UNIVERSIDADE FEDERAL DO PARANÁ

FABRICIO SCHIAVON KOLBERG

CIRCULAR ARC BIGRAPHS AND THEIR HELLY SUBCLASS

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CIRCULAR ARC BIGRAPHS AND THEIR HELLY SUBCLASS

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Área de concentração: *Ciência da Computação*.

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To Joana, my safe harbour in the tormentous tides of these uncertain times

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RESUMO

Um grafo é *arco-circular* se é o grafo de interseção de uma família de arcos em um círculo. Grafos arco-circulares foram extensamente estudados na literatura. Os grafos *bi-arco-circulares* são uma variante bipartida dos grafos arco-circulares. Um modelo bi-arco-circular é uma tripla $(C, \mathbb{I}, \mathbb{E})$ onde C é um círculo, e $\mathbb{I}, \mathbb{E}$ são famílias arcos em C. O grafo correspondente a um modelo bi-arco-circular é montado ao se criar, para cada arco em $\mathbb{I} \cup \mathbb{E}$, um vértice, e arestas entre vértices são criadas se os seus arcos correspondentes intersectam e estão em famílias opostas (ou seja, um em $\mathbb{I}$, outro em $\mathbb{E}$). Um grafo bi-arco-circular é o grafo correspondente de um modelo bi-arco-circular. Em nosso trabalho, estudamos propriedades de várias subclasses de grafos bi-arco-circulares.

Um modelo bi-arco-circular $(C, \mathbb{I}, \mathbb{E})$ é dito ser *Helly* se o par de famílias $\mathbb{I}, \mathbb{E}$ admite a propriedade *bipartido-Helly*. Apresentamos caracterizações por grafos proibidos de três classes de grafos bi-arco-circulares Helly, incluindo grafos bi-arco-circulares Helly que admitem um ciclo sem cordas de comprimento maior que 4, grafos bi-intervalo Helly, e grafos bi-arco-circulares próprio-Helly. As caracterizações se baseiam em apresentar uma estrutura fundamental que todo grafo na classe verifica. Também apresentamos um algoritmo polinomial de reconhecimento para grafos bi-arco-circulares Helly sem vértices isolados, no qual aplicamos uma redução para o problema de reconhecimento de grafos bi-arco-circulares Helly para grafos C_6-free.

Também estudamos as relações de contenção entre várias subclasses de grafos bi-arco-circulares, incluindo grafos bi-arco-circulares Helly, grafos bi-arco-circulares próprios, grafos bi-intervalo Helly, grafos circular convexo bipartidos, e a classe que definimos como *bi-arco-circulares normais*. Também introduzimos as subclasses de grafos bi-arco-circulares *normais, cross-normais* e *cross-próprios*, e exploramos brevemente suas relações de continência.

Um *grafo biclique* de um grafo é o grafo de interseção de suas bicliques. Mostramos que os grafos biclique de grafos bi-arco-circulares Helly são uma subclasse de grafos arco-circulares próprios, e que eles não são comparáveis aos grafos clique de grafos arco-circulares Helly. Porém, também mostramos que os grafos biclique de grafos bi-arco-circulares Helly C_6-free são subclasse dos grafos clique de grafos arco-circulares Helly. Apresentamos uma simples caracterização de grafos biclique de grafos bi-arco-circulares Helly não bicordais, baseada nas mesmas estruturas fundamentais usada na caracterização por grafos proibidos, bem como uma caracterização para seus *grafos de bicliques mutuamente contidas*. Mostramos também que os grafos de bicliques mutuamente contidas de grafos bi-arco-circulares *normal-proper-Helly* são arco-arco circulares próprios.

Também apresentamos uma simples caracterização de uma subclasse de grafos bi-arco-circulares Helly bicordais, e limites superiores para o número de bicliques em diferentes subclasses de grafos bi-arco-circulares.
Palavras-chave: arco circular. Helly. grafo bipartido.

ABSTRACT

A graph is a *circular-arc graph* if it is the intersection graph of a family of arcs on a circle. Circular-arc graphs have been extensively studied in the literature. *Circular-arc bigraphs* are a bipartite variant of circular-arc graphs. A bi-circular-arc model is a triple $(C, \mathbb{I}, \mathbb{E})$ where C is a circle and $\mathbb{I}, \mathbb{E}$ are families of arcs over C. The corresponding graph of a bi-circular-arc model is constructed by creating, for each arc in $\mathbb{I} \cup \mathbb{E}$, a vertex, and edges between vertices are added if their corresponding arcs intersect and are in opposing families (i.e. one is in $\mathbb{I}$ and the other is in $\mathbb{E}$). A circular-arc bigraph is the corresponding graph of a bi-circular-arc model. In our work, we study the properties of several subclasses of circular-arc bigraphs.

A bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ is said to be *Helly* if the pair of families $\mathbb{I}, \mathbb{E}$ admits the *bipartite-Helly* property. We provide forbidden graph characterizations for three subclasses of Helly circular-arc bigraphs, including Helly circular-arc bigraphs that admit a chordless cycle of length greater than 4, Helly interval bigraphs, and proper-Helly circular-arc bigraphs. The characterizations rely on presenting a fundamental structure that every graph in those classes verifies. We also show a polynomial-time recognition algorithm for Helly circular-arc bigraphs without isolated vertices, in which we rely on a reduction to the problem of recognizing Helly circular-arc graphs for input graphs that are C_6-free.

We also study the containment relations between several subclasses of circular-arc bigraphs, including Helly circular-arc bigraphs, proper circular-arc bigraphs, Helly interval bigraphs, circular convex bipartite graphs, and the class we define as *normal circular-arc bigraphs*. We also introduce the classes of *normal, cross-normal,* and *cross-proper* circular-arc bigraphs, and briefly explore their containment relations.

A *biclique graph* of a graph is the intersection graph of its bicliques. We show that biclique graphs of Helly circular-arc bigraphs are a subclass of proper circular-arc graphs, and that they are not comparable to the clique graphs of Helly circular-arc graphs. However, we also show that the biclique graphs of C_6-free Helly circular-arc bigraphs are a subclass of clique graphs of Helly circular-arc graphs. We provide a simple characterization of the biclique graphs of non-bichordal Helly circular-arc bigraphs, based on the very same fundamental structures used in their forbidden graph characterization, as well as one for their *mutually contained biclique graphs*. We also show that the mutually contained biclique graphs of *normal-proper-Helly* circular-arc bigraphs are proper circular-arc graphs.

We also provide a simple characterization of a subclass of bichordal Helly circular-arc bigraphs, and upper bounds for the number of bicliques for different classes of circular-arc bigraphs.
Keywords: circular arc. Helly. bipartite graph.

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LIST OF ACRONYMS

CA	Circular-Arc
CCB	Circular Convex Bipartite
DINF	Departamento de Informática
FBS	Fundamental Bichordal Structure
FCS	Fundamental Circular Structure
FIS	Fundamental Interval Structure
FPHS	Fundamental Proper-Helly Structure
GBS	General Biclique Structure
HIBFF	Helly Interval Bigraph Forbidden-Free
NBHFF	Non-Bichordal Helly Forbidden-Free
NPH	Normal-Proper-Helly
PHFF	Proper-Helly Forbidden-Free
PPGINF	Programa de Pós-Graduação em Informática
UFPR	Universidade Federal do Paraná

CONTENTS

1 INTRODUCTION

A family of sets $\mathbb{F}$ is an *intersection model* of a graph G if there exists a bijection $f : V(G) \rightarrow \mathbb{F}$ such that $vw \in E(G)$ if and only if $f(v) \cap f(w) \neq \emptyset$ for all $v, w \in V(G)$. Graph G is then called the *intersection graph* of $\mathbb{F}$. Similarly, a pair of families $\mathbb{E}, \mathbb{F}$ is a *bi-intersection model* of a bipartite graph $G = (X, Y, E)$ if there exist bijections $f : X \rightarrow \mathbb{E}$ and $g : Y \rightarrow \mathbb{F}$ such that $vw \in E(G)$ if and only if $f(v) \cap g(w) \neq \emptyset$ for all $v \in X, w \in Y$. Graph G is then called the *intersection bigraph* of $\mathbb{E}, \mathbb{F}$.

A graph is a *circular arc graph* (CA graph for short) if it is the intersection graph of a family of arcs on a circle. Similarly, a bipartite graph is a *circular arc bigraph* (CA bigraph for short) if it is the intersection bigraph of a pair of families of arcs on a circle. In this thesis, we study properties of several subclasses of CA bigraphs, with special emphasis on the subclasses of Helly CA bigraphs.

In this chapter, we review existing results pertaining to circular arc graphs and bigraphs, in order to provide an overview of the subject's research history and state of the art. We opt to to go into just enough detail to provide the reader with a solid understanding of what has been done so far. We also outline our contributions alongside the motivation for our work.

The study of the class of circular arc graphs dates back to at least 1969, when Klee [Klee, 1969] mentioned the characterization of the class as an open problem. Since then, several forbidden structure characterizations and efficient recognition algorithms for the class have been discovered. Nowadays, the class of circular arc graphs is a widely studied topic, with a plethora of results on its structural and computational properties being known. In 2003, McConnell and Ross [McConnell, 2003] presented a linear time recognition algorithm for circular arc graphs, and in 2015, Francis et al. [Francis et al., 2015] presented a forbidden structure characterization of the class, alongside an efficient certifying recognition algorithm (i.e. a recognition algorithm that also provides proof of its answer).

Alongside the aforementioned characterizations and recognition algorithms for the general class of CA graphs, various subclasses have also been characterized and provided with efficient recognition algorithms [Lin and Szwarcfiter, 2009]. For instance, in 1996, Deng et al. [Deng et al., 1996] provided linear time representation algorithms (i.e. recognition algorithms that also provide intersection models) for both proper CA graphs and proper interval graphs, alongside a forbidden subgraph characterization of proper interval graphs. In 2006, Lin and Szwarcfiter [Lin and Szwarcfiter, 2006] provided a forbidden graph characterization and linear time recognition algorithm for the class of Helly circular arc graphs. More recently, in 2007 and 2013 respectively, Lin et al. provided characterizations and efficient recognition algorithms for proper-Helly [Lin et al., 2007] and normal-Helly [Lin et al., 2013] CA graphs.

Aside from recognition algorithms and structural characterizations, the computational properties of CA graphs over several computationally hard problems have also been abundantly studied. For instance, in 1980, Garey et al. [Garey et al., 1980] proved that the classical coloring problem is *NP-complete* over circular arc graphs, but the *K-colourability* problem is polynomial. Also, in 1991, Hsu et al. [Hsu and Tsai, 1991] provided a linear time algorithm to solve the *maximum independent set*, *minimum clique cover* and *minimum dominating set* problems over CA graphs.

Circular arc bigraphs arise as a natural bipartite variation of circular arc graphs. Most studies on the class are relatively recent. In 2013, Basu et al. [Basu et al., 2013] provided matrix-based characterizations for the class of CA bigraphs, as well as its proper and unit subclasses.

Later, in 2015, Das and Chakraborty [Das and Chakraborty, 2015] provided characterizations for proper interval bigraphs and proper CA bigraphs based on orderings of their vertex sets. More recently, in 2019, Safe [Safe, 2019] provided a forbidden graph characterization and linear time recognition algorithm for proper CA bigraphs, as a byproduct of his research on proper CA graphs and the circularly compatible 1s property in matrices. Aside from that, the class remains a relatively unexplored subject, with not nearly as many published results as its non-bipartite counterpart.

Interval bigraphs are a subclass of circular arc bigraphs, as well as a bipartite variation of interval graphs. Unlike its circular arc superclass, interval bigraphs and their subclasses have been widely studied in the literature. In 2004, Hell and Huang [Hell and Huang, 2004] presented a forbidden pattern characterization of interval bigraphs, and explored the relationship between interval bigraphs and the complements of two-clique circular arc graphs. In 2012, Rafiey [Rafiey, 2012] provided a quadratic time recognition algorithm for interval bigraphs based on the forbidden pattern characterization presented by Hell and Huang in 2004. In 2010, Brown and Lundgren presented a characterization of unit interval bigraphs based on vertex orderings, alongside several equivalencies between the classes of proper interval bigraphs, unit interval bigraphs, and a plethora of other classes of bipartite graphs [Brown and Lundgren, 2010]. Also, as we previously mentioned, in 2015, Das et al. [Das and Chakraborty, 2015] provided a vertex ordering characterization of proper interval bigraphs.

Aside from the intersection graph and bigraph classes previously mentioned, one concept of particular interest for our research is the *Helly property*, used to define Helly circular arc graphs and several other classes of graphs. The Helly property dates back to 1923, when Eduard Helly's seminal theorem [Helly, 1923] originated the concept of a Helly family. While Helly's original work was on set theoretical geometry, the Helly property has also been extensively studied in the context of combinatorics [Bollobas, 1998] and graph theory.

The usage of the Helly property to define a class of graphs dates back to at least 1981, in a work by Beeri [Beeri et al., 1981] where a class of database structures is defined that is effectively equivalent to what is now known as *Helly graphs*. Since then, many other classes containing the Helly property in its definition have been studied and characterized, such as Helly subtree graphs [Gavril, 1996], clique-Helly graphs [Szwarcfiter, 1997], biclique-Helly graphs [Groshaus and Szwarcfiter, 2007, Groshaus, 2006], and Helly circular arc graphs [Lin and Szwarcfiter, 2006, Lin et al., 2007, 2010, 2013, Soulignac, 2010].

In 2010, Groshaus and Szwarcfiter [Groshaus and Szwarcfiter, 2010] introduced a bipartite analogue to the Helly property of families named *bipartite-Helly*. In 2011, Groshaus [Groshaus and Szwarcfiter, 2011] presented a recognition algorithm for bipartite-Helly hypergraphs. This concept of bipartite-Helly families is applied by us in this thesis to define the class of Helly circular arc bigraphs, albeit modified from their original context of labeled sets to the context of pairs of families.

1.1 MOTIVATION AND OUR CONTRIBUTION

As mentioned in the introduction, circular arc graphs are a widely studied class of intersection graphs. Many important results on the structural and computational properties of the class and several subclasses exist. The class is of great research importance due in large part to its applicability in genetics, traffic control, and several other subjects [Lin and Szwarcfiter, 2009].

Circular arc bigraphs, in contrast, are a relatively unexplored subject. There are matrix characterizations for the class and its proper and unit subclasses [Basu et al., 2013], and a polynomial time algorithm for the recognition of the proper subclass [Safe, 2019], but most results on the class are relatively recent, and very few exist in the literature at the present time.

In particular, no comprehensive study on the containment relations between subclasses of circular arc bigraphs exists. In this thesis, we fill this gap by analyzing the containments between several subclasses of circular arc bigraphs, including proper circular arc bigraphs, circular convex bipartite graphs, proper interval bigraphs, as well as the Helly subclasses for both circular arc bigraphs and interval bigraphs, which we introduce. We also show a simple way to utilize existing algorithms to recognize circular convex bipartite graphs in linear time.

Aside from circular arc graphs being an important class of intersection graphs as previously mentioned, Helly circular arc graphs are a particularly relevant subclass of circular arc graphs. A linear time recognition algorithm [Lin and Szwarcfiter, 2006] as well as forbidden structure characterizations for Helly, proper-Helly and normal-Helly circular arc bigraphs [Lin et al., 2007, 2013] exist. Not only can the class be recognized in linear time, it also has a linear number of cliques, implying many hard clique-related problems are easily solvable when restricted to the class.

Given the structural and computational importance of Helly circular arc graphs, it was a natural question to ask about the potential properties of a bipartite analogue for the class, which led to our introduction of Helly circular arc bigraphs. To define a bipartite analogue to Helly circular arc graphs, we utilize the bipartite-Helly concept introduced by Groshaus [Groshaus and Szwarcfiter, 2011] in the context of pairs of families instead of labeled sets.

In our studies, we found a polynomial time recognition algorithm for Helly circular arc bigraphs, as well as forbidden structure characterizations for subclasses such as Helly interval bigraphs, non-bichordal Helly circular arc bigraphs, and proper-Helly circular arc bigraphs. We also show that the number of bicliques in any Helly circular arc bigraphs is linear over the number of vertices, implying a computational relevance for biclique-related problems similar to that Helly circular arc graphs have for clique-related problems.

1.2 HOW THIS THESIS IS ORGANIZED

The organization of this thesis is as follows. Chapter 2 contains a set of useful definitions and notations we use throughout the rest of the thesis. It is separated into subsections pertaining to the types of structures the definitions are built upon, such as graphs and bicliques in Section 2.1; intersection models in Section 2.2; and sequences, sets and families in Section 2.3.

In Chapter 3, we introduce the results we obtained in our research, separated into four sections.

Section 3.1 contains the characterizations and recognition algorithms we have discovered, including forbidden graph characterizations for non-bichordal Helly CA bigraphs, Helly interval bigraphs, and proper-Helly CA bigraphs. The latter characterization includes, as side results, the facts that non-bichordal Helly CA bigraphs are a subclass of proper-Helly CA bigraphs, that Helly interval bigraphs are equivalent to proper-Helly interval bigraphs, and that Helly and proper CA bigraphs are not comparable. Section 3.1 also includes the recognition algorithm we introduce for Helly CA bigraphs without isolated vertices, a simple description of how existing results can be used to recognize circular convex bipartite graphs in linear time, and a simple characterization of proper CA bigraphs in therms of circular convex bipartite graphs.

Section 3.2 presents containment relations between several subclasses of CA bigraphs. We show that circular convex bipartites are a superclass of proper and Helly CA bigraphs, introduce the classes of cross-proper, normal and cross-normal CA bigraphs, and explore their relationship with the other classes we discuss. A handy diagram is provided in the end of the section to help visualize the containment relations between the subclasses.

Section 3.3 contains results pertaining to the properties of the biclique graphs of Helly CA bigraphs. We show that the class of biclique graphs of Helly CA bigraphs is a subclass of proper CA graphs, and that it is not comparable to the class of clique graphs of Helly CA graphs. We also show simple characterizations for the biclique and mutually contained biclique graphs of non-bichordal Helly CA bigraphs, based on the characterization introduced in Section 3.1, and that the mutually contained biclique graphs of the class of normal-proper-Helly CA bigraphs are a subclass of proper CA graphs.

Finally, Section 3.4 contains results from studies that do not fit the other three categories. We present results pertaining to our as of yet incomplete study of bichordal Helly CA bigraphs, as well as simple upper bounds for the number of bicliques in proper CA bigraphs and Helly CA bigraphs.

Chapter 4 contains the conclusion of the thesis, where we discuss the results we presented, their implications and their relevance. We also delineate potential topics for future research, the motivation behind them, and the facts we already know about them.

2 DEFINITIONS

2.1 GRAPHS, BICLIQUES, ETC.

In this paper, we denote bipartite graphs as triples (X, Y, E) with X, Y being the graph's partite sets and E being its edge set. If $G = (X, Y, E)$ is a bipartite graph, we say that V and W are *opposite* partite sets to each other. For any graph G, we denote by G^* the graph resulting from adding, to G, an isolated vertex. For any integer $n > 0$, C_n denotes the cycle on n vertices.

A graph G is *complete* if, for every $v, w \in V(G)$, $vw \in E(G)$. A *clique* of a graph G is a maximal subset $K \subseteq V(G)$ such that the subgraph induced by K is a complete graph. Similarly, a bipartite graph $H = (X, Y, E)$ is *bipartite-complete* if, for every $v \in X, w \in Y$, $vw \in E(H)$. A *biclique* of a graph G is a maximal subset $K \subseteq V(G)$ which induces a bipartite-complete graph. In this thesis, we consider that a set K is a biclique only if neither partite set is empty in $G[K]$.

To simplify notation, we refer to the set of bicliques of G as $b(G)$, and the set of bicliques of G that contain a vertex $v \in V(G)$ as $b_G(v)$. The graph subscript is omitted when the referred graph is clear from context.

A vertex v in a graph G is said to be *universal* if $N[v] = V(G)$. For a bipartite graph $G = (X, Y, E)$, we call a vertex $v \in X$ ($w \in Y$) *bi-universal* if $N(v) = X$ ($N(w) = Y$).

For any graph G, define the *square* of G as $G^2 = (V(G), E^2(G))$, where $E^2(G) = \{vw | vw \in E(G) \vee \exists x \in V(G) : vx, xw \in E(G)\}$. If two vertices $v, w \in V(G)$ are such that $N(v) = N(w)$, then v and w are called *twins*. Vertices of equal open neighborhoods are commonly called *false twins* in the literature, with vertices of equal closed neighborhood being called twins, but since our study is on bipartite graphs, we use *twins* to refer to vertices of equal open neighborhood. A set $S \subseteq V(G)$ is a *set of twins* if all its elements have equal open neighborhoods. Note that, for any graph G, $V(G)$ can be partitioned into sets of twins. The *twin-free version* of graph G is the graph that results from removing, from every set of twins, every vertex but one, and then repeating the process until no twins remain.

A cycle $(c_1, ..., c_n)$ in a graph is *chordless* if no edges exist between vertices that are not consecutive in the cycle. A graph G is said to be *bichordal* if it does not contain any chordless cycle of length greater than 4. We use the expression *induced cycle* as synonymous to chordless cycle, since the set of vertices that compose a chordless cycle induces a cycle graph.

2.2 INTERSECTION MODELS, BIPARTITE INTERSECTION MODELS

To simplify notation, we treat the circular arc intersection models of graphs as *circular-arc models* as defined by [Soulignac, 2010]. We also introduce our analogous definition of *bi-circular-arc models*. Throughout this thesis, we refer to sets whose elements are themselves sets as *families*.

Given two families $\mathbb{A}, \mathbb{B}$, the *intersection bigraph* of $(\mathbb{A}, \mathbb{B})$ is the bipartite graph constructed by creating a vertex for every element of $\mathbb{A} \cup \mathbb{B}$, and adding edges between vertices corresponding to two elements $A \in \mathbb{A}, B \in \mathbb{B}$ if and only if $A \cap B \neq \emptyset$.

A *circular arc model* is a pair $(C, \mathbb{A})$ such that C is a circle, and $\mathbb{A}$ is a family of arcs over C. The *corresponding graph* of model $(C, \mathbb{A})$ is the intersection graph of $\mathbb{A}$. A graph G is a circular arc graph if and only if it is the corresponding graph of a circular arc model. If $(C, \mathbb{A})$ is a circular arc model for which G is the corresponding graph, we say that G *admits* model $(C, \mathbb{A})$.

A *bi-circular-arc model* is a triple $(C, \mathbb{I}, \mathbb{E})$ such that C is a circle, and $\mathbb{I}, \mathbb{E}$ are arcs over C. The corresponding graph of a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ is the intersection bigraph of

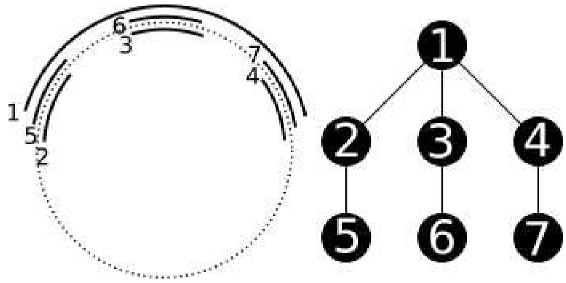

Figure 2.1: The graph T_2 alongside its bi-circular-arc model.

$(\mathbb{I}, \mathbb{E})$. A bipartite graph G is a *circular arc bigraph* if and only if it is the corresponding graph of a bi-circular-arc model.

When graphically representing a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$, we represent C as a dotted circle, $\mathbb{I}$ as arcs inside the circle, and $\mathbb{E}$ as arcs outside the circle, as exemplified in Figure 2.1.

If G is the corresponding graph of a model $(C, \mathbb{I}, \mathbb{E})$, we say that G *admits* the model. Note that families $\mathbb{I}$ and $\mathbb{E}$ are interchangeable, that is, models $(C, \mathbb{I}, \mathbb{E})$ and $(C, \mathbb{E}, \mathbb{I})$ can be considered equal.

If $(C, \mathbb{I}, \mathbb{E})$ is a bi-circular-arc model, we say that $\mathbb{I}$ and $\mathbb{E}$ are *opposite families* to each other. Furthermore, if $\mathbb{I}' \subseteq \mathbb{I}$ and $\mathbb{E}' \subseteq \mathbb{E}$, we say $(C, \mathbb{I}', \mathbb{E}')$ is a *submodel* of $(C, \mathbb{I}, \mathbb{E})$.

In this document, we abbreviate circular arc as *CA* for brevity. Furthermore, if v is a vertex of a graph G that admits a circular arc (bi-circular-arc) model $(C, \mathbb{A})$ ($(C, \mathbb{I}, \mathbb{E})$), we refer to the arc corresponding to v in the model as the v-arc, or $a(v)$.

An *interval model* is a family of intervals on the number line. A graph is an interval graph if and only if it is the intersection graph of an interval model.

A *bi-interval model* is a pair of families $(\mathbb{E}, \mathbb{F})$ of intervals on the number line. The *corresponding graph* of a bi-interval model $(\mathbb{E}, \mathbb{F})$ is constructed by creating a vertex for each element of $\mathbb{E} \cup \mathbb{F}$ and, for every pair of intervals $E \in \mathbb{E}, F \in \mathbb{F}$, an edge between the corresponding vertices of E and F is created if and only if $E \cap F \neq \emptyset$. A graph is an interval bigraph if and only if it admits a bi-interval model.

2.3 SEQUENCES, ARCS, FAMILIES

In this document, we consider all arcs (intervals) to be open unless otherwise stated. If A is an arc on a circle, we use $s(A)$ and $t(A)$ to denote its counter-clockwise and clockwise endpoints, respectively. To simplify notation, we call counter-clockwise endpoints s-endpoints, and clockwise endpoints t-endpoints. If I is an interval on the number line, we denote by $l(I)$ and $r(I)$ its left and right endpoints (i.e. its low and high endpoints).

For any arc A in circle C, we define the *complement* of A as an arc $\bar{A} = C - A$. Note that the complement of an open arc is a closed arc, and vice-versa. Denote the length of arc A as $|A|$.

If p, q are two points in a circle C, we define (p, q) as an open arc such that $s((p, q)) = p, t((p, q)) = q$. We say that a sequence $(p_1, ..., p_n)$ of points in circle C is a clockwise (counter-clockwise) order if, for every $0 < i < n$, the open arc (p_i, p_{i+1}) (the open arc (p_{i+1}, p_i)) does not contain any point in the sequence.

For any indexed set or sequence, index summation is considered cyclic. For example, in a set $\{s_1, ..., s_n\}$, we consider $s_{1-1} = s_n$ and $s_{n+1} = s_1$.

Define, for two points $p, q \in C$, the *distance* between p and q as $d(p, q) = min\{|(p, q)|, |(q, p)|\}$. For $0 \leq c < |C|$, define, for any point p, the points $p - c$ and $p + c$

to be such that $|(p-c,p)| = |(p,p+c)| = c$. That is, the point $p-c$ ($p+c$) is at a counter-clockwise (clockwise) offset of length c from p.

A *rotation* of a sequence $(p_1,\ldots,p_n)$ is a permutation of the form $(p_i,\ldots,p_n,p_1,\ldots,p_{i-1})$ for $1 \le i \le n$. Note that the identity permutation is a rotation. A set $S \subset \{p_1,\ldots,p_n\}$ is said to be *circularly consecutive* in $(p_1,\ldots,p_n)$ if there exists a rotation of the sequence in which S is consecutive. Similarly, a sequence $(p_{a_1},\ldots,p_{a_k})$ with $\{p_{a_1},\ldots,p_{a_k}\} \subset \{p_1,\ldots,p_n\}$ is said to be circularly consecutive if there exists a rotation where it is a consecutive subsequence.

A set of arcs $\{A_1,\ldots,A_n\}$ on a circle C is said to *cover the circle* if $A_1 \cup \ldots \cup A_n = C$. In this paper, assume that, in every circular arc (bi-circular-arc) model over a circle C, there is no individual arc A such that $A = C$. Note that, in a model with a finite number of arcs, any arc that covers the entire circle by itself can be replaced with a shorter one without losing intersections.

A family $\mathbb{F}$ is said to be *proper* if there are no two elements $F, F' \in \mathbb{F}$ such that $F \subset F'$. A family $\mathbb{F}$ of arcs over a circle C is said to be *normal* if there are no two arcs $F, F' \in \mathbb{F}$ such that $F \cup F' = C$.

A family $\mathbb{F}$ is said to be *intersecting* if, for every pair $E, F \in \mathbb{F}$, we have $E \cap F \neq \emptyset$. Similarly, we say that a pair of families $\mathbb{E}, \mathbb{F}$ is *bipartite-intersecting* if, for every $E \in \mathbb{E}, F \in \mathbb{F}$, $E \cap F \neq \emptyset$.

A family $\mathbb{F}$ is *Helly* if, for every intersecting subfamily $\mathbb{F}'$, there exists an element e such that $e \in F$ for every $F \in \mathbb{F}'$. Similarly, a pair of families $\mathbb{E}, \mathbb{F}$ is said to be *bipartite-Helly* if, for every bipartite-intersecting pair of subfamilies $\mathbb{E}', \mathbb{F}'$, there exists an element e such that $e \in F$ for every $F \in \mathbb{E}' \cup \mathbb{F}'$.

3 RESULTS AND PROOFS

3.1 CHARACTERIZATIONS AND RECOGNITION ALGORITHMS

We introduce the class of Helly circular arc bigraphs as a bipartite variation of Helly circular arc graphs, and present characterizations of the class and a few subclasses, as well as a polynomial time recognition algorithm for the class.

We also present a simple recognition algorithm for the class of *circular convex bipartite* graphs, obtained by putting together pre-existing results from the literature.

3.1.1 Non-bichordal Helly CA bigraphs

Definition 1. *A bipartite graph G is a* Helly circular arc bigraph *if it admits a bi-circular-arc model* $(C, \mathbb{I}, \mathbb{E})$ *such that* $\mathbb{I}, \mathbb{E}$ *is a bipartite-Helly pair of families. Equivalently, a bipartite graph G is a Helly circular arc bigraph if it admits a bi-circular-arc model* $(C, \mathbb{I}, \mathbb{E})$ *such that, for every biclique K in G, there exists a point $p \in C$ such that, for every $k \in K$, $p \in a(k)$.*

In this subsection, we characterize non-bichordal Helly CA bigraphs. This specific subclass of Helly circular arc bigraphs is of relevance for the studies of proper-Helly CA bigraphs presented in Subsection 3.1.4, and it also presents multiple structural similarities with Helly interval bigraphs, as will be demonstrated in Subsection 3.1.3. Furthermore, the specific case of non-bichordal Helly circular arc bigraphs that contain an induced C_6 is relevant for the recognition algorithm in Subsection 3.1.2.

Bichordal Helly CA bigraphs, the counterpart subclass to non-bichordal Helly CA bigraphs, have yet to be characterized. Our attempts to find a suitable forbidden graph characterization of it have led to an as of yet incomplete case study, which we discuss in Subsection 3.4.1.

The characterization of non-bichordal Helly CA bigraphs we provide is based on forbidden induced subgraphs. The forbidden graphs we require are T_2, F_1, BW_3, F_2, $L_3 \cup P_2$, as seen in Figure 3.1, as well as $C_n{}^*$ for all $n \geq 6$.

Lemmas 1 and 4 are preliminary results used throughout the section. Lemma 1 is directly analogous to existing results for Helly CA graphs, with the biclique points mentioned in the lemma playing an analogous role to the clique points for Helly CA graphs [Lin and Szwarcfiter, 2006].

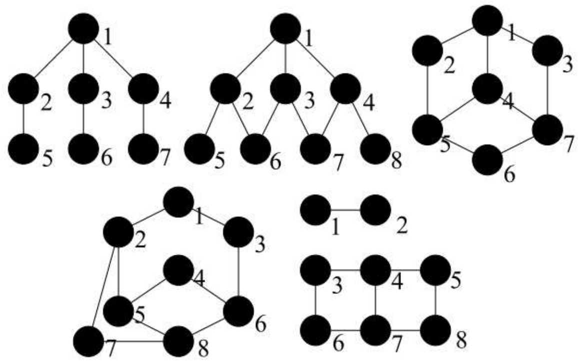

Figure 3.1: Forbidden graphs for the class of Helly CA bigraphs. From left to right, top to bottom, T_2, F_1, BW_3, F_2, $L_3 \cup P_2$. Graphs BW_3 and $L_3 \cup P_2$ are proper CA bigraphs.

Lemma 1. *A bipartite graph G without isolated vertices is a Helly circular arc bigraph if and only if, given a circle C, it is possible to attribute to every biclique K of G a point $p_K \in C$ such that, for all $v \in V(G)$, the points attributed to $b(v)$ are consecutive. We call those points the* biclique points *of each biclique.*

Proof. If we have a Helly bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ of a bipartite graph G, we can easily derive a set of biclique points by picking, for each biclique $K \in b(G)$, one point $p_k \in C$ such that, for every $v \in K$, $p_k \in a(v)$. Such a point must exist, as seen in the definition of Helly CA bigraphs. Note that a set of points chosen in this manner satisfies the properties stated in the lemma.

Conversely, let S be a set of biclique points for $G = (X, Y, E)$ on circle C such that, for every $v \in V \cup W$, the points attributed to the members of $b(v)$ are consecutive. Define $\epsilon = \frac{1}{10} min\{d(a, b) | a, b \in S\}$. Number ϵ is chosen as a small enough value so that no unintentional intersections are added in the construction of the model.

Construct a bi-circular-arc model by making, for each vertex v, a v-arc A_v with the following process: let A be the shortest closed arc that contains every point of $b(v)$ and no point of $b(G) - b(v)$ (with $s(A)$ and $t(A)$ being biclique points), then $A_v = (s(A) - \epsilon, t(A) + \epsilon)$.

Note that, with arcs constructed in this way, all intersections between arcs must always contain at least one biclique point. Also, since there are no isolated vertices, every arc contains at least one biclique point.

We claim that the process creates a bi-circular-arc model of G: if two vertices $v \in X, w \in Y$ are such that $vw \in E(G)$, then they belong to a common biclique K, and $p_K \in a(v) \cap a(w)$. Conversely, if two vertices $v \in X, w \in Y$ are such that $a(v) \cap a(w) \neq \emptyset$, that implies $a(v), a(w)$ have at least one biclique point in common, implying they belong to a common biclique, and are therefore neighbors. Therefore, the model created is a bi-circular-arc model of G. Furthermore, since every biclique is such that the arcs corresponding to its vertices contain its biclique point, the model is also Helly. □

Lemma 1 can be used to prove that the forbidden graphs we cited are, in fact, not in the class of Helly CA bigraphs. Let G be a bipartite graph such that $b(G) = \{K_1, ..., K_n\}$, and $\{p_1, ..., p_n\}$ a set of biclique points of G. We say that a Helly bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ is *compatible* with set of biclique points $\{p_1, ..., p_n\}$ if, for every $K_i \in b(G)$, every vertex $v \in K_i$ is such that $p_i \in a(v)$. It is easy to verify that every Helly bi-circular-arc model is compatible with some set of biclique points.

The proof that the forbidden graphs are not Helly CA bigraphs in Lemma 3 depends on the following auxiliary lemma (Lemma 2).

Lemma 2. *Let G be a bipartite graph, and $v, w \in V(G)$ such that $b(v) \cup b(w) \neq b(G)$ and $|b(v) \cap b(w)| = 2$. If G is a Helly CA bigraph, the biclique points for the pair of bicliques contained in $b(v) \cap b(w)$ must be consecutive in any valid set of biclique points of G.*

Proof. Let $b(v) \cap b(w) = \{A, B\}$, and let $C \in b(G) - (b(v) \cup b(w))$. No matter where the biclique points for A and B are placed around the circle, there is one direction (either clockwise or counter-clockwise) where a walk on the circle from A to B crosses C. Consider the opposite direction (i.e. the one that does not cross C): either there are no points between the points for A, B (i.e. they are consecutive) or there is a point for some biclique D in between A and B, so that every arc that contains both the points for A and B also contains the one for D or the one for C.

Since $b(v) \cap b(w) = \{A, B\}$, then $D \notin b(v)$ or $D \notin b(w)$. Suppose the former w.l.o.g.. In this case, it is impossible for the biclique points corresponding to $b(v)$ to be consecutive, as any arc that crosses both the points for A, B also crosses either the points for C or D, making it impossible to draw an arc that contains all points corresponding to $b(v)$ without also containing points from $b(G) - b(v)$. □

Lemma 3. *Graphs $T_2, F_1, BW_3, F_2, L_3 \cup P_2$ and C_n^*, $n \geq 6$ are not Helly CA bigraphs.*

Proof. For graphs $T_2, F_1, BW_3, F_2, L_3 \cup P_2$, we use Lemma 1 and prove that it is impossible to find a suitable set of biclique points for them. The vertices of those graphs are going to be referred to according to their labels in Figure 3.1.

We start with T_2. The bicliques of T_2 are $A = \{1,2,3,4\}, B = \{1,2,5\}, C = \{1,3,6\}, D = \{1,4,7\}$. Vertex 2 is in bicliques A, B, therefore, in any set of biclique points for T_2, the biclique points corresponding to A and B must be consecutive. Similarly, vertex 3 is in A, C, implying the points corresponding to A, C must also be consecutive. Therefore, the biclique points for C, A, B must be in this order, or in the reverse of this order, around the circle. However, vertex 4 is in A, D, but the point for A cannot be consecutive to more than two other biclique points, implying it is impossible to find a suitable set of biclique points for T_2.

Now for F_1. Its bicliques are $A = \{1,2,3,4\}, B = \{1,2,3,6\}, C = \{1,3,4,7\}, D = \{1,2,5,6\}, E = \{1,3,6,7\}, F = \{1,4,7,8\}$. Bicliques D and F are the only two bicliques that 3 is not in, therefore, the biclique points to D, F must be consecutive. Vertex 7 is contained in C, E, F, implying the points corresponding to those three bicliques must be consecutive in the circle (not necessarily in that order). Also, since vertex 6 is in B, D, E, note that the points C, F must be consecutive: if the points for C, E, F were in this order (i.e. with E immediately in between C and F) in the circle, then the biclique points of $b(6)$ would not be consecutive, as any arc that contained the points for B and E would necessarily contain the point for C or F.

Therefore, the points for D, F, C, E must be consecutive in this exact order (or its reverse) around the circle. However, in this order, it is impossible for the points corresponding to $b(4)$ to be consecutive, as $b(4) = \{A, C, F\}$, and no matter where the biclique point for A appears in the circle, any arc that contained both the point for A and the point for C would have to cross either the point for D or for E.

Now for BW_3. The bicliques are $A = \{1,2,3,4\}, B = \{1,2,4,5\}, C = \{1,3,4,7\}, D = \{4,5,6,7\}, E = \{1,4,5,7\}, F = \{2,4,5,6\}, G = \{3,4,6,7\}$. Note that $b(1) = \{A, B, C, E\}$ and $b(5) = \{B, D, E, F\}$ imply the biclique points for B, E must be consecutive by Lemma 2. Also $b(7) = \{C, D, E, G\}$, implying, together with $b(5)$, that D, E are consecutive by Lemma 2. However, C, E must also be consecutive by Lemma 2 considering $b(1)$ and $b(7)$, implying there is no valid set of biclique points, as the point for E cannot be consecutive to more than two points.

Consider, now, F_2. The bicliques are $A = \{1,2,3\}, B = \{1,3,6\}, C = \{3,4,6,8\}, D = \{4,5,6,8\}, E = \{5,6,7,8\}, F = \{2,5,7,8\}, G = \{2,4,5,8\}$, $H = \{1,2,5,7\}$. Consider the set of bicliques for each vertex: $b(1) = \{A, B, H\}, b(2) = \{A, F, G, H\}, b(3) = \{A, B, C\}, b(4) = \{C, D, G\}, b(5) = \{D, E, F, G, H\}, b(6) = \{B, C, D, E\}, b(7) = \{E, F, H\}, b(8) = \{C, D, E, F, G\}$. We apply Lemma 2 to construct a clockwise (or counter-clockwise) order of the biclique points.

A, B must be consecutive by $b(1), b(3)$. A, H must be consecutive by $b(1), b(2)$. B, C must be consecutive by $b(3), b(6)$. D, C must be consecutive by $b(4), b(6)$. D, E must be consecutive by $b(5), b(6)$.

Therefore, the points corresponding H, A, B, C, D, E must show consecutively up in this order (or its reverse) around the circle, implying the complete order (either clockwise or counter-clockwise) of biclique points around the circle is either (H, A, B, C, D, E, F, G) or (H, A, B, C, D, E, G, F). In both cases, the biclique points of $b(7)$ are not consecutive around the circle.

Now for $L_3 \cup P_2$. The bicliques are $A = \{1,2\}, B = \{3,4,6,7\}, C = \{4,5,7,8\}, D = \{3,4,5,7\}, E = \{4,6,7,8\}$. The points of B, D must be consecutive because $b(3) = \{B, D\}$,

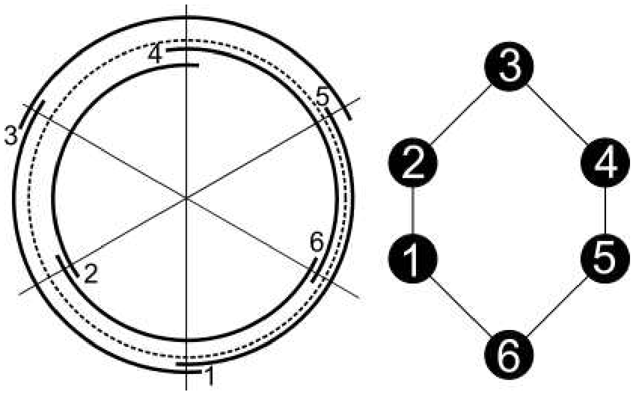

Figure 3.2: A Helly bi-circular-arc model of a C_6 with biclique points indicated by line segments from the center of the circle.

the points of C, D must be consecutive because $b(5) = \{C, D\}$, and the points of C, E must be consecutive because $b(8) = \{C, E\}$.

Therefore, the points for B, D, C, E must be consecutive in this order (or its reverse) around the circle). Therefore, the entire order of biclique points will be (A, B, D, C, E). However, in this situation, the points of $b(6) = \{B, E\}$ are not consecutive.

For the case of $C_n{}^*, n > 4$, Lemma 1 does not apply, as an isolated vertex is present. Note that a Helly bi-circular-arc model of $C_n{}^*$ must be such that the removal of a single arc (i.e. the one corresponding to the isolated vertex) turns it into a Helly bi-circular-arc model of C_n, as the removal of an arc does not break the bipartite-Helly property.

We prove that every Helly bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ of C_n for any $n > 4$ is such that every point in the circle is contained in at least one arc of $\mathbb{I}$ and one arc of $\mathbb{E}$, implying it is impossible to add an arc corresponding to the isolated vertex without accidentally intersecting an arc of the opposing family.

For C_n (without the isolated vertex), Lemma 1 does apply, and any Helly bi-circular-arc model of C_n must be compatible with some set of biclique points. Let us analyze the properties a set of biclique points for C_n must have.

Let $\{c_1, c_2, c_3, ..., c_n\}$ be the set of vertices of a C_n, such that, for every $1 \leq i \leq n$, $N(c_i) = \{c_{i+1}, c_{i-1}\}$ (with $n + 1 = 1, 1 - 1 = n$). The bicliques of C_n are of the form $K_i = \{c_i, c_{i+1}, c_{i+2}\}$ for all $1 \leq i \leq n$ (with $n + 1 = 1, n + 2 = 2$). So, for every $1 \leq i \leq n$, $b(c_i) = \{K_{i-2}, K_{i-1}, K_i\}$. Note that, by Lemma 2, $b(c_i)$ and $b(c_{i+1})$ together imply, for all $1 \leq i \leq n$, that the biclique points corresponding to K_{i-1} and K_i must be consecutive. Therefore, the circular order for any valid set of biclique points of C_n is $(K_1, K_2, ..., K_n)$ or its reverse.

