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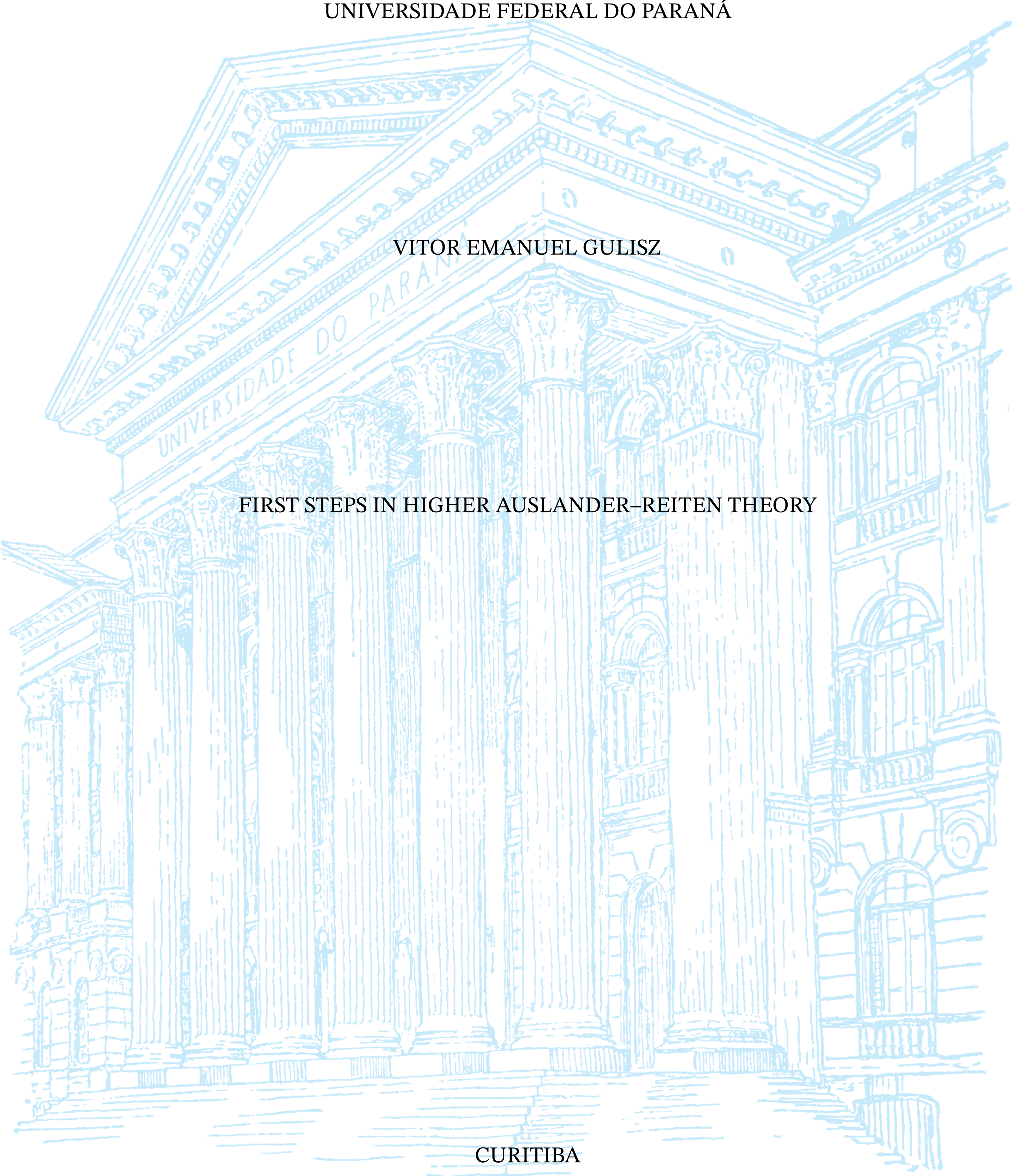

VITOR EMANUEL GULISZ

FIRST STEPS IN HIGHER AUSLANDER–REITEN THEORY

CURITIBA

2021

VITOR EMANUEL GULISZ

FIRST STEPS IN HIGHER AUSLANDER–REITEN THEORY

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Orientador: Edson Ribeiro Alvares

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"Time may change me
But I can't trace time."
David Bowie

RESUMO

Este texto é uma introdução sólida e acessível à teoria de Auslander–Reiten superior no contexto de módulos finitamente gerados sobre álgebras de Artin. Nós introduzimos esta teoria através da abordagem funtorial, passando de conceitos fundamentais de categorias de funtores aos principais resultados da teoria de Auslander–Reiten superior. Em particular, nós discutimos sequências n-quase cindidas, subcategorias n-conglomerado inclinantes, a Correspondência de Auslander Superior e os n-transladados de Auslander–Reiten.

Palavras-chaves: Teoria de Auslander–Reiten superior. Álgebra de Artin. Sequência n-quase cindida.

ABSTRACT

This text is a solid and accessible introduction to higher Auslander–Reiten theory in the context of finitely generated modules over Artin algebras. We introduce this theory through the functorial approach, by passing from fundamental concepts of functor categories to the main results of higher Auslander–Reiten theory. In particular, we discuss n-almost split sequences, n-cluster tilting subcategories, the Higher Auslander Correspondence and the n-Auslander–Reiten translations.

Key-words: Higher Auslander–Reiten theory. Artin algebra. n-Almost split sequence.

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Introduction

Let Λ be an Artin algebra and $\operatorname{mod}\Lambda$ be the category of finitely generated modules over Λ. In the 1970s Maurice Auslander and Idun Reiten have begun the study of almost split sequences in $\operatorname{mod}\Lambda$, which are certain exact sequences of length 1 in $\operatorname{mod}\Lambda$. They have proved their existence, as well as they have introduced certain functors, which are now called the Auslander–Reiten translations, to study the end terms of such sequences. In doing so, they have developed a theory that is nowadays known as Auslander–Reiten theory, which has its origin in [5], [6], [11], [12], [13] and [14].

Surprisingly, in the 2000s Osamu Iyama has introduced in [25] and [24] a generalization of Auslander–Reiten theory, which is now known as higher Auslander–Reiten theory. This generalization is given in the sense that for each positive integer n we can develop an "n-Auslander–Reiten theory" in such a way that Auslander–Reiten theory corresponds to the case $n = 1$. To be more precise, in higher Auslander–Reiten theory, instead of studying almost split sequences in $\operatorname{mod}\Lambda$, we study n-almost split sequences, which are certain exact sequences of length n in $\operatorname{mod}\Lambda$ that generalize the notion of an almost split sequence. In this case, this generalization is given in such a way that the concepts of a 1-almost split sequence and of an almost split sequence coincide.

However, when studying n-almost split sequences for $n \geqslant 2$, one faces a problem, namely, that there are no n-almost split sequences in $\operatorname{mod}\Lambda$ for $n \geqslant 2$. Nevertheless, Osamu Iyama has shown that certain subcategories of $\operatorname{mod}\Lambda$, called n-cluster tilting subcategories of $\operatorname{mod}\Lambda$, in fact have n-almost split sequences. Therefore, it was with these subcategories that Osamu Iyama has developed higher Auslander–Reiten theory. Indeed, he has shown that we can also generalize the Auslander–Reiten translations to n-Auslander–Reiten translations in n-cluster tilting subcategories of $\operatorname{mod}\Lambda$. In doing so, he has also verified that we can use the n-Auslander–Reiten translations to study the end terms of n-almost split sequences in such subcategories, just as was the case for almost split sequences in $\operatorname{mod}\Lambda$.

What is also interesting is that Osamu Iyama has also proved a generalization of a very important result, called the Auslander Correspondence, to the context of higher Auslander–Reiten theory by generalizing Auslander algebras to n-Auslander algebras. This result proved by Osamu Iyama is here referred to as the Higher Auslander Correspondence. This is a very important result not only because of its consequences, but also because it highlights that n-cluster tilting subcategories of $\operatorname{mod}\Lambda$ are indeed appropriate generalizations of the category $\operatorname{mod}\Lambda$, thereby supporting higher Auslander–Reiten theory.

The purpose of this text is to provide a solid and accessible introduction to higher Auslander–Reiten theory in the context of finitely generated modules over Artin algebras.

The main motivation for writing such a text comes from the fact that nowadays there is a lack of introductory texts to higher Auslander–Reiten theory, given that this is a very recent theory. Therefore, we hope that this text will help to fill this gap, as we present here foundational material on this theory and many different proofs for known results of this theory.

We remark that higher Auslander–Reiten theory has also motivated the introduction of higher homological algebra, which has its origin in [22] and [28]. In this theory, we study $(n+2)$-angulated, n-abelian and n-exact categories, that are generalizations of triangulated, abelian and exact categories, respectively, which correspond to the case $n = 1$. Moreover, we may also study higher Auslander–Reiten theory through the viewpoint of higher homological algebra, as it was done in [29], for example. However, for the sake of simplicity, we have chosen not to explicitly discuss higher homological algebra in this text.

The prerequisites of this text are only basic knowledge on category theory and on module theory. We do not assume that the reader knows Auslander–Reiten theory, hence this is a text which allows the reader to learn both Auslander–Reiten theory and higher Auslander–Reiten theory at the same time. We also remark that it is recommended that the reader has some familiarity with homological algebra to read this text, but this is not necessary since we have developed all the theory that we will need from it in Chapter 1. In this chapter the reader may also find all the basic definitions, notations and results that will be necessary to read this text, which include basic notions on modules over Artin algebras.

In this text we follow the philosophy that the best approach to understand higher Auslander–Reiten theory is the functorial approach. Therefore, we devote Chapter 2 to the study of functor categories, which are regarded here as categories of modules over skeletally small preadditive categories. This chapter will be fundamental to establish the language that will be used throughout this text, as well as to prepare the reader to understand the results that will be given in the following chapters. Two subjects of particular interest that are discussed in Chapter 2 in the light of functor categories are the Yoneda Lemma and the Jacobson radical of a category.

It is in Chapter 3 that we start to make a gradual transition from the study of functor categories to higher Auslander–Reiten theory. In this chapter we explain why it is interesting to consider Krull–Schmidt categories, as well as we introduce almost split morphisms by studying when simple modules over certain categories are finitely presented. Moreover, we also show how almost split sequences can be regarded as short exact sequences that induce minimal projective resolutions of simple modules. Through this point of view, we introduce n-almost split sequences as natural generalizations of almost split sequences.

Once we have defined what an n-almost split sequence is, we proceed in Chapter 4 to study their existence. In this chapter we investigate which conditions would be interesting to impose over a category to make it more likely to have n-almost split sequences. Such conditions will lead us to the notion of an n-cluster tilting subcategory. In addition, we also give several characterizations of n-almost split sequences in such subcategories, as well as we prove that the existence of almost split morphisms implies the existence of n-almost split sequences in certain n-cluster tilting subcategories.

From Chapter 5 on we begin to restrict our attention to the case of n-cluster tilting subcategories of $\operatorname{mod}\Lambda$. In this chapter we prove that every n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ has both almost split morphisms and n-almost split sequences. The proof that we give here is a generalization of the proof of the existence of almost split sequences in $\operatorname{mod}\Lambda$ through the functorial approach, via the study of dualizing R-varieties. We also prove the Higher Auslander Correspondence by first passing through a result which we call the Higher Morita–Tachikawa Correspondence, as well as we present some open problems on higher Auslander–Reiten theory.

Finally, in Chapter 6 we introduce the n-Auslander–Reiten translations. By using these concepts we give higher versions of the Auslander–Reiten formulas, which are formulas that play a central role in Auslander–Reiten theory, and we also show how we can use the n-Auslander–Reiten translations to relate the end terms of an n-almost split sequence in an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$. Furthermore, we also describe the n-cluster tilting subcategories of $\operatorname{mod}\Lambda$ for a certain kind of algebra Λ, as well as we give their n-almost split sequences.

To read this text, we suggest the reader a continuous reading, given that for the understanding of the majority of the sections it is essential to first know the content of their preceding sections. Nevertheless, we observe that the reader who has a proper background in homological algebra and in modules over Artin algebras may skip Chapter 1, and only consult it when necessary. Furthermore, in case that the reader would like to read a smaller portion of this text, we would recommend to skip the content concerning the Higher Auslander Correspondence by skipping sections 2.9, 4.4, 5.2 and 5.3. On the other hand, if the aim of the reader is only to achieve the proof of the Higher Auslander Correspondence, then the reader may skip sections 4.3 and 5.1. Finally, we note that the reader interested in learning only what the n-Auslander–Reiten translations are may directly read sections 6.1 and 6.2, without reading the rest of the text and by making a few consultations in it when necessary.

1 Preliminaries

Here we introduce some notations and conventions, as well as we give some basic definitions and results that will be used throughout this text. We also develop some basic notions on homological algebra and on Artin algebras for the convenience of the reader who is not familiarized with these subjects. However, it will be assumed that the reader has some basic knowledge on category theory and on module theory. The material presented in this chapter is intended to provide a minimal background for the reader to understand the theory presented in this text. Considering this, here we often omit some details and proofs.

In this chapter, $\mathcal{A}$ will always stand for an abelian category (see Section 1.1 for the definition of abelian category).

1.1 Basic definitions and notations

In this section we give some basic definitions and results on category theory, as well as we fix some conventions and notations which will be used in this text.

To begin with, let $\mathcal{C}$ be a category. We will usually indicate that X is an object of $\mathcal{C}$ by writing $X \in \mathcal{C}$, and sometimes we may also write $f \in \mathcal{C}$ to indicate that f is a morphism of $\mathcal{C}$, but in doing so, we will always make it clear that f is a morphism and not an object. Moreover, given $X, Y \in \mathcal{C}$, we will denote the collection of morphisms from X to Y by $\mathcal{C}(X, Y)$, and we will always assume that $\mathcal{C}(X, Y)$ is a set, and not a proper class. We will also agree that all subcategories of $\mathcal{C}$ are full subcategories, and sometimes we write $\mathcal{B} \subseteq \mathcal{C}$ to express that $\mathcal{B}$ is a subcategory of $\mathcal{C}$. Furthermore, if $\mathcal{D}$ is another category, then we will denote $\mathcal{D} \approx \mathcal{C}$ to indicate that $\mathcal{D}$ and $\mathcal{C}$ are equivalent categories.

Throughout this text, we also assume that all rings are associative rings with identity, and if Λ is a ring, then the term "Λ-module" will always mean "right Λ-module". In this case, we denote the category of Λ-modules by $\operatorname{Mod}\Lambda$, and given that a left Λ-module is the same as a right Λ^{op}-module, where Λ^{op} is the opposite ring of Λ, we denote the category of left Λ-modules by $\operatorname{Mod}\Lambda^{\mathrm{op}}$. In addition, given $X, Y \in \operatorname{Mod}\Lambda$, we will denote the set of morphisms from X to Y in $\operatorname{Mod}\Lambda$ by $\operatorname{Hom}_\Lambda(X, Y)$.

Given a category $\mathcal{C}$ and a commutative ring R, we say that $\mathcal{C}$ is an **R-category** if $\mathcal{C}(X, Y)$ is an R-module for each $X, Y \in \mathcal{C}$ and if the composition of morphisms in $\mathcal{C}$ is R-bilinear. In this case, if $\mathcal{C}$ is an R-category satisfying that $\mathcal{C}(X, Y)$ is finitely generated as an R-module for every $X, Y \in \mathcal{C}$, then we say that $\mathcal{C}$ is **finitely confined**. Moreover, we say that a functor $F : \mathcal{B} \to \mathcal{C}$ between two R-categories is an **R-functor** if the morphism

$$\mathcal{B}(X, Y) \to \mathcal{C}(F(X), F(Y))$$

induced by F is R-linear for each $X, Y \in \mathcal{B}$. Finally, if $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are R-categories, then we say that a bifunctor $F = F(-,-) : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ is an R-**bifunctor** if the morphism

$$\mathcal{B}(X, Y) \times \mathcal{C}(Z, W) \to \mathcal{D}(F(X, Z), F(Y, W))$$

induced by F is R-bilinear for each $X, Y \in \mathcal{B}$ and $Z, W \in \mathcal{C}$.

In case that $\mathcal{C}$ is an R-category and $R = \mathbb{Z}$ is the ring of integers, we say simply that $\mathcal{C}$ is a **preadditive category**. Moreover, we call a $\mathbb{Z}$-functor and a $\mathbb{Z}$-bifunctor by an **additive functor** and a **biadditive bifunctor**, respectively.

In this text, we will denote the category of abelian groups by Ab. Moreover, if $\mathcal{C}$ is a preadditive category and $X \in \mathcal{C}$, then we will always consider both the covariant functor $\mathcal{C}(X, -)$ and the contravariant functor $\mathcal{C}(-, X)$ as being functors from $\mathcal{C}$ to Ab. Furthermore, if $f : X \to Y$ is a morphism in $\mathcal{C}$, then we let

$$\mathcal{C}(-, f) : \mathcal{C}(-, X) \to \mathcal{C}(-, Y)$$

be the natural transformation whose components are given by $\mathcal{C}(-, f)_Z = \mathcal{C}(Z, f)$ for each $Z \in \mathcal{C}$, and likewise, we let

$$\mathcal{C}(f, -) : \mathcal{C}(Y, -) \to \mathcal{C}(X, -)$$

be the natural transformation whose components are given by $\mathcal{C}(f, -)_Z = \mathcal{C}(f, Z)$ for each $Z \in \mathcal{C}$. In addition, for a given $X \in \mathcal{C}$, we also denote the endomorphism ring of X by $\mathrm{End}_{\mathcal{C}}(X)$. Finally, if $\mathcal{B}$ is a subcategory of $\mathcal{C}$ and if $F : \mathcal{C} \to \mathrm{Ab}$ is a covariant or contravariant functor, then we let $F|_{\mathcal{B}} : \mathcal{B} \to \mathrm{Ab}$ be the restriction of F to $\mathcal{B}$, and if $\alpha : F \to G$ is a natural transformation between covariant or contravariant functors $F, G : \mathcal{C} \to \mathrm{Ab}$, then we let $\alpha|_{\mathcal{B}} : F|_{\mathcal{B}} \to G|_{\mathcal{B}}$ be the restriction of α to $\mathcal{B}$.

We also recall that if $\mathcal{C}$ is a preadditive category and $X \in \mathcal{C}$, then X is called **projective** if for every epimorphism $g : Y \to Z$ in $\mathcal{C}$ and every morphism $f : X \to Z$ in $\mathcal{C}$, there is a morphism $h : X \to Y$ in $\mathcal{C}$ such that $f = gh$.

$$\begin{tikzcd}
 & X \arrow[dl, dashed, "h"'] \arrow[d, "f"] & \\
Y \arrow[r, "g"'] & Z \arrow[r] & 0
\end{tikzcd}$$

Note that X is projective if and only if the functor $\mathcal{C}(X, -) : \mathcal{C} \to \mathrm{Ab}$ preserves epimorphisms. Dually, X is called **injective** if for every monomorphism $g : Z \to Y$ in $\mathcal{C}$ and every morphism $f : Z \to X$ in $\mathcal{C}$, there is a morphism $h : Y \to X$ in $\mathcal{C}$ such that $f = hg$.

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[dl, dashed, "h"] \\
 & X &
\end{tikzcd}$$

Likewise, note that X is injective if and only if the functor $\mathcal{C}(-, X) : \mathcal{C} \to \mathrm{Ab}$ carries monomorphisms into epimorphisms.

For a ring Λ we will denote the subcategory of $\mathrm{Mod}\,\Lambda$ consisting of all finitely generated projective Λ-modules by $\mathcal{P}(\Lambda)$, as well as we will denote the subcategory of $\mathrm{Mod}\,\Lambda$ consisting of all finitely generated injective Λ-modules by $\mathcal{I}(\Lambda)$.

Next, we define an **additive category** to be a preadditive category which has a zero object and finite coproducts. Moreover, if $\mathcal{C}$ is an additive category, then we denote the coproduct of a finite collection of objects $X_1, \dots, X_m \in \mathcal{C}$ by $X_1 \oplus \cdots \oplus X_m$ or simply by $\bigoplus_{i=1}^{m} X_i$. We remark that in this case, finite coproducts coincide with finite direct sums (see [31, Lemma 2.1]). More generally, if $\mathcal{C}$ has infinite coproducts, then we denote the coproduct of a collection of objects $X_i \in \mathcal{C}$ indexed by a set I by $\bigoplus_{i \in I} X_i$.

We say that an idempotent morphism $f : X \to X$ in a category $\mathcal{C}$ **splits** if there are morphisms $g : Y \to X$ and $h : X \to Y$ in $\mathcal{C}$ such that $f = gh$ and $hg = id_Y$. If every idempotent morphism in $\mathcal{C}$ splits, then we call $\mathcal{C}$ an **idempotent complete category**.

If $\mathcal{C}$ is an additive category and $X, Y \in \mathcal{C}$, then we say that Y is a **direct summand** of X if there is some $Z \in \mathcal{C}$ such that $X \simeq Y \oplus Z$. Furthermore, if $\mathcal{C}$ is also idempotent complete, then for a subcategory $\mathcal{B}$ of $\mathcal{C}$ we denote by $\mathrm{add}\,\mathcal{B}$ the subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sums of objects in $\mathcal{B}$. In this case, we remark that $\mathrm{add}\,\mathcal{B}$ is the smallest subcategory of $\mathcal{C}$ containing $\mathcal{B}$ which is additive, idempotent complete and closed under isomorphisms, and we also have that $\mathrm{add}(\mathrm{add}\,\mathcal{B}) = \mathrm{add}\,\mathcal{B}$. If $\mathcal{B}$ has only one object X, then we denote $\mathrm{add}\,\mathcal{B} = \mathrm{add}\,X$, and if we have that $\mathrm{add}\,X = \mathcal{C}$, then we say that X is an **additive generator** for $\mathcal{C}$. We also say that $\mathcal{C}$ **has an additive generator** if there is some $X \in \mathcal{C}$ such that $\mathrm{add}\,X = \mathcal{C}$. Moreover, still assuming that $\mathcal{C}$ is idempotent complete, we say that an object $X \in \mathcal{C}$ is **indecomposable** if $X \neq 0$ and if $X \simeq Y \oplus Z$ in $\mathcal{C}$, then either $Y \simeq 0$ or $Z \simeq 0$. Finally, we denote by $\mathrm{ind}\,\mathcal{C}$ a subcategory of $\mathcal{C}$ whose objects constitute a complete set of nonisomorphic representatives of the indecomposable objects in $\mathcal{C}$, and we say that $\mathcal{C}$ is **of finite type** if $\mathrm{ind}\,\mathcal{C}$ has only finitely many objects.

Next, we say that an additive category $\mathcal{C}$ is a **Krull–Schmidt category** if every nonzero object $X \in \mathcal{C}$ admits a finite direct sum decomposition $X \simeq \bigoplus_{i=1}^{n} X_i$ with $X_i \in \mathcal{C}$ in such a way that each X_i has a local endomorphism ring.

The next result shows how the notions of Krull–Schmidt category and idempotent complete category are related.

Proposition 1.1.1. *Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is a Krull–Schmidt category if and only if $\mathcal{C}$ is idempotent complete and the endomorphism ring of every object of $\mathcal{C}$ is semiperfect.*

Proof. See [31, Corollary 4.4]. □

Now, let $\mathcal{C}$ be a Krull–Schmidt category. Then by Proposition 1.1.1 we have that $\mathcal{C}$ is idempotent complete, and it is not difficult to see that an object in $\mathcal{C}$ is indecomposable if and only if its endomorphism ring is local. Furthermore, note that $\mathcal{C}$ is of finite type if and only if $\mathcal{C}$ has an additive generator.

We also define an **abelian category** to be an additive category satisfying that every morphism has a kernel and a cokernel, and such that every monomorphism is a kernel and every epimorphism is a cokernel (see [31] for an equivalent definition). In this case, we remark that every abelian category is idempotent complete.

We give below some results concerning abelian categories which will be needed in this text. Recall that, in this chapter, $\mathcal{A}$ will always denote an abelian category.

Proposition 1.1.2. *Let $\mathcal{B}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{B}$ is an abelian category such that the inclusion $\mathcal{B} \to \mathcal{A}$ is exact if and only if $\mathcal{B}$ has a zero object of $\mathcal{A}$ and is closed under finite direct sums, kernels and cokernels.*

Proof. See [38, Proposition 5.92]. □

Lemma 1.1.3 (Snake Lemma)**.** *If*

$$\begin{tikzcd}
& X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' &
\end{tikzcd}$$

is a commutative diagram in $\mathcal{A}$ with exact rows, then there is an exact sequence in $\mathcal{A}$ given by the dashed arrows below:

$$\begin{tikzcd}
& \operatorname{Ker} f \arrow[r, dashed] \arrow[d] & \operatorname{Ker} g \arrow[r, dashed] \arrow[d] & \operatorname{Ker} h \arrow[d] \arrow[dddll, dashed, bend left] & \\
& X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] & \\
& \operatorname{Coker} f \arrow[r, dashed] & \operatorname{Coker} g \arrow[r, dashed] & \operatorname{Coker} h &
\end{tikzcd}$$

Proof. See [41, Lemma 1.3.2]. □

Lemma 1.1.4 (Schanuel's Lemma)**.** *If*

$$0 \longrightarrow K \longrightarrow P \longrightarrow X \longrightarrow 0$$

and

$$0 \longrightarrow K' \longrightarrow P' \longrightarrow X \longrightarrow 0$$

are two short exact sequences in $\mathcal{A}$ with P and P' projectives, then $K \oplus P' \simeq K' \oplus P$.

Proof. See [38, Proposition 3.12] and [32, Chapter VI, Metatheorem 7.3]. □

Finally, following [12], we say that a morphism $f : X \to Y$ in a category $\mathcal{C}$ is **right minimal** if every morphism $g : X \to X$ in $\mathcal{C}$ satisfying $fg = f$ is an isomorphism. Dually, we say that a morphism $f : X \to Y$ in $\mathcal{C}$ is **left minimal** if every morphism $g : Y \to Y$ in $\mathcal{C}$ satisfying $gf = f$ is an isomorphism. Note that a morphism in $\mathcal{C}$ is right minimal if and only if its corresponding morphism in $\mathcal{C}^{\mathrm{op}}$ is left minimal.

1.2 Extensions

We begin now to present some concepts and results from homological algebra which will be needed in this text. In this section we discuss what an "extension" of objects in an abelian category is, as well as some other concepts and results related to it. The material presented here is based on a selected content of [32, Chapter VII], and thus the reader is referred to it for more details.

In what follows we fix two objects $X, Z \in \mathcal{A}$.

We say that a short exact sequence

$$\mathtt{E}: \qquad 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{A}$ is an **extension** of X by Z. Moreover, if we have another extension

$$\mathtt{E}': \qquad 0 \longrightarrow X \longrightarrow Y' \longrightarrow Z \longrightarrow 0$$

of X by Z in $\mathcal{A}$, then we say that $\mathtt{E}$ is **equivalent** to $\mathtt{E}'$ if there is a commutative diagram

$$\begin{tikzcd} \mathtt{E}: & 0 \arrow[r] & X \arrow[r] \arrow[d, "id_X"] & Y \arrow[r] \arrow[d, "f"] & Z \arrow[r] \arrow[d, "id_Z"] & 0 \\ \mathtt{E}': & 0 \arrow[r] & X \arrow[r] & Y' \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{A}$, in which case we denote $\mathtt{E} = \mathtt{E}'$ (note that f must be an isomorphism). This gives an equivalence relation between all extensions of X by Z, and we denote by $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ the class of equivalence classes of extensions of X by Z in $\mathcal{A}$ obtained from this relation.

We remark that it could happen that $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ is not a set. Nevertheless, it will indeed be a set for the categories $\mathcal{A}$ which we will be interested in this text, and thus we assume that this is the case for the rest of this section.

We do not describe it here, but we can define an addition operation $+$ between extensions of X by Z in $\mathcal{A}$ which turns $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ into an abelian group. In this case, the zero element of $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ is given by the split exact sequence

$$0 \longrightarrow X \longrightarrow X \oplus Z \longrightarrow Z \longrightarrow 0$$

in $\mathcal{A}$, which we denote by 0. The reader is referred to [32, Chapter VII, Section 1] for details.

We now proceed to show how the extension groups $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ define a biadditive bifunctor $\operatorname{Ext}^1_{\mathcal{A}} = \operatorname{Ext}^1_{\mathcal{A}}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$.

If

$$\mathtt{E}: \qquad 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is an element of $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ and $f : X \to W$ is a morphism in $\mathcal{A}$, then we define $f\mathtt{E} \in \operatorname{Ext}^1_{\mathcal{A}}(Z,W)$ to be the short exact sequence obtained in the pushout diagram below.

$$\begin{tikzcd}
\mathtt{E}: & 0 \arrow[r] & X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "id_Z"] & 0 \\
f\mathtt{E}: & 0 \arrow[r] & W \arrow[r] & V \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

Therefore, we can define a map

$$\operatorname{Ext}^1_{\mathcal{A}}(Z,f) : \operatorname{Ext}^1_{\mathcal{A}}(Z,X) \to \operatorname{Ext}^1_{\mathcal{A}}(Z,W)$$

given by $\operatorname{Ext}^1_{\mathcal{A}}(Z,f)(\mathtt{E}) = f\mathtt{E}$ for each $\mathtt{E} \in \operatorname{Ext}^1_{\mathcal{A}}(Z,X)$.

Dually, if we have a morphism $f : W \to Z$ in $\mathcal{A}$, then we let $\mathtt{E}f \in \operatorname{Ext}^1_{\mathcal{A}}(W,X)$ be the short exact sequence obtained in the pullback diagram below.

$$\begin{tikzcd}
\mathtt{E}f: & 0 \arrow[r] & X \arrow[r] \arrow[d, "id_X"] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "f"] & 0 \\
\mathtt{E}: & 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

In this case, we define a map

$$\operatorname{Ext}^1_{\mathcal{A}}(f,X) : \operatorname{Ext}^1_{\mathcal{A}}(Z,X) \to \operatorname{Ext}^1_{\mathcal{A}}(W,X)$$

by setting $\operatorname{Ext}^1_{\mathcal{A}}(f,X)(\mathtt{E}) = \mathtt{E}f$ for each $\mathtt{E} \in \operatorname{Ext}^1_{\mathcal{A}}(Z,X)$.

With the above definitions, we can conclude that $\operatorname{Ext}^1_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$ is a biadditive bifunctor. The reader is referred to [32, Chapter VII, Section 1] for the proof of this fact.

Next, we consider exact sequences in $\mathcal{A}$ with more terms and we present a generalization of the extensions groups $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$. In what follows, n is a positive integer.

Given an exact sequence

$$\mathtt{E}: \qquad 0 \longrightarrow X \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Z \longrightarrow 0$$

in $\mathcal{A}$, we call n the **length** of $\mathtt{E}$ and we say that X is the **left end term** of $\mathtt{E}$, as well as that Z is the **right end term** of $\mathtt{E}$. Moreover, if we have another exact sequence of length n

$$\mathtt{E}': \qquad 0 \longrightarrow X \longrightarrow Y'_n \longrightarrow \cdots \longrightarrow Y'_1 \longrightarrow Z \longrightarrow 0$$

with left end term X and right end term Z, then we denote $\mathtt{E} = \mathtt{E}'$ if there is a commutative diagram

$$\begin{array}{llllllllllllllll}
\mathtt{E}: & 0 & \longrightarrow & X & \longrightarrow & Y_n & \longrightarrow & \cdots & \longrightarrow & Y_1 & \longrightarrow & Z & \longrightarrow & 0 \\
& & & \big\downarrow{\scriptstyle id_X} & & \big\downarrow{\scriptstyle f_n} & & & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle id_Z} & & \\
\mathtt{E}': & 0 & \longrightarrow & X & \longrightarrow & Y_n' & \longrightarrow & \cdots & \longrightarrow & Y_1' & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

in $\mathcal{A}$ with each f_i being an isomorphism.

More generally, we call a commutative diagram as above, without any assumption over the morphisms f_i, by a **morphism of sequences of length n with fixed ends**, or for short, a **morphism with fixed ends** if there is no danger of confusion. If $\mathtt{E}$ and $\mathtt{E}'$ are exact sequences of length n as above, then we say that they are **equivalent** if there are exact sequences $\mathtt{E} = \mathtt{E}_0, \mathtt{E}_1, \ldots, \mathtt{E}_{k-1}, \mathtt{E}_k = \mathtt{E}'$ in $\mathcal{A}$ of length n such that for each $0 \leqslant i \leqslant k-1$ there is a morphism with fixed ends between $\mathtt{E}_i$ and $\mathtt{E}_{i+1}$. This gives an equivalence relation between all exact sequences of length n with left end term X and right end term Z, and we denote by $\operatorname{Ext}^n_{\mathcal{A}}(Z, X)$ the class of equivalence classes of such exact sequences obtained from this relation.

As it was the case for $n = 1$, it may happen that $\operatorname{Ext}^n_{\mathcal{A}}(Z, X)$ is not a set for a general abelian category $\mathcal{A}$. But again, it will indeed be a set for the categories that we will consider in this text, so that we assume that this is the case for the rest of this section. Moreover, we can also define an addition operation $+$ between exact sequences of length n in $\mathcal{A}$ with left end term X and right end term Z which turns $\operatorname{Ext}^n_{\mathcal{A}}(Z, X)$ into an abelian group. The zero element of this group is given by the direct sum of the exact sequences

$$0 \longrightarrow X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

and

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow Z \xrightarrow{id_Z} Z \longrightarrow 0$$

in $\mathcal{A}$, which we denote by 0. The reader is referred to [32, Chapter VII, Section 3] for details.

Now, suppose that we have two exact sequences

$$\mathtt{E}: \qquad 0 \longrightarrow X \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Z \longrightarrow 0$$

and

$$\mathtt{F}: \qquad 0 \longrightarrow Z \longrightarrow V_m \longrightarrow \cdots \longrightarrow V_1 \longrightarrow W \longrightarrow 0$$

in $\mathcal{A}$ of lengths n and m, respectively, such that the right end term of $\mathtt{E}$ coincides with the left end term of $\mathtt{F}$. Then we define the **splice** of $\mathtt{E}$ and $\mathtt{F}$ to be the induced exact sequence

$$\mathtt{EF}: \qquad 0 \longrightarrow X \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow V_m \longrightarrow \cdots \longrightarrow V_1 \longrightarrow W \longrightarrow 0$$

of length $n + m$ in $\mathcal{A}$. In this case, for each $m, n \geqslant 1$ and $W, X, Z \in \mathcal{A}$ we can define the **splicing operation**

$$\operatorname{Ext}^n_{\mathcal{A}}(Z, X) \times \operatorname{Ext}^m_{\mathcal{A}}(W, Z) \to \operatorname{Ext}^{m+n}_{\mathcal{A}}(W, X)$$

which associates each $\mathtt{E} \in \operatorname{Ext}^n_{\mathcal{A}}(Z,X)$ and $\mathtt{F} \in \operatorname{Ext}^m_{\mathcal{A}}(W,Z)$ to their splice $\mathtt{EF}$. We remark that it follows from [32, Chapter VII, Lemma 3.2] that the splicing operation is bilinear.

The advantage of considering the splice of exact sequences is that if we are given an exact sequence

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\mathcal{A}$ of length n, then we can write it as a splice of n short exact sequences. In fact, for each $1 \leqslant i \leqslant n$ consider the short exact sequence

$$\mathtt{E}_i: \qquad 0 \longrightarrow \operatorname{Ker} f_i \longrightarrow Y_i \longrightarrow \operatorname{Im} f_i \longrightarrow 0$$

in $\mathcal{A}$. Then we have that $\mathtt{E} = \mathtt{E}_n\mathtt{E}_{n-1}\cdots\mathtt{E}_2\mathtt{E}_1$.

As a consequence of the above two paragraphs, we have the following result:

Proposition 1.2.1. *If*

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$ such that f_1 is a split epimorphism or f_{n+1} is a split monomorphism, then $\mathtt{E} = 0$ in $\operatorname{Ext}^n_{\mathcal{A}}(Z,X)$.

Proof. First, write $\mathtt{E} = \mathtt{E}_n\mathtt{E}_{n-1}\cdots\mathtt{E}_2\mathtt{E}_1$ as a splice of n short exact sequences in $\mathcal{A}$. If f_1 is a split epimorphism, then we have that $\mathtt{E}_1 = 0$ in $\operatorname{Ext}^1_{\mathcal{A}}(Z, \operatorname{Ker} f_1)$, and thus we obtain that

$$\mathtt{E} = \mathtt{E}_n\mathtt{E}_{n-1}\cdots\mathtt{E}_2(\mathtt{E}_1 + \mathtt{E}_1) = \mathtt{E}_n\mathtt{E}_{n-1}\cdots\mathtt{E}_2\mathtt{E}_1 + \mathtt{E}_n\mathtt{E}_{n-1}\cdots\mathtt{E}_2\mathtt{E}_1 = \mathtt{E} + \mathtt{E},$$

which implies that $\mathtt{E} = 0$ in $\operatorname{Ext}^n_{\mathcal{A}}(Z,X)$.

The case when f_{n+1} is a split monomorphism is similar. $\square$

With the notion of the splice of exact sequences we can also conclude that the groups $\operatorname{Ext}^n_{\mathcal{A}}(Z,X)$ define a biadditive bifunctor $\operatorname{Ext}^n_{\mathcal{A}} = \operatorname{Ext}^n_{\mathcal{A}}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$. We will show how this is done below.

Consider an exact sequence $\mathtt{E} \in \operatorname{Ext}^n_{\mathcal{A}}(Z,X)$. As we already pointed out, we can write $\mathtt{E} = \mathtt{E}_n\mathtt{E}_{n-1}\cdots\mathtt{E}_2\mathtt{E}_1$ as a splice of n short exact sequences in $\mathcal{A}$. Therefore, if we have a morphism $f : X \to W$ in $\mathcal{A}$, then we let $f\mathtt{E} = (f\mathtt{E}_n)\mathtt{E}_{n-1}\cdots\mathtt{E}_2\mathtt{E}_1$, and thus we can define a map

$$\operatorname{Ext}^n_{\mathcal{A}}(Z,f) : \operatorname{Ext}^n_{\mathcal{A}}(Z,X) \to \operatorname{Ext}^n_{\mathcal{A}}(Z,W)$$

given by $\operatorname{Ext}^n_{\mathcal{A}}(Z,f)(\mathtt{E}) = f\mathtt{E}$ for each $\mathtt{E} \in \operatorname{Ext}^n_{\mathcal{A}}(Z,X)$.

Dually, if we have a morphism $f : W \to Z$ in $\mathcal{A}$, then we let $\mathtt{E}f = \mathtt{E}_n\mathtt{E}_{n-1}\cdots\mathtt{E}_2(\mathtt{E}_1 f)$, and in this case, we define a map

$$\operatorname{Ext}^n_{\mathcal{A}}(f,X) : \operatorname{Ext}^n_{\mathcal{A}}(Z,X) \to \operatorname{Ext}^n_{\mathcal{A}}(W,X)$$

by setting $\operatorname{Ext}^n_{\mathcal{A}}(f,X)(\mathtt{E}) = \mathtt{E}f$ for each $\mathtt{E} \in \operatorname{Ext}^n_{\mathcal{A}}(Z,X)$.

With the above definitions, we can conclude that $\operatorname{Ext}^n_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$ is a biadditive bifunctor. The reader is referred to [32, Chapter VII, Section 3] for the proof of this fact.

We end this section by recalling an important result which says that a short exact sequence in $\mathcal{A}$ can be extended to long exact sequence in Ab after we apply $\mathcal{A}(W,-)$ or $\mathcal{A}(-,W)$ to it, for some $W \in \mathcal{A}$. In order to present this result, we first give some definitions.

If

$$\mathtt{E}: \qquad 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$, then for each $n \geqslant 1$ and $W \in \mathcal{A}$ we define

$$\partial_n(W,\mathtt{E}) : \operatorname{Ext}^n_{\mathcal{A}}(W,Z) \to \operatorname{Ext}^{n+1}_{\mathcal{A}}(W,X)$$

to be the morphism of abelian groups given by $\partial_n(W,\mathtt{E})(\mathtt{F}) = \mathtt{EF}$ for each $\mathtt{F} \in \operatorname{Ext}^n_{\mathcal{A}}(W,Z)$. Furthermore, we also let

$$\partial_0(W,\mathtt{E}) : \mathcal{A}(W,Z) \to \operatorname{Ext}^1_{\mathcal{A}}(W,X)$$

be the morphism of abelian groups given by $\partial_0(W,\mathtt{E})(f) = \mathtt{E}f$ for each $f \in \mathcal{A}(W,Z)$. For each $i \geqslant 0$ we call $\partial_i(W,\mathtt{E})$ the **covariant connecting morphism of degree i at W with respect to the sequence** E, or for short, **covariant connecting morphism**.

Dually, we define

$$\partial_n(\mathtt{E},W) : \operatorname{Ext}^n_{\mathcal{A}}(X,W) \to \operatorname{Ext}^{n+1}_{\mathcal{A}}(Z,W)$$

to be the morphism of abelian groups given by $\partial_n(\mathtt{E},W)(\mathtt{F}) = \mathtt{FE}$ for each $\mathtt{F} \in \operatorname{Ext}^n_{\mathcal{A}}(X,W)$, and we also let

$$\partial_0(\mathtt{E},W) : \mathcal{A}(X,W) \to \operatorname{Ext}^1_{\mathcal{A}}(Z,W)$$

be the morphism of abelian groups given by $\partial_0(\mathtt{E},W)(f) = f\mathtt{E}$ for each $f \in \mathcal{A}(X,W)$. In this case, for each $i \geqslant 0$ we call $\partial_i(\mathtt{E},W)$ the **contravariant connecting morphism of degree i at W with respect to the sequence** E, or for short, **contravariant connecting morphism**.

Theorem 1.2.2. *If*

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$, then for each $W \in \mathcal{A}$ the sequences

$$
\begin{aligned}
&0 \longrightarrow \mathcal{A}(W,X) \xrightarrow{\mathcal{A}(W,f)} \mathcal{A}(W,Y) \xrightarrow{\mathcal{A}(W,g)} \mathcal{A}(W,Z) \xrightarrow{\partial_0(W,\mathrm{E})} \\
&\longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(W,X) \xrightarrow{\mathrm{Ext}^1_{\mathcal{A}}(W,f)} \mathrm{Ext}^1_{\mathcal{A}}(W,Y) \xrightarrow{\mathrm{Ext}^1_{\mathcal{A}}(W,g)} \mathrm{Ext}^1_{\mathcal{A}}(W,Z) \xrightarrow{\partial_1(W,\mathrm{E})} \\
&\longrightarrow \mathrm{Ext}^2_{\mathcal{A}}(W,X) \xrightarrow{\mathrm{Ext}^2_{\mathcal{A}}(W,f)} \mathrm{Ext}^2_{\mathcal{A}}(W,Y) \xrightarrow{\mathrm{Ext}^2_{\mathcal{A}}(W,g)} \mathrm{Ext}^2_{\mathcal{A}}(W,Z) \xrightarrow{\partial_2(W,\mathrm{E})} \\
&\longrightarrow \mathrm{Ext}^3_{\mathcal{A}}(W,X) \xrightarrow{\mathrm{Ext}^3_{\mathcal{A}}(W,f)} \cdots
\end{aligned}
$$

and

$$
\begin{aligned}
&0 \longrightarrow \mathcal{A}(Z,W) \xrightarrow{\mathcal{A}(g,W)} \mathcal{A}(Y,W) \xrightarrow{\mathcal{A}(f,W)} \mathcal{A}(X,W) \xrightarrow{\partial_0(\mathrm{E},W)} \\
&\longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(Z,W) \xrightarrow{\mathrm{Ext}^1_{\mathcal{A}}(g,W)} \mathrm{Ext}^1_{\mathcal{A}}(Y,W) \xrightarrow{\mathrm{Ext}^1_{\mathcal{A}}(f,W)} \mathrm{Ext}^1_{\mathcal{A}}(X,W) \xrightarrow{\partial_1(\mathrm{E},W)} \\
&\longrightarrow \mathrm{Ext}^2_{\mathcal{A}}(Z,W) \xrightarrow{\mathrm{Ext}^2_{\mathcal{A}}(g,W)} \mathrm{Ext}^2_{\mathcal{A}}(Y,W) \xrightarrow{\mathrm{Ext}^2_{\mathcal{A}}(f,W)} \mathrm{Ext}^2_{\mathcal{A}}(X,W) \xrightarrow{\partial_2(\mathrm{E},W)} \\
&\longrightarrow \mathrm{Ext}^3_{\mathcal{A}}(Z,W) \xrightarrow{\mathrm{Ext}^3_{\mathcal{A}}(g,W)} \cdots
\end{aligned}
$$

are exact in Ab, *and all of their morphisms are functorial in W.*

Proof. It is not difficult to see that the morphisms appearing in these sequences are all functorial in W, and we refer the reader to [32, Chapter VII, Theorem 5.1] and [41, Proposition 1.6.8] for the proof that these sequences are exact. □

1.3 Complexes

In this section we present some basic definitions and results concerning “complexes” in abelian categories which will be used in this text.

We say that a sequence

$$(X_\bullet, d_\bullet) : \qquad \cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots$$

in $\mathcal{A}$ is a **chain complex**, or for short, a **complex**, if $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$, and sometimes we denote $(X_\bullet, d_\bullet)$ simply by $X_\bullet$ if no confusion may arise. Furthermore, given two complexes $(X_\bullet, d_\bullet)$ and $(Y_\bullet, c_\bullet)$ in $\mathcal{A}$, we define a **morphism of complexes**

$$f = f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, c_\bullet)$$

to be a collection of morphisms $f_n : X_n \to Y_n$ in $\mathcal{A}$ with $n \in \mathbb{Z}$ such that

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "c_{n+2}"'] & Y_{n+1} \arrow[r, "c_{n+1}"'] & Y_n \arrow[r, "c_n"'] & Y_{n-1} \arrow[r, "c_{n-1}"'] & \cdots
\end{tikzcd}$$

is a commutative diagram.

It is straightforward to see that the composition gf of two morphism of complexes $f = f_\bullet : X_\bullet \to Y_\bullet$ and $g = g_\bullet : Y_\bullet \to Z_\bullet$, which is given by $(gf)_n = g_n f_n$ for each $n \in \mathbb{Z}$, is also a morphism of complexes. In this case, we define the **category of complexes in $\mathcal{A}$**, which we denote by $\mathrm{Ch}(\mathcal{A})$, to be the category whose objects are complexes in $\mathcal{A}$ and whose morphisms are morphisms of complexes.

An important property of $\mathrm{Ch}(\mathcal{A})$ is that it is an abelian category. We refer the reader to [38, Proposition 5.100] for the proof of this fact, as well as for a description of how direct sums, kernels and cokernels are given in $\mathrm{Ch}(\mathcal{A})$. Moreover, the reader should note that a sequence

$$X_\bullet \xrightarrow{f_\bullet} Y_\bullet \xrightarrow{g_\bullet} Z_\bullet$$

of complexes in $\mathrm{Ch}(\mathcal{A})$ is exact if and only if

$$X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n$$

is exact in $\mathcal{A}$ for each $n \in \mathbb{Z}$.

Now, if we fix some $n \in \mathbb{Z}$, then given a complex $X_\bullet = (X_\bullet, d_\bullet) \in \mathrm{Ch}(\mathcal{A})$, we define its ***n*th homology** to be the quotient

$$H_n(X_\bullet) = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}$$

in $\mathcal{A}$. In this case, if $f_\bullet : X_\bullet \to Y_\bullet$ is a morphism in $\mathrm{Ch}(\mathcal{A})$, then we can conclude that $f_\bullet$ induces a morphism $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet)$, and thus we can also verify that these data an additive functor $H_n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$. We refer the reader to [38, Proposition 6.8] for the details. We also remark that $X_\bullet$ is an exact sequence in $\mathcal{A}$ if and only if $H_n(X_\bullet) = 0$ for every $n \in \mathbb{Z}$.

Next, we say that a morphism $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, c_\bullet)$ in $\mathrm{Ch}(\mathcal{A})$ is **null homotopic** if there are morphisms $s_n : X_n \to Y_{n+1}$ in $\mathcal{A}$ such that $f_n = s_{n-1} d_n + c_{n+1} s_n$ for each $n \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, "s_n"] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"] & \cdots \\
\cdots \arrow[r, "c_{n+2}"'] & Y_{n+1} \arrow[r, "c_{n+1}"'] & Y_n \arrow[r, "c_n"'] & Y_{n-1} \arrow[r, "c_{n-1}"'] & \cdots
\end{tikzcd}$$

Moreover, if $g_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, c_\bullet)$ is another morphism in $\mathrm{Ch}(\mathcal{A})$, then we say that $f_\bullet$ and $g_\bullet$ are **homotopic** if $f_\bullet - g_\bullet$ is null homotopic. We remark that the relation of being homotopic gives an equivalence relation between all morphisms from $(X_\bullet, d_\bullet)$ to $(Y_\bullet, c_\bullet)$ in $\mathrm{Ch}(\mathcal{A})$.

Finally, given a morphism of complexes $f = f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, c_\bullet)$ in $\mathrm{Ch}(\mathcal{A})$, we define the **mapping cone** of f to be the complex $(\mathrm{cone}(f)_\bullet, b_\bullet)$ in $\mathrm{Ch}(\mathcal{A})$ given by

$$\mathrm{cone}(f)_n = X_{n-1} \oplus Y_n$$

and

$$b_n = \begin{pmatrix} -d_{n-1} & 0 \\ -f_{n-1} & c_n \end{pmatrix} : X_{n-1} \oplus Y_n \to X_{n-2} \oplus Y_{n-1}$$

for each $n \in \mathbb{Z}$. Furthermore, we say that f is a **quasi-isomorphism** if the morphisms $H_n(f) : H_n(X_\bullet) \to H_n(Y_\bullet)$ induced by f are isomorphisms for every $n \in \mathbb{Z}$.

The following result gives a connection between the two concepts defined above:

Proposition 1.3.1. *A morphism $f \in \mathrm{Ch}(\mathcal{A})$ is a quasi-isomorphism if and only if* $\mathrm{cone}(f)$ *is an exact sequence in $\mathcal{A}$.*

Proof. See [38, Corollary 10.41]. □

1.4 Projective and injective resolutions

We now proceed to give the definition of a "projective resolution" and of an "injective resolution" of an object in $\mathcal{A}$, as well as of some other related concepts. We also give some results and show how we can use these resolutions to compute the groups $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ for $n \geqslant 1$ and $X, Y \in \mathcal{A}$. In general, we present here some results only for the "projective case" since the corresponding results for the "injective case" will be recovered by taking opposite categories.

In what follows, n will denote a positive integer and we fix two objects $X, Y \in \mathcal{A}$.

A **projective n-presentation** of X is an exact sequence

$$P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

in $\mathcal{A}$ such that each P_i is projective, and for simplicity we call a projective 1-presentation of X by a **projective presentation** of X. A **projective resolution** of X is an exact sequence

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

in $\mathcal{A}$ such that each P_i is projective.

Dually, an **injective n-copresentation** of X is an exact sequence

$$0 \longrightarrow X \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_n} I_n$$

in $\mathcal{A}$ such that each I_j is injective, and for simplicity we call an injective 1-copresentation of X by an **injective copresentation** of X. An **injective resolution** of X is an exact sequence

$$0 \longrightarrow X \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \xrightarrow{f_3} \cdots$$

in $\mathcal{A}$ such that each I_j is injective.

In the next result we will see that if we have a morphism between objects in $\mathcal{A}$ which have projective resolutions, then in somehow we can extend it to the projective resolutions of the corresponding objects.

Theorem 1.4.1 (Comparison Theorem). *Let $h : X \to Y$ be a morphism in $\mathcal{A}$ and suppose that there are projective resolutions*

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

and

$$\cdots \xrightarrow{g_3} Q_2 \xrightarrow{g_2} Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} Y \longrightarrow 0$$

of X and Y, respectively. Then we can complete h to a morphism of complexes $h_\bullet$ as described below

$$\begin{tikzcd}
\cdots \arrow[r, "f_3"] & P_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & P_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & P_0 \arrow[r, "f_0"] \arrow[d, "h_0"] & X \arrow[r] \arrow[d, "h"] & 0 \\
\cdots \arrow[r, "g_3"'] & Q_2 \arrow[r, "g_2"'] & Q_1 \arrow[r, "g_1"'] & Q_0 \arrow[r, "g_0"'] & Y \arrow[r] & 0
\end{tikzcd}$$

with $h_{-1} = h$. Furthermore, if there is another morphism of complexes which extends h as above, then it is homotopic to $h_\bullet$.

Proof. See [38, Theorem 6.16] or [41, Theorem 2.2.6]. □

We say that an epimorphism $f : P \to X$ in $\mathcal{A}$ is a **projective cover** of X if P is projective and f is a right minimal morphism. Dually, we say that a monomorphism $f : X \to I$ in $\mathcal{A}$ is an **injective envelope** of X if I is injective and f is a left minimal morphism.

Proposition 1.4.2. *If $f : P \to X$ is an epimorphism in $\mathcal{A}$ with P projective, then f is a projective cover of X if and only if every morphism $g : Z \to P$ in $\mathcal{A}$ is an epimorphism provided that fg is an epimorphism.*

Proof. See [31, Lemma 3.4]. □

Now, we say that a projective n-presentation

$$P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

of X is a **minimal projective n-presentation** of X if each morphism f_i is right minimal. Moreover, we call a projective resolution

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

of X a **minimal projective resolution** of X if each morphism f_i is right minimal.

Dually, we say that an injective n-copresentation

$$0 \longrightarrow X \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_n} I_n$$

of X is a **minimal injective n-copresentation** of X if each morphism f_j is left minimal. Furthermore, we call an injective resolution

$$0 \longrightarrow X \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \xrightarrow{f_3} \cdots$$

of X a **minimal injective resolution** of X if each morphism f_j is left minimal.

The next four results that we present are well known and they can be found, for example, in [18, Section 3, Proposition 8 and its Corollary] and [31, Corollary 3.5]. However, for the convenience of the reader, we give here their proofs, which are slightly different from those of the references. We will show that projective covers are unique up to isomorphism, and also that if X admits a minimal projective resolution, then it is unique up to isomorphism of complexes, as well as that it is a direct summand, as a complex, of all the other projective resolutions of X.

Lemma 1.4.3. *Let*

$$\begin{tikzcd}
\cdots \arrow[r, "f_3"] & P_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & P_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & P_0 \arrow[r, "f_0"] \arrow[d, "h_0"] & X \arrow[r] \arrow[d, "h_{-1}"] & 0 \\
\cdots \arrow[r, "g_3"'] & Q_2 \arrow[r, "g_2"'] & Q_1 \arrow[r, "g_1"'] & Q_0 \arrow[r, "g_0"'] & Y \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram in $\mathcal{A}$ whose top and bottom rows are minimal projective resolutions of X and Y, respectively. If h_{-1} is an isomorphism, then $h_\bullet : P_\bullet \to Q_\bullet$ is an isomorphism of complexes, where $P_{-1} = X$ and $Q_{-1} = Y$.

Proof. First, we will verify that h_0 is an isomorphism. Well, we have that $g_0 h_0 = h_{-1} f_0$ is an epimorphism, so that by Proposition 1.4.2 we obtain that h_0 is an epimorphism. Therefore, given that Q_0 is projective, we conclude that there is some morphism $k_0 : Q_0 \to P_0$ such that $h_0 k_0 = id_{Q_0}$. Hence we have $g_0 = h_{-1} f_0 k_0$, so that $h_{-1} f_0 k_0 h_0 = h_{-1} f_0$, and thus $f_0 k_0 h_0 = f_0$. Then $k_0 h_0$ is an isomorphism, which implies that h_0 is a split monomorphism, and consequently an isomorphism since we already know that it is a split epimorphism.

Next, note that there is a morphism $u_0 : \operatorname{Ker} f_0 \to \operatorname{Ker} g_0$ in $\mathcal{A}$ which makes the diagram with exact rows below commute

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} f_0 \arrow[r] \arrow[d, "u_0"] & P_0 \arrow[r, "f_0"] \arrow[d, "h_0"] & X \arrow[r] \arrow[d, "h_{-1}"] & 0 \\
0 \arrow[r] & \operatorname{Ker} g_0 \arrow[r] & Q_0 \arrow[r, "g_0"'] & Y \arrow[r] & 0
\end{tikzcd}$$

and because h_{-1} and h_0 are isomorphisms, so is u_0. In this case, we can conclude that the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "f_3"] & P_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & P_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & P_1 \arrow[r, "f_1'"] \arrow[d, "h_1"] & \operatorname{Ker} f_0 \arrow[r] \arrow[d, "u_0"] & 0 \\
\cdots \arrow[r, "g_3"'] & Q_3 \arrow[r, "g_3"'] & Q_2 \arrow[r, "g_2"'] & Q_1 \arrow[r, "g_1'"'] & \operatorname{Ker} g_0 \arrow[r] & 0
\end{tikzcd}$$

is commutative, where f_1' and g_1' are the morphisms induced by f_1 and g_1, respectively, and thus by the same reasoning as before we can verify that h_1 is an isomorphism.

Therefore, if we proceed by induction we can conclude that every morphism h_j is an isomorphism, which implies that $h_\bullet : P_\bullet \to Q_\bullet$ is an isomorphism of complexes. □

Proposition 1.4.4. *If X admits two projective covers $f : P \to X$ and $g : Q \to X$, then there is an isomorphism $h : P \to Q$ such that $f = gh$.*

Proof. Since P is projective and g is an epimorphism in $\mathcal{A}$, there is a morphism $h : P \to Q$ in $\mathcal{A}$ such that $f = gh$.

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, dashed, "h"'] & X \arrow[r] \arrow[d, "id_X"] & 0 \\
Q \arrow[r, "g"'] & X \arrow[r] & 0
\end{tikzcd}$$

Furthermore, by following the proof of Lemma 1.4.3 we obtain that h is an isomorphism. □

Proposition 1.4.5. *If X admits two minimal projective resolutions*

$$P_\bullet : \qquad \cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

and

$$Q_\bullet : \qquad \cdots \xrightarrow{g_3} Q_2 \xrightarrow{g_2} Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} X \longrightarrow 0 \ ,$$

where $P_{-1} = Q_{-1} = X$, then $P_\bullet$ and $Q_\bullet$ are isomorphic as complexes.

Proof. It follows from the Comparison Theorem that there is a morphism of complexes $h_\bullet : P_\bullet \to Q_\bullet$ such that $h_{-1} = id_X$, and by Lemma 1.4.3 we obtain that $h_\bullet$ is an isomorphism. □

Proposition 1.4.6. *Assume that X admits a minimal projective resolution*

$$P_\bullet : \qquad \cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0 \ ,$$

where $P_{-1} = X$. If

$$Q_\bullet : \qquad \cdots \xrightarrow{g_3} Q_2 \xrightarrow{g_2} Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} X \longrightarrow 0$$

is another projective resolution of X, where $Q_{-1} = X$, then there is an exact sequence

$$K_\bullet : \qquad \cdots \xrightarrow{k_3} K_2 \xrightarrow{k_2} K_1 \xrightarrow{k_1} K_0 \longrightarrow 0$$

in $\mathcal{A}$ such that $Q_\bullet$ is isomorphic as a complex to

$$K_\bullet \oplus P_\bullet : \qquad \cdots \xrightarrow{\begin{pmatrix} k_3 & 0 \\ 0 & f_3 \end{pmatrix}} K_2 \oplus P_2 \xrightarrow{\begin{pmatrix} k_2 & 0 \\ 0 & f_2 \end{pmatrix}} K_1 \oplus P_1 \xrightarrow{\begin{pmatrix} k_1 & 0 \\ 0 & f_1 \end{pmatrix}} K_0 \oplus P_0 \xrightarrow{(0\ f_0)} X \longrightarrow 0 \ .$$

Proof. It follows from the Comparison Theorem that there are morphisms of complexes $u_\bullet : P_\bullet \to Q_\bullet$ and $v_\bullet : Q_\bullet \to P_\bullet$ such that $u_{-1} = v_{-1} = id_X$. Hence we obtain a morphism $v_\bullet u_\bullet : P_\bullet \to P_\bullet$ which must be an isomorphism by Lemma 1.4.3. Therefore, $v_\bullet$ is a split epimorphism, and this implies that $Q_\bullet \simeq K_\bullet \oplus P_\bullet$, where

$$K_\bullet : \qquad \cdots \xrightarrow{k_3} K_2 \xrightarrow{k_2} K_1 \xrightarrow{k_1} K_0 \longrightarrow 0$$

is the kernel of $v_\bullet$. Finally, because $Q_\bullet$ is an exact sequence in $\mathcal{A}$, it follows that so is $K_\bullet$. □

To end this section, we present another way to compute $\operatorname{Ext}^n_{\mathcal{A}}(X, Y)$ through projective or injective resolutions. But in order to do this, we need to guarantee the existence of such resolutions.

It is not difficult to see that every object in $\mathcal{A}$ admits a projective resolution if and only if for every $Z \in \mathcal{A}$ there is an epimorphism $P \to Z$ in $\mathcal{A}$ with P projective. In case that this happens, we say that $\mathcal{A}$ **has enough projectives**.

Likewise, we also have that every object in $\mathcal{A}$ admits an injective resolution if and only if for every $Z \in \mathcal{A}$ there is a monomorphism $Z \to I$ in $\mathcal{A}$ with I injective. In case that this happens, we say that $\mathcal{A}$ **has enough injectives**.

