UNIVERSIDADE FEDERAL DO PARANÁ
RAFAEL LIMA OLIVEIRA

CONTRIBUTIONS IN PARTIALLY DISSIPATIVE COUPLED SYSTEMS

CURITIBA, 2020

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Tese apresentada ao Programa de Pós- Graduação em Matemática, Setor de Ciências Exatas, Universidade Federal do Paraná, como requisito parcial à obtenção do título de Doutor em Matemática.

Orientador: Prof. Dr. Higidio Portillo Oquendo

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It always seems impossible until it’s done.

(By Nelson Mandela)

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RESUMO

Nesta tese, consideramos três problemas de estabilidade para sistemas elásticos acoplados. Cada sistema possui um mecanismo dissipativo atuando indiretamente. Os principais resultados sobre esses problemas são: a boa colocação do problema, comportamento assintótico com taxas explícitas (decaimento polinomial ou decaimento exponencial) e a otimalidade das taxas de decaimento. A prova dos resultados é baseada na teoria de semigrupos e caracterizações espectrais para estabilidade assintótica (caracterização de Pruss e Borichev-Tomilov).

Palavras-chave: sistema acoplado, amortecimento indireto, estabilidade assintótica, decaimento exponencial, decaimento polinomial, otimalidade.

ABSTRACT

In this thesis, we consider three problems in stability for coupled elastic systems. Each system has one dissipative mechanism acting indirectly. The main results about these problems are: the well-posed, the asymptotic behavior with explicit rates (polynomial decay or exponential decay) and the optimality of the decay rates. The proof of the results are based on semigroup theory and spectral characterizations for asymptotic stability (Pruss and Borichev-Tomilov characterization).

Keywords: coupled system, indirect damping, asymptotic stability, exponential decay, polynomial decay, optimality.

Contents

Introduction

In this work, we propose to study the stability of three coupled systems that present partial dissipative mechanisms. More precisely, we studied coupled systems that involve vibrations of membranes or plates which will be placed in an abstract context. In this sense, we have concentrated our efforts on those problems where the dissipation is caused by frictional type damping or by having a memory type viscoelastic structure. To use semigroup theory, these systems will be placed in an abstract format, establishing the operators that define them in an appropriate Hilbert space, in this way it is possible to guarantee the existence of solutions.

Having resolved the problem about existence of solution, we direct all our efforts in the study of asymptotic behavior. In this sense, we have some characterizations that allow us to have decay results looking only at the behavior of the operator's spectrum associated with the system. These characterizations due to Pruss (exponential decay) and Borichev-Tomilov (polynomial decay) play a fundamental role in the development of this work.

Concerning the results obtained about the asymptotic behavior of the solutions, we observed that certain relations between the structural coefficients of the system cause the decay to become faster or slower. These relations are generally linked to wave propagation speeds or a comparison between the coefficients of the dissipative term and those of the oscillating structure of the system. The decay speed will also be affected by the fractional exponents of the memory effect and we will see that these play a fundamental role in determining the optimal decay rates.

Many problems with dissipative effects have been widely studied in recent years, especially with frictional and memory damping. See, for example, the impressive list of authors who have published articles addressing these subjects: [57], [16], [23], [29], [31], [26], [34], [59], [61], [63], [49], [4], and references therein. Even in the case of one equation (like wave or plate), the investigations of properties in this field are a challenging problem that still have open questions to be considered because it is possible to consider in different ways and with different dissipative effects. In a coupled system, also there exist asks when these systems have some dissipative effects in the following context: if we have one dissipative equation coupled with another conservative one then what will happen? Is it possible to consider any relationship between all coefficient of this system in the asymptotic behavior? If there exist some decay rates these decay are the best? Some research in this way are: [1], [12], [16], [31], [37], [9], [76], [36] and [45].

Finally, this thesis is organized as follows: in Chapter 1 we introduce briefly the notations and preliminary results concerning the Sobolev spaces, some inequalities, spectral properties and semigroup theory. In Chapter 2 we study the coupled system with two waves whose one equation is conservative and the other has a frictional dissipation and delay term. In Chapter 3 we study a class of equations that generalize wave or plate equations with fractional memory. In Chapter 4 we present a study of a coupled system with wave and plate equation endowed with fractional memory dissipation in both equations. We study these effects in many ways. In all chapters, for each coupled system we have shown the well-posed, results of asymptotic behavior with explicit rates and optimal decay rates.

Chapter 1

Preliminaries

At this moment we will introduce some important results to help us understand all the development of this work. For more details about proof or comments of all results presented here, we advise the references [14], [21], [22] and [64].

1.1 Sobolev spaces and inequalities

Initially, let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary denoted by $\partial\Omega$. We define a multi-index $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \in \mathbb{N}^n$, with $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$\mathbf{D}^\alpha u = \frac{\partial^{\alpha_1+\cdots+\alpha_n} u}{\partial x_1^{\alpha_1} + \cdots \partial x_n^{\alpha_n}}.$$

Definition 1.1. *Let's consider $f : \Omega \to \mathbb{C}$ a measurable function. If*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \right)^{\frac{1}{p}} < \infty,$$

where $p \in [1, \infty)$ then $f \in L^p(\Omega)$. For $m \in \mathbb{N}$, we define the space $W^{m,p}(\Omega)$ as

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega); \boldsymbol{D}^\alpha u \in L^p(\Omega), \text{ for each multi-index } |\alpha| \leq m\}.$$

It is possible to prove that $W^{m,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|\boldsymbol{D}^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

When $p = 2$, we denote the space $W^{m,p}(\Omega)$ by $H^m(\Omega)$ (or eventually just H^m) where this is a Hilbert space. Note that $H^0(\Omega) = L^2(\Omega)$. Moreover, $C^k(\Omega)$ $(1 \leq k \leq \infty)$ will denote the space of k times continuously differentiable functions on Ω.

In addiction, if we consider $C_0^k(\Omega) = \left\{u \in C^k(\Omega) |\ supp\ u \subset \Omega\right\}$, *we define* $W_0^{m,p}(\Omega)$ *as the closure of this space in the space* $W^{m,p}(\Omega)$. *In short, we see the space* $W_0^{m,p}(\Omega)$ *as all functions* $u \in W^{m,p}(\Omega)$ *such that "*$\boldsymbol{D}^\alpha u = 0$ *on* $\partial\Omega$*", for* $|\alpha| \leq m-1$*, however this idea is more general because involve traces theory. For a careful analysis, some relevant references are [14] and [22].*

The next theorems are about inequalities that we will use a lot on development of this thesis.

Theorem 1. ***(Hölder inequality)*** *Let* $u \in L^p(\Omega)$ *and* $v \in L^q(\Omega)$*, where* $1 < p < \infty, 1 < q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. We have* $uv \in L^1(\Omega)$ *and also*

$$\int_\Omega |uv| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Proof. See [22]. ∎

Theorem 2. ***(Young inequality)*** *Let* a *and* b *nonnegative real numbers with* $1 < p < \infty$ *and* $1 < q < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. For* $\varepsilon > 0$ *there exists* $C(\varepsilon) > 0$ *such that*

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q.$$

Proof. See [22]. ∎

Theorem 3. ***(Poincaré inequality)*** *If* Ω *is a bounded domain in* $\mathbb{R}^n$ *and* $u \in H_0^1(\Omega)$*, then there exists a positive constant* $C > 0$*, depending only on* Ω*, such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega).$$

Proof. See [22]. ∎

Definition 1.2. ***(Sesquilinear form)*** *Let* H *a complex vector space. A map* $a : H \times H \to \mathbb{C}$ *is a sesquilinear form if for all* $x, y, z, w \in H$ *and* $\forall\ \lambda_1, \lambda_2 \in \mathbb{C}$*,*

i) $a(x + y, z + w) = a(x, z) + a(x, w) + a(y, z) + a(y, w)$;

ii) $a(\lambda_1 x, \lambda_2 y) = \lambda_1 \bar{\lambda}_2 a(x, y)$.

Moreover, we say:

iii) the map is called bounded if there exists $L_1 \geq 0$ *such that*

$$|a(x, y)| \leq L_1 \|x\|_H \|y\|_H, \ \ \forall\ x, y \in H;$$

iv) the map a is coercive if there exists $L_2 > 0$ such that

$$Re\, a(u,u) \geq L_2 \|u\|_H^2, \ \ \forall\, u \in H.$$

Theorem 4. ***(Lax-Milgram)*** *Let H be a complex Hilbert space and $a : H \times H \to \mathbb{C}$ a sesquilinear form, bounded and coercive. If $f \in H'$, where H' denote the dual space of H, then there exists a unique $v \in H$ such that*

$$a(u,v) = f(u), \ \ \forall\, u \in H.$$

Proof. See [40]. ■

1.2 Some definitions about Semigroups Theory and Spectral Properties

In this section, we present some definitions about semigroup theory of operators. These properties are important to develop the well-posed and asymptotic behavior of all problems in this thesis.

Definition 1.3. *Let X be a Banach space with norm $\|\cdot\|$. A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators in X is called a strong continuous semigroup (or $C_0 - semigroup$) if:*

(i) $T(0) = I$, where I is the identity operator in the set of all bounded linear operator in X;

(ii) $T(t+s) = T(t)T(s), \forall\, t, s \in \mathbb{R}^+$;

(iii) For each $x \in X$, $\lim_{t\to 0^+} \|T(t)x - x\| = 0$.

Theorem 5. *If $\{T(t)\}_{t\geq 0}$ is a $C_0 - semigroup$ then there exists $M \geq 1$ and $\omega \geq 0$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \forall t \geq 0.$$

Proof. See [64]. ■

Remark 1. *If $M = 1$ and $\omega = 0$ in Theorem 5, then the semigroup is called semigroup of contractions. This remark is important because the semigroups that are generated by our operators, in this thesis, always will be of contractions. Moreover, the semigroup is said to be exponentially stable if there exist positive constant ω and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{-\omega t}, \forall t \geq 0.$$

Theorem 6. *If* $x \in X$ *and* $\{T(t)\}$ *is a* $C_0 - semigroup$*, then the function* $t \mapsto T(t)x$ *is continuous on* $[0, +\infty)$.

Proof. See [64]. ∎

From Theorem 6 it is possible to define:

Definition 1.4. *Let* X *be a Banach space and* $\{T(t)\}_{t\geq 0}$ *a* $C_0 - semigroup$*. The linear operator* $A : D(A) \subset X \to X$ *defined by*

$$D(A) = \left\{ x \in X \text{ such that } \exists \lim_{t \to 0^+} \frac{T(t)x - x}{t} \right\},$$

and

$$Ax := \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \quad \forall\, x \in D(A),$$

is called the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$.

The next theorem gives us the answer to how it is possible to solve the abstract problem

$$\begin{cases} \dfrac{d}{dt}U(t) = AU(t) \\ \\ U(0) = U_0, \end{cases} \tag{1.1}$$

when A is the infinitesimal generator of the $C_0 - semigroup.$

Theorem 7. *If* $\{T(t)\}_{t\geq 0}$ *is a* $C_0 - semigroup$ *and* A *is the infinitesimal generator, then given* $x \in \mathcal{D}(A)$*, we have* $T(t)x \in \mathcal{D}(A)$*,* $\forall\, t \geq 0$*, and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax. \tag{1.2}$$

In particular, if we take the function $u(t) = T(t)x$ *then it satisfies* $\frac{d}{dt}u(t) = Au(t)$*. So, from Theorem 7 is possible to solve the abstract problem (1.1) if we show that operator* A *is the infinitesimal generator of the* $C_0 - semigroup.$

Proof. See [64]. ∎

The next results can be found at [14], [22] and [44].

Definition 1.5. *Let A be a linear operator in a Hilbert space* H*. It is called the resolvent set of operator* A *the set*

$$\rho(A) = \left\{ \lambda \in \mathbb{C}; \lambda I - A \text{ is injective}; \overline{Im(\lambda I - A)} = H; (\lambda I - A)^{-1} \text{ is bounded} \right\}$$

Moreover, the set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ *is called spectrum of* A.

Definition 1.6. *Let $A : \mathcal{D}(A) \subset H \to H$ a densely defined linear operator in a Hilbert space H. The adjoint operator is given by $A^* : \mathcal{D}(A^*) \subset H \to H$, where*

$$\mathcal{D}(A^*) = \{v \in H; \exists v^* \in H \text{ such that } \forall u \in \mathcal{D}(A), \langle Au, v\rangle_H = \langle u, v^*\rangle_H\},$$

and $A^*v = v^*$. When $A^* = A$ we say that this operator is a self-adjoint.

Definition 1.7. *We say that a operator $A : \mathcal{D}(A) \subset H \to H$ is positive if*

$$\langle Au, u\rangle_H \geq 0, \forall u \in \mathcal{D}(A).$$

Definition 1.8. *Let A a linear operator with $\rho(A) \neq \emptyset$. We say that A has compact resolvent if for one $\lambda_0 \in \rho(A)$ we have that $(\lambda_0 I - A)^{-1}$ is a compact operator.*

From this definition we can enunciate the next result.

Theorem 8 (Spectral Properties). *Let's consider A a positive and self-adjoint operator with compact resolvent on a complex Hilbert space with $dim H = \infty$. We have that H has an orthonormal basis of eigenvectors of A where the eigenvalues are a positive real sequence $(\lambda_n)_{n\in\mathbb{N}}$ satisfying $\lim\limits_{n\to\infty} \lambda_n = \infty$.*

Proof. See [69]. ■

Theorem 9. *If A is a positive self-adjoint operator on H, then there is unique positive self-adjoint operator B on H such that $B^2 = A$.*

Proof. See [69]. ■

Supported by [11], [15],[21] and [69], we will define fractional powers A^α of a positive self-adjoint operator A on a Hilbert space. For this, let's consider the spectral theorem of unbounded self-adjoint operators.

Theorem 10. *Let A be a self-adjoint operator on a Hilbert space H. Then there exists a unique spectral measure $E = E_A$ on the Borel $\sigma-$algebra $\mathcal{B}(\mathbb{R})$ such that*

$$A = \int_{\mathbb{R}} \lambda dE_A(\lambda),$$

in particular, the fractional power A^α, for any α of a positive self-adjoint operator A is defined by

$$A^\alpha = \int_0^\infty \lambda^\alpha dE_A(\lambda),$$

which is also a positive self-adjoint operator.

Proof. See [69] ■

Furthermore, due to [39] we have for any $\alpha \geq \beta$:

(i) $\mathcal{D}(A^\alpha) \subset \mathcal{D}(A^\beta)$,

(ii) $\mathcal{D}(A^\alpha)$ is dense in $\mathcal{D}(A^\beta)$,

(iii) the continuous embedding $\mathcal{D}(A^\alpha) \hookrightarrow \mathcal{D}(A^\beta)$,

(iii) and the fundamental property of powers

$$A^\alpha A^\beta x = A^\beta A^\alpha x = A^{\alpha+\beta} x,$$

for $x \in \mathcal{D}(A^\gamma)$, with $\gamma = \max\{\alpha, \beta, \alpha+\beta\}$.

The next result is about an important inequality for fractional operators. This result is important to conclude the estimate in the polynomial decay rates.

Theorem 11 (Interpolation Inequality). *If $\alpha < \beta < \gamma$, then there exists a constant $L := L(\alpha, \beta, \gamma)$ such that*

$$\|A^\beta x\| \leq L\|A^\alpha x\|^{\frac{\gamma-\beta}{\gamma-\alpha}} \|A^\gamma x\|^{\frac{\beta-\alpha}{\gamma-\alpha}},$$

for every $x \in \mathcal{D}(A^\gamma)$.

Proof. See [21] for the proof and more details about fractional operators. ■

In order to obtain that A is the infinitesimal generator of the $C_0 - semigroup$ we need to the next results.

Definition 1.9. *Let H be a Hilbert space. The operator A is called dissipative operator when for all $x \in \mathcal{D}(A)$,*

$$Re \left\langle Ax, x \right\rangle_X \leq 0.$$

Theorem 12. ***(Lumer-Phillips's Theorem)*** *Let A be a linear operator in a Hilbert space H with dense domain $\mathcal{D}(A)$ in H. If A is dissipative and there exists $\lambda_0 > 0$ such that $Im(\lambda_0 I - A) = X$, then A is the infinitesimal generator of the $C_0 - semigroup$ of contractions on X.*

Proof. See [64]. ■

Normally, the Theorem 12 is used to show the well-posed of partial differential equations. In this case we will use a variant of this theorem.

Lemma 13. *Let $S : H \to H$ be a continuous linear operator with continuous inverse. If $B \in \mathcal{L}(H)$ and*

$$\|B\| < \frac{1}{\|S^{-1}\|},$$

then $S + B$ is a continuous linear operator with continuous inverse.

Proof. See [14]. ■

The following theorem can be found at [44].

Theorem 14. *(**Variant of Lumer-Phillips's Theorem**) Let A be a linear operator with domain $D(A)$ dense in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$, then A is the generator of a C_0-semigroup of contractions on H.*

Proof. By hypothesis we have A invertible. Furthermore $\lambda I - A = A(\lambda A^{-1} - I)$. Taking $S = -I$ and $B = \lambda A^{-1}$ we have

$$\|B\| = \|\lambda A^{-1}\| = |\lambda|\|A^{-1}\| < \frac{1}{\|S^{-1}\|},$$

for $|\lambda| < \frac{1}{\|A^{-1}\|}$. From Lemma 13 we have that $\lambda A^{-1} - I$ is invertible. Moreover, we can assert that $\lambda I - A$ is also invertible because is the composition between two invertible operator. As a conclusion, from Theorem 12 follows that A is the infinitesimal generator of the $C_0 - semigroup$ of contractions on X. ■

Finally, we will present some results about asymptotic behavior.

Theorem 15 (Exponentially stable). *Let A be the generator of a bounded C_0-semigroup of contractions on a Hilbert space H denoted by e^{tA}. Then e^{tA} is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(A) \quad \text{and} \quad \limsup_{|\lambda|\to\infty} \|(i\lambda I - A)^{-1}\| < \infty.$$

Proof. See [35]. ■

Theorem 16 (Borichev and Tomilov's Theorem). *Let A be the generator of a bounded C_0-semigroup on a Hilbert space H such that $i\mathbb{R} \subset \rho(A)$. We have that*

$$\|e^{tA}U_0\| \leq Ct^{-1/\theta}\|U_0\|_{D(A)}, \quad \forall\, t > 0,\ U_0 \in D(A),$$

if and only if,

$$\limsup_{|\lambda|\to\infty} |\lambda|^{-\theta}\|(i\lambda I - A)^{-1}\| < \infty.$$

Proof. See [13]. ■

Chapter 2

Stability and instability results for coupled waves with delay term

In this chapter, we will study a coupled system of two wave equations. One of these equations is conservative and the other has damping and delay terms. If the damping acts with more force than the delay term, we show polynomial stability for strong solutions to the system. Explicit decay rates are found and the optimality of those are discussed. On the other hand, if the damping acts with the same or less force than the delay term, then we obtain a result of asymptotic instability by constructing a sequence of time delays and initial data such that the solutions are not asymptotically stable.

2.1 Motivation

Nowadays, questions about the asymptotic behavior of solutions for PDEs with time delay effects have become interesting for many authors, mainly because this effect appears in many areas of sciences as biology, electrical engineering systems, mechanical applications, and medicine (see [33]). Furthermore, is known that this effect may destroy the stabilizing properties of a well-behaved system. In the literature, several examples illustrate how time delays destabilize some internal or boundary control system. We are going to mention some of them:

Nicaise et al. [53] considered the damped wave equation

$$u_{tt} - \Delta u + a(x)[\mu_1 u_t(t) + \mu_2 u_t(t-\tau)] = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$

satisfying Dirichlet and Neumann conditions on complementary parts of the boundary $\partial\Omega$. The nonnegative coefficient a is strictly positive in a part of $\Omega \subset \mathbb{R}^n$ satisfying the geometric control conditions. They showed that if μ_1, μ_2 are positive numbers such that

$$\mu_1 > \mu_2, \tag{2.1}$$

then, the solutions decay exponentially to zero with the energy norm. When the condition (2.1) does not hold, that is, $\mu_1 \leq \mu_2$, they found an explicit sequence of time delays that destabilize the system.

Following this way, the authors [4] and [30] studied the wave equation with memory and delay term

$$u_{tt} - \Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\,ds + \kappa u_t(t-\tau) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$

satisfying Dirichlet conditions on the boundary $\partial\Omega$. They showed that if the kernel μ of the memory is exponentially decreasing and the coefficient κ is small, then the solutions of this equation decrease exponentially. Instability results for large coefficients κ were not considered.

The asymptotic stability for other damped equations with delay term acting in the interior of Ω was studied by other authors. Some of them can be found in [7, 38, 42, 54, 55, 74, 75, 78]. They all obtained the same result: exponential decay for the solutions of the studied problem.

In some problems, a time delay occurs on the boundary. For example, Xu et al. [77] considered the unidimensional wave equation

$$u_{tt} - u_{xx} = 0, \quad (x,t) \in (0,1) \times \mathbb{R}^+,$$

with following boundary conditions

$$u(0,t) = 0, \quad u_x(1,t) = -\kappa\mu u_t(1,t) - \kappa(1-\mu)u_t(1,t-\tau),$$

were $\kappa > 0$, $\mu \in (0,1)$. They showed that this system is exponentially stable if $\mu > 1/2$ and unstable if $\mu < 1/2$. For the critical case $\mu = 1/2$ they considered $\tau \in (0,1)$ obtaining the following results: the system is unstable when τ is rational and asymptotically stable when τ is irrational. The n-dimensional case was studied by Nicaise et al. [53] obtaining the same result when $\mu > 1/2$. For $\mu \leq 1/2$, they showed that the system is unstable for a sequence of time delays τ_n. Other problems with delay term acting on the boundary can be found in [19, 20, 25, 54, 56, 65].

Other researchers have focused their efforts on studying the asymptotic stability of problems whose damping mechanisms act indirectly. This kind of problem was introduced by Russell in [66]. In these problems, the conservative part of the material absorbs the dissipative properties of indirect damping leading to the possibility of a global stabilization. But the technical difficulties for getting appropriate estimates for the conservative part are not simple; they are difficult and challenging.

In this chapter, we are interested in study the asymptotic behavior of a coupled system with indirect damping and delay time. The system is given by

$$\begin{cases} \rho_1 u_{tt} - \beta_1 \Delta u + \alpha v + \mu_1 u_t(t) + \mu_2 u_t(t-\tau) = 0 \ \text{ in } \ \Omega \times \mathbb{R}^+, \\ \rho_2 v_{tt} - \beta_2 \Delta v + \alpha u = 0 \ \text{ in } \ \Omega \times \mathbb{R}^+, \end{cases} \tag{2.2}$$

satisfying the boundary conditions

$$u = 0, \quad v = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \tag{2.3}$$

and initial data

$$u(0) = u_0,\ v(0) = v_0,\ u_t(0) = u_1,\ v_t(0) = v_1, \quad u_t(-s) = \phi(s),\ s \in (0,\tau),\ \phi(0) = u_1. \tag{2.4}$$

Here Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The coefficients $\rho_1,\ \rho_2,\ \beta_1,\ \beta_2,$ $\mu_1,\ \mu_2$ are positive, the coupling coefficient satisfies the condition

$$0 < |\alpha| < \gamma_1 \sqrt{\beta_1 \beta_2}, \tag{2.5}$$

where γ_1 is the first eigenvalue of the operator $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$.

Alabau et al. [1] studied the coupled system (2.2)-(2.4) for the case $\mu_2 = 0$ and $\rho_1 = \rho_2 = \beta_1 = \beta_2 = 1$. This means that the system does not have delay and the equations have the same wave propagation speeds (define in equation (2.6)). In this case, the coefficient

$$\chi_0 := \frac{\beta_1}{\rho_1} - \frac{\beta_2}{\rho_2} \tag{2.6}$$

is zero. They showed that the semigroup cannot be exponentially stable but it decays polynomially with the rate $t^{-1/2}$ for strong solutions. The optimality of this decay rate was not provided. When the equations have different wave propagation speeds, the situation is completely different, the semigroup decays slower. This study was done by Oquendo et al. [61], who showed that the semigroup decay polynomially with the optimal rate $t^{-1/4}$ for strong solutions. We will see that the relations $\chi_0 = 0$ and $\chi_0 \neq 0$ also play an important role in the stabilization of the system with time delay (2.2)-(2.4). Studies about stabilization of other coupled systems and wave propagation speeds can be found in [2, 12, 24, 60] and [52] respectively.

Our main results are on the asymptotic behavior of the strong solution of system (2.2)-(2.4). We summarize our results in the following items:

- If (2.1) holds, that is, if the damping acts with more force than the delay term, then the solution decay polynomially with the following rates: $t^{-1/2}$ if $\chi_0 = 0$ and $t^{-1/4}$ if $\chi_0 \neq 0$ (see Theorem 18).
- The decay rates in the previous item are the best (see Theorem 21).
- If (2.1) does not hold, that is, $\mu_1 \leq \mu_2$, for suitable time delays, the solutions can oscillate. Consequently, the solutions do not decay to zero. This result is independent of the value that χ_0 assumes (see Theorem 22).

2.2 Well-Posedness

In this section, we will prove that system (2.2)-(2.4) is well-posed by using the semigroup theory. To put this system in an abstract framework, we introduce the function

$$z(x,t,s) = u_t(x,t-s), \quad s \in [0,\tau], \quad x \in \Omega.$$

From this definition we see that z satisfies $z_t = -\partial_s z$. Therefore, if we consider the vector $U(t) = (u(t), v(t), u_t(t), v_t(t), z(t,\cdot))$, the coupled system (2.2)-(2.4) can be written as

$$\frac{d}{dt}U(t) = \mathbb{B}U(t), \quad U(0) = U_0, \tag{2.7}$$

where $U_0 = (u_0, v_0, u_1, v_1, \phi)$ and the operator $\mathbb{B}$ is given by

$$\mathbb{B}U = \left(\dot{u},\ \dot{v},\ \rho_1^{-1}\left\{\beta_1\Delta u - \alpha v - \mu_1\dot{u} - \mu_2 z(\tau)\right\},\ \rho_2^{-1}\left\{\beta_2\Delta v - \alpha u\right\},\ -\partial_s z\right),$$

for $U = (u, v, \dot{u}, \dot{v}, z)$. Here the point on top of this terms is just a notation, it does not mean the time derivative. We will define this operator in an appropriate subspace of the Hilbert space

$$\mathbb{X} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(0,\tau; L^2(\Omega)),$$

provided with inner product

$$\begin{aligned} \langle U_1, U_2\rangle &= \rho_1\langle \dot{u}_1, \dot{u}_2\rangle + \rho_2\langle \dot{v}_1, \dot{v}_2\rangle + \beta_1\langle \nabla u_1, \nabla u_2\rangle + \beta_2\langle \nabla v_1, \nabla v_2\rangle \\ &\quad + \alpha\langle u_1, v_2\rangle + \alpha\langle v_1, u_2\rangle + \mu_1\int_0^\tau \langle z_1(s), z_2(s)\rangle\, ds. \end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\Omega)$. With these considerations, the natural domain of the operator $\mathbb{B}$ is defined by

$$\begin{aligned} D(\mathbb{B}) = \Big\{ U \in \mathbb{X} : \dot{u}, \dot{v} \in H_0^1(\Omega), u, v \in H^2(\Omega) \cap H_0^1(\Omega), \\ \partial_s z \in L^2(0,\tau; L^2(\Omega)),\ z(0) = \dot{u} \Big\}. \end{aligned}$$

To show the well-posedness of the Cauchy problem (2.7) it suffices to prove that the operator $\mathbb{B}$ is the generator of a C_0- semigroup. For this, we will use a variant of Lumer-Phillips's Theorem enunciated in Theorem 14 to conclude the following result.

Theorem 17. *Assume* (2.1) *and* (2.5)*. For initial data* U_0 *in* $D(\mathbb{B})$ *there exists one solution of problem* (2.7) *in the following space*

$$U \in C([0, +\infty[; D(\mathbb{B})) \cap C^1([0, +\infty[; \mathbb{X}).$$

Proof. We are going to verify that $\mathbb{B}$ satisfies the conditions of Theorem 14. First, let us see that $D(\mathbb{B})$ is dense in $\mathbb{X}$. Let $U = (u, v, \dot{u}, \dot{v}, z) \in \mathbb{X}$, then there exists a sequence

$$(u_n, v_n, \dot{u}_n, \dot{v}_n, \eta_n) \in (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2 \times C_0^1(0, \tau; H_0^1(\Omega))$$

such that $(u_n, v_n, \dot{u}_n, \dot{v}_n, \eta_n) \to (u, v, \dot{u}, \dot{v}, z)$ in $\mathbb{X}$. Let us take $\phi_n \in C^1[0, \tau]$ satisfying the conditions $\phi_n(0) = 1$ and $\phi_n \to 0$ in $L^2(0, \tau)$, then considering the sequence $z_n := \phi_n \dot{u}_n + \eta_n$ we have $z_n(0) = \dot{u}_n$ and $z_n \to z$ in $L^2(0, \tau; L^2(\Omega))$. Therefore, $(u_n, v_n, \dot{u}_n, \dot{v}_n, z_n) \in D(\mathbb{B})$ and $(u_n, v_n, \dot{u}_n, \dot{v}_n, z_n) \to (u, v, \dot{u}, \dot{v}, z)$ in $\mathbb{X}$. Consequently, $D(\mathbb{B})$ is dense in $\mathbb{X}$.

On the other hand, considering $U = (u, v, \dot{u}, \dot{v}, z) \in D(\mathbb{B})$, performing a simple computation gives

$$\text{Re}\langle \mathbb{B}U, U\rangle = -\frac{\mu_1}{2}\int_\Omega |\dot{u}|^2 dx - \frac{\mu_1}{2}\int_\Omega |z(\tau)|^2 dx - \mu_2 \text{Re}\int_\Omega z(\tau)\dot{u} dx.$$

Applying Hölder and Young inequalities we get

$$\text{Re}\langle \mathbb{B}U, U\rangle \leq \frac{(\mu_2 - \mu_1)}{2}\left(\int_\Omega |\dot{u}|^2 dx + \int_\Omega |z(\tau)|^2 dx\right), \tag{2.8}$$

which is non-positive because $\mu_2 < \mu_1$. Therefore, the operator $\mathbb{B}$ is dissipative.

Remains to show $0 \in \rho(\mathbb{B})$. Let $F = (f_1, f_2, f_3, f_4, f_5) \in \mathbb{X}$. The vector $U = (u, v, \dot{u}, \dot{v}, z)$ is solution of system $\mathbb{B}U = F$ if and only if

$$\dot{u} = f_1 \quad \text{in} \quad H_0^1(\Omega), \tag{2.9}$$

$$\dot{v} = f_2 \quad \text{in} \quad H_0^1(\Omega), \tag{2.10}$$

$$\rho_1^{-1}(\beta_1 \Delta u - \alpha v - \mu_1 \dot{u} - \mu_2 z(\tau)) = f_3 \quad \text{in} \quad L^2(\Omega), \tag{2.11}$$

$$\rho_2^{-1}(\beta_2 \Delta v - \alpha u) = f_4 \quad \text{in} \quad L^2(\Omega), \tag{2.12}$$

$$-\partial_s z = f_5 \quad \text{in} \quad L^2(0, \tau; L^2(\Omega)). \tag{2.13}$$

Taking the expression

$$z(s) = \dot{u} - \int_0^s f_5(r)dr, \quad s \in [0, \tau], \tag{2.14}$$

it is solution of (2.13) and moreover we have $z(0) = \dot{u}$. Consequently, by Poincaré and Hölder inequalities, there exists $C > 0$ such that

$$\|z(\tau)\|^2_{L^2(\Omega)} \leq C\|F\|^2 \quad \text{and} \quad \int_0^\tau \|z(s)\|^2_{L^2(\Omega)} ds \leq C\|F\|^2. \tag{2.15}$$

Now, using (2.9) and (2.10), the equations (2.11)-(2.12) can be written as

$$\begin{cases} \beta_1 \Delta u - \alpha v = g_1, \\ \beta_2 \Delta v - \alpha u = g_2, \end{cases} \tag{2.16}$$

where $g_1 = \rho_1 f_3 + \mu_1 f_1 + \mu_2 z(\tau)$ and $g_2 = \rho_2 f_4$. Denoting with $W = (u, v)$, this system can be placed in a variational problem

$$a(W, \Phi) = \langle G, \Phi \rangle, \quad \forall\, \Phi = (\varphi, \psi) \in \left(H_0^1(\Omega)\right)^2,$$

where the sesquilinear form $a(\cdot,\cdot)$ and G are defined by

$$\begin{aligned} a(W, \Phi) &= \beta_1 \int_\Omega \nabla u \nabla \bar{\varphi} dx + \alpha \int_\Omega v \bar{\varphi} dx + \beta_2 \int_\Omega \nabla v \nabla \bar{\psi} dx + \alpha \int_\Omega u \bar{\psi} dx, \\ \langle G, \Phi \rangle &= -\int_\Omega g_1 \bar{\varphi} dx - \int_\Omega g_2 \bar{\psi} dx. \end{aligned}$$

From this definition we have that $a(\cdot,\cdot)$ is continuous and $G \in \left(L^2(\Omega)\right)^2$. Therefore, if we verify that $a(\cdot,\cdot)$ is coercive, by Lax-Milgram's Theorem and elliptic regularity we have the existence of strong solutions for (2.16). Consequently, we have a unique solution $U \in D(\mathbb{B})$ of system (2.9)-(2.13).

