For $Re\,\tilde{a}_0$, we have:

$$\begin{aligned} Re\,\tilde{a}_0 &= -Im a_0 - k'(t)\langle x\rangle_h^{1-\sigma} + Re\,a_{1,0} + Re\,r_{\sigma,1} \\ &= -Im a_0 + (-k'(t) - \max\{1, k(0)\}C_T\langle x\rangle_h^{-\sigma} - C_{T,\tilde{\Lambda}}\langle x\rangle_h^{-\sigma})\langle x\rangle_h^{1-\sigma} \\ &\geq (-C_{a_0} + K - \max\{1, KT\}C_T h^{-\sigma} - C_{T,\tilde{\Lambda}}h^{-\sigma})\langle x\rangle_h^{1-\sigma}. \end{aligned}$$

Finally, let us proceed with the choices of M_1, M_2 and K. First we choose K larger than $\max\{C_{a_0}, 1/T\}$, then we set M_2 large in order to obtain $M_2\frac{C_{a_3}}{2} - C_{a_2} - \tilde{C}_{a_3}KT(1-\sigma) > 0$ and after that we take M_1 such that $M_1\frac{C_{a_3}}{2} - C_{a_2} - C_{a_1} - \tilde{C}_{a_2,\lambda_2} > 0$ (choosing M_2, M_1 we determine $\tilde{\Lambda}$). Enlarging the parameter h we may assume

$$KTC_{\lambda_2,\rho_1}h^{-(1-\frac{1}{\theta})} < \frac{1}{4}(M_2\frac{C_{a_3}}{2} - C_{a_2} - \tilde{C}_{a_3}KT(1-\sigma)),$$

$$CKT(1-\sigma)h^{-\frac{\sigma}{2}} + KT(C_{\tilde{\Lambda},\rho_1} + |\rho_1|C_{\lambda_2})h^{-(1-\frac{1}{\theta})} < \frac{1}{4}(M_1\frac{C_{a_3}}{2} - C_{a_2} - C_{a_1} - \tilde{C}_{a_2,\lambda_2}),$$

$$KTC_T h^{-\sigma} + C_{T,\tilde{\Lambda}}h^{-\sigma} < \frac{K - C_{a_0}}{4}.$$

With these choices we obtain that $Re\,\tilde{a}_2 \geq 0$, $Re\,(\tilde{a}_1 + t_1) + M_1\frac{C_{a_3}}{2}\sqrt{2} \geq 0$ and $Re\,\tilde{a}_0 \geq 0$. Let us also remark that the choices of M_2, M_1 and $k(t)$ do not depend of ρ_1 and θ.

4.7 Proof of Theorem 4.1

Let us denote

$$\tilde{P}_\Lambda := e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}}e^{\Lambda}(t, x, D)\, iP\, \{e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}}e^{\Lambda}(t, x, D)\}^{-1}.$$

By (4.21), with the choices of $M_2, M_1, k(t)$ in the previous section, we get

$$i\tilde{P}_\Lambda = \partial_t + ia_3(t, D) + (Re\,\tilde{a}_2 + t_2)(t, x, D) + (\tilde{a}_1 + t_1)(t, x, D) + \tilde{a}_0(t, x, D) + \tilde{r}_0(t, x, D),$$

with

$$Re\,\tilde{a}_2 \geq 0,\ Re\,(\tilde{a}_1 + t_1) + M_1\frac{C_{a_3}}{2}\sqrt{2} \geq 0,\ Re\,\tilde{a}_0 \geq 0. \tag{4.22}$$

Fefferman-Phong inequality applied to $Re\,\tilde{a}_2$ and sharp Gårding inequality applied to $\tilde{a}_1 + t_1 + M_1 \frac{C_{a_3}}{2}\sqrt{2}$ and $\tilde{a}_0$ give

$$\begin{aligned}\mathrm{Re}\langle Re\,\tilde{a}_2(t,x,D)v, v\rangle &\geq -c\|v\|^2_{L^2},\\ \mathrm{Re}\langle (\tilde{a}_1 + t_1)(t,x,D)v, v\rangle &\geq -\left(c + M_1\frac{C_{a_3}}{2}\sqrt{2}\right)\|v\|^2_{L^2},\\ \mathrm{Re}\langle \tilde{a}_0(t,x,D)v, v\rangle &\geq -c\|v\|^2_{L^2}\end{aligned}$$

for some constant c. Now applying Gronwall inequality we come to the following energy estimate

$$\|v(t)\|^2_{L^2} \leq C\left(\|v(0)\|^2_{L^2} + \int_0^t \|(i\tilde{P}_\Lambda v(\tau)\|^2_{L^2} d\tau\right), t \in [0,T],$$

