UNIVERSIDADE FEDERAL DO PARANÁ

WILLIAN GOULART GOMES VELASCO

ALGEBRAS OF EXPANDED STRUCTURES

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WILLIAN GOULART GOMES VELASCO

ALGEBRAS OF EXPANDED STRUCTURES

Tese apresentada como requisito parcial à obtenção do título de Doutor em Matemática, no Curso de Pós-Graduação em Matemática, Setor de Ciências Exatas, da Universidade Federal do Paraná.

Orientador: Prof. Dr. Marcelo Muniz Alves

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Assinatura Eletrônica
30/04/2021 18:53:50.0
MARCELO MUNIZ SILVA ALVES
Presidente da Banca Examinadora

Assinatura Eletrônica
30/04/2021 20:15:32.0
FERNANDO ABADIE VICENS
Avaliador Externo (UNIVERSIDAD DE LA REPÚBLICA - URUGUAY)

Assinatura Eletrônica
01/05/2021 15:09:52.0
ELIEZER BATISTA
Avaliador Externo (UNIVERSIDADE FEDERAL DE SANTA CATARINA)

Assinatura Eletrônica
30/04/2021 22:23:18.0
OLIVIER BRAHIC
Avaliador Interno (UNIVERSIDADE FEDERAL DO PARANÁ)

Assinatura Eletrônica
30/04/2021 22:48:20.0
HÉCTOR EDONIS PINEDO TAPIA
Avaliador Externo (UNIVERSIDAD INDUSTRIAL DE SANTANDER)

Eu não estou interessado
Em nenhuma teoria
Em nenhuma fantasia
Nem no algo mais
...
Amar e mudar as coisas
Me interessa mais.
(Belchior)

Ouça-me bem, amor
Preste atenção, o mundo é um moinho
Vai triturar teus sonhos, tão mesquinhos
Vai reduzir as ilusões a pó.
(Cartola)

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I want to go even further, and I would be glad if you come with me.

RESUMO

O foco desta tese é estudar as álgebras obtidas de produtos semi-diretos definidos por ações globais e parciais, os quais chamamos de expansões. Em particular, estamos interessados em duas familias de álgebras de convolução sobre estas expansões: as álgebras parciais e as álgebras globais. Após a revisão dos aspectos teóricos básicos, desenvolvemos em quatro capítulos a metodologia que chamamos de Bernoulli. Esta consiste em definir uma ação global e uma ação parcial de uma estrutura ágebrica em um conjunto parcialmente ordenado, uma expansão derivada de cada ação e condições que implicam nas álgebras de convolução dessas expansões serem Morita equivalentes. O trabalho aplica esta abordagem com as seguinte estruturas algébricas: grupos, semigrupos inversos, grupoides ordenados e categorias inversas. Além disso desenvolvemos uma fórmula para a álgebra global de um grupo e uma forma de estudar representações de categorias inversas utiliando extensões de Kan. A tese apresenta uma nova forma de interpretar expansões já conhecidas na literatura (o semigrupo universal de Exel ou a expansão de Birget-Rhodes de um grupo, a expansão do préfixo de um semigrupo inverso e a expansão de Birget-Rhodes de um grupoide ordenado) através de produtos semi-diretos das ações parciais de Bernoulli e apresentamos a versão global de cada expansão. Destacamos também as seguintes contribuições: a definição de ações parcias (fibradas) de categorias inversas em conjuntos parcialmente ordenados, a definição do produto semi-direto de uma categoria inversa, uma noção de *enlargements* para categorias inversas e as álgebras global e parcial de uma categoria inversa.

Palavras-chave: Ação parcial. Produto semi-direto. Enlargement. Contexto de Morita. Álgebra parcial. Extensão de Kan.

ABSTRACT

The focus of this thesis is to develop a study of the algebras of the semidirect product defined by global and partial actions, which we call expansions. In particular, we are interested in two families of convolution algebras: the partial algebras and the global algebras. After a review of the theoretical aspects, we develop through four chapters the methodology called the Bernoulli approach. This methodology is developed in the following way: to define a global and a partial action of an algebraic structure on a partially ordered set, then we define the semidirect product derived from each action, finally, we study the conditions which will imply that the convolution algebras of such expansions are Morita equivalent. This thesis applies the previous methodology to the following algebraic structures: groups, inverse semigroups, ordered groupoids, and inverse categories. We also present a formula to compute the global algebra and we study the representation of inverse categories using Kan extensions. The thesis presents a new way to interpret expansions already known in the literature (the Exel universal inverse semigroup, or the Birget-Rhodes expansion of a group, the prefix expansion of an inverse semigroup, and the Birget-Rhodes expansion of an ordered groupoid) through semidirect product from Bernoulli partial actions and the global version of each expansion. We also highlight the following contributions: the definitions of (fibred) partial actions of inverse categories, the definition of the semidirect product of an inverse category, a notion of enlargement of an inverse category, and the global and partial algebras of an inverse category.

Keywords: Partial action. Semidirect product. Enlargement. Morita context. Partial algebra. Kan extension.

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Introduction

Background

Group theory and group actions are classical mathematical objects of study. The abstract concept of a group and the role that group actions play are well established and have many applications in different mathematical research areas. Among many historic beginnings, the mathematicians developed this theory to study symmetries [78][46].

In geometry, a symmetry of an object, such as a vector space, or a manifold, is a bijective function that preserves something which has geometrical meaning, such as distances, areas, or angles. One of the restricting characteristics of symmetries is that the whole object is taken into consideration. However, only small pieces may drive our attention. Then a (global) symmetry should have a restriction that describes partially what is left. To answer this question, specialists developed a new theory generalizing groups and their actions, the theory of inverse semigroups and their actions [51]. An important example is the inverse semigroup of partial bijection of a fixed set X, denoted by $\mathcal{I}(X)$ – following Lawson's [51] notation–, where the composition of two partial bijections f and g is obtained by taking the largest restriction of f such that $g(f(x))$ makes sense for each x in the new domain.

Another generalization of groups are the groupoids, *i.e.* small categories where all the arrows have an inverse. As Lawson [51] exemplifies, if we impose the product of two maps $f, g \in \mathcal{I}(X)$ only when domain$(f) =$ range(g), the structure we achieve is no longer an inverse semigroup, but a groupoid, $\mathbb{G}(X)$ as Gilbert [40] writes. This new algebraic object is much closer to groups than semigroups and translates in a very natural way the study of symmetries, as explained by Weinstein [97] and Brown [13].

At this point, it is important to state that there is a canonical way of restricting the product of an inverse semigroup S that produces a groupoid, the restriction groupoid of S. Conversely, given a topological groupoid $\mathcal{G}$, it is possible to define a structure of inverse semigroup on a particular class of open sets of $\mathcal{G}$ as Paterson [70] shows in Proposition 1.0.1 and Proposition 2.2.3.

There is a purely algebraic – we mean, removing the topology of the groupoid – manner to relate inverse semigroups and groupoids, it is known as the Ehresmann-Schein-Nambooripad Theorem, in [51] Chapter 4. Although, to provide an equivalence, one needs to require an extra hypothesis about the groupoid: it must be an inductive groupoid.

In our short historical motivation, we discussed a few aspects of two generalizations of group theory. Our next milestone is to understand how (partial) actions take place in these new environments. After discussing a little about actions, we will return to inverse semigroups and groupoids and finally to their algebras.

If we interpret group actions as maps, we can relax their definition, and thus we gain a generalization of such a theory. With minor adjustments, we have partial group actions. Ruy Exel, in his expertise area of research, C^*-operator algebras, developed the theory of partial group actions. A few years later, with Mikhailo Dokuchaev, a purely algebraic version was provided. Since then, algebraists around the country and the world have been making progress and contributions to Exel's discovery. Also, by Exel in [32], partial group actions have a relation with semigroups actions. Given a group G, it is possible to construct an inverse semigroup $S(G)$ whose actions are in a one-to-one correspondence with the underlying partial actions of G.

In the year 2000, the above mathematicians in joint work with Paolo Piccione showed ([28]) how to decompose the partial group algebra of any finite group as a direct sum of matrix components. It is important to note that the first version of the formula presented an error, which was later corrected by Dokuchaev and Miles in [30]. They derived a recurrence formula using a particular groupoid and its oriented graph. Recent works of Keunbae Choi [20] and [19], as Dokuchaev comments in his recent survey [26], present a very natural way to write the same formula for the partial algebra. His work uses the richer internal structure of inverse semigroups; using a natural partition into classes (the one associated with Green's relation $\mathcal{D}$) of Exel's universal semigroup [32], Choi showed how to identify the connected components of its associated groupoid.

Now we return to inverse semigroups. Lawson's interpretation of the P-theorem provides us a way to define an E-inverse semigroup from a group acting on a semilattice [53]. More concretely, Alessandra Piske showed in her master thesis [71] (Example 11) this fact applied to a more general case of partial actions, following the generalization made by Kellendonk and Lawson [46]. Moreover, as an example, she showed how to interpret Exel's universal semigroup as a partial semidirect product.

A natural question arises: as partial group actions have globalizations ([34] Theorem 3.5), how does the global action and its associated semigroup relates to Exel's universal semigroup? Well, inverse semigroup theory shows us that this pair of semigroups is related by an enlargement [51] [50]: the global action provides an inverse semigroup that enlarges the partial action case one.

Other two questions rise naturally, one about the groupoid structure and the other one about partial algebras. We are going to deal with them, but a few theoretical aspects came first.

There is a standard construction of a groupoid from (global) group actions, for instance, the transformation groupoid from Renault's thesis [72]. Naturally, this notion has a version for partial actions as firstly introduced by Kellendonk and Lawson [46] and by Abadie [2]. Reinterpreting a few lines in Dokuchaev-Exel-Piccione's work [28], we readily realize that their groupoid (very important to compute the matrix components) is associated with a specific partial group action. The partial action in question is the partial Bernoulli action of a group, using the nomenclature of Exel's book [34] (Definition 5.12). We can characterize the partial algebra as a groupoid algebra, obtained from a partial action with this point of view.

Before delving into the theory, it is essential to introduce the actions of semigroups and the groupoid of germs of an inverse semigroup. Regarding the inverse semigroup theory, one interpretation of the Wagner-Preston Theorem ([51] section 1.5) shows us how to define an inverse semigroup action. Exel uses this type of action ([34] section 4) to provide a cleaner version of Paterson's universal groupoid ([70] theorem 4.3.1) as a germ groupoid. This groupoid plays a vital role in our theory, as we will see next. Because if we use a group, instead of a general inverse semigroup, the groupoid of germs reduces to the action groupoid.

It's time to introduce Benjamin Steinberg's work and his study about ample groupoid algebras [87] and other aspects of inverse semigroups.

Bringing back the work of Dokuchaev-Exel-Piccione ([28]), similar to group algebras, there are inverse semigroup algebras. More concretely, the partial algebra of a group is the inverse semigroup algebra of the associated Exel's universal semigroup. Its formal definition involves the same idea of group algebras, through a free module over the ring with the semigroup elements as the basis. In a series of papers, Steinberg (([85], [86] and [87]) proved that inverse semigroup algebras are closely related to universal groupoid algebras. Since we will restrict ourselves to finite semigroups, the universal groupoid is the same as the restriction groupoid (for instance, in [3] they exhibit the restriction groupoid of Exel's semigroup), and this is a particular case of a Steinberg algebra.

We can realize that the partial algebra of a group is the Steinberg algebra of the action groupoid associated with the Bernoulli partial action; we can use $\mathcal{D}$-classes to write this partial algebra in matrix terms.

The reader may permit us to discuss another algebraic structure. It includes the previous ones (if we adjoint a unit to inverse semigroups). There are categories, in the Mac Lane [57] sense, with behavior very similar to a multi inverse monoid object (or a multi inverse semigroup with unit). Each arrow has a unique inverse, and the "sandwich" identities from inverse semigroups are valid. Linckelmann [56] presents the basics of these "inverse categories" and extend some results from Steinberg – for instance, his algebra isomorphism via Möbius functions.

On the other hand, other perspectives took Cockett-Lack [22] to inverse categories. Their work originates from developing a better setting for categories of partial maps, which they named restriction categories. Using intuitive terms: restriction categories are categories where each morphism has a restriction that plays the role of its domain (rephrasing Dewolf-Pronk [25]). It turns out that inverse categories arise naturally as subcategories of restriction categories. Moreover, these categories have good behavior, in the sense that we can prove a version of the Wagner-Preston Theorem for them (cf. [56]), and even a version of the Ehresmann-Schein-Nambooripad Theorem (cf. [25]).

Inverse categories are a natural environment to develop partial actions. Because it combines the characteristics of inverse semigroups (partial maps) with the categorical aspects of groupoids (convolutions algebras), a more abstract perspective can reveal new connections or provide a better understand of already known results.

A few more facts, and we can wrap up all these ideas. Exel's first conception of the universal semigroup uses generators and relations. Later Kellendonk and Lawson ([46]) realized that his construction is equivalent to the Birget-Rhodes expansion of a group (cf [91] for details of this notion). As Piske showed, as this inverse semigroup is E-unitary, via Lawson's formulation of P-theorem, there is another way to interpret it. This way is more "friendly" because it resembles the Bernoulli partial action groupoid. Also, we can quickly identify the global and its partial Bernoulli actions.

Motivated by this exposition, we may ask:

> Does this approach, *i.e.* to use Bernoulli actions, apply to studying the expansions and algebras of other structures, more specifically inverse semigroups, groupoids, and inverse categories?

The answer is yes, and this thesis will explain this conclusion.

Content of the thesis and methodology

In the following paragraphs, we will explain our motivations and contributions to the theory. We will talk a bit more about the two expansions already present in the literature. The inverse semigroup Prefix of Lawson-Margolis-Steinberg [54] and from Buss-Exel [14]. Also, the Birget-Rhodes expansion of an ordered groupoid from Gilbert [38]. Then we introduce a new expansion in inverse categories that generalizes the previous ones. In each case, we will apply the same reasoning and methodology.

Our first achievement was generalizing Choi's formula for the enlargement of Exel's universal semigroup. In this subject, Steinberg defined a notion of (strong) Morita equivalence of inverse semigroups [88]; more specifically, in this work he proved that if a semigroup is an enlargement of another then these are Morita equivalent in the sense of [88]. We also present an application on the permutation group $\mathbb{S}_3$ and analyze both algebras and compare them.

Next we move to the semigroup case. Two distinct constructions of the expansion of an inverse semigroup are analyzed. One uses a generalization of the P-theorem for inverse semigroups due to the work of Lawson-Margolis-Steinberg in [54]. A few years later, in the same flavor of the generators and relations, Buss-Exel in [14] defined a prefix expansion of an inverse semigroup that includes the group case. It turns out that both constructions are equivalent.

A significant difference from the group case then appears: the richer idempotent structure of inverse semigroups will permit us to define two kinds of Bernoulli actions. One action takes into consideration the natural order of inverse semigroups, and the other imposes equality. These aspects are contemplated by the theory of partial actions, as Khrypchenko [48] shows. Moreover, the prefix expansions we gain are examples of λ- semidirect products from Billhardt [10], and the strict inverse semigroups of O'Carroll [67]. Paying attention to the algebras. Only the ones derived from the Bernoulli strict actions (those with equality) will define a Morita context, as a consequence of which pair of expansions satisfies the enlargement relation.

Afterward, we discuss ordered groupoids. Similar to groups and inverse semigroups, there is a notion of a Birget-Rhodes expansion of an ordered groupoid. Gilbert [38] proposed this new development. It's interesting to note that when the groupoid is inductive, Gilbert's expansion is the restriction groupoid of Lawson-Margolis-Steinberg expansion (cf. [38] page 181). As groupoid theory permits the definition of actions and (convolution) algebras, we managed to work similarly.

We present the expansion of a groupoid using the Bernoulli partial action. In this task, we used as base the generalization of P-theorem to groupoids by Gilbert [39], and Miller's enlargements, from her Ph.D. thesis [62]. Also, the groupoid semidirect product as Steinberg defined [84]. In retrospect, the definition of partial actions of ordered groupoids appear in the work of Bagio and Paques [7] [6] – a general categorical version was proposed by Nystedt [65].

Although groupoids may have several idempotents, to compose arrows, we need additional conditions, which reflects in Bernoulli's actions. These aspects reverberate in its expansions and, despite the earlier case of inverse semigroups where we have four expansions, we find two expansions and they satisfy an enlargement relation. Thus, two algebras Morita equivalent algebras.

The most significant amount of new theory appears in our final chapter, focused on inverse categories. Our primary references are Linckelmann [56], and Deowlf-Pronk [25].

This chapter contains our definition of an inverse category action, which leads us to the Bernoulli actions. The main point is how we can use idempotent morphisms – these arrows are the most crucial ingredient. Also, how to distinguish between "outer and inner" (object) behavior. As one would expect, from the aspects we highlighted, such actions combine distinct parts of inverse semigroups and groupoids actions.

The next move is a slight modification of the semidirect product's definition, which implies eight expansions. Half we termed external, and the other half internal. The first resembles Gilbert's expansions, and later Lawson's/Buss' expansions. Then, we develop enlargements and the study of the algebras.

Finally, we can extrapolate finiteness imposition , and using Kan extensions, study the representations of our expansions.

The above framework depends on the Bernoulli's actions. The methodology of our thesis is:

- first we will define the partial and the global Bernoulli actions;
- next, we use these actions and we will define our expanded structures;
- if it is possible, we also define the action groupoids;
- following, we will define the (convolution) algebras of each expansion and represent them as a direct product of matrix algebra;
- finally, we study the relation that the data from the partial and the global action share.

Graphically this is our methodology in each chapter:

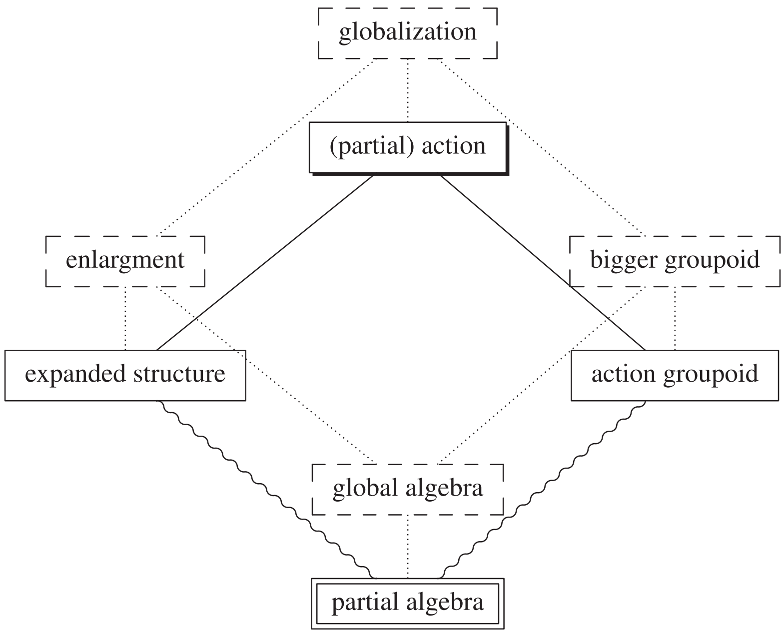


Figure 1: The Bernoulli approach

Even though the arguments are somewhat cyclic they increase in terms of abstraction. This way, we are moving in circles and upward, *i.e.*

This characteristic makes each chapter independent of the preceding ones, although the central theme is always the same.

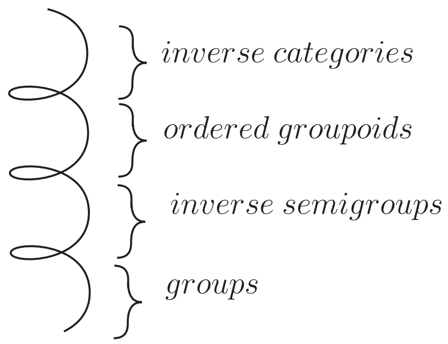


Figure 2: The increase of abstraction

Overview of the thesis and contributions

We give an overview of the structure of the thesis following each chapter. Also, we exhibit the main contributions. At the beginning of each chapter, there are a more detailed introduction and a synopsis. To expose our results more explicitly, they are the underlined theorems, propositions, lemmas, or definitions spread around the text.

Chapter 1: sets the necessary notations, the definitions, and properties of the structures we use.

Chapter 2: has two distinct parts. In the first one, we study the theoretical aspects of the expansion proposed by Exel employing (global and partial) Bernoulli's actions. The second part has a more detailed study of the global and the partial algebra, where we developed Choi's formula. Finally, we exemplify the usage of both formulas.

Bernoulli group actions	2.2.1
Global and partial inverse semigroups	2.2.4 2.2.5
Strong Morita equivalence of part. and glob. inverse semigroups	2.3.9
Morita equivalence of Bernoulli action groupoids	2.3.17
Internal structure of the global inverse semigroup	2.4.14
Global algebra formula	2.5.14

Table 1: The main contributions in Chapter 2

Chapter 3: we start with a summary of the Prefix expansion from Buss-Exel and their formulation of partial inverse semigroup actions. Next, we introduce the work of Lawson-Margolis-Steinberg and O'Carroll. After establishing the existent theory, we move to define the Bernoulli actions, the inverse semigroups we can induce, and its relations (of enlargements and algebras).

Bernoulli inverse semigroup actions	3.2.6 3.2.9 3.2.11 3.2.12
Global, partial and strict prefix expansions	3.3.3 3.3.4
Internal structure of the global prefix expansion	3.3.5
Enlargement relation among strict prefix expansions	3.4.2
Morita equivalence of strict global and partial algebras	3.4.3

Table 2: The main contributions in Chapter 3

Chapter 4: begins with the definitions of ordered and inductive groupoids and their actions. Following Gilbert and Miller's exposition, we set the equivalent formulation of fibred actions and actions by symmetries (or automorphisms). So we reinterpret Gilbert's expansion in Bernoulli's actions and study its relation with the newly ordered groupoid (global expansion). In the end, we use the enlargements from Miller applied to our case.

Bernoulli groupoid fibred and by sym. actions	4.4
Construction of the global Gilbert expansion	4.5.6
Gilbert's expansion via semidirect product	4.5.8
Morita equivalence of Gilbert expansions	4.5.9
Morita equivalence of the global and partial groupoids algebras	4.5.10

Table 3: The main contributions in Chapter 4

Chapter 5: we compile the main results about inverse categories from the perspective of restriction categories and similarities with inverse semigroups. We then introduce the definition of inverse category actions and semidirect product, which leads to Bernoulli's actions and the new expansions. Then we propose an enlargement definition and study how this relation reflects on the expansion's (convolution) algebras. Next, we realize how to use Kan extensions to study inverse category representation and we finish with a far abstract perspective.

Inverse category actions	5.2.1 5.2.3 5.2.6
Bernoulli inverse category actions	5.2.2
inverse category semidirect product	5.3.1
The Szendrei expansions	5.3.1
Enlargements of inverse categories	5.4.6
Equivalency of Cauchy completions	5.4.8
Enlargements of strict Szendrei expansions	5.4.9
The strict convolution algebras are Morita equivalent	5.5.9
Representations of categories using Kan extensions	5.6.6

Table 4: The main contributions in Chapter 5

Chapter 6: contains our final thoughts and points to (possible) future works.

Related works

The subject ”partial action” has a significant number of active research. This thesis addresses to the algebraic structures’ realm, but it is possible to name other areas developing new mathematics based on this topic. Like operator algebras, Hopf algebras, inverse semigroups theory, groupoid theory, category theory, monoid theory, topology, or (co)homology.

To illustrate the relations involving previous areas and ours, we used the software VOSviwer. It is a tool that permits to map scientific production by (co-) authorship’s network and exhibit the data utilizing graphs. [1]

We followed the steps:

- data base: Scopus;
- terms we searched: partial action and Hopf algebra; inverse semigroup, groupoid, groupoid algebra, étale groupoid, Mcalister triples, P-theorem, universal groupoid, partial action, inverse category, category action, ordered groupoid, groupoid action, algebroid;
- area: (only) mathematics;
- period: until December of 2020.

This research returned 3.587 documents. Then, we filtered by the 2.000 most relevant, and the next picture shows our results.

[1] We want to thank Jenifer G. for helping us with this section.

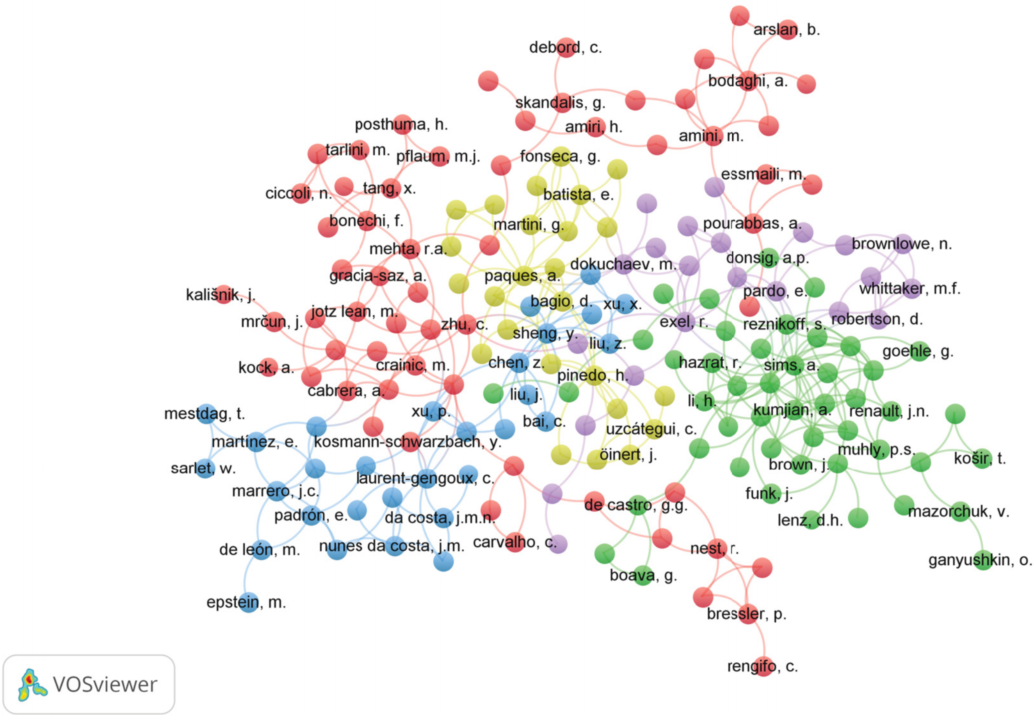


Figure 3: The map of researchers interaction

As the reader may realize, there are too much (in a good way) related researchers. In order to show our niche, we focused the picture on the cluster concerning algebraists. There goes the result.[2]

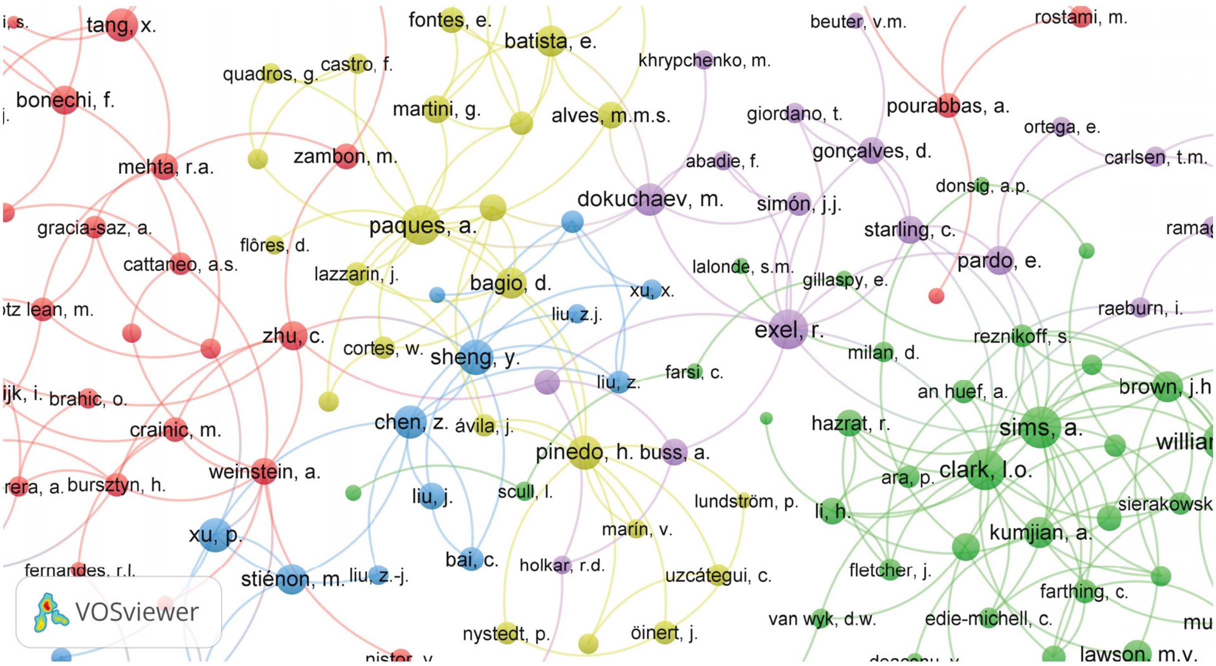


Figure 4: A zoom in the map of researchers interaction

[2]An interative version of this map can be found at https://sites.google.com/site/willianvelasco .

Final considerations

The thesis aims for the largest audience possible. Hence, we present many motivations, explanations, references, and computations. We also use lots of diagrams to summarize contents and some drawings when it is possible.

As we commented, each chapter develops the same methodology with a different structure. We tried to make the main chapters self-contained so that the reader can begin where he/she prefers. However, we recommend the original order.

Technical information: the text template is UFPR-thesis; to construct the diagrams, we use the xy-matrix package, the q.uiver.app web tool, and the tikz package via mathcha.io web tool.

Chapter 1

Basics and general definitions

This chapter contains the basic definitions of partial actions, inverse semigroups, groupoids, algebras of these structures, P-theorem, McAlister triples, and Möbius inversion formula we are going to use. We will briefly discuss each notion and references for further studies. All these concepts are based on a set-theoretical perspective, and we will use it throughout the entire work.

The last chapter of the thesis deals with inverse categories. We decided to present in this chapter the basics of category theory. The specific elements of inverse categories will be presented in Chapter 5.

1.1 Partial actions of groups

Actions give a notable example of a group homomorphism. For more profound results, we suggest the reader [78] Chapter C.

Definition 1.1.1. ([78]) Let G be a group, X be a nonempty set. A group action of G on X is a map $\theta : G \times X \to X$ denoted by $\theta(g, x) = \theta_g(x) = gx$ satisfying

(I) $\theta_g \circ \theta_h(x) = \theta_{gh}(x)$;

(II) $\theta_e(x) = x$.

Remark 1.1.2.

(a) More specifically this is a left action of the group G on X; is the same way one can define a right action $\theta : X \times G \to X$ requiring that $\theta_g(x) = xg$ satisfies

(i)‘ $\theta_g \circ \theta_h(x) = \theta_h g(x)$, and

(ii)‘ $\theta_e(x) = x$.

(b) Any action like in Definition 1.1.4 determines a group homomorphism $G \to Symm(X)$, where $Symm(X)$ is the group of bijections of X.

Indeed, let θ be an action. Fixing $g \in G$ we have a map $\theta_g : X \to X$ with $\theta_g(x) = gx$, whose inverse is $\theta_{g^{-1}}$. Finally the desired homomorphism is $\theta : G \to Symm(X)$ defined by $\theta(g) = \theta_g$ (note that (I) implies $\theta(g) \circ \theta(h) = \theta(gh)$).

(c) We can rephrase Cayley's ([79]) Theorem in terms of actions. Thus G acts on itself by left translations $\theta_g : X \to X$ where $\theta_g(x) = gx$.

We end this section with the following definition.

Definition 1.1.3. ([78]) Let $\theta : G \times X \to X$ be an action.

(I) The *orbit* of $x \in X$ is the subset $\mathcal{O}(x) := \{gx; g \in G\} \subseteq X$.

(II) The action is *transitive* if there is only one orbit.

(III) The *stabilizer* of $x \in X$ is the subset $G_x := \{g \in G; gx = x\} \subset G$.

(IV) The set of all orbits is the *orbit space*; denoted by X/G.

We've only scratched the surface of group action theory, but this is enough for our purposes.

As mentioned in the introduction, the study of partial actions was set in motion by researchers because of Ruy Exel in his field, operator algebras. He wrote a book ([34]) compiling the main concepts and results of this theory, which will be our primary reference for partial actions. Although his focus is partial dynamical systems, the first three chapters suit our algebraic purposes and ambitions. The expert may notice that some definitions are not like in many papers. Despite differences, the heart of the theory remains the same.

Definition 1.1.4. ([34]) Let G be a group and X a nonempty set. A *partial action* of G on X is a pair $\theta = (\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\hat{A}\ \hat{A}\ )$ where $D_g \subseteq X$ and $\theta_g : D_{g^{-1}} \to D_g$ are maps satisfying:

(I) $D_e = X$ and $\theta_e = 1_X : X \to X$, with $e \in G$ the identity element and 1_X is the identity map on X;

(II) $\theta_g \circ \theta_h \subseteq \theta_{gh}$ for all $g, h \in G$.

Where $\subseteq$ stands for map extension.

Some words and remarks.

Remark 1.1.5.

(a) The composition in LHS makes sense on the largest set possible, *i.e.* $\mathrm{dom}(\theta_g \circ \theta_h) = \theta_h^{-1}(D_h \cap D_{g^{-1}})$. Spoiling further sections, the symbol $\subseteq$ stands for function extension, as in Example 1.2.3.

(b) Readily from the definition: θ_g is a bijection for all $g \in G$ and $\theta_{g^{-1}} = \theta_g^{-1}$.

Indeed, if one uses the second axiom of the definition with g and g^{-1}, then $\theta_{g^{-1}} \circ \theta_g \subseteq \theta_e = 1_X$. Hence $\theta_{g^{-1}} \circ \theta_g$ is a restriction of the identity map in its domain $D_{g^{-1}}$. The same arguments hold for $\theta_g \circ \theta_{g^{-1}}$ in D_g. Finally, the equality $\theta_{g^{-1}} = \theta_g^{-1}$ is a straightforward calculation.

(c) The special case when $D_g = X$ for every g in G gives us the Definition 1.1.1. Thus partial actions generalize actions. We will call the group actions of the previous section *global actions* from now to avoid misunderstanding.

The next proposition provides an equivalent definition of a partial action definition (cf. Definition 1.1.4).

Proposition 1.1.6. ([34]) Let G be a group and$\{D_g\}_{g\in G}$ be a family of subsets of a set X and $\{\theta_g : D_{g^{-1}} \to D_g\}_{g\in G}$ be a family of bijections. Then $\theta = (\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\hat{A}\ \hat{A}\ )$ is a partial action if, and only if,

(I) $D_e = X$ and $\theta_e = 1_X : X \to X$, with $e \in G$ being the identity element and 1_X the identity map on X;

(II) $\theta_g(D_{g^{-1}} \cap D_h) \subseteq D_{gh}$ for all $g, h \in G$;

(III) $\theta_g \circ \theta_h(x) = \theta_{gh}(x)$ for $x \in D_{h^{-1}} \cap D_{gh^{-1}}$.

Proof. Before we start, the conditions of composition stated above deserve verification. If $x \in D_{h^{-1}}$, then $\theta_h(x)$ is allowed.Â Using the fact that $x \in D_{(gh)^{-1}}$, the calculation of $\theta_{gh}(x)$ is well defined. Finally $\theta_h(x) \in \theta_h(D_{h^{-1}} \cap D_{(gh)^{-1}}) \subseteq D_{g^{-1}} = \mathrm{dom}(\theta_g)$, by (II). So we can compute the composition and continue the proof. Suppose $\theta = (\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\hat{A}\ \hat{A}\ )$ is a partial action of G in X. By definition the domain of the composition $\theta_g \circ \theta_h$ is the set $\theta_h^{-1}(D_h \cap D_{g^{-1}})$. The second partial actionâs axiom says thatÂ $\theta_g \circ \theta_h$ has θ_{gh} as its extension, whence $\theta_h^{-1}(D_h \cap D_{g^{-1}}) \subseteq D_{(gh)^{-1}}$. Thus, picking $h = g^{-1}$ and $g = h^{-1}$:

$$\theta_g(D_h \cap D_{g^{-1}}) \stackrel{1.1.5}{=} \theta_{g^{-1}}^{-1}(D_h \cap D_{g^{-1}}) \subseteq D_{gh}.$$

Next, let $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$. By (II) of Definition 1.1.4: $\theta_h(D_{h^{-1}} \cap D_{(gh)^{-1}})) \subseteq D_{g^{-1}}$. Hence $\theta_h(x) \in D_{g^{-1}}$ and $x \in D_{h^{-1}}$ implies that $\theta_h(x) \in D_h$. Consequently, $\theta_h(x) \in D_{g^{-1}} \cap D_h$. Therefore, all axioms of this proposition are verified. On other hand, suppose $(I - III)$. We need to show the axioms of partial actionâs definition (cf. Definition 1.1.4). Choosing $h = g^{-1}$ and then replacing g by g^{-1} in (III) we have $\theta_g \circ \theta_{g^{-1}} = \theta_{g^{-1}} \circ \theta_g = \theta_e = 1_X$. This deduction also shows us $\theta_{g^{-1}} = \theta_g^{-1}$. Using this identity and (II):

$$\theta_g(D_{g^{-1}} \cap D_h) = \theta_{g^{-1}}^{-1}(D_{g^{-1}} \cap D_h) \subseteq D_{(gh)^{-1}}.$$

Finally, let $x \in \theta_{h^{-1}}(D_{g^{-1}} \cap D_h)$:

$$x \in \theta_h^{-1}(D_{g^{-1}} \cap D_h) = \theta_{h^{-1}}(D_{g^{-1}} \cap D_h) \subseteq D_{(gh)^{-1}}.$$

Further $\mathrm{dom}(\theta_g \circ \theta_h) \subset \mathrm{dom}(\theta_{gh})$. Also $x \in D_{h^{-1}}$, so $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$. As θ is a partial action, θ_{gh} extends $\theta_g \circ \theta_h$. Consequently $\theta_g \circ \theta_h = \theta_{gh}$. □

We can make an even better improvement.

Proposition 1.1.7. ([34]) Let $\theta = (\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ be a partial action of a group G on a set X. Then $\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$ for all $g, h \in G$.

Proof. Assuming θ a partial action, by previous equivalency we have $\theta_g(D_{g^{-1}} \cap D_h) \subseteq D_{gh}$. In addition $\mathrm{im}(\theta_g) \subseteq D_g$ then $\theta_g(D_{g^{-1}} \cap D_h) \subseteq D_g \cap D_{gh}$. If we apply $\theta_{g^{-1}}$ to both sides

$$D_{g^{-1}} \cap D_h \subseteq \theta_{g^{-1}}(D_g \cap D_{gh}).$$

Replacing g by g^{-1} and h by gh we obtain $D_g \cap D_{gh} \subseteq \theta_g(D_{g^{-1}} \cap D_h)$. Consequently, applying θ_g, we reach the desired relation. □

A few examples will help us to visualize and reinforce the concept explained above. They are from Batista's survey paper [9], and the master thesis of Paiva [68]. Notice that other papers may show some of these examples first, but we cite the papers we used to learn this subject. The reader interested in some analysis flavored type of applications will enjoy the reader list in Ruy Exel's homepage http://mtm.ufsc.br/ exel/http://mtm.ufsc.br/ exel/).

Example 1.1.8.

(1) Let $G = (\mathbb{Z}, +)$ be a group and $X = \mathbb{Z}_+$ be a set. We define a partial action of G in X:

- domains: for each $n > 0$ $D_n := \{m \in \mathbb{Z}_+; m \geqslant n\}$ and $D_{-n} := \mathbb{Z}_+$;
- maps: $\theta_n : D_{-n} \to D_n$ with $\theta_n(m) = n + m$.

Readily one verifies the axioms in Definition 1.1.4.

(2) Let $G = (\mathbb{Z}_4, +) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ and a group and $X = \{(1,0), (0,1), (-1,0)\}$ be a set. The partial action components are:

- domains: $D_{\overline{0}} := X$, $D_{\overline{1}} := \{(0,1), (-1,0)\}$, $D_{\overline{2}} := \{(1,0), (-1,0)\}$ and $D_{\overline{3}} := \{(1,0), (0,1)\}$;
- maps: $\theta_{\overline{g}} : D_{\overline{g}^{-1}} \to D_{\overline{g}}$ with $\theta_{\overline{0}} \equiv 1_X$ and $\theta_{\overline{g}}(x,y) := (x\cos(g\frac{\pi}{2}) - y\sin(g\frac{\pi}{2}), x\sin(g\frac{\pi}{2}) + y\cos(g\cos(g\frac{\pi}{2}))\hat{A}\ )$

The verification is easy using the equivalences of Proposition 1.1.6 and Proposition 1.1.7.

(3) (This example appears in Kellendonk-Lawson [46].) Let $G = PSL_2(\mathbb{C})$ the group of complex projective transformations on $\mathbb{CP}^1 \simeq \mathbb{S}^2$. Although the global action of this group is in the entirely Riemann sphere, the partial action will act on the complex plane only. Hence let $X := \mathbb{C}$: for all $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{C})$

- domains: $D_{g^{-1}} := \begin{cases} \mathbb{C} \setminus \{\frac{-d}{c}\}, & c \neq 0 \\ \mathbb{C}, & c = 0 \end{cases}$ and $D_g := \begin{cases} \mathbb{C} \setminus \{\frac{a}{c}\}, & c \neq 0 \\ \mathbb{C}, & c = 0 \end{cases}$;
- maps: $\theta_g : D_{g^{-1}} \to D_g$ with $\theta_g(z) = \dfrac{az+b}{cz+d}$.

(4) Given a global action $\Theta : G \to S_Y$, of a group G in the set Y, we can restrict this action to a partial action of G in a subset $X \subseteq Y$. Indeed for each $g \in G$ we define:

- domains: $D_g = X \cap \Theta_g(X)$;
- maps: $\theta_g : D_{g^{-1}} \to D_g$ with $\theta_g := \Theta_g D_{g^{-1}}$.

Thus $\theta = (\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ is a partial action of G in X.

A natural question arises from the Example 1.1.8 : when the converse holds? If it is possible to define a global action from a partial given. This question is a problem of *globalization*. The following definition and theorem are from Batista's survey [9].

Definition 1.1.9. ([34]) Given a partial action $\theta = (\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ of a group G on a set X we say (Y, Θ, ϕ) is a *globalization* if:

(I) Θ is a global action of G on Y;

(II) $\phi : X \to Y$ is injective;

(III) for each $g \in G$, $\phi(D_g) = \phi(X) \cap \Theta_g(\phi(X))$;

(IV) for each $x \in D_{g^{-1}}$, $\phi(\theta_g(x)) = \Theta_g(\phi(x))$

(V) $Y = \cup_{g \in G} \Theta_g(\phi(X))$.

Theorem 1.1.10. *([34]) Let $\theta = (\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ be a partial action of the group G on a set X, then there is a globalization for θ.*

Proof. The primary task is to define the larger set and the global action, then verification of $(I - V)$ above is routine.

First we define a relation on $G \times X$ by: $(g, x) \sim (h, y)$ if, and only if $x \in D_{g^{-1}h}$ and $\theta_{h^{-1}g}(x) = y$. This relation is an equivalence relation.

Next, let $Y := (G \times X)/_{\sim}$, whose elements are classes represented by $[g, x]$ where $(g, x) \in G \times X$. Thus the map $\Theta_g([h, x]) = [gh, x]$ for each $g, h \in G$ and $x \in X$ is a well defined global action of G in Y. Finally, the inclusion map is $\phi : X \to Y$ with $\phi(x) = [e, x]$. □

Notice that the problem of globalization related to partial group actions is easy in this context. But if we put extra data, *e. g.* a Hausdorff space, the global space may not be Hausdorff. The globalization of Example 1.1.8-(3) proved by Kellendonk and Lawson [46], coincide with the one-point compactification of the complex plane, *i. e.* the Riemann sphere. One can look at Abadie's thesis [1] for more examples with topology.

1.2 Inverse semigroups

The guideline of the above definitions and results is the first chapter of the book from Lawson [51].

Definition 1.2.1. ([51]) Let S be a nonempty set.

(I) We say the set S is a *semigroup* if there is a binary associative operation $\cdot : S \times S \to S$. By notation $(S, \cdot)$

(II) Also, S is an *inverse semigroup* if S is a semigroup and for every $s \in S$ there is a unique (inverse) element s^* satisfying: $s = s \cdot s^* \cdot s$ and $s^* = s^* \cdot s \cdot s^*$.

To avoid cumbersome notation, we will write $s = ss^*s$ meaning $s = s \cdot s^* \cdot s$.

Remark 1.2.2.

(a) Avoiding too many symbols, we will write only S, instead of $(S, \cdot)$. When necessary we will exhibit the product rule.

(b) Given a semigroup if there is an identity element we call this semigroup a *monoid*.

(c) An *inverse sub-semigroup* A of the inverse semigroup S is a subset of S closed by inversions and by the binary operation.

(d) An alternative way to define inverse semigroups follows.

A nonempty semigroup S is a *inverse semigroup* if

it is regular: for all $a \in S$ there is an element $b \in S$ such that $a = aba$ and $b = bab$, b is called an inverse of a;

and the idempotent elements commute: if $f, g \in S$ satisfy $f^2 = f$ and $g^2 = g$ then $fg = gf$.

(e) Fixing another notation, $\mathcal{E}(S)$ is the set of idempotents of S, that means, the set of elements satisfying $ff = f$. Worth it to mention that $\mathcal{E}(S)$ is a commutative sub-semigroup of S.

Next, let us see examples of semigroups.

Example 1.2.3.

(1) Goups: every group G is an inverse semigroup. The opposite is true if $\mathcal{E}(G) = \{e\}$.

(2) Partial maps: set X and Y non-empty sets, a *partial map* from X to Y is a map $f : A \subseteq X \to B \subseteq Y$. The domain and range in this new context will be denoted by dom(f) and im(f), to avoid confusion. We also highlight two special partial functions from X to Y:

- *empty map*: $0_{XY} : \emptyset \subseteq X \to \emptyset \subseteq Y$ and
- *partial identities*: for each $A \subset X$ the identity on A given by $1_A : A \to A$.

As we composite (ordinary) maps, we can define composition of partial maps on the largest domain where it is possible. For instance, let $g : X \to Y$ and $f : Y \to Z$ partial maps, $f \circ g$ makes sense on

$$dom(f \circ g) = g^{-1}(\text{im}(g) \cap \text{dom}(f)).$$

If $x \in \text{dom}(f \circ g)$ we have $(f \circ g)(x) = f(g(x))$. If $\text{im}(g) \cap \text{dom}(f) = \emptyset$ then we say $f \circ g = 0_{XY}$. In this manner the set of partial maps is a semigroup

(3) Partial symmetries: all precedent discussion is possible if we ask $f : X \to X$ to be a bijective partial map. Thus the set of partial symmetries $\mathcal{I}(X) := \{\text{partial bijections from } X \text{ to } X\}$ is the analogue of the set of symmetries and forms an inverse semigroup.

Once we already know a semigroup, we need to learn some algebraic proprieties of semigroups, define ideal and a (natural) partial order. The following proofs are straightforward calculations, and the reader may check at Lawson [51] Chapter 1 - Section 1.4.

First, we have the properties of inverses. Like groups, the ”inverse operation” on semigroups behave as expected, as we can see in the next proposition.

Proposition 1.2.4. ([51]) Let S be an inverse semigroup and $s \in S$, s^* be the inverse of S and $e \in \mathcal{E}(S)$:

(i) ss^* and s^*s are idempotents and $s(s^*s) = s$ and $(ss^*)s = s$;

(ii) $(s^*)^* = s$;

(iii) $s^*es \in \mathcal{E}(S)$;

(iv) $e^* = e$,

(v) $(s_1 s_2 \ldots s_n)^* = s_n^* \ldots s_1^*$ and

(vi) there is $f \in \mathcal{E}(S)$ such that $es = sf$ and $se = fs$.

Next, we define ideals in semigroups and some of their properties.

Definition 1.2.5. ([51]) Let S be a semigroup, a nonempty subset $I \subseteq S$ is an *left ideal* if: $sa \in I$ for every $a \in I$ and for each $s \in S$. Analogously we define *right ideals* changing the product order, *i.e.* $as \in I$. *Ideals* are subsets which are both left and right ideals.

As usual notation in ring theory, given a semigroup S and an element $a \in S$ the left ideal that $\{a\}$ generates is the set $aS := \{as; s \in S\}$. In the same way we define Sa.

Proposition 1.2.6. ([51]) Let S be an inverse semigroup, $a, b \in S$ and $e, f \in \mathcal{E}(S)$:

(i) $aS = aa^*S$ and $Sa = Sa^*a$;

(ii) $eS \cap fS = efS$ and $Se \cap Sf = Sef$ and

(iii) $(b^*a^*)aS = (b^*a^*)(ab)S$.

Beyond the above algebraic structures, inverse semigroups also have a natural way to relate their elements.

Definition 1.2.7. ([51]) Let S be an inverse semigroup, we define the relation $\leqslant$ between elements of S by: $s \leqslant t \Leftrightarrow \exists e \in \mathcal{E}(S)$ s.t. $s = te$.

Example 1.2.8. The main example, for our purpose, is the natural relation among elements of $\mathcal{I}(X)$, where X is a nonempty set. This order is function extension, *i.e.*, for $f, g \in \mathcal{I}(X)$: $f \leqslant g \Leftrightarrow \text{dom}(f) \subseteq \text{dom}(g)$ and $f(x) = g(x)$ for all $x \in \text{dom}(f)$. This relation among functions will be denoted by: $f \subseteq g$.

One crucial example of inverse semigroup depends on the order relation.

Definition 1.2.9. ([51]) An inverse semigroup S is E-unitary if

$$es \in \mathcal{E}(S) \implies s \in \mathcal{E}(S),$$

for $e \in \mathcal{E}(S)$ and $s \in S$.

This next assertion states properties of the natural relation $\leqslant$.

Proposition 1.2.10. ([51]) Given S an inverse semigroup.

(i) $(S, \leqslant)$ is a partial ordered pair, *i.e.*, $\leqslant$ satisfies reflexivity, skew symmetry and transitivity.

(ii) Given $s, t \in S$ are equivalents:

(α) $s \leqslant t$;

(β) $\exists f \in \mathcal{E}(S)$ such that $s = ft$;

(γ) $s^* \leqslant t^*$;

(δ) $s = ss^*t$;

(ϵ) $s = ts^*s$.

(iii) Let $s_1, s_2, s_3, s_4 \in S$:

- $s_1 \leq s_2$ and $s_3 \leqslant s_4 \Rightarrow s_1s_3 \leqslant s_2s_4$ and $s_3s_1 \leqslant s_4s_2$;
- $s_1 \leqslant s_2 \Rightarrow s_1^* \leqslant s_2^*$;
- $s_1 \leqslant s_2 \Rightarrow s_1s_1^* \leqslant s_2s_2^*$ and $s_1^*s_1 \leqslant s_2^*s_2$.

Definition 1.2.11. ([51]) Let $(P, \leqslant)$ be a poset. We say that P is a *(meet) semilattice* if for every pair of elements $x, y \in P$ there exists the greatest lower bound $z := x \wedge y \in P$.

Using other terms, last definition tells us that z is the largest of the lower bound of x and y, *i.e* $z \leqslant x, y$. For instance, for any inverse semigroup S, we can define the semilattice $(\mathcal{E}(S), \leqslant)$, where $\leqslant$ is the natural order defined in Proposition 1.2.7, and the greatest lower bound of $e, f \in \mathcal{E}(S)$ is $z = ef$.

Next, we show the definition of homomorphism between semigroups.

Definition 1.2.12. ([51]) Let S and T be semigroups, a *semigroup homomorphism* is a map $\theta : S \to T$ such that: $\theta(st) = \theta(s)\theta(t)$, for all $s, t \in S$. If θ is injective and surjective, we say θ is a *semigroup isomorphism*. In addition, if both S and T are equipped with the natural partial relation $\leqslant$ (Â as in Definition 1.2.7), we say θ *preserves relations* if $s \leqslant t$ in S implies $\theta(s) \leqslant \theta(t)$ in T.

The next proposition states relevant and expected calculations with semigroup homomorphisms.

Proposition 1.2.13. ([51]) Given a semigroup homomorphism $\theta : S \to T$:

(i) $\theta(s^*) = \theta(s)^*$;

(ii) $e \in \mathcal{E}(S) \Rightarrow \theta(e) \in \mathcal{E}(T)$;

(iii) if $\theta(s) \in \mathcal{E}(T)$, there is $e \in \mathcal{E}(S)$ such that $\theta(s) = \theta(e)$;

(iv) $\theta(S) \subseteq T$ as a sub semigroup;

(v) if $U \subseteq T$ is a sub semigroup, then $\theta^{-1}(U) \subseteq S$ is a sub semigroup;

(vi) θ preserveÂ relations;

(vii) given $s, t \in S$ satisfying $\theta(s) \leqslant \theta(t)$, there is $a \in S$ such that $a \leqslant t$ and $\theta(a) = \theta(s)$.

Finally, we are about to state the Wagner-Preston theorem, an analog of Cayley's theorem, which asserts that any inverse semigroup is isomorphic to a sub-semigroup of $\mathcal{I}(X)$, for some X.

Theorem 1.2.14 (Wagner-Preston)**.** *([51]) Let S be an inverse semigroup, then exists a set X and an injective inverse semigroup homomorphism $\theta : S \to \mathcal{I}(X)$ such that for all $s, t \in S$: $s \leqslant t \Leftrightarrow \theta(s) \subseteq \theta(t)$.*

1.3 Groupoids

In this sections we discuss basic properties of groupoids in a set theoretical approach, as in [51], [80], [80] and [33]. We will list the definition and basic properties from such references.

Definition 1.3.1. ([80])

A *groupoid* is a set $\mathcal{G}$ with a map (the inversion map) $x \in \mathcal{G} \mapsto x^{-1} \in \mathcal{G}$ a set $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ (the set of composable pairs) and a map (the groupoid product) $(x, y) \in \mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G} \mapsto xy \in \mathcal{G}$ such that:

(I) $(x^{-1})^{-1} = x$ and $(x, x^{-1}) \in \mathcal{G}^{(2)}$ $\forall x \in \mathcal{G}$

(II) $(x, y), (y, z) \in \mathcal{G}^{(2)} \Rightarrow (xy, z), (x, yz) \in \mathcal{G}^{(2)}$ and $(xy)z = x(yz)$;

(III) $\forall (x, y) \in \mathcal{G}^{(2)}$ $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

Also, a groupoid comes equipped with the maps[1] $\mathtt{domain}, \mathtt{range} : \mathcal{G} \to \mathcal{G}^{(0)} := \{xx^{-1}, x \in \mathcal{G}\}$

$$\mathtt{domain}(x) = x^{-1}x, \mathtt{range}(y) = yy^{-1}.$$

Notation: **in our entire work we use $\mathtt{domain} = d$ and $\mathtt{range} = r$.**

We will refer to elements in $\mathcal{G}^{(0)}$, as the *units* of $\mathcal{G}$.

[1] These two maps are also known as source and target maps.

Based on the last definition, we will define categories from a set point of view.

Definition 1.3.2. ([52]) A *(small) category in the sense of Ehresmann* is a groupoid whose elements may have no inverses.

Alternatively, one can define a groupoid as a small category where all arrows are invertible, in the sense of Category theory from Mac Lane [57]. We will return to this topic in the last section of this chapter.

Using this approach, the elements of a groupoid $\mathcal{G}$ are called *arrows*, and the units *objects*. Since these two definitions of groupoids are equivalent, in the following pages we will combine aspects of both; for instance using the product and inversion maps, while referring to the elements as *arrows*.

Next, we define local bissection, which provides a link between groupoid theory and inverse semigroup theory.

We define connected components in the following way.

Definition 1.3.3. ([51]) Two elements x and y in $\mathcal{G}$ are *connected* if there is a sequence of composable elements from $d(x)$ to $r(y)$. The equivalence classes of this relation are the *connected component* of $\mathcal{G}$.

If $\mathcal{G}$ has one connected component, we say it is *connected*.

Likewise groups, groupoids also act on sets.

Definition 1.3.4. ([94]) Let $\mathcal{G}$ be a groupoid, X be a set and $\rho : X \to \mathcal{G}^{(0)}$ be a map. Define the set

$$\mathcal{G}_d \times_\rho X := \{(g, x) \in \mathcal{G} \times X; d(g) = \rho(x)\}.$$

A *ρ-action*, or a *fibred action*, of $\mathcal{G}$ on X is a map $\theta : \mathcal{G}_d \times_\rho X \to X$, with $(g, x) \mapsto \theta_g(x)$ such that:

(I) $\theta_{\rho(x)}(x) = x$ for all $x \in X$;

(II) for $x \in X$, and $g \in \mathcal{G}$ such that there exists $\theta_g(x)$, we have $\rho(\theta_g(x)) = r(g)$;

(III) if $(h, x) \in \mathcal{G}_d \times_\rho X$ and $(g, h) \in \mathcal{G}^{(2)}$ then $(gh, x), (g, \theta_h(x)) \in \mathcal{G}_d \times_\rho X$ and $\theta_g(\theta_h(x)) = \theta_{gh}(x)$.

Notation: $(\rho, \theta) : \mathcal{G} \curvearrowright X$.

The map ρ is known in the literature by the names: anchor map, moment map, or momentum map.

Finally, a notion allows us to define inverse semigroups from groupoids (without any extra structure).

Definition 1.3.5. ([52]) A subset $U \subseteq \mathcal{G}$ is a *partial bissection* (or a *slice*) if $d_{|_U}$ and $r_{|_U}$ are injective.

Notation: $\text{Bis}(\mathcal{G}) := \{U \subseteq \mathcal{G}; U \text{ is a partial bissection }\}$ Another way to characterize this notion is the following. Before we state the results, we will exhibit the definitions of the inversion and product in $\text{Bis}(\mathcal{G})$: given $U, V \in \text{Bis}(\mathcal{G})$

- $U^{-1} = \{g^{-1}; g \in U\}$
- $UV = \{xy; x \in U, y \in V, \exists xy\}$

Lemma 1.3.6. *([52])] A subset $U \subseteq \mathcal{G}$ is a bissection if, and only if, $U^{-1}U, UU^{-1} \subseteq \mathcal{G}^{(0)}$.*

The link mentioned above is due to this proposition.

Proposition 1.3.7. ([52]) Let $U, V \in \text{Bis}(\mathcal{G})$, then

(i) $U^{-1} \in \text{Bis}(\mathcal{G})$,

(ii) $\text{Bis}(\mathcal{G})$ is an inverse semigroup under subset multiplication.

The opposite direction is also available, *i.e.* given an inverse semigroup it is possible to define a groupoid. This result can be found both in Paterson [70] Proposition 1.0.1 and Lawson [51] Section 3 Proposition 4.

Proposition 1.3.8. ([51]) Let S be an inverse semigroup. The set S with structure:

arrows: the set S

units: the set $\mathcal{E}(S)$

inverse map: the involution map of S, only renamed $()^* =: ()^{-1}$

multiplication: there exists the st of elements $s, t \in S$ if, and only if, $s^{-1}s = tt^{-1}$

domain and range: $d(s) = s^{-1}s$ and $r(t) = tt^{-1}$

is a groupoid called the *restriction groupoid* of S and denoted by $\mathcal{G}_S$.

1.3.1 Groupoid of (partial) group actions

Let $\Theta : G \curvearrowright X$ be an action of a group G on a set X. There is a canonical example of a groupoid defined by this action.

Definition 1.3.9. ([51]) The set $\Gamma_\Theta = X \times G$ with structure:

- $\Gamma_\Theta^{(0)} = X \times \{e\} \simeq X$

- $xg^{-1} = \Theta_g(x)g^{-1}$
- $ygxh = xgh$ if $y = \Theta_h(x)$
- $dxg = x$ and $rxg = \Theta_g(x)$

is a groupoid called the *action groupoid* of the global action Θ.

Using a very similar definition, we can transport this construction to partial actions. Consider $\theta = (\{D_g\}_g, \{\theta_g\}_g)$ a partial group action of G on X.

Definition 1.3.10. ([46] [2]) The set $\Gamma_\theta = \{xg \in X \times G; x \in D_{g^{-1}}\}$ with structure:

- $\Gamma_\theta^{(0)} = X \times \{e\} \simeq X$
- $xg^{-1} = \theta_g(x)g^{-1}$
- $ygxh = xgh$ if $y = \theta_h(x)$
- $dxg = x$ and $rxg = \theta_g(x)$

is a groupoid called the *action groupoid of the partial action* θ.

1.3.2 Étale and ample groupoids

The references used in this section are: [73] and the lectures of [52] and [33].

We begin with topological groupoids.

Definition 1.3.11. ([33]) A *topological groupoid* is a groupoid $\mathcal{G}$ such that all structural maps are continuous, where $\mathcal{G}^{(2)}$ and $\mathcal{G}^{(0)}$ carry the induced topology (or the subspace topology). Moreover we say that $\mathcal{G}$ is *open* if d is an open map. And we say that $\mathcal{G}$ is *étale* if d is a local homeomorphism.

A particular example that we use a lot in our work is the action groupoid of a group. As our groups are always discrete, its action groupoid is étale.

In the topological realm, the inverse semigroup Bis(G) plays an important role, as we state below.

Lemma 1.3.12. *([33]) Let $\mathcal{G}$ be a topological groupoid. Then the subset of Bis$(\mathcal{G})$ consisting of the open bissections of $\mathcal{G}$ forms a basis for the topology.*

Despite its heavily topological structure, there is an algebraic equivalence of the étale notion, due to Resende [73].

Theorem 1.3.13. *([73]) Let $\mathcal{G}$ be an étale groupoid and define let $\Omega(\mathcal{G})$ be the lattice of open sets of $\mathcal{G}$. Then $\mathcal{G}$ is étale if, and only if, $\Omega(\mathcal{G})$ is a monoid under subset multiplication.*

The last definition of this subsection is ample groupoids, from Steinberg [87] Section 3.

Definition 1.3.14. ([87]) An étale groupoid is called *ample* if the compact bissection forms a basis for its topology.

1.3.3 Groupoids of germs

Like any (partial) action of a group naturally defines a groupoid, an action of an inverse semigroup defines one too. Nevertheless, this is slightly different in notation. To define such structures, we must deal with some germs (likewise the germs of sheaves). This way, the abundance of idempotents in a semigroup is under control.

The definition we will present follows from the reinterpretation of Paterson's universal groupoid ([70] Section 4.3) made by Exel in [33]. The examples and properties we take from Steinberg [87] Chapter 5.

We begin with the definition of an action of an inverse semigroup.

Definition 1.3.15. ([33]) Let S be an inverse semigroup and X a set. An *action* θ of S on X, or $\theta : S \curvearrowright X$ is a homomorphism $\theta : S \to \mathcal{I}(X)$ with

$$\begin{aligned} s \in S \mapsto \theta_s : D_{s^*s} &\to D_{ss^*} \\ x &\mapsto \theta_s(x) \end{aligned}$$

such that $X = \bigcup_{s\in S} D_{s^*s}$.

It follows from this definition:

$$\theta_s\theta_{s^*}\theta_s = \theta_s \text{ and } \theta_{s^*}\theta_s\theta_{s^*} = \theta_{s^*} \implies \theta_{s^*} = \theta_s^{-1}.$$

And that $\theta_e = 1$ on its domain if $e \in \mathcal{E}(S)$.

There is an important inverse semigroup action (for our work) that we define now.

Definition 1.3.16. ([51]) Let S be an inverse semigroup. The action $\mu : S \curvearrowright \mathcal{E}(S)$ defined for each $s \in S$ by

$$\begin{aligned} \mu_s : D_{s^*s} &\to D_{ss^*} \\ e &\mapsto \mu_s(e) = ses^*, \end{aligned}$$

with $D_{s^*s} = \{t \in S; t \leqslant s^*s\}$ and $D_{ss^*} = \{p \in S; p \leqslant ss^*\}$, is called the *Munn action*.

Next, let us define the groupoid of germs of an action $\theta : S \curvearrowright X$. First, define a set and a relation in this set:

$$\Omega := \{xs \in X \times S; x \in D_{s^*s}\}$$

$$xs \sim yt \iff x = y \text{ and } \exists e \in \mathcal{E}(S) \text{ s.t. } x \in D_e,\ se = te.$$

Proposition 1.3.17. ([33]) Let $\mathcal{G}(\theta, S, X) := \Omega_{/\sim}$ be the set formed by classes xs, together with

<u>units</u>: for $e \in \mathcal{E}(S)$ and $x \in D_e$ we can naturally identify: $xe \in \mathcal{G}(\theta, S, X)^{(0)} \mapsto x \in X$

<u>inversion</u>: $xs^{-1} = \theta_s(x)s^*$

<u>multiplication</u>: $xsyt = yts \iff x = \theta_t(y)$

<u>domain and range</u>: $d(xs) = x$ and $r(xs) = \theta_s(x)$.

Then $\mathcal{G}(\theta, S, X)$ is an étale groupoid whose topological basis is the set

$$\mathcal{O}(s, U) := \{xs \in \Omega_{/\sim}; s \in S, x \in U \subseteq D_{s^*s}\}.$$

Some particular cases deserve our attention:

1. <u>the group case</u>: if $S = G$ a group, then the groupoid of germs is the action groupoid, *i.e.*: $\mathcal{G}(\theta, G, X) = \Gamma_\theta$.

2. <u>the one point set case</u>: if $X = \{*\}$ then $\mathcal{G}(\theta, S, X) = G_S$, *i.e.* the maximal group image of S. This group is obtained by the equivalence on S that relates two elements if they have a lower bound in the natural order. Then G_S is the quotient of this congruence. (cf. Lawson [51] Chapter 2).

3. <u>the restriction groupoid case</u>: if the action is the Munn action, then the groupoid of germs is the restriction groupoid, or $\mathcal{G}(\mu, S, \mathcal{E}(S)) = \mathcal{G}_S$.

Exel provides proof (in [33] Section 5) that every étale groupoid is isomorphic to a groupoid of germs obtained by the natural action of the inverse semigroups of bissections.

1.3.4 Universal groupoid

A particular case of groupoid of germ is a dualization of Munn's action. In this short section, we discuss this case. Our references still the same as last subsection, *i.e.* Steinberg [87] and Exel [33]. We add a new reference Buss–Exel-Meyer [15].

Definition 1.3.18. ([15]) Let $\mathcal{E}$ be a semilattice, a *character* is a homomorphism $\phi : \mathcal{E} \to \{0, 1\}$ not identically zero. We denote the set of characters as $\widehat{\mathcal{E}}$.

For each $e \in E$ we define $U_e := \{\phi \in \widehat{\mathcal{E}}; \phi(e) = 1\}$. Now we dualize the Munn action to by: $\widehat{\mu} : S \curvearrowright \widehat{\mathcal{E}(S)}$ with

$$s \in S \mapsto \widehat{\mu}_s : U_{s^*s} \to U_{ss^*}$$
$$\phi(e) \mapsto \widehat{\mu}_s(\phi)(e) = \phi(s^*es)$$

This action has the property of being a terminal object in the category of actions of S on topological spaces. Buss-Exel-Meyer showed this result in [15].

Following the recipe of Proposition 1.3.17 with $\widehat{\mu}$ we have the *universal groupoid* $\widehat{\mathcal{G}(S)}$. More precisely:

arrows: classes of pair $\phi s \in \widehat{\mathcal{E}(S)} \times S$ where $\phi \in U_{s^*s}$ and the relation is

$$\phi s \sim \psi t \iff \phi = \psi \text{ and } \exists e \in \mathcal{E}(S) \text{ s.t. } se = te \text{ and } \phi \in U_e$$

units: $\widehat{\mathcal{G}(S)}^{(0)} \simeq \widehat{\mathcal{E}(S)}$

inversion: $\phi s^{-1} = \widehat{\mu}_s(\phi)s^*$

product: $\phi s \psi t = \psi st$ if $\phi = \widehat{\mu}_t(\psi)$

domain and range: $d(\phi s) = \phi$ and $r(\phi s) = \widehat{\mu}_s(\phi)$

Like above, this groupoid is étale. Moreover, it is an ample groupoid, but not necessarily Hausdorff.

Finally a particular case that is very useful to our work:

$$|\ \mathcal{E}(S)\ | < \infty \implies \widehat{\mathcal{G}(S)} = \mathcal{G}_S.$$

Finishing this section, we present a diagrammatical interaction between groups, inverse semigroups, and groupoids.

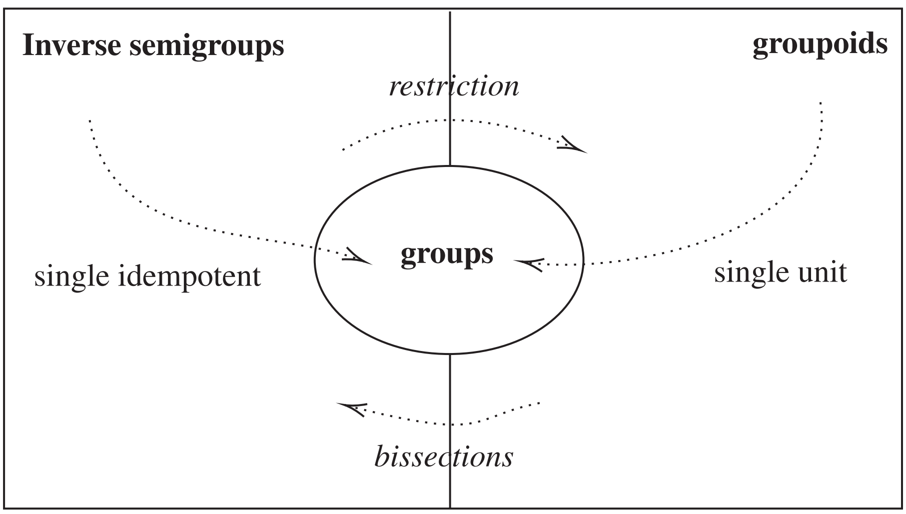


Figure 1.1: The interaction between inverse semigroups and groupoids

1.4 Algebras

We briefly review the definitions of inverse semigroup algebras and groupoid algebras. We interpret groupoid algebras as a particular case of Steinberg algebras. In this section, we also present the relation between universal groupoid algebras and inverse semigroup algebras.

1.4.1 Inverse semigroup and groupoid algebras

Let $\mathbb{K}$ be an associative, commutative an unital ring and S be an inverse semigroup.