Coercivity of $a(\cdot,\cdot)$*:* condition (2.5) implies there exists $\theta \in (0, 1)$ such that $|\alpha| < \theta\gamma_1\sqrt{\beta_1\beta_2}$. Thus, applying Hölder, Poincaré and Young inequalities, we obtain

$$\begin{aligned} 2|\alpha| \int_\Omega |uv| dx &\leq 2|\alpha| \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 dx\right)^{\frac{1}{2}} \\ &\leq 2|\alpha| \frac{1}{\gamma_1} \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}} \\ &< 2\theta \left(\beta_1 \int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}} \left(\beta_2 \int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}} \\ &\leq \theta\beta_1 \left(\int_\Omega |\nabla u|^2 dx\right) + \theta\beta_2 \left(\int_\Omega |\nabla v|^2 dx\right). \end{aligned} \tag{2.17}$$

Using the above estimate we get

$$a(W,W) = \beta_1 \int_\Omega |\nabla u|^2 dx + \beta_2 \int_\Omega |\nabla v|^2 dx + 2\alpha \text{Re} \int_\Omega u\bar{v} dx$$
$$\geq (1-\theta)\beta_1 \int_\Omega |\nabla u|^2 dx + (1-\theta)\beta_2 \int_\Omega |\nabla v|^2 dx, \quad (2.18)$$

that is, $a(\cdot,\cdot)$ is coercive.

On the other hand, applying Young and Poincaré inequalities we have

$$|\langle G, W\rangle| \leq \epsilon \left\{ \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx \right\} + C(\epsilon) \left\{ \int_\Omega |g_1|^2 dx + \int_\Omega |g_2|^2 dx \right\}$$
$$\leq \epsilon \left\{ \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx \right\} + C(\epsilon)\|F\|^2.$$

Since $a(W,W) = \langle G, W\rangle$, computing ϵ sufficiently small in the above estimate and using inequality (2.18) we get

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx \leq C\|F\|^2.$$

From estimates (2.9), (2.10), and (2.15) imply that

$$\|U\| \leq C\|F\|,$$

that is, $0 \in \rho(\mathbb{B})$. Therefore, by Theorem 14, the operator $\mathbb{B}$ is the generator of a semigroup of contractions and therefore we say that the solution of abstract system (2.7) is given by $U(t) = e^{t\mathbb{B}}U_0, t \geq 0$. ■

Remark 2. *For $\mu_1 \leq \mu_2$ also there exists solution using an argument by perturbation on operator.*

2.3 Polynomial decay

In this section, we will see that the semigroup of system (2.2)-(2.4) for initial strong data decays polynomially when the time tends to infinity. The results will be obtained using a spectral characterization for polynomial decay due to Borichev and Tomilov enunciated in Theorem 16.

The main result of this section makes the relations between all coefficients of the system (2.2)-(2.4) and under certain conditions, it shows the decay rates.

Theorem 18. *Let α, χ_0 satisfying (2.5)-(2.6). If the coefficients μ_1, μ_2 satisfy (2.1), then the semigroup $e^{t\mathbb{B}}$ of the system (2.2)-(2.4) has the following asymptotic behavior:*

1. *If $\chi_0 = 0$, the semigroup decays polynomially with decay rate $t^{-1/2}$, that is, there exists $C > 0$ such that*

$$\|e^{t\mathbb{B}}U_0\| \leq \frac{C}{t^{1/2}}\|U_0\|_{D(\mathbb{B})}, \quad \forall\, t > 0,\ U_0 \in D(\mathbb{B}).$$

2. *If $\chi_0 \neq 0$, the semigroup decays polynomially with decay rate $t^{-1/4}$, that is, there exists $C > 0$ such that*

$$\|e^{t\mathbb{B}}U_0\| \leq \frac{C}{t^{1/4}}\|U_0\|_{D(\mathbb{B})}, \quad \forall\, t > 0,\ U_0 \in D(\mathbb{B}).$$

To prove this theorem, we need some prior estimates. In this sense, we will first show some lemmas.

Lemma 19. *Assume the hypothesis of Theorem 18. Suppose that for every $\lambda \in \mathbb{R}$ and $F \in \mathbb{X}$ there exists a solution $U = (u, v, \dot{u}, \dot{v}, z) \in D(\mathbb{B})$ of $(i\lambda I - \mathbb{B})U = F$. Then, there exists a positive constant C such that*

$$\int_\Omega |v|^2 dx \;\; \leq \;\; C\left[|\chi_0|\left(\int_\Omega |\dot{u}|^2 dx\right)^{1/2}\left(\int_\Omega |\dot{v}|^2 dx\right)^{1/2} + \int_\Omega |u|^2 dx + \|F\|\|U\|\right].$$

Proof. First, note that if $F = (f_1, f_2, f_3, f_4, f_5)$, then the components of the system $(i\lambda I - \mathbb{B})U = F$ satisfy

$$i\lambda u - \dot{u} = f_1, \tag{2.19}$$

$$i\lambda v - \dot{v} = f_2, \tag{2.20}$$

$$i\rho_1\lambda\dot{u} - \beta_1\Delta u + \alpha v + \mu_1\dot{u} + \mu_2 z(\tau) = \rho_1 f_3, \tag{2.21}$$

$$i\rho_2\lambda\dot{v} - \beta_2\Delta v + \alpha u = \rho_2 f_4, \tag{2.22}$$

$$i\lambda z(s) + \partial_s z = f_5. \tag{2.23}$$

In the remainder of this thesis, we will make some estimates for the solutions of this system using several constants. In this sense, C will denote a positive constant independent of the solutions and it assumes different values in different places at all times.

First, note that inequality (2.8) implies

$$\left(\frac{\mu_1 - \mu_2}{2}\right)\left[\int_\Omega |\dot{u}|^2 dx + \int_\Omega |z(\tau)|^2 dx\right] \leq C\mathrm{Re}\langle(i\lambda I - \mathbb{B})U, U\rangle \leq C\|F\|\|U\|. \tag{2.24}$$

Using this estimate together with (2.19), we get

$$\int_\Omega |\lambda u|^2 dx \leq C\left(\|F\|\|U\| + \|F\|^2\right). \tag{2.25}$$

Multiplying the equation (2.21) by $\bar{v}$, using (2.20), integrating by parts and taking the real part we obtain

$$\alpha\int_\Omega |v|^2dx = \text{Re}\left\{\rho_1\int_\Omega \dot{u}\bar{\dot{v}}dx + \rho_1\int_\Omega \dot{u}\bar{f}_2dx - \beta_1\int_\Omega \nabla u\nabla\bar{v}dx - \mu_1\int_\Omega \dot{u}\bar{v}dx \right.$$
$$\left. -\mu_2\int_\Omega z(\tau)\bar{v}dx + \rho_1\int_\Omega f_3\bar{v}dx\right\}. \quad (2.26)$$

In a similar way, multiplying equation (2.22) by $\bar{u}$, using (2.19), integrating by parts and taking the real part we have

$$\alpha\int_\Omega |u|^2dx = \text{Re}\left\{\rho_2\int_\Omega \dot{v}\bar{\dot{u}}dx + \rho_2\int_\Omega \dot{v}\bar{f}_1dx - \beta_2\int_\Omega \nabla v\nabla\bar{u}dx + \rho_2\int_\Omega f_4\bar{u}dx\right\}. \quad (2.27)$$

Dividing equation (2.26) by β_1, equation (2.27) by β_2 and performing the subtraction, we obtain

$$\begin{aligned}\frac{\alpha}{\beta_1}\int_\Omega |v|^2dx \;=\; & \frac{\alpha}{\beta_2}\int_\Omega |u|^2dx - \text{Re}\left\{\frac{\rho_2}{\beta_2}\int_\Omega \dot{v}\bar{\dot{u}}dx + \frac{\rho_2}{\beta_2}\int_\Omega \dot{v}\bar{f}_1dx - \int_\Omega \nabla v\nabla\bar{u}dx\right. \\ & +\frac{\rho_2}{\beta_2}\int_\Omega f_4\bar{u}dx - \frac{\rho_1}{\beta_1}\int_\Omega \dot{u}\bar{\dot{v}}dx - \frac{\rho_1}{\beta_1}\int_\Omega \dot{u}\bar{f}_2dx + \int_\Omega \nabla u\nabla\bar{v}dx \\ & \left.+\frac{\mu_1}{\beta_1}\int_\Omega \dot{u}\bar{v}dx + \frac{\mu_2}{\beta_1}\int_\Omega z(\tau)\bar{v}dx - \frac{\rho_1}{\beta_1}\int_\Omega f_3\bar{v}dx\right\}.\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}\frac{\alpha}{\beta_1}\int_\Omega |v|^2dx \;\leq\; & \frac{\alpha}{\beta_2}\int_\Omega |u|^2dx + \frac{\rho_1\rho_2|\chi_0|}{\beta_1\beta_2}\int_\Omega |\dot{u}\dot{v}|dx + C\|F\|\|U\| \\ & +\frac{\mu_1}{\beta_1}\int_\Omega |\dot{u}\bar{v}|dx + \frac{\mu_2}{\beta_1}\int_\Omega |z(\tau)\bar{v}|dx,\end{aligned}$$

where χ_0 is given by (2.6). Applying Young inequality and using (2.24), we have the inequality of this lemma, that is

$$\int_\Omega |v|^2dx \leq C\left[|\chi_0|\left(\int_\Omega |\dot{u}|^2dx\right)^{1/2}\left(\int_\Omega |\dot{v}|^2dx\right)^{1/2} + \int_\Omega |u|^2dx + \|F\|\|U\|\right].$$

■

Lemma 20. *Assume the conditions of lemma 19. Then, there exists a positive constant C such that*

$$\int_\Omega |\dot{v}|^2dx \leq C\left\{\left(\chi_0^2\lambda^4 + \lambda^2 + 1\right)\|F\|\|U\| + \|F\|^2\right\}.$$

Proof. Multiplying the inequality of Lemma 19 by λ^2 and using the estimate (2.25), we get

$$\int_\Omega |\lambda v|^2dx \leq C\left[|\chi_0|\lambda^2\left(\int_\Omega |\dot{u}|^2dx\right)^{1/2}\left(\int_\Omega |\dot{v}|^2dx\right)^{1/2} + (\lambda^2+1)\|F\|\|U\| + \|F\|^2\right] \quad (2.28)$$

On the other hand, from equation (2.20) we have

$$\int_\Omega |\dot{v}|^2 dx \;\; \leq \;\; 2\int_\Omega |\lambda v|^2 dx + 2\int_\Omega |f_2|^2 dx.$$

Using this inequality, the estimate (2.28) and applying Young inequality we obtain

$$\int_\Omega |\dot{v}|^2 dx \leq C\left(|\chi_0|\lambda^2\left(\int_\Omega |\dot{u}|^2 dx\right)^{1/2}\left(\int_\Omega |\dot{v}|^2 dx\right)^{1/2} + (\lambda^2+1)\|F\|\|U\| + \|F\|^2\right)$$
$$\leq C_\varepsilon\left(\chi_0^2\lambda^4\int_\Omega |\dot{u}|^2 dx + (\lambda^2+1)\|F\|\|U\| + \|F\|^2\right) + \varepsilon\left(\int_\Omega |\dot{v}|^2 dx\right).$$

Computing ε small enough and using estimate (2.24) we have the result of this lemma. ■

Proof of Theorem 18. We are going to use Theorem 16 to show this result. Let us see that $i\mathbb{R} \subset \rho(\mathbb{B})$. As $0 \in \rho(\mathbb{B})$, we consider the highest positive number λ_0 such that $]-i\lambda_0, i\lambda_0[\subset \rho(\mathbb{B})$. Then, $i\lambda_0 \in \sigma(\mathbb{B})$ or $-i\lambda_0 \in \sigma(\mathbb{B})$. Suppose that $i\lambda_0 \in \sigma(\mathbb{B})$ (similary if $-i\lambda_0 \in \sigma(\mathbb{B})$). Thus, there exists a sequence of positive real numbers (λ_n) such that $\lambda_n < \lambda_0$, with $\lambda_n \to \lambda_0$, and a sequence $U_n = (u_n, v_n, \dot{u}_n, \dot{v}_n, z_n) \in D(\mathbb{B})$ with $\|U_n\| = 1$ such that

$$\|(i\lambda_n - \mathbb{B})U_n\| = \|F_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

That is, if $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n}, f_{5n})$, then

$$i\lambda_n u_n - \dot{u}_n = f_{1n} \to 0 \quad \text{in } H_0^1(\Omega), \tag{2.29}$$

$$i\lambda_n v_n - \dot{v}_n = f_{2n} \to 0 \quad \text{in } H_0^1(\Omega), \tag{2.30}$$

$$i\rho_1\lambda_n\dot{u}_n - \beta_1\Delta u_n + \alpha v_n + \mu_1\dot{u}_n + \mu_2 z_n(\tau) = \rho_1 f_{3n} \to 0 \text{ in } L^2(\Omega), \tag{2.31}$$

$$i\rho_2\lambda_n\dot{v}_n - \beta_2\Delta v_n + \alpha u_n = \rho_2 f_{4n} \to 0 \text{ in } L^2(\Omega), \tag{2.32}$$

$$i\lambda_n z_n(s) + \partial_s z_n = f_{5n} \to 0 \quad \text{in } L^2(0,\tau; L^2(\Omega)). \tag{2.33}$$

Now, multiplying the equations (2.31) and (2.32) by $\bar{u}_n$ and $\bar{v}_n$ respectively, integrating by parts and summing the results, we obtain

$$\beta_1\int_\Omega |\nabla u_n|^2 dx + \beta_2\int_\Omega |\nabla v_n|^2 dx \leq 2|\alpha||\langle u_n, v_n\rangle| + \rho_1\int_\Omega |\dot{u}_n|^2 dx + \rho_1\int_\Omega |\dot{u}_n\bar{f}_{1n}| dx$$
$$+ \mu_1\int_\Omega |\dot{u}_n\bar{v}_n| dx + \mu_2\int_\Omega |z_n(\tau)\bar{u}_n| dx$$
$$+ \rho_2\int_\Omega |\dot{v}_n|^2 dx + \rho_2\int_\Omega |\dot{v}_n\bar{f}_{2n}| dx$$
$$+ \rho_1\int_\Omega |f_{3n}\bar{u}_n| dx + \rho_2\int_\Omega |f_{4n}\bar{v}_n| dx,$$

and, by a similar argument used in (2.17), we have

$$(1-\theta)\beta_1\int_\Omega|\nabla u_n|^2dx+(1-\theta)\beta_2\int_\Omega|\nabla v_n|^2dx\leq\rho_1\int_\Omega|\dot{u}_n|^2dx+\mu_1\int_\Omega|\dot{u}_n\bar{u}_n|dx$$
$$+\mu_2\int_\Omega|z_n(\tau)\bar{u}_n|dx+\rho_2\int_\Omega|\dot{v}_n|^2dx+C\|F_n\|\|U_n\|.$$

Applying Young and Poincaré inequalities, we get

$$\begin{aligned}(1-\theta-\epsilon C_p)&\left\{\beta_1\int_\Omega|\nabla u_n|^2dx+\beta_2\int_\Omega|\nabla v_n|^2dx\right\}\\ &\leq C(\epsilon)\left\{\int_\Omega|\dot{u}_n|^2dx+\int_\Omega|z_n(\tau)|^2dx+\int_\Omega|\dot{v}_n|^2dx+\|F_n\|\|U_n\|\right\},\end{aligned}\tag{2.34}$$

where C_p is the Poincaré constant and ϵ is positive small number. Here, we fix ϵ such that $(1-\theta-\epsilon C_p)>0$.

On the other hand, the inequality (2.24) implies that

$$\left[\int_\Omega|\dot{u}_n|^2dx+\int_\Omega|z_n(\tau)|^2dx\right]\leq C\|F_n\|\|U_n\|\to 0,\tag{2.35}$$

and from Lemma 20, we obtain

$$\int_\Omega|\dot{v}_n|^2dx\leq C\left\{\left(\chi_0^2\lambda_n^4+\lambda_n^2+1\right)\|F_n\|\|U_n\|+\|F_n\|^2\right\}\to 0.\tag{2.36}$$

Thus, using (2.35) and (2.36) in (2.34), follows that

$$\beta_1\int_\Omega|\nabla u_n|^2dx+\beta_2\int_\Omega|\nabla v_n|^2dx\to 0.\tag{2.37}$$

Finally, solving (2.33) and taking into account $z_n(0)=\dot{u}_n$, we have

$$z_n(s)=\dot{u}_ne^{-i\lambda_n s}+\int_0^s f_{5n}(\sigma)e^{i\lambda_n(\sigma-s)}d\sigma,$$

which implies

$$\int_0^\tau\|z_n(s)\|^2_{L^2(\Omega)}\,ds\leq C\left(\int_\Omega|\dot{u}_n|^2dx+\|f_{5n}\|^2_{L^2(0,\tau;L^2(\Omega))}\right)\to 0.\tag{2.38}$$

Therefore, from estimates (2.35)-(2.38) imply that $\|U_n\|\to 0$, but this is absurd because $\|U_n\|=1$ for all $n\in\mathbb{N}$. Consequently, $i\mathbb{R}\subset\rho(\mathbb{B})$.

Now, consider $U=(u,v,\dot{u},\dot{v},z)$ the solution of the system $(i\lambda-\mathbb{B})U=F$. According to Theorem 16, to show the polynomial decay of the semigroup $e^{t\mathbb{B}}$ it is sufficient to prove that $\|U\|\leq C\lambda^\theta\|F\|$ for $|\lambda|\geq 1$, where $\theta=2$ if $\chi_0=0$, and $\theta=4$ if $\chi_0\neq 0$.

Proof of item 1: Assume that $\chi_0=0$. The inequality in Lemma 20 becomes

$$\int_\Omega|\dot{v}|^2dx\leq C\left\{\left(\lambda^2+1\right)\|F\|\|U\|+\|F\|^2\right\}.\tag{2.39}$$

Using this estimate, estimate (2.24) and proceeding as in (2.34) we have the inequality

$$\beta_1 \int_\Omega |\nabla u|^2 dx + \beta_2 \int_\Omega |\nabla v|^2 dx \quad \leq \quad C\left\{\left(\lambda^2+1\right)\|F\|\|U\| + \|F\|^2\right\}. \qquad (2.40)$$

And also, proceeding as in (2.38) and using (2.24), we obtain

$$\begin{aligned} \int_0^\tau \|z(s)\|^2_{L^2(\Omega)}\, ds \quad &\leq \quad C\left(\int_\Omega |\dot{u}|^2 dx + \|f_5\|^2_{L^2(0,\tau;L^2(\Omega))}\right) \\ &\leq \quad C\left(\|F\|\|U\| + \|F\|^2\right). \qquad (2.41) \end{aligned}$$

Thus, from estimates (2.24), (2.39), (2.40) and (2.41), we get

$$\|U\|^2 \leq C\left\{\lambda^2\|F\|\|U\| + \|F\|^2\right\},$$

for $|\lambda| \geq 1$. Applying Young inequality to the first term on the right side of this inequality, we obtain

$$\|U\| \leq C\lambda^2\|F\|,$$

that means, $\lambda^{-2}\|(i\lambda I - \mathbb{B})^{-1}\|$ is bounded. Then, by Theorem 16 the semigroup $e^{t\mathbb{B}}$ decays polynomially with the rate $t^{-1/2}$.

Proof of item 2: Assume that $\chi_0 \neq 0$. From Lemma 20, we have

$$\int_\Omega |\dot{v}|^2 dx \leq C\left\{\left(\chi_0^2\lambda^4 + \lambda^2 + 1\right)\|F\|\|U\| + \|F\|^2\right\}. \qquad (2.42)$$

Using this inequality instead (2.39), similarly to the previous case, we obtain

$$\|U\| \leq C\lambda^4\|F\|, \qquad (2.43)$$

for $|\lambda| \geq 1$. Therefore, $\lambda^{-4}\|(i\lambda I - \mathbb{B})^{-1}\|$ is bounded. Then, by Theorem 16 implies that the semigroup $e^{t\mathbb{B}}$ decays polynomially with the rate $t^{-1/4}$. This completes the proof of this Theorem. ∎

2.4 Optimality of the decay rates

We will see that the polynomial decay rates found in previous section are the best. The main result is given in the following theorem:

Theorem 21. *The polynomial decay rates found in Theorem 18 are optimal, that is:*

1. *If $\chi_0 = 0$, the semigroup does not decay with the rate $t^{-\sigma}$, for $\sigma > 1/2$.*

2. *If $\chi_0 \neq 0$, the semigroup does not decay with the rate $t^{-\sigma}$, for $\sigma > 1/4$.*

Proof. Since $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is a positive self-adjoint operator with compact resolvent, its spectrum is constituted by positive eigenvalues γ_n, $n \in \mathbb{N}$, with $\gamma_n \to \infty$. Let us denote by (e_n) the corresponding unitary eigenvectors, that is

$$-\Delta e_n = \gamma_n e_n, \quad \|e_n\| = 1, \quad n \in \mathbb{N}. \tag{2.44}$$

If we take $F_n = (0, 0, 0, \rho_2^{-1} e_n, 0)$, the solution of system $(i\lambda - \mathbb{B})U = F_n$ satisfies

$$\begin{aligned} i\lambda u - \dot{u} &= 0, \\ i\lambda v - \dot{v} &= 0, \\ i\rho_1 \lambda \dot{u} - \beta_1 \Delta u + \alpha v + \mu_1 \dot{u} + \mu_2 z(\tau) &= 0, \\ i\rho_2 \lambda \dot{v} - \beta_2 \Delta v + \alpha u &= -e_n, \\ i\lambda z(s) + \partial_s z &= 0. \end{aligned}$$

We look for solutions of the form: $u = \kappa_1 e_n$, $v = \kappa_2 e_n$ and $z(s) = \eta(s) e_n$ with κ_1, κ_2 complex numbers and η a complex function defined in $[0, \tau]$. Then, replacing these terms in the above equations gives

$$\rho_1 \lambda^2 \kappa_1 - \beta_1 \gamma_n \kappa_1 - \alpha \kappa_2 - i\mu_1 \lambda \kappa_1 - \mu_2 \eta(\tau) = 0, \tag{2.45}$$

$$\rho_2 \lambda^2 \kappa_2 - \beta_2 \gamma_n \kappa_2 - \alpha \kappa_1 = 1, \tag{2.46}$$

$$i\lambda \eta(s) + \eta'(s) = 0. \tag{2.47}$$

Solving the ordinary differential equation (2.47) and taking into account $\eta(0) = i\lambda\kappa_1$, we have

$$\eta(s) = i\lambda \kappa_1 e^{-i\lambda s}.$$

Replacing this term in (2.45), the algebraic system (2.45)-(2.46) become

$$\begin{aligned} \rho_1 \lambda^2 \kappa_1 - \beta_1 \gamma_n \kappa_1 - \alpha \kappa_2 - i\mu_1 \lambda \kappa_1 - i\mu_2 \lambda \kappa_1 e^{-i\lambda\tau} &= 0, \\ \rho_2 \lambda^2 \kappa_2 - \beta_2 \gamma_n \kappa_2 - \alpha \kappa_1 &= 1. \end{aligned}$$

Solving this system in κ_2 gives

$$\kappa_2 = \frac{G_1(\lambda^2) - i\lambda(\mu_1 + \mu_2 e^{-i\lambda\tau})}{G_1(\lambda^2) G_2(\lambda^2) - \alpha^2 - i\lambda G_2(\lambda^2)(\mu_1 + \mu_2 e^{-i\lambda\tau})}, \tag{2.48}$$

where G_1 and G_2 are polynomials given by

$$G_1(y) = \rho_1 y - \beta_1 \gamma_n, \quad G_2(y) = \rho_2 y - \beta_2 \gamma_n.$$

Proof of item 1: Assume that $\chi_0 = 0$. We consider the numbers

$$\lambda_n = \sqrt{\frac{\beta_2\gamma_n}{\rho_2}}, \quad n \in \mathbb{N}.$$

Then, we have $G_1(\lambda_n^2) = G_2(\lambda_n^2) = 0$. With this conditions, taking $\lambda = \lambda_n$ in (2.48) the term $\kappa_2 = \kappa_2(n)$ becomes

$$\kappa_2(n) = \frac{i\lambda_n(\mu_1 + \mu_2 e^{-i\lambda_n\tau})}{\alpha^2}.$$

Therefore,

$$|\kappa_2(n)| \geq \text{Im}\big(\kappa_2(n)\big) \geq \frac{\mu_1 - \mu_2}{\alpha^2}\lambda_n.$$

Consequently, if we denote by $U_n = (u_n, v_n, \dot{u}_n, \dot{v}_n, z_n)$ the solution of system $(i\lambda_n I - \mathbb{B})U = F_n$ satisfies

$$\|U_n\| \geq \sqrt{\rho_2}\left(\int_\Omega |\dot{v}_n|^2 dx\right)^{\frac{1}{2}} = \sqrt{\rho_2}\left(\int_\Omega |\lambda_n v_n|^2 dx\right)^{\frac{1}{2}} = \sqrt{\rho_2}\lambda_n|\kappa_2(n)| \geq \sqrt{\rho_2}\frac{\mu_1 - \mu_2}{\alpha^2}\lambda_n^2.$$

Now, if we suppose that the semigroup decays with the rate $t^{-\sigma}$ for $\sigma > \frac{1}{2}$, from Theorem 16 we have $\lambda_n^{-\frac{1}{\sigma}}\|U_n\|$ is bounded. However, from the above inequality we obtain

$$\lambda_n^{-\frac{1}{\sigma}}\|U_n\| \geq \frac{\mu_1 - \mu_2}{\alpha^2}\sqrt{\rho_2}\lambda_n^{2-\frac{1}{\sigma}} \longrightarrow \infty \quad \text{when} \quad n \to \infty,$$

which is absurd. This proves the first part of this theorem.

Proof of item 2: Assume that $\chi_0 \neq 0$. Note that the coefficient (2.48) can be written as

$$\kappa_2 = \frac{G_1(\lambda^2) - i\lambda(\mu_1 + \mu_2 e^{-i\lambda\tau})}{G_0(\lambda^2) - i\lambda G_2(\lambda^2)(\mu_1 + \mu_2 e^{-i\lambda\tau})}, \tag{2.49}$$

where G_0 is given by

$$G_0(y) := G_1(y)G_2(y) - \alpha^2.$$

The roots of this polynomial are

$$y_n^{\pm} := \frac{\beta_M\gamma_n}{2} \pm \sqrt{\left(\frac{\chi_0\gamma_n}{2}\right)^2 + \alpha_1\alpha_2},$$

where

$$\beta_M := \frac{\beta_1}{\rho_1} + \frac{\beta_2}{\rho_2}, \quad \alpha_1 := \frac{\alpha}{\rho_1}, \quad \alpha_2 := \frac{\alpha}{\rho_2}.$$

In view of $\chi_0 \neq 0$, we firstly consider $\chi_0 > 0$. In this case we computing the numbers $\lambda_n = \sqrt{y_n^-}$, $n \in \mathbb{N}$, then

$$\lambda_n^2 = \frac{\beta_M \gamma_n}{2} - \sqrt{\left(\frac{\chi_0 \gamma_n}{2}\right)^2 + \alpha_1 \alpha_2}. \tag{2.50}$$

Consequently, $G_0(\lambda_n^2) = 0$ and $G_2(\lambda_n^2) = \alpha^2 / G_1(\lambda_n^2)$. Taking $\lambda = \lambda_n$ in (2.49), the coefficient $\kappa_2 = \kappa_2(n)$ becomes

$$\begin{aligned} \kappa_2(n) &= \frac{G_1(\lambda_n^2) - i\lambda_n(\mu_1 + \mu_2 e^{-i\lambda_n \tau})}{-i\lambda_n G_2(\lambda_n^2)(\mu_1 + \mu_2 e^{-i\lambda_n \tau})} \\ &= \frac{G_1(\lambda_n^2)}{\alpha^2} + i \frac{G_1^2(\lambda_n^2)}{\lambda_n \alpha^2 (\mu_1 + \mu_2 e^{-i\lambda_n \tau})}. \end{aligned} \tag{2.51}$$

Moreover, from the definition of G_1 and χ_0, we have

$$\frac{1}{\rho_1} G_1(\lambda_n^2) = \frac{-\chi_0 \gamma_n - \sqrt{(\chi_0 \gamma_n)^2 + 4\alpha_1 \alpha_2}}{2}.$$

In what follows, the notation $a_n \approx b_n$ will mean that $\lim\limits_{n \to +\infty} \frac{|a_n|}{|b_n|}$ is a positive real number. Thus, from (2.50) and above equality we have $\lambda_n \approx \gamma_n^{1/2}$ and $G_1(\lambda_n^2) \approx \gamma_n \approx \lambda_n^2$. Then, using these estimates in (2.51), we obtain

$$|\kappa_2(n)| \geq \text{Im}\big(\kappa_2(n)\big) \geq \delta \lambda_n^3, \tag{2.52}$$

for some $\delta > 0$ small and for n large.

Therefore, if we denote by $U_n = (u_n, v_n, \dot{u}_n, \dot{v}_n, z_n)$ the solution of the system $(i\lambda_n I - \mathbb{B})U = F_n$ we get

$$\|U_n\| \quad \geq \quad \delta \sqrt{\rho_2} \lambda_n^4,$$

for n large. Consequently, if we suppose that the semigroup decays with rate $t^{-\sigma}$ with $\sigma > \frac{1}{4}$, from Theorem 16 we have that $\lambda_n^{-\frac{1}{\sigma}} \|U_n\|$ is bounded. However, using the above estimate, we get

$$\lambda_n^{-\frac{1}{\sigma}} \|U_n\| \geq \delta \rho_2 \lambda_n^{4 - \frac{1}{\sigma}} \to \infty \quad \text{when} \quad n \to \infty,$$

which is absurd. Therefore, the decay rate $t^{-1/4}$ is optimal.

If we consider $\chi_0 < 0$ instead of $\chi_0 > 0$ is suffices to consider the sequence $\lambda_n = \sqrt{y_n^+}$ instead of $\sqrt{y_n^-}$. With this sequence, we similarly obtain estimates: $\lambda_n \approx \gamma_n^{1/2}$, $G_1(\lambda_n^2) \approx \lambda_n^2$ and (2.52). Therefore, we obtain the same result for this case. This completes the proof of this theorem. ■

2.5 Asymptotic Instability

In this section, we will see that if the condition (2.1) does not hold, then the system (2.2)-(2.4), for suitable time delays, have solutions that do not decay to zero. This result is enunciated as follows:

Theorem 22. *If $\mu_1 \leq \mu_2$, then there exists a sequence of time delays (arbitrarily small or large) and a sequence of initial data such that the solutions of abstract system do not tend to zero.*

Proof. Let us see that there exist suitable real numbers λ such that $i\lambda$ is eigenvalue of the operator $\mathbb{B}$. In fact, the vector $U = (u, v, \dot{u}, \dot{v}, z)$ is a solution of $(i\lambda I - \mathbb{B})U = 0$ if it satisfies

$$\begin{aligned} i\lambda u - \dot{u} &= 0, \\ i\lambda v - \dot{v} &= 0, \\ i\rho_1 \lambda \dot{u} - \beta_1 \Delta u + \alpha v + \mu_1 \dot{u} + \mu_2 z(\tau) &= 0, \\ i\rho_2 \lambda \dot{v} - \beta_2 \Delta v + \alpha u &= 0, \\ i\lambda z(s) + \partial_s z &= 0, \end{aligned}$$

Solving the last equation we have $z(\tau) = \dot{u} e^{-i\lambda\tau}$. Substituting this term and using the first two equations, the third and fourth equations become

$$\begin{aligned} \rho_1 \lambda^2 u + \beta_1 \Delta u - \alpha v - i\lambda \mu_1 u - i\lambda \mu_2 e^{-i\lambda\tau} u &= 0, \\ \rho_2 \lambda^2 v + \beta_2 \Delta v - \alpha u &= 0. \end{aligned}$$

In this point, let us see what conditions we must impose on the complex numbers κ_1, κ_2 so that $u = \kappa_1 e_n$ and $v = \kappa_2 e_n$ are solutions of this system. Here e_n, $n \in \mathbb{N}$, are the eigenvectors (2.44). Thus, κ_1, κ_2 must satisfy the algebraic linear system

$$\begin{bmatrix} \rho_1 \lambda^2 - \beta_1 \gamma_n - i\lambda(\mu_1 + \mu_2 e^{-i\lambda\tau}) & -\alpha \\ -\alpha & \rho_2 \lambda^2 - \beta_2 \gamma_n \end{bmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.53}$$

This system admits no null solutions if the determinant of the above matrix is zero, that is

$$\left[\rho_1 \lambda^2 - \beta_1 \gamma_n - i\lambda(\mu_1 + \mu_2 e^{-i\lambda\tau})\right] \left(\rho_2 \lambda^2 - \beta_2 \gamma_n\right) - \alpha^2 = 0.$$

Separating the real and the imaginary parts of this equation, we have

$$-\lambda \left(\mu_1 + \mu_2 \cos(\lambda\tau)\right) \left(\rho_2 \lambda^2 - \beta_2 \gamma_n\right) = 0, \tag{2.54}$$

$$\left(\rho_1 \lambda^2 - \beta_1 \gamma_n - \lambda \mu_2 \sin(\lambda\tau)\right) \left(\rho_2 \lambda^2 - \beta_2 \gamma_n\right) = \alpha^2. \tag{2.55}$$

Now, our objective is to ensure the existence of suitable real numbers λ and time delays τ such that these equations are satisfied. At this point, we choose these numbers so that

$$\cos(\lambda\tau) = -\frac{\mu_1}{\mu_2}. \tag{2.56}$$

Note that equation (2.56) is possible because $0 < \mu_1 \leq \mu_2$. In this case the equation (2.54) is satisfied. This condition also implies that $\sin(\lambda\tau)$ is one of the following numbers

$$\pm\sqrt{1-\left(\frac{\mu_1}{\mu_2}\right)^2}.$$

Replacing this term in equation (2.55), it becomes

$$\left\{\rho_1\lambda^2 - \beta_1\gamma_n - \lambda\mu_2\left(\pm\sqrt{1-\left(\frac{\mu_1}{\mu_2}\right)^2}\right)\right\}\left(\rho_2\lambda^2 - \beta_2\gamma_n\right) = \alpha^2.$$

Note that

$$h(\lambda) := \left\{\rho_1\lambda^2 - \beta_1\gamma_n - \lambda\mu_2\left(\pm\sqrt{1-\left(\frac{\mu_1}{\mu_2}\right)^2}\right)\right\}\left(\rho_2\lambda^2 - \beta_2\gamma_n\right)$$

is a continuous function satisfying $h\left(\sqrt{\beta_2\gamma_n/\rho_2}\right) = 0$ and $\lim_{\lambda\to\infty} h(\lambda) = \infty$. Therefore, there exists $\lambda_n \in (\sqrt{\beta_2\gamma_n/\rho_2}, \infty)$, such that $h(\lambda_n) = \alpha^2$.