Now, if we assume that $\mathcal{A}$ has enough projectives, then we can take a projective resolution

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

of X. In this case, if we apply the functor $\mathcal{A}(-, Y)$ to it, then we obtain the complex

$$0 \longrightarrow \mathcal{A}(X, Y) \xrightarrow{\mathcal{A}(f_0, Y)} \mathcal{A}(P_0, Y) \xrightarrow{\mathcal{A}(f_1, Y)} \mathcal{A}(P_1, Y) \xrightarrow{\mathcal{A}(f_2, Y)} \mathcal{A}(P_2, Y) \xrightarrow{\mathcal{A}(f_3, Y)} \cdots$$

in Ab, and we can prove that there is a functorial isomorphism

$$\operatorname{Ext}^n_{\mathcal{A}}(X, Y) \simeq \frac{\operatorname{Ker} \mathcal{A}(f_{n+1}, Y)}{\operatorname{Im} \mathcal{A}(f_n, Y)}.$$

Dually, if we assume that $\mathcal{A}$ has enough injectives, then we can take an injective resolution

$$0 \longrightarrow Y \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \xrightarrow{f_3} \cdots$$

of Y. In this case, if we apply the functor $\mathcal{A}(X,-)$ to it, then we obtain the complex

$$0 \longrightarrow \mathcal{A}(X,Y) \xrightarrow{\mathcal{A}(X,f_0)} \mathcal{A}(X,I_0) \xrightarrow{\mathcal{A}(X,f_1)} \mathcal{A}(X,I_1) \xrightarrow{\mathcal{A}(X,f_2)} \mathcal{A}(X,I_2) \xrightarrow{\mathcal{A}(X,f_3)} \cdots$$

in Ab, and we can also prove that there is a functorial isomorphism

$$\operatorname{Ext}^n_{\mathcal{A}}(X,Y) \simeq \frac{\operatorname{Ker}\mathcal{A}(X,f_{n+1})}{\operatorname{Im}\mathcal{A}(X,f_n)}.$$

We refer the reader to [32, Chapter VII, Section 7] for a proof of the above facts, as well as to [38, Section 6.2] or [41, Section 2.5] for more details about the above discussion on the computation of $\operatorname{Ext}^n_{\mathcal{A}}(X,Y)$.

1.5 Homological dimensions

In this section we end the preliminaries on homological algebra by presenting certain "homological dimensions" which will be fundamental in this text.

In what follows we fix an object $X \in \mathcal{A}$.

We define the **projective dimension** of X, which we denote by $\operatorname{pd}_{\mathcal{A}} X$, to be the minimum nonnegative integer m such that $\operatorname{Ext}^k_{\mathcal{A}}(X,-) = 0$ for every $k > m$ if it exists, and if no such integer exists, then we let $\operatorname{pd}_{\mathcal{A}} X = \infty$.

Dually, we define the **injective dimension** of X, which we denote by $\operatorname{id}_{\mathcal{A}} X$, to be the minimum nonnegative integer m such that $\operatorname{Ext}^k_{\mathcal{A}}(-,X) = 0$ for every $k > m$ if it exists, and if no such integer exists, then we let $\operatorname{id}_{\mathcal{A}} X = \infty$.

It is not difficult to conclude from Proposition 1.2.1 and Theorem 1.2.2 that X is projective if and only if $\operatorname{pd}_{\mathcal{A}} X = 0$, and likewise, that X is injective if and only if $\operatorname{id}_{\mathcal{A}} X = 0$. Considering this, it is interesting to have in mind that the projective dimension measures how far an object is from being projective, and also, that the injective dimension measures how far an object is from being injective.

We will see in the next proposition that if $\mathcal{A}$ has enough projectives, then we can relate the projective dimension of X with the length of its projective resolutions. We remark that the dual result for the case when $\mathcal{A}$ has enough injectives also holds.

Proposition 1.5.1. *Assume that $\mathcal{A}$ has enough projectives and let $m \geqslant 0$. The following are equivalent:*

(a) $\operatorname{pd}_{\mathcal{A}} X \leqslant m$.

(b) $\operatorname{Ext}_{\mathcal{A}}^{m+1}(X,-)=0$.

(c) *If there is an exact sequence*

$$0 \longrightarrow K \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

in $\mathcal{A}$ with each P_i projective, then K is projective.

(d) *There is an exact sequence*

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

in $\mathcal{A}$ with each P_i projective.

Proof. See [32, Chapter VII, Proposition 6.4]. □

The next proposition will be useful to prove some results of this text.

Proposition 1.5.2. *Assume that $\mathcal{A}$ has enough projectives and let* $\operatorname{pd}_{\mathcal{A}} X = d$. *If* $1 \leqslant d < \infty$, *then there is some projective object $P \in \mathcal{A}$ such that* $\operatorname{Ext}_{\mathcal{A}}^{d}(X,P) \neq 0$.

Proof. Let

$$0 \longrightarrow P_d \xrightarrow{f_d} P_{d-1} \xrightarrow{f_{d-1}} P_{d-2} \xrightarrow{f_{d-2}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

be a projective resolution of X. We will verify that $\operatorname{Ext}_{\Lambda}^{d}(X,P_d) \neq 0$.

Well, suppose that $\operatorname{Ext}_{\Lambda}^{d}(X,P_d)=0$. Since we have that

$$\operatorname{Ext}_{\mathcal{A}}^{d}(X,P_d) \simeq \mathcal{A}(P_d,P_d)/\operatorname{Im}\mathcal{A}(f_d,P_d),$$

it follows that

$$\mathcal{A}(f_d,P_d) : \mathcal{A}(P_{d-1},P_d) \to \mathcal{A}(P_d,P_d)$$

is an epimorphism, and this implies that f_d is a split monomorphism. Hence we obtain that $P_{d-1} \simeq P_d \oplus \operatorname{Im} f_{d-1}$, and thus $\operatorname{Im} f_{d-1} = \operatorname{Ker} f_{d-2}$ is projective. Consequently, we obtain a projective resolution

$$0 \longrightarrow \operatorname{Ker} f_{d-2} \longrightarrow P_{d-2} \xrightarrow{f_{d-2}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

of X, and then it follows from Proposition 1.5.1 that $\operatorname{pd}_{\mathcal{A}} X \leqslant d-1$, a contradiction. Therefore, $\operatorname{Ext}_{\Lambda}^{d}(X,P_d) \neq 0$. □

Finally, we define the **global dimension** of $\mathcal{A}$, which we denote by $\operatorname{gl.dim}\mathcal{A}$, to be the minimum nonnegative integer m such that $\operatorname{Ext}_{\mathcal{A}}^{k}(-,-)=0$ for every $k > m$ if it exists, and otherwise, we let $\operatorname{gl.dim}\mathcal{A} = \infty$. In this case, it is not difficult to see that

$$\operatorname{gl.dim}\mathcal{A} = \sup\{\operatorname{pd}_{\mathcal{A}} Z \mid Z \in \mathcal{A}\} = \sup\{\operatorname{id}_{\mathcal{A}} Z \mid Z \in \mathcal{A}\}.$$

The reader should note that $\operatorname{gl.dim}\mathcal{A} = 0$ if and only if every object in $\mathcal{A}$ is projective, which is also equivalent to say that every object in $\mathcal{A}$ is injective.

1.6 Artin algebras

In this last section of preliminaries we define the notion of "Artin algebra", which is the kind of algebra that we will be working with in this text. We also give some basic results concerning finitely generated modules over such algebras, as well as we establish some notations and definitions related to homological algebra.

In what follows we fix a commutative artinian ring R.

We define an **Artin R-algebra**, or more simply, an **Artin algebra** to be an associative R-algebra with identity satisfying that it is finitely generated as an R-module. The reader should note that R is itself an Artin R-algebra.

For the rest of this section we fix an Artin R-algebra Λ.

As a first property, we remark that Λ is both a left and a right artinian ring. Consequently, it is not difficult to conclude that the subcategory of $\operatorname{Mod}\Lambda$ consisting of the finitely presented Λ-modules, which we will denote by $\operatorname{mod}\Lambda$, coincides with the subcategory of $\operatorname{Mod}\Lambda$ consisting of the finitely generated Λ-modules. Moreover, we can also conclude from Proposition 1.1.2 that $\operatorname{mod}\Lambda$ is an abelian category such that the inclusion $\operatorname{mod}\Lambda \to \operatorname{Mod}\Lambda$ is exact.

In this text we will usually restrict our attention to $\operatorname{mod}\Lambda$ rather than to $\operatorname{Mod}\Lambda$. At this point, it is important to observe that we will abuse the notation by denoting the functors $\operatorname{Hom}_\Lambda(X,-)|_{\operatorname{mod}\Lambda}$ and $\operatorname{Hom}_\Lambda(-,X)|_{\operatorname{mod}\Lambda}$ simply by $\operatorname{Hom}_\Lambda(X,-)$ and $\operatorname{Hom}_\Lambda(-,X)$, respectively, for a given $X \in \operatorname{mod}\Lambda$ when no confusion may arise.

In the next result we present some further properties of $\operatorname{mod}\Lambda$.

Proposition 1.6.1. *The following hold for* Λ*:*

(a) $\operatorname{mod}\Lambda$ *is an R-category.*

(b) $\operatorname{Hom}_\Lambda(X,Y)$ *is finitely generated as an R-module for each* $X, Y \in \operatorname{mod}\Lambda$.

(c) $\operatorname{End}_\Lambda(X)$ *is an Artin R-algebra for every* $X \in \operatorname{mod}\Lambda$.

Proof. See [15, Chapter II, Proposition 1.1]. □

We also remark that every object of $\operatorname{mod}\Lambda$ admits a projective cover in $\operatorname{mod}\Lambda$ (see [15, Chapter I, Theorem 4.2]), and thus Λ is a semiperfect ring. Hence it follows from propositions 1.1.1 and 1.6.1 that $\operatorname{mod}\Lambda$ is a Krull–Schmidt category.

Now, it is easy to see that Λ^{op} is also an Artin R-algebra, and a very important property of Λ is that there is a duality between the categories $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$. The duality is given as follows: First, note that because R is an artinian ring, there are only finitely many nonisomorphic simple R-modules $S_1, \dots, S_m$. In this case, let $I(S_i)$ be an injective envelope of S_i for each i, and take $J = \bigoplus_{i=1}^m I(S_i) \in \operatorname{mod} R$. Then we can conclude that $D = \operatorname{Hom}_R(-, J) : \operatorname{mod} R \to \operatorname{mod} R$ is a duality such that $D^2 \simeq id$ (see [15, Chapter II, Theorem 3.1]). Next, note that if $X \in \operatorname{mod} \Lambda$, then $X \in \operatorname{mod} R$ and $\operatorname{Hom}_R(X, J)$ has a structure of left Λ-module given by $(\lambda \cdot g)(z) = g(\lambda z)$ for each $\lambda \in \Lambda$, $g \in \operatorname{Hom}_R(X, J)$ and $z \in X$. Furthermore, we also have that $\operatorname{Hom}_R(X, J) \in \operatorname{mod} \Lambda^{\mathrm{op}}$ since we know by Proposition 1.6.1 that $\operatorname{Hom}_R(X, J)$ is finitely generated as an R-module, and if $f : X \to Y$ is a morphism in $\operatorname{mod} \Lambda$, then it is not difficult to see that $\operatorname{Hom}_R(f, J) : \operatorname{Hom}_R(Y, J) \to \operatorname{Hom}_R(X, J)$ is a morphism in $\operatorname{mod} \Lambda^{\mathrm{op}}$. Therefore, we can conclude that D induces a contravariant functor $\operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{\mathrm{op}}$, which we will also denote by D. Likewise, we have that D induces a contravariant functor $\operatorname{mod} \Lambda^{\mathrm{op}} \to \operatorname{mod} \Lambda$, which we will denote by D, and we can prove that $D : \operatorname{mod} \Lambda \leftrightarrow \operatorname{mod} \Lambda^{\mathrm{op}}$ are dualities such that $D^2 \simeq id$ (see [15, Chapter II, Theorem 3.3]).

Given that $D : \operatorname{mod} \Lambda \leftrightarrow \operatorname{mod} \Lambda^{\mathrm{op}}$ are dualities, we also have that they induce dualities $D : \mathcal{P}(\Lambda) \leftrightarrow \mathcal{I}(\Lambda^{\mathrm{op}})$. Moreover, we can also conclude that there is a duality between $\mathcal{P}(\Lambda)$ and $\mathcal{P}(\Lambda^{\mathrm{op}})$. To obtain such a duality, first note that if $X \in \operatorname{mod} \Lambda$, then $\operatorname{Hom}_\Lambda(X, \Lambda)$ has a structure of left Λ-module given by $(\lambda \cdot g)(z) = g(z)\lambda$ for each $\lambda \in \Lambda$, $g \in \operatorname{Hom}_\Lambda(X, \Lambda)$ and $z \in X$. In this case, we will denote $\operatorname{Hom}_\Lambda(X, \Lambda)$ simply by X^*. Now, by Proposition 1.6.1 we can conclude that $X^* \in \operatorname{mod} \Lambda^{\mathrm{op}}$, and we also have that if $f : X \to Y$ is a morphism in $\operatorname{mod} \Lambda$, then $f^* : Y^* \to X^*$, which is given by $f^* = \operatorname{Hom}_\Lambda(f, \Lambda)$, is a morphism in $\operatorname{mod} \Lambda^{\mathrm{op}}$. Therefore, we obtain a contravariant functor $(-)^* = \operatorname{Hom}_\Lambda(-, \Lambda) : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{\mathrm{op}}$. Likewise, we have a contravariant functor $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(-, \Lambda^{\mathrm{op}}) : \operatorname{mod} \Lambda^{\mathrm{op}} \to \operatorname{mod} \Lambda$, which we will also denote by $(-)^*$, and to simplify the notation, we will indicate the composition $(-)^*(-)^*$ by $(-)^{**}$. Taking this into account, we can prove that if $P \in \mathcal{P}(\Lambda)$, then $P^* \in \mathcal{P}(\Lambda^{\mathrm{op}})$, and also that the induced functors $(-)^* : \mathcal{P}(\Lambda) \leftrightarrow \mathcal{P}(\Lambda^{\mathrm{op}})$ are dualities such that $(-)^{**} \simeq id$ (see [15, Chapter II, Proposition 4.3]).

As a consequence of the above paragraph, we have the following:

Proposition 1.6.2. *The categories $\mathcal{P}(\Lambda)$ and $\mathcal{I}(\Lambda)$ are equivalent.*

Proof. To obtain an equivalence of categories $\mathcal{P}(\Lambda) \to \mathcal{I}(\Lambda)$, it suffices to consider the composition of the dualities $(-)^* : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda^{\mathrm{op}})$ and $D : \mathcal{P}(\Lambda^{\mathrm{op}}) \to \mathcal{I}(\Lambda)$. $\square$

Next, we set some notation. We will denote the global dimension of $\operatorname{Mod} \Lambda$ by $\operatorname{gl.dim} \Lambda$, which by [1, Chapter X, Corollary 2.7] coincides with the global dimension of $\operatorname{mod} \Lambda$. In this case, we call $\operatorname{gl.dim} \Lambda$ the **global dimension** of Λ, and we have that $\operatorname{gl.dim} \Lambda = \operatorname{gl.dim} \Lambda^{\mathrm{op}}$ (see [1, Chapter X, Corollary 2.13]). Furthermore, given $X, Y \in \operatorname{Mod} \Lambda$ and $n \geqslant 1$, we will denote $\operatorname{Ext}^n_{\operatorname{Mod} \Lambda}(X, Y)$ by $\operatorname{Ext}^n_\Lambda(X, Y)$, as well as we will denote $\operatorname{pd}_{\operatorname{Mod} \Lambda} X$ and $\operatorname{id}_{\operatorname{Mod} \Lambda} X$ simply by $\operatorname{pd}_\Lambda X$ and $\operatorname{id}_\Lambda X$, respectively.

We also remark that every finitely generated Λ-module admits a projective cover in $\operatorname{mod}\Lambda$ (see [15, Chapter I, Theorem 4.2]). From this, we can conclude that every $M \in \operatorname{mod}\Lambda$ admits a minimal projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

in $\operatorname{Mod}\Lambda$ with $P_i \in \mathcal{P}(\Lambda)$.

We end this section by introducing some functors which will be used to prove some results of Section 6.2.

Let $X \in \operatorname{Mod}\Lambda$ and $Y \in \operatorname{Mod}\Lambda^{\mathrm{op}}$ and fix $n \geqslant 0$. If we take a projective resolution

$$\cdots \xrightarrow{g_3} Q_2 \xrightarrow{g_2} Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} Y \longrightarrow 0$$

of Y, then by applying the functor $X \otimes_\Lambda - : \operatorname{Mod}\Lambda^{\mathrm{op}} \to \operatorname{Mod}R$ to it, we obtain the complex

$$\cdots \xrightarrow{id \otimes g_3} X \otimes_\Lambda Q_2 \xrightarrow{id \otimes g_2} X \otimes_\Lambda Q_1 \xrightarrow{id \otimes g_1} X \otimes_\Lambda Q_0 \xrightarrow{id \otimes g_0} X \otimes_\Lambda Y \longrightarrow 0$$

in $\operatorname{Mod}R$. In this case, we let

$$\operatorname{Tor}_n^\Lambda(X,Y) = \frac{\operatorname{Ker}(id \otimes g_n)}{\operatorname{Im}(id \otimes g_{n+1})},$$

and we can verify that this construction defines a functor $\operatorname{Tor}_n^\Lambda(X,-) : \operatorname{Mod}\Lambda^{\mathrm{op}} \to \operatorname{Mod}R$.

Likewise, if we take a projective resolution

$$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

of X, then by applying the functor $- \otimes_\Lambda Y : \operatorname{Mod}\Lambda \to \operatorname{Mod}R$ to it, we obtain the complex

$$\cdots \xrightarrow{f_3 \otimes id} P_2 \otimes_\Lambda Y \xrightarrow{f_2 \otimes id} P_1 \otimes_\Lambda Y \xrightarrow{f_1 \otimes id} P_0 \otimes_\Lambda Y \xrightarrow{f_0 \otimes id} X \otimes_\Lambda Y \longrightarrow 0$$

in $\operatorname{Mod}R$. In this case, we let

$$\operatorname{tor}_n^\Lambda(X,Y) = \frac{\operatorname{Ker}(f_n \otimes id)}{\operatorname{Im}(f_{n+1} \otimes id)},$$

and we can show that this construction defines a functor $\operatorname{tor}_n^\Lambda(-,Y) : \operatorname{Mod}\Lambda \to \operatorname{Mod}R$.

We refer the reader to [38, Section 6.2] or [1, Chapter IX, Section 4] for the details about the construction of the functors that we have indicated above.

Finally, we remark that in the above conditions we can prove that there is an isomorphism $\operatorname{Tor}_n^\Lambda(X,Y) \simeq \operatorname{tor}_n^\Lambda(X,Y)$ which is functorial in both variables (see [1, Chapter IX, Theorem 4.4]). Therefore, we usually identify $\operatorname{tor}_n^\Lambda(X,Y)$ with $\operatorname{Tor}_n^\Lambda(X,Y)$, and in doing so, we denote the functor $\operatorname{tor}_n^\Lambda(-,Y)$ by $\operatorname{Tor}_n^\Lambda(-,Y)$.

2 Functor categories

In this chapter we begin the study of functor categories, which will be fundamental for the development of the next chapters. The idea behind functor categories is, as we will see, that it generalizes categories of modules over rings, and we will frequently highlight this idea throughout this text to obtain a better understanding of these categories. We also point out that in some of the next sections we will say that a morphism has a right inverse, a left inverse, or that it is invertible instead of saying that it is a split epimorphism, a split monomorphism and an isomorphism, respectively, in order to make some results more intuitive.

We remark that, throughout this chapter, $\mathcal{B}$ and $\mathcal{C}$ will always denote skeletally small preadditive categories (see Section 2.1 for more details about this assumption).

2.1 Basic notions

Given a category $\mathcal{C}$ with some suitable properties, we are interested in studying the category $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ whose objects are the contravariant additive functors from $\mathcal{C}$ to Ab and whose morphisms are the natural transformations between such functors, which we call the **functor category on** $\mathcal{C}$. In this case, we need to make two assumptions on $\mathcal{C}$. First, we need $\mathcal{C}$ to be preadditive, so that it makes sense to consider contravariant functors that are additive. Second, we need to suppose that $\mathcal{C}$ is **skeletally small**, i.e. that the isomorphism classes of objects in $\mathcal{C}$ form a set, which is the same to say that $\mathcal{C}$ is equivalent to a small category. This last property ensures that the class of morphisms between two objects of $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is a set, and not a proper class. Therefore, from now on in this chapter, the symbols $\mathcal{B}$ and $\mathcal{C}$ will always stand for skeletally small preadditive categories.

An important feature of the category $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is that it is abelian. For the convenience of the reader, we describe below the properties in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ which assure that it is an abelian category.

1. If $F, G \in (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, then the set of morphisms from F to G is an abelian group with the following operation: if $\alpha, \beta \in (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})(F, G)$, then $\alpha + \beta \in (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})(F, G)$ is given by $(\alpha + \beta)_X = \alpha_X + \beta_X$ for each $X \in \mathcal{C}$.

2. The zero object in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is the constant functor which sends all objects in $\mathcal{C}$ to the zero group.

3. If $F, G \in (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, then their direct sum $F \oplus G \in (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is the functor given by $(F \oplus G)(X) = F(X) \oplus G(X)$ for each $X \in \mathcal{C}$ and

$$(F \oplus G)(f) = \begin{pmatrix} F(f) & 0 \\ 0 & G(f) \end{pmatrix} : F(Y) \oplus G(Y) \to F(X) \oplus G(X)$$

for each morphism $f : X \to Y$ in $\mathcal{C}$.

4. If $\alpha : F \to G$ is a morphism in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, then its kernel is given by the functor $\operatorname{Ker} \alpha \in (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ defined by $(\operatorname{Ker} \alpha)(X) = \operatorname{Ker}(\alpha_X)$ for each $X \in \mathcal{C}$, and such that for each morphism $f : X \to Y$ in $\mathcal{C}$, $(\operatorname{Ker} \alpha)(f) : \operatorname{Ker}(\alpha_Y) \to \operatorname{Ker}(\alpha_X)$ is the unique morphism which makes the augmented diagram below commute.

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}(\alpha_Y) \arrow[r] \arrow[d, dashed] & F(Y) \arrow[r, "\alpha_Y"] \arrow[d, "F(f)"] & G(Y) \arrow[d, "G(f)"] \\ 0 \arrow[r] & \operatorname{Ker}(\alpha_X) \arrow[r] & F(X) \arrow[r, "\alpha_X"'] & G(X) \end{tikzcd}$$

5. If $\alpha : F \to G$ is a morphism in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, then its cokernel is given by the functor $\operatorname{Coker} \alpha \in (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ defined by $(\operatorname{Coker} \alpha)(X) = \operatorname{Coker}(\alpha_X)$ for each $X \in \mathcal{C}$, and such that for each morphism $f : X \to Y$ in $\mathcal{C}$, $(\operatorname{Coker} \alpha)(f) : \operatorname{Coker}(\alpha_Y) \to \operatorname{Coker}(\alpha_X)$ is the unique morphism which makes the augmented diagram below commute.

$$\begin{tikzcd} F(Y) \arrow[r, "\alpha_Y"] \arrow[d, "F(f)"] & G(Y) \arrow[r] \arrow[d, "G(f)"] & \operatorname{Coker}(\alpha_Y) \arrow[r] \arrow[d, dashed] & 0 \\ F(X) \arrow[r, "\alpha_X"'] & G(X) \arrow[r] & \operatorname{Coker}(\alpha_X) \arrow[r] & 0 \end{tikzcd}$$

We also point out that $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ has infinite products and coproducts, which are given “pointwisely”. Moreover, it is easy to see that a sequence

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is exact if and only if

$$F(X) \xrightarrow{\alpha_X} G(X) \xrightarrow{\beta_X} H(X)$$

is exact in Ab for each $X \in \mathcal{C}$. Therefore, it follows that a morphism $\alpha : F \to G$ in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is a monomorphism if and only if α_X is a monomorphism for each $X \in \mathcal{C}$, and likewise, that α is an epimorphism if and only if α_X is an epimorphism for each $X \in \mathcal{C}$.

Next, we will investigate an example that will provide us a particular point of view on functor categories, which will be highly explored in this text.

Example 2.1.1. If Λ is a ring, then we can regard it as a preadditive category $\mathcal{A}$ consisting of only one object X and such that $\mathcal{A}(X, X) = \Lambda$, where the addition in $\mathcal{A}(X, X)$ is given by the addition in Λ, and the composition of morphisms in $\mathcal{A}(X, X)$ is given by the ring multiplication, so that $\operatorname{End}_{\mathcal{A}}(X) = \Lambda$. In this case, it is easy to see that for $F \in (\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ the abelian group $F(X)$ has a Λ-module structure given by $z \cdot \lambda = F(\lambda)(z)$ for each $z \in F(X)$ and $\lambda \in \Lambda$, and that for each morphism $\alpha : F \to G$ in $(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$, $\alpha_X : F(X) \to G(X)$ is a morphism of Λ-modules. Furthermore, each $M \in \operatorname{Mod} \Lambda$ defines a functor $F \in (\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ by setting $F(X) = M$ and defining $F(\lambda) : M \to M$ to be the multiplication by λ for each $\lambda \in \mathcal{A}(X, X)$, and we also have that each morphism $f \in \operatorname{Mod} \Lambda$ defines a morphism $\alpha \in (\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ by setting $\alpha_X = f$. Consequently, we can easily conclude that the categories $\operatorname{Mod} \Lambda$ and $(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ are isomorphic.

Example 2.1.1 suggests that skeletally small preadditive categories $\mathcal{C}$ might be generalizations of rings, as well as functor categories $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ might be generalizations of categories of modules over rings. Well, as it was shown by Barry Mitchell in [33], it happens that this is indeed the case, and in this text we adopt this philosophy to achieve a better understanding of functor categories.

The ideas described in the above paragraph were also explored by Maurice Auslander in [5], but with different purposes than those of Barry Mitchell in [33]. In this case, since we are interested in studying a generalization of Auslander–Reiten theory, we follow mainly [5] to develop the results of this chapter.

Taking the preceding discussion into account, from now on we will denote the functor category on $\mathcal{C}$ by $\operatorname{Mod} \mathcal{C}$ and call it the **category of (right) $\mathcal{C}$-modules**, in which case we will call its objects by **(right) $\mathcal{C}$-modules**. Furthermore, for $F, G \in \operatorname{Mod} \mathcal{C}$ we will indicate the set of morphisms from F to G by $\operatorname{Hom}_{\mathcal{C}}(F, G)$. Moreover, we will also abuse the notation and denote $\operatorname{End}_{\operatorname{Mod} \mathcal{C}}(F)$, $\operatorname{pd}_{\operatorname{Mod} \mathcal{C}} F$ and $\operatorname{id}_{\operatorname{Mod} \mathcal{C}} F$ simply by $\operatorname{End}_{\mathcal{C}}(F)$, $\operatorname{pd}_{\mathcal{C}} F$ and $\operatorname{id}_{\mathcal{C}} F$, respectively. In addition, we will usually view a ring Λ as a preadditive category consisting of one object as in Example 2.1.1, which we will also denote by Λ. In this case, we will identify the category of Λ-modules with the category of contravariant additive functors from Λ to Ab, both of which will be indicated by $\operatorname{Mod} \Lambda$.

We remark that we also have the notion of a **left $\mathcal{C}$-module**, which is defined to be a $\mathcal{C}^{\mathrm{op}}$-module, i.e. a covariant additive functor from $\mathcal{C}$ to Ab. Considering this, we define the **category of left $\mathcal{C}$-modules** to be the category $\operatorname{Mod} \mathcal{C}^{\mathrm{op}}$.

At this moment, we point out that, throughout this text, when we say "$\mathcal{C}$-module" we will always mean "right $\mathcal{C}$-module", unless otherwise specified. Furthermore, we will mainly work with right $\mathcal{C}$-modules, given that we will be able to recover the theory for left $\mathcal{C}$-modules by taking $\mathcal{C}^{\mathrm{op}}$ in place of $\mathcal{C}$.

It is interesting to note that as it was the case in Example 2.1.1, if $X \in \mathcal{C}$ and $F \in \operatorname{Mod} \mathcal{C}$, then $F(X)$ has a structure of $\operatorname{End}_{\mathcal{C}}(X)$-module given by $z \cdot f = F(f)(z)$ for $f \in \operatorname{End}_{\mathcal{C}}(X)$ and $z \in F(X)$, and every morphism of $\mathcal{C}$-modules $\alpha : F \to G$ induces a morphism of $\operatorname{End}_{\mathcal{C}}(X)$-modules $\alpha_X : F(X) \to G(X)$. In view of this, we can define a functor $\mathsf{e}_{\mathcal{C},X} : \operatorname{Mod} \mathcal{C} \to \operatorname{Mod} \operatorname{End}_{\mathcal{C}}(X)$ which is given by $\mathsf{e}_{\mathcal{C},X}(F) = F(X)$ and $\mathsf{e}_{\mathcal{C},X}(\alpha) = \alpha_X$ for a

$\mathcal{C}$-module F and a morphism $\alpha \in \operatorname{Mod}\mathcal{C}$. We call $\mathsf{e}_{\mathcal{C},X}$ the **evaluation functor at** X, and we will denote it simply by e_X when no confusion can arise. Therefore, we see that $\operatorname{Mod}\mathcal{C}$ gives rise to a category of modules over a certain ring for each object $X \in \mathcal{C}$.

We end this section by observing that if $\mathcal{C}$ is an R-category for some commutative ring R, then for each $X \in \mathcal{C}$ we have that $\operatorname{End}_{\mathcal{C}}(X)$ is an R-algebra, and consequently $F(X)$ has a structure of R-module for each $F \in \operatorname{Mod}\mathcal{C}$. Therefore, we can view each $F \in \operatorname{Mod}\mathcal{C}$ as being a contravariant functor from $\mathcal{C}$ to $\operatorname{Mod}R$, which in this case turns out to be an R-functor. Hence we can conclude that $\operatorname{Mod}\mathcal{C} = (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ coincides with the category $(\mathcal{C}^{\mathrm{op}}, \operatorname{Mod}R)$ of the contravariant R-functors from $\mathcal{C}$ to $\operatorname{Mod}R$ whose morphisms are the natural transformations between such functors, which is an R-category. The reader may see it as a generalization of the concept of modules over R-algebras, and is referred to [10, Section 1] and [33, Section 11] for more details.

2.2 The Yoneda Lemma

In this section we discuss the Yoneda Lemma and some of its consequences, as well as how to interpret it as a generalization of a known result for modules over rings.

Lemma 2.2.1 (Yoneda Lemma). *For $X \in \mathcal{C}$ and $F \in \operatorname{Mod}\mathcal{C}$, the map $\theta_{F,X}$ defined by*

$$\begin{aligned}\theta_{F,X} : \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),F) &\to F(X)\\ \alpha &\mapsto \alpha_X(id_X)\end{aligned}$$

is a group isomorphism that is functorial in X and in F. More precisely, if $f \in \mathcal{C}(Y,X)$ and $\alpha \in \operatorname{Hom}_{\mathcal{C}}(F,G)$ are morphisms in $\mathcal{C}$ and $\operatorname{Mod}\mathcal{C}$, respectively, then the following diagrams commute:

$$\begin{array}{ccc}\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),F) & \xrightarrow{\theta_{F,X}} & F(X)\\ {\scriptstyle \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,f),F)}\big\downarrow & & \big\downarrow{\scriptstyle F(f)}\\ \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,Y),F) & \xrightarrow[\theta_{F,Y}]{} & F(Y)\end{array} \quad \textit{and} \quad \begin{array}{ccc}\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),F) & \xrightarrow{\theta_{F,X}} & F(X)\\ {\scriptstyle \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_X}\\ \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),G) & \xrightarrow[\theta_{G,X}]{} & G(X)\end{array}.$$

Proof. It is easy to see that $\theta_{F,X}$ is a morphism of groups. To verify the remaining assertions, consider the map

$$\omega_{F,X} : F(X) \to \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),F)$$

which associates to each $z \in F(X)$ the morphism of $\mathcal{C}$-modules $\omega_{F,X}(z) : \mathcal{C}(-,X) \to F$ given by $(\omega_{F,X}(z))_Y(f) = F(f)(z)$ for each $f \in \mathcal{C}(Y,X)$ and $Y \in \mathcal{C}$. It is easy to verify that $\omega_{F,X}$ is a morphism of groups, and using the fact that the diagram

$$\begin{array}{ccc}\mathcal{C}(X,X) & \xrightarrow{\alpha_X} & F(X)\\ {\scriptstyle \mathcal{C}(f,X)}\big\downarrow & & \big\downarrow{\scriptstyle F(f)}\\ \mathcal{C}(Y,X) & \xrightarrow[\alpha_Y]{} & F(Y)\end{array}$$

is commutative for every $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),F)$, $f \in \mathcal{C}(Y,X)$ and $Y \in \mathcal{C}$ we may show that $\omega_{F,X}$ is the inverse of $\theta_{F,X}$, so that $\theta_{F,X}$ is a group isomorphism.

We leave the details for the proof that $\theta_{F,X}$ is functorial in X and in F to the reader (for the first case, use the commutativity of the above diagram). □

We now apply the Yoneda Lemma to give a different proof for a known result on modules over rings, which is mentioned in [33, page 2].

Proposition 2.2.2. *If Λ is a ring and M is a Λ-module, then there is an isomorphism of Λ-modules $\mathrm{Hom}_{\Lambda}(\Lambda, M) \simeq M$ which is funtorial in M.*

Proof. This follows from the Yoneda Lemma and the identification of $\mathrm{Mod}\,\Lambda$ as the category of contravariant additive functors from Λ to Ab as in Example 2.1.1. In fact, the Yoneda Lemma provides an isomorphism of abelian groups $\mathrm{Hom}_{\Lambda}(\Lambda, M) \to M$ which is functorial in M, and from the fact that this isomorphism is functorial in the single object of the preadditive category Λ, we can conclude that it is also a morphism of Λ-modules. □

Next, we proceed to give some definitions and applications of the Yoneda Lemma.

We call a projective object of $\mathrm{Mod}\,\mathcal{C}$ by a **projective $\mathcal{C}$-module**. From the Yoneda Lemma we may conclude that $\mathcal{C}(-,X)$ is a projective $\mathcal{C}$-module for every $X \in \mathcal{C}$. In fact, let $\alpha : F \to G$ be an epimorphism in $\mathrm{Mod}\,\mathcal{C}$. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),F) & \xrightarrow{\theta_{F,X}} & F(X) \\
{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha_X} \\
\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),G) & \xrightarrow[\theta_{G,X}]{} & G(X)
\end{array}$$

where $\theta_{F,X}$ and $\theta_{G,X}$ are isomorphisms, and since α_X is an epimorphism we conclude that $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),\alpha)$ is also an epimorphism. Hence $\mathcal{C}(-,X)$ is a projective $\mathcal{C}$-module.

We say that a $\mathcal{C}$-module F is **representable** if there is an object $X \in \mathcal{C}$ such that $F \simeq \mathcal{C}(-,X)$, and in this case, we say that F is **represented** by X. The next result describes completely the morphism between representable $\mathcal{C}$-modules.

Proposition 2.2.3. *If $X, Y \in \mathcal{C}$, then*

$$\begin{aligned}
\mathcal{C}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),\mathcal{C}(-,Y)) \\
f &\mapsto \mathcal{C}(-,f)
\end{aligned}$$

is a group isomorphism.

Proof. It follows from the proof of the Yoneda Lemma that

$$\omega_{\mathcal{C}(-,Y),X} : \mathcal{C}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),\mathcal{C}(-,Y))$$

is a group isomorphism, and it is easy to see that $\omega_{\mathcal{C}(-,Y),X}(f) = \mathcal{C}(-,f)$ for $f \in \mathcal{C}(X,Y)$. □

The reader should note that by Proposition 2.2.3 we obtain that every morphism $\alpha : \mathcal{C}(-,X) \to \mathcal{C}(-,Y)$ in $\operatorname{Mod}\mathcal{C}$ with $X, Y \in \mathcal{C}$ is given by $\alpha = \mathcal{C}(-,f)$ for some (unique) morphism $f : X \to Y$ in $\mathcal{C}$. We will use this fact throughout this text without reference.

Theorem 2.2.4. *The functor $\mathcal{Y}_{\mathcal{C}} : \mathcal{C} \to \operatorname{Mod}\mathcal{C}$ defined by $\mathcal{Y}_{\mathcal{C}}(X) = \mathcal{C}(-,X)$ on the objects and $\mathcal{Y}_{\mathcal{C}}(f) = \mathcal{C}(-,f)$ on the morphisms is additive, full, faithful and injective on objects.*

Proof. Let $X, Y, Z \in \mathcal{C}$. It is straightforward that $\mathcal{C}(-, id_X) = id_{\mathcal{C}(-,X)}$ and also that if $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$, then $\mathcal{C}(-,g)\mathcal{C}(-,f) = \mathcal{C}(-,gf)$, so that $\mathcal{Y}_{\mathcal{C}}$ is a functor. Furthermore, if $f, h \in \mathcal{C}(X,Y)$, we also have that $\mathcal{C}(-,f) + \mathcal{C}(-,h) = \mathcal{C}(-,f+h)$, and thus $\mathcal{Y}_{\mathcal{C}}$ is additive.

Finally, from Proposition 2.2.3 we know that $\mathcal{Y}_{\mathcal{C}}$ is full and faithful, and it is clear that $\mathcal{Y}_{\mathcal{C}}$ is injective on objects. □

The functor $\mathcal{Y}_{\mathcal{C}}$ defined in Theorem 2.2.4 is called the **Yoneda embedding** of $\mathcal{C}$. As a consequence of Theorem 2.2.4, we conclude that if $X, Y \in \mathcal{C}$ and $f \in \mathcal{C}(X,Y)$, then f is an isomorphism in $\mathcal{C}$ if and only if $\mathcal{C}(-,f)$ is an isomorphism in $\operatorname{Mod}\mathcal{C}$. Hence we have $\mathcal{C}(-,X) \simeq \mathcal{C}(-,Y)$ if and only if $X \simeq Y$, so that if F is a representable $\mathcal{C}$-module, then the object which represents F is unique up to isomorphism. Moreover, if $\mathcal{C}$ is an additive category, then for every $X, Y \in \mathcal{C}$ we have an isomorphism $\mathcal{C}(-, X \oplus Y) \simeq \mathcal{C}(-,X) \oplus \mathcal{C}(-,Y)$ in $\operatorname{Mod}\mathcal{C}$, given that $\mathcal{Y}_{\mathcal{C}}$ is an additive functor.

The Yoneda embedding also allows us to identify $\mathcal{C}$ with the subcategory of $\operatorname{Mod}\mathcal{C}$ consisting of all the $\mathcal{C}$-modules of the form $\mathcal{C}(-,X)$ for some $X \in \mathcal{C}$, and this fact will be very useful in this text. The reader should note that this identification is the generalization of the case when we consider a ring Λ as a Λ-module in $\operatorname{Mod}\Lambda$ as in Example 2.1.1. Furthermore, with this identification we see that the Yoneda Lemma is nothing but a generalization of Proposition 2.2.2 for $\mathcal{C}$-modules.

We end this section by giving a property of the Yoneda embedding of an abelian category. The proof below was adapted from [41, Lemma 1.6.11].

Proposition 2.2.5. *If $\mathcal{A}$ is a skeletally small abelian category, then the Yoneda embedding of $\mathcal{A}$ is a left exact functor which reflects exactness, i.e. we have that a sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{A}$ is exact provided that

$$\mathcal{A}(-,X) \xrightarrow{\mathcal{A}(-,f)} \mathcal{A}(-,Y) \xrightarrow{\mathcal{A}(-,g)} \mathcal{A}(-,Z)$$

is exact in $\operatorname{Mod}\mathcal{A}$.

Proof. In order to verify that $\mathcal{Y}_{\mathcal{A}}$ is a left exact functor, let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$$

be an exact sequence in $\mathcal{A}$ and take $W \in \mathcal{A}$. It follows from Theorem 1.2.2 that

$$0 \longrightarrow \mathcal{A}(W,X) \xrightarrow{\mathcal{A}(W,f)} \mathcal{A}(W,Y) \xrightarrow{\mathcal{A}(W,g)} \mathcal{A}(W,Z)$$

is an exact sequence in Ab, and given that W was taken arbitrarily, we conclude that

$$0 \longrightarrow \mathcal{A}(-,X) \xrightarrow{\mathcal{A}(-,f)} \mathcal{A}(-,Y) \xrightarrow{\mathcal{A}(-,g)} \mathcal{A}(-,Z)$$

is an exact sequence in $\operatorname{Mod}\mathcal{A}$. Hence $\mathcal{Y}_{\mathcal{A}}$ is left exact.

Now, to show that $\mathcal{Y}_{\mathcal{A}}$ reflects exactness, consider a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{A}$ and suppose that

$$\mathcal{A}(-,X) \xrightarrow{\mathcal{A}(-,f)} \mathcal{A}(-,Y) \xrightarrow{\mathcal{A}(-,g)} \mathcal{A}(-,Z)$$

is exact in $\operatorname{Mod}\mathcal{A}$. Write $f = uv$ with u being the image of f and v being an epimorphism. To verify that the initial sequence in $\mathcal{A}$ is exact, we need to show that u is the kernel of g. For this, note that from $0 = \mathcal{A}(-,g)\mathcal{A}(-,f)$ we obtain that $gf = 0$, and thus $gu = 0$. Moreover, if $h : W \to Y$ is a morphism in $\mathcal{A}$ such that $gh = 0$, then we have $\mathcal{A}(W,g)(h) = 0$ and since

$$\mathcal{A}(W,X) \xrightarrow{\mathcal{A}(W,f)} \mathcal{A}(W,Y) \xrightarrow{\mathcal{A}(W,g)} \mathcal{A}(W,Z)$$

is exact in Ab, we conclude that $h = \mathcal{A}(W,f)(k) = fk$ for some morphism $k : W \to X$ in $\mathcal{A}$. Hence we have $h = uvk$, so that h factors through u, and because u is a monomorphism, this factorization is unique. Therefore, u is the kernel of g. □

2.3 Finitely generated projective modules

As a next step to understand the category $\operatorname{Mod}\mathcal{C}$, we now develop the notion of a "finitely generated" $\mathcal{C}$-module, which generalizes the concept of a finitely generated module over a ring. After that, we focus on such $\mathcal{C}$-modules which are also projective, and in doing so, we give the definition of a "variety".

If Λ is a ring, then we know that for every Λ-module M there is an epimorphism $\Lambda^{(I)} \to M$ in $\operatorname{Mod}\Lambda$ for some set I. In this case, if we regard Λ as a preadditive category consisting of only one object X as in Example 2.1.1, and if we consider M as a contravariant functor from Λ to Ab, then we conclude that there is always an epimorphism $\Lambda(-,X)^{(I)} \to M$ for some set I. As we will see in the next result, this fact can be generalized for $\mathcal{C}$-modules. The proof given below is from [36, page 3].

Lemma 2.3.1. *For each $F \in \operatorname{Mod} \mathcal{C}$ there is an epimorphism*

$$\bigoplus_{i \in I} \mathcal{C}(-, X_i) \to F$$

in $\operatorname{Mod} \mathcal{C}$*, where $X_i \in \mathcal{C}$ and I is a set.*

Proof. Let $X \in \mathcal{C}$ and $z \in F(X)$. By the Yoneda Lemma, there is a morphism of $\mathcal{C}$-modules $\alpha_z : \mathcal{C}(-, X) \to F$ satisfying that $(\alpha_z)_X(id_X) = z$. In this case, we obtain that

$$\bigoplus_{z \in F(X)} \alpha_z : \bigoplus_{z \in F(X)} \mathcal{C}(-, X) \to F$$

is a morphism in $\operatorname{Mod} \mathcal{C}$ such that

$$\left(\bigoplus_{z \in F(X)} \alpha_z \right)_X : \bigoplus_{z \in F(X)} \mathcal{C}(X, X) \to F(X)$$

is an epimorphism.

Moreover, if $Y \in \mathcal{C}$ is such that $Y \simeq X$, then

$$\left(\bigoplus_{z \in F(X)} \alpha_z \right)_Y : \bigoplus_{z \in F(X)} (Y, X) \to F(Y)$$

is also an epimorphism. In fact, let $f : Y \to X$ be an isomorphism in $\mathcal{C}$ and let $w \in F(Y)$ be an arbitrary element. Then $F(f) : F(X) \to F(Y)$ is an isomorphism in Ab, so there is some $v \in F(X)$ such that $F(f)(v) = w$. Next, consider the morphism $\alpha_v : \mathcal{C}(-, X) \to F$ in $\operatorname{Mod} \mathcal{C}$ satisfying that $(\alpha_v)_X(id_X) = v$. By the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(X,X) & \xrightarrow{(\alpha_v)_X} & F(X) \\ {\scriptstyle \mathcal{C}(f,X)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathcal{C}(Y,X) & \xrightarrow[(\alpha_v)_Y]{} & F(Y) \end{array}$$

it follows that

$$w = F(f)(v) = F(f)(\alpha_v)_X(id_X) = (\alpha_v)_Y \mathcal{C}(f, X)(id_X) = (\alpha_v)_Y(f),$$

hence $(\bigoplus_{z \in F(X)} \alpha_z)_Y$ is an epimorphism.

Therefore, if S is a complete set of representatives of the isomorphism classes of the objects in $\mathcal{C}$, then

$$\bigoplus_{\substack{z \in F(X) \\ X \in S}} \alpha_z : \bigoplus_{\substack{z \in F(X) \\ X \in S}} \mathcal{C}(-, X) \to F$$

is an epimorphism in $\operatorname{Mod} \mathcal{C}$. $\square$

Using the same notations of the paragraph preceding Lemma 2.3.1, we recall that a Λ-module M is finitely generated if and only if there is an epimorphism $\Lambda^{(n)} \to M$ in $\operatorname{Mod}\Lambda$ for some $n \geqslant 1$, and this is the same to say that there is an epimorphism $\Lambda(-,X)^{(n)} \to M$ in the respective category of functors. Therefore, to generalize this notion for $\mathcal{C}$-modules, we will say that a $\mathcal{C}$-module F is **finitely generated** if there is an epimorphism $\bigoplus_{i=1}^{n} \mathcal{C}(-,X_i) \to F$ in $\operatorname{Mod}\mathcal{C}$ with $X_i \in \mathcal{C}$ and $n \geqslant 1$. We will denote the subcategory of $\operatorname{Mod}\mathcal{C}$ consisting of the finitely generated $\mathcal{C}$-modules by $\operatorname{f.g.}(\mathcal{C})$. Moreover, we will say that F is **cyclic** if there is an epimorphism $\mathcal{C}(-,X) \to F$ in $\operatorname{Mod}\mathcal{C}$ for some $X \in \mathcal{C}$.

It is worth mentioning that if $\mathcal{C}$ is additive, then a $\mathcal{C}$-module F is finitely generated if and only if it is cyclic. Indeed, this follows from the fact that we have that if $\mathcal{C}$ is additive, then

$$\bigoplus_{i=1}^{n} \mathcal{C}(-,X_i) \simeq \mathcal{C}\left(-,\bigoplus_{i=1}^{n} X_i\right)$$

whenever $X_1, \dots, X_n \in \mathcal{C}$. The reader should note that this property does not hold for a ring Λ, given that there are Λ-modules which are finitely generated but are not cyclic.

Now, if a $\mathcal{C}$-module is finitely generated and projective, then we will simply say that it is a **finitely generated projective** $\mathcal{C}$-module, and we will denote by $\mathcal{P}(\mathcal{C})$ the subcategory of $\operatorname{Mod}\mathcal{C}$ consisting of these modules. The reader should note that if $F \in \operatorname{Mod}\mathcal{C}$, then $F \in \mathcal{P}(\mathcal{C})$ if and only if F is a direct summand of a finite direct sum of $\mathcal{C}$-modules of the form $\mathcal{C}(-,X)$ for some $X \in \mathcal{C}$. Therefore, if we view $\mathcal{C}$ as a subcategory of $\operatorname{Mod}\mathcal{C}$ through the Yoneda embedding, then we see that $\mathcal{P}(\mathcal{C}) = \operatorname{add}\mathcal{C}$, which for a ring Λ means that $\mathcal{P}(\Lambda) = \operatorname{add}\Lambda$.

We remark that it is easy to see that a $\mathcal{C}$-module F is finitely generated if and only if there is some epimorphism $P \to F$ in $\operatorname{Mod}\mathcal{C}$ with $P \in \mathcal{P}(\mathcal{C})$. Moreover, we have the following result which is well known for the case of modules over rings:

Proposition 2.3.2. *Let*

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

be a short exact sequence in $\operatorname{Mod}\mathcal{C}$. *If F and H are finitely generated, then so is G.*

Proof. Let $P \to F$ and $Q \to H$ be epimorphisms in $\operatorname{Mod}\mathcal{C}$ with $P, Q \in \mathcal{P}(\mathcal{C})$. It is not difficult to see that with these morphisms we can form a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & P \oplus Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] & G \arrow[r] & H \arrow[r] & 0
\end{tikzcd}$$

in $\operatorname{Mod}\mathcal{C}$ whose top row is a split exact sequence. In this case, it follows from the Snake Lemma that the morphism $P \oplus Q \to G$ is an epimorphism, which implies that G is finitely generated since $P \oplus Q \in \mathcal{P}(\mathcal{C})$. □

We have already noted that if we identify $\mathcal{C}$ as a subcategory of $\operatorname{Mod}\mathcal{C}$ through the Yoneda embedding, then we have that $\mathcal{P}(\mathcal{C}) = \operatorname{add}\mathcal{C}$. However, it would be interesting to have that this identification induces an equivalence between $\mathcal{C}$ and $\mathcal{P}(\mathcal{C})$ since in this case we would have an easier description of the finitely generated projective $\mathcal{C}$-modules, namely, we would have that they coincide with the representable $\mathcal{C}$-modules. Well, note that because $\operatorname{Mod}\mathcal{C}$ is an abelian category, it is in particular an additive and idempotent complete category (see Section 1.1). Therefore, given that $\mathcal{P}(\mathcal{C}) = \operatorname{add}\mathcal{C}$, we obtain that $\mathcal{P}(\mathcal{C})$ is also an additive and idempotent complete category. Furthermore, it is not difficult to verify that $\mathcal{P}(\mathcal{C})$ is skeletally small. Taking this into account, we obtain that if $\mathcal{C}$ is equivalent to $\mathcal{P}(\mathcal{C})$, then $\mathcal{C}$ must be an additive and idempotent complete category. Moreover, as we will see in Proposition 2.3.3, these conditions are also sufficient to guarantee that $\mathcal{C}$ and $\mathcal{P}(\mathcal{C})$ are equivalent.

Considering the above discussion and following [5], we define a **variety** to be a skeletally small category which is additive and idempotent complete. Therefore, by the above paragraph we have that $\mathcal{P}(\mathcal{C})$ is always a variety, as well as that every skeletally small abelian category is a variety. We also note that in case that a variety is also an R-category for some commutative ring R, then we say that it is an R-**variety**.

Next, we will prove that the Yoneda embedding induces an equivalence between $\mathcal{C}$ and $\mathcal{P}(\mathcal{C})$ if and only if $\mathcal{C}$ is a variety. This result is mentioned in [5, Proposition 2.2], and the proof that we present for it is from [21, Chapter 5, Exercise G].

Proposition 2.3.3. *The Yoneda embedding induces an equivalence of categories $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ if and only if $\mathcal{C}$ is a variety.*

Proof. As we have already mentioned above, we have that $\mathcal{P}(\mathcal{C})$ is a variety. Consequently, if the Yoneda embedding induces an equivalence of categories $\mathcal{C} \to \mathcal{P}(\mathcal{C})$, then we obtain that $\mathcal{C}$ is also a variety.

For the converse, it suffices to show that if $\mathcal{C}$ is a variety, then the functor $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ induced by the Yoneda embedding is dense. In other words, we only need to verify that every finitely generated projective $\mathcal{C}$-module is representable.

Well, assume that $\mathcal{C}$ is a variety, take $F \in \mathcal{P}(\mathcal{C})$ and let $\alpha : \mathcal{C}(-, X) \to F$ be an epimorphism in $\operatorname{Mod}\mathcal{C}$ with $X \in \mathcal{C}$. Since F is projective, there is some morphism $\beta : F \to \mathcal{C}(-, X)$ in $\operatorname{Mod}\mathcal{C}$ such that $\alpha\beta = id_F$. In this case, $\beta\alpha$ is an idempotent morphism, and if we let $f : X \to X$ be the morphism in $\mathcal{C}$ such that $\beta\alpha = \mathcal{C}(-, f)$, then it is easy to see that f is also idempotent. Moreover, given that $\mathcal{C}$ is idempotent complete, there are morphisms $g : Y \to X$ and $h : X \to Y$ in $\mathcal{C}$ such that $f = gh$ and $hg = id_Y$, and it is straightforward to see that $\alpha\mathcal{C}(-, g) : \mathcal{C}(-, Y) \to F$ is the inverse of $\mathcal{C}(-, h)\beta : F \to \mathcal{C}(-, Y)$. Hence $F \simeq \mathcal{C}(-, Y)$. $\square$

As a consequence of Proposition 2.3.3, we obtain that $\mathcal{P}(\mathcal{P}(\mathcal{C})) \approx \mathcal{P}(\mathcal{C})$. However, this should be no surprise to the reader since this means that $\operatorname{add}(\operatorname{add}\mathcal{C}) \approx \operatorname{add}\mathcal{C}$, given that $\mathcal{P}(\mathcal{C}) = \operatorname{add}\mathcal{C}$, and we already knew that.

2.4 Submodules and quotient modules

In this section we generalize the notions of submodules and quotient modules of modules over rings for $\mathcal{C}$-modules, as well as we discuss some other concepts which are related to such generalizations.

If $F, G \in \operatorname{Mod}\mathcal{C}$, then we say that F is a **submodule** of G, and we denote it by $F \subseteq G$, if $F(X) \subseteq G(X)$ for all $X \in \mathcal{C}$ and if the canonical inclusions $\eta_X : F(X) \to G(X)$ form a morphism of $\mathcal{C}$-modules $\eta : F \to G$. If this is the case, then we refer to η as the **canonical inclusion**. It is interesting to note that the requested condition on η to be a morphism of $\mathcal{C}$-modules is equivalent to say that η_X is a morphism of abelian groups for each $X \in \mathcal{C}$ and that for each morphism $f : X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{\eta_Y} & G(Y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(X) & \xrightarrow[\eta_X]{} & G(X) \end{array}$$

commutes, and this commutativity means that $F(f) = G(f)|_{F(Y)}$. Therefore, if $F \subseteq G$, then F and G are functors which have essentially the same rule on morphisms.

It is convenient to remark that what we defined as a submodule is also sometimes called a “subfunctor”. Nevertheless, we will use the former terminology in this text.

Example 2.4.1. If $\alpha : F \to G$ is a morphism in $\operatorname{Mod}\mathcal{C}$, then $\operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha$ are easily seen to be submodules of F and G, respectively.

At this moment, we observe that a submodule in $\operatorname{Mod}\mathcal{C}$ is nothing but a subobject in the category $\operatorname{Mod}\mathcal{C}$, or more precisely, a representative of the equivalence class of the corresponding subobject. Furthermore, we can also give an appropriate definition for the quotient objects in the category $\operatorname{Mod}\mathcal{C}$ by choosing certain representatives for their equivalence classes, and this is what we do below.

Given $G \in \operatorname{Mod}\mathcal{C}$ and $F \subseteq G$, we define the **quotient module** of G by F to be the $\mathcal{C}$-module G/F defined by $(G/F)(X) = G(X)/F(X)$ for each $X \in \mathcal{C}$ and such that for each morphism $f : X \to Y$ in $\operatorname{Mod}\mathcal{C}$,

$$(G/F)(f) : G(Y)/F(Y) \to G(X)/F(X)$$

is the morphism induced by $G(f)$ on the quotient groups. In this case, if we consider the canonical projections $\pi_X : G(X) \to G(X)/F(X)$ for each $X \in \mathcal{C}$, then we obtain a morphism of $\mathcal{C}$-modules $\pi : G \to G/F$ which we call the **canonical projection**.

The next result will be useful for us to verify in certain cases when a $\mathcal{C}$-module is a submodule of another $\mathcal{C}$-module.

Proposition 2.4.2. *Let $H \in \operatorname{Mod}\mathcal{C}$ and F, G be two submodules of H. Then $F \subseteq G$ if and only if $F(X) \subseteq G(X)$ for every $X \in \mathcal{C}$.*

Proof. Assume that $F(X) \subseteq G(X)$ for every $X \in \mathcal{C}$, and let $\eta_X : F(X) \to G(X)$ be the corresponding canonical inclusions. To conclude that F is a submodule of G, we only need to show that the morphisms η_X form a morphism of $\mathcal{C}$-modules $\eta : F \to G$. For this, let $\eta^F : F \to H$ and $\eta^G : G \to H$ be the canonical inclusions and take a morphism $f : X \to Y$ in $\mathcal{C}$. By considering the diagram

$$\begin{tikzcd}
F(Y) \arrow[rr, bend left, "\eta_Y"] \arrow[d, "F(f)"'] \arrow[dr, "\eta^F_Y"] & & G(Y) \arrow[d, "G(f)"] \arrow[dl, "\eta^G_Y"'] \\
F(X) \arrow[rr, bend left, "\eta_X"] \arrow[dr, "\eta^F_X"'] & H(Y) \arrow[d, "H(f)"'] & G(X) \arrow[dl, "\eta^G_X"] \\
& H(X) &
\end{tikzcd}$$

we can conclude that $\eta^G_X G(f)\eta_Y = \eta^G_X \eta_X F(f)$, and given that η^G_X is a monomorphism, we obtain that $G(f)\eta_Y = \eta_X F(f)$. Therefore, since η_X is clearly a morphism of abelian groups for each $X \in \mathcal{C}$, we obtain that η is a morphism of $\mathcal{C}$-modules.

The converse is trivial. □

We remark that the set of submodules of a $\mathcal{C}$-module is partially ordered by $\subseteq$. Moreover, if F and G are submodules of a $\mathcal{C}$-module H, then we can consider the submodule $F + G$ of H which is given by $(F + G)(X) = F(X) + G(X)$ for each $X \in \mathcal{C}$, i.e. $(F + G)(X)$ is the subgroup of $H(X)$ generated by $F(X) \cup G(X)$. Likewise, we can consider the submodule $F \cap G$ of H which is given by $(F \cap G)(X) = F(X) \cap G(X)$ for each $X \in \mathcal{C}$. Therefore, it follows that the set of submodules of a $\mathcal{C}$-module is a lattice.

More generally, if $\{F_i\}_{i\in I}$ is a collection of submodules of a $\mathcal{C}$-module H, then we can define the submodules $\sum_{i\in I} F_i$ and $\bigcap_{i\in I} F_i$ of H which are given by $\left(\sum_{i\in I} F_i\right)(X) = \sum_{i\in I} F_i(X)$ and $\left(\bigcap_{i\in I} F_i\right)(X) = \bigcap_{i\in I} F_i(X)$ for each $X \in \mathcal{C}$, where $\sum_{i\in I} F_i(X)$ is the subgroup of $H(X)$ generated by $\bigcup_{i\in I} F_i(X)$. We call $\sum_{i\in I} F_i$ and $\bigcap_{i\in I} F_i$ the **sum** and the **intersection** of $\{F_i\}_{i\in I}$, respectively. Hence the set of submodules of a $\mathcal{C}$-module is a complete lattice.

Next, given $G \in \operatorname{Mod}\mathcal{C}$ and $F \subseteq G$, we say that F is a **proper submodule** of G if $F \neq G$. Furthermore, we say that F is a **maximal submodule** of G if it is a proper submodule of G having the property that if $F \subseteq H \subseteq G$ for some $H \in \operatorname{Mod}\mathcal{C}$, then $H = F$ or $H = G$.

With the above definitions, we can now give the following proposition which is well known for modules over rings:

Proposition 2.4.3. *If G is a finitely generated $\mathcal{C}$-module and F is a proper submodule of G, then F is contained in some maximal submodule of G.*

Proof. As we will see in Proposition 2.7.4, there is a skeletally small additive category $\mathcal{A}$ such that there is an equivalence of categories $\mathtt{s} : \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{A}$. Moreover, as we will also see in Corollary 2.7.3, because G is finitely generated, we also have that $\mathtt{s}(G)$ is finitely generated, and given that $\mathtt{s}$ is an equivalence of categories, it is clear that $\mathtt{s}(F)$ is a proper submodule of $\mathtt{s}(G)$. In addition, we also have that F is contained in some maximal submodule of G if and only if $\mathtt{s}(F)$ is contained in some maximal submodule of $\mathtt{s}(G)$. Therefore, it suffices to prove the result by assuming that $\mathcal{C}$ is additive.

Taking the above paragraph into account, assume below that $\mathcal{C}$ is additive and take an epimorphism $\alpha : \mathcal{C}(-, X) \to G$ in $\operatorname{Mod}\mathcal{C}$ with $X \in \mathcal{C}$.

Let S be the set consisting of the proper submodules of G which contain F. Then $F \in S$, so that S is a nonempty partially ordered set.

Now, let $\{H_i\}_{i\in I}$ be a chain in S. We can define a $\mathcal{C}$-module H by $H(Y) = \bigcup_{i\in I} H_i(Y)$ for each $Y \in \mathcal{C}$ in such a way that H is a submodule of G. Next, we will verify that $H \neq G$. For this, assume by contradiction that $H = G$. Then we can consider $\alpha_X : \mathcal{C}(X, X) \to \bigcup_{i\in I} H_i(X)$, and thus there is some $j \in I$ such that $\alpha_X(id_X) \in H_j(X)$. However, we know by the Yoneda Lemma that there is some $\beta \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-, X), H_j)$ such that $\beta_X(id_X) = \alpha_X(id_X)$. Hence if we let $\eta : H_j \to G$ be the canonical inclusion, then we have that $(\eta\beta)_X(id_X) = \alpha_X(id_X)$, and thus it follows from the Yoneda Lemma that $\eta\beta = \alpha$. Consequently, $G = \operatorname{Im}\alpha = \operatorname{Im}\eta\beta \subseteq H_j$, which implies that $G = H_j$, a contradiction. Therefore, $H \in S$ and since $H_i \subseteq H$ for all $i \in I$ we conclude that H is an upper bound for $\{H_i\}_{i\in I}$.