The $\mathbb{K}$ *algebra of an inverse semigroup*, $\mathbb{K}S$, is the free $\mathbb{K} - module$ generated by elements of S with product

$$(\sum_s a_s\delta_s)(\sum_t b_t\delta_t) = \sum_u(\sum_{st=u} a_s b_t)\delta_u, \forall s,t,u \in S.$$

Similarly if $\mathcal{G}$ is a groupoid, the $\mathbb{K}$-*algebra of the groupoid*, is the free $\mathbb{K}$-module with basis $\mathcal{G}$ and convolution product

$$\delta_x \bullet \delta_y = \begin{cases} \delta_{xy} & , \text{ if } \exists xy \\ 0 & , \text{ if not} \end{cases}$$

It was proved by Steinberg in [87] Theorem 6.3, that $\mathbb{K}S \simeq \mathbb{K}\widehat{\mathcal{G}(S)}$. As the universal groupoid is an étale groupoid, its algebra is a little different. The next subsection presents the basics facts about such algebras.

One nice property of groupoid algebras is that they can be decomposed in matrix components. A motivating discussion of this fact appears in Khalkhali [47] Section 2.2 Example 2.2.2. And there is a proper proof in Dokuchaev-Exel-Piccione [28] Proposition 3.1. *I.e.*, if $\mathcal{G}$ has finite connected components and $\mathbb{K}$ is a associative commutative unital ring, then

$$\mathbb{K}\mathcal{G} \simeq \bigoplus_i \mathbb{K}G_i \otimes \mathbb{M}_{n_i}(\mathbb{K}) \simeq \bigoplus_i \mathbb{M}_{n_i}(\mathbb{K}G_i),$$

where n_i is the cardinality of the connected components, and G_i are the isotropy groups, which means that each G_i is a subgroup of G that fixes one point of X trough the induced action of G on X.

1.4.2 Steinberg algebras

Our interest lies in the algebras of discrete and finite groupoids , but this is a particular case of ample groupoids. We will follow Steinberg [87] Section 4.

Definition 1.4.1. ([87]) Let $\mathcal{G}$ be an ample groupoid and $\mathbb{K}$ be a commutative associative unital ring. The *ample groupoid algebra*, or *Steinberg algebra*, $\mathbb{K}\mathcal{G}$ is the $\mathbb{K}$-submodule of $\mathbb{K}^{\mathcal{G}}$ spanned by the characteristic functions of compact open subsets of $\mathrm{Bis}(\mathcal{G})$.

The most used notation for Steinberg algebras is: $A_{\mathbb{K}}(\mathcal{G})$.

If we deal with discrete groupoids, then it boils down to the previous characterization of groupoid algebras. Because bissections have the form $\{x\}$ for $x \in \mathcal{G}$.

Another nice feature is that the Steinberg algebra of $\mathcal{G}$, an ample groupoid, is unital if, and only if, $\mathcal{G}^{(0)}$ is compact. It is essential to say that Steinberg algebras also satisfy properties such as the decomposition we mentioned above. This fact can be found in Rigby [76] Section 1.3, among many other algebras results.

1.5 P-theorem and McAlister triple

The P-Theorem, which is a structure result for E-unitary inverse semigroups, is an essential result for our work. This result is based on McAlister triples, which are also related to globalizations of partial actions as we explain in the following. Our main reference is Lawson-Margolis [53].

Before the definition of McAlister triples we need the following. A, non empty, subset I of a partially ordered set $(P, \leqslant)$ is an *order ideal*, if the following condition is satisfied:

- for every $x \in I$, $y \in P$ and $y \leqslant x$ implies that $y \in I$.
- for every $x, y \in I$ there exists $z \in I$ such that $x, y \leqslant z$;

Moreover, I is a *principal order ideal* if it is of the form $I = \{x \in I; x \leqslant p\}$ for some $p \in P$.

Definition 1.5.1. ([53]) A *McAlister triple* (G, X, Y) consists of: G a group, X a poset, Y an order ideal of X that is a meet semilattice under the induced order, such that $G \curvearrowright X$ satisfying:

(I) $G \cdot Y = X$

(II) $g \cdot Y \cap Y \neq \emptyset$ for each $g \in G$.

Proposition 1.5.2. ([53])

Let (G, X, Y) be a McAlister triple. The set $P(G, X, Y) := \{(y, g) \in Y \times G; g^{-1} \cdot y \in Y\}$ with binary operation

$$(x, g)(y, h) := (x \wedge g \cdot y, gh)$$

is an E-unitary inverse semigroup with idempotent set isomorphic to Y and maximal group image isomorphic to G.

Inverse semigroups as above are called *P-semigroups*. Next, we state the P-theorem.

Theorem 1.5.3. *([53]) Each E-unitary inverse semigroup is isomorphic to a P-semigroup.*

These notions are related to our work, because if (G, X, Y) is a McAlister triple, then G acts partially on Y, and the action of G on X is its globalization.

Lately, in our work, we will need a more general version of the P theorem and McAlister triples for inverse semigroups and groupoids. To avoid cumbersome notations and the lack of motivation, at this point, we will present such results at an appropriate time.

1.6 Möbius inversion formula

A handy tool for counting problems in our work are Möbius functions. Indeed they appear in the work of Steinberg [85], and [86] where he studied the algebra of semigroup as the algebra of the associated restriction groupoid. Also in the new formulation (and distinct) of the characterization of partial algebras made by Dokuchaev-Milies in [30] and by Choi in [20] and [19].

We will make a short outline of major properties and Möbius inversion. Our references will be Rota [77] and Stanley [83] Section 3.7.

Let $(P, \leqslant)$ be a partial ordered set. We define an interval in P by $[x, y] := \{z \in P; x \leqslant z \leqslant y\}$. The *incidence algebra* of P over the field $\mathbb{K}$ is the set

$$I(P) := \{f : P \times P \to \mathbb{K}; f(x, y) = 0 \text{ if } x \nleqslant y\}.$$

If the poset presents the property that each interval is finite (or even a finite poset), this is an associative algebra with structure:

<u>identity</u>: $\delta(x, y) = \begin{cases} 1 & , x = y \\ 0 & , x \neq y \end{cases}$

<u>addition</u>: $(f + g)(x, y) = f(x, y) + g(x, y)$

<u>convolution</u>: $(f * g)(x, y) = \sum\limits_{x \leqslant z \leqslant y} f(x, z)g(z, y)$

<u>defining function</u>: $\zeta(x, y) = \begin{cases} 1 & , x \leqslant y \\ 0 & , x \nleqslant y \end{cases}$

The zeta function is invertible and its inverse is called the Möbius function, denoted by μ. This function has the recursive formula

$$\mu(x,y) = \begin{cases} 1 & , x = y \\ -\sum\limits_{x \leqslant z < y} \mu(x,z) & , x \neq y \end{cases}.$$

Next to the Möbius inversion theorem.

Theorem 1.6.1 ([83])**.** *Let $(P, \leqslant)$ be a poset such that each every principal order ideal is finite, and let $f, g : P \to \mathbb{K}$. Then the following are equivalent:*

(i) $g(x) = \sum\limits_{y \leqslant x} f(y)$ *for all $x \in P$, and*

(ii) $f(x) = \sum\limits_{y \leqslant x} g(y)\mu(y,x)$.

Where μ is the Möbius function.

In other words, this theorem says: $(g = f * \zeta) \iff (f = g * \mu)$.

There is also its dual form.

Theorem 1.6.2 ([83])**.** *Let $(P, \leqslant)$ be a poset such that each every principal order ideal is finite, and let $f, g : P \to \mathbb{K}$ Then the following are equivalent:*

(i) $g(x) = \sum\limits_{y \geqslant x} f(y)$ *for all $x \in P$, and*

(ii) $f(x) = \sum\limits_{y \geqslant x} g(y)\mu(y,x)$.

Where μ is the Möbius function.

A very useful tool to describe Möbius function follows.

Let P be a finite poset. Elements of the incident algebra $I(P)$ can be related to matrices elements, over $\mathbb{K}$ a field, via

$$\begin{aligned} I(P) &\to \mathbb{M}_{|P|}(\mathbb{K}) \\ f &\mapsto f((x_i, x_j)) \end{aligned}.$$

This map allows the construction of the ζ function. Then computing its inverse gives us the μ function.

Details and more properties of such identification can be found in Spiegel-O'Donnell [81] Section 1.2.

1.7 Categories

The last chapter has a more abstract language. For the sake of completion and to avoid misunderstanding, this section starts with the standard definitions of category theory and functors, following Mac Lane [57] and Riehl [75].

Definition 1.7.1. ([57]) A *category (in the sense of Mac Lane)* is composed by a class of *objects* $\mathcal{C}^{(0)}$ and a class of *morphisms*, *arrows*, $\mathcal{C}$ such that

(I) each $x \in \mathcal{C}$ is called *arrow* and has a *source* and a *target* are, respectively, the objects $d(x), r(x)$;

(II) given two objects $e, f \in \mathcal{C}^{(0)}$, the *hom class* of arrows from e to f is the set $\mathcal{C}(e, f) := \{x \in \mathcal{C}; d(x) = e, r(x) = f\}$;

(III) the composition of $x \in \mathcal{C}(e, f)$ and $y \in \mathcal{C}(f, i)$ is $y \circ x = yx \in \mathcal{C}(e, i)$;

(IV) any object $e \in \mathcal{C}^{(0)}$ determines an *unit arrow* $1_e \in \mathcal{C}(e, e)$ satisfying $1_e x = x$ and $y 1_e = y$ for all $x \in \mathcal{C}(f, e)$ and $y \in \mathcal{C}(e, i)$;

(V) the composition is associative when defined.

Notation: $\mathcal{C}$ denotes the category with object set $\mathcal{C}^{(0)}$.

Eventually we may be interested in composing an arrow with all arrows of a hom class, in this case we will write $x\mathcal{C}(e, f) = \{xy; y : e \to f\}$.

For each category $\mathcal{C}$ there is an associated category $\mathcal{C}^{op}$ composed by the objects of $\mathcal{C}$. Given an arrow $x \in \mathcal{C}(e, f)$, there is a unique $x^{op} \in \mathcal{C}(f, e)$. This category is called the *opposite category* .

Our study is based on sets; this way, we reduce the classes to sets.

Definition 1.7.2. ([57]) A category $\mathcal{C}$ is called *finite* if its object set is finite, and *small* if the arrow class is a set.

The substructures of categories are significant to our study.

Definition 1.7.3. ([57]) Given a category $\mathcal{C}$, then a *subcategory* $\mathcal{D} \subset \mathcal{C}$ is composed by a subclass of arrows $\mathcal{D} \subset \mathcal{C}$ and a subclass of objects $\mathcal{D}^{(0)} \subset \mathcal{C}^{(0)}$, which satisfies the same axioms and with inherited composition.

If $\mathcal{D}$ a subcategory with the additional property: for any $e, f \in \mathcal{D}^{(0)}$, always holds the equality $\mathcal{D}(e, f) = \mathcal{C}(e, f)$; $\mathcal{D}$ is called *full subcategory*.

The morphisms between categories are defined below.

Definition 1.7.4. ([57]) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

(I) A *covariant functor* is a map $F : \mathcal{C} \to \mathcal{D}$ where

$$e \in \mathcal{C}^{(0)} \mapsto F(e) \in \mathcal{D}^{(0)} \text{ and } x \in \mathcal{C}(e,f) \mapsto F(x) \in \mathcal{D}(F(e), F(f))$$

such that:

- $F(1_e) = 1_{F(e)}$ for every $e \in \mathcal{C}$, and
- if there exists $xy \in \mathcal{C}$, then $F(xy) = F(x)F(y)$ exists in $\mathcal{D}$.

(II) A *contravariant functor* is a map $\varrho : \mathcal{C} \to \mathcal{D}$ where

$$e \in \mathcal{C}^{(0)} \mapsto \varrho(e) \in \mathcal{D}^{(0)} \text{ and } x \in \mathcal{C}(e,f) \mapsto \varrho(x) \in \mathcal{D}(\varrho(f), \varrho(e))$$

such that:

- $\varrho(1_e) = 1_{\varrho(e)}$ for every $e \in \mathcal{C}$, and
- if there exists $xy \in \mathcal{C}$, then $\varrho(xy) = \varrho(y)\varrho(x)$ exists in $\mathcal{D}$.

Functors may have different properties and define when two categories are ”the same”, or equivalent as we will term.

An example of functor is the *constant functor* $F_d : \mathcal{C} \to \mathcal{D}$ that maps each object of the category $\mathcal{C}$ to a fixed object d in $\mathcal{D}$ and each morphism of $\mathcal{C}$ to the unit morphism of d.

Definition 1.7.5. ([57]) Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor.

(I) The functor F is *full* if for any two objects $e, f \in \mathcal{C}^{(0)}$, the induced map $\mathcal{C}(e,f) \mapsto \mathcal{D}(F(e), F(f))$ is surjective.

(II) The functor F is *faithful* if for any two objects $e, f \in \mathcal{C}^{(0)}$, the induced map $\mathcal{C}(e,f) \mapsto \mathcal{D}(F(e), F(f))$ is injective.

(III) The functor F is *dense* (or *essentially surjective on objects*) if each object $f \in \mathcal{D}^{(0)}$ is isomorphic to an object $F(e) \in \mathcal{D}^{(0)}$ for $e \in \mathcal{C}^{(0)}$.

If all three previous axioms hold simultaneously, we say that $\mathcal{C}$ is *equivalent* to $\mathcal{D}$, denoted by $\mathcal{C} \simeq \mathcal{D}$.

Some categories present a particular subcategory.

Definition 1.7.6. ([57]) Given a category $\mathcal{C}$, a *skeleton* is a full subcategory $\mathcal{E}$ such that each object of $\mathcal{C}$ is isomorphic, $\mathcal{C}$, to a unique object of $\mathcal{E}$.

Notice that in the case of the existence of skeletons, they are all isomorphic and $\mathcal{E} \simeq \mathcal{C}$.

In category theory, we can establish relations among different functors.

Definition 1.7.7. ([57]) Give the categories $\mathcal{C}$ and $\mathcal{D}$ and the functors $F, \varrho : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \Rightarrow \varrho$ is:

(I) an arrow $\eta_e : F(e) \to \varrho(e)$, for each $e \in \mathcal{C}^{(0)}$, such that for any morphism $x : e \to f$ in $\mathcal{C}$, we have $\eta_f \circ F(x) = \varrho(x) \circ \eta_e$ commuting in $\mathcal{D}$.

(II) a *natural isomorphism* if for each $e \in \mathcal{C}$ the arrows η_e are isomorphisms.

Next, we want to define complete and cocomplete categories, but we must first pass through cones and limits.

Definition 1.7.8. ([57]) Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. A *cone over* F with *appex* $c \in \mathcal{C}$ is a natural transformation $\eta : c \Rightarrow F$ whose domain is the constant functor at c.

Dually, the *cone under*, or *cocone*, under F with *nadir* $c \in \mathcal{C}$ is a natural transformation $\gamma : F \Rightarrow c$.

Some objects in category theory might receive a particular name: let $\mathcal{C}$ be a category and $i, t \in \mathcal{C}^{(0)}$

- i is initial if for every $x \in \mathcal{C}$ there is a unique morphism $i \to x$;
- t is terminal if for every $y \in \mathcal{C}$ there is a unique morphism $y \to t$.

Using these ideas and cones, we can define limits and colimits.

Definition 1.7.9 ([75]). Given a functor $F : \mathcal{D} \to \mathcal{C}$

(I) a *limit* is a terminal object in the category of cones over F;

(II) a *colimit* is an initial object in the category of cocones under F.

If a particular category admits all limits or colimits, it receives a proper definition.

Definition 1.7.10. ([57]) Consider the categories $\mathcal{C}$ and $\mathcal{D}$, where $\mathcal{D}$ is small. We say that $\mathcal{C}$

(I) is *complete* if every functor $F : \mathcal{D} \to \mathcal{C}$ has a limit;

(II) is *cocomplete* if every functor $\varrho : \mathcal{D} \to \mathcal{C}$ has a colimit;

(III) is *bicomplete* if it is complete and cocomplete.

The references we gave have many examples and properties, but we would like to comment on one example. Given an unital and commutative ring $\mathbb{K}$, the category $Mod(\mathbb{K})$ of modules over $\mathbb{K}$ is bicomplete.

A category may define an algebra.

Definition 1.7.11 ([98])**.** Let $\mathcal{C}$ be a small category and $\mathbb{K}$ be a commutative ring. The *category algebra*, or *convolution algebra*, $\mathbb{K}\mathcal{C}$ is a free $\mathbb{K}$-module whose basis is the arrow set $\mathcal{C}$. The product in the basis is defined by

$$x \bullet y = \begin{cases} xy & , \text{ if } \exists xy \\ 0 & , \text{ if not} \end{cases},$$

then we extend linearly to all $\mathbb{K}\mathcal{C}$.

One last comment: if $\mathcal{C}^{(0)}$ is finite then the category algebra is unital, with unit given by $1_{\mathbb{K}\mathcal{C}} = \sum_{e \in \mathcal{C}^{(0)}} 1_e$.

Chapter 2

The expansion of a group

We will first discuss the expansion of a group, as in Exel's 1998 paper on partial actions [32]. Then we will show another equivalent way to achieve the same goal using the P-theorem. Next, we will associate a certain groupoid with Bernoulli's action. Finally, we deal with the partial group algebra. This chapter presents many other results, such as the Morita context for inverse semigroups. Due to our work, new results will appear when we discuss a specific (big) algebra in terms of Choi's algorithm of classes.

2.1 Expansion of a group in terms of generators and relations

As mentioned above, we now focus our attention on the problem of defining a semigroup associated with a group. The primary reference is Exel's paper [32], where this was first proved, and we will follow computations from Paiva's master thesis [68].

Definition 2.1.1 ([32]). Let G be a group, we denote by $S(G)$ the *universal semigroup* of G defined through generators and relations. A set of generators for $S(G)$ is $\{[g]; g \in G\}$ and for each $g, h \in G$ we consider the relations:

(I) $[g][h][h^{-1}] = [gh][h^{-1}]$;

(II) $[g^{-1}][g][h] = [g^{-1}][gh]$;

(III) $[g][e] = [g] = [e][g]$.

First, we take care of the regularity.

Proposition 2.1.2 ([32]). Given G a group and S a semigroup and a map $\phi : G \to S$ such that

(i) $\phi(g)\phi(h)\phi(h^{-1}) = \phi(gh)\phi(h^{-1})$;

(ii) $\phi(g^{-1})\phi(g)\phi(h) = \phi(g^{-1})\phi(gh)$;

(iii) $\phi(g)\phi(e) = \phi(g) = \phi(e)\phi(g)$.

Then there is a unique semigroup homomorphism $\overline{\phi} : S(G) \to S$ such that $\overline{\phi}([g]) = f(g)$.

Notice that $S(G)$, to become an inverse semigroup, needs "inverse elements", the next proposition takes care of this problem.

Proposition 2.1.3 ([32])**.** Let G be a group, there is a unique *anti-homomorphism* $\text{inv} : S(G) \to S(G)$ such that $\text{inv}([g]) = [g^{-1}]$.

Proof. Define $\phi : G \to S(G)^{\text{op}}$ by $\phi(g) = [g^{-1}]$. Readily this map satisfies the previous Proposition 2.1.2. So there is a unique extension $\overline{\phi} : S(G) \to S(G)^{\text{op}}$ such that $\overline{\phi}([g]) = [g^{-1}]$. Hence $\text{inv} : S(G) \to S(G)$ is given by $\text{inv} := \overline{\phi}$. □

Remembering: an element e in a semigroup S is idempotent if $e^2 = e$. Let us give a characterization of the idempotents of $S(G)$.

Proposition 2.1.4. Let $g \in G$ and let $\varepsilon_g = [g][g^{-1}]$. For each $g, h \in G$

(i) $\text{inv}(\varepsilon_g) = \varepsilon_g = \varepsilon_g^2$;

(ii) $[h]\varepsilon_g = \varepsilon_{gh}[h]$;

(iii) $\varepsilon_g \varepsilon_h = \varepsilon_h \varepsilon_g$;

(iv) every $a \in S(G)$ admits a decomposition $a = \varepsilon_{r_1}\varepsilon_{r_2} \dots \varepsilon_{r_n}[g]$ where $n \geqslant 0$ r and $g, r_i \in G$ for all $i = 1, \dots, n$. In addition, one can choose a decomposition with:

- $r_i \neq r_j$ for $i \neq j$;
- $r_i \neq g$ and $r_i \neq e$ for all $i = 1, \dots, n$.

Here, we say a is in *standard form*;

(v) if $f \in S(G)$ is an idempotent with the standard form $f = \varepsilon_{r_1}\varepsilon_{r_2} \dots \varepsilon_{r_n}[g]$ then $f = \varepsilon_{r_1}\varepsilon_{r_2} \dots \varepsilon_{r_n}$.

Proof. Statements (i-iii) are routine calculations. For (iv), let $R \subset S(G)$ of all elements that admit the desired decomposition. Notice that we may have $n = 0$, so $[g] \in R$ for all $g \in G$. Then to show that $R = S(G)$ suffices to prove that R is an ideal of $S(G)$. Foremost for all $g, h \in G$

$$[g][h] = [g][g^{-1}][g][h] = [g][g^{-1}][gh] = \varepsilon_g[gh].$$

Hence if $a = \varepsilon_{r_1}\varepsilon_{r_2} \dots \varepsilon_{r_n}[g] \in S(G)$ and $[h] \in S(G)$

$$a[h] = a = \varepsilon_{r_1}\varepsilon_{r_2} \dots \varepsilon_{r_n}[g][h] = a = \varepsilon_{r_1}\varepsilon_{r_2} \dots \varepsilon_{r_n}\varepsilon_g[gh] \in R.$$

Analogously for $[h_1][h_2] \dots [h_m] \in S(G)$. Thus R is an ideal of $S(G)$.

Regarding non repetition, since the elements εâs commute, they will eliminate any doubles after reordering. If $i = e$, then $\varepsilon_e = [e][e^{-1}] = [e]$ is the identity of $S(G)$. In the last case if $r_i = g$ commuting the elements we have

$$\begin{aligned} a = \varepsilon_{r_1}\dots\varepsilon_{r_i}\dots\varepsilon_{r_n}[g] &= \varepsilon_{r_1}\dots\varepsilon_g\dots\varepsilon_{r_n}[g] \\ &= \varepsilon_{r_1}\dots\varepsilon_{r_n}\varepsilon_g[g] \\ &= \varepsilon_{r_1}\dots\varepsilon_{r_n}[g][g^{-1}][g] \\ &= \varepsilon_{r_1}\dots\varepsilon_{r_n}[gg^{-1}][g] \\ &= \varepsilon_{r_1}\dots\hat{\varepsilon_g}\dots\varepsilon_{r_n}[g]. \end{aligned}$$

Ending the proof, notice that (v) follows from (i-iii). □

Finally, we are ready to prove the regularity.

Proposition 2.1.5 ([32])**.** The universal semigroup $S(G)$, associate with G, is regular.

Proof. Let $a = \varepsilon_{r_1}\dots\varepsilon_{r_n}[g] \in S(G)$ in his standard form. Then the inverse of a is given by $a^* = [g^*]\varepsilon_{r_1}\dots\varepsilon_{r_n}$. Both equalities $a = aa^{-1}a$ and $a^* = a^*aa^*$ are easy to prove. Notice that, as inv : $S(G) \to S(G)$, the second identity is well defined. □

Our next job is to prove that inverses are unique in $S(G)$. This task will demand a little technical effort.

Definition 2.1.6 ([32])**.** Let G be a group with neutral element e, we define:

(I) $\mathcal{P}_e(G) := \{H \subseteq G; e \in H\}$;

(II) $\mathcal{F}(\mathcal{P}_e(G)) := \{\psi; \psi : \mathcal{P}_e(G) \to \mathcal{P}_e(G)\}$;

(III) for each $g \in G$ denote $\psi_g : \mathcal{P}_e(G) \to \mathcal{P}_e(G)$ by $\psi_g(H) = gH \cup \{e\}$, where $gH = \{gh, h \in H\}$.

Two simple remarks are worth to notice:

- $\mathcal{F}(\mathcal{P}_e(G))$ with the operation of composition of maps, is a semigroup;

•• $\psi_g(H) = gH \cup \{e\} = gH \cup \{g, e\}$, since $e \in H$.

In terms of the previous definition and remark:

Proposition 2.1.7 ([32])**.** Let $\Psi : G \to \mathcal{F}(\mathcal{P}_e(G))$ be a map defined by $\Psi(g) = \psi_g$. Then there is a unique semigroup homomorphism $\Phi : S(G) \to \mathcal{F}(\mathcal{P}_e(G))$ such that $\Phi([g]) = \psi_g$.

Proof. Itâs easy to see, using the second point of the above remark, that for every $H \in \mathcal{P}_e$:

(i) $\psi_g\psi_h\psi_{h^{-1}}(H) = \psi_{gh}\psi_{h^{-1}}(H)$;

(ii) $\psi_{g^{-1}}\psi_g\psi_h(H) = \psi_{g^{-1}}\psi_{gh}(H)$;

(iii) $\psi_g\psi_e(H) = \psi_g(H) = \psi_e\psi_g(H)$.

Thus the hypotheses of Proposition 2.1.2 holds, since $\mathcal{P}_e$ is a semigroup and then G is a group, and Φ exists and satisfies the desired conditions. □

The map we gained in the Proposition 2.1.7 provides uniqueness of decomposition in elements of $S(G)$.

Proposition 2.1.8 ([32])**.** The map $\Phi : S(G) \to \mathcal{F}(\mathcal{P}_e(G))$ satisfies:

(i) $\Phi(\varepsilon_g)(H) = H \cup \{g\}$;

(ii) $\Phi(a)(\{e\}) = \Phi(\varepsilon_{r_1}\varepsilon_{r_2}\dots\varepsilon_{r_n}[g]) = \{e, r_1, r_2, \dots, r_n, g\}$.

In addition, every $a \in S(G)$ admits a unique standard decomposition, up to the order of terms ε_{r_i}âs.

Proof. For item (i), using that Φ is a homomorphism:

$$\begin{aligned}
\Psi(\varepsilon_g)(H) &= \Psi([g][g^{-1}])(H)\\
&= \Psi([g])\Psi([g^{-1}])(H)\\
&= \psi_g\psi_{g^{-1}}(E)\\
&= \psi_g(g^{-1}H \cup \{e, g^{-1}\})\\
&= g(g^{-1}H \cup \{e, g^{-1}\}) \cup \{e, g\}\\
&= H \cup \{e, g\}.
\end{aligned}$$

The item (ii) is analogous. For the last statement, define the *degree* map $\mathrm{d} : S(G) \to G$ by $\mathrm{d}(a) = \mathrm{d}(\varepsilon_{r_1}\varepsilon_{r_2}\dots\varepsilon_{r_n}[g]) = g$. Suppose $a \in S(G)$ has two decompositions $a = \varepsilon_{r_1}\varepsilon_{r_2}\dots\varepsilon_{r_n}[g]$ and $a = \varepsilon_{s_1}\varepsilon_{s_2}\dots\varepsilon_{s_m}[h]$. Then

- $g = \mathrm{d}(a) = h \Rightarrow g = h$;
- $\{r_1, r_2, \dots, r_n, g\} = \Phi(a)(\{e\}) = \{s_1, s_2, \dots, s_m, h\}$.

Hence the decomposition is unique. □

Now we are about to prove the main result we have been chasing for so long.

Proposition 2.1.9 ([32])**.** Given a group G, $S(G)$ is an inverse semigroup.

Proof. Remembering we have already shown the regularity of $S(G)$ (Proposition 2.1.5), the uniqueness of the inverse is remaining. Let $a = \varepsilon_{r_1}\varepsilon_{r_2}\dots\varepsilon_{r_n}[g] \in S(G)$ with two inverses $a^* = [g^{-1}]\varepsilon_{r_1}\varepsilon_{r_2}\dots\varepsilon_{r_n}$ (as in Proposition 2.1.5, and b, *i. e.* $a = aba$ and $bab = b$. In addition, suppose $b^* = \varepsilon_{s_1}\varepsilon_{s_2}\dots\varepsilon_{s_m}[h]$, so $b = [h^{-1}]\varepsilon_{s_1}\varepsilon_{s_2}\dots\varepsilon_{s_m} = (b^*)^*$Â . As in Proposition 2.1.8, let d : $S(G) \to S$ the degree map, then

$$g = \mathrm{d}(a) = \mathrm{d}(aba) = gh^{-1}g \Rightarrow g(g^{-1}h) = gh^{-1}g(g^{-1}h) \Rightarrow h = g.$$

Taking the classes on $S(G)$ we have $[g][h^{-1}] = [g][g^{-1}] = \varepsilon_g$. Considering this last equality and Proposition 2.1.4

$$aba = \varepsilon_{r_1}\dots\varepsilon_{r_n}[g][g^{-1}]\varepsilon_{s_1}\dots\varepsilon_{s_m}\varepsilon_{r_1}\dots\varepsilon_{r_n}[g] = \varepsilon_{r_1}\dots\varepsilon_{r_n}\varepsilon_{s_1}\varepsilon_{s_m}[g].$$

But standard forms are unique, so

$$\{r_1, r_2, \dots, r_n\} \cup \{s_1, s_2, \dots, s_m\} = \{r_1, r_2, \dots, r_n\} \Rightarrow \{s_1, s_2, \dots, s_m\} \subseteq \{r_1, r_2, \dots, r_n\}.$$

An analogous argument to $(aba)^* = b^*$ show us that the opposite inclusion. Hence $a^* = b$. □

To conclude, let us compute the "size" of $S(G)$ when G is a finite group. As we will see, the relationship between the number of elements in $S(G)$ and in G is exponential growth.

Proposition 2.1.10 ([32])**.** If G is a finite group with n elements, then $S(G)$ has $2^{n-2}(n+1)$ elements.

Proof. Because of the previous Proposition 2.1.8, standard decomposition is unique up to permutations of εâs. So we have 2^{n-1} elements in $S(G)$ represented as $\varepsilon_{r_1}\varepsilon_{r_2}\dots\varepsilon_{r_n}[e]$. Also, as $g \neq e$ we have $n-1$ choices. So there are 2^{n-2} elements represented as $a = \varepsilon_{r_1}\varepsilon_{r_2}\dots\varepsilon_{r_n}[g] \in S(G)$, with $g \neq e$. Thus the total number of elements in $S(G)$ is

$$2^{n-1} + (n-1)2^{n-2} = (n+1)2^{n-2}.$$

□

We give at least one example to illustrate.

Example 2.1.11. Let $(\mathbb{Z}_2, +)$ be a group, then his universal semigroup is $S(G) = \{\overline{[0]}, \overline{[1]}, \overline{[1]} + \overline{[1]}\}$.

The study we made drove us a step closer to the end. Next, we take a little break from talking about groups and semigroups, and we will study actions and partial actions.

Concluding, we present the main result we are chasing, *i. e.* to describe how the partial actions of a group G are connected to actions of $S(G)$ (definition to come).

Next, we state necessary and sufficient conditions for a semigroup action to be a partial action.Â

Proposition 2.1.12 ([32]). Let G be a group and X be a nonempty set and $\theta : G \to \mathcal{I}(X)$ be a map such that for each $g \in G$ we have that $\theta_g : D_{g^{-1}} \to D_g$ is a bijection. Then θ defines a partial action of G on X if, and only if, for every $g, h \in G$ the following hold,

(i) $\theta_e = 1_X : X \to X$ is the identity map;

(ii) $\theta_{g^{-1}} = (\theta_g)^{-1}$ is the inverse of θ_g;

(iii) $\theta_g \theta_h \theta_{h^{-1}} = \theta_{gh} \theta_{h^{-1}}$;

(iv) $\theta_{g^{-1}} \theta_g \theta_h = \theta_{g^{-1}} \theta_{gh}$.

Â

Proof. First, we suppose θ is a partial action. Then by Definition 1.1.4 and Remark 1.1.5 we have (i) and (ii). Following the equivalence of Proposition 1.1.6 and its improvement in Proposition 1.1.7, the domain of $\theta_{gh}\theta_{h^{-1}}$ is

$$\theta_h(D_{h^{-1}} \cap D_{(gh)^{-1}}) = D_h \cap D_{h(gh^{-1})} = d_h \cap D_{g^{-1}} = \mathrm{dom}(\theta_g \theta_h \theta_{h^{-1}}).$$

So if x belongs to this domain, then

$$\theta_{h^{-1}}(x) \in \theta_{h^{-1}}(D_h \cap D_{g^{-1}}) = D_{h^{-1}} \cap D_{h^{-1}g^{-1}}.$$

The last statement follows using (ii) and (iii):

$$\theta_{g^{-1}} \theta_g \theta_h = (\theta_{h^{-1}} \theta_{g^{-1}} \theta_g)^{-1} = (\theta_{(gh)^{-1}} \theta_g)^{-1} = \theta_g \theta_{gh}.$$

Conversely, if θ satisfies (i-iv) then we define $\mathrm{im}(\theta_g) = D_g$. So, as $\theta_{g^{-1}}$ is the inverse of θ_g^{-1}, we have $\mathrm{im}(\theta_{g^{-1}}) = D_{g^{-1}}$ and $\theta_g : D_{g^{-1}} \to D_g$ is a bijection. The axioms of the Definition 1.1.4 are easy computations from assumption (i-iv). Â ☐

Finally, we can state the last theorem of this section.

Theorem 2.1.13 ([32]). *Let G be a group and X be a nonempty set. There is a one-to-one correspondence between partial actions of G on X and actions of $S(G)$ on X.*

Proof. On the one hand, if $\theta : G \to \mathcal{I}(X)$ is a partial action of G on X by Proposition 2.1.12 the hypothesis of Proposition 2.1.9 are satisfied, then there is an unique $\overline{\theta} : S(G) \to \mathcal{I}(X)$ such that $\overline{\theta}([g]) = \theta_g$. Hence θ is a semigroup action.

On the other hand, if $\Gamma : S(G) \to \mathcal{I}(X)$ is a semigroup homomorphism, the map $\theta : G \to \mathcal{I}(X)$ with $\theta(g) = \theta_g = \Gamma([g])$ satisfies items (i-iv) of Proposition 2.1.12. Then θ is a partial action of G on X. Next, by an analogous computation of above paragraph, there is a homomorphism $\overline{\theta} : S(G) \to \mathcal{I}(X)$ with $\overline{\theta}([g]) = \theta_g$. But the Proposition 2.1.9 provides uniqueness, so $\overline{\theta} = \Gamma$.Â □

From Paiva's dissertation, [68] we present the next and last example of this section.

Example 2.1.14. Let, as in Example 1.1.8, $G = (\mathbb{Z}_4, +) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ be a group and $X = \{(1,0), (0,1), (-1,0)\}$ be a set. The partial action components are:

- domains: $D_{\overline{0}} := X$, $D_{\overline{1}} := \{(0,1), (-1,0)\}$, $D_{\overline{2}} := \{(1,0), (-1,0)\}$ and $D_{\overline{3}} := \{(1,0), (0,1)\}$;
- maps: $\theta_{\overline{g}} : D_{\overline{g}^{-1}} \to D_{\overline{g}}$ with $\theta_{\overline{0}} \equiv 1_X$ and $\theta_{\overline{g}}(x,y) := (x\cos(g\frac{\pi}{2}) - y\sin(g\frac{\pi}{2}), x\sin(g\frac{\pi}{2}) + y\cos(g\cos(g\frac{\pi}{2}))\hat{A}\ )$.

As G has 4 elements, by Proposition 1.2.10, $S(G)$ has 20 elements. Using the relations that define the universal semigroup (Definition 2.1.1) and computations, we must have

$$S(G) = \{\overline{[0]}, \overline{[1]}, \overline{[2]}, \overline{[3]}, \overline{[1][3]}, \overline{[2][2]}, \overline{[3][1]}, \overline{[1][1]}, \overline{[1][2]}, \overline{[2][3]}, \overline{[2][1]}, \overline{[3][3]}, \overline{[3][2]}, \overline{[1][3]},$$

$$\overline{[1][1][2][3]}, \overline{[1][2][1]}, \overline{[1][2][1][2]}, \overline{[2][1][1]}, \overline{[2][1][1][1]}, \overline{[1][1][1][1]}\}.$$

Once we know the classes, by the Correspondence theorem, [78] A-4.79, we have $\overline{\theta}([g]) = \theta_g$ and then each element of $[g] \in S(G)$ give us a map θ_g.

2.2 Bernoulli group actions and related structures

We begin this section by fixing a (discrete) group G, finite or not. We define two actions intrinsic to G:

<u>Definition</u> 2.2.1.

(I) *Bernoulli action*: the action of G on $P(G) := \{A \subset G; 1 \leq\mid A \mid< \infty\}$ via

$$g \in G \mapsto \mathfrak{B}_g : P(G) \to P(G).$$
$$A \mapsto gA$$

(II) *Bernoulli partial action*: the action of G on $P_e(G) := \{A \in P(G); A \ni e\}$ through

$$\begin{aligned} \mathfrak{b}_g : D_{g^{-1}} &\to D_g, \\ A &\mapsto gA \end{aligned}$$

for each $h \in G$ where $D_h = \{A \in P(G); A \ni e, h\} \subseteq P_e(G)$.

We will use the following notation for both actions, respectively: $\mathfrak{B} : G \curvearrowright P(G)$ and $\mathfrak{b} : G \curvearrowright_p P_e(G)$.

Despite the names we used in our definition, we must prove that indeed we have a group action and a partial group action. But readily we verify the axioms of Definition 1.1.1 and Definition 1.1.4. Besides their names, both actions share a nice property: $\mathfrak{B}$ globalizes $\mathfrak{b}$. This fact holds because comparing both maps:

- $P_e(G) \subseteq P(G)$;
- $D_g = P_e(G) \cap \mathfrak{B}_g(P_e(G))$ for each $g \in G$;
- $\mathfrak{b}_g = \mathfrak{B}_g|_{D_{g^{-1}}}$ for each $g \in G$;
- $\bigcup\limits_g \mathfrak{B}_g(P_e(G)) = P(G)$.

In notations of the Definition 1.1.9, our only work is to notice that $\phi : P_e(G) \to P(G)$ is the inclusion of sets. For the sake of organization, let us formalize this statement.

Lemma 2.2.2. In terms of previous definition $\mathfrak{B} : G \curvearrowright P(G)$ globalizes $\mathfrak{b} : G \curvearrowright_p P_e(G)$

More precisely: $\mathfrak{B}$ is the enveloping action of $\mathfrak{b}$, in the sense of Abadie ([1]).

With the Bernoulli actions, we will define their respective inverse semigroups and groupoids.

2.2.1 Inverse semigroups: S_{GB} and S_{PG}

First we need to interpret both subsets $P(G)$ and $P_e(G)$ as semigroups. We establish this fact below as a remark to facilitate further references.

Remark 2.2.3. $P(G)$ is a commutative semigroup with the operation of union of subsets,

$$A \cdot B = A \cup B.$$

Moreover, each element of $P(G)$ is an idempotent, *i.e.*

$$A^2 = A.$$

Also we can notice that $(P_e(G), \cup)$ is a subsemigroup of $(P(G), \cup)$ which has the unit $\{e\}$.

Lemma 2.2.4. Let $S_{GB} := P(G) \rtimes_{\mathfrak{B}} G = \{(A, g) \in P(G) \times G\}$ be a set with:

$$(A, g) \cdot (B, h) := (A \cdot \mathfrak{B}_g(B), gh) = (A \cup gB, gh)$$

and

$$(A, g)^* := (\mathfrak{B}_{g^{-1}}A, g^{-1}) = (g^{-1}A, g^{-1})$$

Then S_{GB} is an inverse semigroup.

Proof. We will use the item (d) of Remark 1.2.2 as our strategy, *i.e.*, we have to study the idempotents and the regularity of S_{GB}.

Idempotents: let $(A, g) \in S_{GB}$ such that $(A, g)^2 = (A, g)$. Rewriting last equation and using the product, we realize that

$$(A, g)^2 = (A \cdot \mathfrak{B}_g(A), gg) = (A, g) \implies g = e.$$

Thus $\mathcal{E}(S_{GB}) = \{(A, e); A \in P(G)\} \simeq P(G)$. As we saw in the Remark 2.2.3 this is a commutative semigroup with respect to the operation of union of subsets.

Regularity: Following the inversion formula, if (A, g), then $(A, g)^* \in S_{GB}$ is such that

$$(A, g) \cdot (A, g)^* \cdot (A, g) = (A, e) \cdot (A, g) = (A, g).$$

A similar computation shows the second equation of regularity. Indeed

$$(A, g)^* \cdot (A, g) \cdot (A, g)^* = (g^{-1}A, e) \cdot (A, g)^* = (A, g)^*.$$

Then S_{GB} is in inverse semigroup. □

Our ambition is to apply the same strategy, replacing $\mathfrak{B}$ by $\mathfrak{b}$. Nevertheless, the structure of partial actions are different, and the domain plays an important role. The product we used above is the global semidirect product (Lawson uses the terminology "classical" in his book [51], page 148). This way, we must find an alternative rule to a partial semidirect product multiplication. We are following Kellendonk-Lawson [46] Section 3.3 and Piske [71] Section 2 in the following result and computations.

Lemma 2.2.5. Let $S_{PB} := P_e(G) \rtimes_{\mathfrak{b}} G = \{(A, g) \in P_e(G) \times G, A \ni g\}$ be a set with

$$(A, g) \cdot (B, h) := ((\mathfrak{b}_g(\mathfrak{b}_{g^{-1}}(A) \cdot B), gh) = (A \cup gB, gh)$$

and

$$(A, g)^* = (\mathfrak{b}_{g^{-1}}(A), g^{-1}) = (g^{-1}A, g^{-1}).$$

Then S_{PB} is an inverse semigroup.

Proof. As the characterization of each operation suggests, the computations needed to show that S_{PB} is an inverse semigroup are similar of Lemma 2.2.4. We must prove that the product is well defined. Indeed, since $A \ni g$ and $B \ni h$, then $A \in D_g$ and $\mathfrak{b}_{g^{-1}}(A) \in D_{g^{-1}}$. So by the Proposition 1.1.7

$$\mathfrak{b}_{g^{-1}}(A) \cdot B \in D_{g^{-1}} \cap D_h \implies \mathfrak{b}_g(\mathfrak{b}_{g^{-1}}(A) \cdot B) \in \mathfrak{b}_g(D_{g^{-1}} \cap D_h) \subset D_{gh}.$$

Next, since $A \ni e$ is clear that $(A \cup gB) \ni e$.

Thus S_{PB} is an inverse semigroup. □

When necessary, we will refer to S_{GB} as the *global inverse semigroup* and to S_{PB} as the *partial inverse semigroup*.

Despite the different presentation, S_{PB} has already appeared in our study. In fact $S_{PB} \simeq S(G)$.

Proposition 2.2.6 ([71])**.** The inverse semigroups $S = P_e(G) \rtimes_{\mathfrak{b}} G$ and $S(G)$ (from the Definition 2.1.1) are isomorphic.

Proof. Before we give the description of the bijection, we would like to point out a computation using the normal form. Let $\alpha = \varepsilon_{r_1} \cdots \varepsilon_{r_n}[g] \in S(G)$, then

$$\begin{aligned}
\alpha &= \varepsilon_e \varepsilon_{r_1} \cdots \varepsilon_{r_n}[g] \\
&= \varepsilon_e \varepsilon_{r_1} \cdots \varepsilon_{r_n}[g][g^{-1}][g] \\
&= \varepsilon_e \varepsilon_{r_1} \cdots \varepsilon_{r_n} \varepsilon_g [g]
\end{aligned}$$

Writing $A := \{e, r_1, \cdots, r_n, g\}$, α has the alternative formula $\alpha = \Pi_{a \in A} \varepsilon_a [g]$, or even shorter only $\alpha = \varepsilon_A[g]$, where $\varepsilon_A = \Pi_{a \in A} \varepsilon_a$.

Now, with the previous motivation, let $\varepsilon_A[g] \overset{\psi}{\mapsto} (A, g)$. This map satisfies:

<u>homomorphism</u>: let $\alpha = \varepsilon_A[g], \beta = \varepsilon_B[h] \in S(G)$ in normal form and not equal; then

$$\psi(\alpha)\psi(\beta) = (A, g)(B, h) = (A \cup gB, gh).$$

On the other hand, once $[g]\varepsilon_B[h] = \varepsilon_{gB}[g][h]$ and $[g][h] = [g][g^{-1}gh] = [g][g^{-1}][gh] = \varepsilon_g[gh]$ and $A \ni g$ we have

$$\alpha \cdot \beta = \varepsilon_A[g]\varepsilon_B[h] = \varepsilon_{A \cup gB \cup \{g\}}[gh] = \varepsilon_{A \cup gB}[gh].$$

Therefore

$$\psi(\alpha \cdot \beta) = (A \cup gB, gh).$$

Injectivity: follows by the uniqueness of the normal form;

Surjectivity: by definition.

Hence $S(G) \simeq P_e(G) \rtimes_{\mathfrak{b}} G$. □

Summarizing what we have accomplished so far, we started with a group and then defined two actions intrinsic, so to speak, of this group on itself. Later we realized that one action is partial and the other one is its globalization. Each action above gave origin to a semigroup. Indeed, the partial action provided us Exel's inverse semigroup.

Remark 2.2.7. We will write the global semidirect product as $\rtimes$ and the partial one as $\rtimes_p$ and omit the action to emphasize the partiality.

2.2.2 Action groupoids: Γ_{PB} and Γ_{GB}

As we discussed in Chapter 2, given group actions, one can define groupoids. The same can work for partial actions in a much easier way.

Let us recall the Bernoulli actions as defined in the Definition 2.2.1. There is the global action

$$\begin{aligned} g \in G \mapsto \mathfrak{B}_g : P(G) &\to P(G) \\ A &\mapsto gA \end{aligned}$$

and the partial action

$$\begin{aligned} g \in G \mapsto \mathfrak{b}_g : D_{g^{-1}} \subseteq P_e(G) &\to D_g \subseteq P_e(G), \\ A &\mapsto gA \end{aligned}$$

where $D_h = \{A \in P(G); A \ni e, h\}$.

Lemma 2.2.8. Let G be a group and $\mathfrak{B} : G \curvearrowright P(G)$ and $\mathfrak{b} : G \curvearrowright_p P_e(G)$ be the Bernoulli actions . Then the following are groupoids:

$\Gamma_{GB} := \{Ag \in P(G) \times G\}$ with

- $\Gamma^0_{GB} = \{Ae; A \subseteq G\} \simeq P(G)$;
- $d, r : \Gamma_{GB} \to \Gamma^0_{GB}$ where $dAg = A$ and $rAg = \mathfrak{B}_g(A) = gA$;

- $Ag \cdot Bh = Bgh$ iff $A = dAg = rBh = hB$
- $()^{-1} : \Gamma_{GB} \to \Gamma_{GB}$ by $Ag^{-1} = \mathfrak{B}_g(A)g^{-1} = gAg^{-1}$.

$\Gamma_{PB} := \{Ag \in P_e(G) \times G; A \ni g^{-1}\}$ with

- $\Gamma_{PB}^0 \simeq P_e(G)$;
- $d, r : \Gamma_{PB} \to \Gamma_{PB}^0$ where $dAg = A$ and $rAg = \mathfrak{b}_g(A) = gA$;
- $Ag \cdot Bh = Bgh$ if $A = dAg = rBh = hB$,
- $()^{-1} : \Gamma_{PB} \to \Gamma_{PB}$ by $Ag^{-1} = \mathfrak{b}_g(A)g^{-1} = gAg^{-1}$.

Next, following Choi [19] Lemma 1.4, we will show how to extend the product in both groupoids.

Lemma 2.2.9 ([19])**.** *For both groupoids Γ_{GB} and Γ_{PB}, the product*

$$Ag \odot Bh = h^{-1}A \cup Bgh$$

is globally defined. This implies that $(\Gamma_{GB}, \odot)$ and $(\Gamma_{PB}, \odot)$ are inverse semigroups.

Proof. Indeed

$\underline{\Gamma_{GB}}$: let $Ag, Bh \in \Gamma_{GB}$; as A, B are finite subsets of G, we also have $h^{-1}A \cup B$ is a finite and a subset of G. So the product $\odot$ is ok in this case.

$\underline{\Gamma_{PB}}$: let $Ag, Bh \in \Gamma_{PB}$; in this case $A \ni e, g^{-1}$ and $B \ni e, h^{-1}$. Thus $h^{-1}A \cup B \ni h^{-1}g^{-1} = (gh)^{-1}$ and $h^{-1}A \cup B \ni e$. Again the product works well.

This fact concludes our proof. $\square$

Remark 2.2.10. The reader familiar with the work of Dokuchaev-Exel-Piccione [28], may have recognized our action groupoid Γ_{PB}. This groupoid is the same groupoid they introduced to compute the partial algebra associated with a group. This groupoid plays an essential role because its connected components provide the structure of the above algebra.

Next, we explore other groupoids.

2.2.3 Restriction groupoids: $\mathcal{G}_{S_{PB}}$ and $\mathcal{G}_{S_{GB}}$

As we commented early in these notes, Chapter 2 - Section 2.3, naturally, one may begin with an inverse semigroup structure and define a groupoid structure on the same underlying set.

This construction is a straightforward computation. In order to remind the reader, if S is an inverse semigroup then $\mathcal{G}_S$ is a groupoid with: units as the subsemigroup of idempotents of S, the same inverse map of S, source and target maps $d(s) = s^*s$ and $r(s) = ss^*$, and product st defined only when $s^*s = d(s) = r(t) = tt^*$. This way, defining the inverse $s^{-1} := s^*$ we can produce two restriction groupoids: $\mathcal{G}_{S_{PB}}$ and $\mathcal{G}_{S_{GB}}$.

Remark 2.2.11. Sometimes we may refer to these two groupoids as restriction groupoids of Bernoulli actions.

In light of the Lemma 2.2.9, and as suspected, we have the next lemma.

Lemma 2.2.12. If we extend the groupoids of Bernoulli actions and next restrict, we recover an isomorphic groupoid. *I.e.*, $\Gamma_{PB} = \mathcal{G}_{(\Gamma_{PB}, \odot)}$ and $\Gamma_{GB} = \mathcal{G}_{(\Gamma_{GB}, \odot)}$.

Proof. We will show only the case of Γ_{PB}, the other is identical. Let $Ag, Bh \in (\Gamma_{PB}, \odot)$. The product in $\Gamma_{PB} = \mathcal{G}_{(\Gamma_{PB}, \odot)}$ exists only if

$$A = dAg = rBg = hB \implies h^{-1}A = B.$$

In this case

$$Ag \odot Bh = h^{-1}A \cup Bgh = Bgh.$$

Thus they are the same. □

Moreover, the groupoid of the partial Bernoulli action with the global product $\odot$ is isomorphic to Exel's universal semigroup. This claim is our next proposition, again following Choi [19] Lemma 1.4.

Proposition 2.2.13. The inverse semigroups, $(S_{PB}, \cdot)$ and $(\Gamma_{PB}, \odot)$ are isomorphic.

Proof. Let a map $\phi : \Gamma_{PB} \to S_{PB}$ with $\phi(Ag) = (gA, g)$. We are interpreting the groupoid Γ_{PB} as an inverse semigroup with $\odot$ and the inverse semigroup S_{PB} with its own product. Next we must check if this map satisfies what we expect :

well defined: if $Ag \in \Gamma_{PB}$ then $A \ni e, g^{-1} \implies gA \ni g, e$; so $(gA, g) \in S_{PB}$

morphism: let $Ag, Bh \in \Gamma_{PB}$, first

$$\phi(Ag)\phi(Bh) = (gA, g) \cdot (hB, h) = (gA \cup ghB, gh);$$

by the other hand

$$\phi(Ag \odot Bh) = \phi(h^{-1}A \cup Bgh) = (gA \cup ghB, gh);$$

surjective: if $(A,g) \in S_{PB}$, as $A \ni e$ we have $g^{-1}A \ni g^{-1}$ and then $g^{-1}Ag \in \Gamma_{PB}$, this way

$$\phi(g^{-1}Ag) = (gg^{-1}A, g) = (A, g);$$

injective: let $Ag, Bh \in \Gamma_{PB}$ such that $\phi(Ag) = \phi(Bh)$, then

$$(gA, g) = (hB, h) \implies g = h \implies A = B.$$

Hence $(\Gamma_{PB}, \odot) \simeq (S_{PB}, \cdot)$. $\square$

Remark 2.2.14. Our approach is closer to S_{PB} then of $S(G)$, although they are isomorphic. This way, we present, as Avila-Marin-Pinedo [3] Section 4.1, $\mathcal{G}_{S(G)}$ using the normal forms of its elements.

Let $\alpha = \varepsilon_{r_1} \cdots \varepsilon_{r_n}[g], \beta = \varepsilon_{s_1} \cdots \varepsilon_{s_m}[h] \in S(G)$ in normal form and not equal

$\mathcal{G}^0_{S(G)}$: the idempotent set is the unit set, so

$$\mathcal{G}^0_{S(G)} = \{e = \varepsilon_{r_1} \cdots \varepsilon_{r_n}; e \in S(G)\};$$

inverse map: is the involution map inherited of $S(G)$, but we rewrite as $()^* =: ()^{-1}$ and using the inverse semigroup structure we see

$$\alpha = \varepsilon_{r_1} \cdots \varepsilon_{r_n}[g] \overset{()^{-1}}{\mapsto} \alpha^{-1} = [g^{-1}]\varepsilon_{r_1} \cdots \varepsilon_{r_n};$$

structural maps: as we saw $d(\alpha) = \alpha^{-1}\alpha$ and $r(\beta) = \beta\beta^{-1}$, now using the normal form

$$\begin{aligned} d(\alpha) &= \alpha^{-1}\alpha \\ &= [g^{-1}]\varepsilon_{r_1} \cdots \varepsilon_{r_n}\varepsilon_{r_1} \cdots \varepsilon_{r_n}[g] \\ &= [g^{-1}]\varepsilon_{r_1} \cdots \varepsilon_{r_n}[g] \\ &= \varepsilon_{g^{-1}r_1} \cdots \varepsilon_{g^{-1}r_n}[g^{-1}][g] \\ &= \varepsilon_{g^{-1}r_1} \cdots \varepsilon_{g^{-1}r_n}\varepsilon_{g^{-1}}; \end{aligned}$$

$$\begin{aligned} r(\beta) &= \beta\beta^{-1} \\ &= \varepsilon_{s_1} \cdots \varepsilon_{s_m}[h][h^{-1}]\varepsilon_{s_1} \cdots \varepsilon_{s_m} \\ &= \varepsilon_{s_1} \cdots \varepsilon_{s_m}\varepsilon_h\varepsilon_{s_1} \cdots \varepsilon_{s_m} \\ &= \varepsilon_{s_1} \cdots \varepsilon_{s_m}\varepsilon_h. \end{aligned}$$

multiplication: we restrict the product of $S(G)$ to the case when $d(-) = r(-)$, using last computation of domain and rang maps we have

$$d(\alpha) = r(\beta)$$
$$\alpha^{-1}\alpha = \beta\beta^{-1}$$
$$\varepsilon_{g^{-1}r_1} \cdots \varepsilon_{g^{-1}r_n}\varepsilon_{g^{-1}} = \varepsilon_{s_1} \cdots \varepsilon_{s_m}\varepsilon_h,$$

Then the subset where arrows are composable is

$$\mathcal{G}^{(2)}_{S(G)} = \{(\varepsilon_{r_1} \cdots \varepsilon_{r_n}[g], \varepsilon_{s_1} \cdots \varepsilon_{s_m}[h]) \in \mathcal{G}^2_{S(G)}; \{g^{-1}r_1, \cdots, g^{-1}r_n, g^{-1}\} = \{s_1, \cdots, s_m, h\}\}.$$

2.2.4 Universal groupoids: $\widehat{\mathcal{G}(S_{PB})}$ and $\widehat{\mathcal{G}(S_{GB})}$

There are other two groupoids we can relate to our Bernoulli actions. We will use this time instead of the actions properly, the inverse semigroups S_{PB} and S_{GB}. The key idea is their germ groupoids.

We have distinct cases: finite and infinite. Both provide us the groupoid of germs $\widehat{\mathcal{G}(S_{PB})}$ and $\widehat{\mathcal{G}(S_{GB})}$. When the inverse S semigroup is finite, as we see in Section 2.3, the universal groupoid $\widehat{\mathcal{G}(S)}$ is the restriction groupoid $\mathcal{G}_S$. In this way we have a nice characterization of such groupoids in this setup, because we combine this information with last section Proposition 2.2.13 and obtain

$$|\, G \,| < \infty \implies \widehat{\mathcal{G}(S_{PB})} = \mathcal{G}_{S_{PB}} \simeq \Gamma_{PB}.$$

The same arguments are true for the other inverse semigroup, *i.e.*

$$|\, G \,| < \infty \implies \widehat{\mathcal{G}(S_{GB})} = \mathcal{G}_{S_{GB}} \simeq \Gamma_{GB}.$$

Let us summarize our constructions with groupoids in a graphical way: for $|\, G \,| < \infty$

we have the dotted line representing isomorphisms

$$
\begin{tikzcd}
 & & & \mathcal{G}_{GB} \arrow[dd, dotted, dash] \\
\Gamma_{GB} \arrow[urrr, dotted, dash, bend left] & & S_{GB} \arrow[ur] \arrow[dr] & \\
 & \mathfrak{B} : G \curvearrowright P(G) \arrow[ul] \arrow[ur] & & \widehat{\mathcal{G}(S_{GB})} \\
 & G \arrow[u] \arrow[d] & & \\
 & \mathfrak{b} : G \curvearrowright_p P(G) \arrow[dl] \arrow[dr] & & \widehat{\mathcal{G}(S_{PB})} \arrow[dd, dotted, dash] \\
\Gamma_{PB} \arrow[drrr, dotted, dash, bend right] & & S_{PB} \arrow[ur] \arrow[dr] & \\
 & & & \mathcal{G}_{PB}
\end{tikzcd}
$$

Figure 2.1: The Bernoulli actions of groups and its related structures

2.3 Enlargements and Morita contexts

The following question is very natural and organic at this moment: given that the Bernoulli global action is a globalization of the Bernoulli partial action, are the corresponding semigroups also related in some way? The same question holds for groupoids. Short and quick, answer: yes. In the rest of this section, we will elaborate on this.

Just to compare, we recall the definition of Morita equivalence for rings (as in Clark-Sims [21] Section 5, also briefly commented by Rotman [79] C-4.6). Let R and S rings, M an $R-S$-bimodule and N an $S-R$-bimodule and bimodules morphisms

$$\psi : M \otimes_S N \to R \text{ and } \phi : N \otimes_R M \to S$$

such that for $n, n_1, n_2 \in N$ and $m, m_1, m_2 \in M$:

$$n_1 \cdot \psi(m \otimes n_2) = \phi(n_1 \otimes m) \cdot n_2,$$

$$m_1 \cdot \phi(n \otimes m_2) = \psi(m_1 \otimes n) \cdot m_2.$$

The 6-uple (R, S, M, N, ψ, ϕ) is called the *Morita context* between R and S. Moreover, if ψ and ϕ are bijective maps we say that (R, S, M, N, ψ, ϕ) is a *strict context*; and we say that R is *Morita equivalent* to S, or $R \simeq_M S$.

Remark 2.3.1. If the maps ϕ and ψ are surjective, then they are bijective maps.

Our interest about the Morita context comes from the following fact: if there exists a strict Morita context (R, S, M, N, ψ, ϕ) between two rings R and S, then the module categories $Mod(R)$ and $Mod(S)$ are equivalent categories, the functors that provide the equivalence are

$$\begin{aligned} Mod(R) &\to Mod(S) \\ X &\mapsto M \otimes_S X \end{aligned} \qquad\qquad \begin{aligned} Mod(S) &\to Mod(R). \\ Y &\mapsto N \otimes_R Y \end{aligned}$$

2.3.1 Semigroups and strong Morita equivalence

Steinberg introduced in [88] the concept of strong Morita equivalences for inverse semigroups. As he explained in his introductory section, strong Morita equivalent inverse semigroups are a generalization of the enlargement relation between semigroups introduced by Lawson in [50], which relates to the globalization of partial group actions. We are more interested in the consequences of these equivalences among semigroups because they carry over to a Morita equivalence between their universal groupoid algebras.

The definition below, of Morita context for inverses semigroups, is based on the work of Steinberg [88] Definition 2.1. We use this construction to provide a Morita context for groupoids and algebras. However, this construction will also provide a Morita context for inverse semigroups. We choose to maintain Steinberg's terminologies. In the next definition, a left action of an inverse semigroup S on the set X is a morphism of inverse semigroups $\theta : S \to \mathcal{I}(X)$ (see Definition 1.3.15). A right action is the same as an antimorphism $\theta : S \to \mathcal{I}(X)$, i.e., $\theta_s\theta_t = \theta_{ts}$.

Definition 2.3.2 ([88])**.** Let S and T be inverse semigroups, X be a set equipped with a left action by S and a right action by T that commute and surjective maps $\langle , \rangle : X \times X \to S$ and $[,] : X \times X \to T$ satisfying the following relations: given $x, y, z, s, t \in T$

(I) $\langle sx, y\rangle = s\langle x, y\rangle$

(II) $\langle y, x\rangle = \langle x, y\rangle^*$

(III) $\langle x, x\rangle x = x$

(IV) $[x, yt] = [x, y]t$

(V) $[y, x] = [x, y]^*$

(VI) $x[x, x] = x$

(VII) $\langle x, y\rangle z = x[y, z]$

We say that S and T are *Strong Morita equivalent*, or denote $S \simeq_{sM} T$, if there exists a 5-uple $(S, X, T, \langle,\rangle, [,])$.

Although the previous definition uses the term "equivalence" carelessly, we will quickly confirm that this is indeed an equivalence relation in the class of inverse semigroups.

Remark 2.3.3. The semigroup literature provides other Morita context flavors, but they are all equivalent, as discussed by Funk-Lawson-Steinberg in [37]. They proved that four different notions of Morita equivalence for inverse semigroups motivated by C^*-algebra theory, topos theory, semigroup theory and the theory of ordered groupoids are equivalent. So, if the reader is used to another Morita definition, there is no need to adopt ours.

We remind the reader that we are interested in the algebras of the semigroups. This detour we that we take will provide a Morita context of such algebras. Also, for the groupoid algebras associated with each action, but this will come in an appropriate moment.

Next we use the idea of enlargements of Lawson [50], or [51] Chapter 8, and show, following [88] Proposition 2.2, that enlargements implies in strong Morita equivalence.

Definition 2.3.4 ([50])**.** Let S be an inverse semigroup and $T \subseteq S$ an inverse subsemigroup. We say that S is an *enlargement* of T, or $T \subseteq_E S$, if $STS = S$ and $TST = T$.

Example 2.3.5. (1) Let S be an inverse semigroup and $e \in \mathcal{E}(S)$. If $SeS = S$, then $eSe \subseteq_E S$.

(2) Let X and Y be semilattices such that $Y \subseteq X$. Consider the P-semigroups $P(G, X, Y)$ and $P(G, X, X)$. Then $P(G, X, Y) \subseteq_E P(G, X, X)$.

A large number of examples can be found in Lawson [50] Section 2.

Notice that we have an easy alternative definition, as Steinberg says.

Lemma 2.3.6. *([88]) The inverse semigroup S is an enlargement of the inverse semigroup T if, and only if, $S\mathcal{E}(T)S = S$ and $\mathcal{E}(T)S\mathcal{E}(T) = T$.*

Proof. First, suppose $T \subseteq_E S$. By definition we have for $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$: $s_1ts_2 = s$. But $t = tt^*t$ and $tt^* \in \mathcal{E}(T)$, so $(s_1)tt^*(ts_2) = s$. Thus we have $STS = S \implies S\mathcal{E}(T)S = S$. The other equation comes from a similar computation, because $t_1st_2 = t$ is the same as $(t_1t_1^*)t_1st_2(t_2^*t_2) = t$, and hence $T \subseteq S$ we have $t_1st_2 \in S$. Thus $TST = T \implies \mathcal{E}(T)S\mathcal{E}(T) = T$.

Now suppose $S\mathcal{E}(T)S = S$ and $\mathcal{E}(T)S\mathcal{E}(T) = T$. The first one is due to the fact $T \ni t = tt^*t$ and $T \subseteq S$: $(s_1t)(tt^*)s_2 = s_1ts_2 \in STS = S$. Finally the last one follows from the fact: $TST \subseteq T\mathcal{E}(T)S\mathcal{E}(T)T \subseteq TTT = T$. $\square$

The notion of enlargements implies a strong Morita context.

Proposition 2.3.7. ([88]) If S is an enlargement of T, then S and T are strong Morita equivalent inverse semigroups.

Proof. We must define a set X and maps $\langle , \rangle$ and $[,]$ as in Definition 2.3.2. As from hypothesis $T \subseteq S$ our strategy is to define $X := ST$ and $\langle x, y \rangle = xy^*$ and $[x, y] = x^*y$, where $x, y \in X$. Now we verify the axioms of Definition 2.3.2, but only surjectivity needs a proper verification, because (I-VI) are readily true.:

surjectivity: for $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$ let $s_1ts_2 = s \in STS = S$. So $(s_1t), (s_2^*tt^*) \in X$, now

$$s = s_1tt^*ts_2 = (s_1t)(s_2^*tt^*)^* = \langle s_1t, s_2^*tt^* \rangle,$$

likewise, for $s \in S$ and $t_1, t_2 \in T$ we have that

$$t = t_1st_2 = t_1ss^*st_2 = (ss^*t_1^*)^*st_2 = [ss^*t_1^*, st_2].$$

Hence $(S, ST, T, \langle , \rangle, [,])$ is the 5-uple we seek. □

Besides the fact we have just shown, the previous result will show that $\simeq_{sM}$ is an equivalence relation. This verification will require a notion of the tensor product. We briefly discuss this construction in the next paragraph.

Let T be a semigroup acting on sets X, Y, respectively, on the right and on the left, *i.e.* $X \curvearrowleft T$ and $T \curvearrowright Y$. The tensor product $X \otimes_T Y$ is the quotient of $X \times Y$ by the equivalence $(xt, y) \sim (x, ty)$ for all $x \in X$, $y \in Y$ and $t \in T$. We denote the class of (x, y), by $x \otimes y$. Also, the map $(x, y) \in X \times Y \mapsto x \otimes y \in X \otimes_T Y$ is the universal map.

Suppose S and T are semigroups and that S has a left action on X and H has a right action on Y such that this actions commute with the actions of T. Then $X \otimes_T Y$ admits well defined actions of S and H given by: $s(x \otimes y) = sx \otimes y$ and $(x \otimes y)h = x \otimes yh$, for $s \in S$, $x \in X$, $y \in Y$ and $h \in H$.

Proposition 2.3.8. ([88]) The relation $\simeq_{sM}$ is an equivalence relation.

Proof. We need to show that $\simeq_{sM}$ is reflexive, symmetric, and transitive.

reflexivity: using Proposition 2.3.7 with only S, we can see that $S \subseteq_E S$, hence $S \simeq_{sM} S$

symmetry: let $(S, X, T, \langle , \rangle, [,])$, we want to show that $(T, X, S, [,], \langle , \rangle)$ is also a Morita context. First from Definition 2.3.2 (I-II) and (IV-V)

$$\langle x, sy \rangle = \langle sy, x \rangle^* = (s\langle x, y \rangle)^* = \langle y, x \rangle^* s^* = \langle x, y \rangle s^*$$
$$[xt, y] = [y, xt]^* = ([y, x]t)^* = t^*[x, y]$$

So these inversion formulas show us that we can write $x.s = s^*x$ and $t.y = yt^*$, which means that we can invert the side of actions of S and T on X. Then we switch the roles of the maps $\langle , \rangle$ and $[,]$.

transitivity: let $T \simeq_{sM} S$ by the 5-uple $(S, X, T, \langle , \rangle_S, [,]_T)$ and suppose $H \simeq_{sM} T$ via $(T, Y, H, \langle , \rangle_T, [,]_H)$. Since by definition all actions commute, we can define the tensor product $X \otimes_T Y$ where $s(x \otimes y) = sx \otimes y$ and $(x \otimes y)h = x \otimes yh$. Thus one can check ([88], 2.5) that $H \simeq_{sM} S$ using the 5-uple $(S, X \otimes_T Y, H, \langle , \rangle, [,])$ with

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x \langle y, y' \rangle_T, x' \rangle_S$$

and

$$[x \otimes y, x' \otimes y'] = [y, [x, x']_T y']_H.$$

Hence $\simeq_{sM}$ is (indeed) an equivalence. □

It will be enough for us now. We refer to Steinberg's paper for more details on our work's scope.

A particular case of strong Morita equivalence of semigroups occurs for S_{PB} and S_{GB}. An idea of the proof is in the following diagram.

$$\begin{tikzcd}
& \mathfrak{B} : G \curvearrowright P(G) \arrow[r] & S_{GB} = P(G) \rtimes G \arrow[dd, dash, dotted, "\subseteq_E \Longrightarrow \simeq_{sM}"] \\
G \arrow[ur] \arrow[dr] & & \\
& \mathfrak{b} : G \curvearrowright_p P_e(G) \arrow[r] & S_{PB} = P_e(G) \rtimes_p G
\end{tikzcd}$$

Figure 2.2: The enlargement relation induced by the Bernoulli group actions

This fact is a simple result.

Lemma 2.3.9. The partial and the global inverse semigroups are strong Morita equivalent, or $S_{PB} \simeq_{sM} S_{GB}$.

Proof. In light of the Lemma 2.3.6 we are going to show $S_{PB} \subseteq_E S_{GB}$, thus the result follows by the Proposition 2.3.7.

As inverse semigroups it is clear that $S_{PB} = P_e(G) \rtimes_p G \subset S_{GB} = P(G) \rtimes G$. Indeed

$\underline{S_{GB}\mathcal{E}(S_{PB})S_{GB} = S_{GB}}$: $(W,t) \in S_{GB}$ let's deal with the two possible cases

1st) $\underline{W \ni e}$: so $(W,e) \in \mathcal{E}(S_{PB})$ and $(W,e) \in S_{GB}$, so

$$(W,e)(W,e)(W,t) = (W \cup eW, et) = (W,t).$$

Moreover, if $W \ni t$ then $(W,t) \in S_{PB}$, in this case we can use $(\{e\},e)$ and

$$(W,t)(\{e\},e)(\{e\},e) = (W \cup \{t\}, e) = (W,t).$$

2nd) $\underline{W \not\ni e}$: pick any $g \in W$, $(W,g), (g^{-1}W, g^{-1}t) \in S_{GB}$ and $(g^{-1}W, e) \in S_{PB}$ and

$$\begin{aligned}(W,g)(g^{-1}W,e)(g^{-1}W,g^{-1}t) &= (W \cup gg^{-1}W, ge)(g^{-1}W,g^{-1}t)\\ &= (W,g)(g^{-1}W,g^{-1}t)\\ &= (W \cup gg^{-1}W, gg^{-1}t)\\ &= (W,t).\end{aligned}$$

This computation shows the inclusion $S_{GB}\mathcal{E}(S_{PB})S_{GB} \supseteq S_{GB}$ and it is enough because the opposite is trivial since we have no constraints in the definition of S_{GB}. We mean, if $(W,t) \in S_{GB}$ then W has no particularity besides being finite and a subset of G. Hence we accomplished our task.