Finally, computing $\lambda = \lambda_n$ in equation (2.56) we have for $\tau = \tau_{n,\ell}$

$$\tau_{n,l} = \frac{\arccos\left(-\frac{\mu_1}{\mu_2}\right) + 2l\pi}{\lambda_n}, \quad n, l \in \mathbb{N}. \tag{2.57}$$

Denoting with $k_1(n,\ell)$, $k_2(n,\ell)$ any nontrivial solution of (2.53) when $\lambda = \lambda_n$ and $\tau = \tau_{n,\ell}$, the vectors $U_{n,l} = (u_{n,l}, v_{n,l}, \dot{u}_{n,l}, \dot{v}_{n,l}, z_{n,l})$ with components

$$u_{n,l} = k_1(n,\ell)e_n, \quad v_{n,l} = k_2(n,\ell)e_n, \quad \dot{u}_{n,l} = i\lambda_n k_1(n,\ell)e_n,$$
$$\dot{v}_{n,l} = i\lambda_n k_2(n,\ell)e_n, \quad z_{n,l}(s) = i\lambda_n e^{-i\lambda s}k_1(n,\ell)e_n,$$

are solutions of the system $(i\lambda_n I - \mathbb{B})U = 0$. Therefore, the functions $U_{n,l}(t) := e^{i\lambda_n t}U_{n,l}$ satisfy

$$\frac{d}{dt}U(t) = \mathbb{B}U(t), \quad U(0) = U_{n,l}.$$

On the other hand, since

$$\|U_{n,l}(t)\| = \|e^{i\lambda_n t}U_{n,l}\| = \|U_{n,l}\|, \quad \forall\, t \geq 0,$$

we conclude that $\|U_{n,l}(t)\| \not\to 0$ when $t \to \infty$. Moreover, since $\lambda_n \to \infty$ when $n \to \infty$, the previous expression for $\tau_{n,l}$ will be arbitrarily small. However, if $l \to \infty$ then we have $\tau_{n,l}$ will be arbitrarily large.

■

Chapter 3

Decay rates for a weakly coupled system with indirect damping given by fractional memory effects

This chapter deals with the asymptotic behavior of a weakly coupled system of two equations which one of them has a dissipative mechanism given by a memory term. This term depends on the fractional operator with exponent $\theta \in [0, 1]$. We show that strong solutions of the system decay polynomially with a rate that depends on both the exponent θ and wave propagation speeds. Optimal decay rates are found and the results show a surprising aspect: more regular damping does not necessarily imply a faster decay.

3.1 Motivation

In the last decades, a variety of viscoelastic systems have caught the attention of many researchers. These systems model the evolution of various problems in applied sciences and studies on the existence of solutions and their asymptotic behavior have been published.

Some of those systems model vibrations of viscoelastic materials that, when placed in an abstract way, they are formulated by the following equation

$$u_{tt} + Au - \int_0^\infty g(s)Bu(t-s)ds = 0. \tag{3.1}$$

Here A and B are unbounded operators. Dafermos [18] was one of the first researchers that studied the stabilizing properties of this equation. He considered A a linear positive self-adjoint operator in a separable Hilbert space and B an operator with less or equal regularity than A. He showed that the solutions to this problem are asymptotically stable provided the kernel g in the memory term is a positive non increasing continuous function.

Since then, many researchers have focused on establishing the best decay rates of solutions for this problem. When both operators A and B coincide, the decay rates depend on how fast the kernel decays, for example, kernels exponentially decreasing (or polynomially decreasing) lead to an exponential (or polynomial) decay of solutions. These results, for example, can be found in the following papers [17, 23, 46, 49, 50, 63]. How is the asymptotic behavior of solutions if the operators B has less regularity than A?

Rivera and Naso [51] studied the equation (3.1) for $B = A^\theta$ where A is a positive self-adjoint operator and the exponent θ stands in interval $[0, 1)$. For kernels g exponentially decreasing, they proved that the semigroup does not decay exponentially. However, they showed that it decays polynomially with almost optimal rate $\ln(t)[\ln(t)/t]^{1/(2-2\theta)}$. A similar problem of this system applied to plate equations with rotational inertia term was studied in [62]. Here, an optimal polynomial decay rate was obtained. Recently, Hao and Wei [34] studied the system considered by Rivera and Naso but with short memory instead the long memory. For kernels satisfying the condition $g'(t) \leq -H(g(t))$, where H is a C^1 positive increasing convex function and $H(0) = 0$, they showed that the energy of this system decay with the rate $s(t)$ where s is the solution of the ODE $s_t + c_0 H(c_1 s^{\frac{1}{\gamma}}) = 0$; c_0, c_1 are positive constants and $\gamma \in]0, 1]$ (see also [41]).

Inspired by the research of Rivera and Naso mentioned above in the scope of the problems with indirect damping, we are interested in study the asymptotic behavior of solutions for the following abstract system

$$\rho_1 u_{tt} + \beta_1 \mathrm{A} u - \int_0^\infty g(s) \mathrm{A}^\theta u(t-s) ds + \alpha(u - v) = 0, \tag{3.2}$$

$$\rho_2 v_{tt} + \beta_2 \mathrm{A} v + \alpha(v - u) = 0, \tag{3.3}$$

subjected to initial conditions

$$u(0) = u_0,\ v(0) = v_0,\ u_t(0) = u_1,\ v_t(0) = v_1,\ u(-s) = \phi(s),\ s > 0,\ \phi(0) = u_1. \tag{3.4}$$

Here the coefficients $\rho_1,\ \rho_2,\ \beta_1,\ \beta_2$ and α are positive constants, the parameter θ is considered in the interval $[0, 1]$ and A is a positive self-adjoint operator with compact inverse on a complex Hilbert space $\mathbb{H}$.

This kind of problem was introduced by Russell in [66] which he called “Problems with indirect damping”. In this case, the memory damping of the first equation of (3.2)-(3.3) acts indirectly in the second through the coupling term. Thus, the vibrations of the conservative part absorb the dissipative properties of the damped equation leading to the possibility of a global stabilization. But the technical difficulties for getting appropriate estimates for this system are not simple; they can be difficult and exhausting.

Doing a search in the literature, we found some studies on the asymptotic behavior of this type of systems which we mention below.

In [45], Matos, Almeida and Santos considered the general abstract system

$$\begin{cases} u_{tt} + A_1 u - \displaystyle\int_0^\infty g(s) A_2 u(t-s)ds + Bv = 0, \\ v_{tt} + A_3 v + Bu = 0, \end{cases} \tag{3.5}$$

with the operator A_2 and A_3 less regular than A_1 and $B = \alpha I$. Considering A_1 a positive self-adjoint operator and the kernel g with exponential decrease, they showed that the solutions can not decay exponentially. However, they exhibited a polynomial decay of the energy with the rate t^{-1} for initial data with additional regularity. The optimality of this result was not discussed.

In [31], Guesmia studied the above system considering $A_3 = A_1$, A_2 less regular than A_1, and B being a bounded linear operator. Considering a boundedness condition on the past history data was found decay rates for kernels that decrease slower than the exponential or polynomial ones. These decay rates depend on the growth of the kernel at infinity, the regularity of the initial data, and some relations between the involved operators.

In 2019, Jin et al [37] studied the above system considering A_3 equal or more regular than A_1 and A_2 with the same regularity than A_1. They showed that the energy of system decays to zero at least with the rate t^{-1} provided the kernel is non-increasing, integrable, and satisfy some conditions linked to the initial data.

In [16], Cavalcanti et al. considered the above system adding a frictional damping Du_t to the first equation. Thus, computing advantage of the two dissipative terms, in a local way, they obtained a result of “weak” stability where the decay rate is given in terms of the general growth of the kernel at infinity and the regularity of the initial data.

Other works about the stability of solutions for similar systems can be found in [5, 29, 36, 76], and for nonlinear systems with memory terms in both equations can be found in [9, 43, 67].

Our main goal here is not only to find some decay rate for the solutions of the system (3.2)-(3.4), but also to find the best checking if possible their optimality. The decay rates will must certainly depend on the parameters ρ_1, ρ_2, β_1, β_2, α and θ. The motivation to consider these parameters in the abstract system is because we want this system to represent vibrations of membranes or plates. This is the case when we consider for example $A = -\Delta$ defined in $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, or $A = \Delta^2$ defined in $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$. When A is the Laplacian operator $-\Delta$, the system (3.2)-(3.4) models a system of two elastic waves with propagation speeds $\sqrt{\frac{\beta_1}{\rho_1}}$ and $\sqrt{\frac{\beta_2}{\rho_2}}$. Thus, the difference of wave propagation speeds given by

$$\chi_0 = \frac{\beta_1}{\rho_1} - \frac{\beta_2}{\rho_2}, \tag{3.6}$$

as in chapter 2, that will play an important role in this process. This term has been decisive in the type of decay rates in other dissipative systems, some of the researches that show these facts can be seen in [61, 47, 70, 52].

In order to obtain exact decay rates for the system (3.2)-(3.4) we will assume, as in [51], the following conditions for the memory kernel

$$\begin{cases} g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \\ g(s) \geqslant 0, \ g'(s) < 0, \forall s \in \mathbb{R}^+, \\ \exists\, c_1 \in \mathbb{R}_*^+; \ g'(s) \leq -c_1 g(s), \forall s \in \mathbb{R}^+, \\ 0 < \kappa := \displaystyle\int_0^\infty g(s)ds < \beta_1 \alpha_1^{1-\theta}, \end{cases} \tag{3.7}$$

where α_1 is the first eigenvalue of operator A.

The main result of this chapter is related to asymptotic behavior of solutions of system (3.2)-(3.4). The results that we find are enunciated in Theorems 24-25 and 38 which synthesizing say:

- For strong initial data and $\chi_0 = 0$ we have the polynomial decay with rate $t^{-\frac{1}{2-2\theta}}$ for $0 \leq \theta \leq \frac{1}{2}$;
- For strong initial data and $\chi_0 = 0$ we have the polynomial decay with rate $t^{-\frac{1}{2\theta}}$ for $\frac{1}{2} \leq \theta \leq 1$;
- For strong initial data and $\chi_0 \neq 0$ we have the polynomial decay with rate $t^{-\frac{1}{6-2\theta}}$;
- The decay rates in the previous items are the best.

Note that if we compare this results with the uncoupled equation studied in [51] where was obtained the rate $t^{-\frac{1}{2-2\theta}}$, we have a loss of speed in the decay of solutions, but these rates are optimal. This loss is due to weak coupling of the system.

In what follows, our results are organized in the following way: in section 3.2, we prove well-posedness of problem (3.2)-(3.4) by semigroup theory. In section 3.3, we show polynomial stability of this problem. Explicit decay rates are found. Finally, in section 3.4, we prove the optimality of decay rates that were found in the previous section.

3.2 Well-Posedness

In this section, we will prove that system (3.2)-(3.4) is well-posed using semigroup theory. As mentioned in introduction we consider A a positive self-adjoint operator with compact inverse on a complex Hilbert space $\mathbb{H}$. The domain of this operator is denoted by $\mathcal{D}(A)$ and the spaces $\mathcal{D}(A^\theta), \theta \leq 1$, are endowed with usual inner product

$$\langle u, v\rangle_{\mathcal{D}(A^\theta)} = \langle A^\theta u, A^\theta v\rangle,$$

where $\langle\cdot,\cdot\rangle$ on the right denotes the inner product in $\mathbb{H}$. Besides, we have the continuous embedding

$$\mathcal{D}(A^{\theta_1}) \hookrightarrow \mathcal{D}(A^{\theta_2}), \text{ for } \theta_1 \geq \theta_2.$$

To consider the system (3.2)-(3.4) in an abstract framework, we introduce the function $\eta := \eta^t(s)$ defined as

$$\eta^t(s) = u(t) - u(t-s),$$

originally used by Dafermos in [18]. Thus, the system (3.2)-(3.3) turn into

$$\begin{cases} \rho_1 u_{tt} + A_0 u + \displaystyle\int_0^\infty g(s) A^\theta \eta^t(s) ds + \alpha(u - v) = 0, \\ \rho_2 v_{tt} + \beta_2 A v + \alpha(v - u) = 0, \\ \eta_t^t(s) + \eta_s^t(s) - u_t = 0, \end{cases} \tag{3.8}$$

where $A_0 := \beta_1 A - \kappa A^\theta$ for $\theta \in [0,1]$ and κ is defined in (3.7). From (3.4), the initial data for this system are

$$u(0) = u_0,\ v(0) = v_0,\ u_t(0) = u_1,\ v_t(0) = v_1,\ \eta^0(s) = \eta_0(s) := u_0 - \phi(s),\ s > 0. \tag{3.9}$$

Note that, as α_1 is the first eigenvalue of A, follows that α_1^γ is the first eigenvalue of A^γ, and consequently, $\|A^{\frac{\gamma}{2}}u\|^2 \geq \alpha_1^\gamma \|u\|^2$. For this, using this estimate we obtain, for $u \in D(A^{\frac{1}{2}})$:

$$\begin{aligned}
\|A_0^{\frac{1}{2}}u\|^2 &= \beta_1\|A^{\frac{1}{2}}u\|^2 - \kappa\|A^{\frac{\theta}{2}}u\|^2 \\
&\geq \beta_1\|A^{\frac{1}{2}}u\|^2 - \kappa\alpha_1^{1-\theta}\|A^{\frac{1-\theta}{2}}(A^{\frac{\theta}{2}}u)\|^2 \\
&= (\beta_1 - \kappa\alpha_1^{\theta-1})\|A^{\frac{1}{2}}u\|^2. \qquad (3.10)
\end{aligned}$$

The condition (3.7) and the above estimate implies that the norms $\|A_0^{\frac{1}{2}}u\|$ and $\|A^{\frac{1}{2}}u\|$ are equivalents in the space $D(A^{\frac{1}{2}})$. With these considerations, if we consider the vector $U(t) = (u(t), v(t), u_t(t), v_t(t), \eta)$, the coupled system (3.8)-(3.9) can be written as

$$\frac{d}{dt}U(t) = \mathbb{B}U(t), \quad U(0) = U_0, \qquad (3.11)$$

where $U_0 = (u_0, v_0, u_1, v_1, \eta_0)$ and operator $\mathbb{B}$ is defined by

$$\mathbb{B}U = \Big(\dot{u},\ \dot{v},\ -\rho_1^{-1}\left\{A_0u + \mathbb{D}\eta + \alpha(u-v)\right\},\ -\rho_2^{-1}\left\{\beta_2Av + \alpha(v-u)\right\}, \dot{u} - \partial_s\eta\Big),$$

for $U = (u, v, \dot{u}, \dot{v}, \eta)$ and $\mathbb{D}\eta = \displaystyle\int_0^\infty g(s)A^\theta\eta(s)\,ds$, where $\mathbb{D} : \mathcal{M}_\theta \to \mathbb{D}(A^{-\frac{\theta}{2}})$. Here the point on top of this terms is just a notation, it does not mean the time derivative. Considering the weighted Sobolev space $\mathcal{M}_\theta := L^2_g(\mathbb{R}^+; \mathcal{D}(A^{\frac{\theta}{2}}))$ with the inner product

$$\langle\eta_1, \eta_2\rangle_{\mathcal{M}_\theta} = \int_0^\infty g(s)\langle A^{\frac{\theta}{2}}\eta_1, A^{\frac{\theta}{2}}\eta_2\rangle\,ds,$$

the operator $\mathbb{B}$ will be defined in a suitable subspace of the phase space

$$\mathbb{X} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathbb{H} \times \mathbb{H} \times \mathcal{M}_\theta.$$

This Hilbert space is endowed with inner product

$$\begin{aligned}
\langle U_1, U_2\rangle_{\mathbb{X}} = \langle A_0^{\frac{1}{2}}u_1, A_0^{\frac{1}{2}}u_2\rangle + \beta_2\langle A^{\frac{1}{2}}v_1, A^{\frac{1}{2}}v_2\rangle + \rho_1\langle\dot{u}_1, \dot{u}_2\rangle + \rho_2\langle\dot{v}_1, \dot{v}_2\rangle \\
+ \alpha\langle u_1 - v_1, u_2 - v_2\rangle + \langle\eta_1, \eta_2\rangle_{\mathcal{M}_\theta},
\end{aligned}$$

for $U_1 = (u_1, v_1, \dot{u}_1, \dot{v}_1, \eta_1)$ and $U_2 = (u_2, v_2, \dot{u}_2.\dot{v}_2, \eta_2)$. The inner product $\langle\cdot,\cdot\rangle$ on the right is considered in $\mathbb{H}$. With these considerations, the natural domain for operator $\mathbb{B}$ is defined by

$$\mathcal{D}(\mathbb{B}) = \{U \in \mathbb{X} : \mathbb{B}U \in \mathbb{X}\} = \left\{U \in \mathbb{X} : \dot{u}, \dot{v} \in \mathcal{D}(A^{\frac{1}{2}}), A_0u + \mathbb{D}\eta \in \mathbb{H}, v \in \mathcal{D}(A), \eta \in \mathcal{D}(\partial_s)\right\},$$

where

$$\mathcal{D}(\partial_s) = \{\eta \in \mathcal{M}_\theta;\ \partial_s\eta \in \mathcal{M}_\theta \text{ and } \eta(0) = 0\}.$$

The well-posedness of Cauchy problem (3.11) is given by the following Theorem.

Theorem 23. *For initial data U_0 in $\mathcal{D}(\mathbb{B})$ there exists only one solution of problem* (3.11) *in following space*

$$U \in C([0,\infty[; D(\mathbb{B})) \cap C^1([0,\infty[; \mathbb{X}).$$

Proof. To show this Theorem we will prove that the $\mathbb{B}$ is the generator of a C_0 - semigroup of contractions. We will use a variant of Lumer-Phillips's Theorem enunciated in preliminaries, that is, just verify that the linear operator $\mathbb{B}$ is dissipative, its domain $D(\mathbb{B})$ is dense in the phase space $\mathbb{X}$ and $0 \in \rho(\mathbb{B})$.

The density of $\mathcal{D}(\mathbb{B})$ follows from the density of set $\mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times C^1_c(\mathbb{R}^+, \mathcal{D}(A^{\theta}))$ in $\mathbb{X}$ and this set is contained in it. On the other hand, considering $U = (u, v, \dot{u}, \dot{v}, \eta) \in D(\mathbb{B})$ we compute the inner product between $\mathbb{B}U$ and U to obtain

$$\text{Re}\langle \mathbb{B}U, U\rangle_{\mathbb{X}} = -\text{Re}\int_0^\infty g(s)\langle \partial_s \eta, \eta\rangle_{\mathcal{D}(A^{\frac{\theta}{2}})}\, ds. \tag{3.12}$$

Performing an integration by parts on the right side of this equation and taking into account that $\eta(0) = 0$ and $\lim_{s\to\infty} g(s) = 0$, we get

$$\text{Re}\langle \mathbb{B}U, U\rangle_{\mathbb{X}} = \frac{1}{2}\int_0^\infty g'(s)\|A^{\frac{\theta}{2}}\eta\|^2 ds. \tag{3.13}$$

From the condition (3.7), the right side of this equality is non-positive. Therefore, $\mathbb{B}$ is a dissipative operator.

Finally, we are going to verify that $0 \in \rho(\mathbb{B})$. For this, let $F = (f_1, f_2, f_3, f_4, f_5) \in \mathbb{X}$. The system $\mathbb{B}U = F$ has solution if and only if the components of $U = (u, v, \dot{u}, \dot{v}, \eta)$ satisfy the following equations

$$\dot{u} = f_1 \quad \text{in} \quad \mathcal{D}(A^{\frac{1}{2}}), \tag{3.14}$$
$$\dot{v} = f_2 \quad \text{in} \quad \mathcal{D}(A^{\frac{1}{2}}), \tag{3.15}$$
$$-\rho_1^{-1}(A_0 u + \mathbb{D}\eta + \alpha(u - v)) = f_3 \quad \text{in} \quad \mathbb{H}, \tag{3.16}$$
$$-\rho_2^{-1}(\beta_2 A v + \alpha(v - u)) = f_4 \quad \text{in} \quad \mathbb{H}, \tag{3.17}$$
$$\dot{u} - \partial_s \eta = f_5 \quad \text{in} \quad \mathcal{M}_\theta. \tag{3.18}$$

If we consider the following expression for η:

$$\eta(s) = s f_1 - \int_0^s f_5(\tau)d\tau,$$

we have $\eta(0) = 0$ and moreover, we conclude that it is a solution for equation (3.18).

To show that $\eta \in \mathcal{M}_\theta$ we use the hypothesis (3.7) to obtain

$$\int_0^\infty g(s)\|A^{\frac{\theta}{2}}\eta(s)\|^2 ds \leq -\frac{1}{c_1}\int_0^\infty g'(s)\|A^{\frac{\theta}{2}}\eta(s)\|^2 ds,$$

and in view of (3.12) and (3.13) we have

$$\int_0^\infty g(s)\|A^{\frac{\theta}{2}}\eta(s)\|^2 \leq \frac{2}{c_1}\int_0^\infty g(s)|\langle A^{\frac{\theta}{2}}\partial_s\eta, A^{\frac{\theta}{2}}\eta\rangle| ds.$$

Applying Cauchy-Schwarz and Young inequalities we obtain the following estimate

$$\int_0^\infty g(s)\|A^{\frac{\theta}{2}}\eta(s)\|^2 \leq \frac{4}{c_1^2}\int_0^\infty g(s)\|A^{\frac{\theta}{2}}\partial_s\eta\|^2 ds. \tag{3.19}$$

From (3.18) we have that $\partial_s\eta \in \mathcal{M}_\theta$, and then the above inequality implies that $\eta \in \mathcal{M}_\theta$. Consequently $\eta \in \mathcal{D}(\partial_s)$.

Furthermore, from equations (3.16) and (3.17) we can be rewritten as follow

$$\begin{cases} A_0 u + \alpha(u - v) = -\rho_1 f_3 - \mathbb{D}\eta, \\ \beta_2 A v + \alpha(v - u) = -\rho_2 f_4. \end{cases} \tag{3.20}$$

Denoting by $W = (u, v)$, the system (3.20) can be placed in a variational problem

$$a(\Phi, W) = \langle G, \Phi\rangle, \quad \forall\, \Phi = (\varphi, \psi) \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}),$$

where the sesquilinear form $a(\cdot,\cdot)$ and G are defined by

$$a(\Phi, W) = \langle A_0^{\frac{1}{2}}\varphi, A_0^{\frac{1}{2}}u\rangle + \beta_2\langle A^{\frac{1}{2}}\psi, A^{\frac{1}{2}}v\rangle + \alpha\langle\varphi - \psi, u - v\rangle,$$

$$\langle G, \Phi\rangle = \langle g_1, \varphi\rangle_{D(A^{-\frac{1}{2}})\times D(A^{\frac{1}{2}})} + \langle g_2, \psi\rangle_{D(A^{-\frac{1}{2}})\times D(A^{\frac{1}{2}})},$$

with $g_1 = -\rho_1 f_3 - \mathbb{D}\eta$ and $g_2 = -\rho_2 f_4$. From this definition we obtain that $a(\cdot,\cdot)$ is continuous and $G \in \mathcal{D}(A^{-\frac{1}{2}}) \times \mathcal{D}(A^{-\frac{1}{2}})$. Moreover, taking into account that

$$a(W, W) = \|A_0^{\frac{1}{2}}u\|^2 + \beta_2\|A^{\frac{1}{2}}v\|^2 + \alpha\|u - v\|^2,$$

we have from estimate (3.10)

$$a(W, W) \geq (\beta_1 - \kappa\alpha_1^{\theta-1})\|A^{\frac{1}{2}}u\|^2 + \beta_2\|A^{\frac{1}{2}}v\|^2,$$

that is, the sesquilinear form $a(\cdot,\cdot)$ is coercive. Therefore, by Lax-Milgram Theorem, there exists an unique solution $(u, v) \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}})$ for system (3.20) in a weak sense. Moreover, from second equation of (3.20) implies that $v \in \mathcal{D}(A)$. Similarly, from first equation we have $A_0 u + \mathbb{D}\eta \in \mathbb{H}$. Consequently, $U \in \mathcal{D}(\mathbb{B})$.

To finish, note that

$$\|U\|^2 = \|A_0^{\frac{1}{2}}u\|^2 + \beta_2\|A^{\frac{1}{2}}v\|^2 + \rho_1\|\dot{u}\|^2 + \rho_2\|\dot{v}\|^2 + \alpha\|u-v\|^2 + \|\eta\|^2_{\mathcal{M}_\theta}$$
$$= a(W,W) + \rho_1\|\dot{u}\|^2 + \rho_2\|\dot{v}\|^2 + \|\eta\|^2_{\mathcal{M}_\theta}$$
$$= \langle G, W\rangle + \rho_1\|\dot{u}\|^2 + \rho_2\|\dot{v}\|^2 + \|\eta\|^2_{\mathcal{M}_\theta}. \tag{3.21}$$

From equations (3.14) and (3.15) we have

$$\rho_1\|\dot{u}\|^2 + \rho_2\|\dot{v}\|^2 \le C\|F\|^2. \tag{3.22}$$

The estimate (3.19) and the equation (3.18) imply that

$$\|\eta\|^2_{\mathcal{M}_\theta} \le \frac{4}{c_1^2}\int_0^\infty g(s)\|A^{\frac{\theta}{2}}\partial_s\eta\|^2 ds \le C\|F\|^2. \tag{3.23}$$

Using the definition of the functional G and the self-adjointness of the operator A we get

$$|\langle G, W\rangle| \le \left|\langle g_1, u\rangle_{D(A^{-\frac{1}{2}})\times D(A^{\frac{1}{2}})}\right| + \left|\langle g_2, v\rangle_{D(A^{-\frac{1}{2}})\times D(A^{\frac{1}{2}})}\right|$$
$$\le |\langle \rho_1 f_3, u\rangle_{D(A^{-\frac{1}{2}})\times D(A^{\frac{1}{2}})}| + |\langle \rho_2 f_4, v\rangle_{D(A^{-\frac{1}{2}})\times D(A^{\frac{1}{2}})}| + \left|\langle \mathbb{D}\eta, u\rangle_{D(A^{-\frac{1}{2}})\times D(A^{\frac{1}{2}})}\right|$$
$$\le C\|F\|\|U\| + \int_0^\infty g(s)\left|\langle A^{\frac{\theta}{2}}\eta(s), A^{\frac{\theta}{2}}u\rangle\right| ds.$$

Applying Young inequality we have

$$|\langle G, W\rangle| \le \varepsilon\|U\|^2 + C_\varepsilon\left\{\|F\|^2 + \|\eta\|^2_{\mathcal{M}_\theta}\right\}, \tag{3.24}$$

for ε positive. Using estimates (3.22), (3.23) and (3.24) in (3.21) and computing ε small enough, we conclude that

$$\|U\| \le C\|F\|,$$

that is, $0 \in \rho(\mathbb{B})$ and then it completes the proof. ■

3.3 Polynomial decay

In this section, we will show the main results of this chapter. For that, it is necessary to invoke an important Theorem about polynomial decay due to Borichev and Tomilov (enunciated in preliminaries). The main purpose of these results is to relate the different types of decay rates of the solution to the system coefficients, in this way, the main difference between both is about the speed propagation that plays an important role.

Theorem 24. *Let $\chi_0 = 0$ satisfying the previous conditions. The semigroup $e^{t\mathbb{B}}$ associated with system* (3.2)-(3.4) *has the following asymptotic behavior:*

(i) If $0 \leq \theta \leq \frac{1}{2}$, the semigroup decays polynomially with decay rate $t^{-1/(2-2\theta)}$, that is, there exists $C > 0$ such that

$$\|e^{t\mathbb{B}}U_0\| \leq \frac{C}{t^{1/(2-2\theta)}}\|U_0\|_{D(\mathbb{B})}, \quad \forall\, t > 0,\ U_0 \in D(\mathbb{B}).$$

(ii) If $\frac{1}{2} \leq \theta \leq 1$, the semigroup decays polynomially with decay rate $t^{-1/2\theta}$, that is, there exists $C > 0$ such that

$$\|e^{t\mathbb{B}}U_0\| \leq \frac{C}{t^{1/2\theta}}\|U_0\|_{D(\mathbb{B})}, \quad \forall\, t > 0,\ U_0 \in D(\mathbb{B}).$$

Theorem 25. *Let $\chi_0 \neq 0$ satisfying the previous conditions. The semigroup $e^{t\mathbb{B}}$ associated with system* (3.2)-(3.4) *decays polynomially with decay rate $t^{-1/(6-2\theta)}$, that is, there exists $C > 0$ such that*

$$\|e^{t\mathbb{B}}U_0\| \leq \frac{C}{t^{1/(6-2\theta)}}\|U_0\|_{D(\mathbb{B})}, \quad \forall\, t > 0,\ U_0 \in D(\mathbb{B}).$$

The proof of these Theorems will be divided into some Lemmas. In view of this, to show these lemmas it is considered many times the stationary problem $(i\lambda I - \mathbb{B})U = F$, where $\lambda \in \mathbb{R}$ and $F = (f_1, f_2, f_3, f_4, f_5)$. Note that for $U = (u, v, \dot{u}, \dot{v}, \eta)$ solution of this problem, we have that are satisfied:

$$i\lambda u - \dot{u} = f_1, \tag{3.25}$$

$$i\lambda v - \dot{v} = f_2, \tag{3.26}$$

$$i\rho_1\lambda\dot{u} + A_0 u + \mathbb{D}\eta + \alpha(u - v) = \rho_1 f_3, \tag{3.27}$$

$$i\rho_2\lambda\dot{v} + \beta_2 A v + \alpha(v - u) = \rho_2 f_4, \tag{3.28}$$

$$i\lambda\eta + \partial_s\eta - \dot{u} = f_5. \tag{3.29}$$

Initially, from set of hypothesis (3.7) and equality (3.13) we have

$$\|\eta\|^2_{\mathcal{M}_\theta} \leq C\int_0^\infty -g'(s)\|A^{\frac{\theta}{2}}\eta\|^2 ds \leq C\text{Re}\langle(i\lambda I - \mathbb{B})U, U\rangle_{\mathbb{X}} \leq C\|F\|\|U\|. \tag{3.30}$$

Lemma 26. *Consider $\theta \in [0, 1]$ and $F \in \mathbb{X}$. Suppose that for every $\lambda \in \mathbb{R}$ such that $0 < \delta \leq |\lambda|$ there exists a solution $U \in \mathcal{D}(\mathbb{B})$ of stationary system $(i\lambda I - \mathbb{B})U = F$. Then, there exists a positive constant C_δ such that:*

(i) $\|A^{\frac{\theta}{2}}\dot{u}\|^2 \leq C_\delta \lambda^2 \left(\|F\|\|U\| + \|F\|^2\right),$

(ii) $\|A^{\frac{\theta}{2}}u\|^2 \leq C_\delta \left(\|F\|\|U\| + \|F\|^2\right).$

Proof. Note that using equation (3.29) we have

$$\begin{aligned}\left(\int_0^\infty g(s)ds\right)\|A^{\frac{\theta}{2}}\dot{u}\|^2 &= \int_0^\infty g(s)\langle A^{\frac{\theta}{2}}(i\lambda\eta(s) + \partial_s\eta(s) - f_5(s)), A^{\frac{\theta}{2}}\dot{u}\rangle ds \\ &= \int_0^\infty g(s)\langle i\lambda A^{\frac{\theta}{2}}\eta(s), A^{\frac{\theta}{2}}\dot{u}\rangle ds + \int_0^\infty g(s)\langle \partial_s A^{\frac{\theta}{2}}\eta(s), A^{\frac{\theta}{2}}\dot{u}\rangle ds \\ &\quad - \int_0^\infty g(s)\langle A^{\frac{\theta}{2}} f_5(s), A^{\frac{\theta}{2}}\dot{u}\rangle ds.\end{aligned}$$

Thus, integrating by parts and using Cauchy-Schwarz inequality follows that

$$\begin{aligned}\left(\int_0^\infty g(s)ds\right)\|A^{\frac{\theta}{2}}\dot{u}\|^2 \leq C|\lambda|\|A^{\frac{\theta}{2}}\dot{u}\| &\left(\int_0^\infty g(s)\|A^{\frac{\theta}{2}}\eta(s)\|^2 ds\right)^{\frac{1}{2}} \\ + C\|A^{\frac{\theta}{2}}\dot{u}\| &\left(-\int_0^\infty g'(s)\|A^{\frac{\theta}{2}}\eta(s)\|^2 ds\right)^{\frac{1}{2}} \\ + C\|A^{\frac{\theta}{2}}\dot{u}\| &\left(\int_0^\infty g(s)\|A^{\frac{\theta}{2}} f_5(s)\|^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

Furthermore, applying Young's inequality we obtain

$$\begin{aligned}\|A^{\frac{\theta}{2}}\dot{u}\|^2 \leq &C\left(\lambda^2\int_0^\infty g(s)\|A^{\frac{\theta}{2}}\eta(s)\|^2 ds - \int_0^\infty g'(s)\|A^{\frac{\theta}{2}}\eta(s)\|^2 ds\right. \\ &\left. + \int_0^\infty g(s)\|A^{\frac{\theta}{2}} f_5(s)\|^2 ds\right),\end{aligned}$$

where C is a positive constant that not depends of λ.

