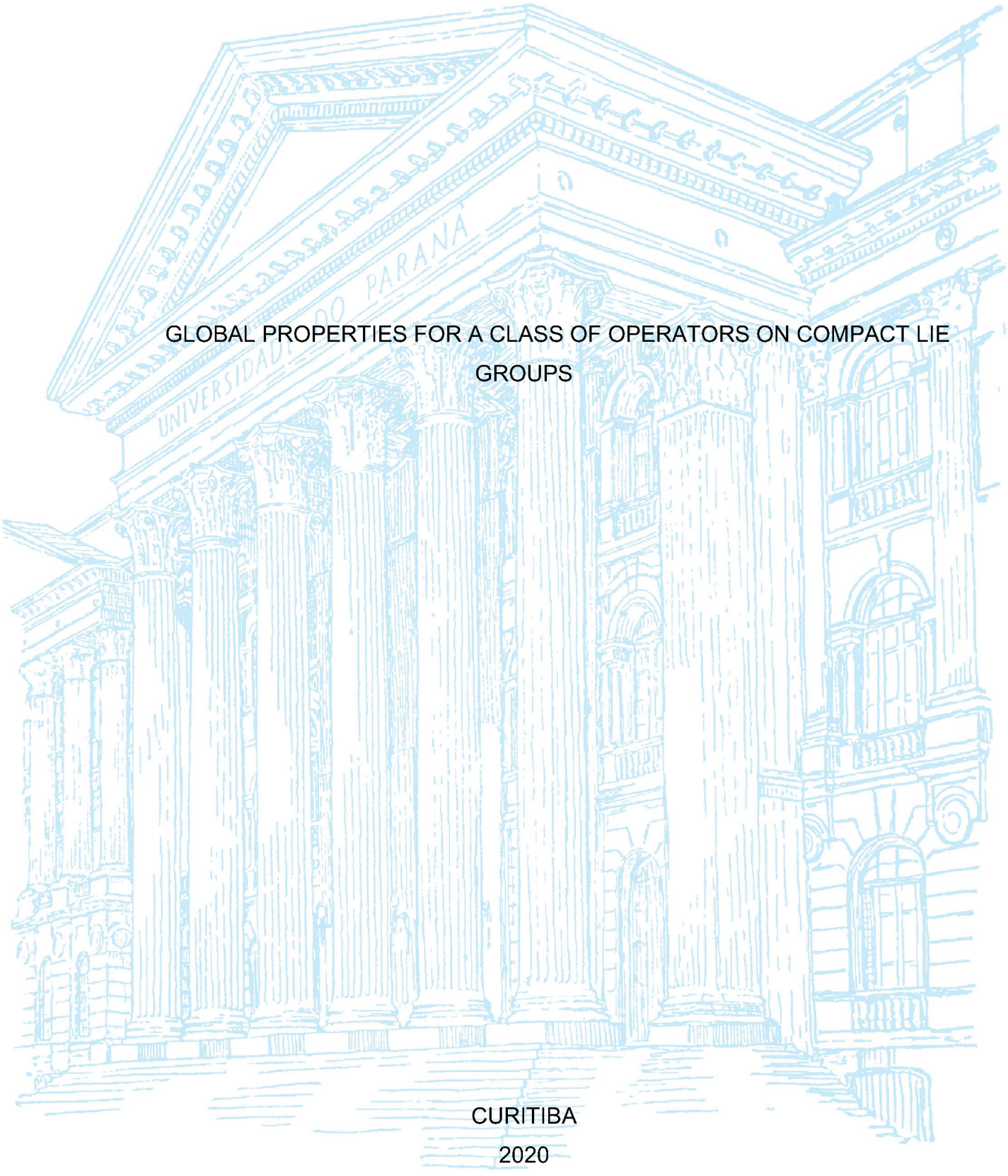
UNIVERSIDADE FEDERAL DO PARANÁ

WAGNER AUGUSTO ALMEIDA DE MORAES



GLOBAL PROPERTIES FOR A CLASS OF OPERATORS ON COMPACT LIE GROUPS

CURITIBA

2020

WAGNER AUGUSTO ALMEIDA DE MORAES

GLOBAL PROPERTIES FOR A CLASS OF OPERATORS ON COMPACT LIE GROUPS

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Orientador: Prof. Dr. Alexandre Kirilov (UFPR-Brasil)

Coorientador: Prof. Dr. Michael Ruzhansky (Ghent University-Bélgica e Queen Mary University of London-Reino Unido)

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"It's the questions we can't answer that teach us the most. They teach us how to think. If you give a man an answer, all he gains is a little fact. But give him a question and he'll look for his own answers."

Patrick Rothfuss, The Wise Man's Fear

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Preface

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2. A. Kirilov, W. A. A. de Moraes, and M. Ruzhansky. Global properties of vector fields on compact Lie groups in Komatsu classes. *arXiv e-prints*, arXiv:1910.01922 [math.AP], Oct 2019.

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4. A. Kirilov, W. A. A. de Moraes, and M. Ruzhansky. Partial Fourier series on compact Lie groups. *Bulletin des Sciences Mathématiques*, 160:102853, 2020.

RESUMO

Esta tese apresenta condições necessárias e suficientes para a obtenção de hipoeliticidade global e resolubilidade global para uma classe de campos vetoriais definidos em um produto de grupos de Lie compactos. Tanto a hipoeliticidade global quanto a resolubilidade global são estudadas no sentido usual das funções suaves, bem como em classes de Komatsu. Em vista da conjectura de Greenfield e Wallach sobre a não existência de campos vetoriais globalmente hipoelíticos senão definidos no toro, é estudada uma classe de exemplos que podem ser considerados como perturbações de ordem zero de campos vetoriais.

Palavras-chave: grupos compactos, hipoeliticidade global, resolubilidade global, classes de Komatsu.

ABSTRACT

In this dissertation we present necessary and sufficient conditions to have global hypoellipticity and global solvability for a class of vector fields defined in a product of compact Lie groups. Both global hypoellipticity and solvability are studied in the usual smooth sense as in the sense of Komatsu. Considering the Greenfield's and Wallach's conjecture, about the non–existence of globally hypoelliptic vector fields out of tori, we also study classes of examples that can be considered as zeros-order perturbations of our vector fields.

Keywords: compact groups, global hypoellipticity, global solvability, Komatsu classes.

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Introduction

In this work we propose to study regularity of solution and solvability of vector fields (and their perturbations by zero order terms) on a compact Lie group G. More precisely, denoting by $\mathcal{D}'(G)$ the space of distributions on G and by $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ a first-order differential operator, we are interested in establishing conditions that ensure that u is smooth whenever Pu is smooth. This property is known as global hypoellipticity. In relation to the global solvability, we want to identified under what conditions it is possible to guarantee that the equation $Pu = f \in \mathcal{D}'(G)$ has a solution, in the sense of distributions.

Both global hypoellipticity and global solvability have been widely studied in recent years, especially in the $n-$dimensional torus $\mathbb{T}^n$. See, for example, the impressive list of authors who have published articles addressing these subjects: [6], [7], [10], [14], [24], [26], [27], [28], [30], [31], [32] and references there in.

Even in the case of $\mathbb{T}^n$, the investigation of these global properties for vector fields is a challenging problem that still has open questions. Perhaps, the most famous and seemingly far-off question of a solution is the Greenfield's and Wallach's conjecture, which states the following: if a closed smooth orientable manifold admits a globally hypoelliptic vector field, then this manifold is $C^\infty-$diffeomorphic to a torus and this vector field is $C^\infty-$conjugated to a constant vector field whose coefficients satisfy a Diophantine condition (see [22] and [27]). S. Greenfield and N. Wallach have proved this conjecture for compact Lie groups in [27]. The conjecture it was also proved for compact manifolds of dimensions 2 and 3, and in some very particular cases, which are described by G. Forni in [22] and by L. Flaminio, G. Forni, and F. Rodriguez Hertz in [21].

Most of the studies that deal with the question of global hypoellipticity and global solvability in the torus make use of Fourier analysis as the main tool to obtain results from conditions imposed on the symbol or on the coefficients of the operator. For example, in [26], S. Greenfield and N. Wallach use only the Fourier series in $\mathbb{T}^n$ to characterize the global hypoellipticity of a

differential operator through its symbol and the famous application: $L = \partial_x + \alpha\partial_y$, $\alpha \in \mathbb{R}$ is globally hypoelliptic in $\mathbb{T}^2$ if, and only if, α is a irrational non-Liouville number appears for the first time. Therefore a natural way of extending such studies to other smooth manifolds would be to consider manifolds where we have a Fourier analysis.

In this direction, based on ideas [28] and [40], J. Delgado and M. Ruzhansky [18] introduced in compact smooth manifold M a notion of Fourier series for operators that commute with a fixed elliptic operator. Using these ideas, a study of global hypoellipticity for such operators was made in [15], [16], and [17]. The obvious disadvantage of this technique is that it works only for operators that commute with a fixed elliptic operator.

In the particular case where the compact manifold is a Lie group G, there is a natural way of introducing a Fourier analysis into G, see for example [11], [12], [13], [20], [36], [37], [38], [39], and [41]. In this work we use the notation and results based on the book by M. Ruzhansky and V. Turunen [35] to study the global hypoellipticity and global solvability of vector fields on Lie groups.

In the development of this project we find natural to begin by extending the results of [26] and [30] to a product of Lie groups $G_1 \times G_2$. In the case of constant coefficients, we observed that the classic results of the torus could be easily recovered and that some interesting novelties appeared. Next, by extending the theory of partial Fourier series to a product of Lie groups, we recover the reduction in the normal form for operators of the form $L = X_1 + \alpha(x_1)X_2$, where $\alpha \in C^\infty(G_1)$ is a real-valued function and each X_j is a vector field on the Lie algebra $\mathfrak{g}_j$. Considering the Greenfield's and Wallach's conjecture, we also analyze the case $L = X_1 + \alpha(x_1)X_2 + q(x_1, x_2)$, where $\alpha \in C^\infty(G_1)$ and $q \in C^\infty(G_1 \times G_2)$.

After analyzing the solvability and hypoellipticity in the smooth sense, we decided to study these same properties in the sense of Gevrey, which naturally led us to a generalization for the Komatsu classes. In this way, we also study global solvability and global hypoellipticity in the sense of Komatsu.

Outline of the dissertation:

This dissertation is organized as follows:

In Chapter 1 we introduce most of the notations and preliminary results concerning the Fourier analysis on compact Lie groups. We also give a brief description of Komatsu classes of Roumieu and Beurling types.

In Chapter 2 we study the global hypoellipticity and global solvability of a constant-co-

efficient vector field defined in compact Lie groups. Moreover, motivated by the presented examples and by Greenfield-Wallach conjecture, we give alternative ways to define global hypoellipticity to obtain examples in groups different from tori. We also investigate the properties of the vector field with a perturbation by a zero-order term.

In Chapter 3 we study a class of vector fields with variable coefficients and give some conditions that relate the global hypoellipticity and global solvability of these equations to constant-coefficient operators.

In Chapter 4 we present a study of the global hypoellipticity of a vector field defined on a product of a one-dimensional torus and a compact Lie group, which imaginary part of the variable coefficient is not constant.

In Chapters 5 and 6 we extend the results of Chapters 2 and 3 to Komatsu classes.

In Appendix A we present precisely the partial Fourier series and in Appendix B we present the proof of some auxiliary results.

Chapter 1

Preliminaries

In this chapter we introduce most of the notations and preliminary results necessary for the development of this dissertation. A presentation of these concepts and the demonstration of all the results presented here can be found in the references [11], [12], [20] (chapters 1 and 2), and [35] (chapters 7, 8 and 10).

1.1 Fourier analysis on compact Lie groups

1.1.1 Representations of topological groups

Let G be a topological group and let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ be a representation of G in a vector space V. We say that ϕ is unitary when $\mathrm{Aut}(V) = \mathcal{U}(V)$ and matrix unitary when $\mathrm{Aut}(V) = \mathrm{U}(n)$. The dimension of ϕ is denoted by

$$d_\phi = \dim \phi := \dim V.$$

A subspace $W \subseteq V$ is said to be ϕ-invariant if $\phi(x)W \subseteq W$, for all $x \in G$. When W is ϕ-invariant, we consider the restricted representation $\phi\big|_W \in \mathrm{Hom}(G, \mathrm{Aut}(W))$ defined by $\phi\big|_W(x)w := \phi(x)w$. In particular, if ϕ is unitary then its restriction is also unitary.

Let $\{V_j\}_{j\in J}$ be a family of mutually orthogonal subspaces of an inner product space V and write $W = \bigoplus_{j\in J} V_j$. If $\phi_j \in \mathrm{Hom}(G, \mathrm{Aut}(V_j))$ and $A_j \in \mathrm{End}(V_j)$ we define

$$\phi = \bigoplus_{j\in J} \phi_j \in \mathrm{Hom}(G, \mathrm{Aut}(W)) \text{ by } \phi\big|_{V_j} = \phi_j,\ j \in J; \text{ and}$$

$$A = \bigoplus_{j\in J} A_j \in \mathrm{End}(W) \text{ and } Av := A_j v, j \in J \text{ and } v \in V_j.$$

Definition 1.1. *Let $\mathcal{H}$ be a Hilbert space and $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ a unitary representation. We say that ϕ is a strongly continuous representation if the map $x \in G \mapsto \phi(x)v \in \mathcal{H}$ is continuous, for every $v \in \mathcal{H}$.*

A strongly continuous representation ϕ is called topologically irreducible if the only closed ϕ-invariant subspaces are the trivial ones ($\{0\}$ and $\mathcal{H}$).

Definition 1.2. *An intertwining operator between the representations $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut}(W))$, denoted $A \in \mathrm{Hom}(\phi, \psi)$, is a linear mapping $A : V \to W$ such that*

$$A\phi(x) = \psi(x)A, \quad \forall x \in G.$$

When the intertwining operator A is invertible, the representations ϕ and ψ are said to be equivalent, and we denote this by $\phi \sim \psi$.

If $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut}(W))$ are irreducible representations and $A \in \mathrm{Hom}(\phi, \psi)$, then it is possible to prove that either $A = 0$ or A is invertible. For equivalent irreducible unitary representations, the operator A is an isometric isomorphism.

When $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ is an irreducible and finite-dimensional representation, by Schur's Lemma, we have $\mathrm{Hom}(\phi, \phi) = \mathbb{C}I = \{\lambda I;\ \lambda \in \mathbb{C}\}$. In particular, if G is commutative, all irreducible finite-dimensional representations of G are one-dimensional.

1.1.2 The Peter-Weyl decomposition

We say that G is a compact group if G is compact as a topological space. In this case, there exists an unique measure μ_G, called Haar measure of G, that satisfies the following properties:

(i) $\int_G \mathbb{1}\, dx = \mu_G(G) = 1$;

(ii) $\int_G f(x)\, dx = \int_G f(yx)\, dx$, for all $y \in G$,

where we write

$$\int_G f(x)dx := \int_G f d\mu_G.$$

From these properties we obtain

(iii) $\int_G f(x)dx = \int_G f(xy)\, dx$, for all $y \in G$;

(iv) $\int_G f(x)dx = \int_G f(x^{-1})\, dx$.

We define the classical spaces $L^p(G)$ as being the set of all complex-valued functions for which the p-th power of their absolute value is integrable with respect to Haar measure μ_G.

The Haar measure of a product of compact groups is the product of the Haar measures of each one of the compact groups and we may write

$$\int_{G\times H} f\, d\mu_{G\times H} = \int_G \int_H f(x,y)\, dxdy.$$

When G is compact, strongly continuous unitary representations can be written as direct sum of finite-dimensional irreducible unitary representations. In particular, strongly continuous irreducible unitary representations of compact groups are finite-dimensional.

We will denote by $\mathrm{Rep}(G)$ the set of all continuous irreducible unitary representation of G.

Definition 1.3. *The unitary dual $\widehat{G}$ of a locally compact group G is the set consisting of all equivalence classes of strongly continuous irreducible unitary representations of G.*

When G is compact, we have

$$\widehat{G} = \{[\phi]; \phi \text{ is a continuous irreducible unitary representation of } G\}\,.$$

For each equivalence class $\xi \in \widehat{G}$, there exists a unitary matrix representation $\phi \in \xi = [\phi]$, that is, there is a homomorphism $\phi = (\phi_{ij})_{i,j=1}^m : G \longrightarrow U(m)$, where the functions $\phi_{ij} : G \longrightarrow \mathbb{C}$ are continuous.

Let $\phi = (\phi_{ij})_{i,j=1}^m$ and $\psi = (\psi_{ij})_{i,j=1}^m$ be irreducible matrix unitary representations such that $\phi \sim \psi$, then there exists a unitary matrix $A \in \mathbb{C}^{m\times m}$ such that

$$\phi(x)A = A\psi(x), \quad \forall x \in G.$$

Lemma 1.4. *Let G be a compact group. Let ϕ and ψ continuous irreducible matrix unitary representations. Then*

$$\langle \phi_{ij}, \psi_{k\ell}\rangle_{L^2(G)} = \begin{cases} 0, & \text{if } \phi \not\sim \psi, \\ \frac{1}{d_\phi}\delta_{ik}\delta_{j\ell}, & \text{if } \phi = \psi. \end{cases} \tag{1.1}$$

Let G be a compact group. We define its left and right regular representations π_L,π_R : $G \longrightarrow \mathcal{U}(L^2(G))$ by

$$(\pi_L(y)\ f)(x) := f(y^{-1}x),$$
$$(\pi_R(y)\ f)(x) := f(xy),$$

for almost every $x \in G$, with respect to μ_G.

Theorem 1.5 (Peter-Weyl). *Let G be a compact group. Then*

$$\mathcal{B} := \left\{ \sqrt{\dim \phi}\, \phi_{ij}\, ;\ \phi = (\phi_{ij})_{i,j=1}^{d_\phi}, [\phi] \in \widehat{G} \right\},$$

is an orthonormal basis for $L^2(G)$, where we pick only one matrix unitary representation in each class of equivalence.

Moreover, let $\phi = (\phi_{ij})_{i,j=1}^{d_\phi}, [\phi] \in \widehat{G}$, then

$$\mathcal{H}_{i,\cdot}^{\phi} := \operatorname{span}\{\phi_{ij};\ 1 \le j \le d_\phi\} \subseteq L^2(G)$$

is π_R-invariant and

$$\phi \sim \pi_R\big|_{\mathcal{H}_{i,\cdot}^{\phi}},$$

$$L^2(G) = \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{i=1}^{d_\phi} \mathcal{H}_{i,\cdot}^{\phi},$$

$$\pi_R \sim \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{i=1}^{d_\phi} \phi.$$

Fourier series on compact Lie groups

Definition 1.6. *Let G be a compact group, $f \in L^1(G)$, and $\phi = (\phi_{ij})_{i,j=1}^{d_\phi}$, $[\phi] \in \widehat{G}$. The ϕ-Fourier coefficient of f is*

$$\widehat{f}(\phi) := \int_G f(x)\phi(x)^*\, dx \in \mathbb{C}^{d_\phi \times d_\phi},$$

more precisely,

$$\widehat{f}(\phi)_{ij} = \int_G f(x)\overline{\phi(x)_{ji}}\, dx = \langle f, \phi_{ji}\rangle_{L^2(G)}.$$

Observe that when $\phi = (\phi_{ij})_{i,j=1}^{m}$ and $\psi = (\psi_{ij})_{i,j=1}^{m}$ are irreducible matrix unitary equivalent representations, there exists a unitary matrix $U \in \mathbb{C}^{m\times m}$ such that

$$\psi(x) = U^*\phi(x)U, \quad \forall x \in G. \tag{1.2}$$

So,

$$\widehat{f}(\psi) = \int_G f(x)\psi(x)^*\, dx = \int_G f(x)(U^*\phi(x)U)^*\, dx = \int_G f(x)U^*\phi(x)U\, dx = U^*\widehat{f}(\phi)U,$$

that is, $\widehat{f}(\phi)$ and $\widehat{f}(\psi)$ are similar matrices.

By the Peter-Weyl Theorem, a Fourier series presentation of $f \in L^2(G)$ is given by

$$\begin{aligned} f(x) &= \sum_{[\phi]\in\widehat{G}} \dim\phi \sum_{i,j=1}^{d_\phi} \langle f, \phi_{ij}\rangle_{L^2(G)} \phi(x)_{ij} \\ &= \sum_{[\phi]\in\widehat{G}} \dim\phi \sum_{i,j=1}^{d_\phi} \widehat{f}(\phi)_{ji}\phi(x)_{ij} \\ &= \sum_{[\phi]\in\widehat{G}} \dim\phi \,\mathrm{Tr}\left(\widehat{f}(\phi)\,\phi(x)\right), \end{aligned}$$

converging for almost every $x \in G$, with respect to μ_G, as well in $L^2(G)$, and the Plancherel identity takes the form

$$\|f\|^2_{L^2(G)} = \sum_{[\phi]\in\widehat{G}} \dim\phi \,\mathrm{Tr}\left(\widehat{f}(\phi)\,\widehat{f}(\phi)^*\right) = \sum_{[\phi]\in\widehat{G}} \dim\phi \,\|\widehat{f}(\phi)\|^2_{\mathtt{HS}}, \tag{1.3}$$

where $\|A\|_{\mathtt{HS}} := \sqrt{\mathrm{Tr}(A^*A)}$.

We point out that by (1.2) and properties of the trace of matrices, the equalities above are independent of the representative of the equivalence class.

1.1.3 Linear Lie groups and Lie algebras

A Lie group is a set endowed with compatible structures of group and C^∞-manifold, that is, the group operation and the inversion are C^∞-functions. A linear Lie group is a Lie group which is a closed subgroup of $\mathrm{GL}(d,\mathbb{C})$.

We will concentrate our study on linear Lie groups because the following characterization of compact Lie groups that can be found in [9] (Chapter III, Theorem 4.1):

Proposition 1.7. *Let G be a compact Lie group. Then there is some $m \in \mathbb{N}$ such that G is isomorphic to a subgroup of $\mathrm{U}(m)$.*

Throughout this work we set $\dim G = d$.

The fundamental tool for studying linear Lie groups is the matrix exponential map. We will endow $\mathbb{C}^{d\times d} \cong \mathcal{L}(\mathbb{C}^d)$ with the operator norm

$$Y \mapsto \|Y\|_{\mathcal{L}(\mathbb{C}^d)} := \sup_{\|x\|_{\mathbb{C}^d}\le 1} \|Yx\|_{\mathbb{C}^d}.$$

Definition 1.8. *Let $X \in \mathbb{C}^{d\times d}$. The exponential $\exp(X) \in \mathbb{C}^{d\times d}$ is defined by the power series*

$$\exp(X) := \sum_{k=0}^{\infty} \frac{1}{k!} X^k,$$

where $X^0 := I$.

Notice that this series converges in the Banach space $\mathbb{C}^{d\times d}$ because

$$\sum_{k=0}^{\infty}\frac{1}{k!}\|X^k\|_{\mathcal{L}(\mathbb{C}^d)}\leq\sum_{k=0}^{\infty}\frac{1}{k!}\|X\|^k_{\mathcal{L}(\mathbb{C}^d)}=e^{\|X\|_{\mathcal{L}(\mathbb{C}^d)}}<\infty.$$

Let $X,Y\in\mathbb{C}^{d\times d}$ and $P\in GL(n,\mathbb{C})$. Then

(i) If $XY=YX$ then

$$\exp(X+Y)=\exp(X)\exp(Y).$$

In particular, $\exp:\mathbb{C}^{d\times d}\longrightarrow GL(n,\mathbb{C})$ satisfies $\exp(-X)=\exp(X)^{-1}$;

(ii) $\exp(X^T)=\exp(X)^T$;

(iii) $\exp(X^*)=\exp(X)^*$;

(iv) $\exp(PXP^{-1})=P\exp(X)P^{-1}$.

We have

$$\mathrm{HOM}(\mathbb{R},\mathrm{GL}(n,\mathbb{C}))=\{t\mapsto\exp(tX);\ X\in\mathbb{C}^{d\times d}\},$$

where $\mathrm{HOM}(\mathbb{R},\mathrm{GL}(n,\mathbb{C}))$ denotes the set of all continuous homomorphism from $\mathbb{R}$ to the group $\mathrm{GL}(n,\mathbb{C})$.

Let $A\in\mathbb{C}^{d\times d}$ be a matrix such that $\|I-A\|_{\mathcal{L}(\mathbb{C}^d)}<1$. The logarithm

$$\log(A):=-\sum_{k=1}^{\infty}\frac{1}{k}(I-A)^k$$

is well defined and $\exp(\log(A))=A$. Moreover, there exists $r>0$ such that

$$\|X\|_{\mathcal{L}(\mathbb{C}^d)}<r\implies\log(\exp(X))=X.$$

Definition 1.9. *A $\mathbb{K}$–Lie algebra is a $\mathbb{K}$-vector space V endowed with a bilinear mapping $[\cdot,\cdot]$ satisfying*

1. $[a,a]=0,\quad\forall a\in V$;
2. *Jacobi identity:* $[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0,\quad\forall a,b,c\in V$.

A vector subspace $W\subseteq V$ of a Lie algebra V is called a Lie subalgebra if $[a,b]\in W$, for all $a,b\in W$.

A linear mapping $A:V_1\longrightarrow V_2$ between Lie algebras V_1,V_2 is called a Lie algebra homomorphism if $[Aa,Ab]_{V_2}=A[a,b]_{V_2}$, for all $a,b\in V_1$.

Let G be a closed subgroup of $GL(n, \mathbb{C})$. The $\mathbb{R}$-vector space

$$\mathfrak{Lie}(G) = \mathfrak{g} := \{X \in \mathbb{C}^{d\times d};\ \exp(tX) \in G,\ \forall t \in \mathbb{R}\}.$$

is a Lie subalgebra of the $\mathbb{R}$–Lie algebra $\mathfrak{Lie}_{\mathbb{R}}(\mathbb{C}^{d\times d}) \cong \mathfrak{gl}(\mathbb{C}^d)$, with respect to the operation $[X, Y] = XY - YX$, for all $X, Y \in \mathbb{C}^{d\times d}$.

Definition 1.10. *Let G be a linear Lie group and $\mathfrak{g} = \mathfrak{Lie}(G)$. The dimension of G is $\dim(G) := \dim(\mathfrak{g}) = k$, hence $\mathfrak{g} \cong \mathbb{R}^k$ as a vector space.*

The mapping $X \in \mathfrak{g} \mapsto \exp(X) \in G$ is a diffeomorphism in a small neighborhood of $0 \in \mathfrak{g}$. Moreover, if G is compact and connected then $\exp(\mathfrak{g}) = G$.

The Lie algebra $\mathfrak{g}$ can be identified with the tangent space of G at the identity $I \in G$. Using left-translations, $\mathfrak{g}$ can be identified with the set of left-invariant vector fields on G, and vector fields have a natural interpretation as first-order partial differential operators on G.

Definition 1.11. *For $x \in G$, $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, define*

$$L_X f(x) := \frac{d}{dt} f(x \exp(tX))\Big|_{t=0}.$$

Notice that the operator L_X is left-invariant. Indeed,

$$\begin{aligned}
\pi_L(y) L_X f(x) &= L_X f(y^{-1}x) \\
&= \frac{d}{dt} f(y^{-1}x \exp(tX))\Big|_{t=0} \\
&= \frac{d}{dt} \pi_L(y) f(x \exp(tX))\Big|_{y=0} \\
&= L_X \pi_L(y) f(x),
\end{aligned}$$

for all $x, y \in G$.

Where there is no possibility of ambiguous meaning, we will write only Xf instead of $L_X f$.

Definition 1.12. *Let G, H be linear Lie groups with respective Lie algebras $\mathfrak{g}, \mathfrak{h}$. The differential homomorphism of $\psi \in \mathrm{HOM}(G, H)$ is the mapping $\psi' = \mathfrak{Lie}(\psi) : \mathfrak{g} \to \mathfrak{h}$ defined by*

$$\psi'(X) := \frac{d}{dt} \psi(\exp(tX))\Big|_{t=0}$$

and satisfies

$$\psi(\exp(tX)) = \exp(t\psi'(X)),$$

that is, the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\psi} & H \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow[\psi']{} & \mathfrak{h} \end{array}$$

Moreover, ψ' is a Lie algebra homomorphism.

The adjoint representation of a linear Lie group G is the mapping $\mathrm{Ad} \in \mathrm{HOM}(G, \mathrm{Aut}(\mathfrak{g}))$ defined by

$$\mathrm{Ad}(A)X := AXA^{-1},$$

where $A \in G$ and $G \in \mathfrak{g}$.

The adjoint representation of the Lie algebra $\mathfrak{g}$ of a linear Lie group G is the differential representation

$$\mathrm{ad} = \mathrm{Ad}' : \mathfrak{g} \to \mathfrak{Lie}(Aut(\mathfrak{g})) \cong \mathfrak{gl}(\mathfrak{g}),$$

that is, $\mathrm{ad}(X) := \mathrm{Ad}'(X)$, so that

$$\mathrm{ad}(X)Y = \mathrm{Ad}'(X)Y = [X, Y].$$

Next we construct a natural associative algebra $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{g}$ modulo an ideal, enabling embedding $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$. Recall that $\mathfrak{g}$ can be interpreted as the vector space of first-order left-translation invariant partial differential operators on G. Consequently, $\mathcal{U}(\mathfrak{g})$ can be interpreted as the vector space of finite-order left-translation invariant partial differential operators on G.

Definition 1.13. *Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra. Let*

$$\mathcal{T} := \bigoplus_{m=0}^{\infty} \otimes^m \mathfrak{g}$$

be the tensor product algebra of $\mathfrak{g}$, where $\otimes^m \mathfrak{g}$ denotes the m-fold tensor product $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$; that is, $\mathcal{T}$ is the linear span of the elements of the form

$$\lambda_{00}\mathbf{1} + \sum_{m=1}^{M} \sum_{k=1}^{K_m} \lambda_{mk} X_{mk1} \otimes \cdots \otimes X_{mkm},$$

where $\mathbf{1}$ is the formal unit element of $\mathcal{T}$, $\lambda_{mk} \in \mathbb{K}$, $X_{mkj} \in \mathfrak{g}$ and $M, K_m \in \mathbb{Z}^+$; the product of $\mathcal{T}$ is begotten by the tensor product, that is,

$$(X_1 \otimes \cdots \otimes X_p)(Y_1 \otimes \cdots Y_q) := X_1 \otimes \cdots \otimes X_p \otimes Y_1 \otimes \cdots Y_q$$

is extended to a unique bilinear mapping $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$. Let $\mathcal{J}$ be the (two-sided) ideal in $\mathcal{T}$ spanned by the set

$$\mathcal{O} := \{X \otimes Y - Y \otimes X - [X,Y] : X, Y \in \mathfrak{g}\}.$$

The quotient algebra

$$\mathcal{U}(\mathfrak{g}) := \mathcal{T}/\mathcal{J}$$

is called the universal enveloping algebra of $\mathfrak{g}$.

Definition 1.14. *The Killing form of the Lie algebra $\mathfrak{g}$ is the bilinear mapping $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$, defined by*

$$B(X,Y) := \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)).$$

A ($\mathbb{R}$ or $\mathbb{C}$)-Lie algebra $\mathfrak{g}$ is called semisimple if its Killing form is non-degenerate, that is, if

$$\forall X \in \mathfrak{g}\backslash\{0\}\ \exists Y \in \mathfrak{g};\ B(X,Y) \neq 0;$$

equivalently, B is non-degenerate if the matrix $(B(X_i, X_j))_{i,j=1}^d$ is invertible, where $\{X_j\}_{j=1}^d \subset \mathfrak{g}$ is a vector space basis.

A connected linear Lie group is called semisimple if its Lie algebra is semisimple.

The Killing form of the Lie algebra of a compact linear Lie group G is negative semi-definite, i.e., $B(X,X) \leq 0$, for all $X \in \mathfrak{g}$. On the other hand, if the Killing form of a Lie group is negative definite, i.e., $B(X,X) < 0$ whenever $X \neq 0$, then the group is compact and semisimple. We point out that there are compact groups which their Killing form is not negative definite. For instance, the Killing form of the torus is identically zero, because of its commutativity.

Let $\mathfrak{g}$ be a semisimple $\mathbb{K}$-Lie algebra with a vector space basis $\{X_j\}_{j=1}^d \subset \mathfrak{g}$. Let B be the Killing form of $\mathfrak{g}$, and define the matrix $R \in \mathbb{K}^{d\times d}$ by $R_{ij} := B(X_i, X_j)$. Let

$$X^i := \sum_{j=1}^{d} (R^{-1})_{ij} X_j,$$

so that $\{X^i\}_{i=1}^d$ is another vector space basis for $\mathfrak{g}$. Then the Casimir element $\Omega \in \mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is defined by

$$\Omega := \sum_{i=1}^{d} X_i X^i.$$

Theorem 1.15. *The Casimir element of a finite-dimensional semisimple $\mathbb{K}$-Lie algebra $\mathfrak{g}$ is independent of the choice of the vector space basis $\{X_j\}_{j=1}^d \subset \mathfrak{g}$. Moreover,*

$$D\Omega = \Omega D,$$

for all $D \in \mathcal{U}(\mathfrak{g})$.

In the case where $\mathfrak{g}$ is semisimple, we can choose a convenient basis $\{X_j\}_{j=1}^d$ such that $B(X_i, X_j) = -\delta_{ij}$. In this case, $R = -I$ and the Casimir element is written as

$$\Omega = -\sum_{i=1}^{d} X_i^2.$$

The Casimir element of a linear semisimple Lie group is also denoted by

$$\mathcal{L}_G := \Omega \in \mathcal{U}(G), \tag{1.4}$$

and viewed as a second-order partial differential operator on G is also called the Laplace operator on G. The Laplace operator $\mathcal{L}_G$ is a negative definite bi-invariant operator on G. If G is equipped with the unique (up to a constant) bi-invariant Riemannian metric, $\mathcal{L}_G$ is its Laplace-Beltrami operator.

Remark 1.16. *In the case where $\mathfrak{g}$ is not semisimple we can construct the Laplace-Beltrami operator as follows. By Theorem 3.6.2 of [19], $\mathfrak{g}$ can be written as*

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z},$$

where $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$ on which the Killing form is negative definite, and $\mathfrak{z}$ is the kernel of the Killing form. Let $\langle \cdot, \cdot \rangle_{\mathfrak{g}'}$ be the inner product induced by the Killing form and let $\{Y_1, \ldots, Y_d\}$ be a orthonormal basis of $\mathfrak{g}'$. For $\mathfrak{z}$, choose any inner product Ad–invariant and consider $\{Z_1, \ldots Z_m\}$ an orthonormal basis of $\mathfrak{z}$. Observe that the sum of these inner products is an inner product Ad–invariant on $\mathfrak{g}$, denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, and we have that $\mathcal{B} = \{Y_1, \ldots, Y_d, Z_1, \ldots, Z_m\}$ is an orthonormal basis of $\mathfrak{g}$. One can shows that

$$\mathcal{L}_G = -\sum_{i=1}^{d} Y_i^2 - \sum_{j=1}^{m} Z_j^2,$$

is the Laplacian-Beltrami operator on G for the metric induced by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ (see [34]). Notice that

$$\mathcal{L}_G = \Omega - \sum_{j=1}^{m} Z_j^2,$$

where Ω is the Casimir element of $\mathfrak{g}$, which implies that $\mathcal{L}_G$ commutes with any element of $\mathfrak{g}$.

Let $\phi = (\phi_{ij})_{i,j=1}^{d_\phi} \in \widehat{G}$, $[\phi] \in \widehat{G}$ and define

$$\mathcal{H}^\phi = \text{span}\{\phi_{ij};\ 1 \leq i, j \leq d_\phi\}.$$

Theorem 1.17. *For every $[\phi] \in \widehat{G}$, the space $\mathcal{H}^\phi$ is an eigenspace of $\mathcal{L}_G$ and*

$$-\mathcal{L}_G \phi_{ij} = \nu_{[\phi]} \phi_{ij}, \quad 1 \leq i, j \leq d_\phi,$$

for some $\nu_{[\phi]} \geq 0$.