$\underline{\mathcal{E}(S_{PB})S_{GB}\mathcal{E}(S_{PB}) = S_{PB}}$: let $(A,g) \in S_{PB}$ so $A \ni e, g$, using $(\{e\},e) \in \mathcal{E}(S_{PB})$ and viewing $(A,g) \in S_{GB}$ we clearly have

$$(\{e\},e)(A,g)(\{e\},e) = (A,g).$$

Fot the opposite inclusion, notice that if $(B,e),(C,e) \in \mathcal{E}(S_{PB})$ and $(X,g) \in S_{GB}$

$$(B,e)(X,g)(C,e) = (B \cup X \cup gC, g) =: (A,g)$$

where $A \ni e, g$ because $B \ni e$ and $gC \ni g$, since $C \ni e$.

Thus we conclude $S_{PB} \subseteq_E S_{GB}$ and consequently $S_{PB} \simeq_{sM} S_{GB}$. $\square$

2.3.2 Groupoids and its equivalences

This subsection's main idea is to translate (strong) Morita equivalence of inverse semigroups to a Morita equivalence of groupoids. In particular to its universal groupoids, as done by Steinberg in [88], Section 4. This discussion will be of crucial importance in Section 3.5.

The reader may know one definition, among many, of equivalence of groupoids. We chose one based on a ”bimodule”, after Jean Renault in the formulation of Farsi-Kumjian-Pask-Sims [35] Definition 3.7.

First, we need to enrich our definition of groupoid action from Definition 1.3.4 with topological aspects. Recall that a topological groupoid is a groupoid $\mathcal{G}$ with a topology and $\mathcal{G}^{(0)}$ with the subspace topology, such that all structural maps are continuous. (cf. the Definition 1.3.11)

Definition 2.3.10 ([94])**.** Let $\mathcal{G}$ be a topological locally compact Hausdorff groupoid and X be a locally Hausdorff compact space. We say that $\mathcal{G}$ *acts on the left* on X, or that X is a *left $\mathcal{G}$-space* if there are

moment map: a continuous open map $\rho : X \to \mathcal{G}^{(0)}$;

action map: a continuous map from $\mathcal{G}_d{\times}_\rho X := \{(g,x) \in \mathcal{G} \times X; d(g) = \rho(x)\}$ to X, $\theta : \mathcal{G}_d{\times}_\rho X \to X$, with $(g,x) \mapsto \theta_g(x)$ such that

(I) if $(h,x) \in \mathcal{G} * X$ and $(g,h) \in \mathcal{G}^{(2)}$ then $(gh,x), (g,\theta_h(x)) \in \mathcal{G}_d{\times}_\rho X$ and $\theta_g(\theta_h(x)) = \theta_{gh}(x)$;

(II) $\theta_{\rho(x)}(x) = x$ for all $x \in X$.

Remark 2.3.11. The map ρ is also termed in the literature as *momentum map*, or *anchor map*.

Analogously one defines right actions, or right $\mathcal{G}$-spaces, with moment map $\sigma : Y \to \mathcal{G}^{(0)}$ and action $\varphi : Y_\sigma{\times}_r\mathcal{G} := \{(y,g) \in Y \times \mathcal{G}; \sigma(y) = r(g)\}$ by $(y,g) \mapsto (y)_g\varphi$.

Notations: $(\rho,\theta) : \mathcal{G} \curvearrowright X$ and $(\sigma,\varphi) : Y \curvearrowleft \mathcal{G}$ are respectively the left and right actions.

Diagrammatically:

$$\begin{tikzcd} \mathcal{G} \arrow[d, shift right, "r"'] \arrow[d, shift left, "d"] & \curvearrowright \\ \mathcal{G}^{(0)} & X \arrow[l, "\rho"] \end{tikzcd} \qquad \begin{tikzcd} \curvearrowleft & \mathcal{G} \arrow[d, shift right, "r"'] \arrow[d, shift left, "d"] \\ Y \arrow[r, "\sigma"'] & \mathcal{G}^{(0)} \end{tikzcd}$$

In addition to the previous definition, we say $(\rho,\theta) : \mathcal{G} \curvearrowright X$

free action: is *free* if $\theta_g(x) = x \implies g = \rho(x)$ for all $x \in X$ such that $d(g) = \rho(x)$;

proper action: is *proper* if the inverse image of θ preserves compact sets.

Now we can define our notion of equivalence for groupoids.

Definition 2.3.12 ([94])**.** Two locally compact groupoids $\mathcal{G}$ and $\mathcal{H}$ are *Morita equivalent*, or $\mathcal{G} \simeq_M \mathcal{H}$ if there is a locally compact space X such that:

(i) X is a free and proper left $\mathcal{G}$-space, *i.e.* $(\rho, \theta) : \mathcal{G} \curvearrowright X$;

(ii) X is a free and proper right $\mathcal{H}$-space, *i.e.* $(\sigma, \varphi) : X \curvearrowleft \mathcal{H}$

(iii) the actions (ρ, θ) and (σ, φ) commute;

(iv) $\rho : X \to \mathcal{G}^{(0)}$ induces a homeomorphism $X/\mathcal{H} \to \mathcal{G}^{(0)}$, where and $X/\mathcal{H}$ is the orbit space by the right action, and

(v) $\sigma : X \to \mathcal{H}^{(0)}$ induces a homeomorphism $\mathcal{G} \setminus X \to \mathcal{H}^{(0)}$, where $\mathcal{G} \setminus X$ is orbit space by the left action.

We may depict these data as:

$$\begin{tikzcd}
\mathcal{G} \arrow[dd, shift right, "r"'] \arrow[dd, shift left, "d"] \arrow[dr, phantom, "\curvearrowright"] & & \mathcal{H} \arrow[dd, shift right, "r"'] \arrow[dd, shift left, "d"] \arrow[dl, phantom, "\curvearrowleft"] \\
& X \arrow[dl, "\rho"] \arrow[dr, "\sigma"'] & \\
\mathcal{G}^{(0)} & & \mathcal{H}^{(0)}
\end{tikzcd}$$

Figure 2.3: The Morita equivalence of groupoids

In a few paragraphs, we will examine a particular example of this equivalence notion. More than that is beyond the scope of this work. We refer the reader to Muhly-Renault-Williams [64] for applications and some examples.

Remark 2.3.13. The above definition is equivalent to that used in Steinberg's article [87], as he observes in that same paper; it follows from Tu [94] Proposition 2.29. There are other notions of equivalence between groupoids. The reader can find in the work of Farsi-Kumjian-Pas-Sims [35] Section 3 a discussion showing that many of these notions coincide for ample étale Hausdorff groupoids.

The following proposition states the Morita relation among universal groupoids; it appears in Steinberg [88] Theorem 4.7.

Proposition 2.3.14 ([88])**.** Let S and T be countable discrete inverse semigroups. If $S \simeq_{sM} T$, then $\widehat{\mathcal{G}(S)} \simeq_M \widehat{\mathcal{G}(T)}$.

We won't present a complete proof of this fact, only the main strategy. Let $S \simeq_{sM} T$ via 5-tuplet $(S, X, T, <,>, [,])$. Then a combination of the following lemmas:

1st) $< x, x > \in \mathcal{E}(S)$ and $[x, x] \in \mathcal{E}(T)$ for all $x \in X$;

2nd) $\alpha_x : D(< x, x >) \subset \widehat{\mathcal{E}(S)} \to D([x, x]) \subset \widehat{\mathcal{E}(T)}$ is a homeomorphism for all $x \in X$;

3rd) define Z as the space of germs at α_x with $x \in X$, formed by equivalent class of pairs $\overline{(x, \phi)}$ where $(x, \phi) \in X \times D([x, x])$ and the equivalence is $(x, \phi) \sim (x', \phi')$ if $\phi = \phi'$ and there exists $y \leqslant x, x'$ such that $\phi \in D([y, y])$. Thus Z is a topological space with basis $(x, U) = \{\overline{(x, \phi)}; x \in X, \phi \in U \subset D([x, x])\}$;

3'rd) equivalently we can use the sets $D(< x, x >)$ and define a topological space Z'. In fact, Z and Z' are homeomorphic spaces;

4th) there are surjective and étale maps $\rho : Z \to \widehat{\mathcal{E}(S)}$ with $\rho(\overline{(x, \phi)}) = x\phi$ and $\sigma : Z \to \widehat{\mathcal{E}(T)}$ with $\sigma(\overline{(y, \psi)}) = \psi$;

will construct the Morita context for $\widehat{\mathcal{G}(S)} \simeq_M \widehat{\mathcal{G}(T)}$, or in diagrams

$$
\begin{tikzcd}
\widehat{\mathcal{G}(S)} \arrow[dd, shift right, "r"'] \arrow[dd, shift left, "d"] \arrow[dr, phantom, "\curvearrowright"] & & \widehat{\mathcal{G}(T)} \arrow[dd, shift right, "r"'] \arrow[dd, shift left, "d"] \arrow[dl, phantom, "\curvearrowleft"] \\
& Z \arrow[dl, "\rho"] \arrow[dr, "\sigma"'] & \\
\widehat{\mathcal{E}(S)} & & \widehat{\mathcal{E}(T)}
\end{tikzcd}
$$

Figure 2.4: The Morita equivalence of universal groupoids

We need one more construction of Steinberg's work, [88] - Section 5, to summarize and use in our study.

For any inverse semigroup S, there is a category $\mathcal{C}_S$ defined by

<u>objects</u>: the set $\mathcal{E}(S)$

<u>morphisms</u>: $\mathcal{H}om(e, f) := \{(f, s, e); e, f \in \mathcal{E}(S), s \in fSe\}$ such that composition is $(f, s_1, e) \circ (e, s_2, d) = (f, s_1 s_2, d)$.

We note that the unit map of each object e of $\mathcal{C}_S$ is given by $1_e = (e, e, e)$; moreover the isomorphisms in $\mathcal{C}_S$ are given by (ss^*, s, s^*s) with inverses (s^*s, s^*, ss^*).

Proposition 2.3.15 ([88])**.** Let S and T be inverse semigroups which are strong Morita equivalent. Then the categories $\mathcal{C}_S$ and $\mathcal{C}_T$ are equivalent. Moreover, the restriction groupoids of S and T are (naturally) Morita equivalent.

Writing in symbols, the part that is relevant to us: $S \simeq_{sM} T \implies \mathcal{G}_S \simeq_M \mathcal{G}_T$.

Wrapping up all these constructions and results, we have the following diagram, for our case of Bernoulli actions and related groupoids

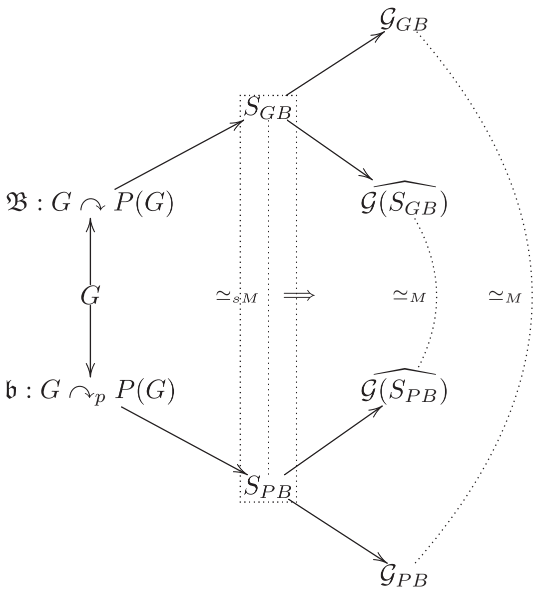


Figure 2.5: The Morita equivalence of groupoids induced by the Bernoulli actions

The reader may notice this diagram has already appeared, but there was one more information about the action groupoids at that time. As commented, if $\mid G \mid < \infty$ then all three groupoids are the same, and clearly, the actions groupoids will also be Morita equivalent. We may picture it as

$$\begin{tikzcd}
\Gamma_{GB} \arrow[r, dash, "\sim"] \arrow[d, dotted, dash, "\simeq_M"] & \mathcal{G}_{GB} \arrow[r, dash, "\sim"] \arrow[d, dash, "\simeq_M", "\Leftarrow"'] & \widehat{\mathcal{G}(S_{GB})} \arrow[d, dash, "\simeq_M"] \\
\Gamma_{PB} \arrow[r, dash, "\sim"] & \mathcal{G}_{PB} \arrow[r, dash, "\sim"] & \widehat{\mathcal{G}(S_{PB})}
\end{tikzcd}$$

Figure 2.6: The isomorphic groupoids associated to the Bernoulli actions

Nevertheless, we can assure that in the infinite case, they are equivalents too. We accomplished this task using the next proposition[1]. We use formulation of Clark-Sims in [21] Lemma 6.1. The main fact is: once we have a groupoid and one subset of its units that are transversal to the action (or meet each orbit in the groupoid), then we can produce a subgroupoid and a set that both provide us the Morita context. The proof is a straight verification of each item in Definition 2.3.12; this way, we choose to omit the proof.

Lemma 2.3.16 ([21])**.** *Let $\mathcal{G}$ be an étale groupoid and let $X \subset \mathcal{G}^{(0)}$, such that X is a topological closed-open set that meets each orbit in $\mathcal{G}$. Then $\mathcal{H} := r^{-1}(X) \cap d^{-1}(X)$ is a topological closed-open subgroupoid of $\mathcal{G}$.*

Moreover, the subspace $Z := X \cdot \mathcal{G} \subseteq \mathcal{G}^{(2)}$, endowed with the actions given by left multiplication by $\mathcal{H}$ and right multiplication by $\mathcal{G}$, satisfies all conditions of Definition 2.3.12, and hence $\mathcal{G} \simeq_M \mathcal{H}$.

Our aim it to apply this result to conclude $\Gamma_{GB} \simeq_M \Gamma_{PB}$. Let's check the hypothesis for $\mathcal{G} = \Gamma_{GB}$ and $X = \Gamma_{PB}^{(0)} \simeq P_e(G)$:

<u>topological properties</u>: as we are considering a groupoid built of an action of a discrete group, Γ_{GB} is étale; also $P_e(G)$ is a discrete topological set, so closed-open;

<u>orbits and transversality</u>: if $Xe \in \Gamma_{GB}^{(0)} = P(G) \times \{e\}$, the orbit of Xe is the subset of $\Gamma_{PB}^{(0)}$

$$\mathcal{O}(Xe) := \{Ye \in \Gamma_{GB}^{(0)} : \exists Zw \in \Gamma_{GB}, d(Zw) = Xe, r(Zw) = Ye\}.$$

Using the definitions of domain and range in Γ_{PB}, and the isomorphism $\Gamma_{PB}^{(0)} \simeq P(G)$ we see:

$$Z = d(Zw) = X \text{ and } wZ = r(Zw) = Y \implies X = Z = w^{-1}Y.$$

Thus the elements in $\mathcal{O}(Xe)$ are written in the form wXe, for all $w \in G$. In particular, one have $w = g^{-1}$ for any $g \in X$. Hence $\mathcal{O}(Xe) \cap P_e(G) \neq \emptyset$.

As the Lemma hypotheses are satisfied, we conclude the existence of:

<u>subgroupoid</u>: $\mathcal{H} = r^{-1}(P_e(G)) \cap d^{-1}(P_e(G)) = P_e(G) \cdot \Gamma_{GB} \cdot P_e(G) \subseteq \Gamma_{PB}^{(2)}$, so an element of such groupoid is

$$Yw = AeXzBe = A \cup z^{-1}AzBe = A \cup z^{-1}Az.$$

Because $A \ni e$, is clear that $Y \ni e, w^{-1}$. Therefore $\mathcal{H} = \Gamma_{PB}$.

<u>Morita context</u>: a space Z such that the Definition 2.3.12 holds.

[1] We would like to thank professor Olivier Brahic that warned us about this property. His lectures and coffee talks, among many other suggestions, were concerned with groupoid theory.

This way, we proved the following corollary.

Corollary 2.3.17. Let $\mathfrak{B} : G \curvearrowright P(G)$ and $\mathfrak{b} : G \curvearrowright_p P_e(G)$ be the Bernoulli actions and let Γ_{GB} and Γ_{PB} be the associated action groupoids, respectively. Then $\Gamma_{GB} \simeq_M \Gamma_{PB}$.

With this result, we finish the study of Morita equivalence in the groupoid setting. Also, we complete the previous diagram with this new piece of information: the dotted line represents the isomorphism when G is finite.

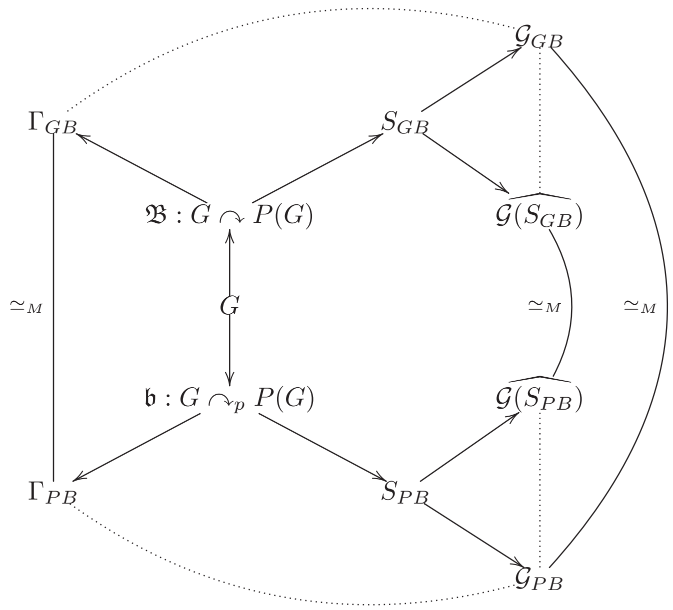


Figure 2.7: The Morita equivalence of action groupoids

2.4 $\mathcal{D}$-classes of inverse semigroups

From this point, we will start to look inside the inverse semigroup structure closely. Digging its core, so to speak, we will find pieces that assemble its algebra in the following sections.

Inverse semigroups are algebraic structures very close to groups, as we commented and used as motivation. Nevertheless, there are significant differences between such theories. For instance, ideals make sense only for inverse semigroups. Also, there are five natural relations: Green's relations. More precisely, the $\mathcal{D}$-classes are the ones that encode the information about the connected components of the associated restriction groupoid. This relation will be the main tool of next pages.

Next we adopt the approach: 1st) define the basic properties of this relation as in Lawson's book [51], Chapter 3- 3.2 ; 2nd) study the particular case of S_{PB}. following Choi [20] and [19]; 3rd) then our computations of S_{GB} classes.

We follow the notations and definitions of Lawson [51], Section 3.2.

Definition 2.4.1 ([51])**.** Let S be an inverse semigroup. We define relations $\mathcal{R}$,$\mathcal{L}$, $\mathcal{J}$,$\mathcal{H}$, $\mathcal{D}$ on S as follows: for $s, t \in S$

(I) $s\mathcal{R}t \Leftrightarrow ss^* = tt^*$;

(II) $s\mathcal{L}t \Leftrightarrow s^*s = t^*t$;

(III) $s\mathcal{J}t \Leftrightarrow SsS = StS$;

(IV) $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$;

(V) $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, where $\circ$ is the composition of relations.

A few remarks and considerations.

Remark 2.4.2.

(a) Sometimes we will write $(s, t) \in \mathcal{R}$ meaning $s\mathcal{R}t$. Also $\mathcal{R}_s$ will indicate the right $\mathcal{R}$-class of s, that is, the set of elements $t \in S$ such that $s\mathcal{R}t$ (*i.e.* such that $tt^* = ss^*$). The same same for the others relations, respecting its definition.

(b) Notice that $\mathcal{H}$ relation highlights groups inside the inverse semigroup. Indeed each $\mathcal{H}_s$, for $s \in S$, is a group.

(c) $\mathcal{H}$ and $\mathcal{D}$ are are defined in terms of operations with relations. What they mean in terms of elements is:

(IV') $s\mathcal{H}t \Leftrightarrow ss^* = tt^*$ and $s^*s = t^*t$

(V') $s\mathcal{D}t \Leftrightarrow \exists c \in S$ s.t. $s^*s = c^*c$ and $cc^* = tt^*$.

(d) Lawson, Proposition 5 in Chapter 3 of [51], provided an equivalent definition for the $\mathcal{D}$ relation : suppose S is an inverse semigroup

(V'a) let $e, f \in \mathcal{E}(S)$, then $e\mathcal{D}f \Leftrightarrow \exists a \in S$ s.t. $a^*a = f$ and $aa^* = e$,

(V'b) let $s, t \in S$, then

$$s\mathcal{D}t \Leftrightarrow \exists a, b \in S \text{ s.t. } \begin{cases} a^*a = t^*t & \text{and } aa^* = s^*s \\ b^*b = tt^* & \text{and } bb^* = ss^* \end{cases},$$

and $t = b^*sa$.

Or, for $s, t \in S$ not idempotents

$$\begin{tikzcd} r(t) \arrow[d, dashed, "b"'] & d(t) \arrow[l, "t"'] \arrow[d, dashed, "a"] \\ r(s) & d(s) \arrow[l, "s"] \end{tikzcd}$$

(e) As we have seen there is a very natural way to define a groupoid from an inverse semigroup S, *i.e.* the restricted groupoid $\mathcal{G}_S$. Its structural maps, domain and range, are: $d, r : \mathcal{G}_S = S \to \mathcal{G}_S^{(0)} = \mathcal{E}(S)$ with $d(s) = s^*s$ and $r(t) = tt^*$. Interpreting the classes in $\mathcal{G}_S$:

$\mathcal{R}_t$ is the set of arrows that end in $r(t)$

$\mathcal{L}_s$ is the set of arrows that start in $d(s)$

$\mathcal{H}_c$ is the set of arrows that start in $d(c)$ and end in $r(c)$

And $\mathcal{D}$-class, will represent the connected components of $\mathcal{G}_S$, as the picture of (d) suggests.

A classic result about groupoids describes their structure in terms of their connected components. As Lawson says in [52] or his book [51] Section 3.3 - Proposition 6, there is a way to describe the structure of all connected groupoids. Let us state this correctly.

Proposition 2.4.3 ([51])**.** Let G be a group and let I be a non empty set. Then the set $I \times G \times I$ with partial multiplication

$$(i, g, j)(l, h, k) = \begin{cases} (i, gh, k) & , j = l \\ & , j \neq l \end{cases},$$

is a connected groupoid. Moreover every connected groupoid is isomorphic to one of this type times an isotropy group..

Another source of this result and one similar and related to groupoids (to appear in the next section) can also be found in Dokuchaev-Exel-Piccione [28] Section 3.

Combining the previous remark and this proposition, we can see how $\mathcal{D}$-classes and connected groupoids are related, as stated by Choi [19] Lemma 1.2.

Lemma 2.4.4 ([19])**.** *Let S be be an inverse semigroup and let D be a $\mathcal{D}$-class. If $e \in D \cap \mathcal{E}(S)$, the $\mathcal{G}_D \simeq \mathcal{G}_D^{(0)} \times \mathcal{G}_D^{(0)} \times G_e$, where the latter has the groupoid structure given in Proposition2.4.3) and $G_e := \{s \in D; d(s) = e = r(s)\}$.*

2.4.1 Green's relations on S_{PB}

In this section, we will present the results of Choi [20] Section 2 about the Green classes of S_{PB}.

First : **throughout this entire section G is a finite group**.

Remembering: an element $(A, g) \in S_{PB} = P_e(G) \rtimes_p G$ has the property $A \ni e, g$ and the product and inversion in this inverse semigroup are:

product: $(A, g)(B, h) = (A \cup gB, gh)$;

inverse: $(A, g)^* = (g^{-1}A, g^{-1})$.

We add a new piece in our puzzle, the *stabilizer*: if $A \subseteq G$ (as subset only) we define

$$\text{Stab}(A) := \{g \in G; gA = A\},$$

which is a subgroup of G.

Next, we present a sequence of results concerning Green's classes of S_{PB}. They are Lemma 2.1 and its corollaries, Lemma 2.6 and Theorem 2.7 from Choi's paper.

Lemma 2.4.5 ([20]). *Let $(A, g), (B, h) \in S_{PB}$:*

(i) $(A, g)\mathcal{R}(B, h) \Leftrightarrow A = B$, *and we have* $\mathcal{R}_{(A,g)} = \{(A, a); a \in A\}$*;*

(ii) $(A, g)\mathcal{L}(B, h) \Leftrightarrow g^{-1}A = h^{-1}B$, *and we have* $\mathcal{L}_{(A,g)} = \{(a^{-1}A, a^{-1}g); a \in A\}$*;*

(iii) $(A, g)\mathcal{H}(B, h) \Leftrightarrow A = B = hg^{-1}A$, *and we have* $\mathcal{H}_{(A,g)} = \{(A, sg); s \in \mathit{Stab}(A)\}$*. In particular* $\mathcal{H}_{(A,e)} \simeq \mathit{Stab}(A)$*.*

(iv) $(A, g)\mathcal{D}(B, h) \Leftrightarrow A = aB$ *for some* $a \in G$*, and we have* $\mathcal{D}_{(A,g)} = \{(a^{-1}A, a'); a \in A, a' \in a^{-1}A\}$*. Moreover* $\mathcal{J} = \mathcal{D}$*;*

As $\text{Stab}(A)$ acts freely on A by multiplications, Orbit and Stabilizer Theorem ([79]) counts its orbits by $\mid A/\text{Stab}(A) \mid = \dfrac{\mid A \mid}{\mid \text{Stab}(A) \mid}$.

We can use the previous characterization of equivalences to provides the counting properties of such classes.

Corollary 2.4.6 ([20]). *Let $(A, g) \in S_{PB}$*

(i) $\mid \mathcal{R}_{(A,g)} \mid = \mid \mathcal{L}_{(A,g)} \mid = \mid A \mid$*;*

(ii) $\mid \mathcal{D}_{(A,g)} \mid = \dfrac{\mid A \mid^2}{\mid \mathit{Stab}(A) \mid}$*;*

(iii) $\mid \mathcal{H}_{(A,g)} \mid = \mid \mathit{Stab}(A) \mid$*.*

Remark 2.4.7. Notice that, with the notation above, A is just a finite subset of G. We mean, there is no structure in A. But by Corollary 2.4.6 the item (ii) can help to decide if A is a subgroup.

Indeed, suppose $\mid \mathcal{D}_{(A,g)} \mid = \mid A \mid$ then

$$\mid A/\text{Stab}(A) \mid = \frac{\mid A \mid}{\mid \text{Stab}(A) \mid} = \frac{\mid \mathcal{D}_{(A,g)} \mid}{\mid \text{Stab}(A) \mid} = (\frac{\mid A \mid}{\mid \text{Stab}(A) \mid})^2$$

$$\Longrightarrow \mid A/\text{Stab}(A) \mid = 1.$$

Let $x, y \in A$, there exists $g \in \text{Stab}(A)$ such that $gx = y$. So

$$xy^{-1} = g^{-1} \implies xy^{-1}A = A.$$

Since $A \ni e$ we have $xy^{-1} \in A$ and A is a subgroup of G.

On the other hand , if A is already a subgroup, $\text{Stab}(A) = A$ and then $\mid A/\text{Stab}(A) \mid = 1$. Hence

$$\mid \mathcal{D}_{(A,g)} \mid = \frac{\mid A \mid^2}{\mid \text{Stab}(A) \mid} = \mid A \mid .$$

Conclusion: $A \leqslant G \Leftrightarrow \mid \mathcal{D}_{(A,g)} \mid = \mid A \mid$.

Next, we investigate more properties of elements $\mathcal{D}$ related.

Corollary 2.4.8 ([20])**.** *Let $(A, g), (B, h) \in S_{PB}$ such that $(A, g)\mathcal{D}(B, h)$, then*

(i) $\mid A \mid = \mid B \mid$,

(ii) $\mid \mathit{Stab}(A) \mid = \mid \mathit{Stab}(B) \mid$;

(iii) if G is Abelian, then $\mathit{Stab}(A) = \mathit{Stab}(B)$.

Note that in light of last corollary, more can be said: if $(A, s)\mathcal{D}(B, h)$, then A is in the orbit of B (by (iv) in Lemma 2.4.5, and $\text{Stab}(A)$ and $\text{Stab}(B)$ are conjugate subgroups of G.

Some further notations are needed. In the computations of Dokuchaev-Exel-Piccione's [28] a very smart move was to work with subgroupoids consisting of pairs (A, g) of Γ_{PB} (in their notation $\Gamma(G)$) in levels depending on the numbers of elements of A. For instance the k-th level of Γ_{PB} is formed by (A, g) such that $\mid A \mid = k$. Another important tool is the subgroup $\text{Stab}(A)$, the stabilizer of A (denoted by $S(A)$ by Dokuchaev et al.), that has already appeared in our notes.

Following Choi, we adopt the nomenclature for such notions: for $1 \leqslant k, m \leqslant \mid G \mid$ and H a subgroup of G we define

- $\mathcal{A}_k := \{(A, g) \in S_{PG}; \mid A \mid = k\}$,

- $d_k(H) :=\mid \{(A,g) \in \mathcal{A}_k;\ \text{Stab}(A) = H\} \mid$ and
- $d_k(m) = \sum\limits_{\substack{H \leqslant G \\ |H|=m}} d_k(H).$

This technology, together with what we have been developing for $\mathcal{D}$-classes, provides us this next Theorem.

Theorem 2.4.9 ([20]). *Using the notations above, the following identities hold:*

(i) $\mid S_{PB}/\mathcal{D} \mid := \{D \in \mathcal{D}; D \subseteq S_{PB}\} = \sum\limits_{k=1}^{|G|} \mid \mathcal{A}_k/\mathcal{D} \mid$ *and*

(ii) $\mid \mathcal{A}_k/\mathcal{D} \mid := \{D \in \mathcal{D}; D \subseteq \mathcal{A}_k\} = \sum\limits_{m=1}^{k} \frac{m}{k^2} d_k(m) = \frac{1}{k^2} \sum\limits_{H \leqslant G} \mid H \mid d_k(H)$ *for each* k.

From previous computation we realize that counting the $\mathcal{D}$-classes, boils down to solve the problem on $d_k(H)$. This is the hint to invoke Möbius inversion arguments. Remembering section 2.6, if we have a function f has a poset as its domain, then

$$\hat{f}(x) = \sum_{x \leqslant y} f(y) \Leftrightarrow f(x) = \sum_{x \leqslant y} \hat{f}(y)\mu(x,y), \text{ where } \mu(x,y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}.$$

With this insight in our mind, let H be a fixed subgroup of G and define

$$\tilde{d}_k(H) :=\mid \{(A,g) \in \mathcal{A}_k; H \leqslant \text{Stab}(A)\} \mid . \tag{2.4.1}$$

In fact

$$\tilde{d}_k(H) = \sum_{H \leqslant L} d_k(L) \Leftrightarrow d_k(H) = \sum_{H \leqslant L} \mu(H,L)\tilde{d}_k(L).$$

Finally we can compute $\tilde{d}_k$.

Lemma 2.4.10 ([20]). *If H is a subgroup G and $1 \leqslant k \leqslant\mid G \mid$, then*

$$\tilde{d}_k(H) = \begin{cases} \begin{pmatrix} \frac{\mid G \mid}{\mid H \mid} - 1 \\ \frac{k}{\mid H \mid} - 1 \end{pmatrix} k & , \mid H \mid \mid k \\ 0 & , \textit{otherwise} \end{cases}.$$

It is important to comment that Choi presents more then we showed above about Green's classes of S_{PB}. For our forthcoming section, this is enough. We end this subsection and generalize such results for S_{GB}.

2.4.2 Green's relations of S_{GB}

This section will present our generalized formula for the matrix decomposition of S_{GB}.

Again: **throughout this entire section G is a finite group**.

Remembering: an element $(A, g) \in S_{GB} = P(G) \rtimes G$ has no particular property, besides A being a finite subset of G. This result indicates more freedom on Green's relations, as we will shortly see.

The product and inversion maps, which behave likewise the previous case, are:

product: $(A, g)(B, h) = (A \cup gB, gh)$;

inverse: $(A, g)^* = (g^{-1}A, g^{-1})$.

Our version of the Lemma 2.4.5 is stated below.

Lemma 2.4.11. Let $(A, g), (B, h) \in S_{GB}$:

(i) $(A, g)\mathcal{R}(B, h) \Leftrightarrow A = B$ and $\mathcal{R}_{(A,g)} = \{(A, k); k \in G\}$;

(ii) $(A, g)\mathcal{L}(B, h) \Leftrightarrow g^{-1}A = h^{-1}B$ and $\mathcal{L}_{(A,g)} = \{(hg^{-1}A, h); g, h \in G\}$;

(iii) $(A, g)\mathcal{H}(B, h) \Leftrightarrow A = B = hg^{-1}A$ and $\mathcal{H}_{(A,g)} = \{(A, sg); s \in \text{Stab}(A)\}$. In particular $G_{(A,e)} \simeq \text{Stab}(A)$.

(iv) $(A, g)\mathcal{D}(B, h) \Leftrightarrow A = aB$ for some $a \in G$ and $\mathcal{D}_{(A,g)} = \{(kA, h); k, h \in G\}$.

Proof. The verification consists of straightforward computations.

(i) We must have $(A, g)(A, g)^* = (B, h)(B, h)^*$, where:

$$(A, g)(A, g)^* = (A, g)(g^{-1}A, g^{-1}) = (A \cup gg^{-1}A, gg^{-1}) = (A, e)$$

In the same manner

$$(B, h)(B, h)^* = (B, e).$$

Thus $A = B$, and finally

$$\mathcal{R}_{(A,g)} = \{(B, h); A = B, h \in G\} = \{(A, k); k \in G\}.$$

(ii) Suppose $(A, g)\mathcal{L}(B, h)$. By definition we must have $(A, g)^*(A, g) = (B, h)^*(B, h)$. Computing the left hand side

$$(A, g)^*(A, g) = (g^{-1}A, g^{-1})(A, g) = (g^{-1}A \cup g^{-1}A, g^{-1}g) = (g^{-1}A, e).$$

Likewise,

$$(B,h)^{-1}(B,h) = (h^{-1}B, e).$$

So $g^{-1}A = h^{-1}B$ and

$$\mathcal{L}_{(A,g)} = \{(B,h); g^{-1}A = h^{-1}B, h \in G\} = \{(hg^{-1}A, h); h \in G\}.$$

(iii) If $(A,g)\mathcal{H}(B,h)$, using (i)-(ii) and the definition of $\mathcal{H}$-class

$$\begin{cases} (A,g)(A,g)^* & = (A,e) \\ (B,h)^*(B,h) & = (h^{-1}B,e) \end{cases} \text{ and } \begin{cases} (A,g)^*(A,g) & = (g^{-1}A,e) \\ (B,h)(B,h)^* & = (B,e) \end{cases}.$$

Hence $A = h^{-1}B = g^{-1}A = B$ implies $A = B = hg^{-1}A$ and $hg^{-1} \in \text{Stab}(A)$. This way we have $h \in \text{Stab}(A)g$. Finally

$$\begin{aligned} \mathcal{H}_{(A,g)} &= \{(B,h); A = B = hg^{-1}A, h \in G\} \\ &= \{(A,h); h \in \text{Stab}(A)g\} \\ &= \{(A,sg); s \in \text{Stab}(A)\}. \end{aligned}$$

Now the second assertion, $G_{(A,e)} \simeq \text{Stab}(A)$:

$$\begin{aligned} G_{(A,e)} &= \{(B,h) \in S_{GB}; d((B,h)) = (A,e) = r((B,h))\} \\ &= \{(B,h) \in S_{GB}; (B,h)^{-1}(B,h) = (A,e) = (B,h)(B,h)^{-1}\} \\ &= \{(B,h) \in S_{GB}; (h^{-1}B,e) = (A,e) = (B,e)\} \\ &= \{(B,h) \in S_{GB}; A = B = h^{-1}B, h \in G\} \\ &= \{(A,h); h \in \text{Stab}(A)\} \\ &= \mathcal{H}_{(A,e)}. \end{aligned}$$

Therefore $G_{(A,e)} \simeq \text{Stab}(A)$.

(iv) Let $(A,g)\mathcal{D}(B,h)$. As $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, there exist $(C,f) \in S_{GB}$ such that

$$(A,g)\mathcal{L}(C,f) \text{ and } (C,f)\mathcal{R}(B,h).$$

By previous items we conclude $C = gf^{-1}A = B$; and the coordinate change $k = gf^{-1}$ implies the characterization

$$\mathcal{D}_{(A,g)} = \{(kA,h); h,k \in G\}.$$

This argument ends the proof. $\square$

Notice that all classes are very similar. The novelty is the fact that now we are indexing the classes by terms of G.

The same occurs with the number of elements within each class, but the $\mathcal{D}$-class is slightly different as we see next.

Corollary 2.4.12. Let $(A, g) \in S_{GB}$

(i) $\mid \mathcal{R}_{(A,g)} \mid = \mid \mathcal{L}_{(A,g)} \mid = \mid G \mid$;

(ii) $\mid \mathcal{D}_{(A,g)} \mid = \dfrac{\mid G \mid^2}{\mid \text{Stab}(A) \mid}$;

(iii) $\mid \mathcal{H}_{(A,g)} \mid = \mid \text{Stab}(A) \mid$.

Proof. The equalities in item (i) and (iii) are clear from Lemma 2.4.11, as the classes are defined with the second entry running over elements of G.

Now we turn our attention to (ii) . Notice that if $a, b \in A$, then

$$a^{-1}A = b^{-1}A \Leftrightarrow ba^{-1}A = A \Leftrightarrow ba^{-1} \in \text{Stab}(A) \Leftrightarrow b \in \text{Stab}(A)a.$$

Using this information and from Orbit and Stabilizer Theorem we have

$$\mid \mathcal{D}_{(A,g)} \mid = \frac{\mid G \mid}{\mid \text{Stab}(A) \mid} \mid G \mid .$$

$\square$

Using the same arguments the Corollary 2.4.8 remains valid for this case , *i.e.*:

$$(A, g)\mathcal{D}(B, h) \text{ in } S_{GB} \implies \begin{cases} \mid A \mid = \mid B \mid \\ \mid \text{Stab}(A) \mid = \mid \text{Stab}(B) \mid \\ \text{if } G \text{ is Abelian, then } \text{Stab}(A) = \text{Stab}(B) \end{cases} .$$

Indeed, suppose $(A, g)\mathcal{D}(B, h)$.

- $\mid A \mid = \mid B \mid$ follows from Corollary 2.4.12 item (i);
- $\mid \text{Stab}(A) \mid = \mid \text{Stab}(B) \mid$ follows from Corollary 2.4.12 item (ii), since

$$\frac{\mid G \mid^2}{\mid \text{Stab}(A) \mid} = \mid \mathcal{D}_{(A,g)} \mid = \mid \mathcal{D}_{(B,h)} \mid = \frac{\mid G \mid^2}{\mid \text{Stab}(B) \mid};$$

- suppose G an abeilian group, from Lemma 2.4.11 item (iv), $A = kB$ for some $k \in G$. If $a \in \text{Stab}(A)$, then $aA = A$ and this implies

$$B = k^{-1}A = k^{-1}aA = ak^{-1}A = aB.$$

Hence $a \in \text{Stab}(B)$. By the other hand, if $b \in \text{Stab}(B)$, then

$$bA = bkB = kbB = kB = A.$$

Thus $b \in \text{Stab}(A)$.

Also, note that the same remarks from Corollary 2.4.8 remain valid. We mean: if $(A, s)\mathcal{D}(B, h)$, then A is in the orbit of B (by (iv) in Lemma 2.4.5, and $\text{Stab}(A)$ and $\text{Stab}(B)$ are conjugate subgroups of G.

Our words in this section are about the levels of S_{GB}. We define:

- $\mathcal{B}_k := \{(A, g) \in S_{GB}; \mid A \mid = k\}$,
- $d_k(H) :=\mid \{(A, g) \in \mathcal{B}_k;\ \text{Stab}(A) = H\} \mid$ and
- $d_k(m) = \sum\limits_{\substack{H \leqslant G \\ |H|=m}} d_k(H).$

We have the analogous version of Lemma 2.4.10.

Lemma 2.4.13. If H is a subgroup G and $1 \leqslant k \leqslant\mid G \mid$, then

$$\tilde{d}_k(H) = \begin{cases} \left(\begin{array}{c} \dfrac{\mid G \mid}{\mid H \mid} \\ \dfrac{k}{\mid H \mid} \end{array} \right) \mid G \mid & , \mid H \mid \mid k \\ 0 & , \text{ otherwise} \end{cases}.$$

Proof. Suppose k with $1 \leqslant k \leqslant\mid G \mid$, and $H \leqslant G$.

First, by the definition of $\tilde{d}_k(-)$, we have

$$\tilde{d}_k(H) =\mid \{(A, g) \in \mathcal{B}_k; H \leqslant \text{Stab}(A)\} \mid\neq \emptyset.$$

If H stabilizes A, then $A = \bigcup\limits_{i \in I} Ha_i$, for $a_i \in A$ and I an index set. Last fact implies: $\mid H \mid \mid k$.

Let $(A, g) \in \mathcal{A}_k$ such that $H \leqslant \text{Stab}(A)$. Notice

- $\text{Stab}(A) = \bigcup\limits_{1 \leqslant i \leqslant \frac{|\text{Stab}(A)|}{|H|}} Hs_i$, where $s_i \in \text{Stab}(A)$ and

- $A = \bigcup_{1 \leqslant j \leqslant \frac{k}{|\text{Stab}(A)|}} \text{Stab}(A)a_j$, for $a_j \in A$.

Combining such identities, we conclude:

$$A = \bigcup_{\substack{1 \leqslant i \leqslant \frac{|\text{Stab}(A)|}{|H|} \\ 1 \leqslant j \leqslant \frac{k}{|\text{Stab}(A)|}}} Hs_i a_j.$$

This computation tells us that A is a union of $\frac{k}{|H|}$ cosets of H.

By the other hand, suppose $B \subset G$ such that $\mid B \mid = k$ and $B = \bigcup_{i=1}^{\frac{k}{|K|}} Hg_i$. Then $H \leqslant \text{Stab}(B)$.

Therefore, for $\mid H \mid |k$ we have

$$\tilde{d}_k(H) = \begin{pmatrix} \dfrac{\mid G \mid}{\mid H \mid} \\ \dfrac{k}{\mid H \mid} \end{pmatrix} k.$$

□

As these last results still valid, we can rephrase the Theorem 2.4.9 with minor differences in the final formula. In this case, we do not need to remove one point of our counting once there is no imposition on $e \in G$ being an element of the sets. This way, the formula is similar.

Theorem 2.4.14. Using the above notations,the following identities holds:

(i) $\mid S_{GB}/\mathcal{D} \mid = \sum_{k=1}^{|G|} \mid \mathcal{B}_k/\mathcal{D} \mid$ and

(ii) $\mid \mathcal{B}_k/\mathcal{D} \mid = \sum_{m=1}^{k} \frac{m}{\mid G \mid^2} d_k(m) = \frac{1}{\mid G \mid^2} \sum_{H \leqslant G} \mid H \mid d_k(H)$ for each k.

Proof. Let $(A, g) \in S_{GB}$. As we've just noticed, if $(B, h) \in S_{GB}$ is such that $(A, g)\mathcal{D}(B, h)$, then $\mid A \mid = \mid B \mid$. It follows from this fact

$$\mid S_{GB}/\mathcal{D} \mid = \sum_{k=1}^{|G|} \mid \mathcal{B}_k/\mathcal{D} \mid .$$

Now we fix k and compute each $\mid \mathcal{B}_k/\mathcal{D} \mid$. As we can write $A = \bigcup_{i=1}^{m} \text{Stab}(A)a_i$, it's possible to describe $\mathcal{B}_k$ as a disjoint union of the sets

$\{(A,g)\in\mathcal{B}_k; \mid \mathrm{Stab}(A)\mid = m\}$. So

$$\mathcal{B}_k = \bigcup_{m=1}^{k}\{(A,g)\in\mathcal{B}_k; \mid \mathrm{Stab}(A)\mid = m\}.$$

Recall that if

$$(A,g)\mathcal{D}(B,h) \implies \mid \mathrm{Stab}(A)\mid = \mid \mathrm{Stab}(B)\mid$$

and

$$\mid \mathcal{D}_{(A,g)}\mid = \frac{\mid G\mid^2}{\mid \mathrm{Stab}(A)\mid}.$$

Also, notice that

$$\bigcup_{\substack{S\leqslant G\\ |S|=m}}\{(A,g)\in\mathcal{B}_k; \mathrm{Stab}(A)=S\} = \{(A,g)\in\mathcal{B}_k; \mid \mathrm{Stab}(A)\mid = m\}.$$

It now follows

$$\begin{aligned}
\mid \mathcal{B}_k/\mathcal{D}\mid &= \sum_{m=1}^{k}\mid \{(A,g)\in\mathcal{B}_k; \mid \mathrm{Stab}(A)\mid = m\}/\mathcal{D}\mid\\
&= \sum_{m=1}^{k}\frac{m}{\mid G\mid^2}\mid \{(A,g)\in\mathcal{B}_k; \mid \mathrm{Stab}(A)\mid = m\}\mid\\
&= \sum_{m=1}^{k}\Big(\sum_{\substack{S\leqslant G\\ |S|=m}}\frac{\mid S\mid}{\mid G\mid^2}\mid \{(A,g)\in\mathcal{B}_k;\ \mathrm{Stab}(A)=S\}\mid\Big)\\
&= \sum_{m=1}^{k}\Big(\sum_{\substack{S\leqslant G\\ |S|=m}}\frac{\mid S\mid}{\mid G\mid^2}d_k(S)\Big).
\end{aligned}$$

We remind the reader that

$$d_k(m) = \sum_{\substack{S\leqslant G\\ |S|=m}} d_k(S) = \sum_{\substack{S\leqslant G\\ |S|=m}}\mid \{(A,g)\in\mathcal{A}_k;\ \mathrm{Stab}(A)=S\}\mid.$$

Hence

$$\mid \mathcal{B}_k/\mathcal{D}\mid = \sum_{m=1}^{k}\Big(\sum_{\substack{S\leqslant G\\ |S|=m}}\frac{\mid S\mid}{\mid G\mid^2}d_k(S)\Big) = \sum_{m=1}^{k}\frac{m}{\mid G\mid^2}d_k(m).$$

Finally, as $\{(A, g) \in \mathcal{A}_k;\ \text{Stab}(A) = S\} = \emptyset$ if $| S |> k$, we have

$$\sum_{m=1}^{k} \sum_{\substack{S \leqslant G \\ |S|=m}} \frac{| S |}{| G |^2} d_k(S) = \frac{1}{| G |^2} \sum_{S \leqslant G} | S | \, d_k(S).$$

Hence the result is proven. □

Remark 2.4.15. The reader more familiar with inverse semigroup theory, may have noticed that since S_{PB} is a subsemigroup of S_{GB} the Green's relations of the former are related to the Green's relations of the latter, as Lawson showed in [50] Lemma 3.1. In a more general fashion, if $S \subseteq T$ then, denoting the classes of each semigroup indexed by S or T:

- $\mathcal{R}^T \cap (S \times S) = \mathcal{R}^S$ and the same for $\mathcal{L}$ and $\mathcal{H}$
- $\mathcal{D}^S \subseteq \mathcal{D}^T \cap (S \times S)$ and the same for $\mathcal{J}$.

Summarizing we compile a table with the results of this subsections:

	S_{PB}	S_{GB}
$\mathcal{R}_{(A,g)}$	$A = B$ $\{(A, a); a \in A\}$ $\mid \mathcal{R}_{(A,g)} \mid = \mid A \mid$	$A = B$ $\{(A, k); k \in G\}$ $\mid \mathcal{R}_{(A,g)} \mid = \mid G \mid$
$\mathcal{L}_{(A,g)}$	$g^{-1}A = h^{-1}B$ $\{(a^{-1}A, a^{-1}g); a \in A\}$ $\mid \mathcal{L}_{(A,g)} \mid = \mid A \mid$	$g^{-1}A = h^{-1}B$ $\{(hg^{-1}A, h); g, h \in G\}$ $\mid \mathcal{L}_{(A,g)} \mid = \mid G \mid$
$\mathcal{H}_{(A,g)}$	$A = B = hg^{-1}A$ $\{(A, sg); s \in \text{Stab}(A)\}$ $\mid \mathcal{H}_{(A,g)} \mid = \mid \text{Stab}(A) \mid$	$A = B = hg^{-1}A$ $\{(A, sg); s \in \text{Stab}(A)\}$ $\mid \mathcal{H}_{(A,g)} \mid = \mid \text{Stab}(A) \mid$
$\mathcal{D}_{(A,g)}$	$A = aB$ for $a \in G$ $\{(a^{-1}A, a'); a \in A, a' \in a^{-1}A\}$ $\mid \mathcal{D}_{(A,g)} \mid = \dfrac{\mid A \mid^2}{\mid \text{Stab}(A) \mid}$	$A = aB$ for $a \in G$ $\{(kA, h); k, h \in G\}$ $\mid \mathcal{D}_{(A,g)} \mid = \dfrac{\mid G \mid^2}{\mid \text{Stab}(A) \mid}$
$\tilde{d}_k(H)$	$\left(\begin{array}{c} \frac{\mid G \mid}{\mid H \mid} - 1 \\ \frac{k}{\mid H \mid} - 1 \end{array}\right)$ k only if $\mid H \mid \mid k$	$\left(\begin{array}{c} \frac{\mid G \mid}{\mid H \mid} \\ \frac{k}{\mid H \mid} \end{array}\right)$ $\mid G \mid$ only if $\mid H \mid \mid k$
$\mid \mathcal{A}_k/\mathcal{D} \mid$, $\mid \mathcal{B}_k/\mathcal{D} \mid$	$\displaystyle\sum_{m=1}^{k} \frac{m}{k^2} d_k(m) = \frac{1}{k^2} \sum_{H \leqslant G} \mid H \mid d_k(H)$	$\displaystyle\sum_{m=1}^{k} \frac{m}{\mid G \mid^2} d_k(m) = \frac{1}{\mid G \mid^2} \sum_{H \leqslant G} \mid H \mid d_k(H)$

Table 2.1: The Green classes of S_{PB} and S_{GB}

2.5 Partial and Global algebras of groups

All tools we have been developing and studying so far will combine its strength in this final section of Chapter 2. We will talk about how the enlargement properties reflect in the respective inverse semigroups algebras and groupoids. Furthermore, we conclude with a characterization of such algebras using Green's $\mathcal{D}$-classes.

In this section we fix: $\mathbb{K}$ **a associative and commutative unital ring**. Moreover, Morita equivalent algebras R and Q will be denoted by $R \simeq_M Q$.

Remark 2.5.1. Our notation for Morita equivalences are pretty similar, but the context will differentiate them. Indeed

for inverse semigroups: $S \simeq_{sM} T$,

for groupoids: $\mathcal{G} \simeq_M \mathcal{H}$,

for algebras: $\mathbb{K}S \simeq_M \mathbb{K}T$ and $\mathbb{K}\mathcal{G} \simeq_M \mathbb{K}\mathcal{H}$.

2.5.1 The Morita relation among the algebras $\mathbb{K}_{glob}(G)$ and $\mathbb{K}_{par}(G)$

We begin with the general setting. Steinberg in [88] showed in Theorem 4.13 the following result.

Theorem 2.5.2 ([88])**.** *Let S and T be strongly Morita equivalent inverse semigroups. Then their algebras are Morita equivalent.*

This relation holds because the Morita context for inverse semigroups is very similar to the context of rings. In fact, $S \simeq_{sM} T$ means that exists a 5-tuple $(S, X, T, <, >, [,])$ together with actions. This way, $\mathbb{K}X$ is naturally a $\mathbb{K}S - \mathbb{K}T$ bimodule.

We introduce a standard notation and define a new one:

$$\mathbb{K}S_{PB} = \mathbb{K}_{par}(G) \text{ and } \mathbb{K}S_{GB} := \mathbb{K}_{glob}(G).$$

The *glob* refers to the fact the inverse semigroup S_{GB} was defined by a map that globalizes the Bernoulli action.

Using our notations and previous computations: $S \subseteq_E T \implies \mathbb{K}S \simeq_M \mathbb{K}T$. This gives them a new version of the diagram before the Lemma 2.3.9

$$\begin{tikzcd}
 & \mathfrak{B} : G \curvearrowright P(G) \arrow[r] & S_{GB} \arrow[r, squiggly] \arrow[dd, dotted, no head, "\subseteq_E \Longrightarrow \simeq_{sM}"] & \mathbb{K}S_{GB} = \mathbb{K}_{glob}(G) \arrow[dd, no head, "\simeq_M"] \\
G \arrow[ur] \arrow[dr] & & & \\
 & \mathfrak{b} : G \curvearrowright_p P_e(G) \arrow[r] & S_{PB} \arrow[r, squiggly] & \mathbb{K}S_{PB} = \mathbb{K}_{par}(G)
\end{tikzcd}$$

Figure 2.8: The inverse semigroup algebras associated to the Bernouli actions

In a proper statement form, we have the Corollary.

Corollary 2.5.3. Let the inverse semigroups defined by the Bernoulli actions and its respective algebras: S_{GB} and $\mathbb{K}_{glob}(G)$, S_{PB} and $\mathbb{K}_{par}(G)$. Then $\mathbb{K}_{par}(G) \simeq_M \mathbb{K}_{glob}(G)$.

We now move to the groupoid case. We want to state a result analogous to the groupoid algebras. We have in the literature the key to solve this problem. Let us elaborate.

As we are dealing with discrete groups, all topological constraints are automatically satisfied. For instance, our groupoids are ample and étale. Thus their Steinberg algebras are just the groupoid algebras. Our Rosetta stone is the Theorem 5.1 of Clark-Sims [21], that we rephrase below.

Theorem 2.5.4 ([21])**.** *If $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent groupoids, then its Steinberg algebras $A_{\mathbb{K}}(\mathcal{G})$ and $A_{\mathbb{K}}(\mathcal{H})$ are Morita equivalent.*

Again, using our notations and applying to our case of study, we can update the final diagram of Section 3.3:

$$\begin{tikzcd}
 & & & & & \mathcal{G}_{GB} \arrow[r, rightsquigarrow] \arrow[dddddd, dotted, bend left, "\simeq_M"] & \mathbb{K}\mathcal{G}_{GB} \arrow[dddddd, no head, bend left=60, "\simeq_M"] \\
\mathbb{K}\Gamma_{GB} \arrow[dddd, no head, "\simeq_M"'] & \Gamma_{GB} \arrow[l, rightsquigarrow] \arrow[dddd, dotted, no head, "\simeq_M"'] & & & S_{GB} \arrow[ur] \arrow[dr] & & \\
 & & \mathfrak{B} : G \curvearrowright P(G) \arrow[ul] \arrow[urr] & & & \widehat{\mathcal{G}(S_{GB})} \arrow[r, rightsquigarrow] \arrow[dd, dotted, no head, bend left, "\simeq_M"] & \mathbb{K}\widehat{\mathcal{G}(S_{GB})} \arrow[dd, no head, bend left, "\simeq_M"] \\
 & & G \arrow[u] \arrow[d] & & & & \\
 & & \mathfrak{b} : G \curvearrowright_p P(G) \arrow[dl] \arrow[drr] & & & \widehat{\mathcal{G}(S_{PB})} \arrow[r, rightsquigarrow] & \mathbb{K}\widehat{\mathcal{G}(S_{PB})} \\
\mathbb{K}\Gamma_{PB} & \Gamma_{PB} \arrow[l, rightsquigarrow] & & & S_{PB} \arrow[ur] \arrow[dr] & & \\
 & & & & & \mathcal{G}_{PB} \arrow[r, rightsquigarrow] & \mathbb{K}\mathcal{G}_{PB}
\end{tikzcd}$$

Figure 2.9: The groupoid algebras associated to the Bernouli actions

Alternatively, more formally and properly written.

Corollary 2.5.5. Let the semigroups S_{GB} and S_{PB} defined by the Bernoulli actions. Consider the action groupoids, restriction groupoids and universal groupoids, respectively: Γ_{GB} and Γ_{PB}, $\mathcal{G}_{GB}$ and $\mathcal{G}_{PB}$, $\widehat{\mathcal{G}(S_{GB})}$ and $\widehat{\mathcal{G}(S_{PB})}$. Then

$$\mathbb{K}\Gamma_{PB} \simeq_M \mathbb{K}\Gamma_{GB}, \quad \mathbb{K}\mathcal{G}_{PB} \simeq_M \mathbb{K}\mathcal{G}_{GB}, \quad \mathbb{K}\widehat{\mathcal{G}(S_{PB})} \simeq_M \mathbb{K}\widehat{\mathcal{G}(S_{GB})}.$$

We can go a little bit further and provide another relation among such algebras, connecting the inverse semigroup algebras and the universal groupoid algebras. Steinberg developed this results in [87], as a generalization of his own previous works [85] and [86]. More precisely, Theorem 6.3, which we state now.

Theorem 2.5.6 ([87]). *Let S be an inverse semigroup. Then the homomorphism $\varphi : S \to \mathbb{K}\widehat{\mathcal{G}(S)}$ defined by $\varphi(s) = \chi_{(s,D(s^*s))}$ (the characteristic map over the bissection) extends to an isomorphism $\tilde{\varphi} : \mathbb{K}S \to \mathbb{K}\widehat{\mathcal{G}(S)}$.*

The proof deals with properties of the bissections of the universal groupoid. We refer to Section 5 of the original article for such results because this deviates from our aim.

Speaking of our interests, these are closer than never. If the reader may allow, we will briefly discuss our motivations. The question that led us to this result was, "If the inverse semigroups S_{PB} and S_{GB} are Morita equivalent, their algebras should share the same (or a kind of) properties".

The literature has already provided an answer to a similar question. The Theorem 5.4 of Dokuchaev-Exel's work paper [27] answers the question of a group acting partially on an unital algebra. This way we guessed there would be a similar formulation for our inverse semigroups algebras. And after digging a little bit, we realized this way of action: action - inverse semigroup - universal groupoid - isomorphism between algebras. As indicated in this diagram:

$$\begin{tikzcd}
 & S_{GB} \arrow[rr, squiggly] \arrow[dr] & & \mathbb{K}S_{GB} = \mathbb{K}_{glob}(G) \arrow[d, dash, "\simeq"] \arrow[dddd, bend left=60, dash, "\simeq_M"] \\
\mathfrak{B} : G \curvearrowright P(G) \arrow[ur] \arrow[dd, leftrightarrow, "G"] & & \widehat{\mathcal{G}(S_{GB})} \arrow[r, squiggly] & \mathbb{K}\widehat{\mathcal{G}(S_{GB})} \arrow[dd, dotted, dash, "\simeq_M"] \\
 & & & \\
\mathfrak{b} : G \curvearrowright_p P(G) \arrow[dr] & & \widehat{\mathcal{G}(S_{PB})} \arrow[r, squiggly] & \mathbb{K}\widehat{\mathcal{G}(S_{PB})} \arrow[d, dash, "\simeq"] \\
 & S_{PB} \arrow[ur] \arrow[rr, squiggly] & & \mathbb{K}S_{PB} = \mathbb{K}_{par}(G)
\end{tikzcd}$$

Figure 2.10: The isomorphism and Morita equivalence of algebras

Using our nomenclature of the algebras arising from Bernoulli's actions, we can conclude the next Corollary.

Corollary 2.5.7. Let S_{PB} and S_{GB} be the inverse semigroups of Bernoulli actions. Then its algebras are Morita equivalent, *i.e.*

$$\mathbb{K}_{par}(G) \simeq_M \mathbb{K}_{glob}(G).$$

The case of G finite provides a matrix realization of such algebras. This realization is the main result of the next subsection.

2.5.2 Characterizing the algebras $\mathbb{K}_{par}(G)$ and $\mathbb{K}_{glob}(G)$

Among other questions, in [28] Dokuchaev-Exel-Piccione was established as a structural decomposition of the partial algebra of the inverse semigroup $S(G)$ when G is finite.

In short terms:

$$\mathbb{K}_{par}(G) \simeq \bigoplus_{\substack{H \leqslant G \\ 1 \leqslant m \leqslant [G:H]}} c_m(H)\mathbb{M}_m(\mathbb{K}H),$$

where $\mathbb{K}H$ is the group algebra of H over the commutative associative and unital ring $\mathbb{K}$; $c_m(H)$ is the number of sets $A \subseteq G$ such that $\mid A \mid = m \mid H \mid$, $A \ni e$, and $H = \{g \in G; gA = A\}$. There is a recursive formula for the terms c_m:

$$c_m(H) = \frac{1}{m}[G : N_G(H)] \left(\begin{pmatrix} [G:H]-1 \\ m-1 \end{pmatrix} - \sum_{\substack{H < B \leqslant G \\ [B:H] \mid m}} \frac{\frac{m}{[B:H]} c_{\frac{m}{[B:H]}}(B)}{[G : N_G(B)]} \right).$$

This version is a correction to the original formula. Here we are basing our discussion on the last Survey by Dokuchaev [26] and reproducing the formula (2) from Dokuchaev-Milies [30]. They achieved this formula based on counting arguments within a graph associated with the groupoid Γ_{PB} (using our notation).

Dokuchaev highlights (in [26] page 33) the existence of the other two proofs of the same formula. Both proofs use Möbius functions. One is due to Dokuchaev and Simon in [29], and the other due to Choi in [19].

We opted for the Choi approach because the technology he developed will help our purposes in chapters to come. This section aims to revise Choi's way to describe partial algebras using $\mathcal{D}$-classes. Next, we generalize his results to our global algebras.

2.5.3 Computing $\mathbb{K}_{par}(G)$ using its $\mathcal{D}$ classes

Before beginning our work, let us establish: G **is a finite group**.

First, we translate the matrix form of connected components of groupoids from Dokuchaev-Exel-Piccione [28] Proposition 3.1, using Green's classes. Choi made this in [19] Lemma 2.1.

Lemma 2.5.8 ([19]). *Let S be an inverse semigroup, D be a $\mathcal{D}$-class and $e \in D \cap \mathcal{E}(S)$. Then there is an isomorphism of matrix algebras $\mathbb{K}\mathcal{G}_D \simeq \mathbb{M}_{|\mathcal{G}^{(0)}|}(\mathbb{K}H)$, where $H = \{s \in D; d(s) = e = r(s)\}$. Moreover $\mathbb{K}S \simeq \bigoplus_i \mathbb{K}\mathcal{G}_{D_i}$, where D_i runs over the Green $\mathcal{D}$-classes.*

The formula we want to present is almost feasible by now. There are two more technicalities we must deal with first.

In last subsection we defined the numbers $d_k(-)$ whose definition we recall:

$$d_k(H) :=\mid \{(A,g) \in \mathcal{A}_k;\ \text{Stab}(A) = H\} \mid .$$

We make a small adjustment in this enumeration, and we will count the pairs (A,e) instead, *i.e.*:

$$d_k^e(H) :=\mid \{(A,e) \in \mathcal{A}_k;\ \text{Stab}(A) = H\} \mid .$$

This implies in a slightly similar rule for $\tilde{d^e}_k(-)$, in comparison to Lemma 2.4.10:

$$\tilde{d^e}_k(H) = \begin{cases} \begin{pmatrix} \dfrac{\mid G \mid}{\mid H \mid} - 1 \\ \dfrac{k}{\mid H \mid} - 1 \end{pmatrix} & , \mid H \mid \mid k \\ 0 & , \text{ otherwise} \end{cases} .$$

It is essential to notice that the above modifications do not change the last section. Only the number of counting arguments are modified.

The next lemma is the last piece we need to state Choi's result. It is after Choi [19] Lemma 2.3.

Lemma 2.5.9 ([19]).

(i) If (A,g) and (B,h) are elements of S_{PB} such that $(A,g)\mathcal{D}(B,h)$, then Stab(A) is conjugate to Stab(B).

(ii) Let H_1 and H_2 be subgroups of G. If H_1 and H_2 are conjugate, then $d_k^e(H_1) = d_k^e(H_2)$.

Before the proper characterization and Choi's approach of the algebra $\mathbb{K}_{par}(G)$ we need to make an adjustment and put in our terms of Bernoulli actions. What we mean is

$$\mathbb{K}_{par}(G) = \mathbb{K}\Gamma_{PB}.$$

This is valid because we have the isomorphisms $(S_{PB},\cdot) \simeq (\Gamma_{PB},\odot)$ (from Proposition 2.2.13) , and as G is finite the universal groupoid is, in fact, isomorphic to the action groupoid and the isomorphism between inverse semigroup algebras and groupoids algebras is due to Steinberg isomorphism of algebras, Theorem 2.5.6. This leads us to

$$\mathbb{K}_{par}(G) = \mathbb{K}S_{PB} \simeq \mathbb{K}\widehat{\mathcal{G}(S_{PB})} \simeq \mathbb{K}\Gamma_{PB}.$$

We do not claim the novelty of the result. On the other hand, this is another approach and way to prove it, with insights from Choi's results and the isomorphism from Steinberg's work. This way, we stated a corollary below for further references.

Corollary 2.5.10. Let S_{PB} be the inverse semigroup defined by the Bernoulli partial action, and let Γ_{PB} be the groupoid of the same action. Then $\mathbb{K}_{par}(G) \simeq \mathbb{K}\Gamma_{PB}$.

Finally, we present Theorem 2.4 from Choi [19], with our notation. This argumentation is another proof of Dokuchaev-Exel-Piccione's [28] Theorem 3.2.

Theorem 2.5.11 ([19]). *The groupoid algebra* $\mathbb{K}_{par}(G)$ *has the form*

$$K_{par}(G) = \bigoplus_{\substack{H \leqslant G \\ 1 \leqslant m \leqslant [G:H]}} \frac{d^{e}_{m|H|}(H)}{m} \mathbb{M}_m(\mathbb{K}H).$$

Moreover, the formula (via Möbius function) is valid:

$$d^{e}_{m|H|}(H) = \sum_{H \leqslant L} \mu(H, L)\tilde{d^{e}}_{m|H|}(L),$$

and

$$\tilde{d^{e}}_{m|H|}(L) = \begin{cases} \begin{pmatrix} \dfrac{|\, G \,|}{|\, L \,|} - 1 \\ \dfrac{m \mid H \mid}{|\, L \,|} - 1 \end{pmatrix} & , \mid L \mid \textit{divides } \ m \mid H \mid \\ 0 & , \textit{ otherwise} \end{cases}.$$

Proof. Let D in $\mathcal{A}_k$ a Green $\mathcal{D}$-class, such that $D \ni (A, g)$.

We are going to need the following previous results:

- by Corollary 2.4.6 we know $\mid \mathcal{D}_{(A,g)} \mid = \dfrac{\mid A \mid^2}{\mid \text{Stab}(A) \mid} = \dfrac{k^2}{\mid \text{Stab}(A) \mid}$;
- and by Lemma 2.5.9 if (B, h) is such that $(A, g)\mathcal{D}(B, h)$, then Stab(A) is conjugate to Stab(B),
- moreover, if we combine this fact with the Corollary 2.4.8, we have $\mid A \mid = \mid B \mid$.

The first half of this proof is based on the following ideas: we relate elements (A, g) and (B, h) of $\mathcal{A}_k$ which have conjugate stabilizers. We will denote this by Stab$(A) \sim_c$ Stab(B). Then we refine the set $\mathcal{A}_k$. So we count elements of $\mathcal{D}$ inside this refinement.

Indeed, let $H \leq G$ fixed. We denote

$$Cjg(H) := \{L \leqslant G; L \text{ is conjugate to } H\};$$

and

$$\mathcal{A}_{k,Cjg(H)} := \{(A,g) \in \mathcal{A}_k;\ \text{Stab}(A) \sim_c H\}.$$

The number of elements in this set is given by

$$\mid \mathcal{A}_{k,Cjg(H)} \mid = \sum_{\substack{L \leqslant G \\ L \sim_c H}} \mid \{(A,g) \in \mathcal{A}_k; \text{Stab}(A) = L\} \mid .$$

As $\mid A \mid = k$

$$\mid \mathcal{A}_{k,Cjg(H)} \mid = \sum_{\substack{L \leqslant G \\ L \sim_c H}} \mid \{(A,e) \in \mathcal{A}_k; \text{Stab}(A) = L\} \mid k.$$

Using the information in our second paragraph in this setting, for each $\mathcal{D}$-class D in $\mathcal{A}_k$ we have

$$\mid D \mid = \frac{k^2}{\mid L \mid} \implies \frac{1}{\mid D \mid} = \frac{\mid L \mid}{k^2}.$$

Recall that we defined $d_k^e(L) := \mid \{(A,e) \in \mathcal{A}_k;\ \text{Stab}(A) = L\} \mid$, so

$$\begin{aligned}
\mid \mathcal{A}_{k,Cjg(H)}/\mathcal{D} \mid &= \frac{\mid L \mid}{k^2} \mid \mathcal{A}_{k,Cjg(H)} \mid \\
&= \sum_{\substack{L \leqslant G \\ L \sim_c H}} \frac{\mid L \mid}{k^2} \mid \{(A,e) \in \mathcal{A}_k;\ \text{Stab}(A) = L\} \mid k \\
&= \sum_{\substack{L \leqslant G \\ L \sim_c H}} \frac{\mid L \mid}{k} d_k^e(L) \\
&= \mid Cjg(H) \mid \frac{\mid H \mid}{k} d_k^e(H).
\end{aligned}$$

The last equality is due to Lemma 2.5.9, because $d_k^e(L) = d_k^e(H)$.

Next, we want to calculate the $\mathbb{K}$ algebra of each $D \in \mathcal{D}$ in the set $\mathcal{A}_{k,Cjg(H)}$. Notice from Lemma 2.5.8 that all of these algebras are isomorphic to $\mathbb{M}_{\frac{k}{|H|}}(\mathbb{K}H)$. The number we've just found above, is the number of copies of $\mathbb{M}_{\frac{k}{|H|}}(\mathbb{K}H)$.