From estimate (3.30) follows that

$$\|A^{\frac{\theta}{2}}\dot{u}\|^2 \leq C\left(\lambda^2\|F\|\|U\| + \|F\|\|U\| + \|F\|^2\right),$$

then recalling that $|\lambda| \geq \delta$, item (i) is obtained. Moreover, computing inner product with $i\lambda A^\theta u$ (note that operator A^θ is self-adjoint), we get

$$\lambda^2\|A^{\frac{\theta}{2}}u\|^2 = -\langle i\lambda A^{\frac{\theta}{2}}u, A^{\frac{\theta}{2}} f_1\rangle - \langle i\lambda A^{\frac{\theta}{2}}u, A^{\frac{\theta}{2}}\dot{u}\rangle.$$

Using Cauchy-Schwarz and Young's inequality follows that

$$\lambda^2\|A^{\frac{\theta}{2}}u\|^2 \leq C(\|A^{\frac{\theta}{2}} f_1\|^2 + \|A^{\frac{\theta}{2}}\dot{u}\|^2).$$

Moreover, by the continuous embedding $\mathcal{D}(A^{\frac{1}{2}}) \hookrightarrow \mathcal{D}(A^{\frac{\theta}{2}})$ (in view of $\theta \in [0,1]$) and estimate obtained in item (i), we have

$$\lambda^2\|A^{\frac{\theta}{2}}u\|^2 \leq C_\delta(\lambda^2\|F\|\|U\| + \|F\|^2),$$

then using that $|\lambda| > \delta$ the result follows. ■

The next lemma is important to understand what is the main term that we need to know the decay rates.

Theorem 27. *The solutions of equations (4.27)-(4.32) satisfy the following result*

$$\|U\|^2 \leq C_\delta \left(\|\dot{u}\|^2 + \|\dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right).$$

Proof. Initially, let's compute the inner product of equation (3.27) with u and equation (3.28) with v; using equations (3.25) and (3.26) respectively we have

$$\|A_0^{\frac{1}{2}}u\|^2 + \alpha\|u\|^2 - \alpha\langle v, u\rangle = \rho_1\|\dot{u}\|^2 - \int_0^\infty g(s)\langle A^{\frac{\theta}{2}}\eta, A^{\frac{\theta}{2}}u\rangle ds$$
$$+ \rho_1\langle \dot{u}, f_1\rangle + \rho_1\langle f_3, u\rangle,$$

and

$$\beta_2\|A^{\frac{1}{2}}v\|^2 + \alpha\|v\|^2 - \alpha\langle u, v\rangle = \rho_2\|\dot{v}\|^2 + \rho_2\langle \dot{v}, f_2\rangle + \rho_2\langle f_4, v\rangle.$$

Performing the sum with both these equations we have

$$\|A_0^{\frac{1}{2}}u\|^2 + \beta_2\|A^{\frac{1}{2}}v\|^2 + \alpha\|u - v\|^2 = \rho_1\|\dot{u}\|^2 - \int_0^\infty g(s)\langle A^{\frac{\theta}{2}}\eta, A^{\frac{\theta}{2}}u\rangle ds + \rho_1\langle \dot{u}, f_1\rangle$$
$$+ \rho_1\langle f_3, u\rangle + \rho_2\|\dot{v}\|^2 + \rho_2\langle \dot{v}, f_2\rangle + \rho_2\langle f_4, v\rangle.$$

Using Young inequality, estimate (3.30) and Lemma 26 we obtain

$$\|A_0^{\frac{1}{2}}u\|^2 + \beta_2\|Av\|^2 + \alpha\|u - v\|^2 \leq C_\delta \left(\|\dot{u}\|^2 + \|\dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right).$$

Furthermore, from inequality (3.30) we can conclude

$$\|\eta\|^2_{\mathcal{M}_\theta} \leq C\|F\|\|U\|.$$

In view of previous estimates, it may be concluded that

$$\|U\|^2 \leq C_\delta \left(\|\dot{u}\|^2 + \|\dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right).$$

■

Lemma 28. *From the same hypothesis of Lemma 26 with $\chi_0 = 0$ we have*

$$\|A^{\frac{\sigma}{2}}v\|^2 \leq \varepsilon\|\lambda^{-1}A^{\sigma+\frac{\theta}{2}}v\|^2 + C\|A^{\frac{\sigma}{2}}u\|^2 + C_\varepsilon(\|U\|\|F\| + \|F\|^2),$$

where $\sigma \leq \frac{1}{2}$.

Proof. From equation (3.27), computing inner product with $A^\sigma v$, using the equation (3.26), definition of A_0 and the fact that the operator is self-adjoint we obtain

$$-\rho_1\langle A^{\frac{\sigma}{2}}\dot{u}, A^{\frac{\sigma}{2}}f_2\rangle - \rho_1\langle A^{\frac{\sigma}{2}}\dot{u}, A^{\frac{\sigma}{2}}\dot{v}\rangle + \beta_1\langle A^{\frac{\sigma+1}{2}}u, A^{\frac{\sigma+1}{2}}v\rangle - \int_0^\infty g(s)\langle A^\theta u, A^\sigma v\rangle ds \\ + \int_0^\infty g(s)\langle A^\theta \eta(s), A^\sigma v\rangle ds + \alpha\langle A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v\rangle - \alpha\|A^{\frac{\sigma}{2}}v\|^2 = \rho_1\langle f_3, A^\sigma v\rangle.$$

In the same way, from equation (3.28), computing the inner product with $A^\sigma u$ and using the equation (3.25), we have

$$-\rho_2\langle A^{\frac{\sigma}{2}}\dot{v}, A^{\frac{\sigma}{2}}f_1\rangle - \rho_2\langle A^{\frac{\sigma}{2}}\dot{v}, A^{\frac{\sigma}{2}}\dot{u}\rangle + \beta_2\langle A^{\frac{\sigma+1}{2}}v, A^{\frac{\sigma+1}{2}}u\rangle + \alpha\langle A^{\frac{\sigma}{2}}v, A^{\frac{\sigma}{2}}u\rangle \\ -\alpha\|A^{\frac{\sigma}{2}}u\|^2 = \rho_2\langle f_4, A^\sigma u\rangle.$$

Divide by β_1 the first equation, by β_2 the second equation and performing the sum between both we get

$$\frac{\alpha}{\beta_1}\|A^{\frac{\sigma}{2}}v\|^2 = -\frac{\rho_1}{\beta_1}\langle f_3, A^\sigma v\rangle - \frac{\rho_1}{\beta_1}\langle A^{\frac{\sigma}{2}}\dot{u}, A^{\frac{\sigma}{2}}f_2\rangle - \frac{\rho_1}{\beta_1}\langle A^{\frac{\sigma}{2}}\dot{u}, A^{\frac{\sigma}{2}}\dot{v}\rangle + \langle A^{\frac{\sigma+1}{2}}u, A^{\frac{\sigma+1}{2}}v\rangle \\ - \frac{1}{\beta_1}\int_0^\infty g(s)\langle A^\theta u, A^\sigma v\rangle ds + \frac{1}{\beta_1}\int_0^\infty g(s)\langle A^\theta\eta(s), A^\sigma v\rangle ds + \frac{\alpha}{\beta_1}\langle A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v\rangle \\ + \frac{\rho_2}{\beta_2}\langle A^{\frac{\sigma}{2}}\dot{v}, A^{\frac{\sigma}{2}}f_1\rangle + \frac{\rho_2}{\beta_2}\langle A^{\frac{\sigma}{2}}\dot{v}, A^{\frac{\sigma}{2}}\dot{u}\rangle - \langle A^{\frac{\sigma+1}{2}}v, A^{\frac{\sigma+1}{2}}u\rangle - \frac{\alpha}{\beta_2}\langle A^{\frac{\sigma}{2}}v, A^{\frac{\sigma}{2}}u\rangle \\ + \frac{\alpha}{\beta_2}\|A^{\frac{\sigma}{2}}u\|^2 + \frac{\rho_2}{\beta_2}\langle f_4, A^\sigma u\rangle.$$

computing the real part and using (3.6) we have

$$\frac{\alpha}{\beta_1}\|A^{\frac{\sigma}{2}}v\|^2 = \frac{\alpha}{\beta_2}\|A^{\frac{\sigma}{2}}u\|^2 + \frac{\rho_1\rho_2}{\beta_1\beta_2}\chi_0\text{Re}\langle A^{\frac{\sigma}{2}}\dot{u}, A^{\frac{\sigma}{2}}\dot{v}\rangle + \frac{1}{\beta_1}\text{Re}\int_0^\infty g(s)\langle A^{\frac{\theta}{2}}(\eta - u), A^{\sigma+\frac{\theta}{2}}v\rangle ds \\ + \frac{\alpha}{\beta_1}\text{Re}\langle A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v\rangle - \frac{\alpha}{\beta_2}\text{Re}\langle A^{\frac{\sigma}{2}}v, A^{\frac{\sigma}{2}}u\rangle - \frac{\rho_1}{\beta_1}\text{Re}\langle f_3, A^\sigma v\rangle - \frac{\rho_1}{\beta_1}\text{Re}\langle \dot{u}, A^\sigma f_2\rangle \\ + \frac{\rho_2}{\beta_2}\text{Re}\langle \dot{v}, A^\sigma f_1\rangle + \frac{\rho_2}{\beta_2}\text{Re}\langle f_4, A^\sigma u\rangle. \tag{3.31}$$

Note that, in view of equation (3.25), we can write (3.29) as

$$(\eta(s) - u) = \frac{-\partial_s\eta(s) - f_1 + f_5(s)}{i\lambda},$$

that is

$$\int_0^\infty g(s)\langle A^{\frac{\theta}{2}}(\eta(s)-u), A^{\sigma+\frac{\theta}{2}}v\rangle ds = \int_0^\infty g(s)\langle A^{\frac{\theta}{2}}\partial_s\eta(s), i\lambda^{-1}A^{\sigma+\frac{\theta}{2}}v\rangle ds$$
$$+\int_0^\infty g(s)\langle A^{\frac{\theta}{2}}f_1, i\lambda^{-1}A^{\sigma+\frac{\theta}{2}}v\rangle ds$$
$$-\int_0^\infty g(s)\langle A^{\frac{\theta}{2}}f_5(s), i\lambda^{-1}A^{\sigma+\frac{\theta}{2}}v\rangle ds$$
$$=-\int_0^\infty g'(s)\langle A^{\frac{\theta}{2}}\eta(s), i\lambda^{-1}A^{\sigma+\frac{\theta}{2}}v\rangle ds$$
$$+\int_0^\infty g(s)\langle A^{\frac{\theta}{2}}(f_1-f_5(s)), i\lambda^{-1}A^{\sigma+\frac{\theta}{2}}v\rangle ds. \quad (3.32)$$

Substituting equation (3.32) in (3.31) and using the hypothesis $\chi_0=0$ we have

$$\frac{\alpha}{\beta_1}\|A^{\frac{\sigma}{2}}v\|^2 = \frac{\alpha}{\beta_2}\|A^{\frac{\sigma}{2}}u\|^2 - \frac{1}{\beta_1}\int_0^\infty g'(s)\langle A^{\frac{\theta}{2}}\eta(s), i\lambda^{-1}A^{\sigma+\frac{\theta}{2}}v\rangle ds$$
$$+\frac{1}{\beta_1}\int_0^\infty g(s)\langle A^{\frac{\theta}{2}}(f_1-f_5(s)), i\lambda^{-1}A^{\sigma+\frac{\theta}{2}}v\rangle ds - \frac{\rho_1}{\beta_1}\text{Re}\langle\dot{u}, A^\sigma f_2\rangle$$
$$+\frac{\alpha}{\beta_1}\text{Re}\langle A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v\rangle - \frac{\alpha}{\beta_2}\text{Re}\langle A^{\frac{\sigma}{2}}v, A^{\frac{\sigma}{2}}u\rangle - \frac{\rho_1}{\beta_1}\text{Re}\langle f_3, A^\sigma v\rangle$$
$$+\frac{\rho_2}{\beta_2}\text{Re}\langle\dot{v}, A^\sigma f_1\rangle + \frac{\rho_2}{\beta_2}\text{Re}\langle f_4, A^\sigma u\rangle.$$

From Young inequality and estimate (3.30) we can conclude

$$\|A^{\frac{\sigma}{2}}v\|^2 \le \varepsilon\|\lambda^{-1}A^{\sigma+\frac{\theta}{2}}v\|^2 + C\|A^{\frac{\sigma}{2}}u\|^2 + C_\varepsilon(\|F\|\|U\| + \|F\|^2).$$

■

Lemma 29. *In the same hypothesis of Lemma 26, we have the following estimates for* $\sigma\le 1$

(i) $\|A^{\frac{\sigma+1}{2}}v\|^2 \le C_\delta\left(\|A^{\frac{\sigma}{2}}\dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right)$,

(ii) $\|\lambda^{-1}A^{\frac{\sigma+1}{2}}v\|^2 \le C_\delta\left(\|A^{\frac{\sigma}{2}}v\|^2 + \|F\|\|U\| + \|F\|^2\right)$.

Proof. Item (i): Computing the inner product between (3.28) and $A^\sigma v$ we have

$$\beta_2\|A^{\frac{\sigma+1}{2}}v\|^2 = \rho_2\langle\dot{v}, A^\sigma(i\lambda v)\rangle - \alpha\|A^{\frac{\sigma}{2}}v\|^2 + \alpha\langle A^{\frac{\sigma}{2}-\frac{1}{2}}u, A^{\frac{\sigma}{2}+\frac{1}{2}}v\rangle + \rho_2\langle f_4, A^\sigma v\rangle.$$

Substituting the equation (3.26) in above equation we obtain

$$\beta_2\|A^{\frac{\sigma+1}{2}}v\|^2 = \rho_2\|A^{\frac{\sigma}{2}}\dot{v}\|^2 - \alpha\|A^{\frac{\sigma}{2}}v\|^2 + \alpha\langle A^{\frac{\sigma}{2}-\frac{1}{2}}u, A^{\frac{\sigma}{2}+\frac{1}{2}}v\rangle$$
$$+\rho_2\langle\dot{v}, A^\sigma f_2\rangle + \rho_2\langle f_4, A^\sigma v\rangle.$$

Applying Young inequality and using Lemma 26 we have

$$\begin{aligned}\|A^{\frac{\sigma+1}{2}}v\|^2 &\leq C\left(\|A^{\frac{\sigma}{2}}\dot{v}\|^2+\|A^{\frac{\sigma-1}{2}}u\|^2+\|F\|\|U\|\right)\\ &\leq C_\delta\left(\|A^{\frac{\sigma}{2}}\dot{v}\|^2+\|F\|\|U\|+\|F\|^2\right).\end{aligned}$$

Item (ii): Multiplying by λ^{-2} item (i) from this Lemma we have

$$\|\lambda^{-1}A^{\frac{\sigma+1}{2}}v\|^2 \leq C_\delta\left(\|\lambda^{-1}A^{\frac{\sigma}{2}}\dot{v}\|^2+\|F\|\|U\|+\|F\|^2\right),$$

however, from equation (3.26) follows that

$$\|\lambda^{-1}A^{\frac{\sigma+1}{2}}v\|^2 \leq C_\delta\left(\|A^{\frac{\sigma}{2}}v\|^2+\|F\|\|U\|+\|F\|^2\right).$$

■

Lemma 30. *In the same hypothesis of Lemma 26, we have the following estimates for* $\chi_0=0$

(i) $\|A^{\frac{\theta-1}{2}}\dot{u}\|^2 \leq C_\delta\left(\|F\|\|U\|+\|F\|^2\right), 0\leq\theta\leq\frac{1}{2}$;

(ii) $\|A^{-\frac{\theta}{2}}\dot{u}\|^2 \leq C_\delta\left(\|F\|\|U\|+\|F\|^2\right), \frac{1}{2}\leq\theta\leq 1$;

(iii) $\|A^{\frac{1-\theta}{2}}\dot{u}\|^2 \leq C_\delta\lambda^2\left(\|F\|\|U\|+\|F\|^2\right), \frac{1}{2}\leq\theta\leq 1$.

Proof. Item (i): Performing the inner product with equation (3.27) and $i\lambda A^{\theta-1}\dot{u}$ we have

$$\begin{aligned}\rho_1\|\lambda A^{\frac{\theta-1}{2}}\dot{u}\|^2 =&\beta_1\|A^{\frac{\theta}{2}}\dot{u}\|^2-\kappa\|A^{\frac{2\theta-1}{2}}\dot{u}\|^2+\int_0^\infty g(s)\langle i\lambda A^{\frac{\theta}{2}}\eta(s),A^{\frac{3\theta-2}{2}}\dot{u}\rangle ds\\ &-\alpha\langle A^{\frac{\theta-1}{2}}u,i\lambda A^{\frac{\theta-1}{2}}\dot{u}\rangle+\alpha\langle A^{\frac{\theta-1}{2}}v,i\lambda A^{\frac{\theta-1}{2}}\dot{u}\rangle-\langle A_0f_1,A^{\theta-1}\dot{u}\rangle.\end{aligned}$$

Using Young inequality, the continuous embedding $\mathcal{D}(A^{\frac{\theta}{2}})\hookrightarrow\mathcal{D}(A^{\frac{3\theta-2}{2}})$ and Lemma 26 (item (i)) we have

$$\begin{aligned}\|\lambda A^{\frac{\theta-1}{2}}\dot{u}\|^2 \leq& C\left(\|A^{\frac{\theta-1}{2}}v\|^2+\|A^{\frac{\theta}{2}}\dot{u}\|^2+\lambda^2\left(\|F\|\|U\|+\|F\|^2\right)\right)\\ \leq& C\left(\|A^{\frac{\theta-1}{2}}v\|^2+\lambda^2\left(\|F\|\|U\|+\|F\|^2\right)\right).\end{aligned}\tag{3.33}$$

From Lemma 28, using $\sigma=\theta-1$ we obtain

$$\|A^{\frac{\theta-1}{2}}v\|^2\leq\varepsilon\|\lambda^{-1}A^{\frac{3\theta}{2}-1}v\|^2+C\|A^{\frac{\theta-1}{2}}u\|^2+C_\varepsilon(\|U\|\|F\|+\|F\|^2).\tag{3.34}$$

Furthermore, from Lemma 29 (item (ii)), using $\sigma=3\theta-3$ we obtain

$$\|\lambda^{-1}A^{\frac{3\theta}{2}-1}v\|^2\leq C_\delta\left(\|A^{\frac{3\theta-3}{2}}v\|^2+\|F\|\|U\|+\|F\|^2\right).\tag{3.35}$$

Therefore, from estimates (3.34)-(3.35), Lemma 26 and ε small enough we can conclude that

$$\|A^{\frac{\theta-1}{2}}v\|^2 \leq C_\delta \left(\|F\|\|U\| + \|F\|^2\right).$$

To finish, from (3.33) we have

$$\|A^{\frac{\theta-1}{2}}\dot{u}\|^2 \leq C_\delta \left(\|F\|\|U\| + \|F\|^2\right).$$

Item (ii): In this item, we proceed computing the inner product with equation (3.27) and $i\lambda A^{-\theta}\dot{u}$ (similarly to the previous item). For this we get

$$\begin{aligned}\rho_1\|\lambda A^{\frac{-\theta}{2}}\dot{u}\|^2 =& \beta_1\|A^{\frac{1-\theta}{2}}\dot{u}\|^2 - \kappa\|\dot{u}\|^2 + \int_0^\infty g(s)\langle i\lambda A^{\frac{\theta}{2}}\eta(s), A^{\frac{-\theta}{2}}\dot{u}\rangle ds \\ & - \alpha\langle A^{\frac{-\theta}{2}}u, i\lambda A^{\frac{-\theta}{2}}\dot{u}\rangle + \alpha\langle A^{\frac{-\theta}{2}}v, i\lambda A^{\frac{-\theta}{2}}\dot{u}\rangle - \langle A_0 f_1, A^{-\theta}\dot{u}\rangle.\end{aligned}$$

The result follows using a similar step to the one performed in item (i).

Item (iii): As $\frac{1}{2} \leq \theta \leq 1$ we have the continuous embedding $\mathcal{D}(A^{\frac{\theta}{2}}) \hookrightarrow \mathcal{D}(A^{\frac{1-\theta}{2}})$. Therefore, from Lemma 26 we obtain the result. ∎

Lemma 31. *In the same hypothesis of Lemma 26, we have the following estimates for* $\chi_0 = 0$

(i) $\|A^{\frac{\theta}{2}}\dot{v}\|^2 \leq C_\delta\lambda^2 \left(\|F\|\|U\| + \|F\|^2\right),\ 0 \leq \theta \leq \frac{1}{2}$;

(ii) $\|A^{\frac{\theta-1}{2}}\dot{v}\|^2 \leq C_\delta \left(\|F\|\|U\| + \|F\|^2\right),\ 0 \leq \theta \leq \frac{1}{2}$;

(iii) $\|A^{\frac{1-\theta}{2}}\dot{v}\|^2 \leq C_\delta\lambda^2 \left(\|F\|\|U\| + \|F\|^2\right),\ \frac{1}{2} \leq \theta \leq 1$;

(iv) $\|A^{-\frac{\theta}{2}}\dot{v}\|^2 \leq C_\delta \left(\|F\|\|U\| + \|F\|^2\right),\ \frac{1}{2} \leq \theta \leq 1$.

Proof. Item (i): From Lemma 28 for $\sigma = \theta$ we have

$$\|A^{\frac{\theta}{2}}v\|^2 \leq \varepsilon\|\lambda^{-1}A^{\frac{3\theta}{2}}v\|^2 + C_\varepsilon(\|F\|\|U\| + \|F\|^2).$$

Moreover, from Lemma 29 (item (ii)) for $\sigma = 3\theta - 1$ we obtain

$$\|\lambda^{-1}A^{\frac{3\theta}{2}}v\|^2 \leq C_\delta(\|A^{\frac{3\theta-1}{2}}v\|^2 + \|F\|\|U\| + \|F\|^2).$$

Applying the continuous embedding $\mathcal{D}(A^{\frac{\theta}{2}}) \hookrightarrow \mathcal{D}(A^{\frac{3\theta-1}{2}})$ (because $\theta \leq \frac{1}{2}$) and computing ε small enough we have

$$\|A^{\frac{\theta}{2}}v\|^2 \leq C_\delta(\|U\|\|F\| + \|F\|^2).$$

Therefore, from equation (3.26) we conclude the result.

Item (ii): Using equation (3.28) to compute the inner product with $i\lambda A^{\theta-1}\dot{v}$ and substituting equation (3.26) in this result we have

$$\begin{aligned}\rho_2\|\lambda A^{\frac{\theta-1}{2}}\dot{v}\|^2 =&\beta_2\langle A^{\frac{\theta}{2}}f_2, A^{\frac{\theta}{2}}\dot{v}\rangle + \beta_2\|A^{\frac{\theta}{2}}\dot{v}\|^2 + \alpha\langle A^{\frac{\theta-1}{2}}f_2, A^{\frac{\theta-1}{2}}\dot{v}\rangle\\ &+ \alpha\|A^{\frac{\theta-1}{2}}\dot{v}\| - \alpha\langle A^{\frac{\theta-1}{2}}u, i\lambda A^{\frac{\theta-1}{2}}\dot{v}\rangle + \rho_2\langle f_4, A^{\theta-1}\dot{v}\rangle\end{aligned} \tag{3.36}$$

To conclude, using Young inequality, Lemma 26 (item (ii)) with $\mathcal{D}(A^{\frac{\theta}{2}}) \hookrightarrow \mathcal{D}(A^{\frac{\theta-1}{2}})$ and previous item (i).

Item (iii): In the same way, from Lemma 28 for $\sigma = 1-\theta$ we have

$$\|A^{\frac{1-\theta}{2}}v\|^2 \leq \varepsilon\|\lambda^{-1}A^{\frac{2-\theta}{2}}v\|^2 + C_\varepsilon(\|U\|\|F\| + \|F\|^2).$$

Moreover, from Lemma 29 for $\sigma = 1-\theta$ we obtain

$$\|\lambda^{-1}A^{\frac{2-\theta}{2}}v\|^2 \leq C_\delta(\|A^{\frac{1-\theta}{2}}v\|^2 + \|F\|\|U\| + \|F\|^2),$$

considering ε small enough we have

$$\|A^{\frac{1-\theta}{2}}v\|^2 \leq C_\delta(\|U\|\|F\| + \|F\|^2).$$

Using the equation (3.26) we conclude the result.

Item (iv): To show this item we use the same way from Item (ii) in this Lemma with equation (3.28) and $i\lambda A^{-\theta}\dot{v}$. ∎

The next results are about the case $\chi_0 \neq 0$.

Lemma 32. *Consider $\theta \in [0,1]$ and $F \in \mathbb{X}$. Suppose that for every $\lambda \in \mathbb{R}$ such that $0 < \delta \leq |\lambda|$ there exists a solution $U \in \mathcal{D}(\mathbb{B})$ of the stationary system $(i\lambda I - \mathbb{B})U = F$. Then, there exists a positive constant C_δ such that*

$$Re\int_0^\infty g(s)\langle A^{\frac{2\theta-1}{2}}(\eta(s)-u), A^{\frac{2\theta-1}{2}}v\rangle ds \leq \frac{C_\delta}{\varepsilon}\left(\|F\|\|U\| + \|F\|^2\right) + \frac{\varepsilon}{\lambda^2}\|A^{\frac{\theta}{2}}v\|^2,$$

where ε is a positive constant that not depends of δ and λ.

Proof. In view of equations (3.25) and 3.29 we have

$$\eta(s) - u = \frac{f_5(s) - \partial_s\eta(s) - f_1}{i\lambda},$$

therefore

$$\text{Re}\int_0^\infty g(s)\langle A^{\frac{2\theta-1}{2}}(\eta(s)-u), A^{\frac{2\theta-1}{2}}v\rangle ds = I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{1}{\lambda}\text{Im}\int_0^\infty g(s)\langle A^{\frac{2\theta-1}{2}} f_5(s), A^{\frac{2\theta-1}{2}} v\rangle ds,$$
$$I_2 = -\frac{1}{\lambda}\text{Im}\int_0^\infty g(s)\langle A^{\frac{2\theta-1}{2}} \partial_s\eta(s), A^{\frac{2\theta-1}{2}} v\rangle ds,$$
$$I_3 = -\frac{1}{\lambda}\text{Im}\int_0^\infty g(s)\langle A^{\frac{2\theta-1}{2}} f_1, A^{\frac{2\theta-1}{2}} v\rangle ds.$$

We will estimate I_1, I_2 and I_3. Using the continuous embedding $\mathcal{D}(A^{\frac{\theta}{2}}) \hookrightarrow \mathcal{D}(A^{\frac{2\theta-1}{2}})$ we conclude that

$$\begin{aligned} I_1 &\leq \frac{1}{4\varepsilon_1}\int_0^\infty g(s)\|A^{\frac{2\theta-1}{2}} f_5(s)\|^2 ds + \frac{\kappa\varepsilon_1}{\lambda^2}\|A^{\frac{2\theta-1}{2}} v\|^2 \\ &\leq \frac{C}{4\varepsilon_1}\int_0^\infty g(s)\|A^{\frac{\theta}{2}} f_5(s)\|^2 ds + \frac{\kappa K_1\varepsilon_1}{\lambda^2}\|A^{\frac{\theta}{2}} v\|^2, \end{aligned}$$

furthermore, integrating I_2 by parts and making some calculus we have

$$\begin{aligned} I_2 &\leq \frac{1}{\lambda}\|A^{\frac{2\theta-1}{2}} v\| \left(\int_0^\infty -g'(s)\|A^{\frac{2\theta-1}{2}}\eta(s)\| ds\right) \\ &\leq \frac{\varepsilon_1}{\lambda^2}\|A^{\frac{2\theta-1}{2}} v\|^2 + \frac{1}{4\varepsilon_1}\left(\int_0^\infty -g'(s)\|A^{\frac{2\theta-1}{2}}\eta(s)\| ds\right)^2 \\ &\leq \frac{\varepsilon_1 K_1}{\lambda^2}\|A^{\frac{\theta}{2}} v\|^2 + \frac{g(0)K_1}{4\varepsilon_1}\int_0^\infty -g'(s)\|A^{\frac{\theta}{2}}\eta(s)\|^2 ds, \end{aligned}$$

and finally

$$I_3 \leq \frac{\kappa^2}{4\varepsilon_1}\|A^{\frac{2\theta-1}{2}} f_1\|^2 + \frac{\varepsilon_1 K_1}{\lambda^2}\|A^{\frac{\theta}{2}} v\|^2.$$

In I_2 estimate we used the hypothesis from (3.7), in all estimates above $K_1 > 0$ is just the constant that appear on continuous embedding $\mathcal{D}(A^{\frac{\theta}{2}}) \hookrightarrow \mathcal{D}(A^{\frac{2\theta-1}{2}})$ and ε_1 is a positive constant that we will choose later

From (3.30) and estimates above we obtain that

$$\begin{aligned} \text{Re}\int_0^\infty g(s)\langle A^{\frac{2\theta-1}{2}}(\eta(s)-u), A^{\frac{2\theta-1}{2}} v\rangle ds \leq &\frac{C}{\varepsilon_1}\left(\|F\|\|U\| + \|F\|^2\right) \\ &+ \frac{\varepsilon_1 K_1}{\lambda^2}(\kappa+2)\|A^{\frac{\theta}{2}} v\|^2. \end{aligned}$$

Computing $\varepsilon_1 K_1(\kappa+2) = \varepsilon$ we obtain the result. ∎

Lemma 33. *The solution (u, v) of system (3.25)-(3.29) obtained in Theorem 23 satisfy the following inequality:*

$$\|A^{\frac{\theta-1}{2}} v\|^2 \leq C|\chi_0|\|A^{\frac{\theta}{2}} u\|\|A^{\frac{\theta}{2}} v\| + \frac{2\varepsilon}{\lambda^2}\|A^{\frac{\theta}{2}} v\|^2 + \frac{C_\delta}{\varepsilon}\left(\|F\|\|U\| + \|F\|^2\right).$$

where ε is as in Lemma 32.

Proof. Computing inner product of equation (3.27) with $A^\sigma v$, using equation (3.25) and considering the fact of A^σ is self-adjoint we obtain

$$\beta_1\langle A^{\frac{\sigma+1}{2}}u, A^{\frac{\sigma+1}{2}}v\rangle - \rho_1\lambda^2\langle A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v\rangle - i\rho_1\lambda\langle A^{\frac{\sigma}{2}}f_1, A^{\frac{\sigma}{2}}v\rangle - \int_0^\infty g(s)\langle A^\theta u, A^\sigma v\rangle ds$$
$$+ \int_0^\infty g(s)\langle A^\theta\eta(s), A^\sigma v\rangle ds + \alpha\langle A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v\rangle - \alpha\|A^{\frac{\sigma}{2}}v\|^2 = \rho_1\langle f_3, A^\sigma v\rangle. \tag{3.37}$$

Similarly, from equation (3.28), computing inner product with $A^\sigma u$ and using equation (3.26), we have

$$\beta_2\langle A^{\frac{\sigma+1}{2}}v, A^{\frac{\sigma+1}{2}}u\rangle - \rho_2\lambda^2\langle A^{\frac{\sigma}{2}}v, A^{\frac{\sigma}{2}}u\rangle - i\rho_2\lambda\langle A^{\frac{\sigma}{2}}f_2, A^{\frac{\sigma}{2}}u\rangle + \alpha\langle A^{\frac{\sigma}{2}}v, A^{\frac{\sigma}{2}}u\rangle$$
$$-\alpha\|A^{\frac{\sigma}{2}}u\|^2 = \rho_2\langle f_4, A^\sigma u\rangle. \tag{3.38}$$

Dividing equations (3.37) and (3.38) by ρ_1 and ρ_2 respectively and computing the difference of results, we obtain

$$\begin{aligned}\frac{\alpha}{\rho_1}\|A^{\frac{\sigma}{2}}v\|^2 = & \frac{\alpha}{\rho_2}\|A^{\frac{\sigma}{2}}u\|^2 + \frac{\beta_1}{\rho_1}\langle A^{\frac{\sigma+1}{2}}u, A^{\frac{\sigma+1}{2}}v\rangle - \frac{\beta_2}{\rho_2}\langle A^{\frac{\sigma+1}{2}}v, A^{\frac{\sigma+1}{2}}u\rangle + \frac{\alpha}{\rho_1}\langle A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v\rangle\\
& - \frac{\alpha}{\rho_2}\langle A^{\frac{\sigma}{2}}v, A^{\frac{\sigma}{2}}u\rangle - \lambda^2\langle A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v\rangle + \lambda^2\langle A^{\frac{\sigma}{2}}v, A^{\frac{\sigma}{2}}u\rangle\\
& + \frac{1}{\rho_1}\int_0^\infty g(s)\langle A^\theta(\eta(s)-u), A^\sigma v\rangle ds + i\lambda\langle A^{\frac{\sigma}{2}}f_2, A^{\frac{\sigma}{2}}u\rangle\\
& - i\lambda\langle A^{\frac{\sigma}{2}}f_1, A^{\frac{\sigma}{2}}v\rangle + \langle f_4, A^\sigma u\rangle - \langle f_3, A^\sigma v\rangle.\end{aligned}$$

Computing real part of above equality and using definition presented in (3.6) follows that

$$\begin{aligned}\frac{\alpha}{\rho_1}\|A^{\frac{\sigma}{2}}v\|^2 = & \frac{\alpha}{\rho_2}\|A^{\frac{\sigma}{2}}u\|^2 + \chi_0\text{Re}\langle A^{\frac{\sigma+1}{2}}u, A^{\frac{\sigma+1}{2}}v\rangle + \left(\frac{\alpha}{\rho_1} - \frac{\alpha}{\rho_2}\right)\text{Re}\langle A^{\frac{\sigma}{2}}u, A^{\frac{\sigma}{2}}v\rangle\\
& + \frac{1}{\rho_1}\text{Re}\int_0^\infty g(s)\langle A^\theta(\eta(s)-u), A^\sigma v\rangle ds - \lambda\text{Im}\langle A^{\frac{\sigma}{2}}f_2, A^{\frac{\sigma}{2}}u\rangle\\
& + \lambda\text{Im}\langle A^{\frac{\sigma}{2}}f_1, A^{\frac{\sigma}{2}}v\rangle - \text{Re}\langle f_3, A^\sigma v\rangle + \text{Re}\langle f_4, A^\sigma u\rangle.\end{aligned}$$

Choosing $\sigma = \theta - 1$ we obtain the identity

$$\begin{aligned}\|A^{\frac{\theta-1}{2}}v\|^2 = & \frac{\rho_1}{\rho_2}\|A^{\frac{\theta-1}{2}}u\|^2 + \chi_0\frac{\rho_1}{\alpha}\text{Re}\langle A^{\frac{\theta}{2}}u, A^{\frac{\theta}{2}}v\rangle + \left(1 - \frac{\rho_1}{\rho_2}\right)\text{Re}\langle A^{\frac{\theta-1}{2}}u, A^{\frac{\theta-1}{2}}v\rangle\\
& + \frac{1}{\alpha}\text{Re}\int_0^\infty g(s)\langle A^{\frac{2\theta-1}{2}}(\eta(s)-u), A^{\frac{2\theta-1}{2}}v\rangle ds - \frac{\rho_1}{\alpha}\text{Im}\langle A^{\frac{\theta-1}{2}}f_2, A^{\frac{\theta-1}{2}}(\lambda u)\rangle\\
& + \frac{\rho_1}{\alpha}\text{Im}\langle A^{\frac{\theta-1}{2}}f_1, A^{\frac{\theta-1}{2}}(\lambda v)\rangle - \frac{\rho_1}{\alpha}\text{Re}\langle f_3, A^{\theta-1}v\rangle + \frac{\rho_1}{\alpha}\text{Re}\langle f_4, A^{\theta-1}u\rangle.\end{aligned} \tag{3.39}$$

From Cauchy-Schwarz inequality, continuous embedding $\mathcal{D}(A^{\frac{\theta}{2}}) \hookrightarrow \mathcal{D}(A^{\frac{\theta-1}{2}})$, Lemma 26 and Lemma 32 we can conclude the following estimates

$$\|A^{\frac{\theta-1}{2}}v\|^2 \leq C|\chi_0|\|A^{\frac{\theta}{2}}u\|\|A^{\frac{\theta}{2}}v\| + \frac{2\varepsilon}{\lambda^2}\|A^{\frac{\theta}{2}}v\|^2 + \frac{C_\delta}{\varepsilon}\left(\|F\|\|U\| + \|F\|^2\right).$$

■

Lemma 34. *The solution* (u,v) *of system* (3.25)-(3.29) *obtained in theorem* 23 *satisfy the following inequality for* $\chi_0 \neq 0$

(i) $\|A^{\frac{\theta}{2}}v\|^2 \leq C_\delta\lambda^4\left(\|F\|\|U\| + \|F\|^2\right)$;

(ii) $\|A^{\frac{\theta}{2}}\dot{v}\|^2 \leq C_\delta\lambda^6\left(\|F\|\|U\| + \|F\|^2\right)$;

(iii) $\|A^{\frac{\theta-1}{2}}\dot{v}\|^2 \leq C_\delta\lambda^4\left(\|F\|\|U\| + \|F\|^2\right)$.