Finally, it follows from Zorn's Lemma that S has a maximal element, which means that F is contained in some maximal submodule of G. □

We end this section by introducing the concept of a "simple" $\mathcal{C}$-module. These $\mathcal{C}$-modules will play an important role in this text, and they will be studied in more detail in Section 3.3 for the case when $\mathcal{C}$ is a Krull–Schmidt category.

We say that a $\mathcal{C}$-module S is **simple** if $S \neq 0$ and its only submodules are 0 and S. The reader should note that if F is a submodule of a $\mathcal{C}$-module G, then G/F is simple if and only if F is a maximal submodule of G.

Lemma 2.4.4. *Let S be a nonzero $\mathcal{C}$-module. The following are equivalent:*

(a) S is simple.

(b) Every nonzero morphism $F \to S$ in $\operatorname{Mod}\mathcal{C}$ *is an epimorphism.*

(c) Every nonzero morphism $S \to F$ in $\operatorname{Mod}\mathcal{C}$ *is a monomorphism.*

Proof. This is straightforward and left to the reader. $\square$

In the next proposition we prove that every simple $\mathcal{C}$-module is cyclic.

Proposition 2.4.5. *If S is a simple $\mathcal{C}$-module and $X \in \mathcal{C}$ satisfies $S(X) \neq 0$, then there is an epimorphism $\mathcal{C}(-,X) \to S$ in* $\operatorname{Mod}\mathcal{C}$.

Proof. If $S(X) \neq 0$, then it follows from the Yoneda Lemma that there is a nonzero morphism of $\mathcal{C}$-modules $\mathcal{C}(-,X) \to S$, which must be an epimorphism by Lemma 2.4.4. $\square$

Finally, we define the **socle** of a $\mathcal{C}$-module F to be the sum of all of its simple submodules, and we denote it by $\operatorname{soc}_{\mathcal{C}} F$. This concept will be explored in Section 6.3.

2.5 Ideals and bimodules

In the sense that a skeletally small preadditive category generalizes the concept of a ring, we now present the generalization of ideals of rings for such categories. In doing so, we also generalize the notion of bimodules over rings.

For a given ring Λ we know that a right ideal of it is nothing but a Λ-module which is a submodule of Λ viewed as a module over itself. In this case, if we consider Λ as a preadditive category consisting of one single object X as in Example 2.1.1, then we obtain that a right ideal of Λ corresponds to a submodule of $\Lambda(-,X)$. Likewise, we also have that a left ideal of Λ corresponds to a submodule of $\Lambda(X,-)$. Therefore, by generalizing this particular case, we define a **right ideal** of $\mathcal{C}$ to be a submodule of $\mathcal{C}(-,X)$ for some $X \in \mathcal{C}$, as well as we define a **left ideal** of $\mathcal{C}$ to be a submodule of $\mathcal{C}(X,-)$ for some $X \in \mathcal{C}$.

If $F \subseteq \mathcal{C}(-,X)$ is a right ideal of $\mathcal{C}$ and if $f : Y \to X$ is a morphism in $\mathcal{C}$, then we will sometimes write $f \in F$ to indicate that $f \in F(Y)$. Considering this, the reader should note that if $f \in F$, then $fg \in F$ for every morphism $g \in \mathcal{C}$ whenever the composition makes sense. Likewise, we will assume the same notation for left ideals of $\mathcal{C}$, and if F is such an ideal, then we also have that $gf \in F$ for every morphisms $f \in F$ and $g \in \mathcal{C}$ whenever the composition makes sense.

Now, if $f : X \to Y$ is a morphism in $\mathcal{C}$, then it is not difficult to see that it generates a submodule $\langle f \rangle_{\mathcal{C}}$ of $\mathcal{C}(-,Y)$ which is defined by

$$\langle f \rangle_{\mathcal{C}}(Z) = \{fg \mid g \in \mathcal{C}(Z,X)\}$$

for each $Z \in \mathcal{C}$. Actually, $\langle f \rangle_{\mathcal{C}}$ is the image of $\mathcal{C}(-,f) : \mathcal{C}(-,X) \to \mathcal{C}(-,Y)$, and in this case, we call it the **right ideal generated by** f. Dually, we define the **left ideal generated by** f, which we denote by ${}_{\mathcal{C}}\langle f \rangle$, to be the image of $\mathcal{C}(f,-) : \mathcal{C}(Y,-) \to \mathcal{C}(X,-)$, i.e. ${}_{\mathcal{C}}\langle f \rangle$ is the submodule of $\mathcal{C}(X,-)$ which is given by

$${}_{\mathcal{C}}\langle f \rangle(Z) = \{gf \mid g \in \mathcal{C}(Y,Z)\}$$

for each $Z \in \mathcal{C}$.

In what follows we only state the next results for right ideals. The dual results for left ideals are recovered by taking $\mathcal{C}^{\mathrm{op}}$ in place of $\mathcal{C}$.

The next proposition, which will be quite useful, is from [36, Corollary 1.3], and here present a simpler proof for it.

Proposition 2.5.1. *Let $X \in \mathcal{C}$ and $F \subseteq \mathcal{C}(-,X)$ be a right ideal of $\mathcal{C}$. If $f \in \mathcal{C}(Y,X)$ for some $Y \in \mathcal{C}$, then $f \in F(Y)$ if and only if $\langle f \rangle_{\mathcal{C}} \subseteq F$.*

Proof. If $f \in F(Y)$, then $fg \in F$ for every morphism $g \in \mathcal{C}$ whenever the composition makes sense, so that $\langle f \rangle_{\mathcal{C}}(Z) \subseteq F(Z)$ for every $Z \in \mathcal{C}$. Hence by Proposition 2.4.2 we conclude that $\langle f \rangle_{\mathcal{C}} \subseteq F$.

The converse is straightforward. □

The next result is a generalization of a well known result for right ideals of rings.

Proposition 2.5.2. *Let $X \in \mathcal{C}$ and $F \subseteq \mathcal{C}(-,X)$ be a right ideal of $\mathcal{C}$. The following are equivalent:*

(a) $F = \mathcal{C}(-,X)$.

(b) $id_X \in F(X)$.

(c) There is some morphism in $F(X)$ that has a right inverse.

Proof. It is clear that (a) implies (b) and (b) implies (c). Therefore, assume that (c) holds and let $f \in F(X)$ be such that there is some $g \in \mathcal{C}(X,X)$ with $fg = id_X$. Then $id_X \in F(X)$ and it follows from Proposition 2.5.1 that $\mathcal{C}(-,X) = \langle id_X \rangle_{\mathcal{C}} \subseteq F$. Hence $F = \mathcal{C}(-,X)$. □

Now, we would like to define what a "two sided" ideal of $\mathcal{C}$ is. In order to do this, we first need to introduce what a "bimodule" and a "subbimodule" are.

A $(\mathcal{B}-\mathcal{C})$-**bimodule** F is a biadditive bifunctor $F = F(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Ab}$. In other words, it is a bifunctor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Ab}$ such that $F(-,Y)$ is a right $\mathcal{C}$-module for every $Y \in \mathcal{B}$, and such that $F(X,-)$ is a left $\mathcal{B}$-module for every $X \in \mathcal{C}$.

We can form the category of $(\mathcal{B}-\mathcal{C})$-bimodules whose objects are the biadditive bifunctors $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Ab}$ and whose morphisms are the natural transformations between such bifunctors. It is interesting to note that if F and G two $(\mathcal{B}-\mathcal{C})$-bimodules, then a collection of morphisms of abelian groups $\alpha_{(X,Y)} : F(X,Y) \to G(X,Y)$ indexed by $(X,Y) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{B}$ forms a morphism of $(\mathcal{B}-\mathcal{C})$-bimodules $\alpha : F \to G$ if and only if $\alpha_{(-,Y)} : F(-,Y) \to G(-,Y)$ is a morphism of right $\mathcal{C}$-modules for every $Y \in \mathcal{B}$, and if $\alpha_{(X,-)} : F(X,-) \to G(X,-)$ is a morphism of left $\mathcal{B}$-modules for every $X \in \mathcal{C}$.

Next, given two $(\mathcal{B}-\mathcal{C})$-bimodules F and G, we say that F is a **subbimodule** of G, and we denote it by $F \subseteq G$, if $F(X,Y) \subseteq G(X,Y)$ for every $(X,Y) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{B}$ and if the canonical inclusions $\eta_{(X,Y)} : F(X,Y) \to G(X,Y)$ form a morphism of $(\mathcal{B}-\mathcal{C})$-bimodules $\eta : F \to G$.

Now, let Λ be a ring and view it as a preadditive category as in Example 2.1.1. Then it is not difficult to see that a two sided ideal of Λ corresponds to a $(\Lambda-\Lambda)$-subbimodule of the $(\Lambda-\Lambda)$-bimodule $\Lambda(-,-)$. Motivated by this, we define a **two sided ideal** of $\mathcal{C}$ to be a subbimodule of the $(\mathcal{C}-\mathcal{C})$-bimodule $\mathcal{C}(-,-)$.

As it was the case for left and right ideals of $\mathcal{C}$, if I is a two sided ideal of $\mathcal{C}$ and if $f : X \to Y$ is a morphism in $\mathcal{C}$, then we will sometimes write $f \in I$ to indicate that $f \in I(X,Y)$. In this case, note that if $f \in I$, then $gfh \in I$ for every morphism $g,h \in \mathcal{C}$ whenever the composition makes sense. Moreover, each morphism $f : X \to Y$ in $\mathcal{C}$ generates a subbimodule ${}_{\mathcal{C}}\langle f\rangle_{\mathcal{C}}$ of $\mathcal{C}(-,-)$ which is defined by

$$ {}_{\mathcal{C}}\langle f\rangle_{\mathcal{C}}(W,Z) = \{gfh \mid g \in \mathcal{C}(Y,Z) \text{ and } h \in \mathcal{C}(W,X)\} $$

for each $W,Z \in \mathcal{C}$. We call ${}_{\mathcal{C}}\langle f\rangle_{\mathcal{C}}$ the **two sided ideal generated by** f.

It is interesting to remark that if I is a two sided ideal of $\mathcal{C}$, then we can form the quotient category $\mathcal{C}/I$ whose objects are those of $\mathcal{C}$, and where $(\mathcal{C}/I)(X,Y) = \mathcal{C}(X,Y)/I(X,Y)$ for each $X,Y \in \mathcal{C}$, with the composition in $\mathcal{C}/I$ being induced by the composition in $\mathcal{C}$.

Next, we proceed to define the "product" of two sided ideals of $\mathcal{C}$, which generalizes the notion of the product of two sided ideals of rings.

If I and J are two sided ideals of $\mathcal{C}$, then for each $X,Y \in \mathcal{C}$ consider

$$ IJ(X,Y) = \left\{ \sum_{i=1}^{m} f_i g_i \mid f_i \in I(Z_i,Y),\ g_i \in J(X,Z_i),\ Z_i \in \mathcal{C},\ m \in \mathbb{N} \right\}. $$

In other words, $IJ(X,Y)$ is the subgroup of $\mathcal{C}(X,Y)$ generated by all morphisms of the form fg with $f \in I(Z,Y)$, $g \in J(X,Z)$ and $Z \in \mathcal{C}$. It is not difficult to see that with these subgroups we obtain a two sided ideal $IJ = IJ(-,-)$ of $\mathcal{C}$, which we call the **product** of I and J. Note that we have $IJ \subseteq I$ and $IJ \subseteq J$.

Now, if I is a two sided ideal of $\mathcal{C}$, then we can define inductively the ***n*th power** of I by $I^n = I^{n-1}I$ for each $n \geqslant 2$. In this case, for each $X,Y \in \mathcal{C}$ we have

$$ I^n(X,Y) = \left\{ \sum_{i=1}^{m} f_{in} \cdots f_{i2} f_{i1} \mid f_{ij} \in I(Z_{i(j-1)}, Z_{ij}),\ Z_{ij} \in \mathcal{C},\ Z_{i0} = X,\ Z_{in} = Y,\ m \in \mathbb{N} \right\}. $$

This means that $I^n(X,Y)$ consists of all finite sums of morphisms $f : X \to Y$ in $\mathcal{C}$ that admit a factorization of the form

$$ X = Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} Z_{n-1} \xrightarrow{f_n} Z_n = Y $$

with $f_j \in I(Z_{j-1}, Z_j)$ for each $1 \leqslant j \leqslant n$.

We will agree that $I^0 = \mathcal{C}(-,-)$ for every two sided ideal I of $\mathcal{C}$, in which case we have $I^n \subseteq I^{n-1}$ for each $n \geqslant 1$. Moreover, we let I^∞ be te two sided ideal of $\mathcal{C}$ defined by $I^\infty(X,Y) = \bigcap_{n\in\mathbb{N}} I^n(X,Y)$ for each $X, Y \in \mathcal{C}$, so that we have

$$\mathcal{C} \supseteq I \supseteq I^2 \supseteq \cdots \supseteq I^n \supseteq \cdots \supseteq I^\infty,$$

where $\mathcal{C} = \mathcal{C}(-,-)$.

We end this section by remarking some features of ideals in additive categories. Therefore, for the rest of this section, let $\mathcal{A}$ be a skeletally small additive category. We first bring attention to the following result:

Proposition 2.5.3. *A right ideal $F \subseteq \mathcal{A}(-,X)$ of $\mathcal{A}$ is finitely generated if and only if there is some morphism $f : Y \to X$ in $\mathcal{A}$ such that $\langle f \rangle_{\mathcal{A}} = F$.*

Proof. Note that every morphism $\mathcal{A}(-,Y) \to F$ in $\operatorname{Mod}\mathcal{A}$ with $Y \in \mathcal{A}$ is induced by $\mathcal{A}(-,f)$ for some morphism $f : Y \to X$ in $\mathcal{A}$. Therefore, F is a finitely generated $\mathcal{A}$-module if and only if there is some morphism $f : Y \to X$ in $\mathcal{A}$ such that $\operatorname{Im}\mathcal{A}(-,f) = F$, and this latter assertion means that $\langle f \rangle_{\mathcal{A}} = F$. □

Next, until the end of this section, let I and J be two sided ideals of $\mathcal{A}$.

Note that we have that

$$I\left(\bigoplus_{i=1}^{m} X_i, \bigoplus_{j=1}^{n} Y_j\right) \simeq \bigoplus_{i,j=1}^{m,n} I(X_i, Y_j)$$

for each $X_i, Y_j \in \mathcal{A}$ and $m, n \geqslant 1$. Indeed, this follows from the fact that I is an additive functor in each variable. In this case, if $X = \bigoplus_{i=1}^m X_i$ and $Y = \bigoplus_{j=1}^n Y_j$, then for a morphism $f \in \mathcal{A}(X,Y)$ let $f_{ij} : X_i \to Y_j$ be the morphism induced by f and the decompositions of X and Y for each pair ij. Then it is easy to see that $f \in I(X,Y)$ if and only if $f_{ij} \in I(X_i, Y_j)$ for each pair ij.

Another particularity that we obtain for $\mathcal{A}$ concerns a simplification for the description of the morphisms in $IJ(X,Y)$ for $X, Y \in \mathcal{A}$. To see this, let $\sum_{i=1}^m f_i g_i \in IJ(X,Y)$ with $f_i \in I(Z_i, Y)$, $g_i \in J(X, Z_i)$ and $Z_i \in \mathcal{A}$. If we take $Z = \bigoplus_{i=1}^m Z_i$, then we can write $\sum_{i=1}^m f_i g_i = fg$ for some suitable morphisms $f \in I(Z,Y)$ and $g \in I(X,Z)$. Therefore, we obtain that

$$IJ(X,Y) = \{fg \mid f \in I(Z,Y),\ g \in J(X,Z),\ Z \in \mathcal{A}\}.$$

In particular, we also conclude that

$$I^n(X,Y) = \left\{f \in \mathcal{A}(X,Y) \mid f \text{ admits a factorization } f = f_n \cdots f_2 f_1 \text{ with } f_j \in I\right\}$$

for each $n \geqslant 2$.

2.6 The Jacobson radical

We now generalize the concepts of the Jacobson radical of a module over a ring for a $\mathcal{C}$-module, and of the Jacobson radical of a ring for a skeletally small preadditive category $\mathcal{C}$, which turns out to be a two sided ideal of $\mathcal{C}$. We remark that this latter generalization was first introduced by Gregory M. Kelly in [30], but here we present it by following the approach given in [33].

If F is a $\mathcal{C}$-module, then we define the **Jacobson radical** of F, which we denote by $\operatorname{rad}_{\mathcal{C}} F$, to be the intersection of all of its maximal submodules, and in case that F has no maximal submodule, we let $\operatorname{rad}_{\mathcal{C}} F = F$. The reader should note that if F is finitely generated and nonzero, then by Proposition 2.4.3 we have that $\operatorname{rad}_{\mathcal{C}} F$ is a proper submodule of F.

In this text, we will be mostly interested in the radical of representable modules, and to simplify the notation, for a given $Y \in \mathcal{C}$, we will denote the radical of $\mathcal{C}(-,Y)$ by $\operatorname{rad}_{\mathcal{C}}(-,Y)$. Moreover, for an object X and a morphism f in $\mathcal{C}$ we will denote $\operatorname{rad}_{\mathcal{C}}(-,Y)(X)$ by $\operatorname{rad}_{\mathcal{C}}(X,Y)$ and $\operatorname{rad}_{\mathcal{C}}(-,Y)(f)$ by $\operatorname{rad}_{\mathcal{C}}(f,Y)$.

In the next result, which is from [33, Lemma 4.1], we give a characterization of the elements of $\operatorname{rad}_{\mathcal{C}}(X,Y)$ for $X, Y \in \mathcal{C}$.

Theorem 2.6.1. *If $X, Y \in \mathcal{C}$, then*

$$\begin{aligned}\operatorname{rad}_{\mathcal{C}}(X,Y) &= \{f \in \mathcal{C}(X,Y) \mid id_Y - fg \textit{ has a right inverse for all } g \in \mathcal{C}(Y,X)\} \\ &= \{f \in \mathcal{C}(X,Y) \mid id_Y - fg \textit{ is invertible for all } g \in \mathcal{C}(Y,X)\}.\end{aligned}$$

Proof. Let $f \in \operatorname{rad}_{\mathcal{C}}(X,Y)$ and suppose that there is some $g \in \mathcal{C}(Y,X)$ such that $id_Y - fg$ does not have a right inverse. In this case, it follows from Proposition 2.5.2 that $\langle id_Y - fg\rangle_{\mathcal{C}}$ is a proper submodule of $\mathcal{C}(-,Y)$. Therefore, by Proposition 2.4.3 we conclude that $\langle id_Y - fg\rangle_{\mathcal{C}}$ is contained in some maximal submodule M of $\mathcal{C}(-,Y)$, and in particular, $id_Y - fg \in M(Y)$. However, we know that $f \in M(X)$, so that $fg \in M(Y)$. Consequently, $id_Y \in M(Y)$, which implies by Proposition 2.5.2 that $M = \mathcal{C}(-,Y)$, a contradiction. Hence $id_Y - fg$ has a right inverse for all $g \in \mathcal{C}(Y,X)$.

Conversely, let $f \in \mathcal{C}(X,Y)$ be a morphism with the property that $id_Y - fg$ has a right inverse for all $g \in \mathcal{C}(Y,X)$. Let M be a maximal submodule of $\mathcal{C}(-,Y)$ and assume that $f \notin M(X)$. Then $\langle f\rangle_{\mathcal{C}} + M$ is a submodule of $\mathcal{C}(-,Y)$ that contains M as a proper submodule, which implies that $\langle f\rangle_{\mathcal{C}} + M = \mathcal{C}(-,Y)$. Therefore, $id_Y \in \langle f\rangle_{\mathcal{C}}(Y) + M(Y)$, so that there are $g \in \mathcal{C}(Y,X)$ and $h \in M(Y)$ such that $id_Y = fg + h$. Thus $id_Y - fg = h \in M(Y)$ has a right inverse, and by Proposition 2.5.2 we obtain that $M = \mathcal{C}(-,Y)$, which is a contradiction. Consequently, $f \in M(X)$ for all maximal submodules M of $\mathcal{C}(-,Y)$ and then $f \in \operatorname{rad}_{\mathcal{C}}(X,Y)$.

To verify the second equality in the statement of the theorem, we only need to show that if $f \in \operatorname{rad}_{\mathcal{C}}(X,Y)$, then $id_Y - fg$ is invertible for all $g \in \mathcal{C}(Y,X)$.

Well, let $f \in \mathrm{rad}_{\mathcal{C}}(X, Y)$ and $g \in \mathcal{C}(Y, X)$. We know that there is some $h \in \mathcal{C}(Y, Y)$ such that $id_Y = (id_Y - fg)h$, and thus we have $h = id_Y - f(-gh)$ with $-gh \in \mathcal{C}(Y, X)$. Hence there is some $k \in \mathcal{C}(Y, Y)$ such that $hk = id_Y$, which gives us $k = (id_Y - fg)hk = id_Y - fg$. Therefore, $id_Y = h(id_Y - fg)$, and we conclude that $id_Y - fg$ is invertible. □

As a consequence of Theorem 2.6.1, we obtain the following:

Corollary 2.6.2. $\mathrm{rad}_{\mathcal{C}}(X, X)$ *is the Jacobson radical of* $\mathrm{End}_{\mathcal{C}}(X)$ *for every* $X \in \mathcal{C}$.

Proof. This follows from Theorem 2.6.1 and [1, Chapter VII, Theorem 3.2]. □

Now, we will show that the radical of the representable left $\mathcal{C}$-modules must in somehow coincide with the radical of the representable right $\mathcal{C}$-modules. To achieve this goal, for each $X \in \mathcal{C}$ we will temporarily denote the radical of $\mathcal{C}(X, -)$ by $\mathrm{rad}^*_{\mathcal{C}}(X, -)$, as well as we will denote $\mathrm{rad}^*_{\mathcal{C}}(X, -)(Y)$ by $\mathrm{rad}^*_{\mathcal{C}}(X, Y)$ for each $Y \in \mathcal{C}$.

First, we give the dual of Theorem 2.6.1:

Theorem 2.6.3. *If* $X, Y \in \mathcal{C}$, *then*

$$\begin{aligned} \mathrm{rad}^*_{\mathcal{C}}(X, Y) &= \{f \in \mathcal{C}(X, Y) \mid id_X - gf \text{ has a left inverse for all } g \in \mathcal{C}(Y, X)\} \\ &= \{f \in \mathcal{C}(X, Y) \mid id_X - gf \text{ is invertible for all } g \in \mathcal{C}(Y, X)\}. \end{aligned}$$

Proof. Since the category of left $\mathcal{C}$-modules coincide with the category of right $\mathcal{C}^{\mathrm{op}}$-modules, the result follows by Theorem 2.6.1. □

Next, we give a lemma which is from [2, Lemma II.1.16] (see also [33, Lemma 4.2]).

Lemma 2.6.4. *Let* $X, Y \in \mathcal{C}$. *If* $f \in \mathcal{C}(X, Y)$ *and* $g \in \mathcal{C}(Y, X)$, *then* $id_Y - fg$ *is invertible if and only if* $id_X - gf$ *is invertible.*

Proof. Assume that $id_Y - fg$ is invertible and let $h \in \mathcal{C}(Y, Y)$ be its inverse. In this case, we can verify that $id_X + ghf$ is the inverse of $id_X - gf$, though we leave the details for the reader.

The converse follows from the above paragraph by renaming X, Y, f and g. □

Well, observe that from theorems 2.6.1 and 2.6.3 and Lemma 2.6.4 we obtain that $\mathrm{rad}^*_{\mathcal{C}}(X, Y) = \mathrm{rad}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}$. Therefore, from now on, for each $X \in \mathcal{C}$ we will denote $\mathrm{rad}^*_{\mathcal{C}}(X, -)$ by $\mathrm{rad}_{\mathcal{C}}(X, -)$, and for an object Y and a morphism f in $\mathcal{C}$ we will denote $\mathrm{rad}_{\mathcal{C}}(X, -)(Y)$ by $\mathrm{rad}_{\mathcal{C}}(X, Y)$ and $\mathrm{rad}_{\mathcal{C}}(X, -)(f)$ by $\mathrm{rad}_{\mathcal{C}}(X, f)$.

From the above discussion we conclude that $\mathrm{rad}_{\mathcal{C}} = \mathrm{rad}_{\mathcal{C}}(-, -)$ is a two sided ideal of $\mathcal{C}$. We call $\mathrm{rad}_{\mathcal{C}}$ the **Jacobson radical** of $\mathcal{C}$.

We end this section by describing the elements of $\mathrm{rad}_{\mathcal{C}}(X,Y)$ by supposing some additional conditions on $X, Y \in \mathcal{C}$. This will be useful when considering the case when $\mathcal{C}$ is a Krull–Schmidt category.

Proposition 2.6.5. *If $X, Y \in \mathcal{C}$, then*

(a) $\mathrm{rad}_{\mathcal{C}}(X,Y) = \{f \in \mathcal{C}(X,Y) \mid f$ *does not have a right inverse*$\}$ *if* $\mathrm{End}_{\mathcal{C}}(Y)$ *is a local ring.*

(b) $\mathrm{rad}_{\mathcal{C}}(X,Y) = \{f \in \mathcal{C}(X,Y) \mid f$ *does not have a left inverse*$\}$ *if* $\mathrm{End}_{\mathcal{C}}(X)$ *is a local ring.*

(c) $\mathrm{rad}_{\mathcal{C}}(X,Y) = \{f \in \mathcal{C}(X,Y) \mid f$ *is not invertible*$\}$ *if* $\mathrm{End}_{\mathcal{C}}(X)$ *and* $\mathrm{End}_{\mathcal{C}}(Y)$ *are local rings.*

(d) $\mathrm{rad}_{\mathcal{C}}(X,Y) = \mathcal{C}(X,Y)$ *if* $\mathrm{End}_{\mathcal{C}}(X)$ *and* $\mathrm{End}_{\mathcal{C}}(Y)$ *are local rings and* $X \not\approx Y$.

Proof. (a) Let $f \in \mathrm{rad}_{\mathcal{C}}(X,Y)$ and suppose that there is some morphism $g \in \mathcal{C}(Y,X)$ such that $fg = id_Y$. Then $0 = id_Y - fg$ is invertible, which implies that $Y = 0$, a contradiction. Hence f does not have a right inverse.

Conversely, suppose that $f \in \mathcal{C}(X,Y)$ does not have a right inverse and consider a morphism $g \in \mathcal{C}(Y,X)$. If $id_Y - fg$ is not invertible, then because $\mathrm{End}_{\mathcal{C}}(Y)$ is local it follows from [1, Chapter VII, Theorem 6.5] that $id_Y - (id_Y - fg) = fg$ is invertible. But this implies that f has a right inverse, which is a contradiction. Therefore, $id_Y - fg$ is invertible and consequently $f \in \mathrm{rad}_{\mathcal{C}}(X,Y)$.

(b) This follows dually from the proof of item (a).

(c) This follows from items (a) and (b).

(d) This follows from item (c). □

2.7 Morita equivalence

In this section we generalize the notion of Morita equivalent rings for skeletally small preadditive categories, as well as we give some results related to this generalization.

As in the case of rings, we will say that $\mathcal{B}$ and $\mathcal{C}$ are **Morita equivalent** if there is an equivalence of categories $\mathrm{Mod}\,\mathcal{B} \to \mathrm{Mod}\,\mathcal{C}$, which in this case is called a **Morita equivalence**.

It is not difficult to verify that if $\mathcal{B} \approx \mathcal{C}$, then $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent, but the converse of this assertion is not true in general. Nevertheless, we will show that if $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent, then $\mathcal{P}(\mathcal{B}) \approx \mathcal{P}(\mathcal{C})$. In order to prove this result, we first need to develop the notion of a "small" object.

Let $\mathcal{A}$ be an additive category with infinite coproducts. An object $X \in \mathcal{A}$ is called **small** if for every collection of objects $Y_i \in \mathcal{A}$ indexed by a set I and every morphism

$f : X \to \bigoplus_{i \in I} Y_i$ in $\mathcal{A}$ there is a finite subset $J \subseteq I$ and a morphism $f' : X \to \bigoplus_{j \in J} Y_j$ in $\mathcal{A}$ such that $f = qf'$, where $q : \bigoplus_{j \in J} Y_j \to \bigoplus_{i \in I} Y_i$ is the canonical injection. In the case that X is small and projective, we say simply that it is a **small projective** object.

The next result is well known.

Lemma 2.7.1. *Every finitely generated $\mathcal{C}$-module is small and every small projective $\mathcal{C}$-module is finitely generated.*

Proof. Let F be a finitely generated $\mathcal{C}$-module and let $\alpha : \bigoplus_{i=1}^{n} \mathcal{C}(-, X_i) \to F$ be an epimorphism in $\operatorname{Mod} \mathcal{C}$ with $X_i \in \mathcal{C}$. Consider a collection of $\mathcal{C}$-modules G_i indexed by a set I and a morphism $\beta : F \to \bigoplus_{i \in I} G_i$ in $\operatorname{Mod} \mathcal{C}$. If $\delta_k : \mathcal{C}(-, X_k) \to \bigoplus_{i=1}^{n} \mathcal{C}(-, X_i)$ is the canonical injection for each $1 \leqslant k \leqslant n$, then since $(\beta\alpha\delta_k)_{X_k}(id_{X_k})$ is an element of $\bigoplus_{i \in I} G_i(X_k)$, there is a finite subset $J_k \subseteq I$ such that $(\beta\alpha\delta_k)_{X_k}(id_{X_k}) \in \bigoplus_{j \in J} G_j(X_k)$. Next, take $J = J_1 \cup \cdots \cup J_n$ and let $\sigma : \bigoplus_{j \in J} G_j \to \bigoplus_{i \in I} G_i$ be the canonical injection and $\rho : \bigoplus_{i \in I} G_i \to \bigoplus_{j \in J} G_j$ be the canonical projection. We claim that $\beta = \sigma\rho\beta$, so that F is small.

To verify the claim, first note that $(\beta\alpha\delta_k)_{X_k}(id_{X_k}) = (\sigma\rho\beta\alpha\delta_k)_{X_k}(id_{X_k})$, thus from the Yoneda Lemma we obtain that $\beta\alpha\delta_k = \sigma\rho\beta\alpha\delta_k$ for each $1 \leqslant k \leqslant n$. Therefore, taking the coproduct of these morphisms we obtain that $\beta\alpha = \sigma\rho\beta\alpha$, and given that α is an epimorphism, we conclude that $\beta = \sigma\rho\beta$.

Now, let F be a small projective $\mathcal{C}$-module. From Lemma 2.3.1 we know that there is an epimorphism $\alpha : \bigoplus_{i \in I} \mathcal{C}(-, X_i) \to F$ in $\operatorname{Mod} \mathcal{C}$ with each $X_i \in \mathcal{C}$, and because F is projective, we obtain that there is a morphism $\beta : F \to \bigoplus_{i \in I} \mathcal{C}(-, X_i)$ in $\operatorname{Mod} \mathcal{C}$ such that $\alpha\beta = id_F$. Thus from the fact that F is small we conclude that there is a finite subset $J \subseteq I$ and a morphism $\gamma : F \to \bigoplus_{j \in J} \mathcal{C}(-, X_j)$ in $\operatorname{Mod} \mathcal{C}$ such that $\beta = \delta\gamma$, where $\delta : \bigoplus_{j \in J} \mathcal{C}(-, X_j) \to \bigoplus_{i \in I} \mathcal{C}(-, X_i)$ is the canonical injection. Hence we have $id_F = \alpha\beta = \alpha\delta\gamma$, so that $\alpha\delta : \bigoplus_{j \in J} \mathcal{C}(-, X_j) \to F$ is an epimorphism, which implies that F is finitely generated. □

With the above discussion on small objects done, we can conclude the following:

Theorem 2.7.2. *If* $\mathbb{s} : \operatorname{Mod} \mathcal{B} \to \operatorname{Mod} \mathcal{C}$ *is an equivalence of categories, then it also induces an equivalence* $\mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{C})$.

Proof. Since the properties of an object being small and projective are preserved by equivalences of categories, we conclude that a $\mathcal{B}$-module F is small projective if and only if $\mathbb{s}(F)$ is a small projective $\mathcal{C}$-module. Moreover, from Lemma 2.7.1 we know that a module is small projective if and only if it is finitely generated projective, hence we conclude that a $\mathcal{B}$-module F is finitely generated projective if and only if the $\mathcal{C}$-module $\mathbb{s}(F)$ is finitely generated projective. Consequently, $\mathbb{s}$ induces an equivalence of categories $\mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{C})$. □

As a first consequence of Theorem 2.7.2, we will prove below that the property of being finitely generated is preserved under Morita equivalences.

Corollary 2.7.3. *If* $\mathtt{s} : \operatorname{Mod}\mathcal{B} \to \operatorname{Mod}\mathcal{C}$ *is an equivalence of categories, then it also induces an equivalence* $\mathrm{f.g.}(\mathcal{B}) \to \mathrm{f.g.}(\mathcal{C})$.

Proof. It is easy to conclude from Theorem 2.7.2 that a $\mathcal{B}$-module F is finitely generated if and only if the $\mathcal{C}$-module $\mathtt{s}(F)$ is finitely generated. Therefore, $\mathtt{s}$ induces an equivalence of categories $\mathrm{f.g.}(\mathcal{B}) \to \mathrm{f.g.}(\mathcal{C})$. □

Next, we will show that if $\mathcal{P}(\mathcal{B}) \approx \mathcal{P}(\mathcal{C})$, then $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent. To achieve this goal, we first need some results.

We start by remarking that we can always extend $\mathcal{C}$ to an additive category $\mathcal{M}(\mathcal{C})$ which contains $\mathcal{C}$ as a subcategory. We define the objects of $\mathcal{M}(\mathcal{C})$ to be the finite sequences $(X_1, \dots, X_n)$ of objects $X_i \in \mathcal{C}$, and for $(X_1, \dots, X_n), (Y_1, \dots, Y_m) \in \mathcal{M}(\mathcal{C})$ we let the morphisms from $(X_1, \dots, X_n)$ to $(Y_1, \dots, Y_m)$ to be $m \times n$ matrices (f_{ij}) with $f_{ij} \in \mathcal{C}(X_j, Y_i)$ for each pair ij. The composition and sum of morphisms in $\mathcal{M}(\mathcal{C})$ are given by standard matrix multiplication and matrix sum, respectively, and the coproduct of two objects $(X_1, \dots, X_n), (Y_1, \dots, Y_m) \in \mathcal{M}(\mathcal{C})$ is given by $(X_1, \dots, X_n, Y_1, \dots, Y_m)$. With this definition, $\mathcal{M}(\mathcal{C})$ is easily seen to be an additive category which is also skeletally small, and it is straightforward that we can view $\mathcal{C}$ as a subcategory of $\mathcal{M}(\mathcal{C})$.

The next result is from [33, page 11], but here we give a more explicit proof for it.

Proposition 2.7.4. *The categories* $\operatorname{Mod}\mathcal{C}$ *and* $\operatorname{Mod}\mathcal{M}(\mathcal{C})$ *are equivalent.*

Proof. It is not difficult to verify that we can always extend a $\mathcal{C}$-module F to a $\mathcal{M}(\mathcal{C})$-module $\widetilde{F}$ by defining $\widetilde{F}(X_1, \dots, X_n) = (F(X_1), \dots, F(X_n))$ on objects and by setting $\widetilde{F}((f_{ij})) = (F(f_{ij}))^{\mathsf{T}}$ for a morphism $(f_{ij}) : (X_1, \dots, X_n) \to (Y_1, \dots, Y_m)$ in $\mathcal{M}(\mathcal{C})$, where $^{\mathsf{T}}$ denotes matrix transposition. Moreover, given a morphism $\alpha : F \to G$ in $\operatorname{Mod}\mathcal{C}$, we can also extend it to a morphism $\widetilde{\alpha} : \widetilde{F} \to \widetilde{G}$ in $\operatorname{Mod}\mathcal{M}(\mathcal{C})$ by defining

$$\widetilde{\alpha}_{(X_1,\dots,X_n)} = \begin{pmatrix} \alpha_{X_1} & & \\ & \ddots & \\ & & \alpha_{X_n} \end{pmatrix}$$

for each object $(X_1, \dots, X_n) \in \mathcal{M}(\mathcal{C})$.

In this case, we obtain a functor $\mathtt{s} : \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{M}(\mathcal{C})$ given by $\mathtt{s}(F) = \widetilde{F}$ and $\mathtt{s}(\alpha) = \widetilde{\alpha}$ for each object F and morphism α in $\operatorname{Mod}\mathcal{C}$.

Now, since we can view $\mathcal{C}$ as a subcategory of $\mathcal{M}(\mathcal{C})$, we can consider the restriction functor $\mathtt{r} : \operatorname{Mod}\mathcal{M}(\mathcal{C}) \to \operatorname{Mod}\mathcal{C}$ which sends an object F and a morphism α in $\operatorname{Mod}\mathcal{M}(\mathcal{C})$ to $F|_{\mathcal{C}}$ and $\alpha|_{\mathcal{C}}$, respectively, and it is easy to verify that $\mathtt{rs} \simeq id_{\operatorname{Mod}\mathcal{C}}$ and $\mathtt{sr} \simeq id_{\operatorname{Mod}\mathcal{M}(\mathcal{C})}$. □

Note that Proposition 2.7.4 can be very useful since it tells us that to study modules over skeletally small preadditive categories, we may choose to work with modules over skeletally small additive categories. Recall that we have already used this idea to prove Proposition 2.4.3, in which case we had the advantage of only needing to consider cyclic modules in its proof.

Next, we give a detailed proof of a result that is mentioned in [5, Proposition 2.5].

Lemma 2.7.5. *If $\mathcal{V}$ is a variety and $\mathcal{A}$ is a subcategory of $\mathcal{V}$ such that* $\operatorname{add}\mathcal{A} = \mathcal{V}$*, then the categories* $\operatorname{Mod}\mathcal{A}$ *and* $\operatorname{Mod}\mathcal{V}$ *are equivalent.*

Proof. Observe that from Proposition 2.7.4 we have that $\operatorname{Mod}\mathcal{A} \approx \operatorname{Mod}\mathcal{M}(\mathcal{A})$, hence to conclude that $\operatorname{Mod}\mathcal{A} \approx \operatorname{Mod}\mathcal{V}$, it suffices to verify that $\operatorname{Mod}\mathcal{M}(\mathcal{A}) \approx \operatorname{Mod}\mathcal{V}$. Moreover, note that $\mathcal{M}(\mathcal{A})$ is equivalent to the subcategory of $\mathcal{V}$ consisting of all finite direct sums of objects in $\mathcal{A}$, and in what follows we will consider $\mathcal{M}(\mathcal{A})$ as being such subcategory.

For each $X \in \mathcal{V}$ we fix a pair of objects $X_0 \in \mathcal{V}$ and $Z_X \in \mathcal{M}(\mathcal{A})$ such that $X \oplus X_0 \simeq Z_X$. Furthermore, for each $X \in \mathcal{V}$ we let $q_X : X \to Z_X$ be the canonical injection $X \to X \oplus X_0$ composed with an isomorphism $X \oplus X_0 \to Z_X$, and $p_X : Z_X \to X$ be the inverse of the chosen isomorphism $X \oplus X_0 \to Z_X$ composed with the canonical projection $X \oplus X_0 \to X$, so that $p_X q_X = id_X$ and $q_X p_X$ is an idempotent morphism.

Now, if F is a $\mathcal{M}(\mathcal{A})$-module, then we extend it to a $\mathcal{V}$-module $\widehat{F}$ by the following procedure: For each $X \in \mathcal{V}$ we know that $F(q_X p_X)$ is an idempotent morphism in Ab, so that $F(Z_X) \simeq \operatorname{Ker} F(q_X p_X) \oplus \operatorname{Im} F(q_X p_X)$, and thus we define $\widehat{F}(X) = \operatorname{Im} F(q_X p_X)$. For a morphism $f : X \to Y$ in $\mathcal{V}$ we set $\widehat{F}(f)$ to be the obvious composition

$$\widehat{F}(Y) \longrightarrow F(Z_Y) \xrightarrow{F(q_Y f p_X)} F(Z_X) \longrightarrow \widehat{F}(X) \ .$$

Moreover, if $\alpha : F \to G$ is a morphism in $\operatorname{Mod}\mathcal{M}(\mathcal{A})$, then we can extend it to a morphism $\widehat{\alpha} : \widehat{F} \to \widehat{G}$ in $\operatorname{Mod}\mathcal{V}$ by defining $\widehat{\alpha}_X$ to be the obvious composition

$$\widehat{F}(X) \longrightarrow F(Z_X) \xrightarrow{\alpha_{Z_X}} G(Z_X) \longrightarrow \widehat{G}(X)$$

for each $X \in \mathcal{V}$.

In this case, we obtain a functor $\mathtt{s} : \operatorname{Mod}\mathcal{M}(\mathcal{A}) \to \operatorname{Mod}\mathcal{V}$ given by $\mathtt{s}(F) = \widehat{F}$ and $\mathtt{s}(\alpha) = \widehat{\alpha}$ for each object F and morphism α in $\operatorname{Mod}\mathcal{M}(\mathcal{A})$.

Finally, if we consider the restriction functor $\mathtt{r} : \operatorname{Mod}\mathcal{V} \to \operatorname{Mod}\mathcal{M}(\mathcal{A})$ which sends an object F and a morphism α in $\operatorname{Mod}\mathcal{V}$ to $F|_{\mathcal{V}}$ and $\alpha|_{\mathcal{V}}$, respectively, then it is not difficult to conclude that $\mathtt{rs} \simeq id_{\operatorname{Mod}\mathcal{M}(\mathcal{A})}$ and $\mathtt{sr} \simeq id_{\operatorname{Mod}\mathcal{V}}$. □

Now, we collect some consequences of the above discussion.

The next proposition is mentioned in [33, page 12]. However, here we present a different approach, as well as a different proof for it.

Proposition 2.7.6. *The categories* $\operatorname{Mod}\mathcal{C}$ *and* $\operatorname{Mod}\mathcal{P}(\mathcal{C})$ *are equivalent.*

Proof. Since $\operatorname{add}\mathcal{C} = \mathcal{P}(\mathcal{C})$, it follows from Lemma 2.7.5 that $\operatorname{Mod}\mathcal{C} \approx \operatorname{Mod}\mathcal{P}(\mathcal{C})$. □

Observe that, as Proposition 2.7.4, we also have that Proposition 2.7.6 can be very useful since it tells us that we may work with modules over varieties instead of modules over skeletally small preadditive categories, given that $\mathcal{P}(\mathcal{C})$ is a variety. The advantage of doing so is that by Proposition 2.3.3 we have an easier description of the finitely generated projective modules over varieties.

Finally, we prove below the result that we had established as our goal, which is from [5, Proposition 2.6].

Theorem 2.7.7. *If* $\mathcal{P}(\mathcal{B})$ *and* $\mathcal{P}(\mathcal{C})$ *are equivalent, then* $\operatorname{Mod}\mathcal{B}$ *and* $\operatorname{Mod}\mathcal{C}$ *are equivalent.*

Proof. This is straightforward from Proposition 2.7.6. □

In addition to the above results, we also have the following:

Proposition 2.7.8. *The following are equivalent for* $\mathcal{B}$ *and* $\mathcal{C}$*:*

(a) $\operatorname{Mod}\mathcal{B} \approx \operatorname{Mod}\mathcal{C}$.

(b) $\mathrm{f.g.}(\mathcal{B}) \approx \mathrm{f.g.}(\mathcal{C})$.

(c) $\mathcal{P}(\mathcal{B}) \approx \mathcal{P}(\mathcal{C})$.

Proof. It follows from theorems 2.7.2 and 2.7.7 that (a) is equivalent to (c), and we also have that (a) implies (b) by Corollary 2.7.3. To end the proof, we verify below that (b) implies (c).

First, we remark that it is not difficult to show that the subcategory consisting of the projective objects in $\mathrm{f.g.}(\mathcal{C})$ coincides with $\mathcal{P}(\mathcal{C})$, and obviously, we have the same for $\mathrm{f.g.}(\mathcal{B})$. Therefore, if $\mathrm{s} : \mathrm{f.g.}(\mathcal{B}) \to \mathrm{f.g.}(\mathcal{C})$ is an equivalence of categories, then given that for each $F \in \mathrm{f.g.}(\mathcal{B})$ we have that F is projective in $\mathrm{f.g.}(\mathcal{B})$ if and only if $\mathrm{s}(F)$ is projective in $\mathrm{f.g.}(\mathcal{C})$, we conclude that s induces an equivalence of categories $\mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{C})$. □

We will also obtain from propositions 3.1.1 and 3.1.2 another equivalent condition for $\mathcal{B}$ and $\mathcal{C}$ to be Morita equivalent, which will improve Proposition 2.7.8.

To end this section, we observe that we have defined $\mathcal{B}$ and $\mathcal{C}$ to be Morita equivalent if their categories of right modules are equivalent. In this case, one could ask if this is equivalent to say that their categories of left modules are equivalent. In what follows we give an affirmative answer to this question.

The lemma and the proposition below are from [5, pages 189–190] and [5, Proposition 2.6], respectively.

Lemma 2.7.9. *The categories $\mathcal{P}(\mathcal{C}^{\mathrm{op}})$ and $\mathcal{P}(\mathcal{C})^{\mathrm{op}}$ coincide.*

Proof. Note that if $\mathcal{A}$ is a subcategory of a variety $\mathcal{V}$, then we have that $\operatorname{add}\mathcal{A} = \mathcal{V}$ if and only if $\operatorname{add}\mathcal{A}^{\mathrm{op}} = \mathcal{V}^{\mathrm{op}}$. Taking this into account and considering that if we view $\mathcal{C}$ as a subcategory of $\mathcal{P}(\mathcal{C})$, then $\operatorname{add}\mathcal{C} = \mathcal{P}(\mathcal{C})$, we obtain that $\operatorname{add}\mathcal{C}^{\mathrm{op}} = \mathcal{P}(\mathcal{C})^{\mathrm{op}}$. However, $\operatorname{add}\mathcal{C}^{\mathrm{op}} = \mathcal{P}(\mathcal{C}^{\mathrm{op}})$, and thus $\mathcal{P}(\mathcal{C}^{\mathrm{op}}) = \mathcal{P}(\mathcal{C})^{\mathrm{op}}$. $\square$

Proposition 2.7.10. *$\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent if and only if so are $\mathcal{B}^{\mathrm{op}}$ and $\mathcal{C}^{\mathrm{op}}$.*

Proof. We know by Proposition 2.7.8 that $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent if and only if $\mathcal{P}(\mathcal{B}) \approx \mathcal{P}(\mathcal{C})$, which is the case if and only if $\mathcal{P}(\mathcal{B})^{\mathrm{op}} \approx \mathcal{P}(\mathcal{C})^{\mathrm{op}}$. But from Lemma 2.7.9 we have that this last condition is equivalent to say that $\mathcal{P}(\mathcal{B}^{\mathrm{op}}) \approx \mathcal{P}(\mathcal{C}^{\mathrm{op}})$, which again, by Proposition 2.7.8, happens if and only if $\mathcal{B}^{\mathrm{op}}$ and $\mathcal{C}^{\mathrm{op}}$ are Morita equivalent. $\square$

2.8 Modules over rings

At this point of the text the reader is probably already convinced that functor categories are indeed natural generalizations of categories of modules over rings. With this idea in mind, one could now ask under which conditions we have that $\operatorname{Mod}\mathcal{C}$ is indeed equivalent to a category of modules over a ring. In what follows we give a complete answer to this problem, as well as we describe how an equivalence between a category of modules over a variety and a category of modules over a ring is given.

We begin by providing an answer to the problem described above.

Theorem 2.8.1. *There is a ring Λ which is Morita equivalent to $\mathcal{C}$ if and only if $\mathcal{P}(\mathcal{C})$ has an additive generator. Moreover, if this is the case, then there is an additive generator P for $\mathcal{P}(\mathcal{C})$ such that* $\operatorname{End}_{\mathcal{C}}(P) \simeq \Lambda$.

Proof. If there is a ring Λ which is Morita equivalent to $\mathcal{C}$, then it follows from Theorem 2.7.2 that there is an equivalence of categories $\mathrm{s} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\Lambda)$. In this case, if we take $P \in \mathcal{P}(\mathcal{C})$ such that $\mathrm{s}(P) \simeq \Lambda$, then given that $\mathcal{P}(\Lambda) = \operatorname{add}\Lambda$, we conclude that $\mathcal{P}(\mathcal{C}) = \operatorname{add}P$. Furthermore, we have that

$$\operatorname{End}_{\mathcal{C}}(P) \simeq \operatorname{End}_{\Lambda}(\mathrm{s}(P)) \simeq \operatorname{End}_{\Lambda}(\Lambda) \simeq \Lambda.$$

Conversely, suppose that $\mathcal{P}(\mathcal{C})$ has an additive generator P, and let $\Lambda = \operatorname{End}_{\mathcal{C}}(P)$. Then, as in Example 2.1.1, we can view Λ as the subcategory of $\mathcal{P}(\mathcal{C})$ whose only object is P, and it follows from Lemma 2.7.5 that $\operatorname{Mod}\Lambda \approx \operatorname{Mod}\mathcal{P}(\mathcal{C})$. Therefore, since we know from Proposition 2.7.6 that $\operatorname{Mod}\mathcal{P}(\mathcal{C}) \approx \operatorname{Mod}\mathcal{C}$, we obtain that $\operatorname{Mod}\Lambda \approx \operatorname{Mod}\mathcal{C}$. $\square$

Theorem 2.8.1 is very interesting, and it gives a complete answer to the problem that we had proposed. Nevertheless, it does not give us an explicit description of a functor that makes an equivalence between $\operatorname{Mod}\mathcal{C}$ and $\operatorname{Mod}\Lambda$, for the case that $\mathcal{P}(\mathcal{C})$ has an additive generator with endomorphism ring Λ. In what follows we describe how such an equivalence is given for the case when $\mathcal{C}$ is a variety.

Let $Z \in \mathcal{C}$ and write $\Lambda = \operatorname{End}_{\mathcal{C}}(Z)$. We already saw in Section 2.1 that we can transform $\mathcal{C}$-modules into Λ-modules by considering the functor $\mathsf{e}_{\mathcal{C},Z} : \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\Lambda$ which is called the evaluation functor at Z. On the other hand, there is also a functor $\mathsf{a}_{\mathcal{C},Z} : \operatorname{Mod}\Lambda \to \operatorname{Mod}\mathcal{C}$ which is given by $\mathsf{a}_{\mathcal{C},Z}(M) = \operatorname{Hom}_{\Lambda}(\mathcal{C}(Z,-),M)$ for $M \in \operatorname{Mod}\Lambda$, and which sends a morphism $f : M \to N$ in $\operatorname{Mod}\Lambda$ to the morphism of $\mathcal{C}$-modules

$$\mathsf{a}_{\mathcal{C},Z}(f) = \operatorname{Hom}_{\Lambda}(\mathcal{C}(Z,-),f) : \operatorname{Hom}_{\Lambda}(\mathcal{C}(Z,-),M) \to \operatorname{Hom}_{\Lambda}(\mathcal{C}(Z,-),N)$$

defined by

$$\mathsf{a}_{\mathcal{C},Z}(f)_X(g) = \operatorname{Hom}_{\Lambda}(\mathcal{C}(Z,X),f)(g) = fg$$

for each $X \in \mathcal{C}$ and $g \in \operatorname{Hom}_{\Lambda}(\mathcal{C}(Z,X),M)$. We call $\mathsf{a}_{\mathcal{C},Z}$ the **assimilation functor at** Z, and we will denote it simply by a_Z when no confusion can arise.

With the above notation, we can easily verify that $\mathsf{e}_Z\mathsf{a}_Z \simeq id_{\operatorname{Mod}\Lambda}$. In fact, for each $M \in \operatorname{Mod}\Lambda$ we have $(\mathsf{e}_Z\mathsf{a}_Z)(M) = \operatorname{Hom}_{\Lambda}(\mathcal{C}(Z,Z),M) = \operatorname{Hom}_{\Lambda}(\Lambda,M)$, and we know from Proposition 2.2.2 that there is an isomorphism of Λ-modules $\operatorname{Hom}_{\Lambda}(\Lambda,M) \simeq M$ which is functorial in M, so that $\mathsf{e}_Z\mathsf{a}_Z \simeq id_{\operatorname{Mod}\Lambda}$.

Now, recall that if $\mathcal{V}$ is a variety, then by Proposition 2.3.3 we have that $\mathcal{V}$ is equivalent to $\mathcal{P}(\mathcal{V})$. Therefore, we obtain from Theorem 2.8.1 that $\mathcal{V}$ is Morita equivalent to some ring Λ if and only if $\mathcal{V}$ has an additive generator $Z \in \mathcal{V}$ such that $\Lambda \simeq \operatorname{End}_{\mathcal{V}}(Z)$. In the next two results we will present another proof of this fact by describing the functors that give an equivalence between $\operatorname{Mod}\mathcal{V}$ and $\operatorname{Mod}\Lambda$.

Proposition 2.8.2. *Let $\mathcal{V}$ be a variety with an additive generator $Z \in \mathcal{V}$ and let $\Lambda = \operatorname{End}_{\mathcal{V}}(Z)$. Then $\mathsf{e}_Z : \operatorname{Mod}\mathcal{V} \to \operatorname{Mod}\Lambda$ and $\mathsf{a}_Z : \operatorname{Mod}\Lambda \to \operatorname{Mod}\mathcal{V}$ are quasi-inverse equivalences.*

Proof. We have already seen that $\mathsf{e}_Z\mathsf{a}_Z \simeq id_{\operatorname{Mod}\Lambda}$, hence we only need to verify that $\mathsf{a}_Z\mathsf{e}_Z \simeq id_{\operatorname{Mod}\mathcal{V}}$. In order to do this, we will define an isomorphism $\psi : \mathsf{a}_Z\mathsf{e}_Z \to id_{\operatorname{Mod}\mathcal{V}}$.

First, for each $X \in \mathcal{V}$ we fix an integer $r(X) \geqslant 1$ and an object $X_0 \in \mathcal{V}$ such that $X \oplus X_0 \simeq Z^{(r(X))}$. Furthermore, for each $X \in \mathcal{V}$ we let $q_X : X \to Z^{(r(X))}$ be the canonical injection $X \to X \oplus X_0$ composed with an isomorphism $X \oplus X_0 \to Z^{(r(X))}$, and $p_X : Z^{(r(X))} \to X$ be the inverse of the chosen isomorphism $X \oplus X_0 \to Z^{(r(X))}$ composed with the canonical projection $X \oplus X_0 \to X$, so that $p_X q_X = id_X$. Moreover, for each $k \geqslant 1$ and $1 \leqslant j \leqslant k$ we denote by $u_j^k : Z \to Z^{(k)}$ and $v_j^k : Z^{(k)} \to Z$ the canonical injection and projection in the jth position, respectively.

If F is a $\mathcal{V}$-module, then we define

$$\psi_F : (\mathtt{a}_Z\mathtt{e}_Z)(F) = \operatorname{Hom}_\Lambda(\mathcal{V}(Z,-), F(Z)) \to F$$

as follows: For each $X \in \mathcal{V}$, $1 \leqslant j \leqslant r(X)$ and $g \in \operatorname{Hom}_\Lambda(\mathcal{V}(Z,X), F(Z))$ we can consider the composition

$$\mathcal{V}(X,X) \xrightarrow{\mathcal{V}(p_X u_j^{r(X)}, X)} \mathcal{V}(Z,X) \xrightarrow{g} F(Z) \xrightarrow{F(v_j^{r(X)} q_X)} F(X) \ ,$$

hence we define

$$(\psi_F)_X : \operatorname{Hom}_\Lambda(\mathcal{V}(Z,X), F(Z)) \to F(X)$$

by

$$(\psi_F)_X(g) = \left(\sum_{j=1}^{r(X)} F(v_j^{r(X)} q_X) g \mathcal{V}(p_X u_j^{r(X)}, X)\right)(id_X) = \sum_{j=1}^{r(X)} F(v_j^{r(X)} q_X)(g(p_X u_j^{r(X)}))$$

for each $g \in \operatorname{Hom}_\Lambda(\mathcal{V}(Z,X), F(Z))$, which is a morphism in Ab.

Now, let

$$\theta_{F,X} : \operatorname{Hom}_{\mathcal{V}}(\mathcal{V}(-,X), F) \to F(X)$$

be the isomorphism from the Yoneda Lemma, and

$$\mathtt{e}_Z^{\mathcal{V}(-,X),F} : \operatorname{Hom}_{\mathcal{V}}(\mathcal{V}(-,X), F) \to \operatorname{Hom}_\Lambda(\mathcal{V}(Z,X), F(Z))$$

be the map induced by the functor $\mathtt{e}_Z$. It is not difficult to verify that the diagram below is commutative,

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(\mathcal{V}(Z,X), F(Z)) \arrow[rr, "(\psi_F)_X"] & & F(X) \\
& \operatorname{Hom}_{\mathcal{V}}(\mathcal{V}(-,X), F) \arrow[ul, "\mathtt{e}_Z^{\mathcal{V}(-,X),F}"] \arrow[ur, "\theta_{F,X}"'] &
\end{tikzcd}$$

hence if we denote $\omega_{F,X} = (\theta_{F,X})^{-1}$, then we obtain that $(\psi_F)_X \mathtt{e}_Z^{\mathcal{V}(-,X),F} \omega_{F,X} = id$. Furthermore, we can show that $\mathtt{e}_Z^{\mathcal{V}(-,X),F} \omega_{F,X} (\psi_F)_X = id$, though we leave the details for the reader, and thus we conclude that $(\psi_F)_X$ is an isomorphism for every $F \in \operatorname{Mod} \mathcal{V}$ and $X \in \mathcal{V}$. Moreover, since $\omega_{F,X}$ and $\mathtt{e}_Z^{\mathcal{V}(-,X),F}$ are functorial in both F and X, we conclude that so is $(\psi_F)_X$.

Consequently, we obtain that ψ_F is an isomorphism in $\operatorname{Mod} \mathcal{V}$ for every $F \in \operatorname{Mod} \mathcal{V}$, and because it is functorial in F, we conclude that $\psi : \mathtt{a}_Z\mathtt{e}_Z \to id_{\operatorname{Mod} \mathcal{V}}$ is an isomorphism. □

Proposition 2.8.3. *If $\mathcal{V}$ is a variety, Λ is a ring and $\mathtt{s} : \operatorname{Mod} \mathcal{V} \to \operatorname{Mod} \Lambda$ is an equivalence of categories, then there is an additive generator Z for $\mathcal{V}$ with* $\operatorname{End}_{\mathcal{V}}(Z) \simeq \Lambda$ *such that* $\mathtt{s} \simeq \mathtt{e}_Z$.

Proof. By Theorem 2.7.2 we know that $\mathtt{s}$ induces an equivalence of categories $\mathcal{P}(\mathcal{V}) \to \mathcal{P}(\Lambda)$. Moreover, from the proof of Theorem 2.8.1 we also know that there is an additive generator P for $\mathcal{P}(\mathcal{V})$ such that $\mathtt{s}(P) \simeq \Lambda$, and in this case, we have $\operatorname{End}_{\mathcal{V}}(P) \simeq \Lambda$.

Now, since $\mathcal{V}$ is a variety, we know by Proposition 2.3.3 that the Yoneda embedding induces an equivalence of categories $\mathcal{V} \to \mathcal{P}(\mathcal{V})$. Therefore, there is some $Z \in \mathcal{V}$ such that $\mathcal{V}(-, Z) \simeq P$, and in this case, Z is an additive generator for $\mathcal{V}$ with

$$\mathrm{End}_{\mathcal{V}}(Z) \simeq \mathrm{End}_{\mathcal{V}}(\mathcal{V}(-, Z)) \simeq \mathrm{End}_{\mathcal{V}}(P) \simeq \Lambda.$$

Finally, note that if $F \in \mathrm{Mod}\,\mathcal{V}$, then we have

$$\mathtt{s}(F) \simeq \mathrm{Hom}_{\Lambda}(\Lambda, \mathtt{s}(F)) \simeq \mathrm{Hom}_{\Lambda}(\mathtt{s}(\mathcal{V}(-, Z)), \mathtt{s}(F)) \simeq \mathrm{Hom}_{\mathcal{V}}(\mathcal{V}(-, Z), F) \simeq F(Z) = \mathtt{e}_Z(F),$$

and since these isomorphisms are functorial in F, we conclude that $\mathtt{s} \simeq \mathtt{e}_Z$. □

The reader should note that by combining propositions 2.8.2 and 2.8.3 we obtain more information than Theorem 2.8.1 provides for us for the case of modules over varieties. In fact, we had just proved that for a variety $\mathcal{V}$, every equivalence $\mathrm{Mod}\,\mathcal{V} \to \mathrm{Mod}\,\Lambda$, where Λ is a ring, is given by the evaluation functor at some object of $\mathcal{V}$, and in this case we also obtain that the quasi-inverse of this equivalence is given by the assimilation functor at the corresponding object of $\mathcal{V}$.

2.9 Some applications

In this section we present some applications of the theory developed in this chapter, some of which will be used to prove some results of Chapter 5.

We begin by presenting a new proof of the Morita Theorem, which is a well known result. For this, we first make some observations.