Notice that $\nu_{[\phi]}$ is independent of the choice of the representative of $[\phi]$, that is, if $\psi = (\psi_{k\ell})_{k\ell}^{d_\phi} \in [\phi]$, then

$$-\mathcal{L}_G \psi_{k\ell} = \nu_{[\phi]} \psi_{k\ell}, \quad 1 \leq k, \ell \leq d_\phi.$$

1.1.4 Function spaces

Let G be a compact Lie group of dimension d and $\{X_i\}_{i=1}^d$ a basis of its Lie algebra. For a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we define the left-invariant differential operator of order $|\alpha|$

$$\partial^\alpha := Y_1 \cdots Y_{|\alpha|},$$

with $Y_j \in \{X_i\}_{i=1}^d$, $1 \leq j \leq |\alpha|$ and $\sum_{j:Y_j = X_k} 1 = \alpha_k$ for every $1 \leq k \leq d$. It means that ∂^α is a composition of left-invariant derivatives with respect to vector $X_1, \ldots, X_d$ such that each X_k enters ∂^α exactly α_k times. We do not specify in the notation ∂^α the order of vectors $X_1, \ldots, X_d$, but this will not be relevant in the arguments that we will use in this work.

Proposition 1.18. *Let G be a compact Lie group of dimension d. The following statements are equivalent:*

(i) $f \in C^k(G)$;

(ii) $\partial^\alpha f \in C(G)$ *for all* $|\alpha| \leq k$;

(iii) $Lf \in C(G)$ *for all* $L \in \mathcal{U}(\mathfrak{g})$ *of degree less or equal* k.

Proposition 1.19. *Let G be a compact Lie group of dimension d. The following statements are equivalent:*

(i) $f \in C^\infty(G)$;

(ii) $\partial^\alpha f \in C(G)$ *for all* $\alpha \in \mathbb{N}_0^d$;

(iii) $(-\mathcal{L}_G)^k f \in C(G)$ *for all* $k \in \mathbb{N}_0$;

(iv) $Lf \in C(G)$ *for all* $L \in \mathcal{U}(\mathfrak{g})$.

We equipped $C^\infty(G)$ with the usual Fréchet space topology defined by seminorms $p_\alpha(f) = \max_{x\in G} |\partial^\alpha f(x)|$. Thus, the convergence on $C^\infty(G)$ is just the uniform convergence of functions and all their derivatives: $f_k \to f$ in $C^\infty(G)$ if $\partial^\alpha f_k(x) \to \partial^\alpha f(x)$, for all $x \in G$, due to the compactness of G.

For all $\phi \in \mathrm{Rep}(G)$, we have $\mathcal{H}^\phi \subset C^\infty(G)$. It follows from Theorem 1.17 that $\phi_{ij} \in C^\infty(G)$, for all $1 \le i, j \le d_\phi$.

Definition 1.20. *We define the space of distributions* $\mathcal{D}'(G)$ *as the space of all continuous linear functionals on* $C^\infty(G)$*, in which we consider the notion of usual convergence: for* $u_j, u \in \mathcal{D}'(G)$*, we write* $u_j \to u$ *in* $\mathcal{D}'(G)$ *as* $j \to \infty$ *if* $u_j(\varphi) \to u(\varphi)$ *in* $\mathbb{C}$ *as* $j \to \infty$*, for all* $\varphi \in C^\infty(G)$.

For $u \in \mathcal{D}'(G)$ and $\varphi \in C^\infty(G)$, we write

$$\langle u, \varphi \rangle_G := u(\varphi).$$

If $u \in L^p(G)$, $1 \le p \le \infty$, we can identify u with a distribution in $\mathcal{D}'(G)$ (which continues to be denoted by u) in a canonical way by

$$\langle u, \varphi \rangle_G := \int_G u(x)\varphi(x)\, dx.$$

In particular, if $u_j \to u$ in $L^p(G)$, then $u_j \to u$ in $\mathcal{D}'(G)$.

For $Y \in \mathfrak{g}$, we can differentiate $u \in \mathcal{D}'(G)$ with respect to the vector field Y:

$$\langle Yu, \varphi \rangle_G := -\langle u, Y\varphi \rangle_G,$$

for all $\varphi \in C^\infty(G)$. Similarly, for $\alpha \in \mathbb{N}_0$, we define

$$\langle \partial^\alpha u, \varphi \rangle_G := (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle_G,$$

for all $\varphi \in C^\infty(G)$.

Definition 1.21. *The space* $\mathcal{M}(\widehat{G})$ *consists of all mappings*

$$F : \widehat{G} \to \bigcup_{[\phi]\in\widehat{G}} \mathcal{L}(\mathcal{H}_\phi) \subset \bigcup_{m=1}^{\infty} \mathbb{C}^{m\times m}$$

satisfying $F([\phi]) \in \mathcal{L}(\mathcal{H}_\phi)$*, for every* $\phi \in \widehat{G}$*. With respect to the matrix representations, we have* $F([\phi]) \in \mathbb{C}^{d_\phi \times d_\phi}$*.*

The space $L^2(\widehat{G})$ *consists of all mappings* $F \in \mathcal{M}(\widehat{G})$ *such that*

$$\|F\|^2_{L^2(\widehat{G})} := \sum_{[\phi]\in\widehat{G}} \dim(\phi)\|F([\phi])\|^2_{\text{HS}} < \infty,$$

where

$$\|F([\phi])\|_{\text{HS}} = \sqrt{\text{Tr}(F([\phi])F([\phi])^*)}.$$

The space $L^2(\widehat{G})$ is a Hilbert space with the inner product

$$\langle E, F\rangle_{L^2(\widehat{G})} := \sum_{[\phi]\in\widehat{G}} \dim(\phi)\text{Tr}(E([\phi])F([\phi])^*).$$

From now on, for every $[\phi] \in \widehat{G}$, we choose a representative matrix $\phi = (\phi_{ij})_{i,j=1}^{d_\phi}$. Notice that for any $f \in L^2(G)$, we can define

$$\begin{array}{rccc} \widehat{f}: & \widehat{G} & \to & \bigcup\limits_{m=1}^{\infty} \mathbb{C}^{m\times m} \\ & [\phi] & \mapsto & \widehat{f}(\phi), \end{array}$$

and by the Plancherel formula on Proposition 1.3, we have $\widehat{f} \in L^2(\widehat{G})$. We have the Parseval's identity

$$\langle f, g\rangle_{L^2(G)} = \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \text{Tr}\left(\widehat{f}(\phi)\widehat{g}(\phi)^*\right) = \left\langle \widehat{f}, \widehat{g}\right\rangle_{L^2(\widehat{G})}.$$

Theorem 1.22. *Let* G *be a compact Lie group. The Fourier transform* $f \mapsto \mathcal{F}_G f = \widehat{f}$ *defines a surjective isometry* $L^2(G) \to L^2(\widehat{G})$*. The inverse Fourier transform is given by*

$$\begin{aligned} (\mathcal{F}_G^{-1}H)(x) &= \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \text{Tr}\left(H([\phi])\phi(x)\right) \\ &= \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{m,n=1}^{d_\phi} H([\phi])_{mn}\phi(x)_{nm} \end{aligned}$$

and we have

$$\mathcal{F}_G^{-1} \circ \mathcal{F}_G = \text{Id}_{L^2(G)} \quad \textit{and} \quad \mathcal{F}_G \circ \mathcal{F}_G^{-1} = \text{Id}_{L^2(\widehat{G})}.$$

Definition 1.23. *Let* G *be a compact Lie group,* $u \in \mathcal{D}'(G)$ *and* $\phi = (\phi_{ij})_{i,j=1}^{d_\phi}$*,* $[\phi] \in \widehat{G}$*. The* ϕ*-Fourier coefficient of* u *is*

$$\widehat{u}(\phi) := \langle u, \phi^*\rangle_G \in \mathbb{C}^{d_\phi \times d_\phi},$$

that is,

$$\widehat{u}(\phi)_{ij} = \left\langle u, \overline{\phi_{ji}}\right\rangle_G.$$

Notice that this definition agrees with Definition 1.6 when the distribution comes from an $L^1(G)$ function.

For $u \in \mathcal{D}'(G)$, we have

$$u = \sum_{[\phi]\in\widehat{G}} \dim(\phi)\mathrm{Tr}(\widehat{u}(\phi)\phi) = \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{i,j=1}^{d_\phi} \widehat{u}(\phi)_{ij}\phi_{ji},$$

where the convergence is in the distribution sense.

Let $\mathcal{L}_G$ be the Laplace-Beltrami operator of G. For each $[\phi] \in \widehat{G}$, its matrix elements are eigenfunctions of $\mathcal{L}_G$ correspondent to the same eigenvalue that we will denote by $-\nu_{[\phi]}$, where $\nu_{[\phi]} \geq 0$. Thus

$$-\mathcal{L}_G\phi_{ij}(x) = \nu_{[\phi]}\phi_{ij}(x), \quad \text{for all } 1 \leq i,j \leq d_\phi, \tag{1.5}$$

and we will denote by

$$\langle\phi\rangle := \left(1 + \nu_{[\phi]}\right)^{1/2}$$

the eigenvalues of $(I - \mathcal{L}_G)^{1/2}$.

Proposition 1.24. *Let G be a compact Lie group. There exists $C > 0$ such that*

$$\nu_{[\phi]} \leq \langle\phi\rangle^2 \leq C\nu_{[\phi]},$$

for all non-trivial $[\phi] \in \widehat{G}$.

Proposition 1.25. *There exists a constant $C > 0$ such that the inequality*

$$\dim(\phi) \leq C\langle\phi\rangle^{\frac{\dim G}{2}}$$

holds for all $\phi \in \mathrm{Rep}(G)$. Moreover, for every integer $M \geq \frac{\dim G}{2}$ there exists $C_M > 0$ such that

$$\|\phi_{ij}\|_{L^\infty(G)} \leq C_M\langle\phi\rangle^M, \tag{1.6}$$

for all $[\phi] \in \widehat{G}$, $1 \leq i,j \leq d_\phi$

Proposition 1.26. *Let G be a compact Lie group. Then*

$$\sum_{[\phi]\in\widehat{G}} d_\phi^2\langle\phi\rangle^{-2t} < \infty \quad \iff \quad t > \frac{n}{2}.$$

Theorem 1.27. *Let G be a compact Lie group. The following statements are equivalent:*

(i) $f \in C^\infty(G)$;

(ii) for each $N > 0$, there exists $C_N > 0$ such that

$$\|\widehat{f}(\phi)\|_{\mathtt{HS}} \leq C_N \langle \phi \rangle^{-N},$$

for all $[\phi] \in \widehat{G}$;

(iii) for each $N > 0$, there exists $C_N > 0$ such that

$$|\widehat{f}(\phi)_{ij}| \leq C_N \langle \phi \rangle^{-N},$$

for all $[\phi] \in \widehat{G}$, $1 \leq i, j \leq d_\phi$.

Theorem 1.28. *Let G be a compact Lie group. The following statements are equivalent:*

(i) $u \in \mathcal{D}'(G)$;

(ii) there exist C, $N > 0$ such that

$$\|\widehat{u}(\phi)\|_{\mathtt{HS}} \leq C \langle \phi \rangle^{N},$$

for all $[\phi] \in \widehat{G}$;

(iii) there exist C, $N > 0$ such that

$$|\widehat{u}(\phi)_{ij}| \leq C \langle \phi \rangle^{N},$$

for all $[\phi] \in \widehat{G}$, $1 \leq i, j \leq d_\phi$.

Definition 1.29. *Let G be a compact Lie group and $A : C^\infty(G) \to C^\infty(G)$ be a continuous linear operator. We define the symbol of the operator A in $x \in G$ and $\phi \in \mathrm{Rep}(G)$, $\phi = (\phi_{ij})_{i,j=1}^{d_\phi}$ as*

$$\sigma_A(x, \phi) := \phi(x)^* (A\phi)(x) \in \mathbb{C}^{d_\phi \times d_\phi},$$

where $(A\phi)(x)_{ij} := (A\phi_{ij})(x)$, for all $1 \leq i, j \leq d_\phi$.

For instance, if we take $A = -\mathcal{L}_G$, we get

$$\sigma_{\mathcal{L}_G}(x, \phi) = \phi(x)^* (-\mathcal{L}_G \phi)(x) = \phi(x)^* (\nu_{[\phi]} \phi)(x) = \nu_{[\phi]} \mathrm{Id}_{d_\phi}.$$

Theorem 1.30. *Let σ_A be the symbol of a continuous linear operator $A : C^\infty(G) \to C^\infty(G)$. Then*

$$Af(x) = \sum_{[\phi] \in \widehat{G}} \dim(\phi) \mathrm{Tr} \left(\phi(x)^* \sigma_A(x, \phi) \widehat{f}(\phi) \right)$$

for every $f \in C^\infty(G)$ and $x \in G$.

Notice that the formula above is independent of the choice of the representative. Indeed, if $\phi \sim \psi$ are matrix representations, there exists a unitary matrix U such that $\phi(x) = U^*\psi(x)U$ for all $x \in G$. By Remark 1.2 we have $\widehat{f}(\phi) = U^*\psi(x)U$ and by the formula of the symbol of the operator A,

$$\sigma_A(x,\phi) = \phi(x)^*(A\phi)(x) = (U^*\psi(x)^*U)(U^*A\psi U)(x)) = U^*\sigma_A(x,\psi)U.$$

Thus $\operatorname{Tr}\left(\phi(x)^*\sigma_A(x,\phi)\widehat{f}(\phi)\right) = \operatorname{Tr}\left(\psi(x)^*\sigma_A(x,\psi)\widehat{f}(\psi)\right)$, for all $x \in G$.

When $A : C^\infty(G) \to \mathbb{C}^\infty(G)$ is a continuous linear left-invariant operator, that is $A\pi_L(y) = \pi_L(y)A$, for all $y \in G$, we have that σ_A is independent of $x \in G$ and

$$\widehat{Af}(\phi) = \sigma_A(\phi)\widehat{f}(\phi),$$

for all $f \in C^\infty(G)$ and $[\phi] \in \widehat{G}$. By duality, this remains true for all $f \in \mathcal{D}'(G)$. For instance, by relation (1.5), we obtain

$$\widehat{\mathcal{L}_G f}(\phi) = -\nu_{[\phi]}\widehat{f}(\phi), \tag{1.7}$$

for all $f \in \mathcal{D}'(G)$ and $[\phi] \in \widehat{G}$.

Proposition 1.31. *Let $A, B : C^\infty(G) \to C^\infty(G)$ be continuous linear operators and $\lambda \in \mathbb{C}$. Then for all $x \in G$ and $[\phi] \in \widehat{G}$ holds:*

1. $\sigma_{A+B}(x,\phi) = \sigma_A(x,\phi) + \sigma_B(x,\phi)$;
2. $\sigma_{\lambda A}(x,\phi) = \lambda\sigma_A(x,\phi)$;
3. *If B is a left-invariant operator, then* $\sigma_{AB}(x,\phi) = \sigma_A(x,\phi)\sigma_B(\phi)$.

Let $Y \in \mathfrak{g}$. Notice that iY is a left-invariant operator and

$$\begin{aligned}
\langle iYf, g\rangle_{L^2(G)} &= \int_G (iYf)(x)\overline{g(x)}\,dx \\
&= i\int_G \frac{d}{dt} f(x\exp(tY))\Big|_{t=0}\overline{g(x)}\,dx \\
&= i\frac{d}{dt}\int_G f(x)\overline{g(x\exp(-tY))}\,dx\Big|_{t=0} \\
&= i\int_G f(x)\overline{(-Yg)(x)}\,dx \\
&= \langle f, iYg\rangle_{L^2(G)},
\end{aligned}$$

that is, the operator iY is symmetric on $L^2(G)$. Hence, for all $[\phi] \in \widehat{G}$ we can choose a representative ϕ such that $\sigma_{iY}(\phi)$ is a diagonal matrix, with entries $\lambda_m(\phi) \in \mathbb{R}$, $1 \leq m \leq$

d_ϕ, which follows because symmetric matrices can be diagonalized by unitary matrices. By Proposition 1.31,

$$\sigma_Y(\phi)_{mn} = i\lambda_m(\phi)\delta_{mn}, \quad \lambda_m \in \mathbb{R}. \tag{1.8}$$

Notice that $\{\lambda_m(\phi)\}_{m=1}^{d_\phi}$ are the eigenvalues of $\sigma_{iY}(\phi)$ and then are independent of the choice of the representative, since the symbol of equivalent representations are similar matrices. We can consider $\mathcal{B} = \{Y_1, \cdots, Y_n\}$ an orthonormal basis of $\mathfrak{g}$ with

$$Y_1 = \frac{Y}{||Y||},$$

where the inner product is took as in Remark 1.16. By the properties of Laplacian operator, we have that $[\mathcal{L}_G - Y_1^2, Y_1^2] = 0$, so we can diagonalize simultaneously $\sigma_{-Y_1^2}(\phi)$ and $\sigma_{-(\mathcal{L}_G - Y_1^2)}(\phi)$, for all $[\phi] \in \widehat{G}$. Notice that

$$\lambda_{[\phi]}\mathrm{Id}_{d_\phi} = \sigma_{-\mathcal{L}_G} = \sigma_{-(\mathcal{L}_G - Y_1^2)}(\phi) + \sigma_{-Y_1^2}(\phi),$$

where $\lambda_{[\phi]} \geq 0$. Since these two operators are positives and Y_1 is left-invariant, we obtain that

$$\lambda_{[\phi]} \geq -(\sigma_{Y_1}(\phi)_{mm})^2,$$

for all $1 \leq m \leq d_\phi$. By (1.8), we have

$$1 + \lambda_{[\phi]} \geq \lambda_{[\phi]} \geq \frac{\lambda_m(\phi)^2}{||Y||^2}.$$

Thus,

$$|\lambda_m(\phi)| \leq ||Y||\langle\phi\rangle, \tag{1.9}$$

for all $1 \leq m \leq d_\phi$. In order to simplify the notation, throughout the text we will assume that the vector fields are normalized.

Proposition 1.32. *Let G be a compact group, $[\phi] \in \widehat{G}$ and $\{Y_1, \cdots, Y_d\}$ be a basis for $\mathfrak{g}$. There exists $C_0 > 0$ such that*

$$\|\sigma_{\partial^\alpha}(\phi)\|_{op} \leq C_0^{|\alpha|}\langle\phi\rangle^{|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^d. \tag{1.10}$$

From Chapter 2 of [23], we have

$$\|\sigma_{\partial^\alpha}(\phi)\|_{op} \leq \|\sigma_{\partial^\alpha}(\phi)\|_{\mathtt{HS}} \leq \sqrt{d_\phi}\|\sigma_{\partial^\alpha}(\phi)\|_{op}. \tag{1.11}$$

1.2 Komatsu classes

In the previous section we have characterized smooth functions on G. The next natural class of functions to study is the class of analytic functions on G, i.e., the class $C^\omega(G)$ of smooth functions ϕ that satisfies the following property: for every $h > 0$, there exists $C_h > 0$ such that

$$\|\partial^\alpha \phi\|_{L^\infty} \leq C h^{|\alpha|} |\alpha|!, \quad \alpha \in \mathbb{N}_0^n.$$

Since $C^\omega(G) \subsetneq C^\infty(G)$, many authors consider intermediary classes of functions between $C^\omega(G)$ and $C^\infty(G)$ (see [1], [2], [3], [4], [8], [25]). An example of such class is the Gevrey class of Roumieu type $\gamma^s(G)$ of order s, with $s \geq 1$ described as follow: $\phi \in \gamma^s(G)$ if for every $h > 0$, there exists $C_h > 0$ such that

$$\|\partial^\alpha \phi\|_{L^\infty} \leq C h^{|\alpha|} |\alpha|!^s, \quad \alpha \in \mathbb{N}_0^n.$$

When $1 \leq s_1 < s_2$, we have $C^\omega(G) \subsetneq \gamma^{s_1}(G) \subsetneq \gamma^{s_2}(G) \subsetneq C^\infty(G)$. Notice that $\gamma^1(G) = C^\omega(G)$.

In [11], A. Dasgupta and M. Ruzhansky have characterized the Gevrey class of functions in terms of their Fourier coefficients.

In this dissertation we will use the characterization given by A. Dasgupta and M. Ruzhansky in [12] to extend our results to the framework of Komatsu classes, which are also classes of functions between $C^\omega(G)$ and $C^\infty(G)$. We point out that our examples will be given mainly in Gevrey classes, which are a particular example of Komatsu classes.

Let $\{M_k\}_{k \in \mathbb{N}_0}$ be a sequence of positive numbers such that there exist $H > 0$ and $A \geq 1$ satisfying

(M.0) $M_0 = 1$

(M.1) (stability) $M_{k+1} \leq A H^k M_k, \quad k = 0, 1, 2, \ldots$.

(M.2) $M_{2k} \leq A H^{2k} M_k^2, \quad k = 0, 1, 2, \ldots$.

(M.3) $\exists \ell, C > 0$ such that $k! \leq C \ell^k M_k$, for all $k \in \mathbb{N}_0$.

(M.4) $\dfrac{M_r}{r!} \dfrac{M_s}{s!} \leq \dfrac{M_{r+s}}{(r+s)!}, \quad \forall r, s \in \mathbb{N}_0.$

We will assume also the logarithmic convexity:

(LC) $M_k^2 \leq M_{k-1} M_{k+1}, \quad k = 1, 2, 3, \ldots$.

Given any sequence $\{M_k\}$ that satisfies (M.0)–(M.3), there exists an alternative sequence that satisfies the logarithmic convexity and defines the same classes that we will study. So assuming (LC) does not restrict the generality compared to (M.0)–(M.3).

From (M.0) and (LC) we have $M_k \leq M_{k+1}$, for all $k \in \mathbb{N}$, that is, $\{M_k\}$ is a non-decreasing sequence. Moreover, for $k \leq n$ we have

$$M_k \cdot M_{n-k} \leq M_n.$$

The condition (M.2) is equivalent to $M_k \leq AH^k \min_{0\leq q\leq k} M_q M_{k-q}$, (see [33], Lemma 5.3).

1.2.1 Associated function

Given a sequence $\{M_k\}$ we define the associated function as

$$M(r) := \sup_{k\in\mathbb{N}_0} \log \frac{r^k}{M_k}, \quad r > 0,$$

and $M(0) := 0$. Notice that M is a non-decreasing function.

Example 1.33. *Let $s \geq 1$ and consider $M_k = (k!)^s$. This sequence satisfies the conditions above and we have*

$$M(r) \simeq r^{1/s}.$$

In the next propositions we present some technical results of the associated function that we will use throughout this chapter.

Follow by the definition that for every $r > 0$ we have

$$\exp\{-M(r)\} = \inf_{k\in\mathbb{N}_0} \frac{M_k}{r^k} \tag{1.12}$$

$$\exp\{M(r)\} = \sup_{k\in\mathbb{N}_0} \frac{r^k}{M_k} \tag{1.13}$$

Proposition 1.34. *For every $r, s > 0$ we have*

(i) $\exp\{-M(r)\}\exp\{-M(s)\} \leq \exp\left\{-M\left(\frac{r+s}{2}\right)\right\}$

(ii) $\exp\{M(r)\}\exp\{M(s)\} \leq A\exp\{M(H(r+s))\}$

Proof. (i) Let $r, s > 0$. By (1.12) we obtain

$$\exp\{-M(r)\}\exp\{-M(s)\} \leq \frac{M_j}{r^j}\frac{M_\ell}{s^\ell} \leq \frac{M_{j+\ell}}{r^j s^\ell},$$

for all $j, \ell \in \mathbb{N}_0$. Let $k \in \mathbb{N}_0$. Thus for $\ell = k - j$ we have

$$\exp\{M(r)\}\exp\{M(s)\} \geq \frac{r^j s^{k-j}}{M_k},$$

so

$$2^k \exp\{M(r)\}\exp\{M(s)\} = \sum_{j=0}^{k} \binom{k}{j} \exp\{M(r)\}\exp\{M(s)\} \geq \sum_{j=0}^{k} \binom{k}{j} \frac{r^j s^{k-j}}{M_k} = \frac{(r+s)^k}{M_k},$$

that is,

$$\exp\{-M(r)\}\exp\{-M(s)\} \leq \frac{M_k}{\left(\frac{r+s}{2}\right)^k},$$

for all $k \in \mathbb{N}_0$. Therefore

$$\exp\{-M(r)\}\exp\{-M(s)\} \leq \exp\left\{-M\left(\frac{r+s}{2}\right)\right\}.$$

(ii) Let $r, s > 0$. We have $M_{k+\ell} \leq AH^{k+\ell} M_k M_\ell$ and $r^k s^\ell \leq (r+s)^{k+\ell}$, for all $k, \ell \in \mathbb{N}_0$. Thus

$$\log \frac{r^k}{M_k} + \log \frac{s^\ell}{M_\ell} = \log \frac{r^k s^\ell}{M_k M_\ell} \leq \log A \frac{H(r+s)^{k+\ell}}{M_{k+\ell}} = \log A + \log \frac{(H(r+s))^{k+\ell}}{M_{k+\ell}}$$
$$\leq \log A + M(H(r+s)).$$

For every $\ell \in \mathbb{N}_0$ fixed we have

$$\log \frac{r^k}{M_k} \leq \log A + M(H(r+s)) - \log \frac{s^\ell}{M_\ell} \implies M(r) \leq \log A + M(H(r+s)) - \log \frac{s^\ell}{M_\ell}.$$

Now,

$$\log \frac{s^\ell}{M_\ell} \leq \log A + M(H(r+s)) - M(r), \quad \forall \ell \in \mathbb{N}_0,$$

which implies that

$$M(s) \leq \log A + M(H(r+s)) - M(r).$$

By the properties of the exponential function we obtain

$$\exp\{M(r)\}\exp\{M(s)\} \leq A \exp\{M(H(r+s))\}.$$

□

Proposition 1.35. *For every $r, s > 0$ and $t \in \mathbb{N}_0$ we have*

(i) $r^t \exp\{-M(sr)\} \leq A\left(Hs^{-1}\right)^t M_t \exp\{-M(H^{-1}sr)\}$;

(ii) $r^t \exp\{M(sr)\} \leq As^{-t} M_t \exp\{M(Hsr)\}$.

Proof. (i) Let $r, s, t > 0$. We have

$$r^t \exp\{-M(sr)\} \leq r^t \frac{M_k}{s^k r^k} = s^{-t} \frac{M_k}{(sr)^{k-t}}, \quad \forall k \geq t.$$

Since $M_k \leq AH^k M_t M_{k-t}$, for all $k \geq t$, we obtain

$$r^t \exp\{-M(sr)\} \leq As^{-t} H^k M_t \frac{M_{k-t}}{(sr)^{k-t}} = A(s^{-1}H)^t M_t \frac{M_{k-t}}{(H^{-1}sr)^{k-t}}, \quad \forall k \geq t,$$

Therefore

$$r^t \exp\{-M(sr)\} \leq A\left(Hs^{-1}\right)^t M_t \exp\{-M(H^{-1}sr)\}.$$

(ii) Let $r, s, t > 0$. We have

$$r^t \exp\{M(sr)\} = r^t \sup_{k\in\mathbb{N}_0} \frac{(sr)^k}{M_k} = \sup_{k\in\mathbb{N}_0} \frac{s^k r^{k+t}}{M_k} = s^{-t} \sup_{k\in\mathbb{N}_0} \frac{(sr)^{k+t}}{M_k}$$

Since $M_{k+t} \leq AH^{k+t} M_k M_t$, we obtain

$$r^t \exp\{M(sr)\} \leq As^{-t} M_t \sup_{k\in\mathbb{N}_0} \frac{(Hsr)^{k+t}}{M_{k+t}} \leq As^{-t} M_t \sup_{\ell\in\mathbb{N}_0} \frac{(Hsr)^{\ell}}{M_{\ell}}.$$

Therefore

$$r^t \exp\{M(sr)\} \leq As^{-t} M_t \exp\{M(Hsr)\}.$$

□

Proposition 1.36. *Let G be a compact Lie group. For every $N, L, \delta > 0$ there exists $C > 0$ such that*

$$\langle\xi\rangle^N \exp\{-\delta M(L\langle\xi\rangle)\} \leq C,$$

for all $[\xi] \in \widehat{G}$.

Proof. Let $N, L, \delta > 0$. Then

$$\langle\xi\rangle^N \exp\{-\delta M(L\langle\xi\rangle)\} = \langle\xi\rangle^N (\exp\{-M(L\langle\xi\rangle)\})^{\delta} \leq \langle\xi\rangle^N \frac{M_k^{\delta}}{(L\langle\xi\rangle)^{k\delta}},$$

for all $k \in \mathbb{N}_0$. In particular, take $k_0 \in \mathbb{N}_0$ such that $k_0\delta > N$. So,

$$\langle\xi\rangle^N \exp\{-\delta M(L\langle\xi\rangle)\} \leq \langle\xi\rangle^{N-k_0\delta} \frac{M_{k_0}^{\delta}}{L^{k_0\delta}} \leq \frac{M_{k_0}^{\delta}}{L^{k_0\delta}} = C.$$

□

Proposition 1.37. *Let G be a compact Lie group and let $L > 0$. Then*

$$\exp\left\{-\tfrac{1}{2} M\left(L\langle\xi\rangle\right)\right\} \leq \sqrt{A} \exp\{-M\left(L_2\langle\xi\rangle\right)\}, \tag{1.14}$$

for all $[\xi] \in \widehat{G}$, where $L_2 = \dfrac{L}{H}$.

Proof. Notice that

$$\exp\{-\tfrac{1}{2}M(L\langle\xi\rangle)\} = \inf_{k\in\mathbb{N}_0}\frac{M_k^{1/2}}{(L\langle\xi\rangle)^{k/2}} \leq \inf_{\ell\in\mathbb{N}_0}\frac{M_{2\ell}^{1/2}}{(L\langle\xi\rangle)^{\ell}}.$$

□

By the property (M.2), we have

$$M_{2\ell} \leq AH^{2\ell}M_\ell^2.$$

This implies

$$\exp\{-\tfrac{1}{2}M(L\langle\xi\rangle)\} \leq \inf_{\ell\in\mathbb{N}_0}\sqrt{A}\frac{H^\ell M_\ell}{(L\langle\xi\rangle)^\ell} = \sqrt{A}\inf_{\ell\in\mathbb{N}_0}\frac{M_\ell}{\left(\frac{L\langle\xi\rangle}{H}\right)^\ell} = \sqrt{A}\exp\{-M(L_2\langle\xi\rangle)\},$$

where $L_2 = \frac{L}{H}$.

1.2.2 Komatsu class of Roumieu type

Definition 1.38. *The Komatsu class of Roumieu type $\Gamma_{\{M_k\}}(G)$ is the space of all complex-valued C^∞ functions f on G such that there exist $h > 0$ and $C > 0$ satisfying*

$$\|\partial^\alpha f\|_{L^2(G)} \leq Ch^{|\alpha|}M_{|\alpha|}, \quad \forall\alpha \in \mathbb{N}_0^d.$$

In the definition above, we could take the L^∞-norm and obtain the same space. The elements of $\Gamma_{\{M_k\}}(G)$ are often called ultradifferentiable functions. Notice that by (M.3) we have that $\Gamma_{\{M_k\}}(G)$ contains the analytic functions on G.

Example 1.39. *Let G be a compact Lie group and $\phi = (\phi_{ij})_{i,j=1}^{d_\phi}$, $[\phi] \in \widehat{G}$. Let us show that $\phi_{ij} \in \Gamma_{\{M_k\}}(G)$ for all sequences $\{M_k\}_{k\in\mathbb{N}_0}$ satisfying the conditions (M.0)–(M.3).*

Let $\beta \in \mathbb{N}_0^d$, so

$$\begin{aligned}
|\partial^\beta\phi_{ij}(x)| &= \left|\sum_{\ell=1}^{d_\phi}\xi_{i\ell}(x)\sigma_{\partial^\beta}(\phi)_{\ell j}\right| \\
&\leq \sum_{\ell=1}^{d_\phi}|\xi_{i\ell}(x)||\sigma_{\partial^\beta}(\phi)_{\ell j}| \\
&\leq C\langle\phi\rangle^M\|\sigma_{\partial^\beta}(\phi)\|_{op} \\
&\overset{(1.10)}{\leq} C\langle\phi\rangle^M(C_0\langle\phi\rangle)^{|\beta|},
\end{aligned}$$

where $M \geq \frac{\dim G}{2}$. *Take* $h = C_0\langle\phi\rangle$ *and by the fact that* $M_k \geq 1$, *for all* $k \in \mathbb{N}_0$, *we conclude that*

$$\|\partial^\beta \phi_{ij}\|_{L^2(G)} \leq Ch^{|\beta|} M_{|\beta|}$$

and then $\phi_{ij} \in \Gamma_{\{M_k\}}(G)$.

Theorem 1.40. *Assume conditions (M.0)–(M.3). The following statements about a function* $f \in C^\infty(G)$ *are equivalent:*

(i) $f \in \Gamma_{\{M_k\}}(G)$;

(ii) There exist constants $C > 0$, $L > 0$ *such that*

$$\|\widehat{f}(\phi)\|_{\mathtt{HS}} \leq C \exp\{-M(L\langle\phi\rangle))\}, \quad \forall [\phi] \in \widehat{G};$$

(iii) There exist constants $C > 0$, $L > 0$ *such that*

$$|\widehat{f}(\phi)_{ij}| \leq C \exp\{-M(L\langle\phi\rangle))\}, \quad \forall [\phi] \in \widehat{G},\ 1 \leq i, j \leq d_\phi.$$

Proof. The equivalence $(i) \Leftrightarrow (ii)$ can be found on [12], Theorem 2.4, page 8487. Let us prove that $(ii) \Leftrightarrow (iii)$. The first implication is trivial because

$$|\widehat{f}(\phi)_{ij}| \leq \|\widehat{f}(\phi)\|_{\mathtt{HS}},$$

for all $[\phi] \in \widehat{G}$, $1 \leq i, j \leq d_\phi$. Conversely, we have

$$\begin{aligned}
\|\widehat{f}(\phi)\|_{\mathtt{HS}}^2 &= \sum_{i,j=1}^{d_\phi} |\widehat{f}(\phi)_{ij}|^2 \\
&\leq \sum_{i,j=1}^{d_\phi} C^2 \exp\{-2M(L\langle\phi\rangle)\} \\
&= d_\phi^2 C^2 \exp\{-2M(L\langle\phi\rangle)\},
\end{aligned}$$

that is,

$$\|\widehat{f}(\phi)\|_{\mathtt{HS}} \leq C d_\phi \exp\{-M(L\langle\phi\rangle)\}.$$

By Proposition 1.25, we have

$$\|\widehat{f}(\phi)\|_{\mathtt{HS}} \leq C\langle\phi\rangle^N \exp\{-M(L\langle\phi\rangle)\},$$

where $N \geq \frac{\dim G}{2}$. Using now Proposition 1.36 and 1.37 we obtain

$$\begin{aligned}\|\widehat{f}(\phi)\|_{\mathtt{HS}} &\leq C\langle\phi\rangle^N \exp\{-M(L\langle\phi\rangle)\} \\ &= C\left(\langle\phi\rangle^N \exp\left\{-\tfrac{1}{2}M(L\langle\phi\rangle)\right\}\right)\exp\left\{-\tfrac{1}{2}M(L\langle\phi\rangle)\right\} \\ &\leq C\exp\{-M(L_2\langle\phi\rangle)\}\end{aligned}$$

□

We also can characterise the elements of the dual $\Gamma'_{\{M_k\}}(G)$ by its Fourier coefficients.