Consider, then, a Helly bi-circular-arc model of C_n. Since it must be compatible with a set of biclique points that follows the order $(K_1, K_2, ..., K_n)$ (or its reverse), every point in the circle is either a biclique point, or is between two biclique points for bicliques K_i, K_{i+1} for some $1 \leq i \leq n$. Consider a point p in between the biclique points for K_i and K_{i+1}. Note that $p \in a(c_{i+1})$ and $p \in a(c_{i+2})$, and that c_{i+1} and c_{i+2} are in opposing partite sets. Therefore, p is contained in at least one arc of each family of the model.

Therefore, it is impossible to add, to a Helly bi-circular-arc model of C_n, an arc corresponding to an isolated vertex. Figure 3.2 contains an example of a Helly bi-circular-arc model of a C_6 with highlighted biclique points, for better visualization.

□

Finally, Lemma 4 shows that twin vertices are irrelevant for the study of the structures of Helly CA bigraphs.

Lemma 4. *If the twin-free version of a graph G is a Helly CA bigraph, then so is G.*

Proof. Let $G = (X, Y, E)$ be a Helly CA bigraph, and let $(C, \mathbb{I}, \mathbb{E})$ be a Helly bi-circular-arc model of G. We claim that adding a twin to G makes it remain a Helly CA bigraph. Suppose that no endpoints of different arcs coincide in the model (it is easy to verify that any model for which this is not the case can be modified into an equivalent one with no coincidences without losing the Helly property).

Let $v \in X$. Suppose we add, to G, a vertex v' such that $N(v') = N(v)$. Let ϵ be a fraction of the smallest distance between a pair of non-coinciding endpoints in the model. Without loss of generality, with d being the smallest distance, make $\epsilon = \frac{d}{100|X \cup Y|}$.

Let $a(v) = (s, t)$ for some $s, t \in C$. Make $a(v') = (s + \epsilon, t + \epsilon)$, and add it to the same family as $a(v)$. It is easy to note that the resulting model is a Helly bi-circular-arc model of $G + v'$. □

Lemma 4 allows us to focus on twin-free graphs when studying the structural properties of Helly CA bigraphs. In the sequence, we present results that lead to the characterization. The proofs of the lemmas and theorems for the remainder of the section use the graphs in Figure 3.1, as well as every ${C_n}^*, n \geq 6$, as forbidden graphs.

Lemma 5. *If G is a twin-free Helly CA bigraph with an induced $C_n, n \geq 6$, then every vertex outside the C_n is neighbor to exactly one vertex of the C_n.*

Proof. Let $C = (c_1, ..., c_n)$ be an induced C_n of G. We prove that, if a vertex from outside the C_n has 0 or more than 1 neighbor in the C_n, then a forbidden graph from Lemma 3 occurs as an induced subgraph.

The proof is separated in two cases: the case for $n = 6$, and the case for $n > 6$. The case for $n = 6$ requires a reduced set of forbidden graphs, which we require for the proof of a later result.

Case for $n = 6$

Suppose $n = 6$. We prove that, if a twin-free bipartite graph with an induced C_6 (in vertex set C) does not contain $BW_3, F_2, {C_6}^*$ as induced subgraphs, then it is such that every $v \in V(G) - C$ has exactly one neighbor in C. Let $v \in V(G) - C$.

If v is neighbor to no vertices in C, then ${C_6}^*$ is induced. If v is neighbor to 3 vertices of C, BW_3 is induced.

Consider, then, the case where v is neighbor to two vertices of C. Without loss of generality, suppose $N(v) \cap C = \{c_2, c_6\}$. Since v and c_1 are not twins, that implies there must exist $w \in V(G)$ such that $w \in N(v) - N(c_1)$ or $w \in N(c_1) - N(v)$. Suppose the former without loss of generality. See Figure 3.3 for a visualization of the cases that follow. If w is neighbor to no vertices of C, then ${C_6}^*$ is induced. If w is neighbor to two vertices of C, then $N(w) \cap C = \{c_3, c_5\}$ as w is not neighbor to c_1. However, in this case, $\{v, c_2, c_3, c_4, c_5, c_6, w\}$ induce a BW_3. If w is neighbor to exactly one vertex of C, say, c_3, then an F_2 is induced with $C \cup \{v, w\}$.

Therefore v has exactly one neighbor in C.

Case for $n > 6$

We prove that every twin-free Helly CA bigraph with an induced $C_n, n > 6$ in set C is such that every $v \in V(G) - C$ has exactly one neighbor in C. Let $v \in V(G) - C$.

If v is neighbor to no elements of C, then the graph contains an induced ${C_n}^*$.

Suppose, then, that v is neighbor to three or more vertices of C. If there are three neighbors of v of the form c_i, c_{i+2}, c_{i+4}, then G contains an induced F_1 (Figure 3.4). Now, if there are no three neighbors of that form, consider three vertices $c_x, c_y, c_z \in N(v) \cap C$ such that c_x is at a distance of at least 4 from c_y and c_z in $G[C]$, with the distance between c_y and c_z

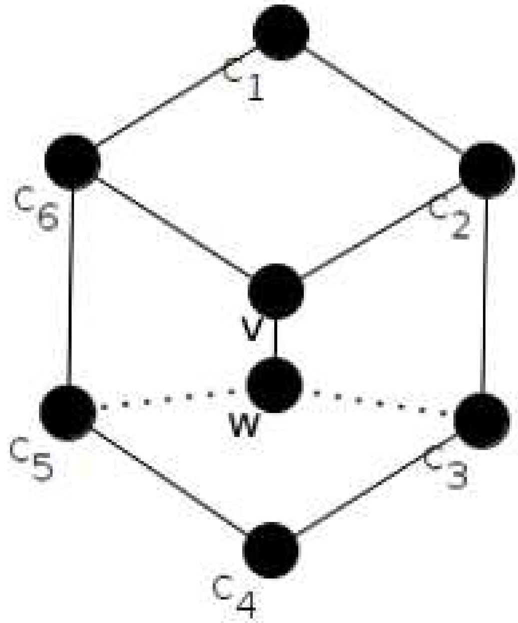

Figure 3.3: Proof of Lemma 5. The dotted edges represent potential adjacencies. If both dotted adjacencies are present, a BW_3 is induced. If only one of them is present, an F_2 is induced. Finally, if neither is present, a $C_6{}^*$ is induced.

being irrelevant. Let $c'_y \in N(c_y) \cap C - N(c_z)$ and $c'_z \in N(c_z) \cap C - N(c_y)$. Vertices c'_y and c'_z necessarily exist, since c_y, c_z are both in C and $n > 6$, and neither are neighbors to c_x, since c_y, c_z are at a distance greater than 2 from c_x in C. Note that a T_2 is induced by $v, c_x, c_y, c_z, c_{x+1}, c'_y, c'_z$ (Figure 3.5).

Consider, now, that v is neighbor to exactly two vertices of C. Suppose, first, that $N(v) \cap C = \{c_x, c_y\}$ with $x \neq y+2, y-2$. In this case, we have it that a T_2 is formed with $c_x, c_{x+1}, v, c_{x-1}, c_{x+2}, c_y, c_{x-2}$.

Now, suppose the two neighbors of v in C are at a distance of 2 in $G[C]$. Suppose w.l.o.g. $N(v) \cap C = \{c_1, c_3\}$. In this case, $(C - c_2) \cup \{v\}$ is also an induced C_n in G, and

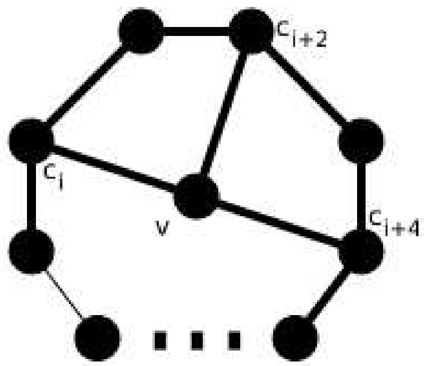

Figure 3.4: Proof of Lemma 5. The induced F_1 is highlighted by the thicker edges. Ellipses represent continuations of the cycle.

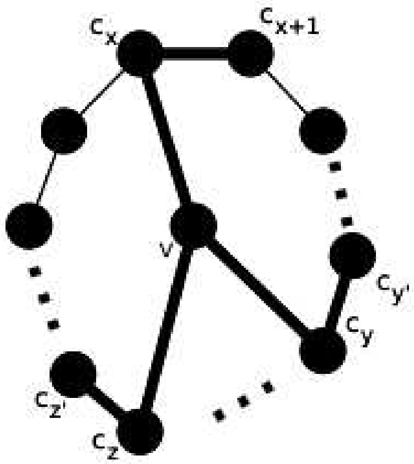

Figure 3.5: Proof of Lemma 5. The induced T_2 is highlighted by the thicker edges. Ellipses represent continuations of the cycle.

$N(v) \cap C = N(c_2) \cap C$. Since G is twin-free, there exists a vertex w that is neighbor to either v or c_2, but not both. Suppose that w is neighbor to v w.l.o.g.

If $N(w) \cap C = \emptyset$, G contains an induced $C_n{}^*$. If w is neighbor to two or more vertices of C, then $C \cup \{v\} - \{c_2\}$ is an induced C_n such that w is neighbor to three or more of its vertices, implying $G[C \cup \{v, w\} - \{c_2\}]$ contains an induced F_1 or T_2 as seen in previous paragraphs. Therefore, let $N(w) \cap C = \{c_x\}$, with $x \neq 4, n$. In this case, $v, c_1, c_3, w, c_n, c_4, c_x$ induce a T_2 (Figure 3.6).

Therefore, if w is neighbor to one vertex of C, it has to be c_n or c_4. Suppose the latter without loss of generality. Note that, in this case, an $L_3 \cup P_2$ is induced with $c_1, c_2, c_3, c_4, v, w, c_6, c_7$ (Figure 3.7).

Therefore, v has exactly one neighbor in C. □

The graph described in Definition 2 is the fundamental structure of every twin-free non-bichordal Helly CA bigraph, and a key element to prove our characterization.

Definition 2. *Let a* Fundamental Circular Structure graph *(FCS graph for short), denoted by* $\mathbb{G}_k(n_1, n_2, \ldots, n_k)$, *for any even* $k \geq 6$ *and any* $n_1, \ldots, n_k \geq 0$, *be a graph defined as follows:*

$V(\mathbb{G}_k(n_1, n_2, \ldots, n_k))$ *is the union of the following sets:*

- $C = \{c_1, c_2, \ldots, c_k\}$.
- $V = \{v_1, v_2, \ldots, v_k\}$.
- $W_i = \{w_{i,1}, \ldots, w_{i,n_i}\}$ *for all* $1 \leq i \leq k$.
- $U_i = \{u_{i,1}, \ldots, u_{i,n_i}\}$ *for all* $1 \leq i \leq k$.

With the neighborhoods of each vertex defined as follows, to form $E(\mathbb{G}_k(n_1, n_2, \ldots, n_k))$:

- $N(c_i) = \{c_{i-1}, c_{i+1}, v_i\} \cup W_i \cup U_{i-1}$.
- $N(v_i) = \{c_i\}$.
- $N(w_{i,j}) = \{c_i\} \cup \{u_{i,l} \in U_i | l \leq j\}$, *for all* $1 \leq j \leq n_i$.

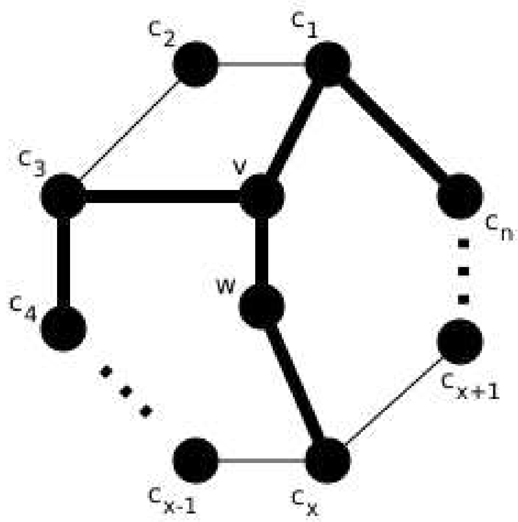

Figure 3.6: Proof of Lemma 5. The induced T_2 is highlighted by the thicker edges. Ellipses represent continuations of the cycle. Note that c_{x-1} may be 5, and c_{x+1} may be $n-1$.

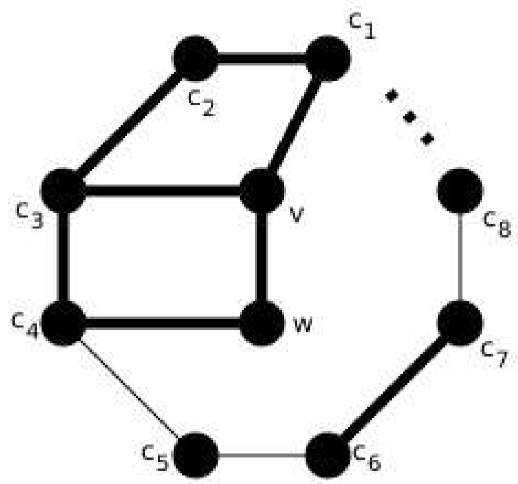

Figure 3.7: Proof of Lemma 5. The induced $L_3 \cup P_2$ is highlighted by the thicker edges. Ellipses represent continuations of the cycle.

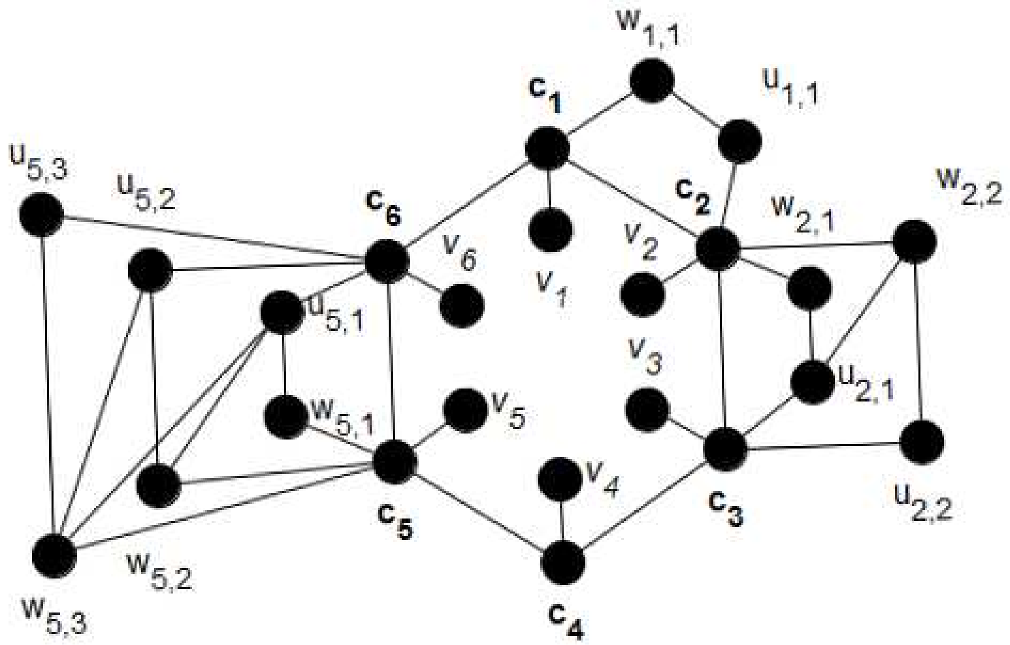

Figure 3.8: Graph $\mathbb{G}_6(1,2,0,0,3,0)$.

- $N(u_{i,j}) = \{c_{i+1}\} \cup \{w_{i,l} \in W_i | l \geq j\}$, *for all* $1 \leq j \leq n_i$.

Figure 3.8 contains an example of an FCS graph from Definition 2.

Theorem 1. *Every FCS graph is a Helly CA bigraph.*

Proof. Let $G = \mathbb{G}_k(n_1, n_2, \ldots, n_k)$. We demonstrate that it is possible to find a permutation of the bicliques of G such that, for every vertex $v \in V(G)$, $b(v)$ is circularly consecutive in the permutation.

The graph's bicliques are the following:

- $A_i = \{c_{i-1}, c_i, c_{i+1}, v_i\} \cup W_i \cup U_{i-1}$ for all $1 \leq i \leq k$.
- $B_{i,j} = \{c_i, c_{i+1}\} \cup \{w_{i,m} | m \geq j\} \cup \{u_{i,l} | l \leq j\}$ for $1 \leq i \leq k$, $1 \leq j \leq n_i$.

It is easy to verify that every set that fits the descriptions on the list is a biclique. We must now prove that every biclique of G is listed.

First, notice that every biclique will contain a vertex from C, since for every vertex $v \in V(G) - C$, there exists a vertex $c \in C$ such that $N(v) \subset N(c)$. Therefore, we look at the bicliques according to the subsets of C they contain. Every biclique contains at least two vertices of C, since for every $c \in C$, every $v \in N(c) - C$ is such that there exists $c' \in C, c' \neq c$ for which $N(v) \subset N(c')$, meaning that if v is present in a biclique with c, then c' also is. There cannot be more than three vertices of C in a biclique, since for any set of four vertices of C, there exists at least one pair of vertices in opposite partite sets that are not neighbors. Therefore, all bicliques will contain at most three vertices of C.

Suppose a biclique B has exactly two vertices of C. The vertices in question must be consecutive, otherwise they are either from the same partite set with no common neighbor in $V(G) - C$, or from opposite partite sets and non-neighbors. Let $B \cap C = \{c_i, c_{i+1}\}$. If any vertex from $\{v_i, v_{i+1}\} \cup W_{i+1} \cup U_{i-1}$ is in B, then either c_{i-1} or c_{i+2} must also be, otherwise it becomes impossible for $G[B]$ to be maximally bipartite-complete. Since $B \cap C = \{c_i, c_{i+1}\}$, it follows that $B - C \subset W_i \cup U_i$. Let some integer j, with $1 \leq j \leq n_i$, be the highest value of j for which $u_{i,j} \in B$. Since $N(u_{i,j}) \subset N(u_{i,l})$ for every $l < j$, all $u_{i,l}$ for $l < j$ must also be in the biclique. The only vertices left to be verified are the vertices in W_i. Every $w_{i,l}$ for $l \geq j$ is neighbor to every vertex in $\{u_{i,1}, ..., u_{i,j}\} \cup \{c_i\}$, meaning they are all contained in the biclique, otherwise $G[B]$ would not be maximally bipartite-complete, implying that $B = B_{i,j}$.

Now, suppose a biclique B has three vertices of C. These three vertices must be consecutive, as seen previously. Let $B \cap C = \{c_{i-1}, c_i, c_{i+1}\}$. Notice that the only common neighbor of c_{i-1} and c_{i+1} is c_i, meaning that $B - C = N(c_i) - C$, making it a star centered in c_i. Since $N(c_i) - C = \{v_i\} \cup W_i \cup U_{i-1}$, $B = A_i$.

To prove that G is a Helly CA bigraph, we apply Lemma 1. Consider a set of points $S = \{a_1, ..., a_k\} \cup \bigcup_{i=1}^{k} \{b_{i,1}, ..., b_{i,n_i}\}$ around a circle C, such that, for all $1 \leq i \leq k, 1 \leq j \leq n_i$, a_i corresponds to biclique A_i and $b_{i,j}$ corresponds to biclique $B_{i,j}$. Let the points be distributed according to the following clockwise order:

$$(a_1, b_{1,1}, ..., b_{1,n_1}, a_2, b_{2,1}, ..., b_{2,n_2}, a_3, b_{3,1}, ..., b_{3,n_3},\\ a_4, b_{4,1}, ..., b_{4,n_4}, a_5, b_{5,1}, ..., b_{k-1,n_{k-1}}, a_k, b_{k,1}, ..., b_{k,n_k}).$$

For clarity's sake, the above permutation is constructed as follows:

1. Make $i = 1$.

2. Add point a_i to the sequence.

3. Add all points of the form $b_{i,j}$ to the sequence in increasing order of j from 1 to n_i.

4. Make $i \leftarrow i + 1$. Return to step 2 if $i \leq k$, and finish the sequence otherwise.

Consider the family of bicliques for each vertex:

- For every $1 \leq i \leq k$, $b(c_i) = \{A_{i-1}, A_i, A_{i+1}\} \cup \{B_{i,j} | 1 \leq j \leq n_i\} \cup \{B_{i-1,j} | 1 \leq j \leq n_{i-1}\}$.
- For every $1 \leq i \leq k$, $b(v_i) = \{A_i\}$.
- For every $1 \leq i \leq k$, and $1 \leq j \leq n_i$, $b(w_{i,j}) = \{A_i\} \cup \{B_{i,l} | 1 \leq l \leq j\}$.
- For every $1 \leq i \leq k$, and $1 \leq j \leq n_i$, $b(u_{i,j}) = \{A_{i+1}\} \cup \{B_{i,l} | j \leq l \leq n_i\}$.

Therefore, for the points distributed according to the order presented, every vertex $v \in V(G)$ is such that the points corresponding to $b(v)$ are circularly consecutive, implying S is a valid set of biclique points. □

Theorem 2 shows that every twin-free non-bichordal Helly CA bigraph is structured like an FCS graph. Its proof depends on Lemma 6.

Lemma 6. *Let G be a twin-free non-bichordal bipartite graph with an induced $C_n, n \geq 6$ in $C = (c_1, ..., c_n)$. Then G is an induced subgraph of an FCS graph if and only if the following statements are true:*

- *Every vertex in $V(G) - C$ has exactly one neighbor in C.*
- *For all $1 \leq i \leq n$, every vertex $v \in N(c_i) - C$ is such that $N(v) - C \subseteq N(c_{i+1}) - C$ or $N(v) - C \subseteq N(c_{i-1}) - C$.*

- *For all $1 \leq i \leq n$, every pair of vertices $v, w \in N(c_i) - C$ such that $N(v) - C$ and $N(w) - C$ are both subsets of $N(c_{i+1}) - C$ (or $N(c_{i-1}) - C$) must be such that $N(v) \subset N(w)$ or $N(w) \subset N(v)$.*

Proof. Let $V_i = N(c_i) - C$ for every $1 \leq i \leq n$, and, for all $1 \leq i \leq n, j \in \{i-1, i+1\}$, let $V_{i,j} = \{v \in V_i | \emptyset \neq N(v_i) - C \subseteq V_j\}$. We show that G is an induced subgraph of $\mathbb{G}_n(m_1, \ldots, m_n)$ for $m_i = |V_{i,i+1}|$ for all $1 \leq i \leq n$.

First of all, it is easy to note that cycle C stands for set C in the definition of the FCS graph. It is also easy to note that, if a vertex $v \in V(G) - C$ is such that $N(v) = \{c_i\}$ for some $1 \leq i \leq n$, then v stands for vertex v_i in the definition. All that is left to show is that $V_{i,i+1}$ stands for W_i, and $V_{i+1,i}$ stands for U_i for all $1 \leq i \leq n$.

Let $V_{i,i+1} = \{w_1, ..., w_{m_i}\}$ with $N(w_h) \subset N(w_{h+1})$ for all $1 \leq h < m_i$, and $V_{i+1,i} = \{u_1, ..., u_l\}$ with $N(u_{h+1}) \subset N(u_h)$ for all $1 \leq h < l$. Note that $m_i = l$ as it is impossible to find $x+1$ pairwise comparable non-empty subsets of a set of size x. We must show that w_h is neighbor to u_j precisely if $j \leq h$ for all $1 \leq h \leq m_i$. We do so by induction.

As a base, note that $N(w_1) = \{c_i, u_1\}$, since if w_1 was neighbor to any u_h for $h > 1$, then $\{u_1, ..., u_h\}$ would all be twins as their neighborhoods would equal $\{c_{i+1}\} \cup V_{i,i+1}$. For the hypothesis, suppose $N(w_h) \cap V_{i+1,i} = \{u_1, ..., u_h\}$ for some $1 \leq h < m_i$. Then $N(w_{h+1}) \cap V_{i+1,i} = \{u_1, ..., u_j\}$ for some $j > h$. If $j > h+1$, then the vertices $\{u_{h+1}, ..., u_j\}$ have neighborhoods equal to $\{c_{i+1}\} \cup \{w_{h+1}, \ldots, w_{m_i}\}$, implying they are twins, leading to a contradiction.

Therefore, $N(w_h) \cap V_{i+1,i} = \{u_1, ..., u_h\}$ for all $1 \leq h \leq m_i$. This, in turn, implies $N(u_h) \cap V_{i,i+1} = \{w_h, ..., w_{m_i}\}$.

Therefore, $V_{i,i+1}$ stands for W_i, and $V_{i+1,i}$ stands for U_i for all $1 \leq i \leq n$ in the definition of FCS graphs, finishing our proof that G is an induced subgraph of $\mathbb{G}_n(m_1, \ldots, m_n)$.

Conversely, note that every FCS graph, by definition, satisfies the conditions in the lemma. □

Theorem 2. *A twin-free non-bichordal bipartite graph is an induced subgraph of of an FCS graph if and only if it does not contain $T_2, F_1, BW_3, F_2, C_n^*$ $(n \geq 6)$ or $L_3 \cup P_2$ as an induced subgraph.*

Proof. Let G be a twin-free non-bichordal bipartite graph that does not contain any of the mentioned induced subgraphs.

Since G is non-bichordal, let $C = \{c_1, ..., c_n\}$ be a $C_n (n \geq 6)$ that G contains. By the proof of Lemma 5, every vertex in $V(G) - C$ contains exactly one neighbor in C, as otherwise, one of the forbidden graphs is induced or G is not twin-free.

Let $V_i = N(c_i) - C$ for every $1 \leq i \leq n$. Let $v_i \in V_i, v_j \in V_j$ for some i, j. If $j \neq i+1, i-1$, then $v_i v_j \notin E(G)$, otherwise, T_2 or an odd cycle is induced. Therefore, elements of V_i have their neighborhoods outside of C entirely contained in $V_{i+1} \cup V_{i-1}$.

Let $v_i \in V_i, v_{i+1} \in V_{i+1}, v_{i-1} \in V_{i-1}$, then either $v_i v_{i-1} \notin E(G)$ or $v_i v_{i+1} \notin E(G)$, otherwise, an F_1 is induced.

Therefore, for all i, every element $v \in V_i$ is such that $N(v) - C \subseteq V_{i+1}$ or $N(v) - C \subseteq V_{i-1}$. For all $1 \leq i \leq n, j \in \{i-1, i+1\}$, let $V_{i,j} = \{v \in V_i | \emptyset \neq N(v_i) - C \subseteq V_j\}$. Suppose two elements $v_1, v_2 \in V_{i,j}$ are such that $N(v_1), N(v_2)$ are not comparable. Without loss of generality, assume $j = i+1$. Let $w_1 \in N(v_1) - N(v_2)$ and $w_2 \in N(v_2) - N(v_1)$. Note that $c_i, c_{i-1}, c_{i-2}, v_1, v_2, w_1, w_2$ induce a T_2.

Therefore, every pair of elements in $V_{i,j}$ has comparable neighborhoods for all $i \leq n, j \in \{i-1, i+1\}$. Note that, with all vertices outside C restricted to one neighbor in C, with vertices in V_i having neighbors outside of C entirely contained in either V_{i+1} or V_{i-1}, and with every set of

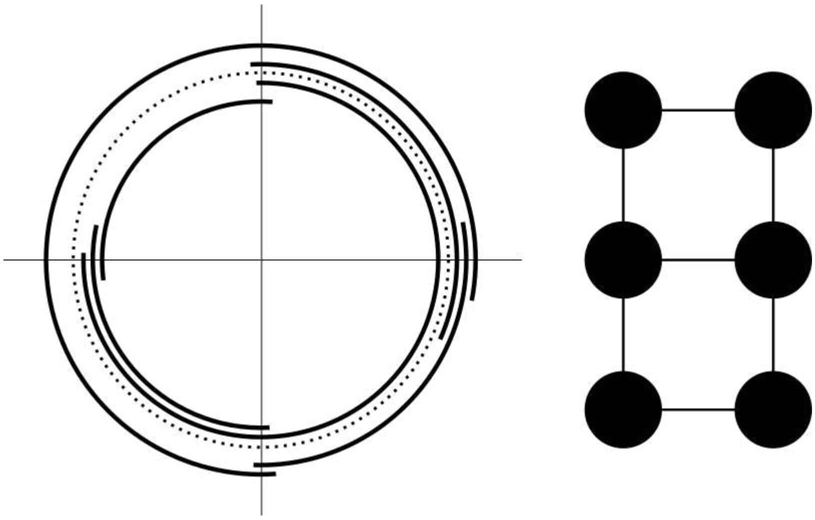

Figure 3.9: The L_3 alongside a bi-circular-arc model that is both proper and Helly. The biclique points are signaled by line segments from the center of the circle.

the form $V_{i,j}$ having pairwise comparable neighborhoods, G is an induced subgraph of an FCS graph by Lemma 6, which is a Helly CA bigraph by Theorem 1.

Conversely, since FCS graphs are Helly CA bigraphs, they cannot contain $T_2, F_1, BW_3, F_2, C_n^*$ $(n \geq 6)$ or $L_3 \cup P_2$ as induced subgraphs. □

The following corollary then concludes our characterization of non-bichordal Helly CA bigraphs by forbidden subgraphs.

Corollary 1. *A non-bichordal bipartite graph is a Helly CA bigraph if and only if it does not contain* $T_2, F_1, BW_3, F_2, C_n^*$ $(n \geq 6)$ *or* $L_3 \cup P_2$ *as an induced subgraph.*

Proof. Let G be a non-bichordal bipartite graph without the induced subgraphs mentioned, and let G^- be its twin-free version. According to Theorem 2, G^- is an induced subgraph of an FCS graph, therefore being a Helly CA bigraph.

Since G^- is a Helly CA bigraph, G is a Helly CA bigraph by Lemma 4. □

Corollary 1 is a relevant characterization in that all of its forbidden graphs are also forbidden for the class of Helly CA bigraphs as a whole, marking an important step in the search for a characterization for the general case. If we allow, however, for forbidden graphs that are only forbidden for the specific case of non-bichordal Helly CA bigraphs, we may conclude a simpler characterization based on it, shown in the sequence.

The forbidden graph we use in this characterization that is not present in Corollary 1 is the L_3, also known as *domino*, shown in Figure 3.9. It is, by itself, a Helly CA bigraph, but its occurrence in a non-bichordal graph makes it not Helly, as will be shown in the sequence.

Corollary 2. *A non-bichordal bipartite graph is a Helly CA bigraph if and only if it does not contain* T_2*,* L_3*, or* $C_n^*, n > 4$ *as an induced subgraph.*

Proof. ($\Leftarrow$) If a graph does not contain T_2, L_3, or $C_n^*, n > 4$ as induced subgraphs, then it also does not contain any of the forbidden graphs from Corollary 1, since all of those forbidden graphs are either the T_2 or the $C_n^*, n > 4$ themselves, or they contain the L_3 as an induced subgraph.

($\Rightarrow$) If a graph G is a non-bichordal Helly CA bigraph, then, by Corollary 1 it does not contain $T_2, F_1, BW_3, F_2, C_n^*$ $(n \geq 6)$ or $L_3 \cup P_2$ as induced subgraphs, which, according to Theorem 2, implies the twin-free version of G is an induced subgraph of an FCS graph. FCS graphs do not contain L_3 as an induced subgraph, therefore G also does not. □

The characterization in Corollary 2 helps us showcase the similarities between the structures of non-bichordal Helly CA bigraphs and those of Helly interval bigraphs, which we discuss in Subsection 3.1.3. It is also an important result for our characterization of proper-Helly CA bigraphs in Subsection 3.1.4.

3.1.2 Helly CA bigraphs without isolated vertices

In this subsection, we characterize and present a recognition algorithm for Helly CA bigraphs without isolated vertices. Our characterization is separated into two different results: one for C_6-free graphs, and the other for graphs with an induced C_6. Lemma 7 is used for this subsection, and for the recognition algorithm we present later on.

Note that the proof of Lemma 7 relies on the non-bipartite analogue of Lemma 1, wherein given a Helly CA graph G with a Helly model $(C, \mathbb{A})$, there exists, for every *clique* in the graph, a point in the circle that all arcs o the clique contain.

Lemma 7. *Let $G = (X, Y, E)$ be a bipartite graph. If G^2 is a Helly circular arc graph, G is a Helly circular arc bigraph.*

Proof. Let $(C, \mathbb{A})$ be a Helly circular arc model of G^2. Let $\mathbb{I}, \mathbb{E} \subset \mathbb{A}$ be such that $\mathbb{I}$ contains the arcs corresponding to X in G, and $\mathbb{E}$ contains the arcs corresponding to Y. We claim that $(C, \mathbb{I}, \mathbb{E})$ is a Helly bi-circular-arc model of G.

Showing that it is a bi-circular-arc model of G is simple: let $vw \in E(G)$. Since they are neighbors, they are also neighbors in G^2, therefore, $a(v) \cap a(w) \neq \emptyset$.

Now, if $v \in X$ and $w \in Y$ are such that $vw \notin E(G)$, then they are also not neighbors in G^2, since they cannot have a common neighbor as they are in opposing partite sets. Therefore, $a(v) \cap a(w) = \emptyset$.

To show that $(C, \mathbb{I}, \mathbb{E})$ is a Helly model, consider a biclique K of G. Note that K will induce a complete subgraph in G^2, implying K either a clique or a subset of a clique in G^2. Since $(C, \mathbb{A})$ is a Helly circular arc model, that implies all the arcs corresponding to vertices of K contain a common point in C.

Therefore, $(C, \mathbb{I}, \mathbb{E})$ is a Helly bi-circular-arc model of G. □

Treatment of the C_6-free case of the characterization depends on Theorem 3, which depends on Lemma 8, a direct implication of results from Groshaus (2006) [Groshaus, 2006].

Lemma 8. *[Groshaus, 2006] If G is a bipartite graph without isolated vertices nor an induced C_6, then the set of cliques of G^2 is equal to the set of bicliques of G.*

Theorem 3. *A C_6-free bipartite graph $G = (X, Y, E)$ with no isolated vertices is a Helly circular arc bigraph if and only if G^2 is a Helly circular arc graph.*

Proof. ($\Leftarrow$) Follows from Lemma 7.

($\Rightarrow$) Construct a Helly bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ based on a set of biclique points S as shown in the proof of Lemma 1. As stated in the lemma's proof, two arcs $A, B \in \mathbb{I} \cup \mathbb{E}$ intersect if and only if $v(A), v(B)$ are contained in a common biclique of G, which happens precisely when $v(A), v(B)$ are neighbors in G^2, as two vertices that are contained in a common biclique must either be neighbors or share a neighbor in G.

Also, for every biclique $K \subset X \cup Y$, there is a point $p \in C$ contained in all arcs corresponding to vertices of K. By Lemma 8, every clique in G^2 is a biclique in G, which in turn implies that every clique in G^2 is such that its corresponding arcs contain a common point. Therefore, circular arc model $(C, \mathbb{I} \cup \mathbb{E})$ is a Helly model of G^2. □

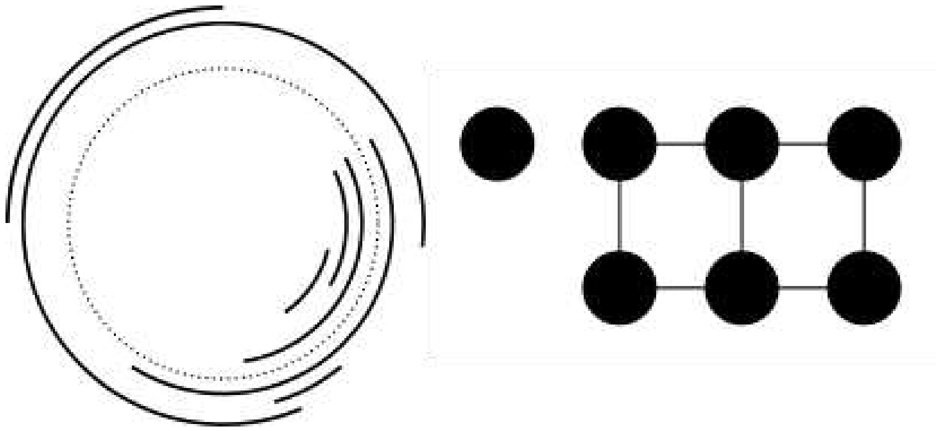

Figure 3.10: Graph $L_3{}^*$ alongside its Helly bi-circular-arc model.

Theorem 3 concludes the C_6-free part of the characterization. Note that it is stated that the graph must not have isolated vertices. That is because, when an isolated vertex is present in a C_6-free bipartite graph, verifying the square of the graph may lead to false negatives. Consider $L_3{}^*$, presented in Figure 3.10 alongside a Helly bi-circular-arc model. The square of $L_3{}^*$ contains a $C_4{}^*$ which is not a CA graph, therefore, $L_3{}^*$ is a C_6-free Helly CA bigraph whose square is not a Helly CA graph.

Theorem 3 does, however, apply to disconnected graphs, as long as none of their components are a single vertex, as a disconnected bipartite graph without isolated vertices is a Helly CA bigraph if and only if all of its connected components are Helly interval bigraphs. That, in turn, happens if and only if its square is an interval graph, as we show later on, in Subsection 3.1.3.

Returning to the characterization, the part on graphs that admit an induced C_6 is a special case of the characterization for non-bichordal Helly CA bigraphs, as shown in the following theorem.

Theorem 4. *Let G be a bipartite graph with no isolated vertices that has an induced C_6. Then G is a Helly circular arc bigraph if and only if it does not contain $T_2, F_1, BW_3, F_2, C_6{}^*$ as an induced subgraph.*

Proof. Let G be a twin-free bipartite graph with no isolated vertices that does not contain $T_2, F_1, BW_3, F_2, C_6{}^*$ as induced subgraphs.

Let $C = (c_1, \ldots, c_6)$ be the induced C_6 in G. According to Case 1 of the proof of Lemma 5, if there is a vertex in $V(G) - C$ with 0 or more than 1 neighbor in C, then either BW_3, F_2 or $C_6{}^*$ will be induced subgraphs.

Furthermore, according to the proof of Theorem 2, since G does not contain T_2 nor F_1, it must verify the following properties:

- Let $V_i = \{v \in V(G) - C | c_i \in N(v)\}$ for every $i \leq 6$. Then, for any i, every vertex $v \in V_i$ is such that $N(v) - C \subseteq V_{i-1}$ or $N(v) - C \subseteq V_{i+1}$.
- Let $V_{i,j} = \{v \in V_i | N(v) - C \subseteq V_j\}$ for every $i, j \leq 6$. Then, for any $i, j \leq 6$, every pair $v, w \in V_{i,j}$ is such that $N(v) \subset N(w)$ or vice-versa.

However, if the vertices verify the aforementioned properties, then, as shown in Lemma 6, the graph is the induced subgraph of an FCS graph. $\square$

With that, the characterization for Helly CA bigraphs without isolated vertices is complete, as shown in Theorem 5.

Theorem 5. *A bipartite graph G without isolated vertices is a Helly CA bigraph if and only if G^2 is a Helly CA graph or G contains an induced C_6 and is $(T_2, F_1, BW_3, F_2, C_6{}^*)$-free.*

Proof. The C_6-free case follows from Theorem 3, and the case for graphs with an induced C_6 follows from Theorem 4. □

3.1.2.1 Recognition algorithm

The recognition algorithm presented in the sequence is based on the characterization in Theorem 5. Instead of directly searching for forbidden graphs in the case where the input graph contains an induced C_6, it verifies if the graph's twin-free version breaks any of the properties pointed out in the proof of Theorem 4. For instance, it verifies whether any vertex outside the cycle has two or more neighbors in the cycle, which, associated with the fact that we are looking at the input graph's twin-free version, implies the presence of an induced BW_3 or F_2 without directly searching for one.

The algorithm consists of the following steps, given an input bipartite graph G without isolated vertices, such that $|V(G)| = n, |E(G)| = m$:

1. Take the square G^2 of G in $O(mn)$.
2. If G^2 is a Helly CA graph, which can be tested in $O(m)$ [Lin and Szwarcfiter, 2006], return *yes*.
3. If G^2 is not a Helly CA graph, search for an induced C_6 in $O(n^4)$. If one cannot be found, return *no*. Otherwise, let $C = \{c_1, ..., c_6\}$ be the C_6 that was found.
4. Take the twin-free version G^- of G in $O(n^3)$. Make sure $c_1, ..., c_6$ are not removed.
5. Compute sets $V_1, ..., V_6$ where $V_i = \{v \in V(G) - C | c_i \in N(v)\}$. This can be done in $O(n)$. If any vertex does not have exactly one neighbor in C, return *no*.
6. For every set V_i, $1 \leq i \leq 6$ compute sets $V_{i,i+1}$ and $V_{i,i-1}$ such that $V_{i,j} = \{v \in V_i | \exists w \in V_j : vw \in E(G)\}$. If any vertex of V_i contains a neighbor in V_{i+3} or contains neighbors in both V_{i+1} and V_{i-1}, return *no*. This step can be computed in $O(n^2)$ time.
7. For every set of the form $V_{i,j}$, check if every pair of vertices $v, w \in V_{i,j}$ have comparable neighborhoods. If there exists one pair that does not have comparable neighborhoods, return *no*. Otherwise, return *yes*. This step concludes the algorithm and can be computed in $O(n^3)$.

Call the above sequence of steps the *main algorithm*. In the sequence, we prove that the main algorithm is correct for the recognition of Helly CA bigraphs without isolated vertices. After that, we demonstrate how each step is computed.

Theorem 6. *For any input bipartite graph G without isolated vertices, the main algorithm correctly determines whether G a Helly CA bigraph.*

Proof. Since every computation will end with a *yes* or *no* answer, it suffices to prove that, every time the algorithm provides one of those answers, the answer is the correct one.

If a *yes* answer on step 2 is reached, that implies G^2 is a Helly CA graph, which, according to Lemma 7, implies G is a Helly CA bigraph. Similarly, if a *no* answer is given on step 3, then G does not contain an induced C_6 and G^2 is not a Helly CA graph, which, by Theorem 3, implies G is not Helly.

Now, if the *no* answer from step 5 is reached, that implies that G^-, the twin-free version of G, is such that there exists a vertex in $v(G^-) - C$ that is either neighbor to no elements of C, implying the existence of an induced ${C_6}^*$, or more than one, implying the existence of an induced F_2 or BW_3 (Lemma 5), which, by Theorem 4, implies G is not Helly.

If the *no* in step 6 is reached, that implies there exists $v \in V_i$, $1 \leq i \leq 6$ that either has a neighbor in V_{i+3}, implying the existence of an induced T_2, or one neighbor in V_{i-1} and one in V_{i+1}, implying an induced F_1.

If the *no* in step 7 is reached, that implies there are two elements $v, w \in V_{i,j}$, with $1 \leq i \leq 6$ and $j \in \{i-1, i+1\}$ such that their neighborhoods are not comparable. That, in turn, implies the existence of an induced T_2. Finally, if the *yes* in step 7 is reached, that implies G^- satisfies every condition necessary to be an induced subgraph of an FCS graph as shown in Lemma 6, which in turn implies G is Helly. □

Step 1 is a trivial squaring operation, and step 2 is taken from [Lin and Szwarcfiter, 2006]. Step 3, the most costly step in the main algorithm, can be done in $O(n^4)$ [Nikolopoulos and Palios, 2007].

Step 4 of the algorithm is calculating by checking, for every pair of vertices v, w of the same partite set, whether $N(v) = N(w)$. In the case of a positive answer, delete one of the vertices.