Let Λ be a ring. Given that $\mathcal{P}(\Lambda) = \mathrm{add}\,\Lambda$ and $\mathrm{End}_{\Lambda}(\Lambda) \simeq \Lambda$, it follows from Proposition 2.8.2 that $\mathtt{e}_{\Lambda} : \mathrm{Mod}\,\mathcal{P}(\Lambda) \to \mathrm{Mod}\,\Lambda$ is an equivalence of categories with a quasi-inverse given by $\mathtt{a}_{\Lambda} : \mathrm{Mod}\,\Lambda \to \mathrm{Mod}\,\mathcal{P}(\Lambda)$. In this case, note that if $M \in \mathrm{Mod}\,\Lambda$, then $\mathtt{a}_{\Lambda}(M) = \mathrm{Hom}_{\Lambda}(\mathcal{P}(\Lambda)(\Lambda, -), M)$, and we have $\mathcal{P}(\Lambda)(\Lambda, -) = \mathrm{Hom}_{\Lambda}(\Lambda, -)|_{\mathcal{P}(\Lambda)}$. Therefore, by Proposition 2.2.2 we obtain that $\mathtt{a}_{\Lambda}(M) \simeq \mathrm{Hom}_{\Lambda}(-, M)|_{\mathcal{P}(\Lambda)}$. Finally, we recall that a Λ-module P is a **progenerator** of $\mathrm{Mod}\,\Lambda$ if P is a finitely generated projective Λ-module such that $\mathrm{add}\,P = \mathcal{P}(\Lambda)$.

Theorem 2.9.1 (Morita Theorem)**.** *Given two rings Λ and Σ, there is an equivalence of categories $\mathtt{s} : \mathrm{Mod}\,\Lambda \to \mathrm{Mod}\,\Sigma$ if and only if there is a progenerator P of $\mathrm{Mod}\,\Lambda$ such that $\mathrm{End}_{\Lambda}(P) \simeq \Sigma$. Under this condition, $\mathtt{s} \simeq \mathrm{Hom}_{\Lambda}(P, -)$, and if $\mathtt{t} : \mathrm{Mod}\,\Sigma \to \mathrm{Mod}\,\Lambda$ is a quasi-inverse of $\mathtt{s}$, then $\mathtt{t} \simeq \mathrm{Hom}_{\Sigma}(\mathrm{Hom}_{\Lambda}(P, \Lambda), -)$.*

Proof. Suppose that there is an equivalence of categories $\mathtt{s} : \mathrm{Mod}\,\Lambda \to \mathrm{Mod}\,\Sigma$. Then we obtain an equivalence $\mathtt{s}\mathtt{e}_{\Lambda} : \mathrm{Mod}\,\mathcal{P}(\Lambda) \to \mathrm{Mod}\,\Sigma$, and thus we conclude from Proposition 2.8.3 that there is a progenerator P of $\mathrm{Mod}\,\Lambda$ such that $\mathrm{End}_{\Lambda}(P) \simeq \Sigma$ and $\mathtt{s}\mathtt{e}_{\Lambda} \simeq \mathtt{e}_P$. Hence we have that $\mathtt{s} \simeq \mathtt{e}_P\mathtt{a}_\Lambda$, and because we know from Proposition 2.8.2 that $\mathtt{e}_P$ and $\mathtt{a}_P$ are quasi-inverses, we conclude that $\mathtt{e}_\Lambda\mathtt{a}_P : \operatorname{Mod}\Sigma \to \operatorname{Mod}\Lambda$ is a quasi-inverse of $\mathtt{e}_P\mathtt{a}_\Lambda$.

Next, note that for each $M \in \operatorname{Mod}\Lambda$ we have $\mathtt{e}_P\mathtt{a}_\Lambda(M) \simeq \operatorname{Hom}_\Lambda(P, M)$, and thus $\mathtt{e}_P\mathtt{a}_\Lambda \simeq \operatorname{Hom}_\Lambda(P, -)$. Furthermore, if $\mathtt{t} : \operatorname{Mod}\Sigma \to \operatorname{Mod}\Lambda$ is a quasi-inverse of $\mathtt{s}$, then $\mathtt{t} \simeq \mathtt{e}_\Lambda\mathtt{a}_P$. In this case, observe that if $N \in \operatorname{Mod}\Sigma$, then $\mathtt{e}_\Lambda\mathtt{a}_P(N) = \operatorname{Hom}_\Sigma(\operatorname{Hom}_\Lambda(P, \Lambda), N)$, and consequently $\mathtt{e}_\Lambda\mathtt{a}_P = \operatorname{Hom}_\Sigma(\operatorname{Hom}_\Lambda(P, \Lambda), -)$.

Conversely, let P be a progenerator of $\operatorname{Mod}\Lambda$ such that $\operatorname{End}_\Lambda(P) \simeq \Sigma$. Then it follows from Proposition 2.8.2 that $\mathtt{e}_P : \operatorname{Mod}\mathcal{P}(\Lambda) \to \operatorname{Mod}\Sigma$ is an equivalence of categories, and thus so is $\mathtt{e}_P\mathtt{a}_\Lambda : \operatorname{Mod}\Lambda \to \operatorname{Mod}\Sigma$, which has $\mathtt{e}_\Lambda\mathtt{a}_P$ as a quasi-inverse. □

As another application of the content of this chapter, we will prove next that we can always view a variety with an additive generator as a category of finitely generated projective modules over a certain ring.

Proposition 2.9.2. *If $\mathcal{V}$ is a variety with an additive generator $Z \in \mathcal{V}$ and $\Lambda = \operatorname{End}_\mathcal{V}(Z)$, then the functor $\mathcal{V}(Z, -) : \mathcal{V} \to \operatorname{Mod}\Lambda$ induces an equivalence of categories $\mathcal{V} \to \mathcal{P}(\Lambda)$.*

Proof. We already know by Proposition 2.8.2 that the evaluation functor at Z gives us an equivalence of categories $\mathtt{e}_Z : \operatorname{Mod}\mathcal{V} \to \operatorname{Mod}\Lambda$, which by Theorem 2.7.2 restricts to an equivalence $\mathcal{P}(\mathcal{V}) \to \mathcal{P}(\Lambda)$. Moreover, we also have by Proposition 2.3.3 that the Yoneda embedding $\mathcal{Y}_\mathcal{V} : \mathcal{V} \to \operatorname{Mod}\mathcal{V}$ induces an equivalence $\mathcal{V} \to \mathcal{P}(\mathcal{V})$. Therefore, the functor $\mathcal{V}(Z, -) = \mathtt{e}_Z\mathcal{Y}_\mathcal{V} : \mathcal{V} \to \operatorname{Mod}\Lambda$ induces an equivalence $\mathcal{V} \to \mathcal{P}(\Lambda)$. □

The above procedure of viewing a variety $\mathcal{V}$ with an additive generator as a category of finitely generated projective modules over a ring Λ, which is sometimes called by "projectivization", is quite interesting since in general it is easier to work with $\mathcal{P}(\Lambda)$ than with $\mathcal{V}$. Furthermore, it may happen that the ring Λ that we obtain has some nice properties. For example, suppose that $\mathcal{V}$ is a Krull–Schmidt category with an additive generator $Z \in \mathcal{V}$ and let $\Lambda = \operatorname{End}_\mathcal{V}(Z)$ (note that in this case $\mathcal{V}$ is skeletally small). If we write $Z \simeq Z_1 \oplus \cdots \oplus Z_n$ as a finite direct sum of objects $Z_i \in \mathcal{V}$ with local endomorphism rings, then by considering the functor $\mathcal{V}(Z, -) : \mathcal{V} \to \operatorname{Mod}\Lambda$, which induces an equivalence $\mathcal{V} \to \mathcal{P}(\Lambda)$ by Proposition 2.9.2, we conclude that we can write Λ viewed as a module over itself as

$$\Lambda = \mathcal{V}(Z, Z) \simeq \mathcal{V}(Z, Z_1) \oplus \cdots \oplus \mathcal{V}(Z, Z_n)$$

with the endomorphism ring of each $\mathcal{V}(Z, Z_i)$ being local. Hence it follows by [31, Proposition 4.1] that Λ is a semiperfect ring.

At this moment, one could ask if it happens that every semiperfect ring can be viewed as the endomorphism ring of an additive generator of a Krull–Schmidt category. Well, this is indeed the case since if Λ is a semiperfect ring, then again by [31, Proposition 4.1] we have that $\mathcal{P}(\Lambda)$ is a Krull–Schmidt category, which has Λ as an additive generator, and we have that $\operatorname{End}_{\mathcal{P}(\Lambda)}(\Lambda) = \operatorname{End}_{\Lambda}(\Lambda) \simeq \Lambda$.

Taking the above discussion into account, we can conclude the following result:

Proposition 2.9.3. *There is a bijection between the equivalence classes of Krull–Schmidt categories of finite type and the Morita equivalence classes of semiperfect rings. The bijection is given as follows:*

(a) If $\mathcal{V}$ is a Krull–Schmidt category of finite type, then take an additive generator $Z \in \mathcal{V}$ of $\mathcal{V}$ and send $\mathcal{V}$ to $\operatorname{End}_{\mathcal{V}}(Z)$*.*

(b) If Λ is a semiperfect ring, then send it to $\mathcal{P}(\Lambda)$.

Proof. The result will follow by combining the previous discussion with the remarks below.

Note that if $\mathcal{V}$ and $\mathcal{W}$ are Krull–Schmidt categories of finite type with additive generators Z and Y, respectively, then it follows from Proposition 2.8.2 that $\operatorname{Mod} \mathcal{V} \approx \operatorname{Mod} \operatorname{End}_{\mathcal{V}}(Z)$ and $\operatorname{Mod} \mathcal{W} \approx \operatorname{Mod} \operatorname{End}_{\mathcal{W}}(Y)$. Moreover, since $\mathcal{P}(\mathcal{V}) \approx \mathcal{V}$ and $\mathcal{P}(\mathcal{W}) \approx \mathcal{W}$, it follows from Proposition 2.7.8 that $\mathcal{V}$ and $\mathcal{W}$ are equivalent if and only if $\operatorname{End}_{\mathcal{V}}(Z)$ and $\operatorname{End}_{\mathcal{W}}(Y)$ are Morita equivalent. This shows that the assignment of the equivalence class of $\mathcal{V}$ to the Morita equivalence class of $\operatorname{End}_{\mathcal{V}}(Z)$ is well defined and does not depend on the choice of the additive generator Z of $\mathcal{V}$, as well as that it is injective.

Furthermore, for two semiperfect rings Λ and Γ we also have by Proposition 2.7.8 that Λ and Γ are Morita equivalent if and only if $\mathcal{P}(\Lambda)$ and $\mathcal{P}(\Gamma)$ are equivalent. Therefore, the assignment of the Morita equivalence class of Λ to the equivalence class of $\mathcal{P}(\Lambda)$ is well defined. □

What is interesting is that we can generalize Proposition 2.9.3 as follows:

Proposition 2.9.4. *Let $\mathbb{V}$ be a collection of varieties which have additive generators. There is a bijection between the equivalence classes of varieties in $\mathbb{V}$ and the Morita equivalence classes of the rings of the form* $\operatorname{End}_{\mathcal{V}}(Z)$*, where $\mathcal{V} \in \mathbb{V}$ and $Z \in \mathcal{V}$ is an additive generator of $\mathcal{V}$. The bijection is given as follows:*

(a) If $\mathcal{V} \in \mathbb{V}$, then take an additive generator $Z \in \mathcal{V}$ of $\mathcal{V}$ and send $\mathcal{V}$ to $\operatorname{End}_{\mathcal{V}}(Z)$*.*

(b) If Λ is a ring that is Morita equivalent to $\operatorname{End}_{\mathcal{V}}(Z)$ *for some $\mathcal{V} \in \mathbb{V}$, where $Z \in \mathcal{V}$ is an additive generator of $\mathcal{V}$, then send Λ to $\mathcal{P}(\Lambda)$.*

Proof. The same reasoning of the proof of Proposition 2.9.3 shows that if $\mathcal{V} \in \mathbb{V}$ and $Z \in \mathcal{V}$ is an additive generator of $\mathcal{V}$, then the assignment of the equivalence class of some $\mathcal{V}$ to the Morita equivalence class of $\operatorname{End}_{\mathcal{V}}(Z)$ is well defined and does not depend on the choice of the additive generator Z of $\mathcal{V}$, as well as that it is injective.

Now, let Λ be a ring that is Morita equivalent to $\operatorname{End}_{\mathcal{V}}(Z)$ for some $\mathcal{V} \in \mathbb{V}$, where $Z \in \mathcal{V}$ is an additive generator of $\mathcal{V}$. Since we know by Proposition 2.8.2 that $\operatorname{Mod} \mathcal{V} \approx \operatorname{Mod} \operatorname{End}_{\mathcal{V}}(Z)$, it follows from Proposition 2.7.8 that $\mathcal{P}(\Lambda) \approx \mathcal{P}(\operatorname{End}_{\mathcal{V}}(Z)) \approx \mathcal{P}(\mathcal{V}) \approx \mathcal{V}$. Therefore, from this fact and also from the same reasoning of the proof of Proposition 2.9.3, we can conclude that the assignment of the Morita equivalence class of Λ to the equivalence class of $\mathcal{P}(\Lambda)$ is well defined.

Finally, to conclude that the above assignments give inverse bijections, let Λ be as above. Because $\mathcal{P}(\Lambda) \approx \mathcal{V}$ and Z is an additive generator of $\mathcal{V}$, we have that $\mathcal{P}(\Lambda)$ is sent to $\operatorname{End}_{\mathcal{V}}(Z)$, which is Morita equivalent to Λ. This completes the proof. $\square$

Proposition 2.9.4 gives us a way to establish correspondences between equivalence classes of varieties with additive generators belonging to a certain collection $\mathbb{V}$ and Morita equivalence classes of rings belonging to a certain collection $\mathbb{R}$. With this in mind, an interesting project would be to search for examples of such correspondences. For instance, one could begin by considering a specific collection $\mathbb{V}$, and then try to describe $\mathbb{R}$. On the other hand, one could also begin by considering a specific collection $\mathbb{R}$, and then try to describe $\mathbb{V}$.

A fundamental and well known example of a correspondence as described above is the Morita–Tachikawa Correspondence, and its higher version which is due to Bruno J. Müller. We will present a proof of these results in Section 5.2. Another very important example is the Auslander Correspondence and its higher version given by Osamu Iyama, which will be proved in Section 5.3. Some other examples of such correspondences are given in [27, Theorem 4.5] and [20, Theorem 3.2]. The reader should note that all these mentioned correspondences are actually restrictions of the Morita–Tachikawa Correspondence and its higher version.

3 Projective resolutions in functor categories

We now begin to make a gradual transition from the subject of functor categories to higher Auslander–Reiten theory by studying minimal projective resolutions of simple modules. In doing so, we pass from the notion of a "finitely presented module" to that of "almost split morphisms" and finally to that of an "n-almost split sequence". This last concept, which is one of the main objects of study in higher Auslander–Reiten theory, will be of fundamental importance to motivate the development of the next chapters.

3.1 Finitely presented modules

As a first step to study projective resolutions in functor categories, we begin in this section by studying the concept of a "finitely presented" module. We also consider the category consisting of these modules, and in particular we study when this category is abelian. At the end we also examine the case of modules over skeletally small abelian categories.

In this section, $\mathcal{B}$ and $\mathcal{C}$ will always stand for skeletally small preadditive categories.

We say that a $\mathcal{C}$-module F is **finitely presented** if it admits a projective presentation

$$P \longrightarrow Q \longrightarrow F \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}$ with $P, Q \in \mathcal{P}(\mathcal{C})$. The reader may easily verify that a $\mathcal{C}$-module F is finitely presented if and only if it admits a projective presentation in $\operatorname{Mod}\mathcal{C}$ of the form

$$\bigoplus_{i=1}^{n} \mathcal{C}(-, X_i) \longrightarrow \bigoplus_{j=1}^{m} \mathcal{C}(-, Y_j) \longrightarrow F \longrightarrow 0$$

with $X_i, Y_j \in \mathcal{C}$. We will denote the subcategory of $\operatorname{Mod}\mathcal{C}$ consisting of all finitely presented $\mathcal{C}$-modules by $\operatorname{mod}\mathcal{C}$.

As a first property, note that a finite direct sum $\bigoplus_{i=1}^{n} F_i$ of $\mathcal{C}$-modules F_i is finitely presented if and only if each F_i is finitely presented. Consequently, $\operatorname{mod}\mathcal{C}$ is an additive subcategory of $\operatorname{Mod}\mathcal{C}$. Moreover, it is not difficult to see that the subcategory of $\operatorname{mod}\mathcal{C}$ consisting of the projective objects of $\operatorname{mod}\mathcal{C}$ coincide with $\mathcal{P}(\mathcal{C})$.

We will see next that the property of being finitely presented is preserved under Morita equivalences.

Proposition 3.1.1. *If* $\mathrm{s} : \operatorname{Mod}\mathcal{B} \to \operatorname{Mod}\mathcal{C}$ *is an equivalence of categories, then it also induces an equivalence* $\operatorname{mod}\mathcal{B} \to \operatorname{mod}\mathcal{C}$.

Proof. Given that we know from Theorem 2.7.2 that s induces an equivalence of categories $\mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{C})$, it is not difficult to see that a $\mathcal{B}$-module F is finitely presented if and only if the $\mathcal{C}$-module $\mathrm{s}(F)$ is finitely presented. Hence s induces an equivalence of categories $\operatorname{mod}\mathcal{B} \to \operatorname{mod}\mathcal{C}$. □

In addition, we will see below that an equivalence between categories of finitely presented modules is sufficient to conclude the existence of a Morita equivalence.

Proposition 3.1.2. *If* $\operatorname{mod}\mathcal{B}$ *and* $\operatorname{mod}\mathcal{C}$ *are equivalent categories, then* $\mathcal{B}$ *and* $\mathcal{C}$ *are Morita equivalent.*

Proof. Because the categories $\operatorname{mod}\mathcal{B}$ and $\operatorname{mod}\mathcal{C}$ are equivalent, it follows that their subcategories of projectives objects are also equivalent. But this means that $\mathcal{P}(\mathcal{B})$ is equivalent to $\mathcal{P}(\mathcal{C})$, and thus it follows by Theorem 2.7.7 that $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent. □

The reader should note that by combining propositions 3.1.1 and 3.1.2 we obtain that $\operatorname{Mod}\mathcal{B} \approx \operatorname{Mod}\mathcal{C}$ if and only if $\operatorname{mod}\mathcal{B} \approx \operatorname{mod}\mathcal{C}$. This gives an additional condition for Proposition 2.7.8.

Next, we give some equivalent conditions for a $\mathcal{C}$-module to be finitely presented, which will be very useful. The result below is from [5, Proposition 4.1].

Proposition 3.1.3. *Let* $F \in \operatorname{Mod}\mathcal{C}$. *The following are equivalent:*

(a) *F is finitely presented.*

(b) *There is a short exact sequence* $0 \longrightarrow H \longrightarrow G \longrightarrow F \longrightarrow 0$ *in* $\operatorname{Mod}\mathcal{C}$ *with G finitely generated projective and H finitely generated.*

(c) *F is finitely generated and if* $0 \longrightarrow H \longrightarrow G \longrightarrow F \longrightarrow 0$ *is a short exact sequence in* $\operatorname{Mod}\mathcal{C}$ *with G finitely generated projective, then H is finitely generated.*

(d) *F is finitely generated and if* $0 \longrightarrow H \longrightarrow G \longrightarrow F \longrightarrow 0$ *is a short exact sequence in* $\operatorname{Mod}\mathcal{C}$ *with G finitely generated, then H is finitely generated.*

Proof. It is easy to see that (a) implies (b). Therefore, we start by proving that (b) implies (c).

Assume that there is a short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow F \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}$ with G finitely generated projective and H finitely generated. Then F is finitely generated, and if

$$0 \longrightarrow H' \longrightarrow G' \longrightarrow F \longrightarrow 0$$

is a short exact sequence in $\operatorname{Mod}\mathcal{C}$ with G' finitely generated projective, then it follows from Schanuel's Lemma that $H \oplus G' \simeq H' \oplus G$. In this case, $H' \oplus G$ is finitely generated, and thus so is H'.

Next, assume that (c) holds and let

$$0 \longrightarrow H \longrightarrow G \longrightarrow F \longrightarrow 0$$

be a short exact sequence in $\operatorname{Mod}\mathcal{C}$ with G finitely generated. Then there are epimorphisms $P \to F$ and $Q \to G$ in $\operatorname{Mod}\mathcal{C}$ with $P, Q \in \mathcal{P}(\mathcal{C})$, and with these morphisms we can obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & P \oplus Q \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "id_F"] & 0 \\
0 \arrow[r] & H \arrow[r] & G \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

in $\operatorname{Mod}\mathcal{C}$ with exact rows, where $P \oplus Q \to G$ is an epimorphism. In this case, K is finitely generated, and it follows from the Snake Lemma that $K \to H$ is an epimorphism, so that H is finitely generated.

Finally, (d) is easily seen to imply (a), and thus we are done. □

The next proposition, which is from [5, Proposition 4.2], will also be very useful to prove some following results. The proof below is from [36, Proposition 3.2].

Proposition 3.1.4. *Let*

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

be a short exact sequence in $\operatorname{Mod}\mathcal{C}$. *If F is finitely generated and G is finitely presented, then H is finitely presented.*

Proof. If we take an epimorphism $P \to G$ in $\operatorname{Mod}\mathcal{C}$ with $P \in \mathcal{P}(\mathcal{C})$, then we can consider a commutative diagram in $\operatorname{Mod}\mathcal{C}$ with exact rows and exact columns as below.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & K \arrow[d] & L \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & P \arrow[r, "id_P"] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d, "id_F"'] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
 & F & 0 & 0 &
\end{tikzcd}$$

Consequently, from the Snake Lemma we obtain a short exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow F \longrightarrow 0$$

in $\operatorname{Mod} \mathcal{C}$. Now, note that since G is finitely presented, if we apply Proposition 3.1.3 in the second column of the above diagram, then we conclude that K is finitely generated. Hence it follows from Proposition 2.3.2 that L is finitely generated, and thus by applying Proposition 3.1.3 in the third column of the above diagram we conclude that H is finitely presented. □

Now, we will prove that every morphism in $\operatorname{mod} \mathcal{C}$ has a cokernel, which coincides with the corresponding cokernel in $\operatorname{Mod} \mathcal{C}$.

Proposition 3.1.5. *If*

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$$

is an exact sequence in $\operatorname{Mod} \mathcal{C}$ *with F and G finitely presented, then H is finitely presented.*

Proof. Consider the short exact sequence

$$0 \longrightarrow \operatorname{Ker} \beta \longrightarrow G \xrightarrow{\beta} H \longrightarrow 0$$

in $\operatorname{Mod} \mathcal{C}$. Since $\operatorname{Ker} \beta = \operatorname{Im} \alpha$ is finitely generated, it follows from Proposition 3.1.4 that H is finitely presented. □

At this moment, the reader may ask if we have that every morphism in $\operatorname{mod} \mathcal{C}$ has a kernel which coincides with the corresponding kernel in $\operatorname{Mod} \mathcal{C}$. As we will see in Proposition 3.1.6, this is the case if we suppose that $\mathcal{C}$ is "right coherent", a concept that we define below.

As a generalization of the correspondent definition for rings, we say that $\mathcal{C}$ is **right coherent** if every finitely generated submodule of a finitely presented right $\mathcal{C}$-module is finitely presented. Dually, we say that $\mathcal{C}$ is **left coherent** if every finitely generated submodule of a finitely presented left $\mathcal{C}$-module is finitely presented. Note that $\mathcal{C}$ is left coherent if and only if $\mathcal{C}^{\mathrm{op}}$ is right coherent.

Proposition 3.1.6. *Assume that $\mathcal{C}$ is right coherent. If*

$$0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

is an exact sequence in $\operatorname{Mod} \mathcal{C}$ *with G and H finitely presented, then F is finitely presented.*

Proof. It follows from Proposition 3.1.5 that $\operatorname{Coker} \beta$ is finitely presented. Therefore, if we apply Proposition 3.1.3 in the short exact sequence

$$0 \longrightarrow \operatorname{Im} \beta \longrightarrow H \longrightarrow \operatorname{Coker} \beta \longrightarrow 0 \ ,$$

then we obtain that $\operatorname{Im}\beta$ is finitely generated. Hence we conclude that $\operatorname{Im}\beta$ is also finitely presented, given that it is a submodule of H and $\mathcal{C}$ is right coherent. Finally, using the same arguments for the short exact sequence

$$0 \longrightarrow F \xrightarrow{\alpha} G \longrightarrow \operatorname{Im}\beta \longrightarrow 0$$

we obtain that F is finitely presented. $\square$

As we will see in the next theorem, which is an adaptation of [4, page 41], if we assume that $\mathcal{C}$ is right coherent, then it is not difficult to conclude from the above results that $\operatorname{mod}\mathcal{C}$ is an abelian category. Furthermore, we will also show that this assumption on $\mathcal{C}$ is not only a sufficient condition for $\operatorname{mod}\mathcal{C}$ to be abelian, but also a necessary condition. In addition, we will also verify that in case that $\operatorname{mod}\mathcal{C}$ is abelian, then the inclusion $\operatorname{mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}$ must be exact.

Theorem 3.1.7. *The following are equivalent for $\mathcal{C}$:*

(a) $\mathcal{C}$ *is right coherent.*

(b) $\operatorname{mod}\mathcal{C}$ *is an abelian category and the inclusion* $\operatorname{mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}$ *is exact.*

(c) $\operatorname{mod}\mathcal{C}$ *is an abelian category.*

Proof. We begin by verifying that (a) is equivalent to (b). For this, first recall that we already know that $\operatorname{mod}\mathcal{C}$ is an additive subcategory of $\operatorname{Mod}\mathcal{C}$ which is also closed under cokernels, by Proposition 3.1.5. Moreover, if $\mathcal{C}$ is right coherent, then it follows from Proposition 3.1.6 that $\operatorname{mod}\mathcal{C}$ is also closed under kernels, and thus we conclude from Proposition 1.1.2 that item (b) holds.

Conversely, assume (b), and let F be a finitely generated submodule of a finitely presented $\mathcal{C}$-module G. If we consider the short exact sequence

$$0 \longrightarrow F \xrightarrow{\eta} G \longrightarrow \operatorname{Coker}\eta \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}$, where η is the canonical inclusion, then we conclude from Proposition 3.1.4 that $\operatorname{Coker}\eta$ is finitely presented. Consequently, F is the kernel of a morphism between finitely presented $\mathcal{C}$-modules and since we are assuming that kernels of morphisms in $\operatorname{mod}\mathcal{C}$ coincide with their correspondent kernels in $\operatorname{Mod}\mathcal{C}$, we conclude that $F \in \operatorname{mod}\mathcal{C}$. Hence $\mathcal{C}$ is right coherent.

Next, we will verify that (b) is equivalent to (c), and because (b) trivially implies (c), we only need to show the converse.

Well, assume that $\operatorname{mod}\mathcal{C}$ is an abelian category, and let

$$\mathrm{E}: \qquad F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

be an exact sequence in $\operatorname{mod}\mathcal{C}$. If $X \in \mathcal{C}$, then we know that $\mathcal{C}(-,X)$ is a projective object of $\operatorname{mod}\mathcal{C}$, and thus we have that

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),F) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),\alpha)} \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),G) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),\beta)} \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),H)$$

is an exact sequence in Ab. However, we know from the Yoneda Lemma that this sequence is isomorphic to

$$F(X) \xrightarrow{\alpha_X} G(X) \xrightarrow{\beta_X} H(X) \ ,$$

which in this case must be exact. Therefore, given that this holds for every $X \in \mathcal{C}$, we conclude that E is an exact sequence in $\operatorname{Mod}\mathcal{C}$. Consequently, the inclusion $\operatorname{mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}$ is exact. □

The reader should note that if $\mathcal{C}$ is right coherent, then by Theorem 3.1.7 we obtain that every $F \in \operatorname{mod}\mathcal{C}$ admits a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow F \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}$ with $P_i \in \mathcal{P}(\mathcal{C})$, given that in this case $\operatorname{mod}\mathcal{C}$ is an abelian category which has enough projectives satisfying that the inclusion $\operatorname{mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}$ is exact, and also because the projective objects of $\operatorname{mod}\mathcal{C}$ coincide with the finitely generated projective $\mathcal{C}$-modules. Considering this, and given that in the next sections we will be interested in studying certain projective resolutions of certain $\mathcal{C}$-modules, the condition that $\mathcal{C}$ is right coherent will be fundamental to guarantee the existence of such resolutions.

For the rest of this section, let $\mathcal{A}$ be a skeletally small abelian category.

We will now prove that $\mathcal{A}$ is both right and left coherent, and for this we first give a lemma. Note that this will provide us with many examples of right and left coherent categories.

Lemma 3.1.8. *If $\mathcal{V}$ is a variety, then $\mathcal{V}$ is right coherent if and only if every finitely generated right ideal of $\mathcal{V}$ is finitely presented.*

Proof. Assume that every finitely generated right ideal of $\mathcal{V}$ is finitely presented and let F be a finitely generated submodule of a finitely presented $\mathcal{C}$-module G. If we consider the short exact sequence

$$0 \longrightarrow F \xrightarrow{\eta} G \longrightarrow H \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{V}$, where η is the canonical inclusion and $H = \operatorname{Coker}\eta$, then by following the proof of Proposition 3.1.4 we can obtain a short exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow F \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{V}$, where K and L are isomorphic to finitely generated submodules of P for some $P \in \mathcal{P}(\mathcal{V})$. But in this case we know from Proposition 2.3.3 that $P \simeq \mathcal{V}(-,X)$ for some $X \in \mathcal{V}$, and thus K and L are isomorphic to finitely generated right ideals of $\mathcal{V}$, which implies that K and L are finitely presented. Hence by Proposition 3.1.5 we conclude that F is finitely presented and consequently $\mathcal{V}$ is right coherent.

The converse is trivial. □

Proposition 3.1.9. *$\mathcal{A}$ is both right and left coherent.*

Proof. We will only prove that $\mathcal{A}$ is right coherent since the proof that $\mathcal{A}$ is left coherent will follow by duality. For this, we will show that every finitely generated right ideal of $\mathcal{A}$ is finitely presented, which is sufficient to conclude that $\mathcal{A}$ is right coherent by Lemma 3.1.8.

Well, let $F \subseteq \mathcal{A}(-,X)$ be a finitely generated right ideal of $\mathcal{A}$ for some $X \in \mathcal{A}$ and $\eta : F \to \mathcal{A}(-,X)$ be the canonical inclusion. Moreover, let $\alpha : \mathcal{A}(-,Y) \to F$ be an epimorphism in $\operatorname{Mod}\mathcal{A}$ with $Y \in \mathcal{A}$. Then we have that $\eta\alpha = \mathcal{A}(-,f)$ for some morphism $f : Y \to X$ in $\mathcal{A}$, and because we know from Proposition 2.2.5 that in this case the Yoneda embedding is left exact, we conclude that there is an exact sequence

$$0 \longrightarrow \mathcal{A}(-,\operatorname{Ker} f) \longrightarrow \mathcal{A}(-,Y) \xrightarrow{\mathcal{A}(-,f)} \mathcal{A}(-,X)$$

in $\operatorname{Mod}\mathcal{A}$. Now, note that $\operatorname{Im}\mathcal{A}(-,f) = F$, and thus we obtain a projective resolution

$$0 \longrightarrow \mathcal{A}(-,\operatorname{Ker} f) \longrightarrow \mathcal{A}(-,Y) \longrightarrow F \longrightarrow 0$$

of F in $\operatorname{Mod}\mathcal{A}$, so that F is finitely presented. □

Finally, note that by combining Proposition 3.1.9 and Theorem 3.1.7 we conclude that both $\operatorname{mod}\mathcal{A}$ and $\operatorname{mod}\mathcal{A}^{\mathrm{op}}$ are abelian categories. In this case, we can consider the global dimensions of these categories, which are, as we will see below, smaller than or equal to 2.

Proposition 3.1.10. $\operatorname{gl.dim}(\operatorname{mod}\mathcal{A}) \leqslant 2$ *and* $\operatorname{gl.dim}(\operatorname{mod}\mathcal{A}^{\mathrm{op}}) \leqslant 2$.

Proof. To begin with, recall from Proposition 2.3.3 that every finitely generated projective $\mathcal{A}$-module is representable. Now, let $F \in \operatorname{mod}\mathcal{A}$ and take a projective presentation

$$\mathcal{A}(-,X) \xrightarrow{\mathcal{A}(-,f)} \mathcal{A}(-,Y) \longrightarrow F$$

of F in $\operatorname{Mod}\mathcal{A}$ with $f : X \to Y$ being a morphism in $\mathcal{A}$. Then since we know from Proposition 2.2.5 that in this case the Yoneda embedding is left exact, we obtain that there is an exact sequence

$$0 \longrightarrow \mathcal{A}(-,\operatorname{Ker} f) \longrightarrow \mathcal{A}(-,X) \xrightarrow{\mathcal{A}(-,f)} \mathcal{A}(-,Y) \longrightarrow F$$

in $\operatorname{Mod}\mathcal{A}$, which implies that $\operatorname{pd}_{\mathcal{A}} F \leqslant 2$. Therefore, $\operatorname{gl.dim}(\operatorname{mod}\mathcal{A}) \leqslant 2$.

The proof that $\operatorname{gl.dim}(\operatorname{mod}\mathcal{A}^{\mathrm{op}}) \leqslant 2$ follows by duality. □

3.2 Minimal projective resolutions

In this section we study minimal projective resolutions of modules whose terms are representable modules. This study will be fundamental for the development of the results of this text when we restrict our attention for modules over varieties. Moreover, we present some equivalent conditions for the existence of projective covers of finitely generated modules, which gives us a reason to consider modules over skeletally small Krull–Schmidt categories.

In what follows, $\mathcal{C}$ will stand for a skeletally small preadditive category.

We begin with a result which is well known for modules over rings:

Proposition 3.2.1. *Let* $\alpha : \mathcal{C}(-,X) \to F$ *be an epimorphism in* $\operatorname{Mod}\mathcal{C}$ *with* $X \in \mathcal{C}$. *The following are equivalent:*

(a) α *is a projective cover of* F.

(b) $\operatorname{Ker}\alpha \subseteq \operatorname{rad}_{\mathcal{C}}(-,X)$.

(c) $\operatorname{Ker}\alpha_X \subseteq \operatorname{rad}_{\mathcal{C}}(X,X)$.

Proof. First, we will prove that (a) implies (b). Suppose that α is a projective cover of F. From Proposition 2.4.2 we know that it suffices to verify that $\operatorname{Ker}\alpha_Y \subseteq \operatorname{rad}_{\mathcal{C}}(Y,X)$ for every $Y \in \mathcal{C}$ to conclude that $\operatorname{Ker}\alpha \subseteq \operatorname{rad}_{\mathcal{C}}(-,X)$. Therefore, let $f \in \operatorname{Ker}\alpha_Y$ for an arbitrary $Y \in \mathcal{C}$. By Theorem 2.6.1, if we show that $id_X - fg$ is an isomorphism for every $g \in \mathcal{C}(X,Y)$, then we will have that $f \in \operatorname{rad}_{\mathcal{C}}(Y,X)$. Hence take $g \in \mathcal{C}(X,Y)$ and consider the morphism

$$\mathcal{C}(-, id_X - fg) : \mathcal{C}(-,X) \to \mathcal{C}(-,X)$$

in $\operatorname{Mod}\mathcal{C}$. Since $\operatorname{Ker}\alpha$ is a right ideal of $\mathcal{C}$, we have that $fg \in \operatorname{Ker}\alpha_X$, and thus

$$(\alpha\mathcal{C}(-, id_X - fg))_X(id_X) = \alpha_X(id_X) - \alpha_X(fg) = \alpha_X(id_X).$$

Consequently, we conclude from the Yoneda Lemma that $\alpha\mathcal{C}(-, id_X - fg) = \alpha$. Now, given that α is a right minimal morphism, we obtain that $\mathcal{C}(-, id_X - fg)$ is an isomorphism in $\operatorname{Mod}\mathcal{C}$, which implies that $id_X - fg$ is an isomorphism, as desired.

Next, since (b) trivially implies (c), it only remains to verify that (c) implies (a). Therefore, assume that $\operatorname{Ker}\alpha_X \subseteq \operatorname{rad}_{\mathcal{C}}(X,X)$ and let $f : X \to X$ be a morphism in $\mathcal{C}$ such that $\alpha = \alpha\mathcal{C}(-,f)$. Then we obtain that $\alpha_X(id_X) = \alpha_X\mathcal{C}(X,f)(id_X)$, so that $\alpha_X(id_X - f) = 0$ and $id_X - f \in \operatorname{rad}_{\mathcal{C}}(X,X)$. Consequently, by Theorem 2.6.1 we have that $id_X - (id_X - f)\, id_X = f$ is an isomorphism, and thus $\mathcal{C}(-,f)$ is an isomorphism, which implies that α is a right minimal morphism. □

Next, we give a necessary and sufficient condition for a projective resolution in $\operatorname{Mod}\mathcal{C}$ whose terms are representable modules to be a minimal projective resolution. For this, we first give a lemma which is well known.

Lemma 3.2.2. *Let*

$$\mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-,Y) \xrightarrow{\alpha} F \longrightarrow 0$$

be a projective presentation of a $\mathcal{C}$-module F with $f : X \to Y$ being a morphism in $\mathcal{C}$. Then α is a projective cover of F if and only if $f \in \mathrm{rad}_{\mathcal{C}}(X,Y)$.

Proof. We know from Proposition 2.5.1 that $f \in \mathrm{rad}_{\mathcal{C}}(X,Y)$ if and only if $\langle f \rangle_{\mathcal{C}} \subseteq \mathrm{rad}_{\mathcal{C}}(-,Y)$. But $\langle f \rangle_{\mathcal{C}} = \mathrm{Ker}\,\alpha$, and thus the result follows by Proposition 3.2.1. □

Theorem 3.2.3. *An exact sequence*

$$\cdots \xrightarrow{\mathcal{C}(-,f_3)} \mathcal{C}(-,Y_2) \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Y_0) \xrightarrow{\alpha} F \longrightarrow 0$$

in $\mathrm{Mod}\,\mathcal{C}$ *with each $f_i : Y_i \to Y_{i-1}$ being a morphism in $\mathcal{C}$ is a minimal projective resolution of F if and only if $f_i \in \mathrm{rad}_{\mathcal{C}}$ for each $i \geqslant 1$.*

Proof. This is a straightforward consequence of Lemma 3.2.2. □

Theorem 3.2.3 will be very important to characterize projective resolutions of modules over varieties which are minimal projective resolutions. In addition, we give bellow an interesting application of this result.

Example 3.2.4. Given a ring Λ, recall from Section 2.9 that $\mathsf{e}_\Lambda : \mathrm{Mod}\,\mathcal{P}(\Lambda) \to \mathrm{Mod}\,\Lambda$ is an equivalence of categories with a quasi-inverse given by $\mathsf{a}_\Lambda : \mathrm{Mod}\,\Lambda \to \mathrm{Mod}\,\mathcal{P}(\Lambda)$. Furthermore, we have that $\mathcal{P}(\Lambda)(-,P) = \mathrm{Hom}_\Lambda(-,P)|_{\mathcal{P}(\Lambda)}$ for each $P \in \mathcal{P}(\Lambda)$ and that $\mathsf{a}_\Lambda(M) \simeq \mathrm{Hom}_\Lambda(-,M)|_{\mathcal{P}(\Lambda)}$ for each $M \in \mathrm{Mod}\,\Lambda$.

Now, suppose that a Λ-module M admits a projective resolution

$$P_\bullet : \qquad \cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

in $\mathrm{Mod}\,\Lambda$ with $P_i \in \mathcal{P}(\Lambda)$. Then, by applying the functor a_Λ in $P_\bullet$ and using Theorem 3.2.3, we can conclude that $P_\bullet$ is a minimal projective resolution of M if and only if $f_i \in \mathrm{rad}_{\mathcal{P}(\Lambda)}$ for each $i \geqslant 1$, which is equivalent to say that $f_i \in \mathrm{rad}_{\mathrm{Mod}\,\Lambda}$ for each $i \geqslant 1$.

As a consequence of the above discussion, we can conclude that if Λ is an Artin algebra whose Jacobson radical is zero, then Λ is semisimple. In fact, since we know from Corollary 2.6.2 that the Jacobson radical of Λ is isomorphic to $\mathrm{rad}_{\mathrm{Mod}\,\Lambda}(\Lambda,\Lambda)$, and given that $\mathcal{P}(\Lambda) = \mathrm{add}\,\Lambda$, we obtain that $\mathrm{rad}_{\mathrm{Mod}\,\Lambda}(P,Q) = 0$ for every $P,Q \in \mathcal{P}(\Lambda)$. Therefore, from the above paragraph we can easily conclude that $\mathrm{pd}_\Lambda M = 0$ for every $M \in \mathrm{mod}\,\Lambda$, which implies that $\mathrm{gl.\,dim}\,\Lambda = 0$. Hence we obtain that every Λ-module is projective, so that Λ is semisimple (see [1, Chapter VI, Theorem 7.1]).

Now that we already know how to decide when a projective resolution whose terms are representable modules is a minimal projective resolution, we proceed to investigate their existence. For this, we begin by giving an appropriate definition.

We say that $\mathcal{C}$ is **semiperfect** if every finitely generated $\mathcal{C}$-module admits a projective cover. We note that, apparently, semiperfect categories were first introduced by Michèle Weidenfeld and Gérard Weidenfeld in [42].

In the next result, which is a generalization of [31, Proposition 4.1], we will present a detailed characterization of when $\mathcal{C}$ is semiperfect.

Theorem 3.2.5. *The following are equivalent:*

(a) $\mathcal{C}$ *is semiperfect.*

(b) Every simple $\mathcal{C}$*-module admits a projective cover.*

(c) $\mathcal{P}(\mathcal{C})$ *is a Krull–Schmidt category.*

(d) Each $P \in \operatorname{ind}\mathcal{P}(\mathcal{C})$ *has a local endomorphism ring and* $\mathcal{P}(\mathcal{C}) = \operatorname{add}(\operatorname{ind}\mathcal{P}(\mathcal{C}))$.

Proof. The proof is omitted since it is quite long and is the same as for modules over rings, which can be found in [31, Proposition 4.1]. □

As a consequence of Theorem 3.2.5, we obtain that $\mathcal{C}$ is semiperfect if and only if $\mathcal{C}^{\mathrm{op}}$ is semiperfect. In fact, note that $\mathcal{P}(\mathcal{C})$ is a Krull–Schmidt category if and only if so is $\mathcal{P}(\mathcal{C})^{\mathrm{op}}$. Furthermore, we know from Lemma 2.7.9 that $\mathcal{P}(\mathcal{C})^{\mathrm{op}} = \mathcal{P}(\mathcal{C}^{\mathrm{op}})$, and thus it follows by Theorem 3.2.5 that every finitely generated $\mathcal{C}$-module admits a projective cover if and only if so does every finitely generated $\mathcal{C}^{\mathrm{op}}$-module.

With the above discussion done, we can now present a few remarks that will give the reader some directions for the next sections.

First of all, remember from Proposition 2.7.6 that $\mathcal{C}$ and $\mathcal{P}(\mathcal{C})$ are Morita equivalent. Consequently, to prove certain results we can choose to work with varieties instead of the more general case of skeletally small preadditive categories. In fact, if we prove a result about modules over varieties concerning only properties that are preserved under Morita equivalences, then we will conclude that the same result holds for modules over skeletally small preadditive categories. Moreover, recall that every finitely generated projective module over a variety is representable by Proposition 2.3.3, and in this case Theorem 3.2.3 turns out to be very useful. That is why we will mainly consider modules over varieties in the next sections. In addition, it would also be interesting to have that $\mathcal{C}$ is not only a variety, but also semiperfect, and this is the case if and only if $\mathcal{C}$ is a skeletally small Krull–Schmidt category, by Proposition 2.3.3, Theorem 3.2.5 and the fact that Krull–Schmidt categories are idempotent complete (see Proposition 1.1.1). Because of this, for the rest of this text we will usually assume that $\mathcal{C}$ is a skeletally small Krull–Schmidt category. Finally, note that if we also suppose that $\mathcal{C}$ is right coherent, then from the paragraph after Theorem 3.1.7 we conclude that every $F \in \operatorname{mod} \mathcal{C}$ has a minimal projective resolution

$$\cdots \longrightarrow \mathcal{C}(-, Y_2) \longrightarrow \mathcal{C}(-, Y_1) \longrightarrow \mathcal{C}(-, Y_0) \longrightarrow F \longrightarrow 0$$

in $\operatorname{Mod} \mathcal{C}$ with $Y_i \in \mathcal{C}$. Therefore, this is a case which will also be of great interest for us.

3.3 Simple modules

By following [6] we now give a complete characterization of the simple modules over a skeletally small Krull–Schmidt category, as well as we describe when these modules admit minimal projective resolutions whose terms are representable modules. The reader is referred to [7, Chapter II, Section 1] for a more general description of the simple modules over a skeletally small preadditive category.

In this section, $\mathcal{C}$ will always stand for a skeletally small Krull–Schmidt category.

We first remark that since we know from Proposition 2.3.3 that the Yoneda embedding induces an equivalence of categories $\mathcal{C} \to \mathcal{P}(\mathcal{C})$, we have that every indecomposable finitely generated projective $\mathcal{C}$-module is isomorphic to $\mathcal{C}(-, X)$ for some $X \in \operatorname{ind} \mathcal{C}$.

Next, we present some properties of indecomposable finitely generated projective $\mathcal{C}$-modules. The following two results are from [6, Proposition 2.1], but here we give different proofs for them.

Proposition 3.3.1. *If $X \in \operatorname{ind} \mathcal{C}$ and $\alpha : \mathcal{C}(-, X) \to F$ is a nonzero epimorphism in* $\operatorname{Mod} \mathcal{C}$*, then α is a projective cover of F.*

Proof. From Proposition 3.2.1 we know that it suffices to verify that $\operatorname{Ker} \alpha_X \subseteq \operatorname{rad}_{\mathcal{C}}(X, X)$ to conclude that α is a projective cover of F. Well, note that $\operatorname{Ker} \alpha_X \neq \mathcal{C}(X, X)$, otherwise we would have that $\alpha_X(id_X) = 0$, which would imply by the Yoneda Lemma that $\alpha = 0$, a contradiction. Therefore, $\operatorname{Ker} \alpha_X$ is a proper right ideal of the local ring $\operatorname{End}_{\mathcal{C}}(X)$, and from Corollary 2.6.2 we conclude that $\operatorname{Ker} \alpha_X \subseteq \operatorname{rad}_{\mathcal{C}}(X, X)$. □

Proposition 3.3.2. *If $X \in \mathcal{C}$, then $\mathcal{C}(-, X)$ is indecomposable if and only if $\mathcal{C}(-, X)$ has a unique maximal submodule, which in this case coincides with* $\operatorname{rad}_{\mathcal{C}}(-, X)$.

Proof. Suppose that $\mathcal{C}(-, X)$ is indecomposable and let F be a maximal submodule of $\mathcal{C}(-, X)$, which exists by Proposition 2.4.3. Then the canonical projection $\pi : \mathcal{C}(-, X) \to \mathcal{C}(-, X)/F$ is a nonzero epimorphism in $\operatorname{Mod} \mathcal{C}$, and thus we conclude from Proposition 3.3.1 that π is a projective cover of $\mathcal{C}(-, X)/F$. Hence it follows from Proposition 3.2.1 that $F = \operatorname{Ker} \pi \subseteq \operatorname{rad}_{\mathcal{C}}(-, X)$, which implies that $F = \operatorname{rad}_{\mathcal{C}}(-, X)$. Therefore, $\operatorname{rad}_{\mathcal{C}}(-, X)$ is the unique maximal submodule of $\mathcal{C}(-, X)$.

Conversely, assume that $\mathcal{C}(-,X)$ has a unique maximal submodule, which in this case coincides with $\operatorname{rad}_{\mathcal{C}}(-,X)$. To conclude that $\mathcal{C}(-,X)$ is indecomposable, it suffices to verify that $\operatorname{End}_{\mathcal{C}}(X)$ is a local ring. In order to do this, let $M \subseteq \operatorname{End}_{\mathcal{C}}(X)$ be a maximal right ideal and suppose that $M \neq \operatorname{rad}_{\mathcal{C}}(X,X)$. By Corollary 2.6.2 we know that $\operatorname{rad}_{\mathcal{C}}(X,X) \subseteq M$, hence there is some $f \in M$ such that $f \notin \operatorname{rad}_{\mathcal{C}}(X,X)$. In this case, it follows from Proposition 2.5.1 that $\langle f \rangle_{\mathcal{C}} \nsubseteq \operatorname{rad}_{\mathcal{C}}(-,X)$, and given that $\operatorname{rad}_{\mathcal{C}}(-,X)$ is the unique maximal submodule of $\mathcal{C}(-,X)$, we conclude from Proposition 2.4.3 that $\langle f \rangle_{\mathcal{C}} = \mathcal{C}(-,X)$. Therefore, $\langle f \rangle_{\mathcal{C}}(X) = \mathcal{C}(X,X)$ and because $\langle f \rangle_{\mathcal{C}}(X) \subseteq M$ we obtain that $M = \mathcal{C}(X,X)$, a contradiction. Hence $M = \operatorname{rad}_{\mathcal{C}}(X,X)$ and we are done. $\square$

We can now use Proposition 3.3.2 to completely characterize the simple modules in $\operatorname{Mod} \mathcal{C}$. The next theorem can be found in [6, Proposition 2.3].

Theorem 3.3.3. *A $\mathcal{C}$-module S is simple if and only if $S \simeq \mathcal{C}(-,X)/\operatorname{rad}_{\mathcal{C}}(-,X)$ for some $X \in \operatorname{ind} \mathcal{C}$.*

Proof. Assume that S is simple. Then there is some $X \in \operatorname{ind} \mathcal{C}$ such that $S(X) \neq 0$ and it follows from Proposition 2.4.5 that there is an epimorphism $\alpha : \mathcal{C}(-,X) \to S$ in $\operatorname{Mod} \mathcal{C}$. Hence $S \simeq \mathcal{C}(-,X)/\operatorname{Ker} \alpha$, so that $\operatorname{Ker} \alpha$ is a maximal submodule of $\mathcal{C}(-,X)$, which coincides with $\operatorname{rad}_{\mathcal{C}}(-,X)$ by Proposition 3.3.2.

The converse follows from Proposition 3.3.2. $\square$

We remark that if we take $\mathcal{C}^{\mathrm{op}}$ in place of $\mathcal{C}$ in Theorem 3.3.3, then we can conclude that a left $\mathcal{C}$-module S is simple if and only if $S \simeq \mathcal{C}(X,-)/\operatorname{rad}_{\mathcal{C}}(X,-)$ for some $X \in \operatorname{ind} \mathcal{C}$. Moreover, since simple modules will play an important role in this text, for each $X \in \operatorname{ind} \mathcal{C}$ we will denote $S_X = \mathcal{C}(-,X)/\operatorname{rad}_{\mathcal{C}}(-,X)$ and $S^X = \mathcal{C}(X,-)/\operatorname{rad}_{\mathcal{C}}(X,-)$.

Next, we observe that even though Theorem 3.3.3 already gives us a characterization of the simple modules in $\operatorname{Mod} \mathcal{C}$, we can also describe these modules in terms of their intrinsic properties, as we do below. The next result is from [6, Proposition 2.2], and here we present a different proof for it, based on [36, Proposition 2.3].

Proposition 3.3.4. *A $\mathcal{C}$-module S is simple if and only if it satisfies the following conditions:*

(a) There is a unique $X \in \operatorname{ind} \mathcal{C}$ such that $S(X) \neq 0$.

(b) If $X \in \operatorname{ind} \mathcal{C}$ is such that $S(X) \neq 0$, then $S(X)$ is a simple $\operatorname{End}_{\mathcal{C}}(X)$-module.

Proof. If S is simple, then we know from Theorem 3.3.3 that $S \simeq S_X$ for some $X \in \operatorname{ind} \mathcal{C}$, and thus it follows from Proposition 2.6.5 that the unique object $Y \in \operatorname{ind} \mathcal{C}$ such that $S(Y) \neq 0$ is $Y = X$. Furthermore, from Corollary 2.6.2 and the fact that $\operatorname{End}_{\mathcal{C}}(X)$ is a local ring we conclude that $S(X) \simeq \mathcal{C}(X,X)/\operatorname{rad}_{\mathcal{C}}(X,X)$ is a simple $\operatorname{End}_{\mathcal{C}}(X)$-module.

Conversely, suppose that S satisfies the conditions (a) and (b), and let $X \in \operatorname{ind} \mathcal{C}$ be such that $S(X) \neq 0$. If $F \subseteq S$ is a nonzero submodule of S, then we have that $F(Y) = 0$ for every $Y \in \operatorname{ind} \mathcal{C}$ with $Y \neq X$, and thus we must have $F(X) \neq 0$. Therefore, given that $S(X)$ is a simple $\operatorname{End}_{\mathcal{C}}(X)$-module and $F(X)$ is a nonzero submodule of $S(X)$, we must have $F(X) = S(X)$. Hence we can conclude that $F(Z) = S(Z)$ for every $Z \in \mathcal{C}$, and by Proposition 2.4.2 we obtain that $F = S$, so that S is a simple $\mathcal{C}$-module. □

As a consequence of the above results, we have the following corollary, which is from [6, Proposition 2.3]:

Corollary 3.3.5. *The map which assigns to each $X \in \operatorname{ind} \mathcal{C}$ the $\mathcal{C}$-module S_X induces a bijection between the isomorphism classes of indecomposable objects in $\mathcal{C}$ and the isomorphism classes of simple $\mathcal{C}$-modules.*

Proof. This is a straightforward consequence of Theorem 3.3.3 and Proposition 3.3.4. □

Now, we start to discuss the existence of minimal projective resolutions of simple $\mathcal{C}$-modules whose terms are representable modules. Such resolutions will be of fundamental importance in this text since, as we will see in Section 3.6, they will lead us to some first notions of higher Auslander–Reiten theory.

To begin with, note that if every simple $\mathcal{C}$-module admits a projective resolution as mentioned above, then we obviously obtain that every simple $\mathcal{C}$-module is finitely presented. Conversely, if every simple $\mathcal{C}$-module is finitely presented and if we also assume that $\mathcal{C}$ is right coherent, then it follows from the last paragraph of Section 3.2 that every simple $\mathcal{C}$-module admits a minimal projective resolution whose terms are representable modules. Therefore, we see that if $\mathcal{C}$ is right coherent, then to obtain the existence of such resolutions for the simple $\mathcal{C}$-modules, it suffices to guarantee that every simple $\mathcal{C}$-module is finitely presented. Likewise, we also have the dual result for left $\mathcal{C}$-modules if we assume that $\mathcal{C}$ is left coherent.

Taking the above paragraph into account, the following result is of great importance:

Proposition 3.3.6. *If $Z \in \operatorname{ind} \mathcal{C}$, then S_Z is finitely presented if and only if there is some morphism $f : Y \to Z$ in $\mathcal{C}$ such that $\langle f \rangle_{\mathcal{C}} = \operatorname{rad}_{\mathcal{C}}(-, Z)$.*

Proof. By applying Proposition 3.1.3 in the short exact sequence

$$0 \longrightarrow \operatorname{rad}_{\mathcal{C}}(-, Z) \longrightarrow \mathcal{C}(-, Z) \longrightarrow S_Z \longrightarrow 0$$

in $\operatorname{Mod} \mathcal{C}$ we obtain that S_Z is finitely presented if and only if $\operatorname{rad}_{\mathcal{C}}(-, Z)$ is finitely generated. Therefore, the result follows from Proposition 2.5.3. □

We remark that if we take $\mathcal{C}^{\mathrm{op}}$ in place of $\mathcal{C}$ in Proposition 3.3.6, then for a given $X \in \operatorname{ind}\mathcal{C}$ we obtain that S^X is finitely presented if and only if there is some morphism $f : X \to Y$ in $\mathcal{C}$ such that ${}_{\mathcal{C}}\langle f\rangle = \operatorname{rad}_{\mathcal{C}}(X,-)$.

Therefore, we see from Proposition 3.3.6 and the above paragraph that given $X, Z \in \mathcal{C}$ satisfying that $\operatorname{End}_{\mathcal{C}}(X)$ and $\operatorname{End}_{\mathcal{C}}(Z)$ are local rings, it would be interesting to have a description of the morphisms which generate $\operatorname{rad}_{\mathcal{C}}(X,-)$ and $\operatorname{rad}_{\mathcal{C}}(-,Z)$ as a left and a right ideal of $\mathcal{C}$, respectively, in case that they exist. We will present such a description in the next section, which will lead us to the notion of "almost split morphisms".

To conclude this section, we observe that if $Z \in \operatorname{ind}\mathcal{C}$ and if

$$\cdots \longrightarrow \mathcal{C}(-,Y_2) \longrightarrow \mathcal{C}(-,Y_1) \longrightarrow \mathcal{C}(-,Y_0) \longrightarrow S_Z \longrightarrow 0$$

is a minimal projective resolution of S_Z in $\operatorname{Mod}\mathcal{C}$, then because we know from Proposition 3.3.1 that the canonical projection $\mathcal{C}(-,Z) \to S_Z$ is a projective cover of S_Z, it follows that from Proposition 1.4.4 that $\mathcal{C}(-,Y_0) \simeq \mathcal{C}(-,Z)$, and thus $Y_0 \simeq Z$. Therefore, in this case we have that a minimal projective resolution of S_Z is given by an exact sequence

$$\cdots \xrightarrow{\mathcal{C}(-,f_3)} \mathcal{C}(-,Y_2) \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z) \longrightarrow S_Z \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}$ with each $f_i : Y_i \to Y_{i-1}$ being a morphism in $\mathcal{C}$, where f_1 is a morphism satisfying that $\langle f_1\rangle_{\mathcal{C}} = \operatorname{rad}_{\mathcal{C}}(-,Z)$.

Dually, we also have that if $X \in \operatorname{ind}\mathcal{C}$ and if S^X admits a minimal projective resolution in $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$ whose terms are representable modules, then it is given by an exact sequence

$$\cdots \xrightarrow{\mathcal{C}(f_3,-)} \mathcal{C}(Y_2,-) \xrightarrow{\mathcal{C}(f_2,-)} \mathcal{C}(Y_1,-) \xrightarrow{\mathcal{C}(f_1,-)} \mathcal{C}(X,-) \longrightarrow S^X \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$ with each $f_i : Y_i \to Y_{i-1}$ being a morphism in $\mathcal{C}$, where f_1 is a morphism such that ${}_{\mathcal{C}}\langle f_1\rangle = \operatorname{rad}_{\mathcal{C}}(X,-)$.

3.4 Almost split morphisms

In this section we present the notion of a "right almost split" morphism and of a "left almost split" morphism, which were introduced by Maurice Auslander and Idun Reiten in [12]. We show that these are precisely the morphisms which generate the Jacobson radical of certain modules, and we continue the discussion of Section 3.3 on minimal projective resolutions of simple modules over skeletally small Krull–Schmidt categories. Moreover, we also prove the existence of these morphisms in some categories.

In what follows, $\mathcal{C}$ will stand for a skeletally small preadditive category.

Following [12], we say that a morphism $f : Y \to Z$ in $\mathcal{C}$ is **right almost split in** $\mathcal{C}$ if f is not a split epimorphism, and given a morphism $g : X \to Z$ in $\mathcal{C}$ which is not a split epimorphism, there is a morphism $h : X \to Y$ such that $g = fh$. Dually, we say that a morphism $f : X \to Y$ in $\mathcal{C}$ is **left almost split in** $\mathcal{C}$ if f is not a split monomorphism, and given a morphism $g : X \to Z$ in $\mathcal{C}$ which is not a split monomorphism, there is a morphism $h : Y \to Z$ such that $g = hf$. Finally, if a morphism is both right almost split in $\mathcal{C}$ and right minimal, then we say that it is **minimal right almost split in** $\mathcal{C}$. Likewise, if a morphism is both left almost split in $\mathcal{C}$ and left minimal, then we say that it is **minimal left almost split in** $\mathcal{C}$.

The reader should note that a morphism in $\mathcal{C}$ is right almost split or minimal right almost split if and only if its corresponding morphism in $\mathcal{C}^{\mathrm{op}}$ is left almost split or minimal left almost split, respectively.

We now present the connection between the morphisms defined above with the discussion around Proposition 3.3.6. The next result is from [7, Chapter II, Proposition 2.3], and here we present a different proof for it.

Proposition 3.4.1. *The following are equivalent for a morphism $f : Y \to Z$ in $\mathcal{C}$:*

(a) $\mathrm{End}_{\mathcal{C}}(Z)$ *is a local ring and* $\langle f \rangle_{\mathcal{C}} = \mathrm{rad}_{\mathcal{C}}(-, Z)$.

(b) *f is right almost split in $\mathcal{C}$.*

Proof. If $\mathrm{End}_{\mathcal{C}}(Z)$ is a local ring and $\langle f \rangle_{\mathcal{C}} = \mathrm{rad}_{\mathcal{C}}(-, Z)$, then it is easy to conclude from Proposition 2.6.5 that f is right almost split in $\mathcal{C}$.

Conversely, assume that f is right almost split in $\mathcal{C}$. First, we will verify that $f \in \mathrm{rad}_{\mathcal{C}}(Y, Z)$. For this, we know from Theorem 2.6.1 that it suffices to show that $id_Z - fg$ is a split epimorphism for every morphism $g \in \mathcal{C}(Z, Y)$. Therefore, let $g \in \mathcal{C}(Z, Y)$ and suppose that $id_Z - fg$ is not a split epimorphism. Then there is a morphism $h : Z \to Y$ such that $id_Z - fg = fh$, which implies that $id_Z = f(g + h)$, a contradiction.

Now, we will prove that $\mathrm{End}_{\mathcal{C}}(Z)$ is a local ring. By [1, Chapter VII, Theorem 6.5], it suffices to verify that if $g \in \mathcal{C}(Z, Z)$ is not an isomorphism, then $id_Z - g$ is an isomorphism. In this case, let $g \in \mathcal{C}(Z, Z)$ and suppose that g is not an isomorphism. Then we have that g is not a split epimorphism. In fact, if there is some morphism $k \in \mathcal{C}(Z, Z)$ with $gk = id_Z$, then k is not a split epimorphism, so that there is a morphism $h : Z \to Y$ such that $k = fh$. Hence $id_Z = gfh$, which implies that $id_Z \in \mathrm{rad}_{\mathcal{C}}(Z, Z)$, and by Proposition 2.5.2 this is a contradiction since $Z \neq 0$. Consequently, there is a morphism $h : Z \to Y$ such that $g = fh$, and thus it follows from Theorem 2.6.1 that $id_Z - g$ is an isomorphism.