for every $v(t,x) \in C^1([0,T];\mathscr{S}(\mathbb{R}))$. This estimate provides well-posedness of the Cauchy problem associated with $\tilde{P}_\Lambda$ in $H^m(\mathbb{R})$ for every $m = (m_1, m_2) \in \mathbb{R}^2$. Being more precise, for any $\tilde{f} \in C([0,T]; H^m(\mathbb{R}))$ and $\tilde{g} \in H^m(\mathbb{R})$, there exists a unique $v \in C([0,T]; H^m(\mathbb{R}))$ such that $\tilde{P}_\Lambda v = \tilde{f}$, $v(0) = \tilde{g}$ and

$$\|v(t)\|^2_{H^m} \leq C\left(\|\tilde{g}\|^2_{H^m} + \int_0^t \|\tilde{f}(\tau)\|^2_{H^m} d\tau\right), \quad t \in [0,T]. \tag{4.23}$$

Let us now turn back to our original Cauchy problem. Taking $f \in C([0,T], H^m_{\rho;s,\theta}(\mathbb{R}))$ and $g \in H^m_{\rho;s,\theta}(\mathbb{R})$ for some $m, \rho \in \mathbb{R}^2$ with $\rho_2 > 0$ and $s, \theta > 1$ such that $\theta > s_0$, we define Λ with $s_0 > 2\mu - 1$ and $M_1, M_2, k(0)$ such that (4.22) holds. Then by Proposition 4.1 we get

$$f_{\rho_1,\Lambda} := e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}} e^{\Lambda}(t,x,D)f \in C([0,T], H^m_{(0,\rho_2-\bar{\delta});s,\theta}(\mathbb{R}))$$

and

$$g_{\rho_1,\Lambda} := e^{\rho_1\langle D\rangle^{\frac{1}{\theta}}} e^{\Lambda}(0,x,D)g \in H^m_{(0,\rho_2-\bar{\delta});s,\theta}(\mathbb{R})$$

for every $\bar{\delta} > 0$, because $1/(1-\sigma) > s$. Since $\bar{\delta}$ can be taken arbitrarily small, we have that $f_{\rho_1,\Lambda} \in C([0,T], H^m)$ and $g_{\rho_1,\Lambda} \in H^m$. Hence the Cauchy problem

$$\begin{cases}\tilde{P}_\Lambda v = f_{\rho_1,\Lambda}\\ v(0) = g_{\rho_1,\Lambda}\end{cases}$$

admits a unique solution $v \in C([0,T], H^m) \cap C^1([0,T], H^{m_1-3, m_2-1+1/\sigma})$ satisfying the energy estimate (4.23). Taking now $u = (e^{\Lambda(t,x,D)})^{-1} e^{-\rho_1\langle D\rangle^{1/\theta}} v$, we get that u solves our original Cauchy problem. Moreover, since $v \in C([0,T]; H^m)$ we obtain $u \in C([0,T], H^m_{(\rho_1,-\tilde{\delta});s,\theta}(\mathbb{R}))$ for every $\tilde{\delta} > 0$ and from (4.23) we get

$$\|u\|_{H^m_{(\rho_1,-\tilde{\delta});s,\theta}} \leq C\|v\|_{H^m} \leq C\left(\|g_{\rho_1,\Lambda}\|^2_{H^m} + \int_0^t \|f_{\rho_1,\Lambda}(\tau)\|^2_{H^m} d\tau\right)$$

$$\leq C\left(\|g\|^2_{H^m_{\rho;s,\theta}} + \int_0^t \|f(\tau)\|^2_{H^m_{\rho;s,\theta}} d\tau\right).$$

This concludes the proof. □

Remark 4.3. *Since in general* $H^m_{(\rho_1,-\delta);s,\theta}(\mathbb{R})$ *is a larger space than* $H^m(\mathbb{R})$*, from our method, we cannot conclude that the solution is unique in* $C([0,T]; H^m_{(\rho_1,-\delta);s,\theta}(\mathbb{R}))$. *On the other hand, it is unique in the following subspace* J *of* $C([0,T]; H^m_{(\rho_1,-\delta);s,\theta}(\mathbb{R}))$

$$J = \{u \in C([0,T]; \mathscr{S}'(\mathbb{R})) : u = \{e^{\Lambda}(t,x,D)\}^{-1} e^{-\rho_1 \langle D \rangle^{\frac{1}{\theta}}} v,\ \textit{for some } v \in C([0,T]; H^m(\mathbb{R}))\}.$$