Next we will proof the decomposition as a directed sum. In this part we need a family of representatives of the conjugate classes of subgroups of G : let $\{H_i\}_{i \in I}$ be such a family, and let $Cjg(H_i)$ be the conjugate class of the subgroup H_i. Now, from Lemma 2.5.9 it follows that

$$\begin{aligned}
\mathbb{K}\Gamma_{PB} &\simeq \bigoplus_{\substack{k \\ i\in I \\ D\in\mathcal{A}_{k,Cjg(H_i)}/\mathcal{D}}} \mathbb{K}D \\
&\simeq \bigoplus_{\substack{k \\ i\in I}} \mid Cjg(H_i) \mid \frac{\mid H_i \mid}{k} d^e_k(H_i)\mathbb{M}_{\frac{k}{|H_i|}}(\mathbb{K}H_i) \\
&\simeq \bigoplus_{\substack{k \\ i\in I}} \bigoplus_{H\in Cjg(H_i)} \frac{\mid H \mid}{k} d^e_k(H)\mathbb{M}_{\frac{k}{|H|}}(\mathbb{K}H) \\
&\simeq \bigoplus_{\substack{k \\ H\leqslant G}} \frac{\mid H \mid}{k} d^e_k(H)\mathbb{M}_{\frac{k}{|H|}}(\mathbb{K}H).
\end{aligned}$$

Finally, letting $m := \dfrac{k}{\mid H \mid}$

$$\mathbb{K}\Gamma_{PB} \simeq \bigoplus_{\substack{H\leqslant G \\ 1\leqslant m\leqslant [G:H]}} \frac{d^e_{m|H|}(H)}{m} \mathbb{M}_m(\mathbb{K}H).$$

We concluded the proof, once the expression of $\tilde{d}^e{}_{m|H|}(H)$ derives from commentary before Lemma 2.5.9. $\square$

The previous proof showed us two ways of computing the algebra of Γ_{PB}:

via conjugate classes:

$$\mathbb{K}\Gamma_{PB} \simeq \bigoplus_{\substack{Cjg(H) \\ 1\leqslant m\leqslant [G:H]}} \mid Cjg(H) \mid \frac{d^e_{m|H|}(H)}{m} \mathbb{M}_m(\mathbb{K}H);$$

using all subgroups:

$$\mathbb{K}\Gamma_{PB} \simeq \bigoplus_{\substack{H\leqslant G \\ 1\leqslant m\leqslant [G:H]}} \frac{d^e_{m|H|}(H)}{m} \mathbb{M}_m(\mathbb{K}H).$$

Depending on the group structure, one formula is more convenient than the other.

Remark 2.5.12. The version we presented above is a correction to Choi's computation. Also, this new version matches the formula presented in [30].

Next, we extend these computations to $\mathbb{K}_{glob}(G)$.

2.5.4 Computing $\mathbb{K}_{glob}(G)$ using its $\mathcal{D}$-classes

Once over: G **is a finite group**.

From the previous subsection discussion, now working with S_{GB}, we have

$$\begin{cases} (S_{GB}, \cdot) \simeq (\Gamma_{GB}, \odot), \\ \widehat{\mathcal{G}(S_{GB})} \simeq \Gamma_{GB} \text{ and} \\ \mathbb{K}S_{GB} \simeq \mathbb{K}\Gamma_{GB}. \end{cases} \implies \mathbb{K}_{glob}(G) \simeq \mathbb{K}\Gamma_{GB}.$$

The remaining result we need is a version of the Lemma 2.5.9. The same statement holds if we substitute S_{PB} by S_{GB}, only the proof is not the same, but very similar. Notice that d_k^e now runs over elements of $\mathcal{B}_k$.

Lemma 2.5.13.

(i) If (A, g) and (B, h) are elements of S_{GB} such that $(A, g)\mathcal{D}(B, h)$, then Stab(A) is conjugate to Stab(B).

(ii) Let H_1 and H_2 subgroups of G. If H_1 and H_2 are conjugate, then $d_k^e(H_1) = d_k^e(H_2)$.

Proof.

(i) Suppose $(A, g)\mathcal{D}(B, h)$, in S_{GB}. By Lemma 2.4.11 item (iv) there exists $k \in G$ such that $A = kB$. Clearly Stab$(A) =$ Stab(kB). Let $a \in$ Stab(A). Notice that $a \in$ Stab(kB) implies $aA = akB = kB$. Hence

$$k^{-1}akB = B \implies k^{-1}ak \in \text{Stab}(B) \implies a \in k\text{Stab}(B)k^{-1}.$$

Thus we've just showed that Stab$(A) \subseteq k$Stab$(B)k^{-1}$. As $(A, g)\mathcal{D}(B, h)$, from Corollary 2.4.12 $\mid$ Stab$(A) \mid=\mid k$Stab$(B)k^{-1} \mid$, we can conclude Stab$(A) = k$Stab$(B)k^{-1}$.

(ii) By hypothesis, as H_1 and H_2 are conjugate, there exists $h \in G$ such that $H_2 = kH_1k^{-1}$.

Let the map $\phi : \{(A, e) \in \mathcal{B}_k;$ Stab$(A) = H_1\} \to \{(B, e) \in \mathcal{B}_k;$ Stab$(B) = H_2\}$ given by $\phi(A, e) = (kAk^{-1}, e)$. Let's verify that this map is well defined by showing that hH_1h^{-1} stabilizes hAt for all t. Indeed if $g \in G$ is satisfies $ghAt = hAt$, then

$$h^{-1}ghAt = At \implies h^{-1}ghA = A.$$

Continuing, the inclusion Stab$(hAl) \subseteq hH_1h^{-1}$ is clear. The opposite is also valid, because given $x \in H_1$

$$hxh^{-1}hAt = hxAt = hAt.$$

Finally notice that this map is bijective. Therefore $d^e_k(H_1) = d^e_k(H_2)$.

$\square$

Now our final result, the counting formula for the global algebra.

Theorem 2.5.14. The groupoid algebra $\mathbb{K}_{glob}(G)$ has the form

$$K_{glob}(G) = \bigoplus_{\substack{k \\ H \leqslant G}} \frac{d^e_k(H)}{[G:H]} \mathbb{M}_{[G:H]}(\mathbb{K}H),$$

where $1 \leqslant k \leqslant | G |$. Moreover, the formula (via Möbius function) is valid:

$$d^e_k(H) = \sum_{H \leqslant L} \mu(H,L) \tilde{d^e}_k(L),$$

and

$$\tilde{d^e}_k(L) = \begin{cases} \begin{pmatrix} \dfrac{| G |}{| L |} \\ \dfrac{k}{| L |} \end{pmatrix} & , | L | \text{ divides } k \\ 0 & , \text{ otherwise} \end{cases}.$$

Proof. Let D in $\mathcal{B}_k$ a Green $\mathcal{D}$-class in S_{GB}, such that $D \ni (A,g)$.

Notice that:

- by Corollary 2.4.12 we have $| \mathcal{D}_{(A,g)} | = \dfrac{| G |^2}{| \text{Stab}(A) |}$;
- and by Lemma 2.5.13 if (B,h) is such that $(A,g)\mathcal{D}(B,h)$, then $\text{Stab}(A)$ is conjugate to $\text{Stab}(B)$,
- moreover $| A | = | B |$, by arguments above the Lemma 2.4.13.

We relate elements (A,g) and (B,h) of $\mathcal{B}_k$ which have conjugate stabilizers, and we will denote this by $\text{Stab}(B) \sim_c \text{Stab}(A)$.

Let $H \leq G$ fixed. We denote

$$Cjg(H) := \{L \leqslant G; L \text{ is conjugate to } H\};$$

and

$$\mathcal{B}_{k,Cjg(H)} := \{(A,g) \in \mathcal{B}_k;\ \text{Stab}(A) \sim_c H\}.$$

The number of elements in this set is given by

$$| \mathcal{B}_{k,Cjg(H)} | = \sum_{\substack{L \leqslant G \\ L \sim_c H}} | \{(A,g) \in \mathcal{B}_k;\ \text{Stab}(A) = L\} | .$$

Noting that there are no conditions on $g \in G$ for the pairs $(A, g) \in B_{k,Cjg(H)}$, we have

$$\mid \mathcal{B}_{k,Cjg(H)} \mid = \sum_{\substack{L \leqslant G \\ L \sim_c H}} \mid \{(A, e) \in \mathcal{B}_k;\ \text{Stab}(A) = L\} \mid\mid G \mid .$$

Using the information in our second paragraph in this setting, for each $\mathcal{D}$-class D in $\mathcal{A}_k$ we have

$$\mid D \mid = \frac{\mid G \mid^2}{\mid L \mid} \implies \frac{1}{\mid D \mid} = \frac{\mid L \mid}{\mid G \mid^2}.$$

Remember that we defined $d_k^e(L) :=\mid \{(A, e) \in \mathcal{B}_k;\ \text{Stab}(A) = L\} \mid$, so

$$\begin{aligned}
\mid \mathcal{B}_{k,Cjg(H)}/\mathcal{D} \mid &= \frac{\mid L \mid}{\mid G \mid^2} \mid \mathcal{B}_{k,Cjg(H)} \mid \\
&= \sum_{\substack{L \leqslant G \\ L \sim_c H}} \frac{\mid L \mid}{\mid G \mid^2} \mid \{(A, e) \in \mathcal{B}_k;\ \text{Stab}(A) = L\} \mid\mid G \mid \\
&= \sum_{\substack{L \leqslant G \\ L \sim_c H}} \frac{\mid L \mid}{\mid G \mid} d_k^e(L) \\
&=\mid Cjg(H) \mid \frac{\mid H \mid}{\mid G \mid} d_k^e(H).
\end{aligned}$$

The last equality is due to Lemma 2.5.13, because $d_k^e(L) = d_k^e(H)$.

Next, we want to calculate the $\mathbb{K}$ algebra of each $D \in \mathcal{D}$ in the set $\mathcal{B}_{k,Cjg(H)}$. Notice from Lemma 2.5.8 that all of these algebras are isomorphic to $\mathbb{M}_{[G:H]}(\mathbb{K}H)$. The number we have just found above is the number of copies of $\mathbb{M}_{[G:H]}(\mathbb{K}H)$.

It's time to begin the second half of this proof. In this part we need a family of representatives of the conjugate classes of subgroups of G : let $\{H_i\}_{i \in I}$ be such a family, and let $Cjg(H_i)$ be the conjugate class of the subgroup H_i. Now, from Lemma 2.5.13 it follows that

$$\begin{aligned}
\mathbb{K}\Gamma_{GB} &\simeq \bigoplus_{\substack{k \\ i \in I \\ D \in \mathcal{B}_{k,Cjg(H_i)}/\mathcal{D}}} \mathbb{K}D \\
&\simeq \bigoplus_{\substack{k \\ i \in I}} \mid Cjg(H_i) \mid \frac{\mid H_i \mid}{\mid G \mid} d_k^e(H_i) \mathbb{M}_{[G:H]}(\mathbb{K}H_i) \\
&\simeq \bigoplus_{\substack{k \\ i \in I}} \bigoplus_{H \in Cjg(H_i)} \frac{\mid H \mid}{\mid G \mid} d_k^e(H) \mathbb{M}_{[G:H]}(\mathbb{K}H) \\
&\simeq \bigoplus_{\substack{k \\ H \leqslant G}} \frac{\mid H \mid}{\mid G \mid} d_k^e(H) \mathbb{M}_{[G:H]}(\mathbb{K}H).
\end{aligned}$$

Finally for $1 \leqslant k \leqslant\mid G \mid$

$$\mathbb{K}\Gamma_{GB} \simeq \bigoplus_{\substack{k \\ H \leqslant G}} \frac{d_k^e(H)}{[G:H]} \mathbb{M}_{[G:H]}(\mathbb{K}H).$$

Concluding the proof, the expression of $\tilde{d}_k^e(-)$ is an application of Lemma 2.4.13 with the commentary before Lemma 2.5.9. □

This formula can be written in terms of conjugates:

$$\mathbb{K}\Gamma_{GB} \simeq \bigoplus_{\substack{Cjg(H) \\ k}} \mid Cjg(H) \mid \frac{d_k^e(H)}{[G:H]} \mathbb{M}_{[G:H]}(\mathbb{K}H).$$

Remark 2.5.15. Notice that both algebras have similar formulas. One important fact is:

for S_{PB}, m depends on $\mid G \mid$ and $\mid H \mid$

for S_{GB}, there is a dependence only on $\mid G \mid$.

2.6 Examples

It is time to use our machinery. In this section, we will study the expansions of the symmetric group of three elements. We will count the $\mathcal{D}$-classes and the respective algebra for each case.

2.6.1 Symmetric group S_3 - Partial Bernoulli case

Let $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$ be the permutation group of three elements. Each element of $S_{PB}(S_3)$, the expansion of S_3, is a pair (A, g), where $A \subseteq G$ and $A \ni e, g$. For instance, $(\{e, (123)\}, (123)) \in S_{PB}(S_3)$ and $(\{e, (123)\}, (132)) \notin S_{PB}(S_3)$.

From Proposition 2.1.10 we know that

$$\mid S_{PB}(S_3) \mid = (6+1)2^{6-2} = 112.$$

First we list subgroups of S_3 for each $\mid A \mid = k$ with $1 \leqslant k \leqslant 6 = \mid G \mid$, because due to Remark 2.4.7 this will help us:

- $k = 2 \implies H_1 = \{e, (12)\},\ H_2 = \{e, (13)\},\ H_3 = \{e, (23)\}$ (which are conjugate)
- $k = 3 \implies H_4 = \{e, (123), (132)\}$
- $k = 6 \implies S_3$

Its lattice diagram of subgroups is:

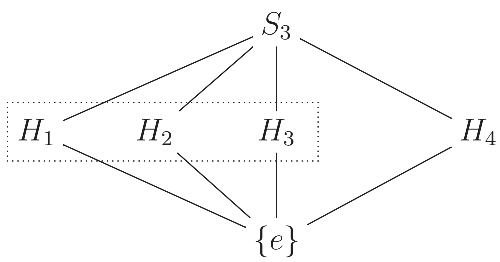


Figure 2.11: The subgroups of S_3

Next, we will compute its $\mathcal{D}$-classes. This example has already appeared in the work (Dissertation thesis) of Valverde [96] Example 3.20. His computations are based on Dokuchaev-Exel-Piccione [28], and we present Choi's approach.

We will use the formulas of Theorem 2.4.9, *i.e.*

(i) $\mid S_{PB}/\mathcal{D} \mid = \displaystyle\sum_{k=1}^{|G|} \mid \mathcal{A}_k/\mathcal{D} \mid$ and

(ii) $\mid \mathcal{A}_k/\mathcal{D} \mid = \displaystyle\sum_{m=1}^{k} \frac{m}{k^2} d_k(m) = \frac{1}{k^2} \sum_{H \leqslant G} \mid H \mid d_k(H)$ for each k.

Where

- $\mathcal{A}_k : \{(A, g) \in S_{PG}; \mid A \mid = k\}$,
- $d_k(H) := \mid \{(A, g) \in \mathcal{A}_k;\ \text{Stab}(A) = H\} \mid$,
- $d_k(m) = \displaystyle\sum_{\substack{H \leqslant G \\ |H| = m}} d_k(H)$

Möbius inversion formula (cf. Section 2.6), provide us

$$\tilde{d}_k(H) = \sum_{H \leqslant L} d_k(L) \Leftrightarrow d_k(H) = \sum_{H \leqslant L} \mu(H, L)\tilde{d}_k(L).$$

moreover, the numbers $\tilde{d}_k$ are computed by Lemma 2.4.10:

$$\tilde{d}_k(H) = \begin{cases} \left(\begin{array}{c} \dfrac{\mid G \mid}{\mid H \mid} - 1 \\ \dfrac{k}{\mid H \mid} - 1 \end{array} \right) k & , \mid H \mid \text{ divides } k \\ 0 & , \text{ otherwise} \end{cases},$$

for H a subgroup G and $1 \leqslant k \leqslant\mid G \mid$.

The next table presents the computation of $\begin{pmatrix} \dfrac{\mid G \mid}{\mid H \mid} - 1 \\ \dfrac{k}{\mid H \mid} - 1 \end{pmatrix} k$ for each k and subgroup of S_3:

$k \backslash H$	$\{e\}$	H_1	H_2	H_3	H_4	S_3
1	1	0	0	0	0	0
2	10	2	2	2	0	0
3	30	0	0	0	3	0
4	40	8	8	8	0	0
5	25	0	0	0	0	0
6	6	6	6	6	6	6

Table 2.2: The values of $\tilde{d}_k(-)$ for S_3

This table will make our computations a little bit easier.

Another information that we need is the Möbius function. This is done using the strategy of matrix computation, as we presented in Section 2.6. Via the lattice structure of S_3 we have ζ and, its inverse, μ:

$$\zeta = \begin{array}{c} \\ \{e\} \\ H_1 \\ H_2 \\ H_3 \\ H_4 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccccc} \{e\} & H_1 & H_2 & H_3 & H_4 & S_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \quad \mu = \begin{array}{c} \\ \{e\} \\ H_1 \\ H_2 \\ H_3 \\ H_4 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccccc} \{e\} & H_1 & H_2 & H_3 & H_4 & S_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now compute each $d_k(-)$ for $\{e\}, H_1, H_2, H_3$ and H_4.

For $\{e\}$ we have: $d_k(\{e\}) = \sum_{\{e\} \leqslant L} \mu(\{e\}, L)\tilde{d}_k(L)$, with $k = 1, \ldots, 6$ and $L \leqslant G$. Notice that $L \in \{\{e\}, H_1, H_2, H_3, H_4, S_3\}$.

When:

- $\underline{k=1}$: If $(A,g) \in \mathcal{A}_1$, then $\mid A \mid = 1$ and $\{e\}$ is the only subgroup of A. So

$$d_1(\{e\}) = \mu(\{e\},\{e\})\tilde{d_1}(\{e\}) = 1 \cdot 1 = 1.$$

- $\underline{k=2}$: If $(A,g) \in \mathcal{A}_2$, then $\mid A \mid = 2$ and $\{e\}, H_1, H_2$ and H_3 are the subgroups of A. As H_1, H_2 and H_3 are conjugates, they have the same values. This way:

$$d_2(\{e\}) = \mu(\{e\},\{e\})\tilde{d_2}(\{e\}) + 3(\mu(\{e\},H_1)\tilde{d_2}(H_1)) = (1 \cdot 10) + 3(-1 \cdot 2) = 4.$$

- $\underline{k=3}$: If $(A,g) \in \mathcal{A}_3$, then $\mid A \mid = 3$ and $\{e\}$ and H_4 are the subgroups of A. So

$$d_3(\{e\}) = \mu(\{e\},\{e\})\tilde{d_3}(\{e\}) + \mu(\{e\},H_4)\tilde{d_3}(H_4) = (1 \cdot 30) + (-1 \cdot 3) = 27.$$

- $\underline{k=4}$: If $(A,g) \in \mathcal{A}_4$, then $\mid A \mid = 4$ and $\{e\}, H_1, H_2$ and H_3 are the subgroups of A. Again, as they are conjugate:

$$d_4(\{e\}) = \mu(\{e\},\{e\})\tilde{d_4}(\{e\}) + 3(\mu(\{e\},H_1)\tilde{d_4}(H_1)) = (1 \cdot 40) + 3(-1 \cdot 8) = 16.$$

- $\underline{k=5}$: The table has only one not zero entry, the first. So:

$$d_5(\{e\}) = \mu(\{e\},\{e\})\tilde{d_5}(\{e\}) = 1 \cdot 25.$$

- $\underline{k=6}$: The table presents values for all entries, so:

$$\tilde{d_6}(\{e\}) = (1 \cdot 6) + 3(-1 \cdot 6) + (-1 \cdot 6) + 3 \cdot 6 = 0.$$

For H_1 we have: $d_k(H_1) = \sum_{H_1 \leqslant L} \mu(H_1, L)\tilde{d_k}(L)$, with $k = 1, \ldots, 6$ and $L \leqslant G$. Notice that $L \in \{H_1, S_3\}$. Moreover, the information from the table and the μ function shows us that only for $k = 2, 4, 6$ and $\mu(H_1, H_1)$, $d_k(-)$ will not vanish. So:

$$\begin{aligned}
\underline{k=2} &\implies d_2(H_1) = \mu(H_1,H_1)\tilde{d_2}(H_1) + \mu(H_1,S_3)\tilde{d_2}(S_3) = 1 \cdot 2 + 0 = 2,\\
\underline{k=4} &\implies d_4(H_1) = \mu(H_1,H_1)\tilde{d_4}(H_1) + \mu(H_1,S_3)\tilde{d_4}(S_3) = 1 \cdot 8 + 0 = 8,\\
\underline{k=6} &\implies d_6(H_1) = \mu(H_1,H_1)\tilde{d_6}(H_1) + \mu(H_1,S_3)\tilde{d_6}(S_3) = 1 \cdot 6 + (-1 \cdot 6) = 0.
\end{aligned}$$

The computations of H_2 and H_3 are the same because the subgroups are conjugate. In light of this fact, we will move to H_4.

Similar arguments (the table and the matrix) provides us, for H_4:

$$\underline{k=3} \implies d_3(H_4) = \mu(H_4,H_4)\tilde{d}_3(H_4) + \mu(H_4,S_3)\tilde{d}_3(S_3) = 1\cdot 3 + 0 = 3,$$
$$\underline{k=6} \implies d_6(H_4) = \mu(H_4,H_4)\tilde{d}_6(H_4) + \mu(H_4,S_3)\tilde{d}_6(S_3) = 1\cdot 6 + (-1\cdot 6) = 0.$$

Finally, S_3 is the easiest one because there is only one case. *I.e.*

$$\underline{k=6} \implies d_6(S_3) = \mu(S_3,S_3)\tilde{d}_6(S_3) = 1\cdot 6 = 6.$$

All values of $d_k(-)$ are displayed below:

k \ H	$\{e\}$	H_1	H_2	H_3	H_4	S_3
1	1	0	0	0	0	0
2	4	2	2	2	0	0
3	27	0	0	0	3	0
4	16	8	8	8	0	0
5	25	0	0	0	0	0
6	0	0	0	0	0	6

Table 2.3: The values of $d_k(-)$ for S_3

Finally, using the formula.

$$\mid \mathcal{A}_k/\mathcal{D} \mid = \frac{1}{k^2}\sum_{H\leqslant G} \mid H \mid d_k(H),$$

we have the number of $\mathcal{D}$-classes for each k:

k	1	2	3	4	5	6
$\mid \mathcal{A}_k/\mathcal{D} \mid$	1	4	4	4	1	1

Table 2.4: The number of $\mathcal{D}$-classes

Conclusion:

$$\mid S_{PB}(S_3)/\mathcal{D} \mid = \sum_{k=1}^{|G|} \mid \mathcal{A}_k/\mathcal{D} \mid = 15.$$

Each $\mathcal{D}$ represents a connected component of the restriction groupoid. This way, we can picture them as a graph. The vertices are elements of the form (A, e) and edges are (B, h) such that $\mid A \mid = \mid B \mid = k$ and, by multiplication, (B, h) connects one edge to other. Following the drawings of Valverde's work ([96] Example 3.20) we compiled the next table:

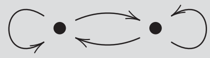
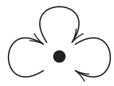
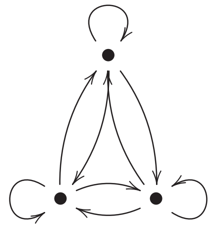
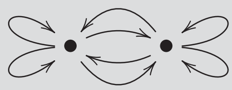
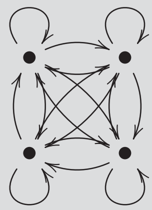
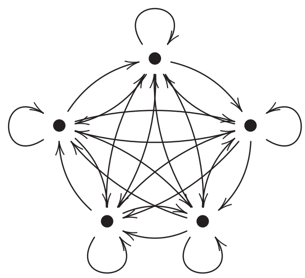
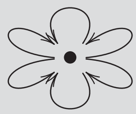
k-th level	Connected components of $\mathcal{G}_{S_{PB}(S_3)}$
1	
2	$3\times($  $)$ and 
3	 and $3\times($  $)$
4	$3\times($  $)$ and 
5	
6	

Table 2.5: Each connected component of $\mathcal{G}_{S_{PB}(S_3)}$

Continuing we will calculate the partial algebra of $S_{PB}(S_3)$. For this purpose, we will use the Theorem 2.5.11,*i.e.* :

- $K_{par}(G) \simeq \bigoplus\limits_{\substack{H\leqslant G\\ 1\leqslant m\leqslant [G:H]}} \dfrac{d^e_{m|H|}(H)}{m}\mathbb{M}_m(\mathbb{K}H)$ or,

- $K_{par}(G) \simeq \bigoplus\limits_{\substack{Cjg(H)\\ 1\leqslant m\leqslant [G:H]}} \mid Cjg(H) \mid \dfrac{d^e_{m|H|}(H)}{m}\,\mathbb{M}_m(\mathbb{K}H),$

- where $d^e_k(H) :=\mid \{(A,e)\in\mathcal{A}_k;\ \mathrm{Stab}(A)=H\}\mid,$

- and recursively $d^e_{m|H|}(H) = \sum\limits_{H\leqslant L}\mu(H,L)\tilde{d^e}_{m|H|}(L)$, and

- $\tilde{d^e}_{m|H|}(L) = \begin{cases} \begin{pmatrix} \frac{|G|}{|L|} - 1 \\ \frac{m|H|}{|L|} - 1 \end{pmatrix} & , |L| \text{ divides } m|H| \\ 0 & , \text{ otherwise} \end{cases}.$

As we did a few paragraphs above, we will present the values of each $d^e{}_k(-)$ for $\{e\}, H_1, H_2, H_3$ and H_4. For this task we need the following information.

First a table with the computation of $\begin{pmatrix} \frac{|G|}{|L|} - 1 \\ \frac{k}{|L|} - 1 \end{pmatrix}$:

k \ L	$\{e\}$	H_1	H_2	H_3	H_4	S_3
1	1	0	0	0	0	0
2	5	1	1	1	0	0
3	10	0	0	0	1	0
4	10	2	2	2	0	0
5	5	0	0	0	0	0
6	1	1	1	1	1	1

Table 2.6: The values of the binomial term

Notice in this table we compile the values of

$$k = m\cdot |H| \text{ for } |H| = 1.$$

Please pay attention: <u>we must multiply by the respective $|H|$, and then pick the right row.</u>

The Möbius function is the same. We are going to write it again, just to make the verification easier:

$$\mu = \begin{array}{c} \\ \{e\} \\ H_1 \\ H_2 \\ H_3 \\ H_4 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccccc} \{e\} & H_1 & H_2 & H_3 & H_4 & S_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

For $\{e\}$ we have: $d^e_{m\cdot 1}(\{e\}) = \sum_{\{e\}\leqslant L} \mu(\{e\},L)\tilde{d}^e_m(L)$, with $m = 1,\dots,6$ and $L \in \{\{e\},H_1,H_2,H_3,H_4,S_3\}$. As H_1, H_2 and H_3 are conjugate, we can skip some computation, multiplying by 3. Moreover $\mid \{e\} \mid = 1$, this way we use the table with $m = k$.

So, the non zero values are:

$$\underline{m=1} \implies d^e_1(\{e\}) = \mu(\{e\},\{e\})\tilde{d}^e_1(\{e\}) = 1\cdot 1 = 1$$
$$\underline{m=2} \implies d^e_2(\{e\}) = \mu(\{e\},\{e\})\tilde{d}^e_2(\{e\}) + 3(\mu(\{e\},H_1)\tilde{d}^e_2(H_1)) = (1\cdot 5) + 3(-1\cdot 1) = 2$$
$$\underline{m=3} \implies d^e_3(\{e\}) = \mu(\{e\},\{e\})\tilde{d}^e_3(\{e\}) + \mu(\{e\},H_4)\tilde{d}^e_3(H_4) = (1\cdot 10) + (-1\cdot 1) = 9$$
$$\underline{m=4} \implies d^e_4(\{e\}) = \mu(\{e\},\{e\})\tilde{d}^e_4(\{e\}) + 3(\mu(\{e\},H_1)\tilde{d}^e_4(H_1)) = (1\cdot 10) + 3(-1\cdot 2) = 4$$
$$\underline{m=5} \implies d^e_5(\{e\}) = \mu(\{e\},\{e\})\tilde{d}^e_5(\{e\}) = 1\cdot 5 = 5$$
$$\underline{m=6} \implies d^e_6(\{e\}) = (1\cdot 1) + 3(-1\cdot 1) + (-1\cdot 1) + (3\cdot 1) = 0.$$

For H_1 the values of m are: $m = 1,2,3$. Because $[G : H_1] = 3$. Notice $\mid H_1 \mid = 2$, so we will need the rows correspondent to: $k = 2,4,6$. And $d^e_{m\cdot 2}(H_1) = \sum_{H_1\leqslant L} \mu(H_1,L)\tilde{d}^e_{m\cdot 2}(L)$, with $L = H_1,H_2,H_3,S_3$.

Thus, the numbers that not vanish are:

$$\underline{m=1} \implies d^e_2(H_1) = \mu(H_1,H_1)\tilde{d}^e_2(H_1) = 1\cdot 1 = 1$$
$$\underline{m=2} \implies d^e_4(H_1) = \mu(H_1,H_1)\tilde{d}^e_4(H_1) = 1\cdot 2 = 2$$
$$\underline{m=3} \implies d^e_6(H_1) = \mu(H_1,H_1)\tilde{d}^e_6(H_1) + \mu(H_1,S_3)\tilde{d}^e_6(S_3) = (1\cdot 1) + (-1\cdot 1) = 0.$$

For H_4, $\mid H_4 \mid = 3$ and $[G : H_4] = 2$. Which implies $m = 1,2$. Also $L = H_4, S_3$. The needed rows are: $k = 3$ and $k = 6$.

Hence, consulting the table and μ:

$$\underline{m=1} \implies d^e_3(H_4) = \mu(H_4,H_4)\tilde{d}^e_3(H_4) = 1\cdot 1 = 1$$
$$\underline{m=3} \implies d^e_6(H_4) = \mu(H_4,H_4)\tilde{d}^e_6(H_4) + \mu(H_4,S_3)\tilde{d}^e_6(S_3) = (1\cdot 1) + (-1\cdot 1) = 0.$$

Finally, for S_3 we have: $\mid S_3 \mid = 6$ and $m = 1$. So we must use the row $k = 6$. The μ has only one not zero value. Therefore:

$$\underline{m=1} \implies d^e_6(S_3) = \mu(S_3,S_3)\tilde{d}^e_6(S_3) = 1\cdot 1 = 1.$$

All these numbers are summarized in the next table; also we add the number of conjugate classes of each subgroup:

$m \mid H \mid$ \ H	$\{e\}$	H_1	H_2	H_3	H_4	S_3
1	1	0	0	0	0	0
2	2	1	1	1	0	0
3	9	0	0	0	1	0
4	4	2	2	2	0	0
5	5	0	0	0	0	0
6	0	0	0	0	0	1
$\mid Cjg(H) \mid$	1	3	3	3	1	1

Table 2.7: Auxiliar values to calculate the algebra

Notice we are going to use the formula with conjugates; only H_1 will be necessary.

In next computation, we use the identification: $H_1 \simeq \mathbb{Z}_2$ and $H_3 \simeq \mathbb{Z}_3$. All these information together, provides us: (each line by subgroup and each parenthesis one level)

$$
\begin{aligned}
K_{par}(S_3) \simeq & \bigoplus_{\substack{Cjg(H) \\ 1 \leqslant m \leqslant [G:H]}} \mid Cjg(H) \mid \frac{d^e_{m|H|}(H)}{m} \mathbb{M}_m(\mathbb{K}H) \\
& = (\mathbb{K}) \oplus (\frac{2}{2}\mathbb{M}_2(\mathbb{K})) \oplus (\frac{9}{3}\mathbb{M}_3(\mathbb{K})) \oplus (\frac{4}{4}\mathbb{M}_4(\mathbb{K})) \oplus (\frac{5}{5}\mathbb{M}_5(\mathbb{K})) \oplus 0 \\
& \oplus (\frac{3}{1}\mathbb{K}\mathbb{Z}_2) \oplus (\frac{3 \cdot 2}{2}\mathbb{M}_2(\mathbb{K}\mathbb{Z}_2)) \oplus 0 \\
& \oplus (\mathbb{K}\mathbb{Z}_3) \\
& \oplus (\mathbb{K}S_3).
\end{aligned}
$$

Concluding our example

$$K_{par}(S_3) \simeq \mathbb{K} \oplus 3\mathbb{K}\mathbb{Z}_2 \oplus \mathbb{K}\mathbb{Z}_3 \oplus \mathbb{M}_2(\mathbb{K}) \oplus 3\mathbb{M}_3(\mathbb{K}) \oplus 3\mathbb{M}_2(\mathbb{K}\mathbb{Z}_2) \oplus \mathbb{M}_4(\mathbb{K}) \oplus \mathbb{M}_5(\mathbb{K}) \oplus \mathbb{K}S_3.$$

2.6.2 Symmetric group S_3 - Global Bernoulli case

This section will follow the same guidelines as the previous example. This time we are going to compute the $\mathcal{D}$-classes and the global algebra of $S_{GB}(S_3)$.

As we have already seen how to manipulate the formulas, this example will be more straightforward.

The formulas needed to compute the $\mathcal{D}$-classes are due to Theorem 2.4.14:

(i) $\mid S_{GB}/\mathcal{D} \mid = \sum_{k=1}^{|G|} \mid \mathcal{B}_k/\mathcal{D} \mid$ and

(ii) $\mid \mathcal{B}_k/\mathcal{D} \mid = \sum_{m=1}^{k} \frac{m}{\mid G \mid^2} d_k(m) = \frac{1}{\mid G \mid^2} \sum_{H \leqslant G} \mid H \mid d_k(H)$ for each k.

Where:

- $\mathcal{B}_k = \{(A, g) \in S_{GB}; \mid A \mid = k\}$,
- $d_k(H) := \mid \{(A, g) \in \mathcal{B}_k;\ \text{Stab}(A) = H\} \mid$,
- $d_k(m) = \sum_{\substack{H \leqslant G \\ |H| = m}} d_k(H)$.

Aiming to use Möbius inversion formula

$$\tilde{d}_k(H) = \sum_{H \leqslant L} d_k(L) \Leftrightarrow d_k(H) = \sum_{H \leqslant L} \mu(H, L) \tilde{d}_k(L),$$

we need the numbers $\tilde{d}_k$ from computed by the Lemma 2.4.13:

$$\tilde{d}_k(H) = \begin{cases} \begin{pmatrix} \frac{\mid G \mid}{\mid H \mid} \\ \frac{k}{\mid H \mid} \end{pmatrix} \mid G \mid & , \mid H \mid \text{ divides } k \\ 0 & , \text{ otherwise} \end{cases},$$

for H a subgroup G and $1 \leqslant k \leqslant \mid G \mid$.

The reader may have noticed that computations will be very similar. We are going to omit the details.

The table below is formed by $\begin{pmatrix} \frac{\mid G \mid}{\mid H \mid} \\ \frac{k}{\mid H \mid} \end{pmatrix} \mid G \mid$ for each k and subgroup of S_3:

k \ H	$\{e\}$	H_1	H_2	H_3	H_4	S_3
1	36	0	0	0	0	0
2	90	18	18	18	0	0
3	120	0	0	0	12	0
4	90	18	18	18	0	0
5	36	0	0	0	0	0
6	6	6	6	6	6	6

Table 2.8: The values of $\tilde{d}_k(-)$ for S_3

The Möbius function is the same as before:

$$\mu = \begin{array}{c} \\ \{e\} \\ H_1 \\ H_2 \\ H_3 \\ H_4 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccccc} \{e\} & H_1 & H_2 & H_3 & H_4 & S_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Then all values of $d_k(-)$ are:

k \ H	$\{e\}$	H_1	H_2	H_3	H_4	S_3
1	36	0	0	0	0	0
2	36	18	18	18	0	0
3	108	0	0	0	12	0
4	36	18	18	18	0	0
5	36	0	0	0	0	0
6	0	0	0	0	0	6

Table 2.9: The values of $d_k(-)$ for S_3

Combining such numbers and the formula:

$$\mid \mathcal{B}_k/\mathcal{D} \mid = \frac{1}{\mid G \mid^2} \sum_{H \leqslant G} \mid H \mid d_k(H),$$

we have the number of $\mathcal{D}$-classes for each k:

k	1	2	3	4	5	6
$\mid \mathcal{B}_k/\mathcal{D} \mid$	1	4	4	4	1	1

Table 2.10: The number of $\mathcal{D}$-classes

Conclusion:

$$\mid S_{GB}(S_3)/\mathcal{D} \mid = \sum_{k=1}^{|G|} \mid \mathcal{B}_k/\mathcal{D} \mid = 15.$$

The connected components of the restriction groupoid are:

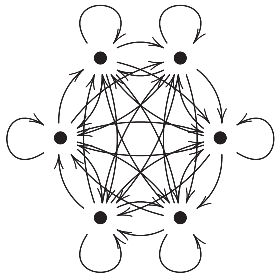
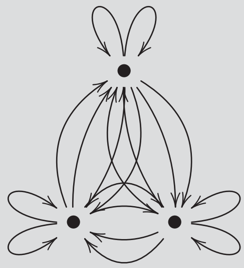
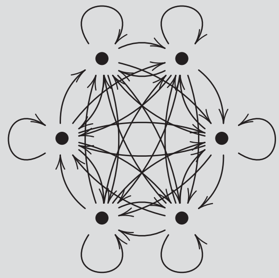
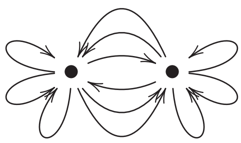
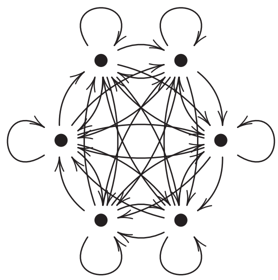
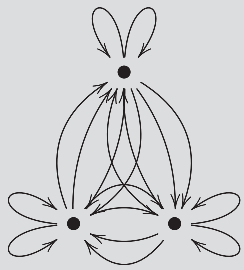
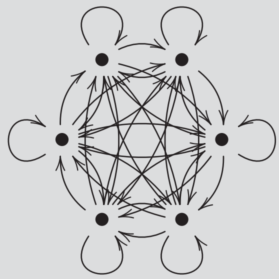
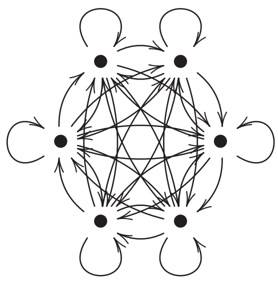
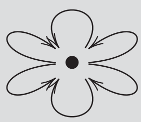
k-th level	Connected components of $\mathcal{G}_{S_{GB}(S_3)}$
1	
2	$3 \times ($  $)$ and 
3	 and $3 \times ($  $)$
4	$3 \times ($  $)$ and 
5	
6	

Table 2.11: Each connected component of $\mathcal{G}_{S_{GB}(S_3)}$

Now the global algebra of S_3. This requires the Theorem 2.5.14:

- $K_{glob}(G) = \bigoplus_{\substack{k \\ H\leqslant G}} \frac{d_k^e(H)}{[G:H]} \mathbb{M}_{[G:H]}(\mathbb{K}H)$, or
- $K_{glob}(G) \simeq \bigoplus_{\substack{Cjg(H) \\ k}} \mid Cjg(H) \mid \frac{d_k^e(H)}{[G:H]} \mathbb{M}_{[G:H]}(\mathbb{K}H)$
- with $d_k^e(H) := \mid \{(A,e) \in \mathcal{B}_k;\ \mathrm{Stab}(A) = H\} \mid$
- where $d_k^e(H) = \sum_{H \leqslant L} \mu(H,L)\tilde{d^e}_k(L)$,
- and $\tilde{d^e}_k(L) = \begin{cases} \begin{pmatrix} \frac{\mid G \mid}{\mid L \mid} \\ \frac{k}{\mid L \mid} \end{pmatrix} & , \mid L \mid \text{ divides } k \\ 0 & , \text{ otherwise} \end{cases}$.

As the table above already has the values of the binomial, the $d_k^e(-)$ in this case are:

$k \backslash H$	$\{e\}$	H_1	H_2	H_3	H_4	S_3
1	6	0	0	0	0	0
2	6	3	3	3	0	0
3	18	0	0	0	2	0
4	6	3	3	3	0	0
5	6	0	0	0	0	0
6	0	0	0	0	0	1
$\mid Cjg(H) \mid$	1	3	3	3	1	1

Table 2.12: The values of $d^e{}_k(-)$ for S_3

The global algebra is:

$$\mathbb{K}_{glob}(S_3) \simeq 7\mathbb{M}_6(\mathbb{K}) \oplus 6\mathbb{M}_3(\mathbb{K}\mathbb{Z}_2) \oplus \mathbb{M}_2(\mathbb{K}\mathbb{Z}_3) \oplus \mathbb{K}S_3.$$

Remark 2.6.1. The formula of the global algebra of the group S_3 was already computed by Abadie in his thesis ([1], page 126). In fact he provided general formulas for the enveloping algebras, which are associated to partial actions.

2.6.3 Comparative of $SPB(S_3)$ and $SGB(S_3)$

We have obtained many numbers, components, and pictures. The purpose of this section is to put such information side by side. Thus we can adequately compare the partial and the global algebra.

$\mathcal{D}$-classes:

k	1	2	3	4	5	6
$\mathcal{A}_k/\mathcal{D}\mid$	1	4	4	4	1	1

k	1	2	3	4	5	6
$\mathcal{B}_k/\mathcal{D}\mid$	1	4	4	4	1	1

Table 2.13: The number of $\mathcal{D}$-classes

Algebras:

$$\mathbb{K}_{par}(S_3) \simeq \mathbb{K} \oplus 3\mathbb{K}\mathbb{Z}_2 \oplus \mathbb{K}\mathbb{Z}_3 \oplus \mathbb{M}_2(\mathbb{K}) \oplus 3\mathbb{M}_3(\mathbb{K}) \oplus 3\mathbb{M}_2(\mathbb{K}\mathbb{Z}_2) \oplus \mathbb{M}_4(\mathbb{K}) \oplus \mathbb{M}_5(\mathbb{K}) \oplus \mathbb{K}S_3$$
$$\mathbb{K}_{glob}(S_3) \simeq \mathbb{M}_2(\mathbb{K}\mathbb{Z}_3) \oplus 6\mathbb{M}_3(\mathbb{K}\mathbb{Z}_2) \oplus 7\mathbb{M}_6(\mathbb{K}) \oplus \mathbb{K}S_3$$

Connected components: we can identify the components of $\mathcal{G}_{S_{PB}(S_3)}$, in red, inside the components of $\mathcal{G}_{S_{GB}(S_3)}$ as follows

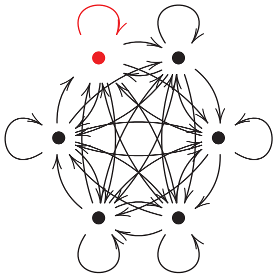
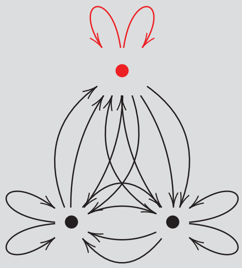
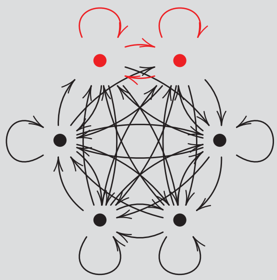
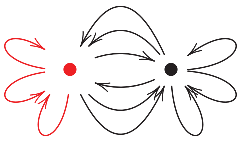
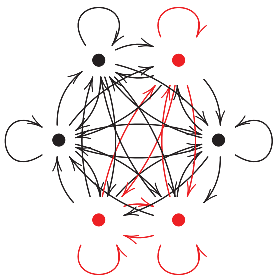
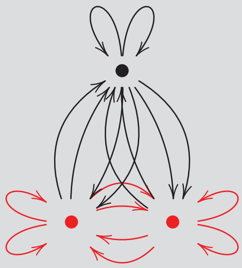
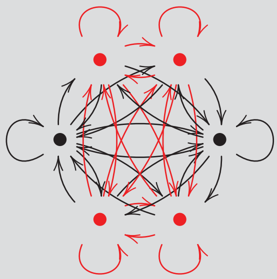
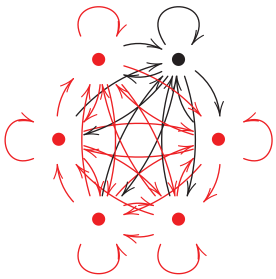
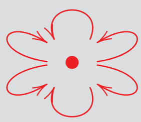
k-th level	Comparative of connected components
1	
2	$3 \times ($  $)$ and 
3	 and $3 \times ($  $)$
4	$3 \times ($  $)$ and 
5	
6	

Table 2.14: Comparative of connected components

Chapter 3

The prefix expansion of an inverse semigroup

Before applying our approach and defining the Bernoulli action of an inverse semigroup, we will present the aspects of Buss-Exel inverse semigroup prefix expansion ([14]). Also, we relate the work of Lawson-Margolis-Steinberg ([54]).

Throughout this chapter, unless we say the otherwise G **will be a group and** S **will be an inverse semigroup**. It is because many times, we make relations and comparisons to motivate the reader.

3.1 Expanding an inverse semigroup in Buss-Exel way

The first expansion in terms of relations and generators we made was taking elements of a group as generators of a free semigroup and then defining relations on them. To refresh the memory of the reader, we will write this definition again.

Let G be a group we defined (cf. Definition 2.1.1): $S(G)$ the universal semigroup of G defined through generators and relations. A set of generators for $S(G)$ is $\{[g]; g \in G\}$ and for each $g, h \in G$ we consider the relations:

(I) $[g][h][h^{-1}] = [gh][h^{-1}]$;

(II) $[g^{-1}][g][h] = [g^{-1}][gh]$;

(III) $[g][e] = [g] = [e][g]$.

What if we change G for an inverse semigroup? The first thing we must notice is that G has a particularity: $\mathcal{E}(G) = \{e\}$. This difference is the main idea we have to keep in mind. Pursuing a similar definition and construction for an inverse semigroup, for example, S, passes through accommodate the large quantity (or at least more than one) of idempotents of S.

A first attempt would be to replace e with elements s^*s and adapt the other relations with the neutral element. Roughly speaking, what is happening is we are chopping S by its idempotents and thus realizing the same expansion. The new computations reinforce the abundance of idempotent elements of an inverse semigroup.

We present the main results concerning this construction after Buss-Exel [14].

Definition 3.1.1 ([14])**.** Let S be an inverse semigroup. We define the semigroup $Pref(S)$ via generator $[s]$, for $s \in S$, and relations: $s, t \in S$

(I) $[s][t][t^*] = [st][t^*]$

(II) $[s^*][s][t] = [s^*][st]$

(III) $[s][s^*][s] = [s]$

The first structure we realize in $Pref(S)$ is being a semigroup. Our next efforts will describe its idempotents and a representation form of an element like we did before in Chapter 2.

Proposition 3.1.2 ([14]). Let $s \in S$ and define $\varepsilon_s = [s][s^*]$. Then the following statements are valid: for $t \in S$

(i) $\varepsilon_s^2 = \varepsilon_s$

(ii) $[t]\varepsilon_s = \varepsilon_{ts}[t]$

(iii) $\varepsilon_s \varepsilon_t = \varepsilon_t \varepsilon_s$.

These identities are very similar to those satisfied by idempotents of $S(G)$. At that time, our only choice of idempotents in G shadowed some technical details that have become evident now. In short terms, for every idempotent of S, the same pattern will appear. It‘s clear that when $S = G$, a group, we recover $Pref(S) = S(G)$.

Now a set of new relations.

Proposition 3.1.3 ([14]). Suppose $e \in \mathcal{E}(S)$ and $s \in S$:

(i) $\varepsilon_e = [e]$

(ii) $[e][s] = [es]$ and $[s][e] = [se]$

(iii) $\varepsilon_e \varepsilon_s = \varepsilon_{es}$.

The above relations were not evident in the group case because we do not write the neutral element in multiplications. Nevertheless, now they are crucial.

Moving toward the analogous of normal forms, we have an interesting way the idempotents relate with themselves.

Lemma 3.1.4 ([14]). *Let* $\{s_1, s_2, \dots, s_n, t\} \subset S$ *and let* $\alpha = \varepsilon_{s_1} \dots \varepsilon_{s_n}[t]$. *If*

- $p = s_1 s_1^* s_2 s_2^* \dots s_n s_n^* t t^* = \Pi_{i=1}^n (s_i s_i^*)(tt^*)$
- $s_i' = p s_i$ *for all* $i = 1, \dots, n$
- $t' = pt$,

then $\alpha = \varepsilon_{s'_1} \dots \varepsilon_{s'_n}[t']$.

This Lemma implies a way to write elements of $Pref(S)$. First, we adjust the definition to make it clear.

Definition 3.1.5 ([14])**.** Let $e \in \mathcal{E}(S)$ and let $A \subset S$, then A is an *e-set* if

(I) $e \in A$

(II) $ss^* = e$ for all $s \in A$.

In notations: A^e will stand for a set A which is an e-set.

Suppose $\alpha = \varepsilon_{s_1} \dots \varepsilon_{s_n}[t]$, with $s_i, t \in S$, for $i = 1, \dots n$. Letting $A = \{s_1, s_2, \dots, s_n, \}$, we will shortly write $\varepsilon_A = \varepsilon_{s_1} \dots \varepsilon_{s_n}$. In this manner we can write $\alpha = \varepsilon_A[t]$.

Proposition 3.1.6 ([14])**.** Every $\alpha \in Pref(S)$ admits a description $\alpha = \varepsilon_A[t]$ where $t \in S$, A is a finite tt^*-set and $A \ni t$.

Every element written in the form above is said to be written in *normal form*. With a little more work, it is possible to conclude that the normal form is unique ([15] Theorem 5.12).

Theorem 3.1.7. *([14]) Let $\alpha = \varepsilon_A[s], \beta = \varepsilon_B[t] \in Pr(S)$ in normal form. If $\alpha = \beta$, then $s = t$ and $A = B$.*

We have crossed a milestone. Next, we discuss the multiplication in $Pref(S)$.

Proposition 3.1.8 ([14])**.** Let $\alpha = \varepsilon_A[t], \beta = \varepsilon_B[s] \in Pr(S)$ in normal form. Then

$$\alpha\beta = \varepsilon_{A \cup (tB)}[ts] = \varepsilon_{(tss^*t^*A) \cup (tB)}[ts].$$

Finally we can prove $Pref(S)$ is regular. It follows from: for each $\alpha = \varepsilon_A[t] \in Pref(S)$ in normal form, the element $\overline{\alpha} = [t^*]\varepsilon_A$ is such that $\alpha\overline{\alpha}\alpha = \alpha$ and $\overline{\alpha}\alpha\overline{\alpha} = \overline{\alpha}$.

Proposition 3.1.9 ([14])**.** The semigroup $Pref(S)$ is an inverse semigroup.

One last proposition, which answers a possible question about E-unitarity of $Pref(S)$. Remembering Definition 1.2.9, an inverse semigroup S is E-unitary if

$$es \in \mathcal{E}(S) \implies s \in \mathcal{E}(S),$$

for $e \in \mathcal{E}(S)$ and $s \in S$.

Proposition 3.1.10 ([14])**.** The inverse semigroup $Pref(S)$ is E-unitary if, and only if, S is E-unitary.

Remark 3.1.11. Notice that all of this construction for $Pref(S)$ reduces to the same relations that defined $S(G)$.

We will end this section with the definition of partial and global inverse semigroup action presented, respectively in Buss-Exel [14] (Section 3) and Cordeiro-Beuter [24] (Section 2).

Definition 3.1.12 ([14])**.** Let S be an inverse semigroup and X be a set, we will say that $\theta = (\{D_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ is a *partial action* of inverse semigroups, where $D_s \subset X$ and $\theta_s \in \mathcal{I}(X)$, if:

(I) for each $s \in S$ the map $\theta_s : D_{s^*} \to D_s$ is a map between subsets of X;

(II) the map $s \in S \mapsto \theta_s$ satisfies:

(i) $\theta_{s^*} = {\theta_s}^{-1}$;

(ii) $s \leqslant t \implies \theta_s \leqslant \theta_t$;

(iii) $\theta_s\theta_t \leqslant \theta_{st}$.

(In this case, we will call this map a *partial homomorphism*);

(III) $X = \displaystyle\bigcup_{e\in\mathcal{E}(S)} D_e$ (we say the action is *non degenerate*).

If the map in (II) is a homomorphism of inverse semigroups – *i.e.* item (iii) has an equal sign –, we say θ is a *global action*. Equivalently, if and only if: $D_{s^*} = D_{s^*s}$.

Being consistent with our notations, we will denote the global action by $S \curvearrowright X$ and the partial action by $S \curvearrowright_p X$.

Similarly to the partial group actions, we have equivalent definitions. We will state these results and recommend Buss-Exel [14] Propositions 3.4 and 3.8.

Proposition 3.1.13 ([14])**.** Let X be a set and let $\theta : S \to \mathcal{I}(X)$ be a map. For each $s \in D$, let X_s be the range of θ_s. Then θ is partial action of inverse semigroups if, and only if: for $s, t \in S$

(i) $\theta_{s^*} = {\theta_s}^{-1}$

(ii) $\theta_s(D_{s^*} \cap D_t) = D_s \cap D_{st}$;

(iii) $\theta_s(\theta_t(x)) = \theta_{st}(x)$ for $x \in D_{t^*} \cap D_{t^*s^*}$.

Finally, making some simplifications, we have the last equivalency.

Proposition 3.1.14 ([14])**.** Let X be a set and let $\theta : S \to \mathcal{I}(X)$ be a map. For each $s \in D$, let X_s be the range of θ_s. Then θ is a partial action of inverse semigroups if, and only if: for $s, t \in S$

(i) $\theta_{s^*} = {\theta_s}^{-1}$

(ii) $\theta_s(D_{s^*} \cap D_t) \subseteq D_s \cap D_{st}$;

(iii) $s \leqslant t \implies D_s \subseteq D_t$

(iv) $\theta_s(\theta_t(x)) = \theta_{st}(x)$ for $x \in D_{t^*} \cap D_{t^*s^*}$.

3.2 Bernoulli semigroup actions

The historical development that culminates in expanding an inverse semigroup, as we did before, was not the one presented. Expansions of semigroups were introduced in the work of Lawson-Margolis-Steinberg [54]. They used the theory of semigroups and O'Carroll, or McAlister-O'Carroll triples to generalize Birget-Rhodes (cf. Szendrei [91] this construction) expansions[1]. This approach is closer to what we did in Section 3.2.

The literature presents, at least, two versions of O'Carroll's construction: there is the original triple, and Lawson uses an (idempotent pure alternate) approach in his book in 8.4. We will use the original definition due to O'Carroll ([67]) and complement with Khrypchenko [48] because he relates partial-global action with the construction of O'Carroll.

Inspired by the paper of Lawson-Margolis-Steinberg, we will present a version of Bernoulli's action for inverse semigroups. We will take an alternative route to past chapters: first, we discuss the sets needed to develop our constructions, then we will present the O'Carroll triples associated, and from such inverse semigroups, we define the actions.

The first question we need to answer is: how to define the power set of an inverse semigroup? A natural attempt would be to pack all finite subsets of S. However, this will not work for our purposes. The idea hidden in group theory, again, is the single idempotent.

The guidelines to provide a way of writing $Pref(S)$ in a similar semidirect product fashion to the Lemma 2.2.5 are:

1st) A version of $P(G)$ and $P_e(G)$ for S,

2nd) A generalization of semidirect product for inverse semigroups and the corresponding McAlister setting,

3rd) The Bernoulli actions.

[1]The term prefix expansion we use and used by Lawson-Margolis-Steinberg is not the same term by Szendrei.

Let us put this plan into action.

Earlier, for groups, we defined: $P_e(G) = \{A \subseteq G; \mid A \mid < \infty, A \ni e\}$ and $P(G) = \{B \subseteq G; \mid B \mid < \infty\}$. This is the model for us. What is happening here is:

- all elements of $P_e(G)$ are finite $\implies$ we need finite subsets of S
- if $g \in A$, with $A \in P_e(G)$, then $gg^{-1} = e$, by definition of groups $\implies$ we need e-sets (cf. the Definition 3.1.5) and $\mathcal{R}$ relations (from the Definition 2.4.1).

So we must slice up the set of finite subsets of an inverse semigroup S using its idempotents. This way, first for each $e \in \mathcal{E}(S)$ we define the set

$$P_e(S) := \{A \subseteq S; \mid A \mid < \infty,\ ss^* = e\, \forall s \in A\}.$$

Then we define the following sets

$$P(S) := \bigcup_{e \in \mathcal{E}(S)} P_e(S)$$

$$P_{\mathcal{E}}(S) := \{B \in P(S); B \cap \mathcal{E}(S) \neq \emptyset\}.$$

In fact: $\forall s \in A\ ss^* = e \iff A \subset \mathcal{R}_e$.

Notice that: if $S = G$, *i.e.* the inverse semigroup is a group and then has only one idempotent e

$$\forall g \in G\ gg^{-1} = e \implies G \subseteq \mathcal{R}_e \text{ and}$$

$$\mathcal{E}(G) = \{e\} \implies P_{\mathcal{E}}(G) = P_e(G),$$

i.e we have the sets of Definition 2.2.1.

The previous sets can be endowed with a semi-lattice structure, as we present in the next Lemma (cf. Lawson-Margolis-Steinberg [54] Section 6.3).

Lemma 3.2.1 ([54]). *Let $A, B \in P(S)$ such that $A \subset \mathcal{R}_e$ and $B \subset \mathcal{R}_f$. Define $A \cdot_{\mathcal{R}} B := fA \cup eB$. Then*

(i) $A \cdot_{\mathcal{R}} B \in P(S)$ and is contained in $\mathcal{R}_{ef}$,

(ii) $A \cdot_{\mathcal{R}} A = A$,

(iii) $A \cdot_{\mathcal{R}} B = B \cdot_{\mathcal{R}} A$ and

(iv) $A \leqslant B \iff e \leqslant f$ *and* $eB \subseteq A$.

We will write AB meaning $A \cdot_{\mathcal{R}} B$, to avoid cumbersome notation. Also, this emphasizes the constructions of Section 3.2.

As simple implication of Lemma 3.2.1: $P_{\mathcal{E}}$ is an order ideal of $P(S)$ and a semilattice.

As we said, before we define the Bernoulli actions for inverse semigroups, we will present the O'Carroll triples. These are the generalization of McAlister P-inverse semigroups for inverse semigroups acting on semi-lattices, and they will provide our definition of the mentioned actions.

As we saw in Chapter 2, these new triples will encode a relation global-partial action by definition. This relation will happen because, similarly to a global group action restriction, we can restrict a global inverse semigroup action.

Let's elaborate this idea, following Khrypchenko [48] Section 4: if $\overline{\theta} = (\{\overline{D_s}\}_{s\in S}, \{\overline{\theta_s}\}_{s\in S})$ is a global action of an inverse semigroup S on a group X and $Y \subset X$, then the map $\theta_s := \overline{\theta}_{s|D_{s^*}}$ where $D_{s^*} = \overline{\theta_s}^{-1}(\overline{D}_s \cap Y) \cap Y$, is a partial action of S on Y.

Next, we present the O'Carroll's triples.

Definition 3.2.2 ([48])**.** Given an inverse semigroup S and two sets X and Y, where $Y \subset X$, an *O'Carroll L- triple* (S, X, Y) is composed by:

(I) X is a down direct poset and a meet-semilattice, and Y it's sub-semilattice and order ideal;

(II) S acts globally on X, such that this action is an order isomorphism between non empty ideals of X;

(III) $X = S \cdot Y$.

We make more clear item (iii): if the global action is $\overline{\theta} : S \curvearrowright X$ with for each $s \in S$ $\overline{\theta_s} : \overline{D}_{s^*} \mapsto \overline{D}_s$, then

$$X = S \cdot Y = \bigcup_{s\in S} \overline{\theta_s}(\overline{D}_{s^*} \cap Y)$$

In his work, [67], O'Carroll defined an inverse semigroup associated with a given L-triple (S, X, Y), as we have just stated, by

$$L(S, X, Y) := \{(x, s) \in X \times S; x \in \overline{D_s} \cap Y, \overline{\theta_s}^{-1}(x) \in Y\}.$$

Its idempotents and structural operations are

- idempotents: $\mathcal{E}(L(S,X,Y))) = \{(x,e) \in X \times \mathcal{E}(S)\}$;
- multiplication: $(x,s)(y,t) = (\overline{\theta_s}(\overline{\theta_s}^{-1}(x) \wedge y), st)$;
- inversion : $(x,s)^* = (\overline{\theta_s}^{-1}(x), s^*)$.

Notice that, in light of the restriction of the global action: if $\theta : S \curvearrowright_p Y$ is the partial action $\theta_s : D_{s^*} \mapsto D_s$ restricted from $\overline{\theta} : S \curvearrowright X$, then

$$L(S,X,Y) := \{(x,s) \in X \times S; x \in D_s\}.$$

This way, we can write L inverse semigroups with the language of partial actions as we did for P inverse semigroups (Chapter 2). Khrypchenko proved this statement in [48] Proposition 4.1.

At this point, we highlighted the similarities between McAlister and O'Carroll constructions. However, a significant difference occurs with the idempotents, since L inverse semigroups have more "options" of idempotents (inhered from the large number of idempotent elements that an inverse semigroup may have). This richness of structure provides a sub semigroup of any L inverse semigroup called *strict*, by O'Carroll [67] Section 2.

Definition 3.2.3 ([67])**.** Let (S,X,Y) be an L-triple as in Definition 3.2.2, we will say this triple is *strict* if: for every $y \in Y$ there exists $m(y) := min\{e \in \mathcal{E}(S); y \in \overline{D_e}\}$ and the map $y \in Y \mapsto m(y)$ is a homomorphism of meet semilattices.

In such conditions, the set

$$L_m(S,X,Y) := \{(x,s) \in L(S,X,Y); m(x) = ss^*\},$$

is an inverse sub semigroup of $L(S,X,Y)$, that we term *strict inverse semigroup*.

We provide more details of strict inverse semigroup, after O'Carroll paper [67]: given $L_m(S,X,Y)$

- $L_m(S,X,Y) = \{(x,t) \in L(S,X,Y); (x,s) \in L(S,X,Y), s \leqslant t \implies s = t\}$;
- $\mathcal{E}(L_m(S,X,Y)) = \{(x,m(x)); x \in Y\}$.

Remark 3.2.4. At the beginning of this section, we commented about the alternative construction credited by Lawson. Next, we present his definition – for future discussion. To make easy reference, we will term Lawson's triples, like he did, by *McAlister-O'Carroll triples*.

Definition 3.2.5 ([54])**.** Let S be an inverse semigroup, X a partially ordered set and Y a meet-semilattice and order ideal of X. A *McAlister-O'Carroll triple* (θ, q, Y) is formed by: $\theta : S \to \mathcal{I}(X)$ a homomorphism and a surjective map $q : Y \to \mathcal{E}(S)$ such that:

(I) $\theta(S) \cdot Y = X$;

(II) $y \in dom(\theta_e) \iff q(y) \leqslant e$ for each $y \in Y$ and $e \in \mathcal{E}(S)$;

(III) for each $s \in S$ there exists $y \in Y$ satisfying $q(y) = ss^*$ and $\theta_{s^*}(y) \in Y$.

He could produce an inverse semigroup from these triples as follows:

$$L(\theta, q, Y) := \{(y, s) \in Y \times S; q(y) = ss^* \text{ and } ss^*, \theta_{s^*}y \in Y\},$$

with product given by the rule

$$(x, s)(y, t) := (\theta_s(\theta_s^*(x) \wedge y), st).$$

Notice that Lawson's triples are, by definition, the strict triples from O'Carroll.

After all theoretical aspects that we will need to combine the expansion proposed by Lawson-Margolis-Steinberg ([54] 6.3) with the global-partial approach of O'Carroll triples stated by Khrypchenko ([48]). Then, our sets and construction nature will reveal four inverse semigroups – besides the two, S_{PB} and S_{GB}, found in Chapter 2.

Given an inverse semigroup S, let $e \in \mathcal{E}(S)$ we defined the set

$$P_e(S) := \{A \subseteq S; \mid A \mid < \infty, \forall\, s \in A,\; ss^* = e,\; e \in \mathcal{E}(S)\}.$$

Next, we define the following sets

$$P(S) := \bigcup P_e(S)$$
$$P_{\mathcal{E}}(S) := \{B \in P(S); B \cap \mathcal{E}(S) \neq \emptyset\};$$

now we will define a global action $S \curvearrowright P(S)$ and then a partial action $S \curvearrowright_p P_{\mathcal{E}}(S)$; finally and a strict action, which will be best explained in appropriate time.

Before state our definition, we make the disclaimer: although the following definition is after Lawson-Margolis-Steinberg [54] Proposition 6.14, we will underline it because this is the inverse semigroup version of Bernoulli action.

<u>Definition</u> 3.2.6. The *Bernoulli action* of the inverse semigroup S on $P(S)$ is the map $s \in S \mapsto \mathfrak{B}_s \in \mathcal{I}(P(S))$ where

$$\mathfrak{B}_s : \overline{D}_{s^*} := \{A \in P(S); A \subset \mathcal{R}_e, e \leqslant s^*s\} \to \overline{D}_s := \{B \in P(S); B \subset \mathcal{R}_f, f \leqslant ss^*\}\,. \\ A \mapsto sA$$

Notation: $\mathfrak{B} : S \curvearrowright P(S)$.

Before moving forward: $a \in A \subset \mathcal{R}_e \implies aa^* = e$, so

$$sa \in sA \implies (sa)(sa)^* = saa^*s^* = ses^* \therefore\ sA \subset \mathcal{R}_{ses^*}.$$

Also, clearly we have

$$s = ss^*s \implies ses^* = (ss^*)se^*s \therefore\ ses^* \leqslant ss^*.$$

Moreover: let $B \in \overline{D}_{s^*}$ such that $B \subset \mathcal{R}_f$ and $A \in P(S)$ with $A \subset \mathcal{R}_e$, satisfying: $A \leqslant B$; by Lemma 3.2.1 (iv) this is equivalent to $e \leqslant f$ and $eB \subseteq A$ and, from $B \in \overline{D}_{s^*}$

$$f \leqslant s^*s \implies e \leqslant s^*s \therefore\ A \in \overline{D}_{s^*}.$$

As well: let $A, B \in \overline{D}_{s^*}$ such that $A \subset \mathcal{R}_e, B \subset \mathcal{R}_f$, so $e, f \leqslant s^*s$; then $e \leqslant f$ implies $ses^* \leqslant sfs^*$ and

$$e \leqslant s^*s \text{ and } eB \subset A \implies ses^*B = seB \subset sA \therefore\ sA \leqslant sB.$$

Finally, is clear that: $\overline{\theta}_{s^*} = \overline{\theta}_s^{-1}$ and $\overline{D}_{s^*} = \overline{D}_{s^*s}$.

The previous information proves that our map is well defined, and its domain and range are order ideals. This result appears in Lawson-Margolis-Steinberg [54] Proposition 6.14, and we will formalize it for references with additional facts.

<u>Lemma</u> 3.2.7. The map $\mathfrak{B}_s : \overline{D_{s^*}} \to \overline{D_s}$, for each $s \in S$, is a well-defined homomorphism, and thus a global action of inverse semigroups.

Proof. Previous argumentation has already shown most of what we need; it remains only to check that the map defines an homomorphism.

Indeed: let $s, t \in S$, then $(st)^*(st) = t^*s^*st$ and

$$\overline{D}_{(st)^*} = \{A \in P(S); A \in \mathcal{R}_e, e \leqslant t^*s^*st\}.$$

By the other hand, given $A \in \overline{D}_{t^*}$

$$tA \in \overline{D}_{s^*} \iff |\, A \,| < \infty, A \subset \mathcal{R}_e \text{ s.t. } e \leqslant t^*t \text{ and } tet^* \leqslant s^*s.$$

The last relation implies: $e = t^*(tet^*)t \leqslant t^*s^*st$. Hence $(st)A = s(tA)$. $\square$

Moving forward, we will restrict $\mathfrak{B}$ to $P_{\mathcal{E}}(S)$ and define the partial action $\mathfrak{b} : S \curvearrowright_p P_{\mathcal{E}}(S)$. Indeed, for each $s \in S$ the domain and range are:

- domain: $D_{s^*} := \mathfrak{B}_{s^*}(\overline{D}_s \cap P_{\mathcal{E}}(S)) \cap P_{\mathcal{E}}(S)$.

  More clearly: if $A \in \overline{D}_{s^*}$ with $A \subset \mathcal{R}_e$, then $sA \in \mathcal{R}_{ses^*}$ and

  $$sA \in \overline{D}_s \cap P_{\mathcal{E}}(S) \implies sA \ni ses^*.$$

  As $\mathfrak{B}_{s^*}(sA) = s^*(sA)$, by the above computation $A \ni s^*(ses^*) = es^*$. Finally

  $$A \in (\overline{D}_s \cap P_{\mathcal{E}}(S)) \cap P_{\mathcal{E}}(S) \implies A \ni e.$$

  Therefore

  $$D_{s^*} = \{A \in P(S); A \ni es^*, e \text{ for } e^2 = e\}.$$

- range: $D_s := \mathfrak{B}_s(\overline{D}_{s^*} \cap P_{\mathcal{E}}(S)) \cap P_{\mathcal{E}}(S)$.

  On elements: for a given $A \in \overline{D}_{s^*}$ with $A \subset \mathcal{R}_e$, and

  $$sA \in \overline{D}_{s^*} \cap P_{\mathcal{E}}(S) \implies A \ni se.$$

  Since $sA \subset \mathcal{R}_{ses^*}$

  $$sA \in (\overline{D}_{s^*} \cap P_{\mathcal{E}}(S)) \cap P_{\mathcal{E}}(S) \implies sA \ni ses^*.$$

  Hence

  $$D_s = \{A \in P(S); A \ni ses^*, se \text{ for } e^2 = e\}.$$

- map: $\mathfrak{b}_s = \mathfrak{B}_{s|D_{s^*}}$.

Remark 3.2.8. Let A be an element in $P(\mathcal{S})$ such that $A \in \mathcal{R}_e$ and $A \ni e, es^*$. If $e = s^*fs$, where $f \leqslant ss^*$, then

- $es^*s = s^*fss^*s = s^*fs = e \implies e \leqslant s^*s$;
- $s^*f = s^*fss^* = es^*$.

In particular, in D_s if we take $f = ses^*$, the set sA satisfies

- $sA \in \mathcal{R}_f$,
- $f \in sA$ and
- $fs = ses^*s = se \in sA$.

Hence the domain and range of the partial action have the same formation rule.