With that out of the way, step 5 simply consists of going through every vertex v outside the cycle, adding them to one of the six sets $V_1, \ldots, V_6$ according to which element of the cycle it is neighbor to, and ending the computation with a negative output if more than one element of the cycle is neighbor to v or if none is. In order to aid in the computation of the other steps, this step also adds a label to each vertex according to the set it was placed in. It is easy to verify this runs in $O(n)$ time and correctly computes the sets described in step 5.

Similarly, step 6 consists of going through each set $V_1, \ldots, V_6$ in turn, and checking, for every $v \in V_i$, whether it has neighbors in V_{i-1}, V_{i+1} or V_{i+3}. Computation is terminated with a negative answer if any vertex v has neighbors in more than one of those sets, or has one neighbor in V_{i+3}. Otherwise, v is added to $V_{i,i-1}$ or $V_{i,i+1}$ and labeled accordingly. This trivially runs in $O(n^2)$ time and is correct for the conditions described in step 6.

Step 7 is slightly more complicated, consisting of going through all sets of the form $V_{i,j}$, and checking, for every pair of vertices $v, w \in V_{i,j}$, whether their neighborhoods in $V_{j,i}$ are comparable. The following algorithm shows how this process can be done in $O(n^3)$.

Note that algorithm *CheckComparable* runs in $O(n^3)$: the two outermost *For*s run a constant number of times, while the third *For* runs in $O(n^2)$ and the innermost *For* runs in $O(n)$, with every other operation being constant.

Lemma 9. *For any input graph G that passed through the first six steps of the main algorithm without terminating, the algorithm* CheckComparable *correctly verifies whether the neighborhoods of every set of the form $V_{i,j}$ are pairwise comparable.*

Proof. It suffices to prove that the answer *no* is given if and only if a pair of non-comparable elements is found.

Suppose that a pair of elements $v, w \in V_{i,j}$ are such that $N(v)$ and $N(w)$ are not comparable. Note that this implies the existence of two vertices $v', w' \in V_{j,i}$ such that $v' \in N(v) - N(w)$ and $w' \in N(w) - N(v)$. Suppose w.l.o.g. that, when running through $V_{j,i}$ in the innermost For loop, v' is investigated before w'. By this point, if *Dominant*$= w$, then the computation terminates

Algorithm 1: CheckComparable(G)

```
G⁻ ← G
for i ∈ {1, ..., 6} do
    for j ∈ {i − 1, i + 1} do
        for v, w ∈ V_{i,j} do
            Dominant←NULL
            for x ∈ V_{j,i} do
                if x ∈ N(v) and x ∉ N(w) then
                    if Dominant== w then
                        return no
                    Dominant← v
                if x ∉ N(v) and x ∈ N(w) then
                    if Dominant== v then
                        return no
                    Dominant← w
return yes
```

with *no*, otherwise, *Dominant*= v after this iteration. Afterwards, when w' or any other vertex from $N(w) - N(v)$ is checked in the innermost *For*, since *Dominant*= v, the computation will terminate with a *no*, since said vertex will be in $N(w)$ but not $N(v)$.

Conversely, suppose the computation has run into a *no* in the *CheckComparable* routine. Suppose w.l.o.g. that the *no* is reached within the if statement that checks if *Dominant*= w. That implies that, at that point in the computation, $x \in N(v) - N(w)$ and *Dominant*= w. Since *Dominant*= w, that implies some previous iteration of the innermost *For* led to this attribution. In that previous iteration, the value of x was some other vertex x' that was in $N(w) - N(v)$. Therefore, since $x \in N(v) - N(w)$ and $x' \in N(w) - N(v)$, the neighborhoods of v and w are not comparable. □

With that, our algorithm for the recognition of Helly CA bigraphs without isolated vertices is concluded, with its correctness and time complexity proven. The algorithm we originally published in [Groshaus et al., 2019] was a naive search for forbidden graphs whose time complexity far exceeds the $O(n^4)$ of the algorithm we present here.

It is important to note, however, that the main algorithm does not guarantee a correct answer if the given graph contains an isolated vertex. For instance, $L_3{}^*$ (Figure 3.10) is a Helly CA bigraph, but due to the fact its square contains an induced $C_4{}^*$, its square is not a Helly CA graph. Since $L_3{}^*$ does not contain an induced C_6, the algorithm returns *no*.

False negatives occur when a C_6-free graph G contains some isolated vertex v, such that $G - v$ is a Helly CA bigraph, $(G - v)^2$ is not an interval graph, and there exists at least one Helly bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ of $G - v$ such that $\mathbb{I}$ does not cover the circle. In that situation, G is a Helly CA bigraph, as it is possible to add, to the aforementioned model, an arc corresponding to v in family $\mathbb{E}$ that is entirely contained within $C - \bigcup_{A \in \mathbb{I}} A$. However, the algorithm will return *no*, as the square of G is not a Helly CA graph, due to the fact that $(G - v)^2$ is not an interval graph, which implies any Helly CA model of $(G - v)^2$ covers the entire circle, making it impossible to add an arc corresponding to an isolated vertex.

A method for solving the recognition problem on graphs that contain an isolated vertex is currently unknown. Attempting to run the algorithm with the isolated vertices removed leads to false positives, like the one in Figure 3.11. False positives in this situation occur when a C_6-free graph G contains an isolated vertex v such that every Helly bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$

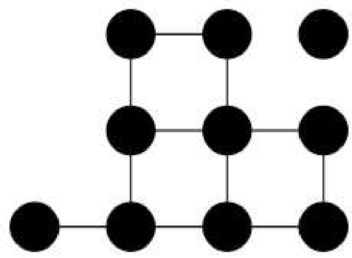

Figure 3.11: A false positive that occurs when removing isolated vertices prior to running the algorithm. The non-isolated component of the graph is such that its square is a Helly CA graph, but any Helly bi-circular-arc model of the non-isolated component necessarily requires every point in the circle to be contained in at least one arc from each family, making it impossible to add an arc corresponding to the isolated vertex.

of $G - v$ has both $\mathbb{I}$ and $\mathbb{E}$ cover the circle. In that situation, G is not a Helly CA bigraph, as it is impossible to add an arc corresponding to v to any Helly bi-circular-arc model of $G - v$ without it intersecting some arc from the opposing family. The algorithm will return *yes*, however, since $G - v$ is a Helly CA bigraph.

Therefore, the algorithm will contain either false negatives or false positives when a C_6-free graph with isolated vertices is presented, depending on whether isolated vertices are removed from the input graph before computation begins.

3.1.3 Helly interval bigraphs

Helly interval bigraphs are a subclass of both Helly CA bigraphs and interval bigraphs. A bi-interval model $(\mathbb{A}, \mathbb{B})$ is said to be Helly if $\mathbb{A}, \mathbb{B}$ is a bipartite-Helly pair of families. A bipartite graph is a Helly interval bigraph if and only if it admits a Helly bi-interval model. Equivalently, a bipartite graph is a Helly interval bigraph precisely if it admits a bi-interval model such that, for every biclique K in the graph, there exists a number p_k in the number line that is contained in every interval corresponding to vertices of K.

Helly interval bigraphs are a particularly interesting subclass of interval bigraphs in that they do not possess a clear non-bipartite analogue. Interval bigraphs are a bipartite variation on interval graphs, but in the case of interval graphs, a Helly subclass is irrelevant, as any family of intervals on the number line trivially verifies the Helly property. In contrast, it is not true that any pair of families of intervals verifies the bipartite-Helly property, and it is also not true that every interval bigraph admits a bi-interval model that is bipartite-Helly. In light of this rather counter-intuitive fact, Helly interval bigraphs are trivially a proper subclass of interval bigraphs.

It is easy to verify that a graph is a Helly interval bigraph if and only if its twin-free version also is. The proof of that fact is similar to that from Lemma 4. Therefore, we may focus our attention on twin-free graphs.

Furthermore, it is also easy to verify that a graph is a Helly interval bigraph precisely if all its connected components are Helly interval bigraphs, as Helly bi-interval models may be placed side by side for each connected component. Therefore, we may focus on connected graphs.

In this section, we include two characterizations of Helly interval bigraphs. The first one, in Theorem 7, allows us to conclude that the recognition of the class can be done in quadratic time, and the second one, in Theorem 9, is a forbidden induced subgraph characterization that helps with the study of proper-Helly CA bigraphs in Subsection 3.1.4, and allows us to draw parallels with non-bichordal Helly CA bigraphs.

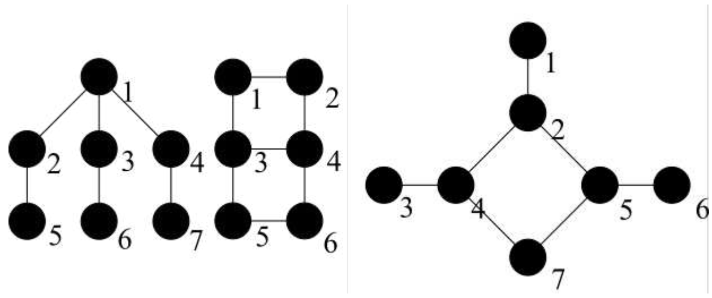

Figure 3.12: Forbidden graphs for the class of Helly interval bigraphs. From left to right, the T_2, the L_3, and the X_2.

Theorem 7. *A bipartite graph G is a Helly interval bigraph if and only if G^2 is an interval graph.*

Proof. ($\Rightarrow$) Let $(\mathbb{A}, \mathbb{B})$ be a Helly bi-interval model of G such that two intervals intersect precisely if they belong to a common biclique.[1]

It is easy to verify that $\mathbb{A} \cup \mathbb{B}$ is an interval model of G^2, as intervals will intersect precisely if their corresponding vertices are neighbors or have a common neighbor in G.

($\Leftarrow$) Let $\mathbb{I}$ be an interval model of G^2. Partition $\mathbb{I}$ into two families $\mathbb{A}, \mathbb{B}$ according to the partite sets of G. Note that $(\mathbb{A}, \mathbb{B})$ is a bi-interval model of G, as two vertices of opposing partite sets are neighbors in G^2 precisely if they are neighbors in G.

Furthermore, every biclique of G is a clique (or subset of a clique) in G^2, implying every biclique has a common point in $(\mathbb{A}, \mathbb{B})$. Therefore, $(\mathbb{A}, \mathbb{B})$ is a Helly bi-interval model. $\square$

Theorem 7 shows that recognizing Helly interval bigraphs can be done in quadratic time, as the squaring operation can be done in $O(nm)$, and recognizing interval bigraphs can be done in linear time [Booth and Lueker, 1976]. It also shows a direct connection between Helly interval bigraphs and interval graphs, as every bipartite square root of an interval graph is a Helly interval bigraph, and vice-versa.

For any interval model of an interval graph G, and every clique $K \subset V(G)$, there exists a *clique point* $p_K \in \mathbb{R}$ such that, for every $v \in K$, the interval corresponding to v in the model contains p_K. This fact, in association with Theorem 7, implies the following corollary.

Corollary 3. *A bipartite graph is a Helly interval bigraph if and only if it is possible to arrange its bicliques in a linear order such that, for every vertex, the bicliques it belongs to are an interval of the order.*

Our forbidden graph characterization in Theorem 9 is, in many ways, analogous to the characterization of non-bichordal Helly CA bigraphs seen in Corollary 1. The forbidden graphs used in the characterization are the ones in Figure 3.12, alongside every cycle $C_n, n > 4$.

Lemma 10. *Graphs T_2, L_3, X_2 and $C_n, n > 4$ are not Helly interval bigraphs.*

Proof. T_2 has been proven, in Lemma 3, to not be a Helly CA bigraph.

Consider, then, L_3. Its bicliques are $A = \{1, 2, 3, 4\}, B = \{1, 3, 4, 5\}, C = \{2, 3, 4, 6\}, D = \{3, 4, 5, 6\}$. We apply Corollary 3.

In a linear order of the bicliques of L_3 as described in the corollary, B and D must be consecutive, as $b(5) = \{B, D\}$. Also, because $b(1) = \{A, B\}$, A, B must be consecutive, and because $b(6) = \{C, D\}$, C, D must be consecutive

[1]Such a model always exists. If you have a Helly model where arcs intersect even if they don't belong to a common biclique, you could build a new model based on that model's biclique points by replacing every interval with the smallest interval that contains the same biclique points the original contained.

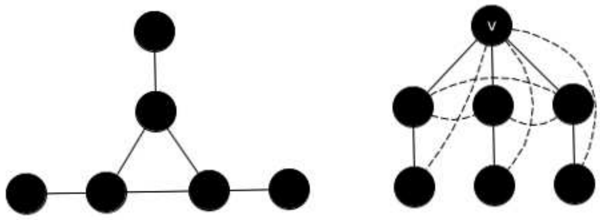

Figure 3.13: The net graph alongside ${T_2}^2$. Note that the removal of vertex v from the ${T_2}^2$ results in the net graph.

Therefore, the order of bicliques must be (A, B, D, C) or its reverse. However, in this order, $b(2) = \{A, C\}$ is not consecutive.

Now for X_2. We again apply Corollary 3. Its bicliques are $A = \{1, 2, 4, 5\}, B = \{2, 3, 4, 7\}, C = \{2, 4, 5, 7\}, D = \{2, 5, 6, 7\}$. Because $b(4) \cap b(5) = \{A, C\}$, A, C must be consecutive. Also, because $b(5) \cap b(7) = \{C, D\}$, C, D must be consecutive. Finally, because $b(4) \cap b(7) = \{B, C\}$, B, C must be consecutive.

However, C cannot be consecutive to more than two bicliques. Therefore, it is impossible to organize the bicliques in an order that satisfies the properties in Corollary 3.

Finally, any chordless cycle of length greater than 4 is not an interval bigraph. □

Note that applying Theorem 7, and showing that the squares of $T_2, L_3, X_2, C_n, n > 4$ contain forbidden graphs for interval graphs, would yield a simpler proof of Lemma 10. However, as in the proof of Lemma 3, we opted for the longer proof to demonstrate how the biclique structure of the forbidden graphs makes it impossible to create a Helly model of them.

Given that every Helly interval bigraph is the square root of an interval graph, one is excused to believe it is trivial to obtain a forbidden graph characterization by simply considering the square roots of forbidden graphs for interval graphs. This approach, however, is not as trivial as it might seem at first.

Firstly, not every minimal forbidden graph for interval graphs is a square graph, but that does not imply it is impossible for a minimal non-Helly interval bigraph to have it as an induced subgraph in its square. For example, the *net* graph (Figure 3.13) is a minimally non-interval graph, but it is not a square graph, and graph T_2 is a non-Helly interval bigraph whose square contains it.

Secondly, there may be minimal forbidden graphs for Helly interval bigraphs whose squares are not minimal forbidden graphs for interval graphs, instead being large square graphs that *contain* a minimal forbidden graph for interval graphs. Figure 3.13 is also an example of that.

Taking that into consideration, if we were to search for a complete set of forbidden graphs for Helly interval bigraphs by checking the forbidden graphs for interval graphs, we would need to either consider several different square graphs that contain forbidden graphs for interval graphs, or consider several different bipartite graphs whose squares contain forbidden induced subgraphs, resulting in a significantly more laborious process than the one we employ here. Furthermore, applying such a process would tell us very little about the fundamental structures of Helly interval bigraphs, whereas our approach provides valuable insights on that.

We call *Helly interval bigraph forbidden free* (HIBFF for short) every bipartite graph that does not admit T_2, L_3, X_2 and $C_n, n > 4$ as induced subgraphs. The characterization in this subsection boils down to showing that the HIBFF class is equal to Helly interval bigraphs.

One might rightfully note that, now that we have the set of forbidden graphs, it should be possible to conclude the characterization by simply applying Theorem 7 and showing that every HIBFF graph is such that its square is an interval graph. In the sequence, we explain why we do not use this approach.

In order to prove that every HIBFF graph has a square that is an interval graph, we would either need to show that every square graph that admits an HIBFF square root has none of the forbidden graphs for interval graphs, or show how to construct interval models of the squares of HIBFF graphs.

The former approach would require us to look into properties of square graphs and bipartite square roots, leading to a potentially much more complex proof. The latter approach, in turn, would require us to study the structural properties of HIBFF graphs in order to construct interval models of their squares, which is similar to what we already do in our approach. However, we conclude our approach not by showing how to build an interval model for the graphs' squares, but by showing how to find a suitable ordering of bicliques as per Corollary 3.

In the sequence, we present results on which the proof of Theorem 9 depends. Let G be a bipartite graph and $P = (p_1, ..., p_n)$ be an induced path in it. Call a *nuisance* on p_1, p_3 (resp. nuisance on p_{n-2}, p_n) a vertex $v \in V(G) - P$ such that $p_1, p_3 \in N(v)$ (resp. $p_{n-2}, p_n \in N(v)$).

Lemma 11. *If G is a connected twin-free HIBFF graph, then there exists in G an induced path of maximum length (i.e. maximum among induced paths) for which no nuisances exist.*

Proof. By contradiction, we suppose that there is no such path. Let $P = (p_1, \ldots, p_n)$ be a maximum length induced path such that the number of nuisances on p_{n-2}, p_n is minimum, that is, of all maximum length induced paths, P is such that the number of vertices $v \in V(G) - P$ such that $N(v) \cap P$ contains $\{p_{n-2}, p_n\}$ is minimum.

For $|P| < 4$, the proof is trivial, as G is twin-free. Therefore, suppose that the length of P is at least 4.

Let $S, T \subset V(G)$ such that $s \in S$ precisely if there exists $t \in V(G)$ such that $(s, t, p_3, \ldots, p_n)$ is an induced path, and $t \in T$ if and only if there exists $s \in S$ such that $(s, t, p_3, \ldots, p_n)$ is an induced path.

The proof is separated in three parts:

1. Proving that every $s \in S$ has at least two neighbors in T.

2. Proving that, for any pair of elements $s, s' \in S$, $N(s) \cap T$ and $N(s') \cap T$ are comparable.

3. Using 1 and 2 to prove that, since every maximum length induced path has a nuisance, G is either not twin-free or contains a forbidden graph.

Part 1: we claim that every element of S has at least two neighbors in T. For that, consider these two cases:

(1) Every induced path of the form $(q_1, \ldots, q_n)$ ($\{q_1, \ldots, q_n\} \subset V(G)$) has nuisances on both q_1, q_3 and q_{n-2}, q_n, or

(2) There exists a path $(q_1, \ldots, q_n)$ that has no nuisances on either q_1, q_3 or q_{n-2}, q_n, implying path $(p_1, \ldots, p_n)$ has no nuisance on p_{n-2}, p_n.

Start with case 1. Let $s \in S$ and $t \in N(s) \cap T$. The induced path $(s, t, p_3, \ldots, p_n)$ must contain a nuisance on s, p_3. Let $x \in V(G)$ be said nuisance. Note that x is not neighbor to any element of $\{p_4, \ldots, p_n\}$, since that would induce either an L_3 or a cycle of length greater than 4. Therefore, $(s, x, p_3, \ldots, p_n)$ is also an induced path, implying $x \in T \cap N(s)$.

Now for case 2. The fact that every element of S has at least two neighbors in T follows analogously to the previous case for $|P| > 4$. The case where $|P| = 4$ is special, however, since changing the first two elements of the path changes the vertex in position $n - 2$ of the path.

Let p_2' be a nuisance on p_1, p_3. Suppose there exists $s \in S$ such that $N(s) \cap T = \{t\}$. The path (s, t, p_3, p_4) must have a nuisance. If there was a nuisance on s, p_3, there would be more

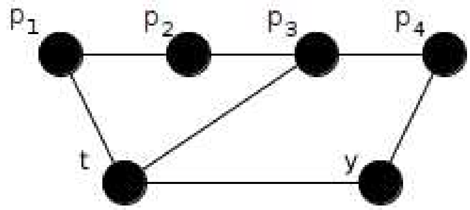

Figure 3.14: Proof of Lemma 11. The L_3 induced by p_1, p_2, p_3, p_4, t, y.

than one vertex in $N(s) \cap T$. Therefore, there is a nuisance on t, p_4. Note that $t \neq p_2$, otherwise, there would be a nuisance in P on p_2, p_4. Also note that s is not neighbor to p_2, otherwise, $p_2 \in N(s) \cap T$.

Let y be a nuisance on t, p_4. If $p_1 t \notin E(G)$, then (s, t, p_3, p_2, p_1) is an induced path, contradicting the premise that P is a maximum length induced path. Therefore, p_1 is neighbor to t. This being the case, consider the graph induced by p_1, p_2, p_3, p_4, t, y. Note that y cannot be neighbor to p_2, otherwise P would have a nuisance in p_2, p_4, and $p_1 p_4 \notin E(G)$, otherwise P would not be an induced path. With that, however, p_1, p_2, p_3, p_4, t, y induce an L_3 (Figure 3.14).

Therefore, in both case 1 and 2, every $s \in S$ is such that $|N(s) \cap T| \geq 2$.

Part 2: we now claim that every pair of elements $s, s' \in S$ is such that $N(s) \cap T$ and $N(s') \cap T$ are comparable. Suppose otherwise. In that case, there exist $t \in T \cap N(s) - N(s')$ and $t' \in T \cap N(s') - N(s)$.

First, suppose $|P| = 4$. In this case, since (s, t, p_3) and (s', t', p_3) are both induced paths, then (s, t, p_3, t', s') is an induced path of length greater than $|P|$, as ss' and tt' are not neighbors, contradicting the assumption that P is a maximum length induced path.

Suppose, therefore, that $|P| > 4$. In this case, the fact that (s, t, p_3) and (s', t', p_3) are induced paths leads to a T_2 in $s, t, s', t', p_3, p_4, p_5$.

Part 3: As mentioned in the previous parts of the proof, we have it that every $s \in S$ has at least two neighbors in T, and that any pair of elements from S has comparable neighborhoods inside T.

Let $S = \{s_1, \ldots, s_k\}$ such that $N(s_i) \cap T \subseteq N(s_j) \cap T$ when $i < j$. Let $t_1, t_2 \in N(s_1) \cap T$. Note that $N(t_1)$ and $N(t_2)$ contain p_3, plus all of S. They must not be twins, however, implying there exists a vertex $x \in V(G)$ that is neighbor to t_1 and not t_2, or vice-versa. Suppose the former w.l.o.g.

If $(x, t_1, p_3, \ldots, p_n)$ is an induced path, then $x \in S$, and therefore x must also be neighbor to t_2. Therefore, $(x, t_1, p_3, \ldots, p_n)$ is not an induced path, implying x is neighbor to some vertex from $\{p_3, \ldots, p_n\}$. If x is neighbor to p_4, then $s_1, t_1, t_2, x, p_3, p_4$ induces an L_3, and if x is neighbor to $p_i, 4 < i \leq n$, then there exists an induced cycle of length greater than 4. See Figure 3.15 for reference.

Therefore, if every maximum length induced path in G has nuisances, then G is either not twin-free, or contains a forbidden induced subgraph. □

That a twin-free HIBFF graph always contains a maximum length induced path with no nuisances is a quite relevant fact. The existence of a path with those properties allows us to conclude several other properties about HIBFF graphs, which we present in the sequence, leading up to our characterization in Theorem 9.

Lemma 12 is analogous to Lemma 5. Where the latter shows every vertex outside the central cycle has exactly one neighbor in the cycle, the former shows that, given a maximum

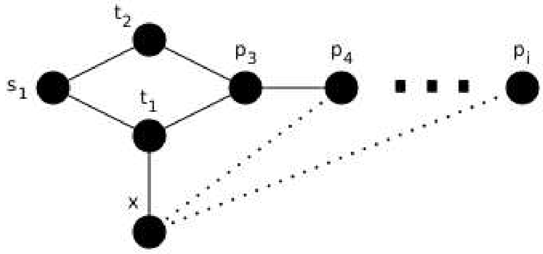

Figure 3.15: Proof of Lemma 11. The dotted edges represent potential adjacencies, and the ellipses represent the continuation of the path between p_4 and p_i, $4 < i \leq n$. If x is neighbor to p_4, an L_3 is induced, and if x is neighbor to p_i, a cycle of length greater than 4 is induced.

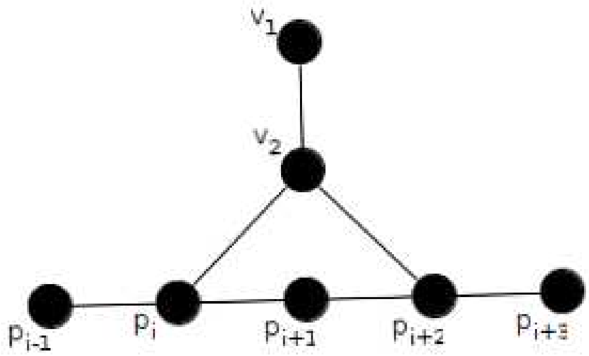

Figure 3.16: Proof of Lemma 12. The X_2 induced by $p_{i-1}, p_i, p_{i+1}, p_{i+2}, v_1, v_2$.

length induced path with no nuisances in a twin-free HIBFF graph, every vertex outside the path is neighbor to exactly one vertex in the path.

Lemma 12. *Let G be a connected twin-free HIBFF graph, and $P = (p_1, ..., p_n)$ a maximum length induced path in G for which no nuisances exist. Then every vertex in $V(G) - P$ has exactly one neighbor in P.*

Proof. Suppose there exists $v \in V(G) - P$ such that $|N(v) \cap P| \geq 3$. If there exist $p_a, p_b, p_c \in P \cap N(v)$ such that $a = b + 2, b = c + 2$, then $p_a, p_{a-1}, p_b, p_{b-1}, p_c, v$ induce an L_3. Otherwise, if there are two elements $p_a, p_b \in P \cap N(v)$ such that $a > b + 2$ and for every $a < i < b$, $p_i \notin N(v)$, then $p_a, p_{a-1}, ..., p_{b+1}, p_b, v$ induce an even cycle of length greater than 4. Therefore, every element of $V(G) - P$ contains fewer than 3 neighbors in P.

Suppose, then, that there exists $v \in V(G) - P$ such that $N(v) \cap P = \emptyset$. Since G is connected, there exists a path Q from v to P. Let v_2 be the first element of Q that has a neighbor in P, and let v_1 be the element that comes before v_2 in Q. Note that v_1 has no neighbors in P, and v_2 has either one or two. If v_2 has two neighbors in P, either it is neighbor to two vertices p_a, p_b with $b > a + 2$, which implies the existence of an induced cycle of length greater than 4, or it is neighbor to two vertices p_i, p_{i+2}, $1 \leq i \leq n - 2$, in which case, either $1 < i < n - 2$, implying X_2 is induced with $p_{i-1}, p_i, p_{i+1}, p_{i+2}, v_1, v_2$ (Figure 3.16), or v_2 is a nuisance.

Now suppose v_2 is neighbor to exactly one vertex of P. If v_2 is neighbor to any vertex in $\{p_1, p_2, p_{n-1}, p_n\}$, that implies P is not a maximum length induced path. If v_2 is neighbor to p_i for $2 < i < n - 1$, a T_2 is induced.

Therefore, every element of $V(G) - P$ must have at least one neighbor in P.

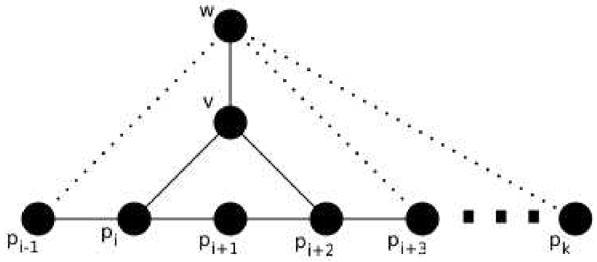

Figure 3.17: Proof of Lemma 12. The dotted edges represent potential adjacencies, and the ellipses represent the continuation of the path from p_{i+3} to p_k for some $k > i + 3$. If none of the dotted adjacencies is present, an X_2 is induced. If w is neighbor to p_{i+3} or p_{i-1}, an L_3 is induced. Finally, if w is neighbor to p_k, a cycle of length greater than 4 is induced.

Now suppose there is a vertex v that is neighbor to exactly two vertices of P. Note that the two neighbors of v in P must be exactly two indices apart, otherwise, a cycle of length greater that 4 is induced. Also, v must not be neighbor to p_1, p_3 or p_{n-1}, p_n, since P has no nuisances. So let v be neighbor to p_i, p_{i+2}, with $1 < i < n - 2$. Since v and p_{i+1} are not twins, either v has a neighbor it does not share with p_{i+1} or vice-versa. Suppose the former w.l.o.g. and let $w \in N(v) - N(p_{i+1})$. See Figure 3.17 for reference.

If w is not neighbor to p_{i-1} nor p_{i+3}, then an X_2 is induced in $v, w, p_{i-1}, p_i,$ $p_{i+1}, p_{i+2}, p_{i+3}$. Now, if w is neighbor to some vertex p_j in P, with $j = i + 3$ or $j = i - 1$, an L_3 is induced with $v, w, p_j, p_i, p_{i+1}, p_{i+2}$. If w is neighbor to any other vertex in the path, a cycle of length greater than 4 is induced.

Therefore, for every $v \in V(G) - P$, $|N(v) \cap P| = 1$. □

Lemmas 13, 14 and 15 present more relevant properties for the characterization. The restrictions imposed by these properties allow us to formulate the fundamental structure in Definition 3 and to prove the characterization in Theorem 9.

Lemma 13. *Let G be a connected twin-free HIBFF graph, and $P = (p_1, ..., p_n)$ a maximum length induced path in G that has no nuisances. Let $v, w \in V(G) - P$, with $\{p_i\} = N(v) \cap P$, and $\{p_j\} = N(w) \cap P$, with i even, j odd, and $i \neq j + 1, j - 1$. Then $vw \notin E(G)$.*

Proof. If $vw \in E(G)$, then $(v, w, p_j, ..., p_i)$ is an induced cycle of length greater than 4. □

Lemma 14. *Let G be a connected twin-free HIBFF graph, and $P = (p_1, ..., p_n)$ a maximum length induced path in G without nuisances. Let $v, w, x \in V(G) - P$, with $\{p_i\} = N(v) \cap P$, $\{p_{i+1}\} = N(w) \cap P$, and $\{p_{i+2}\} = N(x) \cap P$ for some $1 < i < n$. Then $vw \notin E(G)$ or $wx \notin E(G)$.*

Proof. If $vw, wx \in E(G)$, then $v, w, x, p_i, p_{i+1}, p_{i+2}$ induce an L_3. □

Lemma 15. *Let G be a connected twin-free HIBFF graph, and $P = (p_1, ..., p_n)$ a maximum length induced path in G without nuisances. Let $v, w \in V(G) - P$ such that $N(v) \cap P = N(w) \cap P = p_i$, $N(v) \cap N(p_{i+1}) \neq \emptyset$, and $N(w) \cap N(p_{i+1}) \neq \emptyset$ (resp. $N(v) \cap N(p_{i-1}) \neq \emptyset$ and $N(w) \cap N(p_{i-1}) \neq \emptyset$) for some $1 < i < n$. Then $N(v) \subset N(w)$ or $N(w) \subset N(v)$.*

Proof. By Lemma 13, v and w do not have neighbors in $N(p_j), j \neq i + 1, i - 1$. Also, by Lemma 14, we know that, since v and w have neighbors in $N(p_{i+1})$ (resp. $N(p_{i-1})$), they do not have neighbors in $N(p_{i-1})$ (resp. $N(p_{i+1})$). Therefore, $N(v) - P \subset N(p_{i+1})$ and $N(w) - P \subset N(p_{i+1})$ (resp. $N(v) - P \subset N(p_{i-1})$ and $N(w) - P \subset N(p_{i-1})$).

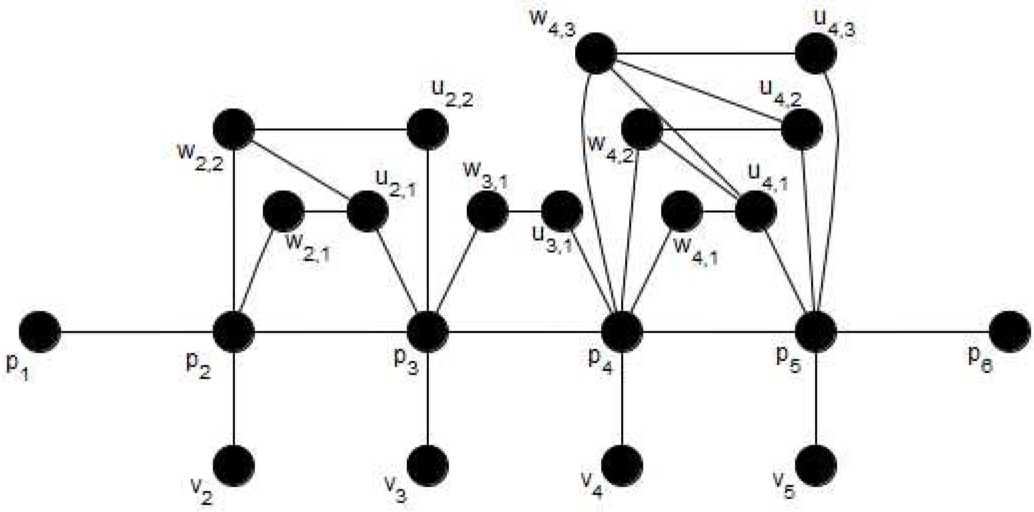

Figure 3.18: Graph $\mathbb{H}_6(2, 1, 3)$.

If $N(v)$ and $N(w)$ are not comparable, that implies there exist $v' \in N(v) - N(w)$ and $w' \in N(w) - N(v)$ such that $v', w' \in N(p_{i+1})$ (resp. $v', w' \in N(p_{i-1})$. The graph induced by $v, w, v', w', p_i, p_{i+1}$ (resp. $v, w, v', w', p_i, p_{i-1}$) is an L_3. □

Definition 3 contains a family of graphs similar to FCS graphs, that plays a role in Helly interval bigraphs similar to that of FCS graphs in non-bichordal Helly CA bigraphs.

Definition 3. *Let a* Fundamental Interval Structure graph *(FIS for short), denoted* $\mathbb{H}_k(n_2, \ldots, n_{k-2})$ *for* $k \geq 1, n_2, \ldots, n_{k-2} \geq 0$*, be a graph defined as follows:*

Let $V(\mathbb{H}_k(n_2, \ldots, n_{k-2}))$ *be the union of the following sets:*

- $P = \{p_1, p_2, \ldots, p_k\}$.
- $V = \{v_2, \ldots, v_{k-1}\}$.
- $W_i = \{w_{i,1}, \ldots, w_{i,n_i}\}$ *for all* $1 < i < k - 1$.
- $U_i = \{u_{i,1}, \ldots, u_{i,n_i}\}$ *for all* $1 < i < k - 1$.

With the neighborhoods of every vertex defined as follows, forming $E(\mathbb{H}_k(n_2, \ldots, n_{k-2}))$*:*

- $N(p_1) = \{p_2\}$, $N(p_k) = \{p_{k-1}\}$, $N(p_i) = \{p_{i-1}, p_{i+1}, v_i\} \cup W_i \cup U_{i-1}$ *for* $1 < i < k$.
- $N(v_i) = \{p_i\}$.
- $N(w_{i,j}) = \{p_i\} \cup \{u_{i,l} \in U_i | l \leq j\}$*, for all* $2 \leq i \leq k-2, 1 \leq j \leq n_i$.
- $N(u_{i,j}) = \{p_{i+1}\} \cup \{w_{i,l} \in W_i | l \geq j\}$*, for all* $2 \leq i \leq k-2, 1 \leq j \leq n_i$.

An example of an FIS graph can be seen in Figure 3.18. The last result we need to prove Theorem 9 is Theorem 8, presented in the sequence.

Theorem 8. *Every FIS graph is a Helly interval bigraph.*

Proof. Let $G = \mathbb{H}_k(n_2, \ldots, n_{k-2})$. The bicliques of G are the following:

1. $A_2 = \{p_1, p_2, p_3\} \cup W_2$;

2. $A_{k-1} = \{p_{k-2}, p_{k-1}, p_k\} \cup U_{k-2}$;

3. $A_i = \{p_{i-1}, p_i, p_{i+1}, v_i\} \cup W_i \cup U_{i-1}$ for all $1 < i < k$;

4. $B_{i,j} = \{p_i, p_{i+1}\} \cup \{w_{i,m} | m \geq j\} \cup \{u_{i,l} | l \leq j\}$ for all $1 < i < k-1$, and $1 \leq j \leq n_i$.

We claim that every biclique in G is of the form A_i or $B_{i,j}$ seen on the list. It is easy to verify that, for every p_i, $1 < i < k$, the star of p_i is the biclique A_i. It is also easy to verify that every other vertex in G will be such that its star alone is not a biclique.

Note that every biclique must include at least two consecutive vertices of P and at most three, and no biclique exists with a non-consecutive subset of vertices of P. Note that every biclique with three vertices of P is of the form A_i since, starting with three elements $\{p_i, p_{i+1}, p_{i+2}\}$, the only vertices that can be added to that set without breaking its bipartite-completeness are neighbors of p_{i+1}, and the set will remain not maximal until all neighbors of p_{i+1} are in it.

Consider, then, a biclique B with only two consecutive vertices p_i, p_{i+1} in P. Note that, if any vertex from U_{i-1} is in B, that implies that every vertex in $B - \{p_i\}$ is neighbor to p_i, since the only vertex that is neighbor to both the elements of U_{i-1} and p_{i+1} is p_i. That would imply that B is either not maximal or that it contains p_{i-1}. For a similar reason, no elements of W_{i+1} are in B, since that would imply every vertex in $B - \{p_{i+1}\}$ is neighbor to p_{i+1}. Also, note that no vertex from V is in B.

Therefore, the only vertices aside from p_i, p_{i+1} that are in B are vertices from W_i, U_i. Let $1 \leq j' \leq n_i$ be the lowest value of j for which $w_{i,j} \in B$. That implies that, for every $l > j'$, $w_{i,l} \in B$, since $N(w_{i,a}) \subset N(w_{i,b})$ when $b > a$. Therefore, we have it that $\{p_i, p_{i+1}, w_{i,j'}, w_{i,j'+1}, ..., w_{i,n_i}\}$ is a subset of B, and only $B \cap U_i$ is left to consider. The only vertices of U_i that may be in B are in the set $\{u_{i,j} | j \leq j'\}$, since the other elements of U_i are not neighbors to all elements of $B \cap W_i$. Furthermore, since $\{p_i, p_{i+1}, w_{i,j'}, w_{i,j'+1}, ..., w_{i,n_i}\} \cup \{u_{i,j} | j \leq j'\}$ is bipartite-complete, B is not maximal unless it contains every vertex in $\{u_{i,j} | j \leq j'\}$. Therefore, B is the biclique $B_{i,j'}$.

We must now prove that it is possible to organize the bicliques in a linear order such that, for every vertex, the bicliques it belongs to are consecutive. Consider the order described as follows:

$$(A_2, B_{2,1}, ..., B_{2,n_2}, A_3, B_{3,1}, ..., B_{3,n_3}, A_4, B_{4,1}, ..., B_{4,n_4},\\ A_5, B_{5,1}, ..., B_{5,n_5}, A_6, B_{6,1}, ..., B_{k-2,n_{k-2}}, A_{k-1}).$$

The above order is constructed in a very similar way to the one in the proof of Theorem 1.

Consider, too, the family of bicliques for each vertex:

- p_1 is only in A_2, p_k is only in A_{k-1}.

- p_2 is in A_2, A_3 and $B_{2,j}$ for every $j \leq n_2$.

- p_{k-1} is in A_{k-2}, A_{k-1} and $B_{k-2,j}$ for every $j \leq n_{k-2}$.

- p_i, for $2 < i < k-1$, is in A_{i-1}, A_i, A_{i+1}, $B_{i,j}$ for every $j \leq n_i$ and $B_{i-1,j}$ for every $j \leq n_{i-1}$.

- For $1 < i < k$, v_i is only in A_i.

- For $1 < i < k-1$, $w_{i,j}$ is in A_i and $B_{i,l}$ for every $l \leq j$.

- For $1 < i < k-1$, $u_{i,j}$ is in A_{i+1} and $B_{i,l}$ for every $l \geq j$.

Note that, for every vertex, the order presented is such that the family of bicliques it belongs to is an interval. □

Having defined a fundamental structure for twin-free Helly interval bigraphs, all that is left to do is proving that every twin-free HIBFF graph follows that structure, as seen in 9.

Theorem 9. *A bipartite graph is a Helly interval bigraph if and only if it does not contain a T_2, an L_3, an X_2, or $C_k, k > 4$ as induced subgraphs (i.e. is an HIBFF graph).*

Proof. We show that every twin-free connected HIBFF graph is an induced subgraph of an FIS graph. Let G be a twin-free, connected bipartite graph without the aforementioned forbidden graphs.

According to Lemma 11, G admits a maximum length induced path $P = (p_1, ..., p_n)$ without nuisances. According to Lemma 12, every vertex in $V(G) - P$ is neighbor to exactly one vertex of P. Note that there are no neighbors in $V(G) - P$ to p_1 or p_n, as that would contradict the maximum length of P. Let $V_i = N(p_i) \cap V(G) - P$ for all $1 < i < n$.

According to Lemmas 13 and 14, every vertex $v \in V_i$ is such that $N(v) - P \subset V_{i-1}$ or $N(v) - P \subset V_{i+1}$. Also, according to Lemma 15, if two vertices $v, w \in V_i$ have neighbors in V_{i+1} (or V_{i-1}), then their neighborhoods must be comparable.

Under these restrictions, G is an induced subgraph of FIS graph $\mathbb{H}_k(n_2, \ldots, n_{k-2})$, with $k = |P|$, and $n_i = |\{v \in V_i | N(v) \cap V_{i+1} \neq \emptyset\}|$ for all $1 < i \leq k-2$: the path P would correspond to set P from the definition, any vertex in $V(G) - P$ that has exactly one neighbor would belong to V, the vertices of $V_i, 1 < i \leq k-2$ that have neighbors in V_{i+1} would belong to set W_i, and the vertices of $V_i, 2 < i \leq k-1$ that have neighbors in V_{i-1} would belong to set U_{i-1}. □

Theorem 9 concludes our forbidden graph characterization.

Note that the similarities between non-bichordal Helly CA bigraphs and Helly interval bigraphs are made explicit by Theorem 8, as every FIS graph is an induced subgraph of an FCS graph. Furthermore, just like the proof of Theorem 9 shows that the twin-free version of every connected Helly interval bigraph is an induced subgraph of an FIS, the proof of Theorem 2 shows that the twin-free version of every non-bichordal Helly CA bigraph is an induced subgraph of an FCS.

In both the definition of FCS graphs and that of FIS graphs, we see very similar elements: a central sequence of vertices (the cycle for FCS graphs and the path for FIS graphs,) such that every other vertex in the graph is neighbor to exactly one element of said sequence, and every vertex outside the main sequence being part of a hierarchy of neighborhood containment. It is easy to notice that an FCS graphs can be constructed from an FIS graph by closing the maximum length induced path of the FIS into a cycle.

In fact, not only is every FIS graph an induced subgraph of an FCS graph, Theorem 10, which we present in the sequence, shows that every bichordal induced subgraph of a non-bichordal Helly CA bigraph is a Helly interval bigraph.

Theorem 10. *Every bichordal induced subgraph of a non-bichordal Helly CA bigraph is a Helly interval bigraph.*

Proof. Let G be a non-bichordal Helly CA bigraph. The twin-free version G^- of G is an induced subgraph of some $\mathbb{G}_k(n_1, \ldots, n_k)$ for certain values of $k, n_1, \ldots, n_k$, which does not contain X_2, an L_3 nor T_2 as induced subgraphs. Therefore, G also does not contain any of those induced subgraphs, and neither does any induced subgraph of G.

Let G' be a bichordal induced subgraph of G. Since G' is bichordal, it does not contain an induced $C_n, n > 4$. Also, since G is $\{X_2, T_2, L_3\}$-free, so is G'. Therefore, G' is a Helly interval bigraph. □

Further similarities between the classes are explored in our studies of proper-Helly CA bigraphs in Subsection 3.1.4.

