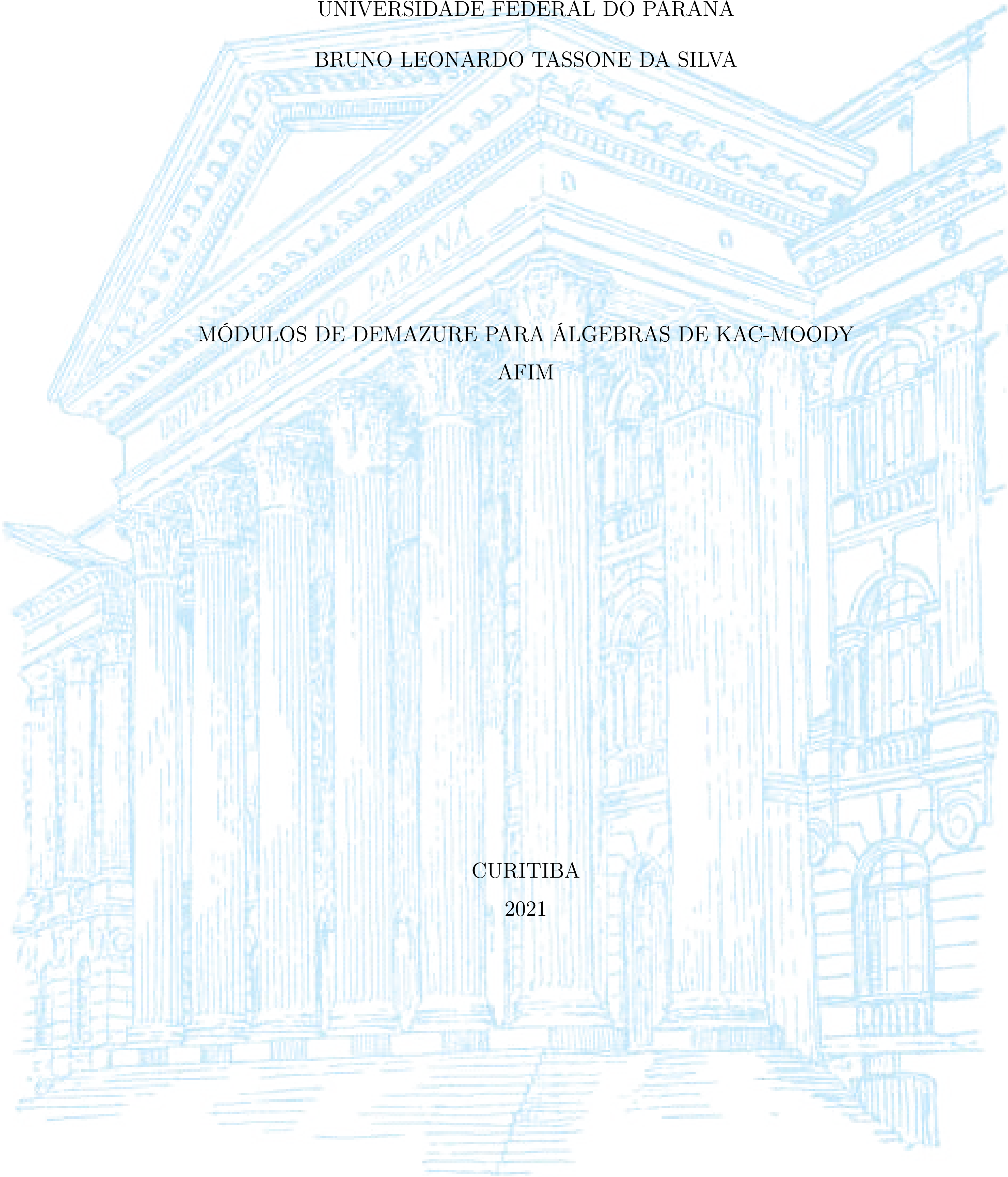

UNIVERSIDADE FEDERAL DO PARANÁ

BRUNO LEONARDO TASSONE DA SILVA

MÓDULOS DE DEMAZURE PARA ÁLGEBRAS DE KAC-MOODY AFIM

CURITIBA
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BRUNO LEONARDO TASSONE DA SILVA

MÓDULOS DE DEMAZURE PARA ÁLGEBRAS DE KAC-MOODY AFIM

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RESUMO

Essa dissertação aborda a teoria de representações para álgebras de correntes associada a uma álgebra de Kac-Moody de tipo finito, com particular interesse em módulos de Weyl e Demazure. Além de desenvolver toda a teoria necessária para o estudo do tema supracitado, são mostradas conexões entre módulos de Weyl e módulos de Demazure $\mathfrak{g}$-estáveis. Através de resultados já conhecidos é apresentado um novo conjunto de relações definidoras para módulos de Demazure de nível um que são $\mathfrak{g}$-estáveis. Parte destes resultados são então aplicados no estudo de produtos de fusão destas classes de representações.

Palavras-chave: Álgebras de correntes; Teoria de representações; Módulos de Weyl; Módulos de Demazure.

ABSTRACT

This dissertation addresses the representation theory for current algebras associated with a finite type Kac-Moody algebra, with a particular interest in Weyl and Demazure modules. In addition to developing all the necessary theories for the study of the aforementioned subject, connections between Weyl modules and Demazure $\mathfrak{g}$-stable modules are shown. Through known results, a new set of defining relations is provided for $\mathfrak{g}$-stable level one Demazure modules. Part of these results is then applied to the study of fusion products for these classes of representations.

Keywords: Current algebras; Representation theory; Weyl modules; Demazure modules.

LIST OF SYMBOLS

Symbol	Definition	Page
$(\, , \,)$	Standard bilinear form	24
$\widehat{\mathfrak{b}}$	Borel subalgebra of $\widehat{\mathfrak{g}}$	31
$\mathfrak{d}$	Subalgebra generated by d_i	24
d_γ	The integer $2((\gamma,\gamma)^{-1})$	27
$\mathcal{D}(\ell,\lambda)$	$\mathfrak{g}$-stable Demazure module	50
$\widehat{\mathfrak{g}}$	Affine extension of a simple Kac-Moody algebra $\mathfrak{g}$	30
h_α	Coroot of α	26
$\mathfrak{g}[t]$	Current algebra associated with $\mathfrak{g}$	30
$\mathfrak{g}_\alpha$	Root space of α	25
$\mathfrak{h}$	Cartan subalgebra of $\mathfrak{g}$	24
$\widehat{\mathfrak{h}}$	Cartan subalgebra of $\widehat{\mathfrak{g}}$	24
$\mathrm{ht}(\gamma)$	The height of γ	28
$M(\Lambda)$	Verma module of weight Λ	41
m_γ	Remainder of the euclidean division of $\lambda(h_\gamma)$	51
$N(\mathfrak{s})$	Normalizer of $\mathfrak{s}$	13
$\mathfrak{n}_\pm$	Triangular subalgebras of $\mathfrak{g}$	24
s_γ	Divisor of the euclidean division of $\lambda(h_\gamma)$	51
$U(\mathfrak{g})$	Universal enveloping algebra of $\mathfrak{g}$	13
$U(\mathfrak{g})^+$	Augmentation ideal of $U(\mathfrak{g})$	14
$V(\Lambda)$	Unique irreducible cyclic module with highest weight Λ	41
$V_\sigma(\Lambda)$	Demazure module	48
$W(\lambda)$	Weyl module	47
$\mathcal{W}$	Weyl group of $\mathfrak{g}$	28
Δ	Root system base	24
$\Delta^\vee$	Dual root system base	27
δ	Imaginary basic root	29
θ	The longest root of Φ	32
Λ_0	Fundamental weight of index 0	33
Φ	Root system of $\mathfrak{g}$	25
Φ_+^{sh}	Roots with short support	53

Contents

INTRODUCTION

This dissertation studies representations of the current algebra $\mathfrak{g}[t]$ associated with a finite Kac-Moody algebra $\mathfrak{g}$. These representations play important roles in many branches of modern mathematics. Examples of such are their connections with Macdonald polynomials (see [2]) and their applications in mathematical physics (see [1, 13, 27]). Aside from those, there are numerous recent papers on this subject, such as [7, 6, 10, 16, 18, 28]. In this text, we mainly focus on two particular families of such representations: Weyl and $\mathfrak{g}$-stable Demazure modules.

The class of Weyl modules for $\mathfrak{g}[t]$ were first introduced in [9] as quotients of a family of level zero integrable $\widehat{\mathfrak{g}}$-modules. We denote by $W(\lambda)$ the Weyl module for $\mathfrak{g}[t]$ associated with $\lambda \in P_+$, where P_+ is the set of dominant integral weight of $\mathfrak{g}$. It was shown in [9] that $W(\lambda)$ exists, for each $\lambda \in P_+$, and they are finite-dimensional modules satisfying the following universal property: any finite-dimensional highest weight module generated by a highest weight vector of weight λ is a quotient of $W(\lambda)$.

The second class of representations that we study in this text is composed of the so-called Demazure modules, which were first introduced in [11]. Let $\widehat{\mathfrak{b}}$ be the standard Borel subalgebra of $\widehat{\mathfrak{g}}$. By definition a Demazure module is a $\widehat{\mathfrak{b}}$-submodule of some $V(\Lambda)$, where Λ is a dominant integral weight of $\widehat{\mathfrak{g}}$. It is well known that Demazure modules are always finite-dimensional, so those structures can be used to study finite parts of $V(\Lambda)$. This is very helpful in the understanding of $V(\Lambda)$, since despite the known presentation of $V(\Lambda)$ via generators and relations its structure remains unknown in general. For example, [12, 20, 26] studies several aspects of this theory.

We are mainly interested in the $\mathfrak{g}$-stable Demazure modules, whose are the Demazure modules that can be regarded as modules for $\mathfrak{g}[t]$. Such modules will be denoted by $\mathcal{D}(\ell, \lambda)$, with $\ell \in \mathbb{Z}_{\geq 0}$ and $\lambda \in P_+$. It was proved in [8], and later using different methods in [18], that those Demazure modules are isomorphic to certain Kirilov-Reshetikhin modules. It was conjectured in [15] that $\mathcal{D}(1, \lambda)$ is isomorphic (as $\mathfrak{g}[t]$-module) to $W(\lambda)$. In [7] this result was proved for $\mathfrak{g} = \mathfrak{sl}_2$, and in [18] proved for any $\mathfrak{g}$ of simply-laced type. When $\mathfrak{g}$ is of type $BCFG$ the result is not necessarily true but rather we have that $W(\lambda)$ admits a flag of $\mathfrak{g}$-stable Demazure modules. In general we have that $\mathcal{D}(\ell, \lambda)$ is a quotient of $W(\lambda)$ ([5, 18, 28]).

Later on, Chari and Venkatesh in [10] provided a simpler presentation of $\mathcal{D}(\ell, \lambda)$ in terms of partitions of $\lambda(h_\alpha)$, $\alpha \in \Phi_+$, where Φ_+ denotes the set of positive roots of $\mathfrak{g}$. In this work we combine the results of [10] and [28] to describe a set of defining relations of $\mathcal{D}(1, \lambda)$ which we believe to be the minimal one (see

Theorem 3.2.1).

In the theory of cyclic $\mathfrak{g}[t]$-modules, the tensor products of representations are replaced by the so-called fusion products, introduced in [16], which consists of the graded module associated with a filtration induced on the tensor product of such representations. The definition of fusion products relies on a choice of distinct parameters which are conjectured to be independent of the choice. In some particular cases, this conjecture has been proved. For instance, it follows from [18, 28] that the fusion product of Demazure modules is isomorphic to a Demazure module and the fusion product of Weyl modules is isomorphic to a Weyl module. These isomorphisms are used, in our text, to describe the fusion product of Demazure and Weyl modules via generators and relations, and hence they are independent of the choice of the fusion parameters.

This dissertation is organized as follows. In the first chapter, we develop the theory of Lie algebras which will be necessary through the text. The second chapter delves into representation theory for Lie algebras, focusing in representations of Kac-Moody algebras of finite and affine types. Both [4, 23] are good references for this chapter, and we follow the first more closely. In the third chapter, we focus on the main families of $\mathfrak{g}[t]$-modules considered in this work; Weyl and Demazure modules. We define such objects and prove the main theorem. The proof of this statement was divided into several minor results, which where spread in few subsections. Lastly, in the fourth chapter, we provide a presentation of certain fusion products related to Weyl modules and level one Demazure modules.

Chapter 1

KAC-MOODY ALGEBRAS

The main goal of this chapter is to introduce an overall context of Lie algebras focussing on Kac-Moody algebras. To do so, we present some elementary concepts, basic results and definitions. Through the chapter, fix $\mathbb{K}$ as a algebraically closed field with characteristic 0, unless otherwise specified.

1.1 Lie Algebras

1.1.1 Basic definitions

A Lie algebra is a pair $(\mathfrak{g}, [\ ,\ ])$, with $\mathfrak{g}$ being a vector space over $\mathbb{K}$ and

$$\begin{aligned}[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g}\\ (x, y) &\to [x, y]\end{aligned}$$

a bilinear binary function, called the *Lie bracket*, which satisfies the following axioms.

(i) For every $x \in \mathfrak{g}$, it holds that $[x, x] = 0$.

(ii) For every $x, y, z \in \mathfrak{g}$, it holds that $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Provided that the characteristic of $\mathbb{K}$ is not 2 one can note that axiom (i) implies $[x, y] = -[y, x]$, and therefore the bracket is an *anti-commutative* operation. Axiom (ii) is called the *Jacobi identity*.

Strictly speaking, each Lie algebra depends on the choice of the bracket. Therefore different choices of brackets yield generally different Lie algebras, as the next examples illustrate. This fact justifies the notation $(\mathfrak{g}, [\ ,\ ])$. However, by abuse of notation, we write $\mathfrak{g}$ for $(\mathfrak{g}, [\ ,\ ])$ when there is no confusion.

Example 1.1.1. Let $\mathfrak{g} = \mathbb{R}^3$, and define $[x, y]$ as the *external product* between x and y. One easily checks that $\mathfrak{g}$ satisfies the three axioms, so it is a Lie algebra. ◆

Example 1.1.2. Let $\mathfrak{g}$ be a vector space, and define $[x,y]=0, \forall x,y\in\mathfrak{g}$. With this choice of bracket $\mathfrak{g}$ trivially satisfies the three axioms and thus is a Lie algebra. In this case we say that $\mathfrak{g}$ is an *abelian* Lie algebra. ◆

Example 1.1.3. Let $\mathfrak{g}$ be a three-dimensional vector space with basis $\{x,y,z\}$. Define the bracket as

$$[x,y]=z, \quad [x,z]=[y,z]=0.$$

It follows that $\mathfrak{g}$ is a Lie algebra, which is said to be the *Heisenberg Lie algebra*. ◆

Example 1.1.4. Let V be a finite-dimensional vector space and let $\mathfrak{gl}(V)$ be the vector space of all endomorphisms of V, that is, the space of all linear transformation $u:V\to V$, and set $[x,y]=xy-yx$. With this bracket $\mathfrak{gl}(V)$ is a Lie algebra. This particular bracket is called the *commutator*. ◆

Definition 1.1.5. We say that a bilinear form $\varphi:\mathfrak{g}\times\mathfrak{g}\to\mathbb{K}$ is *invariant* if $\varphi([x,y],z)=\varphi(x,[y,z])$, $\forall x,y,z\in\mathfrak{g}$.

Given any $x\in\mathfrak{g}$ we can define an application

$$\begin{aligned}\mathrm{ad}x:\mathfrak{g}&\to\mathfrak{g}\\ y&\mapsto[x,y]\end{aligned}$$

which is linear, by the bracket definition. We say that $\mathrm{ad}x$ is the *adjoint* of x. We now bring several basic definitions and some examples, intending to develop the essential theory of Lie algebras. Most of these structures are greatly developed in every Lie algebras book.

Definition 1.1.6. Given $\mathfrak{g}=(\mathfrak{g},[\ ,\ ]_1)$ and $\mathfrak{s}=(\mathfrak{s},[\ ,\ ]_2)$ Lie algebras, a linear transformation $\varphi:\mathfrak{g}\to\mathfrak{s}$ is called the a *homomorphism* (or *Lie homomorphism*) between $\mathfrak{g}$ and $\mathfrak{s}$ if

$$\varphi([x,y]_1)=[\varphi(x),\varphi(y)]_2,\ \forall x,y\in\mathfrak{g}.$$

If φ is a bijective homomorphism, then φ is called an *isomorphism* (or *Lie isomorphism*). In this case we say that the Lie algebras $\mathfrak{g}$ and $\mathfrak{s}$ are *isomorphic*, denoted by $\mathfrak{g}\cong\mathfrak{s}$.

Definition 1.1.7. A vector subspace $\mathfrak{s}$ of a Lie algebra $\mathfrak{g}=(\mathfrak{g},[\ ,\ ])$ is said to be a *Lie subalgebra* of $\mathfrak{g}$ if $[x,y]\in\mathfrak{s},\ \forall x,y\in\mathfrak{s}$.

Example 1.1.8. Let V be a finite-dimensional vector space and recall the Lie algebra $\mathfrak{gl}(V)$ defined in Example 1.1.4. Define $\mathfrak{sl}(V)$ as the vector subspace of $\mathfrak{gl}(V)$ composed of V-endomorphisms with trace 0. Since, for any $x,y\in\mathfrak{sl}(V)$, we have

$$tr(xy-yx)=tr(xy)-tr(yx)=tr(xy)-tr(xy)=0.$$

It follows that $[x,y]\in\mathfrak{sl}(V)$, thus $\mathfrak{sl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. ◆

Remark 1.1.9. *If* $V = \mathbb{K}^n$*, we denote the Lie algebra* $\mathfrak{sl}(V)$ *of last example by* $\mathfrak{sl}_n$*.*

Example 1.1.10. Fix $V = \mathbb{K}^2$. Define $\mathfrak{sl}_2$ as in last example. As a vector space, $\mathfrak{sl}_2$ is generated by

$$x^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; x^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows that $[x^+, x^-] = h$, $[h, x^+] = 2x^+$ and $[h, x^-] = -2x^-$. ◆

Definition 1.1.11. A vector subspace I of a Lie algebra $\mathfrak{g}$ is called an *ideal* if $[x, y] \in I$, for every $x \in \mathfrak{g}$ and $y \in I$.

It is clear that both $\mathfrak{g}$ and $\{0\}$ are ideals of any Lie algebra $\mathfrak{g}$. We say that those are *trivial* ideals of $\mathfrak{g}$. If $\mathfrak{g}$ has only trivial ideals and $\mathfrak{g}$ is not abelian (meaning that $[\mathfrak{g}, \mathfrak{g}] \neq 0$), we say that $\mathfrak{g}$ is a *simple* Lie algebra.

Given a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, define

$$N(\mathfrak{s}) := \{x \in \mathfrak{g} : [x, y] \in \mathfrak{s}, \; \forall y \in \mathfrak{s}\} \tag{1.1}$$

which is called the *normalizer* of $\mathfrak{s}$. Next statement can be found in [4, Lemma 3.1]

Lemma 1.1.12. *If* $\mathfrak{s}$ *is a subalgebra of* $\mathfrak{g}$*, then the following items hold.*

(i) $N(\mathfrak{s})$ *is a subalgebra of* $\mathfrak{g}$*.*

(ii) $\mathfrak{s}$ *is an ideal of* $N(\mathfrak{s})$*.*

(iii) $N(\mathfrak{s})$ *is the largest subalgebra of* $\mathfrak{g}$ *containing* $\mathfrak{s}$ *as an ideal.*

For a Lie algebra $\mathfrak{g}$, let $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and, for $n \in \mathbb{N}$, $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$. We have that $\mathfrak{g}^{(n-1)} \subseteq \mathfrak{g}^{(n)}$, for all n. Moreover each $\mathfrak{g}^{(n)}$ is an ideal of $\mathfrak{g}$. In a similar way let $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$, $n \in \mathbb{N}$. It is easy to check that $\mathfrak{g}^n$ is an ideal of $\mathfrak{g}$, for every $n \in \mathbb{N}$.

A Lie algebra $\mathfrak{g}$ is said to be *solvable* if $\mathfrak{g}^{(n)} = 0$, $n >> 0$. Further, $\mathfrak{g}$ is said to be *nilpotent* if $\mathfrak{g}^n = 0$, $n >> 0$. Finally, $\mathfrak{g}$ is semisimple if it has no non-zero solvable ideals.

Theorem 1.1.13 (Lie's Theorem)**.** *If* $\mathfrak{g}$ *is a solvable subalgebra of* $\mathfrak{gl}(V)$*, with* $dim(V) < \infty$*, then there is a basis of* V *such that every matrix of* $\mathfrak{g}$ *is upper triangular.*

Proposition 1.1.14. *If* $\mathfrak{g}$ *is a semisimple Lie algebra, then* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$*, where* $\mathfrak{g}_i$ *is a simple Lie algebra for each* $1 \leq i \leq n$*.*

Denote $\mathfrak{g}^{\otimes 0} := \mathbb{K}$, $\mathfrak{g}^{\otimes 1} := \mathfrak{g}$ and $\mathfrak{g}^{\otimes n} := \mathfrak{g} \otimes \mathfrak{g}^{\otimes n-1}$. Define

$$T(\mathfrak{g}) := \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}.$$

Fix $I(\mathfrak{g})$ as an ideal of $\mathfrak{g}$ generated by elements of the form $x \otimes y - y \otimes x - [x, y]$, for $x, y \in \mathfrak{g}$. Let $U(\mathfrak{g}) := \dfrac{T(\mathfrak{g})}{I(\mathfrak{g})}$. We say that $U(\mathfrak{g})$ is the *universal enveloping algebra* of $\mathfrak{g}$.

We can fully describe the basis of $U(\mathfrak{g})$ using only monomials of a basis of $\mathfrak{g}$. This very important result can be found in [19, Corollary 17.3-C] and in [4, Theorem 9.4].

Theorem 1.1.15 (Poincaré-Birkhoff-Witt Thorem (PBW))**.** *For a set of indexes K, with a total ordering $<$, let $\{x_i : i \in K\}$ be a basis of $\mathfrak{g}$. If $\pi : \mathfrak{g} \to U(\mathfrak{g})$ is the canonical map and $\pi(x_i) = y_i$, then the elements*

$$\{y_{i_1}^{k_1} \cdots y_{i_n}^{k_n} : n \geq 0,\ i_j < i_{j+1},\ k_j \geq 0\}$$

form a basis of $U(\mathfrak{g})$.

Roughly speaking, for a basis $\{x_i : i \in K\}$ of $\mathfrak{g}$ define $c_{ij,k}$ to be such that $[x_i, x_j] = \sum_k c_{ij,k} x_k$. In this case $U(\mathfrak{g})$ is the algebra generated by $\{y_j : i \in K\}$ with relations

$$y_i y_j - y_j y_i = \sum_k c_{ij,k} y_k. \tag{1.2}$$

Let A and B be unital, associative algebras. We define an *algebra homomorphism* $\phi : A \to B$ as a linear mapping such that $\phi(ab) = \phi(a)\phi(b)$ and $\phi(1_A) = 1_B$. The following proposition can be found in [4, Proposition 9.2].

Proposition 1.1.16. *Let A be an associative algebra with 1 over $\mathbb{K}$ and let $[A]$ be the corresponding Lie algebra. Let $\pi : \mathfrak{g} \to U(\mathfrak{g})$ be the canonical map. Given a Lie algebra homomorphism $\varphi : \mathfrak{g} \to [A]$ there exists a unique assoiative algebra homomorphism $\phi : U(\mathfrak{g}) \to A$ such that $\phi \circ \pi = \varphi$.*

Definition 1.1.17. Let $\epsilon : U(\mathfrak{g}) \to \mathbb{K}$ be the (unique) algebra homomorphism satisfying $\epsilon(x) = 0,\ x \in \mathfrak{g}$. We say that the Kernel of ϵ is the *augmentation ideal* of $U(\mathfrak{g})$, denoted by $U(\mathfrak{g})^+$.

Remark 1.1.18. *For a totally ordered set of indexes K, set $\{x_i : i \in K\}$ as a basis of $\mathfrak{g}$. It follow from the definition that $U(\mathfrak{g})^+$ is the unique maximal ideal of $U(\mathfrak{g})$ that contains x_i, $\forall i \in K$.*

Lastly, suppose that G is an abelian group. We say that $\mathfrak{g}$ is a *graded Lie algebra* if there is a family of vector subspaces $\{\mathfrak{g}_a\}_{a \in G}$ satisfying the following conditions.

(i) We can write $\mathfrak{g} = \bigoplus_{a \in G} \mathfrak{g}_a$.

(ii) For every $a, b \in G$ we have $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$.

A generalization of graded algebras are the filtered algebras. We say that $\mathfrak{g}$ is a *filtered Lie algebra* if it has an increasing sequence of subspaces

$$\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n \subset \cdots \subset \mathfrak{g},$$

such that

$$\mathfrak{g} \cup_{n \geq 0} \mathfrak{g}_n, \quad \text{and} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad i, j \in \mathbb{N}.$$

Given a filtered Lie algebra $\mathfrak{g}$ the *associated graded Lie algebra*, denoted by $\mathcal{G}(\mathfrak{g})$, is s defined as follows: It is the vector space

$$\mathcal{G}(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}} G_n,$$

in which

$$G_0 = \mathfrak{g}_0, \quad \text{and} \quad G_n = \frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}}, \ n \geq 1.$$

Note that the bracket given by

$$[x + \mathfrak{g}_{n-1}, y + \mathfrak{g}_{\ell-1}] = [x, y] + \mathfrak{g}_{n+\ell-1},$$

is well defined and endows $\mathcal{G}(\mathfrak{g})$ with a structure of a graded Lie algebra, with gradation G_n, $n \in \mathbb{N}$.