Definition 3.2.9. The *Bernoulli partial action* of the inverse semigroup S on $P_{\mathcal{E}}(S)$ is the map $s \in S \mapsto \mathfrak{b}_s \in \mathcal{I}(P(S))$ where

$$\begin{aligned} \mathfrak{b}_s : D_{s^*} = \{A \in P(S); A \ni es^*, e, e \in \mathcal{E}(S)\} &\to D_s = \{A \in P(S); A \ni ses^*, se, e \in \mathcal{E}(S)\}\,. \\ A &\mapsto sA \end{aligned}$$

Notation: $\mathfrak{b} : S \curvearrowright_p P_{\mathcal{E}}(S)$.

We aim to show that $(S, P(S), P_{\mathcal{E}})$ is an L-triple, and also study its structure. We are devoting the next section to this task, but we will define one more partial action: the *strict partial action*.

In his paper, [48], Khrypchenko relates the existence of strict inverse semigroups with strict partial actions. Next, we define such actions to apply to our study case.

Definition 3.2.10 ([48])**.** A partial action, $\theta : S \curvearrowright_p X$, of an inverse semigroup on a semilattice X is *strict* if for every $x \in X$ there exists $m(x) := min\{e \in \mathcal{E}(S); x \in \text{dom}(\theta_e)\}$ and the map $x \in X \mapsto m(x) \in \mathcal{E}(S)$ is a homomorphism of meet semilattices.

Notation: $\theta : S \curvearrowright_p X$ or $st(\theta) : S \curvearrowright_p X$ (if necessary, to avoid confusion with general partial actions).

In our case, we have a ”candidate” for strict partial action.

Definition 3.2.11. Let: $s \in S \mapsto \mathfrak{sb}_s \in \mathcal{I}(P_{\mathcal{E}}(S))$ where

$$\begin{aligned} \mathfrak{sb}_s : {D^m}_{s^*} := \{A \in P_{\mathcal{E}}(S); s^*, s^*s \in A\} &\to {D^m}_s := \{B \in P_{\mathcal{E}}(S); s, ss^* \in B\}\,. \\ A &\mapsto sA \end{aligned}$$

We will call this map the *strict Bernoulli partial action* and denote it by $\mathfrak{sb} : S \curvearrowright_p P_{\mathcal{E}}(S)$.

This definition must be checked. Indeed, we split the verification of this claim in a few steps:

(a) This map is well defined: let $A \in D^m{}_{s^*}$, so A is finite; $A \ni s^*s$ and for all $a \in A$ we have the identity $aa^* = s^*s$. For $sA \ni x = sa$

$$xx^* = (sa)(sa)^* = saa^*s = ss^*ss^* = ss^*.$$

So $ss^*, s \in sA$. Hence $sA \in D^m{}_s$.

(b) Suppose $A \in D^m{}_t \cap D^m{}_{s^*}$. Then $t, tt^*, s^*, s^*s \in A$, and for all $a \in A$ holds $aa^* = tt^* = s^*s$.

(c) Observe that $\mathfrak{sb}_{s^*} \circ \mathfrak{sb}_s$ is the identity on $D^m{}_{s^*}$ and $\mathfrak{sb}_s \circ \mathfrak{sb}_{s^*}$ is the identity on $D^m{}_s$.

(d) $\mathfrak{sb}_s^{-1} = \mathfrak{sb}_{s^*}$.

(e) $\mathrm{dom}(\mathfrak{sb}_s \circ \mathfrak{sb}_t) = \mathfrak{b}_t^{-1}(D^m{}_{s^*} \cap D^m{}_t) = \mathfrak{sb}_{t^*}(D^m{}_{s^*} \cap D^m{}_t)$.

(f) If $A \in D^m{}_{s^*}$, by definition $A \ni s^*, s^*s$ and for all $a \in A$ $aa^* = s^*s$. So

$$\mathfrak{sb}_{t^*}(a) = t^*a \implies b_{t^*}(a)(b_{t^*}(a))^* = t^*aa^*t = t^*s^*st.$$

What we concluded is: $\mathfrak{sb}_{t^*}(D^m{}_{s^*} \cap D^m{}_t) \subseteq D^m{}_{(st)^*} \cap D^m{}_{t^*}$.

(g) By (c) , last inclusion and $tt^* = s^*s$:

$$\begin{aligned} &\mathfrak{sb}_t \circ \mathfrak{sb}_{t^*}(D^m{}_{s^*} \cap D^m{}_t) \subseteq \mathfrak{sb}_t(D^m{}_{(st)^*} \cap D^m{}_{t^*}) \subseteq D^m{}_t \cap D^m{}_{t(st)^*} \\ &\implies D^m{}_{s^*} \cap D^m{}_t \subseteq \mathfrak{sb}_t(D^m{}_{(st)^*} \cap D^m{}_{t^*}) \subseteq D^m{}_t \cap D^m{}_{s^*}. \end{aligned}$$

(h) As we showed

$$\mathfrak{sb}_t(D^m{}_{(st)^*} \cap D^m{}_{t^*}) = D^m{}_t \cap D^m{}_{s^*},$$

applying $\mathfrak{sb}_{t^*}$, we get

$$D^m{}_{(st)^*} \cap D^m{}_{t^*} = \mathfrak{sb}_{t^*}(D^m{}_t \cap D^m{}_{s^*}).$$

This concludes the proof, once $\mathrm{dom}(\mathfrak{sb}_{st}) = D^m{}_{(st)^*}$.

(i) On idempotents, this function is the identity: let $e \in \mathcal{E}(S)$, then $\mathfrak{sb}_e : D^m{}_e \to D^m{}_e$. If $A \in D^m{}_e$, for all $a \in A$ $aa^* = e$ and $A \ni e$. So $\mathfrak{sb}_e(A) \ni z = ea$ is such that

$$z = ea = aa^*a = a \implies \mathfrak{sb}_e = id.$$

(j) Finally, given $e \in \mathcal{E}(S)$: $A \in D^m{}_e \iff A \ni e$ and $\forall a \in A$ $aa^* = e$. Hence $min\{e \in \mathcal{E}(S); A \in D^m{}_e\} = \{e\}$. Define the map $\epsilon : P_{\mathcal{E}}(S) \to \mathcal{E}(S)$ by $A \subset \mathcal{R}_e \mapsto \epsilon(A) = e$. Then ϵ is homomorphism.

Conclusion: $\mathfrak{sb} : S \curvearrowright_p P_{\mathcal{E}}(S)$ is a strict partial action.

The way we made our construction, inspired mostly by Khrypchenko [48], guided us from the global action to the partial and the strict partial action.

Next we will describe an action which its restriction to $P_{\mathcal{E}}(S)$ is the strict partial action $\mathfrak{sb} : S \curvearrowright_p P_{\mathcal{E}}(S)$: for $s \in S$ define

$$\begin{aligned} \mathfrak{sB}_s : \overline{D}^m{}_{s^*} := \{A \in P(S); A \subset \mathcal{R}_{s^*s}\} \to \overline{D}^m{}_s := \{B \in P(S); B \subset \mathcal{R}_{ss^*}\}\,. \\ A \mapsto sA \end{aligned}$$

Similar arguments from (a) to (i) shows this map is an action of S on $P(S)$; it remains to verify the global assertion, we will conclude this fact restricting $\mathfrak{sB}$ to $P_{\mathcal{E}}(S)$. Indeed:

- domain: let $A \in \overline{D}^m{}_{s^*}$, so $A \subset \mathcal{R}_{s^*s}$; by definition $sA \in \overline{D}^m{}_s$, which means $sA \subset \mathcal{R}_{ss^*}$ and

  $$sA \in \overline{D}^m{}_s \cap P_{\mathcal{E}}(S) \implies sA \ni ss^*.$$

  As $\mathfrak{sB}_{s^*} = \mathfrak{sB}_s^{-1}$

  $$A = s^*sA \in (\overline{D}^m{}_s \cap P_{\mathcal{E}}(S)) \cap P_{\mathcal{E}}(S) \implies A \ni s^*s.$$

  This computation also shows us:

  $$A \ni s^*(ss^*) \implies A \ni s^*.$$

  Therefore the domain of $\mathfrak{sB}$ restricted to $P_{\mathcal{E}}(S)$ is the set

  $$\mathfrak{sB}_{s^*}(\overline{D}^m{}_s \cap P_{\mathcal{E}}(S)) \cap P_{\mathcal{E}}(S) = \{A \in P_{\mathcal{E}}(S); s^*, s^*s \in A\}.$$

  Furthermore, this is the domain of $\mathfrak{sb}$.

- range: analogous computations reveals

  $$\mathfrak{sB}_s(\overline{D}^m_{s^*} \cap P_{\mathcal{E}}(S)) \cap P_{\mathcal{E}}(S) = \{A \in P_{\mathcal{E}}(S); s, ss^* \in A\}.$$

  Which is precisely the range of $\mathfrak{sb}$.

  Conclusion: the global inverse semigroup action $\mathfrak{sB} : S \curvearrowright P(S)$ is the globalization of the partial (strict) inverse semigroup action $\mathfrak{sb} : S \curvearrowright_p P_{\mathcal{E}}(S)$.

We summarize it in the next definition.

Definition 3.2.12. Let $s \in S \mapsto \mathfrak{s}\mathfrak{B}_s \in \mathcal{I}(P(S))$ be the map

$$\mathfrak{s}\mathfrak{B}_s : \overline{D}^m{}_{s^*} := \{A \in P(S); A \subset \mathcal{R}_{s^*s}\} \to \overline{D}^m{}_s := \{B \in P(S); B \subset \mathcal{R}_{ss^*}\}\,. \\ A \mapsto sA$$

We will term this map by *strict global Bernoulli action* and denote it by $\mathfrak{s}\mathfrak{B} : S \curvearrowright P(S)$.

We will complete this section with a diagram of all actions we constructed: given an inverse semigroup S

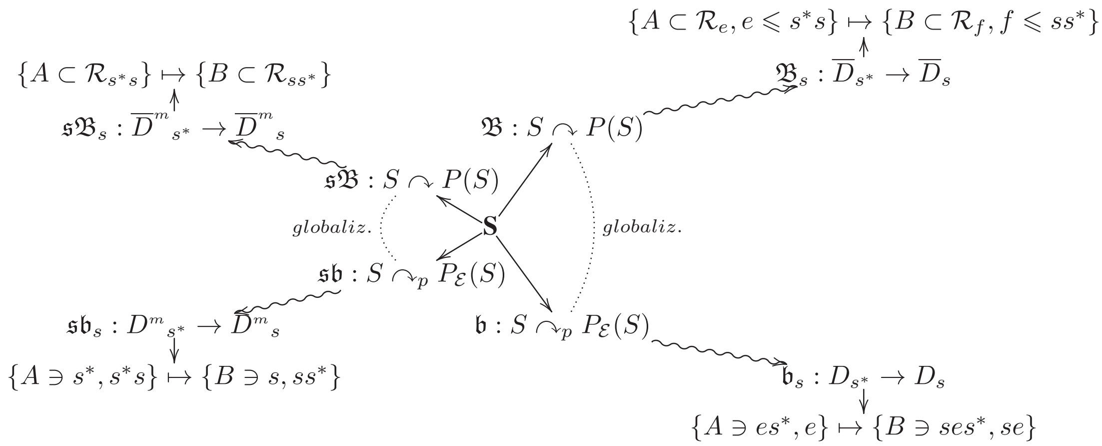


Figure 3.1: The inverse semigroup Bernoulli actions

3.3 Inverse semigroups associated with Bernoulli semigroup actions

Each of the four inverse semigroup actions from the last section will define an inverse semigroup. First, we will introduce the theoretical aspects of our constructions: the λ-semidirect products of inverse semigroups; following Lawson [51] (Section 5.3) and, the source, Billhardt [10].

Lawson says in his book in Section 5.3, rather than partial semidirect products, we must use λ-semidirect products to produce an inverse semigroup from an inverse semigroup action. Its definition will require a notion of action by endomorphisms. We will quickly review his construction.

Definition 3.3.1 ([51])**.** Let S and T be inverse semigroups. We say T *acts by endomorphisms* on S if for every $t \in T$ there is a map $S \to S$ by $s \mapsto t \cdot s$ such that

(I) $t \cdot (s_1 s_2) = (t \cdot s_1)(t \cdot s_2)$ fot $t \in T$ and $s_1, s_2 \in S$,

(II) $(t_1 t_2) \cdot s = t_1 \cdot (t_2 s)$,

(III) if T is a monoid with identity 1_T: $1_T \cdot s = s$ for all $s \in S$.

The modified definition of semidirect product is defined in the following way. Let S and T, with T acting on S by endomorphisms. Defining $r(t) = tt^*$, we set

$$S \rtimes^{\lambda} T =: \{(s,t) \in S \times T; r(t) \cdot s = s\}$$

with product

$$(s,t) \cdot (s',t') := ((r(tt') \cdot s)(t \cdot s'), tt'),\ s, s' \in S,\ t, t' \in T$$

and involution

$$(s,t)^* = (t^* \cdot s^*, t^*).$$

Hence $S \rtimes^{\lambda} T$ is an inverse semigroup drawn from the λ-*semidirect product*.

Remark 3.3.2. In addition to Lawson [51] and Billhardt [10], the reader can learn more about this product in: Gould-Zenad's paper [42], or her talk [100], also in Szendrei's talk [92] and her paper [49], or Gomes article [41].

In our case, we can use this result to construct an inverse semigroup as follows: for a given inverse semigroup S, we defined the set

$$P(S) := \{A \subseteq S; \mid A \mid < \infty, \forall\, s \in A\ ss^* = e,\ e \in \mathcal{E}(S)\}.$$

Notice: if $A \subset \mathcal{R}_e$, for $e \in \mathcal{E}(S)$, and $s \in S$ is such that $r(s)A = A$, for $b \in A$ there exists $a \in A$ such that $r(s)a = b$. Since $ea = a$ and $eb = b$, we have

$$r(s)a = b \iff (ss^*e)a = (e)b \iff ss^*e = e.$$

Therefore

$$r(s)A = A \iff e \leqslant ss^*.$$

Acting on this sets we defined: $\mathfrak{B} : S \curvearrowright P(S)$ and $\mathfrak{l}\mathfrak{B} : S \curvearrowright_l P(S)$, both having the same rule $A \mapsto sA$.

As a result, what we achieve is the next definition.

<u>Definition</u> 3.3.3. Let S be an inverse semigroup, the set $P(S)$ and the global inverse semigroup actions $\mathfrak{B} : S \curvearrowright P(S)$ and $\mathfrak{s}\mathfrak{B} : S \curvearrowright_s P(S)$. We define the λ-inverse semigroups:

(I) The *global prefix*:

$$\begin{aligned}\overline{Pr}(S) := P(S) \rtimes^{\lambda}_{\mathfrak{B}} S &= \{(A,s) \in P(S) \times S; \mathfrak{B}_{r(s)}A = A\}\\ &= \{(A,s) \in P(S) \times S; r(s)A = A\}\\ &= \{(A,s) \in P(S) \times S; A \subset \mathcal{R}_e, e^2 = e \leqslant ss^*\}.\end{aligned}$$

Whose idempotents are:

$$\mathcal{E}(\overline{Pr}(S)) = \{(E,i) \in P(S) \times \mathcal{E}(S); E \subset \mathcal{R}_e, e^2 = e \leqslant i\}.$$

(II) The *strict global prefix*:

$$\begin{aligned}\overline{Pr}(S)_m := P(S) \rtimes^{\lambda}_{s\mathfrak{B}} S &= \{(A,s) \in \overline{Pr}(S); A \subset \mathcal{R}_{ss^*}\}\\ &= \{(A,s) \in P(S) \times S; A \subset \mathcal{R}_e, e^2 = e = ss^*\}.\end{aligned}$$

With idempotents:

$$\mathcal{E}(\overline{Pr}(S)_s) = \{(E,e) \in P(S) \times \mathcal{E}(S); E \subset \mathcal{R}_e\}.$$

The internal structural maps are given by:

- involution: both inverse semigroups share the same expression, so for a given element (A,s) in $\overline{Pr}(S)$, or $\overline{Pr}(S)_s$:
$$(A,s)^* = (s^*A, s^*).$$

- product: the general rule for such construction is
$$(A,s)(B,t) := ((r(st)A)(sB), st).$$

Earlier, in Lemma 3.2.1, we defined the binary operation in $P(S)$ as follows

$$E, F \in P(S), E \subset \mathcal{R}_e, F \subset \mathcal{R}_f \implies EF = fE \cup eF \subset \mathcal{R}_{ef}.$$

Using this result will derive the product in each inverse semigroup:

- global prefix: let $(A,s),(B,s)\in\overline{Pr}(S)$ with $A\subset\mathcal{R}_e$ and $B\subset\mathcal{R}_f$, notice

$$a\in A\implies (r(st)a)(r(st)a)^*=r(st)aa^*r(st)=r(st)e\therefore r(st)A\in\mathcal{R}_{r(st)e}$$

$$b\in B\implies (sb)(sb)^*=sbb^*s^*=sfs^*\therefore sB\subset\mathcal{R}_{sfs^*}.$$

As $r(st)=stt^*s^*$

$$(r(st)A)(sB)=sfs^*(stt^*s^*A)\cup stt^*s^*e(sB)=sftt^*s^*A\cup estt^*B=sftt^*s^*A\cup esB.$$

Where the last equation holds due to: $B\subset\mathcal{R}_f$ and $f\leqslant tt^*$, implies

$$estt^*B=es(tt^*fB)=esB.$$

Finally, since

$$(r(st)A)(sB)\subset\mathcal{R}_{estt^*fs^*},$$

and

$$estt^*fs^*\leqslant(st)(st)^*$$

the product is well defined and has the expression

$$(A,s)(B,t)=((r(st)A)(sB),st)=(sftt^*s^*A\cup esB,st)=(r(sft)A\cup esB,st).$$

- strict global prefix: let $(A,s),(B,s)\in\overline{Pr}(S)_l$ with $A\subset\mathcal{R}_{ss^*}$ and $B\subset\mathcal{R}_{tt^*}$; replacing $e=ss^*$ and $f=tt^*$ in the global prefix product, we discover

$$(A,s)(B,t)=(stt^*s^*A\cup sB,st)=(r(st)A\cup sB,st).$$

Summarizing: we have just associated an λ-inverse semigroup to each global action.

Next we are going to do the same for the partial actions; in this case the reasoning is a bit different. Indeed, following Khrypchenko – [48] Lemma 2.5: given an inverse semigroup action $\theta:S\curvearrowright_p X$, of an inverse semigroup on a semilattice, the set

$$X\rtimes_\theta S:=\{(x,s)\in X\times S; x\in\text{ran}(\theta_s)\},$$

with idempotents

$$\mathcal{E}(X\rtimes_\theta S)=\{(x,e)\in X\times\mathcal{E}(S)\},$$

product
$$(x,s)(y,t) = (\theta_s(\theta_s^{-1}(x) \wedge y), st)$$
and involution
$$(x,s)^* = (\theta_s^{-1}(x), s^*).$$

Then $X \rtimes_\theta S$ is an inverse semigroup with the previous data.

Put together with L-triples. The prior argument is used by Khrypchenko ([48] Section 4) to define inverse semigroups from partial actions that arise from O'Carroll triples.

For our purposes, as we saw in the last section: let S be an inverse semigroup and the sets

$$P(S) := \{A \subseteq S; \mid A \mid < \infty, \forall\, s \in A\; ss^* = e,\; e \in \mathcal{E}(S)\}$$
$$P_{\mathcal{E}}(S) := \{B \in P(S); B \cap \mathcal{E}(S) \neq \emptyset\}.$$

Then $(S, P(S), P_{\mathcal{E}}(S))$ is an L-triple with global action $\mathfrak{B} : S \curvearrowright P(S)$, partial action $\mathfrak{b} : S \curvearrowright_p P_{\mathcal{E}}(S)$ and strict partial action $\mathfrak{sb} : S \curvearrowright_l P_{\mathcal{E}}(S)$. And we will have

$$L(S, P(S), P_{\mathcal{E}}(S)) = P_{\mathcal{E}}(S) \rtimes_{\mathfrak{b}} S$$

and
$$L_m(S, P(S), P_{\mathcal{E}}(S)) = P_{\mathcal{E}}(S) \rtimes_{\mathfrak{sb}} S.$$

In this way, we make the following definition.

Definition 3.3.4. Let $(S, P(S), P_{\mathcal{E}}(S))$ be a L-triple and $\mathfrak{b} : S \curvearrowright_p P_{\mathcal{E}}(S)$ and $\mathfrak{sb} : S \curvearrowright_p P_{\mathcal{E}}(S)$ be the partial actions from the Definitions 3.2.11 and 3.2.9 . We define the inverse semigroups

(I) The *partial prefix*

$$\begin{aligned} Pr(S) :&= L(S, P(S), P_{\mathcal{E}}(S)) \\ &= \{(A, s) \in P_{\mathcal{E}}(S) \times S; A \subset \mathcal{R}_{ses^*}, e^2 = e, A \ni ses^*, se\}. \end{aligned}$$

With

$$\mathcal{E}(Pr(S)) = \{(E, i) \in P_{\mathcal{E}}(S) \times \mathcal{E}(S); E \subset \mathcal{R}_f, f^2 = f = ei, e^2 = e, E \ni f\}.$$

(II) The *strict partial prefix*

$$\begin{aligned} Pr(S)_m &:= L_m(S, P(S), P_{\mathcal{E}}(S)) \\ &= \{(A, s) \in Pr(S); A \subset \mathcal{R}_{ss^*}, A \ni s, ss^*\}. \end{aligned}$$

And its idempotents are

$$\mathcal{E}(Pr(S)_m) = \{(A, e) \in S, P(S), P_{\mathcal{E}}(S); A \subset \mathcal{R}_e, e^2 = e, E \ni e\}.$$

Notice that the involution and multiplication in these inverse semigroup are expressed by:

- involution: as both partial actions are defined by concatenation on the right-hand side, and its inverse is just the same rule, but using the inverse of the element, we have for (A, s) in $Pr(S)$, or $Pr(S)_m$:
$$(A, s)^* = (s^*A, s^*).$$

- multiplication: the given description is
$$(A, s)(B, t) = (s(s^*A \wedge B), st).$$

  In order to compute we invoke the Lemma 3.2.1. Dealing with one inverse semigroup at time:

  - partial prefix: let $(A, s), (B, t) \in Pr(S)$ with $A \subset \mathcal{R}_{ses^*}$ and $B \subset \mathcal{R}_{tft^*}$, notice
  $$a \in A \implies (s^*a)(s^*a)^* = s^*aa^*s = s^*ses^*s = s^*se.$$
  Thus
  $$s^*A \wedge B = tft^*(s^*A) \cup s^*seB,$$
  which implies
  $$s(s^*A \wedge B) = stft^*s^*A \cup seB.$$

This is a well defined operation, because

$$a \in A \implies (stft^*s^*a)(stft^*s^*a)^* = stft^*s^*aa^*stft^*s^* = stft^*s^*(ses^*) = stft^*es^*$$

$$b \in B \implies (seb)(seb)^* = sebb^*es^* = stft^*es^*$$

$$\therefore s(s^*A \wedge B) \subset \mathcal{R}_{stft^*es^*};$$

but we can rewrite $stft^*es^*$ as follows

$$stft^*es^* = stf(t^*tt^*)es^* = (st)(ft^*et)(t^*s^*) = (st)(ft^*et)(st)^*,$$

where $(ft^*et)^2 = (ft^*et)$.

Conclusion:

$$(A, s)(B, t) = (stft^*s^*A \cup seB, st) = (r(stf)A \cup seB, st).$$

– restrict partial prefix: let $(A, s), (B, t) \in Pr(S)$ with $A \subset \mathcal{R}_{ss^*}$ and $B \subset \mathcal{R}_{tt^*}$, notice $sA \in \mathcal{R}_{s^*s}$. So

$$s^*A \wedge B = tt^*(s^*A) \cup s^*sB,$$

which implies

$$s(s^*A \wedge B) = stt^*s^*A \cup sB \subset \mathcal{R}_{stt^*s^*}.$$

For this reason

$$(A, s)(B, t) = (stt^*s^*A \cup sB, st) = (r(st)A \cup sB, st).$$

Observe that the expression of the global and the partial inverse semigroups match. Indeed the strict global and the strict partial share the same formula; by the other hand, replacing e by ses^* and f by tf^* in the global inverse semigroup product we find the product of the partial inverse semigroup, *i.e.*

$$s(tft^*)tt^*s^*A \cup (ses^*)sB = stft^*s^*A \cup seB.$$

Clearly, when $S = G$ is a group

$$\overline{Pr}(G) = \overline{Pr}(G)_m = S_{GB} \text{ and } Pr(G) = Pr(G)_m = S_{PB},$$

where S_{GB} and S_{PB} are the inverse semigroups from Chapter 2 (cf. the Lemmas 2.2.4 and 2.2.5).

Our next effort is to present the internal structure of the four prefix inverse semigroups. We are concerned about the global and partial ones. It is because Lawson-Margolis-Steinberg, in [54] Section 6.4, described the Green classes (from Definition 2.4.1) of (our) strict partial prefix inverse semigroups. Note that it is the same for the strict global – as they have the same product expression.

Proposition 3.3.5. Let $(A, s), (B, t) \in \overline{Pr}(S)$ such that $A \subset \mathcal{R}_e$ and $B \subset \mathcal{R}_f$, the ordering and Green classes of $\overline{Pr}(S)$ are:

(i) $(A, s) \leqslant (B, t) \iff e \leqslant f,\ s \leqslant t,\ eB \subseteq A$;

(ii) $(A, s)\mathcal{R}(B, t) \iff e = f,\ ss^* = tt^*,\ A = B$;

(iii) $(A, s)\mathcal{L}(B, t) \iff e = f,\ s^*s = t^*t,\ s^*A = t^*B$;

(iv) $(A, s)\mathcal{H}(B, t) \iff e = f,\ ss^* = tt^*,\ s^*s = t^*t,\ A = B = ts^*A$;

(v) $(A, s)\mathcal{D}(B, t) \iff e = f, \exists p \in S$ s.t. $s^*s = p^*p,\ tt^* = pp^*,\ A = sp^*B$.

Notice that the same relations work for $Pr(S)$, after adjusting for the appropriate idempotents.

Corollary 3.3.6. Let $(A, s), (B, t) \in \overline{Pr}(S)_m$ such that $A \subset \mathcal{R}_{ss^*}$ and $B \subset \mathcal{R}_{tt^*}$, the ordering and Green classes of $\overline{Pr}(S)$ are:

(i) $(A, s) \leqslant (B, t) \iff s \leqslant t,\ ss^*B \subseteq A$;

(ii) $(A, s)\mathcal{R}(B, t) \iff ss^* = tt^*,\ A = B$;

(iii) $(A, s)\mathcal{L}(B, t) \iff s^*s = t^*t,\ s^*A = t^*B$;

(iv) $(A, s)\mathcal{H}(B, t) \iff ss^* = tt^*,\ s^*s = t^*t,\ A = B = ts^*A$;

(v) $(A, s)\mathcal{D}(B, t) \iff \exists p \in S$ s.t. $s^*s = p^*p,\ tt^* = pp^*,\ A = sp^*B$.

The demonstrations of Proposition 3.3.5 and the Corollary 3.3.6 were omitted since: (i) comes from the natural order of an inverse semigroup, as in Definition 1.2.7; and (ii)-(v) are from the Definition 2.4.1 of the Green relations.

Remark 3.3.7. As the expression of product in $Pr(S)_m$ is the same, Lawson-Margolis-Steinberg [54] Proposition 6.18 turns out to be our last corollary.

The final remarks of this section concern our notation in comparison to the literature.

Remark 3.3.8. The inverse semigroup we termed by $Pr(S)_m$ in Lawson-Margolis-Steinberg, [54] Proposition 6.16, is symbolized by $S^{\mathbf{Pr}}$. Also this inverse semigroup is isomorphic to $Pref(S)$, from Definition 3.1.1.

Indeed, let $\varepsilon : Pref(S) \to Pr(S)_m$ with $\varepsilon_A[s] \overset{\psi}{\mapsto} (A, g)$, where $\varepsilon_A[s]$ is in normal form. This map satisfy:

homomorphism: let $\alpha = \varepsilon_A[s], \beta = \varepsilon_B[t] \in S(G)$ in normal form; then

$$\psi(\alpha)\psi(\beta) = (A, s)(B, t) = (r(st)A \cup sB, st).$$

On the other hand,

$$\psi(\alpha \cdot \beta) = \psi(\varepsilon_{(r(st)A)\cup(sB)[st]}) = (r(st)A \cup sB, st).$$

Injectivity: follows by the uniqueness of the normal form;

Surjectivity: by definition.

Hence $Pref(S) \simeq Pr(S)_m$.

We conclude by presenting a table with all the four inverse semigroups, where al e, f present are such that $e^2 = e$ and $f^2 = f$

	Prefix inverse semigroups	
global	$\overline{Pr}(S) = \{(A, s) \in P(S) \times S; A \subset \mathcal{R}_e, e \leqslant ss^*\}$ $\mathcal{E}(\overline{Pr}(S)) = \{(E, i) \in \overline{Pr}(S); E \subset \mathcal{R}_e, e \leqslant i\}$	$\overline{Pr}(S)_m = \{(A, s) \in P(S) \times S; A \subset \mathcal{R}_e, e = ss^*\}$ $\mathcal{E}(\overline{Pr}(S)_m) = \{(E, e) \in \overline{Pr}(S)_m; E \subset \mathcal{R}_e\}$
partial	$Pr(S) = \{(A, s) \in P_{\mathcal{E}}(S) \times S; A \subset \mathcal{R}_{ses^*}, A \ni ses^*, se\}$ $\mathcal{E}(Pr(S)) = \{(E, i) \in Pr(S); E \subset \mathcal{R}_f, f = ei, E \ni f\}$	$Pr(S)_m = \{(A, s) \in Pr(S); A \subset \mathcal{R}_{ss^*}, A \ni s, ss^*\}$ $\mathcal{E}(Pr(S)_m) = \{(E, e) \in Pr(S)_m; A \subset \mathcal{R}_e, E \ni e\}$
	Structural maps	
involution	$(A, s)^* = (s^*A, s^*)$	
global prd.	$(A, s)(B, t) = (sftt^*s^*A \cup esB, st)$	$(A, s)(B, t) = (stt^*s^*A \cup sB, st)$
partial prd.	$(A, s)(B, t) = (stft^*s^*A \cup seB, st)$	$(A, s)(B, t) = (stt^*s^*A \cup sB, st)$

Table 3.1: The globlal and partial semidirect products

3.4 The algebras of the four semigroups

In this section, we will use Steinberg's isomorphism of algebras (cf. the Theorem 2.5.6); then we study the Morita relations these algebras (might) have. Nevertheless, will not present a decomposition for such algebras as we did in Theorem 2.5.14. First, we recall some facts from Chapter 1.

Let $\mathbb{K}$ be an associative commutative unital ring and S an inverse semigroup. Recall that the $\mathbb{K}$-algebra of an inverse semigroup, $\mathbb{K}S$, is the free $\mathbb{K}$-module generated by elements of S with product

$$(\sum_s a_s\delta_s)(\sum_t b_t\delta_t) = \sum_u(\sum_{st=u} a_s b_t)\delta_u, \forall s,t,u \in S.$$

Similarly, if $\mathcal{G}$ is a groupoid, the $\mathbb{K}$-algebra of the groupoid, is the free $\mathbb{K}$-module with basis $\mathcal{G}$ and convolution product

$$\delta_x * \delta_y = \begin{cases} \delta_{xy} & \text{, if } \exists xy \\ 0 & \text{, if not} \end{cases}$$

It was proved by Steinberg in [87] – which we stated in Theorem 2.5.6 –, that

$$\mid S \mid < \infty \implies \mathbb{K}S \simeq \mathbb{K}\mathcal{G}_S,$$

i.e., the algebra of any finite inverse semigroup is isomorphic to the restrict groupoid algebra associated.

In this section we fix:

- S **is a finite inverse semigroup**,
- and Morita equivalent algebras R and Q will be denoted by $R \simeq_M Q$.

Each prefix inverse semigroup will give origin to an algebra, which we will term by:

global algebra - $\mathbb{K}\overline{Pr}(S) := \mathbb{K}_{glob}(S)$;

strict global algebra - $\mathbb{K}\overline{Pr}(S)_m := \mathbb{K}_{sglob}(S)$;

partial algebra - $\mathbb{K}Pr(S) := \mathbb{K}_{par}(S)$;

<u>*strict partial algebra*</u> - $\mathbb{K}Pr(S)_m := \mathbb{K}_{spar}(S)$.

As we said at the beginning of this section: we would like to provide a Morita context. Similarly to our strategy in Chapter 2, we will take an indirect way and use enlargements.

We have already defined enlargements of inverse semigroups in Chapter 3, Definition 2.3.4. But this time we will use the equivalent definition of Lawson's paper [50] (Section 2).

Proposition 3.4.1 ([50])**.** Let S be an inverse sub semigroup of T. Then T is an enlargement of S if, and only if:

(I) $\mathcal{E}(S)$ is an order ideal of $\mathcal{E}(T)$;

(II) $t \in T$ and $t^*t, tt^* \in S \implies t \in S$;

(III) for every $e \in \mathcal{E}(T)$, there exists $f \in \mathcal{E}(S)$ such that $e\mathcal{D}f$.

In light of Remark 2.4.2 (d), (III) can be replaced by:

$$(III)' \quad e\mathcal{D}f \Leftrightarrow \exists s \in S \text{ s.t. } s^*s = f \text{ and } ss^* = e.$$

Therefore, by Theorem 2.5.2 says: $S \subseteq_E T \implies \mathbb{K}S \simeq_M \mathbb{K}T$.
Without further ado, the main result of this section follows.

<u>Proposition</u> 3.4.2. The strict global prefix is an enlargement of the strict partial prefix, *i.e.*

$$Pr(S)_m \subseteq_E \overline{Pr}(S)_m.$$

Proof. We must verify the axioms of Definition 3.4.1. Indeed:

(I) suppose the idempotents

$$(E, e) \in \mathcal{E}(Pr(S)_m) \iff E \in \mathcal{R}_e,\ e^2 = e,\ E \ni e \text{ and}$$

$$(F, f) \in \mathcal{E}(\overline{Pr}(S)_l) \iff F \in \mathcal{R}_f,\ f^2 = f,$$

such that

$$(F, f) \leqslant (E, e) \iff f \leqslant e \text{ and } fE \subseteq F.$$

So the relation $f \leqslant e \iff f = fe$, implies

$$fE \ni fe = e \text{ and } fE \subseteq F \implies f = ef \in F.$$

Hence $(F, f) \in \mathcal{E}(Pr(S)_m)$.

(II) given $(A, s) \in \overline{Pr}(S)_m$, which means by definition $A \subset R_{ss^*}$, satisfying

$$(s^*A, ss^*), (A, ss^*) \in \mathcal{E}(Pr(S)_m).$$

This inclusion implies

$$A \ni ss^* \text{ and } s^*A \ni s^*s, \text{ and}$$

$$\exists a \in A \text{ s.t. } s^*a = s^*s \implies ss^*a = ss^*s \implies a = s.$$

Therefore $A \subset \mathcal{R}_{ss^*},\ A \ni ss^*, s$, *i.e.* $(A, s) \in Pr(S)_m$.

(III) consider $(F, f) \in \mathcal{E}(Pr(S)_l)$, by definition $F \subset \mathcal{R}_f$ where $f^2 = f$. By the construction of the O'Carroll triple $(S, P(S), P_{\mathcal{E}}(S)$, we have: $S \cdot P_{\mathcal{E}} = P(S)$; as $F \in P(S)$ there are $A \in P_{\mathcal{E}}(S)$ and $s \in S$ such that $sA = F$. Note that

$$\exists sA \iff A \subset \mathcal{R}_{s^*s} \text{ and } A \in P_{\mathcal{E}}(S) \implies A \ni s^*s.$$

This way, the pair (F,s) satisfy:

- $A \subset \mathcal{R}_{s^*s} \iff sA \subset \mathcal{R}_{ss^*}$ and $sA = F \implies \mathcal{R}_f = \mathcal{R}_{ss^*} \therefore (F, s) \in \overline{Pr}(S)_m$;
- $(F, s)^*(F, s) = (s^*F, s^*s) = (A, s^*s) \in \mathcal{E}(Pr(S)_m)$;
- $(F, s)(F, s)^* = (F, ss^*) = (F, f) \in \mathcal{E}(\overline{Pr}(S)_l)$.

Thus $(F, f)\mathcal{D}(A, s^*s)$.

Conclusion: $Pr(S)_m \subseteq_E \overline{Pr}(S)_m$.

□

Corollary 3.4.3. The strict global algebra and the strict partial algebra are Morita equivalent, or

$$\mathbb{K}_{spar}(S) \simeq_M \mathbb{K}_{sglob}(S).$$

Proof. Since the strict global prefix is an enlargement of the strict partial prefix, by Proposition 2.3.7 they are also strong Morita equivalent. Our conclusion comes from Theorem 2.5.2, which states that strong Morita equivalent inverse semigroups have Morita equivalent algebras. □

A particular case is our Corollary 2.5.3: if $S = G$ is a group

$$\mathbb{K}_{par}(G) = \mathbb{K}_{spar}(S) \simeq_M \mathbb{K}_{sglob}(S) = \mathbb{K}_{glob}(G).$$

A natural question is: **does the same relation (Morita equivalence) holds for the global and the partial algebra?** A quick answer: **no**.

We will explain by investigating the axioms of Definition 3.4.1:

(I) let (E,i) and (F,p) be the idempotents

$$(E,i) \in \mathcal{E}(Pr(S)) \iff E \subset \mathcal{R}_e,\ e = ij \in E,\ e,i,j \in \mathcal{E}(S)$$

$$(F,p) \in \mathcal{E}(\overline{Pr}(S)) \iff F \subset \mathcal{R}_f,\ f \leqslant p,\ f,p \in \mathcal{E}(S),$$

such that

$$(F,p) \leqslant (E,i) \iff e \leqslant f,\ p \leqslant i,\ fE \subseteq F.$$

The last equivalence comes from Lemma 3.3.5, and implies:

- $e \leqslant f \implies e = ef$;
- $p \leqslant i \implies p = pi$;
- $E \ni e \implies fE \ni ef$ and $ef = f \in E$.

This items implies

$$F \ni f = fpi = p(fi).$$

Hence $(F,p) \in \mathcal{E}(Pr(S))$.

(II) given $(A,s) \in \overline{Pr}(S)$, which means by definition

$$A \subset \mathcal{R}_g \text{ s.t. } g \leqslant ss^*,\ g \in \mathcal{E}(S),$$

satisfying

$$(s^*A, s^*s), (A, ss^*) \in \mathcal{E}(Pr(S)).$$

Using the definition of the partial prefix inverse semigroup

$$s^*A \subset \mathcal{R}_e,\ e = s^*si, s^*A \ni e,\ e,i \in \mathcal{E}(S)$$

$$A \subset R_f,\ f = ss^*j,\ A \ni f,\ f,j \in \mathcal{E}(S).$$

From these facts, we can conclude:

- $A \subset \mathcal{R}_g$ and $A \ni f$ implies $\forall a \in A\ g = aa^* = f$

$$\therefore A \subset \mathcal{R}_f \text{ and } f = fss^*;$$

- $A \subset \mathcal{R}_f \implies s^*A \subset \mathcal{R}_{s^*fs}$;
- $s^*A \ni e \implies e = s^*fs \iff ses^* = f$

$$\therefore A \subset \mathcal{R}_{ses^*};$$

- $\exists a \in A$ such that $s^*a = e$

$$\implies ss^*a = se \implies a = se \therefore A \ni se;$$

- $e = s^*si$

$$\implies se = si \text{ and } f = ses^* = sis^*.$$

Conclusion: $A \subset \mathcal{R}_{sis^*},\ A \ni sis^*, si$, thus $(A, s) \in Pr(S)$.

(III) by definition, if $(E, i) \in \mathcal{E}(\overline{Pr}(S))$, then

$$E \subset \mathcal{R}_e,\ e = ei,\ e, i \in \mathcal{E}(S).$$

As $(S, \overline{Pr}(S), Pr(S))$ is an O'Carroll triple

$$A \subset \mathcal{R}_p,\ p^2 = p \in A,\ p \leqslant s^*s$$

such that $sA = E$. Notice that

$$sA = spA,\ ps^*E = A \text{ and } sA \subset \mathcal{R}_{sps^*}.$$

Also, as $E \subset \mathcal{R}_e$ and $E = sA \subset \mathcal{R}_{sps^*}$, we must have

$$e = sps^* \implies sps^* = sps^*i.$$

Suppose the pair (E, sp), it satisfies:

- $spps^* = sps^* \implies (E, sp) \in \overline{Pr}(S)$;
- $(E, sp)^*(E, sp) = (ps * E, ps^*sp) = (A, p) \in \mathcal{E}(Pr(S))$;
- $(E, sp)(E, sp)^* = (E, sps^*) = (E, ei)$.

The last item should be (E, i) to conclude the enlargement, and the Morita context, but this will happen only when $e = i$, *i.e.*

$$e \leqslant i \text{ and } i \leqslant e.$$

We conclude with a diagram showing the relation among the algebras:

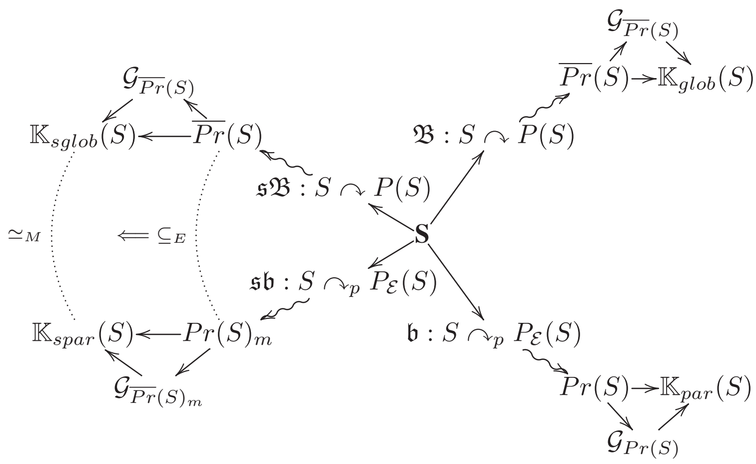


Figure 3.2: The structures induced by the Bernoulli actions

Chapter 4

The expansion of an ordered groupoid

Earlier in Chapter 1, we defined a structure called groupoid (cf. Definition 1.3.1). We have been systematically using it in our algebra computations – due to Steinberg's algebra isomorphism, our Theorem 2.5.6. In this chapter, groupoids are our main structure.

The core of Steinberg's isomorphism is: given any inverse semigroup, there exists a groupoid associated. We called this groupoid the restriction groupoid because it is defined via restricting the inverse semigroup product; precisely in Definition 1.3.8.

When one defines a partial order on a groupoid, a more robust relationship can appear. Every ordered groupoid (whose set of identities forms a semilattice) is isomorphic to an inverse semigroup. This result is the Ehresmann-Schein-Nambooripad, or ESN, Theorem (in Lawson's book [51] Theorem 8 of Chapter 4).

Gilbert proposed the expansion of an ordered groupoid in [38]; he also realized that his expansion isomorphic Lawson-Margolis-Steinberg inverse semigroup expansion when the groupoid presents the additional properties from the last paragraph.

This chapter will be devoted to reinterpreting Gilbert's construction via our Bernoulli approach. The richness of groupoids structure, and its relation with Category theory, will provide a more abstract view of our semidirect products and actions.

Before work begins, we would like to warn the reader about our convention

$$\text{domain/source}(x) := d(x) = x^{-1}x \text{ and range/target}(y) := r(y) = yy^{-1}.$$

This is the opposite of, most of, our references.

4.1 Review of ordered groupoids

We start with the definition of ordered groupoids, as in Lawson's [51] Chapter 4.

Convention: **product is thought as map composition**, *i.e.* $g : d(g) \to r(g)$ composes with $h : r(g) \to r(h)$ in $\mathcal{G}$, with product given by $hg : d(g) \to r(h)$.

Definition 4.1.1 ([51]). A groupoid $\mathcal{G}$, is called an *ordered groupoid* if it is equipped with a partial order $\leqslant$ satisfying: for $g, g', h, h' \in \mathcal{G}$ snd $e, f \in \mathcal{G}^{(0)}$

(I) it $g \leqslant h$ then $g^{-1} \leqslant h^{-1}$;

(II) if there exist the products gh and $g'h'$, the relations $g \leqslant h$ and $g' \leqslant h'$ holds, then $gg' \leqslant hh'$;

(III) if $e \leqslant d(g)$, there exists a unique element ${}_{e|}g$, the *restriction of* g *to* e, such that $d({}_{e|}g) = e$ and ${}_{e|}g \leqslant g$

(IV) if $f \leqslant r(h)$, there exists a unique element $h_{|f}$, the *corestriction of* h *to* f, such that $r(h_{|f}) = f$ and $h_{|f} \leqslant h$.

In addition, if $\mathcal{G}^{(0)}$ is a semilattice, we name it *inductive groupoid*.

Notation: $(\mathcal{G}, \leqslant)$ for ordered groupoids, and $(\mathcal{G}, \leqslant, \wedge)$ for inductive groupoids.

Our notation of restriction is different from most of the literature: thinking of an element as a function, our notation emphasizes the restriction in the domain, this way on the left hand side of the function.

Apart from the product (or composition), inductive groupoids have another binary operation.

Definition 4.1.2. ([51]) Let $(\mathcal{G}, \leqslant)$ be an ordered groupoid and $g, h \in \mathcal{G}$ such that $d(g) \wedge r(h) \in \mathcal{G}^{(0)}$ exists the *pseudo product of* g and h is

$$g \star h = ({}_{d(g)\wedge r(h)|}g)(h_{|d(g)\wedge r(h)}).$$

Before presenting its basic properties, we will present a picture exhibiting an interpretation of the pseudo product:

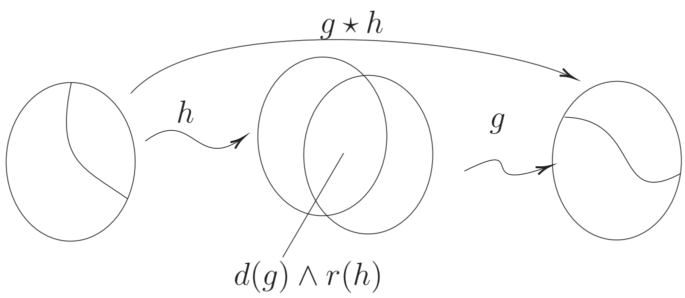


Figure 4.1: The pseudo product

The similarities with the composition of partial functions are not by chance: the restriction and corestriction carry over this ”internal behavior” to the rigid structure of a groupoid – in the sense that there are only domains and ranges.

Remark 4.1.3. A couple of properties of the pseudo product, from Lawson's book [51], Section 4.1:

(i) If there exists gh in $\mathcal{G}$, then $r(h) = d(g)$ and $d(g) = d(g) \wedge r(h) = r(h)$ implies

$$g \star h = ({}_{d(g)|}g)(h_{|r(h)}) = gh$$

by the uniqueness of axiom (III) and (IV) in Definition 4.1.1.

(ii) Given $g \in \mathcal{G}$ and $e \in \mathcal{G}^{(0)}$, the following equations holds

$$g \star e = e \wedge ({}_{d(g)|})g \text{ and } e \star g = (g_{|r(g)}) \wedge e.$$

Definition 4.1.4. ([51]) Given two ordered groupoids $(\mathcal{G}, \leqslant_{\mathcal{G}})$ and $(\mathcal{H}, \leqslant_{\mathcal{H}})$ a *homomorphism of ordered groupoids* is a map $F : \mathcal{G} \to \mathcal{H}$ that satisfies:

$$g \leqslant_{\mathcal{G}} h \implies F(g) \leqslant_{\mathcal{H}} F(h).$$

If the groupoids are inductive, F must preserve the meet structure, also.

Remark 4.1.5. This definition in some references, for instance Lawson's book [51], is termed as *ordered functor*, and *inductive functor*.

Concluding this section, we present the idea of the ESN Theorem's correspondence.

Theorem 4.1.6 (Ehresmann-Schein-Nambooripad [51]). *The category (as in Definition 1.3.2) of inverse semigroups and homomorphisms of inverse semigroups is isomorphic to the category (as in Definition 1.3.2) of inductive groupoids and homomorphisms of inductive groupoids.*

The main points of the demonstration are:

- If S is an inverse semigroup, then $\mathcal{G}_S$ is an inductive groupoid, where the restriction and corestriction are
$${}_{e|}g := ge \text{ and } h_{|f} = fh.$$

- By the other hand, for a given inductive groupoid $\mathcal{G}$, it becomes an inverse semigroup with (global) product as the pseudo product $\star$, *i.e.* $(\mathcal{G}, \star)$ is an inverse semigroup, whose idempotent set is $\mathcal{G}^{(0)}$.

4.2 Partial actions of ordered groupoids

When we presented some basics results in Chapter 1, we defined groupoid actions on sets via a (moment) map , now this construction will be of great importance. Before defining the Bernoulli action, let's review, reinterpret in terms of symmetries and (what we will term) by fibrations. Also we will exhibit the geometric intuition behind this actions, following Ibort-Rodriguez [44] Part I - Sections 4 and 5. Finally we will define partial groupoid actions with references: Gilbert [38] Section 4; Hollings [43] Chapter 8, , Bagio-Flores-Paques [6] Section 2, Dirceu [5] Section 2, and Nystedt [65] Section 4.

Just as groups and inverse semigroups, groupoids also act on sets by symmetries. For a given set X, the ordered groupoid version of the inverse semigroup $\mathcal{I}_{isg}(X)$, of partial symmetries (cf. Example 1.2.3-(3) where it was denoted by $\mathcal{I}(X)$), as defined by Gilbert ([38] Section 4).

Definition 4.2.1 ([38])**.** Let X be a set. The *symmetric groupoid* associated to X is a inductive groupoid $\mathcal{I}_{gpd}(X)$ with structure:

- elements are all the bijections between subsets of X;
- units the identity maps of subsets of X;
- for $g \in \mathcal{I}_{gpd}(X)$ the source and target maps are $d(g) = 1_{dom(g)}$ and $r(g) = 1_{ran(g)}$;
- the product is the map composition, when defined;
- the ordering is given by restriction of mappings, *i.e.* $g \leqslant h \iff \text{dom}(g) \subseteq \text{dom}(h)$;
- for $g \in \mathcal{I}_{gpd}(X)$, its restriction to the identity 1_A is ${}_{1_A|}g = g_{|A}$, or the restriction of the map g to the set $A \subset X$;
- for $h \in \mathcal{I}_{gpd}(X)$, its corestriction to the identity 1_B is $h_{1_A|} = h_{|h^{-1}(B)}$, or the restriction of the map h to the preimage set $h^{-1}(B) \subset X$;
- the meet in its identities is $1_A \wedge 1_B := 1_{A\cap B}$, for $A, B \subset X$.

Notice that, via the ESN Theorem: $\mathcal{I}_{isg}(X) \simeq \mathcal{I}_{gpd}(X)$.

This construction provides a notion of groupoid action, in the sense of what we have already seen.

Definition 4.2.2 ([38])**.** Given and ordered groupoid $\mathcal{G}$ and a set X, a *groupoid action* (via symmetries, or automorphisms) of $\mathcal{G}$ on X is an ordered groupoid homomorphism $\theta : \mathcal{G} \to \mathcal{I}_{gpd}(X)$ such that $X = \bigcup\limits_{e\in\mathcal{G}^{(0)}} \text{dom}(\theta_e)$. If the groupoid is inductive, the homomorphism must preserve the meet.

The previous definition of an action agrees with our intuition about how an structure should act on a set by bijections. But this wasn't our first definition, which we restate for correlation. In the Definition 1.3.4 we said:

Let $\mathcal{G}$ be a groupoid, X a set and a map $\rho : X \to \mathcal{G}^{(0)}$. Define the set $\mathcal{G}_d\times_\rho X := \{(g,x) \in \mathcal{G} \times X; d(g) = \rho(x)\}$. An *action via moment map* ρ of $\mathcal{G}$ on X, is a map from $\mathcal{G}_d\times_\rho X := \{(g,x) \in \mathcal{G} \times X; d(g) = \rho(x)\}$ to X given by $\mathcal{G}_d\times_\rho X \to X$, with $(g,x) \mapsto \theta_g(x)$ such that:

(I) $\theta_{\rho(x)}(x) = x$ for all $x \in X$;

(II) for $x \in X$, and $g \in \mathcal{G}$ such that there exists $\theta_g(x)$, we have $\rho(\theta_g(x)) = r(g)$;

(III) if $(h,x) \in \mathcal{G}_d\times_\rho X$ and $(g,h) \in \mathcal{G}^{(2)}$ then $(gh,x), (g, \theta_h(x)) \in \mathcal{G}_d\times_\rho X$ and $\theta_g(\theta_h(x)) = \theta_{gh}(x)$.

By notation $(\rho,\theta) : \mathcal{G} \curvearrowright X$.

Let us simplify: we will refer to "actions via moment map" as **fibred actions**, from now on. Furthermore, notice that we can replace the source by the target and target by source, and the definition continues to make sense.

In fact, a fibred action is in one-to-one correspondence with an action via symmetries. Indeed, if $\mathcal{G}$ is a groupoid (without order), we have that:

<u>From fibred actions to actions by symmetries</u>: Let $(\rho,\theta) : \mathcal{G} \curvearrowright X$ be a fibred action of a groupoid on a set. For $g \in \mathcal{G}$ define the sets $D(g) := \{x \in X;\ \rho(x) = d(g)\}$ and $R(g) := \{y = \theta_g(x);\ \rho(y) = r(g)\}$ and the bijection $\theta_g : D(g) \to R(g)$ with $x \mapsto \theta_g(x)$. Thus $\theta : \mathcal{G} \to \mathcal{I}_{gpd}(X)$ ruled by $g \mapsto \theta_g$ is an action (by symmetries).

<u>From actions by symmetries fo fibred actions</u>: If $\theta : \mathcal{G} \curvearrowright \mathcal{I}_{gpd}(X)$ is an action and $e \in \mathcal{G}^{(0)}$, we define a moment map for all $x \in \mathrm{dom}(\theta_e)$ by $\rho(x) := e$. So the rule $g \mapsto \theta_g$ defines a map from $\{(g,x) \in \mathcal{G} \times X; x \in \mathrm{dom}(\theta_g)\}$ to X; as $\mathrm{dom}(\theta_g) \subseteq \mathrm{dom}(\theta_{d(g)})$ the element (g,x) satisfies $d(g) = g^{-1}g = \rho(x)$. Finally the pair we seek is (ρ,θ).

This relation appears in the next picture (inspired in Figure 4.4 from [44]), where the meaning of "fibred" in "fibred action" is now evident:

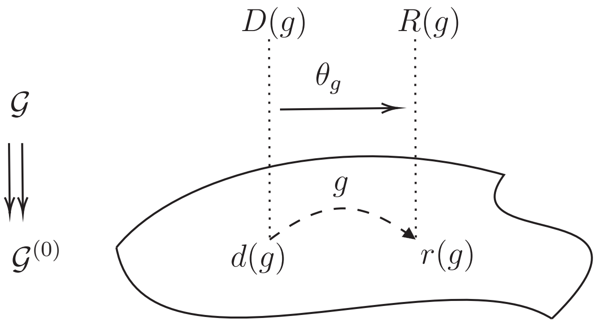


Figure 4.2: The fibred action

As we can see, fibred actions are a collection (or bundle) of bijective maps over the fibers defined by the source and target maps.

It the spirit of inverse semigroup partial actions, Gilbert ([38]) furnished an equivalent definition partial homomorphism – cf. the inverse semigroup case in Definition 3.1.12.

Let $\mathcal{G}$ be ordered groupoid and $\mathcal{H}$ be an inductive groupoid. A *partial homomorphism* of ordered groupoids is a map $\theta : \mathcal{G} \to \mathcal{H}$ such that:

(I) $\theta_g^{-1} = \theta_{g^{-1}}$;

(II) for $g, h \in \mathcal{G}$ satisfying $g \leqslant h$, we have $\theta_g \leqslant \theta_h$;

(III) if the product gh exists then $\theta_g \star \theta_h \leqslant \theta_{(gh)}$.

Clearly, if θ is homomorphism then $\theta_g \star \theta_h = \theta_{(gh)}$, where the star product is the pseudo product as defined in Definition 4.1.2 .

Definition 4.2.3 ([38])**.** A *partial action* of an ordered groupoid $\mathcal{G}$ on a set X is a partial homomorphism $\theta : \mathcal{G} \to \mathcal{I}_{gpd}(X)$. In the case θ is a homomorphism, then it will be called a *global action* of ordered groupoids.

Notation: A partial action will be denoted by $\theta : \mathcal{G} \curvearrowright_p X$ and a global action by $\theta : \mathcal{G} \curvearrowright X$

Although very elegant, this definition does not indicate – at first glance –what is happening with the subsets of X.

The equivalence we are about to present is a combination of results taken from Bagio-Flores-Paques [6] Section 2, Dirceu [5] Section 2, and Nystedt [65] Section 4.

Proposition 4.2.4 ([6][5][65])**.** Let $\mathcal{G}$ be an ordered groupoid and X be a set. Then a mapping $\theta : \mathcal{G} \to \mathcal{I}_{gpd}(X)$ is a partial action of the ordered groupoid $\mathcal{G}$ on X if, and only if, defines a pair $(\{\theta_g\}_{g\in\mathcal{G}}, \{D_g\}_{g\in\mathcal{G}})$ where

(i) for each $g \in \mathcal{G}$ the map $\theta_g : D_{g^{-1}} \subseteq X \to D_g \subseteq X$ is a bijection;

(ii) $X = \bigcup_{e\in\mathcal{G}^{(0)}} D_e$;

(iii) if $e \in \mathcal{G}^{(0)}$ the map θ_e is the identity in its domain D_e;

(iv) given $g, h \in \mathcal{G}$ such that $g \leqslant h$, then $D_{g^{-1}} \subseteq D_{h^{-1}}$ and $\theta_{h|D_{g^{-1}}} = \theta_g$;

(v) if $g \in \mathcal{G}$, then $D_g \subseteq D_{r(g)}$;

(vi) if the product gh exists then $\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$; and $\theta_g(\theta_h(x)) = \theta_{gh}(x)$, for $x \in \theta_{h^{-1}}(D_h \cap D_{g^{-1}})$.

In addition: θ is a *global action* if, and only if, $D_g = D_{r(g)}$ for all $g \in \mathcal{G}$.

Concluding this section, we present a fundamental idea that we use very often in the next section. Gilbert observed that global actions could induce partial action by restriction of domains.

Indeed, let $\overline{\theta} : \mathcal{G} \curvearrowright X$ a global groupoid ordered action on a set X. Suppose $Y \subseteq X$, then there is a partial action $\theta : \mathcal{G} \curvearrowright_p Y$, where: for each $g \in \mathcal{G}$

- the domain is $\mathrm{dom}(\theta_g) := \overline{\theta}_{g^{-1}}(\mathrm{ran}(\overline{\theta}_g) \cap Y) \cap Y$;
- the image is $\mathrm{ran}(\theta_g) = \mathrm{ran}(\overline{\theta}_g) \cap Y$;
- the map is $\theta_g := \overline{\theta}_{g|\mathrm{dom}(\theta_g)}$.

4.3 The Gilbert expansion and the Bernoulli actions

Our objective in this section is to present the Gilbert expansion of an ordered groupoid in two ways. First, the original version. Then we present it via our Bernoulli approach, *i.e.* we will define an intrinsic action of the ordered groupoid on a poset, and next use groupoid theory to construct a new groupoid – using this action and the poset.

The reader should have in mind the approach used for the prefix expansion in Chapter 4. As before, our strategy is:

1. To define sets related to the $\mathcal{R}$ classes of an inverse semigroup endowed with a partial order;

2. To construct a global fibred action and restrict it to a partial fibred action;

3. To present crossed products in the realm of ordered groupoids.

We start recalling the Birget-Rhodes expansion of an ordered groupoid, that we will call **Gilbert's expansion**. Some of his construction needed to define the expansion will guide our work.

4.3.1 Gilbert's expansion

Let $\mathcal{G}$ be an ordered groupoid. Given $e, f \in \mathcal{G}^{(0)}$, in [38] Gilbert defined the following sets:

- $\text{costar}(e) := \{g \in \mathcal{G};\ r(g) = e\}$,
- $star(f) := \{h \in \mathcal{G};\ d(h) = f\}$,
- $\mathcal{F}_e(\text{costar}(e)) := \{U \subset \text{costar}(e); \mid U \mid < \infty,\ U \ni e\}$;
- $U_{|f} := \{g_{|f};\ g \in U\} \in \mathcal{F}_f(\text{costar}(f))$ for each $U \in \mathcal{F}_e(\text{costar}(e))$ and $f \in \mathcal{G}^{(0)}$ such that $f \leqslant e$;
- $\mathcal{F}_*(\mathcal{G}) := \bigcup_{e \in \mathcal{G}^{(0)}} \mathcal{F}_e(\text{costar}(e))$.

Then he showed the following result.

Proposition 4.3.1 ([38])**.** The set

$$\mathcal{G}^{GBR} := \{(U, g);\ U \in \mathcal{F}_*(\mathcal{G}),\ g \in U\}$$

is an ordered groupoid with structure:

<u>units</u>: $\mathcal{G}^{BR^{(0)}} =: \{(E, e) \in \mathcal{G}^{GBR};\ e \in \mathcal{G}^{(0)}\}$;

<u>product</u>: $(U, g)(V, h) := (U, gh)$ if there exists $gh \in \mathcal{G}$ and $gV = U$;

<u>inverse</u>: $(U, g)^{-1} := (g^{-1}U, g^{-1})$;

<u>source, target</u>: $d(U, g) := (g^{-1}U, d(g))$ and $r(U, g) := (U, r(g))$;

<u>ordering</u>: $(U, g) \leqslant (V, h)$ if, and only if, $g \leqslant h$ and $U \supseteq V_{|r(g)}$.

In order to avoid Gilbert's notations in the next sections, notice that

$$(U, g) \in \mathcal{G}^{GBR} \iff \mid U \mid < \infty,\ U \subset \text{costar}(r(g)),\ U \ni r(g), g.$$

The pseudo product in $\mathcal{G}^{GBR}$ takes a little longer to define.

Lemma 4.3.2 ([38]). *Let $f \in \mathcal{G}^{(0)}$, $V \subset costar(f)$ and $g \in \mathcal{G}$ such that $l := d(g) \wedge f$ exists. We define the set:*

(i) $g \star V := \{g \star v;\ v \in V\} = \{({}_{l|}g)(v_{|l});\ v \in V\}$.

For $(E,e),(F,f) \in \mathcal{G}^{GBR^{(0)}}$ there exists$(E,e)\wedge(F,f)$ if, and only if, there exists $e\wedge f$, which is ef, and

(ii) $(E,e) \wedge (F,f) = (E_{|ef} \cup F_{|ef}, ef)$.

For $(U,g) \in \mathcal{G}^{BR}$ and $(E,e) \in \mathcal{G}^{GBR^{(0)}}$ the restriction and corestriction, respectively, are

(iii) ${}_{(E,e)|}(U,g) = ({}_{e|}gE, {}_{e|}g)$ *and* $(U,g)_{|(E,e)} = (E, g_{|e})$*, supposing the unit satisfy the axiom (III) and (IV) of Definition 4.1.1*

For all $(U,g),(V,h) \in \mathcal{G}^{GBR}$, the pseudo product $(U,g)\star(V,h)$ exists if, and only if, $g \star h$ exists, and

(iv) $(U,g)\star(V,h) = (U_{|r({}_{l|}g)} \cup g \star V, g \star h)$*, where* $l = d(g) \wedge r(h)$.

Corollary 4.3.3 ([38]). *If $\mathcal{G}$ is an inductive groupoid, then $\mathcal{G}^{GBR}$ is also inductive.*

After the corollary, we will explain all these sets, anticipating: we are only rewriting some of the inverse semigroup constructions in ”groupoid terms”.

Understanding what we have just stated becomes easier when we compare it with inverse semigroups. If we have an inverse semigroup S, we can form the restricted groupoid $\mathcal{G}_S$. Notice that

$$\mathcal{G}_S{}^{(0)} = \mathcal{E}(S) \implies \mathcal{G}_S \text{ is inductive .}$$

In this case, via the ESN Theorem 4.1.6,

- $(\mathcal{G}_S{}^{GBR}, \star)$ is the strict partial prefix inverse semigroup $Pr(S)_m$ (cf. 3.3.4), and
- $(U,g)\star(V,h) = (r(gh)U \cup gV, gh) = (ghh^*g^*U \cup gV, gh)$.

This relation teaches us how to interpret some identities, for instance: for ordered groupoids the meet of identities is

$$(E,e) \wedge (F,f) = (E_{|ef} \cup F_{|ef}, ef),$$

if the groupoid is inductive. By Lemma 3.2.1, the ESN Theorem 4.1.6, and the fact that $ef = fe$ (because they are idempotents)

$$(E,e) \wedge (F,f) = (E_{|ef} \cup F_{|ef}.ef) = (feE \cup efF, ef) = (fE \cup eF, ef).$$

And the last expression is the product $(E,e)(F,f)$ in the strict prefix inverse semigroup.

4.4 Bernoulli groupoid actions

Based on previous ideas, we will develop the Bernoulli approach for ordered groupoids. Fix an ordered groupoid; for each $e \in \mathcal{G}^{(0)}$ we define the set

$$P_e(\mathcal{G}) := \{A \subset \mathcal{G};\ \mid A \mid < \infty,\ r(a) = e\,\forall a \in A\}.$$

Then we define the following sets

$$P(\mathcal{G}) := \bigcup_{e \in \mathcal{G}} P_e(\mathcal{G})$$
$$P_{\mathcal{U}}(\mathcal{G}) := \{A \in P(\mathcal{G}); A \cap \mathcal{G}^{(0)} \neq \emptyset\}.$$

A few words about these sets:

- if $A \in P(\mathcal{G})$, then A is a finite subset of costar(e), for an unit e;
- for $A \in P_{\mathcal{U}}(\mathcal{G})$ such that $r(a) = e$ for all $a \in A$, then $A \ni e$;
- if $A, B \in P(\mathcal{G})$ are such that $A \subset$ costar(e) and $B \subset$ costar(f), for $e, f \in \mathcal{G}^{(0)}$, then

$$\forall x \in A \cap B \text{ the equality } \ e = r(x) = f \text{ holds .}$$

We endow $P(\mathcal{G})$, and consequently $P_{\mathcal{U}}(\mathcal{G})$, with a partial order: for $A, B \in P(\mathcal{G})$ such that $A \subset$ costar(e) and $B \subset$ costar(f) we define

$$A \leqslant B \iff e \leqslant f \text{ and } B_{|e} \subseteq A, \text{ where } B_{|e} := \{b_{|e};\ b \in B\}.$$

Once we have the sets, we move to the global (fibred) action, which takes the order into account. Its definition is similar to the regular fibred action; we will follow Lawson [51] Chapter 8 in Section 8.4, or Miller [62] Chapter 5 and Subsection 5.3.1 – her notations and exposition are closer to what we present.

Definition 4.4.1 ([62])**.** Let $\mathcal{G}$ be a groupoid, and $(X, \leqslant)$ be a poset. An *ordered fibred groupoid action* of $\mathcal{G}$ on the poset X, is a pair composed by

(ordered) moment: an ordered preserving map $\rho : X \to \mathcal{G}^{(0)}$,

(ordered) action: a map $\theta : \mathcal{G}_d \times_\rho X := \{(g, x) \in \mathcal{G} \times X; \rho(x) \leqslant d(g)\} \to X$ with $(g, x) \mapsto \theta_g(x)$ where for all $g \in \mathcal{G}$ the map θ_g preserves order,

such that:

(I) $\theta_{\rho(x)}(x) = x$ for all $x \in X$;

(II) if there exists $\theta_g(x)$ and $\rho(x) = d(g)$, we have $\rho(\theta_g(x)) = r(g)$;

(III) if $(h, x) \in \mathcal{G}_d \times_\rho X$ and $(g, h) \in \mathcal{G}^{(2)}$ then $(gh, x), (g, \theta_h(x)) \in \mathcal{G}_d \times_\rho X$ and $\theta_g(\theta_h(x)) = \theta_{gh}(x)$.

The other construction that we need to adapt to accomplish the structure of a poset is the groupoid from Definition 4.2.1 and the Proposition 4.2.4.

Following the Lemma 5.3.1 from Miller [62] we have the next couple results.

Definition 4.4.2. ([62]) Let $(X, \leqslant)$ be a poset. The *symmetric groupoid* associated to $(X, \leqslant)$ is an ordered groupoid $\mathcal{I}_{gpd}(X, \leqslant)$ with structure:

- elements are all the order preserver bijections between principal order ideals of $(X, \leqslant)$;
- units the identity maps of principal order ideal of X;
- for $g \in \mathcal{I}_{gpd}(X, \leqslant)$ the source and target maps are $d(g) = 1_{dom(g)}$ and $r(g) = 1_{ran(g)}$;
- given $g : \{x;\ x \leqslant e\} \to \{y;\ y \leqslant f\}$ and $h : \{x;\ x \leqslant e`\} \to \{y;\ y \leqslant f`\}$ the product is the map composition $gh : \{x; x \leqslant e\} \to \{y;\ y \leqslant f\}$ which is defined when $f' = e$;
- the ordering is given by the restriction of mappings, *i.e.* $g \leqslant h \iff \text{dom}(g) \subseteq \text{dom}(h)$;
- for $g \in \mathcal{I}_{gpd}(X, \leqslant)$, its restriction to the identity 1_A is ${}_{1_A|}g = g_{|A}$, or the restriction of the map g to the principal order ideal $A \subset X$;
- for $h \in \mathcal{I}_{gpd}(X, \leqslant)$, its corestriction to the identity 1_B is $h_{1_A|} = h_{|h^{-1}(B)}$, or the restriction of the map h to the preimage set $h^{-1}(B) \subset X$.

As Miller demonstrates in [62] (cf. Lemma 5.3.1), the set $\mathcal{I}_{gpd}(X, \leqslant)$ is indeed an ordered groupoid. Moreover, if $(X, \leqslant)$ is a meet semilattice then the groupoid $\mathcal{I}_{gpd}(X, \leqslant)$ is an inductive groupoid.

Using the previous construction we can state a definition in a similar fashion to Proposition 4.2.4, but now we have an action on a poset.