3.1.4 Proper-Helly CA bigraphs

Proper circular arc graphs are defined as graphs that admit a circular arc model $(C, \mathbb{A})$ such that $\mathbb{A}$ is a proper family. Analogously, proper circular arc bigraphs are graphs that admit a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ where $\mathbb{I}$ and $\mathbb{E}$ are proper families, and proper interval bigraphs are graphs that admit a bi-interval model $(\mathbb{A}, \mathbb{B})$ such that $\mathbb{A}$ and $\mathbb{B}$ are proper families. We call bi-circular-arc models (bi-interval models) with that property proper bi-circular-arc (proper bi-interval) models.

As mentioned in the introduction, proper CA bigraphs and interval bigraphs are relatively well studied subclasses of CA bigraphs. Proper interval bigraphs count with a linear time recognition algorithm [Spinrad et al., 1987], and several characterizations [Das and Chakraborty, 2015]. Furthermore, in [Brown and Lundgren, 2010], equivalences between proper interval bigraphs and several other classes of bipartite graphs are presented.

As for proper CA bigraphs, Basu et al. [Basu et al., 2013] have characterized them in terms of their biadjacency matrices, and Das and Chakraborty [Das and Chakraborty, 2015] presented characterizations related to sequences of vertices. More recently, Safe [Safe, 2019] has presented a forbidden graph characterization alongside a recognition algorithm for the class.

In this subsection, we introduce and characterize a class that combines the proper and Helly properties in the same bi-circular-arc model, called the class of *proper-Helly CA bigraphs*. We also present another, simpler characterization for a subclass of proper-Helly CA bigraphs we call *normal-proper-Helly* CA bigraphs.

Definition 4. *A bipartite graph G is a proper-Helly CA bigraph (proper-Helly interval bigraph) if and only if it admits a bi-circular-arc model (bi-interval model) that is both Helly and proper.*

Proper-Helly CA bigraphs are a bipartite counterpart of the class of proper-Helly CA graphs, defined as graphs which admit CA models that are simultaneously proper and Helly. This class has been quite recently studied and characterized [Lin et al., 2007, 2013] alongside its class of clique graphs [Lin et al., 2010].

Both the classes of proper-Helly CA bigraphs and proper-Helly interval bigraphs are hereditary over induced subgraphs, and such that a graph is in the class precisely if its twin-free version also is.

It is important to note that, while proper-Helly CA bigraphs are contained in the intersection of proper and Helly CA bigraphs, the intersection also contains graphs that are not proper-Helly, such as the $L_3{}^*$: since the L_3 is not a Helly interval bigraph, any Helly bi-circular-arc model of the L_3 covers the circle, meaning it is impossible to add an arc corresponding to its isolated vertex without it being contained in another arc of the same family. Also, since both the L_3 and the isolated vertex are proper interval bigraphs, then so, too, is the $L_3{}^*$ a proper interval bigraph. Another forbidden graph for proper-Helly CA bigraphs that is also in the intersection of proper and Helly is the X_2. It is clearly a proper CA bigraph [Safe, 2019] and a Helly CA bigraph (Figure 3.19). In the sequence, we show that it is not a proper-Helly CA bigraph.

Lemma 16. *The graph X_2 is not a proper-Helly CA bigraph.*

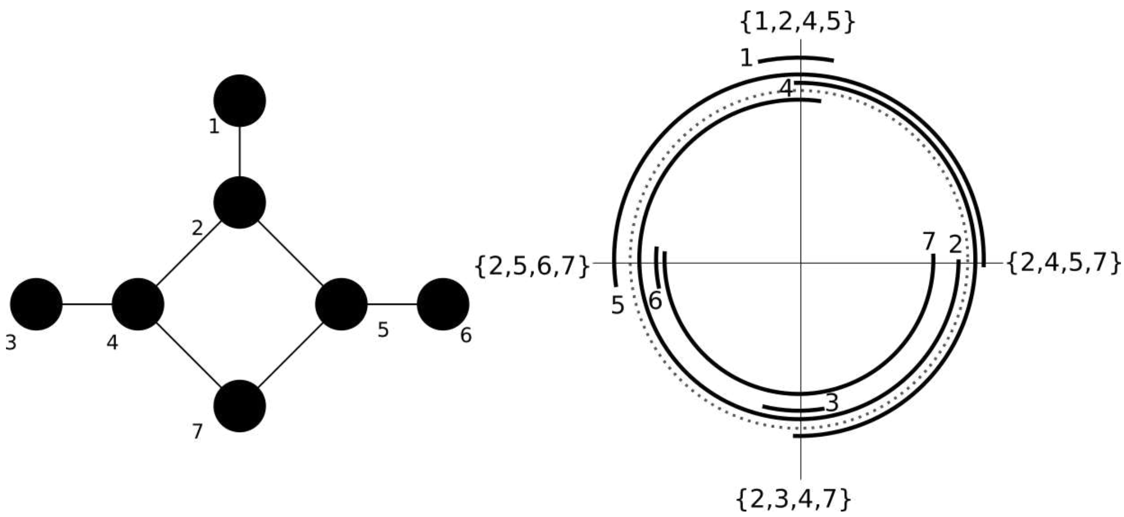

Figure 3.19: Graph X_2 alongside its Helly bi-circular-arc model.

Proof. We show that it is impossible to construct a Helly model of X_2 that is also proper.

Consider the enumeration of the vertices of X_2 as seen in Figure 3.19. The partite sets are $X = \{1, 4, 5\}$ and $Y = \{2, 3, 6, 7\}$, and the bicliques, as shown in Lemma 10, are $A = \{1, 2, 4, 5\}, B = \{2, 3, 4, 7\}, C = \{2, 4, 5, 7\}, D = \{2, 5, 6, 7\}$.

Any Helly model of an X_2 is compatible with some set of biclique points. It suffices to show that, for any valid set of biclique points of X_2, any Helly bi-circular-arc model compatible with it is such that at least two vertices v, w of the same partite set have their corresponding arcs $a(v), a(w)$ be comparable. That, in turn, proves it impossible to build a proper-Helly model compatible with any set of biclique points, implying there is no Helly model that is also proper.

Let p_A, p_B, p_C, p_D be the biclique points of A, B, C, D. Since every vertex of the graph is in either all bicliques, three bicliques or one biclique, any order in which those four points are arranged around the circle yields a valid set of biclique points. Suppose one of those orders yields a set of biclique points for which it is possible to build a proper-Helly model.

Vertex 1 is in biclique A only, whereas vertex 4 is in A, B, C. Therefore, if a distribution of biclique points yields a proper-Helly model, (p_B, p_A, p_C) cannot be circularly consecutive within the order of biclique points, neither clockwise nor counter-clockwise. Therefore, those three biclique points are in circularly consecutive order (p_A, p_C, p_B) or (p_A, p_B, p_C), either clockwise or counter-clockwise. Suppose clockwise without loss of generality.

In the first case, the full clockwise order of the points is p_A, p_C, p_B, p_D. for every Helly bi-circular-arc model compatible with a set of biclique points distributed in that order, $a(3)$ will always be contained in $a(7)$, as $b(3) = \{B\}$ and $b(7) = \{B, C, D\}$, so $a(3)$ will be contained in (p_C, p_D) and $a(7)$ will contain (p_C, p_D).

Consider, then, the second case, where the distribution of biclique points follows the order p_A, p_B, p_C, p_D. Note that, for any model compatible with a set of biclique points in that order, $a(3)$ will be contained in (p_A, p_C) and $a(6)$ will be contained in (p_C, p_A). Consider, then, the placement of arc $a(2)$. Since 2 is in every biclique, then its arc must begin and end within the same gap between two consecutive biclique points. If $a(2)$ has its endpoints within (p_A, p_B) or (p_B, p_C), $a(2)$ contains (p_C, p_A). If $a(2)$ has its endpoints within (p_C, p_D) or (p_D, p_A), $a(2)$ will contain (p_A, p_C).

Therefore, it is impossible to build a Helly model for X_2 that is also proper. □

Graph X_2 is one of the forbidden graphs we utilize in the characterization of proper-Helly CA bigraphs, alongside other graphs that we present further into the subsection.

Proper-Helly CA bigraphs possess a property similar to that of Helly CA bigraphs in general, in that, if a bipartite graph G is such that its square is a proper-Helly CA graph, then G is a proper-Helly CA bigraph, as shown in the sequence.

Lemma 17. *Let G be a bipartite graph. If G^2 is a proper-Helly CA graph, then G is a proper-Helly CA bigraph.*

Proof. Let $(C, \mathbb{A})$ be a proper-Helly circular arc model of G^2. Partition $\mathbb{A}$ into two families $\mathbb{I}, \mathbb{E}$ with arcs corresponding to each partite set of G. It is easy to note that $(C, \mathbb{I}, \mathbb{E})$ is a bi-circular-arc model of G, as two vertices of opposing partite sets are neighbors in G^2 precisely if they are neighbors in G. Also, since $\mathbb{A}$ is a proper family, $\mathbb{I}, \mathbb{E}$ are also proper families.

Finally, as in the proof of Lemma 7, since $(C, \mathbb{A})$ is a Helly CA model, model $(C, \mathbb{I}, \mathbb{E})$ is a Helly bi-circular-arc model. □

The converse is not true, as the L_3 is a proper-Helly CA bigraph (Figure 3.9) and its square is not a proper-Helly CA graph, since it contains a 4-wheel, which is forbidden for proper-Helly CA graphs [Lin et al., 2007].

The next step towards our characterization consists of showing that non-bichordal Helly CA bigraphs are subclass of proper-Helly CA bigraphs, and that Helly interval bigraphs are equivalent to proper-Helly interval bigraphs. These two classes together represent a significant portion of proper-Helly CA bigraphs, as we will show in the sequence.

In order to prove those two containments, we apply Lemma 18.

Lemma 18. *If a bipartite graph $G = (X, Y, E)$ admits a bi-circular-arc (bi-interval) model such that every pair of comparable arcs (intervals) corresponding to vertices from the same partite set share either their beginning or end point, then G is a proper circular arc (interval) bigraph.*

Proof. Start with a bi-circular-arc (bi-interval) model as described in the lemma. Let ϵ be a small fraction of the smallest distance between two non-coinciding endpoints. Without loss of generality, if d is the smallest distance between two non-coinciding endpoints, let $\epsilon = \frac{d}{100|V(G)|}$.

For every point p in the circle (in the number line) that is the s-endpoint (left endpoint) of some arc (interval), let $\mathbb{A}_p = \{A_1, ..., A_n\}$ be the set of all arcs (intervals) such that $s(A_i) = p$ ($l(A_i) = p$) and $v(A_i) \in X$ for all $1 \leq i \leq n$, and let $A_i \subset A_{i+1}$ for all $1 \leq i < n$.

Change the arcs (intervals) of $\mathbb{A}$ by shifting the s-endpoint (left endpoint) of A_i in the clockwise (right) direction by an amount of $(i-1)\epsilon$ for all $1 \leq i \leq n$. Note that the arcs in $\mathbb{A}$ will no longer be properly contained in one another, and the corresponding graph of the model does not change, since ϵ is small enough. Also since ϵ is small enough, no new proper containments are created by the process.

Repeat the same process with partite set Y, and then do the same for t-endpoints (right endpoints), except the t-endpoints (right endpoints) must be shifted in the counter-clockwise (left) direction. The resulting model is a proper bi-circular-arc (bi-interval) model of G. □

In Theorem 11, we apply Lemma 18 to show that every Helly interval bigraph admits a bi-interval model that is simultaneously Helly and proper.

Theorem 11. *Every Helly interval bigraph admits a bi-interval model that is simultaneously Helly and proper.*

Proof. By the proof of Theorem 9, every connected component of a Helly interval bigraph is such that its twin-free version is an induced subgraph of an FIS graph. Therefore, it suffices to prove that, for every $k, n_2, ..., n_{k-2}$, the graph $\mathbb{H}_k(n_2, ..., n_{k-2})$ admits a model that is simultaneously

Helly and proper. It is easy to note that it is then possible to add intervals corresponding to twin vertices without losing either property.

Let $G = \mathbb{H}_k(n_2, ..., n_{k-2})$, with its vertex set partitioned according to the definition of FIS graphs. We can assume that k is even and $k > 4$, since for any case where k is odd or $k \leq 4$, G is an induced subgraph of an FIS for which $k > 4$ and k is even. For the same reason, we may assume that $n_i > 0$ for all $2 \leq i \leq k-2$.

The two partite sets of G are:

- $A = \{p_i \in P | i \text{ odd}\} \cup \{v_i \in V | i \text{ even}\} \cup \bigcup_{i=2,i+2}^{i<k-1} W_i \cup \bigcup_{i=3,i+2}^{i<k-1} U_i.$

- $B = \{p_i \in P | i \text{ even}\} \cup \{v_i \in V | i \text{ odd}\} \cup \bigcup_{i=3,i+2}^{i<k-1} W_i \cup \bigcup_{i=2,i+2}^{i<k-1} U_i.$

For every biclique of G, we attribute a point in the number line, distributed according to the order given by Theorem 8. Every biclique A_i is represented by point a_i, and every biclique $B_{i,j}$ is represented by point $b_{i,j}$. The resulting sequence of points is as follows:

$$(a_2, b_{2,1}, ..., b_{2,n_2}, a_3, b_{3,1}, ..., b_{3,n_3}, a_4, b_{4,1}, ..., b_{4,n_4},$$
$$a_5, b_{5,1}, ..., b_{5,n_5}, a_6, b_{6,1}, ..., b_{k-2,n_{k-2}}, a_{k-1}).$$

Suppose those points are distributed along the number line at a distance of 1 between consecutive points. Note that every vertex $v \in V(G)$ is such that the set of points corresponding to $b(v)$ is consecutive in the sequence.

As shown in Theorem 8, the family of bicliques for each vertex is the following:

- p_1 is only in A_2, p_k is only in A_{k-1}.
- p_2 is in A_2, A_3 and $B_{2,j}$ for every $j \leq n_2$.
- p_{k-1} is in A_{k-2}, A_{k-1} and $B_{k-2,j}$ for every $j \leq n_{k-2}$.
- p_i, for $2 < i < k-1$, is in A_{i-1}, A_i, A_{i+1}, $B_{i,j}$ for every $j \leq n_i$ and $B_{i-1,j}$ for every $j \leq n_{i-1}$.
- For $1 < i < k$, v_i is only in A_i.
- For $1 < i < k-1$, $w_{i,j}$ is in A_i and $B_{i,l}$ for every $l \leq j$.
- For $1 < i < k-1$, $u_{i,j}$ is in A_{i+1} and $B_{i,l}$ for every $l \geq j$.

We show that, for any two vertices v, w from the same partite set such that $b(v), b(w)$ are comparable, the consecutive intervals of the sequence corresponding to $b(v)$ and $b(w)$ either begin or end in the same point. We focus on partite set A, since, with k even, the partite sets are basically analogous to one another.

Consider, first, vertices of the form $w_{i,j}$ for i even. If $i = 2$, then $b(w_{i,j})$ is comparable to $b(p_1), b(p_3), b(v_2)$, and $b(w_{i,l})$ for all $1 \leq l \leq n_2$, all of which have their intervals begin in a_2. If $i > 2$, then $b(w_{i,j})$ is comparable to $b(p_{i+1}), b(v_i)$, and $b(w_{i,l})$ for all $1 \leq l \leq n_2$, all of which begin in a_i. Now, consider, vertices of the form $u_{i,j}$ for j odd. The set $b(u_{i,j})$ is comparable to $b(p_i), b(v_{i+1})$, and $b(u_{i,l})$ for all $1 \leq l \leq n_2$. All of those have their interval end in a_{i+1}.

Now, consider vertices p_i for i odd. If $i = 1$, then $b(p_1) = \{A_2\}$, which is the first biclique in the sequence of points. If $1 < i < k-1$, then $b(p_i)$ is comparable to $b(v_{i-1}), b(v_{i+1}), b(w_{i-1,j})$, for $1 \leq j \leq n_{i-1}$ and $b(u_{i,j})$ for $1 \leq j \leq n_i$. Note that $b(v_{i-1}), b(w_{i-1,j})$ have their intervals start

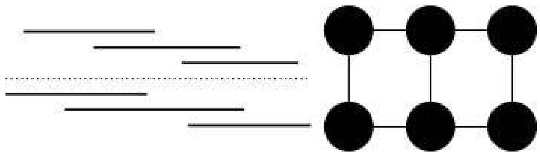

Figure 3.20: Graph L_3 alongside its proper bi-interval model.

in a_{i-1}, and $b(v_{i+1}), b(u_{i,j})$ have their intervals end in a_{i+1}. Finally, if $i = k - 1$, then $b(p_i)$ is comparable to $b(w_{k-2,j}), b(v_{k-2})$ for $1 \leq j \leq n_{k-2}$, both of which have their intervals start in a_{k-2}.

Therefore, every pair of vertices v, w of the same partite set with $b(v) \subset b(w)$ is such that the intervals of points corresponding to $b(v)$ and $b(w)$ either begin or end in the same point.

We can then build a bi-interval model in the following way: for every vertex v such that the sequence of points corresponding to $b(v)$ starts at some point p and ends at some point q, create an interval $I_v = (p - \frac{1}{3}, q + \frac{1}{3})$. Note that, after doing that for every vertex, the resulting model will be a Helly bi-interval model of G, and it will also be such that every pair of comparable intervals of the same partite set share either a left or a right endpoint.

We can then apply the method described in the proof of Lemma 18 to remove proper containments, and with a sufficiently small value for ϵ, no intervals lose any of the points corresponding to their bicliques, therefore also maintaining the Helly property. □

As a side observation, note that Theorem 11 shows, among other things, that Helly interval bigraphs are a subclass of proper interval bigraphs. The converse is not true, however, since the L_3 is a proper interval bigraph, but not a Helly interval bigraph. A proper model for the L_3 can be seen in Figure 3.20. Therefore, Helly interval bigraphs are a proper subclass of proper interval bigraphs.

Another side observation about Theorem 11 is the contrast it implies between interval bigraphs and interval graphs. While non-bipartite interval models are Helly by default, not every interval graph is a proper interval graph. In the case of interval bigraphs, there are graphs that do not admit a Helly bi-interval model, but when a graph has a Helly model, it also admits a proper model.

Furthermore, Theorem 11 yields Corollary 4, which is relevant for our characterization.

Corollary 4. *The class of proper-Helly interval bigraphs is equivalent to the class of Helly interval bigraphs.*

Proof. Theorem 11 shows that every Helly interval bigraph admits a model that is simultaneously proper and Helly, implying every Helly interval bigraph is a proper-Helly interval bigraph. Conversely, every proper-Helly interval bigraph is a Helly interval bigraph by definition. □

We now turn our attention to non-bichordal Helly CA bigraphs. We must show that every non-bichordal Helly CA bigraph is a proper-Helly CA bigraph. The process we apply is much the same as the one for Helly interval bigraphs, involving showing that the fundamental structure (in this case, FCS graphs) admits a model that is simultaneously Helly and proper. Due to the similarities with the proof of Theorem 11, we have omitted some details from the proof of Theorem 12 that can be filled in by making analogies with the proof of the former.

Theorem 12. *Every non-bichordal Helly circular arc bigraph admits a bi-circular-arc model that is simultaneously proper and Helly.*

Proof. It suffices to show that $G = \mathbb{G}_k(n_1, ..., n_k)$ admits a model that is proper and Helly for any $k \geq 6, n_1, ..., n_k \geq 0$. We may assume that $n_i > 0$ for every $1 \leq i \leq k$, since every case where one of those numbers is 0 is an induced subgraph of a case where none is.

Consider the bicliques of G as shown in Theorem 1.

- $A_i = \{c_{i-1}, c_i, c_{i+1}, v_i\} \cup W_i \cup U_{i-1}$ for all $1 \leq i \leq k$.
- $B_{i,j} = \{c_i, c_{i+1}\} \cup \{w_{i,m} | m \geq j\} \cup \{u_{i,l} | l \leq j\}$ for $1 \leq i \leq k$, $1 \leq j \leq n_i$.

Also consider the order of biclique points around the circle given in the theorem, with a_i being the point for A_i and $b_{i,j}$ being the point for $B_{i,j}$ for all $1 \leq i \leq k$ and $1 \leq j \leq n_i$.

$$(a_1, b_{1,1}, ..., b_{1,n_1}, a_2, b_{2,1}, ..., b_{2,n_2}, a_3, b_{3,1}, ..., b_{3,n_3},$$
$$a_4, b_{4,1}, ..., b_{4,n_4}, a_5, b_{5,1}, ..., b_{k-1,n_{k-1}}, a_k, b_{k,1}, ..., b_{k,n_k}).$$

Consider, as well, the list of bicliques for every vertex of G according to the theorem.

- For every $1 \leq i \leq k$, $b(c_i) = \{A_{i-1}, A_i, A_{i+1}\} \cup \{B_{i,j} | 1 \leq j \leq n_i\} \cup \{B_{i-1,j} | 1 \leq j \leq n_{i-1}\}$.
- For every $1 \leq i \leq k$, $b(v_i) = \{A_i\}$.
- For every $1 \leq i \leq k$, and $1 \leq j \leq n_i$, $b(w_{i,j}) = \{A_i\} \cup \{B_{i,l} | 1 \leq l \leq j\}$.
- For every $1 \leq i \leq k$, and $1 \leq j \leq n_i$, $b(u_{i,j}) = \{A_{i+1}\} \cup \{B_{i,l} | j \leq l \leq n_i\}$.

Finally, we must identify the partite sets of G.

Consider the partitioning of the vertices of G according to the definition of FCS graphs. Let $C_1, C_2 \subset C$ be the subset of odd-indexed and even-indexed elements of C, respectively. Also, let $V_1, V_2 \subset V$ be the subsets of odd an even index in V. With that, the partite sets of G are:

- $X = C_1 \cup V_2 \cup \bigcup_{i=2,i+2}^{k} W_i \cup \bigcup_{i=1,i+2}^{k-1} U_i$.
- $Y = C_2 \cup V_1 \cup \bigcup_{i=2,i+2}^{k} U_i \cup \bigcup_{i=1,i+2}^{k-1} W_i$.

It suffices to show that, for any two vertices v, w from the same partite set such that $b(v), b(w)$ are comparable, the circularly consecutive intervals of biclique points corresponding to $b(v)$ and $b(w)$ either begin or end in the same point. Just like in the proof of Theorem 11, we can then construct a Helly model and apply Lemma 18 to turn it into a proper model without losing the Helly property.

Without loss of generality, we use partite set X (the proof for partite set Y is analogous). Consider, first, vertices of the form $w_{i,j}$ for even i. The only vertices of X whose biclique sets are comparable to that of $w_{i,j}$ are $w_{i,l}, l \neq j$, v_i and c_{i+1}. Note that all their biclique point intervals also start in a_i. Consider, now, vertices of the form $u_{i,j}$ for odd i. The vertices of X that have comparable biclique sets are c_i, v_{i+1}, and $u_{i,l}, l \neq j$. All their biclique point intervals end in a_{i+1}.

For vertices of the form c_i for odd i, we have its biclique set comparable to those of $w_{i-1,j}$, $u_{i,j}$, v_{i+1} and v_{i-1}. For $c_i, w_{i-1,j}, v_{i-1}$, all of them have their intervals start in a_{i-1}. For $c_i, u_{i,j}, v_{i+1}$, all their intervals end in a_{i+1}.

Finally, the cases involving v_i for even i are covered in the other cases we cited.

Therefore, by applying a method similar to that of Theorem 11, it is possible to build a bi-circular-arc model that is both proper and Helly for any FCS graph. This, in turn, implies it is possible to construct such a model for any non-bichordal Helly CA bigraph.

□

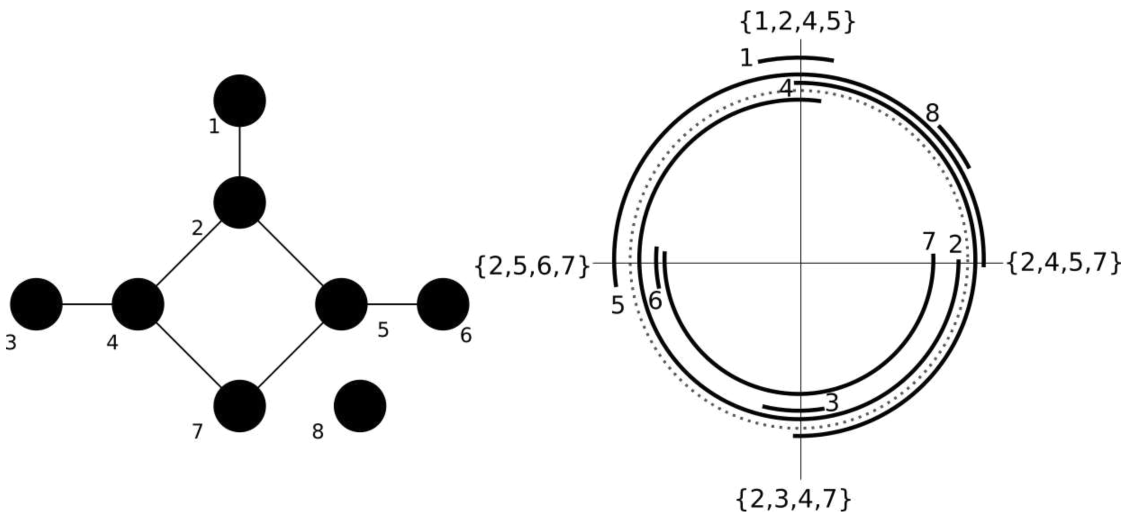

Figure 3.21: Graph $X_2{}^*$ alongside its Helly bi-circular-arc model.

Similarly to Theorem 11, Theorem 12 shows us that non-bichordal Helly CA bigraphs are a subclass of proper CA bigraphs. It is easy to note, however, that not all proper CA bigraphs are Helly CA bigraphs, as $L_3 \cup P_2$ is a proper interval bigraph. Therefore, non-bichordal Helly CA bigraphs are a proper subclass of both Helly CA bigraphs and proper CA bigraphs.

Furthermore, it is important to mention that *bichordal* Helly CA bigraphs are not a subclass of proper CA bigraphs. The graph $X_2{}^*$ is not a proper CA bigraph [Safe, 2019], but it is a Helly CA bigraph, as seen in Figure 3.21. Therefore, proper CA bigraphs and Helly CA bigraphs are not comparable.

With Theorem 12, we conclude another important corollary for our characterization.

Corollary 5. *Non-bichordal Helly CA bigraphs are a proper subclass of proper-Helly CA bigraphs.*

Proof. Theorem 12 shows that every non-bichordal Helly CA bigraph admits a model that is simultaneously proper and Helly. The containment is proper because a P_2, for instance, is both proper and Helly, but not non-bichordal. □

Finally, we bring together the results of Corollary 4 and 5 to formulate the following characterization for L_3-free proper-Helly CA bigraphs, concluding the first main portion of the characterization of the general class.

Theorem 13. *Let G be an L_3-free bipartite graph. Then, the following three claims are equivalent.*

1. *G is a proper-Helly CA bigraph.*
2. *G does not contain T_2, X_2, nor $C_n{}^*, n > 4$ as induced subgraphs.*
3. *G is either a non-bichordal Helly CA bigraph or a Helly interval bigraph.*

Proof. ($1 \Rightarrow 2$) It follows from the fact that T_2, X_2, $C_n{}^*, n > 4$ are not proper-Helly CA bigraphs.
($2 \Rightarrow 3$) Suppose G does not contain T_2, X_2, nor $C_n{}^*, n > 4$ as induced subgraphs. If G is bichordal, that implies it does not contain $C_n, n > 4$ as an induced subgraph, implying it is a a Helly interval bigraph by Theorem 9. If G is non-bichordal, then the lack of the aforementioned forbidden graphs makes it a non-bichordal Helly CA bigraph by Corollary 2.
($3 \Rightarrow 1$) Follows from Corollaries 4 and 5. □

In the sequence, we show that the class of L_3-free proper-Helly CA bigraphs is equivalent to the class of *normal-proper-Helly* (NPH for short) CA bigraphs. A bipartite graph is an NPH CA bigraph if and only if it admits a Helly bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ such that $\mathbb{I}$ and $\mathbb{E}$ are both proper and normal (i.e. no pair of arcs covers the circle). We call such a model an NPH model.

Theorem 14. *A bipartite graph is an NPH CA bigraph if and only if it is L_3-free and a proper-Helly CA bigraph.*

Proof. $(\Leftarrow)$ as shown in Theorem 13, if G is an L_3-free proper-Helly CA bigraph, then it is either a Helly interval bigraph, or a non-bichordal Helly CA bigraph. If G is a Helly interval bigraph, then, by Theorem 11, it admits a bi-interval model that is both proper and Helly. It is easy to verify that a proper-Helly bi-interval model may be turned into a proper-Helly bi-circular-arc model that does not cover the circle, being, therefore, an NPH model.

If G is a non-bichordal Helly CA bigraph instead, then, by Theorem 12, it admits a bi-circular-arc model that is both proper and Helly.

Note that, in a Helly model of a graph G, if two arcs $a(v), a(w)$ (with $v, w \in V(G)$) cover the circle, then $b(v) \cup b(w) = b(G)$. The reason for that is simple: if $a(v), a(w)$ cover the circle, any set of biclique points $\{p_1, ..., p_n\}$ compatible with the model must be such that, for every i, $p_i \in a(v) \cup a(w)$.

Therefore, any Helly model of a non-bichordal Helly CA bigraph $G = (X, Y, E)$ is necessarily normal, since no two vertices $v, w \in X$ $(v, w \in Y)$ in a non-bichordal Helly CA bigraph are such that $b(v) \cup b(w) = b(G)$.

$(\Rightarrow)$ It suffices to show that the L_3 does not admit an NPH model. Consider the labeling of the vertices of an L_3 according to Figure 3.1.

The partite sets are $X = \{1, 4, 5\}, Y = \{2, 3, 6\}$, and the bicliques are $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6\}, C = \{2, 3, 4, 6\}, D = \{1, 3, 4, 5\}$. Let $\{p_A, p_B, p_C, p_D\}$ be a valid set of biclique points around distributed around a circle C. Since $b(1) = \{A, D\}$, $b(5) = \{B, D\}$, and $b(2) = \{A, C\}$, we have it that the biclique points must be distributed around the circle in the circular order (p_A, p_D, p_B, p_C), either clockwise or counter-clockwise. Suppose the former w.l.o.g.

Suppose there is an NPH bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ of the L_3 compatible with that set of biclique points, and consider the arcs $a(1), a(4), a(5)$. Note that $a(1) \subset (p_C, p_B)$ and $a(5) \subset (p_A, p_C)$. Arc $a(4)$, in turn, must cross every biclique point, implying both its s and t endpoints must be in the same gap between two consecutive biclique points.

If the endpoints of $a(4)$ are both in (p_A, p_D), then $a(1), a(4)$ cover the circle, and if they are in (p_D, p_B), $a(4), a(5)$ cover the circle. If they are in (p_B, p_C), $a(4)$ contains $a(1)$, and if they are in (p_C, p_A), $a(4)$ contains $a(5)$.

Therefore, it is impossible to create an NPH model for the L_3. □

Theorem 13 characterizes L_3-free proper-Helly CA bigraphs, and Theorem 14 shows that they are equivalent to NPH CA bigraphs. Therefore, to conclude the characterization of proper-Helly CA bigraphs, all that is left is to consider the graphs that admit an induced L_3.

One forbidden graph for proper-Helly CA bigraphs that admit an induced L_3 is the graph in Figure 3.22.

Lemma 19. *The graph X_{20} is not a proper-Helly CA bigraph.*

Proof. Suppose it is possible to construct a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ of X_{20} that is both proper and Helly. Said model is compatible with some set of biclique points distributed around C. The partite sets are $X = \{1, 3, 5, 7\}, Y = \{2, 4, 6\}$, and the bicliques of the graph are $A = \{1, 2, 3, 5\}, B = \{2, 3, 5, 6\}, C = \{2, 3, 4, 6\}, D = \{3, 4, 6, 7\}, E = \{3, 5, 6, 7\}$,

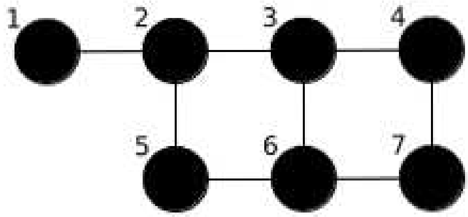

Figure 3.22: The graph X_{20}, a forbidden graph for proper-Helly CA bigraphs.

Let p_A, p_B, p_C, p_D, p_E be the points corresponding to each biclique in a valid set of biclique points. Consider the order in which these points must be distributed around the circle. p_C, p_D must be consecutive, as $b(4) = \{C, D\}$, and p_D, p_E must also be consecutive, as $b(7) = \{D, E\}$. Therefore, (p_C, p_D, p_E) must appear consecutively in this order, either clockwise or counter-clockwise. We assume the former. Therefore, the full clockwise order of points must be either $(p_C, p_D, p_E, p_A, p_B)$ or $(p_C, p_D, p_E, p_B, p_A)$. If the model is proper-Helly, then a set of biclique points distributed in the order $(p_C, p_D, p_E, p_A, p_B)$ is not compatible with it, since it would imply $a(1) \subset (p_E, p_B) \subset a(5)$. Therefore, the set of biclique points follows order $(p_C, p_D, p_E, p_B, p_A)$.

Note that vertex 3 is bi-universal, and that $a(1) \subset (p_B, p_C)$. Since 3 is bi-universal, the arc $a(3)$ must touch all five biclique points, implying it must have both its endpoints occur in the same gap between consecutive biclique points. If they do not occur in (p_B, p_A) nor in (p_A, p_C), then $a(1) \subset a(3)$. However, note that $a(7) \subset (p_C, p_B)$, implying that, if $a(3)$ has both its endpoints in (p_B, p_A) or in (p_A, p_C), $a(7) \subset a(3)$.

Therefore, every Helly model of X_{20} is not proper. □

The forbidden graphs we need for the characterization of proper-Helly CA bigraphs with an induced L_3 are the X_2, X_{20} and $L_3{}^*$ we previously mentioned, as well as the T_2 and BW_3 from Figure 3.1.

Recall that, in our characterization of Helly interval bigraphs, we defined the class of HIBFF graphs. In a similar way, we now define the class of *Proper-Helly Forbidden-Free* (PHFF for short) graphs as bipartite graphs that do not contain $T_2, X_2, BW_3, X_{20}, L_3{}^*$ as induced subgraphs.

To simplify notation in the results we present in the sequence, we say that an L_3 with vertex set $\{1, 2, 3, 4, 5, 6\}$ is *canonically labeled* if its edge set is $\{12, 13, 34, 24, 35, 46, 56\}$. Note that the L_3 in Figure 3.12 is canonically labeled. If a set $\{1, 2, 3, 4, 5, 6\}$ induces a canonically labeled L_3, we say that the set itself is canonically labeled.

The following lemma is particularly useful.

Lemma 20. *Let G be a twin-free PHFF graph with an induced L_3 in set L that is canonically labeled. Then, for every vertex $v \in V(G) - L$, $N(v) \cap L$ is equal to one of the following:*

- $\{1, 4\}$,
- $\{4\}$,
- $\{4, 5\}$,
- $\{2, 3\}$,
- $\{3\}$,

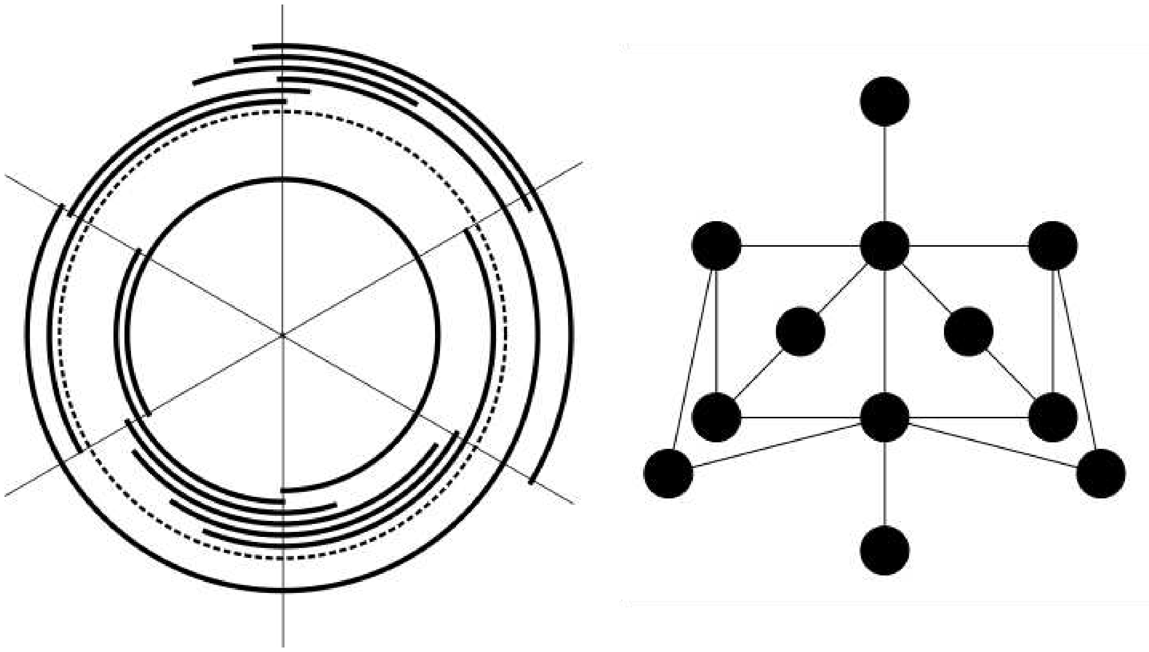

Figure 3.23: A proper-Helly CA bigraph with an induced L_3 where all sets $V_{14}, V_4, V_{45}, V_{23}, V_3, V_{36}$ have at least one vertex, but no adjacencies exist between vertices outside the main L_3. In the bi-circular-arc model, biclique points are indicated by line segments from the center of the circle.

- $\{3, 6\}$.

Proof. Since the graph is $L_3{}^*$-free, every vertex outside L is neighbor to at least one vertex of L.

Let $v \in V(G) - L$ be a vertex that has three neighbors in L. Suppose w.l.o.g. that $\{1, 4, 5\} \subset N(v)$. Since v must not be twin with 3, there must be at least one vertex $w \in N(v) - N(3)$ or $w \in N(3) - N(v)$. Suppose the former without loss of generality.

Vertex w must be neighbor to at least one vertex of L, but not 3. If it is neighbor to both 2 and 6, then a BW_3 is induced by $\{1, 2, 3, 4, 5, 6, w\}$. If it is neighbor to only one of them, say, 2, then an X_{20} is induced, once again, by $\{1, 2, 3, 4, 5, 6, w\}$. Therefore, every vertex outside L is neighbor to at most two vertices of L.

Suppose, then, that a vertex v is neighbor to two vertices of L. If $N(v) \cap L$ is $\{2, 6\}$ or $\{1, 5\}$, then a BW_3 is induced. Therefore, if v has two neighbors in L, then $N(v) \cap L$ is either $\{1, 4\}, \{4, 5\}, \{2, 3\}$ or $\{3, 6\}$.

Finally, if v is neighbor to exactly one vertex of L that is neither 4 nor 3, an X_{20} is induced. □

With the adjacencies between vertices from inside and outside the L_3 described in Lemma 20, we may partition, for every twin-free PHFF graph G with an induced L_3, the vertices of $V(G) - L$ in six sets $V_{14}, V_4, V_{45}, V_{23}, V_3, V_{36}$ for vertices that are neighbors to $\{1, 4\}$, $\{4\}$, $\{4, 5\}$, $\{2, 3\}$, $\{3\}$, and $\{3, 6\}$ respectively. Call these sets the *V_i-sets*.

It is important to note that a graph where every V_i-set has at least one vertex, and no adjacencies exist between pairs of vertices outside the L_3, is a proper-Helly CA bigraph (Figure 3.23). In the sequence, we analyze the situations where adjacencies between vertices from outside the L_3 exist.

Note that, with set L and sets $V_{14}, V_4, V_{45}, V_{23}, V_3, V_{36}$ as defined, sets V_{14}, V_4, V_{45} are in one partite set, whereas sets V_{23}, V_3, V_{36} are in the other. In the sequence, we show results on which sets among V_{14}, V_4, V_{45} can have neighbors in sets V_{23}, V_3, V_{36}.

Lemma 21. *Let G be a PHFF graph with a canonically labeled induced L_3 in set L and the vertices in $V(G) - L$ being partitioned into the V_i-sets. Then no vertices of V_4 are neighbor to vertices of V_3.*

Proof. Let $v \in V_3$ and $w \in V_4$ such that $vw \in E(G)$. A T_2 is induced by $\{1, 2, 4, 5, 6, v, w\}$. □

Lemma 22. *Let G be a PHFF graph with a canonically labeled induced L_3 in set L and the vertices in $V(G) - L$ being partitioned into the V_i-sets. Then no vertex of V_{14} is neighbor to vertices of V_{36}. Analogously, no vertex of V_{45} is neighbor to vertices of V_{23}.*

Proof. Let $v \in V_{14}, w \in V_{36}$ such that $vw \in E(G)$. An X_{20} is induced by $\{1, 2, 4, 5, 6, v, w\}$. □

Lemma 23. *Let G be a PHFF graph with a canonically labeled induced L_3 in set L and the vertices in $V(G) - L$ being partitioned into the V_i-sets. If $v \in V_4$, $w \in V_{23}$ and $x \in V_{36}$, then $vw \notin E(G)$ or $vx \notin E(G)$. Analogously, if $v \in V_3, w \in V_{14}, x \in V_{45}$, then $vw \notin E(G)$ or $vx \notin E(G)$.*

Proof. If $vw, vx \in E(G)$, then an X_{20} is induced by $\{v, w, x, 2, 4, 5, 6\}$. □

Lemmas 21, 22 and 23 show that vertices in V_3 must have their neighbors outside the main L_3 restricted to either V_{14} or V_{45}, but not both, and that vertices in V_{14} have their neighbors outside the L_3 restricted to $V_3 \cup V_{23}$. Those results also apply to other V_i-sets, as described in the lemmas. Other forbidden adjacencies for proper-Helly CA bigraphs with an induced L_3 are more complex, and similar to the cases we encounter in our characterizations of non-bichordal Helly CA bigraphs and Helly interval bigraphs. We explore those prohibitions in the sequence.

Lemma 24. *Let G be a PHFF graph with an induced L_3 in canonically labeled set L and the vertices in $V(G) - L$ being partitioned in the V_i-sets, and let $v \in V_4$ and $w \in V_{23}$. Then, if $vw \in E(G)$, w must be neighbor to every vertex in V_{14}.*

Proof. Suppose $vw \in E(G)$ and let $x \in V_{14}$ such that $x \notin N(w)$. Then a T_2 is induced by $\{1, 4, 5, 6, v, w, x\}$. □

Note that Lemma 24 applies analogously to the relationship between V_4 and V_{36}, as well as the relationship between V_3 and V_{14}, and that between V_3 and V_{45}. With the lemmas we laid out, we may now further partition the sets $V_{23}, V_3, V_{36}, V_{14}, V_4, V_{45}$ of any twin-free PHFF graph into finer sets. For V_3, we may partition it into V_3', $V_{3,14}$, and $V_{3,45}$. The first one being for vertices that are neighbors only to 3, the second for the ones that have neighbors in V_{14}, and the third for the ones that have neighbors in V_{45}. Analogously, we may partition V_4 into the sets $V_4', V_{4,23}, V_{4,36}$.