Proof. Item (i): Computing $\sigma = \theta - 1$ on Lemma 29 and using equation (3.26) we have

$$\|A^{\frac{\theta}{2}}v\|^2 \leq C_\delta\left(\lambda^2\|A^{\frac{\theta-1}{2}}v\|^2 + \|F\|\|U\| + \|F\|^2\right) \tag{3.40}$$

From Lemma 33 we have

$$\lambda^2\|A^{\frac{\theta-1}{2}}v\|^2 \leq C_\delta\lambda^2|\chi_0|\|A^{\frac{\theta}{2}}u\|\|A^{\frac{\theta}{2}}v\| + C_\delta\varepsilon\|A^{\frac{\theta}{2}}v\|^2 + \frac{C_\delta}{\varepsilon}\lambda^2\left(\|F\|\|U\| + \|F\|^2\right) \tag{3.41}$$

Substituting (3.41) in (3.40), using Young inequality and choosing ε small enough we obtain

$$\|A^{\frac{\theta}{2}}v\|^2 \leq C_\delta\lambda^4|\chi_0|^2\|A^{\frac{\theta}{2}}u\|^2 + C_\delta\lambda^2\left(\|F\|\|U\| + \|F\|^2\right).$$

Finally, using Lemma 26 (item (ii)) it may be concluded that

$$\|A^{\frac{\theta}{2}}v\|^2 \leq C_\delta\lambda^4\left(\|F\|\|U\| + \|F\|^2\right).$$

Item (ii): From equation (3.26) we have

$$\|A^{\frac{\theta}{2}}\dot{v}\|^2 \leq C\lambda^2\|A^{\frac{\theta}{2}}v\|^2 + C\|F\|^2.$$

Therefore, the result follows from last item.

Item (iii): Computing the inner product between equation (3.28) and $i\lambda A^{\theta-1}\dot{v}$ we have

$$\begin{aligned}\rho_2\|\lambda A^{\frac{\theta-1}{2}}\dot{v}\|^2 =& \beta_2\|A^{\frac{\theta}{2}}\dot{v}\|^2 + \beta_2\langle A^{\frac{\theta}{2}}f_2, A^{\frac{\theta}{2}}\dot{v}\rangle + \alpha\langle A^{\frac{\theta-1}{2}}(i\lambda v), A^{\frac{\theta-1}{2}}\dot{v}\rangle \\ &+ \alpha\langle A^{\frac{\theta-1}{2}}u, i\lambda A^{\frac{\theta-1}{2}}\dot{v}\rangle + \rho_2\langle f_4, i\lambda A^{\theta-1}\dot{v}\rangle.\end{aligned}$$

Thus, using equation (3.26) and Young inequality we have

$$\|\lambda A^{\frac{\theta-1}{2}}\dot{v}\|^2 \leq C\left\{\|A^{\frac{\theta}{2}}\dot{v}\|^2 + \|A^{\frac{\theta-1}{2}}u\|^2 + \|F\|\|U\| + \|F\|^2\right\}$$

We obtain the result using Lemma 26 and item (ii). ■

Lemma 35. *In the same hypothesis of lemma 26 with $\chi_0 = 0$, the solution of system (3.25)-(3.29) satisfy the following estimates*

(i) $\|\dot{v}\|^2 \leq C_\delta |\lambda|^{2-2\theta}(\|F\|\|U\| + \|F\|^2)$*, for* $0 \leq \theta \leq \frac{1}{2}$;

(ii) $\|\dot{u}\|^2 \leq C_\delta |\lambda|^{2-2\theta}(\|F\|\|U\| + \|F\|^2)$*, for* $0 \leq \theta \leq \frac{1}{2}$;

(iii) $\|\dot{v}\|^2 \leq C_\delta |\lambda|^{2\theta}(\|F\|\|U\| + \|F\|^2)$*, for* $\frac{1}{2} \leq \theta \leq 1$;

(iv) $\|\dot{u}\|^2 \leq C_\delta |\lambda|^{2\theta}(\|F\|\|U\| + \|F\|^2)$*, for* $\frac{1}{2} \leq \theta \leq 1$.

Proof. For items (i) and (ii) we use

$$0 = \theta\left(\frac{\theta - 1}{2}\right) + (1-\theta)\left(\frac{\theta}{2}\right).$$

Item (i) : From interpolation inequality and Lemma 31 (items (i)-(ii)) we obtain

$$\begin{aligned} \|\dot{v}\| &\leq C\|A^{\frac{\theta-1}{2}}\dot{v}\|^{\theta}\|A^{\frac{\theta}{2}}\dot{v}\|^{1-\theta} \\ &\leq C_\delta\left(\sqrt{\|F\|\|U\| + \|F\|^2}\right)^{\theta}\lambda^{(1-\theta)}\left(\sqrt{\|F\|\|U\| + \|F\|^2}\right)^{1-\theta} \\ &\leq C_\delta \lambda^{1-\theta}\sqrt{\|F\|\|U\| + \|F\|^2}. \end{aligned}$$

Item (ii) : We use the same interpolation of item (i), Lemma 26 and Lemma 30.

For items (iii) and (iv) we use

$$0 = (1-\theta)\left(-\frac{\theta}{2}\right) + \theta\left(\frac{1-\theta}{2}\right).$$

Item (iii) : From interpolation inequality and Lemma 31 (items (iii)-(iv)) we have

$$\begin{aligned} \|\dot{v}\| &\leq C\|A^{-\frac{\theta}{2}}\dot{v}\|^{1-\theta}\|A^{\frac{1-\theta}{2}}\dot{v}\|^{\theta} \\ &\leq C_\delta\left(\sqrt{\|F\|\|U\| + \|F\|^2}\right)^{1-\theta}\lambda^{\theta}\left(\sqrt{\|F\|\|U\| + \|F\|^2}\right)^{\theta} \\ &\leq C_\delta \lambda^{\theta}\sqrt{\|F\|\|U\| + \|F\|^2}. \end{aligned}$$

Item (iv) : We use the same interpolation of item (iii) and estimates from Lemma 30. ∎

Lemma 36. *In the same hypothesis of Lemma 26, the solutions of system (3.25)-(3.29) satisfy following estimates*

(i) $\|\dot{v}\|^2 \leq C_\delta |\lambda|^{6-2\theta}(\|F\|\|U\| + \|F\|^2)$*, for* $\chi_0 \neq 0$,

(ii) $\|\dot{u}\|^2 \leq C_\delta |\lambda|^{2}(\|F\|\|U\| + \|F\|^2)$*, for* $\chi_0 \in \mathbb{R}$.

Proof. Item (i) : We use

$$0 = \theta \left(\frac{\theta - 1}{2} \right) + (1 - \theta) \frac{\theta}{2}.$$

Therefore, from interpolation inequality and Lemma 34 we obtain

$$\begin{aligned}
\|\dot{v}\| &\leq C \|A^{\frac{\theta-1}{2}} \dot{v}\|^{\theta} \|A^{\frac{\theta}{2}} \dot{v}\|^{(1-\theta)} \\
&\leq C_\delta \lambda^{2\theta} \left(\sqrt{\|F\|\|U\| + \|F\|^2} \right)^{\theta} \lambda^{3(1-\theta)} \left(\sqrt{\|F\|\|U\| + \|F\|^2} \right)^{1-\theta} \\
&\leq C_\delta \lambda^{3-\theta} \sqrt{\|F\|\|U\| + \|F\|^2}.
\end{aligned}$$

Item (ii) : Using Lemma (26) we have

$$\|\dot{u}\|^2 \leq C \|A^{\frac{\theta}{2}} \dot{u}\|^2 \leq C_\delta \lambda^2 \left(\|F\|\|U\| + \|F\|^2 \right).$$

■

As mentioned earlier we will use Theorem 16 to show Theorem 24 and 25. To proof these theorems we need to show that $i\mathbb{R} \subset \rho(\mathbb{B})$.

Theorem 37. *The operator $\mathbb{B}$ associated with Cauchy problem 3.11 has the property that $i\mathbb{R} \subset \rho(\mathbb{B})$ for χ_0 as (3.6) (considering $\chi_0 = 0$ and $\chi_0 \neq 0$).*

Proof. It will be considered that $i\mathbb{R} \not\subseteq \rho(\mathbb{B})$ and then it will be absurd. As $0 \in \rho(\mathbb{B})$ let's suppose the highest positive number λ_0 such that $]-i\lambda_0, i\lambda_0[\subset \rho(\mathbb{B})$. Then, $i\lambda_0 \in \sigma(\mathbb{B})$ or $-i\lambda_0 \in \sigma(\mathbb{B})$. Supposing that $i\lambda_0 \in \sigma(\mathbb{B})$ (similarly if $-i\lambda_0 \in \sigma(\mathbb{B})$) and fixing a constant $\delta > 0$ with $\delta < \lambda_0$ there exists a sequence of positive real numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that $\delta \leq \lambda_n < \lambda_0$, with $\lambda_n \to \lambda_0$, and a sequence $U_n = (u_n, v_n, \dot{u}_n, \dot{v}_n, \eta_n) \in D(\mathbb{B})$ with $\|U_n\| = 1$ such that

$$\|(i\lambda_n - \mathbb{B})U_n\| = \|F_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

That is, if $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n}, f_{5n})$ then

$$\begin{aligned}
i\lambda_n u_n - \dot{u}_n &= f_{1n} \to 0 \quad \text{in} \quad \mathcal{D}(A^{\frac{1}{2}}), \\
i\lambda_n v_n - \dot{v}_n &= f_{2n} \to 0 \quad \text{in} \quad \mathcal{D}(A^{\frac{1}{2}}), \\
i\rho_1 \lambda_n \dot{u}_n + A_0 u_n + \mathbb{D}\eta_n + \alpha(u_n - v_n) &= \rho_1 f_{3n} \to 0 \quad \text{in} \quad \mathbb{H}, \\
i\rho_2 \lambda_n \dot{v}_n + \beta_2 A v_n + \alpha(v_n - u_n) &= \rho_2 f_{4n} \to 0 \quad \text{in} \quad \mathbb{H}, \\
i\lambda_n \eta_n - \dot{u}_n + \partial_s \eta_n &= f_{5n} \to 0 \quad \text{in} \quad \mathcal{M}_\theta.
\end{aligned}$$

From Lemma 27 we have

$$\|U_n\|^2 \le C_\delta \left(\|\dot{u}_n\|^2 + \|\dot{v}_n\|^2 + \|F_n\|\|U_n\| + \|F_n\|^2\right).$$

Furthermore, from estimates given by Lemma 36 we conclude

$$\|U_n\|^2 \le C_\delta \lambda_n^{6-2\theta} \left(\|F_n\|\|U_n\| + \|F_n\|^2\right).$$

Thus, since $\lambda_n < \lambda_0$ and $\|U_n\| = 1$ follows that

$$1 = \|U_n\|^2 \le C_\delta \lambda_0^{6-2\theta} \left(\|F_n\| + \|F_n\|^2\right) \to 0, \text{when } n \to \infty,$$

that is, absurd. For the case $\chi_0 = 0$ it is necessary just to identify correct decay rates obtained in Lemma 35 and then the result follows for the same argument. So, we conclude $i\mathbb{R} \subset \rho(\mathbb{B})$, $\forall \theta \in [0, 1]$. ■

Proof of Theorem 24: Now, consider $U = (u, v, \dot{u}, \dot{v}, \eta)$ solution of system $(i\lambda - \mathbb{B})U = F$. According for Theorem 16, to show the polynomial decay of semigroup $e^{t\mathbb{B}}$ it is sufficient to prove that $\|U\| \le C\lambda^\sigma \|F\|$ for $|\lambda| \ge 1$.

Proof of item 1: From Lemma 27 and Lemma 35 (items (i) and (ii)) follows that

$$\|U\|^2 \le C_\delta |\lambda|^{2-2\theta} \left(\|F\|\|U\| + \|F\|^2\right),$$

the result follows applying Young inequality to first term on right side of this inequality, that means $\lambda^{-(2-2\theta)}\|(i\lambda I - \mathbb{B})^{-1}\|$ is bounded. Then, for Theorem 16 the semigroup $e^{t\mathbb{B}}$ decays polynomially with rate $t^{-\frac{1}{2-2\theta}}$.

Proof of item 2: We use Lemma 27 and items (iii) and (iv) of Lemma 35.

Proof of Theorem 25: Using a similar way to that done in the previous proof with Lemma 36 we obtain the result.

3.4 Optimality of the decay rates

In this section we will see that the decay rates obtained in Theorem 18 are the best. Our main result is given by the following theorem:

Theorem 38. *Consider $g(t) = g(0)e^{-\delta t}$, $\delta > 0$ and $\theta \in [0, 1]$. The polynomial decay rates found in Theorem 24 and 25 are optimal in the following sense:*

1. *If $\chi_0 = 0$ and $\theta \ge 1/2$, then the semigroup does not decay with the rate $t^{-\sigma}$ for $\sigma > 1/(2\theta)$.*

2. *If* $\chi_0 = 0$ *and* $\theta \leq 1/2$*, then the semigroup does not decay with the rate* $t^{-\sigma}$ *for* $\sigma > 1/(2-2\theta)$.

3. *If* $\chi_0 \neq 0$ *and* θ *is any in the interval* $[0,1]$*, then the semigroup does not decay with the rate* $t^{-\sigma}$ *for* $\sigma > 1/(6-2\theta)$.

Proof. We will use the Borichev and Tomilov's theorem to prove these results. The main idea is to find a sequence of forces (F_n) and a sequence of real numbers (λ_n) tending to infinite in such a way that the solution U_n of the system $(i\lambda_n I - \mathbb{B})U = F_n$ has the same asymptotic behavior as λ_n^k where the exponent k is one of the rates 2θ, $2-2\theta$ or $6-2\theta$.

For this, we will use the spectrum of the operator A. Since it is a positive self-adjoint operator with compact resolvent, the spectrum of this operator is given by a sequence of positive eigenvalues (γ_n) such that $\gamma_n \to \infty$. The corresponding unit eigenvectors are denoted by (e_n), thus

$$Ae_n = \gamma_n e_n, \quad \|e_n\| = 1, \quad n \in \mathbb{N}.$$

We introduce the sequence $F_n = (0, 0, 0, -\rho_2^{-1} e_n, 0)$. The solution $U = (u, v, \dot{u}, \dot{v}, \eta)$ of the system $(i\lambda I - \mathbb{B})U = F_n$, in components reads

$$i\lambda u - \dot{u} = 0, \tag{3.42}$$

$$i\lambda v - \dot{v} = 0, \tag{3.43}$$

$$i\rho_1 \lambda \dot{u} + A_0 u + \mathbb{D}\eta + \alpha(u - v) = 0, \tag{3.44}$$

$$i\rho_2 \lambda \dot{v} + \beta_2 A v + \alpha(v - u) = -e_n, \tag{3.45}$$

$$i\lambda \eta(s) + \partial_s \eta(s) = \dot{u}. \tag{3.46}$$

We substitute $\dot{u}$ from equation (3.42) in (3.46). Solving the resultant equation and using $\eta(0) = 0$, we obtain

$$\eta(s) = u(1 - e^{-i\lambda s}).$$

Substituting this function in (3.44) and taking into account that $A_0 = \beta_1 A - \kappa A^\theta$ we get

$$i\rho_1 \lambda \dot{u} + \beta_1 A u - \left(\int_0^\infty g(s) e^{-i\lambda s} ds \right) A^\theta u + \alpha(u - v) = 0.$$

Now, we use (3.42) in the above equation, and (3.43) in equation (3.45) to obtain the following system

$$\rho_1\lambda^2 u - \beta_1 Au + \left(\int_0^\infty g(s)e^{-i\lambda s}ds\right) A^\theta u - \alpha(u-v) = 0, \tag{3.47}$$

$$\rho_2\lambda^2 v - \beta_2 Av - \alpha(v-u) = e_n. \tag{3.48}$$

In this point, we will look for solutions projected in a one-dimensional space, that is

$$\begin{cases} u = \kappa_1 e_n \\ v = \kappa_2 e_n \end{cases} ; \quad \kappa_1, \kappa_2 \in \mathbb{C}.$$

The substitution of these terms in the system (3.47)-(3.48) gives

$$\left(\rho_1\lambda^2 - \beta_1\gamma_n - \alpha + \gamma_n^\theta \int_0^\infty g(s)e^{-i\lambda s}ds\right)\kappa_1 + \alpha\kappa_2 = 0,$$

$$\left(\rho_2\lambda^2 - \beta_2\gamma_n - \alpha\right)\kappa_2 + \alpha\kappa_1 = 1.$$

Solving this algebraic system, the solution of the second component is

$$\kappa_2 = \frac{P_1(\lambda^2) + I_\lambda\gamma_n^\theta}{P_1(\lambda^2)P_2(\lambda^2) + I_\lambda\gamma_n^\theta P_2(\lambda^2) - \alpha^2}. \tag{3.49}$$

where the polynomials P_1, P_2 are given by

$$P_1(s) = \rho_1 s - \beta_1\gamma_n - \alpha, \quad P_2(s) = \rho_2 s - \beta_2\gamma_n - \alpha, \tag{3.50}$$

and the integral term

$$I_\lambda = \int_0^\infty g(s)e^{-i\lambda s}\,ds = \int_0^\infty g(0)e^{-(\delta+i\lambda)s}\,ds = \frac{g(0)}{\delta + i\lambda}. \tag{3.51}$$

Proof of item 1. Consider $\chi_0 = 0$ and $\theta \geq 1/2$. We define

$$\lambda_n := \sqrt{\frac{\beta_2\gamma_n + \alpha}{\rho_2}}.$$

In what follows, the notation $a_n \approx b_n$ will mean that $\lim\limits_{n\to\infty} \frac{|a_n|}{|b_n|}$ is a positive real number. With this notation we have

$$\lambda_n \approx \gamma_n^{1/2}.$$

Also, from the definitions of the polynomials P_1, P_2 in (3.50) we obtain

$$P_2(\lambda_n^2) = 0 \quad \text{and} \quad P_1(\lambda_n^2) = \frac{\alpha(\rho_1 - \rho_2)}{\rho_2}.$$

Considering $\lambda = \lambda_n$ in (3.49) and taking into account (3.51) we get

$$\kappa_{2,n} = -\frac{\alpha(\rho_1-\rho_2)}{\alpha^2\rho_2} - \frac{I_{\lambda_n}\gamma_n^\theta}{\alpha^2} = -\frac{\alpha(\rho_1-\rho_2)}{\alpha^2\rho_2} - \frac{g(0)(\delta - i\lambda_n)\gamma_n^\theta}{\alpha^2(\delta^2+\lambda_n^2)}$$

whose imaginary part is given by

$$\mathrm{Im}(\kappa_{2,n}) = \frac{g(0)\lambda_n\gamma_n^\theta}{\alpha^2(\delta^2+\lambda_n^2)} \approx \lambda_n^{2\theta-1}.$$

Therefore, if $U_n = (u_n, v_n, \dot{u}_n, \dot{v}_n, \eta_n)$ is the solution of system $(i\lambda_n I - \mathbb{B})U = F_n$ we obtain

$$\|U_n\| \geq \rho_2\|\dot{v}_n\| = \rho_2\lambda_n|\kappa_{2,n}|\|e_n\| \geq \rho_2\lambda_n|\mathrm{Im}(\kappa_{2,n})| \geq \varepsilon\lambda_n^{2\theta}, \tag{3.52}$$

for some $\varepsilon > 0$ and n large enough. In this point, if the semigroup of the system decays polynomially with rate $t^{-\sigma}$, $\sigma > 1/(2\theta)$, we have

$$\varepsilon\lambda_n^{2\theta} \leq \|U_n\| \leq C\lambda_n^{1/\sigma}\|F_n\| \quad \Rightarrow \quad \varepsilon\lambda_n^{2\theta-1/\sigma} \leq C \tag{3.53}$$

which is contradictory because $\lambda_n^{2\theta-1/\sigma} \to \infty$ when $n \to \infty$. Therefore the decay rate $t^{-1/(2\theta)}$ is optimal.

Proof of item 2 and 3. We start rewriting the polynomial that appears in the formula (3.49):

$$\begin{aligned} &P_1(s)P_2(s) - \alpha^2 \\ &\quad = \rho_1\rho_2\left\{s^2 + \left(\beta_0\gamma_n + \frac{\alpha(\rho_1+\rho_2)}{\rho_1\rho_2}\right)s + \frac{(\beta_1\gamma_n+\alpha)}{\rho_1}\frac{(\beta_2\gamma_n+\alpha)}{\rho_2} - \frac{\alpha^2}{\rho_1\rho_2}\right\}, \end{aligned}$$

where $\beta_0 = \frac{\beta_1}{\rho_1} + \frac{\beta_2}{\rho_2}$. The roots of this polynomial are

$$s_n^{\pm} = \frac{\beta_0\gamma_n + \frac{\alpha(\rho_1+\rho_2)}{\rho_1\rho_2} \pm \sqrt{\left(\chi_0\gamma_n + \frac{\alpha(\rho_2-\rho_1)}{\rho_1\rho_2}\right)^2 + \frac{4\alpha^2}{\rho_1\rho_2}}}{2}. \tag{3.54}$$

We define $\lambda_n := \sqrt{s_n^+}$. As $s_n^+ \approx \gamma_n$ we have

$$\lambda_n \approx \gamma_n^{1/2}. \tag{3.55}$$

Evaluating polynomial P_1 in λ_n^2 gives

$$\frac{2}{\rho_1}P_1(\lambda_n^2) = -\chi_0\gamma_n - \frac{\alpha(\rho_2-\rho_1)}{\rho_1\rho_2} + \sqrt{\left(\chi_0\gamma_n + \frac{\alpha(\rho_2-\rho_1)}{\rho_1\rho_2}\right)^2 + \frac{4\alpha^2}{\rho_1\rho_2}}. \tag{3.56}$$

From this formula and using estimate (3.55) we get

$$\begin{cases} P_1(\lambda_n^2) \approx 1 & \text{for} \quad \chi_0 = 0, \\ P_1(\lambda_n^2) \approx \lambda_n^2 & \text{for} \quad \chi_0 < 0. \end{cases} \tag{3.57}$$

Since $P_1(\lambda_n^2)P_2(\lambda_n^2) - \alpha^2 = 0$ we have $P_2(\lambda_n^2) = \frac{\alpha^2}{P_1(\lambda_n^2)}$. Therefore, considering $\lambda = \lambda_n$ in (3.49) we obtain

$$\kappa_{2,n} = \left(\frac{P_1(\lambda_n^2) + I_{\lambda_n}\gamma_n^\theta}{I_{\lambda_n}\gamma_n^\theta} \right) \frac{P_1(\lambda_n^2)}{\alpha^2}. \tag{3.58}$$

Considering the imaginary part of this term and taking into account the estimates (3.55) and (3.57), we conclude that

$$\text{Im}(\kappa_2) = \frac{\lambda_n P_1^2(\lambda_n^2)}{g(0)\gamma_n^\theta \alpha^2} \approx \begin{cases} \lambda_n^{1-2\theta} & \text{for} \quad \chi_0 = 0, \\ \lambda_n^{5-2\theta} & \text{for} \quad \chi_0 < 0. \end{cases}$$

Therefore, performing the same accounts as in (3.52)-(3.53) we can conclude that under the hypothesis of item 2 of this theorem ($\chi_0 = 0$ and $\theta \leq 1/2$) the polynomial decay rate $t^{-1/(2-2\theta)}$ is optimal. On the other hand, if $\chi_0 < 0$ (partial condition in the item 3), then the polynomial decay rate $t^{-1/(6-2\theta)}$ is optimal.

To complete the proof of item 3 we will address the case $\chi_0 > 0$. For this case we define $\lambda_n =: \sqrt{s_n^-}$. Since

$$s_n^+ s_n^- = \frac{(\beta_1\gamma_n + \alpha)}{\rho_1}\frac{(\beta_2\gamma_n + \alpha)}{\rho_2} - \frac{\alpha^2}{\rho_1\rho_2}$$

and $s_n^+ \approx \gamma_n$ we have $s_n^- \approx \gamma_n$. Consequently,

$$\lambda_n \approx \gamma_n^{1/2}.$$

This time, evaluating polynomial P_1 in $\lambda_n^2 = s_n^-$ gives

$$\frac{2}{\rho_1}P_1(\lambda_n^2) = -\chi_0\gamma_n - \frac{\alpha(\rho_2 - \rho_1)}{\rho_1\rho_2} - \sqrt{\left(\chi_0\gamma_n + \frac{\alpha(\rho_2 - \rho_1)}{\rho_1\rho_2}\right)^2 + \frac{4\alpha^2}{\rho_1\rho_2}}.$$

Since $\chi_0 > 0$ we obtain

$$P_1(\lambda_n^2) \approx \gamma_n \approx \lambda_n^2.$$

As the above estimates coincide with (3.55) and (3.57) (case $\chi_0 < 0$), we have again

$$\text{Im}(\kappa_2) = \frac{\lambda_n P_1^2(\lambda_n^2)}{g(0)\gamma_n^\theta \alpha^2} \approx \lambda_n^{5-2\theta}$$

Therefore, under the hypothesis of item 3 ($\chi_0 \neq 0$)), $t^{-1/(6-2\theta)}$ is the optimal decay rate. ∎

Chapter 4

Asymptotic behavior for a coupled wave/plate system with fractional memory dissipation

In this chapter, we consider a coupled system of two equations with different characteristics. This wave-plate system has indirect damping acting in one equation, first in the wave equation, second in the plate equation and we also study what happens when it is considered the damping in two equations. In summary, this research presents asymptotic behavior (exponential and polynomial decay) for the solutions of this system. When it is possible, optimal decay rates are found.

4.1 Motivation

Coupled systems of two or more elastic materials have been studied by many researchers over time. The asymptotic behavior of these systems has aroused special interest mainly when part of it has minimal dissipative properties, or when the components of the system have characteristics of different nature. In the literature, we find several results on the asymptotic behavior where a component of the system transfers its dissipative properties to the other ones, for example, in Timoshenko beams, Bresse systems, thermoelastic systems, coupled systems of wave-wave, plate-plate, or wave-plate interactions, and more. Part of the components of these systems can have conservative or dissipative characteristics, but the important thing in stability studies is that the conservative part can absorb the properties of the dissipative part to stabilize the system. Some of these works can be found in [3, 5, 12, 36, 28, 37, 48, 67, 72].

When the components of a conservative-dissipative system are of the same nature, some studies show that for the system to stabilize or to have a faster stabilization, it is necessary to have a good relationship between the structural coefficients. This feature was first noticed by Soufyane [70] who studied the Timoshenko system

$$\rho_1 u_{tt} - K(u_{xx} - v_x) = 0,$$
$$\rho_2 v_{tt} - b v_{xx} + K(u_x - v) + a(x) v_t = 0.$$

He showed that the solutions of this system decay exponentially if and only if the condition

$$\frac{K}{\rho_1} = \frac{b}{\rho_2},$$

is satisfied. When no relationship is established between the coefficients in a Timoshenko system, depending on the dissipation, the system simply does not decay. This was noted by Bassam et al. [10] who considered this problem with boundary dissipation. They showed that for certain relations between the structural coefficients, the system does not stabilize, but with complementary relations, the system is polynomially stable. This type of behavior has also been observed in couplings between membranes and also between plates. Some of these results have been shown in [1, 8, 57, 61, 62, 58, 48]

Our interest in this work is to study coupled systems where the components are of a different nature and establish whether the decay depends on some relationships between their structural coefficients. For this reason, we consider a weakly coupled system of wave-plate equations with memory damping. For Ω a bounded open set of $\mathbb{R}^n$ the coupled system is given by

$$\begin{cases} \rho_1 u_{tt} - \beta_1 \Delta u - \displaystyle\int_0^\infty g_1(s)(-\Delta)^{\theta_1} u(t-s)ds + \alpha(u-v) = 0, & \text{in} \quad \Omega \times \mathbb{R}^+, \\ \rho_2 v_{tt} + \beta_2 \Delta^2 v - \displaystyle\int_0^\infty g_2(s)\Delta^{2\theta_2} v(t-s)ds + \alpha(v-u) = 0, & \text{in} \quad \Omega \times \mathbb{R}^+, \end{cases} \tag{4.1}$$

subjected to initial conditions

$$u(0) = u_0, \quad v(0) = v_0, \quad u_t(0) = u_1, \quad v_t(0) = v_1, \tag{4.2}$$

$$u(-s) = \phi_1(s), \quad v(-s) = \phi_2(s), \; s > 0, \; \phi_1(0) = u_1, \; \phi_2(0) = u_2 \tag{4.3}$$

and satisfying the boundary condition

$$u = 0, \; v = 0, \; \Delta v = 0, \; \text{on } \partial\Omega \times \mathbb{R}^+. \tag{4.4}$$

For this system we will try to answer several questions: if there is decay, does it depend on any relationships between the structural coefficients? Which dissipative part is dominant, the dissipation of the wave or the plate? what role do the exponents θ_1 and θ_2 play in the decay? If we remove one of the dissipations, what kind of decay do we get?

In the literature we find some studies for similar systems of this problem, we will mention some of them: Tebou [71] considered a plate-wave system with frictional damping

$$y_{tt} + \Delta^2 y + \alpha z + \kappa_1 y_t = 0,$$
$$z_{tt} - \Delta z + \alpha y + \kappa_2 z_t = 0,$$

with clamped boundary conditions for the plate displacement and Dirichlet boundary conditions for the wave displacement. He considered the cases $(\kappa_1, \kappa_2) = (1, 0)$ and $(0, 1)$ and showed that the system cannot decay exponentially. However, he was able to prove that the strong solutions decay polynomially with rates $t^{-1/8}$ and $t^{-1/4}$ respectively.

A coupled plate-wave system with memory damping was considered by Matos et al. [45]. The system they studied was formulated in an abstract way:

$$u_{tt} + A_1 u - \int_0^\infty g(s) A_2 u(t-s)ds + \beta v = 0,$$
$$v_{tt} + A_3 v + \beta u = 0,$$

where the abstract differential operator A_1 is more strong than A_2 and A_3, $A_2 = o(A_1^\alpha)$, $A_2 = o(A_1^\gamma)$, and the kernel is an exponentially decreasing function. They showed that the solution of this system does not decay exponentially, but they showed a polynomial decay for the initial date with appropriate regularity. The decay rates found are not necessarily optimal because the results do not show the α, γ dependence. Also, Guesmia [31] studied this problem considering kernels that decrease in a broader sense than those with exponential or polynomial decay. He found decay rates according to the decay of the kernel. Other results about the stabilization in plate-wave interactions can be view in [6, 16, 27, 32, 68, 72, 73].

We will study the system (4.1)-(4.4) in an abstract format. We consider an unbounded positive self-adjoint operator A with domain $D(A)$ in the Hilbert space $\mathbb{H}$, and we assume that A has a compact inverse. Note that $A = -\Delta$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ satisfy this condition in the Hilbert space $\mathbb{H} = L^2(\Omega)$. Then the system we study is given by

$$\begin{cases} \rho_1 u_{tt} + \beta_1 A u - \displaystyle\int_0^\infty g_1(s) A^{\theta_1} u(t-s)ds + \alpha(u - v) = 0, \\ \rho_2 v_{tt} + \beta_2 A^2 v - \displaystyle\int_0^\infty g_2(s) A^{2\theta_2} v(t-s)ds + \alpha(v - u) = 0, \end{cases} \tag{4.5}$$

satisfying the initial data

$$\begin{cases} u(0) = & u_0, \quad v(0) = v_0, \quad u_t(0) = u_1, \quad v_t(0) = v_1, \\ & u(-s) = \phi_1(s), \quad v(-s) = \phi_2(s) \; s > 0. \end{cases} \tag{4.6}$$

Here, the density and the elasticity coefficients $\rho_1,\ \rho_2,\ \beta_1,\ \beta_2$ are positive constants and the exponents θ_i are considered in the interval $[0,1]$. Moreover, the coupling coefficient α is a positive number.