Remark 1.1.19. *The filtered and graded structure of a Lie algebra* $\mathfrak{g}$ *induces a natural one on* $U(\mathfrak{g})$.

1.1.2 Generalized Cartan matrices

Semisimple finite-dimensional Lie algebras are fully classified and understood. As pointed out in [23], this classification was done by the end of the 19-th century by W. Killing and E. Cartan. Such classification can be done in many ways, and one particularly useful path to do so is via Cartan matrices. Since we are interested in infinite-dimensional Lie algebras we present such classification in terms of Generalized Cartan Matrices, which we define below and are a much broader concept.

Let $m \in \mathbb{N}$ and set $I = \{1, \ldots, m\}$. We say that the matrix $A = (a_{ij})_{i,j \in I} \in M_m(\mathbb{K})$, is a *generalized Cartan matrix* (or simply *GCM*) if the following conditions are satisfied.

(i) $a_{ii} = 2, \ \forall i \in I$.

(ii) If $i \neq j$, then $a_{ij} \in \mathbb{Z}_{\leq 0}$.

(iii) If $a_{ij} = 0$, then $a_{ji} = 0$.

The matrix A is said to be *symmetrisable* if there is a non-singular diagonal matrix

$$B = \operatorname{diag}(b_1, \ldots, b_n) \tag{1.3}$$

and a symmetric matrix S such that $A = BS$.

Definition 1.1.20. Say that a GCM A is *indecomposable* if, for any choice of nonempty disjoint subsets I_1, I_2 of I such that $I = I_1 \cup I_2$, there exists $i \in I_1$ and $j \in I_2$ satisfying $a_{ij} \neq 0$.

From now on, we assume that A is always an indecomposable symmetrisable GCM. Our next goal is to classify such matrices, starting with the following definition.

Definition 1.1.21. A GCM A is said to be

(i) of finite type if it is positive definite;

(ii) of affine type if it is semi positive definite with $\operatorname{corank}(A) = 1$;

(iii) of indefinite type if it is indefinite.

The set of indecomposable GCMs admits a trichotomy, as the next theorem states, and we use this splitting to classify such matrices. Such classification will be enough for our purposes. For more details, we refer [4, Theorem 15.1].

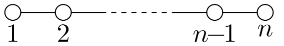
Theorem 1.1.22. *An indecomposable GCM A is either of finite, affine or indefinite type.*

Through our study, we do not delve into indefinite GCMs, so those will be left out of our text. For the finite and affine cases, the GCM can be encoded through the use of Dynkin diagrams, and vice-versa. The reader can find an extensive guide on how to build these diagrams in [4, 19].

A Dynkin diagram is a type of graph with possibly multiple edges between two nodes that, in this case, are directed. The nodes are indexed by I and between two nodes $i \neq j \in I$ we have $a_{i,j}a_{j,i}$ edges. Recall that $a_{i,j} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$, hence $a_{i,j}a_{j,i} \in \mathbb{Z}_{\geq 0}$. If $a_{i,j}a_{j,i} > 1$ we equip the edges between i and j with an arrow pointed toward i if $a_{i,j} < -1$.

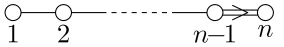
The possible finite Dynkin diagram cases, along with their respective GCMs, are as follows.

A_n:



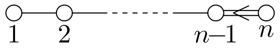
$$\begin{pmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 2 \end{pmatrix}$$

B_n:



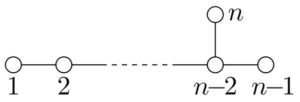
$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -2 & 2 \end{pmatrix}$$

C_n:



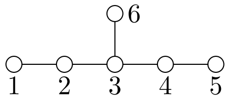
$$\begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 2 & -1 & 0 \\
0 & \cdots & \cdots & 0 & -1 & 2 & -2 \\
0 & \cdots & \cdots & \cdots & 0 & -1 & 2
\end{pmatrix}$$

D_n:



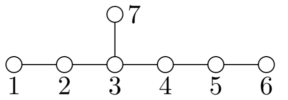
$$\begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & \cdots & 0 \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & & -1 & 2 & -1 & 0 & 0 \\
0 & \cdots & 0 & -1 & 2 & -1 & -1 \\
0 & \cdots & \cdots & 0 & -1 & 2 & 0 \\
0 & \cdots & \cdots & 0 & -1 & 0 & 2
\end{pmatrix}$$

E_6:



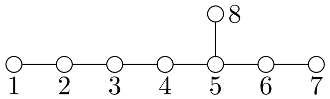
$$\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & -1 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}$$

E_7:



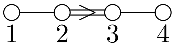
$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

E_8:



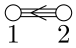
$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

F_4:



$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

G_2:

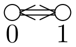


$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

Definition 1.1.23. We say that a GCM A of finite type is a *Cartan matrix.*

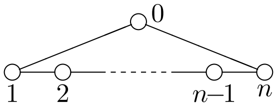
There is a full classification of the affine GCMs, using the same reasoning. If A is a $m \times m$ GCM of affine type set $n := m - 1 = \mathrm{Rank}(A)$ and define the indexation $I := \{0, \cdots, n\}$. In this case, the affine Dynkin diagrams, along with (some of) their respective GCMs are as follows.

$\widehat{A}_1$:



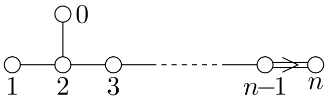
$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$\widehat{A}_n$:
$(n \geq 2)$



$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

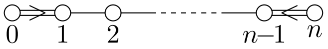
$\widehat{B}_n$:
$(n \geq 3)$



$$\begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & -1 & 2 & -1 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -2 & 2 \end{pmatrix}$$

$\widehat{C}_n$:

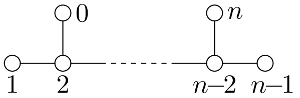
$(n \geq 2)$



$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -2 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & 0 & \cdots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -2 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

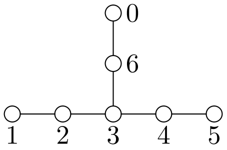
$\widehat{D}_n$:

$(n \geq 4)$



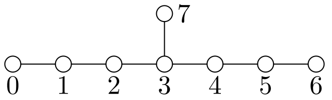
$$\begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & -1 & 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}$$

$\widehat{E}_6$:



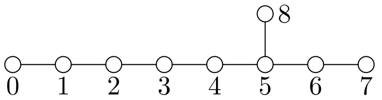
$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$\widehat{E}_7$:



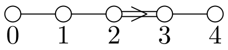
$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$\widehat{E}_8$:



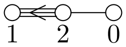
$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$\widehat{F}_4$:



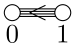
$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$\widehat{G}_2$:

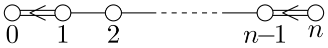


$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}$$

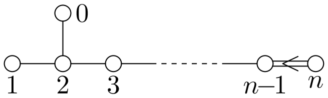
$\widehat{A}'_2$:



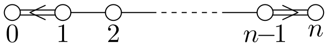
$\widehat{A}'_{2n}$:
($n \geq 2$)



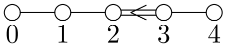
$\widehat{A}'_{2n-1}$:
($n \geq 3$)



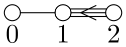
$\widehat{D}'_{n+1}$:
($n \geq 2$)



$\widehat{E}'_6$:



$\widehat{D}'_4$:



We say that A is of *untwisted affine type* if it is one of the following:

$$\widehat{A}_n, \widehat{B}_n, \widehat{C}_n, \widehat{D}_n, \widehat{E}_4, \widehat{E}_5, \widehat{E}_6, \widehat{F}_4, \widehat{G}_2 \tag{1.4}$$

and that A is of *twisted affine type* otherwise. Note that we did not give the matrices of the twisted affine types. This happens because from now on we are only concerned about the GCMs of untwisted type. The reader can find such information in great detail in the Appendix of [4]. Over the next chapters, the theory of GCMs of twisted types will be left out.

The classification above hints that GCMs of untwisted type are extensions of finite ones. Specifically, fix $I := \{1, \ldots, n\}$ and $A = (a'_{ij})_{i,j \in I}$ an indecomposable Cartan matrix. Extend this matrix in the following way. Set $\widehat{I} := \{0, 1, \ldots, n\}$ and define $\widehat{A} = (a_{ij})_{i,j \in \widehat{I}}$ by requiring

(i) $a_{ij} = a'_{ij}$ if $i, j \in I$.

(ii) $a_{i0} = \sum_{j=1}^{n} a'_{ij}$ if $i \in I$.

(iii) $a_{0j} = -\sum_{i=1}^{n} a'_{ij}$ if $j \in I$.

(iv) $a_{00} = 2$.

By construction, $\widehat{A}$ is a GCM of affine type.

Conversely if $\widehat{A} = (a_{ij})_{i,j\in\widehat{I}}$ is a GCM of affine type, as listed in (1.4), define $A = (a_{ij})_{i,j\in I}$, meaning that A is the matrix $\widehat{A}$ without the row/column 0. It is easy to check that A is a Cartan matrix.

Proposition 1.1.24. *The GCM $\widehat{A}$ is of type $\widehat{X}_n$ if, and only if, A is of type X_n.*

For later use, note that since $\mathrm{Ker}(A) = 1$ there are vectors

$$k = (k_0, \dots, k_n) \text{ and } a = (a_0, \dots a_n) \tag{1.5}$$

satisfying $k\widehat{A} = 0$ and $\widehat{A}a = 0$.

Remark 1.1.25. *From the classification it is straightforward to verify that $k_0 = 1$ and $a_0 = 1$.*

Remark 1.1.26. *If $\widehat{A}$ is an affine GCM, then the matrix B from* (1.3) *is such that $b_i = \dfrac{a_i}{k_i}$*

1.2 Kac-Moody algebras

1.2.1 The Lie algebra associated with a GCM

From the data encoded on a GCM, one can define a Lie algebra as follows.

Definition 1.2.1. Let $A = (a_{ij})_{i,j\in I}$ be a $m \times m$ GCM of rank n. Define $J \subseteq I$ such that $|J| = n$ and $C = (a_{i,j})_{i,j\in J}$ is invertible. Set $\mathfrak{g}(A)$ as the Lie algebra generated by $x_i^{\pm}, h_i, d_j$, $i \in I$ and $j \in I \setminus J$, satisfying the following set of relations.

(i) $[h_i, h_j] = 0, \forall i, j \in I$;

(ii) $[x_i^+, x_j^-] = \delta_{ij} h_i, \forall i, j \in I$;

(iii) $[h_i, x_j^{\pm},] = \pm a_{ij} x_j^{\pm}, \forall i, j \in I$;

(iv) $\mathrm{ad}(x_i^{\pm})^{1-a_{ij}}(x_j^{\pm}) = 0, \forall i, j \in I$ with $i \neq j$;

(v) $[d_i, d_j] = 0,\ \forall i, j \in I \setminus J$;

(vi) $[h_i, d_j] = 0, \forall i \in I, \forall j \in I \setminus J$;

(vii) $[d_j, x_i^{\pm}] = \pm\delta_{ij} x_i^{\pm}, \forall i \in I, \forall j \in I \setminus J$.

The Lie algebra $\mathfrak{g}(A)$ is called the *Kac-Moody algebra* associated with A. To simplify the notation we write $\mathfrak{g} = \mathfrak{g}(A)$ when there is no confusion. The elements in $x_i^{\pm}, h_i, d_j$, $i \in I$, $j \in I \setminus J$, are called the *Chevalley generators* of $\mathfrak{g}$.

Proposition 1.2.2. *If $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, then there is an indecomposable Cartan matrix A such that $\mathfrak{g} \cong \mathfrak{g}(A)$.*

Given a GCM A and a Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$, say that $\mathfrak{g}$ is of type X if A is of type X. Moreover $\mathfrak{g}$ is of either finite, affine, or indefinite type. Note that affine and finite Kac-Moody algebras are fully classified using the theory of Section 1.1.2. Say that a Kac-Moody algebra is *simply laced* if its Dynkin diagram is such that between two nodes $i \neq j \in I$ there is at most one edge.

Define the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{d} := \langle d_j : j \in I \setminus J \rangle, \qquad \mathfrak{h} := \langle h_i : i \in I \rangle, \quad \text{and} \quad \widehat{\mathfrak{h}} := \mathfrak{h} \oplus \mathfrak{d}. \tag{1.6}$$

The subalgebra $\widehat{\mathfrak{h}}$ is called a *Cartan subalgebra* of $\mathfrak{g}$. It is easy to verify that Cartan subalgebras are abelian. It follows from the definition that $\widehat{\mathfrak{h}} = N(\widehat{\mathfrak{h}})$ (see (1.1)).

Remark 1.2.3. *If* $\mathfrak{g}$ *is of finite type, then* $\widehat{\mathfrak{h}} = \mathfrak{h}$. *Moreover, if* $\mathfrak{g}$ *is of affine type, then* $dim(\widehat{\mathfrak{h}}) = \dim(\mathfrak{h}) + 1$.

Let $\mathfrak{n}_\pm$ be the Lie subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{n}_\pm := \langle x_i^\pm : i \in I \rangle.$$

The following result can be found in [4, Proposition 14.14].

Proposition 1.2.4. *As vector spaces we have that* $\mathfrak{g} = \mathfrak{n}_+ \oplus \widehat{\mathfrak{h}} \oplus \mathfrak{n}_-$. *This decomposition is called a triangular decomposition of* $\mathfrak{g}$.

Recall the non-singular matrix $B = \mathrm{diag}(b_1, \ldots, b_n)$ from (1.3). Let $(\ , \ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{Q}$ be a bilinear form on $\mathfrak{g}$ satisfying the following set of relations.

(i) $(h_i, h_j) = b_i a_{ij}, \ \forall i, j \in I$;

(ii) $(x_i^+, x_j^-) = \dfrac{\delta_{ij}}{b_j}, \ \forall i, j \in I$;

(iii) $(x_i^\pm, x_j^\pm) = 0, \ \forall i, j \in I$;

(iv) $(h_i, x_j^\pm) = 0, \ \forall i, j \in I$;

(v) $(d_i, d_j) = 0, \forall i, j \in I \setminus J$;

(vi) $(d_j, x_i^\pm) = 0, \forall i \in I$;

(vii) $(h_i, d_j) = \dfrac{\delta_{ij}}{b_j}, \ \forall i \in I, \forall j \in I \setminus J$.

This form is well defined, as $b_j \neq 0, \ \forall j$, and it is clearly symmetric. Moreover $(\ , \ )$ is non-degenerate ([4, Proposition 16.1]) and it is invariant ([4, Proposition 16.2]). Its restriction to $\widehat{\mathfrak{h}}$ is also non-degenerate, symmetric and invariant. Also $(\ , \ )$ is the unique (up to scalar multiple) bilinear symmetric non-degenerate invariant form of $\mathfrak{g}$ ([4, Proposition 16.2]). This allows the following definition.

Definition 1.2.5. The form $(\ , \ )$ is called the *standard bilinear form* of $\mathfrak{g}$. Further, the restriction $(\ , \ )|_{\widehat{\mathfrak{h}}}$ is called the *standard bilinear form* of $\widehat{\mathfrak{h}}$.

From Linear Algebra we know that $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$ are isomorphic. This means, particularly, that we have an induced symmetric, invariant, non-degenerate bilinear form $(\ ,\ ) : \widehat{\mathfrak{h}}^* \times \widehat{\mathfrak{h}}^* \to \mathbb{Q}$. We define such form by doing

$$(\lambda, \mu) := (\varphi^{-1}(\lambda), \varphi^{-1}(\mu)). \tag{1.7}$$

with $\lambda, \mu \in \widehat{\mathfrak{h}}^*$ and $\varphi : \widehat{\mathfrak{h}} \xrightarrow{\sim} \widehat{\mathfrak{h}}^*$.

1.2.2 Root systems

Fix a Kac-Moody algebra $\mathfrak{g}$ and a Cartan subalgebra $\widehat{\mathfrak{h}}$. For each $\alpha \in \widehat{\mathfrak{h}}^*$ define

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h, x] = \alpha(h)x,\ \forall h \in \widehat{\mathfrak{h}}\}.$$

We say that $\alpha \neq 0$ is a *root* of $\mathfrak{g}$ if $\mathfrak{g}_\alpha \neq \{0\}$. The *root system* of $\mathfrak{g}$ is the set

$$\Phi = \{\alpha \in \widehat{\mathfrak{h}}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}.$$

Note that the roots of $\mathfrak{g}$ are not unique, as they vary with the choice of Cartan subalgebra $\widehat{\mathfrak{h}}$. From now on we fix a root system Φ of $\mathfrak{g}$.

Proposition 1.2.6. *There always exists the Cartan decomposition*

$$\mathfrak{g} = \widehat{\mathfrak{h}} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right).$$

Moreover $\widehat{\mathfrak{h}} = \{x \in \mathfrak{g} : [h, x] = 0, \forall h \in \mathfrak{h}\}$.

Proposition 1.2.7. *Assume* $\alpha, \beta \in \Phi$.

- *(i)* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, *if* $\alpha + \beta \in \Phi$.
- *(ii)* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \widehat{\mathfrak{h}}$.
- *(iii)* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$, *if* $\beta \neq -\alpha$ *and* $\alpha + \beta \notin \Phi$.

Let $(\ ,\ )$ be the standard bilinear form of $\widehat{\mathfrak{h}}^*$ (see (1.7)). Given $\lambda \in \widehat{\mathfrak{h}}^*$, set $t_\lambda \in \mathfrak{h}$ as the unique element satisfying

$$\lambda(h) = (t_\lambda, h), \quad \forall h \in \widehat{\mathfrak{h}}.$$

The following proposition gives us a series of properties satisfied by the root system of $\mathfrak{g}$. Its proof can be found in [19, Proposition 8.3].

Proposition 1.2.8. *Assume that* $\alpha, \beta \in \Phi$.

- *(i) The set* Φ *is a generating set of* $\widehat{\mathfrak{h}}^*$ *as vector space.*

(ii) $k\alpha \in \Phi$ *if and only if* $k \in \{-1, 1\}$.

(iii) *If* $x \in \mathfrak{g}_\alpha$ *and* $y \in \mathfrak{g}_{-\alpha}$, *then* $[x, y] = (x, y)t_\alpha$.

(iv) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *is a one dimensional vector space generated by* t_α.

(v) $\beta(t_\alpha) = (\beta, \alpha)$ *and moreover* $\alpha(t_\alpha) \neq 0$.

(vi) *If* $x_\alpha^+ \in \mathfrak{g}_\alpha$ *is nonzero, there is a unique* $x_\alpha^- \in \mathfrak{g}_{-\alpha}$ *such that* x_α^+, x_α^- *and* $h_\alpha := [x_\alpha^+, x_\alpha^-]$ *span a three dimensional Lie algebra isomorphic to* $\mathfrak{sl}_2$ *(see Example 1.1.8).*

Definition 1.2.9. The element $h_\alpha := \dfrac{2t_\alpha}{(t_\alpha, t_\alpha)} \in \widehat{\mathfrak{h}}$ is the *coroot* of α.

Given $\alpha,\ \beta \in \Phi$, we say that a sequence of the form

$$\ldots, \beta - 2\alpha,\ \beta - \alpha,\ \beta,\ \beta + \alpha,\ \beta + 2\alpha, \ldots \tag{1.8}$$

is the *α-string trough β.*

Proposition 1.2.10. *Given* $\alpha,\ \beta \in \Phi$, *there are unique* $r, q \in \mathbb{Z}_{\geq 0}$ *such the terms of the sequence*

$$\beta - r\alpha,\ \ldots,\ \beta - \alpha,\ \beta,\ \beta + \alpha,\ \ldots,\ \beta + q\alpha$$

are the unique roots of the form $\beta + k\alpha$, $k \in \mathbb{Z}$. *Moreover, we have the equality*

$$r - q = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \beta(h_\alpha).$$

Recall the standard bilinear form (1.7). Say that, for $\alpha \in \Phi$, the rational (α, α) is the *length* of α. This allows a classification of Φ in terms of the possible root lengths.

Proposition 1.2.11. *Assume that* $\mathfrak{g}$ *is a Kac-Moody algebra of finite or affine type. If* $\mathfrak{g}$ *is of type* A_n, $\widehat{A}_n$, D_n, $\widehat{D}_n$, E_6, $\widehat{E}_6$, E_7, $\widehat{E}_7$, E_8 *or* $\widehat{E}_8$, *then every root has the same length. If* $\mathfrak{g}$ *has any other type, then there are two possible lengths for the roots of* $\mathfrak{g}$.

If $\mathfrak{g}$ admits two possible lengths, then one must be bigger than the other since the lengths are rationals. We say that $\alpha \in \Phi$ is a *long root* if

$$(\alpha, \alpha) \geq (\beta, \beta),\ \forall \beta \in \Phi.$$

Similarly one defines *short roots.*

Remark 1.2.12. *If* $\mathfrak{g}$ *admits only one root length and* $\mathfrak{g}$ *is not of type* $\widehat{A}_1$, *then* $\mathfrak{g}$ *is simply laced.*

It is convenient, and usual, to normalize the standard bilinear form (1.7) in such a way that $(\alpha, \alpha) = 2$ if $\alpha \in \Phi$ is long, and this normalization fixes the length of short roots. However, the normalized length of short roots vary with the type of $\mathfrak{g}$. From now on set $(\ ,\ )$ as the normalized standard bilinear form of $\widehat{\mathfrak{h}}$ as above.

Lemma 1.2.13. *Suppose that* $\beta \in \Phi$ *is short.*

(i) If $\mathfrak{g}$ is of type B_n, $\widehat{B}_n$, C_n, $\widehat{C}_n$,F_4 or $\widehat{F}_4$, then $(\beta, \beta) = 1$.

(ii) If $\mathfrak{g}$ is of type G_2 or $\widehat{G}_2$ then $(\beta, \beta) = \dfrac{2}{3}$.

Definition 1.2.14. For $\gamma \in \Phi$, define $d_\gamma := \dfrac{2}{(\gamma, \gamma)}$, and set $d_{\alpha_i} = d_i$.

It is clear from Lemma 1.2.13 that $d_\gamma = 1$ if γ is long. If γ is short then $d_\gamma = 2$ if $\mathfrak{g}$ is of type B_n, $\widehat{B}_n$, C_n, $\widehat{C}_n$,F_4 or $\widehat{F}_4$, and that $d_\gamma = 3$ if $\mathfrak{g}$ is of type G_2 or $\widehat{G}_2$.

Definition 1.2.15. If $A = (a_{ij})_{i,j \in I}$ is the GCM associated with $\mathfrak{g}$ and h_i, $i \in I$ are the generators of Definition 1.2.1, say that $\Delta = \{\alpha_i : i \in I\} \subset \widehat{\mathfrak{h}}^*$ is a *base* of Φ if $\alpha_i(h_j) = a_{ij}$, $\forall i, j \in I$, and $\alpha_i(d_j) = \delta_{ij}$, $\forall i \in I,\ j \in I \setminus J$. In this case α_i is called a *basic root*

From now on, we fix a base Δ of Φ. We use the word *basis* of $\mathfrak{g}$ when referring to the vector space basis of $\mathfrak{g}$.

Proposition 1.2.16. *The base Δ is a linearly independent subset of $\widehat{\mathfrak{h}}^*$.*

Proof. Assume that there are $c_i \in \mathbb{K}$ such that $\sum_{i \in I} c_i \alpha_i = 0$. If $j \in I \setminus J$, then

$$c_j = c_j \alpha_j(d_j) = \sum_{i \in I} c_i \alpha_i(d_j) = 0. \tag{1.9}$$

Reindex the set I in such a manner that $I = \{1, \ldots, k, k+1, \ldots, m\}$ and $J = \{1, \ldots, k\}$. Because of (1.9), all that remains is

$$c_1 \alpha_1 + \cdots + c_k \alpha_k = 0.$$

Evaluating this equation in h_r, $r = 1, \ldots, k$, yields

$$\begin{cases} c_1 a_{11} + c_2 a_{12} + & \cdots & + c_k a_{1k} = 0 \\ c_1 a_{21} + c_2 a_{22} + & \cdots & + c_k a_{2k} = 0 \\ & \vdots & \\ c_1 a_{k1} + c_2 a_{k2} + & \cdots & + c_k a_{kk} = 0 \end{cases}$$

hence $(c_1, \ldots, c_k) \in \operatorname{Ker}(C) = 0$, since $C = (a_{ij})_{i,j \in J}$ is an invertible matrix. Therefore $c_i = 0$, $\forall i \in I$. This shows that Δ is a linearly independent subset of $\widehat{\mathfrak{h}}$. □

As a consequence of the last proposition, we may assume that the isomorphism $\varphi : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ satisfies $\varphi(\alpha_i) = h_i$, $\forall i \in I$. Hence the standard bilinear form (1.7) satisfies $(\alpha_i, \alpha_j) = (h_i, h_j)$, $\forall i \in I$.

Define $\Delta^\vee = \{h_i : i \in I\}$. Making the identification $\widehat{\mathfrak{h}} = (\widehat{\mathfrak{h}}^*)^*$, it follows from Linear Algebra that $h_i(\alpha_j) = a_{ij}$. This means that the demonstration of Proposition 1.2.16 proves the following result, with minor adjustments.

Proposition 1.2.17. *The set $\Delta^\vee$ is a linearly independent subset of $\widehat{\mathfrak{h}}$.*

Since $|\Delta^\vee| = n$ and $\dim(\mathfrak{h}) = n$, then we have that as vector spaces

$$\mathrm{Span}(\Delta^\vee) \cong \mathfrak{h}.$$

Further, it can be proved that $\Delta^\vee$ actually is a basis of $\mathfrak{h}$. This means, particularly, that the Chevalley generators $h_i,\ i \in I$ are the coroots of $\alpha_i,\ i \in I$.