We may also partition the vertices of V_{23} into sets $V_{23}', V_{23,14}, V_{23,4}$, defined as follows: $v \in V_{23}'$ if and only if $N(v) = \{2, 3\} \cup \{w \in V_{14} | N(w) \cap V_3 \neq \emptyset\}$ (i.e. the only neighbors of v are $2, 3$, plus every element of V_{14} that has at least one neighbor in V_3, which Lemma 24 shows to be necessary); $v \in V_{23,14}$ if and only if $\{2, 3\} \cup \{w \in V_{14} | N(w) \cap V_3 \neq \emptyset\} \subset N(v)$, $N(v) \cap \{w \in V_{14} | N(w) \cap V_3 = \emptyset\} \neq \emptyset$ and $N(v) \cap V_4 = \emptyset$ (i.e. v is neighbor to 2, 3, every element of V_{14} that has at least one neighbor in V_3, plus at least one element of V_{14} that does not have neighbors in V_3, but no elements of V_4); and $v \in V_{23,4}$ if and only if $\{2, 3\} \cup V_{14} \subset N(v)$ and $N(v) \cap V_4 \neq \emptyset$ (i.e. v is neighbor to $\{2, 3\}$ plus at least one element of V_4 and, therefore, it must also be neighbor to all of V_{14} by Lemma 24).

Analogously to how we partitioned V_{23}, we may partition set V_{36} into $V_{36}', V_{36,45}, V_{36,4}$, set V_{14} into $V_{14}', V_{14,23}, V_{14,3}$, and set V_{45} into $V_{45}', V_{45,36}, V_{45,3}$. Note that, in a twin-free graph, $V_{23}', V_{14}', V_{36}', V_{45}'$ only contain one vertex.

For the remainder of this subsection, we call the sets we just defined, regardless of whether they are of the form V_i' or $V_{i,j}$, the *V-sets*. The following corollary is a direct consequence of Lemmas 21 through 24.

Corollary 6. *Let G be a twin-free PHFF graph with an induced L_3 in canonically labeled set L. Then every vertex in $V(G) - L$ belongs to a V-set.*

In the sequence, we show some more properties of proper-Helly CA bigraphs with an induced L_3 leading up to the definition of a fundamental structure for proper-Helly CA bigraphs with an induced L_3.

Lemma 25. *Let G be a twin-free proper-Helly CA bigraph with an L_3 in canonically labeled set L and the V-sets as defined previously. Then two vertices $v, w \in V_{23,14}$ must be such that $N(v), N(w)$ are comparable. Analogously, any two vertices from the same set set among $V_{14,23}, V_{36,45}$, or $V_{45,36}$ must also have comparable neighborhoods.*

Proof. Suppose v and w have non-comparable neighborhoods. Then there exists a vertex $v' \in N(v) - N(w)$ and a vertex $w' \in N(w) - N(v)$. Since $v, w \in V_{23,14}$, both v' and w' are in V_{14}. Then, a T_2 is induced with $\{v, v', w, w', 3, 5, 6\}$. □

Lemma 26. *Let G be a twin-free proper-Helly CA bigraph with an L_3 in canonically labeled set L and the V-sets as defined previously. Then two vertices $v, w \in V_{3,14}$ must be such that $N(v), N(w)$ are comparable. Analogously, any two vertices from the same set set among $V_{4,23}, V_{3,45}, V_{4,36}$ must also have comparable neighborhoods.*

Proof. Suppose v and w have non-comparable neighborhoods. Then there exists a vertex $v' \in N(v) - N(w)$ and a vertex $w' \in N(w) - N(v)$. Since $v, w \in V_{3,14}$, both v' and w' are in V_{14}. Then, a T_2 is induced with $\{v, v', w, w', 3, 5, 6\}$. □

Corollary 7. *Let G be a twin-free proper-Helly CA bigraph with an L_3 in canonically labeled set L and the V-sets as defined previously. Then two vertices $v, w \in V_{14,3}$ must be such that $N(v), N(w)$ are comparable. Analogously, any two vertices from the same set set among $V_{23,4}, V_{45,3}, V_{36,4}$ must also have comparable neighborhoods.*

Proof. Analogous to the proof of Lemma 26. □

With Lemmas 25 and 26, we now have all we need to formulate a fundamental structure for proper-Helly CA bigraphs with an induced L_3. The structure follows a similar logic to the previously defined FCS and FIS graphs.

Definition 5. *Let a* Fundamental Proper-Helly Structure *(FPHS for short) graph, denoted by $\mathbb{PH}(n_1, n_2, n_3, n_4, n_5, n_6)$ with some $n_1, \ldots, n_6 \geq 0$, be a graph defined as follows.*

The vertex set of the graph is partitioned into the following sets:

- $L = \{1, 2, 3, 4, 5, 6\}$.
- $V' = \{v_{23}, v_{36}, v_{14}, v_{45}, v_3, v_4\}$.
- $V_{14,23} = \{w_{1,1}, \ldots, w_{1,n_1}\}$.
- $V_{23,14} = \{x_{1,1}, \ldots, x_{1,n_1}\}$.
- $V_{36,45} = \{w_{2,1}, \ldots, w_{2,n_2}\}$.
- $V_{45,36} = \{x_{2,1}, \ldots, x_{2,n_2}\}$.
- $V_{3,14} = \{y_{3,1}, \ldots, y_{3,n_3}\}$.

- $V_{14,3} = \{z_{3,1}, ..., z_{3,n_3}\}$.
- $V_{3,45} = \{y_{4,1}, ..., y_{4,n_4}\}$.
- $V_{45,3} = \{z_{4,1}, ..., z_{4,n_4}\}$.
- $V_{4,23} = \{y_{5,1}, ..., y_{5,n_5}\}$.
- $V_{23,4} = \{z_{5,1}, ..., z_{5,n_5}\}$.
- $V_{4,36} = \{y_{6,1}, ..., y_{6,n_6}\}$.
- $V_{36,4} = \{z_{6,1}, ..., z_{6,n_6}\}$.

With the neighborhoods of each vertex being defined as follows:

- $N(1) = \{2, 3, v_{14}\} \cup V_{14,23} \cup V_{14,3}$.
- $N(2) = \{1, 4, v_{23}\} \cup V_{23,14} \cup V_{23,4}$.
- $N(3) = \{1, 4, 5, v_3, v_{23}, v_{36}\} \cup V_{23,14} \cup V_{23,4} \cup V_{36,14} \cup V_{36,4} \cup V_{3,45} \cup V_{3,14}$.
- $N(4) = \{2, 3, 6, v_4, v_{14}, v_{45}\} \cup V_{14,23} \cup V_{45,36} \cup V_{14,3} \cup V_{45,3} \cup V_{4,23} \cup V_{4,36}$.
- $N(5) = \{3, 6, v_{45}\} \cup V_{45,36} \cup V_{45,3}$.
- $N(6) = \{4, 5, v_{36}\} \cup V_{36,45} \cup V_{36,4}$.
- $N(v_{23}) = \{2, 3\} \cup V_{14,3}$.
- $N(v_{36}) = \{3, 6\} \cup V_{45,3}$.
- $N(v_{14}) = \{1, 4\} \cup V_{23,4}$.
- $N(v_{45}) = \{4, 5\} \cup V_{36,4}$.
- $N(v_3) = \{3\}$.
- $N(v_4) = \{4\}$.
- $N(w_{1,i}) = \{1, 4\} \cup \{x_{1,1}, ..., x_{1,i}\} \cup V_{23,4}$ *for all* $1 \le i \le n_1$.
- $N(x_{1,i}) = \{2, 3\} \cup \{w_{1,i}, ..., w_{1,n_1}\} \cup V_{14,3}$ *for all* $1 \le i \le n_1$.
- $N(w_{2,i}) = \{3, 6\} \cup \{x_{2,1}, ..., x_{2,i}\} \cup V_{45,3}$ *for all* $1 \le i \le n_2$.
- $N(x_{2,i}) = \{4, 5\} \cup \{w_{2,i}, ..., w_{2,n_2}\} \cup V_{36,4}$ *for all* $1 \le i \le n_2$.
- $N(y_{3,i}) = \{3\} \cup \{z_{3,1}, ..., z_{3,i}\}$ *for all* $1 \le i \le n_3$.
- $N(z_{3,i}) = \{1, 4, v_{23}\} \cup \{y_{3,i}, ..., y_{3,n_3}\} \cup V_{23,14} \cup V_{23,4}$ *for all* $1 \le i \le n_3$.
- $N(y_{4,i}) = \{3\} \cup \{z_{4,1}, ..., z_{4,i}\}$ *for all* $1 \le i \le n_4$.
- $N(z_{4,i}) = \{4, 5, v_{36}\} \cup \{y_{4,i}, ..., y_{4,n_4}\} \cup V_{36,45} \cup V_{36,4}$ *for all* $1 \le i \le n_4$.
- $N(y_{5,i}) = \{4\} \cup \{z_{5,1}, ..., z_{5,i}\}$ *for all* $1 \le i \le n_5$.

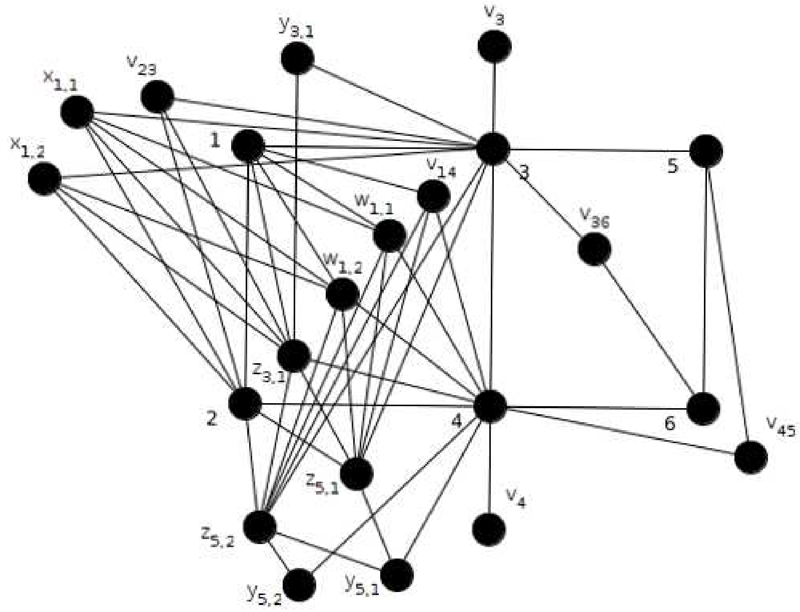

Figure 3.24: Graph $\mathbb{PH}(2, 0, 1, 0, 2, 0)$.

- $N(z_{5,i}) = \{2, 3, v_{14}\} \cup \{y_{5,i}, ..., y_{5,n_5}\} \cup V_{14,23} \cup V_{14,3}$ *for all* $1 \leq i \leq n_5$.
- $N(y_{6,i}) = \{4\} \cup \{z_{6,1}, ..., z_{6,i}\}$ *for all* $1 \leq i \leq n_6$.
- $N(z_{6,i}) = \{3, 6, v_{45}\} \cup \{y_{6,i}, ..., y_{6,n_6}\} \cup V_{45,36} \cup V_{45,3}$ *for all* $1 \leq i \leq n_6$.

Figure 3.24 has an example of an FPHS graph. Similarly to the case with FCS and FIS graphs, we need to prove that FPHS graphs are all proper-Helly CA bigraphs, and that every twin-free PHFF graph with an induced L_3 is an induced subgraph of an FPHS graph. The latter is done in Lemma 27, whereas the former is in Theorem 15. The proof of Theorem 15 is more laborious than that of Theorem 1 or 8 due to the fact that it is harder to generalize sets of vertices on a graph centered around an L_3 than it is for graphs centered around cycles or paths. In essence, however, the proof shares many similarities with that of those other two theorems.

Theorem 15. *Every FPHS graph is a proper-Helly CA bigraph.*

Proof. The proof is separated into three sections. In the first one, we must enumerate the bicliques an FPHS graph contains. In the second, we show that it is possible to find a valid set of biclique points for the graph. Finally, in the third part of the proof, we must show that it is possible to construct a Helly bi-circular-arc model of the graph such that, for any two vertices v, w in the same partite set whose neighborhoods are comparable, $a(v), a(w)$ share either their s or their t endpoint, which makes it possible to change the model into a proper-Helly model by shifting the endpoints of comparable arcs without losing any intersection with biclique points.

Let $G = \mathbb{PH}(n_1, n_2, n_3, n_4, n_5, n_6)$ for some $n_1, ..., n_6 > 0$. We may assume $n_1, ..., n_6 > 0$ as cases where one of those is 0 are induced subgraphs of a case where none is. The partite sets are:

- $X = \{1, 4, 5\} \cup \{v_3, v_{23}, v_{36}\} \cup V_{23,14} \cup V_{36,45} \cup V_{3,14} \cup V_{3,45} \cup V_{23,4} \cup V_{36,4}$.
- $Y = \{2, 3, 6\} \cup \{v_4, v_{14}, v_{45}\} \cup V_{14,23} \cup V_{45,36} \cup V_{14,3} \cup V_{45,3} \cup V_{4,23} \cup V_{4,36}$.

In the sequence, we show that every biclique in G is of one of the following forms:

- $A_3 = \{1,3,4,5,v_3,v_{23},v_{36}\} \cup V_{23,14} \cup V_{23,4} \cup V_{36,45} \cup V_{36,4} \cup V_{3,45} \cup V_{3,14} = N[3]$.
- $A_4 = \{2,3,4,6,v_4,v_{14},v_{45}\} \cup V_{14,23} \cup V_{45,3} \cup V_{45,36} \cup V_{14,3} \cup V_{4,23} \cup V_{4,36} = N[4]$.
- $B'_{14,23} = \{1,2,3,4,v_{14}\} \cup V_{14,23} \cup V_{14,3} \cup V_{23,4}$.
- $B''_{14,23} = \{1,2,3,4,v_{23}\} \cup V_{23,14} \cup V_{23,4} \cup V_{14,3}$.
- $B^i_{14,23} = \{1,2,3,4\} \cup V_{14,3} \cup \{w_{1,i},\dots,w_{1,n_1}\} \cup V_{23,4} \cup \{x_{1,1},\dots,x_{1,i}\}$ for all $1 \le i \le n_1$.
- $B'_{36,45} = \{3,4,5,6,v_{36}\} \cup V_{36,45} \cup V_{36,4} \cup V_{45,3}$.
- $B''_{36,45} = \{3,4,5,6,v_{45}\} \cup V_{45,36} \cup V_{45,3} \cup V_{36,4}$.
- $B^i_{36,45} = \{3,4,5,6\} \cup V_{45,3} \cup \{w_{2,i},\dots,w_{2,n_2}\} \cup V_{36,4} \cup \{x_{2,1},\dots,x_{2,i}\}$ for all $1 \le i \le n_2$.
- $C^i_{3,14} = \{1,3,4,v_{23}\} \cup \{y_{3,i},\dots,y_{3,n_3}\} \cup \{z_{3,1},\dots,z_{3,i}\} \cup V_{23,4} \cup V_{23,14}$ for all $1 \le i \le n_3$.
- $C^i_{3,45} = \{3,4,5,v_{36}\} \cup \{y_{4,i},\dots,y_{4,n_4}\} \cup \{z_{4,1},\dots,z_{4,i}\} \cup V_{36,4} \cup V_{36,45}$ for all $1 \le i \le n_4$.
- $C^i_{4,23} = \{2,3,4,v_{14}\} \cup \{y_{5,i},\dots,y_{5,n_5}\} \cup \{z_{5,1},\dots,z_{5,i}\} \cup V_{14,3} \cup V_{14,23}$ for all $1 \le i \le n_5$.
- $C^i_{4,36} = \{3,4,6,v_{45}\} \cup \{y_{6,i},\dots,y_{6,n_6}\} \cup \{z_{6,1},\dots,z_{6,i}\} \cup V_{45,3} \cup V_{45,36}$ for all $1 \le i \le n_6$.

First, we prove that every biclique must contain at least three vertices of L. Note that $3,4$ must be in every biclique, as they are bi-universal vertices. Suppose a certain biclique B does not contain any other element of L. Then it follows $v_3 \notin B$, as 3 is the only vertex of Y that is neighbor to both 4 and v_3, implying that $B = N[3]$, but $N[3]$ contains other vertices of L. Analogously, $v_4 \notin B$. Also, B must contain elements of at most one set among $V_{3,14}, V_{3,45}, V_{4,23}, V_{4,36}$, as elements of $V_{3,14}, V_{3,45}$ are not neighbors to elements of $V_{4,23}, V_{4,36}$, and the only vertex that is neighbor to elements of both $V_{3,14}$ and $V_{3,45}$ is 3, and the only vertex that is neighbor to elements of both $V_{4,23}$ and $V_{4,36}$ is 4. Suppose, without loss of generality, that B contains elements of $V_{3,14}$.

Note that the only vertices of Y that are neighbors to both 4 and vertices of $V_{3,14}$ are 3 and vertices of $V_{14,3}$, implying $B \cap Y \subset V_{14,3} \cup \{3\}$. However, $V_{14,3} \cup \{3\} \subset N(1)$, implying B is not a biclique unless it includes 1, leading to a contradiction. Suppose, then, that B does not contain any element of $V_{3,14} \cup V_{3,45} \cup V_{4,23} \cup V_{4,36}$. Recall the definitions of the V_i-sets, which, for the purposes of this proof, are $V_{14} = \{v_{14}\} \cup V_{14,23} \cup V_{14,3}$, $V_{45} = \{v_{45}\} \cup V_{45,36} \cup V_{45,3}$, $V_{23} = \{v_{23}\} \cup V_{23,14} \cup V_{23,4}$, and $V_{36} = \{v_{36}\} \cup V_{36,45} \cup V_{36,4}$.

Note that if $v \in V_{23}$ and $w \in V_{36}$, then $v \notin B$ or $w \notin B$, as the only vertex that is neighbor to both vertices of V_{23} and V_{36} is 3. Therefore, B contains either vertices of V_{23}, or vertices of V_{36}, or neither.

Suppose without loss of generality that B contains vertices of V_{23}. Note that the only vertices that are neighbors to both 4 and elements of V_{23} in G are elements of V_{14}, $V_{4,23}$, and $\{2,3\}$, but since B does not contain 2 nor any elements of $V_{4,23}$, that implies $B \cap Y \subseteq V_{14} \cup \{3\}$, which, once again, implies 1 is neighbor to every element of $B \cap Y$, leading to a contradiction.

Therefore, B does not contain elements of V_{23}. Analogously, B must also not contain elements of V_{36}, V_{14} and V_{45}. Therefore, B cannot be a biclique, because $\{3,4\}$ by itself is not a biclique, and, as we have shown, no other vertices of G may be in B. With that, we conclude that every biclique must contain at least three vertices of L.

It is easy to note that 1 and 6 cannot be in a biclique together, neither can 2 and 5, implying every biclique B in the graph falls into one of three situations:

1. $B \cap L$ contains vertices $1, 3, 4, 5$, implying $B = N[3]$, or $2, 3, 4, 6$, implying $B = N[4]$.

2. $B \cap L$ forms a C_4 in the graph (either $1, 2, 3, 4$ or $3, 4, 5, 6$).

3. $B \cap L$ contains three vertices, two of which are 3 and 4.

We must now show that, regardless of which of the three cases a biclique falls in, it is always going to be one of the bicliques on the list. Case 1 invariably leads to bicliques A_3 and A_4. Consider, then, case 2. Suppose without loss of generality that $B \cap L = \{1, 2, 3, 4\}$. Then every element of $B - L$ is in V_{23} or V_{14}. Note that $V_{23,4} \cup V_{14,3} \subset B$, as vertices of $V_{23,4}$ are neighbors to all of V_{14} and vertices $V_{14,3}$ are neighbors to all of V_{23}.

Consider vertices v_{14} and v_{23}. Suppose, first, that $v_{14} \in B$. Then, we have that v_{23} is not in B, nor is any element of $V_{23,14}$, implying $B \cap X = \{1, 4\} \cup V_{23,4}$ and, therefore, $B = \{1, 2, 3, 4, v_{14}\} \cup V_{14,23} \cup V_{14,3} \cup V_{23,4} = B'_{14,23}$. The case where v_{23} is in B is analogous and leads to biclique $B''_{14,23}$.

Suppose, then, that neither v_{23} nor v_{14} are in B. In this case, the graph must have vertices from $V_{23,14}$ and $V_{14,23}$. Let $0 < i < n_1$ be the lowest value of i for which $w_{1,i}$ (from $V_{14,23}$) is in B. Then we have it that $\{w_{1,i}, ..., w_{1,n_1}\} \subset B$, implying $B \cap Y = \{2, 3\} \cup V_{14,3} \cup \{w_{1,i}, ..., w_{1,n_1}\}$, which, in turn, implies $B \cap X = \{1, 4\} \cup V_{23,4} \cup \{x_{1,1}, ..., x_{1,i}\}$, and therefore, B is biclique $B^i_{14,23}$ in the list.

With that, case 2 is concluded, as the case where $B \cap L = \{3, 4, 5, 6\}$ is analogous, and leads to bicliques $B'_{36,45}$, $B''_{36,45}$ and $B^i_{36,45}$ in the list. We now consider case 3.

Without loss of generality, suppose $B \cap L = \{1, 3, 4\}$. Note that $B \cap Y \subset \{3\} \cap V_{14}$, and, therefore, $B \cap X \subset \{1, 4\} \cup V_{23} \cup V_{3,14}$, as those are the only elements of X that are neighbors to $\{3\}$ and at least one element of V_{14}. Also, at least one vertex of $V_{3,14}$ must be in B, as otherwise, B is a subset of a biclique from case 2. Therefore, v_{14} and all of $V_{14,23}$ are not in B, since they have no neighbors in $V_{3,14}$, meaning $B \cap V_{14} \subseteq V_{14,3}$. That also implies that $V_{23} \subset B$, as every element of $V_{14,3}$ is neighbor to all elements of V_{23}.

Let $0 < i \leq n_3$ be the lowest value for which $y_{3,i}$ (from $V_{3,14}$) is in B. Then $\{y_{3,i}, ..., y_{3,n_3}\} \subset B$, implying $B \cap X = \{1, 4, v_{23}, y_{3,i}, ..., y_{3,n_3}\} \cup V_{23,4} \cup V_{23,14}$. Note that this implies $B \cap Y = \{3, z_{3,1}, ..., z_{3,i}\}$, and therefore, B is the biclique $C^i_{3,14}$ on the list.

With that, we conclude our proof that the list of bicliques is complete, as every other case where B contains exactly three vertices of L is analogous to the one we showed, yielding bicliques $C^i_{3,45}$, $C^i_{4,23}$ and $C^i_{4,36}$. In the sequence, we show that it is possible to attribute a valid set of biclique points to the bicliques of G as in Lemma 1, and then show that the order of the biclique points can be used to build a Helly bi-circular-arc model where two comparable arcs of the same partite set necessarily share an endpoint, thus concluding the proof.

For each biclique as defined on the list, attribute a point in circle C denoted by the biclique's name turned into lower case (e.g. the point for biclique A_3 is denoted by a_3, and the point for biclique $C^i_{4,36}$ is denoted by $c^i_{4,36}$.) Suppose the points are distributed around the circle in the following clockwise order:

$$(b''_{36,45}, b^{n_2}_{36,45}, \ldots, b^1_{36,45}, b'_{36,45}, c^{n_4}_{3,45}, \ldots, c^1_{3,45}, a_3, c^1_{3,14}, \ldots, c^{n_3}_{3,14},$$
$$b''_{14,23}, b^{n_1}_{14,23}, \ldots, b^1_{14,23}, b'_{14,23}, c^{n_5}_{4,23}, \ldots, c^1_{4,23}, a_4, c^1_{4,36}, \ldots, c^{n_6}_{4,36})$$

For clarity's sake, the sequence of biclique points is constructed as follows:

1. Add $b''_{36,45}$.

2. Add points of the form $b^i_{36,45}$ from $i = n_2$ to 1 in decreasing order.

3. Add $b'_{36,45}$.
4. Add points of the form $c^i_{3,45}$ from $i = n_4$ to 1 in decreasing order.
5. Add a_3.
6. Add points of the form $c^i_{3,14}$ from $i = 1$ to n_3 in increasing order.
7. Add $b''_{14,23}$.
8. Add points of the form $b^i_{14,23}$ from $i = n_1$ to 1 in decreasing order.
9. Add $b'_{14,23}$.
10. Add points of the form $c^i_{4,23}$ from $i = n_5$ to 1 in decreasing order.
11. Add a_4.
12. Add points of the form $c^i_{4,36}$ from $i = 1$ to n_6 in increasing order.

Suppose that the biclique points are spread around the circle with a distance of 1 between consecutive points. In order to prove that a set of points distributed in that clockwise order is a valid set of biclique points, consider the bicliques to which each vertex belongs:

- $b(1) = \{A_3, B'_{14,23}, B''_{14,23}\} \cup \{B^i_{14,23} | 1 \leq i \leq n_1\} \cup \{C^i_{3,14} | 1 \leq i \leq n_3\}$.
- $b(2) = \{A_4, B'_{14,23}, B''_{14,23}\} \cup \{B^i_{14,23} | 1 \leq i \leq n_1\} \cup \{C^i_{4,23} | 1 \leq i \leq n_5\}$.
- $b(3) = b(4) = b(G)$.
- $b(5) = \{A_3, B'_{36,45}, B''_{36,45}\} \cup \{B^i_{36,45} | 1 \leq i \leq n_2\} \cup \{C^i_{3,45} | 1 \leq i \leq n_4\}$.
- $b(6) = \{A_4, B'_{36,45}, B''_{36,45}\} \cup \{B^i_{36,45} | 1 \leq i \leq n_2\} \cup \{C^i_{4,36} | 1 \leq i \leq n_6\}$.
- $b(v_{14}) = \{A_4, B'_{14,23}\} \cup \{C^i_{4,23} | 1 \leq i \leq n_3\}$.
- $b(v_{23}) = \{A_3, B''_{14,23}\} \cup \{C^i_{3,14} | 1 \leq i \leq n_5\}$.
- $b(v_{36}) = \{A_3, B'_{36,45}\} \cup \{C^i_{3,45} | 1 \leq i \leq n_4\}$.
- $b(v_{45}) = \{A_4, B''_{36,45}\} \cup \{C^i_{4,36} | 1 \leq i \leq n_6\}$.
- $b(v_3) = \{A_3\}$ and $b(v_4) = \{A_4\}$.
- For $1 \leq i \leq n_1$, $b(w_{1,i}) = \{A_4, B'_{14,23}\} \cup \{C^j_{4,23} | 1 \leq j \leq n_5\} \cup \{B^j_{14,23} | 1 \leq j \leq i\}$.
- For $1 \leq i \leq n_1$, $b(x_{1,i}) = \{A_3, B''_{14,23}\} \cup \{C^j_{3,14} | 1 \leq j \leq n_3\} \cup \{B^j_{14,23} | i \leq j \leq n_1\}$.
- For $1 \leq i \leq n_2$, $b(w_{2,i}) = \{A_3, B'_{36,45}\} \cup \{C^j_{3,45} | 1 \leq j \leq n_4\} \cup \{B^j_{36,45} | 1 \leq j \leq i\}$.
- For $1 \leq i \leq n_2$, $b(x_{2,i}) = \{A_4, B''_{36,45}\} \cup \{C^j_{4,36} | 1 \leq j \leq n_6\} \cup \{B^j_{36,45} | i \leq j \leq n_2\}$.
- For $1 \leq i \leq n_3$, $b(y_{3,i}) = \{A_3\} \cup \{C^j_{3,14} | 1 \leq j \leq i\}$.

- For $1 \leq i \leq n_3$, $b(z_{3,i}) = \{A_4, B'_{14,23}, B''_{14,23}\} \cup \{B^j_{14,23} | 1 \leq j \leq n_1\} \cup$
$\cup \{C^j_{4,23} | 1 \leq j \leq n_5\} \cup \{C^j_{3,14} | i \leq j \leq n_3\}$.

- For $1 \leq i \leq n_4$, $b(y_{4,i}) = \{A_3\} \cup \{C^j_{3,45} | 1 \leq j \leq i\}$.

- For $1 \leq i \leq n_4$, $b(z_{4,i}) = \{A_4, B'_{36,45}, B''_{36,45}\} \cup \{B^j_{36,45} | 1 \leq j \leq n_2\} \cup$
$\cup \{C^j_{4,36} | 1 \leq j \leq n_6\} \cup \{C^j_{3,45} | i \leq j \leq n_4\}$.

- For $1 \leq i \leq n_5$, $b(y_{5,i}) = \{A_4\} \cup \{C^j_{4,23} | 1 \leq j \leq i\}$.

- For $1 \leq i \leq n_5$, $b(z_{5,i}) = \{A_3, B'_{14,23}, B''_{14,23}\} \cup \{B^j_{14,23} | 1 \leq j \leq n_1\} \cup$
$\cup \{C^j_{3,14} | 1 \leq j \leq n_3\} \cup \{C^j_{4,23} | i \leq j \leq n_5\}$.

- For $1 \leq i \leq n_6$, $b(y_{6,i}) = \{A_4\} \cup \{C^j_{4,36} | 1 \leq j \leq i\}$.

- For $1 \leq i \leq n_6$, $b(z_{6,i}) = \{A_3, B'_{36,45}, B''_{36,45}\} \cup \{B^j_{36,45} | 1 \leq j \leq n_2\} \cup$
$\cup \{C^j_{3,45} | 1 \leq j \leq n_4\} \cup \{C^j_{4,36} | i \leq j \leq n_6\}$.

Note that, for every family of bicliques cited on the list, the points corresponding to its elements are circularly consecutive in the order we presented. Therefore, a set of points distributed around the circle in that clockwise order is a valid set of biclique points.

For the proof that G is a proper-Helly CA bigraph, note that, in the set of biclique points we presented, every vertex in $X - \{4\}$ has its circularly consecutive interval of biclique points either begin or end in a_3, and every vertex in $Y - \{3\}$ has its interval either begin or end in a_4. For every $v \in V(G) - \{3, 4\}$ whose interval of biclique points starts at p and ends at q, create an arc $a(v) = (p - \frac{1}{2}, q + \frac{1}{2})$. Now, for 4, make an arc $a(4) = (a_3 - \frac{1}{2}, c^1_{3,45} + \frac{1}{2})$, and for 3, make an arc $a(3) = (a_4 - \frac{1}{2}, c^1_{4,23} + \frac{1}{2})$. Note that every arc $a(v)$ for $v \in X$ will either begin at $a_3 - \frac{1}{2}$ or end at $a_3 + \frac{1}{2}$, and every arc $a(v)$ for $v \in Y$ will either begin at $a_4 - \frac{1}{2}$ or end at $a_4 + \frac{1}{2}$.

We show that a bi-circular-arc model constructed in this manner is such that, for any pair of comparable arcs corresponding to vertices of the same partite set, they share either an s or a t endpoint. Suppose two arcs $a(v), a(w)$ for $v, w \in X - \{4\}$ are comparable. If both of them have their s-endpoint in $a_3 - \frac{1}{2}$ or their t-endpoint in $a_3 + \frac{1}{2}$, then they share an endpoint, so suppose that $a(v)$ ends in $a_3 + \frac{1}{2}$ and $a(w)$ starts in $a_3 - \frac{1}{2}$, and neither vertex is v_3, as $a(v_3) = (a_3 - \frac{1}{2}, a_3 + \frac{1}{2})$. Since v, w are not v_3, that implies they are both contained in at least two bicliques in the graph, and therefore, $c^1_{3,45} \in a(v)$ and $c^1_{3,14} \in a(w)$. Since neither of them are 4, neither of them are bi-universal, implying $c^1_{3,45} \notin a(w)$ and $c^1_{3,14} \notin a(v)$, implying they are not comparable, leading to a contradiction. The case for two vertices $v, w \in Y - \{3\}$ is analogous.

Consider, now, the relationship between $a(v)$, with $v \in X - \{4\}$ and $a(4)$. If $s(a(v)) = a_3 - \frac{1}{2}$, then $a(v)$ and $a(4)$ share an endpoint. If, however, $v \neq v_3$ and $t(a(v)) = a_3 + \frac{1}{2}$, then we have it that $(c^1_{3,45}, a_3) \subset a(v)$, implying $a(v)$ and $a(4)$ are not comparable. The case between $a(v), v \in Y - \{3\}$ and $a(3)$ is analogous.

Therefore, every pair of comparable arcs from the same partite set shares an endpoint. By using $\epsilon = \frac{1}{100|V(G)|}$, we may apply the method in Lemma 18 without losing any intersections with biclique points, therefore making the model simultaneously proper and Helly.

Therefore, G is a proper-Helly circular arc bigraph. $\square$

Lemma 27 is the last result we need to finish our characterization of proper-Helly CA bigraphs. Note that its proof has a lot in common with the proof of Lemma 6, with induction being used to prove that certain vertices with pairwise-comparable neighborhoods have certain adjacencies.

Lemma 27. *If a twin-free bipartite graph G that contains an induced L_3 is PHFF, then it is an induced subgraph of some FPHS graph.*

Proof. Let $L = \{1, 2, 3, 4, 5, 6\}$ be the canonically labeled induced L_3 of G. As per Corollary 6, it possible to partition $V(G) - L$ into the sets we call V-sets. It suffices to prove, therefore, that the neighborhoods for vertices within those sets are equal to those for the sets of the same name in the definition of FPHS graphs. Assume all V-sets are non-empty, since any case where one of the sets is empty is an induced subgraph of a case where none are.

First of all, since G is twin-free, then $V'_{23}, V'_{14}, V'_3, V'_4, V'_{36}$, and V'_{45} have only one vertex each. Call those vertices $V' = \{v_{23}, v_{36}, v_{14}, v_{45}, v_3, v_4\}$ as in the definition of FPHS graphs.

Let us consider, then, the adjacencies of the vertices of G, starting with the vertices in set L. Note that $N(1)$ contains $2, 3$ as in the definition of an L_3, and then it also contains all of $V'_{14}, V_{14,23}, V_{14,3}$, since those vertices are neighbors to 1 by definition. No other vertices are in $N(1)$, as every other V-set is composed of vertices that do not have 1 as their neighbor, implying $N(1) = \{2, 3, v_{14}\} \cup V_{14,23} \cup V_{14,3}$, the same as in the definition of FPHS graphs. The cases for 2, 5, and 6 are analogous.

Vertex 3 must be neighbor to $1, 4, 5$ in L, as well as all sets $V'_3, V_{23,14}, V_{23,4}, V_{36,14}, V_{36,4}, V_{3,45}$, and $V_{3,14}$. Once again, all other V-sets are defined as sets of vertices that do not contain 3, implying $N(3) = \{1, 4, 5, v_3, v_{23}, v_{36}\} \cup V_{23,14} \cup V_{23,4} \cup V_{36,14} \cup V_{36,4} \cup V_{3,45} \cup V_{3,14}$. The case for 4 is analogous.

Consider, then, vertex v_{23}. By definition, $N(v_{23}) = \{2, 3\} \cup \{w \in V_{14} | N(w) \cap V_3 \neq \emptyset\}$. Set $\{w \in V_{14} | N(w) \cap V_3 \neq \emptyset\}$, in turn, is exactly set $V_{14,3}$, implying the adjacencies of v_{23} are the same as in the definition of FPHS graphs. The cases for v_{14}, v_{36}, v_{45} are analogous. Also, vertex v_3 is by definition neighbor only to 3, and v_4 is neighbor only to 4.

Consider, then, the relationship between $V_{14,23}$ and $V_{23,14}$. By Lemma 25, two vertices $v, w \in V_{14,23}$ (resp. $v, w \in V_{23,14}$) must have comparable neighborhoods, as otherwise, a T_2 is induced. Let $V_{14,23} = \{w_{1,1}, ..., w_{1,a}\}$ with $a = |V_{14,23}|$ and $N(w_{1,i}) \subset N(w_{1,i+1})$ for all $1 \leq i < a$. Similarly, let $V_{23,14} = \{x_{1,1}, ..., x_{1,b}\}$ with $b = |V_{23,14}|$ and $N(x_{1,i}) \subset N(x_{1,i-1})$ for all $1 < i \leq b$. We must show that $N(w_{1,i}) = \{1, 4\} \cup \{x_{1,1}, ..., x_{1,i}\} \cup V_{23,4}$ for all $1 \leq i \leq a$, and that $a = b$.

Firstly, note that the neighborhood of every vertex in $V_{14,23}$ is contained in $\{1, 4\} \cup V_{23,4} \cup V_{23,14}$, as elements of $V_{14,23}$ do not have neighbors in V_3, must all be neighbors to all of $V_{23,4}$ as per Lemma 24, and must be neighbor to at least one element of $V_{23} - V_{23,4}$. Since every element of $V_{23} - V_{23,4}$ that is neighbor to at least one element of $V_{14} - V_{14,3}$ is in $V_{23,14}$, the containment holds. Similarly, the neighborhood of every vertex in $V_{23,14}$ is contained in $\{2, 3\} \cup V_{14,3} \cup V_{14,23}$. The fact that $a = b$ is a consequence of that, since the only difference between the neighborhoods of vertices in $V_{14,23}$ is which subset of $V_{23,14}$ they contain, and it is impossible to construct a family of a pairwise comparable non-empty subsets from a set of b elements if $b < a$, implying $b \geq a$, and applying the same reasoning to $V_{23,14}$, we conclude $a \geq b$.

In the sequence, we show, by induction, that $N(w_{1,i}) = \{1, 4\} \cup \{x_{1,1}, ..., x_{1,i}\} \cup V_{23,4}$ for all $1 \leq i \leq a$.

For the base, consider $w_{1,1}$. Suppose there is some $i > 1$ such that $x_{1,i} \in N(w_{1,1})$. Then, since $N(x_{1,i}) \subset N(x_{1,i-1})$ for all $1 < i \leq b$, we would have it that $\{x_{1,1}, ..., x_{1,i}\} \subset N(w_{1,1})$. In that case, the neighborhoods of $x_{1,1}$ and $x_{1,i}$ contain all of $\{2, 3\} \cup V_{14,3} \cup V_{14,23}$, implying they are twins, leading to a contradiction. If, on the other hand, $w_{1,1}$ had no neighbors at all in $V_{23,14}$, then it would not belong to $V_{14,23}$. Therefore, $N(w_{1,1}) = \{1, 4\} \cup \{x_{1,1}\} \cup V_{23,4}$.

For the hypothesis, suppose that $N(w_{1,i}) = \{1,4\} \cup \{x_{1,1},...,x_{1,i}\} \cup V_{23,4}$ for some $i < a$. For the step, note that $\{1,4\} \cup \{x_{1,1},...,x_{1,i}\} \cup V_{23,4} \subset N(w_{1.i+1})$, and there is at least one vertex of the form $x_{1,j}, j > i$ such that $x_{1,j} \in N(w_{i,i+1})$. If $j > i+1$, then $N(w_{1,i+1}) = \{1,4\} \cup \{x_{1,1},...,x_{1,j}\} \cup V_{23,4}$, and vertices $x_{1,i+1}, x_{1,j}$ both have $\{2,3\} \cup \{w_{1,i+1},...,w_{1,a}\} \cup V_{14,3}$ as their neighborhood, leading to a contradiction. Therefore, $N(w_{1,i}) = \{1,4\} \cup \{x_{1,1},...,x_{1,i}\} \cup V_{23,4}$ for all $1 \leq i \leq a$. An analogous reasoning can be done to conclude that $N(x_{1,i}) = \{2,3\} \cup \{w_{1,i},...,w_{1,a}\} \cup V_{14,3}$ for all $1 \leq i \leq a$. Make $a = n_1$, and you have the adjacencies of $V_{14,23}$ and $V_{23,14}$ be the same as the ones in the definition of FPHS graphs. The case for $V_{36,45}$ and $V_{45,36}$ is analogous.

Finally, consider the relationship between $V_{3,14}$ and $V_{14,3}$. First, note that, for every $v \in V_{3,14}$, $N(v) \subseteq \{3\} \cup V_{14,3}$, as every element is by definition neighbor to 3, and every other neighbor of v is in V_{14}, and the only elements of V_{14} that are neighbors to elements of V_3 are in $V_{14,3}$. Also note that, for all $v \in V_{14,3}$, $N(v) \subseteq \{1,4\} \cup V_3 \cup V_{23}$, since every element in $V_{14,3}$ must be neighbor to every element in V_{23} by Lemma 24. It is also important to note that $V_{23} = \{v_{23}\} \cup V_{23,14} \cup V_{23,4}$.

By Lemma 26, the neighborhoods of $V_{3,14}$ must be pairwise comparable, and by Corollary 7, so are the neighborhoods of $V_{14,3}$. Let $V_{3,14} = \{y_{3,1},...,y_{3,a}\}$ with $a = |V_{3,14}|$ and $N(y_{3,i}) \subset N(y_{3,i+1})$ for all $1 \leq i < a$, and $V_{14,3} = \{z_{3,1},...,z_{3,b}\}$ with $b = |V_{14,3}|$ and $N(z_{3,i}) \subset N(z_{3,i-1})$ for all $1 < i \leq b$. A similar argument to the one used for the relationship between $V_{14,23}$ and $V_{23,14}$ can be used to prove that $a = b$, that $N(y_{3,i}) = \{3\} \cup \{z_{3,1},...,z_{3,i}\}$ for all $1 \leq i \leq a$, and that $N(z_{3,i}) = \{1,4,v_{23}\} \cup \{y_{3,i},...,y_{3,n_3}\} \cup V_{23,14} \cup V_{23,4}$ for all $1 \leq i \leq b$. By making $a = n_3$, we have that the adjacencies of $V_{14,3}$ and $V_{3,14}$ also perfectly match those of the definition of FPHS graphs. The cases for $V_{3,45}$ and $V_{45,3}$, $V_{4,36}$ and $V_{36,4}$, and $V_{4,23}$ and $V_{23,4}$ are all analogous.

Therefore, G is an induced subgraph of an FPHS graph. □

Lemma 27 allows us to conclude the characterization in Corollary 8, which in turn leads to our final characterization in Theorem 16.

Corollary 8. *A bipartite graph with an induced L_3 is a proper-Helly CA bigraph if and only if it does not contain $T_2, X_2, BW_3, X_{20}, L_3{}^*$ as induced subgraphs.*

Theorem 16. *A bipartite graph G is a proper-Helly CA bigraph if and only if it does not contain $T_2, X_2, BW_3, X_{20}, L_3{}^*$, nor $C_n{}^*, n > 4$ as induced subgraphs.*

Proof. ($\Leftarrow$) Suppose G does not contain any of the forbidden graphs listed. If G does not contain an L_3, then it is a proper-Helly CA bigraph by Theorem 13. If it does contain an L_3, then it is a proper-Helly CA bigraph by Corollary 8.

($\Rightarrow$) The forbidden graphs listed were proven to not be proper-Helly CA bigraphs. □

With Theorem 16, we conclude our forbidden graph characterization of proper-Helly CA bigraphs. Note that the process we use to prove this characterization has striking similarities with the processes applied to prove the characterizations for non-bichordal Helly CA bigraphs and Helly interval bigraphs. We start by narrowing down the possible adjacencies of vertices around a central subgraph (a cycle, a path or an L_3), and discovering the fundamental structure those restrictions impose. We finish by showing that the fundamental structure belongs to the class, and proving that every graph without any of the induced subgraphs mentioned follows the fundamental structure.

We believe that this approach to discover characterizations is a perfect fit for classes with highly restrictive properties, especially ones where it is possible to pinpoint a simple structure around which every other adjacency in the graph occurs. Further discussion of the similarities between these characterizations is done in the conclusion.

3.1.5 Circular Convex Bipartite graphs

A bipartite graph $G = (X, Y, E)$ is a *circular convex bipartite* (CCB for short) graph if X (Y) admits a circular order such that, for every $w \in Y$ ($v \in X$), $N(w)$ ($N(v)$) is a circularly consecutive interval of the order. We call such an order a *CCB order*.

Furthermore, we call a bipartite graph *doubly-CCB* if both partite sets admit their own circular order. Note that the orders for each partite set are independent from one another, with one order not affecting the structure of the other in any way. The graph T_2 from Figure 3.1 is a CCB graph that is not doubly-CCB. Note that both CCB and doubly-CCB are hereditary over induced subgraphs.