Furthermore, the kernel of Volterra equations $g_i(t),\ i = 1, 2$, must have exponential decay in a similar way as [48], that is, $g_i(t)$ satisfy the following set of hypotheses

$$\begin{cases} g_i \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \\ g_i(s) \geqslant 0, g_i'(s) < 0, \forall s \in \mathbb{R}^+, \\ \exists\, c_i \in \mathbb{R}^+;\ g_i'(s) \leq -c_i g_i(s), \forall s \in \mathbb{R}^+, \\ 0 < \kappa_i := \displaystyle\int_0^{+\infty} g_i(s) ds < \beta_i \alpha_1^{i(1-\theta_i)}, \end{cases} \tag{4.7}$$

where α_1 is the first eigenvalue of $-\Delta$.

The main result of this chapter is related to asymptotic behavior of solutions of system (4.5)-(4.6). The results that we found are enunciated in Theorems 40 and 53 which can be synthesized as follow:

- For strong initial data and $\theta_1 = \theta_2 = 1$ we have exponential decay;
- For strong initial data and $\theta_1 < 1$ or $\theta_2 < 1$ we have polynomial decay with rate $t^{-\frac{1}{2-2\theta_0}}$, where $\theta_0 = \min\{\theta_1, \theta_2\}$;
- For strong initial data and if we remove the memory term from the second equation of (4.5)-(4.6) then we have polynomial decay with the rate $t^{-\frac{1}{6-\theta_1}}$;
- For strong initial data and if we remove the memory term from the first equation of (4.5)-(4.6) then we have polynomial decay with the rate $t^{-\frac{1}{10-4\theta_2}}$;
- The decay rates in the previous items are the best.

The results are presented as follows: in section 2, we prove the well-posedness of problem (4.5)-(4.6) by semigroup theory. In section 3, we show asymptotic behavior to the problem where it depends on the exponent θ_i and what the types of memory are considered. Furthermore, in the last section we fount optimal decay rates.

4.2 Well-Posedness

In this section we will show the well-posed of system (4.5)-(4.6). For this, to put this system in an abstract framework, we will use the functions $\eta := \eta^t(s)$ and $\vartheta := \vartheta^t(s)$, originally used by [18], such that:

$$\eta^t(s) = u(t) - u(t-s), \tag{4.8}$$

$$\vartheta^t(s) = v(t) - v(t-s), \tag{4.9}$$

Thus, the system (4.5)-(4.6) turn into

$$\begin{cases} \rho_1 u_{tt} + \beta_1 A u - \kappa_1 A^{\theta_1} u + \displaystyle\int_0^{+\infty} g_1(s) A^{\theta_1} \eta(s) ds + \alpha(u-v) = 0, \\ \rho_2 v_{tt} + \beta_2 A^2 v - \kappa_2 A^{2\theta_2} v + \displaystyle\int_0^{+\infty} g_2(s) A^{2\theta_2} \vartheta(s) ds + \alpha(v-u) = 0, \\ \eta_t(s) + \eta_s(s) - u_t = 0, \\ \vartheta_t(s) + \vartheta_s(s) - v_t = 0, \end{cases}$$

therefore, taking $A_1 = \beta_1 A - \kappa_1 A^{\theta_1}$ and $A_2^2 = \beta_2 A^2 - \kappa_2 A^{2\theta_2}$ we have

$$\begin{cases} \rho_1 u_{tt} + A_1 u + \displaystyle\int_0^{+\infty} g_1(s) A^{\theta_1} \eta(s) ds + \alpha(u-v) = 0, \\ \rho_2 v_{tt} + A_2^2 + \displaystyle\int_0^{+\infty} g_2(s) A^{2\theta_2} \vartheta(s) ds + \alpha(v-u) = 0, \\ \eta_t(s) + \eta_s(s) - u_t = 0, \\ \vartheta_t(s) + \vartheta_s(s) - v_t = 0. \end{cases}$$

Finally, If we denote by $U(t) = (u(t), v(t), u_t(t), v_t(t), \eta, \vartheta)$, we can put this system in an equivalent Cauchy problem given by

$$\frac{d}{dt} U(t) = \mathbb{B} U(t), \; U(0) = U_0, \tag{4.10}$$

where the initial data is $U_0 = (u_0, v_0, u_1, v_1, \eta_0, \vartheta_0)$ and the operator $\mathbb{B}$ is given by

$$\begin{aligned} \mathbb{B} U = \big(\dot{u}, \; \dot{v}, \; -\rho_1^{-1} \left\{ A_1 u + \mathbb{D}_1 \eta + \alpha(u-v) \right\}, \\ -\rho_2^{-1} \left\{ A_2^2 v + \mathbb{D}_2 \vartheta + \alpha(v-u) \right\}, \dot{u} - \partial_s \eta, \dot{v} - \partial_s \vartheta \big), \end{aligned}$$

for $U = (u, v, \dot{u}, \dot{v}, \eta, \vartheta)$. Here the point on top of this terms is just a notation, it does not mean the time derivative. Furthermore, we have $\mathbb{D}_1 \eta = \displaystyle\int_0^{+\infty} g_1(s) A^{\theta_1} \eta(s) ds$ in which $\mathbb{D}_1 : \mathcal{M}_1 \to \mathcal{D}(A^{-\frac{\theta_1}{2}})$; $\mathcal{M}_1 := L^2_{g_1}(\mathbb{R}^+; \mathcal{D}(A^{\frac{\theta_1}{2}}))$; and it is endowed with inner product

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{M}_1} = \int_0^{+\infty} g_1(s) \langle A^{\frac{\theta_1}{2}} \eta_1, A^{\frac{\theta_1}{2}} \eta_2 \rangle ds, \forall \eta_1, \eta_2 \in \mathcal{M}_1,$$

in the same way we have that $\mathbb{D}_2\vartheta = \int_0^{+\infty} g_2(s)A^{2\theta_2}\vartheta(s)ds$ in which $\mathbb{D}_2 : \mathcal{M}_2 \to \mathcal{D}(A^{-\theta_2})$ with $\mathcal{M}_2 := L^2_{g_2}(\mathbb{R}^+;\mathcal{D}(A^{\theta_2}))$ and it is provided with inner product

$$\langle \vartheta_1, \vartheta_2\rangle_{\mathcal{M}_2} = \int_0^{+\infty} g_2(s)\langle A^{\theta_2}\vartheta_1, A^{\theta_2}\vartheta_2\rangle ds, \forall \vartheta_1, \vartheta_2 \in \mathcal{M}_2.$$

We will define this operator in an appropriate subspace of Hilbert space

$$\mathbb{X} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A) \times \mathbb{H} \times \mathbb{H} \times \mathcal{M}_1 \times \mathcal{M}_2,$$

where it is provided with inner product

$$\begin{aligned}\langle U_1, U_2\rangle_{\mathbb{X}} =& \langle A_1^{\frac{1}{2}}u_1, A_1^{\frac{1}{2}}u_2\rangle + \langle A_2v_1, A_2v_2\rangle + \rho_1\langle \dot{u}_1, \dot{u}_2\rangle + \rho_2\langle \dot{v}_1, \dot{v}_2\rangle \\ &+ \alpha\langle u_1 - v_1, u_2 - v_2\rangle + \langle \eta_1, \eta_2\rangle_{\mathcal{M}_1} + \langle \vartheta_1, \vartheta_2\rangle_{\mathcal{M}_2},\end{aligned}$$

for $U_1 = (u_1, v_1, \dot{u}_1, \dot{v}_1, \eta_1, \vartheta_1)$ and $U_2 = (u_2, v_2, \dot{u}_2.\dot{v}_2, \eta_2, \vartheta_2)$ in $\mathbb{X}$. With these considerations, the natural domain of operator $\mathbb{B}$ is defined by

$$\begin{aligned}D(\mathbb{B}) = \Big\{&U \in \mathbb{X} : \dot{u} \in \mathcal{D}(A^{\frac{1}{2}}), \dot{v} \in \mathcal{D}(A), A_1u + \mathbb{D}_1\eta \in \mathbb{H}, \\ &A_2^2v + \mathbb{D}_2\vartheta \in \mathbb{H}, \eta \in \mathcal{D}(\Gamma_1), \vartheta \in \mathcal{D}(\Gamma_2)\Big\},\end{aligned}$$

where

$$\mathcal{D}(\Gamma_1) = \{\eta \in \mathcal{M}_1;\ \partial_s\eta \in \mathcal{M}_1 \text{ and } \eta(0) = 0\},$$

and

$$\mathcal{D}(\Gamma_2) = \{\vartheta \in \mathcal{M}_2;\ \partial_s\vartheta \in \mathcal{M}_2 \text{ and } \vartheta(0) = 0\}.$$

Remark 3. *It is important to know that when the first or second equation is in the absence of memory we have different types to the above features developed, that is, when we have $g_1 = 0$ or $g_2 = 0$ it will imply different result for $A_1, A_2, \mathbb{D}_1$ and $\mathbb{D}_2$. For example, if the second equation of the system* (4.5)-(4.6) *is in the absence of memory term, then since the beginning of the text, we won't need to use change variable* (4.9)*. Consequently, all development would be different, since the start definition of operator $\mathbb{B}$, phases space, abstract system* (4.10) *and domain.*

The next result focus on show the well-posedness of Cauchy problem (4.10).

Theorem 39. *Considering U_0 in $\mathbb{X}$, there exists only one solution U of problem* (4.10) *such that $U \in C([0, +\infty[; \mathbb{X})$. Moreover, if U_0 in $D(\mathbb{B})$, we have*

$$U \in C([0, +\infty[; D(\mathbb{B})) \cap C^1([0, +\infty[; \mathbb{X}).$$

Proof. To get the result we need to show that the operator $\mathbb{B}$ is the generator of a C_0-semigroup. In view of that, we will invoke a consequence of Lummer-Phillips's Theorem (enunciated in preliminaries). Note that from definition of $D(\mathbb{B})$ we can conclude that this domain is dense in $\mathbb{X}$. Furthermore, if we consider $U \in \mathcal{D}(A)$ we have

$$\begin{aligned}\langle \mathbb{B}U, U\rangle_{\mathbb{X}} =& \langle A_1^{\frac{1}{2}}\dot{u}, A_1^{\frac{1}{2}}u\rangle + \langle A_2\dot{v}, A_2 v\rangle - \langle A_1 u + \mathbb{D}_1\eta + \alpha(u-v), \dot{u}\rangle \\ &- \langle A_2^2 v + \mathbb{D}_2\vartheta + \alpha(v-u), \dot{v}\rangle + \alpha\langle \dot{u}-\dot{v}, u-v\rangle \\ &+ \langle \dot{u} - \partial_s\eta, \eta\rangle_{\mathcal{M}_1} + \langle \dot{v} - \partial_s\vartheta, \vartheta\rangle_{\mathcal{M}_2},\end{aligned} \tag{4.11}$$

and then

$$\text{Re}\langle \mathbb{B}U, U\rangle_{\mathbb{X}} = -\text{Re}\int_0^\infty g_1(s)\langle \partial_s\eta, \eta\rangle_{\mathcal{D}(A^{\frac{\theta_1}{2}})} ds - \text{Re}\int_0^\infty g_2(s)\langle \partial_s\vartheta, \vartheta\rangle_{\mathcal{D}(A^{\theta_2})} ds. \tag{4.12}$$

Integrating the first term by parts, we obtain

$$-\text{Re}\int_0^\infty g_1(s)\langle \partial_s\eta, \eta\rangle_{\mathcal{D}(A^{\frac{\theta_1}{2}})} ds = \frac{1}{2}\int_0^\infty g_1'(s)\|A^{\frac{\theta_1}{2}}\eta\|^2 ds. \tag{4.13}$$

In the same way to the second term of equation (4.12) we obtain

$$-\text{Re}\int_0^\infty g_2(s)\langle \partial_s\vartheta, \vartheta\rangle_{\mathcal{D}(A^{\theta_2})} ds = \frac{1}{2}\int_0^\infty g_2'(s)\|A^{\theta_2}\vartheta\|^2 ds. \tag{4.14}$$

As a conclusion, from estimates (4.12), (4.13) and (4.14) we have $\text{Re}\langle \mathbb{B}U, U\rangle_{\mathbb{X}}$ is non-positive by condition (4.7). Therefore, the operator $\mathbb{B}$ is dissipative.

Let's show that $0 \in \rho(\mathbb{B})$. Firstly, we need to check that $\text{Im}(\mathbb{B}) = \mathbb{X}$. To this, let $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathbb{X}$ and we will prove that $U \in D(\mathbb{B})$ where $U = (u, v, \dot{u}, \dot{v}, \eta, \vartheta)$ is solution of system $\mathbb{B}U = F$. The vector $U = (u, v, \dot{u}, \dot{v}, \eta, \vartheta)$ is solution of system $\mathbb{B}U = F$ if and only if

$$\dot{u} = f_1 \quad \text{in} \quad \mathcal{D}(A^{\frac{1}{2}}), \tag{4.15}$$

$$\dot{v} = f_2 \quad \text{in} \quad \mathcal{D}(A), \tag{4.16}$$

$$A_1 u + \mathbb{D}_1\eta + \alpha(u-v) = -\rho_1 f_3 \quad \text{in} \quad \mathbb{H}, \tag{4.17}$$

$$A_2^2 v + \mathbb{D}_2\vartheta + \alpha(v-u) = -\rho_2 f_4 \quad \text{in} \quad \mathbb{H}, \tag{4.18}$$

$$\dot{u} - \partial_s\eta = f_5 \quad \text{in} \quad \mathcal{M}_1, \tag{4.19}$$

$$\dot{v} - \partial_s\vartheta = f_6 \quad \text{in} \quad \mathcal{M}_2. \tag{4.20}$$

Note that, in view of equation (4.19) (and using (4.15)) we can assert that

$$\eta(s) = sf_1 - \int_0^s f_5(\tau)d\tau \tag{4.21}$$

is a solution of this equation that satisfies $\eta(0) = 0$. On the other hand, by hypothesis (4.7) we have

$$\int_0^{+\infty} g_1(s)\|A^{\frac{\theta_1}{2}}\eta(s)\|^2 ds \le -\frac{1}{c_1}\int_0^{+\infty} g_1'(s)\|A^{\frac{\theta_1}{2}}\eta(s)\|^2 ds,$$

and then, from (4.13), we get

$$\int_0^{+\infty} g_1(s)\|A^{\frac{\theta_1}{2}}\eta(s)\|^2 \le \frac{2}{c_1}\int_0^{+\infty} g_1(s)|\langle A^{\frac{\theta_1}{2}}\partial_s\eta, A^{\frac{\theta_1}{2}}\eta\rangle| ds,$$

therefore, using Cauchy-Schwarz and Young's inequalities follows that

$$\int_0^{+\infty} g_1(s)\|A^{\frac{\theta_1}{2}}\eta(s)\|^2 \le \frac{4}{c_1^2}\int_0^{+\infty} g_1(s)\|A^{\frac{\theta_1}{2}}\partial_s\eta\|^2 ds. \tag{4.22}$$

However by (4.19) follows that $\partial_s\eta \in \mathcal{M}_1$, consequently by (4.22) we conclude $\eta \in \mathcal{D}(\Gamma_1)$. Following this same way we conclude that $\vartheta \in \mathcal{D}(\Gamma_2)$. Furthermore, considering equations (4.17) and (4.18) we have that

$$\begin{cases} A_1 u + \alpha(u-v) = -\rho_1 f_3 - \mathbb{D}_1\eta, \\ A_2^2 v + \alpha(v-u) = -\rho_2 f_4 - \mathbb{D}_2\vartheta. \end{cases} \tag{4.23}$$

Let $W = (u, v)$ and consider the variational problem

$$a(W,\Phi) = \langle G, \Phi\rangle, \quad \forall\, \Phi = (\varphi,\psi) \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A),$$

where the sesquilinear form $a(\cdot,\cdot) : [\mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A)]^2 \to \mathbb{C}$ and $G : \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A) \to \mathbb{C}$ are defined by

$$\begin{aligned} a(W,\Phi) &= \langle A_1^{\frac{1}{2}}u, A_1^{\frac{1}{2}}\varphi\rangle + \langle A_2 v, A_2\psi\rangle + \alpha\langle u-v, \varphi-\psi\rangle, \\ \langle G,\Phi\rangle &= \langle g_1,\varphi\rangle + \langle g_2,\psi\rangle. \end{aligned}$$

with $g_1 = -\rho_1 f_3 - \mathbb{D}_1\eta$ and $g_2 = -\rho_2 f_4 - \mathbb{D}_2\vartheta$. From this definition we obtain that $a(\cdot,\cdot)$ is continuous and $G = (g_1, g_2) \in \mathcal{D}(A^{-\frac{1}{2}}) \times \mathcal{D}(A^{-1})$. Furthermore, follows that the sesquilinear form is coercivity. Indeed, we have

$$\begin{aligned} a(W,W) &= \|A_1^{\frac{1}{2}}u\|^2 + \|A_2 v\|^2 + \alpha\|u-v\|^2 \\ &\ge \|A_1^{\frac{1}{2}}u\|^2 + \|A_2 v\|^2, \end{aligned}$$

therefore using the following equivalent norm

$$\|A_1^{\frac{1}{2}}u\|^2 \ge (\beta_1 - \kappa_1\alpha_1^{\theta_1-1})\|A^{\frac{1}{2}}u\|^2, \tag{4.24}$$

and

$$\|A_2 v\|^2 \geq (\beta_2 - \kappa_2 \alpha_1^{2(\theta_2-1)})\|Av\|^2, \tag{4.25}$$

we obtain

$$a(W,W) \geq (\beta_1 - \kappa_1 \alpha_1^{\theta_1-1})\|A^{\frac{1}{2}}u\|^2 + (\beta_2 - \kappa_2 \alpha_1^{2(\theta_2-1)})\|Av\|^2,$$

then we conclude that $a(\cdot,\cdot)$ is coercive. From Lax-Milgram Theorem there exists an unique solution $(u,v) \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A)$ in a weak sense to system (4.23). From the first equation we have $A_1 u + \mathbb{D}_1 \eta \in \mathbb{H}$ and from the second equation of this system we conclude $(A_2^2 v + \mathbb{D}_2 \vartheta) \in \mathbb{H}$. As a conclusion we can affirm that $U \in D(\mathbb{B})$.

Remark 1: The equivalent norm in (4.24) is due to follows calculus: note that for $u \in \mathcal{D}(A^{\frac{1}{2}})$ and $A_1 = \beta_1 A - \kappa_1 A^{\theta_1}$ we have

$$\|A_1^{\frac{1}{2}}u\|^2 = \beta_1 \|A^{\frac{1}{2}}u\|^2 - \kappa_1 \|A^{\frac{\theta_1}{2}}u\|^2.$$

Furthermore,

$$\begin{aligned}\|A^{\frac{1}{2}}u\|^2 &= \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}u\rangle = \langle A^{1-\theta_1}(A^{\frac{\theta_1}{2}}u), A^{\frac{\theta_1}{2}}u\rangle = \langle A^{\frac{1-\theta_1}{2}}(A^{\frac{\theta_1}{2}}u), A^{\frac{1-\theta_1}{2}}(A^{\frac{\theta_1}{2}}u)\rangle \\ &= \|A^{\frac{1-\theta_1}{2}}(A^{\frac{\theta_1}{2}}u)\|^2 \geq \alpha_1^{1-\theta_1}\|A^{\frac{\theta_1}{2}}u\|^2. \end{aligned} \tag{4.26}$$

In the last inequality we are using that $\|A^{\frac{\gamma}{2}}u\|^2 \geq \alpha_1^{\gamma}\|u\|^2, \forall u \in \mathcal{D}(A^{\frac{\gamma}{2}})$ where α_1 is the first eigenvalue of operator A. The result follows immediately from (4.26) and the hypothesis on κ_1 given in 4.7. To obtain the equivalent norm in (4.25) just to follows the same way of remark 1. Finally, let's see that $0 \in \rho(\mathbb{B})$. For this, it suffices to check that $\mathbb{B}^{-1}$ is a bounded operator. From inner product definition we have

$$\begin{aligned}\|U\|^2 =& \|A_1^{\frac{1}{2}}u\|^2 + \|A_2 v\|^2 + \rho_1\|\dot{u}\|^2 + \rho_2\|\dot{v}\|^2 \\ &+ \alpha\|u-v\|^2 + \|\eta\|^2_{\mathcal{M}_1} + \|\vartheta\|^2_{\mathcal{M}_2} \\ \leq& a(W,W) + C\|F\|\|U\| + C\|F\|^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}|\langle G, W\rangle| \leq& \rho_1 |\langle f_3, u\rangle_{D(A^{-\frac{1}{2}})\times D(A^{\frac{1}{2}})}| + |\langle \mathbb{D}_1\eta, u\rangle_{D(A^{-\frac{1}{2}})\times D(A^{\frac{1}{2}})}| + \rho_2|\langle f_4, v\rangle_{D(A^{-\frac{1}{2}})\times D(A)}| \\ &+ |\langle \mathbb{D}_2\vartheta, v\rangle_{D(A^{-\frac{1}{2}})\times D(A)}| \\ \leq& C\|F\|\|U\| + C\|F\|^2 + \varepsilon_1\|A^{\frac{\theta_1}{2}}u\|^2 + \varepsilon_2\|A^{\theta_2}v\|^2.\end{aligned}$$

From continuous embedding $\mathcal{D}(A^{\frac{1}{2}}) \hookrightarrow \mathcal{D}(A^{\frac{\theta_1}{2}})$ and $\mathcal{D}(A) \hookrightarrow \mathcal{D}(A^{\theta_2})$ we have

$$|\langle G, W\rangle| \leq C\|F\|\|U\| + C\|F\|^2 + \varepsilon_1\|A^{\frac{1}{2}}u\|^2 + \varepsilon_2\|Av\|^2.$$

Furthermore, since $a(W, W) = \langle G, W\rangle$ follows from above inequality that

$$\|U\|^2 \leq C\|F\|\|U\| + C\|F\|^2 + \varepsilon_1\|A^{\frac{1}{2}}u\|^2 + \varepsilon_2\|Av\|^2.$$

computing ε_1 and ε_2 small enough we have

$$\|U\|^2 \leq C\|F\|\|U\| + C\|F\|^2.$$

Therefore, using Young's inequality we obtain that $\|U\| \leq C\|F\|$. This conclude the proof that $0 \in \rho(\mathbb{B})$.

Note that in this section we developed the existence of solution when we have $g_1 \neq 0$ and $g_2 \neq 0$ but, for other cases as $g_1 \neq 0$ and $g_2 = 0$ or $g_1 = 0$ and $g_2 \neq 0$ the development is the same. ■

4.3 Asymptotic behavior

In this section, we will see the asymptotic behavior of solution, for that, let's enunciate our main theorems about exponential and polynomial decay.

Theorem 40. *The semigroup $e^{t\mathbb{B}}$ of system (4.10) has the following asymptotic behavior:*

1. *If both memory terms are present in the system (4.5)-(4.6) and $\theta_1 = \theta_2 = 1$, then the semigroup decays exponentially, that is, there exists $C > 0$ and $\mu > 0$ such that*

$$\|e^{t\mathbb{B}}\| \leq Ce^{-\mu t}.$$

2. *If both memory terms are present in the system (4.5)-(4.6) and at least one of the exponents θ_1, θ_2 is strictly less than 1, then the semigroup decays polynomially with decay rate $t^{-1/(2-2\theta_0)}$ for $\theta_0 := \min\{\theta_1, \theta_2\}$. That is, there exists $C > 0$ such that*

$$\|e^{t\mathbb{B}}U_0\| \leq \frac{C}{t^{1/(2-2\theta_0)}}\|U_0\|_{D(\mathbb{B})}, \quad \forall\, t > 0,\ U_0 \in D(\mathbb{B}).$$

3. *If we remove the memory term from the second equation of (4.5)-(4.6), then the semigroup decays polynomially with decay rate $t^{-1/(6-\theta_1)}$, that is, there exists $C > 0$ such that*

$$\|e^{t\mathbb{B}}U_0\| \leq \frac{C}{t^{1/(6-\theta_1)}}\|U_0\|_{D(\mathbb{B})}, \quad \forall\, t > 0,\ U_0 \in D(\mathbb{B}).$$

4. *If we remove the memory term from the first equation of* (4.5)-(4.6)*, then the semigroup decays polynomially with decay rate* $t^{-1/(10-4\theta_2)}$*, that is, there exists* $C>0$ *such that*

$$\|e^{t\mathbb{B}}U_0\| \leq \frac{C}{t^{1/(10-4\theta_2)}}\|U_0\|_{D(\mathbb{B})}, \quad \forall\, t>0,\ U_0\in D(\mathbb{B}).$$

The proof of these theorems will be divided into some lemmas. In view of this, to show these lemmas it is considered many times the stationary problem $(i\lambda I-\mathbb{B})U=F$, where $\lambda\in\mathbb{R}$ and $F=(f_1,f_2,f_3,f_4,f_5,f_6)$. Note that for $U=(u,v,\dot{u},\dot{v},\eta,\vartheta)$ solution of this problem, we have that are satisfied follows equations:

$$i\lambda u-\dot{u}=f_1, \tag{4.27}$$

$$i\lambda v-\dot{v}=f_2, \tag{4.28}$$

$$i\rho_1\lambda\dot{u}+A_1u+\mathbb{D}_1\eta+\alpha(u-v)=\rho_1f_3, \tag{4.29}$$

$$i\rho_2\lambda\dot{v}+A_2^2v+\mathbb{D}_2\vartheta+\alpha(v-u)=\rho_2f_4, \tag{4.30}$$

$$i\lambda\eta+\partial_s\eta-\dot{u}=f_5, \tag{4.31}$$

$$i\lambda\vartheta+\partial_s\vartheta-\dot{v}=f_6. \tag{4.32}$$

Initially, we can conclude from (4.12), (4.13), (4.14) and the current stationary problem

$$\begin{cases} \|\eta\|^2_{\mathcal{M}_1} \leq -\displaystyle\int_0^\infty g_1'(s)\|A^{\frac{\theta_1}{2}}\eta\|^2ds \leq C\text{Re}\langle(i\lambda I-\mathbb{B})U,U\rangle_{\mathbb{X}} \leq C\|F\|\|U\|, \\ \|\vartheta\|^2_{\mathcal{M}_2} \leq -\displaystyle\int_0^\infty g_2'(s)\|A^{\theta_2}\vartheta\|^2ds \leq C\text{Re}\langle(i\lambda I-\mathbb{B})U,U\rangle_{\mathbb{X}} \leq C\|F\|\|U\|. \end{cases} \tag{4.33}$$

To make better organize, we will divide this section into other subsections with each result, where we will explore different cases for g_1 and g_2.

To solve these cases we will start with two lemmas. These results will be used simultaneously when $g_1\neq 0$ and $g_2\neq 0$ otherwise, it will be used separately.

Lemma 41. *Consider* $\theta_1\in[0,1]$ *and* $F\in\mathbb{X}$*. Suppose that for every* $\lambda\in\mathbb{R}$ *such that* $0<\delta\leq|\lambda|$ *there exists a solution* $U\in\mathcal{D}(\mathbb{B})$ *of stationary system* $(i\lambda I-\mathbb{B})U=F$*. Then there exists a positive constant* C_δ *such that:*

(i) $\|A^{\frac{\theta_1}{2}}\dot{u}\|^2 \leq C_\delta\lambda^2\left(\|F\|\|U\|+\|F\|^2\right),$

(ii) $\|A^{\frac{\theta_1}{2}}u\|^2 \leq C_\delta\left(\|F\|\|U\|+\|F\|^2\right).$

Proof. Note that using equation (4.31) we have

$$\begin{aligned}\left(\int_0^\infty g_1(s)ds\right)\|A^{\frac{\theta_1}{2}}\dot{u}\|^2 &= \int_0^\infty g_1(s)\langle A^{\frac{\theta_1}{2}}(i\lambda\eta(s)+\partial_s\eta(s)-f_5(s)), A^{\frac{\theta_1}{2}}\dot{u}\rangle ds\\ &= \int_0^\infty g_1(s)\langle i\lambda A^{\frac{\theta_1}{2}}\eta(s), A^{\frac{\theta_1}{2}}\dot{u}\rangle ds + \int_0^\infty g_1(s)\langle \partial_s A^{\frac{\theta_1}{2}}\eta(s), A^{\frac{\theta_1}{2}}\dot{u}\rangle ds\\ &\quad - \int_0^\infty g_1(s)\langle A^{\frac{\theta_1}{2}}f_5(s), A^{\frac{\theta_1}{2}}\dot{u}\rangle ds.\end{aligned}$$

Thus, integrating by parts, using the hypothesis of function g and Cauchy-Schwarz inequality follows that

$$\begin{aligned}\left(\int_0^\infty g_1(s)ds\right)\|A^{\frac{\theta_1}{2}}\dot{u}\|^2 \leq C|\lambda|\|A^{\frac{\theta_1}{2}}\dot{u}\|\left(\int_0^\infty g_1(s)\|A^{\frac{\theta_1}{2}}\eta(s)\|^2ds\right)^{\frac{1}{2}}\\ + C\|A^{\frac{\theta_1}{2}}\dot{u}\|\left(-\int_0^\infty g_1'(s)\|A^{\frac{\theta_1}{2}}\eta(s)\|^2ds\right)^{\frac{1}{2}}\\ + C\|A^{\frac{\theta_1}{2}}\dot{u}\|\left(\int_0^\infty g_1(s)\|A^{\frac{\theta_1}{2}}f_5(s)\|^2ds\right)^{\frac{1}{2}}.\end{aligned}$$

Now, applying Young's inequality we obtain

$$\begin{aligned}\|A^{\frac{\theta_1}{2}}\dot{u}\|^2 \leq & C\left(\lambda^2\int_0^\infty g_1(s)\|A^{\frac{\theta_1}{2}}\eta(s)\|^2ds - \int_0^\infty g_1'(s)\|A^{\frac{\theta_1}{2}}\eta(s)\|^2ds\right.\\ &\left.+\int_0^\infty g_1(s)\|A^{\frac{\theta_1}{2}}f_5(s)\|^2ds\right),\end{aligned}$$

where C is a positive constant that not depends of λ. From estimate (4.33) follows that

$$\|A^{\frac{\theta_1}{2}}\dot{u}\|^2 \leq C\left(\lambda^2\|F\|\|U\|+\|F\|\|U\|+\|F\|^2\right),$$

then recalling that $|\lambda|\geq\delta$, item (i) is obtained. Furthermore, computing inner product with $i\lambda A^\theta u$ (note that operator A^θ is self-adjoint), we get

$$\lambda^2\|A^{\frac{\theta_1}{2}}u\|^2 = -\langle i\lambda A^{\frac{\theta_1}{2}}u, A^{\frac{\theta_1}{2}}f_1\rangle - \langle i\lambda A^{\frac{\theta_1}{2}}u, A^{\frac{\theta_1}{2}}\dot{u}\rangle.$$

Using Cauchy-Schwartz and Young's inequality follows that

$$\lambda^2\|A^{\frac{\theta_1}{2}}u\|^2 \leq C(\|A^{\frac{\theta_1}{2}}f_1\|^2+\|A^{\frac{\theta_1}{2}}\dot{u}\|^2).$$

Furthermore, by the continuous embedding $\mathcal{D}(A^{\frac{1}{2}})\hookrightarrow\mathcal{D}(A^{\frac{\theta_1}{2}})$ (in view of $\theta_1\in[0,1]$) and the estimate obtained in item (i), we have

$$\lambda^2\|A^{\frac{\theta_1}{2}}u\|^2 \leq C_\delta(\lambda^2\|F\|\|U\|+\|F\|^2),$$

then using that $|\lambda|>\delta$ the result follows. ∎

Lemma 42. *Consider $\theta_2 \in [0,1]$ and $F \in \mathbb{X}$. Suppose that for every $\lambda \in \mathbb{R}$ such that $0 < \delta \leq |\lambda|$ there exists a solution $U \in \mathcal{D}(\mathbb{B})$ of stationary system $(i\lambda I - \mathbb{B})U = F$. Then, there exists a positive constant C_δ such that:*

(i) $\|A^{\theta_2}\dot{v}\|^2 \leq C_\delta \lambda^2 \left(\|F\|\|U\| + \|F\|^2\right),$

(ii) $\|A^{\theta_2}v\|^2 \leq C_\delta \left(\|F\|\|U\| + \|F\|^2\right).$

Proof. The proof follows using a similar way from the last Lemma with equations (4.28) and (4.32). ■

The next theorem will be used in all result from Theorem 40.