Theorem 1.41. *Let $\Gamma'_{\{M_k\}}(G)$ the dual space of $\Gamma_{\{M_k\}}(G)$. The following statements about a linear functional defined on $\Gamma_{\{M_k\}}(G)$ are equivalent:*

(i) $u \in \Gamma'_{\{M_k\}}(G)$;

(ii) For every $B > 0$ there exists $K_B > 0$ such that

$$\|\widehat{u}(\phi)\|_{\mathtt{HS}} \leq K_B \exp\{M(B(\langle\phi\rangle))\}, \quad \forall[\phi] \in \widehat{G};$$

(iii) For every $B > 0$ there exists $K_B > 0$ such that

$$|\widehat{u}(\phi)_{ij}| \leq K_B \exp\{M(B\langle\phi\rangle))\}, \quad \forall[\phi] \in \widehat{G},\ 1 \leq i,j \leq d_\phi.$$

Proof. The equivalence $(i) \Leftrightarrow (ii)$ can be found on [12], Theorem 2.4, page 8488. Let us prove that $(ii) \Leftrightarrow (iii)$. The first implication is trivial because

$$|\widehat{u}(\phi)_{ij}| \leq \|\widehat{u}(\phi)\|_{\mathtt{HS}},$$

for all $[\phi] \in \widehat{G}$, $1 \leq i,j \leq d_\phi$. On the other hand,

$$\begin{aligned}\|\widehat{u}(\phi)\|^2_{\mathtt{HS}} &= \sum_{i,j=1}^{d_\phi} |\widehat{u}(\phi)_{ij}|^2 \\ &\leq \sum_{i,j=1}^{d_\phi} K_B^2 \exp\{2M(B\langle\phi\rangle)\} \\ &= d_\phi^2 K_B^2 \exp\{2M(B\langle\phi\rangle)\},\end{aligned}$$

that is, for all $B > 0$, there exists $K_B > 0$ such that

$$\|\widehat{u}(\phi)\|_{\mathtt{HS}} \leq d_\phi K_B \exp\{M(B\langle\phi\rangle)\}.$$

By Proposition 1.25, we have $d_\phi \leq C\langle\phi\rangle^M$ for some $M > \frac{\dim G}{2}$ and by Proposition 1.36, for all $B > 0$ there exists C_B such that

$$\langle\phi\rangle^M \leq C_B \exp\{M(B\langle\phi\rangle)\},$$

for all $[\phi] \in \widehat{G}$. By Proposition 1.37,

$$\|\widehat{u}(\phi)\|_{\mathtt{HS}} \leq K_B \exp\{2M(B\langle\phi\rangle)\} \leq K_B \exp\{M(\tilde{B}\langle\phi\rangle)\},$$

where $\tilde{B} = BH$. Therefore $u \in \Gamma'_{\{M_k\}}(G)$. □

1.2.3 Komatsu class of Beurling type

Next, to define Komatsu classes of Beurling type, we have to change the condition (M.3) by the following one:

(M.3') $\forall \ell > 0$, $\exists C_\ell$ such that $k! \leq C_\ell \ell^k M_k$, for all $k \in \mathbb{N}_0$.

Notice that the condition (M.3') implies the condition (M.3).

Definition 1.42. *The Komatsu class of Beurling type $\Gamma_{(M_k)}(G)$ is the space of C^∞ functions f on G such that for every $h > 0$ there exists $C_h > 0$ such that we have*

$$\|\partial^\alpha f\|_{L^2(G)} \leq C_h h^{|\alpha|} M_{|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^d.$$

Theorem 1.43. *Assume conditions (M.0)–(M.3'). The following statements about a function $f \in C^\infty(G)$ are equivalent:*

(i) $\phi \in \Gamma_{(M_k)}(G)$;

(ii) For every $L > 0$ there exists $C_L > 0$ such that

$$\|\widehat{f}(\phi)\|_{\mathtt{HS}} \leq C_L \exp\{-M(L(\langle\phi\rangle))\}, \quad \forall [\phi] \in \widehat{G};$$

(iii) For every $L > 0$ there exists $C_L > 0$ such that

$$|\widehat{f}(\phi)_{ij}| \leq C_L \exp\{-M(L\langle\phi\rangle))\}, \quad \forall [\phi] \in \widehat{G},\ 1 \leq i, j \leq d_\phi.$$

Example 1.44. *Let G be a compact group and $\phi = (\phi_{ij})_{i,j=1}^{d_\phi}$, $[\phi] \in \widehat{G}$. We have $\phi_{ij} \in \Gamma_{(M_k)}(G)$ for all sequences $\{M_k\}_{k\in\mathbb{N}_0}$ satisfying the conditions (M.0)–(M.3') because*

$$\widehat{\phi_{ij}}(\xi)_{mn} = \langle \phi_{ij}, \xi_{nm} \rangle_{L^2(G)} = \begin{cases} 0, & \text{if } \xi \neq \phi, \\ \frac{1}{d_\phi}\delta_{in}\delta_{jm}, & \text{if } \xi = \phi, \end{cases}$$

and then ϕ_{ij} satisfies the statement (iii) of the previous theorem.

Theorem 1.45. *Let $\Gamma'_{(M_k)}(G)$ the dual space of $\Gamma_{(M_k)}(G)$. The following statements about a linear functional defined on $\Gamma_{(M_k)}(G)$ are equivalent:*

(i) $u \in \Gamma'_{(M_k)}(G)$;

(ii) There exist $K,\ B > 0$ *such that*

$$\|\widehat{u}(\phi)\|_{\mathtt{HS}} \leq K \exp\{M(B(\langle\phi\rangle))\}, \quad \forall [\phi] \in \widehat{G};$$

(iii) There exist $K,\ B > 0$ *such that*

$$|\widehat{u}(\phi)_{ij}| \leq K \exp\{M(B\langle\phi\rangle))\}, \quad \forall [\phi] \in \widehat{G},\ 1 \leq i, j \leq d_\phi.$$

Part I

Smooth case

Chapter 2

Constant coefficient vector fields

Let G_1 and G_2 be compact Lie groups, $G := G_1 \times G_2$, and consider the linear operator $L : C^\infty(G) \to C^\infty(G)$ defined by

$$L := X_1 + cX_2,$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$ and $c \in \mathbb{C}$. Thus, for each $u \in C^\infty(G)$ we have

$$\begin{aligned} Lu(x_1, x_2) &:= X_1 u(x_1, x_2) + cX_2 u(x_1, x_2) \\ &:= \frac{d}{dt} u(x_1 \exp(tX_1), x_2)\Big|_{t=0} + c\frac{d}{ds} u(x_1, x_2 \exp(sX_2))\Big|_{s=0}. \end{aligned}$$

The operator L extends to distributions in a natural way, that is, if $u \in \mathcal{D}'(G)$, then

$$\langle Lu, \varphi \rangle_G := -\langle u, L\varphi \rangle_G, \quad \varphi \in C^\infty(G).$$

In this chapter, we present necessary and sufficient conditions for the vector field L to be globally hypoelliptic and to be globally solvable. After that, we present examples recovering known results in the torus and presenting examples in $\mathbb{T}^1 \times \mathbb{S}^3$ and $\mathbb{S}^3 \times \mathbb{S}^3$. Because of the presented examples and by the validity of Greenfield-Wallach conjecture on compact Lie groups, we investigate the global properties of perturbations of L by zero-order terms and we also present weaker notions of global hypoellipticity.

The main tool that we will use in the development of our results is the partial Fourier series with respect to each one of the Lie groups. The details and main results about partial Fourier series on compact Lie groups can be found on Appendix A.

2.1 Global hypoellipticity

Definition 2.1. *Let G be a compact Lie group. We say that an operator $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is globally hypoelliptic if the conditions $u \in \mathcal{D}'(G)$ and $Pu \in C^\infty(G)$ imply that $u \in C^\infty(G)$.*

Consider the equation

$$Lu(x_1, x_2) = X_1 u(x_1, x_2) + cX_2 u(x_1, x_2) = f(x_1, x_2),$$

where $f \in C^\infty(G)$. For each $[\xi] \in \widehat{G_1}$, we can choose a representative $\xi \in \mathrm{Rep}(G_1)$ such that

$$\sigma_{X_1}(\xi)_{mn} = i\lambda_m(\xi)\delta_{mn}, \quad 1 \leq m, n \leq d_\xi,$$

where $\lambda_m(\xi) \in \mathbb{R}$ for all $[\xi] \in \widehat{G_1}$ and $1 \leq m \leq d_\xi$ (see Proposition 1.31). Similarly, for each $[\eta] \in \widehat{G_2}$, we can choose a representative $\eta \in \mathrm{Rep}(G_2)$ such that

$$\sigma_{X_2}(\eta)_{rs} = i\mu_r(\eta)\delta_{rs}, \quad 1 \leq r, s \leq d_\eta,$$

where $\mu_r(\eta) \in \mathbb{R}$ for all $[\eta] \in \widehat{G_2}$ and $1 \leq r \leq d_\eta$.

Suppose that $u \in C^\infty(G)$. Thus, taking the partial Fourier coefficient with respect to the first variable at $x_2 \in G_2$ (see Definitions A.1 and A.2) we obtain

$$\begin{aligned}
\widehat{f}(\xi, x_2) &= \widehat{Lu}(\xi, x_2) \\
&= \int_{G_1} Lu(x_1, x_2)\xi(x_1)^* \, dx_1 \\
&= \int_{G_1} X_1 u(x_1, x_2)\xi(x_1)^* \, dx_1 + c\int_{G_1} X_2 u(x_1, x_2)\xi(x_1)^* \, dx_1 \\
&= \widehat{X_1 u}(\xi, x_2) + cX_2 \int_{G_1} u(x_1, x_2)\xi(x_1)^* \, dx_1 \\
&= \sigma_{X_1}(\xi)\widehat{u}(\xi, x_2) + cX_2\widehat{u}(\xi, x_2).
\end{aligned}$$

Hence, for each $x_2 \in G_2$, we have that $\widehat{f}(\xi, x_2) \in \mathbb{C}^{d_\xi \times d_\xi}$ and

$$\widehat{f}(\xi, x_2)_{mn} = i\lambda_m(\xi)\widehat{u}(\xi, x_2)_{mn} + cX_2\widehat{u}(\xi, x_2)_{mn}, \quad 1 \leq m, n \leq d_\xi.$$

Now, taking the Fourier coefficient of $\widehat{f}(\xi, \cdot)_{mn}$ with respect to the second variable, we obtain

$$\begin{aligned}
\widehat{\widehat{f}}(\xi, \eta)_{mn} &= \int_{G_2} \widehat{f}(\xi, x_2)_{mn}\eta(x_2)^* \, dx_2 \\
&= \int_{G_2} (i\lambda_m(\xi)\widehat{u}(\xi, x_2)_{mn} + cX_2\widehat{u}(\xi, x_2)_{mn})\eta(x_2)^* \, dx_2 \\
&= i\lambda_m(\xi)\int_{G_2} \widehat{u}(\xi, x_2)_{mn}\eta(x_2)^* \, dx_2 + c\int_{G_2} X_2\widehat{u}(\xi, x_2)_{mn}\eta(x_2)^* \, dx_2 \\
&= i\lambda_m(\xi)\,\widehat{\widehat{u}}(\xi, \eta)_{mn} + c\sigma_{X_2}(\eta)\,\widehat{\widehat{u}}(\xi, \eta)_{mn}.
\end{aligned}$$

Thus, $\widehat{\widehat{f}}(\xi,\eta)_{mn} \in \mathbb{C}^{d_\eta \times d_\eta}$ and

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = i(\lambda_m(\xi) + c\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}, \quad 1 \leq r,s \leq d_\eta. \tag{2.1}$$

From this we can conclude that

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = 0, \text{ whenever } \lambda_m(\xi) + c\mu_r(\eta) = 0. \tag{2.2}$$

Moreover, if $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$, then

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \frac{1}{i(\lambda_m(\xi) + c\mu_r(\eta))}\,\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}. \tag{2.3}$$

We begin by presenting the following necessary condition for global hypoellipticity of the vector field $L = X_1 + cX_2$.

Proposition 2.2. *Suppose that the set*

$$\mathcal{N} = \{([\xi],[\eta]) \in \widehat{G_1} \times \widehat{G_2};\ \lambda_m(\xi) + c\mu_r(\eta) = 0, \textit{ for some } 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta\} \tag{2.4}$$

has infinitely many elements. Then there exists $u \in \mathcal{D}'(G) \setminus C^\infty(G)$ such that

$$Lu = 0.$$

In particular, L is not globally hypoelliptic.

Proof. Consider the sequence

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 1, & \text{if } \lambda_m(\xi) + c\mu_r(\eta) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that for any $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m,n \leq d_\xi$ and $1 \leq r,s \leq d_\eta$ we have

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq \langle\xi\rangle + \langle\eta\rangle.$$

Thus by the characterization of distributions by Fourier coefficients (Theorem A.3) we conclude that $u \in \mathcal{D}'(G)$, where

$$u = \sum_{[\xi]\in\widehat{G_1}} \sum_{[\eta]\in\widehat{G_2}} d_\xi d_\eta \sum_{m,n=1}^{d_\xi} \sum_{r,s=1}^{d_\eta} \widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} \xi_{nm} \eta_{sr}.$$

Since there exist infinitely many representations such that $\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = 1$, it follows from Theorem A.3 that $u \notin C^\infty(G)$. Furthermore, we have

$$\widehat{\widehat{Lu}}(\xi,\eta)_{mn_{rs}} = i(\lambda_m(\xi) + c\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = 0,$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}, 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta$. Then, by Plancherel formula (1.3), we conclude that $Lu = 0$. □

Theorem 2.3. *The operator $L = X_1 + cX_2$ is globally hypoelliptic if and only if the following conditions are satisfied:*

1. *The set*

$$\mathcal{N} = \{([\xi],[\eta]) \in \widehat{G_1} \times \widehat{G_2};\ \lambda_m(\xi) + c\mu_r(\eta) = 0, \textit{ for some } 1 \le m \le d_\xi, 1 \le r \le d_\eta\}$$

   is finite.

2. *$\exists C,\ M > 0$ such that*

$$|\lambda_m(\xi) + c\mu_r(\eta)| \ge C(\langle\xi\rangle + \langle\eta\rangle)^{-M}, \tag{2.5}$$

   for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \le m \le d_\xi,\ 1 \le r \le d_\eta$, whenever $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$.

Proof. ($\Longleftarrow$) Suppose that $Lu = f \in C^\infty(G)$ for some $u \in \mathcal{D}'(G)$. Let us prove that $u \in C^\infty(G)$. Since the set $\mathcal{N}$ is finite, there exists $C > 0$ such that

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \le C,$$

for all $([\xi],[\eta]) \in \mathcal{N},\ 1 \le m,n \le d_\xi,\ 1 \le r,s \le d_\eta$. Let $N \in \mathbb{N}$. Then, for $([\xi],[\eta]) \in \mathcal{N}$, we have

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &\le C(\langle\xi\rangle + \langle\eta\rangle)^N(\langle\xi\rangle + \langle\eta\rangle)^{-N} \\ &\le C'_N(\langle\xi\rangle + \langle\eta\rangle)^{-N} \end{aligned}$$

where $C'_N = \max\limits_{([\xi],[\eta])\in\mathcal{N}} \left\{C(\langle\xi\rangle + \langle\eta\rangle)^N\right\}$. On the other hand, if $([\xi],[\eta]) \notin \mathcal{N}$, by (2.3) and (2.5) we obtain

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &= \frac{1}{|\lambda_m(\xi) + c\mu_r(\eta)|}|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\le C^{-1}(\langle\xi\rangle + \langle\eta\rangle)^M|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \end{aligned}$$

Since $f \in C^\infty(G)$, there exists $C_{N+M} > 0$ such that

$$|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \le C_{N+M}(\langle\xi\rangle + \langle\eta\rangle)^{-(N+M)}$$

Thus,

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &\le C^{-1}C_{N+M}(\langle\xi\rangle + \langle\eta\rangle)^M(\langle\xi\rangle + \langle\eta\rangle)^{-(N+M)} \\ &= C''_N(\langle\xi\rangle + \langle\eta\rangle)^{-N}, \end{aligned}$$

where $C''_N = C^{-1}C_{N+M}$. Hence, if $([\xi],[\eta]) \notin \mathcal{N}$ we conclude that

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C''_N(\langle\xi\rangle + \langle\eta\rangle)^{-N}.$$

Setting $C_N := \max\{C'_N, C''_N\}$, we have

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$. Therefore by Theorem A.3 we conclude that $u \in C^\infty(G)$.

($\Longrightarrow$) Let us prove the result by contradiction. If Condition 1 were not satisfied, by Proposition 2.2, there would be $u \in \mathcal{D}'(G)\backslash C^\infty(G)$ such that $Lu = 0$, contradicting the hypothesis of global hypoellipticity of L. So, let us assume that Condition 2 is not satisfied, then for every $M \in \mathbb{N}$, we choose $[\xi_M] \in \widehat{G_1}$ and $[\eta_M] \in \widehat{G_2}$ such that

$$0 < |\lambda_m(\xi_M) + c\mu_r(\eta_M)| \leq (\langle\xi_M\rangle + \langle\eta_M\rangle)^{-M}, \tag{2.6}$$

for some $1 \leq m \leq d_{\xi_M}$ and $1 \leq r \leq d_{\eta_M}$.

Let $\mathcal{A} = \{([\xi_j],[\eta_j])\}_{j\in\mathbb{N}}$. It is easy to see that $\mathcal{A}$ has infinitely many elements. Define

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 1, & \text{if } ([\xi],[\eta]) = ([\xi_j],[\eta_j]) \text{ for some } j \in \mathbb{N} \text{ and (2.6) is satisfied,} \\ 0, & \text{otherwise.} \end{cases}$$

In this way, $u \in \mathcal{D}'(G)\backslash C^\infty(G)$. Let us show that $Lu = f \in C^\infty(G)$.

If $([\xi],[\eta]) \notin \mathcal{A}$, then $|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| = 0$. Moreover, for every $M \in \mathbb{N}$, we have

$$\begin{aligned} |\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}| &= |\lambda_m(\xi_M) + c\mu_r(\eta_M)||\widehat{\widehat{u}}(\xi_M,\eta_M)_{mn_{rs}}| \\ &\leq (\langle\xi_M\rangle + \langle\eta_M\rangle)^{-M} \end{aligned}$$

for every element of $\mathcal{A}$.

Fix $N > 0$. If $M > N$, then

$$|\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}| \leq (\langle\xi_M\rangle + \langle\eta_M\rangle)^{-M} \leq (\langle\xi_M\rangle + \langle\eta_M\rangle)^{-N}.$$

For $M \leq N$ we have

$$\begin{aligned} \left|\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}\right| &= \left|\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}\right|(\langle\xi_M\rangle + \langle\eta_M\rangle)^N(\langle\xi_M\rangle + \langle\eta_M\rangle)^{-N} \\ &\leq C'_N(\langle\xi_M\rangle + \langle\eta_M\rangle)^{-N}. \end{aligned}$$

where $C'_N := \max\limits_{\substack{M\leq N \\ m,n,r,s}} \left\{|\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}|(\langle\xi_M\rangle + \langle\eta_M\rangle)^N\right\}$. For $C_N = \max\{C'_N, 1\}$ we obtain

$$|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq C_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m,n \leq d_\xi$, $1 \leq r,s \leq d_\eta$. Therefore $f \in C^\infty(G)$, which contradicts the assumption that L is globally hypoelliptic. □

2.2 Global solvability

In the literature there are several notions for the solvability of an operator, mainly depending on the functional environment in which one is working and what one intends to study. So the first step here is to define precisely what we mean by the global solvability.

Given a function (or distribution) f defined on G, assume that $u \in \mathcal{D}'(G)$ is a solution of $Lu = f$. By taking the partial Fourier coefficient with respect to x_1 and x_2 separately, and following the same procedure of the last subsection, we obtain from (2.2) that

$$\lambda_m(\xi) + c\mu_r(\eta) = 0 \Longrightarrow \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = 0.$$

Therefore, let us consider the following set

$$\mathcal{K} := \{w \in \mathcal{D}'(G);\ \widehat{\widehat{w}}(\xi,\eta)_{mn_{rs}} = 0,\ \text{whenever } \lambda_m(\xi) + c\mu_r(\eta) = 0\}.$$

If $f \notin \mathcal{K}$, then there is no $u \in \mathcal{D}'(G)$ such that $Lu = f$. We call the elements of $\mathcal{K}$ of admissible functions (distributions) for the solvability of L.

Definition 2.4. *We say that the operator L is globally solvable if $L(\mathcal{D}'(G)) = \mathcal{K}$.*

Theorem 2.5. *The operator $L = X_1 + cX_2$ is globally solvable if and only if there exist $C,\ M > 0$ such that*

$$|\lambda_m(\xi) + c\mu_r(\eta)| \geq C(\langle\xi\rangle + \langle\eta\rangle)^{-M}, \tag{2.7}$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m \leq d_\xi,\ 1 \leq r \leq d_\eta$ whenever $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$.

Proof. ($\Longleftarrow$) For each $f \in \mathcal{K}$ define

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 0, & \text{if } \lambda_m(\xi) + c\mu_r(\eta) = 0, \\ -i(\lambda_m(\xi) + c\mu_r(\eta))^{-1}\,\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}, & \text{otherwise.} \end{cases} \tag{2.8}$$

Let us show that $\{\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}\}$ is the sequence of Fourier coefficient of an element $u \in \mathcal{D}'(G)$. Since $f \in \mathcal{D}'(G)$, there exists $N \in \mathbb{N}$ and $C > 0$ such that

$$|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq C(\langle\xi\rangle + \langle\eta\rangle)^N,$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m \leq d_\xi,\ 1 \leq r \leq d_\eta$. So

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &= |\lambda_m(\xi) + c\mu_r(\eta)|^{-1}|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\leq C(\langle\xi\rangle + \langle\eta\rangle)^M|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\leq C(\langle\xi\rangle + \langle\eta\rangle)^{N+M} \end{aligned} \tag{2.9}$$

Therefore $u \in \mathcal{D}'(G)$ and $Lu = f$.

($\Longrightarrow$) Let us proceed by contradiction by constructing an element $f \in \mathcal{K}$ such that there is no $u \in \mathcal{D}'(G)$ satisfying $Lu = f$.

If (2.7) is not satisfied, for each $M \in \mathbb{N}$, there exists $[\xi_M] \in \widehat{G_1}$ and $[\eta_M] \in \widehat{G_2}$ such that

$$0 < |\lambda_{\tilde{m}}(\xi_M) + c\mu_{\tilde{r}}(\eta_M)| < (\langle \xi_M \rangle + \langle \eta_M \rangle)^{-M}, \tag{2.10}$$

for some $1 \leq \tilde{m} \leq d_{\xi_M}$ and $1 \leq \tilde{r} \leq d_{\eta_M}$. We can suppose that $\langle \xi_M \rangle + \langle \eta_M \rangle \leq \langle \xi_N \rangle + \langle \eta_N \rangle$ when $M \leq N$. Let $\mathcal{A} = \{([\xi_j], [\eta_j])\}_{j \in \mathbb{N}}$. Consider $f \in \mathcal{K}$ defined by

$$\widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}} = \begin{cases} 1, & \text{if } ([\xi], [\eta]) = ([\xi_j], [\eta_j]) \text{ for some } j \in \mathbb{N} \text{ and (2.10) is satisfied,} \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that there exits $u \in \mathcal{D}'(G)$ such that $Lu = f$. In this way, its Fourier coefficients must satisfy

$$i(\lambda_m(\xi) + c\mu_r(\eta))\, \widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}} = \widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}}.$$

So

$$\begin{aligned} |\widehat{\widehat{u}}(\xi_M, \eta_M)_{\tilde{m}1_{\tilde{r}1}}| &= |\lambda_{\tilde{m}}(\xi_M) + c\mu_{\tilde{r}}(\eta_M)|^{-1} |\widehat{\widehat{f}}(\xi_M, \eta_M)_{\tilde{m}1_{\tilde{r}1}}| \\ &> (\langle \xi_M \rangle + \langle \eta_M \rangle)^M, \end{aligned}$$

where $\tilde{m}$ and $\tilde{r}$ are coefficients that satisfy (2.10). Thus

$$\| \widehat{\widehat{u}}(\xi_M, \eta_M) \|_{\mathtt{HS}} > (\langle \xi_M \rangle + \langle \eta_M \rangle)^M,$$

for all $M > 0$, which contradicts the fact that $u \in \mathcal{D}'(G)$. Therefore there does not exist $u \in \mathcal{D}'(G)$ such that $Lu = f$. □

Notice that the estimate for the global solvability in the statement of the last theorem is exactly the same as one of the conditions to obtain global hypoellipticity announced in (2.5), thus we have the following corollary.

Corollary 2.6. *If L is globally hypoelliptic, then L is globally solvable.*

A more detailed analysis of the last proof shows that it is possible to obtain a better control on the Fourier coefficients of u when f is smooth, more precisely, we have the following result.

Proposition 2.7. *If L is globally solvable and $f \in \mathcal{K} \cap C^\infty(G)$, then there exists $u \in C^\infty(G)$ such that $Lu = f$.*

Proof. Let $f \in \mathcal{K} \cap C^\infty(G)$ and define u as in (2.8). Since L is globally solvable, it holds (2.7) and then by (2.9)

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &= |\lambda_m(\xi) + c\mu_r(\eta)|^{-1} |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\leq C(\langle\xi\rangle + \langle\eta\rangle)^M |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}|. \end{aligned}$$

In view of the smoothness of f, for every $N > 0$ there exists $C_N > 0$ such that

$$|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq C_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m \leq d_\xi$, $1 \leq r \leq d_\eta$. Hence

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C(\langle\xi\rangle + \langle\eta\rangle)^M |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq C_{N+M}(\langle\xi\rangle + \langle\eta\rangle)^{-N}.$$

Therefore $u \in C^\infty(G)$ and $Lu = f$. □

2.3 Examples

In this section we recover some classical examples of S. Greenfield and N. Wallach (see [26]) on the global hypoellipticity and global solvability in tori ($\mathbb{T}^2$ and $\mathbb{T}^d$) and present a class of examples in $\mathbb{T}^1 \times \mathbb{S}^3$ and in $\mathbb{S}^3 \times \mathbb{S}^3$.

Example 2.8. $G = \mathbb{T}^2$

Set $G_1 = G_2 = \mathbb{T}^1$, where $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Since $\mathbb{T}^1$ is abelian, the irreducible unitary representations of $\mathbb{T}^1$ are unidimensional. Moreover the dual $\widehat{\mathbb{T}^1}$ can be identified to $\mathbb{Z}$. For each $k \in \mathbb{Z}$, the function $e_k : \mathbb{T}^1 \to \mathcal{U}(\mathbb{C})$ defined by

$$e_k(t) := e^{itk}$$

is an element of $\widehat{\mathbb{T}^1}$ and

$$\widehat{\mathbb{T}^1} = \{e_k\}_{k\in\mathbb{Z}}.$$

The Haar measure on $\mathbb{T}^1$ is the normalized Lebesgue measure and

$$\langle k\rangle := \langle e_k\rangle = \sqrt{1+k^2}.$$

Let $c \in \mathbb{C}$ and consider the operator

$$L = \partial_t + c\partial_x, \quad (t,x) \in \mathbb{T}^1 \times \mathbb{T}^1.$$

Notice that

$$\sigma_{\partial_t}(e_k) = e_k(t)^*(\partial_t e_k)(t) = e^{-itk}(ike^{itk}) = ik,$$

that is, $\lambda(e_k) = k$, for all $k \in \mathbb{Z}$. Thus, if $Lu = f$, then

$$\widehat{\widehat{f}}(k,\ell) = i(k + c\ell)\ \widehat{\widehat{u}}(k,\ell).$$

In this case,

$$\mathcal{N} = \{(k,\ell) \in \mathbb{Z}^2;\ k + c\ell = 0\}.$$

By Theorem 2.3, L is globally hypoelliptic if and only if $\mathcal{N}$ is finite and there exist $C, M > 0$ such that

$$|k + c\ell| \geq C(\langle k\rangle + \langle \ell\rangle)^{-M}$$

for all $(k,\ell) \in \mathbb{Z}^2$, whenever $k + a\ell \neq 0$. For $(k,\ell) \neq (0,0)$, we have

$$|k| + |\ell| \leq \langle k\rangle + \langle \ell\rangle \leq 3(|k| + |\ell|),$$

then the second condition of the Theorem 2.3 becomes

$$|k + c\ell| \geq C(|k| + |\ell|)^{-M} \tag{2.11}$$

for all $(k,\ell) \in \mathbb{Z}^2$, whenever $k + c\ell \neq 0$.

Notice that $\mathcal{N}$ is an infinity set if and only if $c \in \mathbb{Q}$. Moreover, if $c \notin \mathbb{Q}$, then $\mathcal{N} = \{(0,0)\}$. Suppose that $\Im(c) \neq 0$. If $\ell \neq 0$, then

$$|k + a\ell| \geq |\Im(c)||\ell| \geq |\Im(c)|(|k| + |\ell|)^{-1}.$$

If $\ell = 0$, we have $k \neq 0$ and

$$|k + a\ell| = |k| \geq (|k| + |\ell|)^{-1}.$$

Take $C = \max\{1, |\Im(c)|\}$. Then

$$|k + c\ell| \geq C(|k| + |\ell|)^{-1},$$

for all $(k,\ell) \in \mathbb{Z}^2\ \{(0,0)\}$. Therefore, if $\Im(c) \neq 0$ then L is globally hypoelliptic.

Suppose now that $\Im(c) = 0$. We recall that an irrational number c is called a Liouville number if it can be approximated by rational numbers to any order. That is, for every positive integer N there is $K > 0$ and infinitely many integer pairs (k,ℓ) so that

$$\left|c - \frac{k}{\ell}\right| < \frac{K}{\ell^N}.$$

Notice that the inequality (2.11) is satisfied if and only if c is an irrational non-Liouville number.

We conclude that $L = \partial_t + c\partial_x$ is globally hypoelliptic if and only if $\Im(c) \neq 0$ or c is an irrational non-Liouville number.

For solvability we need to analyze the condition 2 of the Theorem 2.3 when $c \in \mathbb{Q}$. Suppose that $c = \frac{p}{q}$, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We have

$$|k + c\ell| = \left|k + \frac{p}{q}\ell\right| = \frac{1}{q}|qk + p\ell| \geq \frac{1}{q} \geq \frac{1}{q}(|k| + |\ell|)^{-1},$$

for all $(k, \ell) \in \mathbb{Z}^2$, whenever $qk + p\ell \neq 0$.

Therefore, $L = \partial_t + c\partial_x$ is globally solvable if and only if $\Im(c) \neq 0$, or $c \in \mathbb{Q}$, or c is an irrational non-Liouville number.

Example 2.9. $G = \mathbb{T}^d$

From the above example we can extend the analysis for operators defined on $\mathbb{T}^d$. Let

$$L = \sum_{j=1}^{d} c_j \partial_{t_j}, \quad c_j \in \mathbb{C}$$

If $Lu = f$, then

$$\widehat{f}(k_1, \cdots, k_d) = i\left(\sum_{j=1}^{d} c_j k_j\right) \widehat{u}(k_1, \cdots, k_d).$$

The set $\mathcal{N}$ is

$$\mathcal{N} = \left\{k \in \mathbb{Z}^d; \sum_{j=1}^{d} c_j k_j = 0\right\},$$

and by Theorem 2.3, L is globally hypoelliptic if and only if $\mathcal{N}$ is finite and there exists $C, M > 0$ such that

$$\left|\sum_{j=1}^{d} c_j k_j\right| \geq C\left(\sum_{j=1}^{d} |k_j|\right)^{-M},$$

for all $k \in \mathbb{Z}^d$ whenever $\sum_{j=1}^{d} c_j k_j \neq 0$.

For instance, if some $c_j = 0$, then the set $\mathcal{N}$ is infinity, which implies that L is not globally hypoelliptic. It is easy to see that if all $c_j \in \mathbb{Q}$, them L is globally solvable, even if some of $c_j = 0$.

If $c_j = 1$ for $j = 1, \cdots, d-1$ and $\Im(c_d) \neq 0$, than L is globally hypoelliptic. The same is true if we consider c_d being an irrational non-Liouville number.

Example 2.10. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Let $\widehat{\mathbb{S}^3}$ be the unitary dual of $\mathbb{S}^3$, that is, $\widehat{\mathbb{S}^3}$ consists of equivalence classes $[t^\ell]$ of continuous irreducible unitary representations $t^\ell : \mathbb{S}^3 \to \mathbb{C}^{(2\ell+1)\times(2\ell+1)}$, $\ell \in \frac{1}{2}\mathbb{N}_0$, of matrix-valued functions satisfying $t^\ell(xy) = t^\ell(x)t^\ell(y)$ and $t^\ell(x)^* = t^\ell(x)^{-1}$ for all $x, y \in \mathbb{S}^3$. We will use the standard convention of enumerating the matrix elements t^ℓ_{mn} of t^ℓ using indices m, n ranging between $-\ell$ to ℓ with step one, i.e. we have $-\ell \leq m, n \leq \ell$ with $\ell - m, \ell - n \in \mathbb{N}_0$. For $\ell \in \frac{1}{2}\mathbb{N}_0$ we have

$$\langle \ell \rangle := \langle t^\ell \rangle = \sqrt{1 + \ell(\ell+1)}.$$

The details about the Fourier analysis on $\mathbb{S}^3$ can be found in Chapter 11 of [35].

Let X be a smooth vector field on $\mathbb{S}^3$ and $c \in \mathbb{C}$. Consider the following operator defined on $\mathbb{T}^1 \times \mathbb{S}^3$:

$$L = \partial_t + cX.$$

Using rotation on $\mathbb{S}^3$, without loss of generality, we may assume that the vector field X has the symbol

$$\sigma_X(\ell)_{mn} = im\delta_{mn}, \quad \ell \in \tfrac{1}{2}\mathbb{N}_0,\ -\ell \leq m, n \leq \ell,\ \ell - m, \ell - n \in \mathbb{N}_0$$

with δ_{mn} standing for the Kronecker's delta (see [35], [37], and [38]). Hence, if $Lu = f$, then

$$\widehat{\widehat{f}}(k,\ell)_{mn} = i(k + cm)\,\widehat{\widehat{u}}(k,\ell)_{mn},$$

where $k \in \mathbb{Z}$, $\ell \in \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m, n \leq \ell$ and $\ell - m, \ell - n \in \mathbb{N}_0$. In this case,

$$\mathcal{N} = \{(k,\ell) \in \mathbb{Z} \times \tfrac{1}{2}\mathbb{N}_0;\ k + cm = 0, \text{ for some } -\ell \leq m \leq \ell, \ell - m \in \mathbb{N}_0\}.$$

By Theorem 2.3, L is globally hypoelliptic if and only if $\mathcal{N}$ is finite and there exist $C, M > 0$ such that

$$|k + cm| \geq C(\langle k \rangle + \langle \ell \rangle)^{-M} \tag{2.12}$$

for all $(k,\ell) \in \mathbb{Z} \times \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m \leq \ell$, $\ell - m \in \mathbb{N}_0$ whenever $k + cm \neq 0$. For $\ell \in \frac{1}{2}\mathbb{N}_0$, we have

$$\frac{1}{\sqrt{2}}(1 + \ell) \leq \langle t^\ell \rangle \leq 1 + \ell$$

and we can write (2.12) as

$$|k + cm| \geq C(|k| + 1 + \ell)^{-M}$$

for all $(k,\ell) \in \mathbb{Z} \times \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m \leq \ell$, $\ell - m \in \mathbb{N}_0$ whenever $k + cm \neq 0$.

Notice that $(0,\ell) \in \mathcal{N}$, for all $\ell \in \mathbb{N}_0$, so $\mathcal{N}$ has infinitely many elements and then L is not globally hypoelliptic for any $c \in \mathbb{C}$.

The analysis of the global solvability of L is similar to the $\mathbb{T}^2$ case and we have L globally solvable if and only if $\Im(c) \neq 0$, or $c \in \mathbb{Q}$, or c is an irrational non-Liouville number. For instance, the operator

$$L = \partial_t + \alpha X,$$

where α is the continued fraction $\alpha = \left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$, is not globally solvable because α is an irrational Liouville number (see page 162 of [29]).