Proposition 1.2.18. *Every $\gamma \in \Phi$ can be written as $\gamma = \sum_{i \in I} k_i \alpha_i$, in which $k_i \in \mathbb{Z}$ are all positive or all negative.*

Definition 1.2.19. Given $\gamma = \sum_{i \in I} k_i \alpha_i \in \Phi$, the integer $\mathrm{ht}(\gamma) := \sum_{i \in I} k_i$ is the *height* of γ.

Let Q be the set of $\mathbb{Z}$-linear combinations of Δ. Clearly, $\Phi \subseteq Q$. Define $Q_+ := \{\sum_{\alpha \in \Delta} k_\alpha \alpha : k_\alpha \in \mathbb{Z}_{\geq 0}\}$ and $Q_- := \{\sum_{\alpha \in \Delta} k_\alpha \alpha : k_\alpha \in \mathbb{Z}_{\leq 0}\}$. In this case, $\Phi_+ := \Phi \cap Q_+$ is the set of *positive roots* of Φ and $\Phi_- := \Phi \cap Q_-$ is the set of *negative roots* of Φ. It follows that Φ is a disjoint union of Φ_+ and Φ_-

The set Q_+ allow us to introduce a partial order in $\widehat{\mathfrak{h}}^*$. Given $\alpha, \beta \in \widehat{\mathfrak{h}}^*$ we set

$$\alpha > \beta \iff \alpha - \beta \in Q_+. \tag{1.10}$$

Proposition 1.2.20. *For $\mathfrak{n}_\pm$ as in Proposition 1.2.4 we have*

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha.$$

1.2.3 Weyl group

Given a Kac-Moody algebra $\mathfrak{g}$ with standard bilinear form $(\ ,\ )$ we define, for every $\alpha \in \Phi$, a *reflection* $\sigma_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$\sigma_\alpha(\beta) := \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha. \tag{1.11}$$

Note that every reflection is a linear map. It follows from the definition that $\sigma_\alpha(\alpha) = -\alpha$ and $\sigma_\alpha^2 = \mathrm{Id}$ for every root α. The set of all reflections has a group structure under the composition of functions. This group is called the *Weyl group* of $\mathfrak{g}$, denoted by $\mathcal{W}$.

Now define the form

$$\begin{aligned} \langle\ ,\ \rangle : \Phi \times \Phi &\to \mathbb{Z} \\ \alpha, \beta &\mapsto \frac{2(\beta, \alpha)}{(\alpha, \alpha)}. \end{aligned}$$

It is clear that $\langle \alpha, \alpha \rangle = 2, \forall \alpha \in \Phi$. Note that $\langle\ ,\ \rangle$ is not a bilinear form, but it is linear in the first entry. This form allows the rewriting of a root reflection as $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$.

Proposition 1.2.21. *For every $\alpha, \beta \in \Phi$ we have $\sigma_\alpha(\beta) \in \Phi$.*

Proof. Let $\alpha, \beta \in \Phi$. Then

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \frac{2}{(t_\alpha, t_\alpha)}\beta(t_\alpha)\alpha = \beta - \beta\left(\frac{2t_\alpha}{(t_\alpha,t_\alpha)}\right)\alpha = \beta - \beta(h_\alpha)\alpha \in \Phi.$$

□

From the last proposition, we have the restriction $\sigma_\alpha : \Phi \to \Phi$ is well defined. It is straightforward that such restriction is bijective, hence the Weyl group permutes roots. Moreover, from the proof of the last proposition, we have that $\langle \beta, \alpha \rangle = \beta(h_\alpha) = r - q$, as in (1.8).

Theorem 1.2.22. *The Weyl group $\mathcal{W}$ is generated by the reflections $\{\sigma_{\alpha_i} : \alpha_i \in \Delta\}$.*

Fix $\sigma_{\alpha_i} = \sigma_i$. We say that σ_i is a *basic reflection*. Because of the last theorem, for every $\sigma \in \mathcal{W}$ there must be an integer m such that σ can be written as composition of m basic reflections. However, m does not need to be unique. If such m is minimal, we define $l(\sigma) := m$ as the *length* of σ.

If $\mathfrak{g}$ is of finite type, it follows from [4, Proposition 5.17] that there always exists $w_0 \in \mathcal{W}$ such that $l(w_0) = |\Phi_+|$, and moreover $l(w_0) \geq l(\sigma)$, $\forall \sigma \in \mathcal{W}$. Such w_0 is called the *longest reflection* of $\mathcal{W}$.

1.3 Finite and affine Kac-Moody algebras

1.3.1 Central extension

It is always possible to extend a finite GCM A to an affine GCM $\widehat{A}$, as done in Subsection 1.1.2. This means that the affine Kac-Moody algebra $\mathfrak{g}(\widehat{A})$ can be obtained via the finite Kac-Moody algebra $\mathfrak{g}(A)$. In this section, we provide an explicit realization of $\mathfrak{g}(\widehat{A})$ which connects to $\mathfrak{g}(A)$ more naturally. If needed, Chapter 18 of [4] greatly develops the theory we present in this section.

Fix $\mathfrak{g} = \mathfrak{g}(A)$ as a Kac-Moody algebra of finite type and let $\mathbb{K}[t, t^{-1}]$ be the ring of Laurent polynomials in an indeterminate t. Set the $\mathbb{K}$ vector space $\mathfrak{g}[t, t^{-1}] := \mathfrak{g} \otimes \mathbb{K}[t, t^-1]$ and define the bracket

$$[x \otimes p, y \otimes q] := [x, y] \otimes pq \tag{1.12}$$

in which the bracket on the right side of the equality is the bracket of $\mathfrak{g}$. This defines a Lie algebra structure in $\mathfrak{g}[t, t^{-1}]$. It is clear that (1.12) is bilinear, as both the bracket of $\mathfrak{g}$ and the tensor product are. Also $[x \otimes p, x \otimes p] = [x, x] \otimes p^2 = 0 \otimes p^2 = 0$.

For the Jacobi identity, note that

$$\begin{aligned}
&[x \otimes t^n, [y \otimes t^m, z \otimes t^r]] + [y \otimes t^m, [z \otimes t^r, x \otimes t^n]] + [z \otimes t^r, [x \otimes t^n, y \otimes t^m]] \qquad (1.13)\\
&= [x \otimes t^n, [y, z] \otimes t^{m+r}] + [y \otimes t^m, [z, x] \otimes t^{n+r}] + [z \otimes t^r, [x, y] \otimes t^{n+m}]\\
&= [x, [y, z]] \otimes t^{n+m+r} + [y, [z, x]] \otimes t^{n+m+r} + [z, [x, y]] \otimes t^{n+m+r}\\
&= ([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \otimes t^{n+m+r}\\
&= 0 \otimes t^{n+m+r}\\
&= 0.
\end{aligned}$$

Definition 1.3.1. The Lie algebra $\mathfrak{g}[t,t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$ is called the *loop algebra* of $\mathfrak{g}$. Likewise, $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ is the *current algebra* of $\mathfrak{g}$.

Now we extend the loop algebra via a one-dimensional central extension. Namely, let K' be a formal element and define the Lie algebra $\tilde{\mathfrak{g}}$, whose underlying vector space is $\mathfrak{g}[t,t^{-1}] \oplus \mathbb{K}K'$, with bracket given by

$$[x \otimes t^n + rK', y \otimes t^m + sK'] = [x,y] \otimes t^{n+m} + n\delta_{n,-m}(x,y)K' \tag{1.14}$$

with $(\ ,\ )$ being the standard invariant form of $\mathfrak{g}$.

We show that (1.14) defines a Lie algebra structure in $\tilde{\mathfrak{g}}$. It is clear that (1.14) is bilinear and skew-symmetric. For the Jacobi Identity, we have

$$\begin{aligned}[x \otimes t^n + aK', [y \otimes t^m + bK', z \otimes t^r + cK']] &= [x \otimes t^n + aK', [y,z] \otimes t^{r+m} + m\delta_{m,-r}(y,z)K'] \\ &= [x,[y,z]] \otimes t^{n+m+r} + n\delta_{n,-(m+r)}(x,[y,z])K'. \end{aligned} \tag{1.15}$$

Similarly

$$[y \otimes t^m + bK', [z \otimes t^r + cK', x \otimes t^n + aK']] = [y,[z,x]] \otimes t^{n+m+r} + m\delta_{m,-(n+r)}(y,[z,x])K' \tag{1.16}$$

and

$$[z \otimes t^r + cK', [x \otimes t^n + aK', y \otimes t^m + bK']] = [z,[x,y]] \otimes t^{n+m+r} + r\delta_{r,-(n+m)}(z,[x,y])K'. \tag{1.17}$$

Adding up (1.15), (1.16) and (1.17) we obtain

$$[x,[y,z]] \otimes t^{n+m+r} + [y,[z,x]] \otimes t^{n+m+r} + [z,[x,y]] \otimes t^{n+m+r} = 0$$

by similar computations as in (1.13). If $n \neq -m-r$, then $\delta_{n,-(m+r)} = \delta_{m,-(n+r)} = \delta_{r,-(n+m)} = 0$, so

$$n\delta_{n,-(m+r)}(x,[y,z])K' + m\delta_{m,-(n+r)}(y,[z,x])K' + r\delta_{r,-(n+m)}(z,[x,y])K' = 0.$$

If $n = -m-r$, then $\delta_{n,-(m+r)} = \delta_{m,-(n+r)} = \delta_{r,-(n+m)} = 1$, so

$$\begin{aligned} n(x,[y,z])K' + m(y,[z,x])K' + r(z,[x,y])K' &= n(x,[y,z])K' + m([y,z],x)K' + r([x,y],z)K' \\ &= n(x,[y,z])K' + m(x,[y,z])K' + r(x,[y,z])K' \\ &= (n+m+r)(x,[y,z])K' \\ &= 0. \end{aligned}$$

Hence the Jacobi Identity holds, and $\tilde{\mathfrak{g}}$ is a Lie algebra.

Remark 1.3.2. *Note that* $[x, K'] = 0, \forall x \in \tilde{\mathfrak{g}}$.

Lastly, let d' be a formal element such that $[d', x \otimes t^n] = n(x \otimes t^n)$, and set $\widehat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{K}d'$. In this case $\widehat{\mathfrak{g}}$ is a Lie algebra under the bracket

$$\begin{aligned}&[x \otimes t^n + a_1K' + b_1d', y \otimes t^m + a_2K' + b_2d'] = \\ &\qquad [x,y] \otimes t^{n+m} + b_1m(y \otimes t^m) - b_2n(x \otimes t^n) + \delta_{n,-m}(x,y)K'. \end{aligned} \tag{1.18}$$

Set the GCM $\widehat{A} = (a_{ij})_{i,j \in \widehat{I}}$, such that $A = (a_{ij})_{i,j\in I}$. Let $\mathfrak{g}(\widehat{A})$ be the affine Kac-Moody algebra associated with $\widehat{A}$. Next theorem connects $\mathfrak{g}(\widehat{A})$ to $\widehat{\mathfrak{g}}$ under an explicit realization. Its proof, which is quite extensive, can be found in [4, Theorem 18.5].

Theorem 1.3.3. *For $\widehat{\mathfrak{g}}$ and $\mathfrak{g}(\widehat{A})$ as above, we have that $\widehat{\mathfrak{g}} \cong \mathfrak{g}(\widehat{A})$.*

Remark 1.3.4. *Note that identifying $\mathfrak{g}$ to $\mathfrak{g} \otimes 1$ we have a natural sequence of inclusions:*

$$\mathfrak{g} \subset \mathfrak{g}[t] \subset \widehat{\mathfrak{g}}.$$

Remark 1.3.5. *Fix $\mathcal{W}$ Weyl group of $\mathfrak{g}$ and $\widehat{\mathcal{W}}$ Weyl group of $\widehat{\mathfrak{g}}$. In this case, there is an embedding $\mathcal{W} \hookrightarrow \widehat{\mathcal{W}}$ and hence we say, by abuse of notation, that $\mathcal{W} \subseteq \widehat{\mathcal{W}}$.*

Definition 1.3.6. Given $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ as above, define

(i) $\mathfrak{g}_d = \mathfrak{g} \oplus \mathbb{K}d$, $\mathfrak{g}_d[t] = \mathfrak{g}_d \otimes \mathbb{K}[t]$.

(ii) $\widehat{\mathfrak{n}}^{\pm} = \mathfrak{n}_{\pm} \oplus \mathfrak{g} \otimes t\mathbb{C}[t]$.

(iii) $\widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^{+}$. Say that $\widehat{\mathfrak{b}}$ is the *standard Borel subalgebra* of $\widehat{\mathfrak{g}}$. Likewise, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_{+}$ is the standard Borel subalgebra of $\mathfrak{g}$.

Note that the current algebra $\mathfrak{g}[t]$ (see Definiton 1.3.1) is naturally graded by the degree of the polynomial part of its elements. In this case, we let $G = \mathbb{Z}$ and set $g[t]_n = 0$ for $n < 0$ and for $n \geq 0$

$$\mathfrak{g}[t]_n := \{x \otimes p(t) \in \mathfrak{g}[t] : \deg(p(t)) = n\}. \tag{1.19}$$

One easily checks that the above defines a graded Lie algebra. Moreover, it is trivial to check that $\mathfrak{g}[t]$ is naturally filtered by the degree of the polynomials by setting

$$\mathfrak{g}_n = \oplus_{j=0}^{n} \mathfrak{g}[t]_j, \quad n \in \mathbb{N}.$$

Further, the graded Lie algebra associated with the above filtration is isomorphic to the graded Lie algebra structure of (1.19).

1.3.2 Basic roots of finite and affine Kac-Moody algebras

Fix $\mathfrak{g}$ to be a Kac-Moody algebra of finite type and let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra associated with $\mathfrak{g}$ (see Theorem 1.3.3). Fix Φ a root system of $\mathfrak{g}$ with base Δ. In this case note that

$$\dim(\mathfrak{h}^*) = n = |\Delta|$$

hence Δ is a basis of $\mathfrak{h}^*$ (see Proposition 1.2.16). The following result can be found in [4, Proposition 12.9].

Proposition 1.3.7. *There exists a unique $\theta = \sum_{i\in I} a_i\alpha_i \in \Phi$ such that for any $\alpha = \sum_{i\in I} k_i\alpha_i \in \Phi$ we have $k_i \leq a_i$.*

The root θ is called the *longest* root of Φ. The integers a_i of the definition of θ can be determined in the following way: Extending the Cartan matrix A of $\mathfrak{g}$ to an affine GCM $\widehat{A}$, the integers $a_i,\ i \in I$, are the same as in vector a of (1.5). Fix $h_\theta = \sum_{i \in I_0} k_i h_i$ as the coroot of θ. The integers k_i are the same as in vector k of (1.5).

We now describe the root system $\widehat{\Phi}$ of $\widehat{\mathfrak{g}}$. Fix $\widehat{\Delta} = \{\alpha_i : i \in \widehat{I}\}$ to be a base of $\widehat{\Phi}$. In this case, there is a natural injection $\Delta \hookrightarrow \widehat{\Delta}$ and by abuse of notation we identify $\Delta \subset \widehat{\Delta}$.

Define $\delta := \theta + \alpha_0$. Notice that $\delta \neq 0$, since it can be written as

$$\delta = \sum_{i \in \widehat{I}} a_i \alpha_i$$

with $a_0 = 1$. Thus $\delta = 0$ would imply in $a_i = 0,\ \forall i \in \widehat{I}$, contradiction. Moreover, we have that $\delta \in \widehat{\Phi}$. In fact, since $\delta = \sum_{i \in I} a_i \alpha_i$, it follows that, for every $j \in I$

$$\delta(h_j) = \alpha_0(h_j) + \alpha_1(h_j) + \cdots + a_n \alpha_n(h_j) = a_{j0} + a_1 a_{j1} + \cdots + a_n a_{jn}$$

which is equal to the j-th entry of the vector $Aa = 0$. Hence $\delta(h_j) = 0, \forall j \in I$. As $[h, h'] = 0, \forall h, h' \in \mathfrak{h}$, it follows that $\mathfrak{h} \subseteq \mathfrak{g}_\delta$. So $\mathfrak{g}_\delta \neq \{0\}$, and since $\delta \neq 0$ it follows that $\delta \in \widehat{\Phi}$.

Say that δ is the *basic imaginary root*. Next theorem (which is a combined result of Theorems 16.27 and 17.17 of [4]) uses the basic imaginary root to describe the roots of $\widehat{\Phi}$ in terms of the roots of Φ.

Theorem 1.3.8. *The root system $\widehat{\Phi}$ satisfies*

$$\widehat{\Phi} = \{\alpha + r\delta : \alpha \in \Phi, r \in \mathbb{Z}\} \cup \{r\delta : r \in \mathbb{Z} \setminus \{0\}\}.$$

The elements of $\{\alpha + r\delta : \alpha \in \Phi, r \in \mathbb{Z}\}$ are called the *real roots* of $\widehat{\Phi}$ and the elements of $\{r\delta : r \in \mathbb{Z} \setminus \{0\}\}$ are called the *imaginary roots* of $\widehat{\Phi}$. It is clear that every $\alpha_i \in \widehat{\Delta}$ is real.

Since $\widehat{\Delta}$ and $\widehat{\Delta}^\vee := \{h_i : i \in \widehat{I}\}$ are linearly independent subsets of $\widehat{\mathfrak{h}}^*$ and $\widehat{\mathfrak{h}}$, respectively, they can be completed to a basis (recall Remark 1.2.3). Let us do so, starting with $\widehat{\Delta}^\vee$. Let $d_0 \in \widehat{\mathfrak{h}}$ be the (unique) Chevalley generator in $\mathfrak{d}$ (check (1.6)). From the definition of the basic roots, we have $\alpha_i(d_0) = \delta_{i0}, \forall i \in \widehat{I}$. For simplicity write $d := d_0$.

Proposition 1.3.9. *The set $\{h_0, \ldots, h_n, d\}$ is a basis of $\widehat{\mathfrak{h}}$.*

Proof. Suppose that there are $c_0, \ldots, c_n, c_d \in \mathbb{K}$ such that $c_0 h_0 + \cdots + c_n h_n + c_d d = 0$. If $c_d = 0$, then $c_0 h_0 + \cdots + c_n h_n = 0$ implies that $c_i = 0, \forall i = 0, \ldots, n$, since $\{h_0, \ldots, h_n\}$ is a linearly independent set (see Proposition 1.2.17).

Assume that $c_d \neq 0$. In this case, we can write

$$d = \frac{c_0}{c_d} h_0 + \cdots + \frac{c_n}{c_d} h_n.$$

Define $s_i := \dfrac{c_i}{c_d}$, $\forall i \in \widehat{I}$, and $s := (s_0, \ldots, s_n)$. Hence

$$\alpha_j(d) = \alpha_j\left(\sum_{i=0}^{n} s_i h_i\right) = \sum_{i=0}^{n} s_i \alpha_j(h_i) = \sum_{i=0}^{n} s_i a_{ij}. \tag{1.20}$$

Since $\alpha_j(d) = \delta_{j0}$, equation (1.20) yields $\sum_{i=0}^{n} s_i(a_{i0}, \dots, a_{in}) = (1, 0, \dots, 0)$. Particularly, $\sum_{i=0}^{n} s_i(a_{i1}, \dots, a_{in}) = (0, \dots, 0)$.

Define $A := (a_{ij}), i \in \widehat{I}, j \in I$, which is the GCM $\widehat{A}$ without the first column. Since $\dim(\mathrm{Ker}(A)) = 1$ it follows that $\dim(\mathrm{Ker}(A^{\top})) = 1$. Note also that $k = (k_1, \dots, k_n) \in \mathrm{Ker}(A^{\top})$ (see equation (1.5)). But from our computations, $(s_0, \dots, s_n) \in \mathrm{Ker}(A^{\top})$, meaning that s must be a scalar multiple of k. Hence $s\widehat{A} = 0$. so $\sum_{i=0}^{n} s_i(a_{i0}, \dots, a_{in}) = (0, 0, \dots, 0)$, contradiction.

Thus c_d must be zero, so the set $\{h_0, \dots, h_n, d\}$ needs to be linearly independent. As $\dim(\widehat{\mathfrak{h}}) = n + 2$, it is a basis of $\widehat{\mathfrak{h}}$. ☐

Define an element $\Lambda_0 \in \widehat{\mathfrak{h}}^*$ such that $\Lambda_0(h_i) = \delta_{i0}$, if $i \in \widehat{I}$ and $\Lambda_0(d) = 0$.

Proposition 1.3.10. *The set* $\{\alpha_0, \dots, \alpha_n, \Lambda_0\}$ *forms a basis of* $\widehat{\mathfrak{h}}^*$*.*

Proof. The set $\{\alpha_0, \dots, \alpha_n, \Lambda_0\}$ is a basis of $\widehat{\mathfrak{h}}^*$ if, and only if, the matrix obtained by applying these elements in a basis of $\widehat{\mathfrak{h}}$ is non-singular.

The matrix obtained by applying $\{\alpha_0, \dots, \alpha_n, \Lambda_0\}$ in the basis of $\widehat{\mathfrak{h}}$ given in Proposition 1.3.9 is

$$\left(\begin{array}{c|ccc|c} 2 & * & \cdots & * & 1 \\ \hline * & & & & 0 \\ \vdots & & A & & \vdots \\ * & & & & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \end{array}\right)$$

with A a Cartan matrix. By definition, $\det(A) \neq 0$, hence the determinant of the above matrix is not zero and $\{\alpha_0, \dots, \alpha_n, \Lambda_0\}$ is a basis of $\widehat{\mathfrak{h}}^*$. ☐

Remark 1.3.11. *Note that* Λ_0 *satisfies* $\Lambda_0(K) = \Lambda_0(h_0 + \cdots + k_n h_n) = \Lambda_0(h_0) = 1$.

Say that an element x of $\widehat{\mathfrak{g}}$ is *central* if $[x, y] = 0$, $\forall y \in \widehat{\mathfrak{g}}$. All central elements of $\widehat{\mathfrak{g}}$ can be written as scalar multiples of $K = \sum_{i \in I} k_i h_i$, for k_i as in (1.5), hence K is called the *canonical central element* of $\mathfrak{g}$. Note that $K = h_\theta + h_0$. Moreover, for every $i \in I$ we have

$$\alpha_i(K) = \alpha_i\left(\sum k_j h_j\right) = \sum k_j \alpha_i(h_j) = \sum k_j a_{ji} = 0. \tag{1.21}$$

Remark 1.3.12. *Under the isomorphism of Theorem 1.3.3, we have* $K \mapsto K'$ *and* $d \mapsto d'$*. Hence we settle* $K = K'$ *and* $d = d'$ *from now on, in the sense that* K *and* d *act as in Section 1.3.1.*

Chapter 2

REPRESENTATIONS OF LIE ALGEBRAS

Representation theory of finite-dimensional Lie algebras has been a subject for over a century of work (for a historical review, check [3]). This chapter discusses general aspects of the representation theory of Kac-Moody algebras. For a more detailed discussion on the representation theory of semisimple Lie algebras of finite type, one should check [19]. For the affine case, we recommend [4, 23].

2.1 Modules

Our main goal through this text is to study particular modules (or representations) of current algebras. In this section we define modules over a Lie algebra $\mathfrak{g}$, give some background on the study of $\mathfrak{g}$-modules and present some fundamental results. All notations follow the conventions of Chapter 1.

Definition 2.1.1. A Lie homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, with V a vector space over $\mathbb{K}$, is called a *representation* of $\mathfrak{g}$.

Example 2.1.2. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be such that $\rho(x) = \mathrm{ad}(x)$. It is clear that ρ is well defined. To check that ρ defines a representation we must check that ρ is a Lie homomorphism.

(i) ρ is a linear transformation: For all $x, y \in \mathfrak{g}$ and $a, b \in \mathbb{K}$, we have that $\rho(ax + by) = \mathrm{ad}(ax + by)$, and $\mathrm{ad}(ax+by)(z) = [ax+by, z] = a[x, z]+b[y, z] = (a\mathrm{ad}(x)+b\mathrm{ad}(y))(z)$, so $\mathrm{ad}(ax+by) = a\mathrm{ad}(x)+b\mathrm{ad}(y)$.

(ii) ρ preserves the bracket: $\rho([x,y]) = \mathrm{ad}([x,y])$, and

$$\begin{aligned} \mathrm{ad}([x,y])(z) &= [[x,y],z] \\ &= -[z,[x,y]] \\ &= [x,[y,z]] + [y,[z,x]] \\ &= [x,[y,z]] - [y,[x,z]] \\ &= \mathrm{ad}(x)([y,z]) - \mathrm{ad}(y)([x,z]) \\ &= \mathrm{ad}(x)(\mathrm{ad}(y))(z) - \mathrm{ad}(y)(\mathrm{ad}(x))(z) \\ &= [\mathrm{ad}(x),\mathrm{ad}(y)](z) \\ &= [\rho(x),\rho(y)](z) \end{aligned}$$

So $\rho([x,y]) = [\rho(x),\rho(y)]$.

Therefore, ρ is a representation of $\mathfrak{g}$. This representation is called the *adjoint representation* of $\mathfrak{g}$. ◆

The following proposition, which is proved in [4, Proposition 9.3], relates representations of $\mathfrak{g}$ to representations of $U(\mathfrak{g})$ in a crucial manner.

Proposition 2.1.3. *There is a bijective correspondence between representations of $\mathfrak{g}$ and representations of $U(\mathfrak{g})$. Such representations are related by the following condition: If ρ is a representation of $\mathfrak{g}$, then there is a representation ϕ of $U(\mathfrak{g})$ such that*

$$\phi \circ \pi = \rho$$

in which $\pi : \mathfrak{g} \to U(\mathfrak{g})$ is the natural injection.