Lemma 43. *The solutions of equations* (4.27)-(4.32) *satisfy the following result*

$$\|U\|^2 \leq C_\delta \left(\|\dot{u}\|^2 + \|\dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right).$$

Proof. Initially, let's compute the inner product of equation (4.29) with u and equation (4.30) with v; using equations (4.27) and (4.28) respectively we have

$$\begin{aligned}\|A_1^{\frac{1}{2}}u\|^2 + \alpha\|u\|^2 - \alpha\langle v, u\rangle = \rho_1\|\dot{u}\|^2 - \int_0^\infty g_1(s)\langle A^{\frac{\theta_1}{2}}\eta, A^{\frac{\theta_1}{2}}u\rangle ds \\ + \rho_1\langle \dot{u}, f_1\rangle + \langle f_3, u\rangle\end{aligned}$$

and

$$\begin{aligned}\|A_2 v\|^2 + \alpha\|v\|^2 - \alpha\langle u, v\rangle = \rho_2\|\dot{v}\|^2 - \int_0^\infty g_2(s)\langle A^{\theta_2}\vartheta, A^{\theta_2}v\rangle ds \\ + \rho_2\langle \dot{v}, f_2\rangle + \langle f_4, v\rangle.\end{aligned}$$

Performing the sum with both these equations we have

$$\begin{aligned}\|A_1^{\frac{1}{2}}u\|^2 + \|A_2 v\|^2 + \alpha\|u - v\|^2 = &- \int_0^\infty g_1(s)\langle A^{\frac{\theta_1}{2}}\eta, A^{\frac{\theta_1}{2}}u\rangle ds + \rho_1\|\dot{u}\|^2 + \rho_2\|\dot{v}\|^2 \\ &- \int_0^\infty g_2(s)\langle A^{\theta_2}\vartheta, A^{\theta_2}v\rangle ds + \rho_1\langle \dot{u}, f_1\rangle \\ &+ \rho_2\langle \dot{v}, f_2\rangle + \langle f_3, u\rangle + \langle f_4, v\rangle.\end{aligned}$$

Using Young's inequality, estimate (4.33), Lemma 41 and Lemma 42 we obtain

$$\|A_1^{\frac{1}{2}}u\|^2 + \|A_2 v\|^2 + \alpha\|u - v\|^2 \leq C_\delta \left(\|\dot{u}\|^2 + \|\dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right).$$

Furthermore, from inequality (4.33) we have

$$\|\eta\|^2_{\mathcal{M}_1} \leq C\|F\|\|U\| \text{ and } \|\vartheta\|^2_{\mathcal{M}_2} \leq C\|F\|\|U\|, \tag{4.34}$$

finally, it may be concluded that

$$\|U\|^2 \leq C_\delta \left(\|\dot{u}\|^2 + \|\dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right).$$

∎

4.3.1 Two memory term acting simultaneously

Lemma 44. *Let $\theta_0 = \min\{\theta_1, \theta_2\}$. From the same hypothesis of Lemma 41 and 42 we have*

(i) $\|A^{\frac{\theta_0}{2}}\dot{u}\|^2 \leq C_\delta \lambda^2 (\|F\|\|U\| + \|F\|^2)$,

(ii) $\|A^{\frac{\theta_0-1}{2}}\dot{u}\|^2 \leq C_\delta (\|F\|\|U\| + \|F\|^2)$,

(iii) $\|A^{\theta_0}\dot{v}\|^2 \leq C_\delta \lambda^2 (\|F\|\|U\| + \|F\|^2)$,

(iv) $\|A^{\theta_0-1}\dot{v}\|^2 \leq C_\delta (\|F\|\|U\| + \|F\|^2)$.

Proof. Using Lemma 41 (item(i)); the continuous embedding $\mathcal{D}(A^{\frac{\theta_1}{2}}) \hookrightarrow \mathcal{D}(A^{\frac{\theta_0}{2}})$ we obtain item (i). In the same way from $\mathcal{D}(A^{\theta_1}) \hookrightarrow \mathcal{D}(A^{\theta_0})$ and Lemma 42 we have item (iii).

For item (ii), performing the inner product with 4.29 and $i\lambda A^{\theta_0-1}\dot{u}$; using equation (4.27); we have

$$\begin{aligned}\rho_1\|\lambda A^{\frac{\theta_0-1}{2}}\dot{u}\|^2 =& \beta_1\langle i\lambda A^{\frac{\theta_0}{2}}u, A^{\frac{\theta_0}{2}}\dot{u}\rangle + \kappa_1\langle i\lambda A^{\frac{\theta_1}{2}}u, A^{\theta_0-1+\frac{\theta_1}{2}}\dot{u}\rangle - \alpha\langle A^{\frac{\theta_0-1}{2}}u, i\lambda A^{\frac{\theta_0-1}{2}}\dot{u}\rangle \\ &+ \int_0^\infty g_1(s)\langle i\lambda A^{\frac{\theta_1}{2}}\eta(s), A^{\theta_0-1+\frac{\theta_1}{2}}\dot{u}\rangle ds \\ &+ \alpha\langle A^{\frac{\theta_0-1}{2}}v, i\lambda A^{\frac{\theta_0-1}{2}}\dot{u}\rangle - \rho_1\langle i\lambda f_3, A^{\theta_0-1}\dot{u}\rangle.\end{aligned}$$

Using Young's inequality; suitable continuous embedding; item (i) of this Lemma, estimate (4.33), Lemma 41 and 42 we conclude

$$\|\lambda A^{\frac{\theta_0-1}{2}}\dot{u}\|^2 \leq C_\delta \lambda^2 \left(\|F\|\|U\| + \|F\|^2\right).$$

On the other hand, for item (iv), computing the inner product with equation (4.30) and $i\lambda A^{2\theta_0-2}$ we have

$$\begin{aligned}\rho_2\|\lambda A^{\theta_0-1}\dot{v}\|^2 =& \beta_2\langle i\lambda A^{\theta_0}v, A^{\theta_0}\dot{v}\rangle - \kappa_2\langle i\lambda A^{\theta_2}v, A^{2\theta_0-2+\theta_2}\dot{v}\rangle \\ &+ \int_0^\infty g_2(s)\langle i\lambda A^{\theta_2}\vartheta, A^{2\theta_0-2+\theta_2}\dot{v}\rangle ds - \alpha\langle A^{\theta_0-1}v, i\lambda A^{\theta_0-1}\dot{v}\rangle \\ &\alpha\langle A^{\theta_0-1}u, i\lambda A^{\theta_0-1}\dot{v}\rangle - \rho_2\langle i\lambda f_4, A^{2\theta_0-2}\dot{v}\rangle.\end{aligned}$$

Using Young's inequality; suitable continuous embedding; item (iii) of this Lemma, estimate (4.33), Lemma 41 and 42 we obtain

$$\|\lambda A^{\theta_0-1}\dot{v}\|^2 \leq C_\delta \lambda^2 \left(\|F\|\|U\| + \|F\|^2\right).$$

∎

Lemma 45. *In the same hypothesis of last lemmas, the solution of system* (4.27)-(4.32) *satisfy the following estimates*

(i) $\|\dot{u}\|^2 \leq C_\delta |\lambda|^{2-2\theta_0}(\|F\|\|U\| + \|F\|^2)$,

(ii) $\|\dot{v}\|^2 \leq C_\delta |\lambda|^{2-2\theta_0}(\|F\|\|U\| + \|F\|^2)$.

In particular,

$$\|U\|^2 \leq C_\delta |\lambda|^{2-2\theta_0}(\|F\|\|U\| + \|F\|^2).$$

Proof. For item (i), we use

$$0 = \theta_0 \left(\frac{\theta_0 - 1}{2}\right) + (1-\theta_0)\left(\frac{\theta_0}{2}\right).$$

Then, from interpolation inequality and Lemma 44 (items (i) and (ii)) we obtain

$$\begin{aligned}
\|\dot{u}\| &\leq C\|A^{\frac{\theta_0-1}{2}}\dot{u}\|^{\theta_0}\|A^{\frac{\theta_0}{2}}\dot{u}\|^{1-\theta_0} \\
&\leq C_\delta \left(\sqrt{\|F\|\|U\| + \|F\|^2}\right)^{\theta_0} \lambda^{(1-\theta_0)} \left(\sqrt{\|F\|\|U\| + \|F\|^2}\right)^{1-\theta_0} \\
&\leq C_\delta \lambda^{1-\theta_0}\sqrt{\|F\|\|U\| + \|F\|^2}.
\end{aligned}$$

For item (ii), we use

$$0 = \theta_0(\theta_0 - 1) + (1-\theta_0)(\theta_0).$$

Then, from interpolation inequality and Lemma 44 (items (iii) and (iv)) we have

$$\begin{aligned}
\|\dot{v}\| &\leq C\|A^{\theta_0-1}\dot{v}\|^{\theta_0}\|A^{\theta_0}\dot{u}\|^{1-\theta_0} \\
&\leq C_\delta \left(\sqrt{\|F\|\|U\| + \|F\|^2}\right)^{\theta_0} \lambda^{(1-\theta_0)} \left(\sqrt{\|F\|\|U\| + \|F\|^2}\right)^{1-\theta_0} \\
&\leq C_\delta \lambda^{1-\theta_0}\sqrt{\|F\|\|U\| + \|F\|^2}.
\end{aligned}$$

Therefore, from Lemma 43 we have

$$\|U\|^2 \leq C_\delta |\lambda|^{2-2\theta_0}(\|F\|\|U\| + \|F\|^2).$$

∎

4.3.2 Plate equation in the absence of memory term

In this subsection, we will consider the second equation of (4.5)-(4.6) in absence of the memory term. For this, we have $g_2 = 0$ and naturally, all definitions regarding this case will be adjusted as mentioned before.

Lemma 46. *For $\sigma \leq 1$ we have the follow result*

(i) $\|A^{\sigma+1}v\|^2 \leq C_\delta\left(\|A^\sigma \dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right),$

(ii) $\|\lambda^{-1}A^{\sigma+1}v\|^2 \leq C_\delta\left(\|\lambda A^{\sigma-1}v\|^2 + \|F\|\|U\| + \|F\|^2\right).$

Proof. From equation (4.30), computing inner product with $A^{2\sigma}v$ and using equation (4.28) we have

$$\beta_2\|A^{\sigma+1}v\|^2 = \rho_2\|A^\sigma \dot{v}\|^2 - \alpha\|A^\sigma v\|^2 + \alpha\langle A^{\sigma-1}u, A^{\sigma+1}v\rangle + \rho_2\langle \dot{v}, A^{2\sigma}f_2\rangle + \rho_2\langle f_4, A^{2\sigma}v\rangle$$

then, using Young's inequality follows that

$$\|A^{\sigma+1}v\|^2 \leq C\left(\|A^\sigma \dot{v}\|^2 + \|A^{\sigma-1}u\|^2 + \|F\|\|U\|\right), \tag{4.35}$$

therefore, from Lemma 41 we get the result (item (i)).

On the other hand, multiplying (4.35) by λ^{-2} and using equation (4.28), that is, $\lambda^{-1}\dot{v} = iv - \lambda^{-1}f_2$, then we conclude

$$\begin{aligned}\|\lambda^{-1}A^{\sigma+1}v\|^2 &\leq C_\delta\left(\|\lambda^{-1}A^\sigma \dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right)\\ &\leq C_\delta\left(\|A^\sigma v\|^2 + \|F\|\|U\| + \|F\|^2\right).\end{aligned}$$

Using item (i) of this Lemma, we get

$$\|\lambda^{-1}A^{\sigma+1}v\|^2 \leq C_\delta\left(\|A^{\sigma-1}\dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right).$$

Therefore, from equation (3.26) we can conclude

$$\|\lambda^{-1}A^{\sigma+1}v\|^2 \leq C_\delta\left(\|\lambda A^{\sigma-1}v\|^2 + \|F\|\|U\| + \|F\|^2\right).$$

■

Lemma 47. *If we consider the second equation is in the absence of memory, then it is possible to consider equations (4.27)-(4.31). Furthermore, from the same hypothesis of Lemma 41 we have the follows result* *(i)* $\|A^{\frac{\theta_1}{2}}\dot{v}\|^2 \leq C_\delta \lambda^6 \left(\|F\|\|U\| + \|F\|^2\right),$

(ii) $\|A^{\frac{\theta_1}{2}-1}\dot{v}\|^2 \leq C_\delta \lambda^4 \left(\|F\|\|U\| + \|F\|^2\right).$

Proof. From equation (4.29), computing the inner product with $\lambda^2 A^{\theta_1} v$, using equation (4.28), definition of A_1 and the fact that the operator is self-adjoint we obtain

$$- \rho_1\lambda^2\langle A^{\frac{\theta_1}{2}}\dot{u}, A^{\frac{\theta_1}{2}} f_2\rangle - \rho_1\langle \lambda^2 A^{\frac{\theta_1}{2}}\dot{u}, A^{\frac{\theta_1}{2}}\dot{v}\rangle + \lambda^2\beta_1\langle A^{\frac{\theta_1+1}{2}}u, A^{\frac{\theta_1+1}{2}}v\rangle - \lambda^2\int_0^\infty g_1(s)\langle A^{\theta_1}u, A^{\theta_1}v\rangle ds$$
$$+ \lambda^2\int_0^\infty g_1(s)\langle A^{\theta_1}\eta(s), A^{\theta_1}v\rangle ds + \alpha\langle \lambda A^{\frac{\theta_1}{2}}u, \lambda A^{\frac{\theta_1}{2}}v\rangle - \alpha\|\lambda A^{\frac{\theta_1}{2}}v\|^2 = \lambda^2\rho_1\langle f_3, A^{\theta_1}v\rangle.$$

In the same way, from equation (4.30), computing the inner product with $\lambda^2 A^{\theta_1 - 1}u$ and using equation (4.27), we have

$$-\rho_2\lambda^2\langle A^{\frac{\theta_1-1}{2}}\dot{v}, A^{\frac{\theta_1-1}{2}}f_1\rangle - \rho_2\langle A^{\frac{\theta_1-1}{2}}\dot{v}, \lambda^2 A^{\frac{\theta_1-1}{2}}\dot{u}\rangle + \lambda^2\beta_2\langle A^{\frac{\theta_1+1}{2}}v, A^{\frac{\theta_1+1}{2}}u\rangle$$
$$+\alpha\langle\lambda A^{\frac{\theta_1-1}{2}}v, \lambda A^{\frac{\theta_1-1}{2}}u\rangle - \alpha\lambda^2\|A^{\frac{\theta_1-1}{2}}u\|^2 = \lambda^2\rho_2\langle f_4, A^{\theta_1-1}u\rangle.$$

Divide by β_1 the first equation, by β_2 the second equation and performing the sum between both we get

$$\begin{aligned}\frac{\alpha}{\beta_1}\|\lambda A^{\frac{\theta_1}{2}}v\|^2 = &+\frac{\alpha}{\beta_2}\lambda^2\|A^{\frac{\theta_1-1}{2}}u\|^2 - \frac{\rho_1}{\beta_1}\lambda^2\langle f_3, A^{\theta_1}v\rangle - \frac{\rho_1}{\beta_1}\lambda^2\langle A^{\frac{\theta_1}{2}}\dot{u}, A^{\frac{\theta_1}{2}}f_2\rangle - \frac{\rho_1}{\beta_1}\lambda^2\langle A^{\frac{\theta_1}{2}}\dot{u}, A^{\frac{\theta_1}{2}}\dot{v}\rangle\\ &+\frac{1}{\beta_1}\int_0^\infty g_1(s)\langle\lambda^2 A^{\theta_1}\left(\eta(s)-u\right), A^{\theta_1}v\rangle ds + \frac{\alpha}{\beta_1}\langle\lambda A^{\frac{\theta_1}{2}}u, \lambda A^{\frac{\theta_1}{2}}v\rangle\\ &+\frac{\rho_2}{\beta_2}\lambda^2\langle A^{\frac{\theta_1-1}{2}}\dot{v}, A^{\frac{\theta_1-1}{2}}f_1\rangle + \frac{\rho_2}{\beta_2}\langle A^{\frac{\theta_1-1}{2}}\dot{v}, \lambda^2 A^{\frac{\theta_1-1}{2}}\dot{u}\rangle - \lambda^2\langle A^{\frac{\theta_1+1}{2}}v, A^{\frac{\theta_1+1}{2}}u\rangle\\ &-\frac{\alpha}{\beta_2}\langle\lambda A^{\frac{\theta_1-1}{2}}v, \lambda A^{\frac{\theta_1-1}{2}}u\rangle + \frac{\rho_2}{\beta_2}\lambda^2\langle f_4, A^{\theta_1-1}u\rangle + \lambda^2\langle A^{\frac{\theta_1+1}{2}}u, A^{\frac{\theta_1+1}{2}}v\rangle.\end{aligned} \tag{4.36}$$

Note that, in view of the equation (4.27) and (4.31), we can write it as

$$\eta(s) - u = \frac{-\partial_s\eta(s) - f_1 + f_5(s)}{i\lambda},$$

and then, integrating by parts we obtain

$$\begin{aligned}\int_0^\infty g_1(s)\langle A^{\frac{\theta_1}{2}}(\eta(s)-u), A^{\frac{3\theta_1}{2}}v\rangle ds = &\int_0^\infty g_1(s)\langle A^{\frac{\theta_1}{2}}\partial_s\eta(s), i\lambda^{-1}A^{\frac{3\theta_1}{2}}v\rangle ds\\ &+\int_0^\infty g_1(s)\langle A^{\frac{\theta_1}{2}}f_1, i\lambda^{-1}A^{\frac{3\theta_1}{2}}v\rangle ds\\ &-\int_0^\infty g_1(s)\langle A^{\frac{\theta_1}{2}}f_5(s), i\lambda^{-1}A^{\frac{3\theta_1}{2}}v\rangle ds\\ = &-\int_0^\infty g_1'(s)\langle A^{\frac{\theta_1}{2}}\eta(s), i\lambda^{-1}A^{\frac{3\theta_1}{2}}v\rangle ds\\ &+\int_0^\infty g_1(s)\langle A^{\frac{\theta_1}{2}}(f_1 - f_5(s)), i\lambda^{-1}A^{\frac{3\theta_1}{2}}v\rangle ds.\end{aligned} \tag{4.37}$$

Substituting equations (4.37) in (4.36) we have

$$
\begin{aligned}
\frac{\alpha}{\beta_1}\|\lambda A^{\frac{\theta_1}{2}}v\|^2 =& -\frac{\rho_1}{\beta_1}\lambda^2\langle f_3, A^{\theta_1}v\rangle - \frac{\rho_1}{\beta_1}\lambda^2\langle A^{\frac{\theta_1}{2}}\dot{u}, A^{\frac{\theta_1}{2}}f_2\rangle - \frac{\rho_1}{\beta_1}\lambda^2\langle A^{\frac{\theta_1}{2}}\dot{u}, A^{\frac{\theta_1}{2}}\dot{v}\rangle \\
&+ \frac{1}{\beta_1}\int_0^\infty g_1(s)\langle\lambda^2 A^{\frac{\theta_1}{2}}(f_1 - f_5(s)), i\lambda^{-1}A^{\frac{3\theta_1}{2}}v\rangle ds + \lambda^2\langle A^{\frac{\theta_1+1}{2}}u, A^{\frac{\theta_1+1}{2}}v\rangle \\
&- \frac{1}{\beta_1}\int_0^\infty g_1'(s)\langle\lambda^2 A^{\frac{\theta_1}{2}}\eta(s), i\lambda^{-1}A^{\frac{3\theta_1}{2}}v\rangle ds + \frac{\alpha}{\beta_1}\langle\lambda A^{\frac{\theta_1}{2}}u, \lambda A^{\frac{\theta_1}{2}}v\rangle \\
&+ \frac{\rho_2}{\beta_2}\lambda^2\langle A^{\frac{\theta_1-1}{2}}\dot{v}, A^{\frac{\theta_1-1}{2}}f_1\rangle + \frac{\rho_2}{\beta_2}\langle A^{\frac{\theta_1-1}{2}}\dot{v}, \lambda^2 A^{\frac{\theta_1-1}{2}}\dot{u}\rangle - \lambda^2\langle A^{\frac{\theta_1+1}{2}}v, A^{\frac{\theta_1+1}{2}}u\rangle \\
&- \frac{\alpha}{\beta_2}\langle\lambda A^{\frac{\theta_1-1}{2}}v, \lambda A^{\frac{\theta_1-1}{2}}u\rangle + \frac{\alpha}{\beta_2}\lambda^2\|A^{\frac{\theta_1-1}{2}}u\|^2 + \frac{\rho_2}{\beta_2}\lambda^2\langle f_4, A^{\theta_1-1}u\rangle.
\end{aligned}
$$

Computing the real part we obtain

$$
\begin{aligned}
\frac{\alpha}{\beta_1}\|\lambda A^{\frac{\theta_1}{2}}v\|^2 =& -\lambda^2\frac{\rho_1}{\beta_1}Re\langle f_3, A^{\theta_1}v\rangle - \lambda^2\frac{\rho_1}{\beta_1}Re\langle\dot{u}, A^{\theta_1}f_2\rangle - \frac{\rho_1}{\beta_1}Re\langle\lambda^2 A^{\frac{\theta_1}{2}}\dot{u}, A^{\frac{\theta_1}{2}}\dot{v}\rangle \\
&- \frac{1}{\beta_1}Re\int_0^\infty g_1'(s)\langle\lambda^2 A^{\frac{\theta_1}{2}}\eta(s), i\lambda^{-1}A^{\frac{3\theta_1}{2}}v\rangle ds + \frac{\alpha}{\beta_1}Re\langle\lambda A^{\frac{\theta_1}{2}}u, \lambda A^{\frac{\theta_1}{2}}v\rangle \\
&+ \frac{1}{\beta_1}Re\int_0^\infty g_1(s)\langle\lambda^2 A^{\frac{\theta_1}{2}}(f_1 - f_5(s)), i\lambda^{-1}A^{\frac{3\theta_1}{2}}v\rangle ds + \lambda^2\frac{\alpha}{\beta_2}\|A^{\frac{\theta_1}{2}-\frac{1}{2}}u\|^2 \\
&+ \lambda^2\frac{\rho_2}{\beta_2}Re\langle A^{\frac{\theta_1}{2}-\frac{1}{2}}\dot{v}, A^{\frac{\theta_1}{2}-\frac{1}{2}}f_1\rangle + \frac{\rho_2}{\beta_2}Re\langle A^{\frac{\theta_1}{2}-\frac{1}{2}}\dot{v}, \lambda^2 A^{\frac{\theta_1}{2}-\frac{1}{2}}\dot{u}\rangle \\
&- \frac{\alpha}{\beta_2}Re\langle\lambda A^{\frac{\theta_1}{2}-\frac{1}{2}}v, \lambda A^{\frac{\theta_1}{2}-\frac{1}{2}}u\rangle + \lambda^2\frac{\rho_2}{\beta_2}Re\langle f_4, A^{\theta_1-1}u\rangle.
\end{aligned}
$$

In order to estimate these terms we will use: Young's inequality, Lemma 41, estimate (4.33) and some continuous embedding $\mathcal{D}(A^{r_1}) \hookrightarrow \mathcal{D}(A^{r_2}), r_1 \geq r_2$ we have

$$
\|\lambda A^{\frac{\theta_1}{2}}v\|^2 \leq C\left(\|\lambda^2 A^{\frac{\theta_1}{2}}\dot{u}\|\|A^{\frac{\theta_1}{2}}\dot{v}\| + \varepsilon\|\lambda^{-1}A^{\frac{3\theta_1}{2}}v\|^2\right) + \lambda^4 C_\varepsilon\left(\|F\|\|U\| + \|F\|^2\right) \tag{4.38}
$$

From Lemma 46 (item (ii)), computing the estimate using $\sigma + 1 = \frac{3\theta_1}{2}$ we have

$$
\|\lambda^{-1}A^{\frac{3\theta_1}{2}}v\|^2 \leq C\left(\|\lambda A^{\frac{3\theta_1}{2}-2}v\|^2 + \|F\|\|U\| + \|F\|^2\right) \tag{4.39}
$$

Substituting estimates (4.39) in (4.38) we conclude that

$$
\|\lambda A^{\frac{\theta_1}{2}}v\|^2 \leq C\left(\|\lambda^2 A^{\frac{\theta_1}{2}}\dot{u}\|\|A^{\frac{\theta_1}{2}}\dot{v}\| + \varepsilon\|\lambda A^{\frac{3\theta_1}{2}-2}v\|^2\right) + \lambda^4 C_\varepsilon\left(\|F\|\|U\| + \|F\|^2\right),
$$

therefore, computing ε small enough we get

$$
\|\lambda A^{\frac{\theta_1}{2}}v\|^2 \leq C\|\lambda^2 A^{\frac{\theta_1}{2}}\dot{u}\|\|A^{\frac{\theta_1}{2}}\dot{v}\| + \lambda^4 C\left(\|F\|\|U\| + \|F\|^2\right).
$$

However, from equation (4.28)

$$
\begin{aligned}
\|A^{\frac{\theta_1}{2}}\dot{v}\|^2 &\leq \|\lambda A^{\frac{\theta_1}{2}}v\|^2 + \|F\|^2 \\
&\leq C\left(\|\lambda^2 A^{\frac{\theta_1}{2}}\dot{u}\|\|A^{\frac{\theta_1}{2}}\dot{v}\| + \lambda^4\left(\|F\|\|U\| + \|F\|^2\right)\right).
\end{aligned}
$$

By Young's inequality we have

$$\|A^{\frac{\theta_1}{2}}\dot{v}\|^2 \leq C\left(\lambda^4\|A^{\frac{\theta_1}{2}}\dot{u}\|^2 + \lambda^4\left(\|F\|\|U\| + \|F\|^2\right)\right)$$

Therefore, from Lemma (41) we get

$$\|A^{\frac{\theta_1}{2}}\dot{v}\|^2 \leq \lambda^6 C_\delta\left(\|F\|\|U\| + \|F\|^2\right).$$

For item (ii), computing the inner product between equation (4.30) and $i\lambda A^{\theta_1-2}\dot{v}$; equation (4.28); we have

$$\begin{aligned}\rho_2\|\lambda A^{\frac{\theta_1}{2}-1}\dot{v}\|^2 = \beta_2\|A^{\frac{\theta_1}{2}}\dot{v}\|^2 + \beta_2\langle A^{\theta_1}f_2, \dot{v}\rangle + \alpha\|A^{\frac{\theta_1}{2}-1}\dot{v}\|^2 + \alpha\langle f_2, A^{\theta_1-2}\dot{v}\rangle \\ + \alpha\langle A^{\frac{\theta_1}{2}}u, i\lambda A^{\frac{\theta_1}{2}-1}\dot{v}\rangle + \rho_2\langle f_4, i\lambda A^{\theta_1-2}\dot{v}\rangle.\end{aligned}$$

Using Young's inequality, suitable continuous embedding and Lemma (41) we have

$$\|\lambda A^{\frac{\theta_1}{2}-1}\dot{v}\|^2 \leq C\left(\|A^{\frac{\theta_1}{2}}\dot{v}\|^2 + \|F\|\|U\| + \|F\|^2\right).$$

Therefore, we conclude this Lemma from item (i), that is

$$\|\lambda A^{\frac{\theta_1}{2}-1}\dot{v}\|^2 \leq C_\delta\lambda^6\left(\|F\|\|U\| + \|F\|^2\right),$$

and then

$$\|A^{\frac{\theta_1}{2}-1}\dot{v}\|^2 \leq C_\delta\lambda^4\left(\|F\|\|U\| + \|F\|^2\right).$$

■

Lemma 48. *In the same hypothesis of last lemmas, the solution of system* (4.27)-(4.32) *satisfy the following estimates*

(i) $\|\dot{v}\|^2 \leq C_\delta|\lambda|^{6-\theta_1}(\|F\|\|U\| + \|F\|^2)$;

(ii) $\|\dot{u}\|^2 \leq C_\delta|\lambda|^{6-\theta_1}(\|F\|\|U\| + \|F\|^2)$.

In particular, from Theorem 43 we have

$$\|U\| \leq C_\delta\lambda^{6-\theta_1}\|F\|.$$

Proof. For items (i) and (ii) we use

$$0 = \frac{\theta_1}{2}\left(\frac{\theta_1}{2} - 1\right) + \left(1 - \frac{\theta_1}{2}\right)\left(\frac{\theta_1}{2}\right).$$

Item (i) : Using Lemma 47, we obtain from interpolation inequality

$$
\begin{aligned}
\|\dot{v}\| &\leq C\|A^{\frac{\theta_1}{2}-1}\dot{v}\|^{\frac{\theta_1}{2}}\|A^{\frac{\theta_1}{2}}\dot{v}\|^{1-\frac{\theta_1}{2}} \\
&\leq C_\delta|\lambda|^{\theta_1}\left(\sqrt{\|F\|\|U\|+\|F\|^2}\right)^{\frac{\theta_1}{2}}|\lambda|^{3(1-\frac{\theta_1}{2})}\left(\sqrt{\|F\|\|U\|+\|F\|^2}\right)^{1-\frac{\theta_1}{2}} \\
&\leq C_\delta|\lambda|^{3-\frac{\theta_1}{2}}\sqrt{\|F\|\|U\|+\|F\|^2}.
\end{aligned}
$$

Item (ii) : From Lemma (41) we have

$$
\|\dot{u}\|^2 \leq C\|A^{\frac{\theta_1}{2}}\dot{u}\|^2 \leq C_\delta|\lambda|^2\left(\|F\|\|U\|+\|F\|^2\right) \leq C_\delta|\lambda|^{6-\theta_1}\left(\|F\|\|U\|+\|F\|^2\right).
$$

In particular, from Lemma 43 we have

$$
\|U\|^2 \leq C_\delta\lambda^{6-\theta_1}\|F\|.
$$

■

4.3.3 Wave equation in the absence of memory term

In this subsection, we will consider the first equation of (4.5)-(4.6) in absence of the memory term. For this, we have $g_1 = 0$ and then, the definitions about this case will be modified.

Lemma 49. *From $\sigma \leq 1$ we have*

(i) $\|A^{\frac{\sigma+1}{2}}u\|^2 \leq C_\delta\left(\|A^{\frac{\sigma}{2}}\dot{u}\| + \|F\|\|U\| + \|F\|^2\right)$,

(ii) $\|\lambda^{-1}A^{\frac{\sigma+1}{2}}u\|^2 \leq C_\delta\left(\|A^{\frac{\sigma-1}{2}}\dot{u}\|^2 + \|F\|\|U\| + \|F\|^2\right)$.

Proof. From equation (4.29), computing the inner product with $A^\sigma u$ and using equation (4.27) we have

$$
\beta_1\|A^{\frac{\sigma+1}{2}}u\|^2 = C\left(\|A^{\frac{\sigma}{2}}\dot{u}\|^2 - \alpha\|A^{\frac{\sigma}{2}}u\|^2 + \alpha\langle A^{\sigma-\frac{1}{2}}v, A^{\frac{\sigma+1}{2}}u\rangle + \rho_1\langle\dot{u}, A^\sigma f_2\rangle + \rho_1\langle f_3, A^\sigma u\rangle\right)
$$

therefore, using Young's inequality follows that

$$
\|A^{\frac{\sigma+1}{2}}u\|^2 \leq C\left(\|A^{\frac{\sigma}{2}}\dot{u}\|^2 + \|A^{\sigma-\frac{1}{2}}v\|^2 + \|F\|\|U\| + \|F\|^2\right),
$$

from Lemma 42 it is possible to conclude item (i).