Example 2.11. $G = \mathbb{S}^3 \times \mathbb{S}^3$

Consider the operator

$$L = X_1 + cX_2,$$

where $X_1, X_2 \in \mathfrak{s}^3$ and $c \in \mathbb{C}$. Here, we assume that the vector field X_1 acts only in the first variable, while X_2 acts only in the second variable. Following the ideas of Example 2.10, we may assume that

$$\sigma_{X_1}(\ell)_{mn} = im\delta_{mn}, \quad \ell \in \tfrac{1}{2}\mathbb{N}_0,\ -\ell \leq m, n \leq \ell,\ \ell - m, \ell - n \in \mathbb{N}_0,$$

and

$$\sigma_{X_2}(\kappa)_{rs} = ir\delta_{rs}, \quad \kappa \in \tfrac{1}{2}\mathbb{N}_0,\ -\kappa \leq r, s \leq \kappa,\ \kappa - r, \kappa - s \in \mathbb{N}_0.$$

Hence, if $Lu = f$, we have

$$\widehat{\widehat{f}}(\kappa, \ell)_{mn_{rs}} = i(r + cm)\, \widehat{\widehat{u}}(\kappa, \ell)_{mn_{rs}},$$

where $\kappa, \ell \in \frac{1}{2}\mathbb{N}_0$, $-\kappa \leq r, s \leq \kappa$, $-\ell \leq m, n \leq \ell$, and , $\kappa - r, \kappa - s, \ell - m, \ell - n \in \mathbb{N}_0$. It is easy to see that if $(\kappa, \ell) \in \mathbb{N} \times \mathbb{N}$, then $(\kappa, \ell) \in \mathcal{N}$. So the operator L is not globally hypoelliptic. As in Example 2.10, we conclude that L is globally solvable if and only if $\Im(c) \neq 0$, or $c \in \mathbb{Q}$, or c is an irrational non-Liouville number. For instance, similarly to the previous example, we notice that the operator

$$L = X_1 + \alpha X_2$$

is not globally solvable, because $\alpha = \left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$ is an irrational Liouville number.

2.4 Weaker notions of hypoellipticity

All the known examples of globally hypoelliptic vector fields are set on tori. Actually, in 1973, S. Greenfield and N. Wallach proposed the following conjecture.

Conjecture 2.12 (Greenfield-Wallach). *If a closed, connected, orientable manifold M admits a globally hypoelliptic vector field X, then M is diffeomorphic to a torus and X is smoothly conjugate to a constant Diophantine vector field.*

In [22], G. Forni showed the equivalence between this conjecture and Katok's conjecture, about the existence of C^∞–cohomology free smooth vector fields on closed, connected, orientable smooth manifolds. From this equivalence we will show that on compact connected Lie groups the set $\mathcal{N}$ defined in (2.4) contains only the trivial representation. First, let us define what is a C^∞–cohomology free vector field.

Definition 2.13. *Let M be a closed, connected, orientable smooth manifold. A smooth vector field X on M is C^∞–cohomology free if for all $f \in C^\infty(M)$ there exists a constant $c(f) \in \mathbb{C}$ and $u \in C^\infty(M)$ such that*

$$Xu = f - c(f).$$

Theorem 2.14. *[G. Forni [22]] Let X be a smooth vector field on a closed connected manifold M. Then X is C^∞–cohomology free if and only if X is globally hypoelliptic.*

Proposition 2.15. *If G is a compact connected Lie group and L is globally hypoelliptic, then $\mathcal{N}$ has only one element.*

Proof. Notice that for the trivial representations $\mathbb{1}_{G_1}$ and $\mathbb{1}_{G_2}$ we have $\lambda_1(\mathbb{1}_{G_1}) = \mu_1(\mathbb{1}_{G_2}) = 0$, so $\mathcal{N} \neq \varnothing$. Suppose that there exists a non-trivial representation such that

$$\lambda_m(\xi) + c\mu_r(\eta) = 0.$$

for some $1 \leq m \leq d_\xi$, $1 \leq r \leq d_\eta$. Let $f = \xi_{1m} \times \eta_{1r} \in C^\infty(G)$, so

$$\begin{aligned}
\widehat{\widehat{f}}(\xi,\eta)_{m1_{r1}} &= \int_{G_1}\int_{G_2} f(x_1,x_2)\overline{\xi(x_1)_{1m}}\,\overline{\eta(x_2)_{1r}}dx_2dx_1 \\
&= \int_{G_1}\int_{G_2} \xi(x_1)_{1m}\eta(x_2)_{1r}\overline{\xi(x_1)_{1m}}\,\overline{\eta(x_2)_{1r}}dx_2dx_1 \\
&= \int_{G_1} |\xi(x_1)_{1m}|^2\,dx_1 \int_{G_2} |\eta(x_2)_{1r}|^2\,dx_2 \\
&= (d_\xi d_\eta)^{-1}
\end{aligned}$$

Since L is globally hypoelliptic, by Theorem 2.14 L is C^∞–cohomology free, then there exists $u \in C^\infty(G)$ such that

$$Lu = f - f_0,$$

where $f_0 = \int_G f\, d\mu_G$. We have

$$\widehat{\widehat{Lu}}(\xi,\eta)_{m1_{r1}} = i(\lambda_m(\xi)) + c\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{m1_{r1}} = 0,$$

which implies that

$$\widehat{\widehat{f - f_0}}(\xi,\eta)_{m1_{r1}} = 0.$$

Since $\xi \otimes \eta$ is not the trivial representation, by (1.1) we have $\widehat{\widehat{f}}_0(\xi,\eta)_{m1_{r1}} = 0$, so

$$\widehat{\widehat{f}}(\xi,\eta)_{m1_{r1}} = 0,$$

what is a contradiction because $\widehat{\widehat{f}}(\xi,\eta)_{m1_{r1}} = (d_\xi d_\eta)^{-1}$. Therefore $\mathcal{N}$ contains only the trivial representation. □

In view of Example 2.10 and Proposition 2.15, the following question naturally arises:

Question 2.1. *Does there exist a compact Lie group $G \neq \mathbb{T}^d$ such that there exists $X \in \mathfrak{g}$ satisfying $\sigma_X(\phi)$ singular for only finitely many $[\phi] \in \widehat{G}$, that is, the set*

$$\mathcal{Z} = \{[\phi] \in \widehat{G};\ \lambda_m(\phi) = 0, \textit{ for some } 1 \leq m \leq d_\phi\}$$

is finite, where $\sigma_X(\phi)_{mn} = i\lambda_m(\phi)\delta_{mn}$?

S. Greenfield and N. Wallach have proved this conjecture for compact Lie groups in [27]. The conjecture it was also proved for compact manifolds of dimensions 2 and 3, and in some very particular cases, which are described by G. Forni in [22] and by L. Flaminio, G. Forni, and F. Rodriguez Hertz in [21]. The answer to the above question is a way to obtain an alternative proof for the Greenfield-Wallach conjecture on compact Lie groups.

In view of the validity of the Greenfield-Wallach conjecture on compact Lie groups, the study of the global hypoellipticity of vector fields defined on closed manifolds is restricted to tori. However, the study of the regularity of solutions of such vector fields is yet an interesting subject. For this reason, in this section we will make some considerations looking to weaken the usual concept of the global hypoellipticity and introduce what we will call global hypoellipticity modulo kernel and global $\mathcal{W}$-hypoellipticity.

2.4.1 Global hypoellipticity modulo kernel

First, assuming that the set $\mathcal{N}$ has infinitely many elements, we will show that to reduce the range of the operator does not help us to obtain a weaker version of global hypoellipticity.

Proposition 2.16. *Suppose that $\mathcal{N}$ has infinitely many elements. Then there is no subset $\mathcal{A} \subseteq C^\infty(G)$ that satisfies the condition: $u \in \mathcal{D}'(G)$ and $Lu \in \mathcal{A}$ imply that $u \in C^\infty(G)$.*

Proof. Assume that there exists a subset $\mathcal{A} \subseteq C^\infty(G)$ that satisfies the property above. Let $u \in \mathcal{D}'(G)$ such that $Lu \in \mathcal{A}$, then $u \in C^\infty(G)$. By Proposition 2.2 there exists an element $v \in \ker L$ such that $v \in \mathcal{D}'(G) \backslash C^\infty(G)$. Since $v \in \ker L$, we have $L(u+v) = Lu \in \mathcal{A}$, which implies that $u + v \in C^\infty(G)$. Therefore $v = (u+v) - u \in C^\infty(G)$, a contradiction. □

In view of Proposition 2.16 we give the following definition:

Definition 2.17. *We say that an operator $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is globally hypoelliptic modulo* $\ker P$ *if the conditions $u \in \mathcal{D}'(G)$ and $Pu \in C^\infty(G)$ imply that there exists $v \in C^\infty(G)$ such that $u - v \in \ker P$.*

Clearly, global hypoellipticity implies global hypoellipticity modulo kernel. Our main result here is the equivalence of the concepts of global hypoellipticity modulo kernel and global solvability for constant coefficient vector fields.

Proposition 2.18. *The operator $L = X_1 + cX_2$ is globally hypoelliptic modulo* $\ker L$ *if and only if L is globally solvable.*

Proof. ($\Longrightarrow$) Suppose that L is not globally solvable. Then by Theorem 2.5, for every $M \in \mathbb{N}$, choose $[\xi_M] \in \widehat{G_1}$ and $[\eta_M] \in \widehat{G_2}$ such that

$$0 < |\lambda_m(\xi_M) + c\mu_r(\eta_M)| \leq (\langle \xi_M \rangle + \langle \eta_M \rangle)^{-M},$$

for some $1 \leq m \leq d_{\xi_M}$ and $1 \leq r \leq d_{\eta_M}$. Using the same construction of the proof of Theorem 2.3, we find a $u \in \mathcal{D}'(G) \setminus C^\infty(G)$ such that $Lu = f \in C^\infty(G)$. Notice that if $u - v \in \ker L$, for some $v \in C^\infty(G)$, then

$$i(\lambda_m(\xi) + c\mu_r(\eta))\widehat{\widehat{u-v}}(\xi,\eta)_{mn_{rs}} = 0,$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m,n \leq d_\xi,\ 1 \leq r,s \leq d_\eta$, which implies that

$$\lambda_m(\xi) + c\mu_r(\eta) \neq 0 \implies \widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \widehat{\widehat{v}}(\xi,\eta)_{mn_{rs}}.$$

Since $\widehat{\widehat{u}}(\xi_M,\eta_M)_{mn_{rs}} = 1$, we conclude that $v \notin C^\infty(G)$, so L is not globally hypoelliptic modulo $\ker L$.

($\Longleftarrow$) Let $u \in \mathcal{D}'(G)$ such that $Lu = f \in C^\infty(G)$. Notice that $f \in \mathcal{K} \cap C^\infty(G)$ and by Proposition 2.7 there exists $v \in C^\infty(G)$ such that $Lv = f$. Therefore $u - v \in \ker L$ and then L is globally hypoelliptic modulo $\ker L$. □

Example 2.19. *Let $G = \mathbb{T}^1 \times \mathbb{S}^3$. In Example 2.10 we saw that the operator $L = \partial_t + X$ is not globally hypoelliptic but it is globally solvable. By Proposition 2.18, we conclude that even not being globally hypoelliptic, the operator L is globally hypoelliptic modulo kernel.*

2.4.2 $\mathcal{W}$–global hypoellipticity

In the light of Proposition 2.16, our next notion of hypoellipticity is based on the reduction of the domain of the operator.

Definition 2.20. *Let $\mathcal{W}$ be a subset of $\mathcal{D}'(G)$. We say that an operator $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is $\mathcal{W}$-globally hypoelliptic if the conditions $u \in \mathcal{W}$ and $Pu \in C^\infty(G)$ imply that $u \in C^\infty(G)$.*

Observe that an operator P is always $C^\infty(G)$–globally hypoelliptic, and to say that P is $\mathcal{D}'(G)$-globally hypoelliptic means that P is globally hypoelliptic.

Example 2.21. *Let $L = X_1 + cX_2$ and set*

$$\mathcal{K} := \{u \in \mathcal{D}'(G);\ \widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = 0, \text{whenever } \lambda_m(\xi) + c\mu_r(\eta) = 0\}.$$

If L is globally solvable, then L is $\mathcal{K}$-globally hypoelliptic.

Indeed, by the characterization of the global solvability (Theorem 2.5), there exist $C,\ M > 0$ such that

$$|\lambda_m(\xi) + c\mu_r(\eta)| \geq C(\langle\xi\rangle + \langle\eta\rangle)^{-M},$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m \leq d_\xi,\ 1 \leq r \leq d_\eta$, whenever $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$.

Let $u \in \mathcal{K}$ such that $Lu = f \in C^\infty(G)$. We know that

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = i(\lambda_m(\xi) + c\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}.$$

If $\lambda_m(\xi) + c\mu_r(\eta) = 0$ then $\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = 0$.

If $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$, we have

$$|\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| = \frac{1}{|\lambda_m(\xi) + c\mu_r(\eta)|}|\,\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq C(\langle\xi\rangle + \langle\eta\rangle)^M|\,\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}|.$$

Therefore $u \in C^\infty(G)$.

Proposition 2.22. *If $\mathcal{W}_1 \subseteq \mathcal{W}_2$ and L is $\mathcal{W}_2$–globally hypoelliptic, then L is $\mathcal{W}_1$–globally hypoelliptic.*

Proof. Let $u \in \mathcal{W}_1$ such that $Lu \in C^\infty(G)$. As $\mathcal{W}_1 \subseteq \mathcal{W}_2$, we have $u \in \mathcal{W}_2$ and since L is $\mathcal{W}_2$–globally hypoelliptic, $u \in C^\infty(G)$. Therefore L is $\mathcal{W}_1$–globally hypoelliptic. □

Since we always have $\mathcal{K} \subseteq L(\mathcal{D}'(G))$, where $L(\mathcal{D}'(G))$ denotes the image of L, we obtain the following corollary.

Corollary 2.23. *If L is $\mathcal{K}$–globally hypoelliptic, then L is $L(\mathcal{D}'(G))$–globally hypoelliptic.*

Corollary 2.24. *Suppose that L is globally solvable. If there exists $k \in \mathbb{N}$ such that $L^k u \in C^\infty(G)$, then $Lu \in C^\infty(G)$.*

Proof. Suppose that there exists $k \in \mathbb{N}$ such that $L^k u \in C^\infty(G)$. Since $v = L^{k-1}u \in L(\mathcal{D}'(G))$ and $Lv \in C^\infty(G)$, we have, by the $L(\mathcal{D}'(G))$–global hypoellipticity of L, that $v \in L^{k-1}u \in C^\infty(G)$. We can continue this process to conclude that $Lu \in C^\infty(G)$. □

If L is globally solvable, the previous corollary says that if $Lu \notin C^\infty(G)$, then $L^k u \notin C^\infty(G)$ for all $k \in \mathbb{N}$.

Let

$$\mathcal{M} := \{u \in \mathcal{D}'(G); \forall N \in \mathbb{N}, \exists C_N > 0; \| \widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}} \leq C_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},\ \forall([\xi],[\eta]) \in \mathcal{N}\}.$$

Notice that $C^\infty(G) \subsetneq \mathcal{M}$.

Theorem 2.25. *If L is globally solvable, then L is $\mathcal{M}$–globally hypoelliptic.*

Proof. Let $u \in \mathcal{M}$ such that $Lu \in C^\infty(G)$. We know that

$$\widehat{\widehat{Lu}}(\xi,\eta)_{mn_{rs}} = i(\lambda_m(\xi) + c\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}, 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta$. If $([\xi],[\eta]) \notin \mathcal{N}$, then $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$ and

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \frac{1}{i(\lambda_m(\xi) + c\mu_r(\eta))}\widehat{\widehat{Lu}}(\xi,\eta)_{mn_{rs}}$$

Proceeding similarly as in the proof of Theorem 2.3, it can be proved that for every $N \in \mathbb{N}$, there exists $C'_N > 0$ such that

$$\| \widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}} \leq C'_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $([\xi],[\eta]) \notin \mathcal{N}$. Since $u \in \mathcal{M}$, we can conclude that for every $N \in \mathbb{N}$, there exists $K_N > 0$ such that

$$\| \widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}} \leq K_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $([\xi],[\eta]) \in \widehat{G_1} \times \widehat{G_2}$. Therefore $u \in C^\infty(G)$. □

2.5 Low order perturbations

In view of the Greenfield-Wallach conjecture, a way to obtain example of globally hypoelliptic first order differential operators defined on compact Lie groups other than the torus is to consider perturbations of vector fields by low order terms.

We start by considering the case where the perturbation is given by a constant. In the next chapter we will deal with perturbations by smooth functions. This approach was inspired by the reference [5] of A. Bergamasco. In both situations, perturbations by constant and functions, we characterize the global hypoellipticity and the global solvability.

Let G be a compact Lie group, $X \in \mathfrak{g}$ and $q \in \mathbb{C}$. Define the operator $L_q : C^\infty(G) \to C^\infty(G)$ as:

$$L_q u := Xu + qu, \quad u \in C^\infty(G)$$

We can extend L_q to $\mathcal{D}'(G)$ as:

$$\langle L_q u, \varphi \rangle := -\langle u, X\varphi \rangle + \langle u, q\varphi \rangle = -\langle u, L_{-q}\varphi \rangle, \quad u \in \mathcal{D}'(G),\ \varphi \in C^\infty(G). \tag{2.13}$$

If $L_q u = f \in C^\infty(G)$, the Fourier coefficient of f can be obtained as

$$\widehat{f}(\xi) = \widehat{L_q u}(\xi) = \widehat{Xu}(\xi) + \widehat{qu}(\xi) = \sigma_X(\xi)\widehat{u}(\xi) + q\widehat{u}(\xi),$$

for all $[\xi] \in \widehat{G}$. So

$$\widehat{f}(\xi)_{mn} = i\lambda_m(\xi)\widehat{u}(\xi)_{mn} + q\widehat{u}(\xi)_{mn} = i(\lambda_m(\xi) - iq)\widehat{u}(\xi)_{mn},$$

for all $[\xi] \in \widehat{G}$, $1 \leq m, n \leq d_\xi$.

From this we conclude that

$$\widehat{f}(\xi)_{mn} = 0, \text{ whenever } \lambda_m(\xi) - iq = 0.$$

In addition, if $\lambda_m(\xi) - iq \neq 0$, then

$$\widehat{u}(\xi)_{mn} = \frac{1}{i(\lambda_m(\xi) - iq)}\widehat{f}(\xi)_{mn}.$$

Thus, we obtain the following characterization for the global hypoellipticity and solvability of L_q which is similar to the vector fields case and so its proof will be omitted.

Theorem 2.26. *The operator $L_q = X + q$ is globally hypoelliptic if and only if the following conditions are satisfied:*

1. *The set*

$$\mathcal{N} = \{[\xi] \in \widehat{G};\ \lambda_m(\xi) - iq = 0 \text{ for some } 1 \leq m \leq d_\xi\}$$

is finite.

2. $\exists C,\ M > 0$ *such that*

$$|\lambda_m(\xi) - iq| \geq C\langle\xi\rangle^{-M}, \tag{2.14}$$

for all $[\xi] \in \widehat{G},\ 1 \leq m \leq d_\xi$ *whenever* $\lambda_m(\xi) + iq \neq 0$.

Let $\mathcal{K}_q := \{w \in \mathcal{D}'(G);\ \widehat{w}(\xi)_{mn} = 0, \text{whenever } \lambda_m(\xi) - iq = 0\}$.

Definition 2.27. *We say that* L_q *is globally solvable if* $L_q(\mathcal{D}'(G)) = \mathcal{K}_q$.

Theorem 2.28. *The operator* $L_q = X + q$ *is globally solvable if and only if the condition* (2.14) *is satisfied, that is,* $\exists C,\ M > 0$ *such that*

$$|\lambda_m(\xi) - iq| \geq C\langle\xi\rangle^{-M},$$

for all $[\xi] \in \widehat{G},\ 1 \leq m \leq d_\xi$ *whenever* $\lambda_m(\xi) + iq \neq 0$.

Corollary 2.29. *If* L_q *is globally hypoelliptic, then* L_q *is globally solvable.*

Recall the definition of global hypoellipticity modulo kernel given in Section 2.4. The proof of the next result is similar to Proposition 2.18 and its proof will be omitted.

Proposition 2.30. *The operator* L_q *is globally hypoelliptic modulo* $\ker L_q$ *if and only if* L_q *is globally solvable.*

Example 2.31. $G = \mathbb{T}^1 \times \mathbb{S}^3$

In Example 2.10 we concluded that the operator $L = \partial_t + \sqrt{2}X$ is not globally hypoelliptic, but it is globally solvable, since $\sqrt{2}$ is an irrational non-Liouville number. Consider now the operator

$$L = \partial_t + \sqrt{2}X + i\tfrac{1}{2}.$$

In this case we have

$$\mathcal{N} = \left\{(k,\ell) \in \mathbb{Z} \times \tfrac{1}{2}\mathbb{N}_0; k + \sqrt{2}m + \tfrac{1}{2} = 0, \text{ for some } -\ell \leq m \leq \ell\right\} = \varnothing.$$

Notice that

$$\left|k + \sqrt{2}m + \tfrac{1}{2}\right| = \tfrac{1}{2}\left|(2k+1) + 2\sqrt{2}m\right|.$$

In view of $2\sqrt{2}$ be an irrational non-Liouville number, by Theorem 2.26 we conclude that L is a globally hypoelliptic operator, which implies that L is also a globally solvable operator (Corollary 2.29). Notice that for the operator

$$L = \partial_t + \sqrt{2}X + i,$$

the set $\mathcal{N}$ remains having infinitely many elements. Therefore L is not globally hypoelliptic. However, we have that L is globally solvable because $\sqrt{2}$ is an irrational non-Liouville number and so satisfies (2.14).

We also obtained in Example 2.10 that the operator $L = \partial_t + \alpha X$ is neither globally hypoelliptic nor globally solvable, where $\alpha = \left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$. Consider the perturbation

$$L = \partial_t + \alpha X + i\alpha.$$

We have that

$$\mathcal{N} = \left\{(k,\ell) \in \mathbb{Z} \times \tfrac{1}{2}\mathbb{N}_0; k + \alpha m + \alpha = 0, \text{ for some } -\ell \leq m \leq \ell\right\}$$

has infinitely many elements because $(0,\ell) \in \mathcal{N}$, for every $\ell \in \mathbb{N}$. So L is not globally hypoelliptic. Moreover, we have

$$|k + \alpha m + \alpha| = |k + \alpha(m+1)|.$$

Since α is an irrational Liouville number, we conclude by Theorem 2.28 that L is not globally solvable. Hence, the perturbed operator continues not being neither global hypoelliptic nor globally solvable. With the same argument, we can also conclude that the operator

$$L = \partial_t + \alpha X + i\tfrac{1}{2}$$

is neither globally hypoelliptic nor globally solvable.

Example 2.32. $G = \mathbb{S}^3 \times \mathbb{S}^3$

Similarly to the $\mathbb{T}^1 \times \mathbb{S}^3$ case, we will analyze now what happens to perturbations of the operators that we have studied in Example 2.11. Notice that the operator

$$L = X_1 + \sqrt{2}X_2$$

is globally solvable but it is not globally hypoelliptic. Consider the perturbed operator

$$L_1 = X_1 + \sqrt{2}X_2 + i\tfrac{1}{2}.$$

Differently to the previous example, this operator remains not being globally hypoelliptic because for $\kappa \in \frac{1}{2}\mathbb{N}_0 \setminus \mathbb{N}_0$ and $\ell \in \mathbb{N}_0$, we have $(\kappa, \ell) \in \mathcal{N}$. Notice that L_1 continues being globally solvable.

Consider now the operator

$$L_2 = X_1 + \sqrt{2}X_2 + \tfrac{1}{4}i.$$

For this operator we have $\mathcal{N} = \varnothing$ and it satisfies the conditions of Theorem 2.26 because $\sqrt{2}$ is an irrational non-Liouville number. Therefore L_2 is globally hypoelliptic.

Chapter 3

Variable coefficient vector fields - Real case

3.1 A class of vector fields with variable coefficients

Let G_1 and G_2 compact Lie groups, and set $G = G_1 \times G_2$. In this section we will characterize the global hypoellipticity and the global solvability for operators in the form

$$L_{aq} = X_1 + a(x_1)X_2 + q(x_1, x_2),$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$, $a \in C^\infty(G_1)$ is a real-valued function, and $q \in C^\infty(G)$. First, let us consider the case where $q \equiv 0$.

3.1.1 Normal form

Let

$$L_a = X_1 + a(x_1)X_2,$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$ and $a \in C^\infty(G_1)$ is a real-valued function. If $L_a u = f \in C^\infty(G)$, taking the partial Fourier coefficients with respect to the second variable, we obtain

$$\widehat{L_a u}(x_1, \eta)_{rs} = X_1 \widehat{u}(x_1, \eta)_{rs} + i\mu_r(\eta) a(x_1) \widehat{u}(x_1, \eta)_{rs} = \widehat{f}(x_1, \eta)_{rs},$$

for all $[\eta] \in \widehat{G}_1$, $1 \leq r, s \leq d_\eta$. The idea now is to find $\varphi(\,\cdot\,, \eta)_{rs} \neq 0$ such that

$$v(x_1, \eta)_{rs} = \varphi(x_1, \eta)_{rs} \widehat{u}(x_1, \eta)_{rs}$$

satisfies

$$X_1 v(x_1, \eta)_{rs} + i\mu_r(\eta) a_0 v(x_1, \eta)_{rs} = \varphi(x_1, \eta)_{rs} \widehat{f}(x_1, \eta)_{rs} := g(x_1, \eta)_{rs},$$

for all $[\eta] \in \widehat{G}_1$, $1 \leq r, s \leq d_\eta$, for some $a_0 \in \mathbb{R}$. So

$$\begin{aligned}
\varphi(x,\eta)_{rs}\widehat{f}(x_1,\eta)_{rs} &= X_1(\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs}) + i\mu_r(\eta)a_0\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs} \\
&= X_1(\varphi(x_1,\eta)_{rs})\widehat{u}(x_1,\eta)_{rs} + \varphi(x_1,\eta)_{rs}(X_1\widehat{u}(x_1,\eta)_{rs}) \\
&\quad + i\mu_r(\eta)a_0\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs} \\
&= X_1(\varphi(x_1,\eta)_{rs})\widehat{u}(x_1,\eta)_{rs} - i\mu_r(\eta)(a(x_1)-a_0)\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs} \\
&\quad + \varphi(x_1,\eta)_{rs}\big((X_1\widehat{u}(x_1,\eta)_{rs}) + i\mu_r(\eta)a(x_1)\widehat{u}(x_1,\eta)_{rs}\big) \\
&= X_1(\varphi(x_1,\eta)_{rs})\widehat{u}(x_1,\eta)_{rs} + \varphi(x_1,\eta)_{rs}\widehat{f}(x_1,\eta)_{rs} \\
&\quad - i\mu_r(\eta)(a(x_1)-a_0)\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs}
\end{aligned}$$

Thus, if $\widehat{u}(x_1,\eta)_{rs} \neq 0$, we have

$$X_1\varphi(x_1,\eta)_{rs} = i\mu_r(\eta)(a(x_1)-a_0)\varphi(x_1,\eta)_{rs} \tag{3.1}$$

Suppose that there exists $A \in C^\infty(G_1)$ such that

$$X_1A(x_1) = a(x_1) - a_0. \tag{3.2}$$

We can assume that A is a real-valued smooth function. So

$$0 = \int_{G_1} X_1A(x_1)\,dx_1 = \int_{G_1} (a(x_1)-a_0)\,dx_1$$

Therefore $a_0 = \displaystyle\int_{G_1} a(x_1)\,dx_1$ and the equation (3.1) becomes

$$X_1\varphi(x_1,\eta)_{rs} = i\mu_r(\eta)(X_1A)(x_1)\varphi(x_1,\eta)_{rs} \tag{3.3}$$

and by Lemma 3.15, the function

$$\varphi(x_1,\eta)_{rs} = e^{i\mu_r(\eta)A(x_1)}$$

is a solution of (3.3).

Define the operator Ψ_a as

$$\Psi_a u(x_1,x_2) := \sum_{[\eta]\in\widehat{G_2}} d_\eta \sum_{r,s=1}^{d_\eta} e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}\,\eta_{sr}(x_2). \tag{3.4}$$

Remark 3.1. *When G_1 is the one-dimensional torus, the operator $X_1 = \partial_t$ is globally solvable and $a - a_0$ belongs to the set of admissible functions, therefore the assumption over the existence of such function A satisfying (3.2) is verified, for any $a \in C^\infty(G_1)$. However, for other compact Lie groups, including higher-dimensional torus and the sphere $\mathbb{S}^3$, it is not difficult to construct examples of a function a for which there is no A satisfying (3.2).*

The next lemma is a technical result necessary to show that the operator Ψ_a is well-defined.

Lemma 3.2. *Let G be a compact group, $f \in C^\infty(G)$, and $z \in \mathbb{C}$ with $|z| \geq 1$. Let $\{Y_1, \cdots, Y_d\}$ a basis for $\mathfrak{g}$. For all $\beta \in \mathbb{N}_0^d$, there exists $C_\beta > 0$ such that*

$$|\partial^\beta e^{zf(x)}| \leq C_\beta |z|^{|\beta|} e^{\Re(zf(x))}. \tag{3.5}$$

Proof. Let us proceed by induction on $|\beta|$.

For $|\beta| = 0$, we have

$$|\partial^\beta e^{zf(x)}| = |e^{zf(x)}| = e^{\Re(zf(x))}.$$

Suppose now that (3.5) holds for every $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq k$ and let $\beta \in \mathbb{N}_0^d$ with $|\beta| = k+1$. We can write $\beta = \gamma + e_j$, for some $j = 1, \cdots, d$ and $|\gamma| = k$. So

$$\begin{aligned} |\partial^\beta e^{zf(x)}| = |\partial^\gamma Y_j e^{zf(x)}| &= |\partial^\gamma (z Y_j f(x) e^{zf(x)})| \\ &\leq |z| \sum_{\gamma'+\gamma''=\gamma} |\partial^{\gamma'} Y_j f(x)| \, |\partial^{\gamma''} e^{zf(x)}| \\ &= C_\beta |z|^\beta e^{\Re(zf(x))} \end{aligned}$$

□

Remark 3.3. *We have a similar result for the case where $|z| \leq 1$. In this case, the power of $|z|$ on the estimate (3.5) is equal to 1 for every $\beta \in \mathbb{N}_0$, i.e., for all $\beta \in \mathbb{N}_0^d$ there exists C_β such that*

$$|\partial^\beta e^{zf(x)}| \leq C_\beta |z| e^{\mathrm{Re}(zf(x))}, \quad \forall x \in G.$$

Proposition 3.4. *The operator Ψ_a defined in (3.4) is an automorphism of $C^\infty(G)$ and of $\mathcal{D}'(G)$.*

Proof. First of all, notice that Ψ_{-a} is the inverse of Ψ_a, therefore we only need to prove that $\Psi_a(C^\infty(G)) = C^\infty(G)$ and $\Psi_a(\mathcal{D}'(G)) = \mathcal{D}'(G)$.

Let $\beta \in \mathbb{N}_0$ and $u \in C^\infty(G)$. We will show that $\Psi_a u \in C^\infty(G)$. Notice that $\widehat{\Psi_a u}(x_1, \eta)_{rs} = e^{i\mu_r(\eta) q(x_1)} \widehat{u}(x_1, \eta)_{rs}$ and $\mu_r(\eta) A(x_1) \in \mathbb{R}$, for all $[\eta] \in \widehat{G_2}$, $1 \leq r \leq d_\eta$ and $x_1 \in G_1$. Using (3.5) we obtain

$$\begin{aligned} |\partial^\beta \widehat{\Psi_a u}(x_1, \eta)_{rs}| &= |\partial^\beta (e^{i\mu_r(\eta) A(x_1)} \widehat{u}(x_1, \eta)_{rs})| \\ &= \left| \sum_{\beta'+\beta''=\beta} \partial^{\beta'} e^{i\mu_r(\eta) A(x_1)} \partial^{\beta''} \widehat{u}(x_1, \eta)_{rs} \right| \\ &\leq \sum_{\beta'+\beta''=\beta} \left| \partial^{\beta'} e^{i\mu_r(\eta) A(x_1)} \right| \left| \partial^{\beta''} \widehat{u}(x_1, \eta)_{rs} \right| \\ &\leq \sum_{\beta'+\beta''=\beta} C_{\beta'} |\mu_r(\eta)|^{|\beta'|} \left| \partial^{\beta''} \widehat{u}(x_1, \eta)_{rs} \right| \end{aligned}$$

Since $u \in C^\infty(G)$ and $|\mu_r(\eta)| \leq \langle\eta\rangle$, it is easy to see that given $N > 0$, there exists $C_{\beta N}$ such that

$$|\partial^\beta \widehat{\Psi_a u}(x_1,\eta)_{rs}| \leq C_{\beta N}\langle\eta\rangle^{-N}.$$

Therefore $\Psi_a u \in C^\infty(G)$. The distribution case is analogous. □

Proposition 3.5. *Let $a \in C^\infty(G_1)$, $a_0 := \int_{G_1} a(x_1)\,dx_1$, and consider the operator $L_{a_0} = X_1 + a_0 X_2$. Assume that there exists $A \in C^\infty(G_1)$ such that $X_1 A = a - a_0$. Then we have*

$$L_{a_0} \circ \Psi_a = \Psi_a \circ L_a$$

in both $C^\infty(G)$ and in $\mathcal{D}'(G)$, where Ψ_a is given in (3.4).

Proof. Let us show that for any $u \in C^\infty(G)$ we have

$$\widehat{L_{a_0}(\Psi_a u)}(x_1,\eta)_{rs} = \widehat{\Psi_a(L_a u)}(x_1,\eta)_{rs},$$

for all $x_1 \in G_1,\ [\eta] \in \widehat{G}_2,\ 1 \leq r,s \leq d_\eta$.

Indeed,

$$\begin{aligned}
\widehat{L_{a_0}(\Psi_a u)}(x_1,\eta)_{rs} &= \widehat{X_1\Psi_a u}(x_1,\eta)_{rs} + a_0\widehat{X_2\Psi_a u}(x_1,\eta)_{rs}\\
&= X_1\widehat{\Psi_a u}(x_1,\eta)_{rs} + i\mu_r(\eta)a_0\widehat{\Psi_a u}(x_1,\eta)_{rs}\\
&= X_1(e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}) + i\mu_r(\eta)a_0 e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}\\
&= (X_1 e^{i\mu_r(\eta)A(x_1)})\widehat{u}(x_1,\eta)_{rs} + e^{i\mu_r(\eta)A(x_1)}(X_1\widehat{u}(x_1,\eta)_{rs})\\
&\quad + i\mu_r(\eta)a_0 e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}\\
&= i\mu_r(\eta)(a(x_1)-a_0)e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs} + e^{i\mu_r(\eta)A(x_1)}(X_1\widehat{u}(x_1,\eta)_{rs})\\
&\quad + i\mu_r(\eta)a_0 e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}\\
&= e^{i\mu_r(\eta)A(x_1)}(X_1\widehat{u}(x_1,\eta)_{rs} + i\mu_r(\eta)a(x_1)\widehat{u}(x_1,\eta)_{rs})\\
&= e^{i\mu_r(\eta)A(x_1)}\widehat{L_a u}(x_1,\eta)_{rs}\\
&= \widehat{\Psi_a(L_a u)}(x_1,\eta)_{rs}
\end{aligned}$$

The same is true when $u \in \mathcal{D}'(G)$. □

3.1.2 Global properties

Recall that the operator L_{a_0} is globally solvable if $L_{a_0}(\mathcal{D}'(G)) = \mathcal{K}_{a_0}$, where

$$\mathcal{K}_{a_0} := \{w \in \mathcal{D}'(G);\ \lambda_m(\xi) + a_0\mu_r(\eta) = 0,\ \text{whenever } \widehat{\widehat{w}}(\xi,\eta)_{mn_{rs}} = 0\}.$$

We will say that L_a is globally solvable if $L_a(\mathcal{D}'(G)) = \mathcal{J}_a$, where

$$\mathcal{J}_a := \{v \in \mathcal{D}'(G); \Psi_{-a}v \in \mathcal{K}_{a_0}\}.$$

Proposition 3.6. *The operator L_a is globally hypoelliptic (resp. globally hypoelliptic modulo* $\ker L_a$*) if and only if L_{a_0} is globally hypoelliptic (resp. globally hypoelliptic modulo* $\ker L_{a_0}$*). Similarly, the operator L_a is globally solvable if and only if L_{a_0} is globally solvable.*

Proposition 3.7. *Assume that there exists $A \in C^\infty(G)$ such that $X_1 A = a - a_0$, where $a_0 = \int_G a(x)\,dx$. Then*

1. *L_a is globally hypoelliptic if and only if L_{a_0} is globally hypoelliptic;*

2. *L_a is globally hypoelliptic modulo* $\ker L_a$ *if and only if L_{a_0} is globally hypoelliptic modulo* $\ker L_{q_0}$*;*

3. *L_a is globally solvable if and only if L_{a_0} is globally solvable.*

Proof. 1. Suppose that L_a is globally hypoelliptic. If $L_{a_0}u = f \in C^\infty(G)$ for some $u \in \mathcal{D}'(G)$, then $\Psi_{-a}L_{a_0}u = \Psi_{-a}f \in C^\infty(G)$. Since $\Psi_{-a} \circ L_{a_0} = L_a \circ \Psi_{-a}$, we have $L_a(\Psi_{-a}u) \in C^\infty(G)$ and by global hypoellipticity of L_a we have $\Psi_{-a}u \in C^\infty(G)$, which implies that $u \in C^\infty(G)$ and then L_{a_0} is globally hypoelliptic.