<u>Definition</u> 4.4.3. Let $\mathcal{G}$ be an ordered groupoid and $(X, \leqslant)$ be a poset. Then an order preserver mapping $\theta : \mathcal{G} \to \mathcal{I}_{gpd}(X, \leqslant)$ is a partial action of the ordered groupoid $\mathcal{G}$ on $(X, \leqslant)$ if, and only if, defines a pair $(\{\theta_g\}_{g \in \mathcal{G}}, \{D_g\}_{g \in \mathcal{G}})$ where

(i) for each $g \in \mathcal{G}$ the map $\theta_g : D_{g^{-1}} \subseteq X \to D_g \subseteq X$ is a order preserver bijection between principal order ideals;

(ii) $X = \bigcup_{e \in \mathcal{G}^{(0)}} D_e$;

(iii) if $e \in \mathcal{G}^{(0)}$ the map θ_e is the identity in its domain D_e;

(iv) given $g, h \in \mathcal{G}$ such that $g \leqslant h$, then $D_{g^{-1}} \subseteq D_{h^{-1}}$ and $\theta_{h|D_{g^{-1}}} = \theta_g$;

(v) if $g \in \mathcal{G}$, then $D_g \subseteq D_{r(g)}$;

(vi) if the product gh exists then $\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$; and $\theta_g(\theta_h(x)) = \theta_{gh}(x)$, for $x \in \theta_{h^{-1}}(D_h \cap D_{g^{-1}})$.

In addition: θ is a *global action* if, and only if, $D_g = D_{r(g)}$ for all $g \in \mathcal{G}$.

As we will consider only ordered fibred actions on posets **we will refer to these action by fibred actions**; and use the same notation previously defined.

Definition 4.4.4. The *global ordered fibred Bernoulli action* of $\mathcal{G}$ on $P(\mathcal{G})$ is given by the pair

moment map: $\varepsilon : P(\mathcal{G}) \to \mathcal{G}^{(0)}$ with

$$A \subset \text{costar}(e) \in P(\mathcal{G}) \mapsto \varepsilon(A) = e,$$

where $e \in \mathcal{G}^{(0)}$;

action map: $\mathfrak{B} : \mathcal{G}_d \times_\varepsilon P(\mathcal{G}) = \{(g, A) \in \mathcal{G} \times P(\mathcal{G}); \varepsilon(A) \leqslant d(g)\} \to P(\mathcal{G})$ defined by

$$\mathfrak{B}(g, A) = \mathfrak{B}_g(A) := gA.$$

Notation: $(\varepsilon, \mathfrak{B}) : \mathcal{G} \curvearrowright P(\mathcal{G})$.

Indeed the above pair defines an ordered fibred action:

- moment preserves order: first observe that, as a result of observations made, ε is well defined; also given $A, B \in P(\mathcal{G})$ such that $A \subset \text{costar}(e)$ and $B \subset \text{costar}(f)$, by our previous definition $A \leqslant B \implies e \leqslant f$ so $\varepsilon(A) \leqslant \varepsilon(B)$.

- action is well defined: suppose $(g, A) \in A \in \mathcal{G}_d \times_\varepsilon P(\mathcal{G})$ such that for all $a \in A$ it holds that $aa^{-1} = r(a) = e$, $e \in \mathcal{G}^{(0)}$; then $ga \in gA$ satisfies

$$(ga)(ga)^{-1} = gaa^{-1}g^{-1} = geg^{-1}.$$

Hence this equality holds for all $b \in gA$, that is, $bb^{-1} = geg^{-1}$, and clearly $\mid gA \mid < \infty$. We conclude that $gA = \mathfrak{B}_g(A) \in P(\mathcal{G})$.

- verification of action's axioms: suppose that exists gA with A contained in costar(e), for $e \in \mathcal{G}^{(0)}$, notice that
  - $\mathfrak{B}_{\varepsilon(A)}(A) = \mathfrak{B}_e(A) = eA = A$, as for all $a \in A$ is valid $a = (aa^{-1})a = ea$;
  - if $e = \varepsilon(A) = d(g) = g^{-1}g$, then, by last paragraph's computation, we conclude

$$\varepsilon(\mathfrak{B}_g(A)) = \varepsilon(gA) = geg^{-1} = g(g^{-1}g)g^{-1} = gg^{-1} = r(g);$$

  - assume that the product hg in $\mathcal{G}$ is defined; by basic properties of groupoids, then $r(g) = d(h)$ and $d(hg) = d(g)$, combining these facts with last item,

$$geg^{-1} = \varepsilon(gA) \leqslant g(g^{-1}g)g^{-1} = gg^{-1} = r(g) = d(h).$$

    So $h(gA)$ exists and, as $\varepsilon(A) \leqslant d(g) = d(hg)$, we have

$$h(gA) = (hg)(A) \implies \mathfrak{B}_h(\mathfrak{B}_g(A)) = \mathfrak{B}_{hg}(A);$$

  - given $g \in \mathcal{G}$, suppose $A \subset$ costar(e) and $B \subset$ costar(f) such that $A \leqslant B$ in $P(\mathcal{G})$ and suppose there exist $\mathfrak{B}_g(A)$ and $\mathfrak{B}_g(B)$; by definition

$$A \leqslant B \iff e \leqslant f \text{ and } B_{|e} \subseteq A,$$

    and notice that

$$e \leqslant f \implies geg^{-1} \leqslant gfg^{-1}$$

    since the action of g on A and B is defined, also

$$geg^{-1}, gfg^{-1} \leqslant r(g),$$

    and

$$B_{|e} \subset A \implies (gB)_{|geg^{-1}} \subset gA.$$

    The last implication holds because

$$gb \in (gB)_{|e} \iff e \leqslant r(gb) = gbb^{-1}bg^{-1} = geg^{-1}.$$

    Hence $gA \leqslant gB$, which means that $\mathfrak{B}_g(A) \leqslant \mathfrak{B}_g(B)$.

In conclusion: $(\varepsilon, \mathfrak{B}) : \mathcal{G} \curvearrowright P(\mathcal{G})$ *is* an fibred action.

Once we have a fibred groupoid action, we can define an (ordered) action by symmetries as follows: for each $g \in \mathcal{G}$ let

$$\mathfrak{B}_g : \overline{D}_{g^{-1}} \to \overline{D}_g$$

be the map where

domain/range: the sets where the maps is defined are

$$\overline{D}_{g^{-1}} := \{A \in P(\mathcal{G});\ \varepsilon(A) \leqslant d(g)\} \text{ and } \overline{D}_g := \{B = gA;\ \varepsilon(B) \leqslant r(g)\};$$

map's definition: the map is given by

$$A \in \overline{D}_{g^{-1}} \mapsto \mathfrak{B}_g(A) = gA \in \overline{D}_g.$$

Notice that the following properties are satisfied:

- $\overline{D}_{g^{-1}} = \overline{D}_{g^{-1}g}$ for each $g \in \mathcal{G}$;
- suppose $B \in \overline{D}_{g^{-1}}$ and $A \in P(\mathcal{G})$ such that $A \subset \text{costar}(e)$ and $B \subset \text{costar}(f)$, for $e, f \in \mathcal{G}^{(0)}$, subjected to $A \leqslant B$. By definition

  $$A \leqslant B \iff e \leqslant f \text{ and } B_{|e} \subset A;$$

  since $\varepsilon(A) = e$ and $\varepsilon(B) = f \leqslant d(g)$, we have $A \in \overline{D}_{g^{-1}}$;
- $A, B \in \overline{D}_{g^{-1}}$ with $A \subset \text{costar}(e)$, $B \subset \text{costar}(f)$, and $A \leqslant B$; by definition $e, f \leqslant d(g)$. Notice that

  $$e \leqslant f \implies geg^{-1} \leqslant gfg^{-1},$$

  also

  $$geg^{-1}, gfg^{-1} \leqslant r(g),$$

  and

  $$B_{|e} \subset A \implies (gB)_{|geg^{-1}} \subset gA.$$

  The last implication holds because

  $$gb \in (gB)_{|e} \iff e \leqslant r(gb) = gbb^{-1}bg^{-1} = geg^{-1}.$$

  Hence $gA \leqslant gB$;
- clearly $P(\mathcal{G}) = \bigcup_{e \in \mathcal{G}^{(0)}} \overline{D}_e$ and $\mathfrak{B}_g^{-1} = \mathfrak{B}_{g^{-1}}$;

- and finally, since $r(g) = gg^{-1}$ we have that $\overline{D}_g = \overline{D}_{r(g)}$.

Summarizing: the ordered groupoid $\mathcal{G}$ acts globally on the set $P(\mathcal{G})$. This action, from one point of view is the fibred action $(\varepsilon, \mathfrak{B}) : \mathcal{G} \curvearrowright P(\mathcal{G})$, and by the other hand the action by automorphisms $\mathfrak{B} : \mathcal{G} \curvearrowright \mathcal{I}_{gpd}(P(\mathcal{G}))$. We will refer then as:

$(\varepsilon, \mathfrak{B}) : \mathcal{G} \curvearrowright P(\mathcal{G})$ the *global fibred Bernoulli action*, as we've already stated, and

$\mathfrak{B} : \mathcal{G} \curvearrowright \mathcal{I}_{gpd}(P(\mathcal{G}))$ the *global Bernoulli action by symmetries/automorphisms*.

Notice that we can get intuition from previous chapters; then we can return, from the equivalence of actions to fibred actions.

Our next steps are: first, we will restrict the global action to a partial action on $P_{\mathcal{U}}(\mathcal{G})$; next, we will impose conditions on the domains of both actions, and what we will obtain are the groupoid versions of the strict actions from inverse semigroups – cf. the discussion in Section 4.3 - "Bernoulli semigroup actions".

We recall the definitions:

– the sets are

$$P(\mathcal{G}) := \{A \subset \mathcal{G}; \mid A \mid < \infty,\ \exists e \in \mathcal{G}^{(0)} \text{ s.t. } \forall a \in A,\ r(a) = e\}$$
$$P_{\mathcal{U}}(\mathcal{G}) = \{A \in P(\mathcal{G}); A \cap \mathcal{G}^{(0)} \neq \emptyset\};$$

– for each $g \in \mathcal{G}$, the global Bernoulli action by automorphisms is given by

$$\mathfrak{B}_g : \overline{D}_{g^{-1}} = \{A \in P(\mathcal{G});\ e = \varepsilon(A) \leqslant d(g)\} \to \overline{D}_g = \{B = gA;\ f = \varepsilon(B) \leqslant r(g)\},$$
$$A \mapsto gA$$

where $e, f \in \mathcal{G}^{(0)}$ and $A \subset \text{costar}(e)$ and $B \subset \text{costar}(f)$.

We will omit the next computations because they are very similar to the inverse semigroup case (Section 4-3), indeed:

1st) <u>The restriction of the global action</u>: let $g \in \mathcal{G}$

- the map is $\mathfrak{b}_g := (\mathfrak{B}_g)_{|D_g}$, where
- $\text{dom}(\mathfrak{b}_g) := D_{g^{-1}} = \mathfrak{B}_{g^{-1}}(\overline{D}_g \cap P_{\mathcal{U}}(\mathcal{G})) \cap P_{\mathcal{U}}(\mathcal{G})$, which can be written explicitly as

$$D_{g^{-1}} = \{A \in P(\mathcal{G});\ \forall a \in A\, r(a) = e,\ A \ni e, eg^{-1}, e \in \mathcal{G}^{(0)}\},$$

and

- $\operatorname{ran}(\mathfrak{b}_g) := D_g = \overline{D}_g \cap P_{\mathcal{U}}(\mathcal{G})$, or

$$D_g = \{B \in P(\mathcal{G});\ \forall b \in B\, r(b) = f, B \ni gf, gfg^{-1}, f \in \mathcal{G}^{(0)}\}.$$

Summarizing, we gained the partial action $\mathfrak{b} : \mathcal{G} \curvearrowright_p \mathcal{I}_{gpd}(P_{\mathcal{U}}(\mathcal{G}))$, or for each $g \in \mathcal{G}$

$$\begin{aligned} \mathfrak{b}_g : D_{g^{-1}} = \{A;\ A \ni e, eg^{-1}, e \in \mathcal{G}^{(0)}\} &\to D_g = \{B;\ B \ni gf, gfg^{-1}, f \in \mathcal{G}^{(0)}\}. \\ A &\mapsto gA \end{aligned}$$

2nd) Imposing conditions on domains: for each $g \in \mathcal{G}$

- let $\overline{D}_g^m \subset \overline{D}_g$ be the subset $\overline{D}_g^m = \{A \subset \overline{D}_g; A \subset \operatorname{costar}(d(g))\}$. This means that $A \in \overline{D}_g^m$ if and only if $aa^{-1} = r(a) = d(g)$ for every $a \in A$. Then we have the global action

$$\begin{aligned} \mathfrak{s}\mathfrak{B}_g : \overline{D}_{g^{-1}}^m = \{A;\ A \subset \operatorname{costar}(d(g))\} &\to \overline{D}_g^m = \{B;\ B \subset \operatorname{costar}(r(g))\}. \\ A &\mapsto gA \end{aligned}$$

  This global action will be denoted by $\mathfrak{s}\mathfrak{B} : \mathcal{G} \curvearrowright \mathcal{I}_{gpd}(P(\mathcal{G}))$.

- the restriction of previous global action will differ from $\mathfrak{b} : \mathcal{G} \curvearrowright_p \mathcal{I}_{gpd}(P_{\mathcal{U}}(\mathcal{G}))$ by the imposition upon the units, *i.e.*

$$\begin{aligned} \mathfrak{s}\mathfrak{b}_g : D_{g^{-1}}^m = \{A;\ A \ni d(g), g^{-1}\} &\to D_g^m = \{B;\ B \ni r(g), g\}. \\ A &\mapsto gA \end{aligned}$$

  This partial action will be denoted by $\mathfrak{s}\mathfrak{b} : \mathcal{G} \curvearrowright_p \mathcal{I}_{gpd}(P_{\mathcal{U}}(\mathcal{G}))$.

Before compile all this information, we would like to make one last map restriction: let

$$\epsilon := \varepsilon_{|P_{\mathcal{U}}(\mathcal{G})} \text{ be the map } A \subset \operatorname{costar}(e) \mapsto \epsilon(A) = e \in \mathcal{G}^{(0)}$$

called the *restricted moment*.

Remark 4.4.5. Let A be an element in $P(\mathcal{G})$ such that $\varepsilon(A) = e$ and $A \ni e, eg^{-1}$. If $e = g^{-1}fg$, where $f \leqslant gg^{-1}$, then

- $eg^{-1}g = g^{-1}fgg^{-1}g = g^{-1}fg = e \implies e \leqslant g^{-1}g$;

- $g^{-1}f = g^{-1}fgg^{-1} = eg^{-1}$.

In particular, in D_g if we take $f = geg^{-1}$, the set gA satisfies

- $\varepsilon(gA) = f$,
- $f \in gA$ and
- $fg = geg^{-1}g = ge \in gA$.

Hence the domain and range of the map $\mathfrak{b}$ have the same formation rule.

To present the definitions adequately, we have the following definition.

Definition 4.4.6. Let $\mathcal{G}$ be an ordered groupoid and consider the sets $P(\mathcal{G})$ and $P_{\mathcal{U}}(\mathcal{G})$. From the global Bernoulli action $\mathfrak{B} : \mathcal{G} \curvearrowright \mathcal{I}_{gpd}(P(\mathcal{G}))$ we define the following three actions by automorphisms and their corresponding fibred actions:

(I) the *strict global Bernoulli action* $\mathfrak{sB} : \mathcal{G} \curvearrowright \mathcal{I}_{gpd}(P(\mathcal{G}))$, or for each $g \in \mathcal{G}$

$$\mathfrak{sB}_g : \overline{D}^m_{g^{-1}} := \{A \in P(\mathcal{G});\ A \subset \text{costar}(d(g))\} \to \overline{D}^m_g := \{B \in P(\mathcal{G});\ B \subset \text{costar}(r(g))\};$$
$$A \mapsto gA$$

(I)' the *strict global Bernoulli fibred action* $(\varepsilon, \mathfrak{sB}) : \mathcal{G} \curvearrowright P(\mathcal{G})$ given by the pair pf maps

$$\varepsilon : P(\mathcal{G}) \to \mathcal{G}^{(0)} \text{ where } A \subset \text{costar}(e) \mapsto \epsilon(A) = e \in \mathcal{G}^{(0)},$$
$$\mathfrak{sB} : \{(g, A) \in \mathcal{G} \times P(\mathcal{G}); A \subset \text{costar}(d(g))\} \to P(\mathcal{G}) \text{ where } A \mapsto gA.$$

(II) the *partial Bernoulli action* $\mathfrak{b} : \mathcal{G} \curvearrowright_p \mathcal{I}_{gpd}(P_{\mathcal{U}}(\mathcal{G}))$, or for each $g \in \mathcal{G}$

$$\mathfrak{b}_g : D_{g^{-1}} = \{A \in P(\mathcal{G});\ A \ni e, eg^{-1}, e \in \mathcal{G}^{(0)}\} \to D_g = \{B \in P(\mathcal{G});\ B \ni gf, gfg^{-1}, f \in \mathcal{G}^{(0)}\},$$
$$A \mapsto gA$$

(II)' the *partial Bernoulli fibred action* $(\epsilon, \mathfrak{b}) : \mathcal{G} \curvearrowright_p P_{\mathcal{U}}(\mathcal{G})$ given by the pair pf maps

$$\epsilon : P_{(U)}(\mathcal{G}) \to \mathcal{G}^{(0)} \text{ where } A \subset \text{costar}(e) \mapsto \epsilon(A) = e \in \mathcal{G}^{(0)},$$
$$\mathfrak{b} : \{(g, A) \in \mathcal{G} \times P_{\mathcal{U}}(\mathcal{G});\ A \ni e, eg^{-1}\} \to P_{\mathcal{U}}(\mathcal{G}) \text{ where } A \mapsto gA.$$

(III) the *strict partial Bernoulli action* denoted by $\mathfrak{sb} : \mathcal{G} \curvearrowright_p \mathcal{I}_{gpd}(P_{\mathcal{U}}(\mathcal{G}))$, or for each $g \in \mathcal{G}$

$$\mathfrak{sb}_g : D^m_{g^{-1}} = \{A \in P(\mathcal{G});\ A \ni d(g), g^{-1}\} \to D^m_g = \{B \in P(\mathcal{G});\ B \ni r(g), g\}.$$
$$A \mapsto gA$$

(III)' the *strict partial Bernoulli fibred action* $(\epsilon, \mathfrak{sb}) : \mathcal{G} \curvearrowright_p P_{\mathcal{U}}(\mathcal{G})$ given by the pair pf maps

$$\epsilon : P_{(U)}(\mathcal{G}) \to \mathcal{G}^{(0)} \text{ where } A \subset \text{costar}(e) \mapsto \epsilon(A) = e \in \mathcal{G}^{(0)},$$
$$\mathfrak{sb} : \{(g, A) \in \mathcal{G} \times P_{\mathcal{U}}(\mathcal{G});\ A \ni d(g), g^{-1}\} \to P_{\mathcal{U}}(\mathcal{G}) \text{ where } A \mapsto gA.$$

Summarizing

$$\begin{tikzcd}
 & & \mathfrak{B} : \mathcal{G} \curvearrowright \mathcal{I}_{gpd}(P(\mathcal{G})) \simeq (\varepsilon, \mathfrak{B}) : \mathcal{G} \curvearrowright P(\mathcal{G}) \arrow[dd, dotted, no head, "globaliz."] \\
\mathfrak{sB} : S \curvearrowright \mathcal{I}_{gpd}(P(S)) \simeq (\varepsilon, \mathfrak{sB}) : \mathcal{G} \curvearrowright P(\mathcal{G}) \arrow[d, dotted, no head, "globaliz."'] & \mathcal{G} \arrow[l] \arrow[ur] \arrow[dl] \arrow[dr] & \\
\mathfrak{sb} : \mathcal{G} \curvearrowright_p \mathcal{I}_{gpd}(P_{\mathcal{U}}(\mathcal{G})) \simeq (\epsilon, \mathfrak{sb}) : \mathcal{G} \curvearrowright_p P_{\mathcal{U}}(\mathcal{G}) & & \mathfrak{b} : \mathcal{G} \curvearrowright_p \mathcal{I}_{gpd}(P_{\mathcal{U}}(\mathcal{G})) \simeq (\epsilon, \mathfrak{b}) : \mathcal{G} \curvearrowright_p P_{\mathcal{U}}(\mathcal{G})
\end{tikzcd}$$

Figure 4.3: The groupoid Bernoulli actions

4.5 Odered groupoid semidirect products, enlargements, and algebras

Following the methodology of the preceding chapters, once we have the actions, we move on to the associated structures, *i.e.* the semidirect products and algebras.

4.5.1 Semidirect products and enlargements

The constructions we are about to make were introduced by Brown in [12] called split extensions, for groupoids without any order, and later extended by Steinberg [84] for ordered groupoids, and termed semidirect product. In a few words: a fibred action defines a semidirect product.

Each fibred groupoid action defines a groupoid action, as Miller does in [62] (Chapter 5 - Section 5.3). We choose to adopt the term semidirect product instead because it is consistent with the previous chapter and the next one.[1]

The similarities of ordered groupoids and inverse semigroups go even further. Miller develops, throughout Chapter 4 and 5 of [62], the version of O'Carroll triples for ordered groupoids acting on posets – called by her L-Systems. It turns out that her construction also

[1]Also, there is a subtle matter in the definition of action groupoids: it is definition uses the domain of the actions, for instance, cf. Abadie [2], or Bagio [5]. On the other hand, semidirect products take into account the range.

deals with an ordered groupoid version of enlargements. Worth it to mention: Miller's L-Systems are a generalization of the P-theorem for ordered groupoids proposed by Gilbert, [39].

In the next pages, we will present her results and then apply them to our case. Nevertheless, there will be one difference from the last chapter: as groupoids are more restrictive than inverse semigroups, we will obtain two new groupoids – besides the four prefix inverse semigroups.

Let $(\rho, \theta) : \mathcal{G} \curvearrowright X$ be a fibred action of the ordered groupoid $(\mathcal{G}, \leqslant_{\mathcal{G}})$ on the poset $(X, \leqslant_X)$.

Definition 4.5.1 ([62])**.** The *semidirect product determined by* $(\rho, \theta) : \mathcal{G} \curvearrowright X$ is the ordered groupoid defined by the set

$$X \rtimes_{(\rho,\theta)} \mathcal{G} := \{(x, g) \in X \times \mathcal{G}; \rho(x) = r(g)\},$$

with structure given by

units: $(X \rtimes_{(\rho,\theta)} \mathcal{G})^{(0)} = \{(x, e) \in X \times \mathcal{G}^{(0)}; \rho(x) = e\}$,

product: $(x, g)(y, h) = (x, gh)$ if $x = \theta_g(y)$ and $r(h) = d(g)$,

involution: $(x, g)^{-1} = (\theta_{g^{-1}}(x), g^{-1})$,

source,target: $d(x, g) = (\theta_{g^{-1}}(x), d(g))$ and $r(x, g) = (x, r(g))$.

The ordering is

$$(x, g) \leqslant (y, h) \iff x \leqslant_X y \text{ and } g \leqslant_{\mathcal{G}} h.$$

And the restriction and corestriction are

$${}_{(y,e)|}(x, g) = (\theta_{{}_{e|}g}(y), {}_{e|}g) \text{ and } (x, g)_{|(y,e)} = (y, g_{|e}),$$

supposing the unit satisfies the required axioms.

At this point, the reader may have noticed a similarity with Gilbert's expansion. Soon, it will be re-written in terms of our Bernoulli actions. However, we must introduce a few more technical aspects.

Let $(X, \leqslant_X)$ be a poset, $Y \subseteq X$ be an order ideal, and $\mathcal{G}$ be an ordered groupoid. Suppose $(\rho, \theta) : \mathcal{G} \curvearrowright X$ is a fibred action such that $\mathcal{G} \cdot Y = X$.

Definition 4.5.2 ([62])**.** The triple $(X, Y, \mathcal{G})$ is called an *L-system* and determines an ordered subgroupoid of $X \rtimes_{(\rho,\theta)} \mathcal{G}$ given by

$$L(X, Y, \mathcal{G}) := \{(y, g) \in Y \times \mathcal{G}; \rho(y) = r(g),\ \theta_{g^{-1}}(y) \in Y\}.$$

Now that we have defined the groupoids, we may discuss enlargements. As the reader might expect, following the studies from Chapters 3 and 4, the partial action's semidirect product is more than a substructure of the associated global structure: they share the enlargement relation.

Definition 4.5.3 ([51])**.** Given a sub ordered groupoid $\mathcal{G}$ of the ordered subgroupoid $\mathcal{H}$ satisfying

(I) $\mathcal{G}^{(0)}$ is an ordered ideal of $\mathcal{H}^{(0)}$,

(II) if $h \in \mathcal{H}$ and $d(h), r(h) \in \mathcal{G}$, then $h \in \mathcal{G}$, and

(III) if $e \in \mathcal{H}^{(0)}$, then there exists an $h \in \mathcal{H}$ with $r(h) = e$ and $d(h) \in \mathcal{G}$.

We say that $\mathcal{H}$ is an *enlargement* of $\mathcal{G}$. By notation $\mathcal{G} \subseteq_E \mathcal{H}$.

Finally, it is routine to check the next proposition.

Proposition 4.5.4. Consider the L-system $(X, Y, \mathcal{G})$ defined by the fibred action $(\rho, \theta) : \mathcal{G} \curvearrowright X$, then the semidirect $X \rtimes_{(\rho,\theta)} \mathcal{G}$ is an enlargement of $L(X, Y, \mathcal{G})$, or

$$L(X, Y, \mathcal{G}) \subseteq_E X \rtimes_{(\rho,\theta)} \mathcal{G}.$$

Wrapping up this subsection, we end with the next corollary.

Corollary 4.5.5. The groupoids $L(X, Y, \mathcal{G})$ and $X \rtimes_{(\rho,\theta)} \mathcal{G}$ are Morita equivalent, or in symbols

$$L(X, Y, \mathcal{G}) \simeq_M X \rtimes_{(\rho,\theta)} \mathcal{G}.$$

Proof. By Proposition 4.5.4: $L(X, Y, \mathcal{G}) \subseteq_E X \rtimes_{(\rho,\theta)} \mathcal{G}$. Then the third axiom of Definition 4.5.3 holds. As our groupoids have no topology, the result follows from Lemma 2.3.16. $\square$

Next, we will apply these results to our study case and also examine its associated algebras.

4.5.2 Gilbert expansion via L-system

Let $\mathcal{G}$ be an ordered groupoid and the sets

$$P(\mathcal{G}) := \{A \subset \mathcal{G}; \mid A \mid < \infty,\ \exists e \in \mathcal{G}^{(0)} \text{ s.t. } \forall a \in A,\ r(a) = e\}$$
$$P_{\mathcal{U}}(\mathcal{G}) = \{A \in P(\mathcal{G}).A \cap \mathcal{G}^{(0)} \neq \emptyset\};$$

In Section 5.3 we construct the actions $(\varepsilon, \mathfrak{B}) : \mathcal{G} \curvearrowright P(\mathcal{G})$ and $(\epsilon, \mathfrak{b}) : \mathcal{G} \curvearrowright_p P_{\mathcal{U}}(\mathcal{G})$, for $e \in \mathcal{G}^{(0)}$, respectively

- the global fibred Bernoulli action of $\mathcal{G}$ on $P(\mathcal{G})$ given by the pair
  - $\varepsilon : P(\mathcal{G}) \to \mathcal{G}^{(0)}$ with $A \subset \text{costar}(e) \in P(\mathcal{G}) \mapsto \varepsilon(A) = e$,
  - $\mathfrak{B} : \{(g, A) \in \mathcal{G} \times P(\mathcal{G}); \varepsilon(A) \leqslant d(g)\} \to P(\mathcal{G})$ defined by $\mathfrak{B}_g(A) := gA$,
- the partial fibred Bernoulli action given by
  - $\epsilon : P_{\mathcal{U}}(\mathcal{G}) \to \mathcal{G}^{(0)}$ where $A \subset \text{costar}(e) \mapsto \epsilon(A) = e \in \mathcal{G}^{(0)}$,
  - $\mathfrak{b} : \{(g, A) \in \mathcal{G} \times P_{\mathcal{U}}(\mathcal{G});\ A \ni e, eg^{-1}\} \to P_{\mathcal{U}}(\mathcal{G})$ where $\mathfrak{b}_g(A) := gA$.

Lemma 4.5.6. The triple $(P(\mathcal{G}), P_{\mathcal{U}}(\mathcal{G}), \mathcal{G})$ defined by $(\varepsilon, \mathfrak{B}) : \mathcal{G} \curvearrowright P(\mathcal{G})$ is an L-system.

Proof. We must verify two points: $P_{\mathcal{U}}(\mathcal{G})$ is an order ideal of $P(\mathcal{G})$, and $\mathcal{G} \cdot P_{\mathcal{U}}(\mathcal{G}) = P(\mathcal{G})$. Indeed:

- order ideal: let $A \in P(\mathcal{G})$ and $B \in P_{\mathcal{U}}(\mathcal{G})$ such that $A \subset \text{costar}(e)$ and $B \subset \text{costar}(f)$, also, by definition, $B \ni f$, where $e, f \in \mathcal{G}^{(0)}$. Suppose $A \leqslant B$, this means

  $$e \leqslant f \text{ and } B_{|e} \subset A,$$

  so $f_{|e} \in A$ and as $A \subset \text{costar}(e)$, we have

  $$e = r(f_{|e}) = f_{|e} \in A.$$

- action condition: this assertion follows from the fact that the global fibred Bernoulli is the globalization of the partial fibred Bernoulli action.

Hence the proof is complete. □

Lemma 4.5.7. The ordered groupoid $L(P(\mathcal{G}), P_{\mathcal{U}}(\mathcal{G}), \mathcal{G})$ is equal to the semidirect product $P_{\mathcal{U}}(\mathcal{G}) \rtimes_{(\epsilon,\mathfrak{b})} \mathcal{G}$.

Proof. First, let us write the definition of both groupoids:

$$L(P(\mathcal{G}), P_{\mathcal{U}}(\mathcal{G}), \mathcal{G}) = \{(A, g) \in P_{\mathcal{U}}(\mathcal{G}) \times \mathcal{G};\ \varepsilon(A) = r(g),\ g^{-1}A \in P_{\mathcal{U}}(\mathcal{G})\}$$

$$P_{\mathcal{U}}(\mathcal{G}) \rtimes_{(\epsilon,\mathfrak{b})} \mathcal{G} = \{(B, h) \in P_{\mathcal{U}}(\mathcal{G}) \times \mathcal{G},\ \epsilon(A) = r(g)\}.$$

The similarities are pretty clear and $\epsilon = \varepsilon_{|P_{\mathcal{U}}(\mathcal{G})}$; in order to show the equality we verify that, by the nature of our sets,

$$\epsilon(A) = r(g) \implies g^{-1}A \in P_{\mathcal{U}}(\mathcal{G}).$$

Also, the existence of the action by g is given by the relation

$$\exists g^{-1}A = \mathfrak{B}_g(A) \iff \varepsilon(A) \leqslant d(g).$$

Indeed, let $A \in P_{\mathcal{U}}(\mathcal{G})$ with $\epsilon(A) = e$, where $e \in \mathcal{G}^{(0)}$, by the definition of such set

$$\forall a \in A\ \ aa^{-1} = e \text{ and } A \ni e.$$

Now, if $g \in \mathcal{G}$ satisfies $\epsilon(A) = r(g)$, as $r(g) = d(g^{-1})$,

$$\epsilon(A) = d(g^{-1}) \implies \exists g^{-1}A.$$

Finally, let $g^{-1}a \in g^{-1}A$, then

$$(g^{-1}a)(g^{-1}a)^{-1} = g^{-1}aa^{-1}g = g^{-1}eg = g^{-1}(gg^{-1})g = d(g),$$

and thus $g^{-1}A \in P_{\mathcal{U}}(\mathcal{G})$. □

Corollary 4.5.8. The Gilbert expansion $\mathcal{G}^{GBR}$ is equal to $P_{\mathcal{U}}(\mathcal{G}) \rtimes_{(\epsilon,\mathfrak{b})} \mathcal{G}$.

Proof. By last lemma, if $(A, g) \in P_{\mathcal{U}}(\mathcal{G}) \rtimes_{(\epsilon,\mathfrak{b})} \mathcal{G}$, then this pair satisfy

$$\epsilon(A) = r(g) \text{ and } g^{-1}A \in P_{\mathcal{U}}(\mathcal{G}) \implies \epsilon(A) = r(g) \text{ and } (g^{-1}, A) \in dom(\mathfrak{b}),$$

this means that

$$A \ni r(g), r(g)(g^{-1})^{-1} \iff A \ni r(g), g.$$

This is precisely the characterization of $\mathcal{G}^{GBR}$, from Definition 4.3.1; hence we're done. □

Corollary 4.5.9. The *global Gilbert's expansion* defined by $\overline{\mathcal{G}^{GBR}} := P(\mathcal{G}) \rtimes_{(\varepsilon,\mathfrak{B})} \mathcal{G}$, is Morita equivalent to $\mathcal{G}^{GBR}$, or

$$\mathcal{G}^{GBR} \simeq_M \overline{\mathcal{G}^{GBR}}.$$

Proof. It's only needed to realize that $\mathcal{G}^{GBR} = L(P(\mathcal{G}), P_{\mathcal{U}}(\mathcal{G}), \mathcal{G})$, and then use the corollary of Proposition 4.5.4. □

Before present the algebras of these groupoids, we would like to make a few comments:

- Terminology: In the light of the last Corollary/Definition, we will refer to Gilbert's expansion as *partial Gilbert expansion*.

- Global Gilbert expansion elements characterization: if $(A, g) \in \overline{\mathcal{G}^{GBR}}$, this pair must satisfy

  $$\mid A \mid < \infty \text{ and } A \subset \text{costar(r(g))}.$$

  The main difference from the non global case is the requirement $A \ni r(g), g$.

- The strict actions: In Definition 4.4.6 we established two actions derived from the global and partial fibred Bernoulli action: the strict global Bernoulli fibred action $(\varepsilon, \mathfrak{s}\mathfrak{B})$: $\mathcal{G} \curvearrowright P(\mathcal{G})$, and the strict partial Bernoulli fibred action $(\epsilon, \mathfrak{s}\mathfrak{b}) : \mathcal{G} \curvearrowright_p P_{\mathcal{U}}(\mathcal{G})$. Because those actions share the same moment map with the global and the partial Bernoulli fibred actions, respectively, we will find that

  $$P(\mathcal{G}) \rtimes_{(\varepsilon,\mathfrak{s}\mathfrak{B})} \mathcal{G} = P(\mathcal{G}) \rtimes_{(\varepsilon,\mathfrak{B})} \mathcal{G} \text{ and } P_{\mathcal{U}}(\mathcal{G}) \rtimes_{(\epsilon,\mathfrak{s}\mathfrak{b})} \mathcal{G} = P_{\mathcal{U}}(\mathcal{G}) \rtimes_{(\epsilon,\mathfrak{b})} \mathcal{G}.$$

  This phenomena happens because: the definition of the moment map, of fibred actions, plays an important role in the definition of the semidirect product.

4.5.3 Global and partial algebras from Gilbert expansions

This section is going to be the final one of this chapter, and very short also.

Let $\mathbb{K}$ be an associative commutative unital ring $\mathcal{G}$ a groupoid, the $\mathbb{K}$-algebra of the groupoid, is the free $\mathbb{K}$-module with basis $\mathcal{G}$ and convolution product

$$\delta_x * \delta_y = \begin{cases} \delta_{xy} & , \text{ if } \exists xy \\ 0 & , \text{ if not} \end{cases}$$

In Chapter 3, we stated the Theorem 2.5.4 from Clark-Sims [21]:

> If $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent groupoids, then their Steinberg algebras $A_{\mathbb{K}}(\mathcal{G})$ and $A_{\mathbb{K}}(\mathcal{H})$ are Morita equivalent.

As we are only dealing with groupoids without topology, the Steinberg algebras boil down to groupoid convolution algebras.

Fix $\mathcal{G}$ a **finite** ordered groupoid, combining the strategies of last subsection and the Theorem 2.5.4 we can conclude the next proposition.

Proposition 4.5.10. The convolution algebra of the global and the partial Gilbert expansions are Morita equivalent.

Using the notations $\mathbb{K}_{glob}(\mathcal{G}) := \mathbb{K}\overline{\mathcal{G}^{GBR}}$ and $\mathbb{K}\mathcal{G}^{GBR} =: \mathbb{K}_{par}(\mathcal{G})$, we have last proposition states that

$$\mathbb{K}_{glob}(\mathcal{G}) \simeq_M \mathbb{K}_{par}(\mathcal{G}).$$

The above algebras are, respectively: the *global algebra* and the *partial algebra* of $\mathcal{G}$.

The next diagram compiles all the information we present in this section for $\mathcal{G}$ a finite ordered groupoid.

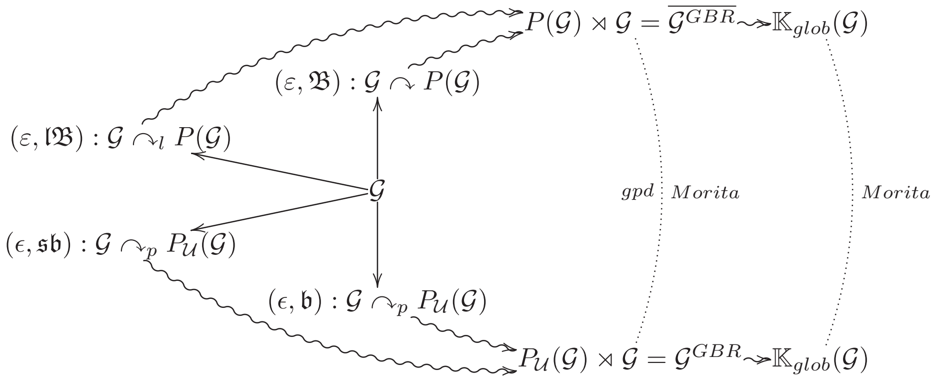


Figure 4.4: The Morita equivalence of algebras associated to the Bernoulli actions

Chapter 5

Expanding inverse categories

The Bernoulli approach we have developed, *i.e.* the diamond-shaped method of think, shows its consistency in different (but somewhat similar) few structures: groups, on inverse semigroups, and ordered groupoids. One possible reason for such success is the capability of these structures to encode partial bijections – which is the core of the definition of partial actions. From the categorical point of view, in the sense of Mac Lane [57], the abstract environment lies on categories that encode the abstraction of map restriction.

Robin Cockett and Stephen Lack attempted to study categories where partial maps are well settled in [22]; they came up with the notion of a restriction category. Intuitively each arrow on a restriction category has a restriction, which plays the domain where this partial map is defined.

Although suitable for our purposes, we need another feature: each map must have a unique inverse. This property leads us to inverse categories, a particular class of restriction categories. Anticipating some comments and definitions, the reader should understand inverse categories as structures in which objects are inverse semigroups with the unit or inverse monoids.

One can study inverse categories inspired only by the idea of multi-object inverse semigroups. This point of view was the motor of Linckelmann in [56] where our intuition takes benefits.

We will take ideas from both, and many others, authors and present a most abstract version of the Bernoulli approach. A brief outline of this chapter follows:

- we begin with the definition of restriction categories and properties of inverse categories;
- next, we establish actions of inverse categories on posets;
- these actions induce semidirect product of categories, and there are other structures associated;
- turning to the associated algebras, we use Kan extensions to understand representations of such categories;
- finally, we present the pair of adjoint functors behind our work.

From this point, our main structures will be categories as defined in Mac Lane [57].

5.1 Restriction and inverse categories

Restriction categories were thought to generalize the idea of ”restrict a map to a subset”, on an abstract level. More than a property that a category must satisfy, a restriction is an extra structure assigned to each arrow an idempotent on its domain – satisfying few axioms. Every category has a trivial restriction that assigns each arrow to an unit arrow; on the other hand, more than one restriction might exist.

This section will present the standard definition and basic properties of restrictions; we will also describe inverse categories and their (strong) relations with inverse semigroup theory.

The next results were taken from: Cockett-Lack [22], Dewolf-Pronk [25] and Linckelmann [56]; where many examples and discussion can be found.

5.1.1 The restriction structure

Definition 5.1.1 ([22])**.** Let $\mathcal{C}$ be a category, a *restriction on* $\mathcal{C}$ assigns to each arrow $(x : A \to B) \in \mathcal{C}$ an arrow $\overline{x} : A \to A$ satisfying the conditions:

(I) $x\overline{x} = x$ for all $x \in \mathcal{C}$;

(II) if $x, y \in \mathcal{C}$ with $dom(x) = dom(y)$, we have $\overline{x}\,\overline{y} = \overline{y}\,\overline{x}$;

(III) if $x, y \in \mathcal{C}$ with $dom(x) = dom(y)$, them $\overline{y\overline{x}} = \overline{y}\,\overline{x}$;

(IV) $x, y \in \mathcal{C}$ with $ran(x) = dom(y)$, implies $\overline{y}x = x\overline{yx}$.

We call a category with a restriction structure by *restriction category*.

Notation: $(\mathcal{C}, \overline{(\ )})$ will stand for a restriction category.

The model we would like the reader to keep in mind is $\mathcal{P}ar$, the category composed by sets and partially defined maps.

Each arrow $X \to Y$ in $\mathcal{P}ar$ is a pair $(f, dom(f))$, where $dom(f) \subseteq X$ and $f : dom(f) \to Y$. Also, the unit of X is $(1_X, X)$ and the composition of $(f, dom(f)) : X \to Y$ with $(g, dom(g)) : Y \to Z$ in $\mathcal{P}ar$ is $(gf_{|dom(f)\cap f^{-1}(dom(g))}, dom(f) \cap f^{-1}(dom(g)))$.

For each arrow $f : dom(f) \subseteq X \to Y$ its restriction is the partially-defined identity $\overline{f} : dom(f) \subseteq X \to X$, *i.e.* $\overline{f}(x)$ exists only if $x \in dom(f)$.

This model permits us to read expressions ”$f\overline{g}$” as ”f restricted to where the map g is defined”.

The following result presents the basics properties of the restrictions.

Lemma 5.1.2 ([22])**.** *Let* $(\mathcal{C}, \overline{(\ )})$ *be a restriction category, then for* $x, y \in \mathcal{C}$ *and* 1 *denoting an unit in* $\mathcal{C}$

(i) $\overline{x}\,\overline{x} = \overline{x}$;

(ii) $\overline{x}\,\overline{yx} = \overline{yx}$;

(iii) $\overline{\overline{y}x} = \overline{yx}$;

(iv) $\overline{\overline{x}} = \overline{x}$;

(v) $\overline{\overline{y}\,\overline{x}} = \overline{y}\,\overline{x}$;

(vi) *if* $xy = xz$ *implies* $y = z$, *then* $\overline{x} = 1$ *and* $\overline{1} = 1$;

(vii) $x\overline{y} = x \implies \overline{x}\,\overline{y} = \overline{x}$.

We can even endow a restriction category with an ordering.

Lemma 5.1.3 ([22])**.** *Let* $(\mathcal{C}, \overline{(\ )})$ *be a restriction category and define, for* $x, y : A \to B$ *the relation*

$$x \leqslant y \iff x = y\overline{x}.$$

Then $\leqslant$ *defines a partial order on* $\mathcal{C}$ *and the following assertions hold:*

(i) $x \leqslant y$ *implies* $\overline{x} \leqslant \overline{y}$;

(ii) *if* $x \leqslant y$ *and* $x' \leqslant y'$, *then* $xy \leqslant x'y'$, *for* $x', y' : B \to C$.

The maps in a restriction category whose restriction is a unit, are called *total maps*, *i.e*

$$x \in (\mathcal{C}, \overline{(\ )}) \text{ is total if } \overline{x} = 1.$$

The last result of this subsection is relative to total maps.

Lemma 5.1.4 ([22])**.** *In a restriction category* $(\mathcal{C}, \overline{(\ )})$ *total maps have the properties:*

(i) *every* x *such that* $xy = xz$ *implies* $y = z$, *is total;*

(ii) *if both* x *and* y *are total, the composition* yx *is total;*

(iii) *if* yx *is total, then* x *is total;*

(iv) *the total maps form a subcategory of* $\mathcal{C}$.

The functor between restriction categories must preserve the restriction structure; the formalization of this sentence follows.

Definition 5.1.5 ([22])**.** Given two restriction categories $(\mathcal{C}, \overline{(\ )})$ and $(\mathcal{D}, \widetilde{(\ )})$ and a functor $F : \mathcal{C} \to \mathcal{D}$, satisfying $F(\overline{x}) = \widetilde{F(x)}$ for all $x \in \mathcal{C}$ is called a *functor of restriction categories*.

5.1.2 Inverses and restrictions

At the beginning of this chapter, we said that we need two ingredients: restrictions of maps and inverses. Categories with a restriction structure answer the first need. For inverses, we must introduce inverse categories.

We will present the basics of inverse categories as in Linckelmann [56], but also present these categories from the point of view of restriction categories. This approach helps us to interpret some old constructions in this utterly abstract study. We will skip proofs since they (in the majority) are similar to proofs of analogous results for inverse semigroups; [56] provides complete arguments for the results below.

To invoke the resemblance between inverse categories and inverse semigroups, we will adopt the following notation: s and t will refer to arrows, and X and Y to objects. We will hold capital letters like A, or B, for the expansion section.

Definition 5.1.6 ([56])**.** A category $\mathcal{C}$ is called an *inverse category* if for each arrow $(s : X \to Y) \in \mathcal{C}$ there exists a unique arrow $(s^\circ : Y \to X) \in \mathcal{C}$, called *inverse*, such that

$$ss^\circ s = s \text{ and } s^\circ s s^\circ = s^\circ.$$

Notation: $(\mathcal{C}, (\ )^\circ)$ will denote an inverse category.

Notice that a given inverse category $(\mathcal{C}, (\ )^\circ)$ induces a restriction structure in $\mathcal{C}$ by $\overline{s} = s^\circ s$; in particular $s^\circ s$ is idempotent.

On the other hand, the equivalences in the next proposition grant the opposite relation.

Proposition 5.1.7 ([22])**.** Let $\mathcal{C}$ be a category, the assertions below are equivalent:

(i) $(\mathcal{C}, \overline{(\ )})$ is a restriction category for which every $s \in \mathcal{C}$ there exists a map t, called *restrict isomorphism*, such that $\overline{s} = ts$ and $\overline{t} = st$;

(ii) for every $(s : X \to Y) \in \mathcal{C}$ there exists a unique $(t : Y \to X) \in \mathcal{C}$, with $sts = s$ and $tst = t$;

(iii) there exists a functor $(\ )^\circ : \mathcal{C} \to \mathcal{C}^{op}$ satisfying for $X \in \mathcal{C}^{(0)}$, and $s, t \in \mathcal{C}$

- $(X)^\circ = X$;
- $(s^\circ)^\circ = s$;
- $ss^\circ s = s$;
- $(ss^\circ)(tt^\circ) = (tt^\circ)(ss^\circ)$.

Also, structures in (i) and (iii) are unique.

One more property in this point of view, idempotents are their own inverses.

Lemma 5.1.8 ([22])**.** *Let* $(\mathcal{C}, (\ )^{\circ})$ *be an inverse category. If* s *is an arrow of* $\mathcal{C}$ *then* $(\overline{s})^{\circ} = \overline{s}$*.*

Some examples of inverse categories are: the category of partial bijections; any inverse semigroup with a unit (an inverse monoid); any groupoid – which is a restriction category whose every arrow is total.[1]

As one would expect, functors between inverse categories are restriction functors preserving restrict isomorphisms.

Next, we will present a collection of algebraic properties of inverse categories, with the flavor of inverse semigroups – there is even a representation theorem in the sense of the Wagner-Preston Theorem 1.2.14.

We will add a hypotheses to the definition of inverse categories:

All of our inverse categories, from now on, are small.

Some basic properties.

Proposition 5.1.9 ([56])**.** Let $(\mathcal{C}, (\ )^{\circ})$ be an inverse category. Given arrows $s, t : X \to Y$, and the idempotent morphisms $e : X \to X$, $f : Y \to Y$ in $\mathcal{C}$, we have that:

(i) $(st)^{\circ} = t^{\circ}s^{\circ}$, and

(ii) the arrows $ses^{\circ} : X \to X$ and $tft^{\circ} : Y \to Y$ are idempotent.

For any pair of arrows $s, t : X \to Y$ in an inverse category, we write

$$s \leqslant t \text{ if } s = te \text{ for the idempotent } e : X \to X.$$

This is the order we defined for restriction categories, and in the case of inverses categories we have the following equivalences.

Proposition 5.1.10 ([56])**.** Let $(\mathcal{C}, (\ )^{\circ})$ be an inverse category and let $s, t : X \to Y$ and $f : Y \to Y$ be arrows. The following are equivalent:

(i) $s \leqslant t$;

[1] There is also an other way to define inverse categories: via Dagger categories, as Karnoven [45] explains. We are aware of such construction, but restrictions are more appealing to our work.

(ii) $s = ft$ for some idempotent morphism f;

(iii) $s = ss^\circ t$;

(iv) $s = ts^\circ s$.

Moreover, if $p, q : Y \to Z$ is another pair of arrows, such that $p \leqslant q$, then $ps \leqslant qt$.

Recall that the Wagner-Preston Theorem (cf. 1.2.14) shows how to embed any inverse semigroup into an inverse monoid of partial bijections on a set. We have a similar statement for inverse categories. This result is made possible by combining the construction of an inverse category (from a given set) and a technical lemma.

Indeed, we start with the lemma.

Lemma 5.1.11 ([56]). *Let $(\mathcal{C}, (\ )^\circ)$ be an inverse category. If $(s : X \to Y) \in \mathcal{C}$ and $Z \in \mathcal{C}^{(0)}$, then $s\mathcal{C}(Z, X) = ss^\circ\mathcal{C}(Z, Y)$.*

Now the construction: let M be a set and $\mathcal{P}_M$ be a partition of M, then $\mathcal{I}_{ic}(\mathcal{P}_M)$ composed by the following data, determines an inverse category

- objects: $\mathcal{I}_{ic}(\mathcal{P}_M)^{(0)} = \mathcal{P}_M$;
- arrows: given subsets $U, V \subset M$ in $\mathcal{P}_M$, morphisms from U to V is the set formed by bijections $s : U' \to V'$, where $U' \subseteq U$ and $V' \subseteq V$;
- composition: let $U, V, W \in \mathcal{P}_M$, its subsets $U' \subseteq U$, $V', V'' \subseteq V$, $W' \subseteq W$, and bijections $s : U' \to V'$ and $t : V'' \to W'$, the composition ts is defined by

$$ts = t_{|V' \cap V''} \circ s_{|s^{-1}(V' \cap V'')} : s^{-1}(V' \cap V'') \to t(V' \cap V'');$$

- inverses: the inverse structure is the inversion of maps.

After the preparation, we can state the theorem.

Theorem 5.1.12. *([56]) Let $(\mathcal{C}, (\ )^\circ)$ be an inverse category. For $X \in \mathcal{C}^{(0)}$ define the set $M_X := \mathcal{C}(-, X)$ of arrows terminating on the object X, and the partition $\mathcal{P}_\mathcal{C} := \bigcup_{X \in \mathcal{C}^{(0)}} M_X$ of $\mathcal{C}$ as a set. There exists a functor of inverse categories $\phi : \mathcal{C} \to \mathcal{I}_{ic}(\mathcal{P}_\mathcal{C})$ with*

(i) $X \in \mathcal{C}^{(0)} \mapsto \phi(X) = M_X \in \mathcal{P}_\mathcal{C}$

(ii) the functor ϕ induces an injection $\mathcal{C}(X, Y) \to \mathcal{I}_{ic}(\mathcal{P}_\mathcal{C})(\phi(X), \phi(Y))$, for $X, Y \in \mathcal{C}^{(0)}$;

(iii) $\phi(s^\circ) = \phi(s)^\circ$ for any arrow $s \in \mathcal{C}$;

(iv) if $s \leqslant t \implies \phi(s) \leqslant \phi(t)$, for any arrows s, t.

Also, Dewolf-Pronk in [25] proved a generalization of the ESN Theorem for inverse categories. They build an equivalence between the category of inverse categories and the category of locally inductive groupoids – a class of groupoids whose object set is a union of meet semilattice.

These last results will give us intuition about how to define actions by automorphisms at an appropriate time. For this purpose, we introduce some notation to help our computations.

For a given inverse category $(\mathcal{C}, (\ )^{\circ})$, we saw that it is possible to induce a restriction structure via $\overline{(\ )} = (\ )^{\circ}(\ )$; in this manner, our category has objects and domains of morphisms. Aiming to differentiate both sets, we will define two kinds of source and target maps.

<u>Definition</u> 5.1.13. Let $\mathcal{C}$ be an inverse category and $s : X \to Y$ be a morphism of $\mathcal{C}$.

(I) The *set of idempotent morphisms* will be denoted by $RId(\mathcal{C})$. Furthermore, if $X \in \mathcal{C}^{(0)}$, the set of idempotent morphisms in X will be denoted by $RId(\mathcal{C}(X))$.

(II) The *outer domain* (or *outer source*) and the *outer range* (or *outer target*) of s are the maps $od, or : \mathcal{C} \to \mathcal{C}^{(0)}$ with

$$od(s) = X \text{ and } or(s) = Y.$$

(III) The *inner domain* (or *inner source*) and the *inner range* (or *inner target*) of s are $id, ir : \mathcal{C} \to RId(\mathcal{C})$ with

$$id(s) = s^{\circ}s \text{ and } ir(s) = ss^{\circ}.$$

Combining (II) and (III) we will denote the domain (or source) and the range (or target) pairs maps by:

$$d = (od, id) \text{ and } r = (or, ir).$$

Notice that if $\mathcal{C}$ is a groupoid, then the outer and inner maps are equal, whenever an object X is identified with 1_X. Both inner maps play the role of source and target maps of groupoids in this setting of inverse categories.

Now we borrow the Definition 2.4.1 from inverse semigroup theory and the definitions of star and costar sets from groupoids, cf. Section 4.3.1.

<u>Definition</u> 5.1.14. Let $\mathcal{C}$ be an inverse category and let s and t be arrows in $\mathcal{C}$.

(I) We say that $s, t \in \mathcal{C}$ are *$\mathscr{L}$-related* if $id(s) = id(t)$, or simply $s^{\circ}s = t^{\circ}t$. The set of arrows $\mathscr{L}$-related to s will be denoted by $\mathscr{L}_s$.

(II) We say that $s, t \in \mathcal{C}$ are *$\mathscr{R}$-related* if $ir(s) = ir(t)$, in other terms $ss^{\circ} = tt^{\circ}$. The set of arrows $\mathscr{R}$-related to s will be denoted by $\mathscr{R}_s$.

Let X and Y be objects in $\mathcal{C}^{(0)}$.

(IV) The set of all arrows of $\mathcal{C}$ starting in X is Star$(X) := \{t \in \mathcal{C}; od(t) = X\}$.

(III) The set of all arrows of $\mathcal{C}$ ending in Y is Costar$(Y) := \{s \in \mathcal{C}; or(s) = Y\}$.

5.2 Bernoulli actions

Once the structural tools and environment are appropriately defined, our next aim is to develop the Bernoulli approach. We will start with a slightly modified definition of category action on a set; this definition will consider the outer and inner structures of an inverse category.

5.2.1 Inverse category actions

First, we introduce the fibred actions.

Definition 5.2.1. Let $(\mathcal{C}, (\ \ )^{\circ})$ be an inverse category and $(P, \leqslant)$ be a partially ordered set, a *fibred ordered action* of $\mathcal{C}$ on P is a pair (ρ, θ), where

- $\rho = (o\rho, i\rho) : P \to \mathcal{C}^{(0)} \times RId(\mathcal{C})$ with $x \mapsto (o\rho(x), i\rho(x))$, where $i\rho(x) : o\rho(x) \to o\rho(x)$ is an idempotent and $i\rho$ is order preserving, the *moment* map;
- $\theta : \mathcal{C}_d {\times}_{\rho} P = \{(s, x) \in \mathcal{C} \times P; o\rho(x) = od(s),\ i\rho(x) \leqslant id(s)\} \to P$ is the action map, whose value on the pair (s, x) will be denoted, as usual, by $\theta_s(x)$ and for each $s \in \mathcal{C}$ preserves order.

Both maps must satisfy:

(I) $\theta_{i\rho(x)}(x) = x$;

(II) $o\rho(\theta_s(x)) = or(s)$ and $i\rho(\theta_s(x)) = ir(s)$ if $id(s) = i\rho(x)$;

(III) $\theta_s(\theta_t(x)) = \theta_{st}(x)$ if there exists the composition st in $\mathcal{C}$.

Notation: $(\rho, \theta) : (\mathcal{C}, (\ \ )^{\circ}) \curvearrowright (P, \leqslant)$ will denote a fibred ordered action.

As we will deal only with ordered fibred actions, we will refer to then by **fibred actions**.

To help the visualization, we present the following picture.

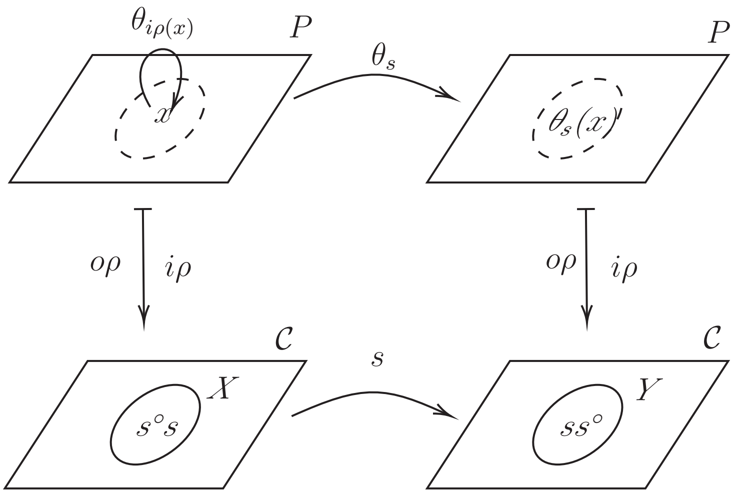


Figure 5.1: The inverse category fibred action

Notice that last definition combines the ordered groupoid fibred actions (cf. Definition 4.4.1 and inverse semigroups actions (as in Definition 1.3.15).

Another point we would like to highlight: as the reader should expect, if we fix an arrow $s \in \mathcal{C}$ each action map θ_s defines a bijection from the set where the function is defined to its range. This fact means that just like groupoid actions, a fibred action induces an action by automorphisms or via symmetries (as we presented in the Definition 4.2.2).

First, we must adapt the construction of the inverse category $\mathcal{I}_{ic}(-)$ for posets. Indeed, let $(P, \leqslant)$ be a poset and $\mathcal{P}_{(P,\leqslant)}$ be a partition of P formed by principal order ideals if it exists, then $\mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})$ composed by the following data, determines an inverse category

- objects: $\mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})^{(0)} = \mathcal{P}_{(P,\leqslant)}$;

- arrows: given sub principal order ideals $U, V \subset P$ in $\mathcal{P}_{(P,\leqslant)}$, morphisms from U to V is the set formed by order preserver bijections $s : U' \to V'$, where $U' \subseteq U$ and $V' \subseteq V$;

- composition: let $U, V, W \in \mathcal{P}_{(P,\leqslant)}$ be sub principal order ideals, its sub principal order ideals $U' \subseteq U, V', V'' \subseteq V, W' \subseteq W$, and order preserver bijections $s : U' \to V'$ and $t : V'' \to W'$, the composition ts is defined by

$$ts = t_{|V' \cap V''} \circ s_{|s^{-1}(V' \cap V'')} : s^{-1}(V' \cap V'') \to t(V' \cap V'');$$

- inverses: the inverse structure is the inversion of maps.

Remark 5.2.2. Note that the choice of principal order ideals leads us to a unique choice of partition, because the top elements of the poset are precisely the ideal generators.

Indeed, let P be a poset given by the disjoint union of principal order ideals $I_i = \langle m_i \rangle$ where $i \in \mathbb{I}$, *i.e.* $P = \bigcup_i I_i$ and suppose that M_P is the set of top elements of P. If $m \in M_P$ then $m \in I_i = \langle m_i \rangle$ for some m_i, and we have $m = m_i$. Moreover, each singleton $\{m_j\}$, where m_j is the generator of I_j, is a subset of M_P, otherwise there would exist any I_i such that $I_j \subset I_i$. Hence $\{m_i; i \in \mathbb{I}\} = M_P$.

The next definition is the analogous of the Definition 4.2.2 for inverse categories.

Definition 5.2.3. Given an inverse category $\mathcal{C}$ and a poset $(P, \leqslant)$, an *inverse category action via symmetries* (or *automorphisms*) of $\mathcal{C}$ on $(P, \leqslant)$ is a functor $\theta : \mathcal{C} \to \mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})$ such that $\mathcal{P}_{(P,\leqslant)}$ is the partition of $(P, \leqslant)$ given by $P = \bigcup\limits_{X \in \mathcal{C}^{(0)}} \mathrm{dom}(\theta_{1_X})$, where each $\mathrm{dom}(\theta_{1_X})$ is a principal order ideal whose top element is $1_X : X \to X$ and θ_s preserves order for each $s \in \mathcal{C}$.

Now, we are one step closer to relate fibred actions and actions by symmetries. The lemma below shows how to define maps among fibers from fibred actions.

Lemma 5.2.4. Each fibred action of an inverse category on a poset is in one-to-one correspondence with an action via symmetries of inverse categories.

Proof. Let $(\rho, \theta) : (\mathcal{C}, (\ )^\circ) \curvearrowright (P, \leqslant)$ be a fibred action of the inverse category $(\mathcal{C}, (\ )^\circ)$ on the poset $(P, \leqslant)$.

For a given $s \in \mathcal{C}$ define the map $\theta_s : D_{s^\circ} \to D_s$ whose domain and range are (respectively) the sets

$$D_{s^\circ} := \{x \in P; o\rho(x) = od(s),\ i\rho(x) \leqslant id(s)\}, \text{ and}$$

$$D_s := \{y \in P; o\rho(y) = or(s),\ i\rho(y) \leqslant ir(s)\}.$$

We need to show that θ_s is a well defined map and preserves the order, also we must show that D_s is a principal order ideal and that P is a union of domains. Indeed

- since (ρ, θ) is a fibred action, by the Definition 5.2.1, for each $s \in \mathcal{C}$ the map θ_s is a well defined order preserving map;

- let $s, t \in \mathcal{C}$ and $x, y \in X$ such that $s \leqslant t$ and $x \leqslant y$; suppose that $\theta_s(x)$ and $\theta_t(y)$ exist; since $i\rho$ is order preserving we have that $i\rho(x) \leqslant i\rho(y)$; so by the definition of the range of each action it is valid that $i\rho(x) \leqslant\leqslant id(t)$, and hence $D_s \subseteq D_t$; finally, as θ_t is order preserving and $x \leqslant y$, we can conclude that $\theta_s(x) \leqslant \theta_t(y)$;

- let $x \in P$ and $y \in D_{s^\circ}$ such that $x \leqslant y$ and $o\rho(x) = o\rho(y)$; since the map $i\rho$ is order preserver, by Definition 5.2.1, we have $i\rho(x) \leqslant i\rho(y)$ and by de definition of D_{s° follows that $i\rho(x) \leqslant i\rho(y) \leqslant id(s)$, hence D_{s° is a principal order ideal;

- let $1_X : X \to X$ be a unit arrow in $\mathcal{C}$, then $D_{1_X} = \{x \in P; o\rho(x) = X,\ i\rho(x) \leqslant 1_X\}$ since each $s \in \mathcal{C}$ satisfies $id(s) \leqslant 1_{od(s)}$ we have that $P = \bigcup_{X \in \mathcal{C}^{(0)}} D_{1_X}$.

Combining the previous computations we define the functor $F : \mathcal{C} \to \mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})$ by

$$X \in \mathcal{C}^{(0)} \mapsto F(X) := D_{1_X} \text{ and } s \in \mathcal{C} \mapsto F(s) := \theta_s.$$

As (θ, ρ) is a fibred action, by Definition 5.2.1 items (I) and (III) this association is indeed a functor.

By the other hand, suppose that $\theta : \mathcal{C} \to \mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})$ is an action by symmetries. Let X, Y be objects of $\mathcal{C}$; as θ is a functor, we have that $\theta(X)$ and $\theta(Y)$ are principal order ideals in $\mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})$. By the Definition 5.2.3, $P = \cup_{X \in \mathcal{C}^{(0)}} \mathrm{dom}(\theta_{1_X})$, so $\theta(X) = dom(\theta_{1_X})$ and $\theta(Y) = dom(\theta_{1_Y})$.

Now, consider an arrow $(s : X \to Y)$, by the definition of the inverse category $\mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})$ we have that $\theta_s : U \subset \theta(X) \to V \subset \theta(Y)$ is a bijection in $\mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})$. Note that for all $s \in \mathcal{C}$ the equality $ss^\circ s = s$ holds, and since θ is a functor we have that $\theta_s = (\theta_{ss^\circ})\theta_s = \theta_s\theta_{s^\circ s}$. Hence θ_{ss° and $\theta_{s^\circ s}$ are the identity maps in its own domains, so we must have that $U = dom(\theta_{s^\circ s})$ and $V = dom(\theta_{ss^\circ})$.

Using the previous constructions, given $(s : X \to Y) \in \mathcal{C}$ we define the outer and inner moment maps $(o\rho, i\rho) : P \to \mathcal{C}^{(0)} \times RId(C)$ where for each $x \in dom(\theta_{s^\circ s})$ $(\subset dom(\theta_{1_X}))$ we have that $o\rho(x) = X$ and $i\rho(x) = s^\circ s$. The action map is given by $\theta : \{(s, x) \in \mathcal{C} \times P; o\rho(x) = od(s),\ i\rho(x) \leqslant id(s)\} \to P$.

Let $s, t \in \mathcal{C}$ such that $s \leqslant t$, by the Proposition 5.1.10 we have that $s = ss^\circ t$ and

$$\theta_s = \theta_{ss^\circ t} = \theta_{ss^\circ}\theta_t = \theta_s\theta_{s^\circ}\theta_t.$$

By the equality $\theta_s = \theta_{ss^\circ s} = \theta_s\theta_{s^\circ}\theta_s$ for all $s \in \mathcal{C}$, follows that $\theta_{s^\circ} = (\theta_s)^\circ$. Hence if $s \leqslant t$ in $\mathcal{C}$

$$\theta_s = \theta_{ss^\circ t} = \theta_s(\theta_s)^\circ\theta_t \implies \theta_s \leqslant \theta_t.$$

From the construction of the inverse category $\mathcal{I}_{ic}(P, \leq)$ we can conclude that the map θ_s is a restriction of the map θ_t, so $dom(\theta_s) \subseteq dom(\theta_t)$, *i.e.* we have that $dom(\theta_{s^\circ s}) \subseteq dom(\theta_{tt^\circ})$. In particular, if $x, y \in P$ satisfy $x \leqslant y$ and $x \in dom(\theta_{s^\circ s})$ and $y \in dom(\theta_{tt^\circ})$, it is clear that $s^\circ s = i\rho(x) \leqslant i\rho(y) = t^\circ t$.

Finally, the conditions (I)-(III) from Definition 5.2.1 are easily verified. $\square$

Remark 5.2.5. The previous calculation uses some particular properties that functors between inverse categories satisfy. These functors are termed as ”partial functors” in Nystedt-Öinert-Pinedo [66] and some of their properties are described.

We want to define partial actions for inverse categories. The motivation is a combination of what we did for inverse semigroups (in Proposition 3.1.13), and the way we specified for ordered groupoids (in Proposition 4.2.4). In order to extend this setup (partial actions) for inverse categories, we have to adapt the corresponding definition given for ordered groupoids in Miller [62] via replacing objects by idempotents and using Lemma 5.2.4.

Definition 5.2.6. Let $\mathcal{C}$ be an inverse category and $(P, \leqslant)$ be an poset. Then a map $\theta : \mathcal{C} \to \mathcal{I}_{ic}((P, \leqslant))$ is an *inverse category partial action* if it defines a pair $(\{\theta_s\}_{s\in\mathcal{C}}, \{D_s\}_{s\in\mathcal{C}})$ satisfying

(i) for each $s \in \mathcal{C}$ the map $\theta_s : D_{s^\circ} \subseteq P \to D_s \subseteq P$ is an order preserving bijection;

(ii) $P = \displaystyle\bigcup_{X\in\mathcal{C}^{(0)}} D_{1_X}$;

(iii) if $e \in RId(\mathcal{C})$ the map θ_e is the identity in his domain D_e;

(iv) given $s, t \in \mathcal{C}$ such that $s \leqslant t$, then $D_{s^\circ} \subseteq D_{t^\circ}$ and $\theta_{t|D_{s^\circ}} = \theta_s$;

(v) if $s \in \mathcal{C}$, then $D_s \subseteq D_{ir(s)}$;

(vi) if exists $st \in \mathcal{C}$, then $\theta_s(D_{s^\circ} \cap D_t) = D_s \cap D_{st}$; and $\theta_s(\theta_t(x)) = \theta_{st}(x)$, for $x \in \theta_{t^\circ}(D_t \cap D_{s^\circ})$.

In addition: θ is a *global action* if $D_s = D_{ir(s)}$ for all $s \in \mathcal{C}$.

The next proposition exhibits a relation between the Definition 5.2.6 and the Definition 5.2.3.

Proposition 5.2.7. Let $\theta : \mathcal{C} \to \mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})$ be an action by symmetries, then θ induces a family of maps $(\{\theta_s\}_{s\in\mathcal{C}}, \{D_s\}_{s\in\mathcal{C}})$ such as in Definition 5.2.6 with $D_s = D_{ir(s)}$ for each $s \in \mathcal{C}$.

Proof. Given $(s : X \to Y) \in \mathcal{C}$, from the definition of functor and $\mathcal{I}_{ic}(\mathcal{P}_{(P,\leqslant)})$ we get

$$\theta_s : dom(\theta_s) \subseteq \theta(X) \to ran(\theta_s) \subseteq \theta(Y),$$

where $\theta_s = \theta(s)$.

Note that, from the proof of Lemma 5.2.4, as $\theta_{s^\circ} = \theta_s^{-1}$ we have that for all $s \in \mathcal{C}$ the equality $dom(\theta_s) = ran(\theta_{s^\circ})$ is valid. In particular if $e^2 = e$ in $\mathcal{C}$, then $\theta_e = \theta_e^{-1}$ and it is the identity map of $dom(\theta_e) = ran(\theta_e)$.