Furthermore, multiplying by λ^{-2} the above inequality and using equation (4.28) we conclude

$$
\begin{aligned}
\|\lambda^{-1}A^{\frac{\sigma+1}{2}}u\|^2 &\leq C_\delta\left(\|\lambda^{-1}A^{\frac{\sigma}{2}}\dot{u}\|^2 + \|F\|\|U\| + \|F\|^2\right) \quad (4.40)\\
&\leq C_\delta\left(\|A^{\frac{\sigma}{2}}u\|^2 + \|F\|\|U\| + \|F\|^2\right). \quad (4.41)
\end{aligned}
$$

Applying item (i) we obtain

$$\|\lambda^{-1}A^{\frac{\sigma+1}{2}}u\|^2 \leq C_\delta\left(\|A^{\frac{\sigma-1}{2}}\dot{u}\|^2 + \|F\|\|U\| + \|F\|^2\right). \tag{4.42}$$

■

Lemma 50. *If we consider the first equation in the absence of memory then from the same hypothesis of Lemma 42 we have the follows result*

(i) $\|A^{\theta_2-1}\dot{u}\|^2 \leq C_\delta\lambda^6\left(\|F\|\|U\| + \|F\|^2\right),$

(ii) $\|A^{\theta_2}\dot{u}\|^2 \leq C_\delta\lambda^{10}\left(\|F\|\|U\| + \|F\|^2\right).$

Proof. From equation (4.29), computing the inner product with $\lambda^2A^{2\theta_2-1}v$, using equation (4.28), definition of A_1 and the fact that operator is self-adjoint we obtain

$$\begin{aligned}-\rho_1\lambda^2\langle A^{\frac{2\theta_2-1}{2}}\dot{u}, A^{\frac{2\theta_2-1}{2}}f_2\rangle - \rho_1\langle A^{\frac{2\theta_2-1}{2}}\dot{u}, \lambda^2A^{\frac{2\theta_2-1}{2}}\dot{v}\rangle + \beta_1\langle\lambda A^{\theta_2}u, \lambda A^{\theta_2}v\rangle\\ +\alpha\langle A^{\frac{2\theta_2-1}{2}}u, \lambda^2A^{\frac{2\theta_2-1}{2}}v\rangle - \alpha\lambda^2\|A^{\frac{2\theta_2-1}{2}}v\|^2 = \lambda^2\rho_1\langle f_3, A^{2\theta_2-1}v\rangle.\end{aligned}$$

In the same way, from equation (4.30), computing the inner product with $\lambda^2A^{2\theta_2-2}u$, definition of A_2 and using equation (4.27), we have

$$\begin{aligned}&-\rho_2\lambda^2\langle A^{\theta_2-1}\dot{v}, A^{\theta_2-1}f_1\rangle - \rho_2\langle\lambda^2A^{\theta_2-1}\dot{v}, A^{\theta_2-1}\dot{u}\rangle + \beta_2\lambda^2\langle A^{\theta_2}v, A^{\theta_2}u\rangle\\ &\quad-\lambda^2\int_0^\infty g_2(s)\langle A^{2\theta_2}v, A^{2\theta_2-2}u\rangle ds + \lambda^2\int_0^\infty g_2\langle A^{2\theta_2}\vartheta(s), A^{2\theta_2-2}u\rangle ds\\ &\quad\quad+\alpha\langle\lambda A^{\theta_2-1}v, \lambda A^{\theta_2-1}u\rangle - \alpha\lambda^2\|A^{\theta_2-1}u\|^2 = \lambda^2\rho_2\langle f_4, A^{2\theta_2-2}u\rangle.\end{aligned}$$

Divide by β_1 the first equation, by β_2 the second equation and performing the sum between both we get

$$\begin{aligned}\frac{\alpha}{\beta_2}\|\lambda A^{\theta_2-1}u\|^2 &= \frac{\alpha}{\beta_1}\|\lambda A^{\frac{2\theta_2-1}{2}}v\|^2 + \frac{\rho_1}{\beta_1}\langle f_3, \lambda^2A^{2\theta_2-1}v\rangle + \frac{\rho_1}{\beta_1}\langle A^{\theta_2-1}\dot{u}, \lambda^2A^{\theta_2}f_2\rangle\\ &+\frac{\rho_1}{\beta_1}\langle A^{\frac{2\theta_2-1}{2}}\dot{u}, \lambda^2A^{\frac{2\theta_2-1}{2}}\dot{v}\rangle - \lambda^2\langle A^{\theta_2}u, A^{\theta_2}v\rangle - \frac{\alpha}{\beta_1}\langle\lambda A^{\frac{2\theta_2-1}{2}}u, \lambda A^{\frac{2\theta_2-1}{2}}v\rangle\\ &-\frac{\rho_2}{\beta_2}\lambda^2\langle A^{\theta_2-1}\dot{v}, A^{\theta_2-1}f_1\rangle - \lambda^2\frac{\rho_2}{\beta_2}\langle A^{\theta_2-1}\dot{v}, A^{\theta_2-1}\dot{u}\rangle + \lambda^2\langle A^{\theta_2}v, A^{\theta_2}u\rangle\\ &+\frac{\alpha}{\beta_2}\langle\lambda A^{\theta_2-1}v, \lambda A^{\theta_2-1}u\rangle - \frac{\rho_2}{\beta_2}\lambda^2\langle f_4, A^{2\theta_2-2}u\rangle\\ &-\frac{1}{\beta_2}\int_0^\infty g_2(s)\langle\lambda^2A^{\theta_2}\left(\vartheta(s)-v\right), A^{3\theta_2-2}u\rangle ds.\end{aligned} \tag{4.43}$$

Note that from equation (4.28) and (4.32), we obtain

$$\vartheta(s) - v = \frac{-\partial_s\vartheta - f_2 + f_6(s)}{i\lambda},$$

and therefore, integrating by parts we have

$$\int_0^\infty g_2(s)\langle A^{\theta_2}(\vartheta(s)-v), A^{3\theta_2-2}u\rangle ds = -\int_0^\infty g_2'(s)\langle A^{\theta_2}\vartheta(s), i\lambda^{-1}A^{3\theta_2-2}u\rangle ds + \int_0^\infty g_2(s)\langle A^{\theta_2}(f_2-f_6(s)), i\lambda^{-1}A^{3\theta_2-2}u\rangle ds. \tag{4.44}$$

Substituting equation (4.44) in (4.43) and computing the real part we have

$$\begin{aligned}\frac{\alpha}{\beta_2}\|\lambda A^{\theta_2-1}u\|^2 &= \frac{\alpha}{\beta_1}\|\lambda A^{\frac{2\theta_2-1}{2}}v\|^2 + \frac{\rho_1}{\beta_1}Re\langle f_3, \lambda^2A^{2\theta_2-1}v\rangle + \frac{\rho_1}{\beta_1}Re\langle A^{\theta_2-1}\dot{u}, \lambda^2A^{\theta_2}f_2\rangle\\ &+ \frac{\rho_1}{\beta_1}Re\langle A^{\theta_2-1}\dot{u}, \lambda^2A^{\theta_2}\dot{v}\rangle - \frac{\alpha}{\beta_1}Re\langle\lambda A^{\theta_2-1}u, \lambda A^{\theta_2}v\rangle - \frac{\rho_2}{\beta_2}\lambda^2Re\langle A^{\theta_2-1}\dot{v}, A^{\theta_2-1}f_1\rangle\\ &- \frac{\rho_2}{\beta_2}Re\langle\lambda^2A^{\theta_2-1}\dot{v}, A^{\theta_2-1}\dot{u}\rangle - \frac{1}{\beta_2}Re\int_0^\infty g_2'(s)\langle\lambda^2A^{\theta_2}\vartheta(s), i\lambda^{-1}A^{3\theta_2-2}u\rangle ds\\ &+ \frac{\alpha}{\beta_2}Re\langle\lambda A^{\theta_2-1}v, \lambda A^{\theta_2-1}u\rangle - \frac{\rho_2}{\beta_2}\lambda^2Re\langle f_4, A^{2\theta_2-2}u\rangle\\ &+ \frac{1}{\beta_2}Re\int_0^\infty g_2'(s)\langle\lambda^2A^{\theta_2}\left(f_2-f_6(s)\right), i\lambda^{-1}A^{3\theta_2-2}u\rangle ds.\end{aligned}$$

Using Young's inequality, suitable continuous embedding and estimates from Lemma 42 we conclude

$$\|\lambda A^{\theta_2-1}u\|^2 \le C\left(\|A^{\theta_2-1}\dot{u}\|\|\lambda^2A^{\theta_2}\dot{v}\| + \varepsilon\|\lambda^{-1}A^{3\theta_2-2}u\|^2\right) + C_\varepsilon\lambda^4\left(\|F\|\|U\| + \|F\|^2\right) \tag{4.45}$$

On the other hand, from Lemma (49) item (ii), for $\frac{\sigma+1}{2} = 3\theta_2 - 2$ we have

$$\|\lambda^{-1}A^{3\theta_2-2}u\|^2 \le C\left(\|A^{3\theta_2-3}\dot{u}\|^2 + \|F\|\|U\| + \|F\|^2\right), \tag{4.46}$$

and then, using equation (4.27) we conclude

$$\|\lambda^{-1}A^{3\theta_2-2}u\|^2 \le C\left(\|\lambda A^{3\theta_2-3}u\|^2 + \|F\|\|U\| + \|F\|^2\right), \tag{4.47}$$

therefore, combine (4.47) with the continuous embedding $\mathcal{D}(A^{\theta_2-1}) \hookrightarrow \mathcal{D}(A^{3\theta_2-3})$, computing ε small enough, we have for estimate (4.45)

$$\|\lambda A^{\theta_2-1}u\|^2 \le C_\delta\left(\|A^{\theta_2-1}\dot{u}\|\|\lambda^2A^{\theta_2}\dot{v}\| + \lambda^4\|F\|\|U\| + \lambda^4\|F\|^2\right) \tag{4.48}$$

Furthermore, from equation (4.27) we can write using estimate (4.48):

$$\begin{aligned}\|A^{\theta_2-1}\dot{u}\|^2 &\le \|\lambda A^{\theta_2-1}u\|^2 + \|F\|^2\\ &\le C_\delta\left(\|A^{\theta_2-1}\dot{u}\|\|\lambda^2A^{\theta_2}\dot{v}\| + \lambda^4\|F\|\|U\| + \lambda^4\|F\|^2\right)\end{aligned}$$

therefore, using Lemma 42 we have

$$\begin{aligned}\|A^{\theta_2-1}\dot{u}\|^2 &\leq C_\delta\left(\|\lambda^2A^{\theta_2}\dot{v}\|^2+\lambda^4\|F\|\|U\|+\lambda^4\|F\|^2\right)\\ &\leq \lambda^4C_\delta\left(\|A^{\theta_2}\dot{v}\|^2+\|F\|\|U\|+\|F\|^2\right)\\ &\leq \lambda^6C_\delta\left(\|F\|\|U\|+\|F\|^2\right).\end{aligned}$$

Item (ii): Furthermore, using Lemma 49 with $\frac{\sigma+1}{2}=\theta_2-\frac{1}{2}$ and applying item (i) from this Lemma we conclude

$$\begin{aligned}\|A^{\theta_2-\frac{1}{2}}u\|^2 &\leq C_\delta\left(\|A^{\theta_2-1}\dot{u}\|^2+\|F\|\|U\|+\|F\|^2\right)\\ &\leq \lambda^6C_\delta\left(\|F\|\|U\|+\|F\|^2\right).\end{aligned}$$

On the other hand, from equation (4.27) we have

$$\|A^{\theta_2-\frac{1}{2}}\dot{u}\|^2\leq C_\delta\lambda^8\left(\|F\|\|U\|+\|F\|^2\right). \tag{4.49}$$

Applying again in the same way Lemma 49 with $\frac{\sigma+1}{2}=\theta$ and estimate 4.49 we have

$$\begin{aligned}\|A^{\theta_2}u\|^2 &\leq C_\delta\left(\|A^{\theta_2-\frac{1}{2}}\dot{u}\|^2+\|F\|\|U\|+\|F\|^2\right)\\ &\leq C_\delta\lambda^8\left(\|F\|\|U\|+\|F\|^2\right).\end{aligned}$$

Therefore, we can conclude from equation (4.27)

$$\|A^{\theta_2}\dot{u}\|^2\leq\lambda^{10}C_\delta\left(\|F\|\|U\|+\|F\|^2\right).$$

■

Lemma 51. *In the same hypothesis of last lemmas, the solution of system (4.27)-(4.32) satisfy the following estimates*

(i) $\|\dot{u}\|^2\leq C_\delta|\lambda|^{10-4\theta_2}(\|F\|\|U\|+\|F\|^2)$,

(ii) $\|\dot{v}\|^2\leq C_\delta|\lambda|^{10-4\theta_2}(\|F\|\|U\|+\|F\|^2)$;

In particular, from Theorem 43 we have

$$\|U\|\leq C_\delta\lambda^{10-4\theta_2}\|F\|.$$

Proof. For items (i) and (ii) we will use

$$0=\theta_2\left(\theta_2-1\right)+\left(1-\theta_2\right)\theta_2:$$

Item (i) : From interpolation inequality and Lemma 50 we obtain

$$\begin{aligned}\|\dot{u}\| &\leq C\|A^{\theta_2-1}\dot{u}\|^{\theta_2}\|A^{\theta_2}\dot{u}\|^{1-\theta_2}\\ &\leq C_\delta\lambda^{3\theta_2}\left(\sqrt{\|F\|\|U\|+\|F\|^2}\right)^{\theta_2}\lambda^{5(1-\theta_2)}\left(\sqrt{\|F\|\|U\|+\|F\|^2}\right)^{1-\theta_2}\\ &\leq C_\delta\lambda^{5-2\theta_2}\sqrt{\|F\|\|U\|+\|F\|^2}.\end{aligned}$$

Item (ii) : From continuous embedding we have

$$\|\dot{v}\|^2 \leq C\|A^{\theta_2}\dot{v}\|^2 \leq C|\lambda|^2\|F\|\|U\| \leq C|\lambda|^{10-4\theta_2}\|F\|\|U\|.$$

In particular, from Lemma 43 we have

$$\|U\|^2 \leq C_\delta\lambda^{10-4\theta_2}\|F\|.$$

■

As mentioned, we are using the Theorem 16 to show our main result, Theorem 40. To finish, let's prove that $i\mathbb{R}\subset\rho(\mathbb{B})$.

Theorem 52. *The operator $\mathbb{B}$ associated with Cauchy problem 4.10 has the property that $i\mathbb{R}\subset\rho(\mathbb{B})$.*

Proof. Let's suppose that $i\mathbb{R}\not\subseteq\rho(\mathbb{B})$. Previously, we show that $0\in\rho(\mathbb{B})$, for this, if we consider the highest positive number λ_0 such that $]-i\lambda_0, i\lambda_0[\subset\rho(\mathbb{B})$, then $i\lambda_0\in\sigma(\mathbb{B})$ or $-i\lambda_0\in\sigma(\mathbb{B})$. Firstly, let's consider $i\lambda_0\in\sigma(\mathbb{B})$ (in the same way for $-i\lambda_0\in\sigma(\mathbb{B})$) and fixing a constant $\delta>0$ with $\delta<\lambda_0$ there exists a sequence of positive real numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that $\delta\leq\lambda_n<\lambda_0$, with $\lambda_n\to\lambda_0$, and a sequence $U_n=(u_n,v_n,\dot{u}_n,\dot{v}_n,\eta_n,\vartheta_n)\in D(\mathbb{B})$ with $\|U_n\|=1$ such that

$$\|(i\lambda_n-\mathbb{B})U_n\|=\|F_n\|\to 0, \quad \text{as} \quad n\to\infty.$$

That is, if $F_n=(f_{1n},f_{2n},f_{3n},f_{4n},f_{5n},f_{6n})$ then

$$\begin{aligned} i\lambda_n u_n-\dot{u}_n=f_{1n}\to 0 \quad &\text{in} \quad \mathcal{D}(A^{\frac{1}{2}}),\\ i\lambda_n v_n-\dot{v}_n=f_{2n}\to 0 \quad &\text{in} \quad \mathcal{D}(A^{\frac{1}{2}}),\\ i\rho_1\lambda_n\dot{u}_n+A_1u_n+\mathbb{D}_1\eta_n+\alpha(u_n-v_n)=\rho_1 f_{3n}\to 0 \quad &\text{in} \quad \mathbb{H},\\ i\rho_2\lambda_n\dot{v}_n+A_2^2v_n+\mathbb{D}_2\vartheta_n+\alpha(v_n-u_n)=\rho_2 f_{4n}\to 0 \quad &\text{in} \quad \mathbb{H},\\ i\lambda_n\eta_n-\dot{u}_n+\partial_s\eta_n=f_{5n}\to 0 \quad &\text{in} \quad \mathcal{M}_1,\\ i\lambda_n\vartheta_n-\dot{v}_n+\partial_s\vartheta_n=f_{6n}\to 0 \quad &\text{in} \quad \mathcal{M}_2.\end{aligned}$$

finally, it may be concluded from Lemma 43 and Lemma 51

$$\|U_n\|^2 \le C_\delta |\lambda_n|^{10-4\theta_2}(\|F_n\|\|U_n\| + \|F_n\|^2). \tag{4.50}$$

Thus, since $\lambda_n < \lambda_0$ and $\|U_n\| = 1$ follows that

$$1 = \|U_n\|^2 \le C_\delta \lambda_0^{10-4\theta_2}\left(\|F_n\| + \|F_n\|^2\right) \to 0,$$

as $n \to \infty$, that is, absurd. So, we conclude $i\mathbb{R} \subset \rho(\mathbb{B})$. Following this way we have similarly the same result for the other cases. ■

4.4 Optimality of decay rates

Theorem 53. *If we consider memory kernels exponentially decreasing in* (4.5) *then all the polynomial decay rates found in Theorem 40 are optimal in the following sense:*

(i) If both memory terms appear in the system (4.5) *and at least one of the exponents* θ_1, θ_2 *is strictly less than 1, then the semigroup does not decay with the rate* $t^{-\sigma}$ *for* $\sigma > 1/(2-2\theta_0)$, $\theta_0 = \min\{\theta_1, \theta_2\}$.

(ii) If we remove the memory term from the second equation of (4.5)*, then the semigroup does not decay with the rate* $t^{-\sigma}$ *for* $\sigma > 1/(6-\theta_1)$.

(iii) If we remove the memory term from the first equation of (4.5)*, then the semigroup does not decay with the rate* $t^{-\sigma}$ *for* $\sigma > 1/(10-4\theta_2)$.

Proof. We are considering memory kernels exponentially decreasing, that is

$$g_1(t) = \nu_1 e^{-\mu_1 t}, \quad g_2(t) = \nu_2 e^{-\mu_2 t}, \quad \nu_1, \nu_2 \ge 0, \quad \mu_1, \mu_2 > 0. \tag{4.51}$$

Thus, the first equation (respectively, the second one) of the system does not have memory term when $\nu_1 = 0$ (respectively, $\nu_2 = 0$).

On the other hand, we have that A is a positive self-adjoint operator with compact resolvent, its spectrum is formed by positive eigenvalues that it is denoted by $\gamma_n, n \in \mathbb{N}$, with $\gamma_n \to \infty$ and the corresponding unitary eigenvectors are (e_n) that satisfies the following equality

$$Ae_n = \gamma_n e_n, \quad \|e_n\| = 1, \quad n \in \mathbb{N}.$$

Let's consider the family vectors given by $F_n = (0, 0, c_1 e_n, c_2 e_n, 0, 0)$ where the constants $c_1, c_2 \in \mathbb{R}$ will be chosen later. The solution $U = (u, v, \dot{u}, \dot{v}, \eta, \vartheta)$ of the system $(i\lambda I - \mathbb{B})U = F_n$ can be consider in the follows way

$$i\lambda u - \dot{u} = 0, \tag{4.52}$$

$$i\lambda v - \dot{v} = 0, \tag{4.53}$$

$$i\lambda \dot{u} + \rho_1^{-1}\left\{A_1 u + \mathbb{D}_1 \eta + \alpha(u - v)\right\} = c_1 e_n, \tag{4.54}$$

$$i\lambda \dot{v} + \rho_2^{-1}\left\{A_2^2 v + \mathbb{D}_2 \vartheta + \alpha(v - u)\right\} = c_2 e_n, \tag{4.55}$$

$$i\lambda \eta(s) + \partial_s \eta(s) = \dot{u}, \tag{4.56}$$

$$i\lambda \vartheta(s) + \partial_s \vartheta(s) = \dot{v}. \tag{4.57}$$

Note that, using the equations (4.52),(4.53), (4.56) and (4.57) we obtain

$$\eta(s) = u(1 - e^{-i\lambda s}),$$

and

$$\vartheta(s) = v(1 - e^{-i\lambda s}).$$

Substituting these equations in (4.54) and (4.55) and using the definition of operator A_1 and A_2 we have

$$\begin{cases} \rho_1 \lambda^2 u - \beta_1 A u + \left(\displaystyle\int_0^\infty g_1(s) e^{-i\lambda s} ds\right) A^{\theta_1} u - \alpha(u - v) = \rho_1 c_1 e_n, \\ \rho_2 \lambda^2 v - \beta_2 A^2 v + \left(\displaystyle\int_0^\infty g_2(s) e^{-i\lambda s} ds\right) A^{2\theta_2} v - \alpha(v - u) = \rho_2 c_2 e_n. \end{cases}$$

At this point we just are going to look for solutions of form

$$\begin{cases} u = \kappa_1 e_n \\ v = \kappa_2 e_n \end{cases}, \kappa_1, \kappa_2 \in \mathbb{C}.$$

In this situation we will have the system

$$\left\{\rho_1 \lambda^2 - \beta_1 \gamma_n - \alpha + \left(\int_0^\infty g_1(s) e^{-i\lambda s} ds\right) \gamma_n^{\theta_1}\right\} \kappa_1 + \alpha \kappa_2 = \rho_1 c_1,$$

$$\left\{\rho_2 \lambda^2 - \beta_2 \gamma_n^2 - \alpha + \left(\int_0^\infty g_2(s) e^{-i\lambda s} ds\right) \gamma_n^{2\theta_2}\right\} \kappa_2 + \alpha \kappa_1 = \rho_2 c_2.$$

and therefore

$$\begin{bmatrix} p_1(\lambda^2) + I_1(\lambda)\gamma_n^{\theta_1} & \alpha \\ \alpha & p_2(\lambda^2) + I_2(\lambda)\gamma_n^{2\theta_2} \end{bmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} \rho_1 c_1 \\ \rho_2 c_2 \end{pmatrix}, \tag{4.58}$$

where we are considering

$$\begin{cases} p_1(s) & = \rho_1 s - \beta_1\gamma_n - \alpha, \\ p_2(s) & = \rho_2 s - \beta_2\gamma_n - \alpha, \\ I_j & = \displaystyle\int_0^\infty g_j(s)e^{-i\lambda s}ds \quad \text{for} \quad j = 1, 2. \end{cases} \tag{4.59}$$

Furthermore, choosing $c_1 = 0$ and $c_2 = \rho_2^{-1}$ the solution of the second component (from (4.58)) is

$$\kappa_2 = \frac{p_1(\lambda^2) + I_1(\lambda)\gamma_n^{\theta_1}}{p_1(\lambda^2)p_2(\lambda^2) - \alpha^2 + p_1(\lambda^2)I_2(\lambda)\gamma_n^{2\theta_2} + p_2(\lambda^2)I_1(\lambda)\gamma_n^{\theta_1} + I_1(\lambda)I_2(\lambda)\gamma_n^{\theta_1+2\theta_2}}, \tag{4.60}$$

where we will assign

$$J_1(\lambda) = p_1(\lambda^2)I_2(\lambda)\gamma_n^{2\theta_2}, \quad J_2(\lambda) = p_2(\lambda^2)I_1(\lambda)\gamma_n^{\theta_1}, \quad J_3(\lambda) = I_1(\lambda)I_2(\lambda)\gamma_n^{\theta_1+2\theta_2}.$$

The polynomial $p_1(s)p_2(s) - \alpha^2$ that appears in (4.60) can be rewritten by

$$\rho_1\rho_2\left\{s^2 - \left(\frac{\beta_2\gamma_n^2}{\rho_2} + \frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha(\rho_1+\rho_2)}{\rho_1\rho_2}\right)s + \left(\frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha}{\rho_1}\right)\left(\frac{\beta_2\gamma_n^2}{\rho_2} + \frac{\alpha}{\rho_2}\right) - \frac{\alpha^2}{\rho_1\rho_2}\right\},$$

whose roots are

$$s_n^{\pm} = \frac{\frac{\beta_2\gamma_n^2}{\rho_2} + \frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha(\rho_1+\rho_2)}{\rho_1\rho_2} \pm \sqrt{\left(\frac{\beta_2\gamma_n^2}{\rho_2} - \frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha(\rho_1-\rho_2)}{\rho_1\rho_2}\right)^2 + \frac{4\alpha^2}{\rho_1\rho_2}}}{2}. \tag{4.61}$$

In this point we consider $\lambda := \lambda_n = \sqrt{s_n^+}$ so the formula in (4.60) becomes

$$\kappa_{2,n} = \frac{p_1(\lambda_n^2) + I_1(\lambda_n)\gamma_n^{\theta_1}}{J_1(\lambda_n) + J_2(\lambda_n) + J_3(\lambda_n)}. \tag{4.62}$$

Introducing the notation $a_n \approx b_n$ when the $\lim_{n\to\infty}\frac{|a_n|}{|b_n|}$ is a positive real number, we check that $s_n^+ \approx \gamma_n^2$ and consequently $\lambda_n \approx \gamma_n$. We will apply this notation to obtain the next results.

On the other hand, note that (from definition of p_1 and λ_n)

$$\frac{2}{\rho_1}p_1(\lambda_n^2) = \frac{\beta_2\gamma_n^2}{\rho_2} - \frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha(\rho_1-\rho_2)}{\rho_1\rho_2} + \sqrt{\left(\frac{\beta_2\gamma_n^2}{\rho_2} - \frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha(\rho_1-\rho_2)}{\rho_1\rho_2}\right)^2 + \frac{4\alpha^2}{\rho_1\rho_2}},$$

however as $p_1(\lambda_n^2)p_2(\lambda_n^2) = \alpha^2$ we get

$$p_1(\lambda_n^2) \approx \gamma_n^2 \quad \text{and} \quad p_2(\lambda_n^2) \approx \gamma_n^{-2}. \tag{4.63}$$

Furthermore, using the expressions from (4.51) in $I_j, j = 1, 2$ we have

$$I_j(\lambda) = \int_0^\infty \nu_j e^{-(\mu_j + i\lambda_n)t} dt = \frac{\nu_j}{\mu_j + i\lambda_n} \approx \nu_i \lambda_n^{-1}, \quad j = 1, 2.$$

Therefore

$$J_1(\lambda_n) \approx \nu_2 \lambda_n^{1+2\theta_2}, \quad J_2(\lambda_n) \approx \nu_1 \lambda_n^{-3+\theta_1}, \quad J_3(\lambda_n) \approx \nu_1 \nu_2 \lambda_n^{-2+\theta_1+2\theta_2}. \tag{4.64}$$

If the two memory terms appear in the system we have $\nu_1 > 0$ and $\nu_2 > 0$. In view of definition of J_1, J_2, J_3, I_1, I_2 and $p_1(\lambda^2)p_2(\lambda^2) = \alpha^2$ we can conclude that

$$\lim_{n\to\infty} \frac{|J_j(\lambda_n)|}{|J_1(\lambda_n)|} = 0, \; j = 2, 3, \quad and \quad \lim_{n\to\infty} \frac{|I_1(\lambda_n)\gamma_n^{\theta_1}|}{|p_1(\lambda_n^2)|} = 0, \tag{4.65}$$

as a conclusion we have from (4.62)

$$\kappa_{2,n} \approx \frac{p_1(\lambda_n^2)}{J_1(\lambda_n)} \approx \lambda_n^{1-2\theta_2}.$$

Now if we assume that $\theta_2 \leq \theta_1$ (the complementary case $\theta_1 \leq \theta_2$ will be addressed later), then we have $\theta_0 = \min\{\theta_1, \theta_2\} = \theta_2$ and $\kappa_{2,n} \approx \lambda_n^{1-2\theta_0}$. Therefore, if $U_n = (u_n, v_n, \dot{u}_n, \dot{v}_n, \eta_n, \vartheta_n)$ is the solution of system $(i\lambda_n I - \mathbb{B})U = F_n$ then we obtain

$$\|U_n\| \geq \rho_2^{1/2}\|\dot{v}_n\| = \rho_2^{1/2}\lambda_n\|v_n\| = \rho_2^{1/2}\lambda_n\kappa_{2,n} \geq \varepsilon_0\lambda_n^{2-2\theta_0},$$

for some $\varepsilon_0 > 0$ and n large enough. In this moment if the semigroup of the system decays polynomially with the rate $t^{-\sigma}$ with $\sigma > 1/(2 - 2\theta_0)$, we have

$$\varepsilon_0\lambda_n^{2-2\theta_0} \leq \|U_n\| \leq C\lambda_n^{1/\sigma}\|F_n\| \;\Rightarrow\; \varepsilon_0\lambda_n^{2-2\theta_0-1/\sigma} \leq C,$$

which is contradictory because $\lambda_n^{2-2\theta_0-1/\sigma} \to \infty$ when $n \to \infty$.

On the other hand, in the absence of the second memory term we have that $\nu_1 > 0$ and $\nu_2 = 0$. In this way from (4.62) (using the second result from (4.65) with (4.63) and (4.64)) we get

$$\kappa_{2,n} = \frac{p_1(\lambda_n^2) + I_1(\lambda_n)\gamma_n^{\theta_1}}{J_2(\lambda_n)} \approx \frac{p_1(\lambda_n^2)}{J_2(\lambda_n)} \approx \lambda_n^{5-\theta_1}. \tag{4.66}$$

In this point, we can conclude for $U_n = (u_n, v_n, \dot{u}_n, \dot{v}_n, \eta_n)$ solution of system $(i\lambda_n I - \mathbb{B})U = F_n$ the follows result

$$\|U_n\| \geq \rho_2^{1/2}\|\dot{v}_n\| = \rho_2^{1/2}\lambda_n\|v_n\| = \rho_2^{1/2}\lambda_n\kappa_{2,n} \geq \varepsilon_1\lambda_n^{6-\theta_1},$$

for some $\varepsilon_1 > 0$ and n large enough. Furthermore, if the semigroup of the system decays polynomially with the rate $t^{-\sigma}$ with $\sigma > 1/(6-\theta_1)$ then we have

$$\varepsilon_1 \lambda_n^{6-\theta_1} \leq \|U_n\| \leq C\lambda_n^{1/\sigma}\|F_n\| \;\Rightarrow\; \varepsilon_1 \lambda_n^{6-\theta_1-1/\sigma} \leq C,$$

which is contradictory to the outcome of the $\lambda_n^{6-\theta_1-1/\sigma} \to \infty$ when $n \to \infty$.

Furthermore, choosing $c_1 = \rho_1^{-1}$, $c_2 = 0$ and using $\lambda := \lambda_n = \sqrt{s_n^-}$ (from (4.61)) the solution of the first component of the system (4.58) is

$$\kappa_{1,n} = \frac{p_2(\lambda_n^2) + I_2(\lambda_n)\gamma_n^{2\theta_2}}{J_1(\lambda_n) + J_2(\lambda_n) + J_3(\lambda_n)}. \tag{4.67}$$

Since

$$s_n^+ s_n^- = \left(\frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha}{\rho_1}\right)\left(\frac{\beta_2\gamma_n^2}{\rho_2} + \frac{\alpha}{\rho_2}\right) - \frac{\alpha^2}{\rho_1\rho_2},$$

we have that $s_n^- \approx \gamma_n$ and therefore $\lambda_n \approx \gamma_n^{1/2}$. On the other hand we have

$$\frac{4}{\rho_1^2} p_1(s_n^-) p_1(s_n^+) = \frac{-4\alpha^2}{\rho_1\rho_2}, \tag{4.68}$$

therefore

$$\begin{aligned}\frac{2}{\rho_1}p_1(\lambda_n^2) &= \frac{\beta_2\gamma_n^2}{\rho_2} - \frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha(\rho_1-\rho_2)}{\rho_1\rho_2} - \sqrt{\left(\frac{\beta_2\gamma_n^2}{\rho_2} - \frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha(\rho_1-\rho_2)}{\rho_1\rho_2}\right)^2 + \frac{4\alpha^2}{\rho_1\rho_2}} \\ &= \frac{-4\alpha^2/(\rho_1\rho_2)}{\frac{\beta_2\gamma_n^2}{\rho_2} - \frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha(\rho_1-\rho_2)}{\rho_1\rho_2} + \sqrt{\left(\frac{\beta_2\gamma_n^2}{\rho_2} - \frac{\beta_1\gamma_n}{\rho_1} + \frac{\alpha(\rho_1-\rho_2)}{\rho_1\rho_2}\right)^2 + \frac{4\alpha^2}{\rho_1\rho_2}}},\end{aligned}$$

and then we conclude that $p_1(\lambda_n^2) \approx \gamma_n^{-2} \approx \lambda_n^{-4}$. Furthermore, in view of $p_1(\lambda_n^2)p_2(\lambda_n^2) = \alpha^2$ we get $p_2(\lambda_n^2) \approx \lambda_n^4$. With this considerations we have

$$J_1(\lambda_n) \approx \nu_2\lambda_n^{-5+4\theta_2}, \quad J_2(\lambda_n) \approx \nu_1\lambda_n^{3+2\theta_1}, \quad J_3(\lambda_n) \approx \nu_1\nu_2\lambda_n^{-2+2\theta_1+4\theta_2}.$$

If the two memory terms appear in the system we have $\nu_1 > 0$ and $\nu_2 > 0$. Taking into account that

$$\lim_{n\to\infty} \frac{|J_j(\lambda_n)|}{|J_2(\lambda_n)|} = 0,\; j = 1,3 \quad \text{and} \quad \lim_{n\to\infty} \frac{|I_2(\lambda_n)\gamma_n^{2\theta_2}|}{|p_2(\lambda_n^2)|} = 0,$$

we conclude from (4.67)

$$\kappa_{1,n} \approx \frac{p_1(\lambda_n^2)}{J_1(\lambda_n)} \approx \lambda_n^{1-2\theta_1}.$$

In this case, if we assume that $\theta_1 \leq \theta_2$ then we have $\theta_0 = \min\{\theta_1, \theta_2\} = \theta_1$ and $\kappa_{1,n} \approx \lambda_n^{1-2\theta_0}$. Therefore, if $U_n = (u_n, v_n, \dot{u}_n, \dot{v}_n, \eta_n, \vartheta_n)$ is the solution of system $(i\lambda_n I - \mathbb{B})U = F_n$ then we obtain

$$\|U_n\| \geq \rho_1^{1/2}\|\dot{u}_n\| = \rho_1^{1/2}\lambda_n\|u_n\| = \rho_1^{1/2}\lambda_n\kappa_{1,n} \geq \varepsilon_2\lambda_n^{2-2\theta_0},$$

for some $\varepsilon_2 > 0$ and n large enough. In this moment if the semigroup of the system decays polynomially with the rate $t^{-\sigma}$ with $\sigma > 1/(2-2\theta_0)$ then we have

$$\varepsilon_2\lambda_n^{2-2\theta_0} \leq \|U_n\| \leq C\lambda_n^{1/\sigma}\|F_n\| \;\Rightarrow\; \varepsilon_2\lambda_n^{2-2\theta_0-1/\sigma} \leq C,$$

which is contradictory because $\lambda_n^{2-2\theta_0-1/\sigma} \to \infty$ when $n \to \infty$.

On the other hand, in the absence of the first memory term we have $\nu_1 = 0$ and $\nu_2 > 0$ and therefore from (4.67) we get

$$\kappa_{1,n} = \frac{p_2(\lambda_n^2) + I_2(\lambda_n)\gamma_n^{2\theta_2}}{J_1(\lambda_n)} \approx \frac{p_2(\lambda_n^2)}{J_1(\lambda_n)} \approx \lambda_n^{9-4\theta_2}. \tag{4.69}$$

In this case we are using $\nu_1 = 0$ and therefore if $U_n = (u_n, v_n, \dot{u}_n, \dot{v}_n, \vartheta_n)$ is solution of system $(i\lambda_n I - \mathbb{B})U = F_n$ then we obtain from (4.66)

$$\|U_n\| \geq \rho_1^{1/2}\|\dot{u}_n\| = \rho_1^{1/2}\lambda_n\|u_n\| = \rho_1^{1/2}\lambda_n\kappa_{1,n} \geq \varepsilon_3\lambda_n^{10-4\theta_2},$$

for some $\varepsilon_3 > 0$ and n large enough. Furthermore, if the semigroup of the system decays polynomially with the rate $t^{-\sigma}$ with $\sigma > 1/(10-4\theta_2)$, we have

$$\varepsilon_3\lambda_n^{10-4\theta_2} \leq \|U_n\| \leq C\lambda_n^{1/\sigma}\|F_n\| \;\Rightarrow\; \varepsilon_3\lambda_n^{10-4\theta_2-1/\sigma} \leq C,$$

which is contradictory to the outcome of the $\lambda_n^{10-4\theta_2-1/\sigma} \to \infty$ when $n \to \infty$. ■

Remark 4. *We can summarize all result in the following table:*

Table 4.1: Wave-Plate equations results

memory dissipation	Optimal decay rate	θ_1	θ_2
just on wave	$t^{-1/(6-\theta_1)}$	$[0,1]$	-
just on plate	$t^{-1/(10-4\theta_2)}$	-	$[0,1]$
on wave and plate	exponential decay	$\theta_1 = 1$	$\theta_2 = 1$
on wave and plate	$t^{-1/(2-2\theta_0)},\ \theta_0 = \min\{\theta_1, \theta_2\}$	$\theta_1 < 1$ and $\theta_2 \leq 1$	$\theta_1 \leq 1$ and $\theta_2 < 1$

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