Assume now that L_{a_0} is globally hypoelliptic. If $L_a u = f \in C^\infty(G)$ for some $u \in \mathcal{D}'(G)$, we can write $L_a(\Psi_{-a}\Psi_a u) = f \in C^\infty(G)$. By the fact that $L_a \circ \Psi_{-a} = \Psi_{-a} \circ L_{a_0}$ we obtain $\Psi_{-a}L_{a_0}(\Psi_a u) = f$, that is, $L_{a_0}(\Psi_a u) = \Psi_a f \in C^\infty(G)$. By global hypoellipticity of L_{a_0} we have that $\Psi_a u \in C^\infty(G)$ and then $u \in C^\infty(G)$.

2. Suppose that L_a is globally hypoelliptic modulo $\ker L_a$. If $L_{a_0}u = f \in C^\infty(G)$ for some $u \in \mathcal{D}'(G)$, then $\Psi_{-a}L_{a_0}u = \Psi_{-a}f \in C^\infty(G)$. Since $\Psi_{-a} \circ L_{a_0} = L_a \circ \Psi_{-a}$, we have $L_a(\Psi_{-a}u) = \Psi_{-a}f \in C^\infty(G)$. By assumption, L_a is globally hypoelliptic modulo $\ker L_a$, so there exists $v \in C^\infty$ such that $\Psi_{-a}u - v \in \ker L_a$, i.e.,

$$L_a\Psi_{-a}(u - \Psi_a v) = L_a(\Psi_{-a}u - v) = 0.$$

Hence, $L_{a_0}(u - \Psi_a v) = 0$. Since $\Psi_a v \in C^\infty(G)$ and $u - \Psi_a v \in \ker L_{a_0}$, we conclude that L_{a_0} is globally hypoelliptic modulo $\ker L_{a_0}$. The other implication is analogous so it is omitted.

3. Assume that L_a is globally solvable and let $f \in \mathcal{K}_{a_0}$. Let us show that there exists $u \in \mathcal{D}'(G)$ such that $L_{a_0}u = f$. We can write $f = \Psi_a\Psi_{-a}f$, so $\Psi_{-a}f \in \mathcal{J}_a$. Since L_a is

globally solvable, there exists $v \in \mathcal{D}'(G)$ such that $L_a v = \Psi_{-a} f$. we can write $v = \Psi_{-a}\Psi_a v$ and then $L_a(\Psi_{-a}\Psi_a v) = \Psi_{-a} f$. By Proposition 3.16, we have

$$\Psi_{-a} L_{a_0} \Psi_a v = L_a(\Psi_{-a}\Psi_a v) = \Psi_{-a} f,$$

that is, $L_{a_0}\Psi_a v = f$.

Suppose now that L_{a_0} is globally solvable and let $f \in \mathcal{J}_a$. By the definition of $\mathcal{J}_a$, we have $\Psi_a f \in \mathcal{K}_{a_0}$ and by the global solvability of L_{a_0}, there exists $u \in \mathcal{D}'(G)$ such that $L_{a_0} u = \Psi_a f$, that is, $\Psi_{-a} L_{a_0} u = f$. By Proposition 3.16, we get $L_a \Psi_{-a} u = f$. □

Hence, the operator L_a inherits the following properties from the operator L_{a_0} that we have proved in Chapter 2.

Corollary 3.8. *If L_a is globally hypoelliptic, then L_a is globally solvable.*

Proof. Suppose that L_a is globally hypoelliptic. By Proposition 3.7 the operator L_{a_0} is globally hypoelliptic, so by Corollary 2.6, L_{a_0} is globally solvable. Finally, by Proposition 3.7, we conclude that L_a is globally solvable. □

Corollary 3.9. *The operator L_a is globally hypoelliptic modulo $\ker L_a$ if and only if L_a is globally solvable.*

Example 3.10. $G = \mathbb{T}^2$

For instance, for $a(t) = \sin(t) + \sqrt{2}$, we have $a_0 = \sqrt{2}$ and $A(t) = -cos(t)$. Take $G_2 = \mathbb{T}^1$. We know by Example 2.8 that the operator

$$L_{a_0} = \partial_t + \sqrt{2}\partial_x$$

is globally hypoelliptic, because $\sqrt{2}$ is an irrational non-Liouville number. Hence,

$$L_a = \partial_t + (\sin(t) + \sqrt{2})\partial_x$$

is globally hypoelliptic and then globally solvable.

Example 3.11. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Take now $G_2 = \mathbb{S}^3$. By Example 2.10, we know that

$$L_{a_0} = \partial_t + \sqrt{2}X$$

is not globally hypoelliptic but it is globally solvable. Therefore

$$L_a = \partial_t + (\sin(t) - \sqrt{2})X$$

is not globally hypoelliptic and it is globally solvable.

Example 3.12. $G = \mathbb{S}^3 \times \mathbb{S}^3$

We can identify $\mathbb{S}^3$ with $\mathrm{SU}(2)$ and the Euler's angle parametrization of $\mathrm{SU}(2)$ is given by

$$x(\phi,\theta,\psi) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{i(\phi+\psi)/2} & i\sin\left(\frac{\theta}{2}\right) e^{i(\phi-\psi)/2} \\ i\sin\left(\frac{\theta}{2}\right) e^{-i(\phi-\psi)/2} & \cos\left(\frac{\theta}{2}\right) e^{-i(\phi+\psi)/2} \end{pmatrix} \in \mathrm{SU}(2), \tag{3.6}$$

where $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$ and $-2\pi \leq \psi < 2\pi$ (see Chapter 11 of [35]). The trace function on $\mathrm{SU}(2)$ in Euler's angles (see (3.6)) is given by

$$\mathrm{tr}(x(\phi,\theta,\psi)) = 2\cos\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\phi+\psi}{2}\right).$$

Consider the operator

$$L = X_1 + a(x_1)X_2,$$

where X_1 acts in the first variable, X_2 acts in the second variable and $a : \mathbb{S}^3 \to \mathbb{R}$ is expressed in Euler's angles by

$$a(x_1(\phi_1,\theta_1,\psi_1)) = -\cos\left(\tfrac{\theta_1}{2}\right)\sin\left(\tfrac{\phi_1+\psi_1}{2}\right) + \sqrt{2} \tag{3.7}$$

The operator X_1 in Euler's angles is the operator ∂_{ψ_1} and then we have

$$X_1\mathrm{tr}(x_1) = a(x_1) - \sqrt{2}.$$

By Proposition 3.7, we conclude that L is not globally hypoelliptic but it is globally solvable, because the operator $L_0 = X_1 + \sqrt{2}X_2$ has this properties (see Example 2.32).

Remark 3.13. *We had supposed that given a function $a \in C^\infty(G_1)$ there exists a function $A \in C^\infty(G_1)$ and $a_0 \in \mathbb{R}$ such that $X_1 A = a - a_0$, that is, X_1 is C^∞–cohomology free on G_1 (see Definition 2.13).*

Conjecture 3.14 (Katok)**.** *If a closed, connected, orientable manifold M admits a C^∞–cohomology free vector field X, then M is diffeomorphic to a torus and X is smoothly conjugate to a Diophantine vector field.*

In [22], G. Forni has proved that the Katok's conjecture is equivalent to the Greenfield's and Wallach's conjecture, mentioned in the previous chapter (see Conjecture 2.12). In view of the proof of this conjecture in dimensions 2 and 3, and its validity in compact Lie groups, it is necessary to add in the hypothesis the existence of such A satisfying $X_1 A = a - a_0$. Otherwise, the above results would be valid only for the case where G_1 is a torus.

3.2 Perturbations of vector fields by functions

In this section we are concerned with the operator $L_q := X + q$, with $q \in C^\infty(G)$. The idea is to establish a connection between the global hypoellipticity and the global solvability of L_q and $L_{q_0} = X + q_0$, where q_0 is the average of q in G.

In [5], Bergamasco proved that the operator

$$L_q = \partial_t + a\partial_x + q,$$

where $a \in \mathbb{R}$ is an irrational non-Liouville number and $q \in C^\infty(\mathbb{T}^2)$, is globally hypoelliptic if and only if it is the operator $L_{q_0} = \partial_t + a\partial_x + q_0$, where $q_0 = \int_{\mathbb{T}^2} q(t,x)\,dxdt$. The key to make this connection is the fact that $L_q \circ e^{-Q} = e^{-Q} \circ L_{q_0}$, where $Q \in C^\infty(\mathbb{T}^2)$ satisfies $(\partial_t + a\partial_x)Q = q - q_0$. The existence of such Q is guaranteed by the global hypoellipticity of the operator $\partial_t + a\partial_x$.

For the study of the operator $L_q = X + q$, with $q \in C^\infty(G)$, we can not assume the global hypoellipticity of X in view of the Greenfield-Wallach's conjecture. Hence, we will assume as hypothesis that there exists $Q \in C^\infty(G)(G)$ such that

$$XQ = q - q_0,$$

where $q_0 = \int_G q(x)\,dx$.

Lemma 3.15. *For any $\varphi \in C^\infty(G)$ we have*

$$Xe^{\varphi} = (X\varphi)e^{\varphi}.$$

Proof. Let $x \in G$, then

$$(Xe^{\varphi})(x) = X\sum_{k=0}^{\infty}\frac{\varphi(x)^k}{k!} = \sum_{k=0}^{\infty}\frac{X\varphi(x)^k}{k!} = \sum_{k=1}^{\infty}\frac{k\varphi(x)^{k-1}(X\varphi)(x)}{k!} = (X\varphi)(x)\sum_{k=1}^{\infty}\frac{\varphi(x)^{k-1}}{(k-1)!}$$
$$= (X\varphi)(x)e^{\varphi(x)}$$

□

Let $L_q : C^\infty(G) \to C^\infty(G)$ defined by

$$L_q u = Xu(x) + qu, \quad u \in C^\infty(G).$$

We can extend L_q to $\mathcal{D}'(G)$ as in (2.13).

Proposition 3.16. *Assume that there exists $Q \in C^\infty(G)$ such that $XQ = q - q_0$, where $q_0 = \int_G q(x)\,dx$. Then*

1. *$L_q \circ e^{-Q} = e^{-Q} \circ L_{q_0}$, in both $C^\infty(G)$ and in $\mathcal{D}'(G)$;*

2. *L_q is globally hypoelliptic if and only if L_{q_0} is globally hypoelliptic;*

3. *L_q is globally hypoelliptic modulo $\ker L_q$ if and only if L_{q_0} is globally hypoelliptic modulo $\ker L_{q_0}$.*

Proof. 1. Let $u \in C^\infty(G)$. Then

$$\begin{aligned}(L_q \circ e^{-Q})u &= L_q(e^{-Q}u)\\ &= X(e^{-Q}u) + qe^{-Q}u = (Xe^{-Q})u + e^{-Q}Xu + qe^{-Q}u\\ &= (-XQ)e^{-Q}u + e^{-Q}Xu + qe^{-Q}u\\ &= -(q - q_0)e^{-Q}u + e^{-Q}Xu + qe^{-Q}u\\ &= e^{-Q}(Xu + q_0 u)\\ &= (e^{-Q} \circ L_{q_0})u\end{aligned}$$

The same is true when we have $u \in \mathcal{D}'(G)$.

The proof of 2. and 3. is similar to what was done in Proposition 3.7. □

Now assume that $L_q u = f \in \mathcal{D}'(G)$ for some $u \in \mathcal{D}'(G)$. We may write $u = e^{-Q}(e^Q u)$, so $L_q(e^{-Q}(e^Q u)) = f$. By Proposition 3.16, we have $e^{-Q}L_{q_0}e^Q u = f$, that is,

$$L_{q_0}e^Q u = e^Q f.$$

This implies that $e^Q f \in \mathcal{K}_{q_0}$.

Definition 3.17. *We say that the operator L_q is globally solvable if:*

1. *there is Q such that $XQ = q - q_0$, where $q_0 = \int_G q(x)\,dx$; and*

2. *$L_q(\mathcal{D}'(G)) = \mathcal{J}_q$, where*

$$\mathcal{J}_q := \{v \in \mathcal{D}'(G);\ e^Q v \in \mathcal{K}_{q_0}\}.$$

Proposition 3.18. *L_q is globally solvable if and only if L_{q_0} is globally solvable.*

The proof is omitted because it is analogous to the proof of Proposition 3.7.

Corollary 3.19. *If L_q is globally hypoelliptic then L_q is globally solvable.*

Corollary 3.20. *L_q is globally hypoelliptic modulo $\ker L_q$ if and only if L_q is globally solvable.*

Example 3.21. $G = \mathbb{T}^2$

Consider the operator

$$L_q = \partial_t + \partial_x + q(t,x),$$

where $q(t,x) = \sin(t+x)$. For $Q(t,x) = -\frac{1}{2}\cos(t+x)$ we have $(\partial_t+\partial_x)Q(t,x) = q(t,x)-q_0$, where $q_0 = 0$. Since $L_{q_0} = \partial_t+\partial_x$ is not globally hypoelliptic, we conclude by Proposition 3.16 that L_q is not globally hypoelliptic. On the other hand, the operator L_{q_0} is globally solvable, then L_q is globally solvable.

For $q(t,x) = \sin(t+x)+1$, we have $q_0 = 1$ and by Theorem 2.26 we have that L_{q_0} is globally hypoelliptic and then L_q is globally hypoelliptic, which implies that L_q is also globally solvable.

Example 3.22. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Recall from Example 3.12 that the trace function on SU(2) in Euler's angles (see (3.6)) is given by

$$\operatorname{tr}(x(\phi,\theta,\psi)) = 2\cos\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\phi+\psi}{2}\right).$$

Consider the operator

$$L_q = \partial_t + \sqrt{2}X + q(t,x),$$

where X is the same vector field from Example 2.10 and

$$q(t,x) = -\sin(t)\operatorname{tr}(x) + \sqrt{2}\cos(t)h(x) + i\tfrac{1}{2},$$

where $h : \mathbb{S}^3 \to \mathbb{C}$ is expressed in Euler's angles by

$$h(x(\phi,\theta,\psi)) = -\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\phi+\psi}{2}\right). \tag{3.8}$$

We have that X in Euler's angles is the operator ∂_ψ, so

$$X\operatorname{tr}(x) = h(x).$$

Let $Q(t,x) = \cos(t)\mathrm{tr}(x)$. Notice that

$$(\partial_t + \sqrt{2}X)Q(t,x) = q(t,x) - i\tfrac{1}{2}.$$

By Example 2.31, the operator $L_{q_0} = \partial_t + \sqrt{2}X + \frac{1}{2}$ is not globally hypoelliptic but it is globally solvable. By Proposition 3.16 we conclude that

$$L_q = \partial_t + \sqrt{2}X - \sin(t)\mathrm{tr}(x) + \sqrt{2}\cos(t)h(x) + i\tfrac{1}{2}$$

is not globally hypoelliptic, but it is globally solvable.

Example 3.23. $G = \mathbb{S}^3 \times \mathbb{S}^3$

Consider

$$L = X_1 + \sqrt{2}X_2 + q(x_1, x_2),$$

where X_1 acts in the first variable, X_2 acts in the second variable, and $q : \mathbb{S}^3 \to \mathbb{C}$ is expressed in Euler's angles by

$$q(x_1, x_2) = p_1(x_1) + \sqrt{2}\,p_2(x_2) + \tfrac{1}{2}i,$$

where p_1 and p_2 are the projections of $\mathrm{SU}(2) \simeq \mathbb{S}^3$ given in Euler's angle by

$$p_1(x(\phi,\theta,\psi)) = \cos\left(\tfrac{\theta}{2}\right) e^{i(\phi+\psi)/2} \quad \text{and} \quad p_2(x(\phi,\theta,\psi)) = i\sin\left(\tfrac{\theta}{2}\right) e^{i(\phi-\psi)/2},$$

with $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$, $-2\pi \leq \psi < 2\pi$. Notice that the function $Q(x_1, x_2) = 2i(p_2(x_2) - p_1(x_1))$ satisfies

$$(X_1 + \sqrt{2}X_2)Q(x_1, x_2) = q(x_1, x_2) - \tfrac{1}{2}i.$$

By Proposition 3.16 and 3.18, we conclude that L is not globally hypoelliptic but it is globally solvable, because $L_0 = X_1 + \sqrt{2}X_2 + \frac{1}{2}i$ has this properties (see Example 2.32). Similarly, we conclude that

$$L = X_1 + \sqrt{2}X_2 + p_1(x_1) + \sqrt{2}\,p_2(x_2) + \tfrac{1}{4}i$$

is globally hypoelliptic because $L_0 = X_1 + \sqrt{2}X_2 + \frac{1}{4}i$ is globally hypoelliptic (see Example 2.32).

3.3 The general case

We can now combine what was made in Sections 3.1 and 3.2 to study the operator

$$L_{aq} = X_1 + a(x_1)X_2 + q(x_1, x_2),$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$, $a \in C^\infty(G_1)$ is a real-valued function, and $q \in C^\infty(G)$. Furthermore, we will assume that there exists $Q \in C^\infty(G)$ satisfying

$$L_a Q = (X_1 + a(x_1)X_2)Q = q - q_0.$$

By Proposition 3.16 we have

$$L_{aq} \circ e^{-Q} = e^{-Q} \circ L_{aq_0},$$

where $L_{aq_0} = X_1 + a(x_1)X_2 + q_0$.

It follows from Proposition 3.5 that

$$L_{a_0 q_0} \circ \Psi_a = \Psi_a \circ L_{aq_0},$$

where $L_{a_0 q_0} = X_1 + a_0 X_2 + q_0$. Thus,

$$L_{aq} \circ e^{-Q} \circ \Psi_a = e^{-Q} \circ L_{aq_0} \circ \Psi_a = e^{-Q} \circ \Psi_a \circ L_{a_0 q_0}.$$

We say that L_{aq} is globally solvable if $L_{aq}(\mathcal{D}'(G)) = \mathcal{J}_{aq}$, where

$$\mathcal{J}_{aq} := \{v \in \mathcal{D}'(G);\ \Psi_{-a} e^{Q} v \in \mathcal{K}_{a_0 q_0}\}$$

and

$$\mathcal{K}_{a_0 q_0} := \{w \in \mathcal{D}'(G); \widehat{\widehat{w}}(\xi, \eta)_{mn_{rs}} = 0 \text{ whenever } \lambda_m(\xi) + a_0 \mu_r(\eta) - i q_0 = 0\}.$$

The next results are consequences of what was done previously.

Proposition 3.24. *The operator L_{aq} is globally hypoelliptic (resp. globally hypoelliptic modulo $\ker L_{aq}$) if and only if $L_{a_0 q_0}$ is globally hypoelliptic (resp. globally hypoelliptic modulo $\ker L_{a_0 q_0}$). Similarly, the operator L_{aq} is globally solvable if and only if $L_{a_0 q_0}$ is globally solvable.*

Corollary 3.25. *If L_{aq} is globally hypoelliptic, then $L_{a_0 q_0}$ is globally solvable.*

Corollary 3.26. *The operator L_{aq} is hypoelliptic modulo* $\ker L_{aq}$ *if and only if $L_{a_0q_0}$ is globally solvable.*

Example 3.27. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Let $G = \mathbb{T}^1 \times \mathbb{S}^3$ and $X \in \mathfrak{s}^3$ as in the Example 2.10. Let $a(t) = \sin(t) + \sqrt{2}$ and $q(t,x) = \cos(t) + (\sin(t) + \sqrt{2})h(x) + 1$, with h as in Example 3.2. Here, $a_0 = \sqrt{2}$ and $q_0 = 1$. Notice that the function $Q(t,x) = \sin(t) + tr(x)$ satisfies $(\partial_t + a(t)X)Q(t,x) = q(t,x)$. By Theorem 2.26 the operator

$$L_{a_0q_0} = \partial_t + \sqrt{2}X + 1$$

is globally hypoelliptic (see Example 2.10) and then the operator

$$L_{aq} = \partial_t + (\sin(t) + \sqrt{2})X + (\cos(t) + (\sin(t) + \sqrt{2})h(x) + 1)$$

is globally hypoelliptic, which implies that L_{aq} is globally solvable.

For $q(t,x) = \cos(t) + (\sin(t) + \sqrt{2})h(x) + i$, the operator

$$L_{a_0q_0} = \partial_t + \sqrt{2}X + i$$

is not globally hypoelliptic (see Example 2.31) and then the operator

$$L_{aq} = \partial_t + (\sin(t) + \sqrt{2})X + (\cos(t) + (\sin(t) + \sqrt{2})h(x) + i)$$

is not globally hypoelliptic. However, since $\sqrt{2}$ is an irrational non-Liouville number, the operator $L_{a_0q_0}$ is globally solvable, which implies that L_{aq} is globally solvable.

Example 3.28. $G = \mathbb{S}^3 \times \mathbb{S}^3$

In this example we will analyze a perturbation of the operator studied in Example 3.12. Consider

$$L = X_1 + a(x_1)X_2 + q(x_1, x_2),$$

where X_1 acts in the first variable, X_2 acts in the second variable, $a : \mathbb{S}^3 \to \mathbb{R}$ expressed in Euler's angles by

$$a(x_1(\phi_1, \theta_1, \psi_1)) = -\cos\left(\tfrac{\theta_1}{2}\right)\sin\left(\tfrac{\phi_1+\psi_1}{2}\right) + \sqrt{2},$$

and $q : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{C}$ is given by

$$q(x_1, x_2) = p_1(x_1) + a(x_1)\, p_2(x_2) + \tfrac{1}{2}i,$$

where p_1 and p_2 are the projections of $\mathrm{SU}(2) \simeq \mathbb{S}^3$ (see Example 3.23). Notice that $Q(x_1, x_2) = 2i(p_2(x_2) - p_1(x_1))$ satisfies

$$(X_1 + a(x_1)X_2)Q(x_1, x_2) = q(x_1, x_2) - \tfrac{1}{2}i.$$

By Proposition 3.24, we can extract the global properties of L_{aq} from the operator

$$L_{a_0 q_0} = X_1 + \sqrt{2}X_2 + \tfrac{1}{2}i,$$

that we already have studied in Example 3.23. Therefore, the operator

$$L = X_1 + a(x_1)X_2 + p_1(x_1) + a(x_1)\, p_2(x_2) + \tfrac{1}{2}i$$

is not globally hypoelliptic but it is globally solvable.

Analogously, with a slight change in the definition of q, we conclude that

$$L = X_1 + a(x_1)X_2 + p_1(x_1) + a(x_1)\, p_2(x_2) + \tfrac{1}{4}i$$

is globally hypoelliptic, since the operator $L = X_1 + \sqrt{2}X_2 + \frac{1}{4}i$ is globally hypoelliptic (see Example 2.32).

Chapter 4

Variable coefficient vector fields - Complex case

Let G be a compact Lie group and consider the operator $L_q : \mathcal{D}'(\mathbb{T}^1 \times G) \to \mathcal{D}'(\mathbb{T}^1 \times G)$ defined by

$$L_q := \partial_t + c(t)X + q,$$

where $X \in \mathfrak{g}$, $c \in C^\infty(\mathbb{T}^1)$, $c(t) = a(t) + ib(t)$, and $q \in \mathbb{C}$. In this chapter, we will study the necessary and sufficient conditions for the global hypoellipticity of this operator. Here we are assuming that the first group is the one-dimensional torus because the study of the operator L_q leads us to solve a system of ordinary differential equations, which we only can solve in $\mathbb{T}^1$, for now. The case where either c is a constant function or $b \equiv 0$ was completely characterized in Chapters 2 and 3. Recall that for each $[\eta] \in \widehat{G}$, we can choose a representative $\eta \in \mathrm{Rep}(G)$ such that

$$\sigma_X(\eta)_{rs} = i\mu_r(\eta)\delta_{rs}, \quad 1 \leq r, s \leq d_\eta,$$

where $d_\eta = \dim \eta$, and $\mu_r(\eta) \in \mathbb{R}$ for all $[\eta] \in \widehat{G}$. In Chapter 3, we have seen that when $b \equiv 0$, the global hypoellipticity of L_q is strictly related to the global hypoellipticity of the operator

$$L_{q_0} = \partial_t + a_0 X + q$$

because these two operator are conjugated by the automorphism

$$\Psi_a u(t,x) := \sum_{[\eta]\in\widehat{G}} d_\eta \sum_{r,s=1}^{d_\eta} e^{i\mu_r(\eta)A(t)} \widehat{u}(t,\eta)_{rs}\, \eta_{sr}(x)$$

defined in (3.4). If we wanted to do the same in the case where $b \not\equiv 0$ we would not have the growth control of the term $e^{-\mu_r(\eta)B(t)}$ that appears in the definition of Ψ_c Therefore Ψ_c is not an

automorphism. However, we will prove in Proposition 4.2 that the global hypoellipticity of L_{q_0} is a necessary condition for the global hypoellipticity of L_q and at the end of the chapter we give an example where this is not a sufficient condition. First, observe that by the automorphism Ψ_a we may assume that $a(t)$ is a constant function, so

$$L_q = \partial_t + (a_0 + ib(t))X + q.$$

4.1 Global hypoellipticity

Consider the equation $L_q u = f \in C^\infty(\mathbb{T}^1 \times G)$. Taking the partial Fourier coefficient with respect to the second variable we obtain

$$\widehat{f}(t,\eta) = \widehat{L_q u}(t,\eta) = \partial_t \widehat{u}(t,\eta) + c(t)\sigma_X(\eta)\widehat{u}(t,\eta) + q\widehat{u}(t,\eta),$$

that is,

$$\widehat{f}(t,\eta)_{rs} = \widehat{L_q u}(t,\eta)_{rs} = \partial_t \widehat{u}(t,\eta)_{rs} + i(\mu_r(\eta)c(t) - iq)\widehat{u}(t,\eta)_{rs}, \tag{4.1}$$

for $1 \leq r, s \leq d_\eta$.

Let

$$C(t) = \int_0^t c(\tau)d\tau - c_0 t, \quad \text{where} \quad c_0 = \frac{1}{2\pi}\int_0^{2\pi} c(\tau)d\tau.$$

Multiplying by $e^{i\mu_r(\eta)C(t)}$, we obtain

$$\partial_t \widehat{u}(t,\eta)_{rs} e^{i\mu_r(\eta)C(t)} + i(\mu_r(\eta)c(t) - iq)\widehat{u}(t,\eta)_{rs} e^{i\mu_r(\eta)C(t)} = \widehat{f}(t,\eta)_{rs} e^{i\mu_r(\eta)C(t)}$$

Then

$$\partial_t\left[\widehat{u}(t,\eta)_{rs} e^{i\mu_r(\eta)C(t)}\right] + i(\mu_r(\eta)c_0 - iq)\widehat{u}(t,\eta)_{rs} e^{i\mu_r(\eta)C(t)} = \widehat{f}(t,\eta)_{rs} e^{i\mu_r(\eta)C(t)},$$

that is, for each $\eta \in \widehat{G}$ and $1 \leq r, s \leq d_\eta$, we have that $\widehat{u}(t,\eta)_{rs} e^{i\mu_r(\eta)C(t)}$ is a solution of

$$\partial_t v(t,\eta)_{rs} + i(\mu_r(\eta)c_0 - iq)v(t,\eta)_{rs} = g(t,\eta)_{rs}, \tag{4.2}$$

where $g(t,\eta)_{rs} = \widehat{f}(t,\eta)_{rs} e^{i\mu_r(\eta)C(t)}$. It follows from Lemma B.1, from Appendix B, that (4.2) has a unique solution given by

$$v(t,\eta)_{rs} = \frac{1}{1 - e^{-2\pi i(\mu_r(\eta)c_0 - iq)}} \int_0^{2\pi} e^{-i(\mu_r(\eta)c_0 - iq)\tau} g(t-\tau,\eta)_{rs} d\tau,$$

whenever $\mu_r(\eta)c_0 - iq \notin \mathbb{Z}$, or equivalently by

$$v(t,\eta)_{rs} = \frac{1}{e^{2\pi i(\mu_r(\eta)c_0 - iq)} - 1} \int_0^{2\pi} e^{i(\mu_r(\eta)c_0 - iq)\tau} g(t+\tau,\eta)_{rs} d\tau.$$

Therefore, we obtain

$$\widehat{u}(t,\eta)_{rs} = \frac{1}{1-e^{-2\pi i(\mu_r(\eta)c_0 - iq)}} \int_0^{2\pi} e^{-q\tau} e^{-i\mu_r(\eta)(c_0\tau - C(t-\tau)+C(t))} \widehat{f}(t-\tau,\eta)_{rs} d\tau, \tag{4.3}$$

or equivalently,

$$\widehat{u}(t,\eta)_{rs} = \frac{1}{e^{2\pi i(\mu_r(\eta)c_0 - iq)}-1} \int_0^{2\pi} e^{q\tau} e^{-i\mu_r(\eta)(-c_0\tau - C(t+\tau)+C(t))} \widehat{f}(t+\tau,\eta)_{rs} d\tau. \tag{4.4}$$

In the remainder of this chapter we will need to control the behavior of the numerical sequence that precedes the integral in the expression above. For this end we will use the following technical lemma, the proof of which can be found in Appendix B.

Lemma 4.1. *Are equivalent:*

1. *There exist $C, M > 0$ such that*

$$|k + c_0\mu_r(\eta) - iq| \geq C(|k| + \langle\eta\rangle)^{-M},$$

for all $k \in \mathbb{Z}$, $[\eta] \in \widehat{G}$, $1 \leq r \leq d_\eta$, whenever $k + c_0\mu_r(\eta) - iq \neq 0$.

2. *There exist $C, M > 0$ such that*

$$\left|1 - e^{\pm 2\pi i(c_0\mu_r(\eta) - iq)}\right| \geq C\langle\eta\rangle^{-M}, \tag{4.5}$$

for all $[\eta] \in \widehat{G}$, $1 \leq r \leq d_\eta$, whenever $c_0\mu_r(\eta) - iq \notin \mathbb{Z}$.

4.1.1 Necessary conditions

Proposition 4.2. *If L_q is globally hypoelliptic, then L_{q_0} is globally hypoelliptic.*

Proof. Assume that L_{q_0} is not globally hypoelliptic. By Theorem 2.26 we have two cases to consider:

(i) The set

$$\mathcal{N} = \{(k,[\eta]) \in \mathbb{Z} \times \widehat{G}; k + c_0\mu_r(\eta) - iq = 0, \text{ for some } 1 \leq r \leq d_\eta\}$$

has infinitely many elements or;

(ii) for all $M > 0$, there exists $k_M \in \mathbb{Z}$ and $[\eta_M] \in \widehat{G}$ satisfying

$$0 < |k_M + c_0\mu_r(\eta_M) - iq| \leq (|k| + \langle\eta_M\rangle)^{-M},$$

for some $1 \leq r \leq d_{\eta_M}$.

Case (i): Assume that there exists a sequence $[\eta_k] \in \widehat{G}$ such that $c_0\mu_r(\eta_k) - iq \in \mathbb{Z}$, for some $1 \leq r \leq d_{\eta_k}$. for all $k \in \mathbb{N}$. We may assume without loss of generality that $r = 1$ for all $[\eta_k] \in \widehat{G}$. For each $k \in \mathbb{N}$, let $t_k \in [0, 2\pi]$ such that

$$m_k := \max_{t\in[0,2\pi]} \int_0^t (\mathrm{Re}\,(q) - \mu_1(\eta_k)b(s))\,ds = \int_0^{t_k} (\mathrm{Re}\,(q) - \mu_1(\eta_k)b(s))\,ds.$$

Set

$$\widehat{u}(t,\eta)_{rs} = \begin{cases} e^{m_k} \exp\left\{-\int_0^t (i\mu_1(\eta_k)c(s) + q)\,ds\right\}, & \text{if } [\eta] = [\eta_k] \text{ and } r = s = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $c_0\mu_1(\eta_k) - iq \in \mathbb{Z}$, for all $k \in \mathbb{N}$, the sequence of functions $\{\widehat{u}(t,\eta)_{rs}\}$ is well-defined on $\mathbb{T}^1$. Notice that

$$|\widehat{u}(t,\eta_k)_{11}| = \left| e^{m_k} \exp\left\{-\int_0^t (\mathrm{Re}\,(q) - \mu_1(\eta_k)b(s))\,ds\right\}\right| \leq 1,$$

by the definition of m_k. By Theorem A.5, we have that $u \in \mathcal{D}'(\mathbb{T}^1 \times G)$. Moreover, we have

$$|\widehat{u}(t_k,\eta_k)_{11}| = 1,$$

for all $k \in \mathbb{N}$. By Theorem A.4 we conclude $u \notin C^\infty(\mathbb{T}^1 \times G)$. Since each element of the sequence $\{\widehat{u}(t,\eta)_{rs}\}$ satisfies

$$\partial_t\{\widehat{u}(t,\eta)_{rs}\} + i(\mu_r(\eta)c(t) - iq)\{\widehat{u}(t,\eta)_{rs}\} = 0,$$

for all $[\eta] \in \widehat{G}$, $1 \leq r, s \leq d_\eta$, we conclude that $L_q u = 0$, which implies that L_q is not globally hypoelliptic.

Case (ii): By the equivalence given in Lemma 4.1, we can construct a sequence $[\eta_k]$ satisfying for all $k \in \mathbb{N}$

$$0 < |1 - e^{-2\pi i(c_0\mu_r(\eta_k) - iq)}| < \langle \eta_k \rangle^{-k}, \tag{4.6}$$

for some $1 \leq r \leq d_{\eta_k}$. We may assume $r = 1$ for convenience of notation and $c_0\mu_1(\eta_k) - iq \notin \mathbb{Z}$ for all $k \in \mathbb{Z}$, because $\mathcal{N}$ is finite.

For each $k \in \mathbb{N}$, choose $t_k \in [0, 2\pi]$ such that

$$\max_{t\in[0,2\pi]} \int_0^t (\mu_1(\eta_k)b(s) - \mathrm{Re}\,(q))\,ds = \int_0^{t_k} (\mu_1(\eta_k)b(s) - \mathrm{Re}\,(q))\,ds.$$

Notice that with this choice we have

$$\int_{t_k}^t (\mu_1(\eta_k)b(s) - \mathrm{Re}\,(q))\,ds \leq 0, \quad \text{for all } t \in [0, 2\pi]. \tag{4.7}$$

By the compactness of the torus, we may assume, by passing to a subsequence, that there exists $t_0 \in [0, 2\pi]$ such that $t_k \to t_0$, as $k \to \infty$.

Let $\varphi \in C^\infty(\mathbb{T}^1)$ be a real-valued smooth function satisfying $\operatorname{supp}(\varphi) \subseteq I$, $0 \leq \varphi(t) \leq 1$, and $\int_t^{2\pi} \varphi(s)\, ds > 0$, where I is a closed interval in $(0, 2\pi)$ such that $t_0 \notin I$.