As $\mathcal{C}$ is an inverse category, $s = ss^\circ s$ for all $s \in \mathcal{C}$ and as θ is a functor, we obtain

$$\theta_s = \theta_{ss^\circ}\theta_s = \theta_s\theta_{s^\circ s}.$$

Hence

$$\begin{aligned}\theta_s &= \theta_s\theta_{s^\circ s}\\ &= \theta_{s|ran(\theta_{s^\circ s})\cap dom(\theta_s)} \circ \theta_{s^\circ s|(\theta_{s^\circ s})^{-1}(ran(\theta_{s^\circ s})\cap dom(\theta_s))}\\ &= \theta_{s|ran(\theta_{s^\circ s})\cap dom(\theta_s)}\end{aligned}$$

and it follows that $ran(\theta_{s^\circ s}) \cap dom(\theta_s) = dom(\theta_s)$, *i.e.* that

$$dom(\theta_s) \subseteq dom(\theta_{s^\circ s}) = ran(\theta_{s^\circ s}).$$

Using that $dom(\theta_{s^\circ}) = ran(\theta_s)$, we have that

$$\begin{aligned}\theta_{s^\circ s} &= \theta_{s^\circ}\theta_s\\ &= \theta_{s^\circ|dom(\theta_{s^\circ})\cap ran(\theta_s)} \circ \theta_{s|\theta_{s^\circ}(dom(\theta_{s^\circ})\cap ran(\theta_s))}\\ &= \theta_{s^\circ|dom(\theta_{s^\circ})} \circ \theta_{s|dom(\theta_s)}\\ &= \theta_{s^\circ}\theta_s\end{aligned}$$

and since the domain of the last composition is the domain of θ_s, it follows that

$$dom(\theta_s) = dom(\theta_{s^\circ s}).$$

Therefore we have the equalities

$$dom(\theta_s) = d(\theta_{s^\circ s}) = ran(\theta_{s^\circ s}),$$

$$ran(\theta_s) = ran(\theta_{ss^\circ}) = dom(\theta_{ss^\circ}).$$

Now if we define $D_s = ran(\theta_s)$, which implies that $D_{s^\circ} = dom(\theta_s)$, we have:

- by construction the item (i) is satisfied;
- as $D_{1_X} = ran(\theta_{1_X}) = dom(\theta_{1_X})$, each D_e is and ideal and $P = \bigcup_X dom(\theta_{1_X})$ (because θ is an action by symmetries), (ii) holds;
- if $e^2 = e$ the map θ_e is the identity map in its domain D_e, so (iii) holds;
- since $D_s = ran(\theta_s) = ran(\theta_{ss^\circ})$, we have $D_s = D_{ir(s)}$ for all s, (iv) holds;
- if s, t are arrow such that $s \leqslant t$, then $s = ts^\circ s$; combining this equivalency with the fact that $D_{s^\circ} = dom(\theta_s) = D_{s^\circ s}$ we have (vi);

- suppose $x \in D_s \cap D_t$, then there exists $y \in D_{t^\circ}$ such that $x = \theta_t(y)$; hence $\theta_s(x) = \theta_{st}(y) \in D_s \cap D_{st}$ and therefore

$$\theta_s(D_{s^\circ} \cap D_t) \subseteq D_s \cap D_{st}.$$

It follows then that

$$\theta_{s^\circ}(D_s \cap D_{st}) \subseteq D_{s^\circ} \cap D_{s^\circ st} = D_{s^\circ} \cap D_t$$

and applying θ_s, we obtain

$$D_s \cap D_{st} \subseteq \theta_s(D_{s^\circ} \cap D_t).$$

Therefore we concluded the proof. $\square$

Remark 5.2.8. Note that if instead of asking $\theta : \mathcal{C} \to \mathcal{I}_{ic}((P, \leqslant))$ to be an action by symmetries in the previous computation,we ask it only to be a functor, then only (ii) would not be possible to obtain. As we saw, (ii) will be satisfied if instead of P we consider the poset $P' = \bigcup_X D_{1_X}$. Since each D_e is an ideal of P, $\mathcal{I}ic(P') \subseteqq \mathcal{I}_{ic}(P)$, and θ factors trough this inclusion.

Proposition 5.2.9. Let $(\mathcal{C}, ()^\circ)$ be an inverse category and let $(P, \leqslant)$ be a poset. If $(\{\theta_s\}_{s\in\mathcal{C}}, \{D_s\}_{s\in\mathcal{C}})$ is a family of maps as in Definition 5.2.6 with $D_s = D_{ir(s)}$ such that for each $s \in \mathcal{C}$, then there exists a fibred action of $\mathcal{C}$ on $(P, \leqslant)$.

Proof. Let $s : X \to Y$ be a morphism of $\mathcal{C}$ and consider the bijection $\theta_s : D_{s^\circ} \to D_s$. Take $x \in D_{s^\circ}$ and note that by (ii) in Definition 5.2.6 and since $D_s = D_{ir(s)}$, we obtain

$$x \in D_{s^\circ} = D_{s^\circ s} \subseteq D_{1_X}.$$

Define the map $\rho = (o\rho, i\rho) : P \to \mathcal{C}^{(0)} \times RId(\mathcal{C})$ where

$$x \in D_{s^\circ s} \subseteq D_{1_X} \mapsto \rho(x) = (o\rho(x), i\rho(x)) = (X, s^\circ s).$$

Note that since by (i), (ii) and (iv) in Definition 5.2.6 the map ρ is well defined and $i\rho$ is order preserving.

Also note that the since $x \in D_{s^\circ} = D_{s^\circ s} \subseteq D_{1_X}$ we have that $s^\circ s \leqslant 1_X$, and the pair (s, x) satisfies

$$op(x) = od(s) \text{ and } i\rho(x) \leqslant id(s).$$

Define $\theta : \mathcal{C}_d \times_\rho P = \{(s, x) \in \mathcal{C} \times P; o\rho(x) = od(s),\ i\rho(x) \leqslant id(s)\} \to P$ by $(s, x) \mapsto \theta_s(x)$. This map is well defined because we are assuming a family $\{\theta_s : D_{s^\circ} \to D_s\}_s$ with $D_s = D_{ir(s)}$ such that for each $s \in \mathcal{C}$ of maps. Then we obtain

- (i) from Definition 5.2.6 that θ is order preserving;
- (iii) from Definition 5.2.6 that $\theta_{i\rho(x)}(x) = x$, since $i\rho(x)$ is an idempotent;
- if $x \in D_{s^\circ s} = D_{s^\circ}$, then $\theta_s(x) \in D_{ss^\circ} = D_s$ and since $i\rho(s) = s^\circ s$ we have that $o\rho(\theta_s(x)) = or(s)$ and $i\rho(\theta_s(x)) = ir(s)$ if $id(s) = i\rho(x)$;
- (iv) from Definition 5.2.6 that $\theta_s(\theta_t(x)) = \theta_{st}(x)$ if there exists the composition st in $\mathcal{C}$.

Thus (ρ, θ) is a fibred action of $\mathcal{C}$ on $(P, \leqslant)$. $\square$

We can present the last results condensed in the following "commuting diagram"

$$\begin{tikzcd}
\text{Fibred actions(5.2.1)} \arrow[rr, Leftrightarrow, "5.2.4"] & & \text{Action by symmetries(5.2.3} \arrow[dl, Rightarrow, "5.2.7"] \\
& \text{Partial actions with } D_- = D_{ir(-)}\text{(5.2.6)} \arrow[ul, Rightarrow, "5.2.9"] &
\end{tikzcd}$$

Fixing the notations: given an inverse category $\mathcal{C}$ and a poset P ,

- the symbol $\overline{\theta} : (\mathcal{C}, (\ )^\circ) \curvearrowright \mathcal{I}_{ic}(P)$, denotes a global action by symmetries, and
- the symbol $\theta : (\mathcal{C}, (\ )^\circ) \curvearrowright_p \mathcal{I}_{ic}(P)$ denotes a partial action by symmetries.

About the terminology: we deal only with actions on posets, so we will say action by symmetries and suppress the "ordered".

Remark 5.2.10. Once more, we fall back on the idea of restricting a global action. This idea has been escorting us since the beginning of the work. Indeed, let $\overline{\theta} : (\mathcal{C}, (\ )^\circ) \curvearrowright \mathcal{I}_{ic}(P)$ a global inverse category ordered action on a poset P. Suppose $Q \subseteq P$, then there is a partial action $\theta : (\mathcal{C}, (\ )^\circ) \curvearrowright_p \mathcal{I}_{ic}(Q)$, where: for each $s \in \mathcal{C}$ s

- the domain is $\mathrm{dom}(\theta_s) := \overline{\theta}_{s^\circ}(\mathrm{ran}(\overline{\theta}_s) \cap Q) \cap Q$,
- the image is $\mathrm{ran}(\theta_s) = \mathrm{ran}(\overline{\theta}_s) \cap Q$,
- the map is $\theta_s := \overline{\theta}_{s|\mathrm{dom}(\theta_s)}$.

In the next section, we will apply this construction to a particular action: the Bernoulli action.

5.2.2 Bernoulli inverse category actions

We will introduce Bernoulli actions in a similar way as it was done in the previous cases of groups, semigroups and groupoids. One of the main differences is that now we must take objects and arrows into account.

Let $(\mathcal{C}, (\ )^{\circ})$ be an inverse category. Given $X \in \mathcal{C}^{(0)}$ and $e \in \mathcal{C}(X,X)$, $e = e^2$, we define the set

$$P_{e,X} := \{A \subset \mathcal{C}; \mid A \mid < \infty,\ or(a) = X,\ ir(a) = e\ \forall a \in A\}.$$

Then we define the following sets

$$P(\mathcal{C}) := \bigcup_{\substack{X \in \mathcal{C}^{(0)} \\ e^2 = e \in \mathcal{C}(X,X)}} P_{e,X}$$

$$P_{\circ}(\mathcal{C}) := \{A \in P(\mathcal{C}); A \cap RId(\mathcal{C}) \neq \emptyset\}.$$

An equivalent characterization is possible if we use costars and right relations. For instance, if $X \in \mathcal{C}^{(0)}$ and $e = e^2 \in \mathcal{C}(X,X)$

$$A \in P(\mathcal{C}) \iff \mid A \mid < \infty,\ A \subset Costar(X),\ A \subset \mathscr{R}_e, \text{ and}$$

$$A \in P_{\circ}(\mathcal{C}) \iff A \ni e.$$

In a few paragraphs, we will define actions of $\mathcal{C}$ on these sets. First, it is essential to realize that we can imbue these sets with a partial order.

Definition 5.2.11. Let $\mathcal{C}$ be an inverse category and consider $A, B \in P(\mathcal{C})$ such that $A \subset Costar(X)$, $A \subset \mathscr{R}_e$, and $B \subset Costar(Y)$, $B \subset \mathscr{R}_f$, for $X, Y \in \mathcal{C}^{(0)}$ and $e = e^2 \in \mathcal{C}(X,X)$, $f = f^2 \in \mathcal{C}(Y,Y)$; then we define an order on $P(\mathcal{C})$ by

$$A \leqslant B \iff X = Y,\ e \leqslant f,\ eB \subseteq A.$$

Where the order between the idempotents is the natural order of inverse categories – which was stated in Proposition 5.1.10.

Notation: $(P(\mathcal{C}), \leqslant)$.

Clearly, this order restricts to $P_{\circ}(\mathcal{C})$.

This definition is the same we used earlier for inverse semigroups, in Lemma 3.2.1 item (iv), but now for each object of $\mathcal{C}$.

Using the tools we have developed in this section, we define the Bernoulli action of an inverse category.

Definition 5.2.12. The (*global*) *Bernoulli fibred action* of an inverse category $\mathcal{C}$ on $P(\mathcal{C})$ is the pair $(\varepsilon, \mathfrak{B})$, where

- $\varepsilon = (o\varepsilon, i\varepsilon)$ is the moment map $\varepsilon : P(\mathcal{C}) \to \mathcal{C}^{(0)} \times RId(\mathcal{C})$, which is defined as follows : $A \subset Costar(X)$ and $A \subset \mathscr{R}_e$ then $\varepsilon(A) = (o\varepsilon(A), i\varepsilon(A)) = (X, e)$;
- the action map $\mathfrak{B} : \mathcal{C}_d \times_\varepsilon P(\mathcal{C}) \to P(\mathcal{C})$ with $(s, A) \mapsto \mathfrak{B}(s, A) = \mathfrak{B}_s(A) := sA$.

Although we already used the word "action", being such is a property that we now verify. Indeed, following Definition 5.2.1: suppose $A \subset Costar(X)$ and $A \subset \mathscr{R}_e$, such that $e = e^2 : X \to X$.

The action map is well defined: if there exists $\mathfrak{B}_s(A) = sA$, then sA is a finite set such that $sA \subset Costar(or(s))$ and $sA \subset \mathscr{R}_{ses^\circ}$. Next, we will verify the order preserving conditions of the moment map and the action map

- by construction, if $A \leqslant B$, then $i\varepsilon(A) \leqslant i\varepsilon(B)$, so the inner moment preserves order;
- suppose $s, t \in \mathcal{C}$ such that $s \leqslant t$ and let $A, B \in P(\mathcal{C})$ such that $\mathfrak{B}_s(A)$ and $\mathfrak{B}_t(B)$ are defined; also, suppose $B \subset \mathcal{R}_f$ and $B \subset Costar(Y)$, $s : X \to U$ and $t : Y \to V$; if $s \leqslant t$ and $A \leqslant B$ we will show that $\mathfrak{B}_s(A) \leqslant \mathfrak{B}_t(B)$:
  - since $o\varepsilon(sA) = or(s) = U$, $o\varepsilon(tB) = or(t) = V$, and from $s \leqslant t$ we have that the composition $s^\circ t$ exists, so $or(t) = od(s^\circ)$ which implies in $U = V$;
  - as $i\varepsilon(sA) = ses^\circ$ and $i\varepsilon(tb) = tft^\circ$, and from $s \leqslant t$ and $e \leqslant f$ we have $ses^\circ \leqslant sfs^\circ \leqslant tft^\circ$;
  - as $eB \subseteq A$, $e \leqslant id(s)$ and $f \leqslant id(t)$, $e \leqslant f$ and $s \leqslant t$ we can show that $sef = ses^\circ sf = ses^\circ st^\circ tf = sefs^\circ st^\circ t = ses^\circ t$; hence $ses^\circ(tB) = seB \subseteq sA$.

  The conclusion is that $sA \leqslant tB$, *i.e.* $\mathfrak{B}_s(A) \leqslant \mathfrak{B}_t(B)$.

The following computations are the verification of the axioms (I)-(III) from Definition 5.2.1:

(I)
- $od(i\varepsilon(A)) = od(e) = X$, and
- $\mathfrak{B}_{i\varepsilon(A)}(A) = i\varepsilon(A)A = eA = A$, since for all $a \in A$ $a = (aa^\circ)a = ea$.

(II) $\varepsilon(\mathfrak{B}_s(A)) = (o\varepsilon(sA), i\varepsilon(sA))$, where

- $o\varepsilon(sA) = or(s)$, since $sA = \{sa \in \mathcal{C}; a \in A\}$, and
- $i\varepsilon(sA) = ss^\circ = ir(s)$, because $(sa)(sa)^\circ = saa^\circ s^\circ = ss^\circ$, if $s^\circ s = id(s) = i\varepsilon(A) = e$;

(III) $\mathfrak{B}_s(\mathfrak{B}_t(A)) = \mathfrak{B}_{st}(A)$ follows directly from the existence of the composition st in $\mathcal{C}$;

Now that we constructed the global Bernoulli action, we may restrict it to the poset $P_\circ(\mathcal{C})$ and thus acquire a partial action. To achieve this goal, we deal with the global action by symmetries derived from $(\varepsilon, \mathfrak{B})$.

Fix $s \in \mathcal{C}$, the *Bernoulli action by symmetries* is the map

$$\mathfrak{B}_s : \overline{D}_{s^\circ} \to \overline{D}_s \text{ with } A \mapsto \mathfrak{B}_s(A) = sA,$$

where

$$\overline{D}_{s^\circ} := \{A \in P(\mathcal{C}); o\varepsilon(A) = od(s),\ i\varepsilon(A) \leqslant id(s)\} \text{ and}$$

$$\overline{D}_s := \{B = sA \in P(\mathcal{C}); o\varepsilon(B) = or(s),\ i\varepsilon(B) \leqslant ir(s)\}.$$

Applying the method of Remark 5.2.10, *i.e.* the restriction of a global action to a partial action, we can define the *Bernoulli partial action by symmetries* $\mathfrak{b}_s : D_{s^\circ} \to D_s$., via

<u>domain</u>: $D_{s^\circ} = \mathfrak{B}_{s^\circ}(\overline{D}_s \cap P_\circ(\mathcal{C})) \cap P_\circ(\mathcal{C})$;

<u>range</u>: $D_s = \overline{D}_s \cap P_\circ(\mathcal{C})$;

<u>action map</u>: $\mathfrak{b}_s := (\mathfrak{B}_s)_{|D_s}$.

Describing in details: let $A \in \overline{D}_{s^\circ}$, such that $A \subset Costar(X), A \subset \mathscr{R}_e$, $e \leqslant id(s) = s^\circ s$ and $s : X \to Y$ in $\mathcal{C}$. See that

- from $sA = B \in \overline{D}_s$, results $B \subset Costar(Y)$ and $B \subset \mathscr{R}_{ses^\circ}$;
- if $sA = B \in \overline{D}_s \cap P_\circ(\mathcal{C})$, then $B \ni ses^\circ$;
- composing with s°, reveals $s^\circ B = s^\circ sA = A$, since $A \in \overline{D}_{s^\circ}$, and last item shows us $A \ni s^\circ(ses^\circ) = es^\circ$;
- finally, $s^\circ B = A \in (\overline{D}_s \cap P_\circ(\mathcal{C})) \cap P_\circ(\mathcal{C})$, if $A \ni e = es^\circ s$.

Previous data makes the structure of the partial action by symmetries clearer:

$$D_{s^\circ} = \{A \in P(\mathcal{C}); o\varepsilon(A) = od(s),\ i\varepsilon(A) \leqslant id(s),\ A \ni i\varepsilon(A)s^\circ, i\varepsilon(A)\}$$

$$D_s = \{B = sA \in P(\mathcal{C}); o\varepsilon(B) = or(s),\ i\varepsilon(B) \leqslant ir(s),\ B \ni s(i\varepsilon(A))s^\circ,\ s(i\varepsilon(A))\}.$$

Remark 5.2.13. Let A be an element in $P(\mathcal{C})$ such that $o\varepsilon(A) = e$ and $A \ni e, es^\circ$. If $e = s^\circ fs$, where $f \leqslant ss^\circ$, then

- $es^\circ s = s^\circ fss^\circ s = s^\circ fs = e \implies e \leqslant s^\circ s$;

- $s^\circ f = s^\circ f s s^\circ = e s^\circ$.

In particular, in D_s if we take $f = ses^\circ$, the set sA satisfies

- $o\varepsilon(sA) = f$,
- $f \in sA$ and
- $fs = ses^\circ s = se \in sA$.

Hence the domain and range of the map $\mathfrak{b}$ have the same formation rule.

Although we have been calling both maps ”actions” carelessly, we will provide the appropriate validation.

Lemma 5.2.14. Given an inverse category $(\mathcal{C}, (\ \ )^\circ)$ and the posets $P(\mathcal{C})$ and $P_\circ(\mathcal{C})$, the pairs $(\{\mathfrak{B}_s\}_{s\in\mathcal{C}}, \{\overline{D}_s\}_{s\in\mathcal{C}})$ and $(\{\mathfrak{b}\}_{s\in\mathcal{C}}, \{D_s\}_{s\in\mathcal{C}})$ are, respectively, a global and a partial inverse category actions. In addition, $\mathcal{C} \cdot P_\circ(\mathcal{C}) = P(\mathcal{C})$.

Proof. We will check the axioms of Definition 5.2.6 for $\mathfrak{B}$. The other case is analogous.

(i) From Lemma 5.2.4, the map $\mathfrak{B}$ is a order preserving bijection.

(ii) By construction $P(\mathcal{C}) = \bigcup\limits_{X\in\mathcal{C}^{(0)}} \overline{D}_{1_X}$;

(iii) If $A \in \overline{D}_e$, by definition $i\varepsilon(A) \leqslant e$, so $eA = A$.

(iv) Suppose $s \leqslant t$, by inverse category properties we have $s = ts^\circ s$ and $id(s) \leqslant ir(t)$. The existence of ts° implies in $od(t) = or(s^\circ) = od(s)$, in this case $\overline{D}_{s^\circ} \subset \overline{D}_{t^\circ}$. Moreover, $i\varepsilon(A) \leqslant id(s)$ and $s = ts^\circ s$, so $i\varepsilon(A) \leqslant s^\circ st^\circ$, and for $A \in \overline{D}_{s^\circ}$, thus

$$tA = ti\varepsilon(A)A = t(s^\circ st^\circ ti\varepsilon(A))A = ts^\circ sA = sA.$$

(v) Since $or(ss^\circ) = or(s)$ and $ir(ss^\circ) = ir(s)$, we can see $\overline{D}_s = \overline{D}_{ss^\circ}$.

(vi) Presume to be true the existence of the composition st and take $A \in \overline{D}_{s^\circ} \cap \overline{D}_t$. Last assumption asserts that

$$o\varepsilon(A) = od(s) \text{ and } o\varepsilon(A) = or(t),$$

and

$$i\varepsilon(A) \leqslant is(A) \text{ and } i\varepsilon(A) \leqslant ir(t).$$

If we calculate sA, observe that

$$o\varepsilon(sA) = or(s) = or(st),$$

$$i\varepsilon(sA) = si\varepsilon(A)s^{\circ} \leqslant s(id(s))s^{\circ} = ir(s)$$

and

$$i\varepsilon(sA) \leqslant s(ir(t))s^{\circ} = ir(st).$$

Therefore $sA \in \overline{D}_s \cap \overline{D}_{st}$. The last claim follows from the fact that $\mathcal{B}$ is the action map of a fibred action.

By proving the items above we have just shown that $\mathfrak{B}$ defines a global inverse category action by symmetries. $\square$

Notation:

$\mathfrak{B} : (\mathcal{C}, (\ )^{\circ}) \curvearrowright \mathcal{I}_{ic}(P(\mathcal{C}))$ is a global action, and

$\mathfrak{b} : (\mathcal{C}, (\ )^{\circ}) \curvearrowright_p \mathcal{I}_{ic}(P_{\circ}(\mathcal{C}))$ is a partial action.

Summing up: from the (global) Bernoulli fibred action, we were able to define a global and a partial action by symmetries. In the same manner, we did in the chapter on groupoids (Chapter 4), the partial action by symmetries induces a fibred action. Applying to our case we have the pair $(\epsilon, \mathfrak{b})$, where $\epsilon = \varepsilon_{P_{\circ}\mathcal{C}}$. More details will follow below.

After so many computations, we will write both actions side by side to reinforce their definitions:

global fibred Bernoulli action: $(\varepsilon, \mathfrak{B})$, where

- $\varepsilon : P(\mathcal{C}) \to \mathcal{C}^{(0)} \times RId(\mathcal{C})$, with $P(\mathcal{C}) \mapsto \varepsilon(A) = (o\varepsilon(A), i\varepsilon(A)) = (X, e)$;
- $\mathfrak{B} : \mathcal{C}_d \times_{\varepsilon} P(\mathcal{C}) \to P(\mathcal{C})$ with $(s, A) \mapsto \mathfrak{B}(s, A) = \mathfrak{B}_s(A) := sA$;

global Bernoulli action by symmetries: given $s \in \mathcal{C}$

$$\mathfrak{B}_s : \left\{ A \in P(\mathcal{C}); \begin{cases} o\varepsilon(A) = od(s), \\ i\varepsilon(A) \leqslant id(s) \end{cases} \right\} \to \left\{ B = sA \in P(\mathcal{C}); \begin{cases} o\varepsilon(B) = or(s), \\ i\varepsilon(B) \leqslant ir(s) \end{cases} \right\};$$

partial fibred Bernoulli action: $(\epsilon, \mathfrak{b})$, where

- $\epsilon : P_{\circ}(\mathcal{C}) \to \mathcal{C}^{(0)} \times RId(\mathcal{C})$, with $P(\mathcal{C}) \mapsto \epsilon(A) = (o\epsilon(A), i\epsilon(A)) = (X, e)$;
- $\mathfrak{b} : \mathcal{C}_d \times_{\epsilon} P_{\circ}(\mathcal{C}) \to P_{\circ}(\mathcal{C})$ with $(s, A) \mapsto \mathfrak{b}(s, A) = \mathfrak{b}_s(A) := sA$.

<u>partial Bernoulli action by symmetries</u>: given $s \in \mathcal{C}$

$$\mathfrak{b}_s : \left\{ A \in P(\mathcal{C}); \begin{cases} o\varepsilon(A) = od(s), \\ i\varepsilon(A) \leqslant id(s) \\ A \ni i\varepsilon(A)s^\circ, i\varepsilon(A) \end{cases} \right\} \to \left\{ B = sA \in P(\mathcal{C}); \begin{cases} o\varepsilon(B) = or(s), \\ i\varepsilon(B) \leqslant ir(s) \\ B \ni s(i\varepsilon(A))s^\circ, s(i\varepsilon(A)) \end{cases} \right\}.$$

There are two more actions we would like to define. These new actions will arise if we change the inequality by equality in both global and partial Bernoulli actions. We were inspired by O'Carroll's strict inverse semigroups (as we did in Chapter 3 Section 4.3).

In order to avoid cluttering the text with too much information we introduce new notations:

$$\mathcal{C}_d \overline{\times}_\varepsilon P(\mathcal{C}) := \{(A, s) \in \mathcal{C} \times P(\mathcal{C}); o\varepsilon(A) = od(s),\ i\varepsilon(A) = id(s)\},$$

$$\mathcal{C}_d \overline{\times}_\epsilon P_\circ(\mathcal{C}) := \{(A, s) \in \mathcal{C} \times P_\circ(\mathcal{C}); o\epsilon(A) = od(s),\ i\epsilon(A) = id(s)\}.$$

Next we define the *strict Bernoulli actions*.

<u>strict global fibred Bernoulli action</u> : $(\varepsilon, \mathfrak{s}\mathfrak{B})$, where

- $\varepsilon : P(\mathcal{C}) \to \mathcal{C}^{(0)} \times RId(\mathcal{C})$, with $P(\mathcal{C}) \mapsto \varepsilon(A) = (o\varepsilon(A), i\varepsilon(A)) = (X, e)$;
- $\mathfrak{s}\mathfrak{B} : \mathcal{C}_d \overline{\times}_\varepsilon P(\mathcal{C}) \to P(\mathcal{C})$ with $(s, A) \mapsto \mathfrak{s}\mathfrak{B}(s, A) = \mathfrak{s}\mathfrak{B}_s(A) := sA$;

<u>strict global Bernoulli action by symmetries</u>: given $s \in \mathcal{C}$

$$\mathfrak{s}\mathfrak{B}_s : \left\{ A \in P(\mathcal{C}); \begin{cases} o\varepsilon(A) = od(s), \\ i\varepsilon(A) = id(s) \end{cases} \right\} \to \left\{ B = sA \in P(\mathcal{C}); \begin{cases} o\varepsilon(B) = or(s), \\ i\varepsilon(B) = ir(s) \end{cases} \right\};$$

<u>strict partial fibred Bernoulli action</u>: $(\epsilon, \mathfrak{s}\mathfrak{b})$, where

- $\epsilon : P_\circ(\mathcal{C}) \to \mathcal{C}^{(0)} \times RId(\mathcal{C})$, with $P(\mathcal{C}) \mapsto \epsilon(A) = (o\epsilon(A), i\epsilon(A)) = (X, e)$;
- $\mathfrak{s}\mathfrak{b} : \mathcal{C}_d \overline{\times}_\epsilon P_\circ(\mathcal{C}) \to P_\circ(\mathcal{C})$ with $(s, A) \mapsto \mathfrak{s}\mathfrak{b}(s, A) = \mathfrak{s}\mathfrak{b}_s(A) := sA$.

<u>strict partial Bernoulli action by symmetries</u>: given $s \in \mathcal{C}$

$$\mathfrak{s}\mathfrak{b}_s : \left\{ A \in P(\mathcal{C}); \begin{cases} o\varepsilon(A) = od(s), \\ i\varepsilon(A) = id(s) \\ A \ni s^\circ, s^\circ s \end{cases} \right\} \to \left\{ B = sA \in P(\mathcal{C}); \begin{cases} o\varepsilon(B) = or(s), \\ i\varepsilon(B) = ir(s) \\ B \ni ss^\circ, s \end{cases} \right\}.$$

We conclude with a diagram relating all the partial and global Bernoulli actions presented in this section:

$$
\begin{tikzcd}
 & & & & \mathfrak{B}_s : \overline{D}_{s^*} \to \overline{D}_s \\
\mathfrak{sB}_s : \overline{D}^m{}_{s^*} \to \overline{D}^m{}_s & & & \mathfrak{B} : (\mathcal{C}, (\ )^\circ) \curvearrowright P(\mathcal{C}) \arrow[ur, squiggly] \arrow[dddd, dotted, no head, bend left] & \\
 & \mathfrak{sB} : (\mathcal{C}, (\ )^\circ) \curvearrowright P(\mathcal{C}) \arrow[ul, squiggly] \arrow[dd, dotted, no head, bend right] & & & \\
 & & (\mathcal{C}, (\ )^\circ) \arrow[ul] \arrow[dl] \arrow[uur] \arrow[ddr] & & \\
 & \mathfrak{sb} : (\mathcal{C}, (\ )^\circ) \curvearrowright_p P_\circ(\mathcal{C}) \arrow[dl, squiggly] & & & \\
\mathfrak{sb}_s : D^m{}_{s^*} \to D^m{}_s & & & \mathfrak{b} : (\mathcal{C}, (\ )^\circ) \curvearrowright_p P_\circ(\mathcal{C}) \arrow[dr, squiggly] & \\
 & & & & \mathfrak{b}_s : D_{s^*} \to D_s
\end{tikzcd}
$$

Figure 5.2: The inverse category Bernoulli actions

5.3 Categorical semidirect product

In earlier chapters, we constructed structures from Bernoulli actions; inverse categories' actions will not be different. We will explore these constructions in the next paragraphs, many properties are similar to inverse semigroups and groupoids, and we will show new ones.

First, we will define inverse categories for a generic inverse category fibred action. Next, we specialize it to our study of Bernoulli's actions.

Let $(\rho, \theta) : (\mathcal{C}, (\ )^\circ) \curvearrowright (P, \leqslant)$ be a fibred action of an inverse category on a poset.

Definition 5.3.1. The *semidirect product* determined by (ρ, θ) is the category $P \rtimes_{(\rho,\theta)} \mathcal{C}$ with structure

arrows: $P \rtimes_{(\rho,\theta)} \mathcal{C} := \{(x, s) \in P \times \mathcal{C}; o\rho(x) = or(s),\ i\rho(x) \leqslant ir(s)\}$

objects: $(P \rtimes_{(\rho,\theta)} \mathcal{C})^{(0)} := \{(x, 1) \in P \times Rid(\mathcal{C}); o\rho(x) = or(1),\ i\rho(x) \leqslant 1\}$ where 1 is an unit of the object $o\rho(x)$ in $\mathcal{C}^{(0)}$;

composition: $(x, s)(y, t) = (x, st)$ if, and only if, $x = \theta_s(y)$ and $\exists st \in \mathcal{C}$.

A routine verification shows that $P \rtimes_{(\rho,\theta)} \mathcal{C}$ is indeed a category and its objects are in correspondence with the set P via $(x, 1) \mapsto x$.

It is important to describe the set of restriction idempotents of this category

$$RId(P \rtimes_{(\rho,\theta)} \mathcal{C}) := \{(x, e) \in P \times Rid(\mathcal{C}); o\rho(x) = or(e),\ i\rho(x) \leqslant e\}.$$

Remark 5.3.2. From definition all elements of the semidirect product $P \rtimes_{(\rho,\theta)} \mathcal{C}$ are arrows, so to identify its objects we used the "Nonobjective approach" from Mitchell's book [63].

Also note that this methodology already appeared in our first definition of groupoids in Definition 1.3.1, where we followed a set theoretical point of view from Eilenberg categories (cf. [51]).

Naturally, we can define a forgetful functor $\pi : P \rtimes \mathcal{C} \to \mathcal{C}$, with

$$((x, s) : (a, 1_X) \to (b, 1_Y)) \in \mathcal{P} \rtimes \mathcal{C} \mapsto (s : X \to Y \in \mathcal{C}).$$

When there is no space for misreading, we will write just $P \rtimes \mathcal{C}$.

Lemma 5.3.3. The category $P \rtimes \mathcal{C}$ with $(x, s)^\circ := (\theta_{s^\circ}(x), s^\circ)$, is an inverse category.

Proof. We will verify the axioms of item (iii) of Proposition 5.1.7. Indeed: let (x, s) and (y, t) be elements of $P \rtimes \mathcal{C}$.

(I) Applying two times the involution to an arrow, we find

$$[(x, s)^\circ]^\circ = (\theta_{s^\circ}(x), s^\circ)^\circ = (\theta_s(\theta_{s^\circ}(x)), (s^\circ)^\circ).$$

Note that

$$\theta_s(\theta_{s^\circ}(x)) = \theta_{ss^\circ}(x) = \theta_{ir(s)}(x) = \theta_{i\rho(s)}(x) = x \text{ and } (s^\circ)^\circ = s.$$

Thus $[(x, s)^\circ]^\circ = (x, s)$.

(II) Using same arguments of previous lines, and properties of inverse categories arrows, we can assure

$$(x, s)(x, s)^\circ(x, s) = (x, s)(\theta_{s^\circ}(x), s^\circ)(x, s) = (x, ss^\circ)(x, s) = (x, ss^\circ s) = (x, s).$$

Similar arguments holds for $(x, s) = (y, t)^\circ$.

(III) Suppose there exists the product

$$[(x, s)(x, s)^\circ][(y, t)(y, t)^\circ] = (x, ss^\circ)(y, tt^\circ),$$

so there exists

$$(x, ss^\circ)(y, tt^\circ) = (x, ss^\circ tt^\circ) \iff x = \theta_{ss^\circ}(y).$$

Also, consider the existence of

$$[(y, t)(y, t)^\circ][(x, s)(x, s)^\circ] = (y, tt^\circ)(x, ss^\circ),$$

then

$$(y, tt^\circ)(x, ss^\circ) = (y, tt^\circ ss^\circ) \iff y = \theta_{tt^\circ}(x).$$

Since $\mathcal{C}$ is an inverse category, is true that $ss^\circ tt^\circ = tt^\circ ss^\circ$; from $x = \theta_{ss^\circ}(x)$ and $y = \theta_{tt^\circ}(y)$ we deduce

$$\theta_{tt^\circ}(x) = \theta_{tt^\circ}(\theta_{ss^\circ}(x)) = \theta_{ss^\circ}(\theta_{tt^\circ}(x)) = \theta_{ss^\circ}(y).$$

Finally, wrapping up all prior arguments we have the computation

$$(x, ss^\circ)(y, tt^\circ) = (x, ss^\circ tt^\circ) = (\theta_{ss^\circ}(y), tt^\circ ss^\circ) = (\theta_{tt^\circ}(x), tt^\circ ss^\circ) = (y, tt^\circ ss^\circ),$$

since $(y, tt^\circ ss^\circ) = (y, tt^\circ)(x, ss^\circ)$, we achieve our goal

$$(x, ss^\circ)(y, tt^\circ) = (y, tt^\circ)(x, ss^\circ).$$

We confirmed the axioms, them $P \rtimes \mathcal{C}$ is an inverse category, and the proof is complete. □

Finishing the discussion about this topic, by Definition 5.1.13 and using the involution: for $(x, s) \in P \rtimes \mathcal{C}$,

the inner source of (x,s) is the idempotent map $id(x, s) = (\theta_{s^\circ}(x), s^\circ s)$, and

the inner target of (x,s) is the idempotent map $ir(x, s) = (x, ss^\circ)$.

5.3.1 The categories associated to Bernoulli actions

The machinery from the last paragraphs is put in motion in this subsection, where we will define the expansion of an inverse category.

Definition 5.3.4. Let $(\varepsilon, \mathfrak{B}) : (\mathcal{C}, ()^\circ) \curvearrowright (P(\mathcal{C}), \leqslant)$ be the Bernoulli fibred action. The *Szendrei expansion* of the inverse category $\mathcal{C}$ is the inverse category

$$\overline{Sz}(\mathcal{C}) := P(\mathcal{C}) \rtimes_{(\varepsilon, \mathfrak{B})} \mathcal{C}.$$

Remark 5.3.5. Before we proceed, we will explain the terminology we chose. Inspired by Hollings [43], we decided to name our expansion after Maria B. Szendrei.

As the description of the ”International Conference on Semigroups - On the occasion of the 65th birthday of MÃ¡ria Szendrei” (Univ. Szeged, Hungary) says

Professor Maria B. Szendrei is world recognized leader on the theory of regular semigroups and their generalization. Her research develops and deepens several classic directions in studying various classes of regular semigroups; at the same time, she has invented many novel and inherently original approaches that opened new avenues of research. Her outstanding services to community include maintaining excellent scientific quality of the Hungarian mathematical journals and organizing a series of semigroup conferences that had proved to be important meeting points between semigroupists from Western countries and those from former Eastern block countries.

This conference occurred at the Faculty of Sciences, University of Lisbon, 11-14 July 2018.

Aiming further computations, and to make the definition clear, we will write this semidirect product explicitly:

<u>arrows</u>: $\overline{Sz}(\mathcal{C}) = \{(A,s) \in P(\mathcal{C}) \times \mathcal{C}; o\varepsilon(A) = or(s),\ i\varepsilon(A) \leqslant ir(s)\}$;

<u>objects</u>: $\overline{Sz}(\mathcal{C})^{(0)} = \{(A,1) \in P(\mathcal{C}) \times RId(\mathcal{C}); o\varepsilon(A) = or(1),\ i\varepsilon(A) \leqslant 1\}$;

<u>res. idemp.</u>: $RId(\overline{Sz}(\mathcal{C})) = \{(E,e) \in P(\mathcal{C}) \times RId(\mathcal{C}); o\varepsilon(E) = or(e),\ i\varepsilon(E) \leqslant e\}$;

<u>composition</u>: $(A,s)(B,t) = (A,st)$ if, and only if, $A = sB$ and there exists $st \in \mathcal{C}$;

<u>involution</u>: $(A,s)^\circ = (s^\circ A, s^\circ)$;

<u>outer source</u>: $od(A,s) = (s^\circ A, 1_{o\varepsilon(s^\circ A)})$;

<u>outer target</u>: $or(A,s) = (A, 1_{o\varepsilon(A)})$;

<u>inner source</u>: $id(A,s) = (s^\circ A, s^\circ s)$;

<u>inner target</u>: $ir(A,s) = (A, ss^\circ)$.

We can summarize last identities and think about an arrow in $\overline{Sz}(\mathcal{C})$ as

$$(A,s) : s^\circ A \to A \text{ where } (A,s)(s^\circ A, s^\circ s) = (A,s) = (A,ss^\circ)(A,s).$$

The next theoretical aspect we will develop is an ordering of our expansion. Some words of motivation: there exists a natural order for inverse semigroups. On the other hand, one must demand ordering for groupoids; this aspect is an example of a general definition of ordered categories. Remember that pseudo products are the link between inverse semigroups and inductive groupoids – the ESN Theorem's (4.1.6) critical idea.

We will conclude this section exposing the study of the ordering on $\overline{Sz}(\mathcal{C})$. To establish the theory, we will follow Hollings [43] Definition 7.7.2.

Definition 5.3.6 ([43]). Let $\mathcal{C}$ be a small category endowed with a partial order relation $\leqslant$ – induced by unitary maps on objects –, we say that $(\mathcal{C}, \leqslant)$ is an *ordered category* if the next axioms hold: for $s, t, s‘, t‘ \in \mathcal{C}$ and $X, Y \in \mathcal{C}^{(0)}$

(I) $s \leqslant s‘,\ t \leqslant t‘,\ \exists st,\ \exists s't' \implies st \leqslant s‘t‘$;

(II) $s \leqslant t \implies d(s) \leqslant d(t),\ r(s) \leqslant r(t)$;

(III) $X \leq d(s) \implies \exists!\ {}_{X|}s \in \mathcal{C}$ s.t. ${}_{X|}s \leqslant s,\ d({}_{X|}s) = X$;

(IV) $Y \leq r(t) \implies \exists!\ t_{|Y} \in \mathcal{C}$ s.t. $t_{|Y} \leqslant t,\ r(t_{|Y}) = Y$.

The arrow of item (III) is the *restriction*, and the arrow of (IV) the *corestriction*.

The previous definition is originated from inverse semigroup and ordered groupoid theories, as Hollings [43] explained in his Ph.D. thesis. In both theories, the characterization of objects uses idempotent arrows or the source and target maps. The inverse categories theory is richer in terms of idempotents, so we adjust previous definitions to accomplish such structures replacing source and target maps with its internal versions and our restriction are on idempotents, rather than objects.

Definition 5.3.7. We say that an inverse category $(\mathcal{C}, (\ )^{\circ})$ endowed with a partial ordering $\leqslant$ is an *ordered inverse category* when for $s, t, s‘, t‘ \in \mathcal{C}$ and idempotent arrows $e, f \in \mathcal{C}$

(I) if $s \leqslant s'$, $t \leqslant t‘$, $\exists st$ and $\exists s't'$ then $st \leqslant s‘t‘$;

(II) if $s \leqslant t$, then $id(s) \leqslant id(t)$ and $ir(s) \leqslant ir(t)$;

(III) if $e \leqslant id(s)$, then there exists a unique ${}_{e|}s \in \mathcal{C}$ s.t. ${}_{e|}s \leqslant s$ and $id({}_{e|}s) = e$;

(IV) $f \leqslant ir(t)$, then there exists a unique $t_{|f} \in \mathcal{C}$ s.t. $t_{|f} \leqslant t$ and $ir(t_{|f}) = f$.

Note that when the inverse category $\mathcal{C}$ is a groupoid, *i.e.* for all $s \in \mathcal{C}$ the idempotents $s^{\circ}s$ and ss° are unit maps, we have the Definition 4.1.1. Indeed for groupoids an object can be associated to its unit map, so there is no distinction between idempotent maps and unit maps.

Remark 5.3.8. Every inverse category is an ordered inverse category by its natural ordering. Surely Lemma 5.1.3) will guarantee axioms (I), (II) and (III); for the last one if $e \leqslant id(s)$ and $f \leqslant ir(t)$ define ${}_{e|}s := se$ and $t_{|f} := ft$.

Each element of the Szendrei expansion has two entries: the first comes from the set $P(\mathcal{C})$, and the second is an arrow of $\mathcal{C}$. So, the natural candidate for a partial order in our expansion is the combination of both.

<u>Definition</u> 5.3.9. For each pair of elements $(A,s),(B,t) \in \overline{Sz}(\mathcal{C})$, we define the partial order $\leqslant$, by

$$(A,s) \leqslant (B,t) \iff A \leqslant_{P(\mathcal{C})} B,\ s \leqslant_{\mathcal{C}} t.$$

We will drop the inequality's labels because the context determines the nature of each one.

Let us make the definitions more clear: suppose $(A,s),(B,t) \in \overline{Sz}(\mathcal{C})$ are pairs with

$$A \subset Costar(X),\ A \subset \mathscr{R}_e \text{ and } B \subset Costar(Y),\ B \subset \mathscr{R}_f,$$

where $X,Y \in \mathcal{C}^{(0)}$ and $e^2 = e : X \to X, f^2 = f : Y \to Y \in \mathcal{C}$.

By Definition 5.2.11 and Proposition 5.1.10,

$$(A,s) \leqslant (B,t) \iff \begin{cases} X = Y \\ e \leqslant f \\ eB \subseteq A \end{cases} \quad \text{and } s = ss^\circ t.$$

Remark 5.3.10. The partial order that we have just defined is finer than the natural order of inverse categories, in the following sense: let (A,s) and (B,t) be two arrows in $\overline{Sz}(\mathcal{C})$ and suppose $(A,s) \leqslant (B,t)$.

If we use the natural order, as in Proposition 5.1.9, then the order is equivalent to the equality $(A,s) = (A,s)(A,s)^*(B,t)$, which implies $i\varepsilon(A)B = A$.

By the other hand, if $(A,s) \leqslant (B,t)$, then our definition leads us to $i\varepsilon(A)B \subseteq A$.

Thus the natural order also satisfies our definition of order, but the converse is not true.

The following lemma is the most important result of this subsection.

<u>Lemma</u> 5.3.11. The Szendrei expansion $(\overline{Sz}(\mathcal{C}), (\ )^\circ)$ with $\leqslant$ is an ordered inverse category.

Proof. We will verify each condition of Definition 5.3.6. Indeed

(I) Let $(A,s),(A^‘,s^‘),(B,t),(B^‘,t^‘)$ be elements in $\overline{Sz}(\mathcal{C})$ with

$$\begin{cases} A \subset Costar(X),\ A \subset \mathscr{R}_e \\ A^‘ \subset Costar(X^‘),\ A^‘ \subset \mathscr{R}_{e^‘} \end{cases} \text{and} \begin{cases} B \subset Costar(Y),\ B \subset \mathscr{R}_f \\ B^‘ \subset Costar(Y^‘),\ B^‘ \subset \mathscr{R}_{f^‘} \end{cases},$$

where $e^2 = e : X \to X, e^{‘2} = e^‘ : X^‘ \to X^‘, f^2 = f : Y \to Y, f^{‘2} = f^‘ : Y^‘ \to Y^‘ \in \mathcal{C}$.

Consider $(A,s) \leqslant (A^‘,s^‘)$ and $(B,t) \leqslant (B^‘,t^‘)$, or equivalently

$$\begin{cases} X = X^‘,\ e \leqslant e^‘,\ eA^‘ \subseteq A,\ s \leqslant s^‘ \\ Y = Y^‘,\ f \leqslant f^‘,\ fB^‘ \subseteq B,\ t \leqslant t^‘ \end{cases}.$$

Also, assume that the compositions $(A, s)(B, t)$ and $(A', s')(B', t')$ exist, *i.e.*

$$\begin{cases} (A, s)(B, t) = (A, st) \iff A = sB,\ \exists st \\ (A', s')(B', t') = (A', s't') \iff A' = s'B',\ \exists s't' \end{cases}.$$

We must verify if $(A, st) \leqslant (A', s't')$, that is, if $A \leqslant A'$ and $st \leqslant s't'$. By construction we already have $A \leqslant A'$, and by Lemma 5.1.3

$$s \leqslant t,\ s' \leqslant t' \implies st \leqslant s't'.$$

This last computation concludes the first item.

(II) Take $(A, s), (B, t) \in \overline{Sz}(\mathcal{C})$, with $A \subset Costar(X), A \subset \mathscr{R}_e, B \subset Costar(Y), B \subset \mathscr{R}_f$, and $s : U \to X,\ t : V \to Y$. The previous objects and arrows are in $\mathcal{C}$.

Suppose $(A, s) \leqslant (B, t)$. This hypothesis implies

$$X = Y,\ e \leqslant f, eB \leqslant A \text{ and } s \leqslant t.$$

Also, by the definition of elements in $\overline{Sz}(\mathcal{C})$, we must have

$$\begin{cases} o\varepsilon(A) = or(s) \\ i\varepsilon(A) \leqslant ir(s) \end{cases} \text{and} \begin{cases} o\varepsilon(B) = or(t) \\ i\varepsilon(B) \leqslant ir(t) \end{cases}.$$

We want to confirm $id(A, s) \leqslant id(B, t)$ and $ir(A, s) \leqslant ir(B, t)$.

- First we will show: $id(A, s) \leqslant id(B, t)$, *i.e.* $(s^\circ A, s^\circ s) \leqslant (t^\circ B, t^\circ t)$. To prove the last inequality, by Definition 5.3.9, we must assure that

  $$s^\circ A \leqslant t^\circ B \text{ and } s^\circ s \leqslant t^\circ t.$$

  Beginning by the ordering in $P(\mathcal{C})$. As $\mathcal{C}$ is an inverse category, each arrow has a unique inverse, so

  $$s : U \to X,\ t : V \to Y \implies s^\circ : X \to U,\ t^\circ : Y \to V.$$

  In addition, $o\varepsilon(s^\circ A) = or(s^\circ)$ and $o\varepsilon(t^\circ B) = or(t^\circ)$. These facts imply

  $$s^\circ A \subset Costar(U) \text{ and } t^\circ B \subset Costar(V).$$

By construction, the arrows s and t obey the relation $s \leqslant t$. Using properties of the ordering in inverse categories (cf. Proposition 5.1.10), we infer

$$s^\circ \leqslant t^\circ \iff s^\circ = s^\circ st^\circ \implies \exists st^\circ \in \mathcal{C}.$$

Hence $or(t^\circ) = od(s)$, which means that $U = V$. At this moment we have already showed that $s^\circ A, t^\circ B \subset Costar(U)$, that is the first item of Definition 5.2.11.

Subsequently, we will prove that $i\varepsilon(s^\circ A) \leqslant i\varepsilon(t^\circ B)$. By assumption $A \subset \mathscr{R}_e$ and $B \subset \mathscr{R}_f$, implying

$$s^\circ A \subset \mathscr{R}_{s^\circ es} \text{ and } t^\circ B \subset \mathscr{R}_{t^\circ ft}.$$

The initial hypothesis also states that $e \leqslant f$ and $s \leqslant t$. From the first relation we conclude that

$$\exists ef = fe \in \mathcal{C} \implies X = Y,$$

and this equality grants the existence of the compositions: es, fs and ft in $\mathcal{C}$. The last inequality implies $s^\circ \leqslant t^\circ$. Using Lemma 5.1.3, just proven, we can compute

$$e \leqslant f \implies s^\circ es \leqslant s^\circ fs \leqslant t^\circ ft \implies s^\circ es \leqslant t^\circ ft.$$

Thus

$$i\varepsilon(s^\circ A) = s^\circ es \leqslant t^\circ ft = i\varepsilon(t^\circ B)$$

To finalize this stage of the proof, we must show that $(s^\circ es)(t^\circ B) \subset s^\circ A$. Since $i\varepsilon(A) \leqslant ir(s)$, where $i\varepsilon(A) = e$ and $ir(s) = ss^\circ$, we obtain $e = ess^\circ$. Also, the inequality $s^\circ \leqslant t^\circ$ implies $s^\circ = s^\circ st^\circ$. The initial supposition of $A \leqslant B$, permits us to conclude

$$eB \subset A \implies s^\circ(eB) \subset s^\circ A,$$

however

$$s^\circ e = s^\circ(ess^\circ) = (s^\circ es)s^\circ = (s^\circ es)s^\circ st^\circ = s^\circ est^\circ.$$

As a result

$$(s^\circ es)t^\circ B = s^\circ(eB) \subset s^\circ A.$$

The amount of computation we have made, shows $s^\circ A \leqslant t^\circ B$. The last piece we need comes from Lemma 5.1.3, because

$$s \leqslant t \implies s^\circ s \leqslant t^\circ t.$$

<u>Conclusion</u>: $s^\circ A \leqslant t^\circ B$ and $id(s) \leqslant id(t)$, *i.e.* $id(A, s) \leqslant id(B, t)$.

- The validity of $ir(A, s) \leqslant ir(B, t)$ follows. This one is tautological, since $ir(A, s) = (A, ss^\circ)$ and $ir(B, t) = (B, tt^\circ)$.

So, we have finished with (II).

(III) We will deal with restrictions and corestrictions, in this order.

Suppose (E, f) a restriction idempotent in $\in \overline{Sz}(\mathcal{C})$ and $(A, s) \in \overline{Sz}(\mathcal{C})$, such that

$$\begin{cases} E \subset Costar(Y) \\ E \subset \mathscr{R}_i \\ f : Y \to Y \\ i \leqslant f \end{cases} \text{and} \begin{cases} A \subset Costar(X) \\ A \subset \mathscr{R}_e \\ s : U \to X \\ e \leqslant ss^\circ \end{cases}.$$

Also, suppose $(E, f) \leqslant id(A, s)$, which means that

$$(E, f) \leqslant (s^\circ A, s^\circ s) \iff E \leqslant s^\circ A \text{ and } f \leqslant s^\circ s.$$

We will prove that

$$_{(E,f)|}(A, s) := (sE, sf)$$

is the restriction of (A, s) to (E, f).

We have to verify some technical details. Initially, we see it is a proper element of the Szendrei expansion. Then we will check the axioms of restrictions.

- There exists the composition sf in $\mathcal{C}$, by the reason of $s^\circ A \subset Costar(U)$ and $E \leqslant s^\circ A$, implying in $Y = U$.
- The composition sfE exists, because $od(sf) = od(f) = o\varepsilon(E) = Y$.
- The pair (sE, sf) is an element of $\overline{Sz}(\mathcal{C})$, due to the fact that

$$or(sf) = or(s) = X = o\varepsilon(sE),$$

and $i \leqslant f$ implies

$$i\varepsilon(sE) = sis^\circ \leqslant sfs^\circ = ir(sf).$$

- The inner source of (sE, sf) is

$$id(sE, sf) = ((sf)^\circ sE, (sf)^\circ(sf)) = (E, f),$$

since $f \leqslant s^\circ s$, and $E \subset \mathscr{R}_i$ and $i \leqslant f$ it is possible to resolve

$$(sf)^\circ sE = fs^\circ sE = fE = fiE = iE = E.$$

Furthermore,

$$(sf)^\circ(sf) = fs^\circ sf = f.$$

- To prove the relation $(sE, sf) \leqslant (A, s)$, we need to go over the inequalities $sE \leqslant A$ and $sf \leqslant s$, *i.e.* show that

$$o\varepsilon(sE) = o\varepsilon(A),\ i\varepsilon(sE) \leqslant i\varepsilon(A),\ i\varepsilon(sE)A \subset sE,\ sf \leqslant s.$$

Indeed, respectively

  - the outer moment map condition follows from $o\varepsilon(sE) = or(s) = o\varepsilon(A)$
  - the initial hypothesis $E \leqslant s^\circ A$ implies $i \leqslant s^\circ es$, so $i = s^\circ esi$, and $ss^\circ es = es$ implies $sis^\circ = esis^\circ$; so $i\varepsilon(sE) = sis^\circ \leqslant e$
  - since $E \leqslant s^\circ A$, we have $is^\circ A \subset E$, and multiplying by s on both sides we see $sis^\circ A \subset sE$.
  - the last item is a consequence of $(sf)(sf)^\circ s = sfs^\circ s = sf$, since $f \leqslant s^\circ s$.

- The remaining fact is the uniqueness of the restriction. Let (B, t) be another arrow in $\overline{Sz}(\mathcal{C})$, such that

$$(B, t) \leqslant (A, s) \text{ and } id(B, t) = (E, f).$$

Since $id(B, t) = (t^\circ B, t^\circ t)$, last line means

$$B \leqslant A,\ t \leqslant s,\ t^\circ B = E \text{ and } t^\circ t = f.$$

Using Proposition's 5.1.10 last inequality

$$t \leqslant s \implies t = st^\circ t = sf.$$

Furthermore, as $(B,t) \in \overline{Sz}(\mathcal{C})$, $tt^\circ B = B$, so

$$t^\circ B = E \implies B = tt^\circ B = tE.$$

Finally, $tE = sfE = sE$, because $(E,f) \in \overline{Sz}(\mathcal{C})$. In conclusion

$$(B,t) = (sE, sf).$$

Partial conclusion: ${}_{(E,f)|}(A,s) := (sE, sf)$ is the restriction.

We devote the next lines to exhibit the corestriction. For such purpose, consider (E,f) a restriction idempotent in $\overline{Sz}(\mathcal{C})$ and $(A,s) \in \overline{Sz}(\mathcal{C})$, such that

$$\begin{cases} E \subset Costar(Y) \\ E \subset \mathscr{R}_i \\ f : Y \to Y \\ i \leqslant f \end{cases} \text{and} \begin{cases} A \subset Costar(X) \\ A \subset \mathscr{R}_e \\ s : U \to X \\ e \leqslant ss^\circ \end{cases}.$$

In addition, let $(E,f) \leqslant ir(A,s)$, which means

$$(E,f) \leqslant (A, ss^\circ) \iff E \leqslant A \text{ and } f \leqslant ss^\circ.$$

These inequalities signify that

$$X = Y,\ i \leqslant e,\ iA \subset E \text{ and } f = fss^\circ.$$

We claim that the following arrow is the corestriction

$$(A,s)_{|(E,f)} = (E, fs).$$

Let us prove the last statement. Breaking it into intermediate steps:

- The arrow (E, fs) is well defined, since $X = Y$ implies the existence of fs in $\mathcal{C}$.
- (E, fs) is an element of Szendrei's expansion, for the reason that

  $$or(fs) = or(e) = X = Y = o\varepsilon(E)$$

  and

  $$ir(fs) = fss^\circ = f \implies i\varepsilon(E) = i \leqslant f = ir(fs).$$

- The corestriction's axioms:
  - We must confirm

$$(E, fs) \leqslant (A, s) \iff E \leqslant A \text{ and } fs \leqslant s$$

. The first item is true by the initial hypothesis. For the second one

$$(fs)(fs)^{\circ}s = fss^{\circ}fs = fs \implies fs \leqslant s.$$

- The inner target is

$$ir(E.fs) = (E, ir(fs)) = (E, sf),$$

using last item's computation.

- Finally, let us prove uniqueness. Suppose $(B, t) \in \overline{Sz}(\mathcal{C})$, such that

$$(B, t) \leqslant (A, s) \text{ and } ir(B, t) = (E, f).$$

The last line is equivalent to

$$B \leqslant A,\ t \leqslant s \text{ and } (B, tt^{\circ}) = (E, f.)$$

By Proposition 5.1.10,

$$t \leqslant s \text{ and } tt^{\circ} = f \implies t = tt^{\circ}s = fs.$$

Thus

$$(B, t) = (E, fs).$$

Since all the axioms are accurate, we can say $(A, s)_{|(E,f)} = (E, sf)$ is the corestriction.

In conclusion $((\overline{Sz}(\mathcal{C}), (\ )^{\circ}), \leqslant)$ is an ordered inverse category. □

Our next move is towards to define another operation among arrows. Likewise ordered groupoids' Definition 4.1.2, we aim for the pseudo products, but we do not have meets within idempotents yet.

After Dewolf-Pronk [25] Proposition 3.2, we state the next lemma.

Lemma 5.3.12 ([25]). *Let X be an object of the inverse category $(\mathcal{C}, (\ )^{\circ})$. The set $RId(\mathcal{C}(X))$, of idempotent morphisms in X, is a meet semilattice with the natural partial order from $\mathcal{C}$ and $e \wedge f := ef$.*

Proof. Let $e, f \in RId(\mathcal{C}(X))$, therefore

$$(e \wedge f)e = (ef)e = ef \implies e \wedge f \leqslant e.$$

The same arguments works for f. So $e \wedge f$ is a lower bound.

For the uniqueness, suppose m such that $m \leqslant e, f$ and $e \wedge f \leqslant m$. Immediately

$$m = mf = (me)f = m(ef) = ef = e \wedge f \implies m = e \wedge f.$$

□

It is worth to mention that Dewolf and Pronk also showed that each $RId(\mathcal{C}(X))$ has 1_X as top element.

Continuing, we extend the pseudo product's definition for an inverse category.

Definition 5.3.13. Let $\mathcal{C}$ be an ordered inverse category such that for each object X the set $RId(X)$, of restriction idempotents in X, is a meet semilattice. Let $s, t \in \mathcal{C}$ such that there exists $id(s) \wedge ir(t)$. The *pseudo product* of s and t is

$$s \star t := ({}_{id(s) \wedge ir(t)|}s)(t_{|id(s) \wedge ir(t)}).$$

Remark 5.3.14. Considering the natural order of the inverse category $\mathcal{C}$ and by the Lemma 5.3.12 $id(s) \wedge ir(t) = s^\circ s t t^\circ$; so the Remark 5.3.8 implies

$$s \star t = s(s^\circ s t t^\circ)(s^\circ s t t^\circ)t = st.$$

Also, pay attention to the fact that

$$\exists st \iff \exists s^\circ s t t^\circ = id(s) \wedge ir(t).$$

As the groupoid case, the pseudo product when defined extends the composition of arrows.

Before all else, to define the pseudo product for $\overline{Sz}(\mathcal{C})$, we must deal with wedges of idempotents. Inspired by Gilbert's Lemma 4.3.2 we state the following lemma.

Lemma 5.3.15. Let $(E, i), (F, j)$ be idempotent morphisms in $\overline{Sz}(\mathcal{C})$, such that there exists the composition ij in $\mathcal{C}$. Then the wedge product

$$(E, i) \wedge (F, j) := (i\varepsilon(F)E \cup i\varepsilon(E)F, ij)$$

is an idempotent arrow in Szendrei's expansion, and is the greatest lower bound of (E, i) and (F, j).

Proof. Given $(E, i), (F, j) \in \overline{Sz}(\mathcal{C})$, suppose that

$$\begin{cases} E \subset Costar(X) \\ E \subset \mathscr{R}_e \\ i : X \to X \\ e \leqslant i \end{cases} \quad \text{and} \quad \begin{cases} F \subset Costar(Y) \\ F \subset \mathscr{R}_f \\ j : Y \to Y \\ f \leqslant j \end{cases} .$$

By hypothesis

$$\exists ij \implies X = Y.$$

In addition, as $i\varepsilon(E) = e$ and $i\varepsilon(F) = f$, the wedge product is

$$(E, i) \wedge (F, j) := (fE \cup eF, ij).$$

We may now prove that the wedge product is an element of $\overline{Sz}(\mathcal{C})$, and that it is the greatest lower bound of (E, i) and (F, j):

- the existence of ij implies the existence of $ef = fe$, showing that the sets fE and eF are well defined;
- the pair $(fE \cup eF, ij)$ belongs to $\overline{Sz}(\mathcal{C})$, due to the following facts
  - $or(ij) = or(i) = X = Y = o\varepsilon(fE \cup eF)$;
  - $i\varepsilon(fE) = i\varepsilon(eF) = ef$, since $E \subset (R)_e$ and $F \subset (R)_f$;
  - on account of $e \leqslant i$ and $f \leqslant j$, Lemma 5.1.3 implies $ef \leqslant ij$, so $i\varepsilon(fE \cup eF) = ef \leqslant ij$.
- note that $ef(fE \cup eF) = fE \cup eF$ and $(ij)^2 = ij$, so $(fE \cup eF, ij)^2 = (fE \cup eF, ij)$.
- the relation $(fE \cup eF, ij) \leqslant (E, i)$, is equivalent to the couple of inequalities $(fE \cup eF) \leqslant E$ and $ij \leqslant i$, and these are valid by reason of
  - $(fE \cup eF)$ and E are in $Costar(X) = Costar(Y)$;
  - $(fe)E = fE \subset fE \cup eF$;
  - $ij \leqslant i$, since $ij = i \wedge i$.
- by similar arguments $(fE \cup eF, ij) \leqslant (F, j)$.

Thus, we have finished the proof. $\square$

After the wedge construction, we can define expansion's pseudo product.

Proposition 5.3.16. The pseudo product of two arrows $(A, s), (B, t)$ in $\overline{Sz}(\mathcal{C})$ is

$$(A, s) \star (B, t) = (s(i\varepsilon(B))s^\circ A \cup (i\varepsilon(A))sB, st)$$

if there exists $id(A, s) \wedge ir(B, t)$ in $\overline{Sz}(\mathcal{C})$.

Proof. Let $(A, s), (B, t) \in \overline{Sz}(\mathcal{C})$ with

$$\begin{cases} A \subset Costar(X) \\ A \subset \mathscr{R}_e \\ s : U \to X \\ e \leqslant ss^\circ \end{cases} \quad \text{and} \quad \begin{cases} B \subset Costar(Y) \\ B \subset \mathscr{R}_f \\ t : V \to Y \\ f \leqslant tt^\circ \end{cases}.$$

Note that if there exists (L, l) in $\overline{Sz}(\mathcal{C})$ such that $(L, l) \leqslant id(A, s), ir(B, t)$, then the composition st is defined.

Indeed, if $(L, l) \leqslant id(A, s) = (s^\circ A, s^\circ s)$ then, by the Definition 5.3.9, we have that $L \leqslant s^\circ A$ and $l \leqslant s^\circ s$; the first condition gives us

$$o\varepsilon(L) = o\varepsilon(s^\circ A) = od(s).$$

Analogously, if $(L, l) \leqslant ir(B, t) = (B, tt^\circ)$ then

$$o\varepsilon(L) = o\varepsilon(B) = or(t).$$

Hence

$$od(s) = o\varepsilon(L) = or(t) \implies \exists st.$$

Using previous data, and assuming the existence of $id(A, s) \wedge ir(B, t)$, we want to prove that

$$(A, s) \star (B, t) = (sfs^\circ A \cup esB, st).$$

Define the pair

$$(L, l) := id(A, s) \wedge ir(B, t) = (s^\circ A, s^\circ s) \wedge (B, tt^\circ).$$

We can compute such wedge, since the existence of st implies the existence of $s^\circ stt^\circ$. Moreover, we have $s^\circ A \subset \mathscr{R}_{s^\circ es}$, and therefore by Lemma 5.3.15

$$(L, l) = (\,f(s^\circ A) \cup s^\circ esB, s^\circ stt^\circ).$$

We can utilize Definition 5.3.13 and (the characterization of restriction, and corestriction, from) Lemma 5.3.11 to compute

$$(A, s) \star (B, t) = ({}_{(L,l)|}(A, s))((B, t)_{|(L,l)}) = (sL, sl)(L, lt).$$

Indeed

- as $s(s^\circ es) = es$ and $s(s^\circ stt^\circ) = stt^\circ$ it comes to light that

$$ {}_{(L,l)|}(A, s) = (sL, sl) = (sfs^\circ A \cup esB, stt^\circ);$$

- since $lt = (s^\circ stt^\circ)t = s^\circ st$, clearly

$$(B, t)_{|(L,l)} = (L, lt) = (fs^\circ A \cup s^\circ esB, s^\circ st);$$

- finally, as there exists the composition of previous arrows, and $slt = s(s^\circ stt^\circ)t = st$, we conclude

$$(sL, sl)(L, lt) = (sL, slt) = (sfs^\circ A \cup esB, st).$$

The last step to conclude the proof is to guarantee that $(A, s) \star (B, t)$ is an element of $\overline{Sz}(\mathcal{C})$. Actually, $o\varepsilon(sfs^\circ A) = or(s) = X$, $o\varepsilon(es) = or(e) = X$ and $or(st) = or(s) = X$ implies

$$o\varepsilon(sfs^\circ A \cup esB) = or(st).$$

In addition, since $sfs^\circ A, esB \subset \mathscr{R}_{esfs^\circ}$, $(st)(st)^\circ = stt^\circ s^\circ$ and $f \leqslant tt^\circ$, we have

$$(esfs^\circ)(st)(st)^\circ = (esfs^\circ)(stt^\circ s^\circ) = esfs^\circ \implies i\varepsilon(sfs^\circ A \cup esB) \leqslant ir(st).$$

$\square$

Remark 5.3.17. With the same conditions of Proposition 5.3.16, the equation $ir(sft) = sftt^\circ fs$ and the inequality $f \leqslant tt^\circ$, allows us to write

$$(A, s) \star (B, t) = (\,ir(sft)A \cup esB, st).$$

Now that we have developed two operations among arrows, it might help to analyze its differences. Indeed, if $(A, s), (B, t) \in \overline{Sz}(\mathcal{C})$, then composition: $(A,s)(B,t) = (A,st)$ if, and only if, $A = sB$ and there exists $st \in \mathcal{C}$;

pseudo product: $(A,s) \star (B,t) = (s(i\varepsilon(B))s^{\circ}A \cup (i\varepsilon(A))sB, st)$ if here exists $st \in \mathcal{C}$.

A natural question is if the pseudo product extends, when it exists, the composition. We address a lemma for an answer.

Lemma 5.3.18. If there exists the composition of (A,s) and (B,t) in $\overline{Sz}(\mathcal{C})$, then

$$(A,s) \star (B,t) = (A,s)(B,t).$$

Proof. Suppose $A \subset \mathscr{R}_e$ and $B \subset \mathscr{R}_f$. By assumption

$$\exists (A,s)(B,t) = (A,st) \iff A = sB \text{ and } \exists st.$$

From the equality $A = sB$, and since $sB \subset \mathscr{R}_{sfs^{\circ}}$, we conclude that $e = sfs^{\circ}$. Using this information in the formula of the pseudo product we arrive at

$$(A,s) \star (B,t) = (sfs^{\circ}A \cup esB, st) = (eA, st) = (A, st).$$

The last passage is due to the fact that $A \subset \mathscr{R}_e$. This concludes the proof. $\square$

So far, we have discovered many features of the Szendrei expansion; it is an excellent moment to present a summary: for an inverse category $\mathcal{C}$,

- we showed that $\overline{Sz}(\mathcal{C})$ is an inverse category (cf. Lemma 5.3.3 and Definition 5.3.4;
- given $(A,s),(B,t) \in \overline{Sz}(\mathcal{C})$, we defined $(A,s) \leqslant (B,t)$ if, and only if, $A \leqslant B$ and $s \leqslant t$ (cf. Definition 5.3.9);
- in particular, as $A \leqslant B$ exists only when $o\varepsilon(A) = o\varepsilon(B)$, we were able to define wedges among idempotent morphisms (cf. Definition5.2.11, and Lemma 5.3.15);
- with the definition of restrictions, corestrictions and wedges we were able to compute $(A,s) \star (B,t)$, under the existence of the composition st (cf. Lemma 5.3.18).

By beholding all information simultaneously, we can comprehend that we have an "external expansion" using the standard composition and an "internal expansion" using the pseudo product. The next definition formalizes this idea.

Definition 5.3.19. Fix X an object of the inverse category $\mathcal{C}$, the *inner Szendrei expansion* related to $\overline{Sz}(\mathcal{C})$ is the set

$$\overline{Sz}(\mathcal{C}(X)) := P(\mathcal{C}) \rtimes_{(\varepsilon,\mathfrak{B})} \mathcal{C}(X) := \{(A,s) \in \overline{Sz}(\mathcal{C}); s : X \to X\}.$$

Making more explicit this definition: if $(A, s) \in \overline{Sz}(\mathcal{C}(X))$, then

$$o\varepsilon(A) = or(s),\ i\varepsilon(A) \leqslant ir(s) \text{ and } s : X \to X.$$

It is essential to pay attention to the fact: we are not demanding s to be an idempotent.

Actually, it might not be possible to compute the composition of (A, s) and (B, t) in $\overline{Sz}(\mathcal{C}(X))$, but st always exists. So, each inner expansion admits the pseudo product. The set $\overline{Sz}(\mathcal{C}(X))$ has a more richer structure.

Proposition 5.3.20. For $X \in \mathcal{C}^{(0)}$, the inner Szendrei expansion $\overline{Sz}(\mathcal{C}(X))$ with the pseudo product is an inverse semigroup.