Finally, it follows from Proposition 2.5.1 that $\langle f \rangle_{\mathcal{C}} \subseteq \mathrm{rad}_{\mathcal{C}}(-, Z)$, and by Proposition 2.6.5 we have that $\mathrm{rad}_{\mathcal{C}}(-, Z) \subseteq \langle f \rangle_{\mathcal{C}}$, so that $\langle f \rangle_{\mathcal{C}} = \mathrm{rad}_{\mathcal{C}}(-, Z)$. □

We point out that if we take $\mathcal{C}^{\mathrm{op}}$ in place of $\mathcal{C}$ in Proposition 3.4.1, then we can conclude that for a morphism $f : X \to Y$ in $\mathcal{C}$, we have that $\mathrm{End}_{\mathcal{C}}(X)$ is a local ring and ${}_{\mathcal{C}}\langle f \rangle = \mathrm{rad}_{\mathcal{C}}(X, -)$ if and only if f is left almost split in $\mathcal{C}$.

For the rest of this section, let $\mathcal{B}$ denote a skeletally small Krull–Schmidt category.

We see from propositions 3.3.6 and 3.4.1 that almost split morphisms in $\mathcal{B}$ are of great importance, given that their existence or nonexistence determine whether or not simple $\mathcal{B}$-modules are finitely presented. Therefore, it would be interesting to study when these morphisms exist in $\mathcal{B}$, and this is what we begin to discuss now.

Following [17], we say that $\mathcal{B}$ **has right almost split morphisms** if for every $Z \in \mathrm{ind}\,\mathcal{B}$ there is a right almost split morphism $Y \to Z$ in $\mathcal{B}$. Dually, we say that $\mathcal{B}$ **has left almost split morphisms** if for every $X \in \mathrm{ind}\,\mathcal{B}$ there is a left almost split morphism $X \to Y$ in $\mathcal{B}$. Finally, we say that $\mathcal{B}$ **has almost split morphisms** if $\mathcal{B}$ has both right and left almost split morphisms.

With the above definitions, we can summarize the discussion started in Section 3.3 in the next two results:

Proposition 3.4.2. *The following are equivalent:*

(a) $\mathcal{B}$ has right almost split morphisms.

(b) $\mathrm{rad}_{\mathcal{B}}(-, Z)$ is finitely generated for every $Z \in \mathrm{ind}\,\mathcal{B}$.

(c) Every simple $\mathcal{B}$-module is finitely presented.

Proof. First, recall from Proposition 2.5.3 that if $Z \in \mathcal{B}$, then $\mathrm{rad}_{\mathcal{B}}(-, Z)$ is finitely generated if and only if there is some morphism $f : Y \to Z$ in $\mathcal{B}$ such that $\langle f \rangle_{\mathcal{B}} = \mathrm{rad}_{\mathcal{B}}(-, Z)$. Therefore, it is easy to see by Proposition 3.4.1 that items (a) and (b) are equivalent, as well as that we have by Proposition 3.3.6 and Theorem 3.3.3 that items (b) and (c) are equivalent. □

Proposition 3.4.3. *Assume that $\mathcal{B}$ is right coherent. Then every simple $\mathcal{B}$-module admits a minimal projective resolution whose terms are representable modules if and only if $\mathcal{B}$ has right almost split morphisms.*

Proof. This follows from Proposition 3.4.2 and the last paragraph of Section 3.2. □

It is important to remark that if there is a right almost split morphism $Y \to Z$ in $\mathcal{B}$ for some $Z \in \mathrm{ind}\,\mathcal{B}$, then there is also a minimal right almost split morphism $W \to Z$ in $\mathcal{B}$. In fact, we obtain from Proposition 3.4.1 that $\mathrm{rad}_{\mathcal{B}}(-, Z)$ is a finitely generated $\mathcal{B}$-module, and because $\mathcal{B}$ is semiperfect by Theorem 3.2.5, we conclude that $\mathrm{rad}_{\mathcal{B}}(-, Z)$ admits a projective cover in $\mathrm{Mod}\,\mathcal{B}$ which is given by $\mathcal{B}(-, f) : \mathcal{B}(-, W) \to \mathrm{rad}_{\mathcal{B}}(-, Z)$ for some morphism $f : W \to Z$ in $\mathcal{B}$. In this case, it follows from Proposition 3.4.1 that f is right almost split in

$\mathcal{B}$ since $\langle f \rangle_{\mathcal{B}} = \mathrm{rad}_{\mathcal{B}}(-, Z)$, and we also have that f is right minimal, given that $\mathcal{B}(-, f)$ is right minimal. Dually, we can also verify that if there is a left almost split morphism $X \to Y$ in $\mathcal{B}$ for some $X \in \mathrm{ind}\, \mathcal{B}$, then there is a minimal left almost split morphism $X \to W$ in $\mathcal{B}$.

We end this section by studying the existence of almost split morphisms in $\mathcal{B}$ in case that $\mathcal{B}$ has an additive generator.

Proposition 3.4.4. *Suppose that $\mathcal{B}$ has an additive generator $X \in \mathcal{B}$, and let $\Lambda = \mathrm{End}_{\mathcal{B}}(X)$ and* $\mathrm{rad}\,\Lambda$ *be the Jacobson radical of Λ. Then we have the following:*

(a) $\mathcal{B}$ has right almost split morphisms if and only if $\mathrm{rad}\,\Lambda$ *is finitely generated as a right Λ-module.*

(b) $\mathcal{B}$ has left almost split morphisms if and only if $\mathrm{rad}\,\Lambda$ *is finitely generated as a left Λ-module.*

Proof. (a) Since we have an equivalence of categories $\mathsf{e}_X : \mathrm{Mod}\, \mathcal{B} \to \mathrm{Mod}\, \Lambda$ by Proposition 2.8.2 and we know from Corollary 2.6.2 that $\mathsf{e}_X(\mathrm{rad}_{\mathcal{B}}(-, X)) = \mathrm{rad}\,\Lambda$, it follows from Corollary 2.7.3 that the $\mathcal{B}$-module $\mathrm{rad}_{\mathcal{B}}(-, X)$ is finitely generated if and only if $\mathrm{rad}\,\Lambda$ is finitely generated as a Λ-module. Now, write $X = X_1 \oplus \cdots \oplus X_n$ with each X_i being indecomposable. We have that $\mathrm{rad}_{\mathcal{B}}(-, X)$ is finitely generated if and only if so is each $\mathrm{rad}_{\mathcal{B}}(-, X_i)$, and this is the case if and only if $\mathcal{B}$ has right almost split morphisms by Proposition 3.4.2.

(b) This follows dually from the proof of item (a). □

As a consequence of Proposition 3.4.4, we have the following:

Corollary 3.4.5. *Suppose that $\mathcal{B}$ is a finitely confined R-category for some commutative noetherian ring R. If $\mathcal{B}$ has an additive generator, then $\mathcal{B}$ has almost split morphisms.*

Proof. If $X \in \mathcal{B}$ is an additive generator of $\mathcal{B}$, then $\Lambda = \mathrm{End}_{\mathcal{B}}(X)$ is both a right and a left noetherian ring since it is finitely generated as an R-module. Therefore, the Jacobson radical of Λ is both finitely generated as a right and a left Λ-module, and we conclude from Proposition 3.4.4 that $\mathcal{B}$ has almost split morphisms. □

3.5 Functorially finite subcategories

In this section we give the notions of "contravariantly finite", "covariantly finite" and "functorially finite" subcategory, which were first introduced in [16] and subsequently explored in [17] by Maurice Auslander and Sverre O. Smalø. Such subcategories will play an important role in this text, and we show here that they behave very well with the properties of being right or left coherent and of having right or left almost split morphisms.

In what follows, $\mathcal{V}$ is a variety and $\mathcal{C}$ is a subcategory of $\mathcal{V}$ such that add $\mathcal{C} = \mathcal{C}$, so that $\mathcal{C}$ is also a variety.

We say that $\mathcal{C}$ is **contravariantly finite in** $\mathcal{V}$ if every finitely generated $\mathcal{V}$-module is also finitely generated when viewed as a $\mathcal{C}$-module, i.e. if $F \in \mathrm{f.g.}(\mathcal{V})$ implies that $F|_{\mathcal{C}} \in \mathrm{f.g.}(\mathcal{C})$. This is equivalent to say that $\mathcal{V}(-,Y)|_{\mathcal{C}}$ is a finitely generated $\mathcal{C}$-module for every $Y \in \mathcal{V}$, and this happens if and only if for every $Y \in \mathcal{V}$ there is some morphism $f : X \to Y$ in $\mathcal{V}$ with $X \in \mathcal{C}$ such that $\mathcal{V}(-,f)|_{\mathcal{C}} : \mathcal{C}(-,X) \to \mathcal{V}(-,Y)|_{\mathcal{C}}$ is an epimorphism in Mod $\mathcal{C}$, in which case we call f a **right $\mathcal{C}$-approximation of** Y. Dually, we say that $\mathcal{C}$ is **covariantly finite in** $\mathcal{V}$ if every finitely generated left $\mathcal{V}$-module is also finitely generated when viewed as a left $\mathcal{C}$-module, i.e. if $F \in \mathrm{f.g.}(\mathcal{V}^{\mathrm{op}})$ implies that $F|_{\mathcal{C}} \in \mathrm{f.g.}(\mathcal{C}^{\mathrm{op}})$, and this happens if and only if for every $Y \in \mathcal{V}$ there is some morphism $f : Y \to X$ in $\mathcal{V}$ with $X \in \mathcal{C}$ such that $\mathcal{V}(f,-)|_{\mathcal{C}} : \mathcal{C}(X,-) \to \mathcal{V}(Y,-)|_{\mathcal{C}}$ is an epimorphism in Mod $\mathcal{C}^{\mathrm{op}}$, in which case we call f a **left $\mathcal{C}$-approximation of** Y. Finally, we say that $\mathcal{C}$ is **functorially finite in** $\mathcal{V}$ if $\mathcal{C}$ is both contravariantly and covariantly finite in $\mathcal{V}$.

The reader should note that $\mathcal{C}$ is covariantly finite in $\mathcal{V}$ if and only if $\mathcal{C}^{\mathrm{op}}$ is contravariantly finite in $\mathcal{V}^{\mathrm{op}}$. Therefore, we develop some results in this section only for the case of contravariantly finite subcategories, since the covariantly case will be recovered by considering opposite categories.

We first prove that the subcategories described above behave well with the notions of being right or left coherent. The result below is a generalization of [17, Proposition 2.1(b)].

Proposition 3.5.1. *Suppose that $\mathcal{V}$ is right coherent and that $\mathcal{C}$ is contravariantly finite in $\mathcal{V}$. Then $\mathcal{C}$ is also right coherent and if $F \in \mathrm{mod}\,\mathcal{V}$, then $F|_{\mathcal{C}} \in \mathrm{mod}\,\mathcal{C}$.*

Proof. We begin by verifying that $F|_{\mathcal{C}} \in \mathrm{mod}\,\mathcal{C}$ for every $F \in \mathrm{mod}\,\mathcal{V}$, and for this, we first show that $\mathcal{V}(-,Y)|_{\mathcal{C}} \in \mathrm{mod}\,\mathcal{C}$ for each $Y \in \mathcal{V}$.

Let $Y \in \mathcal{V}$ be arbitrary. Since $\mathcal{C}$ is contravariantly finite in $\mathcal{V}$, there is a morphism $f : X \to Y$ in $\mathcal{V}$ with $X \in \mathcal{C}$ such that $\mathcal{V}(-,f)|_{\mathcal{C}} : \mathcal{C}(-,X) \to \mathcal{V}(-,Y)|_{\mathcal{C}}$ is an epimorphism in Mod $\mathcal{C}$. In this case, if we consider the exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{V}(-,X) \xrightarrow{\mathcal{V}(-,f)} \mathcal{V}(-,Y)$$

in Mod $\mathcal{V}$, then we conclude by Proposition 3.1.6 that the K is a finitely generated $\mathcal{V}$-module. Consequently, $K|_{\mathcal{C}}$ is a finitely generated $\mathcal{C}$-module and from the short exact sequence

$$0 \longrightarrow K|_{\mathcal{C}} \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{V}(-,f)|_{\mathcal{C}}} \mathcal{V}(-,Y)|_{\mathcal{C}} \longrightarrow 0$$

in Mod $\mathcal{C}$ we obtain by Proposition 3.1.3 that $\mathcal{V}(-,Y)|_{\mathcal{C}} \in \mathrm{mod}\,\mathcal{C}$.

Now, take $F \in \mathrm{mod}\,\mathcal{V}$ and let

$$\mathcal{V}(-,X) \longrightarrow \mathcal{V}(-,Y) \longrightarrow F \longrightarrow 0$$

be an exact sequence in $\operatorname{Mod} \mathcal{V}$ with $X, Y \in \mathcal{V}$. Then we obtain an exact sequence

$$\mathcal{V}(-,X)|_{\mathcal{C}} \longrightarrow \mathcal{V}(-,Y)|_{\mathcal{C}} \longrightarrow F|_{\mathcal{C}} \longrightarrow 0$$

in $\operatorname{Mod} \mathcal{C}$, and given that $\mathcal{V}(-,X)|_{\mathcal{C}}, \mathcal{V}(-,Y)|_{\mathcal{C}} \in \operatorname{mod} \mathcal{C}$, we conclude by Proposition 3.1.5 that $F|_{\mathcal{C}} \in \operatorname{mod} \mathcal{C}$.

Next, we will verify that $\mathcal{C}$ is right coherent. In order to show this, let $F \subseteq \mathcal{C}(-,Y)$ be a finitely generated right ideal of $\mathcal{C}$ for some $Y \in \mathcal{C}$. If we prove that F is finitely presented, then we will conclude from Lemma 3.1.8 that $\mathcal{C}$ is right coherent. Well, we know from Proposition 2.5.3 that there is some morphism $f : X \to Y$ in $\mathcal{C}$ such that $F = \langle f \rangle_{\mathcal{C}}$. Therefore, consider $G = \langle f \rangle_{\mathcal{V}}$ in $\operatorname{Mod} \mathcal{V}$. Since $\mathcal{V}$ is right coherent, we have that $G \in \operatorname{mod} \mathcal{V}$, and thus $F = G|_{\mathcal{C}} \in \operatorname{mod} \mathcal{C}$ by the above paragraph. □

As we will see in the next result, which is from [16, Proposition 3.10], we also have that contravariantly and covariantly finite subcategories preserve the properties of having right and left almost split morphisms, respectively.

Proposition 3.5.2. *If $\mathcal{V}$ is a Krull–Schmidt category which has right almost split morphisms and $\mathcal{C}$ is contravariantly finite in $\mathcal{V}$, then $\mathcal{C}$ also has right almost split morphisms.*

Proof. This follows from Proposition 3.4.2 and the fact that $\operatorname{rad}_{\mathcal{C}}(-,Z) = \operatorname{rad}_{\mathcal{V}}(-,Z)|_{\mathcal{C}}$ for every $Z \in \mathcal{C}$. □

At this moment, recall from Proposition 3.4.3 that for a variety, the properties of being Krull–Schmidt, right coherent and of having right almost split morphisms are very important. Considering this, it is interesting to point out that as a consequence of propositions 3.5.1 and 3.5.2, we conclude that if $\mathcal{V}$ has these three properties, then so does $\mathcal{C}$ provided that $\mathcal{C}$ is contravariantly finite in $\mathcal{V}$. Likewise, we also have that if $\mathcal{V}$ is Krull–Schmidt, left coherent and has left almost split morphisms, then so does $\mathcal{C}$ provided that $\mathcal{C}$ is covariantly finite in $\mathcal{V}$. Therefore, we see that subcategories which are contravariantly or covariantly finite are indeed of great interest if one would like to study minimal projective resolutions of simple modules, as is our case.

To end this section, we present a case when we can verify that a subcategory is functorially finite. The next result is a generalization of [16, Proposition 4.2], and here we give a different approach to prove it.

Proposition 3.5.3. *Suppose that $\mathcal{V}$ is a finitely confined R-category for some commutative ring R. If $\mathcal{C}$ has an additive generator, then $\mathcal{C}$ is functorially finite in $\mathcal{V}$.*

Proof. We only show that $\mathcal{C}$ is contravariantly finite in $\mathcal{V}$, since the proof that $\mathcal{C}$ is covariantly finite in $\mathcal{V}$ will follow by duality.

Let $X \in \mathcal{C}$ be such that $\operatorname{add} X = \mathcal{C}$ and write $\Lambda = \operatorname{End}_{\mathcal{C}}(X)$. We know from Proposition 2.8.2 that $\mathsf{e}_X : \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\Lambda$ is an equivalence of categories, and thus for each $Y \in \mathcal{V}$ we have by Corollary 2.7.3 that $\mathcal{V}(-,Y)|_{\mathcal{C}}$ is a finitely generated $\mathcal{C}$-module if and only if $\mathcal{V}(X,Y)$ is a finitely generated Λ-module. But we know that $\mathcal{V}(X,Y)$ is a finitely generated R-module for each $Y \in \mathcal{V}$, and because Λ is an R-algebra, we obtain that $\mathcal{V}(X,Y)$ is also finitely generated as a Λ-module. Therefore, $\mathcal{C}$ is contravariantly finite in $\mathcal{V}$. $\square$

3.6 n-Almost split sequences

We now begin to take the first steps into higher Auslander–Reiten theory by giving the definition and some properties of an "n-almost split sequence", which is an exact sequence satisfying certain conditions. This kind of sequence plays a central role in higher Auslander–Reiten theory, and generalizes the notion of an "almost split sequence", one of the main objects of study in Auslander–Reiten theory.

In what follows, $\mathcal{B}$ is a skeletally small Krull–Schmidt category.

Before we define what an "n-almost split sequence" is, we first develop some thoughts on minimal projective resolutions of simple $\mathcal{B}$-modules.

If $\mathcal{B}$ is right coherent and has right almost split morphisms, then it follows from Proposition 3.4.3 that for each $Z \in \operatorname{ind}\mathcal{B}$ the simple $\mathcal{B}$-module S_Z admits a minimal projective resolution

$$\cdots \xrightarrow{\mathcal{B}(-,f_3)} \mathcal{B}(-,Y_2) \xrightarrow{\mathcal{B}(-,f_2)} \mathcal{B}(-,Y_1) \xrightarrow{\mathcal{B}(-,f_1)} \mathcal{B}(-,Z) \longrightarrow S_Z \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{B}$ with each $f_i : Y_i \to Y_{i-1}$ being a morphism in $\mathcal{B}$, where $Y_0 = Z$. Therefore, for each $Z \in \operatorname{ind}\mathcal{B}$ we obtain a complex

$$\mathtt{C}_Z : \qquad \cdots \xrightarrow{f_3} Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\mathcal{B}$ with $f_i \in \operatorname{rad}_{\mathcal{B}}$ for each $i \geqslant 1$ by Theorem 3.2.3, where f_1 is right almost split in $\mathcal{B}$ by Proposition 3.4.1. Moreover, this complex is unique up to isomorphism among all other complexes which induce minimal projective resolutions of S_Z, given that we know from Proposition 1.4.5 that minimal projective resolutions are unique up to isomorphism of complexes.

Likewise, if $\mathcal{B}$ is left coherent and has left almost split morphisms, then for each $X \in \operatorname{ind}\mathcal{B}$ there is, up to isomorphism, a unique complex

$$\mathtt{C}_X : \qquad 0 \longrightarrow X \xrightarrow{g_1} W_1 \xrightarrow{g_2} W_2 \xrightarrow{g_3} \cdots$$

in $\mathcal{B}$ which induces a minimal projective resolution

$$\cdots \xrightarrow{\mathcal{B}(g_3,-)} \mathcal{B}(W_2,-) \xrightarrow{\mathcal{B}(g_2,-)} \mathcal{B}(W_1,-) \xrightarrow{\mathcal{B}(g_1,-)} \mathcal{B}(X,-) \longrightarrow S^X \longrightarrow 0$$

of S^X in $\operatorname{Mod} \mathcal{B}^{\mathrm{op}}$. In this case, we can also conclude that $g_i \in \operatorname{rad}_{\mathcal{B}}$ for each $i \geqslant 1$, as well as that g_1 is left almost split in $\mathcal{B}$.

Now, suppose that $\mathcal{B}$ is an abelian category. Then we know from Proposition 3.1.9 that $\mathcal{B}$ is both right and left coherent, and it follows from Proposition 3.1.10 that $\operatorname{pd}_{\mathcal{B}} F \leqslant 2$ for every $F \in \operatorname{mod} \mathcal{B}$, as well as that $\operatorname{pd}_{\mathcal{B}^{\mathrm{op}}} G \leqslant 2$ for every $G \in \operatorname{mod} \mathcal{B}^{\mathrm{op}}$. Therefore, if $\mathcal{B}$ has almost split morphisms and $X, Z \in \operatorname{ind} \mathcal{B}$, then there are complexes $\mathtt{C}_Z$ and $\mathtt{C}_X$ as described above, and they satisfy that $Y_i = 0$ and $W_i = 0$ for all $i \geqslant 3$.

Under the conditions and notations of the above paragraph, an interesting case to have would be that $\mathtt{C}_Z = \mathtt{C}_X$ for some pair $X, Z \in \operatorname{ind} \mathcal{B}$. Well, if this is indeed the case, then we have a complex

$$\mathtt{C}: \qquad 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\mathcal{B}$ such that

$$0 \longrightarrow \mathcal{B}(-,X) \xrightarrow{\mathcal{B}(-,f)} \mathcal{B}(-,Y) \xrightarrow{\mathcal{B}(-,g)} \mathcal{B}(-,Z) \longrightarrow S_Z \longrightarrow 0$$

is a minimal projective resolution of S_Z in $\operatorname{Mod} \mathcal{B}$ and

$$0 \longrightarrow \mathcal{B}(Z,-) \xrightarrow{\mathcal{B}(g,-)} \mathcal{B}(Y,-) \xrightarrow{\mathcal{B}(f,-)} \mathcal{B}(X,-) \longrightarrow S^X \longrightarrow 0$$

is a minimal projective resolution of S^X in $\operatorname{Mod} \mathcal{B}^{\mathrm{op}}$. Furthermore, we obtain from Proposition 2.2.5 that $\mathtt{C}$ is a short exact sequence in $\mathcal{B}$, and by the above comments we also have that g is right almost split in $\mathcal{B}$ and that f is left almost split in $\mathcal{B}$. In this case, $\mathtt{C}$ is a sequence which was called by an "almost split sequence" by Maurice Auslander and Idun Reiten in [12, Section 2] (see also [7, Chapter II, Section 4]) and which was first introduced in [11] for the case when $\mathcal{B} = \operatorname{mod} \Lambda$ for an Artin algebra Λ.

With the above discussion done, we can now proceed to present the definition of an "n-almost split sequence", given by Osamu Iyama in [25], which generalizes the notion of an "almost split sequence" introduced above.

From now on in this section, $\mathcal{A}$ will stand for a skeletally small abelian category, $\mathcal{C}$ will be a subcategory of $\mathcal{A}$ which is also a Krull–Schmidt category and n will denote a positive integer.

Following [25, Definition 3.1], we say that an exact sequence

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\mathcal{A}$ with terms in $\mathcal{C}$ is an **n-almost split sequence in $\mathcal{C}$** if X and Z are indecomposable and if it induces both a minimal projective resolution

$$0 \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f_{n+1})} \mathcal{C}(-,Y_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z) \longrightarrow S_Z \longrightarrow 0$$

of S_Z in $\operatorname{Mod}\mathcal{C}$ and a minimal projective resolution

$$0 \longrightarrow \mathcal{C}(Z,-) \xrightarrow{\mathcal{C}(f_1,-)} \mathcal{C}(Y_1,-) \xrightarrow{\mathcal{C}(f_2,-)} \cdots \xrightarrow{\mathcal{C}(f_n,-)} \mathcal{C}(Y_n,-) \xrightarrow{\mathcal{C}(f_{n+1},-)} \mathcal{C}(X,-) \longrightarrow S^X \longrightarrow 0$$

of S^X in $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$. Moreover, if $n = 1$, then we call a 1-almost split sequence in $\mathcal{C}$ simply by an **almost split sequence in** $\mathcal{C}$.

Briefly, an n-almost split sequence in $\mathcal{C}$ is an exact sequence in $\mathcal{A}$ of length n with terms in $\mathcal{C}$ which induces simultaneously a minimal projective resolution of a simple right $\mathcal{C}$-module, and also of a simple left $\mathcal{C}$-module.

As we will see in the next proposition, we can also describe n-almost split sequences in $\mathcal{C}$ without explicitly mentioning minimal projective resolutions of simple $\mathcal{C}$-modules.

Proposition 3.6.1. *An exact sequence*

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\mathcal{A}$ with terms in $\mathcal{C}$ is an n-almost split sequence in $\mathcal{C}$ if and only if f_1 is right almost split in $\mathcal{C}$, f_{n+1} is left almost split in $\mathcal{C}$, $f_i \in \operatorname{rad}_{\mathcal{C}}$ for each $2 \leqslant i \leqslant n$ and the sequences

$$0 \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f_{n+1})} \mathcal{C}(-,Y_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z)$$

and

$$0 \longrightarrow \mathcal{C}(Z,-) \xrightarrow{\mathcal{C}(f_1,-)} \mathcal{C}(Y_1,-) \xrightarrow{\mathcal{C}(f_2,-)} \cdots \xrightarrow{\mathcal{C}(f_n,-)} \mathcal{C}(Y_n,-) \xrightarrow{\mathcal{C}(f_{n+1},-)} \mathcal{C}(X,-)$$

are exact in $\operatorname{Mod}\mathcal{C}$ *and in* $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$*, respectively.*

Proof. This follows from Theorem 3.2.3 and Proposition 3.4.1. □

In particular, it follows from Proposition 3.6.1 that if

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$, then Z is nonprojective and X is noninjective in $\mathcal{C}$. Indeed, this follows from the fact that f_1 is an epimorphism in $\mathcal{C}$ but not a split epimorphism and f_{n+1} is a monomorphism in $\mathcal{C}$ but not a split monomorphism since f_1 and f_{n+1} are right and left almost split in $\mathcal{C}$, respectively.

We also remark that by propositions 2.2.5 and 3.6.1 we can conclude that a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\mathcal{A}$ with terms in $\mathcal{C}$ is an almost split sequence in $\mathcal{C}$ if and only if g is right almost split in $\mathcal{C}$ and f is left almost split in $\mathcal{C}$. Therefore, we see that by taking $n = 1$ we recover the original definition of "almost split sequence" given by Maurice Auslander and Idun Reiten, which we have already mentioned before.

It is interesting to note that if there is an n-almost split sequence in $\mathcal{C}$, then it is uniquely determined up to isomorphism of complexes by each of its end terms, as we will prove below. The next result is from [25, Proposition 3.1.1], which is a generalization of the case $n = 1$ in [12, Proposition 2.12].

Proposition 3.6.2. *If*

$$0 \longrightarrow X \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Z \longrightarrow 0$$

and

$$0 \longrightarrow X' \longrightarrow Y_n' \longrightarrow \cdots \longrightarrow Y_1' \longrightarrow Z' \longrightarrow 0$$

are two n-almost split sequences in $\mathcal{C}$, then the following are equivalent:

(a) The sequences are isomorphic as complexes.

(b) $Z \simeq Z'$.

(c) $X \simeq X'$.

Proof. This is an easy consequence of Proposition 1.4.5, given that n-almost split sequences in $\mathcal{C}$ induce minimal projective resolutions of simple right and left $\mathcal{C}$-modules. □

Now, the reader should note that so far we have seen some properties of n-almost split sequences in $\mathcal{C}$, but we still do not know if they exist. Well, in general, if we do not impose any conditions on $\mathcal{C}$, then we cannot guarantee their existence, and actually a great part of Chapter 4 will be devoted to study which conditions would be interesting to impose on $\mathcal{C}$ to make $\mathcal{C}$ more likely to have such sequences. But before we proceed to investigate these conditions, first we need to be more precise about what we mean by the existence of such sequences in $\mathcal{C}$.

We say that $\mathcal{C}$ **has n-almost split sequences** if for each indecomposable and nonprojective object $Z \in \mathcal{C}$ there is an n-almost split sequence in $\mathcal{C}$ whose right end term is Z, and also if for each indecomposable and noninjective object $X \in \mathcal{C}$ there is an n-almost split sequence in $\mathcal{C}$ whose left end term is X. The reader should note that, for the case $n = 1$, the notion of having almost split sequences that we defined here does not coincide with the corresponding notion given in [17].

In Theorem 5.1.12 we will prove that if Λ is an Artin algebra and if $\mathcal{C}$ is a subcategory of mod Λ which satisfies certain conditions, then $\mathcal{C}$ has n-almost split sequences. However, before we achieve this result we first need to understand what these conditions are, which are described in the next chapter.

4 Where to look for n-almost split sequences

In this chapter, $\mathcal{A}$ will stand for a skeletally small abelian category and $\mathcal{C}$ will be a subcategory of $\mathcal{A}$ such that add $\mathcal{C} = \mathcal{C}$, so that $\mathcal{C}$ is a variety. Moreover, n will always denote a positive integer.

Our aim now is to establish appropriate conditions over $\mathcal{C}$ to make it more likely to have n-almost split sequences in case that it is Krull–Schmidt. A subcategory satisfying the conditions that we introduce here will be called an "n-cluster tilting subcategory of $\mathcal{A}$". Moreover, we also present several equivalent conditions for an exact sequence in $\mathcal{A}$ of length n with terms in $\mathcal{C}$ to be an n-almost split sequence in $\mathcal{C}$, in case that $\mathcal{C}$ is an "n-cluster tilting subcategory of $\mathcal{A}$" which is also Krull–Schmidt.

4.1 n-Rigid subcategories

In this section we begin by presenting the condition of $\mathcal{C}$ being "n-rigid", which as we will see, gives an easier description of n-almost split sequences in $\mathcal{C}$, in case that $\mathcal{C}$ is Krull–Schmidt. Furthermore, we also prove that there are no m-almost split sequences in $\mathcal{C}$ for $0 < m < n$ if $\mathcal{C}$ is Krull–Schmidt and satisfies such condition.

In view of Proposition 3.6.1, it is interesting to make the following definitions, which are from [29, Definition 2.3]. We say that a sequence

$$\mathrm{S}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\mathcal{C}$ is a **left n-exact sequence in $\mathcal{C}$**, or for short, **left n-exact in $\mathcal{C}$**, if

$$0 \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f_{n+1})} \mathcal{C}(-,Y_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z)$$

is an exact sequence in Mod $\mathcal{C}$. Dually, we say that it is a **right n-exact sequence in $\mathcal{C}$**, or for short, **right n-exact in $\mathcal{C}$**, if

$$0 \longrightarrow \mathcal{C}(Z,-) \xrightarrow{\mathcal{C}(f_1,-)} \mathcal{C}(Y_1,-) \xrightarrow{\mathcal{C}(f_2,-)} \cdots \xrightarrow{\mathcal{C}(f_n,-)} \mathcal{C}(Y_n,-) \xrightarrow{\mathcal{C}(f_{n+1},-)} \mathcal{C}(X,-)$$

is an exact sequence in Mod $\mathcal{C}^{\mathrm{op}}$. If S is both left and right n-exact in $\mathcal{C}$, then we say that it is an **n-exact sequence in $\mathcal{C}$**, or for short, **n-exact in $\mathcal{C}$**.

The reader should note that it follows from Proposition 2.2.5 that a sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\mathcal{A}$ is 1-left exact in $\mathcal{A}$ if and only if

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$$

is an exact sequence in $\mathcal{A}$, as well as that it is 1-right exact in $\mathcal{A}$ if and only if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$. Therefore, the concept of a 1-exact sequence in $\mathcal{A}$ coincides with that of a short exact sequence in $\mathcal{A}$.

In case that $\mathcal{C}$ is a Krull–Schmidt category, from the above definitions we see that a necessary condition for an exact sequence in $\mathcal{A}$ of length n with terms in $\mathcal{C}$ to be an n-almost split sequence in $\mathcal{C}$ is that it must be n-exact in $\mathcal{C}$. Because of this, it would be interesting to have some property over $\mathcal{C}$ which guarantees that every exact sequence in $\mathcal{A}$ of length n with terms in $\mathcal{C}$ is n-exact in $\mathcal{C}$. Well, as we will see in Proposition 4.1.2, a sufficient condition for this to happen is that $\mathcal{C}$ is "n-rigid", a concept which we define next.

We say that $\mathcal{C}$ is n-**rigid** if $\operatorname{Ext}^i_{\mathcal{A}}(X,Y) = 0$ for every $X, Y \in \mathcal{C}$ and $0 < i < n$. It is important to remark that saying that $\mathcal{C}$ is 1-rigid does not impose any additional condition on $\mathcal{C}$, so that this is a trivial case. Furthermore, note that if $\mathcal{C}$ is n-rigid, then it is also m-rigid for every $0 < m < n$.

Before we proceed to prove Proposition 4.1.2, we define the following subcategories of $\mathcal{A}$, which will be useful for the development of the text:

$$\mathcal{C}^{\perp n} = \{X \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(Y,X) = 0 \text{ for all } 0 < i < n \text{ and } Y \in \mathcal{C}\},$$
$${}^{\perp n}\mathcal{C} = \{X \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(X,Y) = 0 \text{ for all } 0 < i < n \text{ and } Y \in \mathcal{C}\}.$$

Observe that with the above definitions we have that $\mathcal{C}$ is n-rigid if and only if $\mathcal{C} \subseteq \mathcal{C}^{\perp n}$, which is also equivalent to say that $\mathcal{C} \subseteq {}^{\perp n}\mathcal{C}$.

To prove Proposition 4.1.2 we need the following technical lemma, which was motivated by [25, Item 2.2.1(1)].

Lemma 4.1.1. *Suppose that $n \geqslant 3$ and let*

$$\mathtt{E}: \qquad 0 \longrightarrow X \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_j \longrightarrow X_j \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$ with $X, Y_i \in \mathcal{C}$ for each i, where $j \geqslant 3$. Then $\operatorname{Ext}^1_{\mathcal{A}}(W, X_j) = 0$ for each $W \in {}^{\perp n}\mathcal{C}$.

Proof. First, write $\mathtt{E} = \mathtt{E}_n\mathtt{E}_{n-1}\cdots\mathtt{E}_j$ as a splice of $n - j + 1$ short exact sequences

$$\mathtt{E}_i: \qquad 0 \longrightarrow X_{i+1} \longrightarrow Y_i \longrightarrow X_i \longrightarrow 0$$

in $\mathcal{A}$, where $X_{n+1} = X$. From Theorem 1.2.2 we obtain that $\operatorname{Ext}^k_{\mathcal{A}}(W, X_i) = 0$ for each $0 < k < i - 1$ and $W \in {}^{\perp n}\mathcal{C}$, hence $\operatorname{Ext}^1_{\mathcal{A}}(W, X_j) = 0$ for each $W \in {}^{\perp n}\mathcal{C}$. $\square$

The next result is from [29, Proposition 2.2] (see also [25, Lemma 3.2]).

Proposition 4.1.2. *Assume that $\mathcal{C}$ is n-rigid and let*

$$\mathrm{E}: \qquad X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z$$

be an exact sequence in $\mathcal{A}$ with terms in $\mathcal{C}$.

(a) If f_{n+1} is a monomorphism in $\mathcal{A}$, then E is left n-exact in $\mathcal{C}$.

(b) If f_1 is an epimorphism in $\mathcal{A}$, then E is right n-exact in $\mathcal{C}$.

Therefore, every exact sequence in $\mathcal{A}$ of length n with terms in $\mathcal{C}$ is n-exact in $\mathcal{C}$.

Proof. (a) Write $\mathrm{E} = \mathrm{E}_n\mathrm{E}_{n-1}\cdots\mathrm{E}_1$ as a splice of the exact sequences

$$\mathrm{E}_1: \qquad 0 \longrightarrow X_2 \longrightarrow Y_1 \longrightarrow Z$$

and

$$\mathrm{E}_i: \qquad 0 \longrightarrow X_{i+1} \longrightarrow Y_i \longrightarrow X_i \longrightarrow 0$$

in $\mathcal{A}$, where $2 \leqslant i \leqslant n$ and $X_{n+1} = X$.

It follows from Proposition 2.2.5 that

$$0 \longrightarrow \mathcal{A}(-,X_2)|_{\mathcal{C}} \longrightarrow \mathcal{C}(-,Y_1) \longrightarrow \mathcal{C}(-,Z)$$

is an exact sequence in $\operatorname{Mod}\mathcal{C}$, hence if $n = 1$, then we are done. Otherwise, if $n \geqslant 2$, then since we know from Lemma 4.1.1 that $\operatorname{Ext}^1_{\mathcal{A}}(W,X_j) = 0$ for each $W \in \mathcal{C}$ and $j \geqslant 3$ (note that if $n = 2$, then $X_3 = X$, so that we also have this equality), we conclude by Theorem 1.2.2 that

$$0 \longrightarrow \mathcal{A}(-,X_{i+1})|_{\mathcal{C}} \longrightarrow \mathcal{C}(-,Y_i) \longrightarrow \mathcal{A}(-,X_i)|_{\mathcal{C}} \longrightarrow 0$$

is also an exact sequence in $\operatorname{Mod}\mathcal{C}$ for each $2 \leqslant i \leqslant n$. Hence by taking the splice of the above sequences, we conclude that

$$0 \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f_{n+1})} \mathcal{C}(-,Y_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z)$$

is an exact sequence in $\operatorname{Mod}\mathcal{C}$, which means that E is left n-exact in $\mathcal{C}$.

(b) This follows dually from the proof of item (a). $\square$

As a consequence of Proposition 4.1.2, if $\mathcal{C}$ is a Krull–Schmidt category which is also n-rigid, then we obtain the following characterization of n-almost split sequences in $\mathcal{C}$:

Corollary 4.1.3. *If $\mathcal{C}$ is Krull–Schmidt and n-rigid, then an exact sequence*

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

in $\mathcal{A}$ with terms in $\mathcal{C}$ is an n-almost split sequence in $\mathcal{C}$ if and only if f_1 is right almost split in $\mathcal{C}$, f_{n+1} is left almost split in $\mathcal{C}$ and $f_i \in \operatorname{rad}_{\mathcal{C}}$ for each $2 \leqslant i \leqslant n$.

Proof. This follows from propositions 3.6.1 and 4.1.2. □

We see then from Corollary 4.1.3 that it is indeed interesting to assume that $\mathcal{C}$ is n-rigid, and in fact, this is one of the conditions that $\mathcal{C}$ will need to satisfy to be an “n-cluster tilting subcategory of $\mathcal{A}$”, as we will see in Section 4.2. Because of this, we now investigate some consequences that we obtain by assuming that $\mathcal{C}$ is n-rigid.

To begin with, note that if $\mathcal{C}$ is Krull–Schmidt and

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$ with terms in $\mathcal{C}$, then a necessary condition that $\mathtt{E}$ need to satisfy to be an n-almost split sequence in $\mathcal{C}$ is that $\operatorname{Coker} \mathcal{C}(-, f_1)$ and $\operatorname{Coker} \mathcal{C}(f_{n+1}, -)$ must be simple right and left $\mathcal{C}$-modules, respectively. Therefore, it could be useful to have a description of both $\operatorname{Coker} \mathcal{C}(-, f_1)$ and $\operatorname{Coker} \mathcal{C}(f_{n+1}, -)$ in terms of $\mathtt{E}$, and this is what we present in Proposition 4.1.4 for the case when $\mathcal{C}$ is n-rigid. But before we give this result, we first need to establish some notation.

For each $i \geqslant 0$ and $X \in \mathcal{A}$ we define $\operatorname{Ext}^i_{\mathcal{A}}(-, X) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ to be the $\mathcal{A}$-module which is given by $\operatorname{Ext}^i_{\mathcal{A}}(-, X)(W) = \operatorname{Ext}^i_{\mathcal{A}}(W, X)$ for each $W \in \mathcal{A}$, with the agreement that $\operatorname{Ext}^0_{\mathcal{A}}(W, X) = \mathcal{A}(W, X)$. Moreover, if

$$\mathtt{E}: \qquad 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$, then we let

$$\partial_i(-, \mathtt{E}) : \operatorname{Ext}^i_{\mathcal{A}}(-, Z) \to \operatorname{Ext}^{i+1}_{\mathcal{A}}(-, X)$$

be the morphism of $\mathcal{A}$-modules whose components are defined by $\partial_i(-, \mathtt{E})_W = \partial_i(W, \mathtt{E})$ for each $W \in \mathcal{A}$ (see Section 1.2). Likewise, we define $\operatorname{Ext}^i_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathrm{Ab}$ to be the left $\mathcal{A}$-module which is given by $\operatorname{Ext}^i_{\mathcal{A}}(X, -)(W) = \operatorname{Ext}^i_{\mathcal{A}}(X, W)$ for each $W \in \mathcal{A}$, and we let

$$\partial_i(\mathtt{E}, -) : \operatorname{Ext}^i_{\mathcal{A}}(X, -) \to \operatorname{Ext}^{i+1}_{\mathcal{A}}(Z, -)$$

be the morphism of left $\mathcal{A}$-modules whose components are defined by $\partial_i(\mathtt{E}, -)_W = \partial_i(\mathtt{E}, W)$ for each $W \in \mathcal{A}$ (see Section 1.2).

The proposition below was motivated by [25, Lemma 3.2].

Proposition 4.1.4. *Assume that $\mathcal{C}$ is n-rigid and let*

$$\mathrm{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$ with terms in $\mathcal{C}$. Then

(a) The cokernel of $\mathcal{C}(-, f_1)$ in Mod $\mathcal{C}$ *is given by the image of the morphism*

$$\phi(-,\mathrm{E}) : \mathcal{C}(-,Z) \to \mathrm{Ext}^n_{\mathcal{A}}(-,X)|_{\mathcal{C}}$$

whose components $\phi(-,\mathrm{E})_W = \phi(W,\mathrm{E})$ are given by $\phi(W,\mathrm{E})(g) = \mathrm{E}g$ for each $g \in \mathcal{C}(W,Z)$ and $W \in \mathcal{C}$.

(b) The cokernel of $\mathcal{C}(f_{n+1}, -)$ in Mod $\mathcal{C}^{\mathrm{op}}$ *is given by the image of the morphism*

$$\phi(\mathrm{E},-) : \mathcal{C}(X,-) \to \mathrm{Ext}^n_{\mathcal{A}}(Z,-)|_{\mathcal{C}}$$

whose components $\phi(\mathrm{E},-)_W = \phi(\mathrm{E},W)$ are given by $\phi(\mathrm{E},W)(g) = g\mathrm{E}$ for each $g \in \mathcal{C}(X,W)$ and $W \in \mathcal{C}$.

Proof. (a) First, write $\mathrm{E} = \mathrm{E}_n\mathrm{E}_{n-1}\cdots\mathrm{E}_1$ as a splice of n short exact sequences

$$\mathrm{E}_i: \qquad 0 \longrightarrow X_{i+1} \longrightarrow Y_i \longrightarrow X_i \longrightarrow 0$$

in $\mathcal{A}$, where $X_{n+1} = X$ and $X_1 = Z$. From Theorem 1.2.2 we know that

$$0 \longrightarrow \mathcal{A}(-,X_2)|_{\mathcal{C}} \longrightarrow \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z) \xrightarrow{\partial_0(-,\mathrm{E}_1)|_{\mathcal{C}}} \mathrm{Ext}^1_{\mathcal{A}}(-,X_2)|_{\mathcal{C}}$$

is an exact sequence in Mod $\mathcal{C}$, and thus if $n = 1$, then we have the result. Otherwise, from the fact that $\mathcal{C}$ is n-rigid we obtain by Theorem 1.2.2 that

$$\partial_{i-1}(-,\mathrm{E}_i)|_{\mathcal{C}} : \mathrm{Ext}^{i-1}_{\mathcal{A}}(-,X_i)|_{\mathcal{C}} \to \mathrm{Ext}^i_{\mathcal{A}}(-,X_{i+1})|_{\mathcal{C}}$$

is an isomorphism in Mod $\mathcal{C}$ for each $2 \leqslant i \leqslant n-1$, and also that it is a monomorphism for $i = n$. Consequently, if we take

$$\phi = \partial_{n-1}(-,\mathrm{E}_n)|_{\mathcal{C}} \cdots \partial_1(-,\mathrm{E}_2)|_{\mathcal{C}}\partial_0(-,\mathrm{E}_1)|_{\mathcal{C}} : \mathcal{C}(-,Z) \to \mathrm{Ext}^n_{\mathcal{A}}(-,X)|_{\mathcal{C}},$$

then we have that

$$\mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z) \xrightarrow{\phi} \mathrm{Ext}^n_{\mathcal{A}}(-,X)|_{\mathcal{C}}$$

is exact in Mod $\mathcal{C}$, and it is easy to see that $\phi = \phi(-,\mathrm{E})$.

(b) This follows dually from the proof of item (a). □

As a consequence of Proposition 4.1.4, we obtain the following result, which was motivated by [25, Proposition 3.3]:

Corollary 4.1.5. *Suppose that $\mathcal{C}$ is Krull–Schmidt and n-rigid. If*

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$, then $\mathtt{E}$ generates both a simple submodule of the right $\mathrm{End}_{\mathcal{C}}(Z)$-module $\mathrm{Ext}^n_{\mathcal{A}}(Z,X)$ and a simple submodule of the left $\mathrm{End}_{\mathcal{C}}(X)$-module $\mathrm{Ext}^n_{\mathcal{A}}(Z,X)$.

Proof. This follows from propositions 3.3.4 and 4.1.4. □

We observe that in Proposition 6.3.3 we will present an improved version of Corollary 4.1.5 for the case when $\mathcal{C}$ is an "n-cluster tilting subcategory of mod Λ", where Λ is an Artin algebra.

Now, taking Corollary 4.1.5 into account, we see that if $\mathcal{C}$ is Krull–Schmidt and n-rigid, and if

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$, then $\mathtt{E}$ is somehow very close of being zero in $\mathrm{Ext}^n_{\mathcal{A}}(Z,X)$. This should help the reader to gain some intuition on n-almost split sequences in $\mathcal{C}$.

The next result may shed some light on the idea presented in the above paragraph.

Proposition 4.1.6. *Assume that $\mathcal{C}$ is n-rigid and let*

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$ with terms in $\mathcal{C}$. Then the following are equivalent:

(a) $\mathtt{E}$ *is zero in* $\mathrm{Ext}^n_{\mathcal{A}}(Z,X)$.

(b) f_1 *is a split epimorphism.*

(c) f_{n+1} *is a split monomorphism.*

Proof. To begin with, note that if f_1 is a split epimorphism or f_{n+1} is a split monomorphism, then we already know from Proposition 1.2.1 that $\mathtt{E}$ is zero in $\mathrm{Ext}^n_{\mathcal{A}}(Z,X)$.

Conversely, suppose that $\mathtt{E}$ is zero in $\mathrm{Ext}^n_{\mathcal{A}}(Z,X)$. Well, since we know from Proposition 4.1.4 that

$$\mathcal{C}(Z,Y_1) \xrightarrow{\mathcal{C}(Z,f_1)} \mathcal{C}(Z,Z) \xrightarrow{\phi(Z,\mathtt{E})} \mathrm{Ext}^n_{\mathcal{A}}(Z,X)$$

is exact in Ab, and given that $\mathtt{E} = \mathtt{E}\, id_Z = \phi(Z,\mathtt{E})(id_Z)$, we obtain that $id_Z \in \mathrm{Im}\,\mathcal{C}(Z,f_1)$. From this, it is easy to see that f_1 is a split epimorphism. Dually, we can conclude that f_{n+1} is a split monomorphism. □

To conclude this section, we give the following consequence of Proposition 4.1.6:

Corollary 4.1.7. *If $\mathcal{C}$ is Krull–Schmidt and n-rigid, then there is no m-almost split sequence in $\mathcal{C}$ for* $0 < m < n$.

Proof. Note that if

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{m+1}} Y_m \xrightarrow{f_m} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$ with terms in $\mathcal{C}$ and $0 < m < n$, then $\mathtt{E} \in \operatorname{Ext}^m_{\mathcal{A}}(Z,X) = 0$. Consequently, given that $\mathcal{C}$ is m-rigid, it follows from Proposition 4.1.6 that f_1 is a split epimorphism, and thus $\mathtt{E}$ is not an m-almost split sequence in $\mathcal{C}$. $\square$

Therefore, we see from Corollary 4.1.7 that if $\mathcal{C}$ is Krull–Schmidt and n-rigid, then the least positive integer m for which $\mathcal{C}$ may have m-almost split sequences is $m = n$.

4.2 n-Cluster tilting subcategories

We now introduce other conditions on $\mathcal{C}$ that facilitate the existence of n-almost split sequences in $\mathcal{C}$, in case that $\mathcal{C}$ is Krull–Schmidt, which will lead us to the notion of an "n-cluster tilting subcategory". In particular, we prove that if $\mathcal{C}$ is such a subcategory which is also Krull–Schmidt, then the only positive integer m such that $\mathcal{C}$ may have m-almost split sequences is $m = n$.

To begin with, recall that if $\mathcal{C}$ is Krull–Schmidt, then we have by Proposition 3.4.3 a useful condition to decide whether or not the simple $\mathcal{C}$-modules admit minimal projective resolutions whose terms are in $\mathcal{P}(\mathcal{C})$ if we assume that $\mathcal{C}$ is right coherent. Likewise, we also have the dual condition if we suppose that $\mathcal{C}$ is left coherent. Therefore, it would be interesting to have that $\mathcal{C}$ is both right and left coherent to study the existence of n-almost split sequences in $\mathcal{C}$. In this case, note that it follows from propositions 3.1.9 and 3.5.1 that to assure this property, it suffices to demand that $\mathcal{C}$ is functorially finite in $\mathcal{A}$. The reader should note that demanding this condition on $\mathcal{C}$ is very convenient since in this case we would be able to use the results of Section 3.5, which can be very useful.

Now, suppose that $\mathcal{C}$ is Krull–Schmidt, functorially finite in $\mathcal{A}$ and that $\mathcal{C}$ also has almost split morphisms. Then we know from the above paragraph and Proposition 3.4.3 that for a given $Z \in \operatorname{ind}\mathcal{C}$ the simple $\mathcal{C}$-module S_Z admits a minimal projective resolution whose terms are representable modules. Suppose that such resolution has finitely many nonzero terms, being described below,

$$0 \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f_{n+1})} \mathcal{C}(-,Y_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z) \longrightarrow S_Z \longrightarrow 0$$

and also that the object X appearing in its last term is indecomposable. In this case, if

$$0 \longrightarrow \mathcal{C}(Z,-) \xrightarrow{\mathcal{C}(f_1,-)} \mathcal{C}(Y_1,-) \xrightarrow{\mathcal{C}(f_2,-)} \cdots \xrightarrow{\mathcal{C}(f_n,-)} \mathcal{C}(Y_n,-) \xrightarrow{\mathcal{C}(f_{n+1},-)} \mathcal{C}(X,-) \longrightarrow S^X \longrightarrow 0$$

is a minimal projective resolution of S^X in Mod $\mathcal{C}^{\mathrm{op}}$, then to conclude that

$$0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$, we would only need to verify that it is an exact sequence in $\mathcal{A}$. Hence it would be convenient to have some property on $\mathcal{C}$ which guarantees that sequences as above, i.e. n-exact sequences in $\mathcal{C}$, are always exact in $\mathcal{A}$. As we will see in Proposition 4.2.1, one such property is that $\mathcal{C}$ is both "generating" and "cogenerating" in $\mathcal{A}$, which are concepts that we define below.

We say that $\mathcal{C}$ is **generating in** $\mathcal{A}$ if for every $Y \in \mathcal{A}$ there is some epimorphism $X \to Y$ in $\mathcal{A}$ with $X \in \mathcal{C}$. Dually, we say that $\mathcal{C}$ is **cogenerating in** $\mathcal{A}$ if for every $Y \in \mathcal{A}$ there is some monomorphism $Y \to X$ in $\mathcal{A}$ with $X \in \mathcal{C}$. The reader should note that $\mathcal{C}$ is cogenerating in $\mathcal{A}$ if and only if $\mathcal{C}^{\mathrm{op}}$ is generating in $\mathcal{A}^{\mathrm{op}}$. Therefore, we present here some results only for the case when $\mathcal{C}$ is generating in $\mathcal{A}$, given that the cogenerating case will be recovered by taking opposite categories.

We thank Gustavo Jasso for pointing out the next result.

Proposition 4.2.1. *If $\mathcal{C}$ is generating in $\mathcal{A}$, then a sequence*

$$\mathrm{S}: \qquad X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{A}$ is exact provided that

$$\mathrm{S}_{\mathcal{A},\mathcal{C}}: \qquad \mathcal{A}(-,X)|_{\mathcal{C}} \xrightarrow{\mathcal{A}(-,f)|_{\mathcal{C}}} \mathcal{A}(-,Y)|_{\mathcal{C}} \xrightarrow{\mathcal{A}(-,g)|_{\mathcal{C}}} \mathcal{A}(-,Z)|_{\mathcal{C}}$$

is exact in Mod $\mathcal{C}$.

Proof. Assume that $\mathrm{S}_{\mathcal{A},\mathcal{C}}$ is exact in Mod $\mathcal{C}$. To prove that S is exact in $\mathcal{A}$, we begin by showing that $gf = 0$.

Well, let $u : V \to X$ be an epimorphism in $\mathcal{A}$ with $V \in \mathcal{C}$. Then because we have $0 = \mathcal{A}(V,g)\mathcal{A}(V,f)(u) = gfu$, we conclude that $gf = 0$.

Next, take an epimorphism $W \to \operatorname{Ker} g$ in $\mathcal{A}$ with $W \in \mathcal{C}$ and consider the induced exact sequence

$$W \xrightarrow{h} Y \xrightarrow{g} Z$$

in $\mathcal{A}$. Since

$$\mathcal{A}(W,X) \xrightarrow{\mathcal{A}(W,f)} \mathcal{A}(W,Y) \xrightarrow{\mathcal{A}(W,g)} \mathcal{A}(W,Z)$$

is exact in Ab and $\mathcal{A}(W,g)(h) = 0$, we obtain that there is some morphism $k \in \mathcal{A}(W,X)$ such that $h = fk$. Using this fact the reader may easily verify that

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is an exact sequence in $\mathcal{A}$ in case that $\mathcal{A} = \mathrm{Ab}$, which is sufficient to complete the proof by [32, Chapter VI, Metatheorem 7.3]. $\square$

Therefore, it follows from Proposition 4.2.1 and its dual that if $\mathcal{C}$ is generating and cogenerating in $\mathcal{A}$, then every n-exact sequence in $\mathcal{C}$ is exact in $\mathcal{A}$.

Now, suppose that $\mathcal{C}$ is Krull–Schmidt, n-rigid, generating, cogenerating and functorially finite in $\mathcal{A}$. Then, recall from Corollary 4.1.7 that there is no m-almost split sequence in $\mathcal{C}$ for $0 < m < n$. Furthermore, given that we would like to introduce appropriate conditions on $\mathcal{C}$ so that it is more likely to have n-almost split sequences, it would be convenient to impose some condition on $\mathcal{C}$ which eliminates the possibility of the existence of m-almost split sequences in $\mathcal{C}$ for $m > n$, so that the only possible positive integer m for which it would be possible that there is an m-almost split sequence in $\mathcal{C}$ would be $m = n$. Well, one way of doing this is by demanding that $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}) \leqslant n+1$ or $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}^{\mathrm{op}}) \leqslant n+1$, and as we will see in Proposition 4.2.3, to have these properties, it is sufficient to suppose that $\mathcal{C}^{\perp n} \subseteq \mathcal{C}$ and ${}^{\perp n}\mathcal{C} \subseteq \mathcal{C}$, respectively.

Before we prove Proposition 4.2.3, we first need the following lemma, which was motivated by [25, Item 2.2.1(2)]:

Lemma 4.2.2. *Suppose that $\mathcal{C}$ is n-rigid, generating and contravariantly finite in $\mathcal{A}$. If $\mathcal{C}^{\perp n} \subseteq \mathcal{C}$, then* $\operatorname{pd}_{\mathcal{C}} \mathcal{A}(-,X)|_{\mathcal{C}} \leqslant n-1$ *for every $X \in \mathcal{A}$.*

Proof. Note that if $n = 1$, then $\mathcal{C}^{\perp n} = \mathcal{A}$, and thus we have the result. Therefore, assume in the following that $n \geqslant 2$.

To begin with, recall from propositions 3.1.9 and 3.5.1 and Theorem 3.1.7 that $\operatorname{mod}\mathcal{C}$ is an abelian category and such that the inclusion $\operatorname{mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}$ is exact. Now, if $X \in \mathcal{A}$, then we know by Proposition 3.5.1 that $\mathcal{A}(-,X)|_{\mathcal{C}} \in \operatorname{mod}\mathcal{C}$, and thus it admits a projective resolution

$$\cdots \xrightarrow{\mathcal{C}(-,f_3)} \mathcal{C}(-,Y_2) \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Y_0) \xrightarrow{\mathcal{A}(-,f_0)|_{\mathcal{C}}} \mathcal{A}(-,X)|_{\mathcal{C}} \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}$ with $Y_i \in \mathcal{C}$. In this case, it follows from Proposition 4.2.1 that

$$\mathtt{E}: \qquad \cdots \xrightarrow{f_3} Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$. Next, consider the short exact sequences

$$\mathtt{E}_i: \qquad 0 \longrightarrow X_{i+1} \longrightarrow Y_i \longrightarrow X_i \longrightarrow 0$$

in $\mathcal{A}$ induced by $\mathtt{E}$ for each $i \geqslant 0$, where $X_0 = X$. It is not difficult to see that for each i the sequence

$$0 \longrightarrow \mathcal{A}(-,X_{i+1})|_{\mathcal{C}} \longrightarrow \mathcal{C}(-,Y_i) \longrightarrow \mathcal{A}(-,X_i)|_{\mathcal{C}} \longrightarrow 0$$

obtained by $\mathtt{E}_i$ is exact in $\operatorname{Mod}\mathcal{C}$ since it coincides with the short exact sequences induced by the initial projective resolution of $\mathcal{A}(-,X)|_{\mathcal{C}}$. Therefore, we can conclude by Theorem 1.2.2 that for each $0 < i < n$ we have that $\operatorname{Ext}^k_{\mathcal{A}}(W,X_i) = 0$ for every $0 < k < i+1$ and $W \in \mathcal{C}$, and thus we obtain that $X_{n-1} \in \mathcal{C}^{\perp n}$. Hence $\mathcal{A}(-,X_{n-1})|_{\mathcal{C}} = \mathcal{C}(-,X_{n-1}) \in \mathcal{P}(\mathcal{C})$, which implies that $\operatorname{pd}_{\mathcal{C}} \mathcal{A}(-,X)|_{\mathcal{C}} \leqslant n-1$. □

Proposition 4.2.3. *Suppose that $\mathcal{C}$ is n-rigid, generating and contravariantly finite in $\mathcal{A}$. If $\mathcal{C}^{\perp n} \subseteq \mathcal{C}$, then* $\operatorname{gl.dim}(\operatorname{mod} \mathcal{C}) \leqslant n+1$.

Proof. Let $F \in \operatorname{mod} \mathcal{C}$ and take a projective presentation

$$\mathcal{C}(-,Y) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-,Z) \longrightarrow F \longrightarrow 0$$

of F in $\operatorname{Mod} \mathcal{C}$ with $Y, Z \in \mathcal{C}$. If we consider the exact sequence

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{f} Z$$

in $\mathcal{A}$, then by Proposition 2.2.5 we obtain that

$$0 \longrightarrow \mathcal{A}(-,X)|_{\mathcal{C}} \longrightarrow \mathcal{C}(-,Y) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-,Z)$$

is exact in $\operatorname{Mod} \mathcal{C}$, and thus it follows from Lemma 4.2.2 that $\operatorname{pd}_{\mathcal{C}} F \leqslant n+1$. Therefore, $\operatorname{gl.dim}(\operatorname{mod} \mathcal{C}) \leqslant n+1$. □

We can now give the definition of an "n-cluster tilting subcategory of $\mathcal{A}$", which is, as we will see, a subcategory of $\mathcal{A}$ satisfying the conditions that we have discussed so far.

Following [28, Definition 3.14], we say that $\mathcal{C}$ is an **n-cluster tilting subcategory of** $\mathcal{A}$ if $\mathcal{C}$ is generating, cogenerating, functorially finite in $\mathcal{A}$ and if $\mathcal{C}$ also satisfies that $\mathcal{C} = \mathcal{C}^{\perp n}$ and $\mathcal{C} = {}^{\perp n}\mathcal{C}$. We remark that these subcategories were first introduced by Osamu Iyama in [25, Definition 2.2].

The reader should note that the unique 1-cluster tilting subcategory of $\mathcal{A}$ is $\mathcal{A}$ itself. Moreover, if $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$, then it is also n-rigid, and by $\mathcal{C} = {}^{\perp n}\mathcal{C}$ we obtain that $\mathcal{C}$ contains all projective objects of $\mathcal{A}$, as well as that by $\mathcal{C} = \mathcal{C}^{\perp n}$ we conclude that $\mathcal{C}$ contains all injective objects of $\mathcal{A}$.

As a consequence of what we have done so far, we obtain the following:

Corollary 4.2.4. *If $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$ which is also Krull–Schmidt, then there is no m-almost split sequence in $\mathcal{C}$ for $m \neq n$.*

Proof. This is straightforward from Corollary 4.1.7 and Proposition 4.2.3. □

Therefore, we see from Corollary 4.2.4 that the study of m-almost split sequences in n-cluster tilting subcategories of $\mathcal{A}$ which are also Krull–Schmidt is restricted to the case $m = n$. This gives us hope that such subcategories may have n-almost split sequences, and in fact, we will prove in Theorem 5.1.12 that this is indeed the case when $\mathcal{A} = \operatorname{mod} \Lambda$ for an Artin algebra Λ.