Indeed, if $u_j \in J$*,* $j = 1,2$*, are solutions for*

$$\begin{cases} Pu_j = f \\ u_j(0) = g, \end{cases}$$

then the functions u_j *also satisfy*

$$\begin{cases} \tilde{P}_\Lambda u_{j,\rho_1,\Lambda} = f_{\rho_1,\Lambda} \\ u_{j,\rho_1,\Lambda}(0) = g_{\rho_1,\Lambda}, \end{cases}$$

where $u_{j,\rho_1,\Lambda} = e^{\rho_1 \langle D \rangle^{\frac{1}{\theta}}} e^{\Lambda}(t,x,D) u_j$. *Using that* $u_j \in J$*, we have* $v_j \in C([0,T]; H^m)$ *such that* $u_j = \{e^{\Lambda}(t,x,D)\}^{-1} e^{-\rho_1 \langle D \rangle^{\frac{1}{\theta}}} v_j$, $j = 1,2$. *Hence*

$$\begin{cases} \tilde{P}_\Lambda v_j = f_{\rho_1,\Lambda} \\ v_j = g_{\rho_1,\Lambda}. \end{cases}$$

By the $H^m(\mathbb{R})$ *well-posedness of the Cauchy problem associated with* $\tilde{P}_\Lambda$ *we get* $v_1 = v_2$. *Therefore* $u_1 = u_2$.

Chapter 5

Further Research

Before introducing possible new directions of research, let us summarize the two main results of the present thesis. In the last two Chapters, we studied the Cauchy problem

$$\begin{cases} Pu(t,x) = f(t,x), \quad t \in [0,T],\, x \in \mathbb{R}, \\ u(0,x) = g(x), \quad x \in \mathbb{R}, \end{cases}$$

with P being a $3-$evolution operator of the following shape

$$P(t,x,D_t,D_x) = D_t + a_3(t)D_x^3 + a_2(t,x)D_x^2 + a_1(t,x)D_x + a_0(t,x),$$

where the coefficients are always continuous with respect to time and $s_0 > 1$ Gevrey regular with respect to the space variable. We also assume that $a_3(t)$ is real-valued and never vanishes, meanwhile we allow the lower order terms $a_j(t,x)$, $j = 0,1,2$, to be complex-valued. Under suitable decay conditions on the lower order terms, we were able to obtain existence and well-posedness results. Namely, we ask that a_2 behaves like $\langle x\rangle^{-\sigma}$ and a_1 behaves like $\langle x\rangle^{-\frac{\sigma}{2}}$, for some $\sigma \in (0,1)$. We have achieved the following:

- if $\sigma \in (\frac{1}{2},1)$ and $s_0 < \frac{1}{2(1-\sigma)}$, then for any $f \in C([0,T];H^m_{\rho;\theta})$ and $g \in H^m_{\rho;\theta}$, with $\rho > 0$ and $s_0 \leq \theta < \frac{1}{2(1-\sigma)}$, there exists a unique solution $u \in C([0,T];H^m_{\rho';\theta})$, for some $\rho' < \rho$. In particular, we have $\mathcal{H}^\infty_\theta$ well-posedness;
- if $s_0 < \frac{1}{1-\sigma}$, then for any $f \in C([0,T];H^{(m_1,m_2)}_{(\rho_1,\rho_2);s,\theta})$ and $g \in H^{(m_1,m_2)}_{(\rho_1,\rho_2);s,\theta}$, with $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $s_0 \leq s < \frac{1}{1-\sigma}$ and $\theta > s_0$, there exists a solution $u \in C([0,T];H^{(m_1,m_2)}_{(\rho_1,-\delta);s,\theta})$, for every $\delta > 0$. Here the solution has the same Gevrey regularity with respect to the initial data, but with a possible different behavior at infinity.

Now we point out three further research directions, which we believe to be interesting problems.

Generalization of the Space Dimension and of the Degree of Evolution

Concerning the Cauchy problem for $p-$evolution equations with general $p \geq 3$ we have the two following works [3] and [5]. The first one is set in the standard Hörmander classes and $H^\infty(\mathbb{R})$ well-posedness is achieved, whereas the second is set in the SG context and is obtained $\mathscr{S}(\mathbb{R})$ well-posedness. As we saw in the two previous Chapters, to treat the case $p = 3$ it suffices to apply the sharp Gårding inequality only once, meanwhile for $p > 3$ we need to apply it several times (cf. [3, 5]).

The Cauchy problem for $p-$evolution operators with general $p \geq 3$ in the Gevrey and Gelfand-Shilov functional settings is still an open problem. We hope that the ideas developed in Section 2.5, Chapter 2, will be useful in the treatment of this question.

Another issue is to consider our main theorems (Theorems 3.1 and 4.1) in higher dimensions for the spatial variable. At this moment, results of this type exist only for the case $p = 2$, see [6, 14, 33]. We believe that what we have proved for $\mathbb{R}^1$ should work in a similar way for the $\mathbb{R}^n$. The main difficulty is the choice of the functions λ_1 and λ_2 defining the change of variable, which, in higher space dimension, must be chosen in order to satisfy certain partial differential inequalities. These may be nontrivial and some technical effort should be required.