Lemma 28. *A graph G is CCB (doubly-CCB) if and only if its twin-free version is.*

Proof. Consider $G = (X, Y, E)$ such that V admits a CCB order. If we add, to G, a new vertex v' that is twin to $v \in X$, it suffices to add v' immediately before (or immediately after) v in the CCB order, as that will make all neighborhoods in Y remain circularly consecutive: vertices whose neighborhoods do not contain v will be unaffected, and vertices whose neighborhood contains v will still have their neighborhood be a circularly consecutive interval. □

While many studies exist about the class's computational properties [Liang and Blum, 1995, Liu et al., 2014], little has been published about its relationship with different subclasses of CA bigraphs, and the recognition of the class itself. It is a class of interest for us as a subclass of CA bigraphs and a superclass of both Helly and proper CA bigraphs, being sort of an "in-between" subclass within CA bigraphs. In the Containment Relations section, the containments mentioned here are proven.

3.1.5.1 Characterization and linear time recognition

A $(0, 1)$ matrix M is said to have the *circular 1s property for rows* if, for every row in M, the 1s in the row are circularly consecutive, that is, the positions that have a value of 1 form a circularly consecutive sequence. In Theorem 17, we use this definition to characterize CCB graphs.

Theorem 17. *A bipartite graph is a CCB graph if and only if it admits a biadjacency matrix with the circular* 1*s property for rows.*

Proof. ($\Leftarrow$) Let $G = (X, Y, E)$ be a bipartite graph that admits a biadjacency matrix M that has the circular 1s property for rows. Let $Y = \{w_1, \dots, w_k\}$ such that the ith column of M represents w_i for all $1 \leq i \leq k$. We show that order $(w_1, \dots, w_k)$ is a CCB order. Consider any vertex $v \in X$. Since the row that represents v has the circular 1s property, that implies $N(v)$ is circularly consecutive in $(w_1, \dots, w_k)$. Therefore, $(w_1, \dots, w_k)$ is a CCB order of Y.

($\Rightarrow$) Let $G = (X, Y, E)$ be a CCB graph, and let $(w_1, \dots, w_k)$ be a CCB order of Y. Construct a biadjacency matrix M with X in the rows, Y in the columns such that, for every $1 \leq i \leq k$, column i represents vertex w_i. Note that, for every $v \in X$, the row that represents v will have the circular 1s property, as $N(v)$ is circularly consecutive in $(w_1, \dots, w_k)$. □

A result from [Booth and Lueker, 1976] shows that the circular 1s property can be recognized in linear time, which, associated with Theorem 17, implies CCB graphs can be recognized in linear time.

3.1.5.2 Simple characterization of proper CA bigraphs

It is possible to use CCB graphs to characterize, in a very simple manner, proper CA bigraphs without isolated vertices nor bi-universal vertices. In this subsection, we present this simple characterization.

Let M be a matrix with n columns that has the circular 1s property for rows. Call the *circular stretch of* 1*s* $[a, b]$ of a row i the circularly consecutive sequence of column indices such that, for any index $j \in [a, b]$, $M_{i,j} = 1$, and for any index $j \notin [a, b]$, $M_{i,j} = 0$. Note that b does not have to be greater than a, since if $b < a$ then $[a, b] = [a, n] \cup [1, b]$.

A $(0, 1)$ $m \times n$ matrix M is said to have a *proper circular arrangement* (PCA for short) if:

1. It has the circular 1s property for rows.
2. For every row i, Let $[a_i, b_i]$ be the circular stretch of 1s in it. If two rows i, j with $i < j$ are such that there exists at least one column c where $M_{i,c} = M_{j,c} = 1$, then there exists some rotation of the columns of M in which $a_i \leq a_j \leq b_i \leq b_j$ or $a_i \leq b_j \leq a_j \leq b_i$.

In [Basu et al., 2013] it is proven that a bipartite graph is a proper CA bigraph if and only if it admits a biadjacency matrix M with a PCA. We use this result to help us prove the following theorem.

Theorem 18. *A bipartite graph $G = (X, Y, E)$ without isolated vertices and bi-universal vertices is a proper CA bigraph if and only if X admits a CCB order in which, for every pair $w_1, w_2 \in Y$ such that $N(w_1)$ and $N(w_2)$ are comparable, the intervals corresponding to $N(w_1)$ and $N(w_2)$ in the CCB order either begin or end in the same vertex.*

Proof. ($\Rightarrow$) Let $G = (X, Y, E)$ be a proper CA bigraph without. Let M be a matrix of G with a PCA. Let $X = \{v_1, ..., v_n\}$ such that for all $1 \leq i \leq n$, column i represents v_i. We prove that $(v_1, ..., v_n)$ is a CCB order with the property.

Consider two vertices $w, w' \in Y$ such that $N(w) \subset N(w')$. Let $[a, b]$ be the stretch of 1s in the row corresponding to w, and $[a', b']$ the stretch for w'. Since w is not an isolated vertex, there is a rotation of the columns of M for which either $a \leq a' \leq b \leq b'$ or $a \leq b' \leq a' \leq b$. If the former is true, since $N(w) \subset N(w')$, then either $a = a'$ or $b = b'$, implying the intervals corresponding to $N(w), N(w')$ in $(v_1, ..., v_n)$ begin or end in the same vertex. However, if the latter is true, that implies $N(w) \not\subset N(w')$, as the stretch $[b' + 1, a' - 1]$ must contain at least 1 column since w' is not a bi-universal vertex.

Therefore, $(v_1, ..., v_n)$ is a CCB order where, for every pair of vertices $w, w' \in Y$ such that $N(w) \subset N(w')$, the intervals corresponding to their neighborhoods share a common beginning or end point.

($\Leftarrow$) Let $G = (X, Y, E)$ and $(v_1, ..., v_n)$ be a CCB order of V as described in the theorem. Let M be a biadjacency matrix where X is represented by the columns in the order given by the CCB order (i.e. v_i being column i). We show that M has a PCA. Let w, w' be two vertices whose neighborhoods intersect. Let $[a, b]$ and $[a', b']$ be the stretch of 1s in the rows corresponding to w and w' in M, respectively.

If $N(w), N(w')$ are not comparable, then two possibilities exist: either a and b are both in $[a', b']$, or one of them is not. In the first case, it is evident that there is a rotation of the columns of M in which $a \leq b' \leq a' \leq b$, as the two stretches of 1s contain each other's endpoints but are not comparable. In the second case, we have it that either $b \notin [a', b']$ or $a \notin [a', b']$. Consider the latter w.l.o.g.. In this case, a rotation of M that places a in the first column will be such that $a \leq a' \leq b \leq b'$.

Now, if $N(w), N(w')$ are comparable, that implies the intervals corresponding to the two neighborhoods share a common beginning or end point in $(v_1, ..., v_n)$, which in turn implies that $[a, b]$ and $[a', b']$ are such that $a = a'$ or $b = b'$. That implies there is a rotation of the columns of M where $a \leq a' \leq b \leq b'$.

Therefore, M has a PCA. □

3.2 CONTAINMENT RELATIONS

In this section, we present a series of containment relations between subclasses of CA bigraphs and interval bigraphs.

3.2.1 Of CCB graphs and CA bigraphs

As previously mentioned, CCB graphs are of relevance to our study due to their role as an "in-between" subclass of CA bigraphs. In this section, we prove that CCB graphs are a subclass of CA bigraphs, and that both the proper and the Helly classes of CA bigraphs are subclasses of doubly-CCB graphs.

We start with Theorem 19, whose proof shows a simple way to construct a bi-circular-arc model given a CCB order.

Theorem 19. *Every CCB graph is a CA bigraph.*

Proof. Consider a bipartite graph $G = (X, Y, E)$ such that X admits a CCB order $(v_1, ..., v_n)$. Construct a family of arcs to correspond to X around a circle C in the following way: make n mutually disjoint arcs of the same size, and attribute them to the vertices of X in the order given by the CCB order (i.e. $a(v_1)$ is immediately consecutive to $a(v_2)$, which is immediately consecutive to $a(v_3)$, and so on). Let ϵ be a small fraction of the smallest distance between two of those arcs. Without loss of generality, if d is said distance, let $\epsilon = \frac{d}{100n}$.

Consider, then, the neighborhood of some vertex $y \in Y$. Since $(v_1, ..., v_n)$ is a CCB order, then $N(y)$ is circularly consecutive in that order. Therefore, either $N(y) = \{v_a, ..., v_b\}$ with $a \geq b$ or $N(y) = \{v_a, ..., v_n\} \cup \{v_1, ..., v_b\}$ with $b < a$. In both cases, we may create an arc $a(y) = (s(a(v_a)) - \epsilon, t(a(v_b)) + \epsilon)$ for y. Note that $a(y)$ intersects precisely the arcs corresponding to $N(y)$ and none of the arcs corresponding to $X - N(y)$.

Therefore, if we create, for every element of Y, an arc in that way, we have a valid bi-circular-arc model of G. □

The fact that CCB graphs are a proper subclass of CA bigraphs comes from the fact that the graph from Figure 3.25 is not a CCB graph, as shown in Lemma 29.

Lemma 29. *The graph in Figure 3.25 is not a CCB graph.*

Proof. We must prove that neither the partite set $X = \{2, 3, 4, 10, 11, 12\}$ nor the partite set $Y = \{1, 5, 6, 7, 8, 9\}$ admit a CCB order.

Start with X. Note that $N(5) = \{2, 10\}, N(6) = \{2, 11\}$ and $N(7) = \{2, 12\}$, implying that, if there was a CCB order of X, then 2 would have to be simultaneously consecutive to 10, 11 and 12, which is impossible.

Consider, then, set Y. Note that $N(2) = \{1, 5, 6, 7\}, N(3) = \{1, 8\}$ and $N(4) = \{1, 9\}$. Therefore, if there was a CCB order of Y, 1 would be consecutive with 8 on one side, and 9 on the other, making it impossible for $\{1, 5, 6, 7\}$ to be circularly consecutive in the order. □

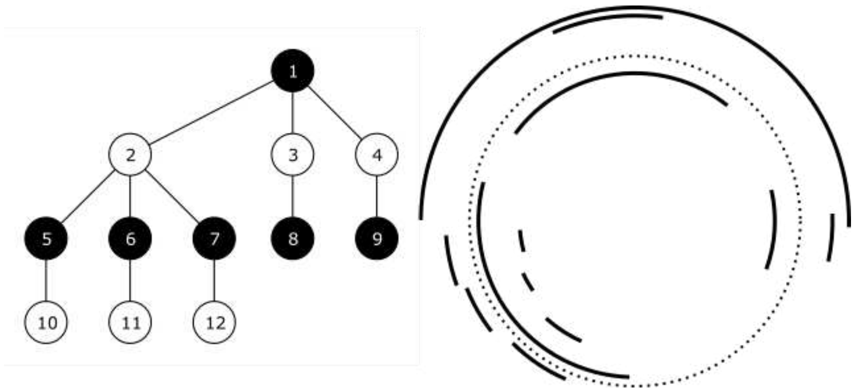

Figure 3.25: A forbidden graph for the CCB class, alongside its bi-circular-arc model. Note that, in the model presented, no pair of arcs cover the circle.

In the sequence, we present Theorems 20 and 21, showing that proper and Helly CA bigraphs are both subclasses of doubly-CCB graphs. The proofs of both theorems consist in showing how to construct a CCB order of one partite set, which is applicable analogously to the other.

To prove that the containment is proper, Figure 3.31 shows, among other things, a graph that is doubly-CCB but neither a Helly nor a proper CA bigraph.

Theorem 20. *If G is a proper CA bigraph, then it is doubly-CCB.*

Proof. Let $G = (X, Y, E)$ be a proper CA bigraph with proper bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$, with $\mathbb{I}$ representing set X and $\mathbb{E}$ representing set Y. Let $X = \{v_1, ..., v_n\}$ such that the s-endpoints of the arcs in $\mathbb{I}$ are distributed around C according to clockwise order $(s(a(v_1)), s(a(v_2)), ..., s(a(v_n)))$. Since $\mathbb{I}$ is a proper family, we have that $(t(a(v_1)), t(a(v_2)), ..., t(a(v_n)))$ is also a clockwise order. Consider, then, the circular ordering $(v_1, ..., v_n)$ of X. We show that, for every $y \in Y$, $N(y)$ is a circularly consecutive interval of that order.

Suppose $w \in Y$ is such that $N(w)$ is not a circularly consecutive interval in $(v_1, ..., v_n)$. Let $v_a, v_b \in N(w)$ with $a < b$ such that there is a vertex $v_c \notin N(w)$ with $c < a$ or $c > b$ and a vertex $v_d \notin N(w), a < d < b$. All four vertices v_a, v_b, v_c, v_d are guaranteed to exist since $N(w)$ is not an interval in the circular order. Note that $(s(a(v_a)), s(a(v_d)), s(a(v_b)), s(a(v_c)))$ is a clockwise order, and that $a(v_c), a(v_d)$ must be disjoint, otherwise, $a(v_c) \cup a(v_d)$ would contain either $a(v_a)$ or $a(v_b)$, making it impossible for w to be neighbor to neither of them.

Consider the two arcs $(t(a(v_c)), s(a(v_d)))$ and $(t(a(v_d)), s(a(v_c)))$. We have it that $a(w)$ must be entirely contained within one of them, since it must not intersect $a(v_d)$ nor $a(v_c)$. Suppose w.l.o.g. that $a(w)$ is in the former.

Note that $s(a(v_b))$ is contained within the arc $(s(a(v_d)), s(a(v_c)))$. If $a(v_b)$ intersects $a(w)$, that implies $t(a(v_b))$ is contained in $(t(a(v_c)), s(a(v_d)))$, which in turn implies $a(v_b)$ contains $a(v_c)$, leading to a contradiction (see Figure 3.26 for a graphical representation). Therefore, every vertex $w \in Y$ is such that $N(w)$ is an interval in $(v_1, ..., v_n)$, implying it is a CCB order of X.

The proof that Y admits a CCB order is perfectly analogous, therefore, G is doubly-CCB. □

Theorem 21. *If G is a Helly CA bigraph, then it is doubly-CCB.*

Proof. Let $G = (X, Y, E)$ be a Helly CA bigraph, and $(C, \mathbb{I}, \mathbb{E})$ be a Helly bi-circular-arc model of G, with $\mathbb{I}$ containing the arcs corresponding to X, and $\mathbb{E}$ containing the arcs corresponding to Y. We may assume that G is twin-free (Lemma 28).

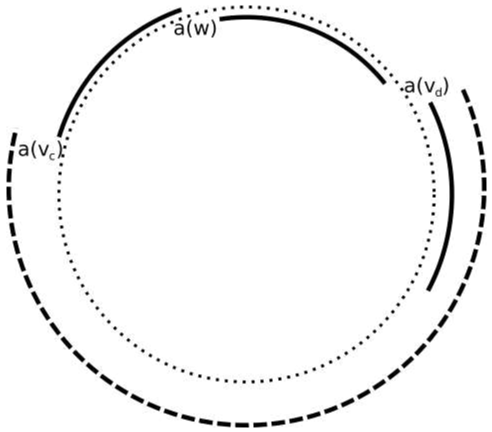

Figure 3.26: Proof of Theorem 20. The thick traced arc is the arc within which $s(a(v_b))$ is contained. Note that, no matter where in the arc it is, it is impossible for $a(v_b)$ to intersect $a(w)$ without containing all of $a(v_c)$.

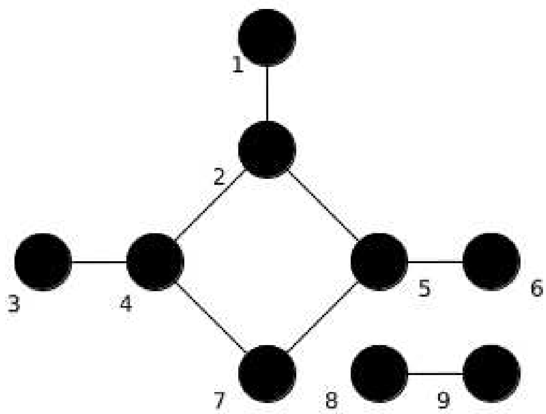

Figure 3.27: Graph $X_2 \cup P_2$.

Note that, for every vertex $v \in X$, the set $\{v\} \cup N(v) \cup \{v' \in X | N(v) \subseteq N(v')\}$ is a biclique in G. Call that biclique K_v for all $v \in X$. Also note that, in $(C, \mathbb{I}, \mathbb{E})$, there exists a point p_{K_v} that all arcs corresponding to biclique K_v contain. Let $X = \{v_1, ..., v_n\}$ such that $(p_{K_{v_1}}, p_{K_{v_2}}, ..., p_{K_{v_n}})$ is a clockwise order. We show that $(v_1, ..., v_n)$ is a CCB order.

Suppose $w \in Y$ is such that $N(w)$ is not a circularly consecutive interval in $(v_1, ..., v_n)$. Note that, for all $v \in N(w)$, $w \in K_v$, and therefore $p_{K_v} \in a(w)$. Let $v_a, v_b, v_c, v_d \in X$ such that $v_a, v_c \in N(w)$, $v_b, v_d \notin N(w)$ and $a < b < c < d$ or $d < a < b < c$ (suppose the former w.l.o.g.). Note that the arc $(p_{K_{v_a}}, p_{K_{v_c}})$ contains $p_{K_{v_b}}$, and the arc $(p_{K_{v_c}}, p_{K_{v_a}})$ contains $p_{K_{v_d}}$. Note $a(w)$ must contain one of these two arcs, as otherwise it cannot intersect both $p_{K_{v_c}}$ and $p_{K_{v_a}}$. In that case, however, $a(w)$ must contain either $p_{K_{v_b}}$ or $p_{K_{v_d}}$, leading to a contradiction.

The proof that Y also admits a CCB order is analogous. Therefore, G is doubly-CCB. □

To prove that the containments in Theorems 20 and 21 are proper, consider graph $X_2 \cup P_2$ in Figure 3.27. Lemma 30 shows that the graph is doubly-CCB, but does belongs to neither the Helly nor the proper subclass of CA bigraphs.

Lemma 30. *The graph $X_2 \cup P_2$ is a doubly-CCB graph, but not a Helly CA bigraph nor a proper CA bigraph.*

Proof. The graph $X_2 \cup P_2$ contains $X_2{}^*$ as an induced subgraph, making it not a proper CA bigraph [Safe, 2019]. To show that it is not a Helly CA bigraph, note that X_2 is not a Helly interval bigraph (Lemma 10), implying every Helly bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ of the X_2 is such that $\mathbb{I} \cup \mathbb{E}$ covers the circle, and therefore, that it is impossible to add two arcs corresponding to a P_2 to the model without at least one arc of the P_2 intersecting an arc of the X_2 from the opposing partite set.

To show that $X_2 \cup P_2$ is doubly-CCB, consider the labeling of the vertices according to Figure 3.27. The partite sets are $X = \{1, 4, 5, 8\}$ and $Y = \{2, 3, 6, 7, 9\}$. Note that $(1, 4, 5, 8)$ is a CCB order of X and $(3, 2, 7, 6, 9)$ is a CCB order of Y. □

And with that, we conclude that CCB graphs are, in fact, a middle of the road subclass of CA bigraphs, containing both proper and Helly CA bigraphs, but being contained in the general class of CA bigraphs.

3.2.2 Of proper, cross-proper, normal and cross-normal CA bigraphs

Recall that proper CA bigraphs are defined as bipartite graphs that admit a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ such that $\mathbb{I}$ and $\mathbb{E}$ are proper families. Proper CA bigraphs are a bipartite analogue of proper CA graphs, defined as the intersection graphs of proper families of arcs on a circle.

To provide a different approach to the concept of prohibiting proper containments between arcs, we define the class of *cross-proper* CA bigraphs. A graph is a cross-proper CA bigraphs if it admits a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ in which there are no two arcs $I \in \mathbb{I}, E \in \mathbb{E}$ such that $I \subset E$ or $E \subset I$.

Consider, as well, the class of *normal CA graphs*, defined as graphs that admit a circular arc model $(C, \mathbb{A})$ where $\mathbb{A}$ is a normal family (i.e. a family where no pair of arcs covers the circle). While little is known about the class's structural and computational properties, it has been shown that proper CA graphs are a subclass of normal CA graphs [Bonomo et al., 2009], and the subclass of normal Helly CA graphs (i.e. graphs that admit an intersection model a family of arcs on a circle that is simultaneously Helly and normal) has been thoroughly studied [Lin et al., 2013].

In analogy to the definition of normal CA graphs, we define the class of *normal CA bigraphs* as bipartite graphs that admit a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ such that both $\mathbb{I}$ and $\mathbb{E}$ are normal families. Note that the class of normal-proper-Helly CA bigraphs, which we define in Subsection 3.1.4, is a subclass of normal CA bigraphs.

We also define the class of *cross-normal CA bigraphs* as graphs that admit a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ such that no two arcs $I \in \mathbb{I}, E \in \mathbb{E}$ cover the circle.

Since normal CA graphs are a superclass of proper CA graphs, a natural question is whether there is a containment relation between proper, cross-proper, normal, and cross-normal CA bigraphs. A result from [Das and Chakraborty, 2015] implies that proper CA bigraphs are a subclass of cross-normal CA bigraphs. Also, in our studies, we have proven that cross-proper CA bigraphs are a subclass of normal CA bigraphs.

It is important to note that the result in Lemma 31 originally states that every proper CA bigraph admits a proper bi-circular-arc model such that no two arcs from opposing families cover the circle. The paper itself never explicitly names or defines the class of cross-normal CA bigraphs.

Lemma 31. *[Das and Chakraborty, 2015] Every proper circular arc bigraph admits a bi-circular-arc model that is both proper and cross-normal.*

Lemma 31 then allows us to conclude that proper CA bigraphs are a proper subclass of cross-normal CA bigraphs, as seen in Corollary 9

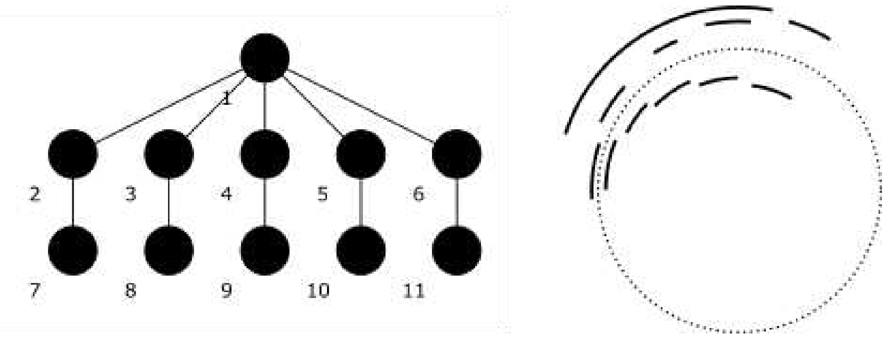

Figure 3.28: A graph that is not a cross-proper CA bigraph, alongside its bi-circular-arc model. Note that the model is cross-normal, as no pair of arcs covers the circle.

Corollary 9. *Proper CA bigraphs are a proper subclass of cross-normal CA bigraphs.*

Proof. Lemma 31 shows that proper CA bigraphs are a subclass of cross-normal CA bigraphs. The fact that the containment is proper is then implied by the fact that the T_2 admits a model where no pair of arcs covers the circle, as seen in Figure 2.1, but is not a proper CA bigraph [Basu et al., 2013]. □

Analogously, we prove that cross-proper CA bigraphs are a subclass of normal CA bigraphs in Lemma 32.

Lemma 32. *Every cross-proper circular arc bigraph admits a bi-circular-arc model that is both cross-proper and normal.*

Proof. Let $(C, \mathbb{I}, \mathbb{E})$ be a cross-proper bi-circular-arc model of graph G. Suppose there are two arcs $A, B \in \mathbb{I}$ such that $A \cup B = C$. Let $S = \{s(A) | A \in \mathbb{I} \cup \mathbb{E}\} \cup \{t(A) | A \in \mathbb{I} \cup \mathbb{E}\}$, define $\epsilon = \frac{1}{10} min\{d(a, b) | a, b \in S\}$.

Since the model is cross-proper, every arc in $\mathbb{E}$ intersects both A and B. Furthermore, no arc in $\mathbb{E}$ is properly contained in A nor B. Therefore, it is possible to replace arc A with arc $(t(B) + \epsilon, s(B) - \epsilon)$ without changing the corresponding graph of the model, and without the model losing the cross-proper property.

If the same process is applied to every pair of arcs of the same family that cover the circle, redefining ϵ after every replacement, the resulting model will be normal when no pairs remain. The process necessarily halts, since the number of pairs of arcs of the same family that cover the circle diminishes by at least 1 after every iteration. □

To prove that the containment between cross-proper CA bigraphs and normal CA bigraphs is proper, we must prove that the graph in Figure 3.28 is not a cross-proper CA bigraph.

Lemma 33. *The graph from Figure 3.28 is not a cross-proper CA bigraph.*

Proof. We show that, in any bi-circular-arc model of the graph, the arc corresponding to 1 must contain the arc corresponding to 2, 3, 4, 5 or 6.

Let $(C, \mathbb{I}, \mathbb{E})$ be a bi-circular-arc model of the graph. Note that $\{a(2), a(3), a(4), a(5), a(6)\}$ is a proper family, as the neighborhoods of 2, 3, 4, 5, 6 are pairwise non-comparable. Furthermore, since, for every $i \in \{2, 3, 4, 5, 6\}$, there exists a vertex $w \in N(i)$ that is not neighbor to any vertex of $\{2, 3, 4, 5, 6\} - i$, we have it that, for all $i \in \{, 2, 3, 4, 5, 6\}$, $a(i)$ is not contained in the union of all elements of the family $\{a(2), ..., a(6)\} - \{a(i)\}$ for all $2 \leq i \leq 6$.

Suppose w.l.o.g. that $(s(a(2)), s(a(3)), ..., s(a(6)))$ is a clockwise order. Note that $a(1)$ must intersect all arcs from $\{a(2), ..., a(6)\}$. Let $p_2, p_3, p_4, p_5, p_6 \in C$ such that $p_i \in a(i) \cap a(1)$

for all $2 \leq i \leq 6$. Since $a(1) \neq C$, there is exactly one value $2 \leq i \leq 6$, for which (p_i, p_{i+1}) (with $6 + 1 = 2$ for cyclic indexation) that is not entirely contained in $a(1)$. Suppose w.l.o.g. it is (p_6, p_2). Then, moving clockwise from $s(a(1))$ to $t(a(1))$, the points p_2, p_3, p_4, p_5, p_6 are encountered in this order.

Note that $a(1)$ contains $(t(a(2)), s(a(6)))$. Also, $a(4)$ is contained in $(t(a(2)), s(a(6)))$: $s(a(4))$ must be contained in $(s(a(3)), s(a(5)))$. Therefore, if $s(a(4))$ was not in $(t(a(2)), s(a(6)))$, that would imply $a(3) \subset a(2) \cup a(4)$. Similarly, $t(a(4))$ must be in $(t(a(3)), t(a(5)))$. If it was not in $(t(a(2)), s(a(6)))$, that would imply $a(5) \subset a(4) \cup a(6)$.

Therefore, $a(4) \subset a(1)$. □

Another fact worthy of mention in this subsection is that CCB graphs, discussed in Subsection 3.2.1, are a proper subclass of cross-normal CA bigraphs.

Lemma 34. *Every CCB graph admits a cross-normal bi-circular-arc model.*

Proof. Let $G = (X, Y, E)$ be such that $X = \{x_1, \ldots, x_n\}$ admits CCB order $(x_1, \ldots, x_n)$. We construct a cross-normal bi-circular-arc model of G.

Construct a family $\mathbb{I}$ of n pairwise disjoint arcs on a circle C to represent X, such that $(s(a(x_1)), s(a(x_2)), \ldots, s(a(x_n)))$ is a clockwise order. We prove that it is possible to compose a family $\mathbb{E}$ to correspond to Y without any pair of arcs $E \in \mathbb{E}, I \in \mathbb{I}$ covering the circle. Let ϵ be a tenth of the smallest length of any arc in $\mathbb{I}$.

If $y \in Y$ is such that $N(y) = \{x_a, \ldots, x_b\}$ with $1 \leq a < b \leq n$ or $N(y) = \{x_a, \ldots, x_n, x_1, \ldots, x_b\}$ with $1 \leq b < a \leq n$, make $a(y) = (t(a(x_a)) - \epsilon, s(a(x_b)) + \epsilon)$. If $y \in Y$ is such that $N(y) = \{x_i\}, 1 \leq i \leq n$, make $a(y) = (s(a(x_i)) + \epsilon, t(a(x_i)) - \epsilon)$.

We show that, with every arc corresponding to vertices of Y being constructed in this way, every $y \in Y$ is such that there is no $x_i \in X, 1 \leq i \leq n$ for which $a(y) \cup a(x_i) = C$. For any $y \in Y$ with a neighborhood of size 1, this is trivial, so suppose $|N(y)| > 1$.

Suppose w.l.o.g. that $N(y) = \{x_a, \ldots, x_b\}$ with $1 \leq a < b \leq n$. Note that, for an arc $a(x_i)$ to cover the circle with $a(y)$, it is necessary that $a(x_i) \cap a(y) \neq \emptyset$, as all arcs are open. If $a(x_i)$ intersects $a(y)$, then either $a < i < b$, and $a(x_i) \subset a(y)$, or $i \in \{a, b\}$. In the first case, the pair $a(x_i), a(y)$ can only cover the circle if $a(y) = C$, which is false.

In the second case, suppose w.l.o.g. that $i = a$. Note that $a(y) \cup a(x_a) = (s(a(x_a)), s(a(x_b)) + \epsilon)$, which does not contain $t(a(x_b))$. Therefore, $a(y) \cup a(x_a) \neq C$. □

Lemma 34 shows that CCB graphs are a subclass of cross-normal circular arc bigraphs. Corollary 10 shows the containment is proper.

Corollary 10. *CCB graphs are a proper subclass of cross-normal CA bigraphs.*

Proof. The fact that every CCB graph is a cross-normal CA bigraph is established in Lemma 34.

To prove that the containment is proper, note that the graph in Figure 3.25 is not a CCB graph. Furthermore, as noted in the figure's description, the bi-circular-arc model accompanying the graph does not have two arcs that, together, cover the circle, implying that the graph is cross-normal. □

Other containment relations between the four classes discussed in this subsection are as of yet unknown. We currently believe that proper CA bigraphs are a subclass of cross-proper CA bigraphs, but have not yet arrived at a conclusive proof. We are led to believe in this conjecture by the fact we did not find any forbidden graphs for cross-proper CA bigraphs that were not forbidden for proper CA bigraphs, and by the fact that it holds true when restricted to interval bigraphs, which we discuss later in this subsection.

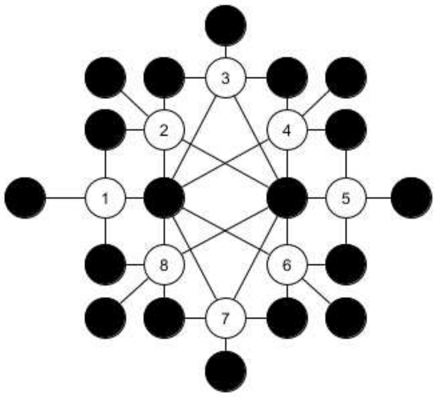

Figure 3.29: A graph that is not a normal circular arc bigraph. Only the vertices from one of the partite sets are labeled, as these are the only labels the proof necessitates.

We know, however, that all four classes described in this subsection are proper subclasses of CA bigraphs. Lemma 35 shows that there exists a CA bigraph that is not normal and, by extension, not cross-proper. Similarly, Lemma 36 shows that there exists a CA bigraph that is not cross-normal.

Lemma 35. *The graph in figure 3.29 is a CA bigraph, but not a normal CA bigraph.*

Proof. To prove that the graph is a CA bigraph, note that $(1, 2, 3, 4, 5, 6, 7, 8)$ is a valid CCB order for the partite set highlighted in white in the figure: every vertex in the black partite set is such that its neighborhood is either $\{i\}$, $\{1, ..., 8\} - \{i\}$, or $\{i, i+1\}$ for $1 \le i \le 8$.

We now show that the graph is not a normal CA bigraph. Let $(C, \mathbb{I}, \mathbb{E})$ be a bi-circular-arc model of the graph, and let $\mathbb{I}$ correspond to the $\{1, 2, 3, 4, 5, 6, 7, 8\}$ partite set. First, note that, for every vertex $v \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, there is a vertex $w \in N(v)$ such that w is not neighbor to any element of $\{1, 2, 3, 4, 5, 6, 7, 8\} - v$, implying $\mathbb{I}$ is a proper family, and that no arc $a(i)$ for $1 \le i \le 8$ is contained in $\bigcup_{A \in \mathbb{I} - \{a(i)\}} A$. Furthermore, note that, for every $1 \le i \le 8$, there exists a vertex whose neighborhood is exactly $\{i, i+1\}$, which in turn implies that the s-endpoints of the arcs of $\mathbb{I}$ are distributed around the circle in the order $(s(a(1)), ...s(a(8)))$, either clockwise or counter-clockwise. Suppose the former without loss of generality. Since $\mathbb{I}$ is a proper family, $(t(a(1)), t(a(2)), ..., t(a(8)))$ is also a clockwise order.

Also note that, since no arc $a(i)$ for $1 \le i \le 8$ is contained in $\bigcup_{A \in \mathbb{I} - \{a(i)\}} A$, that implies $a(i)$ does not intersect $a(j)$ if $j \ne i+1, i-1$ for $1 \le i \le 8$.

Consider, now, the two vertices in the center of the figure. One of them, let us call it v, is such that $N(v) = \{1, 2, 3, 4, 6, 7, 8\}$, and the other, call it w, is such that $N(w) = \{2, 3, 4, 5, 6, 7, 8\}$. Consider the properties of the arcs $a(v), a(w)$. Note that $(t(a(v)), (s(a(v)))$ is contained in $(s(a(4)), t(a(6)))$, as $a(v)$ must not intersect $a(5)$ but must intersect $a(4), a(6)$. Similarly, $(t(a(w)), s(a(w)))$ is contained in $(s(a(8)), t(a(2)))$. That implies, however, that $a(w)$ contains $(s(a(4)), t(a(6)))$, implying $a(v) \cup a(w) = C$. □

Lemma 36. *The graph in Figure 3.30 is not a cross-normal CA bigraph.*

Proof. We show that, in any bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ of the graph, $a(9) \cup a(v) = C$.

Let $(C, \mathbb{I}, \mathbb{E})$ be a bi-circular-arc model of the graph, with $\mathbb{I}$ representing the partite set $\{1, 2, ..., 9\}$. Note that the subfamily $\mathbb{A} = \{a(1), ..., a(8)\}$ must be a proper family, as the

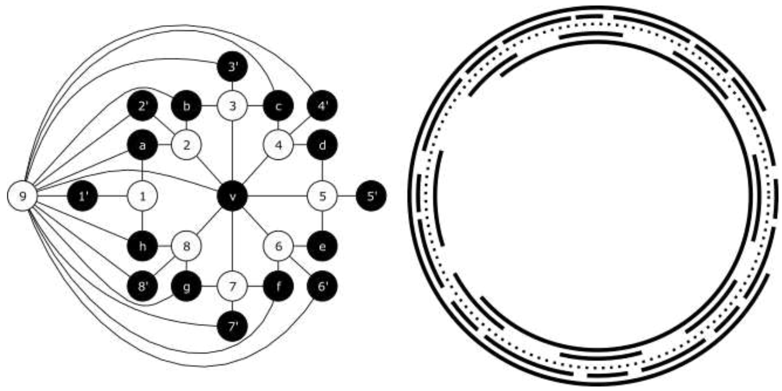

Figure 3.30: A graph that is not a cross-normal circular arc bigraph, alongside its bi-circular-arc model.

neighborhoods of $\{1, \ldots, 8\}$ are not comparable. Furthermore, for every element $v \in \{1, \ldots, 8\}$, there's a vertex $w \in N(v)$ such that w is not neighbor to any vertex in $\{1, \ldots, 8\} - v$, implying that $a(i) \not\subset \bigcup_{A \in \mathbb{A} - \{a(i)\}} A$ for all $1 \leq i \leq 8$. Also, since for every $1 \leq i \leq 8$, there is a vertex x such that $N(x) \cap \{1, \ldots, 8\} = \{i, i+1\}$ (with 8+1=1), we have it that $(s(a(1)), s(a(2)), \ldots, s(a(8)))$ is either a clockwise or counter-clockwise order in C. Suppose the former w.l.o.g.

Consider the vertices $\{1', \ldots, 8'\}$ from the graph. Note that, since $N(i') = \{i\}$ for every $1 \leq i \leq 8$, every arc $a(i')$ must be entirely contained in $(t(a(i-1)), s(a(i+1)))$.

Consider, then, the relationship between arcs $a(v)$ and $a(9)$. Since $a(v)$ must intersect every arcs of $\mathbb{A}$ except $a(1)$, that implies $a(v)$ contains $(t(a(2)), s(a(8)))$. Also, $a(9)$ must intersect $a(4')$ and $a(6')$ but not $a(5')$. Since $a(4') \subset (t(a(3)), s(a(5)))$ and $a(6') \subset (t(a(5)), s(a(7)))$, $(t(a(9)), s(a(9)))$ must be contained within $(t(a(3)), s(a(7)))$.

However, $(t(a(3)), s(a(7))) \subset (t(a(2)), s(a(8)))$, implying $a(v) \cup a(9) = C$. □

3.2.2.1 *Of proper and cross-proper interval bigraphs*

Recall that proper interval bigraphs are defined as bipartite graphs that admit a bi-interval model $(\mathbb{A}, \mathbb{B})$ such that $\mathbb{A}$ and $\mathbb{B}$ are proper families. Analogously to cross-proper CA bigraphs, define *cross-proper interval bigraphs* as graphs that admit a bi-interval model $(\mathbb{A}, \mathbb{B})$ such that there are no two intervals $A \in \mathbb{A}, B \in \mathbb{B}$ such that $A \subset B$ or $B \subset A$.

Brown and Lundgren (2010) [Brown and Lundgren, 2010] have shown that every proper interval bigraph admits a model that is both proper and cross-proper, implying that proper interval bigraphs are a subclass of cross-proper interval bigraphs. It is important to note that, in Brown and Lundgren's work, the class of proper interval bigraphs is defined as graphs which admit a bi-interval model $(\mathbb{A}, \mathbb{B})$ where $\mathbb{A} \cup \mathbb{B}$ is a proper family, but that is not a problem, since the article also shows that the class defined in that manner is equivalent to the class under the definition we employ here.

Since T_2 is not a proper interval bigraph, but is a cross-proper interval bigraph, the class of proper interval bigraphs is a proper subclass of cross-proper interval bigraphs. This proper containment provides a semblance of evidence to our conjecture that proper CA bigraphs are a subclass of cross-proper CA bigraphs.

3.2.3 Venn diagram of containment relations

In the sequence, we provide a large Venn diagram of the containment relations explored in this section so far, with every space containing an example graph, in Figure 3.31. The labels in the diagram are as follows:

CAB stands for CA bigraphs.

CN stands for cross-normal CA bigraphs.

CCB stands for CCB graphs.

D-CCB stands for doubly-CCB graphs.

P stands for proper CA bigraphs.

H stands for Helly CA bigraphs.

PH stands for Proper-Helly CA bigraphs.

PI stands for proper interval bigraphs.

HI stands for Helly interval bigraphs.

NBH stands for non-bichordal Helly CA bigraphs.

The classes of cross-proper and normal CA bigraphs are not included, as their relationships with several other classes is currently unknown. Furthermore, the class of normal-proper-Helly CA bigraphs is not included, as it is the union of **NBH** and **HI**.

3.3 BICLIQUE GRAPH PROPERTIES

The *biclique graph* of a graph is the intersection graph of its family of bicliques. Similarly, the *clique graph* of a graph is the intersection graph of its family of cliques. For every class of graphs, the clique (biclique) graphs of the class represent graph classes on their own right.

Biclique graphs, as well as clique graphs, can be thought of in therms of the biclique (clique) operator. The *biclique operator KB* (*clique operator K*) is the operator that relates a graph to its biclique graph (clique graph). In simple terms, if G is a graph, then $KB(G)$ $(K(G))$ is the biclique (clique) graph of G.

Biclique graphs were first defined in [Groshaus and Szwarcfiter, 2010], as a bipartite variant of the concept of clique graphs. Since then, the biclique graphs of several classes have been explored, such as proper interval bigraphs [Cruz et al., 2020], triangle-free graphs [Groshaus and Guedes, 2020] and others.

Furthermore, in 2014, Groshaus et al. [Groshaus et al., 2014] have demonstrated that most graphs diverge under the biclique operator.

Our studies on the biclique graphs of CA bigraphs is motivated, in part, by existing results on the clique graphs of CA graphs. In 2010, Lin et al. [Lin et al., 2010] have demonstrated several results pertaining to the clique graphs of CA graphs, including particularly interesting results on the clique graphs of Helly CA bigraphs and proper-Helly CA bigraphs.

Since CA bigraphs are a bipartite analogue to CA graphs, and bicliques are to CA bigraphs what cliques are to CA graphs, the question of what the biclique graphs of CA bigraphs are like arises naturally. In this section, we present results we have found on the properties of the biclique graphs of Helly CA bigraphs and some of their subclasses.

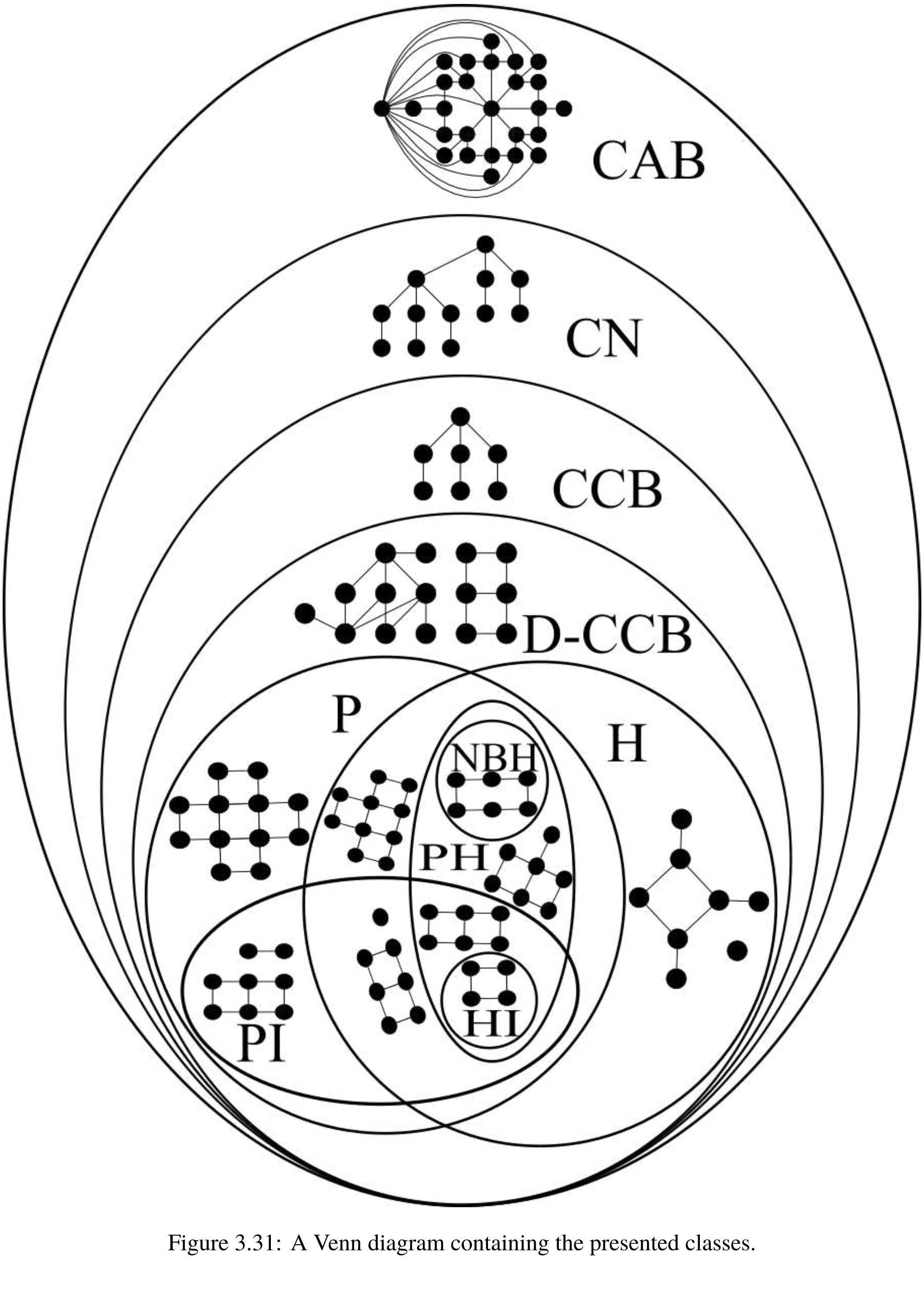

Figure 3.31: A Venn diagram containing the presented classes.