We end this section by giving some propositions which may be useful to decide whether or not a certain $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$. The next results were motivated by [25, Proposition 2.2.2] (see also [26, Proposition 2.2]).

Lemma 4.2.5. *If $\mathcal{C}$ is generating and contravariantly finite in $\mathcal{A}$ and satisfies that $\mathcal{C} = \mathcal{C}^{\perp n}$, then $\mathcal{C} = {}^{\perp n}\mathcal{C}$.*

Proof. First, note that if f $n = 1$, then $\mathcal{C}^{\perp n} = \mathcal{A} = {}^{\perp n}\mathcal{C}$, and there is nothing to prove. Therefore, assume in the following that $n \geqslant 2$.

Since $\mathcal{C} = \mathcal{C}^{\perp n}$, we have that $\mathcal{C}$ is n-rigid, and thus $\mathcal{C} \subseteq {}^{\perp n}\mathcal{C}$. Hence to conclude that $\mathcal{C} = {}^{\perp n}\mathcal{C}$, we only need to verify that ${}^{\perp n}\mathcal{C} \subseteq \mathcal{C}$.

Well, let $X \in {}^{\perp n}\mathcal{C}$. It follows from Lemma 4.2.2 that $\operatorname{pd}_{\mathcal{C}} \mathcal{A}(-,X)|_{\mathcal{C}} \leqslant n-1$, so that there is a projective resolution

$$0 \longrightarrow \mathcal{C}(-,Y_{n-1}) \xrightarrow{\mathcal{C}(-,f_{n-1})} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Y_0) \xrightarrow{\mathcal{A}(-,f_0)|_{\mathcal{C}}} \mathcal{A}(-,X)|_{\mathcal{C}} \longrightarrow 0$$

of $\mathcal{A}(-,X)|_{\mathcal{C}}$ in $\operatorname{Mod} \mathcal{C}$ with $Y_i \in \mathcal{C}$. In this case, we conclude by Proposition 4.2.1 that

$$\mathrm{E}: \qquad 0 \longrightarrow Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} X \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$.

Now, consider the short exact sequence

$$\mathrm{E}_0: \qquad 0 \longrightarrow X_1 \longrightarrow Y_0 \xrightarrow{f_0} X \longrightarrow 0$$

in $\mathcal{A}$. If $n = 2$, then $X_1 = Y_1$ and from $\operatorname{Ext}^1_{\mathcal{A}}(X, Y_1) = 0$ we obtain that E_0 splits, so that $X \in \mathcal{C}$ and consequently ${}^{\perp n}\mathcal{C} \subseteq \mathcal{C}$. Otherwise, write $\mathrm{E} = \mathrm{E}'\mathrm{E}_0$ as a splice of the exact sequence

$$\mathrm{E}': \qquad 0 \longrightarrow Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} Y_1 \longrightarrow X_1 \longrightarrow 0$$

and E_0. Then by applying Lemma 4.1.1 in E' we obtain that $\operatorname{Ext}^1_{\mathcal{A}}(X, X_1) = 0$, and as above we conclude that ${}^{\perp n}\mathcal{C} \subseteq \mathcal{C}$. □

Proposition 4.2.6. *If $\mathcal{C}$ is generating, cogenerating and functorially finite in $\mathcal{A}$, then the following are equivalent:*

(a) *$\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$.*

(b) *$\mathcal{C} = \mathcal{C}^{\perp n}$.*

(c) *$\mathcal{C} = {}^{\perp n}\mathcal{C}$.*

Proof. This follows from Lemma 4.2.5 and its dual. □

For the next result, note that if $\mathcal{A}$ has enough projectives, then $\mathcal{C}$ is generating in $\mathcal{A}$ if and only if $\mathcal{C}$ contains all projective objects of $\mathcal{A}$. Dually, observe that if $\mathcal{A}$ has enough injectives, then $\mathcal{C}$ is cogenerating in $\mathcal{A}$ if and only if $\mathcal{C}$ contains all injective objects of $\mathcal{A}$.

Proposition 4.2.7. *Assume that $\mathcal{A}$ has enough projectives and injectives. If $\mathcal{C}$ is functorially finite in $\mathcal{A}$, then the following are equivalent:*

(a) $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$.

(b) $\mathcal{C} = \mathcal{C}^{\perp n}$ and $\mathcal{C}$ is generating in $\mathcal{A}$.

(c) $\mathcal{C} = {}^{\perp n}\mathcal{C}$ and $\mathcal{C}$ is cogenerating in $\mathcal{A}$.

Proof. Note that (a) implies both (b) and (c). Therefore, to prove the result, it suffices to verify that (b) is equivalent to (c). Considering this, in what follows we only prove that (b) implies (c) since the converse will follow by duality. Moreover, observe that if $n = 1$, then $\mathcal{C}^{\perp n} = \mathcal{A} = {}^{\perp n}\mathcal{C}$, and there is nothing to prove. Hence assume in the following that $n \geqslant 2$.

Well, if $\mathcal{C} = \mathcal{C}^{\perp n}$ and $\mathcal{C}$ is generating in $\mathcal{A}$, then from Lemma 4.2.5 we obtain that $\mathcal{C} = {}^{\perp n}\mathcal{C}$. Furthermore, from $\mathcal{C} = \mathcal{C}^{\perp n}$ we obtain that $\mathcal{C}$ contains all injective objects of $\mathcal{A}$, and thus $\mathcal{C}$ is cogenerating in $\mathcal{A}$. □

4.3 Characterizations of n-almost split sequences

In this section we give many equivalent conditions for an exact sequence in $\mathcal{A}$ of length n with terms in $\mathcal{C}$ to be an n-almost split sequence in $\mathcal{C}$, by assuming that $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$ which is also Krull–Schmidt. Under these assumptions, we also prove that if $\mathcal{A}$ has enough projectives and injectives and if $\mathcal{C}$ has almost split morphisms, then $\mathcal{C}$ has n-almost split sequences.

Before we characterize n-almost split sequences in n-cluster tilting subcategories of $\mathcal{A}$ which are also Krull–Schmidt, we first need some lemmas.

The next result is from [28, Proposition 4.8] and the proof that we present here is based on that of [29, Proposition 2.18].

Lemma 4.3.1. *Suppose that $\mathcal{C}$ is generating, contravariantly finite in $\mathcal{A}$ and that $\mathcal{C} = \mathcal{C}^{\perp n}$. If*

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$ with terms in $\mathcal{C}$, then every morphism $u : W \to Z$ in $\mathcal{C}$ can be extended to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "g_{n+1}"] \arrow[d, "id"] & W_n \arrow[r, "g_n"] \arrow[d, "u_n"] & \cdots \arrow[r, "g_2"] & W_1 \arrow[r, "g_1"] \arrow[d, "u_1"] & W \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & X \arrow[r, "f_{n+1}"'] & Y_n \arrow[r, "f_n"'] & \cdots \arrow[r, "f_2"'] & Y_1 \arrow[r, "f_1"'] & Z \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{C}$ whose top row is exact in $\mathcal{A}$.

Proof. Note that if $n = 1$, then $\mathcal{C}^{\perp n} = \mathcal{A}$, and thus it suffices to take the diagram formed by the pullback of u and f_1. Therefore, assume in the following that $n \geqslant 2$.

To obtain a commutative diagram as described above, first let

$$\begin{tikzcd} U_1 \arrow[r, "g_1'"] \arrow[d, "u_1'"'] & W \arrow[d, "u"] \\ Y_1 \arrow[r, "f_1"'] & Z \end{tikzcd}$$

be the pullback of f_1 and u in $\mathcal{A}$. Since we have that $f_1 f_2 = 0 = u0$, we know that there is a unique morphism $h_2 : Y_2 \to U_1$ in $\mathcal{A}$ such that $u_1' h_2 = f_2$ and $g_1' h_2 = 0$.

$$\begin{tikzcd} Y_2 \arrow[dr, dashed, "h_2"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "f_2"'] & & \\ & U_1 \arrow[r, "g_1'"] \arrow[d, "u_1'"'] & W \arrow[d, "u"] \\ & Y_1 \arrow[r, "f_1"'] & Z \end{tikzcd}$$

Moreover, using the uniqueness of h_2 and the fact that $\mathtt{E}$ is exact, we can conclude that h_2 factors through the cokernel of f_3, though we leave the details for the reader, so that we have $h_2 f_3 = 0$. Next, take a right $\mathcal{C}$-approximation $w_1 : W_1 \to U_1$ of U_1 and consider the pullback

$$\begin{tikzcd} U_2 \arrow[r, "g_2'"] \arrow[d, "u_2'"'] & W_1 \arrow[d, "w_1"] \\ Y_2 \arrow[r, "h_2"'] & U_1 \end{tikzcd}$$

of h_2 and w_1 in $\mathcal{A}$. If we proceed with the above steps, then we obtain a commutative diagram

$$\begin{tikzcd} & & W_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "w_{n-1}"] & \cdots \arrow[r, "g_3"] & W_2 \arrow[r, "g_2"] \arrow[d, "w_2"] & W_1 \arrow[r, "g_1"] \arrow[d, "w_1"] & W \arrow[d, "u"] \\ & U_n \arrow[ur, "g_n'"] \arrow[d, "u_n'"] & U_{n-1} \arrow[ur, "g_{n-1}'"'] \arrow[d, "u_{n-1}'"] & \cdots \arrow[ur, "g_3'"'] & U_2 \arrow[ur, "g_2'"'] \arrow[d, "u_2'"] & U_1 \arrow[ur, "g_1'"'] \arrow[d, "u_1'"] & Z \\ X \arrow[ur, "h_{n+1}"] \arrow[r, "f_{n+1}"'] & Y_n \arrow[ur, "h_n"'] \arrow[r, "f_n"'] & Y_{n-1} \arrow[ur, "h_{n-1}"'] \arrow[r, "f_{n-1}"'] & \cdots \arrow[ur, "h_3"'] \arrow[r, "f_3"'] & Y_2 \arrow[ur, "h_2"'] \arrow[r, "f_2"'] & Y_1 \arrow[ur, "f_1"'] & \end{tikzcd}$$

in $\mathcal{A}$, where $g_i = g_i' w_i$, which give us a commutative diagram

$$\mathtt{D}: \quad \begin{tikzcd} 0 \arrow[r] & X \arrow[r, "h_{n+1}"] \arrow[d, "id"] & U_n \arrow[r, "g_n'"] \arrow[d, "u_n'"] & W_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "u_{n-1}"] & \cdots \arrow[r, "g_3"] & W_2 \arrow[r, "g_2"] \arrow[d, "u_2"] & W_1 \arrow[r, "g_1"] \arrow[d, "u_1"] & W \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & X \arrow[r, "f_{n+1}"'] & Y_n \arrow[r, "f_n"'] & Y_{n-1} \arrow[r, "f_{n-1}"'] & \cdots \arrow[r, "f_3"'] & Y_2 \arrow[r, "f_2"'] & Y_1 \arrow[r, "f_1"'] & Z \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{A}$ whose top row is a complex, where $u_i = u_i' w_i$.

Now, we will verify that $U_n \in \mathcal{C}$. For this, note that

$$\mathtt{F}_i : \qquad 0 \longrightarrow U_{i+1} \xrightarrow{\begin{pmatrix} g'_{i+1} \\ u'_{i+1} \end{pmatrix}} W_i \oplus Y_{i+1} \xrightarrow{(-w_i \ h_{i+1})} U_i \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$ for each $1 \leqslant i \leqslant n-1$. Indeed, this follows from the fact that

$$\begin{tikzcd} U_{i+1} \arrow[r, "g'_{i+1}"] \arrow[d, "u'_{i+1}"'] & W_i \arrow[d, "w_i"] \\ Y_{i+1} \arrow[r, "h_{i+1}"'] & U_i \end{tikzcd}$$

is a pullback diagram, and also because w_i is an epimorphism, given that w_i is a right $\mathcal{C}$-approximation, and thus we know that it is an epimorphism by Proposition 4.2.1. Furthermore, it follows from Proposition 2.2.5 and the fact that w_i is a right $\mathcal{C}$-approximation that the sequence

$$0 \longrightarrow \mathcal{A}(-, U_{i+1})|_{\mathcal{C}} \longrightarrow \mathcal{C}(-, W_i \oplus Y_{i+1}) \longrightarrow \mathcal{A}(-, U_i)|_{\mathcal{C}} \longrightarrow 0$$

induced by $\mathtt{F}_i$ is exact in Mod $\mathcal{C}$ for each $1 \leqslant i \leqslant n-1$. Therefore, if we consider the exact sequence

$$0 \longrightarrow U_n \longrightarrow W_{n-1} \oplus Y_n \longrightarrow \cdots \longrightarrow W_1 \oplus Y_2 \longrightarrow U_1 \longrightarrow 0$$

obtained by the splice of the sequences $\mathtt{F}_i$, then we conclude by the proof of Lemma 4.2.2 that $U_n \in \mathcal{C}$. Hence $\mathtt{D}$ is a commutative diagram in $\mathcal{C}$.

Finally, we will verify that the top row of $\mathtt{D}$ is an exact sequence in $\mathcal{A}$. For this, note that by the same reasoning as before, we have that

$$\mathtt{F}_0 : \qquad 0 \longrightarrow U_1 \xrightarrow{\begin{pmatrix} g'_1 \\ u'_1 \end{pmatrix}} W \oplus Y_1 \xrightarrow{(-u \ f_1)} Z \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$, and it is also easy to see that

$$\mathtt{F}_n : \qquad 0 \longrightarrow X \xrightarrow{\begin{pmatrix} -h_{n+1} \\ -id \end{pmatrix}} U_n \oplus X \xrightarrow{(-id \ h_{n+1})} U_n \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$. Hence the sequence

$$0 \longrightarrow X \longrightarrow U_n \oplus X \longrightarrow W_{n-1} \oplus Y_n \longrightarrow \cdots \longrightarrow W_1 \oplus Y_2 \longrightarrow W \oplus Y_1 \longrightarrow Z \longrightarrow 0$$

obtained by the splice of all the sequences $\mathtt{F}_i$ is exact in $\mathcal{A}$. But observe that the above sequence is precisely the mapping cone of the morphism of complexes which appears in the diagram $\mathtt{D}$. Therefore, it follows from Proposition 1.3.1 that this morphism of complexes is a quasi-isomorphism. Consequently, since $\mathtt{E}$ is an exact sequence, we conclude that so is the top row of $\mathtt{D}$. $\square$

Lemma 4.3.2. *If*

$$\begin{tikzcd} 0 \arrow[r] & Y_{n+1} \arrow[r, "f_{n+1}"] & Y_n \arrow[r, "f_n"] & \cdots \arrow[r, "f_3"] & Y_2 \arrow[r, "f_2"] & Y_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & Y_0 \arrow[r] \arrow[d, "h_0"] & 0 \\ 0 \arrow[r] & W_{n+1} \arrow[r, "g_{n+1}"'] & W_n \arrow[r, "g_n"'] & \cdots \arrow[r, "g_3"'] & W_2 \arrow[r, "g_2"'] & W_1 \arrow[r, "g_1"'] & W_0 \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram in $\mathcal{C}$ whose rows are left n-exact sequences in $\mathcal{C}$, then we can extend it to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "h_{n+1}"] & Y_n \arrow[r, "f_n"] \arrow[d, "h_n"] & \cdots \arrow[r, "f_3"] & Y_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & Y_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & Y_0 \arrow[r] \arrow[d, "h_0"] & 0 \\ 0 \arrow[r] & W_{n+1} \arrow[r, "g_{n+1}"'] & W_n \arrow[r, "g_n"'] & \cdots \arrow[r, "g_3"'] & W_2 \arrow[r, "g_2"'] & W_1 \arrow[r, "g_1"'] & W_0 \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{C}$.

Proof. It is not difficult to see that we can consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}(-,Y_{n+1}) \arrow[r, "\mathcal{C}(-,f_{n+1})"] & \mathcal{C}(-,Y_n) \arrow[r, "\mathcal{C}(-,f_n)"] & \cdots \arrow[r, "\mathcal{C}(-,f_3)"] & \mathcal{C}(-,Y_2) \arrow[r, "\mathcal{C}(-,f_2)'"] & F \arrow[r] \arrow[d, "\mathcal{C}(-,h_1)'"] & 0 \\ 0 \arrow[r] & \mathcal{C}(-,W_{n+1}) \arrow[r, "\mathcal{C}(-,g_{n+1})"'] & \mathcal{C}(-,W_n) \arrow[r, "\mathcal{C}(-,g_n)"'] & \cdots \arrow[r, "\mathcal{C}(-,g_3)"'] & \mathcal{C}(-,W_2) \arrow[r, "\mathcal{C}(-,g_2)'"'] & G \arrow[r] & 0 \end{tikzcd}$$

in Mod $\mathcal{C}$ whose rows are exact, where $\mathcal{C}(-,f_2)'$, $\mathcal{C}(-,g_2)'$ and $\mathcal{C}(-,h_1)'$ are the morphisms induced by $\mathcal{C}(-,f_2)$, $\mathcal{C}(-,g_2)$ and $\mathcal{C}(-,h_1)$, respectively. Therefore, it follows from the Comparison Theorem that there are morphisms $h_i : Y_i \to W_i$ in $\mathcal{C}$ such that

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}(-,Y_{n+1}) \arrow[r, "\mathcal{C}(-,f_{n+1})"] \arrow[d, "\mathcal{C}(-,h_{n+1})"] & \mathcal{C}(-,Y_n) \arrow[r, "\mathcal{C}(-,f_n)"] \arrow[d, "\mathcal{C}(-,h_n)"] & \cdots \arrow[r, "\mathcal{C}(-,f_3)"] & \mathcal{C}(-,Y_2) \arrow[r, "\mathcal{C}(-,f_2)'"] \arrow[d, "\mathcal{C}(-,h_2)"] & F \arrow[r] \arrow[d, "\mathcal{C}(-,h_1)'"] & 0 \\ 0 \arrow[r] & \mathcal{C}(-,W_{n+1}) \arrow[r, "\mathcal{C}(-,g_{n+1})"'] & \mathcal{C}(-,W_n) \arrow[r, "\mathcal{C}(-,g_n)"'] & \cdots \arrow[r, "\mathcal{C}(-,g_3)"'] & \mathcal{C}(-,W_2) \arrow[r, "\mathcal{C}(-,g_2)'"'] & G \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram in Mod $\mathcal{C}$, which gives us a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}(-,Y_{n+1}) \arrow[r, "\mathcal{C}(-,f_{n+1})"] \arrow[d, "\mathcal{C}(-,h_{n+1})"] & \mathcal{C}(-,Y_n) \arrow[r, "\mathcal{C}(-,f_n)"] \arrow[d, "\mathcal{C}(-,h_n)"] & \cdots \arrow[r, "\mathcal{C}(-,f_3)"] & \mathcal{C}(-,Y_2) \arrow[r, "\mathcal{C}(-,f_2)"] \arrow[d, "\mathcal{C}(-,h_2)"] & \mathcal{C}(-,Y_1) \arrow[d, "\mathcal{C}(-,h_1)"] \\ 0 \arrow[r] & \mathcal{C}(-,W_{n+1}) \arrow[r, "\mathcal{C}(-,g_{n+1})"'] & \mathcal{C}(-,W_n) \arrow[r, "\mathcal{C}(-,g_n)"'] & \cdots \arrow[r, "\mathcal{C}(-,g_3)"'] & \mathcal{C}(-,W_2) \arrow[r, "\mathcal{C}(-,g_2)"'] & \mathcal{C}(-,W_1) \end{tikzcd}$$

in Mod $\mathcal{C}$. Hence by Theorem 2.2.4 we obtain that

$$\begin{tikzcd} 0 \arrow[r] & Y_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "h_{n+1}"] & Y_n \arrow[r, "f_n"] \arrow[d, "h_n"] & \cdots \arrow[r, "f_3"] & Y_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & Y_1 \arrow[d, "h_1"] \\ 0 \arrow[r] & W_{n+1} \arrow[r, "g_{n+1}"'] & W_n \arrow[r, "g_n"'] & \cdots \arrow[r, "g_3"'] & W_2 \arrow[r, "g_2"'] & W_1 \end{tikzcd}$$

is a commutative diagram in $\mathcal{C}$, and thus we are done. $\square$

We can now give the main result of this section, whose proof is based on a generalization of the proof of [12, Theorem 2.14] (see also [25, Proposition 3.3]).

Theorem 4.3.3. *Suppose that $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$ and let*

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$ with terms in $\mathcal{C}$ and $I = \{i \in \mathbb{N} \mid 2 \leqslant i \leqslant n\}$. The following are equivalent:

(a) *$f_i \in \mathrm{rad}_{\mathcal{C}}$ for each $i \in I$, f_1 is right almost split in $\mathcal{C}$ and f_{n+1} is left almost split in $\mathcal{C}$.*

(b) *$f_i \in \mathrm{rad}_{\mathcal{C}}$ for each $i \in I$, f_1 is right almost split in $\mathcal{C}$ and $f_{n+1} \in \mathrm{rad}_{\mathcal{C}}$.*

(c) *$f_i \in \mathrm{rad}_{\mathcal{C}}$ for each $i \in I$, $f_1 \in \mathrm{rad}_{\mathcal{C}}$ and f_{n+1} is left almost split in $\mathcal{C}$.*

(d) *f_i is right minimal for each $i \in I$ and f_1 is minimal right almost split in $\mathcal{C}$.*

(e) *f_i is left minimal for each $i \in I$ and f_{n+1} is minimal left almost split in $\mathcal{C}$.*

Furthermore, if $\mathcal{C}$ is Krull–Schmidt, then the above conditions are equivalent to

(f) *$\mathtt{E}$ is an n-almost split sequence in $\mathcal{C}$.*

Proof. First of all, note that we already know from Corollary 4.1.3 that if $\mathcal{C}$ is Krull–Schmidt, then (a) is equivalent to (f). Moreover, it follows from Proposition 4.1.2 that $\mathtt{E}$ is an n-exact sequence in $\mathcal{C}$, and thus by Lemma 3.2.2 and its dual we can conclude that (b) is equivalent to (d), as well as that (c) is equivalent to (e). Now, note that by Proposition 3.4.1 and its dual we also have that (a) implies (b), as well as that (a) implies (c). Therefore, to complete the proof, it suffices to show that (b) implies (a) and that (c) implies (a), and in what follows we only prove the former implication since the latter will follow by duality.

Assume that item (b) holds. To prove that item (a) is true we only need to show that f_{n+1} is left almost split in $\mathcal{C}$. Well, since f_1 is not a split epimorphism, it follows from Proposition 4.1.6 that f_{n+1} is not a split monomorphism. Next, let $u : X \to W$ be a morphism in $\mathcal{C}$ such that $u \neq v f_{n+1}$ for every morphism $v : Y_n \to W$ in $\mathcal{C}$. If we show that u is a split monomorphism, then we are done. Well, in order to verify this, first observe that it follows from the dual of Lemma 4.3.1 that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f_{n+1}"] \arrow[d, "u"] & Y_n \arrow[r, "f_n"] \arrow[d, "u_n"] & \cdots \arrow[r, "f_2"] & Y_1 \arrow[r, "f_1"] \arrow[d, "u_1"] & Z \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & W \arrow[r, "g_{n+1}"'] & W_n \arrow[r, "g_n"'] & \cdots \arrow[r, "g_2"'] & W_1 \arrow[r, "g_1"'] & Z \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{C}$ whose bottom row is exact in $\mathcal{A}$. In this case, we have that g_{n+1} is not a split monomorphism, otherwise there would exist a morphism $v : Y_n \to W$ in $\mathcal{C}$ such that $u = v f_{n+1}$.

Hence it follows from Proposition 4.1.6 that g_1 is not a split epimorphism, and thus there is a morphism $h_1 : W_1 \to Y_1$ in $\mathcal{C}$ such that $g_1 = f_1h_1$. Therefore, by Proposition 4.1.2 and Lemma 4.3.2 we conclude that there is a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & W & \xrightarrow{g_{n+1}} & W_n & \xrightarrow{g_n} & \cdots & \xrightarrow{g_2} & W_1 & \xrightarrow{g_1} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_{n+1}} & & \downarrow{\scriptstyle h_n} & & & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & X & \xrightarrow[f_{n+1}]{} & Y_n & \xrightarrow[f_n]{} & \cdots & \xrightarrow[f_2]{} & Y_1 & \xrightarrow[f_1]{} & Z & \longrightarrow & 0
\end{array}$$

in $\mathcal{C}$, and by combining the two diagrams above, we obtain a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X & \xrightarrow{f_{n+1}} & Y_n & \xrightarrow{f_n} & \cdots & \xrightarrow{f_2} & Y_1 & \xrightarrow{f_1} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_{n+1}u} & & \downarrow{\scriptstyle h_nu_n} & & & & \downarrow{\scriptstyle h_1u_1} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & X & \xrightarrow[f_{n+1}]{} & Y_n & \xrightarrow[f_n]{} & \cdots & \xrightarrow[f_2]{} & Y_1 & \xrightarrow[f_1]{} & Z & \longrightarrow & 0
\end{array}$$

in $\mathcal{C}$. Finally, because $\mathbb{E}$ is left n-exact in $\mathcal{C}$, we obtain a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathcal{C}(-,X) & \xrightarrow{\mathcal{C}(-,f_{n+1})} & \mathcal{C}(-,Y_n) & \xrightarrow{\mathcal{C}(-,f_n)} & \cdots & \xrightarrow{\mathcal{C}(-,f_2)} & \mathcal{C}(-,Y_1) & \xrightarrow{\mathcal{C}(-,f_1)'} & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{C}(-,h_{n+1}u)} & & \downarrow{\scriptstyle \mathcal{C}(-,h_nu_n)} & & & & \downarrow{\scriptstyle \mathcal{C}(-,h_1u_1)} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \mathcal{C}(-,X) & \xrightarrow[\mathcal{C}(-,f_{n+1})]{} & \mathcal{C}(-,Y_n) & \xrightarrow[\mathcal{C}(-,f_n)]{} & \cdots & \xrightarrow[\mathcal{C}(-,f_2)]{} & \mathcal{C}(-,Y_1) & \xrightarrow[\mathcal{C}(-,f_1)']{} & F & \longrightarrow & 0
\end{array}$$

in $\operatorname{Mod}\mathcal{C}$ with exact rows, where $\mathcal{C}(-,f_1)'$ is the morphism induced by $\mathcal{C}(-,f_1)$. Therefore, given that $f_i \in \operatorname{rad}_{\mathcal{C}}$ for each $2 \leqslant i \leqslant n+1$, we conclude by Theorem 3.2.3 that the rows of the above diagram are minimal projective resolutions of F in $\operatorname{Mod}\mathcal{C}$. Hence it follows from Lemma 1.4.3 that $\mathcal{C}(-,h_{n+1}u)$ is an isomorphism in $\operatorname{Mod}\mathcal{C}$, which implies that $h_{n+1}u$ is an isomorphism in $\mathcal{C}$, and thus u is a split monomorphism, as desired. □

We now proceed to study the existence of n-almost split sequences in n-cluster tilting subcategories of $\mathcal{A}$ which are Krull–Schmidt in the case that $\mathcal{A}$ has enough projectives or injectives.

First, we need the following lemma, which is well known:

Lemma 4.3.4. *Assume that $\mathcal{A}$ has enough projectives and that $\mathcal{C}$ is generating in $\mathcal{A}$. If a morphism $f : Y \to Z$ in $\mathcal{C}$ is right almost split in $\mathcal{C}$ and if Z is nonprojective, then f is an epimorphism in $\mathcal{A}$.*

Proof. Let $g : P \to Z$ be an epimorphism in $\mathcal{A}$ with P being projective. Since $\mathcal{C}$ is generating in $\mathcal{A}$, it follows that $P \in \mathcal{C}$, and thus g is a morphism in $\mathcal{C}$. Moreover, from the fact that Z is nonprojective we obtain that g is not a split epimorphism. Consequently, there is a morphism $h : P \to Y$ in $\mathcal{C}$ such that $g = fh$, and because g is an epimorphism in $\mathcal{A}$ we conclude that so is f. □

The next result is a generalization of [7, Chapter II, Corollary 4.5].

Theorem 4.3.5. *Assume that $\mathcal{A}$ has enough projectives and that $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$ which is also Krull–Schmidt. If $Z \in \mathcal{C}$ is indecomposable and nonprojective and if S_Z admits a minimal projective resolution*

$$0 \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f_{n+1})} \mathcal{C}(-,Y_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,Y_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,Z) \longrightarrow S_Z \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}$ *with each f_i being a morphism in $\mathcal{C}$, then*

$$\mathtt{E}: \qquad 0 \longrightarrow X \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$.

Proof. It follows from Proposition 3.4.1 that f_1 is right almost split in $\mathcal{C}$, and thus by Lemma 4.3.4 we obtain that f_1 is an epimorphism in $\mathcal{A}$. Hence we conclude by Proposition 4.2.1 that E is an exact sequence in $\mathcal{A}$. Finally, by Theorem 3.2.3 we also have that $f_i \in \operatorname{rad}_{\mathcal{C}}$ for each i, so that E is an n-almost split sequence in $\mathcal{C}$ by Theorem 4.3.3. □

As a consequence of Theorem 4.3.5 and its dual, we have the following result:

Theorem 4.3.6. *Assume that $\mathcal{A}$ has enough projectives and injectives and that $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$ which is also Krull–Schmidt. If $\mathcal{C}$ has almost split morphisms, then $\mathcal{C}$ has n-almost split sequences.*

Proof. Let $Z \in \mathcal{C}$ be indecomposable and nonprojective. It follows from Proposition 3.4.3 that the simple $\mathcal{C}$-module S_Z has a minimal projective resolution in $\operatorname{Mod}\mathcal{C}$ whose terms are in $\mathcal{P}(\mathcal{C})$. Therefore, by Proposition 4.2.3 and Theorem 4.3.5 we conclude that there is an n-almost split sequence in $\mathcal{C}$ whose right end term is Z.

Dually, we can verify that if $X \in \mathcal{C}$ is indecomposable and noninjective, then there is an n-almost split sequence in $\mathcal{C}$ whose left end term is X. □

Theorem 4.3.6 may be very useful to prove that certain n-cluster tilting subcategories have n-almost split sequences. In fact, we will use it to prove in Theorem 5.1.12 that if Λ is an Artin algebra, then every n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ has n-almost split sequences, and for this we will study the notion of a "dualizing R-variety" in Section 5.1. However, with the theory developed so far, we can give a proof of this fact for the case when such a subcategory is of finite type, as we show below.

Corollary 4.3.7. *If Λ is an Artin algebra and if $\mathcal{C}$ is an n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$ *of finite type, then $\mathcal{C}$ has n-almost split sequences.*

Proof. This follows from Proposition 1.6.1, Corollary 3.4.5 and Theorem 4.3.6. □

4.4 More on n-cluster tilting subcategories

To end this chapter, we now present some miscellaneous results on n-cluster tilting subcategories for the case when $\mathcal{A}$ has enough projectives or injectives, some of which will be useful for the development of the next chapter.

We begin with the following:

Proposition 4.4.1. *Assume that $\mathcal{A}$ has enough projectives and that $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$. If $X \in \mathcal{C}$, then either X is projective or* $\operatorname{pd}_{\mathcal{A}} X \geqslant n$.

Proof. This is straightforward from Proposition 1.5.2. $\square$

Before we give the next result, it is interesting to observe that if $\operatorname{gl.dim} \mathcal{A} = 0$, then $\operatorname{Ext}^i_{\mathcal{A}}(X, Y) = 0$ for every $i \geqslant 1$ and $X, Y \in \mathcal{A}$, hence every n-cluster tilting subcategory of $\mathcal{A}$ coincides with $\mathcal{A}$. Moreover, in this case we also have that $\mathcal{A}$ is an m-cluster tilting subcategory of itself for every positive integer m. Therefore, we see that this is an uninteresting case.

Proposition 4.4.2. *Assume that $\mathcal{A}$ has enough projectives and injectives and that $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$. If $d =$ $\operatorname{gl.dim} \mathcal{A}$ satisfies that $1 \leqslant d < \infty$, then $n \leqslant d$.*

Proof. Let $X \in \mathcal{A}$ be such that $\operatorname{id}_{\mathcal{A}} X = d$. Then it follows from the dual of Proposition 1.5.2 that there is some injective object $I \in \mathcal{A}$ such that $\operatorname{Ext}^d_{\mathcal{A}}(I, X) \neq 0$, and in particular we obtain that I is nonprojective. Therefore, given that $I \in \mathcal{C}$, we obtain from Proposition 4.4.1 that $n \leqslant \operatorname{pd}_{\mathcal{A}} I$, and thus $n \leqslant d$. $\square$

As a consequence of Proposition 4.4.2, we obtain that if $\mathcal{A}$ satisfies its assumptions, then there are only finitely many positive integers m such that there is an m-cluster tilting subcategory of $\mathcal{A}$.

Next, we give a result which will be needed to prove the Higher Morita–Tachikawa Correspondence, as well as the Higher Auslander Correspondence. But first we establish some notation.

For a skeletally small preadditive category $\mathcal{B}$, we will denote by $\mathcal{PI}(\mathcal{B})$ the subcategory of $\operatorname{mod} \mathcal{B}$ consisting of all objects which are both projective and injective in $\operatorname{mod} \mathcal{B}$.

Observe that, as we already know, the projective objects in $\operatorname{mod} \mathcal{C}$ coincide with $\mathcal{P}(\mathcal{C})$, so that every object in $\mathcal{PI}(\mathcal{C})$ is isomorphic to $\mathcal{C}(-, X)$ for some $X \in \mathcal{C}$. In the next proposition we will describe such objects $X \in \mathcal{C}$ by assuming that $\mathcal{A}$ has enough injectives and also by imposing some conditions on $\mathcal{C}$, which include the case when $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathcal{A}$.

The result below is a generalization of [4, Lemma, page 50].

Proposition 4.4.3. *Assume that $\mathcal{A}$ has enough injectives and that $\mathcal{C}$ is generating, cogenerating and contravariantly finite in $\mathcal{A}$. Given $F \in \operatorname{mod} \mathcal{C}$, we have that $F \in \mathcal{PI}(\mathcal{C})$ if and only if $F \simeq \mathcal{C}(-, W)$ for some $W \in \mathcal{C}$ which is injective in $\mathcal{C}$.*

Proof. If $F \in \mathcal{PI}(\mathcal{C})$, then we already know that there is some $W \in \mathcal{C}$ such that $F \simeq \mathcal{C}(-, W)$. To verify that W is injective in $\mathcal{C}$, let $f : X \to Y$ be a monomorphism in $\mathcal{C}$. It is easy to see that $\mathcal{C}(-, f) : \mathcal{C}(-,X) \to \mathcal{C}(-,Y)$ is a monomorphism in $\operatorname{Mod} \mathcal{C}$, hence if we apply the functor $\operatorname{Hom}_{\mathcal{C}}(-, \mathcal{C}(-, W))$ in this morphism, then we obtain an epimorphism in Ab. Furthermore, it follows from the Yoneda Lemma that we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,Y), \mathcal{C}(-,W)) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X), \mathcal{C}(-,W)) \arrow[r] \arrow[d, "\simeq"] & 0 \\
\mathcal{C}(Y, W) \arrow[r, "\mathcal{C}(f,W)"'] & \mathcal{C}(X, W) \arrow[r] & 0
\end{tikzcd}$$

in Ab whose vertical arrows are isomorphisms, and thus we conclude that $\mathcal{C}(f, W)$ is an epimorphism. Consequently, W is injective in $\mathcal{C}$.

Conversely, suppose that $F \simeq \mathcal{C}(-, W)$ for some $W \in \mathcal{C}$ which is injective in $\mathcal{C}$. To verify that $F \in \mathcal{PI}(\mathcal{C})$ we only need to show that F is injective in $\operatorname{mod} \mathcal{C}$ since we already know that F is projective in $\operatorname{mod} \mathcal{C}$. For this, observe that because $\mathcal{C}$ is contravariantly finite in $\mathcal{A}$, we have by propositions 3.1.9 and 3.5.1, and Theorem 3.1.7 that $\operatorname{mod} \mathcal{C}$ is an abelian category, so that for every $G, H \in \operatorname{mod} \mathcal{C}$ and $i \geqslant 1$ we can consider $\operatorname{Ext}^i_{\operatorname{mod} \mathcal{C}}(G, H)$, which we denote here simply by $\operatorname{Ext}^i(G, H)$. Therefore, to conclude that F is injective in $\operatorname{mod} \mathcal{C}$, it suffices to show that $\operatorname{Ext}^1(G, \mathcal{C}(-, W)) = 0$ for every $G \in \operatorname{mod} \mathcal{C}$.

Well, let $G \in \operatorname{mod} \mathcal{C}$ and take a projective resolution

$$\cdots \longrightarrow \mathcal{C}(-,X_2) \longrightarrow \mathcal{C}(-,X_1) \longrightarrow \mathcal{C}(-,X_0) \longrightarrow G \longrightarrow 0$$

of G in $\operatorname{mod} \mathcal{C}$ with $X_i \in \mathcal{C}$. In this case, it follows from Theorem 3.1.7 that the above sequence is exact in $\operatorname{Mod} \mathcal{C}$, and thus by Proposition 4.2.1 we obtain that

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

is an exact sequence in $\mathcal{A}$. Moreover, given that $\mathcal{A}$ has enough injectives and $\mathcal{C}$ is cogenerating in $\mathcal{A}$, we can conclude that W is injective in $\mathcal{A}$, though we leave the details for the reader. Hence we obtain that

$$\mathcal{C}(X_0, W) \longrightarrow \mathcal{C}(X_1, W) \longrightarrow \mathcal{C}(X_2, W) \longrightarrow \cdots$$

is an exact sequence in Ab. However, it follows from the Yoneda Lemma that the above sequence is isomorphic as a complex to

$$(\mathcal{C}(-,X_0), \mathcal{C}(-,W)) \longrightarrow (\mathcal{C}(-,X_1), \mathcal{C}(-,W)) \longrightarrow (\mathcal{C}(-,X_2), \mathcal{C}(-,W)) \longrightarrow \cdots ,$$

where $(\mathcal{C}(-,X_i), \mathcal{C}(-,W)) = \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X_i), \mathcal{C}(-,W))$, which then turns out to be exact in Ab. But this implies that $\operatorname{Ext}^i(G, \mathcal{C}(-, W)) = 0$ for every $i \geqslant 1$, and then we are done. □

5 The case of modules over Artin algebras

In this chapter, R will stand for a commutative artinian ring and n will denote a positive integer.

From now on in this text, we restrict our attention to n-cluster tilting subcategories of categories of finitely generated modules over Artin algebras. We show in this chapter, through the study of “dualizing R-varieties”, that such subcategories have both almost split morphisms and n-almost split sequences. Furthermore, we prove generalizations of both the Morita–Tachikawa Correspondence and of the Auslander Correspondence, as well as we present some open problems concerning n-cluster tilting subcategories in this setting of modules over Artin algebras.

5.1 Dualizing R-varieties

In this section we prove that if Λ is an Artin R-algebra and $\mathcal{C}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$, then $\mathcal{C}$ has almost split morphisms and n-almost split sequences. To achieve this goal, we study the notion of a “dualizing R-variety”, which was introduced by Maurice Auslander and Idun Reiten in [10]. We observe that the presentation of the content of this section is mainly based in [10].

First of all, it is convenient to tell the reader what the idea behind the concept of a “dualizing R-variety” is. Well, briefly, it is a categorical generalization of the concept of Artin R-algebra. We now proceed to give its definition and also to explain in which sense this generalization is given.

Remember from Section 1.6 that an R-algebra Λ is an Artin R-algebra if Λ is finitely generated as an R-module. In this case, if we view Λ as a preadditive category consisting of only one object as in Example 2.1.1, then we have that Λ is a finitely confined R-category. Now, recall from Proposition 2.7.6 that Λ and $\mathcal{P}(\Lambda)$ are Morita equivalent, so that to study the category $\operatorname{Mod}\Lambda$ we can consider instead the category $\operatorname{Mod}\mathcal{P}(\Lambda)$, and the advantage of doing so is that $\mathcal{P}(\Lambda)$ is a variety. From this point of view, we can regard $\mathcal{P}(\Lambda)$ as a nice categorical version of the Artin R-algebra Λ. Moreover, we know by Proposition 1.6.1 that $\mathcal{P}(\Lambda)$ is an R-category such that $\mathcal{P}(\Lambda)(X,Y) = \operatorname{Hom}_\Lambda(X,Y)$ is finitely generated as an R-module for every $X, Y \in \mathcal{P}(\Lambda)$, so that $\mathcal{P}(\Lambda)$ is also finitely confined. Therefore, if we want to develop a categorical generalization of the concept of Artin R-algebra, it would be interesting to at least define it as a finitely confined R-variety. Nevertheless, to develop such a generalization, Maurice Auslander and Idun Reiten have required in [10] that it must satisfy an additional condition which we explain below.

A very important property of an Artin R-algebra Λ is that the usual duality functor $D : \operatorname{mod} R \to \operatorname{mod} R$ induces dualities $D_\Lambda : \operatorname{mod} \Lambda \leftrightarrow \operatorname{mod} \Lambda^{\mathrm{op}}$, which we usually denote by D. Therefore, it would be reasonable that a categorical generalization of the concept of Artin R-algebra should satisfy a similar property. Well, if $\mathcal{C}$ is a skeletally small R-category, then we already know from Section 2.1 that $\operatorname{Mod} \mathcal{C} = (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ coincides with the category $(\mathcal{C}^{\mathrm{op}}, \operatorname{Mod} R)$ of the contravariant R-functors from $\mathcal{C}$ to $\operatorname{Mod} R$. Hence we can consider the subcategory $(\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$ of $\operatorname{Mod} \mathcal{C}$ consisting of all $\mathcal{C}$-modules F such that $F(X)$ is a finitely generated R-module for every $X \in \mathcal{C}$. Consequently, we can define a contravariant functor $D_{\mathcal{C}} : (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R) \to (\mathcal{C}, \operatorname{mod} R)$ induced by D, which is given by $D_{\mathcal{C}}(F) = D \circ F$ for each $F \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$ and such that for a morphism $\alpha \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$, we have that $D_{\mathcal{C}}(\alpha)$ is given by $D_{\mathcal{C}}(\alpha)_X = D(\alpha_X)$ for each $X \in \mathcal{C}$. Similarly, we can also define a contravariant functor $D_{\mathcal{C}} : (\mathcal{C}, \operatorname{mod} R) \to (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$ induced by D, which we also denote by $D_{\mathcal{C}}$. In this case, it is not difficult to see that $D_{\mathcal{C}} : (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R) \leftrightarrow (\mathcal{C}, \operatorname{mod} R)$ are dualities satisfying that $D_{\mathcal{C}}^2 \simeq id$. Now, if $\mathcal{C}$ is also finitely confined, then it is easy to see that $\mathrm{f.g.}(\mathcal{C}) \subseteq (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$, and consequently we also have that $\operatorname{mod} \mathcal{C} \subseteq (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$. At this moment, the reader might agree that the property that we would like to have for $\mathcal{C}$ corresponding to the dualities $D_\Lambda : \operatorname{mod} \Lambda \leftrightarrow \operatorname{mod} \Lambda^{\mathrm{op}}$ for an Artin R-algebra Λ is that the functors $D_{\mathcal{C}}$ induce dualities $D_{\mathcal{C}} : \operatorname{mod} \mathcal{C} \leftrightarrow \operatorname{mod} \mathcal{C}^{\mathrm{op}}$. Therefore, considering this and the discussion of the preceding paragraph and following [10], we say that a category $\mathcal{C}$ is a **dualizing R-variety** if it is a finitely confined R-variety such that the functors $D_{\mathcal{C}}$ induce dualities $D_{\mathcal{C}} : \operatorname{mod} \mathcal{C} \leftrightarrow \operatorname{mod} \mathcal{C}^{\mathrm{op}}$.

The reader should note that if $\mathcal{C}$ is a finitely confined R-variety, then to say that $\mathcal{C}$ is a dualizing R-variety is equivalent to say that a right $\mathcal{C}$-module $F \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$ is finitely presented if and only if the left $\mathcal{C}$-module $D_{\mathcal{C}} F$ is finitely presented.

In what follows we will present some properties of dualizing R-varieties. But first we will verify that these categories are indeed generalizations of Artin R-algebras in the sense that we have discussed above.

The next result is from [10, Proposition 2.5], and here we give a simpler proof for it.

Proposition 5.1.1. *If Λ is an Artin R-algebra, then $\mathcal{P}(\Lambda)$ is a dualizing R-variety.*

Proof. To begin with, note that we know from Proposition 2.8.2 that the evaluation functors $\mathsf{e}_{\mathcal{P}(\Lambda),\Lambda} : \operatorname{Mod} \mathcal{P}(\Lambda) \to \operatorname{Mod} \Lambda$ and $\mathsf{e}_{\mathcal{P}(\Lambda)^{\mathrm{op}},\Lambda} : \operatorname{Mod} \mathcal{P}(\Lambda)^{\mathrm{op}} \to \operatorname{Mod} \Lambda^{\mathrm{op}}$ are equivalence of categories. Here, to simplify the notation, we will write $\mathsf{e}_{\mathcal{P}(\Lambda),\Lambda} = \mathsf{e}_\Lambda$ and $\mathsf{e}_{\mathcal{P}(\Lambda)^{\mathrm{op}},\Lambda} = \mathsf{e}_\Lambda^{\mathrm{op}}$. Moreover, recall from Proposition 3.1.1 that e_Λ and $\mathsf{e}_\Lambda^{\mathrm{op}}$ induce equivalences of categories $\operatorname{mod} \mathcal{P}(\Lambda) \to \operatorname{mod} \Lambda$ and $\operatorname{mod} \mathcal{P}(\Lambda)^{\mathrm{op}} \to \operatorname{mod} \Lambda^{\mathrm{op}}$, respectively.

Now, remember from the previous paragraphs that we have already pointed out that

$\mathcal{P}(\Lambda)$ is a finitely confined R-variety. Therefore, to conclude that $\mathcal{P}(\Lambda)$ is a dualizing R-variety, it suffices to show that for a given $F \in (\mathcal{P}(\Lambda)^{\mathrm{op}}, \operatorname{mod} R)$ we have that $F \in \operatorname{mod} \mathcal{P}(\Lambda)$ if and only if $D_{\mathcal{P}(\Lambda)}(F) \in \operatorname{mod} \mathcal{P}(\Lambda)^{\mathrm{op}}$.

Well, let $F \in (\mathcal{P}(\Lambda)^{\mathrm{op}}, \operatorname{mod} R)$. Then it follows from the above comments that $F \in \operatorname{mod} \mathcal{P}(\Lambda)$ if and only if $\mathsf{e}_\Lambda(F) = F(\Lambda) \in \operatorname{mod} \Lambda$, which is the case if and only if $D(F(\Lambda)) \in \operatorname{mod} \Lambda^{\mathrm{op}}$ since we have dualities $D : \operatorname{mod} \Lambda \leftrightarrow \operatorname{mod} \Lambda^{\mathrm{op}}$. On the other hand, we have that $D(F(\Lambda)) = \mathsf{e}_\Lambda^{\mathrm{op}}(D_{\mathcal{P}(\Lambda)}(F))$, and by the above comments we also have that $\mathsf{e}_\Lambda^{\mathrm{op}}(D_{\mathcal{P}(\Lambda)}(F)) \in \operatorname{mod} \Lambda^{\mathrm{op}}$ if and only if $D_{\mathcal{P}(\Lambda)}(F) \in \operatorname{mod} \mathcal{P}(\Lambda)^{\mathrm{op}}$. $\square$

Before we give the next results, it is interesting to make a few remarks. First, we observe that it is easy to see that a category $\mathcal{C}$ is a dualizing R-variety if and only if so is $\mathcal{C}^{\mathrm{op}}$. Second, note that if $\mathcal{C}$ is a dualizing R-variety, then the endomorphism ring of each object of $\mathcal{C}$ is an Artin R-algebra, and thus a semiperfect ring, hence it follows from Proposition 1.1.1 that $\mathcal{C}$ is a Krull–Schmidt category. Finally, we remark that it is not difficult to conclude from Proposition 1.1.2 that if $\mathcal{C}$ is a skeletally small R-category, then $(\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$ is an abelian category such that the inclusion $(\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R) \to \operatorname{Mod} \mathcal{C}$ is exact.

Now, we will prove that if $\mathcal{C}$ is a dualizing R-variety, then it is both right and left coherent. This will be a very important result, since we will obtain by Theorem 3.1.7 that if $\mathcal{C}$ is a dualizing R-variety, then both $\operatorname{mod} \mathcal{C}$ and $\operatorname{mod} \mathcal{C}^{\mathrm{op}}$ are abelian categories such that the inclusions $\operatorname{mod} \mathcal{C} \to \operatorname{Mod} \mathcal{C}$ and $\operatorname{mod} \mathcal{C}^{\mathrm{op}} \to \operatorname{Mod} \mathcal{C}^{\mathrm{op}}$ are exact.

Proposition 5.1.2. *If $\mathcal{C}$ is a dualizing R-variety, then $\mathcal{C}$ is both right and left coherent.*

Proof. Given that $\mathcal{C}$ is a dualizing R-variety if and only if so is $\mathcal{C}^{\mathrm{op}}$, and also that $\mathcal{C}$ is left coherent if and only if $\mathcal{C}^{\mathrm{op}}$ is right coherent, it suffices to show that $\mathcal{C}$ is right coherent.

Well, let $G \in \operatorname{mod} \mathcal{C}$ and F be a finitely generated submodule of G. If we consider the short exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow G/F \longrightarrow 0$$

in $\operatorname{Mod} \mathcal{C}$, then it follows from Proposition 3.1.4 that $G/F \in \operatorname{mod} \mathcal{C}$, and in particular we conclude that this sequence is in $(\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$. Therefore, if we apply the duality $D_{\mathcal{C}}$ in the above sequence, then we obtain a short exact sequence

$$0 \longrightarrow D_{\mathcal{C}}(G/F) \longrightarrow D_{\mathcal{C}}(G) \longrightarrow D_{\mathcal{C}}(F) \longrightarrow 0$$

in $(\mathcal{C}, \operatorname{mod} R)$, which is also exact in $\operatorname{Mod} \mathcal{C}^{\mathrm{op}}$ since the inclusion $(\mathcal{C}, \operatorname{mod} R) \to \operatorname{Mod} \mathcal{C}^{\mathrm{op}}$ is exact. Hence by Proposition 3.1.5 we obtain that $D_{\mathcal{C}}(F) \in \operatorname{mod} \mathcal{C}^{\mathrm{op}}$, which gives us that $F \in \operatorname{mod} \mathcal{C}$. $\square$

Next, we proceed to show that if Λ is an Artin R-algebra and if $\mathcal{C}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$, then $\mathcal{C}$ is a dualizing R-variety. But to achieve this goal, we first need to give some other results.

The next lemma is a generalization of Proposition 1.6.1.

Lemma 5.1.3. *Let $\mathcal{C}$ be a skeletally small R-category. If $F \in \mathrm{f.g.}(\mathcal{C})$ and $G \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$, then $\operatorname{Hom}_{\mathcal{C}}(F, G)$ is finitely generated as an R-module.*

Proof. Let $\bigoplus_{i=1}^{m} \mathcal{C}(-, X_i) \to F$ be an epimorphism in $\operatorname{Mod}\mathcal{C}$ with $X_i \in \mathcal{C}$. Given that $\operatorname{Hom}_{\mathcal{C}}(-, G)$ is a left exact functor, if we apply it in this epimorphism, then we obtain a monomorphism

$$\operatorname{Hom}_{\mathcal{C}}(F, G) \to \operatorname{Hom}_{\mathcal{C}}\left(\bigoplus_{i=1}^{m} \mathcal{C}(-, X_i), G\right)$$

in $\operatorname{Mod} R$. Furthermore, it follows from the Yoneda Lemma that

$$\operatorname{Hom}_{\mathcal{C}}\left(\bigoplus_{i=1}^{m} \mathcal{C}(-, X_i), G\right) \simeq \bigoplus_{i=1}^{m} \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-, X_i), G) \simeq \bigoplus_{i=1}^{m} G(X_i),$$

and because $G \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$, we obtain that $\bigoplus_{i=1}^{m} G(X_i) \in \operatorname{mod} R$. Therefore, $\operatorname{Hom}_{\mathcal{C}}(F, G)$ is isomorphic to a submodule of a finitely generated R-module, which implies that it is a finitely generated R-module since R is a noetherian ring. $\square$

For the next lemma, we remark that if $\mathcal{C}$ is a skeletally small category and if $F \in \operatorname{mod}\mathcal{C}$, then it is usual to write $\operatorname{Hom}_{\mathcal{C}}(F, -)$ for $\operatorname{Hom}_{\mathcal{C}}(F, -)|_{\operatorname{mod}\mathcal{C}}$ and $\operatorname{Hom}_{\mathcal{C}}(-, F)$ for $\operatorname{Hom}_{\mathcal{C}}(-, F)|_{\operatorname{mod}\mathcal{C}}$ when no confusion may arise. In what follows we commit this abuse of notation.

Lemma 5.1.4. *If $\mathcal{C}$ is a dualizing R-variety and $X \in \mathcal{C}$, then there is an isomorphism*

$$D_{\operatorname{mod}\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-, D_{\mathcal{C}}(\mathcal{C}(X, -)))) \simeq \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-, X), -)$$

in $\operatorname{Mod}(\operatorname{mod}\mathcal{C})^{\mathrm{op}}$.

Proof. If $F \in \operatorname{mod}\mathcal{C}$, then

$$D_{\operatorname{mod}\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-, D_{\mathcal{C}}(\mathcal{C}(X, -))))(F) = D(\operatorname{Hom}_{\mathcal{C}}(F, D_{\mathcal{C}}(\mathcal{C}(X, -)))).$$

However, given that $D_{\mathcal{C}} : \operatorname{mod}\mathcal{C} \leftrightarrow \operatorname{mod}\mathcal{C}^{\mathrm{op}}$ are dualities, and also considering the covariant version of the Yoneda Lemma, we have that

$$\operatorname{Hom}_{\mathcal{C}}(F, D_{\mathcal{C}}(\mathcal{C}(X, -))) \simeq \operatorname{Hom}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}(X, -), D_{\mathcal{C}}(F)) \simeq D_{\mathcal{C}}(F)(X) = D(F(X)).$$

Therefore, we obtain that

$$D_{\operatorname{mod}\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-, D_{\mathcal{C}}(\mathcal{C}(X, -))))(F) \simeq D(D(F(X))) \simeq F(X) \simeq \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-, X), F),$$

and because all these isomorphisms are functorial, we conclude that

$$D_{\operatorname{mod}\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-,D_{\mathcal{C}}(\mathcal{C}(X,-)))) \simeq \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),-)$$

in $\operatorname{Mod}(\operatorname{mod}\mathcal{C})^{\operatorname{op}}$. $\square$

The next result is from [10, Proposition 2.6], but the proof that we present here is a generalization of the proof of [36, Proposition 4.1] (see also [8, Proposition 3.3]).

Proposition 5.1.5. *If $\mathcal{C}$ is a dualizing R-variety, then so is* $\operatorname{mod}\mathcal{C}$.

Proof. It is not difficult to see that $\operatorname{mod}\mathcal{C}$ is an R-variety, and from Lemma 5.1.3 we also have that $\operatorname{mod}\mathcal{C}$ is finitely confined. Therefore, to conclude that $\operatorname{mod}\mathcal{C}$ is a dualizing R-variety, we only need to show that for a given $\mathbb{F} \in ((\operatorname{mod}\mathcal{C})^{\operatorname{op}}, \operatorname{mod}R)$ we have that $\mathbb{F} \in \operatorname{mod}(\operatorname{mod}\mathcal{C})$ if and only if $D_{\operatorname{mod}\mathcal{C}}(\mathbb{F}) \in \operatorname{mod}(\operatorname{mod}\mathcal{C})^{\operatorname{op}}$.

Well, let $\mathbb{F} \in ((\operatorname{mod}\mathcal{C})^{\operatorname{op}}, \operatorname{mod}R)$ and suppose that $\mathbb{F} \in \operatorname{mod}(\operatorname{mod}\mathcal{C})$. Then there is an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(-,G) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-,H) \longrightarrow \mathbb{F} \longrightarrow 0$$

in $\operatorname{Mod}(\operatorname{mod}\mathcal{C})$ with $G, H \in \operatorname{mod}\mathcal{C}$, which in this case is also in $((\operatorname{mod}\mathcal{C})^{\operatorname{op}}, \operatorname{mod}R)$. Hence we obtain that

$$0 \longrightarrow D_{\operatorname{mod}\mathcal{C}}(\mathbb{F}) \longrightarrow D_{\operatorname{mod}\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-,H)) \longrightarrow D_{\operatorname{mod}\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-,G))$$

is exact in $(\operatorname{mod}\mathcal{C}, \operatorname{mod}R)$, so that it is also exact in $\operatorname{Mod}(\operatorname{mod}\mathcal{C})^{\operatorname{op}}$. Now, note that we know from Proposition 5.1.2 and Theorem 3.1.7 that $\operatorname{mod}\mathcal{C}$ is an abelian category, and thus so is $(\operatorname{mod}\mathcal{C})^{\operatorname{op}}$. Hence by Proposition 3.1.9 we obtain that $(\operatorname{mod}\mathcal{C})^{\operatorname{op}}$ is right coherent. Consequently, it follows from Proposition 3.1.6 that to conclude that $D_{\operatorname{mod}\mathcal{C}}(\mathbb{F}) \in \operatorname{mod}(\operatorname{mod}\mathcal{C})^{\operatorname{op}}$, it suffices to verify that $D_{\operatorname{mod}\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-,F)) \in \operatorname{mod}(\operatorname{mod}\mathcal{C})^{\operatorname{op}}$ for every $F \in \operatorname{mod}\mathcal{C}$.

Well, let $F \in \operatorname{mod}\mathcal{C}$. Given that $D_{\mathcal{C}}(F) \in \operatorname{mod}\mathcal{C}^{\operatorname{op}}$, it follows that there is an exact sequence

$$\mathcal{C}(X,-) \longrightarrow \mathcal{C}(Y,-) \longrightarrow D_{\mathcal{C}}(F) \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}^{\operatorname{op}}$ with $X, Y \in \mathcal{C}$, which is also in $\operatorname{mod}\mathcal{C}^{\operatorname{op}}$. Hence we obtain an exact sequence

$$0 \longrightarrow F \longrightarrow D_{\mathcal{C}}(\mathcal{C}(Y,-)) \longrightarrow D_{\mathcal{C}}(\mathcal{C}(X,-))$$

in $\operatorname{mod}\mathcal{C}$, which gives us by Proposition 2.2.5 an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-,F) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-,D_{\mathcal{C}}(\mathcal{C}(Y,-))) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-,D_{\mathcal{C}}(\mathcal{C}(X,-)))$$

in $\operatorname{Mod}(\operatorname{mod}\mathcal{C})$. Furthermore, because the above sequence is in $((\operatorname{mod}\mathcal{C})^{\operatorname{op}}, \operatorname{mod}R)$, we can apply the functor $D_{\operatorname{mod}\mathcal{C}}$ to it, and in doing so, by Lemma 5.1.4 we obtain an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,X),-) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-,Y),-) \longrightarrow D_{\operatorname{mod}\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-,F)) \longrightarrow 0$$

in $(\operatorname{mod}\mathcal{C}, \operatorname{mod} R)$. Consequently, since this sequence is exact in $\operatorname{Mod}(\operatorname{mod}\mathcal{C})^{\mathrm{op}}$, we conclude by Proposition 3.1.5 that $D_{\operatorname{mod}\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-,F)) \in \operatorname{mod}(\operatorname{mod}\mathcal{C})^{\mathrm{op}}$.

Therefore, we obtain that $D_{\operatorname{mod}\mathcal{C}}(\mathbb{F}) \in \operatorname{mod}(\operatorname{mod}\mathcal{C})^{\mathrm{op}}$.

Dually, we can prove that if $D_{\operatorname{mod}\mathcal{C}}(\mathbb{F}) \in \operatorname{mod}(\operatorname{mod}\mathcal{C})^{\mathrm{op}}$, then $\mathbb{F} \in \operatorname{mod}(\operatorname{mod}\mathcal{C})$. □

As a consequence of the above results, we have the following:

Corollary 5.1.6. *If Λ is an Artin R-algebra, then* $\operatorname{mod}\Lambda$ *is a dualizing R-variety.*

Proof. We already know from Proposition 5.1.1 that $\mathcal{P}(\Lambda)$ is a dualizing R-variety, and thus it follows from Proposition 5.1.5 that $\operatorname{mod}\mathcal{P}(\Lambda)$ is also a dualizing R-variety. Furthermore, by propositions 2.8.2 and 3.1.1 we have that the categories $\operatorname{mod}\mathcal{P}(\Lambda)$ and $\operatorname{mod}\Lambda$ are equivalent, hence $\operatorname{mod}\Lambda$ is a dualizing R-variety. □

Finally, to conclude that if Λ is an Artin R-algebra, then every n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ is a dualizing R-variety, we only need the result below, which is from [24, Proposition 1.2] (see also [17, Theorem 2.3]).

Proposition 5.1.7. *Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{B}$ be a subcategory of $\mathcal{C}$ such that* $\operatorname{add}\mathcal{B} = \mathcal{B}$. *If $\mathcal{B}$ is functorially finite in $\mathcal{C}$, then $\mathcal{B}$ is a dualizing R-variety.*

Proof. It is clear that $\mathcal{B}$ is a finitely confined R-variety. Moreover, if $F \in \operatorname{mod}\mathcal{B}$, then there is an exact sequence

$$\mathcal{B}(-,X) \xrightarrow{\mathcal{B}(-,f)} \mathcal{B}(-,Y) \longrightarrow F \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{B}$ with $f : X \to Y$ being a morphism in $\mathcal{B}$, so that if we consider the exact sequence

$$\mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-,Y) \longrightarrow G \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{C}$, then we obtain that $F \simeq G|_{\mathcal{B}}$. Now, because $\mathcal{C}$ is a dualizing R-variety, we obtain that $D_{\mathcal{C}}(G) \in \operatorname{mod}\mathcal{C}^{\mathrm{op}}$, and it is easy to see that $D_{\mathcal{C}}(G)|_{\mathcal{B}} \simeq D_{\mathcal{B}}(F)$. Therefore, it follows from propositions 5.1.2 and 3.5.1 that $D_{\mathcal{B}}(F) \in \operatorname{mod}\mathcal{B}^{\mathrm{op}}$.