Necessary Conditions

Let us consider the Cauchy problem associated with the following model Schrödinger type operator

$$P = D_t - \partial_x^2 + i\langle x \rangle^{-\sigma} D_x,$$

where $x \in \mathbb{R}^1$ and $\sigma \in (0, 1)$. In this case we have well-posedness in Gevrey spaces with index $1 < \theta < \frac{1}{1-\sigma}$ (see [33]). On the other hand, Theorem 2 of [19] implies ill-posedness when $\theta > \frac{1}{1-\sigma}$. Finally, when $\theta = \frac{1}{1-\sigma}$ one can prove local in time well-posedness. Hence we obtain a nice relation between the decay rate of the coefficients σ and the Gevrey index θ for which we find well-posedness.

Now if we consider the $3-$evolution operator

$$P = D_t + D_x^3 + i\langle x \rangle^{-\sigma} D_x^2 + i\langle x \rangle^{-\frac{\sigma}{2}} D_x,$$

where $x \in \mathbb{R}$ and $\sigma \in (\frac{1}{2}, 1)$, from our Theorem 3.1 we obtain well-posedness in Gevrey spaces with index $1 < \theta < \frac{1}{2(1-\sigma)}$ for the Cauchy problem associated with P. But what happens in the cases $\theta > \frac{1}{2(1-\sigma)}$ or $0 < \sigma \leq \frac{1}{2}$? This is another interesting question which we should consider in the future. Maybe a first step in this direction is trying apply similar techniques presented in the works [4, 19, 28] to $3-$evolution equations, in order to obtain necessary conditions for well-posedness in Gevrey type spaces.

Semilinear $p-$evolution Equations

Last but not least, we would like to study the Cauchy problem

$$\begin{cases} P_u(D)u(t,x) = f(t,x), & t \in [0,T], x \in \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}, \end{cases}$$

for the semilinear $p-$evolution ($p = 2, 3$) operator

$$P_u(D) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t,x,u(t,x))D_x^j$$

in the Gevrey functional setting. We believe that the standard hypotheses should be $a_p(t)$ real-valued, continuous and never vanishes and $a_j(t,x,u(t,x)) \in C([0,T]; G^{s_0}(\mathbb{R} \times \mathbb{R}^2))$ (here we are identifying $\mathbb{C}$ with $\mathbb{R}^2$) with suitable decay and bound conditions on the variables x and $w = u(t,x)$, respectively.

We note that the $H^\infty(\mathbb{R})$ case (for general p) was considered by A. Ascanelli and C. Boiti in [2]. There the authors proved $H^\infty(\mathbb{R})$ local in time well-posedness. The strategy of the proof is first to consider the linearized problem obtained by fixing the variable u, that is, by considering the Cauchy problem for the linear operator

$$\tilde{P} = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} \tilde{a}_j(t,x)D_x^j,$$

where $\tilde{a}(t,x) := a_j(t,x,u(t,x))$ for a fixed u. The second step is to obtain L^2 energy estimates for the Cauchy problem associated with $\tilde{P}$, with a precise dependence of the constants on the variable u. To finish, an argument based on the powerful Nash-Moser inversion theorem is applied.

The Nash-Moser theorem (cf. [23]) is a generalization of the standard inversion function theorem in Banach spaces to the so-called Fréchet tame spaces. Roughly speaking, a tame Fréchet space is a Fréchet space that is isomorphic (in a suitable sense) to a sequence space like

$$\Sigma(B) := \{\{f_k\}_{k\in\mathbb{N}_0} \subset B : \sum_{k\in\mathbb{N}_0} e^{nk}\|f_k\|_B < \infty, \forall n \in \mathbb{N}_0\},$$

where B is a given Banach space.

We recall that $\mathcal{H}^{\infty}_{s_0}(\mathbb{R})$ is not a tame Fréchet space, in fact, it is not even Fréchet. But it should be possible to prove that the space

$$H^{\infty}_{s_0}(\mathbb{R}) := \bigcap_{\rho>0} H^{0}_{\rho;s_0}(\mathbb{R})$$

is a tame Fréchet space (see the proof that $H^{\infty}(\mathbb{R})$ is tame in [15]). Therefore, a first natural step to consider the semilinear problem in the Gevrey setting, is to try obtain $H^{\infty}_{s_0}(\mathbb{R})$ well-posedness results. We believe that it is possible if in addition, we assume that the coefficients are projective Gevrey regular, that is $a_j(t,x) \in C([0,T];\gamma^{s_0}(\mathbb{R}))$ where

$$\gamma^{s_0}(\mathbb{R}) := \bigcap_{A>0} G^{s_0}(\mathbb{R};A).$$

After this first step, the ideas developed in [2] should be tested in the (projective) Gevrey framework.

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