Definition 2.1.4. A *left $\mathfrak{g}$-module* is a vector space V over $\mathbb{K}$ with an external bilinear operation

$$\begin{aligned} \cdot : \mathfrak{g} \times V &\to V \\ (x,v) &\mapsto x \cdot v \end{aligned}$$

called an *action*, which satisfies $[x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v), \forall x,y \in \mathfrak{g}$, $\forall v \in V$.

As one can expect, there is a similar definition of *right $\mathfrak{g}$-module*. All the following results are valid for both cases, given the proper adjustments. Thus we only work with left $\mathfrak{g}$-modules from now on, and simply call them $\mathfrak{g}$-modules.

Example 2.1.5. For every Lie algebra $\mathfrak{g}$, we can regard $\mathfrak{g}$ as a $\mathfrak{g}$-module with the action given by $x \cdot v = [x,v]$, for all $x,v \in \mathfrak{g}$. From the defining axioms of Lie algebras, it is straightforward that this defines an action. ◆

Proposition 2.1.6. *Representations of $\mathfrak{g}$ and $\mathfrak{g}$-modules are equivalent structures. Namely:*

(i) If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then V is a $\mathfrak{g}$-module with action $x \cdot v = \rho(x)(v)$.

(ii) If V is a $\mathfrak{g}$-module with action $x \cdot v$, then $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, with $\rho(x)v = x \cdot v$, is a representation of $\mathfrak{g}$.

We do not distinguish between $\mathfrak{g}$-modules and representations of $\mathfrak{g}$ on the remainder of the text, and we only use the terminology of modules instead of representations.

Given two $\mathfrak{g}$-modules V and W one can define the tensor product as the $\mathbb{K}$-vector space $V \otimes W$. In the representation theory of generic algebraic structures, it is not true that $V \otimes W$ is again a module. However, in the case of Lie algebras one has a natural way to define an action on $V \otimes W$ related to the actions on V and W. This happens because $U(\mathfrak{g})$ is a Hopf algebra.

For $x \in \mathfrak{g}$, $v \in V$ and $w \in W$, define the function $\mathfrak{g} \times V \otimes W \to V \otimes W$ given by

$$x \cdot (v \otimes w) \mapsto \underbrace{(x \cdot v)}_{V \text{ action}} \otimes w + v \otimes \underbrace{(x \cdot w)}_{W \text{ action}} \tag{2.1}$$

and extend it linearly to $V \otimes W$. This defines an action of $\mathfrak{g}$ in $V \otimes W$. We show that Definition 2.1.4 holds. It is easy to check it is bilinear. For $x, y \in \mathfrak{g}$ and $v \otimes w \in V \otimes W$

$$\begin{aligned}
[x,y] \cdot (v \otimes w) &= ([x,y] \cdot v) \otimes w + v \otimes ([x,y] \cdot w) \\
&= (x \cdot (y \cdot v) - y \cdot (x \cdot v)) \otimes w + v \otimes (x \cdot (y \cdot w) - y \cdot (x \cdot w)) \\
&= x \cdot (y \cdot v) \otimes w - y \cdot (x \cdot v) \otimes w + v \otimes x \cdot (y \cdot w) - v \otimes y \cdot (x \cdot w) \\
&= x \cdot (y \cdot v) \otimes w + v \otimes x \cdot (y \cdot w) - (y \cdot (x \cdot v) \otimes w + v \otimes y \cdot (x \cdot w)).
\end{aligned}$$

On the other hand

$$\begin{aligned}
x \cdot (y \cdot (v \otimes w)) &= x \cdot ((y \cdot v) \otimes w + v \otimes (y \cdot w)) \\
&= x \cdot ((y \cdot v) \otimes w) + x \cdot (v \otimes (y \cdot w)) \\
&= x \cdot (y \cdot v) \otimes w + (y \cdot v) \otimes (x \cdot w) + (x \cdot v) \otimes (y \cdot w) + v \otimes x \cdot (y \cdot w)
\end{aligned}$$

and

$$\begin{aligned}
y \cdot (x \cdot (v \otimes w)) &= y \cdot ((x \cdot v) \otimes w + v \otimes (x \cdot w)) \\
&= y \cdot ((x \cdot v) \otimes w) + y \cdot (v \otimes (x \cdot w)) \\
&= y \cdot (x \cdot v) \otimes w + (x \cdot v) \otimes (y \cdot w) + (y \cdot v) \otimes (x \cdot w) + v \otimes y \cdot (x \cdot w).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
x \cdot (y \cdot (v \otimes w)) - y \cdot (x \cdot (v \otimes w)) &= x \cdot (y \cdot v) \otimes w + v \otimes x \cdot (y \cdot w) - (y \cdot (x \cdot v) \otimes w + v \otimes y \cdot (x \cdot w)) \\
&= [x,y] \cdot (v \otimes w).
\end{aligned}$$

Definition 2.1.7. Let $W \subseteq V$ be a vector subspace of a $\mathfrak{g}$-module V with action $x \cdot v$. We say that W is a $\mathfrak{g}$-*submodule* of V if $x \cdot w \in W$, $\forall x \in \mathfrak{g}$ and $\forall w \in W$.

Note that, for a $\mathfrak{g}$-module V, both V and $\{0\}$ are submodules of V. A submodule W of V is said to be *proper* if $W \neq \{0\}$ and $W \neq V$. The module $\{0\}$ is often called the *trivial module.*

Definition 2.1.8. A $\mathfrak{g}$-module V is said to be *irreducible* if is has no proper submodule. If V has irreducible submodules $W_1, \dots, W_n$ such that $V = \bigoplus_{i=1}^n W_i$, then we say that V is *completely reducible.* If there are no proper submodules W_1, W_2 of V such that $V = W_1 \oplus W_2$ then V is an *indecomposable* module.

Example 2.1.9. Set $\mathfrak{g}$ as the Heisenberg Lie algebra (see Example 1.1.3), and regard $\mathfrak{g}$ as a $\mathfrak{g}$-module via the adjoint action. In this case, the only proper submodules of $\mathfrak{g}$ are $\mathfrak{g}_1 = \langle z \rangle$, $\mathfrak{g}_2 = \langle x, z \rangle$ and $\mathfrak{g}_3 = \langle y, z \rangle$. It is clear that it is not possible to write $\mathfrak{g}$ as a direct sum of $\mathfrak{g}_i$, $i = 1, 2, 3$, hence $\mathfrak{g}$ is indecomposable. ◆

Example 2.1.10. Fix $\mathfrak{g} = \mathfrak{sl}_2$ (see Example 1.1.10) and assume $\mathrm{char}(\mathbb{K}) \neq 2$. Fix V as the vector space of 2×2 matrices over $\mathbb{K}$. It is clear that V is a $\mathfrak{g}$-module with action being the matrix multiplication. Moreover, V is completely reducible, since $V = V_1 \oplus V_2$, with

$$V_1 = \{M = (a_{ij}) \in V : a_{12} = a_{22} = 0\} \text{ and } V_2 = \{M = (a_{ij}) \in V : a_{11} = a_{21} = 0\}.$$

It is easy to check that both V_1 and V_2 are irreducible submodules of V. ◆

Example 2.1.11. Fix $\mathfrak{g} = \mathfrak{sl}_2$ and assume $\mathrm{char}(\mathbb{K}) \neq 2$. If we regard V as a $\mathfrak{g}$-module under the adjoint action, then V is reducible. This happens because $\mathfrak{g}$ is a vector subspace of V, and since $\mathfrak{g}$ is closed under the action of $\mathfrak{g}$ it follows that $\mathfrak{g}$ is a $\mathfrak{g}$-submodule of V.

When we see $\mathfrak{g}$ as a $\mathfrak{g}$-module under the adjoint action (see Example 2.1.5) then $\mathfrak{g}$ is an irreducible module. This happens because any submodule of $\mathfrak{g}$ is also an ideal of $\mathfrak{g}$. Since $\mathfrak{sl}_2$ is a simple Lie algebra, there are no proper submodules of $\mathfrak{g}$.

If we define $W = \mathbb{K}I$, with I as the identity matrix, then W is closed under the action of $\mathfrak{g}$, because $[\mathfrak{g}, I] = 0$, which means that W is a $\mathfrak{g}$-submodule of V, which is clearly irreducible. Hence $V = \mathfrak{g} \oplus W$, thus W is completely reducible. ◆

In the representation theory of any algebra, irreducible modules always play an important role. In the case of finite-dimensional simple Lie algebras, in particular, the study of any finite-dimensional representations reduces to the study of the irreducible ones as the next result states.

Theorem 2.1.12 (Weyl Theorem)**.** *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, and V be a finite-dimensional $\mathfrak{g}$-module. Then V is completely reducible.*

Definition 2.1.13. Let V and W be two $\mathfrak{g}$-modules, with respective actions $x \cdot v$ and $x * w$. A linear transformation $\varphi : V \to W$ is a *homomorphism of* $\mathfrak{g}$*-modules* (or $\mathfrak{g}$-homomorphism) if, for every $x \in \mathfrak{g}$ and $v \in V$, $\varphi(x \cdot v) = x * \varphi(v)$. If φ is a vector space isomorphism, say that φ is a $\mathfrak{g}$*-module isomorphism*, and V and W are *isomorphic* $\mathfrak{g}$-modules, denoted by $V \cong_{\mathfrak{g}} W$, or simply $V \cong W$ when there is no confusion.

Let $\mathfrak{g}$ be a graded Lie algebra, with gradation $\{\mathfrak{g}_a\}_{a \in G}$. A $\mathfrak{g}$-module V is a *graded module* of $\mathfrak{g}$ (or a graded $\mathfrak{g}$-module) if there is a family of vector subspaces $\{V_a\}_{a \in G}$ of V satisfying the following.

(i) $V = \bigoplus_{a \in G} V_a$.

(ii) For every $a, b \in G$ it holds that $\mathfrak{g}_a \cdot V_b \subset V_{a+b}$.

Similarly, if $\mathfrak{g}$ is a filtered Lie algebra, with filtration $\{\mathfrak{g}_i\}_{i\in\mathbb{N}}$, a $\mathfrak{g}$-module V is a *filtered module* of $\mathfrak{g}$ (or a filtered $\mathfrak{g}$-module) if there is a family of vector subspaces $\{V_n\}_{n\in\mathbb{N}}$ of V satisfying the following.

(i) If $i < j$, then $V_i \subseteq V_j$.

(ii) We can write $V = \bigcup_{n\geq 0} V_n$.

(iii) For every $i, j \in \mathbb{N}$ we have $\mathfrak{g}_i \cdot V_j \subset V_{i+j}$.

2.2 Weight spaces

This section develops some basic information about weights and weight vectors, which are generalizations of eigenvalues and eigenvectors. For a deeper look into these subjects, one can check Chapters 7 and 13 of [19] (for the classical case) and Chapters 10 and 19 of [4] (for the finite and affine Kac-Moody case, respectively).

Definition 2.2.1. Let V be a $\mathfrak{g}$-module. For all $\Lambda \in \widehat{\mathfrak{h}}^*$, set $V_\Lambda := \{v \in V : h \cdot v = \Lambda(h)v, \ \forall h \in \widehat{\mathfrak{h}}\}$. If $V_\Lambda \neq \{0\}$ then V_Λ is an *weight space* and Λ is an *weight* of V. The non-zero elements $v \in V_\lambda$ are called the *weight vectors* of V associated with the weight Λ. A module V is called an *weight module* if

$$V = \bigoplus_{\Lambda\in\widehat{\mathfrak{h}}^*} V_\Lambda \tag{2.2}$$

The decomposition (2.2) is the *weight space decomposition* of V.

Example 2.2.2. Fix $\mathfrak{g} = \mathfrak{sl}_2$ (see Example 1.1.10). Let V be an arbitrary finite-dimensional $\mathfrak{g}$-module. It follows from the definition that h acts diagonally on V, hence we can decompose V as a sum of eigenspaces $V = \bigoplus_{\lambda\in\mathbb{K}} V_\lambda$, with $V_\lambda = \{v \in \mathfrak{g} : h \cdot x = \lambda x\}$. This means, particularly, that every finite-dimensional $\mathfrak{g}$-module is a weight module. Note that each V_λ is one-dimensional,

For a weight λ of $\mathfrak{g}$ and $v \in V_\lambda$, we have that

$$h \cdot (x^+ \cdot v) = [h, x^+] \cdot v + x \cdot (h \cdot v) = 2x^+ \cdot v + \lambda x \cdot v = (\lambda + 2)x \cdot v$$

meaning that $x^+ \cdot v \in V_{\lambda+2}$. Likewise, $x^- \cdot v \in V_{\lambda-2}$. Since we assume that $\dim(V) < \infty$, there must be some $\lambda \in \mathbb{K}$ such that $V_\lambda \neq \{0\}$, but $V_{\lambda+2} = \{0\}$. It can be shown (see [19, Lemma 7.2]) that such λ is equal to $\dim(V) - 1$.

The same reasoning shows that there must be some $\mu \in \mathbb{K}$ such that $V_\mu \neq \{0\}$, but $V_{\mu-2} = \{0\}$. It follows that $\mu = -m$, so we can decompose

$$V = V_m \oplus V_{m-2} \oplus \cdots \oplus V_{-(m-2)} \oplus V_{-m}.$$

◆

Example 2.2.3. Recall the root space decomposition as in Proposition 1.2.6. Regard $\mathfrak{g}$ as a $\mathfrak{g}$-module via the adjoint action (see Example 2.1.5). We prove that the root spaces of $\mathfrak{g}$ correspond to the weight spaces of the adjoint module.

For $\Lambda \in \widehat{\mathfrak{h}}^*$, since the action in this case is given by

$$h \cdot v = [h, v]$$

it follows that the weight space of Λ is $\mathfrak{g}_\Lambda = \{v \in \mathfrak{g} : [h, v] = \Lambda(h)v,\ \forall h \in \widehat{\mathfrak{h}}\}$. Hence $\Lambda \in \Phi \cup \{0\}$. Conversely, if $\alpha \in \Phi$ then it is clear that α is also a weight of the adjoint module $\mathfrak{g}$. Thus every non-zero weight space is a root space, and every root space is a non-zero weight space.

Note also that the weight space of $\Lambda = 0$ satisfy $\mathfrak{g}_0 = \{v \in \mathfrak{g} : [h, v] = 0,\ \forall h \in \widehat{\mathfrak{h}}\}$. By the defining relations of $\mathfrak{g}$ it follows that $\widehat{\mathfrak{h}} \subseteq \mathfrak{g}_0$. Now we show that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$: If $x, y \in \mathfrak{g}_0$, then

$$[h, [x, y]] = [x, [h, y]] + [y, [x, h]] = [x, 0] + [y, 0] = 0$$

hence $[x, y] \in \mathfrak{g}_0$. Also, $\widehat{\mathfrak{h}}$ is an ideal of $\mathfrak{g}_0$, since $x \in \mathfrak{g}_0$ and $h \in \widehat{\mathfrak{h}}$ imply $[x, h] = 0 \in \widehat{\mathfrak{h}}$. But Lemma 1.1.12(iii) forces $\mathfrak{g}_0 \subseteq N(\widehat{\mathfrak{h}}) = \widehat{\mathfrak{h}}$, thus $\mathfrak{g}_0 = \widehat{\mathfrak{h}}$. ◆

Lemma 2.2.4. *If V is a $\mathfrak{g}$-module and $V' := \sum_{\Lambda \in \widehat{\mathfrak{h}}^*} V_\Lambda$, then the following items hold.*

(i) $\mathfrak{g}_\alpha \cdot (V_\Lambda) \subset V_{\Lambda+\alpha}$, $\forall \alpha \in \Phi$ and $\lambda \in \widehat{\mathfrak{h}}^*$, and hence V' is a $\mathfrak{g}$-submodule of V.

(ii) The sum $\sum_{\Lambda \in \widehat{\mathfrak{h}}^*} V_\Lambda$ is direct.

(iii) If $\mathfrak{g}$ is a finite Kac-Moody algebra and dim$(V) < \infty$, then $V = V'$.

Proof. (i) Given $h \in \widehat{\mathfrak{h}}$, $v \in V_\Lambda$ and $x \in \mathfrak{g}_\alpha$ recall that $[h, x] = \alpha(h)x$ and $h \cdot v = \Lambda(h)v$. Therefore

$$\begin{aligned} h \cdot (x \cdot v) &= x \cdot (h \cdot v) + [h, x] \cdot v \\ &= x \cdot (\Lambda(h)v) + \alpha(h)x \cdot v \\ &= (\Lambda(h) + \alpha(h))x \cdot v \\ &= (\Lambda + \alpha)(h)x \cdot v \end{aligned}$$

so $x \cdot v \in V_{\Lambda+\alpha}$. Thus V' is closed by the action of $\mathfrak{g}$, and since $V' \subseteq V$ it follows that V' is a $\mathfrak{g}$-submodule of V.

(ii) To show that the sum $\sum_{\Lambda \in \widehat{\mathfrak{h}}^*} V_\Lambda$ is direct it suffices to verify that any finite set of weight vectors associated with different weights is always linearly independent. Set $\Lambda_1, \dots, \Lambda_n \in \widehat{\mathfrak{h}}^*$ weights such that $\Lambda_i \neq \Lambda_j$ if $i \neq j$, and fix $v_i \in V_{\Lambda_i} \setminus \{0\}$. Suppose that for certain $c_1, \dots, c_n \in \mathbb{K}$ one has

$$c_1 v_1 + \cdots + c_n v_n = 0.$$

We show that $c_i = 0, \forall i = 1, \ldots, n$, by induction in the number n of weights. If $n = 1$, there is nothing to be done. If $n = 2$, set Λ_1, Λ_2 weights such that $\Lambda_1 \neq \Lambda_2$. In this case, there is $h \in \widehat{\mathfrak{h}}$ such that $\Lambda_1(h) \neq \Lambda_2(h)$. Fix $v_1 \in V_{\Lambda_1} \setminus \{0\}$ and $v_2 \in V_{\Lambda_2} \setminus \{0\}$ and assume that

$$c_1 v_1 + c_2 v_2 = 0, \text{ with } c_1, c_2 \in \mathbb{K}.$$

Then we have

$$0 = h(0) = h(c_1 v_1 + c_2 v_2) = \Lambda_1(h) c_1 v_1 + \Lambda_2(h) c_2 v_2. \tag{2.3}$$

Since $\Lambda_1(h) \in \mathbb{K}$ we also have

$$0 = \Lambda_1(h)(c_1 v_1 + c_2 v_2) = \Lambda_1(h) c_1 v_1 + \Lambda_1(h) c_2 v_2. \tag{2.4}$$

Subtracting (2.3) from (2.4) yields

$$0 = (\Lambda_2(h) - \Lambda_1(h)) c_2 v_2$$

Since $v_2 \neq 0$ and $\Lambda_2(h) - \Lambda_1(h) \neq 0$, we must have $c_2 = 0$, hence $c_1 = 0$, as desired.

As the induction hypothesis, suppose that the result is true for any integer smaller than n. Then, for n distinct weights $\Lambda_1, \ldots, \Lambda_n$, assume that $c_1 v_1 + \cdots + c_n v_n = 0$. We show that $c_i = 0$, $\forall i = 1, \ldots, n$.

Fix an arbitrary $h \in \widehat{\mathfrak{h}}$. Multiplying this equation by $\Lambda_1(h)$ yields

$$\Lambda_1(h) c_1 v_1 + \cdots + \Lambda_n(h) c_n v_n = 0. \tag{2.5}$$

Acting with h in $v_1 + \cdots v_n$ yields

$$\begin{aligned} 0 &= h \cdot (c_1 v_1 + \cdots + c_n v_n) \\ &= (h \cdot c_1 v_1) + \cdots + (h \cdot c_n v_n) \\ &= \Lambda_1(h) c_1 v_1 + \cdots + \Lambda_n(h) c_n v_n \end{aligned} \tag{2.6}$$

It follows from equations (2.5) and (2.6) that

$$(\Lambda_1(h) - \Lambda_2(h)) c_2 v_2 + \cdots (\Lambda_1(h) - \Lambda_n(h)) c_n v_n = 0.$$

By the induction hypothesis, $(\Lambda_1(h) - \Lambda_i(h)) c_i = 0$, $\forall i = 2, \ldots, n$. Since $h \in \widehat{\mathfrak{h}}$ is arbitrary we may chose $h_i \in \widehat{\mathfrak{h}}$ such that $\Lambda_1(h_i) \neq \Lambda_i(h_i)$, $\forall i = 2, \ldots, n$. Hence $c_i = 0$, $\forall i = 2, \ldots, n$, which imply in $c_1 = 0$, completing the proof.

(iii) Since $\mathfrak{g}$ is a finite semisimple Lie algebra and V is finite-dimensional, by the Weyl Theorem V is completely reducible. We may assume, without loss of generality, that V is irreducible, as the general case can be extended from this one. Recall that in this case $\widehat{\mathfrak{h}} = \mathfrak{h}$ (see Remark 1.2.3).

If V is irreducible, then all that its left to do is show that $V' \neq \{0\}$, because of item (ii). Since $\mathfrak{h}$ is abelian, it is solvable, so Lie's Theorem ensures that there is some $v \in V \setminus \{0\}$ such that v is an eigenvector for the action of $\mathfrak{h}$. It follows that $v \in V'$, hence $V' \neq \{0\}$ and $V = V'$. □

It follows from item (i) of last Lemma that if $v \in V_\Lambda$, then $\mathfrak{g}_\alpha \cdot v$ is either zero or a subset of $V_{\Lambda+\alpha}$, $\forall \alpha \in \Phi$. Thus there are cases in which a certain weight vector v satisfies $\mathfrak{g}_\alpha \cdot v = 0$ for every $\alpha \in \Phi_+$. The vector v is called the *highest weight vector* of weight Λ and Λ is called a *highest weight.*

Definition 2.2.5. If a $\mathfrak{g}$-module V can be written as $V = U(\mathfrak{g}) \cdot v$, for a given highest weight vector $v \in V_\Lambda$, we say that V is a *cyclic* $\mathfrak{g}$*-module* of highest weight Λ.

To simplify the notation, we sometimes refer to a cyclic $\mathfrak{g}$-module V of highest weight Λ as a cyclic $\mathfrak{g}$-module V of highest weight vector $v \in V(\Lambda)$, or just cyclic $\mathfrak{g}$-module V of highest weight vector v, when there is no confusion.

Theorem 2.2.6. *Let V be a cyclic $\mathfrak{g}$-module with highest weight vector $v \in V_\Lambda$. Set $\Phi_+ = \{\beta_1, \dots, \beta_m\}$. Then the following items hold.*

(i) V is spanned by the vectors $(x^-_{\beta_1})^{i_1} \cdots (x^-_{\beta_m})^{i_m} \cdot v$, with $x^-_{\beta_i} \in \mathfrak{g}_{-\beta_i}$ and $i_j \in \mathbb{Z}_{>0}$. In particular, V is a weight module.

(ii) The weights of V are of the form $\mu = \Lambda - \sum_{i\in I} k_i\alpha_i$, with $k_i \in \mathbb{Z}_{>0}$ and $\alpha_i \in \Delta$. Hence Λ is a highest weight.

(iii) For every $\mu \in \widehat{\mathfrak{h}}^*$, $dim V_\mu < \infty$ and $dim V_\Lambda = 1$.

(iv) Each submodule of V is again a weight module.

(v) V is an indecomposable $\mathfrak{g}$-module, with a unique maximal proper submodule and a corresponding unique irreducible quotient.

Now we show that cyclic modules always exists for any $\Lambda \in \widehat{\mathfrak{h}}^*$. For $\alpha \in \Phi_+$ and $x_\alpha \in \mathfrak{g}_\alpha$, define

$$I_\Lambda := \sum_{\alpha\in\Phi_+} U(\mathfrak{g})x_\alpha + \sum_{i\in I} U(\mathfrak{g})(h_i - \Lambda(h_i)).$$

Hence I_Λ is a left ideal of $U(\mathfrak{g})$ generated by the elements x_α and $(h_i - \Lambda(h_i))$. Define

$$M(\Lambda) := \frac{U(\mathfrak{g})}{I_\Lambda}.$$

It follows that $M(\Lambda)$ is a $U(\mathfrak{g})$-module, which we call *Verma module* of highest weight Λ.

For a given weight $\Lambda \in \widehat{\mathfrak{h}}^*$ the Verma module $M(\Lambda)$ is a cyclic $\mathfrak{g}$-module of highest weight Λ: Set $\phi : U(\mathfrak{g}) \to M(\Lambda)$ as the quotient map and fix $v = \phi(1)$. It is clear that $\mathfrak{n}_+ \cdot v = 0$ and that $(h - \Lambda(h)) \cdot v = 0$, $\forall h \in \widehat{\mathfrak{h}}$. This ensures the existence of a cyclic $\mathfrak{g}$-module of highest weight Λ, for any weight Λ.

Notice that from Theorem 2.2.6 (v) there must exist a unique maximal proper submodule $J(\Lambda)$ of $M(\Lambda)$, which is again a cyclic $\mathfrak{g}$-module. Hence cyclic modules are not unique in general. Furthermore Theorem 2.2.6 (v) ensures that there is a corresponding unique quotient

$$V(\Lambda) := \frac{M(\Lambda)}{J(\Lambda)}$$

which is irreducible. It is clear that $V(\Lambda)$ is again a cyclic $\mathfrak{g}$-module of highest weight Λ.