Consider

$$\widehat{f}(t,\eta)_{rs} = \begin{cases} c_k \exp\left\{-i \int_{t_k}^t (\mu_1(\eta_k)c(w) - iq)\, dw\right\} \varphi(t), & \text{if } [\eta] = [\eta_k], \text{ and } r = s = 1, \\ 0, & \text{otherwise,} \end{cases}$$

for $t \in [0, 2\pi]$, where $c_k := 1 - e^{-2\pi i(c_0\mu_1(\eta_k) - iq)}$. Since $\operatorname{supp}(\varphi) \subseteq I$, the sequence $\{\widehat{f}(t,\eta)_{rs}\}$ is well-defined on $\mathbb{T}^1$. Let us show that $\{\widehat{f}(t,\eta)_{rs}\}$ defines a smooth function on $\mathbb{T}^1 \times G$. For $\alpha \in \mathbb{N}$ we have

$$\left|\partial_t^\alpha \widehat{f}(t,\eta_k)_{11}\right| = \left|1 - e^{-2\pi i(c_0\mu_1(\eta_k) - iq)}\right| \left|\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial_t^\beta \exp\left\{-i \int_{t_k}^t (\mu_1(\eta_k)c(w) - iq)\, dw\right\} \partial^{\alpha-\beta}\varphi(t)\right|.$$

By Faà di Bruno's formula we have

$$\partial_t^\beta \exp\left\{-i \int_{t_k}^t (\mu_1(\eta_k)c(w) - iq)\, dw\right\} = \sum_{\gamma \in \Delta(\beta)} \frac{\beta!}{\gamma!} \exp\left\{-i \int_{t_k}^t (\mu_1(\eta_k)c(w) - iq)\, dw\right\} \times \prod_{j=1}^{\beta} \left(\frac{-i\partial_t^j \int_{t_k}^t (\mu_1(\eta_k)c(w) - iq)\, dw}{j!}\right)^{\gamma_j},$$

where $\Delta(\beta) = \left\{\gamma \in \mathbb{N}_0^\beta;\ \sum_{j=1}^{\beta} j\gamma_j = \beta\right\}$.

Since for all $k \in \mathbb{N}$ we have $|\mu_1(\eta_k)| \leq \langle \eta_k \rangle$ and $\int_{t_k}^t (\operatorname{Re}(q) - \mu_1(\eta_k)b(w))\, dw \leq 0$, for all $t \in [0, 2\pi]$, we obtain

$$\partial_t^\beta \exp\left\{-i \int_{t_k}^t (\mu_1(\eta_k)c(w) - iq)\, dw\right\} \leq C_\beta \langle \eta_k \rangle^\beta,$$

for some $C_\beta > 0$. By (4.6) we obtain

$$\left|\partial_t^\alpha \widehat{f}(t,\eta_k)_{11}\right| \leq C_\alpha \langle \eta_k \rangle^{\alpha - k},$$

for some $C_\alpha > 0$. By Theorem A.4 we conclude that $f \in C^\infty(\mathbb{T}^1 \times G)$.

Let us construct now a distribution $u \in \mathcal{D}'(\mathbb{T}^1 \times G) \setminus C^\infty(\mathbb{T}^1 \times G)$ satisfying $L_q u = f$. By (4.3), set

$$\widehat{u}(t,\eta_k)_{11} = \frac{1}{1 - e^{-2\pi i(c_0\mu_1(\eta_k) - iq)}} \int_0^{2\pi} \exp\left\{-i \int_{t-\tau}^t (c(w)\mu_1(\eta_k) - iq)\, dw\right\} \widehat{f}(t - \tau, \eta_k)_{11}\, d\tau$$

and $\widehat{u}(t,\eta)_{rs}=0$ for all the other cases. For $t-\tau\geq 0$ we have

$$\begin{aligned}\widehat{u}(t,\eta_k)_{11} &= \int_0^{2\pi}\exp\left\{-i\int_{t-\tau}^{t}(c(w)\mu_1(\eta_k)-iq)\,dw - i\int_{t_k}^{t-\tau}(\mu_1(\eta_k)c(w)-iq)\,dw\right\}\varphi(t-\tau)\,d\tau\\ &= \exp\left\{-i\int_{t_k}^{t}(c(w)\mu_1(\eta_k)-iq)\,dw\right\}\int_0^{2\pi}\varphi(t-\tau)\,d\tau. \end{aligned}\tag{4.8}$$

We have $0\leq\varphi(t)\leq 1$, so by (4.7) we obtain

$$|\widehat{u}(t,\eta_k)_{11}|\leq 2\pi\exp\left\{\int_{t_k}^{t}(\mu_1(\eta_k)b(w)-\operatorname{Re}(q))\,dw\right\}\leq 2\pi.$$

For $t-\tau<0$ we need to use the 2π–periodic extension of f on de definition of $\widehat{u}(t,\eta_k)_{11}$. Hence,

$$\begin{aligned}\widehat{u}(t,\eta_k)_{11} &= \int_0^{2\pi}\exp\left\{-i\int_{t-\tau}^{t}(c(w)\mu_1(\eta_k)-iq)\,dw - i\int_{t_k}^{t-\tau+2\pi}(\mu_1(\eta_k)c(w)-iq)\,dw\right\}\\ &\quad\times\varphi(t-\tau+2\pi)\,d\tau\\ &= \exp\left\{-i\int_{t_k}^{t}(c(w)\mu_1(\eta_k)-iq)\,dw + 2\pi i(c_0\mu_1(\eta_k)-iq)\right\}\int_0^{2\pi}\varphi(t-\tau+2\pi)\,d\tau. \end{aligned}\tag{4.9}$$

Similar to the previous case, we have

$$|\widehat{u}(t,\eta_k)_{11}|\leq 2\pi\exp\{2\pi(\operatorname{Re}(q)-b_0\mu_1(\eta_k))\}\leq 4\pi,$$

for sufficiently large k, where the last inequality comes from the fact that by (4.6) we have $|e^{-2\pi i(c_0\mu_1(\eta_k)-iq)}|\to 1$, when $k\to\infty$. By Theorem A.5, we have $u\in\mathcal{D}'(\mathbb{T}^1\times G)$. Notice that if $t_0>\sup I$, then $t_k>\sup I$, for k sufficiently large, which implies that $t_k-\tau\geq 0$, for every $\tau\in\operatorname{supp}(\varphi)$. By (4.8) we obtain

$$|\widehat{u}(t_k,\eta_k)_{11}| = \int_0^{2\pi}\varphi(t_k-\tau)\,d\tau = \|\varphi\|_{L^1(\mathbb{T}^1)}>0.$$

On the other hand, if $t_0<\inf I$, we have for k sufficiently large. that $t_k<\tau$, for every $\tau\in\operatorname{supp}(\varphi)$. By (4.9) we have

$$|\widehat{u}(t_k,\eta_k)_{11}| = \exp\{2\pi(\operatorname{Re}(q)-b_0\mu_1(\eta_k))\}\int_0^{2\pi}\varphi(t_k-\tau+2\pi)\,d\tau > \tfrac{1}{2}\|\varphi\|_{L^1(\mathbb{T}^1)}>0.$$

By Theorem A.4 we conclude that $u\notin C^\infty(\mathbb{T}^1\times G)$. Therefore L_q is not globally hypoelliptic.

□

For the next theorem we will assume an additional hypothesis about the eigenvalues of the symbol of X. Precisely:

Hypothesis A: Assume that there exist $0 < C < 1$ and a sequence $\{[\eta_j]\}_{j\in\mathbb{N}}$ in $\widehat{G}$ such that for all $j \in \mathbb{N}$ we have

$$C\langle\eta_j\rangle \leq |\mu_r(\eta_j)|,$$

for some $1 \leq r \leq d_{\eta_j}$. We may assume without loss of generality that $r = d_{\eta_j}$ and $\mu_{d_{\eta_j}}(\eta_j) > 0$, for all $j \in \mathbb{N}$. Hence, $\mu_{d_{\eta_j}}(\eta_j) \overset{j\to\infty}{\longrightarrow} \infty$.

Remark 4.3. *When $G = \mathbb{T}^1$ and $X = \partial_x$, we have $\mu_1(k) = k$ and $\langle k\rangle = \sqrt{1+k^2}$, for all $k \in \mathbb{Z}$. Thus*

$$\tfrac{1}{2}\langle k\rangle \leq |k| \leq \langle k\rangle, \quad \forall k \in \mathbb{Z}\setminus\{0\}.$$

For $G = \mathbb{S}^3$ and X the usual vector field that we are studying, we have $\mu_\ell(\ell) = \ell$ and $\langle\ell\rangle = \sqrt{1+\ell(\ell+1)}$, for all $\ell \in \frac{1}{2}\mathbb{N}_0$. Therefore

$$\tfrac{1}{2}\langle\ell\rangle \leq \ell \leq \langle\ell\rangle, \quad \forall \ell \in \tfrac{1}{2}\mathbb{N}.$$

Theorem 4.4. *Assume that $b \not\equiv 0$ and that Hypothesis A holds. If $L_q = \partial_t + (a_0 + ib(t))X + q$ is globally hypoelliptic then b does not change sign.*

Proof. Suppose that b change sign and $b_0 > 0$. Consider

$$G(t,\tau) = \int_t^{t+\tau} (a_0 + ib(w))\,dw = a_0\tau + i\int_t^{t+\tau} b(w)dw, \quad t,\tau \in [0,2\pi]$$

and define

$$B = \min_{0\leq t,\tau\leq 2\pi} \operatorname{Im}(G(t,\tau)) = \operatorname{Im}(G(t_0,\tau_0)) = \int_{t_0}^{t_0+\tau_0} b(w)\,dw.$$

Since b change sign, we have $B < 0$. Moreover, we can consider $t_0, \tau_0 \in (0,2\pi)$ and $b(0) \neq 0$. It can be shown that $b(t_0+\tau_0) = 0$, which implies that $t_0 + \tau_0 \in (0,2\pi)$.

Let $\varphi \in C^\infty(\mathbb{T}^1)$ such that $\operatorname{supp}(\varphi) \subset [t_0+\tau_0-\delta, t_0+\tau_0+\delta] \subset (0,t_0)$ with $\varphi(t) \equiv 1$ for $t \in [t_0+\tau_0-\delta/2, t_0+\tau_0+\delta/2]$ and $0 \leq \varphi(t) \leq 1$.

Let us construct a distribution $u \in \mathcal{D}'(\mathbb{T}^1\times G)\backslash C^\infty(\mathbb{T}^1\times G)$ such that $L_q u = f \in C^\infty(\mathbb{T}^1\times G)$. From Hypothesis A, there exist $0 < C < 1$ and a sequence $\{[\eta_j]\}_{j\in\mathbb{N}}$ in $\widehat{G}$ such that

$$C\langle\eta_j\rangle \leq \mu_{d_{\eta_j}}(\eta_j), \tag{4.10}$$

for all $j \in \mathbb{N}$. By Proposition 4.2 we have that L_{q_0} is globally hypoelliptic. In particular, the set

$$\mathcal{N} = \{[\eta] \in \widehat{G}; \mu_r(\eta)c_0 - iq \in \mathbb{Z}, \text{ for some } 1 \leq r \leq d_\eta\}$$

is finite and we may assume that $[\eta_j] \notin \mathbb{Z}$, for all $j \in \mathbb{N}$. Define

$$\widehat{f}(t,\eta)_{rs} = \begin{cases} \tilde{c}_k e^{B\mu_r(\eta_j)} \varphi(t) e^{-i\mu_r(\eta_j)a_0(t-t_0)}, & \text{if } [\eta] = [\eta_j], \text{ for some } j \in \mathbb{N}, \text{ and } r = s = d_{\eta_j}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\tilde{c}_k := e^{2\pi i(\mu_r(\eta_j)c_0 - iq)} - 1$. In order to prove that the sequence $\{\widehat{f}(t,\eta)_{rs}\}$ defines a smooth function in $\mathbb{T}^1 \times G$, it is enough to consider the representations $[\eta_j]$ and the components $r = s = d_{\eta_j}$.

Notice that

$$\begin{aligned} |\partial_t^\alpha \widehat{f}(t,\eta_j)_{d_{\eta_j}d_{\eta_j}}| &= \left| (e^{2\pi i(\mu_{d_{\eta_j}}(\eta_j)c_0 - iq)} - 1) e^{B\mu_{d_{\eta_j}}(\eta_j)} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial_t^\beta e^{-i\mu_{d_{\eta_j}}(\eta_j)a_0(t-t_0)} \partial_t^{\alpha-\beta}\varphi(t) \right| \\ &\le \left| e^{2\pi i(\mu_{d_{\eta_j}}(\eta_j)c_0 - iq)} - 1 \right| e^{B\mu_{d_{\eta_j}}(\eta_j)} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left| \partial_t^\beta e^{-i\mu_{d_{\eta_j}}(\eta_j)a_0(t-t_0)} \right| \left| \partial_t^{\alpha-\beta}\varphi(t) \right|. \end{aligned}$$

Observe that

$$\left| e^{2\pi i(\mu_{d_{\eta_j}}(\eta_j)c_0 - iq)} - 1 \right| \le \left| e^{2\pi i(\mu_{d_{\eta_j}}(\eta_j)c_0 - iq)} \right| + 1 \le e^{2\pi(-\mu_{d_{\eta_j}}(\eta_j)b_0 + \mathrm{Re}(q))} + 1 \le C,$$

for some $C > 0$, because $b_0 \ge 0$ and $\mu_{d_{\eta_j}}(\eta_j) \to \infty$. Notice that

$$\left| \partial_t^\beta e^{-i\mu_{d_{\eta_j}}(\eta_j)a_0(t-t_0)} \right| = \left| (-i\mu_{d_{\eta_j}}(\eta_j)a_0)^\beta e^{-i\mu_{d_{\eta_j}}(\eta_j)a_0(t-t_0)} \right| \le C_\beta \langle \eta_j \rangle^\beta.$$

Moreover, since $B < 0$, we obtain from (4.10)

$$e^{B\mu_{d_{\eta_j}}(\eta_j)} \le e^{CB\langle \eta_j \rangle}.$$

Hence,

$$|\partial_t^\alpha \widehat{f}(t,\eta_j)_{d_{\eta_j}d_{\eta_j}}| \le C_\alpha e^{CB\langle \eta_j \rangle} \langle \eta_j \rangle^\alpha.$$

Since $B < 0$, for any $N > 0$ there exists $C_{\alpha N}$ such that

$$|\partial_t^\alpha \widehat{f}(t,\eta)_{d_\eta d_\eta}| \le C_{\alpha N} \langle \eta \rangle^{-N}.$$

By Theorem A.4, we conclude that $f \in C^\infty(\mathbb{T}^1 \times G)$. If $L_q u = f$, then

$$\widehat{f}(t,\eta)_{rs} = \widehat{L_q u}(t,\eta)_{rs} = \partial_t \widehat{u}(t,\eta)_{rs} + i\mu_r(\eta)(a_0 + ib(t) - iq)\widehat{u}(t,\eta)_{rs}, \tag{4.11}$$

for $1 \le r, s, \le d_\eta$. Since $\mu_{d_{\eta_j}}(\eta_j)c_0 - iq \notin \mathbb{Z}$, by (4.4) we obtain

$$\begin{aligned} \widehat{u}(t,\eta_j)_{d_{\eta_j}d_{\eta_j}} &= \frac{1}{e^{2\pi i(\mu_{d_{\eta_j}}(\eta_j)c_0 - iq)} - 1} \int_0^{2\pi} e^{q\tau} e^{i\mu_{d_{\eta_j}}(\eta_j)G(t,\tau)} \widehat{f}(t+\tau,\eta_j)_{rs}\, d\tau \\ &= e^{-i\mu_{d_{\eta_j}}(\eta_j)a_0(t-t_0)} \int_0^{2\pi} e^{q\tau} e^{\mu_{d_{\eta_j}}(\eta_j)(B - \mathrm{Im}(G(t,\tau)))} \varphi(t+\tau)\, d\tau. \end{aligned}$$

In all the other cases set $\widehat{u}(t,\eta)_{rs}=0$. First, let us show that the sequence $\{\widehat{u}(t,\eta)_{rs}\}$ defines a distribution $u\in\mathcal{D}'(\mathbb{T}^1\times G)$, where

$$u=\sum_{[\eta]\in\widehat{G}} d_\eta \sum_{r,s=1}^{d_\eta}\widehat{u}(t,\eta)_{rs}\eta_{sr}.$$

In order to apply the Theorem A.5, it is enough to consider the case where $[\eta]=[\eta_j]$ for some $j\in\mathbb{N}$ and $r=s=d_{\eta_j}$ because the other cases are well-controlled.

Let $\psi\in C^\infty(\mathbb{T}^1)$, then

$$\begin{aligned}
|(\widehat{u}(t,\eta_j)_{rs},\psi)| &= \left|\int_0^{2\pi} e^{-i\mu_{d_{\eta_j}}(\eta_j)a_0(t-t_0)}\int_0^{2\pi} e^{q\tau}e^{\mu_{d_{\eta_j}}(\eta_j)(B-\mathrm{Im}(G(t,\tau)))}\varphi(t+\tau)\,d\tau\,\psi(t)\,dt\right| \\
&\leq \int_0^{2\pi}\int_0^{2\pi} e^{\mathrm{Re}(q)\tau}e^{\mu_{d_{\eta_j}}(\eta_j)(B-\mathrm{Im}(G(t,\tau)))}|\varphi(t+\tau)|\,|\psi(t)|\,d\tau dt \\
&\leq (2\pi)^2\|\varphi\|_\infty\|\psi\|_\infty \\
&\leq Kp_1(\psi)\langle\eta_j\rangle.
\end{aligned}$$

Notice that here we have used the fact that $\mu_{d_{\eta_j}}(\eta_j)(B-\mathrm{Im}\,(G(t,\tau)))\leq 1$. Therefore $u\in\mathcal{D}'(\mathbb{T}^1\times G)$. Consider the function

$$\theta(\tau)=B-\mathrm{Im}\,(G(t_0,\tau))=B-\int_{t_0}^{t_0+\tau} b(w)\,dw.$$

We may consider δ small enough in the properties of φ such that either $\cos(\mathrm{Im}\,(q))$ or $\sin(\mathrm{Im}\,(q))$ does not change sign on $(\tau_0+\delta,\tau_0+\delta)$. Assume without loss of generality that $\sin(\mathrm{Im}\,(q))\geq 0$ on $(\tau_0+\delta,\tau_0+\delta)$. Thus

$$\begin{aligned}
|\widehat{u}(t_0,\eta_j)_{d_{\eta_j}d_{\eta_j}}| &= \left|\int_0^{2\pi} e^{q\tau}e^{\mu_{d_{\eta_j}}(\eta_j)\theta(\tau)}\varphi(t_0+\tau)\,d\tau\right| \\
&\geq \int_{\tau_0-\delta}^{\tau_0+\delta} e^{\mathrm{Re}(q)\tau}\sin(\mathrm{Im}\,(q)\,\tau)e^{\mu_{d_{\eta_j}}(\eta_j)\theta(\tau)}\varphi(t_0+\tau)\,d\tau \\
&\geq \int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^{\mathrm{Re}(q)\tau}\sin(\mathrm{Im}\,(q)\,\tau)e^{\mu_{d_{\eta_j}}(\eta_j)\theta(\tau)}\,d\tau \\
&\geq K\int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^{\langle\eta_j\rangle\theta(\tau)}\,d\tau,
\end{aligned}$$

where we use the fact that $\theta(\tau)\leq 0$, for all $\tau\in[0,2\pi]$, $\mu_{d_{\eta_j}}(\eta_j)\leq\langle\eta_j\rangle$ for all $[\eta_j]\in\widehat{G}$, and there exists $K>0$ such that $e^{\mathrm{Re}(q)\tau}\sin(\mathrm{Im}\,(q)\,\tau)\geq K$ on $[\tau_0-\delta/2,\tau_0+\delta/2]$.

Let us analyze the behavior of the function

$$J(\eta_j)=\int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^{\langle\eta_j\rangle\theta(\tau)}\,d\tau$$

when $\langle \eta_j \rangle \to \infty$. We have

$$\theta(\tau_0) = B - \int_{t_0}^{t_0+\tau_0} b(w)\, dw = B - B = 0$$

and

$$\theta'(\tau_0) = -b(t_0 + \tau_0) = 0.$$

Thus by Taylor's formula, we have

$$\theta(\tau_0 + h) = \theta(\tau_0) + \theta'(\tau_0)h + \frac{1}{2}\theta''(\tau_0 + \theta(h))h^2 = \frac{1}{2}\theta''(\tau_0 + \theta(h))h^2,$$

for $h \in (\tau_0 - \delta/2, \tau_0 + \delta/2)$ and $\theta(h) \in [\tau_0 - \delta/2, \tau_0 + \delta/2]$. Let

$$M := \sup_{\tau_0 - \delta \le y \le \tau_0 + \delta} \left| \frac{\theta''(y)}{2} \right|.$$

If $M = 0$ then $\theta \equiv 0$ in $[\tau_0 - \delta/2, \tau_0 + \delta/2]$. Thus

$$\int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^{\langle \eta_j \rangle \theta(\tau)}\, d\tau = \int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^0\, d\tau = \delta \ge \frac{C_1}{\sqrt{\langle \eta_j \rangle}},$$

for some $C_1 > 0$.

If $M > 0$, then

$$-\theta(\tau_0 + h) = -\frac{1}{2}\theta''(\tau_0 + \theta(h))h^2 \le Mh^2.$$

So

$$\langle \eta_j \rangle \theta(\tau_0 + h) \ge -M\langle \eta_j \rangle h^2.$$

Thereby

$$\int_{\tau_0-\delta/2}^{\tau_0+\delta/2} e^{\langle \eta_j \rangle \theta(\tau)}\, d\tau = \int_{-\delta/2}^{\delta/2} e^{\langle \eta_j \rangle \theta(\tau_0+h)}\, dh \ge \int_{-\delta/2}^{\delta/2} e^{-M\langle \eta_j \rangle h^2}\, dh \ge \frac{C_2}{\sqrt{\langle \eta_j \rangle}},$$

for some $C_2 > 0$. Considering $C = \max\{KC_1, KC_2\}$, we have

$$|\widehat{u}(t_0, \eta_j)_{d_{\eta_j} d_{\eta_j}}| \ge \frac{C}{\sqrt{\langle \eta_j \rangle}},$$

for all $[\eta_j] \in \widehat{G}$ such that $o_X(\eta_j) = \mu_{d_{\eta_j}}(\eta_j)$. Therefore $u \notin C^\infty(\mathbb{T}^1 \times G)$.

The case where $b_0 < 0$ is analogous to the previous one, but needs some adaptions. Here we take

$$\widetilde{B} := \max_{0 \le t, \tau \le 2\pi} \mathrm{Im}\,(H(t, \tau)) = \mathrm{Im}\,(H(t_1, \tau_1)) = \int_{t_1-\tau_1}^{t_1} b(w)\, dw.$$

Since b change sign, then $\widetilde{B} > 0$. For $r = s = d_{\eta_j}$, define

$$\widehat{f}(t, \eta_j)_{rs} = (1 - e^{-2\pi i(\mu_r(\eta_j)c_0 - iq)})e^{-\widetilde{B}\mu_r(\eta_j)}\widetilde{\varphi}(t)e^{-i\mu_r(\eta_j)a_0(t-t_1)},$$

where $\widetilde{\varphi} \in C^\infty(\mathbb{T}^1)$ satisfies similar properties of φ. One can shows that $f \in C^\infty(\mathbb{T}^1 \times G)$ and there exists $u \in \mathcal{D}'(\mathbb{T}^1 \times G) \backslash C^\infty(\mathbb{T}^1 \times G)$ such that $L_q u = f$. For this, define for $r = s = d_{\eta_j}$

$$\widehat{u}(t, \eta_j)_{rs} = e^{-i\mu_r(\eta_j)a_0(t-t_1)} \int_0^{2\pi} e^{\mu_r(\eta_j)(\mathrm{Im}(H(t,\tau))-\widetilde{B})} \widetilde{\varphi}(t-\tau)\, d\tau.$$

The proof that $u \in \mathcal{D}'(\mathbb{T}^1 \times G) \backslash C^\infty(\mathbb{T}^1 \times G)$ is similar to the previous case and it will be omitted. □

4.1.2 Sufficient conditions

In view of Proposition 4.2, from now we will assume that L_{q_0} is global hypoelliptic. By Theorem 2.26, this assumption implies that the set

$$\mathcal{N} = \{(k, [\eta]) \in \mathbb{Z} \times \widehat{G}; k + c_0\mu_r(\eta) - iq = 0, \text{ for some } 1 \leq r \leq d_\eta\} \tag{4.12}$$

is finite and there exist $C, M > 0$ such that

$$|k + c_0\mu_r(\eta) - iq| \geq C(|k| + \langle\eta\rangle)^{-M}, \tag{4.13}$$

for all $k \in \mathbb{Z}$, $[\eta] \in \widehat{G}$, $1 \leq r \leq d_\eta$, whenever $k + c_0\mu_r(\eta) - iq \neq 0$.

Theorem 4.5. *Assume that L_{q_0} is globally hypoelliptic and $b \not\equiv 0$. If b does not change sign then L_q is globally hypoelliptic.*

Proof. Assume that $b(t) \geq 0$ for all $t \in \mathbb{T}^1$. By hypothesis, $b_0 \neq 0$, where $c_0 = a_0 + ib_0$. Notice that the global hypoellipticity of L_{q_0} implies that $\mu_r(\eta)c_0 - iq \in \mathbb{Z}$ for only finitely many representations. So there is no loss of generality to assume that $\mu_r(\eta)c_0 - iq \notin \mathbb{Z}$. Let $f \in C^\infty(\mathbb{T}^1 \times G)$ such that $L_q u = f$, for some $u \in \mathcal{D}'(\mathbb{T}^1 \times G)$. Let us show that $u \in C^\infty(\mathbb{T}^1 \times G)$.

Define

$$H(\tau, t) = c_0\tau - C(t-\tau) + C(t).$$

For $\mu_r(\eta) < 0$, consider the solution (4.3):

$$\widehat{u}(t, \eta)_{rs} = \frac{1}{1 - e^{-2\pi i(\mu_r(\eta)c_0 - iq)}} \int_0^{2\pi} e^{-q\tau} e^{-i\mu_r(\eta)H(\tau,t)} \widehat{f}(t-\tau, \eta)_{rs} d\tau \tag{4.14}$$

and for $\mu_r(\eta) \geq 0$, consider the solution (4.4):

$$\widehat{u}(t, \eta)_{rs} = \frac{1}{e^{2\pi i(\mu_r(\eta)c_0 - iq)} - 1} \int_0^{2\pi} e^{q\tau} e^{-i\mu_r(\eta)H(-\tau,t)} \widehat{f}(t+\tau, \eta)_{rs} d\tau. \tag{4.15}$$

Notice that

$$\begin{aligned} H(\tau,t) &= c_0\tau - C(t-\tau) + C(t) \\ &= c_0\tau - \int_0^{t-\tau} c(w)dw + (t-\tau)c_0 + \int_0^t c(w)dw - tc_0 \\ &= \int_{t-\tau}^t c(w)dw \end{aligned}$$

So, using the fact that $b(t) \geq 0$, for all $t \in \mathbb{T}^1$, we obtain

$$\operatorname{Im}\left(H(\tau,t)\right) = \operatorname{Im}\left(\int_{t-\tau}^t c(w)dw\right) = \int_{t-\tau}^t b(w)dw \geq 0. \tag{4.16}$$

$$\operatorname{Im}\left(H(-\tau,t)\right) = \operatorname{Im}\left(\int_{t+\tau}^t c(w)dw\right) = \int_{t+\tau}^t b(w)dw \leq 0. \tag{4.17}$$

Notice that there exist $K > 0$ such that

$$\left|e^{\pm q\tau}\right| \leq K,$$

because $0 \leq \tau \leq 2\pi$. Let $\alpha \in \mathbb{N}_0$ and $\mu_r(\eta)c_0 - iq \notin \mathbb{Z}$. If $\mu_r(\eta) < 0$, by (4.14) we have

$$\begin{aligned} \left|\partial_t^\alpha \widehat{u}(t,\eta)_{rs}\right| &= \left|\frac{1}{1-e^{-2\pi i(\mu_r(\eta)c_0 - iq)}}\int_0^{2\pi} \partial_t^\alpha\left[e^{-q\tau}e^{-i\mu_r(\eta)H(\tau,t)}\widehat{f}(t-\tau,\eta)_{rs}\right]d\tau\right| \\ &\leq \left|\frac{1}{1-e^{-2\pi i(\mu_r(\eta)c_0 - iq)}}\right| \int_0^{2\pi} |e^{-q\tau}| \sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\left|\partial_t^\beta e^{-i\mu_r(\eta)H(\tau,t)}\right|\left|\partial_t^{\alpha-\beta}\widehat{f}(t-\tau,\eta)_{rs}\right|d\tau. \end{aligned}$$

By the assumption of the global hypoellipticity of L_{q_0}, we obtain from Lemma 4.1 constants $C, M > 0$ satisfying

$$|1 - e^{-2\pi i(\mu_r(\eta)c_0 - iq)}|^{-1} \leq C\langle\eta\rangle^M, \tag{4.18}$$

for all $[\eta] \in \widehat{G}$, $1 \leq r \leq d_\eta$, whenever $c_0\mu_r(\eta) - iq \notin \mathbb{Z}$. By Faà di Bruno's Formula, we have

$$\partial_t^\beta e^{-i\mu_r(\eta)H(t,t)} = \sum_{\gamma\in\Delta(\beta)} \frac{\beta!}{\gamma!}(-i\mu_r(\eta))^{|\gamma|}e^{-i\mu_r(\eta)H(\tau,t)}\prod_{j=1}^{\beta}\left(\frac{\partial_t^j H(\tau,t)}{j!}\right)^{\gamma_j},$$

where $\Delta(\beta) = \left\{\gamma \in \mathbb{N}_0^\beta;\ \sum_{j=1}^{\beta} j\gamma_j = \beta\right\}$. Hence,

$$\left|\partial_t^\beta e^{-i\mu_r(\eta)H(t,t)}\right| \leq \sum_{\gamma\in\Delta(\beta)} \frac{\beta!}{\gamma!}|\mu_r(\eta)|^{|\gamma|}e^{\mu_r(\eta)\operatorname{Im}(H(\tau,t))}\prod_{j=1}^{\beta}\left|\frac{\partial_t^j H(\tau,t)}{j!}\right|^{\gamma_j}.$$

Notice that by (1.9) we have

$$|\mu_r(\eta)|^{|\gamma|} \leq \langle\eta\rangle^{|\gamma|} \leq \langle\eta\rangle^{\beta},$$

for all $[\eta] \in \widehat{G}$, $1 \leq r \leq d_\eta$, and $\gamma \in \Delta(\beta)$. Moreover, by (4.16) we have

$$e^{\mu_r(\eta)\,\mathrm{Im}(H(\tau,t))} \leq 1.$$

Thus,

$$\left|\partial_t^\beta e^{-i\mu_r(\eta)H(t,t)}\right| \leq \langle\eta\rangle^\beta \sum_{\gamma\in\Delta(\beta)} \frac{\beta!}{\gamma!}\prod_{j=1}^{\beta}\left|\frac{\partial_t^j H(\tau,t)}{j!}\right|^{\gamma_j}.$$

By the continuity of the function H and the compactness of $\mathbb{T}^1$, for all $\beta \in \mathbb{N}_0$ there exists $C_\beta > 0$ such that

$$\sum_{\gamma\in\Delta(\beta)} \frac{\beta!}{\gamma!}\prod_{j=1}^{\beta}\left|\frac{\partial_t^j H(\tau,t)}{j!}\right|^{\gamma_j} \leq C_\beta,$$

for all $0 \leq t, \tau \leq 2\pi$.

Let $N > 0$. Since $f \in C^\infty(\mathbb{T}^1 \times G)$, by Theorem A.4 for every $\beta \leq \alpha$ there exists $C_{\beta N} > 0$ such that

$$|\partial_t^{\alpha-\beta}\widehat{f}(t,\eta)_{rs}| \leq C_{\beta N}\langle\eta\rangle^{-(N+\beta+M)},$$

for all $t \in \mathbb{T}^1$, with M as in (4.18). Therefore,

$$\begin{aligned}
\left|\partial_t^\alpha \widehat{u}(t,\eta)_{rs}\right| &\leq \left|\frac{1}{1-e^{-2\pi i(\mu_r(\eta)c_0 - iq)}}\right| \int_0^{2\pi} |e^{-q\tau}| \sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\left|\partial_t^\beta e^{-i\mu_r(\eta)H(\tau,t)}\right|\left|\partial_t^{\alpha-\beta}\widehat{f}(t-\tau,\eta)_{rs}\right| d\tau. \\
&\leq KC\langle\eta\rangle^M \int_0^{2\pi}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta} C_\beta\langle\eta\rangle^\beta C_{\beta N}\langle\eta\rangle^{-(N+\beta+M)}\, d\tau \\
&\leq C_{\alpha N}\langle\eta\rangle^{-N}.
\end{aligned}$$

We can obtain the same type of estimate when $\mu_r(\eta) \geq 0$. In this case, it is enough to consider the expression (4.15) to take the derivatives. We can adjust $C_{\alpha N}$, if necessary, to obtain

$$\left|\partial_t^\alpha \widehat{u}(t,\eta)_{rs}\right| \leq C_{\alpha N}\langle\eta\rangle^{-N},$$

for every $[\eta] \in \widehat{G}$, $1 \leq r, s \leq d_\eta$. By Theorem A.4 we conclude that $u \in C^\infty(\mathbb{T}^1 \times G)$. The case $b(t) \leq 0$, for all $t \in \mathbb{T}^1$, is totally analogue, just use (4.14) for $\mu_r(\eta) \geq 0$ and (4.15) for $\mu_r(\eta) < 0$. □

We can summarize the results obtained in this chapter as follows:

Theorem 4.6. *Let G be a compact Lie group and consider the operator $L_q : \mathcal{D}'(\mathbb{T}^1 \times G) \to \mathcal{D}'(\mathbb{T}^1 \times G)$ defined as*

$$L_q := \partial_t + c(t)X + q,$$

where $X \in \mathfrak{g}$, $c \in C^\infty(\mathbb{T}^1)$, and $q \in \mathbb{C}$. Assume that

a) $b \not\equiv 0$;

b) $L_{q_0} := \partial_t + c_0 X + q$ *is globally hypoelliptic, where* c_0 *is the average of* c;

c) there exist $0 < C < 1$ *and a sequence* $\{[\eta_j]\}_{j\in\mathbb{N}}$ *in* $\widehat{G}$ *such that for all* $j \in \mathbb{N}$ *we have*

$$C\langle \eta_j \rangle \leq |\mu_r(\eta_j)|,$$

for some $1 \leq r \leq d_{\eta_j}$.

Then L_q *is globally hypoelliptic if and only if* b *does not change sign.*

Example 4.7. *Let* G *be a compact Lie group and* $q \in i(\mathbb{R} \setminus \mathbb{Z})$. *The operator*

$$L_q := \partial_t + (e^{it} + i)X + q$$

is globally hypoelliptic. Indeed, we have $\operatorname{Im}(e^{it} + i) = \sin(t) + 1 \not\equiv 0$ *and the operator* $L_{q_0} = \partial_t + iX + q$ *is globally hypoelliptic by Theorem 2.26 because in this case we have*

$$\mathcal{N} = \{[\eta] \in \widehat{G}; i\mu_r(\eta) - iq \in \mathbb{Z}\} = \varnothing,$$

and

$$|k + i\mu_r(\eta) - iq| \geq |k - iq| \geq C,$$

for some $C > 0$, *for all* $k \in \mathbb{Z}$, $[\eta] \in \widehat{G}$, $1 \leq r \leq d_\eta$. *Since* $\operatorname{Im}(e^{it} + i) = \sin(t) + 1$ *does not change sign, by Theorem 4.5 we conclude that* L_q *is globally hypoelliptic.*

Example 4.8. *Let* $G = \mathbb{S}^3$ *and* $X \in \mathfrak{s}^3$ *a normalized vector field on* $\mathbb{S}^3$. *Let* $q \in i(\mathbb{R} \setminus \mathbb{Z})$, *and consider the operator*

$$L_q := \partial_t + (2e^{it} + i)X + q.$$

Notice that $\operatorname{Im}(2e^{it} + i) = 2\sin(t) + 1 \not\equiv 0$ *and the operator* $L_{q_0} = \partial_t + iX + q$ *is globally hypoelliptic (see previous example). Moreover, we have seen in Remark 4.3 that condition c) from Theorem 4.6 holds. Since* $\operatorname{Im}(2e^{it} + i) = 2\sin(t) + 1$ *changes sign, we conclude that* L_q *is not globally hypoelliptic.*

Part II

Komatsu classes case

Chapter 5

Constant coefficient vector fields

In Part I we have studied global properties for a class of operators defined in smooth functions. The next step is to extend these results for other classes of functions. For instance, in [1], [2], [3], [4], [8], [25] it was consider the Gevrey classes of functions.