Similarly, we can verify that if $D_{\mathcal{B}}(F) \in \operatorname{mod}\mathcal{B}^{\mathrm{op}}$, then $F \in \operatorname{mod}\mathcal{B}$.

Hence $\mathcal{B}$ is a dualizing R-variety. □

Corollary 5.1.8. *If Λ is an Artin R-algebra and $\mathcal{C}$ is an n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$*, then $\mathcal{C}$ is a dualizing R-variety.*

Proof. This is straightforward from Corollary 5.1.6 and Proposition 5.1.7. □

Next, to prove that if Λ is an Artin R-algebra, then every n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ has almost split morphisms and n-almost split sequences, we proceed to show that every dualizing R-variety has almost split morphisms. But for this, we first need some other propositions.

The next result is from [10, Proposition 3.1].

Proposition 5.1.9. *Let $\mathcal{C}$ be a dualizing R-variety and $F \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$. Then F is a finitely presented $\mathcal{C}$-module if and only if F and $D_{\mathcal{C}}(F)$ are finitely generated right and left $\mathcal{C}$-modules, respectively.*

Proof. If $F \in \operatorname{mod}\mathcal{C}$, then given that $\mathcal{C}$ is a dualizing R-variety, we conclude that $D_{\mathcal{C}}(F) \in \operatorname{mod}\mathcal{C}^{\mathrm{op}}$. Hence we have that $F \in \mathrm{f.g.}(\mathcal{C})$ and $D_{\mathcal{C}}(F) \in \mathrm{f.g.}(\mathcal{C}^{\mathrm{op}})$.

Conversely, suppose that $F \in \mathrm{f.g.}(\mathcal{C})$ and $D_{\mathcal{C}}(F) \in \mathrm{f.g.}(\mathcal{C}^{\mathrm{op}})$. Then there is an epimorphism $\mathcal{C}(X,-) \to D_{\mathcal{C}}(F)$ in $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$ with $X \in \mathcal{C}$, and since this morphism is in $(\mathcal{C}, \operatorname{mod} R)$ we can apply the functor $D_{\mathcal{C}}$ to it. In doing so, we obtain a monomorphism $F \to D_{\mathcal{C}}(\mathcal{C}(X,-))$ in $(\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$, which is also a monomorphism in $\operatorname{Mod}\mathcal{C}$. Now, note that $D_{\mathcal{C}}(\mathcal{C}(X,-)) \in \operatorname{mod}\mathcal{C}$, and thus F is isomorphic to a finitely generated submodule of a finitely presented $\mathcal{C}$-module. Therefore, it follows from Proposition 5.1.2 that $F \in \operatorname{mod}\mathcal{C}$. $\square$

We will now show that every simple module over a dualizing R-variety is finitely presented. The result below is from [10, Proposition 3.2].

Proposition 5.1.10. *If $\mathcal{C}$ is a dualizing R-variety and S is a simple $\mathcal{C}$-module, then we have the following:*

(a) $S \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$.

(b) $D_{\mathcal{C}}(S)$ *is a simple $\mathcal{C}^{\mathrm{op}}$-module.*

(c) $S \in \operatorname{mod}\mathcal{C}$.

Proof. (a) It follows from Proposition 2.4.5 that $S \in \mathrm{f.g.}(\mathcal{C})$, and thus $S \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$.

(b) First, note that by item (a) we have that $D_{\mathcal{C}}(S) \in (\mathcal{C}, \operatorname{mod} R)$ and that $D_{\mathcal{C}}(S) \neq 0$. Moreover, if F is a nonzero submodule of $D_{\mathcal{C}}(S)$ and if $\eta : F \to D_{\mathcal{C}}(S)$ is the canonical inclusion, then it is easy to see that $F \in (\mathcal{C}, \operatorname{mod} R)$, and consequently we can consider $D_{\mathcal{C}}(\eta) : D_{\mathcal{C}}(D_{\mathcal{C}}(S)) \to D_{\mathcal{C}}(F)$. Furthermore, given that $D_{\mathcal{C}}(D_{\mathcal{C}}(S)) \simeq S$ is simple, we obtain by Lemma 2.4.4 that $D_{\mathcal{C}}(\eta)$ is a monomorphism. Hence η is an epimorphism in $(\mathcal{C}, \operatorname{mod} R)$, and thus it is also an epimorphism in $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$. Therefore, we obtain that $F = D_{\mathcal{C}}(S)$, which proves that $D_{\mathcal{C}}(S)$ is simple.

(c) This follows from Proposition 5.1.9 since we have that $S \in \mathrm{f.g.}(\mathcal{C})$ and also that $D_{\mathcal{C}}(S) \in \mathrm{f.g.}(\mathcal{C}^{\mathrm{op}})$ by item (b). $\square$

Corollary 5.1.11. *If $\mathcal{C}$ is a dualizing R-variety, then $\mathcal{C}$ has almost split morphisms.*

Proof. It follows from propositions 5.1.10 and 3.4.2 that $\mathcal{C}$ has right almost split morphisms. Furthermore, because $\mathcal{C}^{\mathrm{op}}$ is a dualizing R-variety, we also conclude that it has right almost split morphisms, which implies that $\mathcal{C}$ has left almost split morphisms. □

We can now prove the main result of this section, which is from [25, Theorem 3.3.1]. However, the proof that we present for it is based on the proof of the existence of almost split sequences given in [14].

Theorem 5.1.12. *If Λ is an Artin R-algebra and $\mathcal{C}$ is an n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$*, then we have the following:*

(a) $\mathcal{C}$ has almost split morphisms.

(b) $\mathcal{C}$ has n-almost split sequences.

Proof. (a) This follows from corollaries 5.1.8 and 5.1.11.

(b) This follows from item (a) and Theorem 4.3.6. □

5.2 Higher Morita–Tachikawa Correspondence

In this section we present the Higher Morita–Tachikawa Correspondence, which is an example of a correspondence as described in Proposition 2.9.4. The material given here will be fundamental to prove the Higher Auslander Correspondence in Section 5.3.

Before we state and prove the Higher Morita–Tachikawa Correspondence, we first need to make some definitions, as well as to give some lemmas.

If Γ is an Artin algebra and $Q \in \mathcal{P}(\Gamma)$, then we denote by $\mathcal{P}_Q$ the subcategory of $\operatorname{mod}\Gamma$ consisting of all $X \in \operatorname{mod}\Gamma$ which have a projective presentation

$$Q_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow 0$$

in $\operatorname{mod}\Gamma$ with $Q_0, Q_1 \in \operatorname{add} Q$.

The lemma below is from [15, Chapter II, Proposition 2.5], and the proof that we present here is slightly different from that of the reference.

Lemma 5.2.1. *Let Γ be an Artin algebra, $Q \in \mathcal{P}(\Gamma)$ and $\Lambda = \operatorname{End}_\Gamma(Q)$. Then the functor* $\operatorname{Hom}_\Gamma(Q, -) : \mathcal{P}_Q \to \operatorname{mod}\Lambda$ *is an equivalence of categories.*

Proof. Note that we have by propositions 2.8.2 and 3.1.1 that the evaluation functor at Q induces an equivalence of categories $\mathsf{e}_Q : \operatorname{mod}\mathcal{C} \to \operatorname{mod}\Lambda$, where $\mathcal{C} = \operatorname{add} Q$. In this case, if we prove that the functor $\mathcal{Y} : \mathcal{P}_Q \to \operatorname{mod}\mathcal{C}$ given by $\mathcal{Y}(X) = \operatorname{Hom}_\Gamma(-,X)|_\mathcal{C}$ for an object $X \in \mathcal{P}_Q$ and $\mathcal{Y}(f) = \operatorname{Hom}_\Gamma(-,f)|_\mathcal{C}$ for a morphism $f \in \mathcal{P}_Q$ is an equivalence of categories, then we will have that $\operatorname{Hom}_\Gamma(Q,-) = \mathsf{e}_Q\mathcal{Y} : \mathcal{P}_Q \to \operatorname{mod}\Lambda$ is an equivalence. Therefore, in what follows we will prove that $\mathcal{Y}$ is an equivalence.

First of all, note that if $P \in \mathcal{C}$, then $\mathcal{C}(-,P) = \operatorname{Hom}_\Gamma(-,P)|_\mathcal{C}$. Now, to see that $\mathcal{Y}$ is well defined, let $X \in \mathcal{P}_Q$ and take a projective presentation

$$Q_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow 0$$

of X in $\operatorname{mod}\Gamma$ with $Q_0, Q_1 \in \mathcal{C}$. If we take $P \in \mathcal{C}$, then we obtain that

$$\operatorname{Hom}_\Gamma(P,Q_1) \longrightarrow \operatorname{Hom}_\Gamma(P,Q_0) \longrightarrow \operatorname{Hom}_\Gamma(P,X) \longrightarrow 0$$

is exact in Ab since P is projective in $\operatorname{mod}\Gamma$. Consequently, we obtain that the sequence

$$\operatorname{Hom}_\Gamma(-,Q_1)|_\mathcal{C} \longrightarrow \operatorname{Hom}_\Gamma(-,Q_0)|_\mathcal{C} \longrightarrow \operatorname{Hom}_\Gamma(-,X)|_\mathcal{C} \longrightarrow 0$$

is exact in $\operatorname{Mod}\mathcal{C}$, and thus $\operatorname{Hom}_\Gamma(-,X)|_\mathcal{C} \in \operatorname{mod}\mathcal{C}$.

Next, to verify that $\mathcal{Y}$ is dense, let $F \in \operatorname{mod}\mathcal{C}$ and take a projective presentation

$$\operatorname{Hom}_\Gamma(-,P_1)|_\mathcal{C} \longrightarrow \operatorname{Hom}_\Gamma(-,P_0)|_\mathcal{C} \longrightarrow F \longrightarrow 0$$

of F in $\operatorname{mod}\mathcal{C}$ with $P_0, P_1 \in \mathcal{C}$. If we let W be the cokernel of the induced morphism $P_1 \to P_0$ in $\operatorname{mod}\Gamma$, then we have that $W \in \mathcal{P}_Q$ and by the same reasoning as before we obtain an exact sequence

$$\operatorname{Hom}_\Gamma(-,P_1)|_\mathcal{C} \longrightarrow \operatorname{Hom}_\Gamma(-,P_0)|_\mathcal{C} \longrightarrow \operatorname{Hom}_\Gamma(-,W)|_\mathcal{C} \longrightarrow 0$$

in $\operatorname{mod}\mathcal{C}$. Therefore, $F \simeq \operatorname{Hom}_\Gamma(-,W)|_\mathcal{C}$.

Finally, to see that $\mathcal{Y}$ is full and faithful, first note that the morphism

$$\operatorname{Hom}_\Gamma(P,Y) \to \operatorname{Hom}_\mathcal{C}(\operatorname{Hom}_\Gamma(-,P)|_\mathcal{C}, \operatorname{Hom}_\Gamma(-,Y)|_\mathcal{C})$$

induced by $\mathcal{Y}$ is an isomorphism for every $P \in \mathcal{C}$ and $Y \in \mathcal{P}_Q$, given that it is the inverse of the isomorphism from the Yoneda Lemma. For the general case, let $X, Y \in \mathcal{P}_Q$ and

$$Q_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow 0$$

be a projective presentation of X in $\operatorname{mod}\Gamma$ with $Q_0, Q_1 \in \mathcal{C}$. In this case, it is not difficult to see that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_\Gamma(X,Y) \arrow[r] \arrow[d] & \operatorname{Hom}_\Gamma(Q_0,Y) \arrow[r] \arrow[d] & \operatorname{Hom}_\Gamma(Q_1,Y) \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_\mathcal{C}(\mathcal{Y}(X),\mathcal{Y}(Y)) \arrow[r] & \operatorname{Hom}_\mathcal{C}(\mathcal{Y}(Q_0),\mathcal{Y}(Y)) \arrow[r] & \operatorname{Hom}_\mathcal{C}(\mathcal{Y}(Q_1),\mathcal{Y}(Y))
\end{tikzcd}$$

in Ab with exact rows, whose vertical arrows are the morphisms induced by $\mathcal{Y}$. Therefore, since the two rightmost vertical arrows are both isomorphisms, we conclude that leftmost vertical arrow is also an isomorphism. □

Now, let Γ be an Artin algebra and

$$0 \longrightarrow \Gamma \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

be a minimal injective resolution of Γ in $\operatorname{mod}\Gamma$. We define the **dominant dimension** of Γ, which we denote by $\operatorname{dom.dim}\Gamma$, to be the maximum integer m such that I_j is projective for all $j \leqslant m-1$, if such integer exists, and otherwise, if all I_j are projective, then we let $\operatorname{dom.dim}\Gamma = \infty$. We remark that if $\operatorname{dom.dim}\Gamma \geqslant n$, then $\operatorname{dom.dim}\Gamma^{\mathrm{op}} \geqslant n$ (see [23]).

The next lemma is a generalization of a fact which is mentioned in the proof of [15, Chapter VI, Proposition 5.6].

Lemma 5.2.2. *Let* Γ *be an Artin algebra with* $\operatorname{dom.dim}\Gamma \geqslant n+1$*. If* $X \in \mathcal{I}(\Gamma)$*, then* X *admits a projective* n*-presentation*

$$Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow 0$$

in $\operatorname{mod}\Gamma$ *with* $Q_i \in \mathcal{PI}(\Gamma)$ *for each* i*.*

Proof. To begin with, note that there is some $Y \in \mathcal{I}(\Gamma)$ such that $X \oplus Y \simeq (D\Gamma^{\mathrm{op}})^{(r)}$ for some $r \geqslant 1$ since $\mathcal{I}(\Gamma) = \operatorname{add} D\Gamma^{\mathrm{op}}$. In this case, if we take minimal projective n-presentations

$$Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow 0$$

and

$$P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Y \longrightarrow 0$$

of X and Y, respectively, in $\operatorname{mod}\Gamma$, then their direct sum

$$Q_n \oplus P_n \longrightarrow \cdots \longrightarrow Q_1 \oplus P_1 \longrightarrow Q_0 \oplus P_0 \longrightarrow X \oplus Y \longrightarrow 0$$

gives us a minimal projective n-presentation of $(D\Gamma^{\mathrm{op}})^{(r)}$.

Now, if we take a minimal injective n-copresentation

$$0 \longrightarrow \Gamma^{\mathrm{op}} \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n$$

of Γ^{op} in $\operatorname{mod}\Gamma^{\mathrm{op}}$, then because $\operatorname{dom.dim}\Gamma^{\mathrm{op}} \geqslant n+1$, it follows that $I_j \in \mathcal{PI}(\Gamma^{\mathrm{op}})$ for each j. Therefore, if we apply D in the above sequence, then we obtain a minimal projective n-presentation

$$DI_n \longrightarrow \cdots \longrightarrow DI_1 \longrightarrow DI_0 \longrightarrow D\Gamma^{\mathrm{op}} \longrightarrow 0$$

of $D\Gamma^{\mathrm{op}}$ in $\operatorname{mod}\Gamma$ with $DI_j \in \mathcal{PI}(\Gamma)$ for each j. Consequently, if we take the direct sum of r copies of the above sequence, then we obtain a minimal projective n-presentation

$$(DI_n)^{(r)} \longrightarrow \cdots \longrightarrow (DI_1)^{(r)} \longrightarrow (DI_0)^{(r)} \longrightarrow (D\Gamma^{\mathrm{op}})^{(r)} \longrightarrow 0$$

of $(D\Gamma^{\mathrm{op}})^{(r)}$ in $\operatorname{mod}\Gamma$.

Finally, taking the above two paragraphs into account, we conclude from Proposition 1.4.5 that $Q_j \oplus P_j \simeq (DI_j)^{(r)}$ for each j, which implies that $Q_j \in \mathcal{PI}(\Gamma)$ for each j. $\square$

Next, let Λ be an Artin R-algebra and $M \in \operatorname{mod}\Lambda$. We say that M is an **n-rigid module over** Λ if $\operatorname{add} M$ is an n-rigid subcategory of $\operatorname{mod}\Lambda$. Moreover, we say that M is a **generator module over** Λ if $\operatorname{add} M$ is generating in $\operatorname{mod}\Lambda$, and that it is a **cogenerator module over** Λ if $\operatorname{add} M$ is cogenerating in $\operatorname{mod}\Lambda$. The reader should note that M is a generator module over Λ if and only if $\mathcal{P}(\Lambda) \subseteq \operatorname{add} M$, as well as that is a cogenerator module over Λ if and only if $\mathcal{I}(\Lambda) \subseteq \operatorname{add} M$. In case that M is an n-rigid, generator and cogenerator module over Λ, then we say that M is a **generator-cogenerator n-rigid module over** Λ. Finally, if Σ is another Artin R-algebra and $N \in \operatorname{mod}\Sigma$, then we say that M and N are **equivalent** if the categories $\operatorname{add} M$ and $\operatorname{add} N$ are equivalent.

Now we are ready to prove the Higher Morita–Tachikawa Correspondence. This is a generalization of the Morita–Tachikawa Correspondence, which has its origin in [34] and [39], and which was also proved in [35, Theorem 2]. The generalization that we give here is due to Bruno J. Müller (see [35, Lemma 3]), but the proof that we present is based on a generalization of the proof of the Auslander Correspondence (see the paragraph before the statement of the Higher Auslander Correspondence for more details). We remark that the Morita–Tachikawa Correspondence will be recovered by taking $n = 1$ in the result below.

Theorem 5.2.3 (Higher Morita–Tachikawa Correspondence). *There is a bijection between the equivalence classes of generator-cogenerator n-rigid modules over Artin R-algebras and the Morita equivalence classes of Artin R-algebras with dominant dimension at least $n+1$. The bijection is given as follows:*

(a) If M is a generator-cogenerator n-rigid module over Λ for some Artin R-algebra Λ, then send M to $\operatorname{End}_\Lambda(M)$.

(b) If Γ is an Artin R-algebra with $\operatorname{dom.dim}\Gamma \geqslant n+1$, then take $Q \in \operatorname{mod}\Gamma$ such that $\operatorname{add} Q = \mathcal{PI}(\Gamma)$ and send Γ to $\operatorname{Hom}_\Gamma(Q, D\Gamma^{\mathrm{op}})$ viewed as a module over the Artin R-algebra $\operatorname{End}_\Gamma(Q)$.

Proof. First of all, note that we already know by Proposition 2.9.4 that the assignment of the equivalence class of a generator-cogenerator n-rigid module over some Artin R-algebra to the Morita equivalence class of its endomorphism ring is well defined and is injective.

Therefore, we begin by showing that these endomorphism rings are Artin R-algebras with dominant dimension at least $n+1$.

Let M be a generator-cogenerator n-rigid module over Λ for some Artin R-algebra Λ and denote $\mathcal{C} = \operatorname{add} M$ and $\Gamma = \operatorname{End}_\Lambda(M)$. Note that we already know from Proposition 1.6.1 that Γ is an Artin R-algebra. Moreover, to verify that $\operatorname{dom.dim} \Gamma \geqslant n+1$, take a minimal injective n-copresentation

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_n} I_n$$

of M in $\operatorname{mod} \Lambda$. Given that the above sequence has all of its terms in $\mathcal{C}$, it follows from Proposition 4.1.2 that

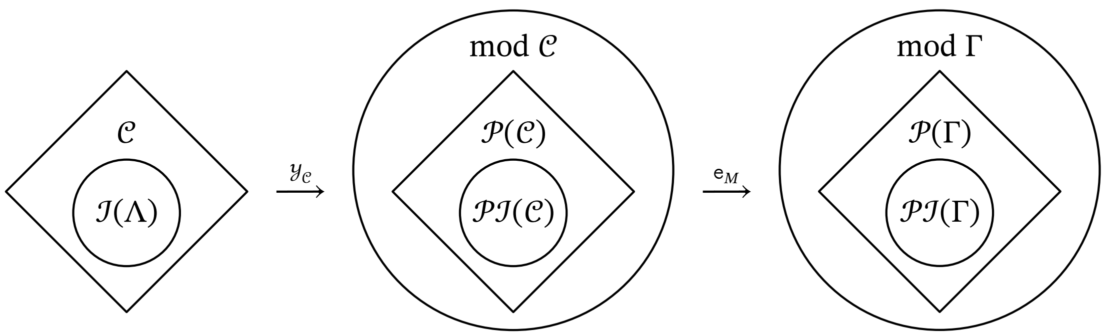
$$0 \longrightarrow \mathcal{C}(-,M) \xrightarrow{\mathcal{C}(-,f_0)} \mathcal{C}(-,I_0) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,I_1) \xrightarrow{\mathcal{C}(-,f_2)} \cdots \xrightarrow{\mathcal{C}(-,f_n)} \mathcal{C}(-,I_n)$$

is an exact sequence in $\operatorname{Mod} \mathcal{C}$, so that it is also exact in $\operatorname{mod} \mathcal{C}$. In this case, it is easy to see that each morphism $\mathcal{C}(-,f_j)$ is left minimal since each f_j is left minimal. Furthermore, note that by propositions 1.6.1 and 3.5.3 we have that $\mathcal{C}$ is functorially finite in $\operatorname{mod} \Lambda$, and it is not difficult to see that the injective objects of $\mathcal{C}$ coincide with $\mathcal{I}(\Lambda)$, so that we have by Proposition 4.4.3 that $\mathcal{C}(-,I_j) \in \mathcal{PI}(\mathcal{C})$ for each j. At this moment, observe that by propositions 2.8.2 and 3.1.1 we have that the evaluation functor at M induces an equivalence of categories $\mathsf{e}_M : \operatorname{mod} \mathcal{C} \to \operatorname{mod} \Gamma$. Consequently, if we apply e_M in the above sequence, then we obtain an exact sequence

$$0 \longrightarrow \mathcal{C}(M,M) \xrightarrow{\mathcal{C}(M,f_0)} \mathcal{C}(M,I_0) \xrightarrow{\mathcal{C}(M,f_1)} \mathcal{C}(M,I_1) \xrightarrow{\mathcal{C}(M,f_2)} \cdots \xrightarrow{\mathcal{C}(M,f_n)} \mathcal{C}(M,I_n)$$

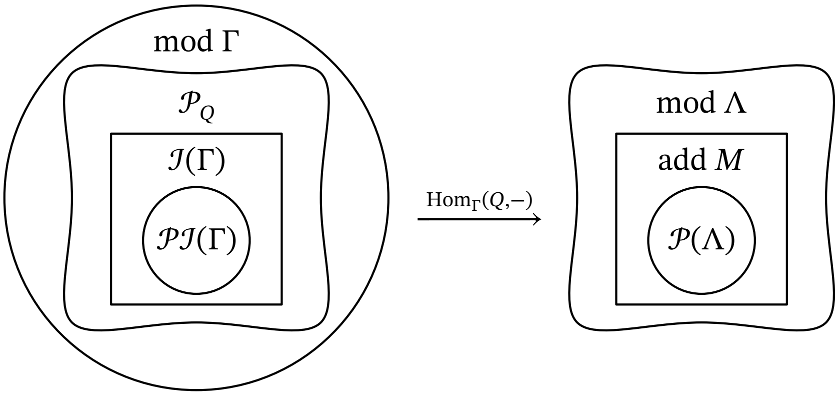
in $\operatorname{mod} \Gamma$ which is a minimal injective n-copresentation of Γ satisfying that $\mathcal{C}(M,I_j)$ is projective for each j. Hence $\operatorname{dom.dim} \Gamma \geqslant n+1$.

The diagram below may be helpful to understand the above reasoning.



Now, let Γ be an Artin R-algebra with $\operatorname{dom.dim} \Gamma \geqslant n+1$ and take $Q \in \operatorname{mod} \Gamma$ such that $\operatorname{add} Q = \mathcal{PI}(\Gamma)$. If we let $\Lambda = \operatorname{End}_\Gamma(Q)$, which is an Artin R-algebra by Proposition 1.6.1, then it follows from Lemma 5.2.1 that the functor $\operatorname{Hom}_\Gamma(Q,-) : \mathcal{P}_Q \to \operatorname{mod} \Lambda$ is an equivalence of categories. In this case, by Proposition 2.9.2 we also have that this functor induces an equivalence $\mathcal{PI}(\Gamma) \to \mathcal{P}(\Lambda)$. Furthermore, we know by Lemma 5.2.2 that $\mathcal{I}(\Gamma) \subseteq \mathcal{P}_Q$, and because $\mathcal{I}(\Gamma) = \operatorname{add} D\Gamma^{\mathrm{op}}$, we conclude that $\operatorname{Hom}_\Gamma(Q, -)$ induces an equivalence $\mathcal{I}(\Gamma) \to \operatorname{add} M$, where $M = \operatorname{Hom}_\Gamma(Q, D\Gamma^{\mathrm{op}})$.

The diagram below sums up the discussion of the above paragraph.



We will prove that M is a generator-cogenerator n-rigid module over Λ. For this, we only need to verify that $\operatorname{add} M$ is an n-rigid subcategory of $\operatorname{mod} \Lambda$ and that $\mathcal{I}(\Lambda) \subseteq \operatorname{add} M$ since we already have that $\mathcal{P}(\Lambda) \subseteq \operatorname{add} M$.

To see that $\operatorname{add} M$ is an n-rigid subcategory of $\operatorname{mod} \Lambda$, it suffices to show that $\operatorname{Ext}^i_\Lambda(M, M) = 0$ for each $0 < i < n$. For this, we first obtain a projective n-presentation of M in $\operatorname{mod} \Lambda$.

Well, by Lemma 5.2.2 we know that $D\Gamma^{\mathrm{op}}$ has a projective n-presentation

$$Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow D\Gamma^{\mathrm{op}} \longrightarrow 0$$

in $\operatorname{mod} \Gamma$ with $Q_j \in \mathcal{PI}(\Gamma)$ for each j. Moreover, given that Q is projective in $\operatorname{mod} \Gamma$, if we apply $\operatorname{Hom}_\Gamma(Q, -)$ in the above sequence, then we obtain a projective n-presentation

$$\operatorname{Hom}_\Gamma(Q, Q_n) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Gamma(Q, Q_1) \longrightarrow \operatorname{Hom}_\Gamma(Q, Q_0) \longrightarrow M \longrightarrow 0$$

of M in $\operatorname{mod} \Lambda$, as $\operatorname{Hom}_\Gamma(Q, Q_j) \in \mathcal{P}(\Lambda)$ for each j.

Now, if we apply $\operatorname{Hom}_\Lambda(-, M)$ in the above projective n-presentation, then we obtain a complex

$$\operatorname{Hom}_\Lambda(K_0, M) \longrightarrow \operatorname{Hom}_\Lambda(K_1, M) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Lambda(K_n, M) \ ,$$

where $K_j = \operatorname{Hom}_\Gamma(Q, Q_j)$. But because we have an equivalence of categories $\operatorname{Hom}_\Gamma(Q, -) : \mathcal{P}_Q \to \operatorname{mod} \Lambda$, it follows that the above complex is isomorphic to

$$\operatorname{Hom}_\Gamma(Q_0, D\Gamma^{\mathrm{op}}) \longrightarrow \operatorname{Hom}_\Gamma(Q_1, D\Gamma^{\mathrm{op}}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Gamma(Q_n, D\Gamma^{\mathrm{op}}) \ ,$$

which is an exact sequence in Ab since $D\Gamma^{\mathrm{op}}$ is injective in $\operatorname{mod} \Gamma$. Hence we obtain that $\operatorname{Ext}^i_\Lambda(M, M) = 0$ for each $0 < i < n$, as desired.

Next, we will verify that $\mathcal{I}(\Lambda) \subseteq \operatorname{add} M$. For this, let $X \in \mathcal{I}(\Lambda)$ and take $Y \in \mathcal{P}_Q$ such that $X \simeq \operatorname{Hom}_\Gamma(Q, Y)$. Given that X is injective in $\operatorname{mod} \Lambda$, we also have that Y is injective in $\mathcal{P}_Q$. However, it is not difficult to see that every injective object of $\mathcal{P}_Q$ is in $\mathcal{I}(\Gamma)$, so that $Y \in \mathcal{I}(\Gamma)$. In this case, as $\operatorname{Hom}_\Gamma(Q, -)$ induces an equivalence $\mathcal{I}(\Gamma) \to \operatorname{add} M$, we conclude that $X \in \operatorname{add} M$, and thus $\mathcal{I}(\Lambda) \subseteq \operatorname{add} M$.

We will now verify that the assignment of the Morita equivalence class of Γ to the equivalence class of the generator-cogenerator n-rigid module $\operatorname{Hom}_\Gamma(Q, D\Gamma^{\mathrm{op}})$ over $\operatorname{End}_\Gamma(Q)$ is well defined, as well as that it does not depend on the choice of the additive generator Q of $\mathcal{PI}(\Gamma)$. For this, let Σ be an Artin R-algebra with $\operatorname{dom.dim} \Sigma \geqslant n+1$ which is Morita equivalent to Γ and take $P \in \operatorname{mod} \Sigma$ such that $\operatorname{add} P = \mathcal{PI}(\Sigma)$. It follows from Proposition 2.7.8 that $\mathcal{P}(\Gamma) \approx \mathcal{P}(\Sigma)$, and since we also have by Proposition 1.6.2 that $\mathcal{P}(\Gamma) \approx \mathcal{I}(\Gamma)$ and $\mathcal{P}(\Sigma) \approx \mathcal{I}(\Sigma)$, we conclude that $\mathcal{I}(\Gamma) \approx \mathcal{I}(\Sigma)$. Therefore, given that we already know that $\operatorname{add} \operatorname{Hom}_\Gamma(Q, D\Gamma^{\mathrm{op}}) \approx \mathcal{I}(\Gamma)$ and $\operatorname{add} \operatorname{Hom}_\Sigma(P, D\Sigma^{\mathrm{op}}) \approx \mathcal{I}(\Sigma)$, we obtain that $\operatorname{Hom}_\Gamma(Q, D\Gamma^{\mathrm{op}})$ and $\operatorname{Hom}_\Sigma(P, D\Sigma^{\mathrm{op}})$ are equivalent.

Finally, to conclude that the assignments that were defined give inverse bijections, it only remains to show that $\operatorname{End}_\Lambda(M)$ is Morita equivalent to Γ. Well, because $\operatorname{Hom}_\Gamma(Q, -) : \mathcal{P}_Q \to \operatorname{mod} \Lambda$ is an equivalence of categories, it follows that $\operatorname{End}_\Lambda(M) \simeq \operatorname{End}_\Gamma(D\Gamma^{\mathrm{op}})$. But given that $D : \operatorname{mod} \Gamma \to \operatorname{mod} \Gamma^{\mathrm{op}}$ is a duality, we also have that $\operatorname{End}_\Gamma(D\Gamma^{\mathrm{op}}) \simeq \operatorname{End}_{\Gamma^{\mathrm{op}}}(\Gamma^{\mathrm{op}})^{\mathrm{op}}$. Therefore, since $\operatorname{End}_{\Gamma^{\mathrm{op}}}(\Gamma^{\mathrm{op}})^{\mathrm{op}} \simeq (\Gamma^{\mathrm{op}})^{\mathrm{op}} \simeq \Gamma$ we conclude that $\operatorname{End}_\Lambda(M) \simeq \Gamma$, and consequently $\operatorname{End}_\Lambda(M)$ and Γ are Morita equivalent. □

The reader should note that the Higher Morita–Tachikawa Correspondence in fact agrees with Proposition 2.9.4 since if Γ is an Artin R-algebra with $\operatorname{dom.dim} \Gamma \geqslant n+1$ and $Q \in \operatorname{mod} \Gamma$ is such that $\operatorname{add} Q = \mathcal{PI}(\Gamma)$, then $\operatorname{add} \operatorname{Hom}_\Gamma(Q, D\Gamma^{\mathrm{op}}) \approx \mathcal{I}(\Gamma)$ and we have that $\mathcal{I}(\Gamma) \approx \mathcal{P}(\Gamma)$ by Proposition 1.6.2.

5.3 Higher Auslander Correspondence

We now present the Higher Auslander Correspondence, which is a particular case of Proposition 2.9.4 and of the Higher Morita–Tachikawa Correspondence. In doing so, we also discuss the concept of “n-Auslander algebra” and we see how it is related to a certain kind of n-cluster tilting subcategory.

In order to prove the Higher Auslander Correspondence, we first need to establish some definitions and give some results.

To begin with, if Λ is an Artin algebra and $M \in \operatorname{mod} \Lambda$, then we say that M is an ***n*-cluster tilting module over** Λ if $\operatorname{add} M$ is an n-cluster tilting subcategory of $\operatorname{mod} \Lambda$.

We give below some conditions to determine if a module over an Artin algebra is n-cluster tilting or not.

Proposition 5.3.1. *Let Λ be an Artin algebra and $M \in \operatorname{mod}\Lambda$. The following are equivalent:*

(a) *M is an n-cluster tilting module over Λ.*

(b) $\operatorname{add} M = (\operatorname{add} M)^{\perp_n}$ *and* $\operatorname{add} M = {}^{\perp_n}(\operatorname{add} M)$.

(c) $\operatorname{add} M = (\operatorname{add} M)^{\perp_n}$ *and* $\mathcal{P}(\Lambda) \subseteq \operatorname{add} M$.

(d) $\operatorname{add} M = {}^{\perp_n}(\operatorname{add} M)$ *and* $\mathcal{I}(\Lambda) \subseteq \operatorname{add} M$.

Proof. Note that by propositions 1.6.1 and 3.5.3 we have that $\operatorname{add} M$ is functorially finite in $\operatorname{mod}\Lambda$. Therefore, it is easy to see that the result follows from Proposition 4.2.7. □

Now, following [26], we say that an Artin R-algebra Γ is an **n-Auslander R-algebra**, or for short, an **n-Auslander algebra** if no confusion may arise, if Γ satisfies that

$$\operatorname{gl.dim}\Gamma \leqslant n+1 \leqslant \operatorname{dom.dim}\Gamma.$$

In the case that $n = 1$, we call a 1-Auslander R-algebra simply by an **Auslander R-algebra**, or for short, an **Auslander algebra**, which is a kind of algebra that plays an important role in Auslander–Reiten theory (see [15, Chapter VI, Section 5]).

In the next result, which is mentioned in [26, page 358], we will see that there are not too many possibilities for the global and the dominant dimension of an n-Auslander algebra. We observe that this result will not be used to prove the Higher Auslander Correspondence.

Proposition 5.3.2. *If Γ is an n-Auslander algebra, then either we have that* $\operatorname{gl.dim}\Gamma = 0$ *and* $\operatorname{dom.dim}\Gamma = \infty$ *or we have that* $\operatorname{gl.dim}\Gamma = n+1 = \operatorname{dom.dim}\Gamma$.

Proof. First of all, we remark that $\operatorname{gl.dim}\Gamma = \operatorname{id}_\Gamma \Gamma$ (see [15, Chapter VI, Lemma 5.5]).

Now, note that if $\operatorname{gl.dim}\Gamma = 0$, then Γ is injective, and thus $\operatorname{dom.dim}\Gamma = \infty$.

Next, suppose that $\operatorname{gl.dim}\Gamma \neq 0$, and by considering that $\operatorname{gl.dim}\Gamma \leqslant n+1$, take a minimal injective resolution

$$0 \longrightarrow \Gamma \xrightarrow{f_0} I_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} I_n \xrightarrow{f_{n+1}} I_{n+1} \longrightarrow 0$$

of Γ in $\operatorname{mod}\Gamma$. Then from $\operatorname{dom.dim}\Gamma \geqslant n+1$ we obtain that I_j is both injective and projective for each $0 \leqslant j \leqslant n$. In this case, observe that if I_{n+1} is projective, then we obtain that $I_n \simeq \operatorname{Ker} f_{n+1} \oplus I_{n+1}$, which implies that $\operatorname{Im} f_n = \operatorname{Ker} f_{n+1}$ is both projective and injective. Therefore, by repeating this argument we can conclude that $\operatorname{Im} f_0$ is both projective and injective, so that Γ is injective, and thus $\operatorname{gl.dim}\Gamma = \operatorname{id}_\Gamma \Gamma = 0$, a contradiction. Hence I_{n+1} is not projective, which implies that $\operatorname{dom.dim}\Gamma = n+1$, as well as that $\operatorname{id}_\Gamma \Gamma = n+1$ since we obtain in particular that $I_{n+1} \neq 0$, and consequently $\operatorname{gl.dim}\Gamma = n+1$. □

Next, to prove the Higher Auslander Correspondence, we only need one more result.

Lemma 5.3.3. *Let* Γ *be an Artin algebra with* $\operatorname{gl.dim}\Gamma \leqslant n+1$ *and* $Q \in \operatorname{mod}\Gamma$ *be such that* $\operatorname{add} Q = \mathcal{PI}(\Gamma)$. *If* $X \in \mathcal{P}_Q$, *then* $\operatorname{id}_\Gamma X \leqslant n-1$.

Proof. Let

$$Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} X \longrightarrow 0$$

be a projective presentation of X in $\operatorname{mod}\Gamma$ with $Q_1, Q_0 \in \operatorname{add} Q$ and

$$0 \longrightarrow X \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots$$

be an injective resolution of X in $\operatorname{mod}\Gamma$. Then we obtain an injective resolution

$$0 \longrightarrow \operatorname{Ker} g_1 \longrightarrow Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{f_0 g_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots$$

of $\operatorname{Ker} g_1$ in $\operatorname{mod}\Gamma$, and given that $\operatorname{id}_\Gamma \operatorname{Ker} g_1 \leqslant n+1$, it follows from the dual of Proposition 1.5.1 that $\operatorname{Coker} f_{n-1} \in \mathcal{I}(\Gamma)$. Therefore, we obtain an injective resolution

$$0 \longrightarrow X \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} I_{n-2} \longrightarrow \operatorname{Coker} f_{n-1} \longrightarrow 0$$

of X in $\operatorname{mod}\Gamma$, which gives us that $\operatorname{id}_\Gamma X \leqslant n-1$. □

The theorem below, which is from [24, Theorem 0.2], is a generalization of the Auslander Correspondence, which was first proved in [4, Chapter III, Section 4]. The proof that we give here for it is basically an extension of the proof of the Higher Morita–Tachikawa Correspondence that we have presented in Section 5.2. The main steps of the whole proof are generalizations of the corresponding steps in the proof of the Auslander Correspondence given in [4, Chapter III, Section 4] and [15, Chapter VI, Section 5]. We also observe that we will recover the Auslander Correspondence in the result below by taking $n = 1$.

Theorem 5.3.4 (Higher Auslander Correspondence)**.** *There is a bijection between the equivalence classes of n-cluster tilting modules over Artin R-algebras and the Morita equivalence classes of n-Auslander R-algebras. The bijection is given as follows:*

(a) If M *is an n-cluster tilting module over* Λ *for some Artin R-algebra* Λ, *then send* M *to* $\operatorname{End}_\Lambda(M)$.

(b) If Γ *is an n-Auslander R-algebra, then take* $Q \in \operatorname{mod}\Gamma$ *such that* $\operatorname{add} Q = \mathcal{PI}(\Gamma)$ *and send* Γ *to* $\operatorname{Hom}_\Gamma(Q, D\Gamma^{\mathrm{op}})$ *viewed as a module over the Artin R-algebra* $\operatorname{End}_\Gamma(Q)$.

Proof. Since the assignments that were defined are restrictions of the assignments from the Higher Morita–Tachikawa Correspondence, we only need to verify two things. First, that if M is an n-cluster tilting module over Λ for some Artin R-algebra, then $\operatorname{End}_\Lambda(M)$ is an

n-Auslander R-algebra. Second, that if Γ is an n-Auslander R-algebra and if $Q \in \operatorname{mod}\Gamma$ is such that $\operatorname{add} Q = \mathcal{PI}(\Gamma)$, then $\operatorname{Hom}_\Gamma(Q, D\Gamma^{\mathrm{op}})$ is an n-cluster tilting module over $\operatorname{End}_\Gamma(Q)$.

Let M be an n-cluster tilting module over Λ for some Artin R-algebra Λ and denote $\mathcal{C} = \operatorname{add} M$ and $\Gamma = \operatorname{End}_\Lambda(M)$. Because we already know from the Higher Morita–Tachikawa Correspondence that Γ is an Artin R-algebra with $\operatorname{dom.dim}\Gamma \geqslant n+1$, to conclude that it is an n-Auslander R-algebra, we only need to verify that $\operatorname{gl.dim}\Gamma \leqslant n+1$.

Well, recall from the proof of the Higher Morita–Tachikawa Correspondence that the evaluation functor at M induces an equivalence of categories $\mathsf{e}_M : \operatorname{mod}\mathcal{C} \to \operatorname{mod}\Gamma$. In this case, given that we know by Proposition 4.2.3 that $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}) \leqslant n+1$, we obtain that $\operatorname{gl.dim}(\operatorname{mod}\Gamma) \leqslant n+1$, which implies that $\operatorname{gl.dim}\Gamma \leqslant n+1$.

Now, let Γ be an n-Auslander R-algebra and take $Q \in \operatorname{mod}\Gamma$ such that $\operatorname{add} Q = \mathcal{PI}(\Gamma)$. If we let $\Lambda = \operatorname{End}_\Gamma(Q)$, then we already know from the Higher Morita–Tachikawa Correspondence that $M = \operatorname{Hom}_\Gamma(Q, D\Gamma^{\mathrm{op}})$ satisfies that $\operatorname{add} M$ is an n-rigid subcategory of $\operatorname{mod}\Lambda$, so that $\operatorname{add} M \subseteq (\operatorname{add} M)^{\perp n}$, as well as that $\mathcal{P}(\Lambda) \subseteq \operatorname{add} M$. Consequently, it follows from Proposition 5.3.1 that to conclude that M is an n-cluster tilting module over Λ, it suffices to verify that $(\operatorname{add} M)^{\perp n} \subseteq \operatorname{add} M$.

Well, first recall from the proof of the Higher Morita–Tachikawa Correspondence that we have an equivalence of categories $\operatorname{Hom}_\Gamma(Q,-) : \mathcal{P}_Q \to \operatorname{mod}\Lambda$ which induces an equivalence $\mathcal{I}(\Gamma) \to \operatorname{add} M$. Moreover, remember that there is a projective n-presentation

$$Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow D\Gamma^{\mathrm{op}} \longrightarrow 0$$

of $D\Gamma^{\mathrm{op}}$ in $\operatorname{mod}\Gamma$ with $Q_j \in \mathcal{PI}(\Gamma)$ for each j, which induces a projective n-presentation

$$\operatorname{Hom}_\Gamma(Q, Q_n) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Gamma(Q, Q_1) \longrightarrow \operatorname{Hom}_\Gamma(Q, Q_0) \longrightarrow M \longrightarrow 0$$

of M in $\operatorname{mod}\Lambda$.

Now, let $X \in (\operatorname{add} M)^{\perp n}$ and take $Y \in \mathcal{P}_Q$ such that $X \simeq \operatorname{Hom}_\Gamma(Q, Y)$. In this case, to conclude that $X \in \operatorname{add} M$, we only need to show that $Y \in \mathcal{I}(\Gamma)$. Well, note that if we apply the functor $\operatorname{Hom}_\Lambda(-, X)$ in the above projective n-presentation of M, then we obtain an exact sequence

$$\operatorname{Hom}_\Lambda(K_0, X) \longrightarrow \operatorname{Hom}_\Lambda(K_1, X) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Lambda(K_n, X)$$

in Ab since $\operatorname{Ext}^i_\Lambda(M, X) = 0$ for each $0 < i < n$, where $K_j = \operatorname{Hom}_\Gamma(Q, Q_j)$. Consequently, by considering the equivalence given by $\operatorname{Hom}_\Gamma(Q,-)$, we conclude that the above sequence is isomorphic as a complex to

$$\operatorname{Hom}_\Gamma(Q_0, Y) \longrightarrow \operatorname{Hom}_\Gamma(Q_1, Y) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Gamma(Q_n, Y) \ ,$$

which then turns out to be exact in Ab. But this implies that $\operatorname{Ext}^i_\Gamma(D\Gamma^{\mathrm{op}}, Y) = 0$ for each $0 < i < n$, and thus we obtain that $\operatorname{Ext}^i_\Gamma(I, Y) = 0$ for every $I \in \mathcal{I}(\Gamma)$ and $0 < i < n$. Therefore, because we know by Lemma 5.3.3 that $\mathrm{id}_\Gamma Y \leqslant n-1$, it follows from the dual of Proposition 1.5.2 that $Y \in \mathcal{I}(\Gamma)$. Hence $(\operatorname{add} M)^{\perp n} \subseteq \operatorname{add} M$. □

By considering the Higher Auslander Correspondence, the reader should note the importance of n-Auslander algebras, and also how appropriate the notion of an n-cluster tilting subcategory is, given that it allows us to generalize the Auslander Correspondence.

5.4 Some open problems

To end this chapter, we now give some open problems concerning n-cluster tilting subcategories for the case of modules over Artin algebras.

In this section, Λ will denote an Artin algebra.

To begin with, the reader should note that even if we have developed so far many interesting results for n-cluster tilting subcategories of $\operatorname{mod}\Lambda$, we have not discussed at any moment the existence of such subcategories. Well, it turns out that their existence is still not so well understood. Considering this, we can state the following open problem, which is from [26, page 366]:

Problem 5.4.1. When does $\operatorname{mod}\Lambda$ contain an n-cluster tilting subcategory or an n-cluster tilting module?

Recently, some approaches to solve Problem 5.4.1 have been to consider it first for particular kinds of Artin algebras. This is what was done in [40], for example.

Another very relevant open problem is the following, which is from [26, page 367]:

Problem 5.4.2. Is every n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ with $n \geqslant 2$ of finite type?

The reader should note that a positive answer to Problem 5.4.2 would imply that every n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ is obtained from an n-cluster tilting module over Λ. Furthermore, this would also simplify a lot the study of such subcategories since we already know from the Higher Auslander Correspondence that n-cluster tilting modules over Artin algebras correspond to n-Auslander algebras.

Taking Problem 5.4.2 into account, we give below a condition for an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ to be of finite type.

Proposition 5.4.3. *If $\mathcal{C}$ is an n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$*, then $\mathcal{C}$ is of finite type if and only if there is some morphism $f \in \mathcal{C}$ such that* $\operatorname{rad}_{\mathcal{C}} = {}_{\mathcal{C}}\langle f\rangle_{\mathcal{C}}$.

Proof. If $\mathcal{C}$ is of finite type, then let $\{Z_1, \dots, Z_m\}$ be a complete set of nonisomorphic representatives of the indecomposable objects in $\mathcal{C}$. For each i let $f_i : Z_i \to X_i$ be a left almost split morphism in $\mathcal{C}$ and $g_i : Y_i \to Z_i$ be a right almost split morphism in $\mathcal{C}$, which exist by Theorem 5.1.12. Since we know from Proposition 3.4.1 that ${}_{\mathcal{C}}\langle f_i \rangle = \operatorname{rad}_{\mathcal{C}}(Z_i, -)$ and $\langle g_i \rangle_{\mathcal{C}} = \operatorname{rad}_{\mathcal{C}}(-, Z_i)$ for each i, we can conclude that the morphism

$$h = \begin{pmatrix} f_1 & & & & & \\ & \ddots & & & & \\ & & f_m & & & \\ & & & g_1 & & \\ & & & & \ddots & \\ & & & & & g_m \end{pmatrix} : \left(\bigoplus_{i=1}^{m} Z_i\right) \oplus \left(\bigoplus_{i=1}^{m} Y_i\right) \to \left(\bigoplus_{i=1}^{m} X_i\right) \oplus \left(\bigoplus_{i=1}^{m} Z_i\right)$$

in $\mathcal{C}$ satisfies that $\operatorname{rad}_{\mathcal{C}}(Z_i, Z_j) = {}_{\mathcal{C}}\langle h \rangle_{\mathcal{C}}(Z_i, Z_j)$ for every i, j, though we leave the details for the reader. Therefore, given that $\mathcal{C}$ is a Krull–Schmidt category, this implies that $\operatorname{rad}_{\mathcal{C}} = {}_{\mathcal{C}}\langle h \rangle_{\mathcal{C}}$.

Conversely, suppose that there is a morphism $f : X \to Y$ in $\mathcal{C}$ satisfying that $\operatorname{rad}_{\mathcal{C}} = {}_{\mathcal{C}}\langle f \rangle_{\mathcal{C}}$. If we take an indecomposable and nonprojective object $Z \in \mathcal{C}$, then there is a minimal right almost split morphism $g : W \to Z$ in $\mathcal{C}$, and because we know by Lemma 4.3.4 that g is an epimorphism in $\operatorname{mod}\Lambda$, we must have that $W \neq 0$. In this case, it follows from Proposition 3.4.1 that $g \in \operatorname{rad}_{\mathcal{C}}(W, Z)$, and thus there are morphisms $u : Y \to Z$ and $v : W \to X$ such that $g = ufv$. Now, if u is not a split epimorphism, then there is some morphism $h : Y \to W$ such that $u = gh$. Hence we have that $g = ghfv$ and because g is right minimal we obtain that hfv is an isomorphism. Moreover, given that $f \in \operatorname{rad}_{\mathcal{C}}$, we also have that $hfv \in \operatorname{rad}_{\mathcal{C}}$, and from Proposition 2.5.2 we conclude that $\operatorname{rad}_{\mathcal{C}}(-, W) = \mathcal{C}(-, W)$, a contradiction since $W \neq 0$. Therefore, u is a split epimorphism, so that Z is a direct summand of Y. Consequently, there must be only a finite number of objects in $\mathcal{C}$ which are both indecomposable and nonprojective, and once we already know that there is a finite number of indecomposable objects in $\mathcal{C}$ which are projective, we conclude that $\mathcal{C}$ is of finite type. □

Finally, we present one more open problem, which is from [19, Conjecture 6.2]. But before we state it, first remember that we already know from Proposition 4.4.2 that if $1 \leqslant \operatorname{gl.dim}\Lambda < \infty$, then there are only finitely many positive integers m such that there is an m-cluster tilting module over Λ.

Problem 5.4.4. If $\operatorname{gl.dim}\Lambda = \infty$, then are there only finitely many positive integers m such that there is an m-cluster tilting module over Λ?

6 Higher Auslander–Reiten translations

In this chapter, R will always stand for a commutative artinian ring and Λ will be an Artin R-algebra. Moreover, n will denote a positive integer.

We now present higher versions of the "Auslander–Reiten translations" for finitely generated Λ-modules and their applications to the study of n-almost split sequences in n-cluster tilting subcategories of $\operatorname{mod}\Lambda$. Our main aim here is to show how we can use these higher versions of the "Auslander–Reiten translations" to relate the end terms of an n-almost split sequence in an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$. In addition, we also study some examples of n-cluster tilting subcategories, as well as we describe their n-almost split sequences.

6.1 The n-transpose

In this section we present a higher version of the "Auslander–Bridger transpose", which will be fundamental to define higher versions of the "Auslander–Reiten translations".

To begin with, we need to make some definitions.

Let $\mathcal{B}$ be a subcategory of $\operatorname{mod}\Lambda$ which is closed under finite direct sums. Given $X, Y \in \operatorname{mod}\Lambda$ and a morphism $f : X \to Y$ in $\operatorname{mod}\Lambda$, we say that f **factors through** $\mathcal{B}$ if there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

in $\operatorname{mod}\Lambda$ with $Z \in \mathcal{B}$. In this case, we define

$$\langle\mathcal{B}\rangle(X, Y) = \{f \in \operatorname{Hom}_\Lambda(X, Y) \mid f \text{ factors through } \mathcal{B}\},$$

and it is easy to see that these sets define a two sided ideal $\langle\mathcal{B}\rangle = \langle\mathcal{B}\rangle(-,-)$ of $\operatorname{mod}\Lambda$. The reader should note that every object in $\mathcal{B}$ becomes isomorphic to zero in the quotient category $(\operatorname{mod}\Lambda)/\mathcal{B}$.

Now, following [3], in case that $\mathcal{B} = \mathcal{P}(\Lambda)$, we write simply $\langle\mathcal{P}(\Lambda)\rangle = \mathcal{P}$ and we denote the quotient category $(\operatorname{mod}\Lambda)/\mathcal{P}$ by $\underline{\operatorname{mod}}\,\Lambda$, which we call the **projectively stable category**. Moreover, given $X, Y \in \operatorname{mod}\Lambda$, we write

$$\operatorname{Hom}_\Lambda(X, Y)/\mathcal{P}(X, Y) = \underline{\operatorname{Hom}}_\Lambda(X, Y).$$

Dually, we write $\langle \mathcal{I}(\Lambda)\rangle = \mathcal{I}$, we denote the quotient category $(\operatorname{mod}\Lambda)/\mathcal{I}$ by $\overline{\operatorname{mod}}\,\Lambda$, which we call the **injectively stable category**, and given $X, Y \in \operatorname{mod}\Lambda$, we write

$$\operatorname{Hom}_\Lambda(X,Y)/\mathcal{I}(X,Y) = \overline{\operatorname{Hom}}_\Lambda(X,Y).$$

Finally, we note that if $\mathcal{C}$ is a subcategory of $\operatorname{mod}\Lambda$, then we denote by $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ the subcategories of $\underline{\operatorname{mod}}\,\Lambda$ and $\overline{\operatorname{mod}}\,\Lambda$, respectively, induced by $\mathcal{C}$.

Our goal now is to define a contravariant R-functor $\operatorname{Tr}_n : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ which will restrict to a duality for appropriate subcategories of $\underline{\operatorname{mod}}\,\Lambda$ and $\underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$, respectively. Furthermore, this will be done in such a way that if we take $n = 1$, then we will obtain that $\operatorname{Tr}_1 : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ is the functor obtained from the "Auslander–Bridger transpose".

At this moment, the reader should remember from Section 1.6 that we have contravariant functors $(-)^* : \operatorname{mod}\Lambda \leftrightarrow \operatorname{mod}\Lambda^{\mathrm{op}}$ that induce dualities $(-)^* : \mathcal{P}(\Lambda) \leftrightarrow \mathcal{P}(\Lambda^{\mathrm{op}})$ such that $(-)^*(-)^* = (-)^{**} \simeq id$.

Following [25], we now define the "n-transpose" of a finitely generated Λ-module. For each $X \in \operatorname{mod}\Lambda$ fix a minimal projective n-presentation

$$P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

of X. If we apply $(-)^*$ in the above sequence, then we obtain the complex

$$0 \longrightarrow X^* \xrightarrow{f_0^*} P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{f_2^*} \cdots \xrightarrow{f_{n-1}^*} P_{n-1}^* \xrightarrow{f_n^*} P_n^*$$

in $\operatorname{mod}\Lambda^{\mathrm{op}}$, and thus we define $\operatorname{Tr}_n X = \operatorname{Coker} f_n^*$ and we call it the **n-transpose of** X.

We remark that we can conclude from Proposition 1.4.5 that if $X \in \operatorname{mod}\Lambda$, then $\operatorname{Tr}_n X$ is uniquely determined up to isomorphism by X. Furthermore, we observe that if we take $n = 1$, then $\operatorname{Tr}_1 X$ coincides with what is called the "Auslander–Bridger transpose of X", which will be denoted here simply by $\operatorname{Tr} X$ and called by the **transpose of** X.

Next, we proceed to show how we can obtain a contravariant functor from the assignment $X \mapsto \operatorname{Tr}_n X$. For this, let $h : X \to Y$ be a morphism in $\operatorname{mod}\Lambda$. We will define a morphism $\operatorname{Tr}_n h : \operatorname{Tr}_n Y \to \operatorname{Tr}_n X$ in $\operatorname{mod}\Lambda^{\mathrm{op}}$. Well, let $(P_\bullet, f_\bullet)$ and $(Q_\bullet, g_\bullet)$ be minimal projective resolutions of X and Y, respectively, where $P_{-1} = X$ and $Q_{-1} = Y$. It follows from the Comparison Theorem that we can complete h to a morphism of complexes $h_\bullet : P_\bullet \to Q_\bullet$

$$\begin{tikzcd}
\cdots \arrow[r, "f_{n+2}"] & P_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "h_{n+1}"] & P_n \arrow[r, "f_n"] \arrow[d, "h_n"] & P_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "h_{n-1}"] & \cdots \arrow[r, "f_2"] & P_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & P_0 \arrow[r, "f_0"] \arrow[d, "h_0"] & X \arrow[r] \arrow[d, "h"] & 0 \\
\cdots \arrow[r, "g_{n+2}"'] & Q_{n+1} \arrow[r, "g_{n+1}"'] & Q_n \arrow[r, "g_n"'] & Q_{n-1} \arrow[r, "g_{n-1}"'] & \cdots \arrow[r, "g_2"'] & Q_1 \arrow[r, "g_1"'] & Q_0 \arrow[r, "g_0"'] & Y \arrow[r] & 0
\end{tikzcd}$$

with $h_{-1} = h$. In this case, if we apply $(-)^*$ in the above diagram, then we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y^* \arrow[r, "g_0^*"] \arrow[d, "h^*"] & Q_0^* \arrow[r, "g_1^*"] \arrow[d, "h_0^*"] & Q_1^* \arrow[r, "g_2^*"] \arrow[d, "h_1^*"] & \cdots \arrow[r, "g_{n-1}^*"] & Q_{n-1}^* \arrow[r, "g_n^*"] \arrow[d, "h_{n-1}^*"] & Q_n^* \arrow[r, "g_{n+1}^*"] \arrow[d, "h_n^*"] & Q_{n+1}^* \arrow[r, "g_{n+2}^*"] \arrow[d, "h_{n+1}^*"] & \cdots \\
0 \arrow[r] & X^* \arrow[r, "f_0^*"'] & P_0^* \arrow[r, "f_1^*"'] & P_1^* \arrow[r, "f_2^*"'] & \cdots \arrow[r, "f_{n-1}^*"'] & P_{n-1}^* \arrow[r, "f_n^*"'] & P_n^* \arrow[r, "f_{n+1}^*"'] & P_{n+1}^* \arrow[r, "f_{n+2}^*"'] & \cdots
\end{tikzcd}$$

in $\operatorname{mod} \Lambda^{\mathrm{op}}$. Therefore, we conclude that there is a unique morphism $u : \operatorname{Tr}_n Y \to \operatorname{Tr}_n X$ in $\operatorname{mod} \Lambda^{\mathrm{op}}$ which makes the augmented diagram below with exact rows commute.

$$\begin{tikzcd}
Q_{n-1}^* \arrow[r, "g_n^*"] \arrow[d, "h_{n-1}^*"] & Q_n^* \arrow[r] \arrow[d, "h_n^*"] & \operatorname{Tr}_n Y \arrow[r] \arrow[d, dashed, "u"] & 0 \\
P_{n-1}^* \arrow[r, "f_n^*"'] & P_n^* \arrow[r] & \operatorname{Tr}_n X \arrow[r] & 0
\end{tikzcd}$$

Considering this, we define $\operatorname{Tr}_n h : \operatorname{Tr}_n Y \to \operatorname{Tr}_n X$ to be $\operatorname{Tr}_n h = u$.

Now, note that in the above construction the morphism $\operatorname{Tr}_n h$ is uniquely determined by the morphism of complexes $h_\bullet$, but on the other hand, $h_\bullet$ is not uniquely determined by h. Because of this, the assignment $X \mapsto \operatorname{Tr}_n X$ does not define in general a contravariant functor $\operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{\mathrm{op}}$. Nevertheless, we know from the Comparison Theorem that $h_\bullet$ is uniquely determined "up to homotopy" by h, and thus, as we will see in propositions 6.1.2 and 6.1.3, the assignment $X \mapsto \operatorname{Tr}_n X$ defines contravariant functors $\operatorname{mod} \Lambda \to \underline{\operatorname{mod}}\, \Lambda^{\mathrm{op}}$ and $\underline{\operatorname{mod}}\, \Lambda \to \underline{\operatorname{mod}}\, \Lambda^{\mathrm{op}}$.

Before we prove propositions 6.1.2 and 6.1.3, we first need the following lemma:

Lemma 6.1.1. *Let $X, Y \in \operatorname{mod} \Lambda$ and let $(P_\bullet, f_\bullet)$ and $(Q_\bullet, g_\bullet)$ be minimal projective resolutions of X and Y, respectively, where $P_{-1} = X$ and $Q_{-1} = Y$. If a morphism of complexes $h_\bullet : P_\bullet \to Q_\bullet$ is null homotopic, where $h_{-1} = h$, then $\operatorname{Tr}_n h$ factors through $\mathcal{P}(\Lambda^{\mathrm{op}})$.*

Proof. Given that $h_\bullet$ is null homotopic, there are morphisms $s_i : P_i \to Q_{i+1}$ in $\operatorname{mod} \Lambda$ such that $h_i = s_{i-1} f_i + g_{i+1} s_i$ for each $i \in \mathbb{Z}$. Now, consider the diagram

$$\begin{tikzcd}
 & & Q_{n+1}^* \arrow[dl, "s_n^*"] & & \\
Q_{n-1}^* \arrow[r, "g_n^*"] \arrow[d, "h_{n-1}^*"'] & Q_n^* \arrow[r, "g'"] \arrow[d, "h_n^*"] \arrow[ur, "g_{n+1}^*"] \arrow[dl, "s_{n-1}^*"] & \operatorname{Tr}_n Y \arrow[r] \arrow[d, "\operatorname{Tr}_n h"] & 0 \\
P_{n-1}^* \arrow[r, "f_n^*"'] & P_n^* \arrow[r, "f'"'] & \operatorname{Tr}_n X \arrow[r] & 0
\end{tikzcd}$$

in $\operatorname{mod} \Lambda^{\mathrm{op}}$ whose rows are exact. Because $g_{n+1}^* g_n^* = 0$, it follows that there is a morphism $k : \operatorname{Tr}_n Y \to Q_{n+1}^*$ in $\operatorname{mod} \Lambda^{\mathrm{op}}$ such that $g_{n+1}^* = k g'$. Hence from $h_n^* = f_n^* s_{n-1}^* + s_n^* g_{n+1}^*$ and $(\operatorname{Tr}_n h) g' = f' h_n^*$ we obtain that

$$(\operatorname{Tr}_n h) g' = f' f_n^* s_{n-1}^* + f' s_n^* k g' = f' s_n^* k g',$$

which implies that $\operatorname{Tr}_n h = f' s_n^* k$ since g' is an epimorphism. □

The next two results are generalizations of the construction of the functor obtained from the transpose of a finitely generated Λ-module given in [11, Section 2] (see also [3, Chapter IV, Proposition 2.2]).