Proposition 2.2.7. *If V is an irreducible cyclic module with highest weight vector $v \in V_\Lambda$, then*

$$V \cong V(\Lambda)$$

Proof. Since V is a cyclic module, there must be a highest weight vector $v \in V_\Lambda$ such that $\mathfrak{g}_\alpha \cdot v = 0$, for all $\alpha \in \Phi_+$. The map $\phi : M(\Lambda) \to V$, $\phi(m) = v$, where $m \in M(\Lambda)_\Lambda$ non-zero, is then an $U(\mathfrak{g})$-modules homomorphism. Since V is irreducible the homomorphism ϕ must be surjective and hence

$$V \cong \frac{M(\Lambda)}{\mathrm{Ker}(\phi)}.$$

But then V is an irreducible quotient of $M(\Lambda)$. By the uniqueness of $V(\Lambda)$ ensured by Theorem 2.2.6 (v) it follows that $V(\Lambda) \cong V$. □

Definition 2.2.8. Given any Lie algebra $\mathfrak{g}$, say that an application $f : \mathfrak{g} \to \mathfrak{g}$ is *locally nilpotent* if for every $x \in \mathfrak{g}$ there exists $n(x) \in \mathbb{Z}_{\geq 0}$ such that

$$f^{n(x)}(x) = 0.$$

If V is a weight $\mathfrak{g}$-module such that the actions of $x_i^+ : V \to V$ and $x_i^- : V \to V$ are locally nilpotent, then V is said to be an *integrable* module.

Example 2.2.9. Suppose that $\mathfrak{g}$ is of finite type and regard $\mathfrak{g}$ as a $\mathfrak{g}$-module under the adjoint action (see Example 2.1.5). We show that $\mathfrak{g}$ is an integrable $\mathfrak{g}$-module. From Example 2.2.3 we already know that $\mathfrak{g} = \bigoplus_{\Lambda \in \widehat{\mathfrak{h}}^*} \mathfrak{g}_\Lambda$, so we prove that $\mathrm{ad}x_i^+ : \mathfrak{g} \to \mathfrak{g}$ and $\mathrm{ad}x_i^- : \mathfrak{g} \to \mathfrak{g}$ are locally nilpotent.

Suppose that $\mathrm{ad}x_i^+$ acts locally nilpotent on certain $x, y \in \mathfrak{g}$. This means that there are some $k, s \in \mathbb{Z}_{\geq 0}$ such that $(\mathrm{ad}x_i^+)^k(x) = 0$ and $(\mathrm{ad}x_i^+)^s(y)$. In this case, since for every $n \in \mathbb{Z}_{\geq 0}$ we have

$$(\mathrm{ad}x_i^+)^n([x, y]) = \sum_{j=0}^{n} \binom{n}{j} \left[(\mathrm{ad}x_i^+)^j(x), (\mathrm{ad}x_i^+)^{n-j}(y)\right].$$

it follows that $(\mathrm{ad}x_i^+)^m([x, y]) = 0$ for $m = k + s + 1$. Thus the set $\mathfrak{s} \subseteq \mathfrak{g}$ of elements $x \in \mathfrak{g}$ on which $\mathrm{ad}x_i^+$ acts locally nilpotently is a Lie subalgebra. But we have that

(i) $\mathrm{ad}x_i^+(x_i^+) = 0$.

(ii) $(\mathrm{ad}x_i^+)^{1-a_{ij}}(x_j^+) = 0$, $\forall j \neq i$.

(iii) $(\mathrm{ad}x_i^+)^2(h_j) = 0$, $\forall j$.

(iv) $(\mathrm{ad}x_i^+)^3(x_i^-) = 0$.

(v) $(\mathrm{ad}x_i^+)(x_j^-) = 0$, if $i \neq j$.

Hence every Chevalley generator of $\mathfrak{g}$ is in the Lie subalgebra $\mathfrak{s}$, thus $\mathfrak{s} = \mathfrak{g}$, so $\mathrm{ad}x_i^+$ acts locally nilpotently in $\mathfrak{g}$. The same reasoning shows that $\mathrm{ad}x_i^-$ acts locally nilpotently in $\mathfrak{g}$. ◆

The next proposition is proved in [4, Proposition 19.13]. We do not give its proof as it demands theories we do not delve into. Nevertheless, it is an important result that plays a crucial role in Chapter 3.

Proposition 2.2.10. *If* $\mathcal{W}$ *is the Weyl group of* $\mathfrak{g}$ *and* V *is an integrable* $\mathfrak{g}$*-module then* $dim(V_\Lambda) = dim(V_{\sigma(\Lambda)})$, $\forall \Lambda \in \widehat{\mathfrak{h}}$, $\forall \sigma \in \mathcal{W}$.

Remark 2.2.11. *Since* $dim(V(\Lambda)_\Lambda) = 1$ *the above proposition ensures that* $dim(V(\Lambda)_{\sigma(\Lambda)}) = 1$, $\forall \sigma \in \mathcal{W}$.

Definition 2.2.12. For $i \in I$ define $\Lambda_i \in \widehat{\mathfrak{h}}^*$ by requiring

$$\Lambda_i(h_j) = \delta_{ij},\ \forall j \in I.$$

The elements Λ_i, $i \in I$, are called the *fundamental weights* of $\mathfrak{g}$. Moreover fix $P = \{\Lambda \in \widehat{\mathfrak{h}}^* : \Lambda(h) \in \mathbb{Z},\ \forall h \in \widehat{\mathfrak{h}}\}$ and $P_+ = \{\Lambda \in P : \Lambda(h) \geq 0,\ \forall h \in \widehat{\mathfrak{h}}\}$. Say that Λ is *integral* if $\Lambda \in P$ and that Λ is *integral dominant* if $\Lambda \in P_+$.

Our next goal is to study the fundamental weights of finite and affine Kac-Moody algebras. From now on, fix $\mathfrak{g}$ as a finite Kac-Moody algebra with affine realization $\widehat{\mathfrak{g}}$ and recall all the terminology used in this scenario.

Set $\{\omega_i : i \in I\}$ as the set of fundamental weights of $\mathfrak{g}$. It follows from the definition that $\{\omega_i : i \in I\}$ is a linearly independent subset of $\mathfrak{h}^*$, since $\{h_i : i \in I\}$ is a linearly independent subset of $\mathfrak{h}$. Hence $\{\omega_i : i \in I\}$ is a basis of $\widehat{\mathfrak{h}}$.

The next proposition establishes an extremely useful relationship between fundamental weights and basic roots of $\mathfrak{g}$.

Proposition 2.2.13. *If* $A = (a_{ij})$ *is the Cartan matrix relative to* $\mathfrak{g}$ *and* α_j *is a basic root, then* $\alpha_j = \sum_i a_{ij}\omega_i$.

Proof. We know that $\{\omega_i : i \in I_0\}$ is a basis of $\mathfrak{h}^*$, so there must be $k_{j1}, \ldots, k_{jn} \in \mathbb{K}$ such that $\alpha_j = \sum_i k_{ji}\omega_i$. Thus, $\alpha_j(h_i) = \sum_i k_{ji}\omega_i(h_i) = k_{ji}$. Therefore,

$$k_{ji} = \alpha_j(h_i) = \alpha_j\left(\frac{2t_i}{(t_i,t_i)}\right) = \left(t_j, \frac{2t_i}{(t_i,t_i)}\right) = 2\frac{(t_j,t_i)}{(t_i,t_i)} = 2\frac{(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} = a_{ij}.$$

□

Example 2.2.14. Let $\mathfrak{g}$ be of B_3 type. Suppose that $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ are the basic roots and $\{\omega_1, \omega_2, \omega_3\}$ are the fundamental weights. Since the Cartan matrix A in this case is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

we have that the transpose of A is

$$A^\top = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

so the basic roots are $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = -\omega_1 + 2\omega_2 - 2\omega_3$ and $\alpha_3 = -\omega_2 + 2\omega_3$. ♦

We now determine the fundamental weights of $\widehat{\mathfrak{g}}$ in terms of $\omega_i,\ i \in I$. Recall the basis $\{h_0, \dots, h_n, d\}$ from Proposition 1.3.9. Set $\widehat{\Phi}$ root system of $\widehat{\mathfrak{g}}$ with base $\widehat{\Delta}$. For each $i \in I_0$, define

$$\Lambda_i := \omega_i + k_i \Lambda_0 \in \mathfrak{h}^* \tag{2.7}$$

with ω_i a fundamental weight of $\mathfrak{g}$, k_i as in (1.5) and Λ_0 as in Proposition 1.3.10. Note that Λ_0 is by definition a fundamental weight of $\widehat{\mathfrak{g}}$.

Lemma 2.2.15. *The elements Λ_i, $i \in I$, are fundamental weights of $\widehat{\mathfrak{g}}$.*

Proof. For all $i, j \in I$

$$\Lambda_i(h_j) = \omega_i(h_j) + k_i \Lambda_0(h_j) = \omega_i(h_j) = \delta_{ij}.$$

To ensure that $\Lambda_i(h_0) = 0,\ \forall i \in I$, recall that $\omega_i = \sum_{j=1}^n a_{ij}\alpha_j$ (due to Theorem 2.2.13). Hence

$$\omega_i(K) = \sum a_{ij}\alpha_j(K) = 0$$

because of (1.21). In this case, as $k_0 = 1$ (Remark 1.1.25) we have

$$0 = \omega_i(K) = \omega_i(h_0) + \omega_i\Big(\sum_{i=1}^n k_i h_i\Big) = \omega_i(h_0) + k_i$$

thus $\omega_i(h_0) = -k_i$. This implies that

$$\Lambda_i(h_0) = \omega_i(h_0) + k_i \Lambda_0(h_0) = \omega_i(h_0) + k_i = -k_i + k_i = 0.$$

It follows that Λ_i is a fundamental weight, for all $i \in I$. □

Remark 2.2.16. *Note that $\Lambda_i(d) = \omega_i(d) + k_i \Lambda_0(d) = \omega_i(d) = 0$.*

Unlike the finite case, the set of fundamental weights of $\widehat{\mathfrak{g}}$ is not a basis for $\widehat{\mathfrak{h}}^*$, as it has not enough elements. To complete it to a basis, we use the fundamental imaginary root δ.

Proposition 2.2.17. *The set $\{\Lambda_0, \dots, \Lambda_n, \delta\}$ is a basis of $\widehat{\mathfrak{h}}^*$.*

Proof. Recall that $\delta(h_i) = 0, \forall i \in I$, $\delta = \sum_{i \in I} a_i \alpha_i$ and $\alpha_i(d) = \delta_{i0}$. Using the same reasoning as in Proposition 1.3.10, applying $\{\Lambda_0, \dots, \Lambda_n, \delta\}$ in $\{h_0, \dots, h_n, d\}$ yields the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

which is non-singular, and therefore $\{\Lambda_0, \dots, \Lambda_n, \delta\}$ is a basis of $\widehat{\mathfrak{h}}^*$. □

Much like in the finite case, the GCM $\widehat{A}$ of $\widehat{\mathfrak{g}}$ can also be used to write basic roots in terms of fundamental weights. However, in the affine case, the relation is not as straightforward as before.

Proposition 2.2.18. *Let $\widehat{A} = (a_{ij})_{i,j\widehat{I}}$ be the GCM relative to $\widehat{\mathfrak{g}}$. If $i \in I$ then $\alpha_i = \sum_j a_{ji}\Lambda_j$, and $\alpha_0 = \sum_j a_{j0}\Lambda_j - \delta$.*

Set P and P_+ as the sets of integral and dominant integral (respectively) of $\mathfrak{g}$. Likewise set $\widehat{P}$ and $\widehat{P}_+$ as the sets of integral and dominant integral (respectively) of $\widehat{\mathfrak{g}}$.

Proposition 2.2.19. *Assume that $\Lambda \in \widehat{\mathfrak{h}}^*$. Then:*

(i) $\Lambda \in \widehat{P} \iff \Lambda = \sum_{i\in\widehat{I}} r_i\Lambda_i + \xi\delta$, *for certain $r_i \in \mathbb{Z}$ and $\xi \in \mathbb{K}$.*

(ii) $\Lambda \in \widehat{P}_+ \iff \Lambda = \sum_{i\in\widehat{I}} r_i\Lambda_i + \xi\delta$, *for certain $r_i \in \mathbb{Z}_{\geq 0}$ and $\xi \in \mathbb{K}$.*

Proof. We prove only item (i), as item (ii) is analogous. Suppose that $\Lambda \in P_+$. By its definition (see Definition 2.2.12) we have $\Lambda(h_i) \in \mathbb{Z}$. By Proposition 2.2.17 we have $\Lambda = \sum r_i\Lambda_i + \xi\delta$, for certain $r_i, \xi \in \mathbb{K}$.

Since $\Lambda(h_j) = \sum r_i\Lambda_i(h_j) + \xi\delta(h_j) = r_j$, it follows that $r_j \in \mathbb{Z}$. The converse is immediate. ☐

Remark 2.2.20. *The previous proposition holds for the finite case, suppressing the term δ.*

From now on we focus on the study of irreducible cyclic modules associated with integral dominant weights. The next proposition follows from Theorem 10.20, Proposition 19.14, and Theorem 19.19 of [4].

Proposition 2.2.21. *Given the $\mathfrak{g}$-module $V(\lambda)$, $\lambda \in \mathfrak{h}^*$, the following items are equivalent.*

(i) $V(\lambda)$ is integrable.

(ii) $\lambda \in P_+$.

(iii) $V(\lambda)$ is finite-dimensional.

Moreover, in this case, $V(\lambda) = \dfrac{M(\Lambda)}{J(\Lambda)}$, with $J(\Lambda)$ being the submodule of $M(\Lambda)$ generated by elements of the form $(x_i^-)^{\lambda(h_i)+1} \cdot v$, $\forall i \in I$.

Corollary 2.2.22. *Let v be the generating vector of Theorem 2.2.21. If $\beta \in \Phi_+$ then*

$$(x_\beta^-)^{\lambda(h_\beta)+1} \cdot v = 0.$$

There is a statement similar to Proposition 2.2.22 for $\widehat{\mathfrak{g}}$. However, it should be noted that it is a weaker result, as stated in the next proposition. It is a combined result of Proposition 19.14 and Theorem 19.19 of [4].

Proposition 2.2.23. *Given the $\widehat{\mathfrak{g}}$-module $V(\Lambda)$, $\Lambda \in \widehat{\mathfrak{h}}$, the following items are equivalent.*

(i) $V(\Lambda)$ is integrable.

(ii) $\Lambda \in \widehat{P}_+$.

Moreover, in this case, $V(\Lambda)$ is isomorphic to the $\widehat{\mathfrak{g}}$-module generated by non-zero vector v satisfying the following relation

(i) $(x_i^+ \otimes t^s) \cdot v = 0, \forall i \in I, \ \forall s \in \mathbb{Z}$.

(ii) $(h \otimes t^s) \cdot v = \delta_{s0}\Lambda(h)v, \forall h \in \mathfrak{h}$.

(iii) $(x_i^-)^{\lambda(h_i)+1} \cdot v = 0, \forall i \in I$.

For every integrable $V(\Lambda)$ we have that $\dim(V(\Lambda)) = \infty$. This follows because $\widehat{\Phi}$ has infinite roots (see Theorem 1.3.8), hence $\widehat{\mathcal{W}}$ has infinite reflections. From Remark 2.2.11 we have $\dim(V(\Lambda))_{\sigma(\Lambda)} = 1, \ \forall \sigma \in \widehat{\mathcal{W}}$. Hence there are infinite non-zero weight spaces as subspaces of $V(\Lambda)$, all generated by linearly independent vectors. This forces $\dim(V(\Lambda)) = \infty$.

Chapter 3

DEMAZURE MODULES

We now focus on the main two families of representations we address in this work, namely Weyl modules and $\mathfrak{g}$-stable Demazure modules. The main result of this chapter comes in Section 3.2. We show that level one Demazure modules are generated by a vector v that needs to satisfy a smaller set of relations compared to the previously known results coming from [10] and [28]. From now on fix $\mathbb{K} = \mathbb{C}$.

3.1 Weyl and Demazure modules

Weyl and Demazure modules were first introduced, respectively, in [9] and [11]. Over this section, we define these modules using a somewhat modern approach, hence the definitions of both structures will differ from the ones found in the original articles. Throughout this chapter, fix $\mathfrak{g}$ a simple finite Kac-Moody algebra with affine realization $\widehat{\mathfrak{g}}$. Recall that $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ is the current algebra of $\mathfrak{g}$, and make the identification $x \otimes 1 = x$, $\forall x \in \mathfrak{g}$. We retain the notation used in Chapters 1 and 2.

3.1.1 Weyl modules

Definition 3.1.1. For $\lambda \in P_+$, let $W(\lambda)$ be the $\mathfrak{g}[t]$-module generated by a non-zero element w satisfying the following set of relations:

(i) $(x_i^+ \otimes \mathbb{C}[t]) \cdot w = 0, \quad i \in I.$

(ii) $(h_i \otimes t^s) \cdot w = \delta_{s0}\lambda(h_i)w, \quad i \in I, \; s \in \mathbb{Z}_{\geq 0}.$

(iii) $(x_i^-)^{\lambda(h_i)+1} \cdot w = 0, \quad i \in I.$

The $\mathfrak{g}[t]$-module $W(\lambda)$ is called the *local Weyl module* of $\mathfrak{g}[t]$ of weight λ, and we refer to the defining relations of $W(\lambda)$ as *Weyl relations.*

Remark 3.1.2. *Note that the generator w of $W(\lambda)$ is a highest weight vector of weight λ by definition. Moreover $W(\lambda)$ is a graded $\mathfrak{g}[t]$-module under the degree of the polynomial part of $\mathfrak{g}[t]$ (see (1.19)).*

Remark 3.1.3. *Similarly, one can define global Weyl modules. Since our work does not need such a general structure, we omit its definition and refer to local Weyl modules simply by Weyl modules.*

The next theorem states some basic information about Weyl modules. The reader should be aware that these results are far from trivial to prove, and such demonstration can be found in [9].

Theorem 3.1.4. *For a given* $\lambda \in P_+$*:*

(i) The Weyl module $W(\lambda) \neq 0$*.*

(ii) The Weyl module $W(\lambda)$ *is finite-dimensional.*

(iii) If V *is a finite-dimensional* $\mathfrak{g}[t]$*-module generated by a highest weight vector* $v \in V_\lambda$*, then* V *is quotient of* $W(\lambda)$*.*

Let $\lambda \in P_+$. For a non-zero $w \in W(\lambda)_\lambda$, define

$$W' := U(\mathfrak{g}) \cdot w.$$

By definition W' is a cyclic $\mathfrak{g}$-module generated by a highest weight vector $w \in W'_\lambda$, hence w satisfies the first two relations of $V(\lambda)$ given in Proposition 2.2.21. Further, the last relation of $V(\lambda)$ coincides with the last relations of $W(\lambda)$, which is satisfied by w as well. This means that W' is a quotient of $V(\lambda)$. By the irreducibility of $V(\lambda)$ we have that $W' \cong V(\lambda)$, so next lemma follows from Corollary 2.2.22.

Lemma 3.1.5. *If* $\gamma \in \Phi_+$ *and* w *is the generator of* $W(\lambda)$*, then*

$$(x_\gamma^-)^{\lambda(h_\gamma)+1} \cdot w = 0. \tag{3.1}$$

3.1.2 Demazure modules

Since the original definitions of Demazure modules were very demanding and not so useful for what we will do, we present a modern approach to defining these structures. Hence, we follow closely the definitions given in [28].

Let $\Lambda \in \widehat{P}_+$ and $\sigma \in \widehat{\mathcal{W}}$. Choose $v \in V(\Lambda)_{\sigma(\Lambda)}$ non-zero, and define

$$V_\sigma(\Lambda) := U(\widehat{\mathfrak{b}}) \cdot v. \tag{3.2}$$

The $U(\widehat{\mathfrak{b}})$-module $V_\sigma(\Lambda)$ is called the *Demazure module* associated with σ and Λ. Let $w \in V_\sigma(\Lambda)_\mu$ a weight vector for some weight μ. Note that $\mu \geq \sigma(\Lambda)$ and by Theorem 2.2.6 (ii) we have that $\sigma(\Lambda) = \Lambda - \sum_{i \in \widehat{I}} k_i \alpha_i$, with $k_i \in \mathbb{Z}_{\geq 0}$, so there are finitely-many weights μ satisfying $\mu \geq \sigma(\Lambda)$. Since $\dim(V(\Lambda)_\mu) < \infty$ for every μ (Proposition 2.2.6(iii)) it follows that $V_\sigma(\Lambda)$ is finite-dimensional.

Remark 3.1.6. *Recall that* $dim(V(\Lambda)_{\Lambda'}) = dim(V(\Lambda)_{\sigma(\Lambda')})$*, for every* $\Lambda' \in \widehat{P}$*,* $\sigma \in \mathcal{W}$ *(see Proposition 2.2.10). Particularly,* $dim(V(\Lambda)_\Lambda) = 1$ *implies that* $dim(V(\Lambda)_{\sigma(\Lambda)}) = 1$*. So, up to scalar multiple, there is a unique weight vector* $v \in V(\Lambda)$ *with weight* $\sigma(\Lambda)$*, meaning that the generator* v *of* $V_\sigma(\Lambda)$ *is unique up to scalar.*

Example 3.1.7. Let $\widehat{\mathfrak{g}}$ be of type $\widehat{A}_2$, $\Lambda = \Lambda_1$ and $\sigma \in \widehat{\mathcal{W}}$. Recall from Proposition 2.2.18 that each root can be written as a sum of fundamental weights and that such decomposition can be done using the associated GCM. Also, recall from Lemma 2.2.4 that the action of x_i^- in some $v \in V_\Lambda \setminus \{0\}$ yields an element $v' \in V_{\Lambda-\alpha_i} \setminus \{0\}$.

Let $v_\sigma \in (V(\Lambda))_{\sigma\Lambda} \setminus \{0\}$. If $\sigma = \text{id}$ then

$$V_\sigma(\Lambda) = \mathbb{C}v_{\text{id}}.$$

Suppose that $\sigma = \sigma_1$. Then

$$\sigma\Lambda = \Lambda - \Lambda(h_1)\alpha_1 = -\Lambda_1 + \Lambda_2 + \Lambda_0.$$

In particular $(\sigma\Lambda)(h_j) \geq 0$ for $i = 0, 2$. Since v_σ is an extremal vector, using the representation theory of $\mathfrak{sl}_2$ it follows that

$$x_0^+ v_\sigma = 0 = x_2^+ v_\sigma = (x_1^+)^2 v_\sigma, \quad x_1^+ v_\sigma \neq 0.$$

Since $x_1^+ v_\sigma \in V(\Lambda)_\Lambda$, it follows that

$$V_\sigma(\Lambda) = \mathbb{C}v_\sigma + \mathbb{C}x_1^+ v_\sigma.$$

Suppose now that $\sigma = \sigma_2\sigma_1$. Then

$$\sigma(\Lambda) = -\Lambda_2 + 2\Lambda_0.$$

Again using that v_σ is an extremal vector we obtain

$$x_0^+ v_\sigma = 0 = x_1^+ v_\sigma = (x_2^+)^2 v_\sigma, \quad x_2^+ v_\sigma \neq 0.$$

Straightforward computations show that $x_2^+ v_\sigma \in V(\Lambda)_{\sigma_1\Lambda}$ and hence we can use the previous case to conclude that

$$V_\sigma(\Lambda) = \mathbb{C}v_\sigma + \mathbb{C}x_2^+ v_\sigma + \mathbb{C}x_1^+ x_2^+ v_\sigma.$$

◆

Recall that $V(\Lambda)$ is generated by a vector $w \neq 0$ satisfying a set of relations (see Theorem 2.2.21). Despite its description by generators and relations, its structure is far from being completely understood. Among other uses the Demazure modules play an important role, providing pieces of information of $V(\Lambda)$. However its original definition does not provide a practical way to work with them. Therefore a natural task would be to describe such modules via generators and relations. The following theorem, which is Theorem 1 of [18], gives a way to do so.

Theorem 3.1.8. *For $\Lambda \in \widehat{P}_+$ and $\sigma \in \widehat{\mathcal{W}}$, the Demazure module $V_\sigma(\Lambda)$ is isomorphic to the $U(\widehat{\mathfrak{b}})$-module generated by an element $v \neq 0$ satisfying the following set of relations, $\forall \gamma \in \Phi_+$, $\forall s \in \mathbb{Z}_{\geq 0}$.*

(i) $(x_\gamma^+ \otimes t^s)^{k_\gamma+1} \cdot v = 0$, *with* $k_\gamma = max\{0, -\sigma(\Lambda)(h_{\gamma+s\delta})\}$.

(ii) $(x_\gamma^- \otimes t^s)^{k_\gamma+1} \cdot v = 0$, *with* $k_\gamma = max\{0, -\sigma(\Lambda)(h_{-\gamma+s\delta})\}$.

(iii) $(h_i \otimes t^s) \cdot v = \delta_{0s}\sigma(\Lambda)(h_i)v$, *for* $i \in I$.

(iv) $d \cdot v = \sigma(\Lambda)(d)v$.

(v) $K \cdot v = \Lambda(K)v$.

Notice that $\forall i \in I$

$$x_i^- \cdot V(\Lambda)_{\sigma(\Lambda)} = 0 \iff \sigma(\Lambda)(h_i) \leq 0. \tag{3.3}$$

We can use this information to determine whenever $V_\sigma(\Lambda)$ is $\mathfrak{g}$-stable, meaning that $V_\sigma(\Lambda)$ can be regarded as a $\mathfrak{g}$[t]-module.

Lemma 3.1.9. *A Demazure module* $V_\sigma(\Lambda)$ *is* $\mathfrak{g}$*-stable if, and only if,* $\sigma(\Lambda)(h_i) \leq 0$, $\forall i \in I$.