3.3.1 Helly CA bigraphs

In this subsection, we present results on the biclique graphs of Helly CA bigraphs, and their relationship with the clique graphs of Helly CA graphs.

For the remainder of this subsection, we refer to the class of biclique graphs of Helly CA bigraphs as KB(HCAB), and the class of clique graphs of Helly CA graphs as K(HCA).

The simple inclusion proven in Lemma 37 is also true of K(HCA) [Lin et al., 2010].

Lemma 37. *Every graph in KB(HCAB) is a proper CA graph.*

Proof. Let $G = (X, Y, E)$ be a Helly CA bigraph and $(C, \mathbb{I}, \mathbb{E})$ be its Helly model. Let $b(G) = \{K_1, K_2, ..., K_n\}$ and $\{p_1, ..., p_n\} \subset C$ a set of biclique points compatible with $(C, \mathbb{I}, \mathbb{E})$ where p_i corresponds to K_i for every $1 \leq i \leq n$. Suppose w.l.o.g. that $(p_1, ..., p_n)$ is a clockwise order.

We construct a proper CA model of $KB(G)$. Let $S = \{s(A)|A \in \mathbb{I} \cup \mathbb{E}\} \cup \{t(A)|A \in \mathbb{I} \cup \mathbb{E}\} \cup \{p_1, ..., p_n\}$ and d the smallest distance between two points of S. With that, let $\epsilon = d\frac{1}{100(|X \cup Y| + n)}$ and, for all i, let $\lambda_i = |\{K \in b(G)|N_{KB(G)}(K_i) \subseteq N_{KB(G)}(K)\}|$ (i.e. the number of bicliques whose neighborhoods in $KB(G)$ contain that of K_i).

For every K_i, let $v_i \in K_i$ such that $|(p_i, t(a(v_i)))| = max\{|(p_i, t(a(v)))| : v \in K_i\}$ (i.e. the arc $a(v_i)$ is such that its t-endpoint is the farthest from p_i in the clockwise direction). Let $(p_{a_i}, ..., p_{b_i})$ be the clockwise order of biclique points that $a(v_i)$ contains ($1 \leq a_i, b_i \leq n$). Create arc $A_i = (p_i - \epsilon, p_{b_i} + \epsilon\lambda_i)$ to represent biclique K_i for all i. We show that $(C, \{A_1, ..., A_n\})$ is a proper CA model of $KB(G)$.

Suppose, first, that bicliques K_i, K_j intersect for some $1 \leq i, j \leq n$. That implies that there exists an arc in $(C, \mathbb{I}, \mathbb{E})$ that contains (p_i, p_j) or (p_j, p_i). In the first case, $|(p_i, p_j)| \leq |(p_i, t(a(v_i)))|$, implying $p_j \in A_i$, and in the second, $p_i \in A_j$, therefore, $A_i \cap A_j \neq \emptyset$.

Now, suppose A_i, A_j intersect for some $1 \leq i, j \leq n$. First, note that, given how ϵ is defined, A_i, A_j intersect if and only if they contain a biclique point in common.

Therefore, either p_j is in A_i or p_i is in A_j. Suppose w.l.o.g. that $p_j \in A_i$. Since A_i is entirely contained in an arc A of $\mathbb{I} \cup \mathbb{E}$, that implies $v(A) \in K_i \cap K_j$ since A contains both p_i and p_j. Therefore, K_i, K_j intersect, implying they are neighbors in $KB(G)$. Therefore, $(C, \{A_1, ..., A_n\})$ is a CA model of $KB(G)$. We now prove that no proper containments occur.

Suppose two arcs A_i, A_j are such that $A_j \subset A_i$. Note that, this being the case, $A_j \cap \{p_1, ..., p_n\} \subset A_i \cap \{p_1, ..., p_n\}$. Since $p_i \in A_i$ and $p_j \in A_j$, that implies that $v_i \in K_j$, as $A_i \subset a(v_i)$ and $p_j \in A_i$.

Now, let $(p_i, ..., p_x)$ be the clockwise order of biclique points A_i crosses and $(p_j, ..., p_y)$ the order for A_j, with $1 \leq x, y \leq n$. Note we have two possibilities: either the two clockwise orders end in the same point (i.e. $p_x = p_y$) or $(p_i, ..., p_x)$ ends after $(p_j, ..., p_y)$ (i.e. $p_x \neq p_y$).

Suppose, first, that $p_y \neq p_x$. Since $A_i \subseteq a(v_i)$, that implies $a(v_i)$ is such that $|(p_j, t(a(v_i)))| > |(p_j, t(a(v_j)))|$, which leads to a contradiction, since, by definition, v_j is such that $|(p_j, t(a(v_j)))|$ is maximum.

Suppose, then, that $p_y = p_x$. In that case, since $N_{KB(G)}(K_j) \subset N_{KB(G)}(K_i)$, that implies $\lambda_j > \lambda_i$. Therefore, since $t(A_i) = p_x + \epsilon\lambda_i$ and $t(A_j) = p_x + \epsilon\lambda_j$, we have it that A_j ends further away from p_x than A_i does, implying A_i, A_j are not comparable.

Therefore, $(C, \{A_1, ..., A_n\})$ is a proper CA model of $KB(G)$. □

Given that KB(HCAB) is a subclass of proper CA graphs, and so is K(HCA), that raises the question of what the relationship between KB(HCAB) and K(HCA) is. Given a Helly CA bigraph G without an induced C_6, we have it that $KB(G) = K(G^2)$ since, as shown in Lemma 8, the set of cliques of G^2 is equal to the set of bicliques of G. Therefore, the biclique graphs of Helly CA bigraphs without an induced C_6 are all in K(HCA). Furthermore, the fact that P_3 is

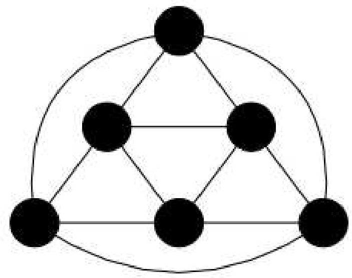

Figure 3.32: The biclique graph of C_6.

trivially in K(HCA), but is not a biclique graph [Groshaus and Szwarcfiter, 2010], implies that the class of biclique graphs of Helly CA bigraphs without an induced C_6 is a proper subclass of K(HCA).

It is not true, however, that KB(HCAB) as a whole is a subclass of K(HCA). Consider the biclique graph of a C_6, as shown in Figure 3.32. Since C_6 is a Helly CA bigraph, $KB(C_6)$ is in KB(HCAB).

However, $KB(C_6)$ is not in K(HCA), as shown in the following lemma.

Lemma 38. *The graph from Figure 3.32 is not the clique graph of any Helly CA graph.*

Proof. According to Lemma 5 from [Lin et al., 2010], a graph that contains an induced 4-wheel can only be the clique graph of a Helly CA graph if it contains at least two universal vertices. The graph from Figure 3.32 contains an induced 4-wheel but no universal vertices. □

Lemma 38 allows us to conclude that the classes K(HCA) and KB(HCAB) are not comparable, implying each of them is a different subclass of proper CA graphs. In particular, it is known that the biclique graph of any bipartite graph is a square graph [Groshaus and Guedes, 2020], implying that KB(HCAB) is in the intersection between proper CA graphs and square graphs. It is an open problem to determine whether KB(HCAB) includes all square proper CA graphs or not.

Class K(HCA), on the other hand, has been fully characterized in [Lin et al., 2010]. A graph is in K(HCA) if and only if it is a proper-Helly circular arc graph, or if it has at least two universal vertices and the removal of all universal vertices results in a graph that is co-bipartite and proper-Helly circular-arc.

In the sequence, we present a simple characterization of the biclique graphs of non-bichordal Helly CA bigraphs.

Recall that every twin-free non-bichordal Helly CA bigraph. The characterization is divided in two parts: first, in Lemma 39, we show the general structure every biclique graph of an FCS graph must follow. Then, in Lemma 40, we show that every twin-free non-bichordal Helly CA bigraph has a biclique graph that is equal to that of the smallest FCS graph that contains it as an induced subgraph. Theorem 22 then puts those two results together to conclude the characterization.

The general structure employed in Lemma 39 is the one in the following definition.

Definition 6. *Let $k \geq 6$ be an even number, and $n_1, ..., n_k \geq 0$. We define the* general biclique structure *(GBS for short) graph, denoted by $\mathbb{G}'_k(n_1, ..., n_k)$, in the following way:*

Let $V(\mathbb{G}'_k(n_1, ..., n_k)$ be partitioned in the following vertex sets:

- *$A = \{a_1, ..., a_k\}$, and*

- $B_i = \{b_{i,1}, ..., b_{i,n_i}\}$ *for* $1 \leq i \leq k$.

With the neighborhoods of the vertices defined as follows (consider cyclic summation, e.g. $k + 1 = 1$*):*

- $N(b_{i,j}) = B_{i-1} \cup B_{i+1} \cup (B_i - \{b_{i,j}\}) \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2}\}$ *for* $1 \leq i \leq k$*, and* $1 \leq j \neq n_k$.
- $N(a_i) = B_{i-2} \cup B_{i-1} \cup B_i \cup B_{i+1} \cup \{a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}\}$.

Lemma 39. *Let G be the FCS graph $\mathbb{G}_k(n_1, ..., n_k)$, for any valid values of $k, n_1, ..., n_k$. Then $KB(G) = \mathbb{G}'_k(n_1, ..., n_k)$.*

Proof. The bicliques of G, as seen in the proof of Theorem 1, are the following:

- $A_i = \{c_{i-1}, c_i, c_{i+1}, v_i\} \cup W_i \cup U_{i-1}$ for all $1 \leq i \leq k$.
- $B_{i,j} = \{c_i, c_{i+1}\} \cup \{w_{i,m} | m \geq j\} \cup \{u_{i,l} | l \leq j\}$ for all $1 \leq i \leq k$, and $1 \leq j \leq n_i$.

For ease of notation, let $H = \mathbb{G}'_k(n_1, ..., n_k)$.

We claim that H is the biclique graph of G, with a_i being the corresponding vertex of A_i, and $b_{i,j}$ being the corresponding vertex of $B_{i,j}$, for all $1 \leq i \leq k$, and $1 \leq j \leq n_i$. We show that the adjacencies between the vertices of H perfectly correspond to the intersections between bicliques of G, and vice-versa.

Consider, first, bicliques of the form $B_{i,j}$. Every biclique of that form intersects every other biclique of the form $B_{i,l}$ for $1 \leq l \leq n_i$ as they all contain vertex c_i. Similarly, $B_{i,j}$ intersects every biclique of the form $B_{i+1,l}$ $(1 \leq l \leq n_{i+1})$ and $B_{i-1,l}$ $(1 \leq l \leq n_{i-1})$, as those contain c_{i+1} and c_i, respectively. Also, note that $B_{i,j}$ intersects $A_{i-1}, A_i, A_{i+1}, A_{i+2}$, as A_{i-1}, A_i contain c_i, and A_{i+1}, A_{i+2} contain c_{i+1}. Therefore, for every $v \in N_H(b_{i,j})$, $B_{i,j}$ intersects the biclique corresponding to v in G.

Conversely, note that $B_{i,j} \subset \{c_i, c_{i+1}\} \cup W_i \cup U_i$. Therefore, $B_{i,j}$ does not intersect any biclique of the form $B_{x,y}$ where $x \notin \{i-1, i, i+1\}$. Furthermore, note that $B_{i,j}$ does not intersect any biclique of the form A_x where $x \notin \{i-i, i, i+1, i+2\}$, as those will be contained in $\{c_{x-1}, c_x, c_{x+1}\} \cup W_x \cup U_{x-1}$. Therefore, for every $v \in V(H) - N_H(b_{i,j})$, $B_{i,j}$ does not intersect the corresponding biclique of v in G.

As for bicliques of the form A_i, it has already been shown in the previous paragraphs that they intersect every biclique of the form $B_{x,y}$ for $x \in \{i-2, i-1, i, i+1\}, 1 \leq y \leq n_x$, and no bicliques of the form $B_{x,y}$ for $x \notin \{i-2, i-1, i, i+1\}, 1 \leq y \leq n_x$. All that is left to verify is that $a_i a_j \in E(H)$ if and only if $A_i \cap A_j \neq \emptyset$, for all $1 \leq i, j \leq k$. Note that A_i intersects $A_{i-2}, A_{i-1}, A_{i+1}, A_{i+2}$, as A_{i-2}, A_{i-1} contain c_{i-1}, and A_{i+1}, A_{i+2} contain c_{i+1}. As for bicliques of the form A_x for $x \notin \{i-2, i-1, i+1, i+2\}$, note that A_i does not intersect A_x, as none of their elements in C are the same, and for every pair of bicliques of the form $A_i, A_j, i \neq j$, we have it that $(A_i - C) \cap (A_j - C) = \emptyset$, since every element in $V(G) - C$ is neighbor to exactly one element of C, and biclique A_i represents the star around c_i.

Therefore, H is the biclique graph of G. ◻

For the next part of the characterization, recall that we have shown that, for all twin-free non-bichordal Helly CA bigraphs, there exists an FCS graph of which it is an induced subgraph (Theorem 2 and Corollary 1).

With that in mind, let G be a twin-free non-bichordal Helly CA bigraph, and $\mathbb{G}_k(n_1, ..., n_k)$ be the smallest FCS graph to which G is either an induced subgraph or isomorphic. We call $\mathbb{G}_k(n_1, ..., n_k)$ the *minimum containing supergraph* of G, denoted by $MCS(G)$.

Lemma 40. *Let G be a twin-free non-bichordal Helly CA bigraph. Then $KB(G) = KB(MCS(G))$*

Proof. Let $MCS(G) = \mathbb{G}_k(n_1, ..., n_k)$. If G and $MCS(G)$ are isomorphic, the proof is trivial. Suppose, then, that G is an induced subgraph of $MCS(G)$.

Let H be an induced subgraph of $MCS(G)$ that is isomorphic to G. Consider the labels given to the vertices of $\mathbb{G}_k(n_1, ..., n_k)$ according to Definition 2. It is easy to verify that every vertex from subset C has to be contained in $V(H)$, as otherwise, G would be bichordal. Note that any vertex of degree 1 in H is neighbor to an element of cycle C, meaning that, if it is not a vertex of set V (i.e. the set of vertices whose only neighbor is an element of cycle C), it may be replaced with a vertex of V without breaking its isomorphism with G. Therefore, suppose every vertex of degree 1 in H is from the set V.

We claim that, for every $1 \le i \le k$, the sets W_i and U_i from $MCS(G)$ are contained in $V(H)$. Suppose otherwise. For some i, let $W' = W_i \cap V(H)$ such that $|W'| = m < n_i$. Recall, from the definition of FCS graphs, that $W_i = \{w_{i,1}, ..., w_{i,n_i}\}$. Therefore, $W' = \{w_{i,j_1}, ..., w_{i,j_m}\}$ where, for every $1 \le l \le m$, we have $1 \le j_l \le n_i$ and $j_l < j_{l+1}$. It is important to note that j_{l+1} does not need to equal $j_l + 1$.

Consider, then, the subset of U_i that is contained in $V(H)$. Without loss of generality, suppose $|U_i \cap V(H)| \ge m$ (the opposite case is perfectly analogous), and let $U' = U_i \cap V(H)$.

Let $1 \le l < m$ such that $j_l \ne j_{l+1} - 1$. Recall, from the definition of FCS graphs, that $U_i = \{u_{i,1}, ..., u_{i,n_i}\}$, and for any $1 \le i \le n_i$, $N(u_{i,x}) \cap W_i = \{w_{i,x}, ..., w_{i,n_i}\}$. Therefore, every element of $\{u_{i,j_l+1}, ..., u_{i,j_{l+1}}\} \subset U_i$ has the same neighbors within W' (namely, they are neighbors to $\{w_{i,j_{l+1}}, ..., w_{i,j_m}\}$). That implies only one of them can be in $V(H)$, as otherwise, H would not be twin-free. However, it is also true that at least one element in $\{u_{i,j_l+1}, ..., u_{i,j_{l+1}}\}$ must be in $V(H)$, as otherwise, $w_{i,j_l}, w_{i,j_{l+1}}$ would be twins. Therefore, exactly one element of $\{u_{i,j_l+1}, ..., u_{i,j_{l+1}}\}$ is in $V(H)$.

Similarly, every element of $\{u_{i,1}, ..., u_{i,j_1}\}$ has the same neighbors within W' (they are neighbors to all of W'), also implying that exactly one of them is in $V(H)$, since if more than one of them was in $V(H)$, they would be twins, and if none of them were in $V(H)$, then w_{i,j_1} would have c_i as its only neighbor, being, therefore, replaceable with v_i in H.

Finally, every element of $\{u_{i,j_m+1}, ..., u_{i,n_i}\}$ has zero neighbors in W', implying none of them is present in $V(H)$, since their only neighbor in $V(H)$ would be c_{i+1}, implying they'd be replaceable v_{i+1} in H. Therefore, $|U'| = |W'|$. Without loss of generality, then, suppose $U' = \{u_{i,j_1}, ..., u_{i,j_m}\}$.

Note that, in this case, for all $1 \le l \le m$, we have $N(u_{i,j_l}) = \{c_{i+1}\} \cup \{w_{i,j'_l} | l \le l' \le m\}$ and $N(w_{i,j_l}) = \{c_i\} \cup \{u_{i,j'_l} | 1 \le l' \le l\}$, implying that, if we replaced w_{i,j_l} with $w_{i,l}$ and u_{i,j_l} with $u_{i,l}$ for all $1 \le l \le m$, the resulting graph would still be isomorphic to H (and therefore to G). However, in that case, we have it that $\mathbb{G}_k(n_1, ..., n_{i-1}, m, n_{i+1}, ..., n_k)$ also contains G as an induced subgraph (or is isomorphic to G), implying $\mathbb{G}_k(n_1, ..., n_k)$ is not the smallest FCS graph that G is an induced subgraph of, since $m < n_i$, leading to a contradiction.

Therefore, for every $1 \le i \le k$, U_i and W_i are both contained in $V(H)$.

Thus, the only set of vertices from $MCS(G)$ that may not be completely contained in $V(H)$ is set V from Definition 2. However, removing elements of V from $MCS(G)$ does not change its biclique graph: every $v \in V$ is a leaf vertex that is only contained in biclique $\{c_i\} \cup N(c_i)$ for some $1 \le i \le k$. Since no other biclique of $MCS(G)$ contains the set $\{c_i\} \cup N(c_i) - \{v\}$, we have that $\{c_i\} \cup N(c_i) - \{v\}$ is a biclique in $MCS(G) - v$. Furthermore, since v only belongs to that one biclique, that implies the neighborhood of $\{c_i\} \cup N(c_i) - \{v\}$ in $KB(MCS(G) - v)$ is equal to the neighborhood of $\{c_i\} \cup N(c_i)$ in $KB(MCS(G))$. Other than that, the neighborhoods of other bicliques do not change at all when v is removed.

Therefore, $KB(G) = KB(MCS(G))$. □

We use Lemmas 39 and 40 to prove the characterization in Theorem 22.

Theorem 22. *A bipartite graph is the biclique graph of a non-bichordal Helly CA bigraph if and only if it is graph $\mathbb{G}'_k(n_1, ..., n_k)$ for some $k, n_1, ..., n_k$.*

Proof. ($\Leftarrow$) We have shown that $\mathbb{G}'_k(n_1, ..., n_k)$ is the biclique graph of $\mathbb{G}_k(n_1, ..., n_k)$ in Lemma 39.

($\Rightarrow$) Let G be a non-bichordal Helly CA bigraph. Note that its biclique graph is equal to the biclique graph of its twin-free version G^-, which, in turn, is such that $KB(G^-) = KB(MCS(G^-))$ as per Lemma 40. Graph $MCS(G^-)$ is an FCS graph, implying its biclique graph is a GBS graph by Lemma 39. □

3.3.1.1 Mutually contained biclique graphs

A concept related to biclique graphs is the concept of *mutually contained biclique graphs*. In a bipartite graph $G = (X, Y, E)$, two bicliques K_1, K_2 are *mutually contained* if $K_1 \cap X \subset K_2 \cap X$ and $K_2 \cap Y \subset K_1 \cap Y$. We call the relationship between mutually contained bicliques a *mutual containment*.

The mutually contained biclique graph of G, represented as $KB_m(G)$, is the graph where the vertex set is $b(G)$, and two bicliques are neighbors if and only if they are mutually contained.

In 2020, Groshaus and Guedes [Groshaus and Guedes, 2020] proved that, for any bipartite graph G, $KB(G) = KB_m(G)^2$. That is, the biclique graph is the square of the mutually contained biclique graph. In the sequence, we present a characterization of the mutually contained biclique graphs of non-bichordal Helly CA bigraphs. Similarly to the case for biclique graphs, the mutually contained biclique graph of a graph is equal to the mutually contained biclique graph of its twin-free version.

The characterization is analogous to the one we presented for biclique graphs. We show that, for every non-bichordal Helly CA bigraph G, there exists an FCS graph H such that $KB_m(G) = KB_m(H)$ (Lemma 43), and demonstrate the general structure of mutually contained biclique graphs of FCS graphs (Lemma 42).

To simplify proofs, we use the following Lemma.

Lemma 41. *[Groshaus and Guedes, 2020] Let $G = (X, Y, E)$ be a bipartite graph. Then two bicliques K_1, K_2 are mutually contained if and only if $K_1 \cap X \subset K_2 \cap X$ or vice-versa.*

In the sequence, we present the fundamental structure for the mutually contained biclique graphs of the non-bichordal Helly CA bigraphs.

Definition 7. *Let $k \geq 6$ be an even number, and $n_1, ..., n_k \geq 0$. We define the* general mutually contained biclique structure *(GMCBS for short) graph, denoted by $\mathbb{G}^m_k(n_1, ..., n_k)$, in the following way:*

Let $V(\mathbb{G}^m_k(n_1, ..., n_k))$ be partitioned into the following sets:

- *$A = \{a_1, ..., a_k\}$, and*
- *$B_i = \{b_{i,1}, ..., b_{i,n_i}\}$ for $1 \leq i \leq k$.*

With the neighborhoods of the vertices defined as follows (consider cyclic summation, e.g. $k + 1 = 1$):

- *$N(b_{i,j}) = (B_i - \{b_{i,j}\}) \cup \{a_i, a_{i+1}\}$ for $1 \leq i \leq k$, and $1 \leq j \neq n_k$.*

- $N(a_i) = B_{i-1} \cup B_i \cup \{a_{i-1}, a_{i+1}\}$.

Lemma 42. *Let $G = \mathbb{G}_k(n_1, ..., n_k)$ with $k \geq 6$, $n_1, ..., n_k \geq 0$. Then $KB_m(G) = \mathbb{G}_k^m(n_1, ..., n_k)$.*

Proof. For ease of notation, let $F = \mathbb{G}_k^m(n_1, ..., n_k)$. The bicliques of G, as seen in the proof of Theorem 1, are the following:

- $A_i = \{c_{i-1}, c_i, c_{i+1}, v_i\} \cup W_i \cup U_{i-1}$ for all $1 \leq i \leq k$.
- $B_{i,j} = \{c_i, c_{i+1}\} \cup \{w_{i,m} | m \geq j\} \cup \{u_{i,l} | l \leq j\}$ for all $1 \leq i \leq k$, and $1 \leq j \leq n_i$.

Attribute to every biclique A_i the vertex a_i from, and to every biclique $B_{i,j}$ the vertex $b_{i,j}$ from F, for all $1 \leq i \leq k, 1 \leq j \leq n_i$. It suffices to show that two bicliques are mutually contained if and only if their corresponding vertices intersect.

Consider the separation of $V(G)$ according to Definition 2. Let $C_1, C_2 \subset C$ be the subset of odd-indexed and even-indexed elements of C, respectively. Also, let $V_1, V_2 \subset V$ be the subsets of odd an even index in V. The two partite sets of G, as seen in the proof of Theorem 12, are the following:

- $X = C_1 \cup V_2 \cup \bigcup_{i=2,i+2}^{k} W_i \cup \bigcup_{i=1,i+2}^{k-1} U_i$.
- $Y = C_2 \cup V_1 \cup \bigcup_{i=2,i+2}^{k} U_i \cup \bigcup_{i=1,i+2}^{k-1} W_i$.

Consider a biclique of the form $B_{i,j}$. The neighborhood of $b_{i,j}$ is $N(b_{i,j}) = (B_i - \{b_{i,j}\}) \cup \{a_i, a_{i+1}\}$. First, we prove that every edge that contains $b_{i,j}$ represent a mutual containment of bicliques. Suppose w.l.o.g that i is even.

Consider the intersection between $B_{i,j}$ and a biclique of the form $B_{i,j'}$ where $j' \neq j$. If $j' < j$, then $B_{i,j} \cap X \subset B_{i,j'} \cap X$. If $j < j'$, then $B_{i,j'} \cap X \subset B_{i,j} \cap X$. Therefore, the edge of the form $b_{i,j}b_{i,j'}$ corresponds to a mutual containment.

Consider, now, the intersection with biclique A_i. The only element of Y contained in A_i is c_i, implying $A_i \cap Y \subset B_{i,j} \cap Y$. Similarly, for biclique A_{i+1}, the only element in X is c_{i+1}, implying $A_{i+1} \cap X \subset B_{i,j} \cap X$. Therefore $b_{i,j}a_i, b_{i,j}a_{i+1}$ also represent valid mutual containments.

Now, we show that every vertex in $V(F) - N(b_{i,j})$ corresponds to a biclique that is not mutually contained with $B_{i,j}$. For bicliques of the form $B_{i',j'}$ with $i' \neq i$, and i' even, $w_{i,n_i} \in (B_{i,j} \cap X) - B_{i',j'}$ and $w_{i',n_{i'}} \in (B_{i',j'} \cap X) - B_{i,j}$, implying their intersections with X are not comparable. For i' odd, $w_{i,n_i} \in (B_{i,j} \cap X) - B_{i',j'}$ and $u_{i',1} \in (B_{i',j'} \cap X) - B_{i,j}$.

Now for a biclique of the form $A_x, x \neq i, i+1$, note that $w_{i,n_i} \in (B_{i,j} \cap X) - A_x$. If x is odd, then $c_x \in (A_x \cap X) - B_{i,j}$. If x is even, then $c_{x+1} \in (A_x \cap X) - B_{i,j}$.

Consider, now, the adjacencies of vertices of the form a_x, $1 \leq x \leq k$, corresponding to biclique A_x. As seen in previous paragraphs, the edges of the form $a_x b_{x,j}$ and $a_x b_{x-1,j}$ correspond to mutual containments, and the non-adjacent vertices of the form $b_{y,j}, y \neq x, x-1$ correspond to bicliques that are not mutually contained with A_x. All that is left to check, therefore, are the adjacencies between vertices of A.

For two vertices a_x, a_{x+1}, if x is odd, then $A_x \cap X = \{c_x\} \subset A_{x+1} \cap X$. If x is even, $A_x \cap Y = \{c_x\} \subset A_{x+1} \cap Y$. Now for two vertices $a_x, a_y, y \neq x-1, x+1$, if x, y are both odd or both even, the presence of c_x in $A_x - A_y$ and c_y in $A_y - A_x$ proves they are not mutually contained.

Suppose, then, that x is even and y is odd. Then $c_x \in (A_x \cap Y) - A_y$ and $c_{y-1} \in (A_y \cap Y) - A_x$.

Therefore, every edge in F corresponds to a mutual containment of bicliques in $b(G)$, and every pair of mutually contained bicliques in $b(G)$ are such that their corresponding vertices are neighbors in F. $\square$

Lemma 43. *For every twin-free non-bichordal Helly CA bigraph G, $KB_m(G) = KB_m(MCS(G))$.*

Proof. Consider the labeling of the vertices of $V(MCS(G))$ according to Definition 2. As seen on the proof of Lemma 40, it is possible to obtain G from $MCS(G)$ by removing a subset $V' \subseteq V$ from $V(MCS(G))$. Consider the bicliques of $MCS(G) = \mathbb{G}_k(n_1, ..., n_k)$.

- $A_i = \{c_{i-1}, c_i, c_{i+1}, v_i\} \cup W_i \cup U_{i-1}$ for all $1 \leq i \leq k$.
- $B_{i,j} = \{c_i, c_{i+1}\} \cup \{w_{i,m} | m \geq j\} \cup \{u_{i,l} | l \leq j\}$ for all $1 \leq i \leq k$, and $1 \leq j \leq n_i$.

Also consider the bipartition of the vertices of $MCS(G)$ as seen in the proof of Lemma 42:

- $X = C_1 \cup V_2 \cup \bigcup\limits_{i=2,i+2}^{k} W_i \cup \bigcup\limits_{i=1,i+2}^{k-1} U_i$.
- $Y = C_2 \cup V_1 \cup \bigcup\limits_{i=2,i+2}^{k} U_i \cup \bigcup\limits_{i=1,i+2}^{k-1} W_i$.

It suffices to show that, for all A_i, $1 \leq i \leq k$ and all $K \in b(MCS(G))$, A_i and K are mutually contained if and only if $A_i - \{v_i\}$ and K also are. Biclique A_i is mutually contained with $A_{i-1}, A_{i+1}, B_{i,j}, B_{i-1,j}$ for all $1 \leq j \leq n_i$. Suppose w.l.o.g. that i is even. Then $v_i \in X$.

Note that $\{c_i\} = A_i - \{v_i\} \cap Y$, and c_i is contained in all bicliques of the forms $A_{i-1}, A_{i+1}, B_{i,j}, B_{i-1,j}$. Therefore, all of those bicliques are mutually contained with A_i. Note that this still holds true if A_{i-1} and A_{i+1} are replaced with $A_{i-1} - \{v_{i-1}\}$ and $A_{i+1} - \{v_{i+1}\}$.

We now show that $A_i - \{v_i\}$ is not mutually contained with any other biclique in $MCS(G)$.

For $B_{l,j}$, $l \neq i, i-1, 1 \leq j \leq n_l$, $\{c_i\} = (A_i - \{v_i\} \cap Y) - B_{l,j}$. If l is even, $c_l \in B_{l,j} \cap Y - (A_i - \{v_i\})$. If l is odd, $c_{l+1} \in B_{l,j} \cap Y - (A_i - \{v_i\})$.

Consider, then, $A_j, j \neq i-1, i+1$. If j is even, then $c_i \in (A_i - \{v_i\}) \cap Y - A_j$ and $c_j \in A_j \cap Y - (A_i - \{v_i\})$. If j is odd, then $c_i \in (A_i - \{v_i\}) \cap Y - A_j$ and $c_{j+1} \in A_j \cap Y - (A_i - \{v_i\})$. Once again, A_j can be replaced with $A_j - \{v_j\}$ without changing the proof.

Therefore, $A_i - \{v_i\}$ is mutually contained with a biclique of $MCS(G)$ if and only if A_i also is, implying $KB_m(G) = KB_m(MCS(G))$. $\square$

With Lemma 43, we conclude our characterization.

Theorem 23. *A graph is the mutually contained biclique graph of a non-bichordal Helly CA bigraph if and only if it is a GMCBS graph.*

Proof. ($\Leftarrow$) graph F is the mutually contained biclique graph of $\mathbb{G}_k(n_1, ..., n_k)$.

($\Rightarrow$) let G be a non-bichordal Helly CA bigraph and G^- its twin-free version. $KB_m(G^-) = KB_m(MCS(G^-))$ as seen in Lemma 43, which in turn is equal to graph $\mathbb{G}_k^m(n_1, ..., n_k)$ for some values of $k, n_1, ..., n_k$. $\square$

Recall the class of normal-proper-Helly CA bigraphs defined in Subsection 3.1.4, and characterized in Theorem 14. In the sequence, we show that the mutually contained biclique graphs of NPH CA bigraphs are proper CA graphs.

Let $(C, \mathbb{I}, \mathbb{E})$ be an NPH model of a graph $G = (X, Y, E)$ with $\mathbb{I}$ representing X, and let $K \in b(G)$. Let $a_{\mathbb{I}}(K)$ ($a_{\mathbb{E}}(K)$) be the set of all points in C such that, for every $v \in K \cap X$ ($v \in K \cap Y$), $a(v)$ contains said point. Since the model is normal, $a_{\mathbb{I}}(K)$ and $a_{\mathbb{E}}(K)$ are arcs. Furthermore, since the model is Helly, $a_{\mathbb{I}}(K)$ and $a_{\mathbb{E}}(K)$ intersect, implying $a_{\mathbb{I}}(K) \cap a_{\mathbb{E}}(K)$ is an arc. We call that arc the *intersection arc* A_K of K. Note that, in a set of biclique points compatible with $(C, \mathbb{I}, \mathbb{E})$, every intersection arc of a biclique also contains its biclique point.

Lemma 44. *Let $(C, \mathbb{I}, \mathbb{E})$ be an NPH model of a graph $G = (X, Y, E)$, and for every biclique $K \in b(G)$, let p_K be a biclique point of K. Then two bicliques K_1, K_2 are mutually contained if and only if $p_{K_2} \in A_{K_1}$.*

Proof. As shown in [Groshaus and Guedes, 2020], since G is bipartite, K_1 and K_2 are mutually contained if and only if $K_1 \cap X \subset K_2 \cap X$ or $K_1 \cap Y \subset K_2 \cap Y$.

($\Rightarrow$) Suppose w.l.o.g. $K_1 \cap X \subset K_2 \cap X$. That implies every arc in $a(K_1 \cap X)$ contains point p_K, which implies $p_K \in a_{\mathbb{I}}(K)$.

($\Leftarrow$) Suppose w.l.o.g. $p_K \in a_{\mathbb{I}}(K)$. That implies that for every $v \in K_1 \cap X$, $v \in K_2$, implying K_1, K_2 are mutually contained. □

We use Lemma 44 in our proof that the mutually contained biclique graphs of NPH CA bigraphs are proper CA graphs. The arcs we use to build the proper models are defined in the sequence.

Let $(C, \mathbb{I}, \mathbb{E})$ be an NPH model of a graph $G = (X, Y, E)$, and for every biclique $K \in b(G)$, let p_K be a biclique point of K. Let the arc A'_K be defined as the largest arc that has the following properties:

- $s(A'_K) = p_K$.
- There is at least one vertex $v \in K$ such that $A'_K \subseteq a(v)$.
- No arcs of $\mathbb{I}$ whose corresponding vertices do not belong to K intersect A'_K.

We then define the *mutual containment arc* M_K of K as the largest arc between A'_K and $(p_K, t(a_{\mathbb{I}}(K)))$. Figure 3.33 demonstrates situations where $M_K = A'_K$ and $M_K = (p_K, t(a_{\mathbb{I}}(K)))$.

Lemma 45. *Let $(C, \mathbb{I}, \mathbb{E})$ be an NPH model of a graph $G = (X, Y, E)$ with $\mathbb{I}$ representing set X. Then the pair $(C, \{M_K | K \in b(G)\})$ is a circular arc model of $KB_m(G)$.*

Proof. It suffices to prove that, for two bicliques K_1, K_2, M_{K_1} and M_{K_2} intersect precisely if K_1 and K_2 are mutually contained.

($\Rightarrow$) Suppose M_{K_1} and M_{K_2} intersect. Without loss of generality, suppose $s(M_{K_1}) \in M_{K_2}$, and recall that $s(M_{K_1}) = p_{K_1}$. Two possibilities exist:

1. $M_{K_2} = A'_{K_2}$.
2. $M_{K_2} = (p_{K_2}, t(a_{\mathbb{I}}(K_2)))$.

Consider, first, case 1. Note that, by definition, every point p that A'_{K_2} crosses is such that all arcs of $\mathbb{I}$ that cross p correspond to vertices of $K_2 \cap X$. Therefore, the arcs of $\mathbb{I}$ who cross p_{K_1} all correspond to vertices of $K_2 \cap X$. Therefore, $K_1 \cap X \subset K_2 \cap X$.

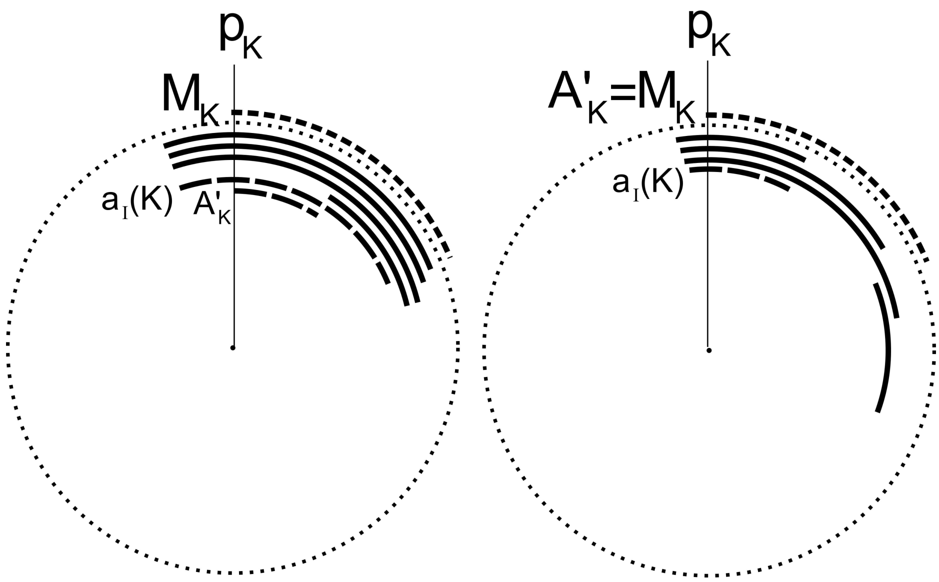

Figure 3.33: Examples of cases when $M_K = (p_K, t(a_{\mathbb{I}}(K)))$ (left) and $M_K = A'_K$ (right). In both examples, arcs A'_K and $a_{\mathbb{I}}(K)$ are shown.

Now for the second case. In this case, by definition, $M_{K_2} \subset a_{\mathbb{I}}(K_2)$. That implies that every single arc corresponding to a vertex of $K_2 \cap X$ crosses point p_{K_1}. Therefore $K_2 \cap X \subset K_1 \cap X$.

($\Longleftarrow$) Suppose without loss of generality that $K_1 \cap X \subset K_2 \cap X$. Then $p_{K_2} \in a_{\mathbb{I}}(K_1)$, as every arc corresponding to a vertex of $K_1 \cap X$ must also contains p_{K_2}. If $p_{K_2} \in (p_{K_1}, t(a_{\mathbb{I}}(K_1)))$, then $p_{K_2} \in M_{K_1}$. Suppose, then, that $p_{K_2} \in (s(a_{\mathbb{I}}(K_1)), p_{K_1})$.

In this case, let ϵ be a fraction of the smallest distance between two consecutive endpoints. Consider the arc $(p_{K_2}, p_{K_1} + \epsilon)$. Since the model is proper, no arc of $\mathbb{I}$ can be entirely contained in it, implying every arc of $\mathbb{I}$ that intersects it must cross one of the two biclique points. Since $K_1 \cap X \subset K_2 \cap X$, every arc that crosses p_{K_1} must also cross p_{K_2}, therefore, every arc of $\mathbb{I}$ that intersects (p_{K_2}, p_{K_1}) corresponds to a vertex of $K_2 \cap X$.

Note that, therefore, the arc $(p_{K_2}, p_{K_1} + \epsilon)$ is such that p_{K_2} is its s-endpoint, every vertex $v \in K_1$ is such that $a(v)$ contains it, and no arcs $I \in \mathbb{I}$ such that $v(I) \notin K_2$ intersect it. That implies $(p_{K_2}, p_{K_1} + \epsilon) \subset A'_{K_2}$.

Therefore $p_{K_1} \in A'_{K_2}$, implying $p_{K_1} \in M_{K_2}$, which in turn implies M_{K_1} and M_{K_2} intersect. □

Lemma 45 shows that the mutually contained biclique graphs of NPH circular arc bigraphs are circular arc graphs. Moreover, it is possible to create a proper circular arc model from the model described in the lemma, as every pair of arcs that are comparable share an endpoint. We go into more detail in Corollary 11.

Corollary 11. *Every NPH graph G is such that $KB_m(G)$ admits a proper circular arc model.*

Proof. It suffices to show that every pair of comparable arcs in $\{M_K | K \in b(G)\}$ shares an endpoint. We can then apply Lemma 18 to conclude that $KB_m(G)$ is proper.

Let $M_{K_1} \subset M_{K_2}$. If $M_{K_2} = A'_{K_2}$, then note that the arc $(p_{K_1}, t(A'_{K_2}))$ does not contain any s-endpoints of arcs from $\mathbb{I}$: if there was an s-endpoint, it would either belong to an arc corresponding to a vertex that is not in K_2, which may not happen according to the definition of A'_{K_2}, or it would belong to an arc that corresponds to a vertex of K_2, in which case the model is not normal. Therefore, every arc of $\mathbb{I}$ that is intersected by $(p_{K_1}, t(A'_{K_2}))$ contains p_{K_1}, implying their corresponding vertices are in K_1 and therefore A'_{K_1} and A'_{K_2} share their t-endpoint.

If $M_{K_2} = (p_{K_2}, t(a_{\mathbb{I}}(K_2)))$, then the arc $(p_{K_1}, t(a_{\mathbb{I}}(K_2)))$ intersects all arcs of $\mathbb{I}$ whose corresponding vertices are in K_1, implying, once again, that M_{K_1}, M_{K_2} share their t-endpoint. $\square$

Recall that, for any bipartite graph G, $(KB_m(G))^2 = KB(G)$. Since $KB(HCAB)$ is a subclass of proper CA graphs, we have it $KB_m(NPH)$ is a subclass of proper CA graphs whose squares are also proper CA graphs. Furthermore, since non-bichordal Helly CA bigraphs are NPH CA bigraphs, that implies the biclique graphs of non-bichordal Helly CA bigraphs are proper CA graphs whose square roots are also proper CA graphs.

That is an interesting fact, and raises the question of which other proper CA graphs have that same property. The characterization of proper CA graphs whose squares are also proper CA graphs is an open problem. Another open problem is determining the subclass of $KB(HCAB)$ in which every graph has a proper CA graph as a square root.

3.4 OTHER STUDIES

3.4.1 Study on bichordal Helly CA bigraphs

As mentioned in Subsection 3.1.1, our study on bichordal Helly CA bigraphs (i.e. Helly CA bigraphs that do not admit am induced cycle of length greater than 4) has so far led to an incomplete case study. This happens because, counter-intuitively, prohibiting large cycles actually leads to a larger number of permitted structures than focusing on the cases where one is present. In this subsection, we present what we have discovered in our attempt to characterize bichordal Helly CA bigraphs, and demonstrate the situation where the incomplete case study arises.