Proof. Let $(A, s), (B, t), (C, u) \in \overline{Sz}(\mathcal{C}(X))$, where $A.B, C \subset Costar(X)$, $s, t, u : X \to X$ and

$$A \subset \mathscr{R}_e,\ B \subset \mathscr{R}_f,\ C \subset \mathscr{R}_g,$$

and by the definition of $\overline{Sz}(\mathcal{C})$ we must have $e \leqslant ss^\circ,\ f \leqslant tt^\circ, g \leqslant uu^\circ$.

Since s, t, u are automorphisms in $\mathcal{C}$, we do not need to worry about associativity on the second component. We are about to verify inverse semigroup axioms:

- We will show $[(A, s) \star (B, t)] \star (C, u) = (A, s) \star [(B, t) \star (C, u)]$, computing it by pieces
  - by reason of $(A, s) \star (B, t) = (sfs^\circ A \cup esB, sr)$ and $i\varepsilon(sfs^\circ A \cup esB) = esfs^\circ$, the left hand side of the equation is equal to

    $$(sfs^\circ A \cup esB, sr) \star (C, u) = (stg(st)^\circ(sfs^\circ A \cup esB) \cup esfs^\circ(stC), stu);$$

  - repeating the same method, since $(B, t)\star(C, u) = (tgt^\circ B \cup ftC, tu)$ and $i\varepsilon(tgt^\circ B \cup ftC) = tst^\circ f$, the right hand side becomes

    $$(A, s) \star (tgt^\circ B \cup ftC, tu) = (s(tgt^\circ f)s^\circ A \cup es(tgt^\circ B \cup ftC), stu).$$

  The remaining step is to compare the elements of the first component. It is easier to observe that

  - $stg(st)^\circ[sfs^\circ] = stgt^\circ s^\circ sfs^\circ = stgt^\circ fs^\circ$;
  - $stg(st)^\circ[es] = stgt^\circ s^\circ es = ss^\circ estgt^\circ t = estgt^\circ$;
  - $stgt^\circ s^\circ[st] = esfs^\circ st = esft$.

  Thus we have the equality

  $$stg(st)^\circ(sfs^\circ A \cup esB) \cup esfs^\circ(stC) = stgt^\circ fs^\circ A \cup estgt^\circ B \cup esftC$$

and we can finish this proof's stage.

- In the sequence, we will ensure that $(A,s)\star(A,s)^{\circ}\star(A,s) = (A,s)$. Without further computations, as $ss^{\circ}A = A$, $i\varepsilon(A) \leqslant ir(s)$ and $ss^{\circ}s = s$, it follows that

$$[(A,s)\star(A,s)^{\circ}]\star(A,s) = (A,ss^{\circ})\star(A,s) = (A,s).$$

- The last axiom is hat idempotents commute. Let $(E,i),(F,j) \in \overline{Sz}((C(X))$ be idempotent arrows, *i.e.* with i,j idempotent in $\mathcal{C}$, $E \subset \mathscr{R}_e$, $F \subset \mathscr{R}_f$, by definition $e \leqslant i$ and $f \leqslant j$.

  What we want is to guarantee the equality $(E,i)\star(F,j) = (F,j)\star(E,i)$. Indeed

  - $(E,i)\star(F,j) = (ifiE \cup eiF, ij)$;
  - $(F,j)\star(E,i) = (jejF \cup fjF, ji)$

  The conditions $e \leqslant i$ and $f \leqslant j$, are equivalent to $e = ie$ and $f = fj$, also e,f,i and j are idempotent in $\mathcal{C}$ and commute. Also, by hypothesis $E \subset \mathscr{R}_e$ and $F \subset \mathscr{R}_f$. As a result we obtain

$$ifi = fiE = fE = fjE \text{ and } jejF = ejF = eF = eiF.$$

  Therefore

$$(E,i)\star(F,j) = (F,j)\star(E,i).$$

□

We want to shed light on the internal structure of $\overline{Sz}(\mathcal{C})$.

Consider a restriction idempotent $(E,e) \in \overline{Sz}(\mathcal{C})$ with $E \subset Costar(X)$ and, by definition, $e^2 = e : X \to X$. Take an arrow (A,s) such that

$$id(A,s) = (E,e) = ir(A,s).$$

The definitions of internal source and internal target map, provides

$$(s^{\circ}A, s^{\circ}s) = (E,e) = (A, ss^{\circ}).$$

One can see, the equation $s^{\circ}s = e = ss^{\circ}$ implies $s : X \to X$, since $od(s^{\circ}) = od(e) = od(s)$ and $or(s^{\circ}) = or(e) = or(s)$. With this computation, we have discovered $(A,s) = (E,s)$ satisfies $s^{\circ}s = e = ss^{\circ}$, and it is an element of $\overline{Sz}(\mathcal{C}(X))$.

The set $\overline{Sz}(\mathcal{C})((E,e))$ of such arrows is a group inside the inner expansion $\overline{Sz}(\mathcal{C}(X))$.

The same behavior appears for each idempotent of an inverse category $\mathcal{C}$. For instance, the cases where $\mathcal{C}$ is a group or a groupoid are fascinating. Truthfully

- If $\mathcal{C} = G$ is a group, then there exists a single object, which is the neutral element $e \in G$. Employing previous arguments

$$\overline{Sz}(G) = \overline{Sz}(G(e)) = S_{GB},$$

i.e. the inner expansion of a group coincides with the external one, and it is equal to the inverse semigroup S_{GB}, from Chapter 3 (cf. Lemma 2.2.4).

- If $\mathcal{C} = \mathcal{G}$ is an inductive groupoid and $e \in \mathcal{G}^{(0)}$, as $\mathcal{G}$ is an inverse category where for every arrow s the idempotent map $s^{\circ}s$ is a unit , our inner and outer sources/targets coincide, follows

$$\overline{Sz}(\mathcal{G}(e)) = \overline{Sz}(\mathcal{G})(e),$$

because every arrow has source and target equal to e.

We can carry over all the work we have been doing in this section to the other Bernoulli actions defined in the last section. We must introduce a new notation first.

Let $(\rho, \theta) : (\mathcal{C}, (\ )^{\circ}) \curvearrowright (P, \leqslant)$ be a fibred action of an inverse category on a poset, then we define

$$P \overline{\rtimes}_{(\rho,\theta)} \mathcal{C} := \{(x, s) \in P \times \mathcal{C}; o\rho(x) = or(s),\ i\rho(x) = ir(s)\}.$$

Lower down, we list all the Bernoulli structures we can construct:

<u>global Szendrei expansion</u>: $(\varepsilon, \mathfrak{B}) : (\mathcal{C}, (\ )^{\circ}) \curvearrowright P(\mathcal{C})$ induces

- <u>outer</u>: $\overline{Sz}(\mathcal{C}) = P(\mathcal{C}) \rtimes_{(\varepsilon,\mathfrak{B})} \mathcal{C}$,
- <u>inner</u>: $\overline{Sz}(\mathcal{C}(-)) = P(\mathcal{C}) \rtimes_{(\varepsilon,\mathfrak{B})} \mathcal{C}(-)$;

<u>partial Szendrei expansion</u>: $(\epsilon, \mathfrak{b}) : (\mathcal{C}, (\ )^{\circ}) \curvearrowright_p P_{\circ}(\mathcal{C})$ induces

- <u>outer</u>: $Sz(\mathcal{C}) := P_{\circ}(\mathcal{C}) \rtimes_{(\epsilon,\mathfrak{b})} \mathcal{C}$,
- <u>inner</u>:$Sz(\mathcal{C}(-)) := P_{\circ}(\mathcal{C}) \rtimes_{(\epsilon,\mathfrak{b})} \mathcal{C}(-)$;

<u>strict global Szendrei expansion</u>: $(\varepsilon, \mathfrak{s}\mathfrak{B}) : (\mathcal{C}, (\ )^{\circ}) \curvearrowright P(\mathcal{C})$ induces

- <u>outer</u>: $\overline{Sz}(\mathcal{C})_m = P(\mathcal{C}) \overline{\rtimes}_{(\varepsilon,\mathfrak{s}\mathfrak{B})} \mathcal{C}$,
- <u>inner</u>: $\overline{Sz}(\mathcal{C}(-))_m = P(\mathcal{C}) \overline{\rtimes}_{(\varepsilon,\mathfrak{s}\mathfrak{B})} \mathcal{C}(-)$;

strict partial Szendrei expansion: the action $(\epsilon, \mathfrak{sb}) : (\mathcal{C}, (\ )^\circ) \curvearrowright_p P_\circ(\mathcal{C})$ induces

- outer: $Sz(\mathcal{C})_m = P_\circ(\mathcal{C})\overline{\rtimes}_{(\epsilon,\mathfrak{sb})}\mathcal{C}$,
- inner: $Sz(\mathcal{C}(-))_m = P_\circ(\mathcal{C})\overline{\rtimes}_{(\epsilon,\mathfrak{sb})}\mathcal{C}(-)$.

Another way to characterize the Szendrei expansions is by using the respective actions' domains.

Remark 5.3.21. The reader might be confused, so we will explicit an element of each (outer) expansion. Let $(A, s) \in P(\mathcal{C}) \times \mathcal{C}$, the objects $U, X \in \mathcal{C}^{(0)}$ and $e^2 = e : X \to X$:

$$(A.s) \in \overline{Sz}(\mathcal{C}) \iff A \subset Costar(X),\ A \subset \mathscr{R}_e,\ s : U \to X,\ e \leqslant ss^\circ;$$

$$(A.s) \in Sz(\mathcal{C}) \iff A \subset Costar(X),\ A \subset \mathscr{R}_{ses^\circ},\ s : U \to X \text{ and } A \ni ses^\circ, se;$$

$$(A.s) \in \overline{Sz}(\mathcal{C})_m \iff A \subset Costar(X),\ A \subset \mathscr{R}_e,\ s : U \to X,\ e = ss^\circ;$$

$$(A.s) \in Sz(\mathcal{C})_m \iff A \subset Costar(X),\ A \subset \mathscr{R}_{ss^\circ},\ s : U \to X \text{ and } A \ni ss^\circ, s.$$

We similarly characterize inner expansion elements. The difference is that in this case $s : X \to X$ – if we use the previous example.

We state the properties of these sets in the next theorem.

Theorem 5.3.22. Let $\mathcal{C}$ be an inverse category. Then

(i) $Sz(\mathcal{C})$ is an ordered inverse subcategory of the ordered inverse category $\overline{Sz}(\mathcal{C})$;

(ii) $Sz(\mathcal{C})_m$ is an ordered inverse subcategory of the ordered inverse category $\overline{Sz}(\mathcal{C})_m$.

Furthermore, for each $X \in \mathcal{C}^{(0)}$

(iii) $Sz(\mathcal{C}(X))$ is a inverse monoid and a subset of the inverse semigroup $\overline{Sz}(\mathcal{C}(X))$;

(iv) $Sz(\mathcal{C}(X))_m$ is a inverse monoid and a subset of the inverse semigroup $\overline{Sz}(\mathcal{C}(X))_m$.

Proof. We start by pointing that, as $P_\circ(\mathcal{C})$ is a subset of $P(\mathcal{C})$, it becomes a poset with the induced order. Also, Definition's 5.3.1 data , grants the inverse categorical structure of the expansions on items (i) and (ii). Combining both facts and the arguments used to prove Lemma 5.3.11, we have the claims of the first two items.

For item (iii), the claim that $Sz(\mathcal{C}(X))$ is an inverse semigroup has the same proof as Proposition 5.3.20. It only remains to be proved that it is also an inverse monoid.

Let $(A, s) \in Sz(\mathcal{C}(X))$, by definition $A \subset Costar(X),\ A \subset \mathscr{R}_{ses^\circ},\ s : X \to X$ and $A \ni ses^\circ, se$. Since $s \in \mathcal{C}(X, X)$, it is possible to compute $s1_X = s = 1_X s$. Moreover, clearly $(\{1_X\}, 1_X) \in Sz(\mathcal{C}(X))$. So there exists $(A, s) \star (\{1_X\}, 1_X)$ and $(\{1_X\}, 1_X) \star (A, s)$, and

$$(A, s) \star (\{1_X\}, 1_X) = (s1_X s^\circ A \cup (ses^\circ)s\{1_X\}, s1_X) = (ss^\circ A \cup se\{1_X\}.s) = (A, s),$$

and by the other hand

$$(\{1_X\}, 1_X) \star (A, s) = (1_X(ses^\circ)1_X A \cup 1_X 1_X A, 1_X s) = (ses^\circ A \cup A, s) = (A, s).$$

Since the strict partial case is analogous, we have finished the proof. $\square$

Remark 5.3.23. We owe the reader a few words relative to the pseudo product. Only this operation deserves special attention since the usual composition will not change from expansion to expansion. The restricted cases will have a different form because their definition uses equality instead of inequality.

Let $(A, s), (B, t) \in \overline{Sz}(\mathcal{C}(X))$ with

$$\begin{cases} A \subset Costar(X) \\ A \subset \mathscr{R}_e \\ e \leqslant ss^\circ \end{cases} \quad \text{and} \quad \begin{cases} B \subset Costar(Y) \\ B \subset \mathscr{R}_f \\ f \leqslant tt^\circ \end{cases}.$$

As we saw, the pseudo product is

$$(A, s) \star (B, t) = (sfs^\circ A \cup esB, st).$$

On the other hand, if $(A, s), (B, t) \in \overline{Sz}(\mathcal{C}(X))_m$ with

$$\begin{cases} A \subset Costar(X) \\ A \subset \mathscr{R}_e \\ e = ss^\circ \end{cases} \quad \text{and} \quad \begin{cases} B \subset Costar(Y) \\ B \subset \mathscr{R}_f \\ f = tt^\circ \end{cases}.$$

As we saw, the pseudo product is

$$(A, s) \star (B, t) = (s(tt^\circ)s^\circ A \cup (ss^\circ)sB, st) = (ir(st)A \cup sB, st).$$

An attempt to interpret the Szendrei expansion graphically follows:

- the grid dots represent objects and arrows between then the ordinary composition;
- the cone over dots indicates the idempotents below the object's defining idempotent;

- the plane depicts the set of arrows with the pseudo product, this is an inverse semigroup;
- the dotted line above the dot is the automorphism group of arrows of such object.

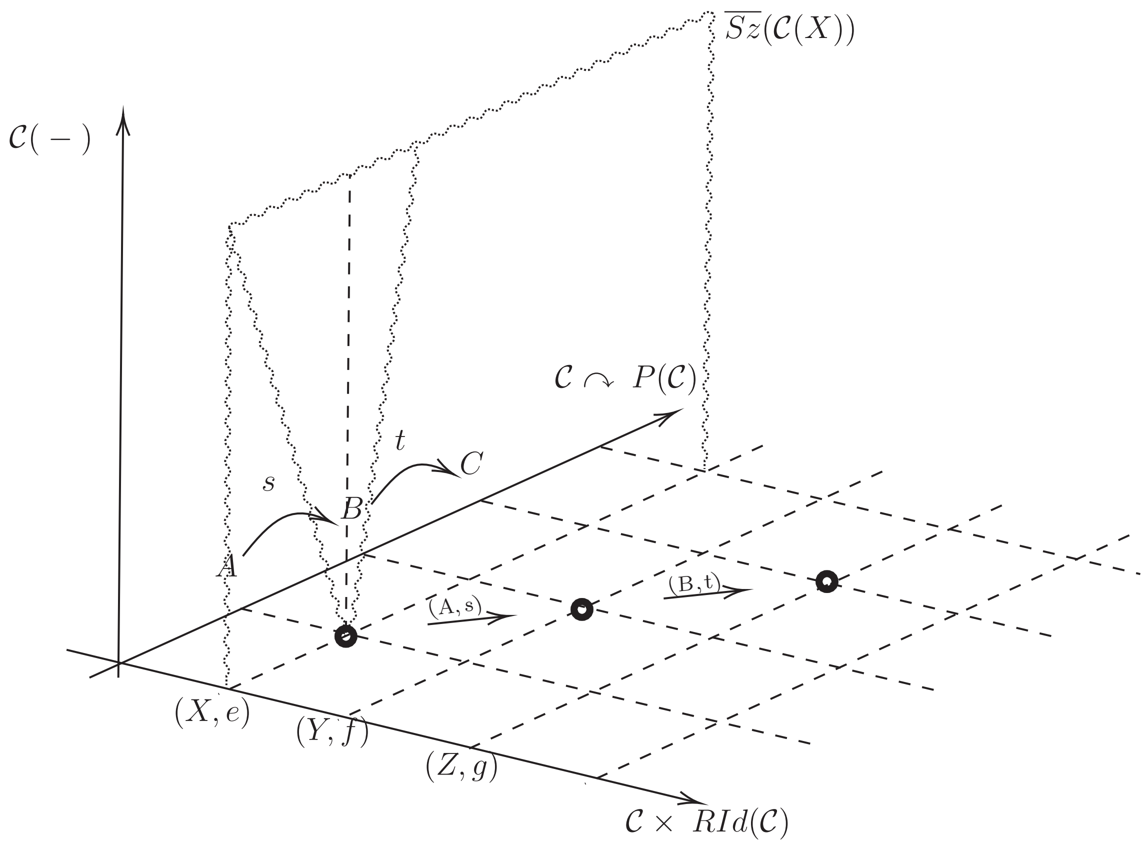


Figure 5.3: The Szendrei expasion of an inverse category

5.4 Idempotent completion, the associated groupoid, and enlargements

Any category $\mathcal{C}$ may be embedded in an auxiliary category $\widehat{\mathcal{C}}$ constructed upon the idempotent arrows of $\mathcal{C}$. This construction has many names in the literature, such as idempotent completion, Karoubi completion, Karoubienne completion, and Cauchy completion. We will stick with the last name because it has more occurrence in the literature of category theory.

Once our category has inverses, these bigger categories have a full subcategory, which turns out to be a groupoid. This motivation is the crucial theoretical idea behind the restriction of an inverse semigroup into a groupoid. The inverse categories have an even stronger relation since a category is inverse if, and only if, its idempotent completion is an inverse category.

In the next pages, we will present the main results of our work, relative to idempotent completions. The references for this section are Linckelmann [56] and Borceux [11] Section

5 of Chapter 6; in particular we follow the construction of Cauchy completion as presented by Linckelmann in [56], where he calls it Karoubienne completion.

5.4.1 Cauchy completion

Let $\mathcal{C}$ be a small category.

Definition 5.4.1. ([11]) Suppose $X \in \mathcal{C}^{(0)}$ and $e : X \to X$ an idempotent, *i.e.* $e^2 = e$, we say that the *idempotent* e *splits* if there are an object $Y \in \mathcal{C}^{(0)}$ and arrows $s : X \to Y$, $t : Y \to X$, such that

$$e = ts \text{ and } st = 1_Y.$$

When all idempotents split, we say that $\mathcal{C}$ is a *Cauchy complete* category.

As we said earlier, a category can be embedded in its Cauchy completion. Indeed, let $\mathcal{C}$ be a category, its Cauchy completion $\widehat{\mathcal{C}}$ can be constructed as follows:

- objects: an object of $\widehat{\mathcal{C}}^{(0)}$ is a pair (X, e), where $X \in \mathcal{C}$ and $e^2 = e \in \mathcal{C}(X, X)$;
- arrows: a morphism is a triple $(e, s, f) : (X, e) \to (Y, f)$, where $s : X \to Y$ is an arrow of $\mathcal{C}$ satisfying: $se = s = fs$;
- composition: the composition of $(e, s, f) : (X, e) \to (Y, f)$ and $(f, t, g) : (Y, f) \to (Z, g)$ is the arrow $(f, t, g)(e, s, f) = (e, ts, g) : (X, e) \to (Z, g)$.

Notice that

- (e, f, e) is an idempotent in $\widehat{\mathcal{C}}$ if $ef = fe = f$ and $f^2 = f$
- the unit map on (X, e) is $1_{(X,e)} = (e, e, e)$
- an idempotent (e, f, e) in $\widehat{\mathcal{C}}$ splits since $(e, f, e) = (e, f, f)(f, f, e)$ and $(f, f, f) = 1_{(X,f)} = (f, f, e)(e, f, f)$
- two objects (X, e) and (Y, f) are isomorphic if there are two arrows $(e, s, f) : (X, e) \to (Y, f)$ and $(f, t, e) : (Y, f) \to (X, e)$ such that $s : X \to Y$ and $t : Y \to X$ are arrows in $\mathcal{C}$ satisfying
$$se = s = fs, \;\; tf = t = et, \;\; ts = e, \;\; st = f.$$
- an automorphism $(e, s, e) : (X, e) \to (X, e)$ has the inverse $(e, t, e) : (X, e) \to (X, e)$ with
$$st = e = ts, \;\; sts = s, \;\; tst = t;$$

In the particular case of $(\mathcal{C}, (\ )^\circ)$ being an inverse category, things are a bit easier for isomorphisms, *i.e.* (X, e) is isomorphic to (Y, f) if, and only if, there is an arrow $s : X \to Y$ such that

$$s^\circ s = e \text{ and } ss^\circ = f.$$

We have an even nicer equivalence.

Remark 5.4.2. As we commented, in the introduction of this section, Cauchy completions are known by other names in the literature and many equivalent approaches. In particular there is the notion of "Karoubi envelope" which deals with components . We recommend the reader the nLab entry about this topic: https://ncatlab.org/nlab/show/Karoubi+envelope.

Proposition 5.4.3. ([56]) Let $\mathcal{C}$ be a small category. The category $\mathcal{C}$ is an inverse category if, and only if, $\widehat{\mathcal{C}}$ is an inverse category.

Proof. Let $X, Y \in \mathcal{C}^{(0)}$ and let $e^2 = e : X \to X$ and $f^2 = f : Y \to Y$ be idempotents. If $s : X \to Y$ is an arrow of $\mathcal{C}$ with $se = s = fs$, then $(e, s, f) \in \widehat{\mathcal{C}}$.

Supposing $(\mathcal{C}, (\ )^\circ)$ is an inverse category, we may take inverses on both sides of the last equality, obtaining

$$es^\circ = s^\circ = s^\circ f,$$

since $e^\circ = e$ and $f^\circ = f$. As $s^\circ : Y \to X$, the triple (f, s°, e) is the inverse of (e, s, f) in $\widehat{\mathcal{C}}$.

The opposite assertion can be proved using a similar argument. Hence we are done. $\square$

Our final considerations about Cauchy completions, for now, are three properties compiled in the next proposition.

Proposition 5.4.4. ([11]) Let $\mathcal{C}$ be a small category, its Cauchy completion $\widehat{\mathcal{C}}$ satisfies:

(i) $\widehat{\mathcal{C}}$ is small;

(ii) $\mathcal{C}$ is a full subcategory of $\widehat{\mathcal{C}}$, where the inclusion functor is defined by $X \in \mathcal{C}^{(0)} \mapsto (X, 1_X) \in \widehat{\mathcal{C}}^{(0)}$ and $(s : X \to Y) \in \mathcal{C} \mapsto (1_X, s, 1_Y) \in \widehat{\mathcal{C}}$;

(iii) the inclusion of $\mathcal{C}$ in $\widehat{\mathcal{C}}$, which sends $X \in \mathcal{C}^{(0)}$ to $(X, 1_X) \in \widehat{\mathcal{C}}^{(0)}$ is an equivalence if, and only if, every idempotent in $\mathcal{C}$ splits.

We will return to Cauchy completions further ahead, when we study algebras of Szendrei expansions.

5.4.2 The restriction groupoid

Let $\mathcal{C}$ be a small category; we can define a groupoid via its Cauchy completion.

Definition 5.4.5. ([56]) The *restriction groupoid* associated to $\mathcal{C}$, by notation $\mathcal{G}_{\mathcal{C}}$, is the subcategory of $\widehat{\mathcal{C}}$ defined by

- objects: $\mathcal{G}_{\mathcal{C}}^{(0)} = \widehat{\mathcal{C}}^{(0)}$;
- arrows: $\mathcal{G}_{\mathcal{C}} = \{x \in \widehat{\mathcal{C}}; x \text{ is an isomorphism }\}$;

In particular, we will write with more details the particular case of inverse categories. Indeed, suppose $(\mathcal{C}, (\ \ )^{\circ})$ an inverse category, its associated groupoid is given by

- $\mathcal{G}_{\mathcal{C}}^{(0)} = \{(X, e); X \in \mathcal{C}^{(0)},\ e^2 = e \in \mathcal{C}(X, X)\}$
- $\mathcal{G}_{\mathcal{C}} \ni (e, s, f) : (X, e) \to (Y, f)$ such that $(s : X \to X) \in \mathcal{C}$ satisfying $s^{\circ}s = e$ and $ss^{\circ} = f$.

Notice that, if $e^2 = e \in \mathcal{C}(X, X)$, we can define a group in $\mathcal{C}$ by the set

$$\mathcal{C}_e := \{s \in \mathcal{C}(X, X); X \in \mathcal{C}^{(0)},\ ss^{\circ} = e = s^{\circ}s\}.$$

It turns out that this group is isomorphic to the automorphism group $\mathcal{G}_{\mathcal{C}}((X, e), (X, e))$ via

$$s \in \mathcal{C}_e \mapsto (e, s, e) \in \mathcal{G}_{\mathcal{C}}((X, e), (X, e)).$$

Lately, when we discuss algebras of Szendrei expansions, these groups will be necessary. Also, in Chapter 3 Section3.4, we have already used this same idea when discussing the $\mathcal{D}$ Green classes of an inverse semigroup.

If the reader wants to see an equivalent construction of the restriction, in the case of inverse categories, we recommend the work of Dewolf-Pronk ([25]). In their ESN Theorem, the restriction groupoid presentation characterizes the groupoid based on the category's arrow structure; also, they showed how the ordered structure could be defined.

For us, restriction groupoids will be the key to understand algebras of inverse categories.

5.4.3 Enlargements of inverse categories

Our purpose in this section is to develop a notion of enlargements for inverse categories so that it includes the ordered groupoid case – and by consequence of the ESN Theorem 4.1.6, inverse semigroups.

With the intention of comparison/motivation, we will write again Lawson's definitions of enlargements of inverse semigroups and ordered groupoids, as we did in Definition 3.4.1 and Definition 4.5.3.

> An inverse semigroup T is an enlargement of the inverse semigroup S if it is an inverse subsemigroup satisfying
>
> (I) $\mathcal{E}(S)$ is an order ideal of $\mathcal{E}(T)$;
>
> (II) if $s \in T$ and $s^*s, ss^* \in \mathcal{E}(S)$, implies $s \in S$;
>
> (III) for each $f \in \mathcal{E}(T)$, there exists an unique $e \in \mathcal{E}(S)$, and $s \in T$ such that $s^*s = e$ and $ss^* = f$ (or equivalently $e\mathcal{D}f$).
>
> An ordered groupoid $\mathcal{H}$ is an enlargement of the ordered groupoid $\mathcal{G}$ if $\mathcal{G}$ is an ordered subgroupoid of $\mathcal{H}$ and fulfill the axioms
>
> (I) $\mathcal{G}^{(0)}$ is an order ideal of $\mathcal{H}^{(0)}$;
>
> (II) for $g \in \mathcal{H}$ and $d(g), r(g) \in \mathcal{G}^{(0)}$, then $g \in \mathcal{G}$;
>
> (III) given $f \in \mathcal{H}^{(0)}$, there exists $e \in \mathcal{G}^{(0)}$ and $g \in \mathcal{H}$ such that $d(g) = e$ and $r(g) = f$.

We realize the aspects that our definition must present if we take inductive groupoids. The list below emphasizes these points, putting in the perspective of inverse categories:

(I) substructure and order ideal relation among idempotent morphisms;

(II) full subcategory;

(III) subcategory with the inclusion functor essentially surjective on objects.

After the discussion and motivation, we properly define the notion we seek for inverse categories.

Definition 5.4.6. Let $\mathcal{C}$ be an ordered inverse subcategory of the ordered inverse category $\mathcal{D}$. We say that $\mathcal{D}$ is an *enlargement* of $\mathcal{C}$, if

(I) for each $X \in \mathcal{C}^{(0)}$ the set $RId(\mathcal{C}(X))$ is an order ideal of $RId(\mathcal{D}(X))$, *i.e.* if $(E, e) \in \overline{Sz}(\mathcal{C})$ and $(F, f) \in Sz(\mathcal{C})$ are idempotent arrows such that $(E, e) \leqslant (F, f)$ then $(E, e) \in Sz(\mathcal{C})$;

(II) let $X, Y \in \mathcal{C}^{(0)}$, $e^2 = e \in \mathcal{C}(X, X)$ and $f^2 = f \in \mathcal{C}(Y, Y)$: if $(s : X \to Y) \in \mathcal{D}$, and $se = s = fs$, then we have that $s \in \mathcal{C}$;

(III) suppose $Y \in \mathcal{D}$ and $f^2 = f \in \mathcal{D}(Y, Y)$: there exists $X \in \mathcal{C}$ and $e^2 = e \in \mathcal{C}(X, X)$, and there exists $s \in \mathcal{D}$ with $s : X \to Y$ satisfying $s^\circ s = e$ and $ss^\circ = f$.

Notation: $\mathcal{C} \subseteq_E \mathcal{D}$.

When we treat inductive groupoids or inverse categories, the two notions of enlargement are equivalent.

Comparing the definition with the motivation, some terms – like full subcategory – are not explicit, but the next lemma will shed light on these aspects.

Notice that the Cauchy completion of an inverse category turns all idempotent restrictions maps into unit maps. This way the idempotent maps of the completion are the unit maps of the objects in the completion.

Lemma 5.4.7. Let the inverse category $\mathcal{D}$ be an enlargement of the inverse category $\mathcal{C}$, then the inclusion functor $inc : \mathcal{C} \to \mathcal{D}$ is a fully and faithful functor.

Proof. Axiom (II) implies that $\mathcal{C}(X, Y) = \mathcal{D}(X, Y)$ for each pair of objects $X, Y \in \mathcal{C}$, and hence the inclusion functor $inc : \mathcal{C} \to \mathcal{D}$ is a full and faithful functor. □

Note that, with the same conditions of Lemma 5.4.7 we can not prove that $inc : \mathcal{C} \to \mathcal{D}$ is a functor essentially surjective on objects. Indeed, we may begin with $Y \in \mathcal{D}$ and $1_Y \in \mathcal{D}(Y, Y)$. Axiom (III) provides an object $X \in \mathcal{C}$, an idempotent $e \in RId(\mathcal{C}(X))$ and a morphism $s : X \to Y$ such that $s^\circ s = e$ and $ss^\circ = 1_Y$. But in order to inc be an equivalence s should be an isomorphism, *i.e.* $s^\circ s$ should be equal to 1_X.

The conclusion is that $\mathcal{C}$ and $\mathcal{D}$ might not be equivalent categories, but we show in the next proposition that its Cauchy completions are equivalent categories.

Proposition 5.4.8. Suppose $\mathcal{C}$ and $\mathcal{D}$ are inverse categories satisfying $\mathcal{C} \subseteq_E \mathcal{D}$. Then the inclusion functor of its Cauchy completions, $\widehat{\mathcal{C}} \hookrightarrow \widehat{\mathcal{D}}$, is an equivalence.

Proof. We begin with the definition of $\hat{inc} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ using $inc : \mathcal{C} \to \mathcal{D}$ (which we already shown to be an equivalence):

$$(X, e) \in \widehat{\mathcal{C}}^{(0)} \mapsto \hat{inc}(X, e) = (inc(X), inc(e)) \in \widehat{\mathcal{D}}^{(0)}$$

$$(e, s, f) \in \widehat{\mathcal{C}} \mapsto \hat{inc}(e, s, f) = (inc(e), inc(s), inc(f)) \in \widehat{\mathcal{D}}$$

Now we verify the axioms of equivalence (Definition 1.7.5):

<u>full functor</u>: Take two objects $(X, e), (Y, f) \in \widehat{\mathcal{C}}^{(0)}$, and let $(e, s, f) : \hat{inc}(X, e) \to \hat{inc}(Y, f)$ be a map in $\widehat{\mathcal{D}}$. From the definition of $\widehat{\mathcal{D}}$ and since $\hat{inc}(X, e) = (X, e)$ and $\hat{inc}(Y, f) = (Y, f)$ we have that $(e, s, f) \in \widehat{\mathcal{C}}$.

<u>faithful functor</u>: It is a direct conclusion, since $\hat{inc}$ is an inclusion.

<u>e.s.o. functor</u>: Let $(Y, f) \in \widehat{\mathcal{D}}^{(0)}$, *i.e.* $f^2 = f \in \mathcal{D}(Y, Y)$. As $\mathcal{C} \subseteq_E \mathcal{D}$: there exists $X \in \mathcal{C}$, $e^2 = e \in \mathcal{C}(X, X)$ and $s : X \to Y$ satisfying $e = s^\circ s$ and $f = ss^\circ$.

With this data, we define $(s^\circ s, s, ss^\circ) : (X, s^\circ s) \to (Y, ss^\circ)$. This arrow is an isomorphism in $\widehat{\mathcal{D}}$ with inverse $(ss^\circ, s^\circ, s^\circ s) : (Y, ss^\circ) \to (X, s^\circ s)$. Since

$$(ss^\circ, s^\circ, s^\circ s)(s^\circ s, s, ss^\circ) = (s^\circ s, s^\circ s, s^\circ s) = 1_{(X, s^\circ s)}$$

$$(s^\circ s, s, ss^\circ)(ss^\circ, s^\circ, s^\circ s) = (ss^\circ, ss^\circ, ss^\circ) = 1_{(Y, ss^\circ)}.$$

Hence, Cauchy completions are equivalent categories. □

Naturally, one may ask if any pair of our Szendrei expansions shares enlargement's relation. With this purpose in our mind, we address the following pages to study the question: is $\overline{Sz}(\mathcal{C})$ an enlargement of $Sz(\mathcal{C})$?

The answer we are about to develop with the reader will demand many computations to break it into small claims and slow down the pace. Indeed, let us verify if enlargement's axioms hold.

(I) <u>substructure</u>: Readily, the definition of $Sz(\mathcal{C})$ confirms that it is a subcategory of $\overline{Sz}(\mathcal{C})$.

(I) <u>order ideal</u>: let $(E, e) \in \overline{Sz}(\mathcal{C})$ and $(F, f) \in Sz(\mathcal{C})$ be idempotent arrows satisfying $(E, e) \leqslant (F, f)$. We will show that $(E, e) \in Sz(\mathcal{C})$.

Indeed, let us make the conditions that we are assuming more clear:

- since $(E, e)^2 = (E, e)$ we have that $e^2 = e$ and $eE = E$;
- since $(E, e) \leqslant (F, f)$ follows from the Definition 5.3.9 that
  - $E, F \subset Costar(X)$ for an object $X \in \mathcal{C}^{(0)}$
  - $E \subseteq \mathscr{R}_{i\varepsilon(E)}$, $F \subseteq \mathscr{R}_{i\varepsilon(F)}$ and $i\varepsilon(E) \leqslant i\varepsilon(F)$, *i.e.* $i\varepsilon(E)i\varepsilon(F) = i\varepsilon(E)$
  - $i\varepsilon(E)F \subseteq E$
- as $(F, f) \in Sz(\mathcal{C})$ and $f^2 = f$ we have that $i\varepsilon(F) \leqslant ff^\circ = f$, *i.e.* $i\varepsilon(F) = fi\varepsilon(F)$, also the remaining conditions about F are

$$fi\varepsilon(F)f^\circ \in F, \ fi\varepsilon(F)$$

  which are equivalent to $i\varepsilon(F) \in F$.

Note that, since (E, e) is an idempotent arrow, in order to conclude that $(E, e) \in Sz(\mathcal{C})$ it is enough to verify that $i\varepsilon(E) \in E$.

As we are supposing that $(E, e) \leqslant (F, f)$, we have that $i\varepsilon(E)F \subseteq E$, and since $i\varepsilon(F) \in F$, we obtain that

$$E \ni i\varepsilon(E)i\varepsilon(F) = i\varepsilon(E).$$

Then, we concluded the first axiom with a positive answer.

(II) Consider the restriction idempotents $(E, i), (F, j) \in Sz(\mathcal{C})$, with the data

$$\begin{cases} E \subset Costar(X) \\ E \subset \mathscr{R}_e \\ e, i : X \to X \\ e \leqslant i \\ e \in E \end{cases} \quad \text{and} \quad \begin{cases} F \subset Costar(Y) \\ F \subset \mathscr{R}_f \\ f, j : Y \to Y \\ f \leqslant j \\ f \in F \end{cases}.$$

Let $((A, s) : (E, 1_X) \to (F, 1_Y)) \in \overline{Sz}(\mathcal{C})$ such that

$$(A, s)(E, i) = (A, s) = (F, j)(A, s).$$

We can infer a few facts:

- as $(E, i) = id(A, s) = (s^\circ A, s^\circ s)$, then

$$E = s^\circ A,\ X = od(s) \text{ and } i = s^\circ s = id(s);$$

- similarly, since $(F, j) = ir(A, s) = (A, ss^\circ)$, we can see

$$F = A,\ Y = or(s) \text{ and } j = ss^\circ = ir(s).$$

Together, both facts results in

$$\begin{cases} s : X \to Y,\ si = s = js \\ A \subset \mathscr{R}_f,\ A \ni f = fj \\ s^\circ A \subset \mathscr{R}_e,\ s^\circ A \ni e = ei \end{cases}.$$

Moving forward, we would like to verify $(A, s) \in Sz(\mathcal{C})$. This is indeed true, since

- $o\varepsilon(A) = o\varepsilon(F) = Y = or(s)$;
- $i\varepsilon(A) = i\varepsilon(F) = f \leqslant j = ir(s)$;
- $A \ni f \implies s^\circ A \ni s^\circ f s$;

- $s^\circ A = E \subset \mathscr{R}_e \implies e = s^\circ f s \implies s e s^\circ f s s^\circ = jfj = f \therefore f = ses^\circ$;
- $A = s(s^\circ A) \implies A \ni se = s(s^\circ f s) = fs$.

This information is what we need. Hence, the second axiom of enlargements holds.

(III) Consider the restriction idempotent $(F, j) \in \overline{Sz}(\mathcal{C})$, with

$$F \subset Costar(Y),\ F \subset \mathscr{R}_f \text{ and } f \leqslant j.$$

From Lemma 5.2.14, $\mathcal{C} \cdot P_\circ(\mathcal{C}) = P(\mathcal{C})$, so there are $(s : X \to U) \in \mathcal{C}$ and $A \in P_\circ(\mathcal{C})$, such that

$$\exists sA \text{ and } sA = F.$$

Notice

$$Y = o\varepsilon(F) = o\varepsilon(sA) = or(s) = U \implies Y = U,$$

which implies $s \in \mathcal{D}(X, Y)$. Moreover

$$sA = F \subset \mathscr{R}_f \implies sA \subset \mathscr{R}_f.$$

If $A \subset \mathscr{R}_p$, then $sA \subset \mathscr{R}_{sps^\circ}$ and $A \ni p$. By last calculations $f = sps^\circ$ and

$$\exists sA \iff A \in dom(\mathfrak{B}_s) \textit{ i. e. } X = od(s) = o\varepsilon(A) \text{ and } p = i\varepsilon(A) \leqslant id(s) = s^\circ s.$$

Another conclusion is that $p : X \to X$, because

$$or(p) = or(s^\circ sp) = or(s^\circ) = X \text{ and } p^2 = p.$$

Consider the pair (F, sp). It has following the properties:

- (F, sp) is an arrow of $\overline{Sz}(\mathcal{C})$, due to the fact that

$$\begin{cases} or(sp) = or(s) = Y = o\varepsilon(F) \\ i\varepsilon(F) = f = sps^\circ \leqslant ir(sp) \end{cases};$$

- its inner source is (A, p), because

$$\begin{cases} (sp)^\circ F = ps^\circ F = ps^\circ(sA) = pA = A \\ od(sp) = od(p) = X \\ id(sp) = (sp)^\circ(sp) = ps^\circ sp = p \end{cases};$$

- also, its inner target is (F, sps°), since

$$\begin{cases} or(sp) = or(s) = Y \\ ir(sp) = (sp)(sp)^\circ = sps^\circ \end{cases}.$$

Summarizing, $(F, sp) : A \to F$ is an arrow in $\overline{Sz}(\mathcal{C})$ such that

$$(F, sp)^\circ(F, sp) = (A, p) \text{ and } (F, sp)(F, sp)^\circ = (F, f).$$

But, to conclude the third axiom, we need the equality

$$(F, sp)(F, sp)^\circ = (F, j), \textit{ i.e. } f = ir(sp) = j$$

and we have only $f \leqslant j$.

Conclusion: the pair $(Sz(\mathcal{C}), \overline{Sz}(\mathcal{C}))$ satisfies the axioms (I) and (II), but the previous computations do not ensure that (III) holds.

However, observe that they do show that (III) holds whenever the idempotent (F, j) satisfies $j = i\varepsilon(F)$. So we have the next theorem.

Theorem 5.4.9. The strict global expansion $\overline{Sz}(\mathcal{C})_m$ is an enlargement of the strict partial expansion $Sz(\mathcal{C})_m$. Moreover, the same relation is true if we substitute each outer expansion with its respective inner version.

Stenographically, the last theorem says

$$Sz(\mathcal{C})_m \subseteq_E \overline{Sz}(\mathcal{C})_m \text{ and } Sz(\mathcal{C}(-))_m \subseteq_E \overline{Sz}(\mathcal{C}(-))_m.$$

Observe that, in particular we have Lemma 2.3.9, Proposition 3.4.2 and a particular case of Proposition 4.5.4.

5.5 Convolution algebras of finite inverse categories

The algebras of categories, or convolution algebras, are similar to groupoids algebras, and we defined them in Definition 1.7.11, back in Chapter 2.

Recalling

Let $\mathcal{C}$ be a small category and $\mathbb{K}$ be a commutative ring. The *category algebra*, or *convolution algebra*, $\mathbb{K}\mathcal{C}$ is a free $\mathbb{K}$-module whose basis is the arrow set $\mathcal{C}$. The product in the basis is defined by

$$x \bullet y = \begin{cases} xy & , \text{ if } \exists xy \\ 0 & , \text{ if not} \end{cases},$$

then we extend linearly to all $\mathbb{K}\mathcal{C}$.

Moreover, if $\mathcal{C}^{(0)}$ is finite, in above definition, $1_{\mathbb{K}\mathcal{C}} = \sum_{e \in \mathcal{C}^{(0)}} 1_e$.

The particular case of inverse categories is treated below, where we present Linckelmann's isomorphism of algebras, from [56].

5.5.1 Linckelmann's isomorphism

Before discussing the main case, we will reinforce the similarities between inverse semigroups and inverse categories.

In our chapters dedicated to groups and inverse semigroups, one of the essential results – which linked the global structures with the partial ones –is Steinberg's algebra isomorphism (cf. Theorem 2.5.4). We presented an idea of his proof using the universal groupoid construction. Speaking about Steinberg's works, the formulation we presented is an alternative proof of a previous work due to Steinberg himself. In the papers [85], and [86], Steinberg used Möbius functions (as we defined in Chapter 2) to relate inverse semigroup and groupoid algebras. Lately, as we saw, he reformulated this isomorphism in terms of universal groupoids.

We are presenting these facts again because Linckelmann ([56]) realized that an generalization of Steinberg's original strategy, of Möbius functions, would work for inverse categories. Hence he established an isomorphism between the convolution algebra of an inverse category and the algebra of its restriction groupoid. We will call it Linckelmann's isomorphism.

We will not present the proof of this result because it is quite long, and involves the development of techniques which are of no use for this thesis. Therefore we will just present the statement of this theorem and refer the interested reader to [56] for the proof; if the reader is familiar with Steinberg's use of Möbius functions, the understanding will be more comfortable.

The main idea is: the map that sends $(e, s, f) : (X, e) \to (Y, f)$ in $\mathcal{G}_\mathcal{C}$ to $\sum_{t \leqslant s} \mu(t, s)t$ (μ being the Möbius function of $\mathcal{C}$ with the natural partial order and with coefficients in $\mathbb{K}$) in the subspace $\mathbb{K}\mathcal{C}(X, Y)$, of $\mathbb{K}\mathcal{C}$, determines an $\mathbb{K}$-algebra isomorphism

We state Linckelmann's isomorphism.

Theorem 5.5.1. *([56]) Let $(\mathcal{C}, (\ \ )^\circ)$ be a finite inverse category and $\mathbb{K}$ be a commutative ring. The convolution algebra of $\mathcal{C}$, $\mathbb{K}\mathcal{C}$, and the groupoid restriction algebra, $\mathbb{K}\mathcal{G}_\mathcal{C}$, are isomorphic.*

This theorem allows us to write $\mathbb{K}\mathcal{C}$ as a groupoid algebra, and groupoid algebras can be realized as direct products of matrix algebras with coefficientes in group algebras. Those groups correspond to automorphism groups / isotropy groups of objects of the groupoid. We have the following corollary.

Corollary 5.5.2. *([56]) Let $(\mathcal{C}, (\ )^\circ)$ be a finite inverse category and $\mathbb{K}$ be a commutative ring. Let $\mathcal{E}$ be the set of representatives of the isomorphism classes of idempotent endomorphisms in $\mathcal{C}$; denote by n_e the number of idempotents in $\mathcal{C}$ which are isomorphic to e. Then the convolution algebra of $\mathcal{C}$ has the representation*

$$\mathbb{K}\mathcal{C} \simeq \bigoplus_{e \in \mathcal{E}} \mathbb{M}_{n_e}(\mathbb{K}\mathcal{C}_e).$$

This corollary also implies that every functor $F : \mathcal{C} \to Mod(\mathbb{K})$ decomposes (naturally) as a direct sum of functors $F = \oplus_{e \in \mathcal{E}} F_e$. More details can be found in [56].

5.5.2 Theoretical aspects of inverse category algebras

As we are dealing with the abstract category, we have a piece of the vast machinery that permits us to realize theoretical aspects due to the big picture. Our work's title suggests we are trying to understand the algebras of expanded structures in general.

We will present some aspects of symmetric algebras, as in Linckelmann [56], that provide a significant understatement of our algebras. Finishing this subsection, we will shortly comment on work of Todea [93] that generalizes and shed lights in further applications.

Definition 5.5.3. ([93]) Let A be an algebra over a commutative ring $\mathbb{K}$, we say that A is *symmetric* if A is finitely generated and projective as a $\mathbb{K}$-module and there is a $\mathbb{K}$-linear map $\tau : A \to \mathbb{K}$ satisfying $\tau(ab) = \tau(ba)$, for all $a, b \in A$, which induces an isomorphism, of $A - A$-bimodules

$$\widetilde{\tau} : A \to A^*, \;\; \widetilde{\tau}(a)(b) = \tau(ab)$$

for $a, b \in A$, and A^* is the $\mathbb{K}$-dual of A.

The map τ is called the *symmetric form*, or *symmetrizing form*, of A. In particular if its onto, we say that τ is *principally symmetric*.

The prominent example of such algebras are the matrix algebras; therefore, convolution algebras of finite inverse categories. Hence the next result.

Proposition 5.5.4. ([56]) Let $(\mathcal{C}, (\ )^\circ)$ be a finite inverse category and $\mathbb{K}$ be a commutative ring. Then $\mathbb{K}\mathcal{C}$ is a symmetric algebra whose symmetrizing form is the map $\tau : \mathbb{K}\mathcal{C} \to \mathbb{K}$ which sends an arrow $s \in \mathcal{C}$ to the number of idempotent arrows $e \in \mathcal{C}$ satisfying $e \leqslant s$.

Concerning the enlargement process, Linckelmann also realized how subcategories of inverse categories fit into his isomorphism.

Theorem 5.5.5. *([56]) Let $(\mathcal{C}, (\ )^\circ)$ be a finite inverse category and $\mathcal{D} \subset \mathcal{C}$ a subcategory such that for any $X \in \mathcal{D}^{(0)}$ the arrow set $\mathcal{D}(X, X)$ contains all the idempotent arrows of $\mathcal{C}(X, X)$. Also, let $\mathbb{K}$ be a commutative ring.*

(i) The, following diagram of non-unitary algebra homomorphism is commutative

$$\begin{tikzcd} \mathbb{K}\mathcal{G}_{\mathcal{C}} \arrow[r, "\simeq"] & \mathbb{K}\mathcal{C} \\ \mathbb{K}\mathcal{G}_{\mathcal{D}} \arrow[u, hook] \arrow[r, "\simeq"'] & \mathbb{K}\mathcal{D} \arrow[u, hook] \end{tikzcd} .$$

(ii) The identity $1_{\mathbb{K}\mathcal{C}} = \sum_{X \in \mathcal{D}^{(0)}} 1_X$ implies that the $\mathbb{K}\mathcal{C} - \mathbb{K}\mathcal{D}$-bimodule $\mathbb{K}\mathcal{C}1_{\mathbb{K}\mathcal{D}}$ and the $\mathbb{K}\mathcal{D} - \mathbb{K}\mathcal{C}$-bimodule $1_{\mathbb{K}\mathcal{D}}\mathbb{K}\mathcal{C}$ are both finitely generated and projective left and right modules.

(iii) The symmetrizing form of $\mathbb{K}\mathcal{D}$ is the restriction of the symmetrizing form of $\mathbb{K}\mathcal{C}$ to $\mathbb{K}\mathcal{D}$.

(iv) The functor $1_{\mathbb{K}\mathcal{D}}\mathbb{K}\mathcal{C} \otimes_{\mathbb{K}\mathcal{C}} -$ is the right adjoint to the functor $\mathbb{K}\mathcal{C}1_{\mathbb{K}\mathcal{D}} \otimes_{\mathbb{K}\mathcal{D}} -$.

We mention that Todea, in [93], using ideas of Linckelmann, generalizes symmetric algebras to what he termed *inverse-symmetric* algebras. He also defines the skew category algebra associated with an inverse category, which is a symmetric algebra.

5.5.3 Morita equivalence for convolution algebras

The last piece of our puzzle is to define Morita equivalences for categories and to understand how it carries over to its convolution algebras.

The definition we seek was already provided by Borceux [11], Section 7.9. Also by Elkins-Zilber in [31]. The extension of the Morita equivalence from categories to their algebras is presented in the work of Xu [98] Proposition 2.2.4, or in his notes [99], with further explanation and more examples.

Definition 5.5.6. ([11]) Two small categories $\mathcal{C}$ and $\mathcal{D}$ are called *Morita equivalent* if their Cauchy completion $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are equivalent.

Notation: $\mathcal{C} \simeq_M \mathcal{D}$.

Now the proposition of Xu, which connects Morita equivalent categories and Morita equivalent convolution algebras.

Proposition 5.5.7. ([98]) Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories, and $\mathbb{K}$ be a commutative unital ring. If $\mathcal{C}^{(0)}$ and $\mathcal{D}^{(0)}$ are finite and $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent categories then the convolution category algebras $\mathbb{K}\mathcal{C}$ and $\mathbb{K}\mathcal{D}$ are Morita equivalent (as algebras).

We remark that (by Proposition 5.4.3) if $\mathcal{C}$ and $\mathcal{D}$ are categories then

$$\mathcal{C} \text{ and } \mathcal{D} \text{ are inverse categories } \iff \widehat{\mathcal{C}} \text{ and } \widehat{\mathcal{D}} \text{ are inverse categories }.$$

Continuing, due to Linckelmann's isomorphism and the fact that the Cauchy completion is equivalent to its own Cauchy completion (cf. Proposition 5.4.4)

$$\mathbb{K}\widehat{\mathcal{C}} \simeq \mathbb{K}\mathcal{G}_{\mathcal{C}} \text{ and } \mathbb{K}\widehat{\mathcal{D}} \simeq \mathbb{K}\mathcal{G}_{\mathcal{D}},$$

so

$$\mathcal{C} \simeq_M \mathcal{D} \implies \widehat{\mathcal{C}} \simeq_M \widehat{\mathcal{D}} \;\therefore\; \mathbb{K}\mathcal{C} \simeq \mathbb{K}\mathcal{G}_{\mathcal{C}} \simeq_M \mathbb{K}\mathcal{G}_{\mathcal{D}} \simeq \mathbb{K}\mathcal{D}.$$

We can provide a sufficient condition for the Morita equivalency of two convolution algebras, using the new concept of enlargements for inverse categories. Let us explain it better: we showed earlier, in Proposition 5.4.8, that

$$\mathcal{C} \subseteq_E \mathcal{D} \implies \widehat{\mathcal{C}} \simeq \widehat{\mathcal{D}},$$

i.e. if we have an enlargement relation of a pair of inverse categories, then we can guarantee equivalence of their Cauchy completions. In particular, the associated restriction groupoids must be Morita equivalent.

We have just proved the following theorem.

Theorem 5.5.8. Let $\mathcal{C}$ and $\mathcal{D}$ be two finite inverse categories and let $\mathbb{K}$ be a commutative unital ring. If $\mathcal{D}$ is an enlargement of $\mathcal{C}$, then the convolution algebras $\mathbb{K}\mathcal{C}$ and $\mathbb{K}\mathcal{D}$ are Morita equivalent.

In a succinct form, the last theorem says

$$\mathcal{C} \subseteq_E \mathcal{D} \implies \mathbb{K}\mathcal{C} \simeq_M \mathbb{K}\mathcal{D}.$$

Finally, we can apply it to our case. Nevertheless, first, we will denominate the algebras.

Given a commutative and unital ring $\mathbb{K}$, each Szendrei expansion will give origin to an algebra, which we now define:

<u>global algebra</u>:

- <u>outer</u>: $\mathbb{K}\overline{Sz}(\mathcal{C}) = \mathbb{K}_{glob}\mathcal{C}$,
- <u>inner</u>: $\mathbb{K}\overline{Sz}(\mathcal{C}(-)) = \mathbb{K}_{glob}\mathcal{C}(-)$;

<u>partial algebra</u>:

- <u>outer</u>: $\mathbb{K}Sz(\mathcal{C}) := \mathbb{K}_{par}\mathcal{C}$,
- <u>inner</u>: $\mathbb{K}Sz(\mathcal{C}(-)) := \mathbb{K}_{par}\mathcal{C}(-)$;

<u>strict global algebra</u>:

- <u>outer</u>: $\mathbb{K}\overline{Sz}(\mathcal{C})_m = \mathbb{K}_{sglob}\mathcal{C}$,
- <u>inner</u>: $\mathbb{K}\overline{Sz}(\mathcal{C}(-))_m = \mathbb{K}_{sglob}\mathcal{C}(-)$;

<u>strict partial algebra</u>:

- <u>outer</u>: $\mathbb{K}Sz(\mathcal{C})_m = \mathbb{K}_{spar}\mathcal{C}$,
- <u>inner</u>: $\mathbb{K}Sz(\mathcal{C}(-))_m = \mathbb{K}_{spar}\mathcal{C}(-)$.

<u>Corollary</u> 5.5.9. The outer (resp. inner) strict global algebra is Morita equivalent to the outer (resp. inner) strict partial algebra. In other terms

$$\mathbb{K}_{spar}\mathcal{C} \simeq_M \mathbb{K}_{sglob}\mathcal{C} \text{ and } \mathbb{K}_{spar}\mathcal{C}(-) \simeq_M \mathbb{K}_{sglob}\mathcal{C}(-).$$

These explanations show our study's consistency since one condition that permitted us to conclude Morita equivalence of algebras in previous chapters was the Morita relation of restriction groupoids.

5.6 A brief application of Kan extensions

We have just seen how to handle Szendrei convolution algebras, mainly when the inverse category is finite. However, the techniques used will not work in the non finite case. We shall invoke Kan extensions to elucidate the question.

Briefly, we will present the definition of the Kan extension and how they fit in our case. To found more results about Kan extensions, please check: Mac Lane [57] Chapter X; or Riehl [75] Chapter 6; Milewski [61] Part three; and also the great survey from Lehner [55].

The reader who is not familiar with Kan's theory might ask: what is the motivation behind this tool's choice? Well, Mac Lane, in his book [57], has a famous quote where he says that all concepts in category theory are Kan extension; we will not go that far. Nevertheless, Milewski is very educational and motivates the usage of Kan extension, likewise extensions of maps.

Anticipating the main result, we want to use Kan extensions to show that the representations of the strict global Szendrei expansion are extensions of representations of the strict partial Szendrei expansion.

Definition 5.6.1. ([75]) Let $F : \mathcal{C} \to \mathcal{E}$ and $\wp : \mathcal{C} \to \mathcal{D}$ be functors, the *left Kan extension* of F along $\wp$ is a functor $Lan_{\wp}F : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta : F \Rightarrow$

$Lan_{\wp}F \circ \wp$, such that for any other pair $(\varrho : \mathcal{D} \to \mathcal{E}, \gamma : F \Rightarrow \varrho \circ \wp)$, then γ factors uniquely through η.

In diagrams, this definition says

<u>functor and natural transformation</u>:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\wp"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dotted, "Lan_{\wp}F"'] & \end{tikzcd}$$

<u>uniqueness of the natural transformation</u>:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\wp"'] & \arrow[d, phantom, "\Downarrow\gamma"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dotted, "\varrho"'] & \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\wp"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dotted, "Lan_{\wp}F"] \arrow[ur, bend right, "\varrho"'] & \end{tikzcd}$$

Analogously, we define the right Kan extension.

Definition 5.6.2. ([75]) Let $F' : \mathcal{C} \to \mathcal{E}$ and $\kappa' : \mathcal{C} \to \mathcal{D}$ be functors, the *right Kan extension* of F' along κ' is a functor $Ran_{\kappa'}F' : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta' : F' \Leftarrow Ran_{\kappa'}F' \circ \kappa'$, such that for any other pair $(\varrho' : \mathcal{D} \to \mathcal{E}, \gamma' : F' \Leftarrow \varrho' \circ \kappa')$, then γ' factors uniquely through η'.

We desire to use the Kan extensions in a particular case. This way, we need a few technical results. The first proposition addresses its existence, from Lehner [55] Corollary 3.9.

Proposition 5.6.3. ([57][11]) In the terms of Kan's definitions with $\mathcal{C}$ a small category:

(i) if $\mathcal{E}$ is complete, then the right Kan extension exists;

(ii) if $\mathcal{E}$ is cocomplete, then the left Kan extension exists.

The next proposition provides sufficient conditions for the natural transformations in the definition of left/right Kan extensions to be natural isomorphisms. The proof for the left Kan extension is in Borceoux [11] Chapter 3, and for the right Kan extensions in Mac Lane [57].

Proposition 5.6.4. ([57][11]) Let $\mathcal{C}$ be a small category, $\mathcal{E}$ be a bicomplete category, $\mathcal{D}$ be a category and $\wp : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor. Given a functor $F : \mathcal{C} \to \mathcal{E}$, let $Lan_{\wp}F$ and $Ran_{\wp}F$ be the left (respectively) right Kan extensions of F along $\wp$. Then

(i) the natural transformation $\digamma \Rightarrow Lan_{\wp}\digamma \circ \wp$ is an isomorphism;

(ii) the natural transformation $\digamma \Leftarrow Ran_{\wp}\digamma \circ \wp$ is an isomorphism.

Let us make a brief computation: suppose two functors $\digamma : \mathcal{C} \to \mathcal{E}$ and $\wp : \mathcal{C} \to \mathcal{D}$

$$\begin{cases} \mathcal{C} \text{ small} \\ \mathcal{E} \text{ bicomplete} \\ \wp \text{ full and faithful} \end{cases} \Longrightarrow \begin{cases} \exists \overline{\digamma}_R = Ran_{\wp}\digamma \\ \exists \overline{\digamma}_L = Lan_{\wp}\digamma \\ \digamma \Leftarrow \overline{\digamma}_L \circ \wp \text{ is an isomorphism} \\ \digamma \Rightarrow \overline{\digamma}_R \circ \wp \text{ is an isomorphism} \end{cases} .$$

We can say a bit more, observe:

$$(\overline{\digamma}_R \circ \wp) \overset{\sim}{\Rightarrow} \digamma \overset{\sim}{\Rightarrow} (\overline{\digamma}_L \circ \wp) \therefore \overline{\digamma}_R \overset{\sim}{\Rightarrow} \overline{\digamma}_L,$$

since $\wp$ is full and faithful. In simple terms, with those starting hypothesis the right and left Kan extensions are naturally isomorphic. In this case we will denote the Kan extensions of $\digamma$ by the same symbol $\overline{\digamma}$.

There is one last definition we need, and it is about categorical representation. We are following Xu [98].

Definition 5.6.5. ([98]) Let $\mathcal{C}$ be a category and $\mathbb{K}$ a commutative ring, a *representation* of $\mathcal{C}$ over $\mathbb{K}$ is a covariant functor $\digamma : \mathcal{C} \to Mod(\mathbb{K})$.

Now we are able to construct our application. Fix $\mathcal{C}$ an inverse category and $\mathbb{K}$ a unital and commutative ring. From Theorem 5.4.9 and Lemma 5.4.7

$$Sz\mathcal{C}_m \subseteq_E \overline{Sz}(\mathcal{C})_m \implies inc : Sz(\mathcal{C})_m \hookrightarrow \overline{Sz}(\mathcal{C})_m \text{ is an equivalence .}$$

Since $\mathcal{C}$ is small by definition, and $Mod(\mathbb{K})$ is a bicomplete category, given any functor $\digamma : Sz(\mathcal{C})_m \to Mod(\mathbb{K})$, there exists its Kan extension $\overline{\digamma}$.

Writing in a diagram form:

$$\begin{tikzcd} Sz(\mathcal{C})_m \arrow[rr, "\digamma"] \arrow[dr, "inc"'] & \arrow[d, phantom, "\Downarrow\eta"] & Mod(\mathbb{K}) \\ & \overline{Sz}(\mathcal{C})_m \arrow[ur, dotted, "\overline{\digamma}"'] & \end{tikzcd}$$

Theorem 5.6.6. The representations of $\overline{Sz}(\mathcal{C})_m$ over $\mathbb{K}$, a commutative unital ring, are Kan extensions of the representations of $Sz(\mathcal{C})_m$ over $\mathbb{K}$.

We can exhibit this fact with a picture, inspired by one of B. Milewski's drawings:

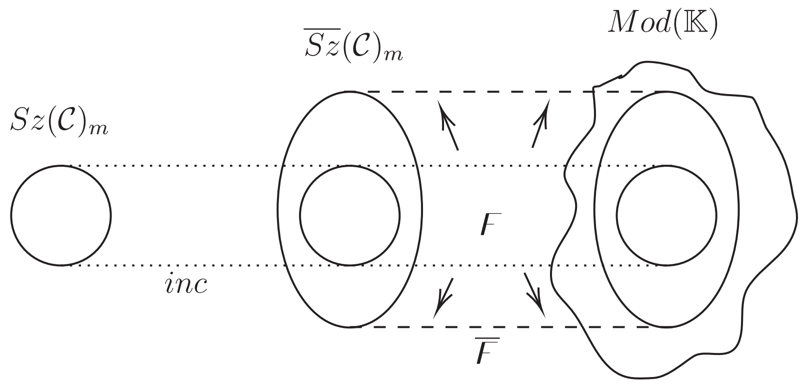


Figure 5.4: The Kan extension of the strict partial Szendrei expansion

5.7 The adjunction perspective

This section is the last section of our final chapter. We will present (very quickly) the categorical perspective behind the semidirect product of categories; more than the proper definitions, we want to present the concepts and an indication of bibliography.

5.7.1 Grothendieck constructions

As Steinberg and Tilson commented, in [90], the semidirect product of categories is – most known – by the name of Grothendieck construction. [2] In some references, these constructions are also called: category of objects, as in Riehl [75] and [74]; or in Awdoey [4].

Following Barr-Wells [8], Chapter 12, and Spivak [82], Chapter 6: the categorical construction that corresponds to a category acting on a category is a functor taking values on the category and taking it to the category of categories, *i.e.* $F : \mathcal{C} \to Cat$. Then the *Grothendieck construction* for F is a category, denoted by $\int_{\mathcal{C}} F$, or $F \rtimes \mathcal{C}$, where

- objects are pairs (a, X) where $X \in \mathcal{C}^{(0)}$ and $a \in F(X)^{(0)}$;
- arrows are pairs $(u, f) : (a, X) \to (b, Y)$ such that $(f : X \to Y) \in \mathcal{C}$ and $(u : F(f(a)) \to b) \in F(Y)^{(0)}$;

[2] A bit of controversy about the name: Mac Lane and Moerdijk argue, in [58], that this construction appeared first in the work of Yoneda and was developed by Mac Lane, "way before" Grothendieck.

- given two arrows $(u, f) : (a, X) \to (b, Y)$ and $(v, g) : (b, Y) \to (c, Z)$, the composition $(v, g)(u, f) : (a, X) \to (c, Z)$ obeys the rule $(v, g)(u, f) = (vF(g(u)), gf)$.

When $\mathcal{C}$ is a set (or a discrete category) and we replace Cat by Set, the category of sets, then $\int_{\mathcal{C}} F = \cup_{x \in \mathcal{C}} F(x)$ is a disjoint union formed by identity morphisms.

Another example that justifies the semidirect name product: if $\mathcal{C}$ is a group viewed as a category over $\{e\}$, the single object, then F being a functor is equivalent to defining a group homomorphism $\phi : G \to Aut(H)$, where $F(e) := H$ is a group. So $\int_{\mathcal{C}} F = H \rtimes_{\phi} G$ is the standard semidirect product of groups. The same idea holds for monoids.

In our study categories actions on sets are equivalent to functors $F : \mathcal{C} \to Set$, so $\int_{\mathcal{C}} F$ has objects and arrows given by

- $(\int_{\mathcal{C}} F)^{(0)} = \{(a, X); X \in \mathcal{C}^{(0)},\ a \in F(X)\}$;
- $\int_{\mathcal{C}} F((a, X), (b, Y)) = \{f : X \to Y \in \mathcal{C}; F(f)(a) = b\}$.

There are many other properties in the cited works, but what we present is sufficient for us.

5.7.2 Comma categories

The comma category construction allows one to produce a new category whose objects are arrows. In particular, the presentation we are about to give is the most general case. A more intuitive approach appears in Mac Lane [57] Section 6 of Chapter II. Or Spivak [82], Chapter 6.

Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories and $F : \mathcal{D} \to \mathcal{C}$ and $\varrho : \mathcal{E} \to \mathcal{C}$ be functors. The *comma category* of $\mathcal{C}$ morphisms from F to ϱ, denoted by $(F \downarrow \varrho)$, is the category formed by

- $(F \downarrow \varrho)^{(0)} = \{(X, A, f); X \in \mathcal{D},\ A \in \mathcal{E};\ f : F(X) \to \varrho(A) \in \mathcal{C}\}$,
- $(F \downarrow \varrho)((X.A.f), (Y, B, g)) = \{(q, r); q : X \to Y \in \mathcal{D}, r : A \to B \in \mathcal{E}$ s.t. $gF(q) = \varrho(r)f\}$;
- composition of morphisms is given on components by composition in $\mathcal{D}$ and $\mathcal{E}$.

This construction generalizes slice categoriescf. [57], or [82]. Next we present and example from Spivak, [82], which will give some hint about the relation between comma categories and Grothendieck constructions.

Indeed, let $\mathcal{C}$ be a category and $F : \mathcal{C} \to Set$ a functor. Suppose the unital set $\mathbb{1} = \{\,1\}$ and the category $Discrete(1)$, the discrete category on $\mathbb{1}$ with one object and one isomorphism

– or the terminal category. Suppose a functor $\varrho : Discrete(\mathbb{1}) \to Set$ sending $1 \mapsto \mathbb{1}$. Then it can be shown – cf. [82] Example 6.2.4.3. – there is an isomorphism of categories

$$\int_{\mathcal{C}} F \simeq (\varrho \downarrow F).$$

A final example to reinforce the construction of comma categories. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and suppose the (unique) functor to the terminal category $\mathbb{1}$, $F : \mathcal{C}, \mathcal{D} \to \mathbb{1}$. Then there is an isomorphism between the comma category and the product category $(\mathcal{C} \downarrow \mathcal{D} \simeq \mathcal{C} \times \mathcal{D})$.

5.7.3 The adjunction

We follow the argumentation of Steinberg-Tilson [90], and from each construction, we define a functor. First, we need to introduce some notation.

Fix a category $\mathcal{A}$; we define the category $\mathcal{A} - Cat$, which has as objects categories under the action of $\mathcal{A}$ and the morphism are functor that respects the action of $\mathcal{A}$ – also known as equivariant maps, in the particular case of a group acting on a set.

Indeed, we have the functors

- right adjoint: is given by $(- \rtimes \mathcal{A}) : \mathcal{A} - Cat \to (Cat \downarrow \mathcal{A})$, *i.e.* the Grothendieck construction viewed as a functor;
- left adjoint: suppose $F : \mathcal{C} \to \mathcal{A}$ a functor; associated to F there is the *derived category* (in the sense of monoid, and kernel theory) defined by $Der(F) := \coprod\{(A \downarrow F); A \in \mathcal{A}^{(0)}\}$, *i.e.* the disjoint union of comma categories, one for each object of $\mathcal{A}$. The operation $F \mapsto Der(F)$ defines a functor $Der : (Cat \downarrow \mathcal{A}) \to \mathcal{A} - Cat$.

Therefore, the pair $\langle Der(-); (- \rtimes \mathcal{A})\rangle$ is an adjoint.

The details of profs, examples, and even a generalization can be found in Steinberg-Tilson [90].

Once we establish the most abstract characterization (at this point) of our constructions, we conclude our work.

Chapter 6

Conclusion and further directions

Returning to the initial question:

> Does this approach, *i.e.* to use Bernoulli actions, apply to studying the expansions and algebras of other structures, more specifically inverse semigroups, groupoids, and inverse categories?

The main contribution of our thesis is the methodology used to provide an affirmative answer. Along with its execution, we produced new results and gave new interpretations of already known facts. We want to stress the study of inverse categories expansions, which generalizes the previous expansions and pointed new directions to study representations.

Let us talk about subsequent studies, with no particular order.

We want to use the Haskell functional language to analyze examples. The choice of this particular language is since it uses category theory. For instance, cf. Milewski [61].

Another and more abstract objective would be studying semi groupoids (a multi inverse semigroup object) extending the work of Cordeiro [23], and extend our findings to semi (inverse) categories, or semi groupoids .

In this categorical direction, a natural, but not yet explored, study would be defining partial actions for ∞-groupoids and next to higher categories. There are few works in this direction, mainly by Buss and Meyer as in [17].

Another direction is to follow Cordeiro-Beuter [24] and Buss-Meyer [16] and expand the notion of partial actions on other structures. Also, we would like to provide a proper definition and application in differential geometry.

Finally there is a formulation of partial actions of (weak) Hopf algebras and coalgebras (cf. [9] [18][36] [89]) that we would like to translate to string diagrammatic language in he realm of monoidal categories, following the work of Marsden [59] and Turaev-Verilizier [95], McCurdy [60] and Pastro-Street [69].

Our final words are thank you for your time and attention.

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