Proof. Suppose that $V_\sigma(\Lambda)$ is $\mathfrak{g}$-stable. This means that $x_i^- \cdot v = 0$, $\forall i \in I$, which forces $\sigma(\Lambda)(h_i) \leq 0$ because of (3.3). Conversely, if $V_\sigma(\Lambda)$ is not $\mathfrak{g}$-stable, then there must be some index $j \in I$ such that $x_j^- \cdot v \neq 0$. Again by (3.3) this imply that $\sigma(\Lambda)(h_j) > 0$. The result follows. ☐

Since $\sigma(\Lambda) \in \widehat{P}$ we can write $\sigma(\Lambda) = \mu + \ell\Lambda_0 + m\delta$, for some $\mu \in \widehat{P}$, $\ell \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}$. By the previous lemma it follows that $V_\sigma(\Lambda)$ is $\mathfrak{g}$-stable if, and only if, $\mu \in P_- := \{\lambda = \sum k_i\omega_i \in P : k_i \leq 0,\ \forall i \in I\}$ and hence $\lambda := w_0(\mu) \in P_+$, for w_0 the longest element of $\widehat{W}$. This leads us to reformulate the definition of $\mathfrak{g}$-stable Demazure modules as follows. Given $\lambda \in P_+$, $\ell \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$, there are unique $\Lambda \in \widehat{P}_+$ and $\sigma \in \widehat{\mathcal{W}}$ satisfying

$$\sigma(\Lambda) = w_0(\lambda) + \ell\Lambda_0 + m\delta.$$

For this particular choice of Λ and σ, we denote the $\mathfrak{g}[t]$-module $V_\sigma(\Lambda)$ by $\mathcal{D}(\ell,\lambda)[m]$. Although it is a $\mathfrak{g}_d[t] \oplus \mathbb{C}K$-module, we usually regard only the $\mathfrak{g}_d[t]$ structure (as the action of K on $\mathcal{D}(\ell,\lambda)[m]$ is just a scalar product by ℓ). We say that the integer ℓ is the *level* of $\mathcal{D}(\ell,\lambda)[m]$. The following proposition simplifies the description of the $\mathfrak{g}$-stable Demazure modules.

Proposition 3.1.10. *For every* $n, m \in \mathbb{Z}_{\geq 0}$

$$\mathcal{D}(\ell,\lambda)[m] \cong_{\mathfrak{g}[t]} \mathcal{D}(\ell,\lambda)[n].$$

Proof. Let $m, n \in \mathbb{Z}_{\geq 0}$ be such that $m \neq n$ (otherwise the result is obvious) and assume $n = 0$. It follows from Proposition 2.2.23 that

$$V(\Lambda) \cong_{\mathfrak{g}[t]} V(\Lambda + m\delta)$$

since $\delta(h) = 0$, $\forall h \in \widehat{\mathfrak{h}}$. Let $\varphi : V(\Lambda) \to V(\Lambda + m\delta)$ be such $\mathfrak{g}[t]$-isomorphism.

By definition there are unique $\sigma \in \widehat{\mathcal{W}}$ and $\Lambda \in \widehat{P}_+$ such that $\mathcal{D}(\ell,\lambda)[0] = U(\widehat{\mathfrak{b}}) \cdot v_1$, for some non-zero $v_1 \in V(\Lambda)_{\sigma(\Lambda)}$. It is straightforward that $\sigma \in \widehat{\mathcal{W}}$ and $\Lambda - m\delta \in \widehat{P}_+$ are the unique elements satisfying $\mathcal{D}(\ell,\lambda)[m] = U(\widehat{\mathfrak{b}}) \cdot v_2$, for some non-zero $v_2 \in V(\Lambda + m\delta)_{\sigma(\Lambda - m\delta)}$.

Note that

$$h \cdot \varphi(v_1) = \varphi(h \cdot v_1) = \varphi(\sigma(\Lambda)(h)v_1) = \varphi(\sigma(\Lambda - m\delta)(h)v_1) = \sigma(\Lambda - m\delta)(h)\varphi(v_1)$$

hence $\varphi(v_1) \in V(\Lambda + m\delta)_{\sigma(\Lambda+m\delta)}$. From Remark 3.1.6 it follows that $\varphi(v_1) = kv_2$, for some non-zero scalar k, so there is an $\mathfrak{g}[t]$-isomorphism $\varphi' : V(\Lambda) \to V(\Lambda + m\delta)$ such that $\varphi'(v_1) = v_2$, hence $\mathcal{D}(\ell, \lambda)[0] \cong_{\mathfrak{g}[t]} \mathcal{D}(\ell, \lambda)[m]$. ☐

Remark 3.1.11. *Although* $\mathcal{D}(\ell, \lambda)[n] \cong_{\mathfrak{g}[t]} \mathcal{D}(\ell, \lambda)[m]$, $\forall n, m \in \mathbb{Z}_{\geq 0}$, *the isomorphism does not hold in* $\mathfrak{g}[t] \oplus \mathbb{C}d$. *For example, given* $\mathcal{D}(1, \lambda)[0]$, *with generator* v_0, *and* $\mathcal{D}(1, \lambda)[1]$, *with generator* v_1, *one has that*

$$d \cdot v_0 = 0 \text{ and } d \cdot v_1 = 1.$$

This means that the generator v_0 *does not satisfy the relations of* $\mathcal{D}(1, \lambda)[1]$, *as in Theorem 3.1.8(iv), thus* $\mathcal{D}(1, \lambda)[0]$ *and* $\mathcal{D}(1, \lambda)[1]$ *cannot be isomorphic as* $\mathfrak{g}[t] \oplus \mathbb{C}d$*-modules.*

From now on we mainly focus on the $\mathfrak{g}[t]$-structure of $\mathfrak{g}$-stable Demazure modules, and we denote by $\mathcal{D}(\ell, \lambda)$ the module $\mathcal{D}(\ell, \lambda)[0]$. As $\mathfrak{g}[t]$-module one can further refine the presentation of $\mathcal{D}(\ell, \lambda)$, following [28, Proposition 3.6]. Recall from Definition 1.2.14 that $d_\gamma = \dfrac{2}{(\gamma, \gamma)}$ and that $d_\gamma = 1$ if γ is long and $d_\gamma > 1$ otherwise.

Proposition 3.1.12. *The Demazure module* $\mathcal{D}(\ell, \lambda)$ *is isomorphic to the* $\mathfrak{g}[t]$*-module generated by a vector* v *satisfying the following set of relations.*

(i) $(x_i^+ \otimes \mathbb{C}[t]) \cdot v = 0, \ i \in I.$

(ii) $(h_i \otimes t^s) \cdot v = \delta_{0s}\lambda(h_i)v, \ i \in I, \ s \in \mathbb{Z}_{\geq 0}0.$

(iii) $(x_\gamma^- \otimes t^s)^{k+1} \cdot v = 0$, *with* $k = max\{0, \lambda(h_\gamma) - d_\gamma \ell s\}, \ \gamma \in \Phi_+.$

Also, $\mathcal{D}(1, \lambda)[m]$ *is isomorphic to the* $\mathfrak{g}_d$*-module generated by a vector* v *with the above relations and* $d \cdot v = mv$.

If v is the generator of $\mathcal{D}(\ell, \lambda)$ as in Proposition 3.1.12, then v satisfies the Weyl relations. The first two are the very same as in the definition of $W(\lambda)$. Fixing $s = 0$ and $\gamma = \alpha_i$ in the third relation of $\mathcal{D}(\ell, \lambda)$ yields $\lambda(h_i) - d_i \ell s = \lambda(h_i)$, for any $i \in I$. Hence $(x_1^-)^{\lambda(h_i)+1} = 0$, which is the third defining relation of $W(\lambda)$. This means that there must be a surjective homomorphism $\varphi : W(\lambda) \to \mathcal{D}(\ell, \lambda)$ such that $\varphi(w) = v$, with w generator of $W(\lambda)$, and particularly that $\mathcal{D}(\ell, \lambda)$ is a quotient of $W(\lambda)$.

Definition 3.1.13. Given $\lambda \in P_+$ and $\gamma \in \Phi_+$, define s_γ, m_γ as the unique non-negative integers such that, if $\lambda(h_\gamma) > 0$, then

$$\lambda(h_\gamma) = (s_\gamma - 1)d_\gamma \ell + m_\gamma \tag{3.4}$$

with $0 < m_\gamma \leq d_\gamma$ and d_γ as in Definition 1.2.14. If $\lambda(h_\gamma) = 0$, set $s_\gamma = m_\gamma = 0$. Moreover fix $s_{\alpha_i} = s_i$ and $m_{\alpha_i} = m_i$.

In [10], Chari and Venkatesh developed an entire theory on certain $\mathfrak{g}[t]$-modules which turns out to simplify the set of relations that the generator v of $\mathcal{D}(\ell, \lambda)$ must satisfy. To prove such a result we need several additional theories, which would imply a deviation from what we are doing. This being the case, we just enunciate the following theorem, which is an adaptation of several results of [10].

Theorem 3.1.14. *The Demazure module $\mathcal{D}(\ell, \lambda)$ is isomorphic to the $\mathfrak{g}[t]$-module generated by a vector v satisfying the following set of relations:*

(i) Weyl relations for λ.

(ii) $(x_\gamma^- \otimes t^{s_\gamma}) \cdot v = 0$, for $\gamma \in \Phi_+$.

(iii) $(x_\gamma^- \otimes t^{s_\gamma - 1})^{m_\gamma + 1} \cdot v = 0$, for $\gamma \in \Phi_+$ and $m_\gamma < d_\gamma \ell$.

Particularly, $\mathcal{D}(1, \lambda)$ is isomorphic to the $\mathfrak{g}[t]$-module generated by a vector v satisfying the following set of relations:

(i) Weyl relations for λ.

(ii) $(x_\gamma^- \otimes t^{s_\gamma}) \cdot v = 0$, for $\gamma \in \Phi_+$ such that $d_\gamma > 1$.

(iii) $(x_\gamma^- \otimes t^{s_\gamma - 1})^2 \cdot v = 0$, for $\gamma \in \Phi_+$ such that $d_\gamma = 3$ and $m_\gamma = 1$.

Recall from Proposition 3.1.12 that the generator v of $\mathcal{D}(\ell, \lambda)$ must satisfy, for every $s \in \mathbb{Z}_{\geq 0}$, a set of relations. This means that, potentially, there are infinitely many relations that v must satisfy, which is very impractical computation-wise. The last theorem changes this, as it fixes, for every $\gamma \in \Phi_+$, up to two relations that v must satisfy. Until now, the last theorem represents the very best description of $\mathfrak{g}$-stable Demazure modules.

It follows from Theorem 3.1.14 that if $\mathfrak{g}$ is simply laced, or if $\lambda(h_i) = 0$ for all $i \in I$ such that $\alpha_i \in \Delta$ is a short root, then the only relations that the generator v of $\mathcal{D}(1, \lambda)$ must satisfy are the Weyl relations for λ. In this case the generator w of $W(\lambda)$ satisfies all relations of $\mathcal{D}(1, \lambda)$, so there must be an surjetive homomorphism $\varphi' : \mathcal{D}(1, \lambda) \to W(\lambda)$ such that $\varphi'(v) = w$, for v generator of $\mathcal{D}(1, \lambda)$, hence $W(\lambda)$ is a quotient of $\mathcal{D}(1, \lambda)$. Since $\mathcal{D}(1, \lambda)$ is always a quotient of $W(\lambda)$, these modules are actually isomorphic. This proves the following theorem.

Theorem 3.1.15. *Given $\lambda \in P_+$, if either $\mathfrak{g}$ is simply laced or $\lambda(h_i) = 0$, for all $i \in I$ such that $\alpha_i \in \Delta$ is a short root, then*

$$\mathcal{D}(1, \lambda) \cong_{\mathfrak{g}[t]} W(\lambda)$$

Remark 3.1.16. *Last theorem can be troublesome to prove if we do not have Theorem 3.1.14. For instance, check Theorem 7 of [18].*

3.2 A smaller set of relations for level one Demazure modules

Given $\alpha = \sum_{i\in I} k_i\alpha_i \in Q^+$ let $\operatorname{supp}(\alpha) = \{i \in I : k_i > 0\}$. Define $I^{\mathrm{sh}} := \{i \in I : d_i > 1\}$ and $\Phi_+^{\mathrm{sh}} = \{\gamma \in \Phi_+ : \operatorname{supp}(\gamma) \subset I^{\mathrm{sh}}\}$. In [28], Katsuyuki Naoi shows that the generator v of $\mathcal{D}(1,\lambda)$ needs to satisfy the relations of Proposition 3.1.12 only for $\gamma \in \Phi_+^{\mathrm{sh}}$. In this section, we combine the theories of [10, 28] to prove a more sophisticated result: The generator of $\mathcal{D}(1,\lambda)$ must satisfy the set of relations given in Theorem 3.1.14, but only for $\gamma \in \Phi_+^{\mathrm{sh}}$.

Given $\lambda \in P_+$, recall from (3.4) that for every $\gamma \in \Phi_+^{\mathrm{sh}}$ there are $s_\gamma,\ m_\gamma \in \mathbb{Z}_{\geq 0}$ such that $\lambda(h_\gamma) = (s_\gamma - 1)d_\gamma + m_\gamma$. Define M_λ as the cyclic $\mathfrak{g}[t]$-module generated by a vector $v_M \neq 0$ satisfying the following set of relations.

(i) Weyl relations for λ.

(ii) $(x_\gamma^- \otimes t^{s_\gamma}) \cdot v_M = 0$, for $\gamma \in \Phi_+^{\mathrm{sh}}$ such that $d_\gamma > 1$.

(iii) $(x_\gamma^- \otimes t^{s_\gamma - 1})^2 \cdot v_M = 0$, for $\gamma \in \Phi_+^{\mathrm{sh}}$ such that $d_\gamma = 3$ and $m_\gamma = 1$.

Our goal in this section is to prove the following.

Theorem 3.2.1. *Let $\mathfrak{g}$ be a simple Kac-Moody algebra of finte type and $\lambda \in P_+$. Then*

$$\mathcal{D}(1,\lambda) \cong_{\mathfrak{g}[t]} M_\lambda.$$

Fix v_D as the generator of $\mathcal{D}(1,\lambda)$, as given in Theorem 3.1.14. As both M_λ and $\mathcal{D}(1,\lambda)$ are cyclic modules, the Theorem 3.2.1 will be proved if we show that v_D satisfies the defining relations of M_λ and if v_M satisfies the defining relations of $\mathcal{D}(1,\lambda)$. It is clear that v_D satisfies the relations of M_λ, since the set of defining relations of M_λ is a subset of the set of defining relations of $\mathcal{D}(1,\lambda)$, hence to prove the Theorem 3.2.1 it suffices to prove the following.

Proposition 3.2.2. *The generator v_M of M_λ satisfies the defining relations of $\mathcal{D}(1,\lambda)$ given in Theorem 3.1.14.*

The proof of the last proposition is divided into this section, accordingly to the type of $\mathfrak{g}$. If $\mathfrak{g}$ is simply laced the result follows from Theorem 3.1.15.

Before we continue we prove some preliminary results that play important roles in all the remaining cases.

Lemma 3.2.3. *Let v_M be the generator of M_λ. If $\gamma \in \Phi_+$ is a long root, then*

$$(x_\gamma^- \otimes t^s)^{k_\gamma + 1} \cdot v_M = 0. \tag{3.5}$$

with $k_\gamma = max\{0, \lambda(h_\gamma) - d_\gamma s\}$.

Proof. Let h_γ be the coroot of γ. Define the Lie subalgebra of $\mathfrak{g}$

$$\mathfrak{sl}_{2,\gamma} := \mathbb{C}x_\gamma^+ \oplus \mathbb{C}h_\gamma \oplus \mathbb{C}x_\gamma^-.$$

It is clear that $\mathfrak{sl}_{2,\gamma}$ is isomorphic to $\mathfrak{sl}_2$. Set $\mathfrak{sl}_{2,\gamma}[t] := \mathfrak{sl}_{2,\gamma} \otimes \mathbb{C}[t]$ and let $W(\lambda(h_\gamma))$ be the Weyl module of $\mathfrak{sl}_{2,\gamma}[t]$ associated with the weight $\lambda(h_\gamma)$. Let w to be a generator of $W(\lambda(h_\gamma))$ and let $V := U(\mathfrak{sl}_{2,\gamma}[t]) \cdot v_M$. Since v_M satisfies the Weyl relations of $W(\lambda)$ in particular it implies that V is a quotient of $W(\lambda(h_\gamma))$. Denote by $\mathcal{D}(1, \lambda(h_\gamma))$ the Demazure module of $\mathfrak{sl}_{2,\gamma}[t]$ associated with $\lambda(h_\gamma)$. Since $\mathfrak{sl}_{2,\gamma}[t]$ is simply laced, $\mathcal{D}(1, \lambda(h_\gamma))$ must be isomorphic to $W(\lambda(h_\gamma))$ (see Theorem 3.1.15). Thus v_M satisfies the relation of Proposition 3.1.12 (iii), which is identical to (3.5). ☐

For any long root γ it holds that $\lambda(h_\gamma) - d_\gamma \ell s_\gamma = 0$. Using this information in Lemma 3.2.3 immediately proves the following corollary.

Corollary 3.2.4. *Let v_M be the generator of M_λ. If $\gamma \in \Phi$ long, then*

$$(x_\gamma^- \otimes t^{s_\gamma}) \cdot v_M = 0. \tag{3.6}$$

Lemma 3.2.5. *Assume that $\mathfrak{g}$ is a Lie algebra of rank 2 and fix $\Delta = \{\alpha, \beta\}$ base of Φ. If V is a $U(\mathfrak{n}_-)$-module and $v \in V$ satisfies*

$$(x_\alpha^-)^{a+1} \cdot v = 0 = (x_\beta^-)^{b+1} \cdot v, \quad \text{for some } a, b \in \mathbb{Z}_{\geq 0}$$

then

$$(x_\gamma^-)^{(n_1 a + n_2 b + 1)} \cdot v = 0, \quad \gamma \in \Phi_+, \; h_\gamma = n_1 h_\alpha + n_2 h_\beta. \tag{3.7}$$

Proof. Let ω_α and ω_β be the fundamental weights corresponding to α and β, respectively, and fix the weight $\lambda = a\omega_\alpha + b\omega_\beta$. As $U(\mathfrak{n}_-)$-modules we have

$$V(\lambda) \cong \frac{U(\mathfrak{n}_-)}{U(\mathfrak{n}_-)(\mathbb{C}(x_\alpha^-)^{a+1} + \mathbb{C}(x_\beta^-)^{b+1})}$$

hence there must be a homomorphism of $U(\mathfrak{n}_-)$-modules from $V(\lambda)$ to V such that the highest weight vector v_λ of $V(\lambda)$ is mapped onto v. Since v_λ satisfies equation (3.7) so does v. ☐

3.2.1 The B_n, C_n, F_4 cases

In this subsection, we assume that $\mathfrak{g}$ is of type B_n, C_n or F_4. Thus the generator v_M must satisfy relations (i) and (ii) of Theorem 3.1.14, since for every short root $\gamma \in \Phi_+$ we have $d_\gamma = 2$. Note that these are the only relations in this case, as $d_\gamma \neq 3$, $\forall \gamma \in \Phi_+$. To show this result, the following lemmas are needed.

Lemma 3.2.6. *If $\gamma \in \Phi_+ \setminus \Phi_+^{sh}$ is short, then there are $\alpha, \beta \in \Phi_+$ such that α is short, β is long and $\gamma = \alpha + \beta$.*

Proof. We prove by induction on $\mathrm{ht}(\gamma)$. Since γ is short and $\gamma \notin \Phi_+^{\mathrm{sh}}$, we must have that $\gamma \neq \alpha_i$, for every $i \in I$. So the first step of induction occurs for $\mathrm{ht}(\gamma) = 2$. By hypothesis, there are $\alpha_i, \alpha_j \in \Delta$ such that $\gamma = \alpha_i + \alpha_j$. One of those must be long, otherwise $\gamma \in \Phi_+^{\mathrm{sh}}$, so the result follows.

For the induction step, fix $n > 2$ and suppose that $2 \leq \mathrm{ht}(\gamma) < n$ imply the existence of $\alpha, \beta \in \Phi_+$ satisfying $\gamma = \alpha + \beta$. with α short and β long.

Let $\gamma \in \Phi_+$ be such that $\mathrm{ht}(\gamma) = n$ and write $\gamma = \sum_{i \in I} k_i \alpha_i$, $k_i \geq 0$. Since $(\gamma, \gamma) > 0$ there exists $i \in I$ such that $(\gamma, \alpha_i) > 0$ and moreover γ being short implies $\gamma(h_i) = 1$. Then $\sigma_i(\gamma) = \gamma - \alpha_i$ and hence $\gamma = \sigma_i(\gamma) + \alpha_i$. If $\sigma_i(\gamma)$ is a long root, the result follows, so suppose that $\sigma_i(\gamma)$ is short.

Since $\gamma \notin \Phi_+^{\mathrm{sh}}$, we must have that $\sigma_i(\gamma) \notin \Phi_+^{\mathrm{sh}}$ as well. In fact, by our assumptions if $\gamma = \sum_{i \in I} k_i \alpha_i$ then $k_j > 0$ for some j such that α_j is long. Then $\sigma_i(\gamma) = \gamma - \alpha_i = (k_i - 1)\alpha_i + \sum_{p \neq i} k_p \alpha_p$, and hence $\sigma_i(\gamma) \notin \Phi_+^{\mathrm{sh}}$. Since $\mathrm{ht}(\sigma_i(\gamma)) < n$, by the induction hypothesis there are $\alpha, \beta \in \Phi_+$ such that α is short, β is long and $\sigma_i(\gamma) = \alpha + \beta$.

It follows that $\gamma = \alpha + \beta + \alpha_i$. If $\alpha = \alpha_i$, then $\gamma = 2\alpha_i + \beta$, so

$$(\gamma, \gamma) = 2(\alpha_i, \alpha_i) + 2(\alpha_i, \beta) + (\beta, \beta) = 2 - 2 + 2 = 2$$

which contradicts that γ is short. Thus, $\alpha \neq \alpha_i$ and hence $\sigma_i(\alpha) \in \Phi_+$ is again short. Also, $\sigma_i(\beta)$ must be long, so $\gamma = \sigma_i(\sigma_i(\gamma)) = \sigma_i(\alpha + \beta) = \sigma_i(\alpha) + \sigma_i(\beta)$ finishes the proof. □

Lemma 3.2.7. *Let $\gamma \in \Phi_+$ be a short root. If $\gamma = \alpha + \beta$, with $\alpha \in \Phi_+$ short and $\beta \in \Phi_+$ long, then $s_\gamma = s_\alpha + s_\beta$.*

Proof. If $\gamma = \alpha + \beta$, then $h_\gamma = h_\alpha + 2h_\beta$. So

$$\begin{aligned} 2(s_\gamma - 1) + m_\gamma &= \lambda(h_\gamma) \\ &= \lambda(h_\alpha) + 2\lambda(h_\beta) \\ &= 2(s_\alpha - 1) + m_\alpha + 2(s_\beta - 1) + 2. \end{aligned}$$

Therefore

$$\begin{aligned} m_\gamma &= 2(s_\alpha - 1) + 2(s_\beta - 1) + 2 - 2(s_\gamma - 1) + m_\alpha \\ &= 2(s_\alpha + s_\beta - s_\gamma) + m_\alpha. \end{aligned}$$

Since $m_\gamma, m_\alpha \in \{1, 2\}$ we have $|m_\gamma - m_\alpha| \leq 1$ and hence

$$2|s_\alpha + s_\beta - s_\gamma| \leq 1 \implies s_\gamma = s_\alpha + s_\beta.$$

□

Finally, we are set to prove Proposition 3.2.2 when $\mathfrak{g}$ is of type B_n, C_n or F_4.

Proof. Recall that relation (i) is one of the defining relations of M_λ, so v_M satisfies it from the definition. To prove (ii) we proceed by induction on $\mathrm{ht}(\gamma)$. If $\mathrm{ht}(\gamma) = 1$, then $\gamma \in \Delta$ and so $(x_\gamma^- \otimes t^{s_\gamma}) \cdot v_M = 0$ by hypothesis. Fix $n > 1$ and assume, as the induction hypothesis, that for any $\gamma \in \Phi_+$

$$1 \leq \mathrm{ht}(\gamma) < n \implies (x_\gamma^- \otimes t^{s_\gamma}) \cdot v_M = 0.$$

Set $\gamma \in \Phi_+$ such that $\mathrm{ht}(\gamma) = n$ and $\gamma \notin \Phi_+^{\mathrm{sh}}$ (otherwise there is nothing to be done). By Lemma 3.2.6 there exists $\alpha, \beta \in \Phi_+$ such that α is short, β is long and $\gamma = \alpha + \beta$. As $\mathrm{ht}(\alpha) < \mathrm{ht}(\gamma)$, the induction hypothesis ensures $(x_\alpha^- \otimes t^{s_\alpha}) \cdot v_M = 0$.

Since $(x_\beta^- \otimes t^{s_\beta}) \cdot v = 0$ (see Lemma 3.2.4) and $s_\gamma = s_\alpha + s_\beta$ (see Lemma 3.2.7), it follows that

$$\begin{aligned}(x_\gamma^- \otimes t^{s_\gamma}) \cdot v_M &= (x_{\alpha+\beta}^- \otimes t^{s_\alpha + s_\beta}) \cdot v_M \\ &= [x_\alpha^- \otimes t^{s_\alpha}, x_\beta^- \otimes t^{s_\beta}] \cdot v_M \\ &= (x_\alpha^- \otimes t^{s_\alpha})(x_\beta^- \otimes t^{s_\beta}) \cdot v_M - (x_\beta^- \otimes t^{s_\beta})(x_\alpha^- \otimes t^{s_\alpha}) \cdot v_M \\ &= 0.\end{aligned} \tag{3.8}$$

The result is proved. □

3.2.2 The G_2 case

In the remainder of this section, we fix $\mathfrak{g}$ to be of type G_2. In this case, the generator v_M must satisfy all three relations of Theorem 3.1.14, since for every short root $\gamma \in \Phi_+$ we have $d_\gamma = 3$. Fix α_1 basic short root and α_2 basic long root.