Proposition 6.1.2. *The assignment* $X \mapsto \operatorname{Tr}_n X$ *defined for each* $X \in \operatorname{mod}\Lambda$ *induces a contravariant R-functor* $\operatorname{Tr}_n : \operatorname{mod}\Lambda \to \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$.

Proof. Let $h : X \to Y$ be a morphism in $\operatorname{mod}\Lambda$ and suppose that $h_\bullet$ and $h'_\bullet$ are two extensions of h to a morphism of complexes between the minimal projective resolutions of X and Y. If we denote by u and u' the induced morphisms $\operatorname{Tr}_n Y \to \operatorname{Tr}_n X$ by $h_\bullet$ and $h'_\bullet$, respectively, then it is easy to see that the morphism $\operatorname{Tr}_n Y \to \operatorname{Tr}_n X$ induced by $h_\bullet - h'_\bullet$ is $u - u'$. Moreover, we know from the Comparison Theorem that $h_\bullet - h'_\bullet$ is null homotopic, so that it follows from Lemma 6.1.1 that $u - u'$ factors through $\mathcal{P}(\Lambda^{\mathrm{op}})$, which means that u and u' coincide in $\underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$. Therefore, $\operatorname{Tr}_n : \operatorname{mod}\Lambda \to \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ is a well defined assignment on the objects and morphisms of $\operatorname{mod}\Lambda$, which is easily seen to be a contravariant R-functor. □

Proposition 6.1.3. *The contravariant R-functor* $\operatorname{Tr}_n : \operatorname{mod}\Lambda \to \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ *induces a contravariant R-functor* $\operatorname{Tr}_n : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$.

Proof. We only need to verify that $\operatorname{Tr}_n : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ is well defined on the morphisms. For this, it suffices to show that if $h : X \to Y$ is a morphism in $\operatorname{mod}\Lambda$ which factors through $\mathcal{P}(\Lambda)$, then $\operatorname{Tr}_n h$ factors through $\mathcal{P}(\Lambda^{\mathrm{op}})$.

Well, let $h : X \to Y$ be a morphism in $\operatorname{mod}\Lambda$ such that there is some $W \in \mathcal{P}(\Lambda)$ and morphisms $v : X \to W$ and $u : W \to Y$ in $\operatorname{mod}\Lambda$ with $h = uv$. Moreover, let $(P_\bullet, f_\bullet)$ and $(Q_\bullet, g_\bullet)$ be minimal projective resolutions of X and Y, respectively, where $P_{-1} = X$ and $Q_{-1} = Y$, and complete h to a morphism of complexes $h_\bullet : P_\bullet \to Q_\bullet$ with $h_{-1} = h$. Given that W is projective and g_0 is an epimorphism, there is some morphism $w : W \to Q_0$ such that $u = g_0 w$.

$$\begin{tikzcd}
\cdots \arrow[r, "f_{n+2}"] & P_{n+1} \arrow[r, "f_{n+1}"] \arrow[dd, "h_{n+1}"] & P_n \arrow[r, "f_n"] \arrow[dd, "h_n"] & P_{n-1} \arrow[r, "f_{n-1}"] \arrow[dd, "h_{n-1}"] & \cdots \arrow[r, "f_2"] & P_1 \arrow[r, "f_1"] \arrow[dd, "h_1"] & P_0 \arrow[r, "f_0"] \arrow[dd, "h_0"] & X \arrow[r] \arrow[dd, "h"] \arrow[dr, "v"] & 0 \\
 & & & & & & & & W \arrow[dl, "u"] \arrow[dll, dashed, "w"'] \\
\cdots \arrow[r, "g_{n+2}"'] & Q_{n+1} \arrow[r, "g_{n+1}"'] & Q_n \arrow[r, "g_n"'] & Q_{n-1} \arrow[r, "g_{n-1}"'] & \cdots \arrow[r, "g_2"'] & Q_1 \arrow[r, "g_1"'] & Q_0 \arrow[r, "g_0"'] & Y \arrow[r] & 0
\end{tikzcd}$$

Therefore, if we define $s_{-1} : X \to Q_0$ by $s_{-1} = wv$ and if we follow the proof of the Comparison Theorem, then we can conclude that there are morphisms $s_i : P_i \to Q_{i+1}$ in $\operatorname{mod}\Lambda$ such that $h_i = s_{i-1} f_i + g_{i+1} s_i$ for each $i \in \mathbb{Z}$, which means that $h_\bullet$ is null homotopic. Hence we obtain by Lemma 6.1.1 that $\operatorname{Tr}_n h$ factors through $\mathcal{P}(\Lambda^{\mathrm{op}})$, as desired. □

We remark that we will also denote the contravariant R-functor $\underline{\mathrm{mod}}\,\Lambda^{\mathrm{op}} \to \underline{\mathrm{mod}}\,\Lambda$ obtained from the n-transpose of each $X \in \mathrm{mod}\,\Lambda^{\mathrm{op}}$ by Tr_n.

Next, we proceed to show that the functors $\mathrm{Tr}_n : \underline{\mathrm{mod}}\,\Lambda \leftrightarrow \underline{\mathrm{mod}}\,\Lambda^{\mathrm{op}}$ induce contravariant R-functors $\mathrm{Tr}_n : \underline{{}^{\perp n}\mathcal{P}(\Lambda)} \leftrightarrow \underline{{}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})}$ which are dualities satisfying that $\mathrm{Tr}_n^2 \simeq id$. This will be fundamental to study higher versions of the "Auslander–Reiten translations" in n-cluster tilting subcategories of $\mathrm{mod}\,\Lambda$ in the next section.

We first give some properties of the subcategory ${}^{\perp n}\mathcal{P}(\Lambda)$ of $\mathrm{mod}\,\Lambda$.

To begin with, observe that $\mathcal{P}(\Lambda) \subseteq {}^{\perp n}\mathcal{P}(\Lambda)$, and that if we take $n = 1$, then we obtain that ${}^{\perp 1}\mathcal{P}(\Lambda) = \mathrm{mod}\,\Lambda$. In addition, we also have the following result, which generalizes some of the items of [3, Chapter IV, Proposition 2.1]:

Proposition 6.1.4. *Let $X \in {}^{\perp n}\mathcal{P}(\Lambda)$,*

$$P_\bullet : \qquad P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

be a minimal projective n-presentation of X, and consider the sequence

$$P_\bullet^* : \qquad P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{f_2^*} \cdots \xrightarrow{f_{n-1}^*} P_{n-1}^* \xrightarrow{f_n^*} P_n^* \xrightarrow{f'} \mathrm{Tr}_n X \longrightarrow 0$$

in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$, where f' is the cokernel of f_n^*. Then we have the following:

(a) Either X is projective or $\mathrm{pd}_\Lambda X \geqslant n$.

(b) X is projective if and only if $\mathrm{Tr}_n X = 0$.

(c) $P_\bullet^*$ is a projective n-presentation of $\mathrm{Tr}_n X$.

(d) $\mathrm{Tr}_n X \in {}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})$.

Furthermore, if X is indecomposable and nonprojective, then we also have the following:

(e) $P_\bullet^*$ is a minimal projective n-presentation of $\mathrm{Tr}_n X$.

(f) $\mathrm{Tr}_n(\mathrm{Tr}_n X) \simeq X$.

Proof. (a) This is straightforward from Proposition 1.5.2.

(b) If X is projective, then we have $P_0 = X$ and $P_i = 0$ for each $i \geqslant 1$, and thus it is clear that $\mathrm{Tr}_n X = 0$. On the other hand, if $\mathrm{Tr}_n X = 0$, then we have that f_n^* is a split epimorphism, which implies that f_n is a split monomorphism. Hence we obtain that $P_{n-1} \simeq P_n \oplus \mathrm{Im}\, f_{n-1}$, and thus $\mathrm{Im}\, f_{n-1} = \mathrm{Ker}\, f_{n-2}$ is projective. Therefore, we obtain a projective resolution

$$0 \longrightarrow \mathrm{Ker}\, f_{n-2} \longrightarrow P_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

of X, so that $\mathrm{pd}_\Lambda X \leqslant n-1$, and we conclude by item (a) that X is projective.

(c) We already know that $P_i^* \in \mathcal{P}(\Lambda^{\mathrm{op}})$ for each i, and because

$$\frac{\operatorname{Ker} f_{i+1}^*}{\operatorname{Im} f_i^*} \simeq \operatorname{Ext}_\Lambda^i(X, \Lambda) = 0$$

for each $1 \leqslant i \leqslant n-1$, it follows that $P_\bullet^*$ is an exact sequence in $\operatorname{mod} \Lambda^{\mathrm{op}}$, so that it is a projective n-presentation of $\operatorname{Tr}_n X$.

(d) If we apply $(-)^*$ in $P_\bullet^*$, then we can consider the sequence

$$P_n^{**} \xrightarrow{f_n^{**}} P_{n-1}^{**} \xrightarrow{f_{n-1}^{**}} \cdots \xrightarrow{f_2^{**}} P_1^{**} \xrightarrow{f_1^{**}} P_0^{**}$$

in $\operatorname{mod} \Lambda$, which is exact since it is isomorphic to

$$P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \ .$$

Therefore, we have that

$$\operatorname{Ext}_{\Lambda^{\mathrm{op}}}^i(\operatorname{Tr}_n X, \Lambda^{\mathrm{op}}) \simeq \frac{\operatorname{Ker} f_{n-i}^{**}}{\operatorname{Im} f_{n-i+1}^{**}} = 0$$

for each $1 \leqslant i \leqslant n-1$, which implies that $\operatorname{Tr}_n X \in {}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})$.

Next, to verify that (e) and (f) hold, assume from now on that X is indecomposable and nonprojective.

(e) Given that we already know from item (c) that $P_\bullet^*$ is a projective n-presentation of $\operatorname{Tr}_n X$, it follows from Proposition 1.4.6 that we can write $P_\bullet^* \simeq Q_\bullet^* \oplus K_\bullet^*$, where

$$Q_\bullet^* : \qquad Q_0^* \xrightarrow{g_1^*} Q_1^* \xrightarrow{g_2^*} \cdots \xrightarrow{g_{n-1}^*} Q_{n-1}^* \xrightarrow{g_n^*} Q_n^* \longrightarrow \operatorname{Tr}_n X \longrightarrow 0$$

is a minimal projective n-presentation of $\operatorname{Tr}_n X$ and

$$K_\bullet^* : \qquad K_0^* \xrightarrow{k_1^*} K_1^* \xrightarrow{k_2^*} \cdots \xrightarrow{k_{n-1}^*} K_{n-1}^* \xrightarrow{k_n^*} K_n^* \longrightarrow 0 \longrightarrow 0$$

is an exact sequence in $\operatorname{mod} \Lambda^{\mathrm{op}}$, where the morphisms g_i^* and k_j^* are induced by morphisms $g_i : Q_i \to Q_{i-1}$ and $k_j : K_j \to K_{j-1}$ in $\mathcal{P}(\Lambda)$, respectively. In this case, by applying the functor $(-)^*$ in the morphisms g_i^* and k_j^* we can conclude that $P_\bullet$ is isomorphic to the direct sum of the sequences

$$Q_\bullet : \qquad Q_n \xrightarrow{g_n} Q_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_2} Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g'} Y \longrightarrow 0$$

and

$$K_\bullet : \qquad K_n \xrightarrow{k_n} K_{n-1} \xrightarrow{k_{n-1}} \cdots \xrightarrow{k_2} K_1 \xrightarrow{k_1} K_0 \xrightarrow{k'} Z \longrightarrow 0$$

in $\operatorname{mod}\Lambda$, where g' and k' are the cokernel of g_1 and k_1, respectively. In particular, we obtain that $Q_\bullet$ and $K_\bullet$ are exact in $\operatorname{mod}\Lambda$, and also that $X \simeq Y \oplus Z$, which implies that either $X \simeq Y$ or $X \simeq Z$ since X is indecomposable.

Well, if $X \simeq Z$, then $K_\bullet$ is a projective n-presentation of X. Hence using the fact that $P_\bullet \simeq Q_\bullet \oplus K_\bullet$ and that $P_\bullet$ is a minimal projective n-presentation of X we can conclude by Proposition 1.4.6 that $Q_\bullet = 0$. But then we have $Q_\bullet^* = 0$, so that $\operatorname{Tr}_n X = 0$, which implies that X is projective by item (b), a contradiction. Therefore, $X \simeq Y$ and by the same reasoning as above we can conclude that $K_\bullet = 0$, which implies that $P_\bullet \simeq Q_\bullet$. Consequently, $P_\bullet^* \simeq Q_\bullet^*$ and then $P_\bullet^*$ is a minimal projective n-presentation of $\operatorname{Tr}_n X$.

(f) Note that $Y \simeq \operatorname{Tr}_n(\operatorname{Tr}_n X)$, hence it follows from the above paragraph that $X \simeq \operatorname{Tr}_n(\operatorname{Tr}_n X)$. □

The result below is from [25, Corollary 1.1.2], and here we present a different proof for it.

Theorem 6.1.5. *The contravariant R-functors* $\operatorname{Tr}_n : \underline{\operatorname{mod}}\,\Lambda \leftrightarrow \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ *induce contravariant R-functors* $\operatorname{Tr}_n : \underline{{}^{\perp n}\mathcal{P}(\Lambda)} \leftrightarrow \underline{{}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})}$ *which are dualities such that* $\operatorname{Tr}_n^2 \simeq id$.

Proof. First of all, recall that we already know by Proposition 6.1.4 that $\operatorname{Tr}_n X \in {}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})$ for every $X \in {}^{\perp n}\mathcal{P}(\Lambda)$, hence $\operatorname{Tr}_n : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ induces a contravariant R-functor $\operatorname{Tr}_n : \underline{{}^{\perp n}\mathcal{P}(\Lambda)} \to \underline{{}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})}$. Furthermore, by taking Λ^{op} in place of Λ, we also obtain that $\operatorname{Tr}_n : \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}} \to \underline{\operatorname{mod}}\,\Lambda$ induces a contravariant R-functor $\operatorname{Tr}_n : \underline{{}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})} \to \underline{{}^{\perp n}\mathcal{P}(\Lambda)}$.

Now, to verify that $\operatorname{Tr}_n^2 \simeq id$ in $\underline{{}^{\perp n}\mathcal{P}(\Lambda)}$, take an arbitrary $X \in {}^{\perp n}\mathcal{P}(\Lambda)$ and write it as a finite direct sum $X \simeq X_1 \oplus \cdots \oplus X_r \oplus P$ with each X_i being indecomposable and nonprojective and P being projective. We have $X \simeq X_1 \oplus \cdots \oplus X_r$ in $\underline{{}^{\perp n}\mathcal{P}(\Lambda)}$, so that

$$\operatorname{Tr}_n X \simeq \operatorname{Tr}_n X_1 \oplus \cdots \oplus \operatorname{Tr}_n X_r$$

in $\underline{{}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})}$. Moreover, we know by Proposition 6.1.4 that $\operatorname{Tr}_n(\operatorname{Tr}_n X_i) \simeq X_i$ for each i, and it is not difficult to see that these isomorphisms are functorial since they are induced by the functorial isomorphisms $Q^{**} \simeq Q$ for $Q \in \mathcal{P}(\Lambda)$. Therefore, we obtain a functorial isomorphism $\operatorname{Tr}_n(\operatorname{Tr}_n X) \simeq X$ in $\underline{{}^{\perp n}\mathcal{P}(\Lambda)}$.

Dually, we also have that there is a functorial isomorphism $\operatorname{Tr}_n(\operatorname{Tr}_n X) \simeq X$ for each $X \in \underline{{}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})}$. Consequently, $\operatorname{Tr}_n : \underline{{}^{\perp n}\mathcal{P}(\Lambda)} \leftrightarrow \underline{{}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})}$ are dualities such that $\operatorname{Tr}_n^2 \simeq id$. □

6.2 The n-Auslander–Reiten translations

We now present higher versions of the "Auslander–Reiten translations", as well as higher versions of the "Auslander–Reiten formulas". In addition, we also show that these notions can be restricted to n-cluster tilting subcategories of $\operatorname{mod}\Lambda$.

To begin with, recall that the dualities $D : \operatorname{mod}\Lambda \leftrightarrow \operatorname{mod}\Lambda^{\mathrm{op}}$ induce dualities $D : \mathcal{P}(\Lambda) \leftrightarrow \mathcal{I}(\Lambda^{\mathrm{op}})$. Therefore, it is not difficult to see that $D : \operatorname{mod}\Lambda \leftrightarrow \operatorname{mod}\Lambda^{\mathrm{op}}$ also induce dualities $D : \underline{\operatorname{mod}}\,\Lambda \leftrightarrow \overline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$. In this case, following [25], we define

$$\tau_n = D\,\mathrm{Tr}_n : \underline{\operatorname{mod}}\,\Lambda \to \overline{\operatorname{mod}}\,\Lambda \quad \text{and} \quad \tau_n^- = \mathrm{Tr}_n\,D : \overline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda.$$

We call τ_n and τ_n^- the n-**Auslander–Reiten translations**. Furthermore, if $n = 1$, then we denote $\tau_1 = D\,\mathrm{Tr}$ and $\tau_1^- = \mathrm{Tr}\,D$ simply by τ and τ^-, respectively, and we call them the **Auslander–Reiten translations**.

As we will see in Theorem 6.3.2, the n-Auslander–Reiten translations will enable us to relate the end terms of an n-almost split sequence in an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$. Hence the reader may realize from this the importance of the n-Auslander–Reiten translations, given that they will provide us a new approach to study n-almost split sequences.

It is interesting to remark that we can also define the n-Auslander–Reiten translations by considering only τ and τ^-. In fact, for each $X \in \operatorname{mod}\Lambda$, let ΩX be the kernel of a projective cover of X, and $\Omega^- X$ be the cokernel of an injective envelope of X. Then it is not difficult to see that by considering ΩX and $\Omega^- X$ for each $X \in \operatorname{mod}\Lambda$ we can obtain functors $\Omega : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda$ and $\Omega^- : \overline{\operatorname{mod}}\,\Lambda \to \overline{\operatorname{mod}}\,\Lambda$, which are called the **syzygy functor** and the **cosyzygy functor**, respectively. In this case, note that $\mathrm{Tr}_n = \mathrm{Tr}_j\,\Omega^{n-j}$ for each $1 \leqslant j \leqslant n$, with the agreement that $\Omega^0 = id$, and thus we obtain that $\mathrm{Tr}_n = \mathrm{Tr}\,\Omega^{n-1}$. Therefore, we have that

$$\tau_n = D\,\mathrm{Tr}_n = D\,\mathrm{Tr}\,\Omega^{n-1} = \tau\Omega^{n-1},$$

and because $\Omega D = D\Omega^-$, we also obtain that

$$\tau_n^- = \mathrm{Tr}_n\,D = \mathrm{Tr}\,\Omega^{n-1}D = \mathrm{Tr}\,D\Omega^{-(n-1)} = \tau^-\Omega^{-(n-1)}.$$

We prove next that we can restrict the n-Auslander–Reiten translations to certain subcategories of $\underline{\operatorname{mod}}\,\Lambda$ and $\overline{\operatorname{mod}}\,\Lambda$, respectively, to obtain quasi-inverse equivalences. The result below is from [25, Theorem 1.4.1].

Theorem 6.2.1. *The n-Auslander–Reiten translations induce quasi-inverse equivalences*

$$\tau_n : \underline{{}^{\perp n}\mathcal{P}(\Lambda)} \to \overline{\mathcal{I}(\Lambda)^{\perp n}} \quad \textit{and} \quad \tau_n^- : \overline{\mathcal{I}(\Lambda)^{\perp n}} \to \underline{{}^{\perp n}\mathcal{P}(\Lambda)}.$$

Therefore, τ_n gives a bijection from the nonprojective objects in $\operatorname{ind}({}^{\perp n}\mathcal{P}(\Lambda))$ *to the noninjective objects in* $\operatorname{ind}(\mathcal{I}(\Lambda)^{\perp n})$*, with the inverse given by τ_n^-.*

Proof. It is not difficult to see that for a given $X \in \operatorname{mod}\Lambda$ we have that $X \in {}^{\perp n}\mathcal{P}(\Lambda)$ if and only if $DX \in \mathcal{I}(\Lambda^{\mathrm{op}})^{\perp n}$, so that $D : \operatorname{mod}\Lambda \leftrightarrow \operatorname{mod}\Lambda^{\mathrm{op}}$ induce dualities $D : {}^{\perp n}\mathcal{P}(\Lambda) \leftrightarrow \mathcal{I}(\Lambda^{\mathrm{op}})^{\perp n}$. Therefore, $D : \underline{\operatorname{mod}}\,\Lambda \leftrightarrow \overline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ also induce dualities $D : \underline{{}^{\perp n}\mathcal{P}(\Lambda)} \leftrightarrow \overline{\mathcal{I}(\Lambda^{\mathrm{op}})^{\perp n}}$ with

$D^2 \simeq id$. Consequently, because we know from Theorem 6.1.5 that $\mathrm{Tr}_n : \underline{{}^{\perp n}\mathcal{P}(\Lambda)} \leftrightarrow \underline{{}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})}$ are dualities with $\mathrm{Tr}_n^2 \simeq id$, we obtain that

$$\tau_n : \underline{{}^{\perp n}\mathcal{P}(\Lambda)} \to \overline{\mathcal{I}(\Lambda)^{\perp n}} \quad \text{and} \quad \tau_n^- : \overline{\mathcal{I}(\Lambda)^{\perp n}} \to \underline{{}^{\perp n}\mathcal{P}(\Lambda)}$$

are quasi-inverse equivalences.

The last assertion in the statement of the theorem follows immediately. □

We now proceed to present higher versions of the "Auslander–Reiten formulas". For this, we first give some results.

Following [3], for each $X, Y \in \operatorname{mod}\Lambda$ we define φ_Y^X to be the morphism of R-modules

$$\begin{aligned} \varphi_Y^X : Y \otimes_\Lambda X^* &\to \operatorname{Hom}_\Lambda(X, Y) \\ y \otimes f &\mapsto (x \mapsto yf(x)) \end{aligned}$$

which is easily seen to be functorial in both X and Y.

Lemma 6.2.2. *Let $X, Y \in \operatorname{mod}\Lambda$. If X or Y is projective, then φ_Y^X is an isomorphism.*

Proof. It is not difficult to see that if φ_Y^Λ is an isomorphism, then φ_Y^X is an isomorphism whenever X is projective, and likewise, that if φ_Λ^X is an isomorphism, then φ_Y^X is an isomorphism whenever Y is projective. Therefore, it suffices to show that both φ_Y^Λ and φ_Λ^X are isomorphisms, and for this we can verify that

$$\begin{aligned} \operatorname{Hom}_\Lambda(\Lambda, Y) &\to Y \otimes_\Lambda \Lambda^* \\ g &\mapsto g(1) \otimes id \end{aligned}$$

is the inverse of φ_Y^Λ, as well as that

$$\begin{aligned} \operatorname{Hom}_\Lambda(X, \Lambda) &\to \Lambda \otimes_\Lambda X^* \\ g &\mapsto 1 \otimes g \end{aligned}$$

is the inverse of φ_Λ^X, though we leave the details for the reader. □

The next result is from [3, Chapter IV, Lemma 2.12].

Lemma 6.2.3. *If $X, Y \in \operatorname{mod}\Lambda$, then* $\operatorname{Coker}\varphi_Y^X = \underline{\operatorname{Hom}}_\Lambda(X, Y)$.

Proof. We need to prove that $\operatorname{Im}\varphi_Y^X = \mathcal{P}(X, Y)$. For this, let $f : P \to Y$ be an epimorphism in $\operatorname{mod}\Lambda$ with $P \in \mathcal{P}(\Lambda)$ and consider the commutative diagram

$$\begin{tikzcd} P \otimes_\Lambda X^* \arrow[r, "\varphi_P^X"] \arrow[d, "f\otimes id"'] & \operatorname{Hom}_\Lambda(X, P) \arrow[d, "\operatorname{Hom}_\Lambda(X,f)"] \\ Y \otimes_\Lambda X^* \arrow[r, "\varphi_Y^X"'] & \operatorname{Hom}_\Lambda(X, Y) \end{tikzcd}$$

in Mod R. We know by Lemma 6.2.2 that φ_P^X is an isomorphism, and because $-\otimes_\Lambda X^*$ is a right exact functor, we also have that $f\otimes id$ is an epimorphism. Consequently, we obtain that $\operatorname{Im}\varphi_Y^X = \operatorname{Im}\operatorname{Hom}_\Lambda(X,f)$, and it is easy to verify that $\operatorname{Im}\operatorname{Hom}_\Lambda(X,f) = \mathcal{P}(X,Y)$. □

With the above lemmas being given, we can now prove the next result, which is from [25, Proposition 1.1.3].

Proposition 6.2.4. *For each $X \in {}^{\perp n}\mathcal{P}(\Lambda)$, $Y \in \operatorname{mod}\Lambda$ and $0 < i < n$, there are isomorphisms*

$$\operatorname{Tor}^\Lambda_{n-i}(Y,\operatorname{Tr}_n X) \simeq \operatorname{Ext}^i_\Lambda(X,Y) \quad \textit{and} \quad \operatorname{Tor}^\Lambda_n(Y,\operatorname{Tr}_n X) \simeq \underline{\operatorname{Hom}}_\Lambda(X,Y)$$

that are functorial in both variables.

Proof. To begin with, let

$$P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

be a minimal projective n-presentation of X, so that

$$P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{f_2^*} \cdots \xrightarrow{f_{n-1}^*} P_{n-1}^* \xrightarrow{f_n^*} P_n^* \longrightarrow \operatorname{Tr}_n X \longrightarrow 0$$

is a projective n-presentation of $\operatorname{Tr}_n X$ by Proposition 6.1.4. Then we can consider the following commutative diagram

$$\begin{tikzcd}
Y\otimes_\Lambda P_0^* \arrow[r, "id\otimes f_1^*"] \arrow[d, "\simeq"] & Y\otimes_\Lambda P_1^* \arrow[r, "id\otimes f_2^*"] \arrow[d, "\simeq"] & \cdots \arrow[r, "id\otimes f_{n-1}^*"] & Y\otimes_\Lambda P_{n-1}^* \arrow[r, "id\otimes f_n^*"] \arrow[d, "\simeq"] & Y\otimes_\Lambda P_n^* \arrow[d, "\simeq"] \\
\operatorname{Hom}_\Lambda(P_0,Y) \arrow[r, "(f_1{,}Y)"'] & \operatorname{Hom}_\Lambda(P_1,Y) \arrow[r, "(f_2{,}Y)"'] & \cdots \arrow[r, "(f_{n-1}{,}Y)"'] & \operatorname{Hom}_\Lambda(P_{n-1},Y) \arrow[r, "(f_n{,}Y)"'] & \operatorname{Hom}_\Lambda(P_n,Y)
\end{tikzcd}$$

whose vertical arrows $\varphi_Y^{P_i}$ are isomorphisms by Lemma 6.2.2, where $(f_i,Y) = \operatorname{Hom}_\Lambda(f_i,Y)$. In this case, since

$$\operatorname{Tor}^\Lambda_{n-i}(Y,\operatorname{Tr}_n X) = \frac{\operatorname{Ker}(id\otimes f_{i+1}^*)}{\operatorname{Im}(id\otimes f_i^*)} \quad \text{and} \quad \operatorname{Ext}^i_\Lambda(X,Y) \simeq \frac{\operatorname{Ker}(\operatorname{Hom}_\Lambda(f_{i+1},Y))}{\operatorname{Im}(\operatorname{Hom}_\Lambda(f_i,Y))}$$

we conclude that there is an isomorphism

$$\operatorname{Tor}^\Lambda_{n-i}(Y,\operatorname{Tr}_n X) \simeq \operatorname{Ext}^i_\Lambda(X,Y)$$

which is functorial in both variables for each $0 < i < n$.

To prove the remaining isomorphism, first note that because $(-)^*$ is a left exact functor, the sequence

$$0 \longrightarrow X^* \xrightarrow{f_0^*} P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{f_2^*} \cdots \xrightarrow{f_{n-1}^*} P_{n-1}^* \xrightarrow{f_n^*} P_n^* \longrightarrow \operatorname{Tr}_n X \longrightarrow 0$$

is exact in $\operatorname{mod}\Lambda^{\mathrm{op}}$, and thus we can take an epimorphism $g : Q \to X^*$ with $Q \in \mathcal{P}(\Lambda^{\mathrm{op}})$ to obtain a projective $(n+1)$-presentation of $\operatorname{Tr}_n X$. Therefore, if we apply $Y \otimes_\Lambda -$ in this presentation, then we obtain the complex

$$Y \otimes_\Lambda Q \xrightarrow{id \otimes f_0^* g} Y \otimes_\Lambda P_0^* \xrightarrow{id \otimes f_1^*} Y \otimes_\Lambda P_1^* \xrightarrow{id \otimes f_2^*} \cdots \xrightarrow{id \otimes f_n^*} Y \otimes_\Lambda P_n^*$$

and because $\operatorname{Im}(id \otimes f_0^* g) = \operatorname{Im}(id \otimes f_0^*)$, we conclude that

$$\operatorname{Tor}_n^\Lambda(Y, \operatorname{Tr}_n X) = \frac{\operatorname{Ker}(id \otimes f_1^*)}{\operatorname{Im}(id \otimes f_0^*)}.$$

Now, from the commutative diagram

$$\begin{tikzcd}
 & Y \otimes_\Lambda X^* \arrow[r, "id \otimes f_0^*"] \arrow[d, "\varphi_Y^X"] & Y \otimes_\Lambda P_0^* \arrow[r, "id \otimes f_1^*"] \arrow[d, "\simeq"] & Y \otimes_\Lambda P_1^* \arrow[d, "\simeq"] \\
0 \arrow[r] & \operatorname{Hom}_\Lambda(X, Y) \arrow[r, "\operatorname{Hom}_\Lambda(f_0, Y)"'] & \operatorname{Hom}_\Lambda(P_0, Y) \arrow[r, "\operatorname{Hom}_\Lambda(f_1, Y)"'] & \operatorname{Hom}_\Lambda(P_1, Y)
\end{tikzcd}$$

in $\operatorname{Mod} R$ whose bottom row is exact we can conclude that $\operatorname{Ker}(id \otimes f_1^*) \simeq \operatorname{Hom}_\Lambda(X, Y)$ and $\operatorname{Im}(id \otimes f_0^*) \simeq \operatorname{Im} \varphi_Y^X$. Hence by Lemma 6.2.3 we obtain an isomorphism

$$\operatorname{Tor}_n^\Lambda(Y, \operatorname{Tr}_n X) \simeq \underline{\operatorname{Hom}}_\Lambda(X, Y)$$

which is functorial in both variables. □

We are now ready to present higher versions of the "Auslander–Reiten formulas", that are the isomorphisms given in the next result, which is from [25, Theorem 1.5]. We remark that the so called "Auslander–Reiten formulas" are obtained below by taking $n = 1$.

Theorem 6.2.5. *For each $X \in {}^{\perp n}\mathcal{P}(\Lambda)$, $Y \in \mathcal{I}(\Lambda)^{\perp n}$, $Z \in \operatorname{mod}\Lambda$ and $0 < i < n$ there are isomorphisms*

$$\operatorname{Ext}_\Lambda^i(X, Z) \simeq D \operatorname{Ext}_\Lambda^{n-i}(Z, \tau_n X), \qquad \underline{\operatorname{Hom}}_\Lambda(X, Z) \simeq D \operatorname{Ext}_\Lambda^n(Z, \tau_n X)$$

$$\operatorname{Ext}_\Lambda^i(Z, Y) \simeq D \operatorname{Ext}_\Lambda^{n-i}(\tau_n^- Y, Z), \qquad \overline{\operatorname{Hom}}_\Lambda(Z, Y) \simeq D \operatorname{Ext}_\Lambda^n(\tau_n^- Y, Z)$$

which are functorial in both variables.

Proof. It follows from [1, Chapter IX, Theorem 4.11] that for every $W, Z \in \operatorname{mod}\Lambda$ and $j \geqslant 0$ we have an isomorphism

$$\operatorname{Tor}_j^\Lambda(Z, DW) \simeq D \operatorname{Ext}_\Lambda^j(Z, W)$$

which is functorial in both variables. Therefore, if we take $W = D\operatorname{Tr}_n X = \tau_n X$, then we obtain that

$$\operatorname{Tor}_{n-i}^\Lambda(Z, \operatorname{Tr}_n X) \simeq D \operatorname{Ext}_\Lambda^{n-i}(Z, \tau_n X)$$

for each $i \leqslant n$. Consequently, we conclude by Proposition 6.2.4 that there are isomorphisms

$$\operatorname{Ext}^i_\Lambda(X,Z) \simeq D\operatorname{Ext}^{n-i}_\Lambda(Z,\tau_n X) \quad \text{and} \quad \underline{\operatorname{Hom}}_\Lambda(X,Z) \simeq D\operatorname{Ext}^n_\Lambda(Z,\tau_n X)$$

which are functorial in both variables, where $0 < i < n$.

The proof of the other isomorphisms follows by duality since $DY \in {}^{\perp n}\mathcal{P}(\Lambda^{\mathrm{op}})$. □

To end this section, we show that we can restrict the n-Auslander–Reiten translations for n-cluster tilting subcategories of $\operatorname{mod}\Lambda$, as well as that the higher versions of the Auslander–Reiten formulas also hold for objects of such subcategories. To achieve these goals, the reader should first note that if $\mathcal{C}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$, then $\mathcal{C} \subseteq {}^{\perp n}\mathcal{P}(\Lambda)$ and $\mathcal{C} \subseteq \mathcal{I}(\Lambda)^{\perp n}$.

The result below is from [25, Theorem 2.3].

Theorem 6.2.6. *Let $\mathcal{C}$ be an n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$.

(a) $\tau_n X \in \mathcal{C}$ and $\tau_n^- X \in \mathcal{C}$ for all $X \in \mathcal{C}$.

(b) The n-Auslander–Reiten translations induce quasi-inverse equivalences

$$\tau_n : \underline{\mathcal{C}} \to \overline{\mathcal{C}} \quad \textit{and} \quad \tau_n^- : \overline{\mathcal{C}} \to \underline{\mathcal{C}}.$$

(c) τ_n gives a bijection from the nonprojective objects in $\operatorname{ind}\mathcal{C}$ *to the noninjective objects in* $\operatorname{ind}\mathcal{C}$*, with the inverse given by τ_n^-.*

Proof. (a) Let $X \in \mathcal{C}$. To prove that $\tau_n X \in \mathcal{C}$, it suffices to show that $\tau_n X \in \mathcal{C}^{\perp n}$ since $\mathcal{C} = \mathcal{C}^{\perp n}$. Therefore, take $Z \in \mathcal{C}$. It follows from Theorem 6.2.5 that

$$D\operatorname{Ext}^{n-i}_\Lambda(Z,\tau_n X) \simeq \operatorname{Ext}^i_\Lambda(X,Z) = 0$$

for each $0 < i < n$, which implies that $\operatorname{Ext}^j_\Lambda(Z,\tau_n X) = 0$ for each $0 < j < n$, and thus $\tau_n X \in \mathcal{C}^{\perp n}$. Similarly, we can verify that $\tau_n^- X \in \mathcal{C}$.

(b) This follows immediately from item (a) and Theorem 6.2.1.

(c) This is a straightforward consequence of item (b). □

Finally, we also have the following result, which is from [25, Theorem 2.3.1], that gives "higher Auslander–Reiten formulas" for n-cluster tilting subcategories of $\operatorname{mod}\Lambda$:

Theorem 6.2.7. *Let $\mathcal{C}$ be an n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$. *For each $X, Y \in \mathcal{C}$ there are isomorphisms*

$$\underline{\operatorname{Hom}}_\Lambda(X,Y) \simeq D\operatorname{Ext}^n_\Lambda(Y,\tau_n X) \quad \textit{and} \quad \overline{\operatorname{Hom}}_\Lambda(X,Y) \simeq D\operatorname{Ext}^n_\Lambda(\tau_n^- Y,X)$$

which are functorial in both variables.

Proof. This follows from Theorem 6.2.5. □

6.3 n-Almost split sequences and the socle

By using the n-Auslander–Reiten translations, we now show how we can relate the end terms of an n-almost split sequence in an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$, as well as we verify that these sequences generate the socle of certain modules.

In this section, $\mathcal{C}$ will always denote an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$.

First of all, note that if

$$0 \longrightarrow X \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Z \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$, then we already know by Proposition 4.1.4 that the right $\mathcal{C}$-module $\operatorname{Ext}^n_\Lambda(-,X)|_\mathcal{C}$ has a simple submodule which is isomorphic to S_Z, as well as that the left $\mathcal{C}$-module $\operatorname{Ext}^n_\Lambda(Z,-)|_\mathcal{C}$ has a simple submodule which is isomorphic to S^X. In the next proposition we will verify that these submodules are the unique simple submodules of $\operatorname{Ext}^n_\Lambda(-,X)|_\mathcal{C}$ and $\operatorname{Ext}^n_\Lambda(Z,-)|_\mathcal{C}$, respectively.

The result below is from [25, Corollary 2.4.1].

Proposition 6.3.1. *Let $X \in \mathcal{C}$ be indecomposable and noninjective and $Z \in \mathcal{C}$ be indecomposable and nonprojective. Then we have the following:*

(a) The right $\mathcal{C}$-module $\operatorname{Ext}^n_\Lambda(-,X)|_\mathcal{C}$ has a simple socle which is isomorphic to $S_{\tau_n^- X}$.

(b) The left $\mathcal{C}$-module $\operatorname{Ext}^n_\Lambda(Z,-)|_\mathcal{C}$ has a simple socle which is isomorphic to $S^{\tau_n Z}$.

Proof. We only prove item (a) since (b) will follow by duality.

To begin with, observe that because we know from Theorem 5.1.12 that there is an n-almost split sequence in $\mathcal{C}$ whose left end term is X, then as we have already pointed out, it follows from Proposition 4.1.4 that $\operatorname{Ext}^n_\Lambda(-,X)|_\mathcal{C}$ has at least one simple submodule. Moreover, note that by theorems 6.2.6 and 6.2.7 and Proposition 1.6.1 we have that $\operatorname{Ext}^n_\Lambda(-,X)|_\mathcal{C} \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$ and we also have an isomorphism of left $\mathcal{C}$-modules

$$D_\mathcal{C}(\operatorname{Ext}^n_\Lambda(-,X)|_\mathcal{C}) \simeq \underline{\operatorname{Hom}}_\Lambda(\tau_n^- X, -)|_\mathcal{C} = \frac{\mathcal{C}(\tau_n^- X, -)}{\mathcal{P}(\tau_n^- X, -)|_\mathcal{C}}.$$

Now, let $F \in \operatorname{Mod}\mathcal{C}$ be a simple submodule of $\operatorname{Ext}^n_\Lambda(-,X)|_\mathcal{C}$. To prove item (a), it suffices to show that $F \simeq S_{\tau_n^- X}$. Well, we have that $F \in (\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$, so that the canonical inclusion $\eta : F \to \operatorname{Ext}^n_\Lambda(-,X)|_\mathcal{C}$ is in $(\mathcal{C}^{\mathrm{op}}, \operatorname{mod} R)$. Hence we can consider the epimorphism $D_\mathcal{C}(\eta) : D_\mathcal{C}(\operatorname{Ext}^n_\Lambda(-,X)|_\mathcal{C}) \to D_\mathcal{C}(F)$ in $(\mathcal{C}, \operatorname{mod} R)$, which is also an epimorphism in $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$. Consequently, we obtain an epimorphism $\underline{\operatorname{Hom}}_\Lambda(\tau_n^- X, -)|_\mathcal{C} \to D_\mathcal{C}(F)$ in $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$. But recall from Corollary 5.1.8 that $\mathcal{C}$ is a dualizing R-variety, and thus we obtain by Proposition 5.1.10 that $D_\mathcal{C}(F)$ is a simple left $\mathcal{C}$-module. Therefore, we conclude that

$$D_\mathcal{C}(F) \simeq \frac{\underline{\operatorname{Hom}}_\Lambda(\tau_n^- X, -)|_\mathcal{C}}{G}$$

for some maximal submodule $G \subseteq \underline{\mathrm{Hom}}_{\Lambda}(\tau_n^- X, -)|_{\mathcal{C}}$. In this case, note that by Theorem 6.2.6 we have that $\tau_n^- X$ is indecomposable and nonprojective, and this implies that $\mathcal{P}(\tau_n^- X, -)|_{\mathcal{C}}$ is a proper submodule of $\mathcal{C}(\tau_n^- X, -)$ since $id_{\tau_n^- X} \notin \mathcal{P}(\tau_n^- X, \tau_n^- X)$. Furthermore, we also obtain by Proposition 3.3.2 that $\mathrm{rad}_{\mathcal{C}}(\tau_n^- X, -)$ is the unique maximal submodule of $\mathcal{C}(\tau_n^- X, -)$, and because we know from Proposition 2.4.3 that $\mathcal{P}(\tau_n^- X, -)|_{\mathcal{C}} \subseteq \mathrm{rad}_{\mathcal{C}}(\tau_n^- X, -)$, we conclude that

$$G \simeq \frac{\mathrm{rad}_{\mathcal{C}}(\tau_n^- X, -)}{\mathcal{P}(\tau_n^- X, -)|_{\mathcal{C}}}.$$

Therefore, we have that

$$D_{\mathcal{C}}(F) \simeq \frac{\mathcal{C}(\tau_n^- X, -)}{\mathrm{rad}_{\mathcal{C}}(\tau_n^- X, -)} = S^{\tau_n^- X},$$

which implies that $F \simeq D_{\mathcal{C}}(S^{\tau_n^- X})$. Thus, given that $F(\tau_n^- X) \neq 0$, it follows from Theorem 3.3.3 and Proposition 3.3.4 that $F \simeq S_{\tau_n^- X}$. □

As a consequence of Proposition 6.3.1, we obtain the theorem below, which is from [25, Theorem 3.3.1], that shows how we can use the n-Auslander–Reiten translations to determine the right and left end terms of an n-almost split sequence one from each other.

Theorem 6.3.2. *If*

$$0 \longrightarrow X \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Z \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$, then $Z \simeq \tau_n^- X$ and $X \simeq \tau_n Z$.

Proof. We already know from Proposition 4.1.4 that the $\mathcal{C}$-module $\mathrm{Ext}^n_{\Lambda}(-, X)|_{\mathcal{C}}$ has a simple submodule which is isomorphic to S_Z. Moreover, it follows from Proposition 6.3.1 that $S_Z \simeq S_{\tau_n^- X}$. Therefore, we obtain from Corollary 3.3.5 that $Z \simeq \tau_n^- X$.

Similarly, we can conclude that $X \simeq \tau_n Z$. □

By using the above results, we can now give an improved version of Corollary 4.1.5. The proposition below was based on [25, Corollary 2.4.1].

Proposition 6.3.3. *If*

$$\mathtt{E}: \qquad 0 \longrightarrow X \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Z \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$, then we have the following:

(a) *The right $\mathrm{End}_{\Lambda}(Z)$-module $\mathrm{Ext}^n_{\Lambda}(Z, X)$ has a simple socle which is generated by $\mathtt{E}$ and which satisfies that*

$$\mathrm{soc}_{\mathrm{End}_{\Lambda}(Z)} \mathrm{Ext}^n_{\Lambda}(Z, X) = \mathrm{soc}_{\mathcal{C}} \left(\mathrm{Ext}^n_{\Lambda}(-, X)|_{\mathcal{C}} \right)(Z).$$

(b) *The left* $\operatorname{End}_\Lambda(X)$*-module* $\operatorname{Ext}^n_\Lambda(Z,X)$ *has a simple socle which is generated by* $\mathbb{E}$ *and which satisfies that*

$$\operatorname{soc}_{\operatorname{End}_\Lambda(X)^{\mathrm{op}}} \operatorname{Ext}^n_\Lambda(Z,X) = \operatorname{soc}_{\mathcal{C}^{\mathrm{op}}} \left(\operatorname{Ext}^n_\Lambda(Z,-)|_{\mathcal{C}}\right)(X).$$

Proof. We only prove item (a) since (b) will follow by duality.

First of all, recall that we already know from Corollary 4.1.5 that $\mathbb{E}$ generates a simple submodule of the right $\operatorname{End}_\Lambda(Z)$-module $\operatorname{Ext}^n_\Lambda(Z,X)$. Moreover, observe that by Theorem 6.3.2 we have that $Z \simeq \tau_n^- X$ and $X \simeq \tau_n Z$.

Now, note that from Proposition 6.3.1 we obtain that $\operatorname{soc}_{\mathcal{C}}\left(\operatorname{Ext}^n_\Lambda(-,X)|_{\mathcal{C}}\right) \simeq S_Z$, and thus it follows from Proposition 3.3.4 that $\operatorname{soc}_{\mathcal{C}}\left(\operatorname{Ext}^n_\Lambda(-,X)|_{\mathcal{C}}\right)(Z)$ is a simple $\operatorname{End}_\Lambda(Z)$-module. Consequently, we have that $\operatorname{soc}_{\mathcal{C}}\left(\operatorname{Ext}^n_\Lambda(-,X)|_{\mathcal{C}}\right)(Z) \subseteq \operatorname{soc}_{\operatorname{End}_\Lambda(Z)} \operatorname{Ext}^n_\Lambda(Z,X)$.

Finally, to conclude the proof of item (a), it suffices to show that $\operatorname{Ext}^n_\Lambda(Z,X)$ has only one simple submodule. To verify it, let $N \in \operatorname{Mod}\operatorname{End}_\Lambda(Z)$ be a simple submodule of $\operatorname{Ext}^n_\Lambda(Z,X)$. Since we know from theorems 6.2.6 and 6.2.7 that $\underline{\operatorname{Hom}}_\Lambda(Z,Z) \simeq D\operatorname{Ext}^n_\Lambda(Z,X)$, we can conclude from Proposition 1.6.1 that $\operatorname{Ext}^n_\Lambda(Z,X) \in \operatorname{mod}\operatorname{End}_\Lambda(Z)$, given that $\operatorname{End}_\Lambda(Z)$ is an Artin algebra. Therefore, the canonical inclusion $N \to \operatorname{Ext}^n_\Lambda(Z,X)$ is in $\operatorname{mod}\operatorname{End}_\Lambda(Z)$, so that we obtain an epimorphism $D\operatorname{Ext}^n_\Lambda(Z,X) \to DN$ in $\operatorname{mod}\operatorname{End}_\Lambda(Z)^{\mathrm{op}}$. Hence we obtain an epimorphism $\underline{\operatorname{Hom}}_\Lambda(Z,Z) \to DN$ in $\operatorname{mod}\operatorname{End}_\Lambda(Z)^{\mathrm{op}}$, and because DN is simple, we conclude that

$$DN \simeq \frac{\underline{\operatorname{Hom}}_\Lambda(Z,Z)}{M}$$

for some maximal submodule $M \subseteq \underline{\operatorname{Hom}}_\Lambda(Z,Z)$. In this case, note that since Z is nonprojective, we have that $id_Z \notin \mathcal{P}(Z,Z)$, which implies that $\mathcal{P}(Z,Z)$ is a proper submodule of $\operatorname{Hom}_\Lambda(Z,Z)$. Furthermore, given that Z is indecomposable, we have that $\operatorname{Hom}_\Lambda(Z,Z)$ has a unique maximal submodule which coincides with the Jacobson radical $\operatorname{rad}\operatorname{End}_\Lambda(Z)$ of $\operatorname{End}_\Lambda(Z,Z)$. Consequently, we obtain that $\mathcal{P}(Z,Z) \subseteq \operatorname{rad}\operatorname{End}_\Lambda(Z)$, and thus

$$M \simeq \frac{\operatorname{rad}\operatorname{End}_\Lambda(Z)}{\mathcal{P}(Z,Z)},$$

which implies that

$$DN \simeq \frac{\operatorname{Hom}_\Lambda(Z,Z)}{\operatorname{rad}\operatorname{End}_\Lambda(Z)}.$$

Therefore,

$$N \simeq D\left(\frac{\operatorname{Hom}_\Lambda(Z,Z)}{\operatorname{rad}\operatorname{End}_\Lambda(Z)}\right)$$

and we are done. □

6.4 Examples

In this section we describe the n-cluster tilting subcategories of $\operatorname{mod}\Lambda$ for a certain kind of algebra Λ, thereby providing the reader with many examples of n-cluster tilting subcategories. Furthermore, we study the behavior of the n-Auslander–Reiten translations in such subcategories of $\operatorname{mod}\Lambda$, as well as we give their n-almost split sequences.

We remark that here we will make use of some concepts and results about quivers, path algebras and modules over such algebras. However, the reader who is not familiarized with these subjects may still benefit from the content of this section, and is referred to [3] for further details on them.

In what follows, K will denote a field and Q will be the quiver

$$Q: \qquad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow m-1 \longrightarrow m \ ,$$

where $m \geqslant 0$. Moreover, KQ will denote the path algebra of Q, $\operatorname{rad} KQ$ will stand for the Jacobson radical of KQ and we will let Λ be the quotient algebra $KQ/(\operatorname{rad} KQ)^2$.

Our aim here is to give a complete characterization of the n-cluster tilting subcategories of $\operatorname{mod}\Lambda$. For this, we first present some properties of the objects of $\operatorname{mod}\Lambda$.

To begin with, for each $0 \leqslant j \leqslant m$ let P_j, I_j and S_j be the indecomposable finitely generated projective Λ-module, the indecomposable finitely generated injective Λ-module and the simple Λ-module, respectively, corresponding to the vertex j of Q. In this case, we have that these Λ-modules are actually the only indecomposable objects of $\operatorname{mod}\Lambda$, and we also have that $I_j = P_{j-1}$ for each $1 \leqslant j \leqslant m$, as well as that $S_0 = I_0$ and $S_m = P_m$.

Next, we give some results which will be useful to determine the n-cluster tilting subcategories of $\operatorname{mod}\Lambda$.

Lemma 6.4.1. *For integers $0 \leqslant j, k \leqslant m$, we have that $\operatorname{Hom}_\Lambda(P_j, S_k) \neq 0$ if and only if $j = k$.*

Proof. This is straightforward and left to the reader. □

Lemma 6.4.2. *For each integer $0 \leqslant j \leqslant m$ we have that the minimal projective resolution of S_j in $\operatorname{mod}\Lambda$ is given by the exact sequence*

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_{j+1} \longrightarrow P_j \longrightarrow S_j \longrightarrow 0 \ .$$

Proof. This is easy and left to the reader. □

Proposition 6.4.3. *Given integers $0 \leqslant j, k \leqslant m$ and $i \geqslant 1$, we have that $\operatorname{Ext}^i_\Lambda(S_j, S_k) \neq 0$ if and only if $i + j = k$.*

Proof. By Lemma 6.4.2 we know that the minimal projective resolution of S_j in $\operatorname{mod}\Lambda$ is given by the exact sequence

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_{j+1} \longrightarrow P_j \longrightarrow S_j \longrightarrow 0 \ .$$

Therefore, in order to compute $\operatorname{Ext}^i_\Lambda(S_j, S_k)$, we can apply the functor $\operatorname{Hom}_\Lambda(-, S_k)$ in this sequence and consider the complex

$$\operatorname{Hom}_\Lambda(P_j, S_k) \longrightarrow \operatorname{Hom}_\Lambda(P_{j+1}, S_k) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Lambda(P_m, S_k) \longrightarrow 0 \ .$$

In this case, it follows from Lemma 6.4.1 that $\operatorname{Ext}^i_\Lambda(S_j, S_k) \neq 0$ if and only if $i + j = k$. □

We are now ready to characterize the n-cluster tilting subcategories of $\operatorname{mod}\Lambda$. The result below is from [28, Proposition 6.2] and is due to Gustavo Jasso and Martin Herschend.

Theorem 6.4.4. *There is an n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$ *if and only if there is some integer $q \geqslant 0$ such that $m = qn$. Furthermore, if this is the case, then there is only one n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$*, which is given by*

$$\operatorname{add}\left(P_0 \oplus P_1 \oplus \cdots \oplus P_{m-1} \oplus S_0 \oplus S_n \oplus S_{2n} \oplus \cdots \oplus S_{qn}\right).$$

Proof. First of all, note that it is easy to conclude from Lemma 6.4.2 that $\operatorname{gl.dim}\Lambda = m$. In particular, we have that the case $m = 0$ is trivial since we obtain that Λ is semisimple. Therefore, from now on in this proof, assume that $m \geqslant 1$.

Suppose that there is an n-cluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$. Then we obtain by Proposition 4.4.2 that $n \leqslant m$, and because $\mathcal{P}(\Lambda) \subseteq \mathcal{C}$ and $\mathcal{I}(\Lambda) \subseteq \mathcal{C}$, we also have that $P_0, P_1, \dots, P_{m-1}, S_0, S_m \in \mathcal{C}$. Moreover, given that $\mathcal{C}$ is n-rigid and $n \leqslant m$, we conclude from Proposition 6.4.3 that $S_i \notin \mathcal{C}$ for each $0 < i < n$.

Now, let q be the positive integer such that $qn \leqslant m < (q+1)n$. We will show below that $S_{jn} \in \mathcal{C}$ for each $1 \leqslant j \leqslant q$, as well as that $S_k \notin \mathcal{C}$ if k is not a multiple of n.

Well, suppose that $S_n \notin \mathcal{C}$. Then $S_n \notin \mathcal{C}^{\perp n}$, which implies that there is some $S_k \in \mathcal{C}$ such that $\operatorname{Ext}^i_\Lambda(S_k, S_n) \neq 0$ for some integer $0 < i < n$. Hence we obtain by Proposition 6.4.3 that $i + k = n$, and consequently we have that $0 < k < n$, which is a contradiction. Therefore, $S_n \in \mathcal{C}$, and because $\mathcal{C}$ is n-rigid, we also obtain by Proposition 6.4.3 that $S_{n+i} \notin \mathcal{C}$ for each $0 < i < n$ such that $n + i \leqslant m$.

Clearly, if we proceed with the reasoning of the above paragraph, then we conclude that $S_{jn} \in \mathcal{C}$ for each $1 \leqslant j \leqslant q$, as well as that $S_k \notin \mathcal{C}$ if k is not a multiple of n.

Consequently, given that $S_m \in \mathcal{C}$, we obtain from the above discussion that m must be a multiple of n, and thus $m = qn$. In addition, we also have proved that

$$\mathcal{C} = \operatorname{add}\left(P_0 \oplus P_1 \oplus \cdots \oplus P_{m-1} \oplus S_0 \oplus S_n \oplus S_{2n} \oplus \cdots \oplus S_{qn}\right).$$

To conclude the proof, it remains to verify that if there is some integer $q \geqslant 1$ such that $m = qn$, then

$$\mathcal{B} = \operatorname{add}\left(P_0 \oplus P_1 \oplus \cdots \oplus P_{m-1} \oplus S_0 \oplus S_n \oplus S_{2n} \oplus \cdots \oplus S_{qn}\right)$$

is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$. In this case, note that because $\mathcal{P}(\Lambda) \subseteq \mathcal{B}$, it follows from Proposition 5.3.1 that to conclude that $\mathcal{B}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$, it suffices to show that $\mathcal{B} = \mathcal{B}^{\perp n}$.

Well, assume the conditions of the above paragraph. We will prove that $\mathcal{B} = \mathcal{B}^{\perp n}$.

We first verify that $\mathcal{B} \subseteq \mathcal{B}^{\perp n}$, which is equivalent to show that $\mathcal{B}$ is n-rigid. For this, we only need to prove that $\operatorname{Ext}^i_\Lambda(X, Y) = 0$ for each integer $0 < i < n$ and each pair of indecomposable objects $X, Y \in \operatorname{mod}\Lambda$. However, note that $P_0, P_1, \dots, P_{m-1}$ are both injective and projective in $\operatorname{mod}\Lambda$, and thus, to conclude that $\mathcal{B}$ is n-rigid, it suffices to verify that $\operatorname{Ext}^i_\Lambda(S_{jn}, S_{kn}) = 0$ for each integer $0 < i < n$ and each pair of integers $0 \leqslant j, k \leqslant q$. But this follows easily from Proposition 6.4.3, hence $\mathcal{B} \subseteq \mathcal{B}^{\perp n}$.

Finally, to conclude that $\mathcal{B} = \mathcal{B}^{\perp n}$, we only need to show that if an integer $0 < k < m$ is not a multiple of n, then $S_k \notin \mathcal{B}^{\perp n}$. Therefore, let k be such an integer and take the integers $j \geqslant 0$ and $0 < i < n$ such that $jn + i = k$. Then we obtain by Proposition 6.4.3 that $\operatorname{Ext}^i_\Lambda(S_{jn}, S_k) \neq 0$, and this implies that $S_k \notin \mathcal{B}^{\perp n}$, as desired. □

Next, we will study the behavior of the n-Auslander–Reiten translations in the n-cluster tilting subcategories of $\operatorname{mod}\Lambda$, as well as we will show what their n-almost split sequences are. For this, we will consider for the rest of this section that $m \geqslant 1$ since the case $m = 0$ is uninteresting, as we have already observed in the proof of Theorem 6.4.4.

In what follows, let n and q be positive integers such that $m = qn$, and let $\mathcal{C}$ be the unique n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ that was described in Theorem 6.4.4.

First, note that $S_0, S_n, \dots, S_{(q-1)n}$ are all the nonprojective objects in $\operatorname{ind}\mathcal{C}$, and that $S_n, S_{2n}, \dots, S_{qn}$ are all the noninjective objects in $\operatorname{ind}\mathcal{C}$. Therefore, the n-almost split sequences in $\mathcal{C}$ have $S_0, S_n, \dots, S_{(q-1)n}$ as their right end terms, as well as they have $S_n, S_{2n}, \dots, S_{qn}$ as their left end terms. In fact, we have the following characterization of such sequences:

Proposition 6.4.5. *The n-almost split sequences in $\mathcal{C}$ are precisely the exact sequences*

$$\mathtt{E}_j: \qquad 0 \longrightarrow S_{(j+1)n} \longrightarrow P_{jn+(n-1)} \longrightarrow \cdots \longrightarrow P_{jn+1} \longrightarrow P_{jn} \longrightarrow S_{jn} \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ *with* $0 \leqslant j \leqslant q-1$.

Proof. Observe that we can conclude from Lemma 6.4.2 that if we take the minimal projective resolution of $S_{(j+1)n}$ and consider the splice of it with the exact sequence $\mathtt{E}_j$, then we obtain the minimal projective resolution of S_{jn}. Therefore, we have that the morphisms $P_{i+1} \to P_i$

and $P_{jn} \to S_{jn}$ of the sequence E_j are right minimal. Furthermore, it is not difficult to see that the morphism $P_{jn} \to S_{jn}$ is right almost split in $\mathcal{C}$, hence we conclude from Theorem 4.3.3 that E_j is an n-almost split sequence in $\mathcal{C}$.

In addition, note that it follows from Proposition 3.6.2 that the sequences E_j with $0 \leqslant j \leqslant q-1$ are indeed the unique n-almost split sequences in $\mathcal{C}$, up to isomorphism of complexes, given that $S_0, S_n, \dots, S_{(q-1)n}$ are all the nonprojective objects in $\operatorname{ind}\mathcal{C}$. □

Now, recall from Theorem 6.2.6 that τ_n gives a bijection from the nonprojective objects in $\operatorname{ind}\mathcal{C}$ to the the noninjective objects in $\operatorname{ind}\mathcal{C}$, with the inverse given by τ_n^-. What is interesting is that we can use Proposition 6.4.5 to determine how this bijection is given, as we show below.

Proposition 6.4.6. *We have that* $\tau_n S_{jn} = S_{(j+1)n}$ *in* $\operatorname{ind}\mathcal{C}$ *for each integer* $0 \leqslant j \leqslant q-1$.

Proof. This is straightforward from Proposition 6.4.5 and Theorem 6.3.2. □

Therefore, by Proposition 6.4.6 we may describe the behavior of τ_n in $\operatorname{ind}\mathcal{C}$ as below.

$$\begin{tikzcd} S_0 \arrow[r, bend right, "\tau_n"] & S_n \arrow[r, bend right, "\tau_n"] & S_{2n} \arrow[r, bend right, "\tau_n"] & \cdots \arrow[r, bend right, "\tau_n"] & S_{(q-1)n} \arrow[r, bend right, "\tau_n"] & S_{qn} \end{tikzcd}$$

To end this section, we remark that the study of the n-cluster tilting subcategories of $\operatorname{mod}\Lambda$ for $\Lambda = KQ/(\operatorname{rad}KQ)^2$ has also been generalized by Laertis Vaso in [40] for the case of the algebras $KQ/(\operatorname{rad}KQ)^r$ with $r \geqslant 3$. Indeed, he has shown in [40, Theorem 2] that for $r \geqslant 3$ the category of finitely generated modules over $KQ/(\operatorname{rad}KQ)^r$ has an n-cluster tilting subcategory if and only if n is even and $m = \frac{n}{2}r + q(nr - r + 2)$ for some integer $q \geqslant 0$. The reader is referred to [40] for a description of these subcategories, as well as for some particular examples of them.

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