Definition 5.1. *Let G be a compact Lie group and $s \geq 1$. The Gevrey–Roumieu class $\gamma^s(G)$ is the space of functions $f \in C^\infty(G)$ for which there exist constants $h > 0$ and $C > 0$ such that*

$$\|\partial^\alpha f\|_{L^2(G)} \leq Ch^{|\alpha|}|\alpha|!^s, \quad \alpha \in \mathbb{N}_0^d.$$

We refer [11] for a detail study of these spaces on compact Lie groups. Notice that when $s = 1$ we obtain the space of analytic functions on G. These spaces are well-defined on G because the compact Lie group G is an analytic manifold.

The Gevrey classes $\gamma^s(G)$ is an example of a Komatsu class when we consider the sequence $M_k = k!^s$ (see Section 1.2) , so we have decided to extend the results from Part I to Komatsu classes.

5.1 Global properties in Komatsu classes of Roumieu type

In this section we will study global hypoellipticity of the operator

$$L = X_1 + cX_2$$

on Komatsu Classes of Roumieu type.

If we restrict the operator $L = X_1 + cX_2$ to the Komatsu class of Roumieu type $\Gamma_{\{M_k\}}(G)$ we obtain an endomorphism, that is, $L : \Gamma_{\{M_k\}}(G) \to \Gamma_{\{M_k\}}(G)$. In this way, we can extend

the operator L to $u \in \Gamma'_{\{M_k\}}(G)$ as

$$\langle Lu, \varphi \rangle := -\langle u, L\varphi \rangle, \quad \forall \varphi \in \Gamma_{\{M_k\}}(G).$$

Definition 5.2. *Let G be a compact Lie group. We say that an operator $P : \Gamma'_{\{M_k\}}(G) \to \Gamma'_{\{M_k\}}(G)$ is globally $\Gamma_{\{M_k\}}$-hypoelliptic if the conditions $u \in \Gamma'_{\{M_k\}}(G)$ and $Pu \in \Gamma_{\{M_k\}}(G)$ imply that $u \in \Gamma_{\{M_k\}}(G)$.*

Theorem 5.3. *The operator $L = X_1 + cX_2$ is globally $\Gamma_{\{M_k\}}-$hypoelliptic if and only the following conditions are satisfied:*

1. *The set*

$$\mathcal{N} = \{([\xi], [\eta]) \in \widehat{G_1} \times \widehat{G_2};\ \lambda_m(\xi) + c\mu_r(\eta) = 0, \text{ for some } 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta\}$$

   is finite.

2. $\forall B > 0, \exists K_B > 0$ *such that*

$$|\lambda_m(\xi) + c\mu_r(\eta)| \geq K_B \exp\{-M(B(\langle \xi \rangle + \langle \eta \rangle))\}, \tag{5.1}$$

   for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m \leq d_\xi,\ 1 \leq r \leq d_\eta$ whenever $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$.

Proof. ($\Longleftarrow$) Suppose $Lu = f \in \Gamma_{\{M_k\}}(G)$ for some $u \in \Gamma_{\{M_k\}}(G)$. Since $\mathcal{N}$ is finite, it is enough to study the behavior of $\widehat{u}(\xi, \eta)_{mn_{rs}}$ outside of $\mathcal{N}$. If $([\xi], [\eta]) \notin \mathcal{N}$, by (2.3) we have

$$\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}} = -i(\lambda_m(\xi) + c\mu_r(\eta))^{-1} \widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}},$$

for all $1 \leq m \leq d_\xi$ and $1 \leq r \leq d_\eta$. Thus

$$\begin{aligned} |\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}}| &= |\lambda_m(\xi) + c\mu_r(\eta)|^{-1} |\widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}}| \\ &\leq C_N \exp\{M(N(\langle \xi \rangle + \langle \eta \rangle))\} |\widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}}| \end{aligned}$$

Since $f \in \Gamma_{\{M_k\}}(G)$, by (1.40) there exist constants $C, N' > 0$ such that

$$|\widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}}| \leq C \exp\{-M(N'(\langle \xi \rangle + \langle \eta \rangle))\}.$$

Hence

$$|\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}}| \leq C_N \exp\{M(N(\langle \xi \rangle + \langle \eta \rangle))\} \exp\{-M(N'(\langle \xi \rangle + \langle \eta \rangle))\}.$$

From (1.14), for $N = \frac{N'}{H}$, we obtain

$$\exp\{-M(N'(\langle\xi\rangle + \langle\eta\rangle))\} \leq A\exp\{-2M(N(\langle\xi\rangle + \langle\eta\rangle))\}.$$

Thus

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C\exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\},$$

Therefore $u \in \Gamma_{\{M_k\}}(G)$.

($\Longrightarrow$) Let us prove the result by contradiction. If (1) were not satisfied, by Lemma 2.2, there would be $u \in \mathcal{D}'(G)\backslash C^\infty(G)$, which implies that $u \in \Gamma'_{\{M_k\}}(G) \setminus \Gamma_{\{M_k\}}(G)$, such that $Lu = 0$, contradicting the hypothesis of global $\Gamma_{\{M_k\}}$-hypoellipticity of L. So, let us assume that 2. is not satisfied, then there exists $B > 0$ such that for all $K \in \mathbb{N}$ there exist $[\xi_K] \in \widehat{G_1}$ and $[\eta_K] \in \widehat{G_2}$ satisfying

$$0 < |\lambda_m(\xi_K) + c\mu_r(\eta_K)| < \tfrac{1}{K}\exp\{-M(N(\langle\xi_K\rangle + \langle\eta_K\rangle))\}, \tag{5.2}$$

for some $1 \leq m \leq d_{\xi_K}$ and $1 \leq r \leq d_{\eta_K}$. We can suppose that $([\xi_K],[\eta_K]) \notin \mathcal{N}$ and that $\langle\xi_j\rangle + \langle\eta_j\rangle \leq \langle\xi_\ell\rangle + \langle\eta_\ell\rangle$ when $j \leq \ell$.

Let $K \in \mathbb{N}$ and $\tilde{m}$ and $\tilde{r}$ such that (5.2) holds. Define

$$\widehat{\widehat{f}}(\xi_K,\eta_K)_{mn_{rs}} = \begin{cases} (\lambda_m(\xi_K) + c\mu_r(\eta_K))(\langle\xi_K\rangle + \langle\eta_K\rangle), & \text{if } mn = \tilde{m}1, rs = \tilde{r}1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $C > 0$ be obtained from (1.36) satisfying

$$(\langle\xi_K\rangle + \langle\eta_K\rangle)\exp\left\{-\tfrac{1}{2}M(N(\langle\xi_K\rangle + \langle\eta_K\rangle))\right\} < C,$$

for all $K \in \mathbb{N}$. Hence

$$\begin{aligned} |\widehat{\widehat{f}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}}| &= |\lambda_{\tilde{m}}(\xi_K) + c\mu_{\tilde{r}}(\eta_K)|(\langle\xi_K\rangle + \langle\eta_K\rangle) \\ &\leq \frac{1}{K}\exp\left\{-M(N(\langle\xi_K\rangle + \langle\eta_K\rangle))\right\}(\langle\xi_K\rangle + \langle\eta_K\rangle) \\ &\leq C\exp\{-M(N(\langle\xi_K\rangle + \langle\eta_K\rangle))\}\exp\{\tfrac{1}{2}M(N(\langle\xi_K\rangle + \langle\eta_K\rangle))\} \\ &\leq C\exp\{-M(\tilde{N}(\langle\xi_K\rangle + \langle\eta_K\rangle))\}, \end{aligned}$$

where $\tilde{N} = \frac{N}{H}$. Thus $f \in \Gamma_{\{M_k\}}(G)$.

By (2.3) and (2.2), if $Lu = f$ for some $u \in \Gamma'_{\{M_k\}}(G)$, we have

$$\widehat{\widehat{u}}(\xi_K,\eta_K)_{mn_{rs}} = \begin{cases} -i(\langle\xi_K\rangle + \langle\eta_K\rangle), & \text{if } mn = \tilde{m}1, rs = \tilde{r}1 \\ 0, & \text{otherwise.} \end{cases}$$

In particular,

$$\left|\widehat{\widehat{u}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}}\right| = \langle\xi_K\rangle + \langle\eta_K\rangle, \tag{5.3}$$

for all $K \in \mathbb{N}$. Thus

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq \langle\xi\rangle + \langle\eta\rangle,$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m,n \leq d_\xi$ and $1 \leq r,s \leq d_\eta$. Therefore $u \in \mathcal{D}'(G)$ and then $u \in \Gamma'_{\{M_k\}}(G)$. By (5.3) $u \notin C^\infty(G)$. Consequently $u \notin \Gamma_{\{M_k\}}(G)$, which contradicts the fact that L is globally $\Gamma_{\{M_k\}}$-hypoelliptic. □

Corollary 5.4. *If L is globally hypoelliptic, then L is globally $\Gamma_{\{M_k\}}$-hypoelliptic.*

Proof. By Theorem 2.3, if L is globally hypoelliptic, the set $\mathcal{N}$ is finite and there exist $C, N' > 0$ such that

$$|\lambda_m(\xi) + c\mu_r(\eta)| \geq C(\langle\xi\rangle + \langle\eta\rangle)^{-N'},$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m \leq d_\xi,\ 1 \leq r \leq d_\eta$, whenever $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$.

By (1.36), for every $N > 0$, there exits $C_N > 0$ such that

$$(\langle\xi\rangle + \langle\eta\rangle)^{N'} \exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\} \leq C_N.$$

Thus,

$$|\lambda_m(\xi) + c\mu_r(\eta)| \geq C_N \exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle)),$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m \leq d_\xi,\ 1 \leq r \leq d_\eta$, whenever $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$. By Theorem 5.3 the operator L is globally $\Gamma_{\{M_k\}}$-hypoelliptic.

□

For the case where $M_k = k!$, we obtain the class of analytic functions on G and we have $M(r) \simeq r$. Hence, we have the following characterization for the global analytic hypoellipticity of the operator L:

Theorem 5.5. *The operator $L = X_1 + cX_2$ is globally analytic hypoelliptic if and only the following conditions are satisfied:*

1. *The set*

$$\mathcal{N} = \{([\xi],[\eta]) \in \widehat{G_1} \times \widehat{G_2};\ \lambda_m(\xi) + c\mu_r(\eta) = 0, \textit{ for some } 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta\}$$

is finite.

2. $\forall B > 0, \exists K_B > 0$ *such that*

$$|\lambda_m(\xi) + c\mu_r(\eta)| \geq K_B \exp\{-B(\langle\xi\rangle + \langle\eta\rangle))\}, \tag{5.4}$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m \leq d_\xi$, $1 \leq r \leq d_\eta$ *whenever* $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$.

Now, to define global solvability for the operator L in the sense of Komatsu classes, observe that given an ultradifferentiable function (or ultradistribution) f defined on G, if $u \in \mathcal{D}'(G)$ is a solution of $Lu = f$, we obtain from (2.2) that

$$\lambda_m(\xi) + c\mu_r(\eta) = 0 \Longrightarrow \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = 0.$$

Therefore, let us consider the following set

$$\mathcal{K} := \{f \in \Gamma'_{\{M_k\}}(G);\ \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = 0 \text{ whenever } \lambda_m(\xi) + c\mu_r(\eta) = 0\}.$$

Clearly there are no $u \in \Gamma'_{\{M_k\}}(G)$ satisfying $Lu = f$ when $f \notin \mathcal{K}$.

Definition 5.6. *We say that the operator L is globally $\Gamma'_{\{M_k\}}$–solvable if $L(\Gamma'_{\{M_k\}}(G)) = \mathcal{K}$.*

Notice that $L(\Gamma'_{\{M_k\}}(G)) \subseteq \mathcal{K}$ and the next result give us the condition to obtain the other inclusion.

Theorem 5.7. *The operator $L = X_1 + cX_2$ is globally $\Gamma'_{\{M_k\}}(G)$-solvable if and only if (5.1) holds, that is, for all $N > 0$ there exists $C_N > 0$ such that*

$$|\lambda_m(\xi) + c\mu_r(\eta)| \geq C_N \exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}, 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta$ *whenever* $\lambda_m(\xi)+c\mu_r(\eta) \neq 0$. *Moreover, if L is globally $\Gamma'_{\{M_k\}}(G)$-solvable, for any admissible ultradifferentiable function $f \in \Gamma_{\{M_k\}}(G)$, there exists $u \in \Gamma_{\{M_k\}}(G)$ such that $Lu = f$.*

Proof. ($\Longleftarrow$) For each $f \in \mathcal{K}$ define

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 0, & \text{if } \lambda_m(\xi) + c\mu_r(\eta) = 0, \\ -i(\lambda_m(\xi) + c\mu_r(\eta))^{-1}\, \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}, & \text{otherwise.} \end{cases} \tag{5.5}$$

Let us show that $\{\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}\}$ is the sequence of Fourier coefficient of an ultradistribution $u \in \Gamma'_{\{M_k\}}(G)$. If $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$, by (5.1) we have

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &= |\lambda_m(\xi) + c\mu_r(\eta)|^{-1} |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\leq C_N \exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\} |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \end{aligned}$$

Using the fact that $f \in \Gamma'_{\{M_k\}}(G)$, we conclude that for all $N > 0$ and $N' > 0$, there exist $C_{NN'} > 0$ such that

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C_{NN'} \exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\} \exp\{M(N'(\langle\xi\rangle + \langle\eta\rangle))\},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m, n \leq d_\xi$ and $1 \leq r, s \leq d_\eta$.

Let $D > 0$. Choose $N = N' = \frac{D}{H}$. Using (1.14) we obtain

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &\leq C_D \exp\left\{2M\left(\tfrac{D}{H}(\langle\xi\rangle + \langle\eta\rangle)\right)\right\} \\ &\leq C_D \exp\{M(D(\langle\xi\rangle + \langle\eta\rangle))\}. \end{aligned}$$

Therefore $u \in \Gamma'_{\{M_k\}}(G)$ and $Lu = f$.

($\Longrightarrow$) Suppose that is not true, then there exists $N > 0$ such that for all $K \in \mathbb{N}$ there exist $[\xi_K] \in \widehat{G_1}$ and $[\eta_K] \in \widehat{G_2}$ satisfying

$$0 < |\lambda_{\tilde{m}}(\xi_K) + c\mu_{\tilde{r}}(\eta_K)| \leq \frac{1}{K} \exp\{-M(N(\langle\xi_K\rangle + \langle\eta_K\rangle))\}, \tag{5.6}$$

for some $1 \leq \tilde{m} \leq d_{\xi_K}$ and $1 \leq \tilde{r} \leq d_{\xi_K}$. We can assume that $\langle\xi_j\rangle + \langle\eta_j\rangle \leq \langle\xi_\ell\rangle + \langle\eta_\ell\rangle$ when $j \leq \ell$. Consider $f \in \mathcal{K}$ defined by

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 1, & \text{if } ([\xi],[\eta]) = ([\xi_j],[\eta_j]) \text{ for some } j \in \mathbb{N} \text{ and (5.6) is satisfied,} \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that there exits $u \in \Gamma'_{\{M_k\}}(G)$ such that $Lu = f$. In this way, its Fourier coefficients must satisfy

$$i(\lambda_m(\xi) + c\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}.$$

So

$$\begin{aligned} |\widehat{\widehat{u}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}}| &= |\lambda_{\tilde{m}}(\xi_K) + c\mu_{\tilde{r}}(\eta_K)|^{-1} |\widehat{\widehat{f}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}}| \\ &\geq K \exp\{M(N(\langle\xi_K\rangle + \langle\eta_K\rangle))\}, \end{aligned}$$

which, by Proposition 1.41, implies that $u \notin \Gamma'_{\{M_k\}}(G)$. Therefore L is not globally solvable.

Let us now prove the last part of the theorem. Let $f \in \mathcal{K} \cap \Gamma_{\{M_k\}}(G)$ and define u as in (5.5). Since L is globally $\Gamma'_{\{M_k\}}$–solvable, it holds (5.1) and then

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C_N \exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\} |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}|,$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m \leq d_\xi$, $1 \leq r \leq d_\eta$. By (1.40), there exist $C, N' > 0$ such that

$$|\widehat{\widehat{f}}\,(\xi,\eta)_{mn_{rs}}| \leq C \exp\{-M(N'(\langle\xi\rangle + \langle\eta\rangle))\}.$$

By (1.14), we have for $N = \frac{N'}{H}$ that

$$\begin{aligned}|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &\leq C\exp\left\{M\left(\tfrac{N'}{H}(\langle\xi\rangle+\langle\eta\rangle)\right)\right\}\exp\{-M(N'(\langle\xi\rangle+\langle\eta\rangle))\}\\ &\leq C\exp\left\{M\left(\tfrac{N'}{H}(\langle\xi\rangle+\langle\eta\rangle)\right)\right\}C\exp\left\{-2M\left(\tfrac{N'}{H}(\langle\xi\rangle+\langle\eta\rangle)\right)\right\}\\ &\leq C\exp\left\{-M\left(\tfrac{N'}{H}(\langle\xi\rangle+\langle\eta\rangle)\right)\right\}\end{aligned}$$

for all $[\xi]\in\widehat{G_1},\ [\eta]\in\widehat{G_2}, 1\leq m\leq d_\xi, 1\leq r\leq d_\eta$. Therefore $Lu=f$ and $u\in\Gamma_{\{M_k\}}(G)$. □

As in the smooth case we obtain the following corollary in Komatsu classes:

Corollary 5.8. *If L is globally $\Gamma_{\{M_k\}}$-hypoelliptic, then L is globally $\Gamma'_{\{M_k\}}$-solvable.*

With the same proof of Corollary 5.4 we obtain the following class of globally $\Gamma'_{\{M_k\}}$-solvable operators:

Corollary 5.9. *If L is globally C^∞-solvable, then L is globally $\Gamma'_{\{M_k\}}$-solvable.*

5.2 Global properties in Komatsu classes of Beurling type

Analogously to the Roumieu type case, restricting the operator $L=X_1+cX_2$ to the Komatsu class of Beurling type $\Gamma_{(M_k)}(G)$ we obtain an endomorphism, that is,

$$L:\Gamma_{(M_k)}(G)\to\Gamma_{(M_k)}(G).$$

In this way, we can extend the operator L to $u\in\Gamma'_{(M_k)}(G)$ as

$$\langle Lu,\varphi\rangle := -\langle u, L\varphi\rangle,\quad \forall\varphi\in\Gamma_{(M_k)}(G).$$

Definition 5.10. *Let G be a compact Lie group. We say that $P:\Gamma'_{(M_k)}(G)\to\Gamma'_{(M_k)}(G)$ is globally $\Gamma_{(M_k)}$-hypoelliptic if the conditions $u\in\Gamma'_{(M_k)}(G)$ and $Pu\in\Gamma_{(M_k)}(G)$ imply that $u\in\Gamma_{(M_k)}(G)$.*

Theorem 5.11. *The operator $L=X_1+cX_2$ is globally $\Gamma_{(M_k)}$-hypoelliptic on $G_1\times G_2$ if and only if the following conditions are satisfied:*

1. *The set*

$$\mathcal{N}=\{([\xi],[\eta])\in\widehat{G_1}\times\widehat{G_2};\ \lambda_m(\xi)+c\mu_r(\eta)=0, \text{ for some } 1\leq m\leq d_\xi, 1\leq r\leq d_\eta\}$$

is finite.

2. $\exists B > 0, K > 0$ *such that*

$$|\lambda_m(\xi) + c\mu_r(\eta)| \geq K \exp\{-M(B(\langle\xi\rangle + \langle\eta\rangle))\}, \tag{5.7}$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2}, 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta$ *whenever* $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$.

Proof. ($\Longleftarrow$) Suppose $Lu = f \in \Gamma_{(M_k)}(G)$ for some $u \in \Gamma'_{(M_k)}(G)$. Since $\mathcal{N}$ is finite, it is enough to study the behaviour of $\widehat{u}(\xi,\eta)_{mn_{rs}}$ outside of $\mathcal{N}$. If $([\xi],[\eta]) \notin \mathcal{N}$, then

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = -i(\lambda_m(\xi) + c\mu_r(\eta)^{-1}\, \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}},$$

for all $1 \leq m \leq d_\xi$ and $1 \leq r \leq d_\eta$. Thus

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &= |\lambda_m(\xi) + c\mu_r(\eta)|^{-1} |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\leq C \exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\} |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \end{aligned}$$

Since $f \in \Gamma_{(M_k)}(G)$, for every $N' > 0$, there exists $C_{N'} > 0$ such that

$$\|\widehat{\widehat{f}}(\xi,\eta)\|_{\mathtt{HS}} \leq C_{N'} \exp\{-M(N'(\langle\xi\rangle + \langle\eta\rangle))\}.$$

Hence

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C_{N'} \exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\} \exp\{-M(N'(\langle\xi\rangle + \langle\eta\rangle))\}.$$

Fix $D > 0$. If $N \leq D$, then

$$\exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\} \leq \exp\{M(D(\langle\xi\rangle + \langle\eta\rangle))\},$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2}$, because M is a non-decreasing function, as well the exponential. So

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C_{N'} \exp\{M(D(\langle\xi\rangle + \langle\eta\rangle))\} \exp\{-M(N'(\langle\xi\rangle + \langle\eta\rangle))\}.$$

Choose $N' = DH$. By (1.14) we have

$$\exp\{-M(DH(\langle\xi\rangle + \langle\eta\rangle))\} \leq A \exp\{-2M(D(\langle\xi\rangle + \langle\eta\rangle))\}.$$

Thus

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &\leq C_D \exp\{M(D(\langle\xi\rangle + \langle\eta\rangle))\} \exp\{-M(DH(\langle\xi\rangle + \langle\eta\rangle))\} \\ &\leq C_D \exp\{M(D(\langle\xi\rangle + \langle\eta\rangle))\} \exp\{-2M(D(\langle\xi\rangle + \langle\eta\rangle))\} \\ &\leq C_D \exp\{-M(D(\langle\xi\rangle + \langle\eta\rangle))\}. \end{aligned}$$

If $N > D$, choose $N' = NH$. Again by (1.14),

$$\exp\{-M(NH(\langle\xi\rangle + \langle\eta\rangle))\} \leq A\exp\{-2M(N(\langle\xi\rangle + \langle\eta\rangle))\}.$$

So

$$\begin{aligned}|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &\leq C\exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\}\exp\{-M(NH(\langle\xi\rangle + \langle\eta\rangle))\}\\ &\leq C\exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\}\exp\{-2M(N(\langle\xi\rangle + \langle\eta\rangle))\}\\ &\leq C\exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\}\\ &\leq C\exp\{-M(D(\langle\xi\rangle + \langle\eta\rangle))\}.\end{aligned}$$

Hence, for every $D > 0$, there exists $C_D > 0$ such that

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C_D\exp\{-M(D(\langle\xi\rangle + \langle\eta\rangle)),$$

for all $([\xi],[\eta]) \notin \mathcal{N}$. Therefore $u \in \Gamma_{(M_k)}(G)$.

$(\implies)$ Let us prove the result by contradiction. If (1) were not satisfied, by Lemma 2.2, there would be $u \in \mathcal{D}'(G)\backslash C^\infty(G)$, which implies that $u \in \Gamma'_{\{M_k\}}(G) \setminus \Gamma_{(M_k)}(G)$, such that $Lu = 0$, contradicting the hypothesis of global $\Gamma_{(M_k)}$-hypoellipticity of L. So, let us assume that (2) is not satisfied, then for every $K \in \mathbb{N}$, we can choose a $[\xi_K] \in \widehat{G_1}$ and a $[\eta_K] \in \widehat{G_2}$ such that

$$0 < |\lambda_{\tilde{m}}(\xi_K) + c\mu_{\tilde{r}}(\eta_K)| \leq \exp\{-M(K(\langle\xi_K\rangle + \langle\eta_K\rangle))\}, \tag{5.8}$$

for some $1 \leq \tilde{m} \leq d_{\xi_K}$ and $1 \leq \tilde{r} \leq d_{\xi_K}$. We can assume that $\langle\xi_j\rangle + \langle\eta_j\rangle \leq \langle\xi_\ell\rangle + \langle\eta_\ell\rangle$ when $j \leq \ell$.

Let $\mathcal{A} = \{([\xi_j],[\eta_j])\}_{j\in\mathbb{N}}$. It is easy to see that $\mathcal{A}$ has infinitely many elements. Define

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 1, & \text{if } ([\xi],[\eta]) = ([\xi_j],[\eta_j]) \text{ for some } j \in \mathbb{N} \text{ and (5.8) is satisfied,}\\ 0, & \text{otherwise.}\end{cases}$$

By (1.43) and (1.45), it is easy to see that $u \in \Gamma'_{(M_k)}(G)\backslash\Gamma_{(M_k)}(G)$. Let us show that we have $Lu = f \in \Gamma_{(M_k)}(G)$.

If $([\xi],[\eta]) \neq ([\xi_j],[\eta_j])$ for any $j \in \mathbb{N}$ then $\widehat{\widehat{f}}(\xi,\eta) = 0$. In the other hand, for every $K \in \mathbb{N}$, we have

$$\begin{aligned}|\widehat{\widehat{f}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}}| &= |\lambda_{\tilde{m}}(\xi_K) + c\mu_{\tilde{r}}(\eta_K)||\widehat{\widehat{u}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}}|\\ &\leq \exp\{-M(K(\langle\xi_K\rangle + \langle\eta_K\rangle))\}\end{aligned}$$

Therefore $Lu = f \in \Gamma_{(M_k)}(G)$, which contradicts the hypothesis. □

Notice that the conditions for the global $\Gamma_{\{M_k\}}-$hypoellipticity of the Theorem 5.3 imply the conditions for the global $\Gamma_{(M_k)}-$hypoellipticity of the Theorem 5.11. In this way, we have the following corollary:

Corollary 5.12. *If L is globally $\Gamma_{\{M_k\}}-$hypoelliptic, then L is globally $\Gamma_{(M_k)}-$hypoelliptic.*

For the study of global solvability in Komatsu classes of Beurling type, define

$$\mathcal{K} := \{f \in \Gamma'_{(M_k)}(G);\ \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = 0 \text{ whenever } \lambda_m(\xi) + c\mu_r(\eta) = 0\}.$$

So, if $f \notin \mathcal{K}$ then there are no $u \in \Gamma'_{(M_k)}(G)$ satisfying $Lu = f$.

Definition 5.13. *We say the operator L is globally $\Gamma'_{(M_k)}$-solvable if $L(\Gamma'(M_k)(G)) = \mathcal{K}$.*

We always have $L(\Gamma'_{(M_k)}(G)) \subseteq \mathcal{K}$. The next result give us the condition for the other inclusion.

Theorem 5.14. *The operator $L = X_1 + cX_2$ is globally $\Gamma'_{(M_k)}$-solvable if and only if (5.7) holds, that is, there exist $C,\ N > 0$ such that*

$$|\lambda_m(\xi) + c\mu_r(\eta)| \geq C\exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\},$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2}, 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta$, whenever $\lambda_m(\xi) + c\mu_r(\eta) \neq 0$. Moreover, if L is globally $\Gamma'_{(M_k)}$-solvable, for any admissible ultradifferentiable function $f \in \Gamma_{(M_k)}(G)$, there exists $u \in \Gamma_{(M_k)}(G)$ such that $Lu = f$.

Proof. ($\Longleftarrow$) For each $f \in \mathcal{K}$ define

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 0, & \text{if } \lambda_m(\xi) + c\mu_r(\eta) = 0, \\ -i(\lambda_m(\xi) + c\mu_r(\eta))^{-1}\,\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}, & \text{otherwise.} \end{cases}$$

Let us show that $\{\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}\}$ is the sequence of Fourier coefficient of an ultradistribution $u \in \Gamma'_{(M_k)}(G)$. We have by hypothesis that

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &= |\lambda_m(\xi) + c\mu_r(\eta)|^{-1}|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\leq C\exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\}|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \end{aligned}$$

Using the fact that $f \in \Gamma'_{(M_k)}(G)$, we conclude that there exist $C,\ N' > 0$ such that

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C\exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\}\exp\{M(N'(\langle\xi\rangle + \langle\eta\rangle))\},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m, n \leq d_\xi$ and $1 \leq r, s \leq d_\eta$. Let $D = \max\{B, N\}$, so

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C \exp\{2M(D(\langle\xi\rangle + \langle\eta\rangle))\} \leq C \exp\{M(\widetilde{D}(\langle\xi\rangle + \langle\eta\rangle))\},$$

where $\widetilde{D} = DH$. Therefore $u \in \Gamma'_{(M_k)}(G)$ and $Lu = f$.

($\Longrightarrow$) Suppose that is not true, then for any $K \in \mathbb{N}$ there exist $[\xi_K] \in \widehat{G_1}$ and $[\eta_K] \in \widehat{G_2}$ satisfying

$$0 < |\lambda_{\tilde{m}}(\xi_K) + c\mu_{\tilde{r}}(\eta_K)| \leq \frac{1}{K} \exp\{-M(K(\langle\xi_K\rangle + \langle\eta_K\rangle))\}, \tag{5.9}$$

for some $1 \leq \tilde{m} \leq d_{\xi_K}$ and $1 \leq \tilde{r} \leq d_{\xi_K}$. We can assume that $\langle\xi_j\rangle + \langle\eta_j\rangle \leq \langle\xi_\ell\rangle + \langle\eta_\ell\rangle$ when $j \leq \ell$.

Define

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 1, & \text{if } ([\xi],[\eta]) = ([\xi_j],[\eta_j]) \text{ for some } j \in \mathbb{N} \text{ and (5.9) is satisfied,} \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $f \in \mathcal{K}$. If $Lu = f$ for some $u \in \Gamma'_{\{M_k\}}(G)$, then

$$\widehat{\widehat{u}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}} = -i(\lambda_{\tilde{m}}(\xi_K) + c\mu_{\tilde{r}}(\eta_K))^{-1}\, \widehat{\widehat{f}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}}.$$

So

$$\begin{aligned} |\widehat{\widehat{u}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}}| &= |\lambda_{\tilde{m}}(\xi_K) + c\mu_{\tilde{r}}(\eta_K)|^{-1} |\widehat{\widehat{f}}(\xi_K,\eta_K)_{\tilde{m}1_{\tilde{r}1}}| \\ &\geq K \exp\{M(K(\langle\xi_K\rangle + \langle\eta_K\rangle))\}, \end{aligned}$$

which implies that $u \notin \Gamma'_{(M_k)}(G)$, a contradiction.

The proof of the last part of the theorem is analogous to the proof of Theorem 5.7 and then its proof is omitted.

□

Similar to the smooth and Roumieu cases, we have the following corollaries:

Corollary 5.15. *If L is globally $\Gamma_{(M_k)}$-hypoelliptic, then L is globally $\Gamma'_{(M_k)}$-solvable.*

Corollary 5.16. *If L is globally $\Gamma'_{\{M_k\}}$-solvable, then L is globally $\Gamma'_{(M_k)}$-solvable.*

We can summarize the last corollaries about the operator L in the following diagram:

$$\begin{array}{ccccc} GH & \Longrightarrow & G\Gamma_{\{M_k\}}H & \Longrightarrow & G\Gamma_{(M_k)}H \\ \Downarrow & & \Downarrow & & \Downarrow \\ GS & \Longrightarrow & G\Gamma'_{\{M_k\}}S & \Longrightarrow & G\Gamma'_{(M_k)}S \end{array} \tag{5.10}$$

5.2.1 Komatsu levels

We can prove that the global $\Gamma_{\{M_k\}}(G)$–hypoellipticity of the operator L implies its global $\Gamma_{(M_k)}$–hypoellipticity using what we will call Komatsu levels.

Definition 5.17. *Let $\{M_k\}_{k\in\mathbb{N}}$ be a sequence satisfying the conditions (M.0)–(M.3') and let $N > 0$. The Komatsu Level N of ultradifferentiable functions $\Gamma^N_{M_k}(G)$ is the space of C^∞ functions f on G such that there exists $C > 0$ satisfying*

$$\|\widehat{f}(\phi)\|_{\mathtt{HS}} \leq C\exp\{-M(N\langle\phi\rangle)\},$$

for all $[\phi] \in \widehat{G}$, $1 \leq i, j \leq d_\phi$.

Notice that this definition is independent of the choice of the representative of $[\phi] \in \widehat{G}$. Moreover, we have

$$\Gamma_{\{M_k\}}(G) = \bigcup_{N>0} \Gamma^N_{M_k}(G) \qquad \text{and} \qquad \Gamma_{(M_k)}(G) = \bigcap_{N>0} \Gamma^N_{M_k}(G). \tag{5.11}$$

Let us investigate how the operator L acts on Komatsu levels. For $u \in \Gamma^N_{M_k}(G)$, we obtain from (2.1)

$$\widehat{\widehat{Lu}}(\xi,\eta)_{mn_{rs}} = i(\lambda_m(\xi) + c\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}.$$

By (1.11), we have $|\lambda_m(\xi)| \leq \langle\xi\rangle$ and $|\mu_r(\eta)| \leq \langle\eta\rangle$, so we have

$$\|\widehat{\widehat{Lu}}(\xi,\eta)\|_{\mathtt{HS}} \leq C(\langle\xi\rangle + \langle\eta\rangle)\|\,\widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}}.$$

By (1.36), there exists $C > 0$ such that $\langle\xi\rangle + \langle\eta\rangle \leq C\exp\{\frac{1}{2}M(N(\langle\xi\rangle + \langle\eta\rangle))\}$. Using now (1.14), we obtain

$$\|\widehat{\widehat{Lu}}(\xi,\eta)\|_{\mathtt{HS}} \leq C\exp\left\{-M\left(\tilde{N}(\langle\xi\rangle + \langle\eta\rangle)\right)\right\},$$

where $\tilde{N} = \frac{N}{H}$, which implies that $Lu \in \Gamma^{\tilde{N}}_{M_k}(G)$.

Assume that L is globally $\Gamma_{\{M_k\}}$–hypoelliptic. In the proof of Theorem 5.3 we showed that if $Lu \in \Gamma^N_{M_k}(G)$, then $u \in \Gamma^{\tilde{N}}_{M_k}(G)$, where $\tilde{N} = \frac{N}{H}$. Let us prove that L is globally $\Gamma_{(M_k)}(G)$–hypoelliptic. If $Lu \in \Gamma_{(M_k)}(G)$, by (5.11) we get $Lu \in \Gamma^N_{M_k}(G)$, for all $N > 0$ and then $u \in \Gamma^{\tilde{N}}_{M_k}(G)$, for all $\tilde{N} > 0$. Therefore $u \in \Gamma_{(M_k)}(G)$ and L is globally $\Gamma_{(M_k)}$–hypoelliptic.

We also can prove that global $\Gamma_{\{M_k\}}$–solvability implies global $\Gamma_{(M_k)}$–solvability for the operator L using Komatsu levels of ultradistributions.

Definition 5.18. *Let $\{M_k\}_{k\in\mathbb{N}}$ be a sequence satisfying the conditions (M.0), (M.1), (M.2) and (M.3') and $N > 0$. The Komatsu Level N of ultradistributions $\Gamma'^{N}_{M_k}(G)$ is the space of linear functionals u such that there exists $C > 0$ satisfying*

$$\|\widehat{u}(\phi)\|_{\mathtt{HS}} \leq C\exp\{M(N\langle\phi\rangle)\},$$

for all $[\phi] \in \widehat{G}$, $1 \leq i, j \leq d_\phi$.