Note that $\Phi_+ = \{\alpha_1,\ \alpha_2,\ \alpha + \alpha_2,\ 2\alpha_1 + \alpha_2,\ 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2\}$. Also, the positive short roots are $\alpha_1,\ \alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$, and $\Phi_+^{\mathrm{sh}} = \{\alpha_1\}$. For easy of notation we fix

$$\beta := \alpha_1 + \alpha_2 \text{ and } \sigma := 2\alpha_1 + \alpha_2 = \alpha_1 + \beta.$$

It follows from the definition of M_λ that the generator v_M satisfies $(x_1^- \otimes t^{s_1}) \cdot v_M = (x_1^- \otimes t^{s_1 - 1})^2 \cdot v_M = 0$. Hence we need to check if v_M satisfies the relations of Theorem 3.1.14 for $\gamma = \beta$ and $\gamma = \sigma$. Namely we shall prove that

$$(x_\gamma^- \otimes t^{s_\gamma}) \cdot v_M = 0, \quad \text{and} \quad (x_\gamma^- \otimes t^{s_\gamma - 1})^2 \cdot v_M, \quad \text{if} \quad m_\gamma = 1, \quad \gamma \in \{\beta, \sigma\}. \tag{3.9}$$

Lemma 3.2.8. *The relations* (3.9) *hold for* β.

Proof. Note that $h_\beta = h_1 + 3h_2$ and hence

$$\begin{aligned}3(s_\beta - 1) + m_\beta &= \lambda(h_\beta) \\ &= \lambda(h_1) + 3\lambda(h_2) \\ &= 3(s_1 - 1) + m_1 + 3(s_2 - 1) + 3m_2 \\ &= 3(s_1 - 1) + 3(s_2 - 1) + 3 + m_1.\end{aligned}$$

In particular

$$\begin{aligned}m_\beta &= 3(s_1 - 1) + 3(s_2 - 1) - 3(s_\beta - 1) + 3 + m_1 \\ &= 3(s_1 - 1 + s_2 - 1 - s_\beta + 1 + 1) + m_1 \\ &= 3(s_1 + s_2 - s_\beta) + m_1.\end{aligned}$$

Since $m_1, m_\beta \in \{1,2,3\}$ it follows that

$$3|s_1 + s_2 - s_\beta| = |m_\beta - m_1| \leq 2 \implies |s_1 + s_2 - s_\beta| = 0 \text{ and } |m_\beta - m_1| = 0.$$

Therefore we must have $s_\beta = s_1 + s_2$ and $m_\beta = m_1$.

If $m_1 = 2$ or $m_1 = 3$, computations analogous to (3.8) can be done, and so the result is proved. If $m_1 = 1$, then relation (iii) of M_λ ensures that $(x_1^- \otimes t^{s_1 - 1})^2 \cdot v_M = 0$. Also, Lemma 3.2.4 establishes that $(x_2^- \otimes t^{s_2}) \cdot v_M = 0$. Then using Lemma 3.2.5 if follows that

$$(x_{\alpha_1+\alpha_2} \otimes t^{s_1+s_2-1})^2 \cdot v_M = (x_\beta \otimes t^{s_\beta - 1})^2 \cdot v_M = 0$$

which ends the proof. □

To prove that (3.9) holds for σ it demands much more labor and some technical results are needed.

Lemma 3.2.9. *Suppose that $m_1 \in \{1,3\}$ where $\lambda(h_1) = 3(s_{\alpha_1} - 1) + m_1$, with $1 \leq m_1 \leq 3$. Then (3.9) holds.*

Proof. Recall that $h_\sigma = h_\beta + h_1$ and that $s_\beta = s_1 + s_2$ (see Lemma 3.2.7), so

$$\begin{aligned} 3(s_\sigma - 1) + m_\sigma &= \lambda(h_\sigma) \\ &= \lambda(h_1 + h_\beta) \\ &= 3(s_1 - 1) + m_1 + 3(s_\beta - 1) + m_\beta \\ &= 3(s_1 - 1) + 3(s_1 + s_2 - 1) + 2m_1. \end{aligned}$$

Therefore

$$m_\sigma = 3(2s_1 + s_2 - s_\sigma - 1) + 2m_1.$$

Recall that $m_\sigma \in \{1,2,3\}$ and, by hypothesis, $m_1 \in \{1,3\}$. If $m_1 = 1$, then

$$3|2s_1 + s_2 - s_\sigma - 1| = |m_\sigma - 2| \leq 1$$

so $s_\sigma = 2s_1 + s_2 - 1 = s_1 + s_\beta - 1$ and $m_\sigma = 2$. In this case, by hypothesis we have that $(x_1^- \otimes t^{s_1}) \cdot v_M = 0$, and since $(x_\beta^- \otimes t^{s_\beta}) \cdot v_M = 0$ (by Lemma 3.2.8) it follows from Lemma 3.2.5 that $(x_\sigma^- \otimes t^{s_\sigma}) \cdot v = 0$.

If $m_1 = 3$, then

$$3|2s_1 + s_2 - s_\sigma| = |m_\sigma - 3| \leq 2$$

so $s_\sigma = 2s_1 + s_2$, and $m_\sigma = 3$. Similarly as in the case $m_1 = 1$, Lemma 3.2.5 and Lemma 3.2.8 ensures that $(x_\sigma^- \otimes t^{s_\sigma}) \cdot v_M = 0$. □

The only possible scenario left off of the previous lemma is $m_1 = 2$. In this case

$$3|2s_1 + s_2 - s_\sigma| = |m_\sigma - 1| \leq 2$$

which yields $s_\sigma = 2s_1 + s_2$ and $m_\sigma = 1$. It is a little troublesome, as we only have that $(x_1^- \otimes t^{s_1}) \cdot v_M = 0$ and wish to show that $(x_\sigma \otimes t^{s_\sigma}) \cdot v_M = 0$ and $(x_\sigma \otimes t^{s_\sigma - 1})^2 \cdot v_M = 0$. The first relation follows the same lines as in Lemma 3.2.8. To prove that $(x_\sigma \otimes t^{s_\sigma-1})^2 \cdot v_M = 0$ we need the following lemma.

Lemma 3.2.10. *Let $\{x^+, h, x^-\}$ be the Chevalley basis of $\mathfrak{sl}_2$, $a \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$. Define a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{sl}_2[t]$ by*

$$\mathfrak{a} := x^+ \otimes t\mathbb{C}[t] + h \otimes \mathbb{C}[t] + x^- \otimes \mathbb{C}[t].$$

Let I be the subspace of $U(\mathfrak{a})$ defined by

$$I := x^+ \otimes t\mathbb{C}[t] + h \otimes t\mathbb{C}[t] + \mathbb{C}(h - a) + \sum_{s \geq 0} \mathbb{C}(x^- \otimes t^s)^{k_s+1}$$

with $k_s = max\{0, a - \ell s\}$. Then,

$$(x^+)^p \cdot (x^- \otimes t^s)^{k_s+1} \in U(\mathfrak{a})I + U(\mathfrak{sl}_2)x^+, \quad p \in \mathbb{Z}_{\geq 0}, \; s \in \mathbb{Z}_{\geq 0}.$$

Proof. Note that $U(\mathfrak{sl}_2[t]) = U(\mathfrak{a}) \oplus U(\mathfrak{sl}_2[t]) \cdot x^+$. For arbitrary fixed $p, s \in \mathbb{Z}_{\geq 0}$ there exists $X \in U(\mathfrak{sl}_2[t])$ such that

$$(x^+)^p \cdot (x^- \otimes t^s)^{k_s+1} - Xx^+ \in U(\mathfrak{a}).$$

Set the $\mathfrak{sl}_2[t]$-Demazure module $\mathcal{D}(\ell, a\omega)$ with generator v. By Propostion 3.1.12 the annihilating ideal of v in $U(\mathfrak{a})$ is $U(\mathfrak{a})I$. But

$$((x^+)^p \cdot (x^- \otimes t^s)^{k_s+1} - Xx^+) \cdot v = 0$$

from relations (i) and (iii) of Proposition 3.1.12. Thus $(x^+)^p \cdot (x^- \otimes t^s)^{k_s+1} - Xx^+ \in U(\mathfrak{a})I$, ending the proof. ☐

Set $\lambda = a\omega_1 + b\omega_2$, with $a, b \in \mathbb{Z}_{\geq 0}$ such that $a \equiv_3 2$. Define a subspace $I_{a,b}$ of $U(\mathfrak{g}[t])$ by

$$\begin{aligned} I_{a,b} = \mathfrak{n}_+ \otimes \mathbb{C}[t] + \mathfrak{h} \otimes t\mathbb{C}[t] + \mathbb{C}(h_1 - a) + \mathbb{C}(h_2 - b) \\ + \mathbb{C}(x_1^-)^{a+1} + \mathbb{C}(x_1^- \otimes t^{s_1-1})^2 + \mathbb{C}(x_1^- \otimes t^{s_1}) + \mathbb{C}(x_2^-)^{b+1}. \end{aligned}$$

In this case, notice that

$$a = \lambda(h_1) = 3(s_1 - 1) + 2 = 3s_1 - 1. \tag{3.10}$$

The following lemma is immediate from the definition of M_λ.

Lemma 3.2.11. *Given $I_{a,b}$ as above and v_M generator of M_λ, it follows that $X \cdot v_M = 0$ if, and only if, $X \in U(\mathfrak{g}[t])I_{a,b}$.*

Last lemma ensures that

$$(x_\sigma^- \otimes t^{s_\sigma - 1})^2 \cdot v_M = 0 \iff (x_\sigma^- \otimes t^{s_\sigma - 1})^2 \in U(\mathfrak{g}[t])I_{a,b}.$$

In this case, we show the second statement. Fix an arbitrary $b \in \mathbb{Z}_{\geq 0}$. We proceed by induction on a. It is clear that the first step of induction must be taken at $a = 2$ (as $a \equiv_3 2$). Assuming that this is the case, set $\mu = 3\alpha_1 + \alpha_2 \in \Phi_+$, which is long and satisfies $h_\mu = \dfrac{1}{3}h_1 + \dfrac{1}{3}h_\sigma$. Therefore

$$\begin{aligned}
s_\mu &= \lambda(h_\mu) \\
&= \lambda\left(\frac{1}{3}h_1 + \frac{1}{3}h_\sigma\right) \\
&= \frac{1}{3}\lambda(h_1) + \frac{1}{3}\lambda(h_\sigma) \\
&= \frac{1}{3}(3(s_1 - 1) + m_1 + 3(s_\sigma - 1) + m_\sigma) \\
&= \frac{1}{3}(3(s_1 - 1) + 3(s_\sigma - 1) - 3) \\
&= s_1 + s_\sigma - 1 \\
&= s_1 + 2s_1 + s_2 - 1 \\
&= 3s_1 + s_2 - 1.
\end{aligned}$$

Since $a = 2$ it follows that $s_1 = 1$, so $s_\mu = b + 2$ and hence Lemma 3.2.3 ensures that

$$(x_\mu^- \otimes t^{b+1})^2 \in U(\mathfrak{g}[t])I_{2,b}$$

since $\lambda(h_\mu) - d_\mu(b + 1) = b + 2 - b - 1 = 1$. Moreover, $s_\beta = b + 1$ implies that

$$(x_\beta^- \otimes t^{b+1}) \in U(\mathfrak{g}[t])I_{2,b}$$

by Lemma 3.2.8. Notice that $(x_\mu^- \otimes t^{b+1})^2 \in U(\mathfrak{g}[t])I_{2,b}$ implies that $(x_1^+)^2(x_\mu^- \otimes t^{b+1})^2 \in U(\mathfrak{g}[t])I_{2,b}$, and

$$\begin{aligned}
(x_1^+)^2(x_\mu^- \otimes t^{b+1})^2 = (x_1^+)(x_\mu^- \otimes t^{b+1})^2(x_1^+) + 2[x_1^+, x_\mu^- \otimes t^{b+1}]^2 \\
+ 2(x_\mu^- \otimes t^{b+1})[x_1^+, x_\mu^- \otimes t^{b+1}](x_1^+) + 2(x_\mu^- \otimes t^{b+1})\mathrm{ad}(x_1^+)^2(x_\mu^- \otimes t^{b+1}).
\end{aligned}$$

This is translated as

$$\begin{aligned}
0 &= (x_1^+)(x_\mu^- \otimes t^{b+1})^2(x_1^+) \cdot v + 2[x_1^+, x_\mu^- \otimes t^{b+1}]^2 \cdot v \\
&\quad + 2(x_\mu^- \otimes t^{b+1})[x_1^+, x_\mu^- \otimes t^{b+1}](x_1^+) \cdot v + 2(x_\mu^- \otimes t^{b+1})\mathrm{ad}(x_1^+)^2(x_\mu^- \otimes t^{b+1}) \cdot v \\
&= 2[x_1^+, x_\mu^- \otimes t^{b+1}]^2 \cdot v \\
&= 2(x_\sigma^- \otimes t^{b+1})^2 \cdot v
\end{aligned}$$

so $(x_\sigma^- \otimes t^{b+1})^2 \in U(\mathfrak{g}[t])I_{2,b}$.

For the inductive step, suppose we have proved $(x_\sigma^- \otimes t^{s_\sigma - 1})^2 \in U(\mathfrak{g}[t])I_{a,b}$. We show that

$$(x_\sigma^- \otimes t^{s'_\sigma - 1})^2 \in U(\mathfrak{g}[t])I_{a+3,b}, \tag{3.11}$$

with s'_σ being such that $\lambda'(h_\sigma) = 3(s'_\sigma - 1) + m'_\sigma$, for $\lambda' = (a+3)\omega_1 + b\omega_2$. Note that $m'_\sigma = m_\sigma$ and hence

$$3(s'_\sigma - 1) + m'_\sigma = \lambda'(h_\sigma) = \lambda(h_\sigma) + 3\omega_1(h_\sigma) = 3(s_\sigma - 1) + m_\sigma + 6 = 3(s_\sigma + 1) + m_\sigma \tag{3.12}$$

which implies $s'_\sigma = s_\sigma + 2$. Therefore (3.11) is equivalent to

$$(x_\sigma^- \otimes t^{s_\sigma + 1})^2 \in U(\mathfrak{g}[t])I_{a+3,b}. \tag{3.13}$$

Assuming (3.13) holds it finishes the proof of Proposition 3.2.3. The remainder of this chapter is devoted to prove (3.13). Fix

$$\mathfrak{s} := \sum_{\beta \in \Phi_+} (x_\beta^+ \otimes t^{\omega_1^\vee(\beta)} \mathbb{C}[t])$$

with $\omega_1^\vee \in \mathfrak{h}$ such that $\omega_1^\vee(\alpha_j) = \delta_{1,j}$ Define the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{a}_0$ of $\mathfrak{g}[t]$ as

$$\mathfrak{a} = \mathfrak{n}_- \otimes \mathbb{C}[t] \oplus \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathfrak{s} \quad \text{and} \quad \mathfrak{a}_0 = \sum_{\substack{\beta \in \Phi_+ \setminus \{\alpha_2\} \\ 0 \le s < \omega_1^\vee(\beta)}} \mathbb{C}(x_\beta^+ \otimes t^s).$$

Lemma 3.2.12. *For $\mathfrak{a}$ and $\mathfrak{a}_0$ as above we have $\mathfrak{g}[t] = \mathfrak{a} \oplus \mathfrak{a}_0$.*

Proof. Since $\mathfrak{g}[t] = \mathfrak{n}_+[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}_-[t]$ and $\mathfrak{a}_0 \subset \mathfrak{n}_+[t]$, we have that $\mathfrak{g}[t] = \mathfrak{a} \oplus \mathfrak{a}_0$ if $\mathfrak{n}_+[t] = \mathfrak{s} \oplus \mathfrak{a}_0$. Recall that for a root system of type G_2 we have $\Phi_+ = \{\alpha_1, \alpha_2, \beta, \sigma, \mu, \theta\}$, with $\beta = \alpha_1 + \alpha_2$, $\sigma = 2\alpha_1 + \alpha_2$, $\mu = 3\alpha_1 + \alpha_2$ and $\theta = 3\alpha_1 + 2\alpha_2$. More explicitly we have

$$\mathfrak{s} = x_1^+ \otimes t\mathbb{C}[t] + x_2^+ \otimes \mathbb{C}[t] + x_\beta^+ \otimes t\mathbb{C}[t] + x_\sigma^+ \otimes t^2\mathbb{C}[t] + x_\mu^+ \otimes t^3\mathbb{C}[t] + x_\theta^+ \otimes t^3\mathbb{C}[t].$$

and

$$\begin{aligned}\mathfrak{a}_0 = \mathbb{C}x_1^+ &+ \mathbb{C}x_\beta^+ + \mathbb{C}x_\sigma^+ + \mathbb{C}(x_\sigma^+ \otimes t) + \mathbb{C}x_\mu^+ \\ &+ \mathbb{C}(x_\mu^+ \otimes t) + \mathbb{C}(x_\mu \otimes t^2) + \mathbb{C}x_\theta^+ + \mathbb{C}(x_\theta^+ \otimes t) + \mathbb{C}(x_\theta^+ \otimes t^2).\end{aligned}$$

It is now clear that $\mathfrak{n}_0^+[t] = \mathfrak{s} \oplus \mathfrak{a}_0$. $\square$

Define a subspace $I'_{a,b}$ of $I_{a,b}$ by

$$I'_{a,b} := I_{a,b} \cap U(\mathfrak{a}).$$

It follows that

$$\begin{aligned}I'_{a,b} = \mathfrak{s} &+ \mathfrak{h} \otimes t\mathbb{C}[t] + \mathbb{C}(h_1 - a) + \mathbb{C}(h_2 - b) \\ &+ \mathbb{C}(x_1^-)^{a+1} + \mathbb{C}(x_1^- \otimes t^{s_1 - 1})^2 + \mathbb{C}(x_1^- \otimes t^{s_1}) + \mathbb{C}(x_2^-)^{b+1}.\end{aligned}$$

Lemma 3.2.13. *For $I'_{a,b}$, $\mathfrak{a}$ and $\mathfrak{a}_0$ as above we have*

$$U(\mathfrak{g}[t])I_{a,b} \subseteq U(\mathfrak{a})I'_{a,b} \oplus U(\mathfrak{g}[t])\mathfrak{a}_0.$$

Proof. To simplify notation, we write (during this proof only) $I = I_{a,b}$, $I' = I'_{a,b}$ and $J = U(\mathfrak{a})I'_{a,b} \oplus U(\mathfrak{g}[t])\mathfrak{a}_0$. By Lemma 3.2.12 we have $U(\mathfrak{g}[t]) = U(\mathfrak{a}_0)U(\mathfrak{a})$ and it follows from the definition that $U(\mathfrak{a})J = J$. Hence it suffices to show that $U(\mathfrak{a}_0)I \subseteq J$. Define the following vector subspace of $I_{a,b}$

$$I_1 := \mathfrak{n}_+ \otimes \mathbb{C}[t] + \mathfrak{h} \otimes t\mathbb{C}[t] + \mathbb{C}(h_1 - a) + \mathbb{C}(h_2 - b).$$

We first show that $U(\mathfrak{a}_0)I_1 \subseteq J$. Let $k \geq 0$ and $X_1, \ldots, X_k \in \mathfrak{a}_0$. We prove that $X_1 \cdots X_k I_1 \subseteq J$ by induction on k. This follows by proving the following stronger statement

$$X_1 \cdots X_k I_1 \subseteq (I' \cap I_1) \oplus U(\mathfrak{g}[t])\mathfrak{a}_0. \tag{3.14}$$

If $k = 0$, then (3.14) is equivalent to $I_1 \subseteq (I' \cap I_1) \oplus U(\mathfrak{g})\mathfrak{a}_0$, which is true because $I_1 = (I_1 \cap I') \oplus \mathfrak{a}_0 \subseteq (I' \cap I_1) \oplus U(\mathfrak{g}[t])\mathfrak{a}_0$. Also, note that

(i) The inclusion $\operatorname{ad}(\mathfrak{a}_0)I_1 \subseteq \mathfrak{n}_+ \otimes \mathbb{C}[t]$ holds: Recall that $x, y \in \mathfrak{n}_+$ and $h \in \mathfrak{h}$ imply that $[x, h], [y, h] \in \mathfrak{n}_+$. Since every element $y \in I_1$ is written as $y = y' \otimes t^r + h \otimes t^s + c_1(h_1 - a) + c_2(h_2 - b)$, $c_1, c_2 \in \mathbb{C}$, given $x \otimes t^k \in \mathfrak{a}_0$ we have that

$$\begin{aligned}
&[x \otimes t^k, y' \otimes t^r] = [x, y'] \otimes t^{k+r} \in \mathfrak{n}_+ \otimes \mathbb{C}[t],\\
&[x \otimes t^k, h \otimes t^s] = [x, h] \otimes t^{k+s} \in \mathfrak{n}_+ \otimes \mathbb{C}[t],\\
&[x \otimes t^k, (h_1 - a)] = [x, (h_1 - a)] \otimes t^k \in \mathfrak{n}_+ \otimes \mathbb{C}[t], \text{ and}\\
&[x \otimes t^k, (h_1 - a)] = [x, (h_1 - a)] \otimes t^k \in \mathfrak{n}_+ \otimes \mathbb{C}[t].
\end{aligned}$$

Thus the inclusion holds by linearity.

(ii) We have $\mathfrak{n}_+ \otimes \mathbb{C}[t] \subseteq (I_1 \cap I') \oplus \mathfrak{a}_0$, as $\mathfrak{n}_+ \otimes \mathbb{C}[t] \subseteq I_1$ by definition and $I_1 = (I_1 \cap I') \oplus \mathfrak{a}_0$.

Hence

$$\operatorname{ad}(\mathfrak{a}_0)I_1 \subseteq \mathfrak{n}_+ \otimes \mathbb{C}[t] \subseteq (I_1 \cap I') \oplus \mathfrak{a}_0. \tag{3.15}$$

Given an integer $k > 0$, suppose as the induction step that (3.14) holds for every positive integer smaller than k. In this case,

$$\begin{aligned}
X_1 \cdots X_k I_1 &\subseteq X_1 \cdots X_{k-1} I_1 X_k + X_1 \cdots X_{k-1} \left(\operatorname{ad}(X_k) I_1\right) && \text{(property of the adjoint action)}\\
&\subseteq X_1 \cdots X_{k-1}(I' \cap I_1) + U(\mathfrak{g}[t])\mathfrak{a}_0 && \text{(because of (3.15))}\\
&\subseteq (I' \cap I_1) \oplus U(\mathfrak{g}[t])\mathfrak{a}_0 && \text{(by the induction hypothesis).}
\end{aligned}$$

In particular, it implies that $U(\mathfrak{a}_0)I_1 \subseteq J$. Now we show that $U(\mathfrak{a}_0)(x_2^-)^{b+1} \subseteq J$; by definition $(x_2^-)^{b+1} \in I'$, so all that must be done is to show that $U(\mathfrak{a}_0)^+(x_2^-)^{b+1} \subseteq J$, with $U(\mathfrak{a}_0)^+$ being the augmentation ideal of $U(\mathfrak{a}_0)$ (see Definition 1.1.17). For the adjoint action, the set of $\mathfrak{h}$-weights in $U(\mathfrak{a}_0)^+(x_2^-)^{b+1}$ satisfy

$$\operatorname{wt}_{\mathfrak{h}}(U(\mathfrak{a}_0)^+(x_2^-)^{b+1}) \subseteq \mathbb{Z}_{>0}\alpha_1 + \mathbb{Z}\alpha_2. \tag{3.16}$$

Since $\mathfrak{a}_0 \oplus \mathbb{C}(x_2^-)$ is a Lie subalgebra and $U(\mathfrak{a}_0 \oplus \mathbb{C}(x_2^-)) = \mathbb{C}[x_2^-] \oplus \mathbb{C}[x_2^-]U(\mathfrak{a}_0)^+$, equation (3.16) implies that

$$U(\mathfrak{a}_0)^+(x_2^-)^{b+1} \subseteq \mathbb{C}[x_2^-]U(\mathfrak{a}_0)^+ \subseteq J$$

as desired. Finally, set

$$I_2 = \mathbb{C}(x_1^-)^{a+1} + \mathbb{C}(x_1^- \otimes t^{s_1-1})^2 + \mathbb{C}(x_1^- \otimes t^{s_1})$$

and note that $I = I_1 \oplus I_2 \oplus \mathbb{C}(x_2^-)^{b+1}$. All that is left to do now is check if $U(\mathfrak{a}_0)I_2 \subseteq J$. To do so define

$$\mathfrak{a}_0' := \sum_{\substack{\beta \in \Phi_0^+ \setminus \{\alpha_1, \alpha_2\} \\ 0 \le s < \omega_1^\vee(\beta)}} \mathbb{C}(x_\beta^+ \otimes t^s).$$

Since $\mathfrak{a}_0 = \mathfrak{a}_0' \oplus \mathbb{C}(x_1^+)$, the set of $\mathfrak{h}$-weights

in $U(\mathfrak{a}_0')^+ I_2$ with respect to the adjoint action satisfy

$$\mathrm{wt}(U(\mathfrak{a}_0')^+ I_2) \subseteq \mathbb{Z}\alpha_1 + \mathbb{Z}_{>0}\alpha_2. \tag{3.17}$$

Fix $\mathfrak{n}_+^1 := \sum_{\beta \in \Phi_0^+ \setminus \{\alpha_1\}} \mathfrak{g}_\beta$. In this case, $x_1^- \otimes \mathbb{C}[t] \oplus \mathfrak{n}_+^1 \otimes \mathbb{C}[t]$ is a Lie subalgebra of $\mathfrak{g}[t]$ and

$$U(x_1^- \otimes \mathbb{C}[t] \oplus \mathfrak{n}_+^1 \otimes \mathbb{C}[t]) = U(x_1^- \otimes \mathbb{C}[t]) \oplus U(x_1^- \otimes \mathbb{C}[t])U(\mathfrak{n}_+^1 \otimes \mathbb{C}[t])^+.$$

By (3.17) we have that

$$U(\mathfrak{a}_0')^+ I_2 \subseteq U(x_1^- \otimes \mathbb{C}[t])U(\mathfrak{n}_+^1 \otimes \mathbb{C}[t])^+ \subseteq U(\mathfrak{g}[t])I_1.$$

Fixing $\ell = 3$ in Lemma 3.2.10 yields $\mathbb{C}[x_1^+]I_2 \subseteq J$. Hence

$$U(\mathfrak{a}_0)I_2 \subseteq \mathbb{C}[x_1^+]I_2 \oplus \mathbb{C}[x_1^+]U(\mathfrak{a}_0')^+ I_2 \subseteq J \oplus U(\mathfrak{g}[t])I_1 \subseteq J$$

which ends the proof. □

Finally, we are set to prove that (3.13) holds. Given integers a and b, suppose that we have

$$(x_\sigma^- \otimes t^{s_\sigma - 1})^2 \in U(\mathfrak{g}[t])I_{a,b}.$$

Recall that $U(\mathfrak{g}[t]) = U(\mathfrak{a}) \oplus U(\mathfrak{g}[t])\mathfrak{a}_0$. Also, Lemma 3.2.13 gives that $U(\mathfrak{g}[t])I_{a,b} \subseteq U(\mathfrak{a})I_{a,b}' \oplus U(\mathfrak{g}[t])\mathfrak{a}_0$, so

$$(x_\sigma^- \otimes t^{s_\sigma - 1})^2 \in U(\mathfrak{a})I_{a,b}'. \tag{3.18}$$

Define a $\mathbb{C}$-linear map $\varphi : \mathfrak{a} \to U(\mathfrak{g}[t])$ as

(i) $\varphi(x_\gamma^+ \otimes t^k) = (x_\gamma^+ \otimes t^{k - \omega_1^\vee(\beta)})$, $\varphi(x_\gamma^- \otimes t^k) = (x_\gamma^- \otimes t^{k + \omega_1^\vee(\gamma)})$ if $\gamma \in \Phi_+$.