When a Helly CA bigraph G does not contain an induced cycle of length greater than 4, that implies one of two things:

- There exists a Helly bi-circular-arc model of G where no cycle covers the circle.
- For every Helly bi-circular-arc model of G, there exists a C_4 whose arcs cover the circle.

In the first case, the graph is a Helly interval bigraph, which we study in detail in Subsection 3.1.3. For the remainder of this subsection, we focus on the second case.

We define a *centralized bichordal graph* as a bichordal bipartite graph G for which there exists a set $C \subset V(G)$ which induces a C_4, such that every element of $V(G) - C$ is neighbor to exactly one vertex of C. We call C the *central cycle* of G. For centralized bichordal Helly CA bigraphs, we have a forbidden structure characterization similar to that we presented for non-bichordal Helly CA bigraphs.

The forbidden graphs necessary for this characterization are the T_2 seen in previous sections, as well as the F_3, which we introduce here, as seen in Figure 3.34.

Lemma 46. *Graph F_3 is not a Helly CA bigraph.*

Proof. We apply Lemma 1, showing that it is impossible to attribute a valid set of biclique points to F_3.

The bicliques are the following:

- $A = \{1, 3, 4, 6\}$
- $B = \{2, 4, 5, 7\}$
- $C = \{4, 6, 7, 9\}$

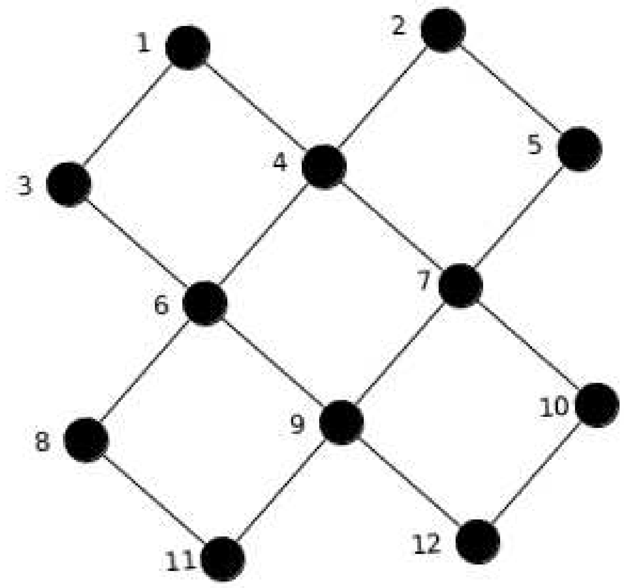

Figure 3.34: Forbidden graph F_3.

- $D = \{6, 8, 9, 11\}$
- $E = \{7, 9, 10, 12\}$
- $F = \{3, 4, 6, 8, 9\}$
- $G = \{4, 5, 7, 9, 10\}$
- $H = \{1, 2, 4, 6, 7\}$
- $I = \{6, 7, 9, 11, 12\}$

Firstly, note that $b(3) = \{A, F\}$ and $b(1) = \{A, H\}$, implying the points corresponding to F, A, H must be found consecutively in this clockwise order around the circle (or its reverse). Suppose F, A, H are in clockwise order w.l.o.g.

Now, note that $b(2) = \{B, H\}$ and $b(5) = \{B, G\}$. Therefore, the points corresponding to F, A, H, B, G must be consecutive in this clockwise order. Similarly, $b(10) = \{E, G\}$ and $b(12) = \{E, I\}$, implying order F, A, H, B, G, E, I. Furthermore, $b(11) = \{D, I\}$ and $b(8) = \{D, F\}$, implying the points corresponding to F, A, H, B, G, E, I, D must show up consecutively in this clockwise order around the circle.

Consider, now, the situation of vertex 4, with $b(4) = \{A, B, C, F, G, H\}$. For the points corresponding to the bicliques of $b(4)$ to be consecutive, the points corresponding to all bicliques must be found consecutively in clockwise order $C, F, A, H, B, G, E, I, D$. However, in this order, the bicliques in $b(7) = \{B, C, E, G, H, I\}$ are such that their corresponding points are not circularly consecutive.

Therefore, F_3 is not a Helly CA bigraph. □

In the sequence, we present results leading to the characterization of centralized bichordal Helly CA bigraphs.

Lemma 47. *Let G be a centralized bichordal graph, and let $C = \{c_1, c_2, c_3, c_4\}$ be its central cycle. If $u, v, w \in V(G) - C$ such that $N(u) \cap C = \{c_{i-1}\}$, $N(v) \cap C = \{c_i\}$ and $N(w) \cap C = \{c_{i+1}\}$ for some $1 \leq i \leq 4$, then $uv \notin E(G)$ or $vw \notin E(G)$.*

Proof. If uv, vw are both in $E(G)$, then $c_{i-1}, c_{i-2}, c_{i+1}, w, v, u$ induce a C_6, implying the graph is not bichordal. □

Lemma 48. *Let G be a centralized bichordal graph without an induced T_2, and let $C = \{c_1, c_2, c_3, c_4\}$ be its central cycle. If v_1, v_2 are such that $N(v_1) \cap C = N(v_2) \cap C = \{c_i\}$ for some $1 \leq i \leq 4$, and both contain neighbors in $N(c_{i+1}) - C$, then $N(v_1), N(v_2)$ are comparable.*

Proof. If $N(v_1)$ and $N(v_2)$ are not comparable, that implies there exist $w_1 \in N(C_{i+1}) \cap N(v_1) - N(v_2)$ and $w_2 \in N(C_{i+1}) \cap N(v_2) - N(v_1)$. However, in that case, $c_{i-1}, c_i, c_{i-2}, v_1, v_2, w_1, w_2$ induce a T_2. □

Lemma 49. *Let G be a centralized bichordal graph without induced T_2, F_3 subgraphs, and let $C = \{c_1, c_2, c_3, c_4\}$ be its central cycle. Let $V_i \subset V(G)$, with $1 \leq i \leq 4$ such that $v \in V_i$ if and only if $v \in N(c_i) - C$ and $(N(c_{i+1}) - C) \cap N(v) \neq \emptyset$. Then at least one set of the form V_i is empty.*

Proof. If no set of the form V_i is empty, an F_3 is induced. □

In the sequence, we present the definition of a structure analogous to the previously introduced FCS and FIS, which is the fundamental structure of every twin-free centralized bichordal Helly CA bigraph.

Definition 8. *Let a* Fundamental Bichordal Structure graph *(FBS for short), denoted by $g(n_1, n_2, n_3)$ for any $n_1, n_2, n_3 \geq 0$, be a graph defined as follows:*

Let $V(g(n_1, n_2, n_3))$ be separated into the following subsets:

- $C = \{c_1, c_2, c_3, c_4\}$.
- $V = \{v_1, v_2, v_3, v_4\}$.
- $W_i = \{w_{i,1}, ..., w_{i,n}\}$ *for all* $1 \leq i \leq 3$.
- $U_i = \{u_{i,1}, ..., u_{i,n}\}$ *for all* $1 \leq i \leq 3$.

With the neighborhoods of every vertex being defined as follows:

- $N(c_i) = \{c_{i-1}, c_{i+1}, v_i\} \cup W_i \cup U_{i-1}$ *for all* $i \in \{2, 3\}$.
- $N(c_4) = \{v_4, c_3, c_1\} \cup U_3$.
- $N(c_1) = \{v_1, c_4, c_2\} \cup W_1$.
- $N(v_i) = \{c_i\}$.
- $N(w_{i,j}) = \{c_i\} \cup \{u_{i,l} \in U_i | l \leq j\}$, *for all* $1 \leq j \leq n_i$.
- $N(u_{i,j}) = \{c_{i+1}\} \cup \{w_{i,l} \in W_i | l \geq j\}$, *for all* $1 \leq j \leq n_i$.

Figure 3.35 has an example of an FBS graph. Note the fundamental differences between an FBS and an FCS graph. Firstly, there is the trivial facts that FBS graphs always have the same size for their central cycle (set C) and are always bichordal. Second, there is the fact that, in an FBS graph, c_4, c_1 are two consecutive elements of the central cycle such that there are no two vertices $v \in N(c_4) - C, w \in N(c_1) - C$ such that v, w are neighbors. That is, of course, due to the fact that F_3 must not be an induced subgraph.

Note that Lemma 50 is analogous to Theorem 1.

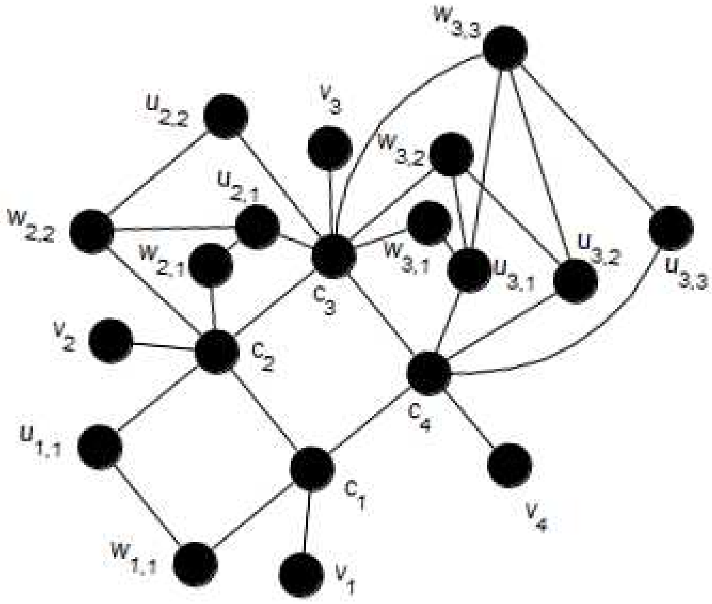

Figure 3.35: Graph $g(1,2,3)$.

Lemma 50. *Every FBS graph is a Helly CA bigraph.*

Proof. The bicliques are the following:

- $Z = \{c_1, c_2, c_3, c_4\}$.
- $A_i = \{c_{i-1}, c_i, c_{i+1}, v_i\} \cup W_i \cup U_{i-1}$ for all $i \in \{2, 3\}$.
- $A_1 = \{c_4, c_1, c_2, v_1\} \cup W_1$.
- $A_4 = \{c_3, c_4, c_1, v_4\} \cup U_3$.
- $B_{i,j} = \{c_i, c_{i+1}\} \cup \{w_{i,m} | m \geq j\} \cup \{u_{i,l} | l \leq j\}$ for $1 \leq i \leq 3$, $1 \leq j \leq n_i$.

It is easy to verify that every element on the list is a biclique. The proof that every biclique of the graph is on the list is analogous to the proof in Theorem 1, except citing that Z is the only biclique where every element of C is contained.

Let $S = \{z, a_1, ..., a_4\} \cup \bigcup_{i=1}^{3} \{b_{i,1}, ..., b_{i,n_i}\}$ be a set of points around a circle. Attribute point z to biclique Z, point a_i to biclique A_i for all $1 \leq i \leq 4$, and point $b_{i,j}$ to biclique $B_{i,j}$ for all $1 \leq i \leq 3$, $1 \leq j \leq n_i$. Suppose the points are distributed around the circle in the following clockwise order:

$$(z, a_1, b_{1,1}, ..., b_{1,n_1}, a_2, b_{2,1}, ..., b_{2,n_2}, a_3, b_{3,1}, ..., b3, n_3, a_4$$

Consider the family of bicliques for each vertex:

- For every $1 \leq i \leq 4$, $b(v_i) = \{A_i\}$.
- For every $1 \leq i \leq 3$, and $1 \leq j \leq n_i$, $b(w_{i,j}) = \{A_i\} \cup \{B_{i,l} | 1 \leq l \leq j\}$.
- For every $1 \leq i \leq 3$, and $1 \leq j \leq n_i$, $b(u_{i,j}) = \{A_{i+1}\} \cup \{B_{i,l} | j \leq l \leq n_i\}$.

- For every $i \in \{2,3\}$, $b(c_i) = \{Z, A_{i-1}, A_i, A_{i+1}\} \cup \{B_{i,j} | 1 \leq j \leq n_i\} \cup \{B_{i-1,j} | 1 \leq j \leq n_{i-1}\}$.
- $b(c_1) = \{Z, A_4, A_1, A_2\} \cup \{B_{1,j} | 1 \leq j \leq n_i\}$.
- $b(c_4) = \{Z, A_3, A_4, A_1\} \cup \{B_{3,j} | 1 \leq j \leq n_i\}$.

Note that, for every vertex, the points corresponding to the bicliques it is contained in are consecutive in the cycle. □

We use Lemmas 47, 48, 49 and 50 to prove the characterization.

Lemma 51. *A centralized bichordal graph G is a Helly CA bigraph if and only if it contains no T_2, F_3 as an induced subgraph.*

Proof. Let $C = (c_1, c_2, c_3, c_4)$ be the central C_4 of G. Let $V_i = N(c_i) - C$ for $1 \leq i \leq 4$. Suppose G is twin-free, as a graph is a Helly CA bigraph if and only if its twin-free version is.

By Lemma 47, every element of $v \in V_i$ must be such that $N(v) - C \subset V_{i-1}$ or $N(v) - C \subset V_{i+1}$, as otherwise, a C_6 would be induced. By Lemma 48, every pair of elements $v, w \in V_i$ such that $N(w) - C, N(v) - C \subset V_{i-1}$ ($N(w) - C, N(v) - C \subset V_{i+1}$) must be such that their neighborhoods are comparable, as otherwise, a T_2 is induced.

Let $V_{i,j} = \{v \in N(c_i) - C | \emptyset \neq N(v) - C \subset V_j\}$ for all $1 \leq i, j \leq 4$. By Lemma 49, at least one set of the form $V_{i,i+1}$ must be empty, as otherwise, there would be an induced F_3. Suppose w.l.o.g. that $V_{4,1}$ is empty. Symmetrically, $V_{1,4}$ is also empty.

Note that, under those restrictions, G is an induced subgraph of an FBS, with every non-empty $V_{i,i+1}$ being mapped to W_i, every non-empty $V_{i+1,i}$ being mapped to U_i, set C from graph G being mapped to set C from the FBS's definition, and any vertices of degree 1 being mapped to set V. Therefore, G is a Helly CA bigraph. □

Therefore, for centralized bichordal graphs, the problem of characterizing and recognizing Helly CA bigraphs is solved. The unfinished case study arises when we remove the requirement that the graph be centralized bichordal. The possibility of the existence of vertices that are neighbors to two or zero elements of a "central" C_4 leads to a plethora of permitted adjacencies, leading to an explosion in the number of cases to consider.

We say that a bichordal bipartite graph G is *quasi-centralized* if there exists a set $C \subset V(G)$ that induces a C_4 such that every vertex in $V(G) - C$ is neighbor to *at least* one element of C. It is in the study of quasi-centralized Helly CA bigraphs that the explosion in potential cases takes place.

Let $V_i = \{v \in V(G) - C | N(v) \cap C = \{c_i\}\}$, $1 \leq i \leq 4$, $V_{13} = \{v \in V(G) - C | N(v) \cap C = \{c_1, c_3\}\}$ and $V_{24} = \{v \in V(G) - C | N(v) \cap C = \{c_2, c_4\}\}$. For instance, it is possible for a vertex in V_1 to be neighbor to vertices in V_2 and V_{24} at the same time, and vertices in V_{24} and V_{13} may also be neighbors, all of that without breaking the Helly property. Not only that, but several of those situations may also happen simultaneously. This leads to a massive number of cases to consider. In the sequence, we show a list of known properties of these graphs.

Lemma 52. *Let G be a quasi-centralized bichordal Helly CA bigraph with $C = (c_1, c_2, c_3, c_4)$ being its central C_4. Then:*

1. *If $v, w \in V_1$ are such that $\emptyset \neq N(v) \cap V_2 \subset N(w) \cap V_2$, then $N(v) \cap V_{24} \subseteq N(w) \cap V_{24}$.*
2. *If $v, w \in V_1$ are such that $N(v) \cap V_{24}$ and $N(w) \cap V_{24}$ are not comparable, then every element of V_3 is neighbor to either all elements of $(N(v) \cap V_{24}) - N(w)$ or all elements of $(N(w) \cap V_{24}) - N(v)$.*

3. *If $v, w \in V_1$ are such that $N(v) \cap V_{24}$ and $N(w) \cap V_{24}$ are not comparable, then every element of V_3 is neighbor to either no elements of $(N(v) \cap V_{24}) - N(w)$ or no elements of $(N(w) \cap V_{24}) - N(v)$.*

4. *If $v \in V_1, w \in V_2$ are such that $vw \in E(G)$, then $V_{24} \cap N(v)$ or $V_{13} \cap N(w)$ is empty.*

Proof. **(1)** Since $N(v) \cap V_2 \neq \emptyset$, there exists $v' \in V_2$ such that $vv' \in E(G)$. Since $N(v) \cap V_2 \subset N(w) \cap V_2$, $v'w \in E(G)$ and there exists $w' \in (V_2 \cap N(w)) - N(v)$.

Suppose $N(v) \cap V_{24} \not\subseteq N(w) \cap V_{24}$. That implies there exists a vertex $x \in (V_{24} \cap N(v)) - N(w)$. In this situation, however, an F_1 is induced by $\{c_1, c_4, c_3, v, w, v', w', x\}$.
(2) Suppose there is a vertex $x \in V_3$ that is neighbor to neither $v' \in (V_{24} \cap N(v)) - N(w)$ nor $w' \in (V_{24} \cap N(w)) - N(v)$. Then a T_2 is induced by $\{c_2, v, v', w, w', c_3, x\}$.
(3) Suppose there is a vertex $x \in V_3$ that is neighbor to both $v' \in (V_{24} \cap N(v)) - N(w)$ and $w' \in (V_{24} \cap N(w)) - N(v)$. Then a C_6 is induced by $\{x, v', v, c_1, w, w'\}$.
(4) Suppose there is $x \in V_{24} \cap N(v)$ and $y \in V_{13} \cap N(w)$. Then a C_6 is induced by $\{x, v, w, y, c_3, c_4\}$. □

The items cited in Lemma 52 serve to exemplify how complex certain situations that lead to forbidden graphs may get. There are several cases not covered by any of the items cited, leading us to believe we might need another approach that does not involve this case study.

As of the time of writing, we have yet to discover an approach that allows us to simplify the search for a forbidden graph characterization of bichordal Helly CA bigraphs.

3.4.2 Upper bounds for the number of bicliques in different CA bigraph subclasses

In this subsection, we present upper bounds for the number of bicliques in certain CA bigraph subclasses. It is important to note that the bounds presented here are not necessarily optimal.

In 2000, Prisner [Prisner, 2000] demonstrated that every bipartite graph of n vertices has at most $2^{\frac{n}{2}}$ bicliques. That upper bound, however, is based on the fact that Prisner's paper allows bicliques in which one partite set is empty. Since we only consider bicliques for which both partite sets are non-empty, the basic upper bound we use for bipartite graphs with 3 or more vertices is $2^{\frac{n}{2}} - 2$ for even n and $2^{\lfloor \frac{n}{2} \rfloor} - 1$ for odd n, as per Lemma 53.

Lemma 53. *Every bipartite graph of $n \geq 3$ vertices has at most $2^{\frac{n}{2}} - 2$ bicliques for even n and at most $2^{\lfloor \frac{n}{2} \rfloor} - 1$ for odd n.*

Proof. Let $G = (X, Y, E)$ be a bipartite graph such that $|X| \leq |Y|$ and $|X| + |Y| = n$. Note that, for any non-empty subset $X' \subseteq X$, there is at most one biclique K such that $K \cap X = X'$, implying G has at most $2^{|X|} - 1$ bicliques. Consider, then, the following possibilities:

1. there exists a non-empty subset $X' \subseteq X$ for which there is no biclique $K \in b(G)$ such that $K \cap X = X'$.

2. for every non-empty subset $X' \subseteq X$, there exists a biclique $K \in b(G)$ such that $K \cap X = X'$.

Case 1 is simple: it implies G has at most $2^{|X|} - 2$ bicliques which, since $|X| \leq \lfloor \frac{n}{2} \rfloor$, is at most $2^{\lfloor \frac{n}{2} \rfloor} - 2$.

Consider, then, case 2. In this case, $|b(G)| = 2^{|X|} - 1$. We claim that $|X| < |Y|$. If $|X| = 1$, this is trivial, so suppose $|X| > 1$.

Enumerate $X = \{x_1, ..., x_m\}, m \leq n$. Let $K_0 \in b(G)$ such that $K_0 \cap X = X$, and, for every $1 \leq i \leq m$, let $K_i \in b(G)$ such that $K_i \cap X = X - \{x_i\}$.

Note that, since K_0 is a biclique, there exists a vertex $y_0 \in Y$ such that $N(y_0) = X$, and since K_i is a biclique for every $1 \leq i \leq m$, there is a vertex $y_i \in Y$ such that $N(y_i) = X - \{x_i\}$. Furthermore, for all $0 \leq i, j \leq m$, y_i, y_j are different vertices precisely if $i \neq j$. Therefore, $|Y| \geq |X| + 1$. If n is even, then $|Y| \geq |X| + 2$, and $2^m - 1 \leq 2^{\frac{n}{2}} - 2$. If n is odd, then $m \leq \lfloor \frac{n}{2} \rfloor$ and therefore, $2^m - 1 \leq 2^{\lfloor \frac{n}{2} \rfloor} - 1$. □

It is possible to find extremal cases for both even and odd values of n: *crown graphs* (Definition 9) are extremal cases for even n, which is also the case for Prisner's upper bound [Prisner, 2000], and a crown graph with a bi-universal vertex being added to one of the partite sets is an extremal case for odd n.

3.4.2.1 Proper and cross-proper CA bigraphs

In the sequence, we demonstrate an upper bound for proper CA bigraphs and, consequently, for CCB, doubly-CCB and CA bigraphs in general. The bound we present depends on the fact that all *crown graphs* are proper CA bigraphs for an even number of vertices.

Definition 9. *A* crown graph $S_n = (X, Y, E), n \geq 1$ *is a bipartite graph with* $X = \{x_1, ..., x_n\}$*,* $Y = \{y_1, ..., y_n\}$*, such that, for* $1 \leq i, j \leq n$*,* $x_i y_j \in E(G)$ *precisely when* $i \neq j$*.*

Lemma 54 demonstrates that the $2^{\lceil \frac{n}{2} \rceil} - 2$ upper bound is tight for crown graphs of at least 3 vertices. That, in association with the fact that every crown graph admits a proper bi-circular-arc model, allows us to conclude that the upper bound is also tight for proper CA bigraphs, and every subclass of it.

Lemma 54. *The crown graph* S_n *has* $2^n - 2$ *bicliques for any* $n \geq 2$*.*

Proof. In Prisner's work [Prisner, 2000], an upper bound of $2^{\frac{n}{2}}$ bicliques is shown for graphs of n vertices, and crown graphs of index greater than 3 are proven to be extremal cases for that bound. In our upper bound, however, we count two fewer bicliques. The reason for that, as stated before, is that we do not count subgraphs where one partite set is empty as bicliques. □

We must now prove that every crown graph has a proper bi-circular-arc model.

Lemma 55. *Crown graphs are proper CA bigraphs.*

Proof. Consider graph S_n with its partite sets being $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$ as described in Definition 9. We apply Theorem 18. Since no pair of neighborhoods in Y is comparable, it suffices to show X admits a CCB order.

The order $(x_1, ..., x_n)$ is trivially a CCB order, as every vertex in Y is neighbor to every element of X but one. □

For the case of graphs with an odd number of vertices, we show that adding a bi-universal vertex to one of the partite sets of a crown graph leads to an extremal graph for the odd case. We then show how to construct a simple proper bi-circular-arc model of such a graph.

Lemma 56. *The graph resulting from adding, to* S_n*, a bi-universal vertex on either partite set has* $2^n - 1$ *bicliques for any* $n \geq 2$*.*

Proof. Let $G = (X, Y, E)$ with $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n, v\}$ such that $G - v$ is an S_n and v is bi-universal. We show that, for every non-empty subset $X' \subseteq X$, there exists a biclique $K \in b(G)$ such that $K \cap X = X'$.

For $X' = X$, the biclique is $X \cup \{v\}$. Consider, then, that X' is a proper subset of X. Similar to the proof of Lemma 54, $X' \cup \{y_i \in Y | x_i \notin X'\} \cup \{v\}$ is a biclique. □

Lemma 57. *The graph resulting from adding, to S_n, a bi-universal vertex on either partite set is a proper CA bigraph.*

Proof. Let $G = (X, Y, E)$ with $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n, v\}$ such that $G - v$ is an S_n and v is bi-universal. Let $(C, \mathbb{I}, \mathbb{E})$ be a bi-circular-arc model of G with $\mathbb{E}$ representing Y as a set of $n + 1$ disjoint arcs with $(s(a(v)), s(a(y_1)), ..., s(a(y_n))$ being a clockwise order. Note that $\mathbb{E}$ is a proper family.

Since every vertex $x \in X$ is such that its neighborhood equals all elements of Y but one, that implies it is possible to draw an arc $a(x) \in \mathbb{I}$ on C that intersects all arcs corresponding to vertices of $N(x)$ without intersecting any arcs corresponding to $Y - N(x)$. Since no two neighborhoods in X are comparable, $\mathbb{I}$ is necessarily a proper family. Therefore, $(C, \mathbb{I}, \mathbb{E})$ is a proper bi-circular-arc model of G. □

With Lemmas 54, 55, 56 and 57, we conclude the following corollary:

Corollary 12. *The upper bound of $2^{\lfloor \frac{n}{2} \rfloor} - 2$ for even n and $2^{\lfloor \frac{n}{2} \rfloor} - 1$ for odd n is tight for proper CA bigraphs of $n \geq 3$ vertices.*

Corollary 12 allows us to conclude that this upper bound is also tight for CCB graphs, doubly-CCB graphs, cross-normal CA bigraphs and CA bigraphs in general, as per the containment relations presented in Section 3.2.

Furthermore, the graphs described in Lemma 57 are also cross-proper CA bigraphs, as shown in Lemma 58. This allows us to conclude the upper bounds for cross-proper and normal CA bigraphs.

Lemma 58. *The graph resulting from adding, to S_n, a bi-universal vertex on either partite set is a cross-proper CA bigraph.*

Proof. Let $G = (X, Y, E)$ with $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n, v\}$ such that $G - v$ is an S_n and v is bi-universal. We construct a cross-proper bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ of G. Once again, $\mathbb{I}$ stands for set X, and $\mathbb{E}$ for set Y.

Distribute the endpoints of $\mathbb{I}$ around the circle in the clockwise order $(s(a(x_1)), ..., s(a(x_n)), t(a(x_1)), ..., t(a(x_n)))$. In simple therms, every s-endpoint is placed, in order of their index, and then every t-endpoint, also ordered by their index. Suppose w.l.o.g. that each pair of consecutive endpoints is at a distance of 1 from one another, including $t(a(x_n))$ and $s(a(x_1))$. Note that the arc $(t(a(x_n)), s(a(x_1)))$ is not intersected by any element of $\mathbb{I}$.

For $\mathbb{E}$, start by making $a(v) = (s(a(x_1)) - \frac{1}{2}, t(a(x_1)) - \frac{1}{2}$. Note that $a(v)$ intersects every arc of $\mathbb{I}$ but contains none, and is also contained by none.

For any $1 \leq i \leq n$, create arc $a(y_i) = (t(a(x_i)) + \frac{1}{2}, s(a(x_i) - \frac{1}{2})$. Arc $a(y_i)$ is not contained in any arc of $\mathbb{I}$ as it intersects the portion of the circle no element of $\mathbb{I}$ crosses, and it also does not contain any arc of $\mathbb{I}$ as all of them intersect $a(x_i)$, which $a(y_i)$ does not. Furthermore, $a(y_i)$ intersects all arcs of $\mathbb{I}$ except $a(x_i)$, as it contains $t(a(x_j))$ for all $j < i$ and $s(a(x_k))$ for all $k > i$.

Therefore, $(C, \mathbb{I}, \mathbb{E})$ is a cross-proper bi-circular-arc model of G. □

Lemma 58 also proves, by consequence, that crown graphs are cross-proper CA bigraphs, as the class is hereditary over induced subgraphs. Therefore, it allows us to conclude Corollary 13.

Corollary 13. *The upper bound of* $2^{\lfloor \frac{n}{2} \rfloor} - 2$ *for even* n *and* $2^{\lfloor \frac{n}{2} \rfloor} - 1$ *for odd* n *is tight for cross-proper CA bigraphs of* $n \geq 3$ *vertices.*

Corollary 13 also shows that the upper bounds are tight for normal CA bigraphs as well, as per the containment relations presented in Section 3.2.

3.4.2.2 Helly CA bigraphs

Since we have shown that proper CA bigraphs and, therefore, all its superclasses have an exponential number of bicliques, we divert our attention to Helly CA bigraphs. In this subsection, we show that the number of bicliques in a Helly CA bigraph of n vertices is linear. This result is separated into two parts: graphs that contain an induced C_6, and graphs that are C_6-free. For the first part, it suffices to reference results from Subsection 3.3.1, and for the second part, we apply the knowledge that the number of bicliques in a C_6-free Helly CA bigraph is equal to the number of cliques in its square.

We start with the C_6 case in Lemma 59.

Lemma 59. *A Helly CA bigraph* G *with* n *vertices that contains an induced* C_6 *has at most* n *bicliques, with* C_6 *itself being an extremal case.*

Proof. Suppose G is twin-free, since any non-twin-free graph whose twin-free version is G has the exact same number of bicliques as G but more vertices than G.

By Lemma 40, $KB(G) = KB(MCS(G))$. Let $MCS(G) = \mathbb{G}_6(n_1, \dots, n_6)$ for some value of $n_1, \dots, n_6$. As shown in the proof of Lemma 40, $|V(G)| \geq |V(MCS(G))| - 6$.

Furthermore, by Lemma 39, the number of vertices in $KB(MCS(G))$ (i.e. the number of bicliques in $MCS(G)$) is $n_1 + \dots + n_6 + 6$. By Definition 2, $MCS(G)$ has $2(n_1 + \dots + n_6) + 12$ vertices, implying $|b(MCS(G))| \leq |V(MCS(G))| - 6 \leq |V(G)|$.

The fact that C_6 is an extremal case is trivial, as it has exactly 6 bicliques. □

For the C_6-free case, we apply the fact that, for any C_6-free Helly CA bigraph G without isolated vertices, the set of bicliques of G is equal to the set of bicliques of G^2.

Lemma 60. *Let* G *be a* C_6*-free Helly circular arc bigraph with* n *vertices. Then* G *has at most* n *bicliques.*

Proof. Let $V' \subset V(G)$ be the set of non-isolated vertices of G, and $G' = G[V']$. Let $|V'| = n' \leq n$. By Theorem 3, $(G')^2$ is a Helly circular arc graph. Furthermore, according to [Durán et al., 2006], $(G')^2$ has at most n' cliques. As seen in Lemma 8, that implies the number of bicliques in G' is also at most n'.

Since isolated vertices do not form bicliques, G has at most $n' \leq n$ bicliques. □

And with that, we conclude the following corollary.

Corollary 14. *A Helly circular arc bigraph with* n *vertices has at most* n *bicliques.*

Note that Corollary 14 could also be proven by utilizing an argument similar to the one presented in [Durán et al., 2006], and showing that the intersection region of every biclique in a Helly bi-circular-arc model ends on a t-endpoint, and every t-endpoint ends at most one biclique's intersection region. We chose our approach because it employs mostly results presented in the thesis, making for a clearer and more self-contained proof.

4 CONCLUSION

As mentioned in the introduction, the class of circular arc bigraphs is relatively unexplored, with most results on it being very recent. In our work, we provided our contribution to the studies of the class.

One of the central concepts in our thesis are the Helly subclasses of circular-arc and interval bigraphs. The definition of those classes relies on the bipartite Helly property, introduced by [Groshaus and Szwarcfiter, 2011], but adapted to our context of pairs of families. We have provided forbidden graph characterizations for Helly interval bigraphs, and for the non-bichordal and proper-Helly subclasses of Helly CA bigraphs.

In all of those characterizations, we introduce a fundamental structure, and prove that every graph in the class is an induced subgraph of that structure. The fact that Helly interval bigraphs, non-bichordal Helly circular arc bigraphs, and proper-Helly circular arc bigraphs are all variations of a fundamental structure is fascinating and useful, as it allows any problem on those classes to be studied in terms of those structures. For instance, we apply the fundamental structures of non-bichordal Helly CA bigraphs and Helly interval bigraphs to prove that these classes are subclasses of proper-Helly CA bigraphs, and we also apply the fundamental structure for our simple characterization of the biclique graphs of non-bichordal Helly circular arc bigraphs.

The approach used in those characterizations, however, is of little use in studying classes with less restrictive definitions. First of all, it is necessary that an easily identifiable "central subgraph" is present in all graphs of the class. Second, the adjacencies between the so-called central subgraph and the other vertices in the graph must fall into easily identifiable sets. For any class of graphs in which no intuitive central subgraph is present, it is nearly impossible to find a suitable fundamental structure.

For instance, recall that we have also characterized the subclass of bichordal Helly circular arc bigraphs that we call the *centralized* subclass. Just like in the other characterizations, we apply a fundamental structure. That class, however, does not include all bichordal Helly circular arc bigraphs, and our search for a complete set of forbidden graphs for Helly circular arc bigraphs remains to be concluded, as our attempt to apply the fundamental structure approach led us to an as of yet incomplete case study, implying we might need to change our approach.

We have also provided a polynomial time ($O(n^4)$) recognition algorithm for Helly circular arc bigraphs without isolated vertices. The algorithm relies on a reduction to the recognition of Helly circular-arc graphs for the cases where the input graph is C_6-free, and on our characterization of non-bichordal Helly circular arc bigraphs for the cases where the input graph contains an induced C_6. The algorithm may not be optimal, and it does not cover cases where the input graph may contain isolated vertices. Both the problems of finding an optimal algorithm and one that covers the isolated vertex cases are open.

Another part of our work involved looking into the intersections and containments between several different subclasses of circular arc bigraphs. The classes we studied include the previously mentioned Helly interval bigraphs and Helly circular arc bigraphs, as well as circular convex bipartite graphs, proper interval bigraphs and proper circular arc bigraphs, as well as the classes of normal, cross-normal, and cross-proper circular arc bigraphs, which we introduced. We have shown that the Helly and proper subclasses of circular arc bigraphs are not comparable, while Helly interval bigraphs are a subclass of proper interval bigraphs. We have also proven that non-bichordal Helly circular arc bigraphs are a subclass of proper circular arc bigraphs, and that both proper and Helly circular arc bigraphs are subclasses of circular convex bipartite graphs.

As for the other normal and cross-proper classes we introduced, we have shown that every proper circular arc bigraph is cross-normal, and every cross-proper circular arc bigraph is normal. We have also shown that cross-normal circular arc bigraphs are a superclass of circular convex bipartite graphs. It remains an open problem to know whether the proper and cross-proper subclasses are comparable.

On the subject of biclique graphs, we have studied the biclique graphs and mutually contained biclique graphs of Helly circular arc bigraphs. We have shown that the class of biclique graphs of C_6-free Helly circular arc bigraphs are a subclass of the clique graphs of Helly circular arc graphs applying some of the same arguments we use for the C_6-free case of our recognition algorithm. Furthermore, we have shown that, due to the biclique graph of C_6 not being the clique graph of any Helly circular arc graph, the class of biclique graphs of Helly circular arc bigraphs and that of clique graphs of Helly circular arc graphs are not comparable.

Also, just like the clique graphs of Helly circular arc graphs, we have shown the biclique graphs of Helly circular arc bigraphs are proper circular arc graphs. We have also provided simple characterizations of the biclique graphs and mutually-contained biclique graphs of non-bichordal Helly circular arc bigraphs based on the fundamental structure we use for our forbidden graph characterization.

Still on the subject of biclique graphs, we showed that the mutually contained biclique graphs of bipartite graphs that admit a bi-circular-arc model that is simultaneously proper, normal and Helly are proper circular arc graphs. To prove this inclusion, we had to rely on the normal, proper and Helly properties to ensure that the points corresponding to mutually contained bicliques of any given biclique are all circularly consecutive, and that the mutual containment arc of a biclique only contains points of bicliques that are mutually contained with it. There is a possibility that this proof may be generalized for a larger subclass of NPH circular arc bigraphs, as a model only needs to be normal and Helly for the mutually contained biclique points of a given biclique to be circularly consecutive, and the proper property is only relevant to ensure that every biclique point crossed by a biclique's mutual containment arc corresponds to a mutually contained biclique. It may be possible to construct proper circular-arc models for the mutually contained biclique graphs of any normal-Helly circular arc bigraph with some different form of mutual containment arc.

Open problems for the subject of biclique graphs involve a general characterization of the biclique graphs and mutually contained biclique graphs of Helly circular arc bigraphs, as well as checking which other subclasses of Helly circular arc bigraphs are such that their mutually biclique graphs are proper circular-arc.

Aside from that, we have also found several other simple results on circular arc bigraphs, including:

- Demonstrating upper bounds for the numbers of bicliques in Helly and proper circular arc bigraphs.

- Showing that it is possible to recognize circular convex bipartite graphs in linear time by checking whether a graph's biadjacency matrix has the circular 1s property, using Booth and Lueker's algorithm [Booth and Lueker, 1976].

- Presenting structural properties of proper-Helly circular arc bigraphs.

Table 4.1 showcases the current state of problems pertaining to complexity of recognition and biclique numbers in the subclasses of CA bigraphs we have studied. In the table, every biclique number bound is strict, with n being the number of vertices. For each entry on the table, we reference a theorem from the thesis that proves it, or a publication that contains the proof. For the recognition column, *Poly* means the class is recognizable in polynomial time, and *Linear* means the class is recognizable in linear time. Entries marked **OPEN** imply that the complexity of that entry is an open problem.

The abbreviations used in the table are the ones used in the Venn diagram in Figure 3.31, as well as the following additional ones:

H-H* stands for Helly CA bigraphs without isolated vertices.

N stands for normal CA bigraphs.

CP stands for cross-proper CA bigraphs.

NPH stands for normal-proper-Helly CA bigraphs.

In the table, classes for which we have a forbidden graph characterization are marked as recognizable in polynomial time. That is due to the fact that testing for a fixed finite set of induced subgraphs can be done in polynomial time, and testing the presence of induced cycles of length greater than 4 in bipartite graphs also can [Nikolopoulos and Palios, 2007]. Furthermore, for classes that have $C_n{}^*, n > 4$ as forbidden graphs, if an induced cycle of length greater than 4 is found, the vertices from outside the cycle can be easily tested for neighbors in the cycle in quadratic time.

Table 4.1: Current state of problems pertaining to recognition and biclique numbers in CA bigraph subclasses.

	Recognition	Biclique number
`CAB`	**OPEN**	$O(2^{n/2})$ (Corollary 12)
`CN`	**OPEN**	$O(2^{n/2})$ (Corollary 12)
`CCB`	*Linear* (Theorem 17)	$O(2^{n/2})$ (Corollary 12)
`D-CCB`	*Linear* (Theorem 17)	$O(2^{n/2})$ (Corollary 12)
`H`	**OPEN**	$O(n)$ (Corollary 14)
`H-H*`	*Poly* (Theorem 6)	$O(n)$ (Corollary 14)
`P`	*Linear* [Safe, 2019]	$O(2^{n/2})$ (Corollary 12)
`PH`	*Poly* (Theorem 16)	$O(n)$ (Corollary 14)
`NPH`	*Poly* (Theorem 14)	$O(n)$ (Corollary 14)
`CP`	**OPEN**	$O(2^{n/2})$ (Corollary 13)
`HI`	*Poly* (Theorem 7)	$O(n)$ (Corollary 14)
`PI`	*Linear* [Spinrad et al., 1987]	$O(n)$ [Brown and Lundgren, 2010]
`NBH`	*Poly* (Corollary 2)	$O(n)$ (Corollary 14)

4.1 FUTURE WORK

In the sequence, we delineate a handful of research topics and open questions to be studied in future works. The list is not exhaustive, as there may be many other potential topics we have not considered.

4.1.1 Studying intersection subclasses

In the circular arc graph context, graphs that admit a model that simultaneously verifies multiple relevant properties (e.g. Helly, normal, proper) have yielded several interesting results, such as characterizations and efficient recognition algorithms for Normal-Helly [Lin et al., 2011] and proper-Helly [Lin et al., 2007] circular arc graphs, as well as a proof that a graph is a proper-Helly circular-arc graph if and only if it is the clique graph of a proper-Helly circular arc graph [Lin et al., 2010]. The question of what may be discovered by applying this concept to circular arc bigraphs arises naturally from that.

Corollaries 4, 5, and 11, as well as Subsection 3.1.4, are examples of that concept applied to circular arc bigraphs, but they only scrape the surface of what may be learned with this concept. A more in-depth study of circular arc bigraph subclasses defined in this manner has yet to be made. Such a study would involve looking into the structural and computational properties of these graphs, as well as their biclique graphs.

4.1.2 A complete characterization and recognition algorithm for Helly circular arc bigraphs

As shown in this thesis, we have forbidden graph characterizations for several relevant subclasses of Helly circular arc bigraphs, including Helly interval bigraphs, non-bichordal Helly circular arc bigraphs, and bichordal Helly circular arc bigraphs that admit a *central cycle* as seen in Lemma 51. Those three characterizations rely on very similar arguments, always involving a simple structure that every graph in the subclasses conform to.

Our attempt to characterize Helly circular arc bigraphs that belong to neither of those classes has led to an as of yet incomplete case study with an extensive set of cases, as seen in Subsection 3.4.1. Future attempts to characterize this subclass involve either trying to finish the aforementioned case study, or discovering another approach that allows us to reach a characterization without going through it. The arguments used for the other cases do not apply in this, since there is no central cycle or path around which all other adjacencies occur.

As for the algorithm, the one standing problem is that it does not apply to all bipartite graphs that contain isolated vertices. This problem does not exist on the non-bipartite case, since Helly circular-arc graphs that are not interval graphs are always connected. For Helly circular arc bigraphs, we know that a bipartite graph $G = (X, Y, E)$ with isolated vertices is a Helly circular arc bigraph if and only if the subgraph induced by its non-isolated vertices admits a Helly bi-circular-arc model where one of the families does not cover the circle. That implies one of the partite sets, say, V, is such that $G^2[V]$ is an interval graph. That is not a sufficient condition, though, since both partite sets of a C_6 induce triangles, but a $C_6{}^*$ is not a Helly circular arc bigraph.

4.1.3 Studying containments and intersections between other subclasses of circular arc bigraphs

In Section 3.2, we present containment and intersection relations between several different subclasses of circular arc bigraphs, including circular convex bipartite graphs, proper circular-arc bigraphs and Helly circular arc bigraphs. We also introduce cross-proper, normal and cross-normal circular arc bigraphs, and prove some containments between them. Expanding this study to different subclasses of circular arc bigraphs, and answering any open questions that remain about the ones we presented, are interesting topics for future research.

On the subject of open problems, it is currently unknown whether the proper subclass is contained in the cross-proper subclass, and very little is known about the relationship between circular convex bipartite graphs and the cross-proper and normal subclasses. We currently believe that cross-proper circular arc bigraphs are a superclass of proper circular arc bigraphs, as we have not found any forbidden graph for the cross-proper class that is not forbidden for the proper class, and in the case of interval bigraphs, the classes are equivalent [Brown and Lundgren, 2010]. It is important to note that, for circular arc bigraphs, the classes are not equivalent, as the T_2 is a cross-proper circular arc bigraph, but not a proper circular arc bigraph, as it is the bipartite complement of C_6^* [Safe, 2019].

Aside from that, including other subclasses of circular arc bigraphs to our study of containment relations is potentially interesting. One set of subclasses to consider are the ones mentioned in Subsection 4.1.1, as well as unit circular arc bigraphs [Basu et al., 2013] and others.

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