(ii) $\varphi(h_1 \otimes t^k) = h_1 \otimes t^k - 3\delta_{k,0}$.

(iii) $\varphi(h_2 \otimes t^k) = h_2 \otimes t^k$.

Note that φ satisfies $\varphi([X_1, X_2]) = [\varphi(X_1), \varphi(X_2)]$, for every $X_1, X_2 \in \mathfrak{a}$, thus φ induces a $\mathbb{C}$-algebra homomorphism $\varphi : U(\mathfrak{a}) \to U(\mathfrak{g}[t])$. Applying φ to (3.18) yields

$$\begin{aligned}
(x_\sigma^- \otimes t^{s_\gamma+1})^{-1} &\in \varphi(U(\mathfrak{a})I'_{a,b}) \\
&\subseteq U(\mathfrak{g}[t])\varphi(I'_{a,b}) \\
&\subseteq U(\mathfrak{g}[t])(\mathfrak{n}_+ \otimes \mathbb{C}[t] \oplus \mathfrak{h} \otimes t\mathbb{C}[t] \oplus \mathbb{C}(h_1 - (a+3)) \\
&\oplus \mathbb{C}(h_2 - b) \oplus \mathbb{C}(x_1^- \otimes t^{s_1+1}) \oplus \mathbb{C}(x_2^- \otimes t^{s_2})) \\
&\subseteq U(\mathfrak{g})I_{a+3,b}
\end{aligned}$$

as desired.

Chapter 4

FUSION PRODUCTS

In general, given $V_1, \dots, V_k$ cyclic $\mathfrak{g}[t]$-modules, the tensor product $V_1 \otimes \cdots \otimes V_k$ is not cyclic. As a workaround, we replace such tensor with the fusion product defined in [16], which are graded modules associated with certain filtrations related to $V_1 \otimes \cdots \otimes V_k$ depending on a choice of parameters.

From now on assume that $\mathfrak{g}$ is a Kac-Moody algebra of finite type. If V is a cyclic $\mathfrak{g}[t]$-module with highest weight vector v, then there is an induced filtration

$$F^n V = U^{\leq n}(\mathfrak{g}[t])v \tag{4.1}$$

in which $F^0 V$ is the cyclic $\mathfrak{g}$-module V with highest weight vector v and $U^{\leq n}(\mathfrak{g}[t])$ denotes the filtered component of $U(\mathfrak{g}[t])$ of degree at most n. Following this induced filtration we define the *associated graded module* by

$$\mathrm{gr}(V) := \bigoplus_{i \geq 0} \frac{F^i V}{F^{i-1} V} \tag{4.2}$$

where $F^{-1}V := \{0\}$.

Fix $a \in \mathbb{C} \setminus \{0\}$ and define $\varphi_a : \mathfrak{g}[t] \to \mathfrak{g}[t]$ as

$$\varphi_a(x \otimes t^s) = x \otimes (t-a)^s. \tag{4.3}$$

Note that φ_a is a Lie homomorphism, since $\forall x, y \in \mathfrak{g}$ and $s, r \geq 0$ we have

$$\begin{aligned} \varphi_a([x \otimes t^s, y \otimes t^r]) &= \varphi_a([x,y] \otimes t^{s+r}) \\ &= [x,y] \otimes (t-a)^{s+r} \\ &= [x \otimes (t-a)^s, y \otimes (t-a)^r] \\ &= [\varphi_a(x \otimes t^s), \varphi_a(y \otimes t^r)]. \end{aligned}$$

Given a $\mathfrak{g}[t]$-module W, set $\varphi_a^*(W)$ as a $\mathfrak{g}[t]$-module defined by the action

$$(x \otimes t^s) \cdot w := (x \otimes (t-a)^s) \cdot w \tag{4.4}$$

in which the right-side action is the original action of W. We say that the module $\varphi_a^*(W)$ is the *pullback* of W. To simplify the notation we denote $W_a := \varphi_a^*(W)$. The proof we present follows the same lines as the proof of [16, Proposition 1.4].

Lemma 4.0.1. *Fix $V_1, \ldots, V_k$ cyclic, graded, finite-dimensional $\mathfrak{g}[t]$-modules with highest weight vectors $v_1, \ldots, v_s$. If $C = \{c_1, \ldots, c_k\}$ is a set of pairwise different complex numbers, then $(V_1)_{c_1} \otimes \cdots \otimes (V_k)_{c_k}$ is a cyclic $\mathfrak{g}[t]$-module generated by $v_1 \otimes \cdots \otimes v_k$.*

Proof. Given $u_i \in V_i,\ \forall i = 1, \ldots, k$, fix $u := u_1 \otimes \cdots \otimes u_k$. In this case

$$(x \otimes t^s) \cdot u = \sum_{i=1}^{k} \sum_{r=1}^{s} (-1)^{s+r} \begin{bmatrix} s \\ r \end{bmatrix} c_i^{s-r} (u_1 \otimes \cdots \otimes (x \otimes t^r) \cdot u_i \otimes \cdots \otimes u_k). \tag{4.5}$$

By hypothesis each V_i is a finite-dimensional graded module. Hence there must be some $N >> 0$ such that

$$(x \otimes t^s) \cdot u = 0,\ \forall s > N.$$

Define the matrix

$$A := \left((-1)^{s+r} \begin{bmatrix} s \\ r \end{bmatrix} c_i^{s-r} \right)_{s,(i,r)}$$

with $i = 1, \ldots, k,\ r = 1, \ldots, N,\ s = 1, \ldots, kN$. Here the elements (i, r) are under the lexicographic order, meaning that $(i, r) < (i', r')$ if $i < i'$, or $i = i'$ and $r < r'$. Thus we may rewrite the equation (4.5) as

$$(x \otimes t^s) \cdot u = \sum_{i=1}^{k} \sum_{r=1}^{s} A_{s,(i,r)} (u_1 \otimes \cdots \otimes (x \otimes t^r) \cdot u_i \otimes \cdots \otimes u_k).$$

The proof that Vandermonde matrixes are invertible can be adapted to show that A is invertible. If A^{-1} is the inverse of A we have, for every $i = 1, \ldots k$ and $r = 1, \ldots N$

$$u_1 \otimes \cdots \otimes (x \otimes t^r) \cdot u_i \otimes \cdots \otimes u_k = \sum_{s=1}^{kN} (A^{-1})_{s,(i,r)} ((x \otimes t^s) \cdot u).$$

This means that the elements $u_1 \otimes \cdots \otimes (x \otimes t^s) \cdot u_i \otimes \cdots \otimes u_k$ can be obtained as linear combinations of $(x \otimes t^s) \cdot u$. If we follow these steps for $v = v_1 \otimes \cdots \otimes v_k$ we have that

$$U(\mathfrak{g}) v_1 \otimes \cdots \otimes U(\mathfrak{g}) v_k \subset U(\mathfrak{g}[t]) v.$$

Since v_i are cyclic vectors for all $i = 1, \ldots, k$ the result is proved. $\square$

It follows from the last lemma that $(V_1)_{c_1} \otimes \cdots \otimes (V_k)_{c_k}$ is a cyclic module generated by a highest weight vector $v_1 \otimes \cdots \otimes v_k$. In this case, there is an induced filtration in $(V_1)_{c_1} \otimes \cdots \otimes (V_k)_{c_k}$, as in (4.1). This means that the associated graded $\mathfrak{g}[t]$-module $\operatorname{gr}((V_1)_{c_1} \otimes \cdots \otimes (V_k)_{c_k})$ exists (see (4.2)), so the following definition makes sense.

Definition 4.0.2. For $V_1, \ldots, V_k$ cyclic, graded, finite-dimensional $\mathfrak{g}[t]$-modules and $C = \{c_1, \ldots, c_k\}$ pairwise distinct complexes, the $\mathfrak{g}[t]$-module

$$V_1 * \cdots * V_k := \mathrm{gr}_C((V_1)_{c_1} \otimes \cdots \otimes (V_k)_{c_k})$$

is called the *fusion product* of $V_1, \ldots, V_k$.

Remark 4.0.3. *We denote by $v_1 * \cdots v_k$ the image of $v_1 \otimes \cdots \otimes v_k$ in $V_1 * \cdots * V_k$ and so we have*

$$V_1 * \cdots * V_k = U(\mathfrak{g}[t])(v_1 * \cdots * v_k)$$

Note that there is a dependence of choice of C in the last definition, which we have omitted. Feigin and Loktev have conjectured in [16] that the fusion product does not depend on the choice of pairwise distinct complexes. In this same article, the authors were able to prove the result in several cases. We enunciate such conjecture below and refer to Section 2 of [16] for further information.

Conjecture 4.0.4. *Let $V_1, \ldots, V_k$ be cyclic graded $\mathfrak{g}[t]$-modules.*

(i) If $C = \{c_1, \ldots, c_n\}$ and $C' = \{c'_1, \ldots, c'_k\}$ are sets of pairwise different complexes, then

$$\mathrm{gr}_C((V_1)_{c_1} \otimes \cdots \otimes (V_k)_{c_k}) = \mathrm{gr}_{C'}((V_1)_{c'_1} \otimes \cdots \otimes (V_k)_{c'_k}).$$

Meaning that the fusion product does not depend on the choice of C.

(ii) The fusion product is associative, up to isomorphism. Namely

$$(V_1 * \cdots * V_k) \cong V_1 * (V_2 \cdots * V_k) \cong (V_1 * \cdots V_{n-1}) * V_k.$$

Several papers have also proved specific cases in which the last conjecture holds. For example, in [7] it was shown that the conjecture holds for $V(\omega_1) * \cdots * V(\omega_j)$ if $\mathfrak{g} = \mathfrak{sl}_n$, and in [1] it was proved that the conjecture hold for the fusion product of Kirilov-Reshetikhin modules if $\mathfrak{g}$ is simple.

In the context of $\mathfrak{g}$-stable Demazure modules there is a contribution from Fourier and Littelmann as the next proposition states. Since its proof heavily depends on theories we do not approach (particularly quantum Weyl modules), we omit it from this work. Details can be found in [18, Corollaries 5 and 6].

Proposition 4.0.5. *For every $\ell \geq 1$, if $\lambda_i \in P_+$, for $i = 1, \ldots, k$ and $\ell\lambda = \sum_{i=1}^{k} \ell\lambda_i \in P_+$ we have*

$$\mathcal{D}(\ell, \ell\lambda_1) * \cdots * \mathcal{D}(\ell, \ell\lambda_k) \cong_{\mathfrak{g}[t]} \mathcal{D}(\ell, \ell\lambda).$$

Moreover the fusion product

$$\mathcal{D}(\ell, \ell\lambda_1) * \cdots * \mathcal{D}(\ell, \ell\lambda_k) \tag{4.6}$$

is associative and does not depend on the pairwise distinct complexes $c_1, \ldots, c_k$.

Last proposition has an immediate consequence. Given $\lambda \in P_+$ write $\lambda = \sum_{i=1}^{n} k_i \omega_i$, with $k_i \in \mathbb{Z}_{\geq 0}$ and ω_i fundamental weights. Then

$$\mathcal{D}(1, \lambda) \cong \mathcal{D}(1, k_1\omega_1) * \cdots * \mathcal{D}(1, k_n\omega_n).$$

Furthermore, the last proposition can be used to give a description of the fusion product of Demazure modules via generators and relations.

Theorem 4.0.6. *For $\lambda_1, \ldots, \lambda_k \in P_+$, the fusion product $\mathcal{D}(1,\lambda_1) * \cdots * \mathcal{D}(1,\lambda_k)$ is a cyclic module isomorphic to the $\mathfrak{g}[t]$-module generated by a non-zero vector v satisfying the following set of relations.*

(i) The Weyl relations, for $\lambda = \sum_{i=1}^{k} \lambda_i$.

(ii) $(x_\gamma^- \otimes t^{s_\gamma}) \cdot v = 0$, $\forall \gamma \in \Phi_+^{sh}$ such that $d_\gamma > 1$.

(iii) $(x_\gamma^- \otimes t^{s_\gamma - 1})^2 \cdot v = 0$, $\forall \gamma \in \Phi_+^{sh}$ such that $d_\gamma = 3$ and $m_\gamma = 1$.

Proof. It follows from Proposition 4.0.5 that $\mathcal{D}(1,\lambda_1) * \cdots * \mathcal{D}(1,\lambda_k) \cong_{\mathfrak{g}[t]} \mathcal{D}(1,\lambda)$, for $\lambda = \sum_{i=1}^{k} \lambda_i$. From Theorem 3.2.1 $\mathcal{D}(1,\lambda) \cong_{\mathfrak{g}[t]} M_\lambda$, with M_λ being a $\mathfrak{g}[t]$-module generated by a non-zero vector v with the above relations, hence $\mathcal{D}(1,\lambda_1) * \cdots * \mathcal{D}(1,\lambda_k)$ is isomorphic to M_λ, which ends the proof. ☐

We now show that Conjecture 4.0.4 holds for the fusion product of Weyl modules. This result was conjectured in [18] and later proved in [28]. In order to present its proof, we prepare the following proposition, which was proved in [28, Corollary 9.5 (i)].

Proposition 4.0.7. *If $\lambda = \sum_{i=1}^{k} r_i \omega_i \in P_+$, then*

$$\dim(W(\lambda)) = \prod_{i=s}^{n} \dim(W(\omega_i))^{r_i}.$$

Remark 4.0.8. *Given $r_i \in \mathbb{Z}_{\geq 0}$, $i \in I$, it follows from last proposition that* $\dim(W(r_i\omega_i)) = \dim(W(\omega_i))^{r_i}$

Theorem 4.0.9. *Let $\lambda_i \in P_+$, for $i = 1, \ldots, s$. In this case*

$$W(\lambda) \cong_{\mathfrak{g}[t]} W(\lambda_1) * \cdots * W(\lambda_k).$$

Moreover the fusion product

$$W(\lambda_1) * \cdots * W(\lambda_k) \tag{4.7}$$

is associative and does not depend on the pairwise distinct complexes $c_1, \ldots, c_k$.

Proof. Let $\lambda_1, \ldots, \lambda_k \in P_+$ and $\{c_1, \ldots, c_k\}$ be pairwise different complexes. First we show that the fusion product

$$W(\lambda_1)_{c_1} * \cdots * W(\lambda_k)_{c_k}$$

is a quotient of $W(\lambda)$, with $\lambda = \sum_{i=1}^{k} \lambda_i$. Fix w_i generator of $W(\lambda_i)$ as in its definition and recall that $w_1 * \cdots * w_k$ is the image of $w_1 \otimes \cdots \otimes w_k$ in the fusion product. For every $1 \leq i \leq k$ we have

$$(x_j^+ \otimes \mathbb{C}[t]) \cdot w_i = 0, \quad (h_j \otimes t^s) \cdot v_i = \delta_{s0}(h_j) w_i, \quad (x_j^-)^{\lambda_i(h_j)+1} \cdot w_i = 0$$

with $j \in I$ and $s \in \mathbb{Z}_{\geq 0}$. It is straightforward to verify that

(i) $(x_j^+ \otimes p(t)) \cdot (w_1 \otimes \cdots \otimes w_k) = 0$, $j \in I$.

(ii) $(h_j \otimes 1) \cdot (w_1 \otimes \cdots \otimes w_k) = \lambda(h_j)(w_1 \otimes \cdots \otimes w_k)$, $j \in I$, $s \in \mathbb{Z}_{\geq 0}$.

(iii) $(x_j^-)^{\lambda(h_j)+1} \cdot (w_1 \otimes \cdots \otimes w_k) = 0$, $j \in I$.

If $s > 0$ and $j \in I$, then

$$\begin{aligned}(h_j \otimes t^s) \cdot (w_1 \otimes \cdots \otimes w_k) &= \sum_{i=1}^{k} \sum_{r=0}^{s} (-1)^{s+r} \begin{bmatrix} s \\ r \end{bmatrix} c_j^{s-r} (u_1 \otimes \cdots \otimes (h_j \otimes t^r) \cdot u_i \otimes \cdots \otimes u_k) \\ &= \sum_{i=1}^{k} (-1)^s c_j^s (u_1 \otimes \cdots \otimes \lambda_i(h_j) u_i \otimes \cdots \otimes u_k) \\ &= \sum_{i=1}^{k} (-1)^s c_j^s \lambda_i(h_j)(u_1 \otimes \cdots \otimes u_k) \\ &= (-1)^s c_j^s \lambda(h_j)(u_1 \otimes \cdots \otimes u_k).\end{aligned}$$

Fix $V = (W_1)_{c_1} \otimes \cdots \otimes (W_k)_{c_k}$. It follows from the definition of $W(\lambda_1) * \cdots * W(\lambda_k)$ that

$$(h_j \otimes t^s) \cdot (w_1 \otimes \cdots \otimes w_k) \in \frac{F^s V}{F^{s-1} V}.$$

Moreover, $w_1 \otimes \cdots \otimes w_k \in F^0 V$, and since $(h_j \otimes t^s) \cdot (w_1 \otimes \cdots \otimes w_k)$ is a scalar multiple of $w_1 \otimes \cdots \otimes w_k$ it follows that $(h_j \otimes t^s) \cdot (w_1 * \cdots * w_k) = 0$.

Thus $w_1 * \cdots * w_k$ satisfies the defining relations of $W(\lambda)$, so there is a surjective $\mathfrak{g}[t]$-homomorphism $\varphi : W(\lambda) \to W(\lambda_1) * \cdots * W(\lambda_k)$ such that $\varphi(w) = w_1 * \cdots * w_k$, with w being the highest weight vector of $W(\lambda)$. By Proposition 4.0.7 and Remark 4.0.7 it follows that φ is an isomorphism, meaning that

$$W(\lambda) \cong_{\mathfrak{g}[t]} W(\lambda_1) * \cdots * W(\lambda_k).$$

If $C' = \{c_1', \ldots, c_k'\}$ is a different set of pairwise distinct complexes and $W(\lambda_1)' * \cdots * W(\lambda_k)'$ is the corresponding fusion product, then

$$W(\lambda_1)' * \cdots * W(\lambda_k)' \cong_{\mathfrak{g}[t]} W(\lambda) \cong_{\mathfrak{g}[t]} W(\lambda_1) * \cdots * W(\lambda_k)$$

ensuring that, in this case, the fusion product does not depend on the choice of C. $\square$

BIBLIOGRAPHY

[1] E. Ardonne, R. Kedem. *Fusion products of Kirillov-Reshetikhin modules and Fermionic multiplicity formulas.* Journal of Algebra 308.1, 2007: 270-294. math/RT: 0602177.

[2] M. Bennett, V. Chari, N. Manning. *BGG reciprocity for current algebras.* Advances in Mathematics 231.1, 2012: 276-305. math/RT: 1106.0347.

[3] L. Boza, E. Fedriani, R. Núñes, A. Tenorio. *A historical review of the classifications of Lie algebras.* Revista de la Unión Matemática Argentina, 54 2013.

[4] R. Carter. *Lie algebras of finite and affine type.* Cambridge University Press, New York, 2005.

[5] V. Chari, G. Fourier, T. Khandai. *A categorical aproach to Weyl modules.* Transformation Groups 15.3, 2010: 517 - 549. math/RT: 0906.2014.

[6] V. Chari, J. Greenstein. *Current algebras, highest weight modules categories and quivers.* math/RT: 0612206.

[7] V. Chari, S. Loktev. *Weyl, Demazure and Fusion modules for the current algebra of $\mathfrak{sl}_{r+1}$.* Advances in Mathematics 207.2, 2006: 928-960. math/RT: 0502165.

[8] V. Chari, A. Moura. *The restricted Kirillov-Reshetikhin modules for the current and twisted current algebras.* Communications in mathematical physics 266-2, 2006: 431-454. math/RT: 0507584.

[9] V. Chari, A. Pressley. *Weyl modules for classical and quantum affine algebras.* Representation Theory of the American Mathematical Society 5.9, 2001: 191-223. math/RT: 0004174.

[10] V. Chari, R. Venkatesh. *Demazure modules, fusion products and Q-systems.* Communications in Mathematical Physics 333.2, 2015: 799-830. math/RT: 1305.2523.

[11] M. Demazure. *Désingularisation des variétés de Schubert généralisées.* Annales scientifiques de l'É.N.S. 4, 1974: 53-88.

[12] G. Dhillon, A. Khare *Faces of highest weight modules and the universal Weyl polhyedron.* Advances in Mathematics, 319, 2017: 111-152. math/RT: 1611.00114.

[13] P. Di Francesco, R. Kedem. *Proof of the combinatorial Kirillov-Reshetikhin conjecture.* International Mathematics Research Notices, 2008.9, 2008: rnn006-rnn006.. math/RT: 0710.4415.

[14] B. Feigin, E. Feigin. *Q-characters of the tensor products in $\mathfrak{sl}_2$-case,.* Moscow Mathematical Journal, 2(3), 2002: 567-588 math/RT: 0201111.

[15] B. Feigin, A.N. Kirillov, S. Loktev. *Combinatorics and Geometry of Higher Level Weyl Modules.* arXiv preprint. math/RT: 0503315.

[16] B. Feigin, S. Loktev. *On generalized Kostka polynomials and quantum Verlinde rule.* Differential topology, infinite-dimensional Lie algebras, and applications, 1999: 61-79. math/RT: 9812093.

[17] G. Fourier, P. Littlelmann. *Tensor product structure of affine Demazure modules and limite constructions.* Nagoya Math Journal 182, 2006: 182: 171-198. math/RT: 0412432.

[18] G. Fourier, P. Littelmann. *Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions.* Advances in Mathematics 211(2), 2007: 566-593. math/RT: 0509276.

[19] J. E. Humphreys. *Introduction to Lie algebras and Representation theory*, third printing. Springer-Verlag New York Inc., New York. 1980.

[20] M. Ishii. *Demazure character formulas for generalized Kac–Moody algebras.* Communications in Algebra 43(7), 2015: 2785-2801. math/RT: 1304.3867.

[21] N. Jacobson. *Lie algebras.* Dover Publications Inc., New York. 1979.

[22] A. Joseph. *On the Demazure character formula.* Annales scientifiques de l'É.N.S. 4, tome 18, no 3, 1985: 389-419.

[23] V. G. Kac. *Infinite-dimensional Lie algebras.* Progress in Math, Vol. 44, Birkhäuser. 1983.

[24] D. Kuz, P. Littelmann. *Fusion products and toroidal algebras.* Pacific Journal of Mathematics, 278(2), 2015: 427-445. math/RT: 1411.5272.

[25] O. Mathieu. *Construction d'un groupe de Kac-Moody et applications.* Compositio Mathematica, tome 69, no 1, 1989: 37-60.

[26] V. Mazorchuk, K. Zhao. *Characterization of simple highest weight modules.* Canadian mathematical bulletin, 56(3), 2013: 606-614. math/RT: 1105.1123.

[27] K. Naoi. *Fusion products of Kirillov-Reshetikhin modules and the $X = M$ conjecture.* Advances in Mathematics, 231(3-4), 2012: 1546-1571. math/RT: 1109.2450.

[28] K. Naoi. *Weyl modules, Demazure modules and finite crystals for non-simply laced type.* Advances in Mathematics, 229(2), 2012: 875-934. math/RT: 1012.5480.