Similarly, we have

$$\Gamma'_{\{M_k\}}(G) = \bigcap_{N>0} \Gamma'^{N}_{M_k}(G) \qquad \text{and} \qquad \Gamma'_{(M_k)}(G) = \bigcup_{N>0} \Gamma'^{N}_{M_k}(G). \tag{5.12}$$

Suppose that L is globally $\Gamma'_{\{M_k\}}$–solvable. In the proof of Theorem 5.7 we showed that if f is an admissible ultradistribution and $f \in \Gamma'^{N}_{M_k}(G)$, then there exists $u \in \Gamma'^{\tilde{N}}_{M_k}(G)$ such that $Lu = f$, where $\tilde{N} = NH$.

Let us prove that L is globally $\Gamma'_{(M_k)}$–solvable. Let $f \in \Gamma'_{(M_k)}(G)$ an admissible ultradistribution. By (5.12), $f \in \Gamma'^{N}_{M_k}(G)$ for some $N > 0$ and then there exists $u \in \Gamma'^{\tilde{N}}_{M_k}(G)$ such that $Lu = f$, where $\tilde{N} = NH$. Therefore L is globally $\Gamma'_{(M_k)}$–solvable.

5.3 Examples

In this section we will consider the sequence $\{M_k\}_{k\in\mathbb{N}_0}$ given by $M_k = (k!)^s$, with $s \geq 1$. So, the Komatsu class of Roumieu type associated to this sequence is the Gevrey space $\gamma^s(G)$ and we have that the associated function satisfies

$$M(r) \simeq r^{1/s},$$

for all $r \geq 0$.

In this framework we present a class of examples in $\mathbb{T}^1 \times \mathbb{S}^3$ and in $\mathbb{S}^3 \times \mathbb{S}^3$. Examples of operators defined on tori in Gevrey spaces can be found on [3, 8].

Example 5.19. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Let

$$L = \partial_t + cX,$$

where $c \in \mathbb{C}$ and $X \in \mathfrak{s}^3$ is a normalized vector field on $\mathbb{S}^3$. With a similar analysis to that Example 2.10, we may assume that

$$\sigma_X(\ell)_{mn} = im\delta_{mn}, \quad \ell \in \tfrac{1}{2}\mathbb{N}_0,\ -\ell \leq m, n \leq \ell,\ \ell - m, \ell - n \in \mathbb{N}_0.$$

In this case, we have

$$\mathcal{N} = \{(k,\ell) \in \mathbb{Z} \times \tfrac{1}{2}\mathbb{N}_0;\ k + cm = 0,\ \text{for some } -\ell \leq m \leq \ell, \ell - m \in \mathbb{N}_0\}.$$

Notice that $\mathcal{N}$ has infinitely many elements, so by Theorem 5.3 the operator L is not globally γ^s–hypoelliptic, for any $s \geq 1$. Let us analyze the global γ^s–solvability of L. In order to L satisfies Condition 2 of Theorem 5.3, for any $B > 0$ must there exist $K_B >$ such that

$$|k + cm| \geq K_B e^{-(|k|+\ell)^{1/s}}, \tag{5.13}$$

for all $k \in \mathbb{Z}$, $\ell \in \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m \leq \ell$, $\ell - m \in \mathbb{N}$, whenever $k + cm \neq 0$. Notice that this is satisfied when either $\mathrm{Im}(c) \neq 0$ or $c \in \mathbb{Q}$. In the case where $c \in \mathbb{R} \setminus \mathbb{Q}$, the condition (5.13) is satisfied if and only if c is not an exponential Liouville number of order s. For instance, in Example 2.10 we showed that

$$L = \partial_t + \sqrt{2}X$$

is globally solvable. By Corollary 5.9, we conclude that L is globally γ^s–solvable, for any $s \geq 1$.

Consider now the operator

$$L = \partial_t + \alpha X,$$

where α is the continued fraction $\left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$. We proved in Example 2.10 that L is neither globally hypoelliptic nor globally solvable, because in this case $\mathcal{N}$ has infinitely many elements and α is an irrational Liouville number. However, α is not an exponential Liouville number, for any $s \geq 1$ (see Proposition 6.2 of [3]). By Theorem 5.7, we conclude that L is globally γ^s–solvable, for any $s \geq 1$.

Example 5.20. $G = \mathbb{S}^3 \times \mathbb{S}^3$

Consider the operator

$$L = X_1 + cX_2,$$

where $X_1, X_2 \in \mathfrak{s}^3$ and $c \in \mathbb{C}$. Here, we assume that the vector field X_1 acts only in the first variable, while X_2 acts only in the second variable. As seen in Example 2.11, the analysis of this operator is similar to the analysis of the operator studied in Example 5.19. Hence, the operator L is not globally γ^s–hypoelliptic, for any $s \geq 1$. If $\mathrm{Im}(c) \neq 0$ or $c \in \mathbb{Q}$, the operator L is globally γ^s–solvable, for any $s \geq 1$. When $c \in \mathbb{R} \setminus \mathbb{Q}$, the operator L is globally γ^s–solvable if and only if c is not an exponential Liouville number of order s. For instance, the operators

$$L = X_1 + \sqrt{2}X_2 \quad \text{and} \quad L = X_1 + \alpha X_2,$$

where $\alpha = \left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$, are globally γ^s–solvable, for any $s \geq 1$.

5.4 Low order perturbations

We can characterize the global hypoellipticity and global solvability of the operator

$$L = X + q,$$

where $X \in \mathfrak{g}$ and $q \in \mathbb{C}$, on Komatsu classes, both Roumieu and Beurling type, similarly to the vector field case. We say that L_q is globally $\Gamma_{\{M_k\}}(G)$–solvable if $L_q(\Gamma_{\{M_k\}}(G)) = \mathcal{K}_q$, where

$$\mathcal{K}_q := \{w \in \Gamma_{\{M_k\}}(G);\ \widehat{w}(\xi)_{mn} = 0, \text{whenever } \lambda_m(\xi) - iq = 0\}.$$

Analogously we define de global $\Gamma_{(M_k)}(G)$–solvability of L_q.

Theorem 5.21. *The operator $L_q = X + q$ is globally $\Gamma_{(M_k)}$-hypoelliptic (respectively, globally $\Gamma_{\{M_k\}}$-hypoelliptic) if and only if the following conditions hold:*

1. *The set*

$$\mathcal{N} = \{[\xi] \in \widehat{G}; \lambda_m(\xi) - iq = 0, \textit{ for some } 1 \leq m \leq d_\xi\}$$

is finite.

2. *$\exists N > 0$ (respectively, $\forall N > 0$) and $\exists C > 0$ such that*

$$|\lambda_m(\xi) - iq| \geq C \exp\{-M(N\langle\xi\rangle)\},$$

for all $[\xi] \in \widehat{G}$, $1 \leq m \leq d_\xi$, whenever $\lambda_m(\xi) - iq \neq 0$.

Moreover, the operator L_q is globally $\Gamma_{(M_k)}$-solvable (respectively, globally $\Gamma_{\{M_k\}}$-solvable) if and only if Condition 2. above is satisfied.

The proof is similar to the vector field case and it will be omitted. We also have (6.13) for this case.

Example 5.22. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Consider the following operator defined on $\mathbb{T}^1 \times \mathbb{S}^3$:

$$L = \partial_t + \sqrt{2}X + i\tfrac{1}{2}.$$

In Example 2.31 we have seen that L is globally hypoelliptic. By Corollary 5.4, we conclude that L is globally γ^s–hypoelliptic, for any $s \geq 1$.

Consider now the operator

$$L = \partial_t + \alpha X + i\tfrac{1}{2}.$$

We have seen in Example 2.31 that this operator is neither globally hypoelliptic nor globally solvable because although the set $\mathcal{N}$ is finite, the fact that α is an irrational Liouville number implies that Condition 2 of Theorem 2.3 is not satisfied. However, since α is not an exponential Liouville number, we conclude that L is globally γ^s–solvable, for any $s \geq 1$. Similarly, we can conclude that

$$L = \partial_t + \alpha X + i\alpha$$

is not globally γ^s-hypoelliptic, but it is globally γ^s–solvable, for any $s \geq 1$.

Example 5.23. $G = \mathbb{S}^3 \times \mathbb{S}^3$

Let

$$L = X_1 + \alpha X_2 + i\tfrac{1}{2}$$

be an operator defined on $\mathbb{S}^3 \times \mathbb{S}^3$, where $\alpha = \left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$. In Example 2.32 we have seen that the set $\mathcal{N}$ for this operator has infinitely many elements, which implies, by Theorem 5.21, that L is not globally γ^s–solvable, for any $s \geq 1$. However, although L is not globally solvable, by the fact that α is not an exponential Liouville number of order s, for any $s \geq 1$, we conclude that L is globally γ^s–solvable, for any $s \geq 1$.

Consider now the operator

$$L = X_1 + \alpha X_2 + i\tfrac{1}{4}.$$

In this case the set $\mathcal{N}$ is empty and, again by the fact that α is not an exponential Liouville number of order s, for any $s \geq 1$, we conclude by Theorem 5.21 that L is globally γ^s–hypoelliptic, for any $s \geq 1$.

Chapter 6

Variable coefficient vector fields - Real case

6.1 Normal form

Let G_1 and G_2 be compact Lie groups and consider the operator L_a defined on $G := G_1 \times G_2$ by

$$L_a = X_1 + a(x_1)X_2,$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$, and $a \in \Gamma_{\{M_k\}}(G_1)$ is a real-valued function. Recall that for each $[\xi] \in \widehat{G_1}$, we can choose a representative $\xi \in \mathrm{Rep}(G_1)$ such that

$$\sigma_{X_1}(\xi)_{mn} = i\lambda_m(\xi)\delta_{mn}, \quad 1 \leq m, n \leq d_\xi,$$

where $\lambda_m(\xi) \in \mathbb{R}$ for all $[\xi] \in \widehat{G_1}$ and $1 \leq m \leq d_\xi$. Similarly, for each $[\eta] \in \widehat{G_2}$, we can choose a representative $\eta \in \mathrm{Rep}(G_2)$ such that

$$\sigma_{X_2}(\eta)_{rs} = i\mu_r(\eta)\delta_{rs}, \quad 1 \leq r, s \leq d_\eta,$$

where $\mu_r(\eta) \in \mathbb{R}$ for all $[\eta] \in \widehat{G_2}$ and $1 \leq r \leq d_\eta$.

Now assume that there exists $A \in \Gamma_{\{M_k\}}(G_1)$ such that

$$X_1 A(x_1) = a(x_1) - a_0,$$

for all $x_1 \in G_1$, where

$$a_0 := \int_{G_1} a(x_1)\, dx_1.$$

By the definition of ultradifferentiable functions, there exist $K, \ell > 0$ such that for all $\alpha \in \mathbb{N}_0^{d_1}$ holds

$$|\partial^\alpha A(x_1)| \leq K'\ell'^{|\alpha|} M_{|\alpha|}, \quad \forall x_1 \in G_1.$$

Since $M_{|\alpha|} \leq AH^{|\alpha|}M_1M_{|\alpha|-1}$, we obtain for all non-zero $\alpha \in \mathbb{N}_0^{d_1}$

$$|\partial^\alpha A(x_1)| \leq K\ell^{|\alpha|-1}M_{|\alpha|-1}, \quad \forall x_1 \in G_1, \tag{6.1}$$

where $K = K'\ell' HAM_1$ and $\ell = \ell' H$.

Similarly, if $A \in \Gamma_{(M_k)}(G_1)$, for any $\ell > 0$ there exists $K_\ell > 0$ such that for all non-zero $\alpha \in \mathbb{N}_0^{d_1}$ holds

$$|\partial^\alpha A(x_1)| \leq K_\ell\ell^{|\alpha|-1}M_{|\alpha|-1}, \quad \forall x_1 \in G_1. \tag{6.2}$$

Define the operator Ψ_a as:

$$\Psi_a u := \sum_{[\eta]\in\widehat{G_2}} d_\eta \sum_{r,s=1}^{d_\eta} e^{i\mu_r(\eta)A(\cdot)}\widehat{u}(\,\cdot\,,\eta)_{rs}\,\eta_{sr}. \tag{6.3}$$

In Section 3.1 of Chapter 3 it was proved that Ψ_a is an automorphism of $C^\infty(G)$ and $\mathcal{D}'(G)$, with inverse Ψ_{-a}. Moreover, it holds

$$\Psi_a \circ L_a = L_{a_0} \circ \Psi_a, \tag{6.4}$$

where $L_{a_0} = X_1 + a_0X_2$.

Since the operator L_a is the same as in Chapter 3, the expression (6.4) remains valid in Komatsu classes. In the next results, we present sufficient conditions for the operator Ψ_a be an automorphism in the space of ultradifferentiable functions and ultradistributions of both Roumieu and Beurling types.

Proposition 6.1. *Let $a \in \Gamma_{\{M_k\}}(G_1)$. Then the operator Ψ_a, defined in (6.3), is an automorphism of $\Gamma_{\{M_k\}}(G_1 \times G_2)$.*

Proof. It is enough to show that $\Psi_a u \in \Gamma_{\{M_k\}}(G_1 \times G_2)$ when $u \in \Gamma_{\{M_k\}}(G_1 \times G_2)$. By the characterization of ultradifferentiable functions of Roumieu type from their partial Fourier coefficients, there exist $C, h, \varepsilon > 0$ such that

$$|\partial^\alpha\widehat{u}(x_1,\eta)_{rs}| \leq Ch^{|\alpha|}M_{|\alpha|}\exp\{-M(\varepsilon\langle\eta\rangle)\}, \tag{6.5}$$

for all $\alpha \in \mathbb{N}_0^{d_1}$, $x_1 \in G_1$, $[\eta] \in \widehat{G_2}$ and $1 \leq r, s \leq d_\eta$. Notice that

$$\widehat{\Psi_a u}(x_1,\eta)_{rs} = e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}$$

Thus, for $\alpha \in \mathbb{N}_0^{d_1}$ we have

$$|\partial^\alpha\widehat{\Psi_a u}(x_1,\eta)_{rs}| = \left|\partial^\alpha\left(e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}\right)\right| \leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left|\partial^\beta e^{i\mu_r(\eta)A(x_1)}\right|\left|\partial^{\alpha-\beta}\widehat{u}(x_1,\eta)_{rs}\right|.$$

Using that $|\mu_r(\eta)| \leq \langle\eta\rangle$ and (6.1), we have by Faà di Bruno's Formula that

$$|\partial^\beta e^{i\mu_r(\eta)A(x_1)}| \leq \sum_{k=1}^{|\beta|} K^k \langle\eta\rangle^k \ell^{|\beta|-k} \left(\sum_{\lambda\in\Delta(|\beta|,k)} \binom{|\beta|}{\lambda} \frac{1}{r(\lambda)!} \prod_{j=1}^{k} M_{\lambda_j-1} \right),$$

where $\Delta(|\beta|,k) = \{\lambda \in \mathbb{N}^k; |\lambda| = |\beta| \text{ and } \lambda_1 \geq \cdots \geq \lambda_k \geq 1\}$ and $r(\lambda) \in \mathbb{N}_0^{d_1}$, where $r(\lambda)_j$ counts how many times j appears on λ.

By property (M.4) of the sequence $\{M_k\}_{k\in\mathbb{N}_0}$ we obtain

$$\binom{|\beta|}{\lambda} \prod_{j=1}^{|\beta|} M_{\lambda_j-1} = |\beta|! \prod_{j=1}^{|\beta|} \frac{M_{\lambda_j-1}}{\lambda_j!} \leq |\beta|! \prod_{j=1}^{|\beta|} \frac{M_{\lambda_j-1}}{(\lambda_j-1)!} \leq |\beta|! \frac{M_{|\beta|-k}}{(|\beta|-k)!}, \tag{6.6}$$

for $\lambda \in \Delta(|\beta|,k)$. Using the fact that

$$\sum_{\lambda\in\Delta(|\beta|,k)} \frac{1}{r(\lambda)!} = \binom{|\beta|-1}{k-1} \frac{1}{k!},$$

we have

$$|\partial^\beta e^{i\mu_r(\eta)A(x_1)}| \leq \sum_{k=1}^{|\beta|} \binom{|\beta|-1}{k-1} \frac{1}{k!} K^k \langle\eta\rangle^k \ell^{|\beta|-k} |\beta|! \frac{M_{|\beta|-k}}{(|\beta|-k)!} \tag{6.7}$$

By (6.5), we have

$$|\partial^\alpha \widehat{\Psi_a u}(x_1,\eta)_{rs}| \leq C \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{k=1}^{|\beta|} \binom{|\beta|-1}{k-1} \frac{1}{k!} K^k \langle\eta\rangle^k \ell^{|\beta|-k} \tag{6.8}$$

$$\times |\beta|! \frac{M_{|\beta|-k}}{(|\beta|-k)!} h^{|\alpha|-|\beta|} M_{|\alpha|-|\beta|} \exp\{-M(\varepsilon\langle\eta\rangle)\} \tag{6.9}$$

By Proposition 1.35,

$$\langle\eta\rangle^k \exp\{-M(\varepsilon\langle\eta\rangle)\} \leq A \left(\frac{H}{\varepsilon}\right)^k M_k \exp\{-M(\varepsilon H^{-1}\langle\eta\rangle)\}.$$

So,

$$|\partial^\alpha \widehat{\Psi_a u}(x_1,\eta)_{rs}| \leq AC \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{k=1}^{|\beta|} \binom{|\beta|-1}{k-1} \left(\frac{KH}{\ell\varepsilon}\right)^k \ell^{|\beta|} h^{|\alpha|-|\beta|}$$

$$\times |\beta|! \frac{M_{|\beta|-k}}{(|\beta|-k)!} \frac{M_k}{k!} M_{|\alpha|-|\beta|} \exp\{-M(\varepsilon\langle\eta\rangle)\}.$$

Notice that

$$|\beta|! \frac{M_{|\beta|-k}}{(|\beta|-k)!} \frac{M_k}{k!} M_{|\alpha|-|\beta|} \leq |\beta|! \frac{M_{|\beta|}}{|\beta|!} M_{|\alpha|-|\beta|} \leq M_{|\alpha|}.$$

Denote by $S = \max\{\frac{KH}{\varepsilon}, \ell\}$. Thus

$$|\partial^\alpha \widehat{\Psi_a u}(x_1,\eta)_{rs}| \leq AC \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} S^{|\beta|} h^{|\alpha|-|\beta|} M_{|\alpha|} \exp\{-M(\varepsilon H^{-1}\langle\eta\rangle)\} \sum_{k=1}^{|\beta|} \binom{|\beta|-1}{k-1}.$$

We have $\sum_{k=1}^{|\beta|}\binom{|\beta|-1}{k-1} = 2^{|\beta|-1}$. Moreover,

$$\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(2S)^{|\beta|}h^{|\alpha|-|\beta|} = \sum_{|\beta|=0}^{|\alpha|}\binom{|\alpha|}{|\beta|}(2S)^{|\beta|}h^{|\alpha|-|\beta|} = (2S+h)^{|\alpha|}. \tag{6.10}$$

In this way

$$|\partial^{\alpha}\widehat{\Psi_a u}(x_1,\eta)_{rs}| \leq AC\,(2S+h)^{|\alpha|}\,M_{|\alpha|}\exp\{-M(\varepsilon H^{-1}\langle\eta\rangle)\}.$$

By Theorem A.8 we conclude that $\Psi_a u \in \Gamma_{\{M_k\}}(G_1\times G_2)$. □

Proposition 6.2. *Assume that $a\in\Gamma_{(M_k)}(G_1)$. Then Ψ_a is an automorphism of $\Gamma_{(M_k)}(G_1\times G_2)$.*

Proof. Let $u\in\Gamma_{(M_k)}(G_1\times G_2)$. By (6.1) we have that

$$|\partial^{\alpha}A(x_1)| \leq K_{\ell}\ell^{|\alpha|-1}M_{|\alpha|-1}, \quad \forall x_1\in G_1.$$

By Theorem A.9 for all $h,\varepsilon>0$ there exists $C_{h\varepsilon}>0$ such that

$$|\partial^{\alpha}\widehat{u}(x_1,\eta)_{rs}| \leq C_{h\varepsilon}h^{|\alpha|}M_{|\alpha|}\exp\{-M(\varepsilon\langle\eta\rangle)\}, \tag{6.11}$$

for all $\alpha\in\mathbb{N}_0^{d_1}$, $x_1\in G_1$, $[\eta]\in\widehat{G_2}$ and $1\leq r,s\leq d_\eta$. We can follow the proof of Roumieu type case and obtain

$$|\partial^{\alpha}\widehat{\Psi_a u}(x_1,\eta)_{rs}| \leq C_{h\varepsilon}\,(2S+h)^{|\alpha|}\,M_{|\alpha|}\exp\{-M(\varepsilon H^{-1}\langle\eta\rangle)\},$$

where $S=\max\{\frac{K_\ell H}{\varepsilon},\ell\}$. Given $j,\delta>0$, choose $\ell=\frac{j}{4}$ and $\varepsilon=\max\left\{\delta H,\frac{4K_jH}{j}\right\}$. Thus $S=\frac{j}{4}$ and

$$\exp\{-M(\varepsilon H^{-1}\langle\eta\rangle)\}\leq\exp\{-M(\delta\langle\eta\rangle)\},$$

for all $[\eta]\in\widehat{G_2}$. Hence

$$|\partial^{\alpha}\widehat{\Psi_a u}(x_1,\eta)_{rs}| \leq AC_{h\delta}\left(\tfrac{j}{2}+h\right)^{|\alpha|}M_{|\alpha|}\exp\{-M(\delta\langle\eta\rangle)\},$$

Choose now $h=\frac{j}{2}$. Therefore

$$|\partial^{\alpha}\widehat{\Psi_a u}(x_1,\eta)_{rs}| \leq C_{j\delta}j^{|\alpha|}M_{|\alpha|}\exp\{-M(\delta\langle\eta\rangle)\},$$

which implies that $\Psi_a u\in\Gamma_{(M_k)}(G_1\times G_2)$. □

Proposition 6.3. *For $a\in\Gamma_{\{M_k\}}(G_1)$, the operator Ψ_a is an automorphism of $\Gamma'_{\{M_k\}}(G_1\times G_2)$.*

Proof. Most of the estimate that we will use here was proved in the demonstration of Theorem 6.1. Let us show that $\Psi_a u \in \Gamma'_{\{M_k\}}(G_1 \times G_2)$ when $u \in \Gamma'_{\{M_k\}}(G_1 \times G_2)$. By the characterization of ultradistributions of Roumieu type (Theorem A.10) for all $h, \varepsilon > 0$, there exists $C_{h\varepsilon} > 0$ such that

$$|\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \varphi\rangle| \leq C_{h\varepsilon}\|\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\}, \quad \forall \varphi \in \Gamma_{\{M_k\}}(G_1).$$

In this way, for $\varphi \in \Gamma_{\{M_k\}}(G_1)$, we have

$$\left\langle \widehat{\Psi_a u}(\cdot,\eta)_{rs}, \varphi \right\rangle = \left\langle e^{i\mu_r(\eta)A(\cdot)}\widehat{u}(\cdot,\eta)_{rs}, \varphi \right\rangle = \left\langle \widehat{u}(\cdot,\eta)_{rs}, e^{i\mu_r(\eta)A(\cdot)}\varphi \right\rangle.$$

Hence,

$$\left\langle \widehat{u}(\cdot,\eta)_{rs}, e^{i\mu_r(\eta)A(\cdot)}\varphi \right\rangle \leq C_{h\varepsilon}\|e^{i\mu_r(\eta)A(\cdot)}\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\}.$$

Notice that

$$\left|\partial^\alpha \left(e^{i\mu_r(\eta)A(x_1)}\varphi(x_1)\right)\right| \leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left|\partial^\beta e^{i\mu_r(\eta)A(t)}\right|\left|\partial^{\alpha-\beta}\varphi(t)\right|.$$

By (6.7), using that $|\partial^{|\alpha|}A(x_1)| \leq K\ell^{|\alpha|-1}M_{|\alpha|-1}$, we obtain

$$|\partial^\beta e^{i\mu_r(\eta)A(x_1)}| \leq \sum_{k=1}^{|\beta|}\binom{|\beta|-1}{k-1}\frac{1}{k!}K^k\langle\eta\rangle^k\ell^{|\beta|-k}|\beta|!\frac{M_{|\beta|-k}}{(|\beta|-k)!}$$

By Proposition 1.35,

$$\langle\eta\rangle^k \exp\{M(\varepsilon\langle\eta\rangle)\} \leq A\varepsilon^{-k}M_k \exp\{M(H\varepsilon\langle\eta\rangle)\}$$

and then by the property (M.4) we obtain

$$\begin{aligned}\left|\partial^\alpha \left(e^{i\mu_r(\eta)A(x_1)}\varphi(x_1)\right)\right| \exp\{M(\varepsilon\langle\eta\rangle)\} \leq A\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\sum_{k=1}^{|\beta|}\binom{|\beta|-1}{k-1}\left(\frac{K}{\ell\varepsilon}\right)^k \ell^{|\beta|}M_{|\beta|} \\ \times \left|\partial^{|\alpha|-|\beta|}\varphi(x_1)\right| \exp\{M(H\varepsilon\langle\eta\rangle)\}\end{aligned} \tag{6.12}$$

Let $S = \max\left\{\frac{K}{\varepsilon}, \ell\right\}$, then for any $j > 0$ we have

$$\begin{aligned}\left|\partial^\alpha \left(e^{i\mu_r(\eta)A(x_1)}\varphi(x_1)\right)\right| \exp\{M(\varepsilon\langle\eta\rangle)\} &\leq A\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}S^{|\beta|}M_{|\beta|}\left|\partial^{|\alpha|-|\beta|}\varphi(x_1)\right|\sum_{k=1}^{|\beta|}\binom{|\beta|-1}{k-1} \\ &\quad \times \exp\{M(H\varepsilon\langle\eta\rangle)\} \\ &\leq A\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(2S)^{|\beta|}\,M_{|\beta|}\,\|\varphi\|_j\, j^{|\alpha|-|\beta|}M_{|\alpha|-|\beta|} \\ &\quad \times \exp\{M(H\varepsilon\langle\eta\rangle)\}\end{aligned}$$

Using the fact that $M_{|\alpha|-|\beta|}M_{|\beta|} \leq M_{|\alpha|}$ and (6.10), we obtain

$$\left|\partial^\alpha \left(e^{i\mu_r(\eta)A(t)}\varphi(t)\right)\right| \exp\{M(\varepsilon\langle\eta\rangle)\} \leq A\,(2S+j)^{|\alpha|}\,\|\varphi\|_j\, M_{|\alpha|} \exp\{M(H\varepsilon\langle\eta\rangle)\}$$

Given $j, \delta > 0$, choose $\varepsilon = \frac{\delta}{H}$ and then $h = 2S + j$. Notice that

$$\left\|e^{i\mu_r(\eta)A(\cdot)}\varphi\right\|_h \exp\{M(\varepsilon\langle\eta\rangle)\} \leq A\|\varphi\|_j \exp\{M(\delta\langle\eta\rangle)\},$$

then we conclude that

$$\begin{aligned}\left|\left\langle \widehat{\Psi_a u}(\cdot,\eta)_{rs}, \varphi\right\rangle\right| &\leq C_{h\varepsilon}\|e^{i\mu_r(\eta)A(\cdot)}\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\} \\ &\leq C_{j\delta}\|\varphi\|_j \exp\{M(\delta\langle\eta\rangle)\}.\end{aligned}$$

Therefore $\Psi_a u \in \Gamma'_{\{M_k\}}(G_1 \times G_2)$ and then Ψ_a is an automorphism. □

Proposition 6.4. *For $a \in \Gamma_{(M_k)}(G_1)$, the operator Ψ_a is an automorphism of $\Gamma'_{(M_k)}(G_1 \times G_2)$.*

Proof. Let us show that $\Psi_a u \in \Gamma'_{(M_k)}(G_1 \times G_2)$ when $u \in \Gamma'_{(M_k)}(G_1 \times G_2)$. By the characterization of ultradistributions of Beurling type (Theorem A.11) there exist $h, \varepsilon, C > 0$ such that

$$|\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \varphi\rangle| \leq C\|\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\}, \quad \forall \varphi \in \Gamma_{(M_k)}(G_1).$$

In this way, for $\varphi \in \Gamma_{(M_k)}(G_1)$,

$$\left\langle \widehat{\Psi_a u}(\cdot,\eta)_{rs}, \varphi\right\rangle = \left\langle e^{i\mu_r(\eta)A(\cdot)}\widehat{u}(\cdot,\eta)_{rs}, \varphi\right\rangle = \left\langle \widehat{u}(\cdot,\eta)_{rs}, e^{i\mu_r(\eta)A(\cdot)}\varphi\right\rangle.$$

We have

$$\left\langle \widehat{u}(\cdot,\eta)_{rs}, e^{i\mu_r(\eta)A(\cdot)}\varphi\right\rangle \leq C\|e^{i\mu_r(\eta)A(\cdot)}\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\}.$$

Following the proof of the Proposition 6.3, by the fact that $a \in \Gamma_{(M_k)}(G_1)$ we obtain

$$\|e^{i\mu_r(\eta)A(\cdot)}\varphi\|_{2S+j} \exp\{M(\varepsilon\langle\eta\rangle)\} \leq A\|\varphi\|_j \exp\{M(H\varepsilon\langle\eta\rangle)\},$$

where $S = \max\left\{\frac{K_\ell}{\varepsilon}, \ell\right\}$. Now, choose $\ell = \frac{h}{4}$ and consider ε sufficiently large such that $S = \ell$. For $j = \frac{h}{2}$, we obtain

$$\begin{aligned}\left\langle \widehat{\Psi_a u}(\cdot,\eta)_{rs}, \varphi\right\rangle &\leq C\|e^{i\mu_r(\eta)A(\cdot)}\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\} \\ &\leq C\|\varphi\|_{\frac{h}{2}} \exp\{M(H\varepsilon\langle\eta\rangle)\},\end{aligned}$$

which implies that $\Psi_a u \in \Gamma'_{(M_k)}(G_1 \times G_2)$. □

6.1.1 Global Komatsu hypoellipticity and solvability

Let us turn our attention to the study of global properties of the operator L_a defined on the compact Lie group $G := G_1 \times G_2$ by

$$L_a = X_1 + a(x_1)X_2,$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$, and $a \in \Gamma_{\{M_k\}}(G_1)$ (or $a \in \Gamma_{(M_k)}(G_1)$) is a real-valued function.

Recall that $L_{a_0} = X_1 + a_0 X_2$, where $a_0 := \int_{G_1} a(x_1)\, dx_1$. Now, if $L_{a_0}u = f \in \Gamma'_{\{M_k\}}(G)$, for some $u \in \Gamma'_{\{M_k\}}(G)$, then

$$i(\lambda_m(\xi) + a_0\mu_r(\eta))\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \widehat{\widehat{f}}\,(\xi,\eta)_{mn_{rs}},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m \leq d_\xi$, and $1 \leq r \leq d_\eta$. In particular, f belongs to the following set

$$\mathcal{K}_{a_0} := \{g \in \Gamma'_{\{M_k\}}(G_1 \times G_2);\ \widehat{\widehat{g}}(\xi,\eta)_{mn_{rs}} = 0,\ \text{whenever } \lambda_m(\xi) + a_0\mu_r(\eta) = 0\}.$$

In order to study the solvability of the operator L_a, assume that $L_a u = f \in \Gamma'_{\{M_k\}}(G_1 \times G_2)$ for some $u \in \Gamma'_{\{M_k\}}(G_1 \times G_2)$. We can write $u = \Psi_{-a}(\Psi_a u)$, so $L_a(\Psi_{-a}(\Psi_a u)) = f$. Thus, using the fact that $\Psi_a \circ L_a = L_{a_0} \circ \Psi_a$, we obtain $\Psi_{-a}L_{a_0}\Psi_a u = f$, that is,

$$L_{a_0}\Psi_a u = \Psi_a f.$$

This implies that $\Psi_a f \in \mathcal{K}_{a_0}$ and motivates the following definition:

Definition 6.5. *We say that the operator L_a is globally $\Gamma'_{\{M_k\}}$–solvable if $L_a(\Gamma'_{\{M_k\}}(G_1 \times G_2)) = \mathcal{J}_a$, where*

$$\mathcal{J}_a := \{v \in \Gamma'_{\{M_k\}}(G_1 \times G_2);\ \Psi_a v \in \mathcal{K}_{a_0}\}.$$

Similarly is defined these global properties for Komatsu classes of Beurling type. Using the results from the previous section, we obtain the following connection between the operator L_a and its normal form, which proof will be omitted because is the same of the smooth case (see Proposition 3.7).

Proposition 6.6. *Let $a \in \Gamma_{\{M_k\}}(G_1)$ (respectively, $a \in \Gamma_{(M_k)}(G_1)$) then:*

1. *the operator L_a is globally $\Gamma_{\{M_k\}}$-hypoelliptic (respectively, $\Gamma_{(M_k)}$-hypoelliptic) if and only if L_{a_0} is globally $\Gamma_{\{M_k\}}$-hypoelliptic (respectively, $\Gamma_{(M_k)}$-hypoelliptic);*

2. *the operator L_a is globally $\Gamma_{\{M_k\}}$-solvable (respectively, $\Gamma_{(M_k)}$-solvable) if and only if L_{a_0} is globally $\Gamma_{\{M_k\}}$-solvable (respectively, $\Gamma_{(M_k)}$-solvable).*

From the automorphism Ψ_a we recover for the operator L_a the connection between the different notions of global hypoellipticity and global solvability, obtained in Chapter 5, for constant coefficients vector fields, summarized in the following diagram:

$$\begin{array}{ccccc} GH & \Longrightarrow & G\Gamma_{\{M_k\}}H & \Longrightarrow & G\Gamma_{(M_k)}H \\ \Downarrow & & \Downarrow & & \Downarrow \\ GS & \Longrightarrow & G\Gamma'_{\{M_k\}}S & \Longrightarrow & G\Gamma'_{(M_k)}S \end{array} \tag{6.13}$$

Notice that we need to assume that $a \in \Gamma_{(M_k)}(G)$ for the implications involving Komatsu classes of Beurling type.

Example 6.7. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Consider the continued fraction $\alpha = \left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$ and a normalized vector field $X \in \mathfrak{s}^3$. Let L_a be the operator defined as

$$L_a = \partial_t + a(t)X,$$

where $a(t) = \sin(t)+\alpha$. Notice that $a \in \gamma^s(\mathbb{T}^1)$, for all $s \geq 1$ and the function $A : t \mapsto -\cos(t)$ satisfies $\partial_t A(t) = a(t) - \alpha$. By Proposition 6.6, we can study the global properties of L_a from the operator

$$L_{a_0} = \partial_t + \alpha X.$$

In Example 5.19 we have seen that the operator L_{a_0} is globally γ^s–solvable, for any $s \geq 1$. In addition, since α is a Liouville number, the operator L_{a_0} is not globally solvable in the smooth sense (Example 2.10).

We conclude then that the operator L_a is neither globally γ^s-hypoelliptic nor globally solvable in the smooth sense, but L_a is globally γ^s–solvable, for any $s \geq 1$.

Example 6.8. $G = \mathbb{S}^3 \times \mathbb{S}^3$

Consider the operator

$$L_h = X_1 + h(x_1)X_2,$$

where $X_1, X_2 \in \mathfrak{s}^3$, h is expressed in Euler's angle by

$$h(x_1(\phi_1, \theta_1, \psi_1)) = -\cos\left(\tfrac{\theta_1}{2}\right)\sin\left(\tfrac{\phi_1+\psi_1}{2}\right) + \alpha,$$

where $0 \leq \phi_1 < 2\pi$, $0 \leq \theta_1 \leq \pi$, $-2\pi \leq \psi_1 < 2\pi$, and α is the continued fraction $\left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$. Moreover, we will assume that the vector field X_1 acts only in the first variable, while X_2 acts only in the second variable. Since $X_1 \mathrm{tr}(x_1) = h(x_1) - \alpha$, with tr as in Example 3.12, it is enough to understand the global properties of the operator

$$L_{h_0} = X_1 + \alpha X_2$$

for the study of the global properties of L_h. In Example 2.32 we have seen that the operator L_{h_0} is globally γ^s–solvable, for any $s \geq 1$. In addition, since α is a Liouville number, the operator L_{h_0} is not globally solvable in the smooth sense (Example 2.11).

Therefore, the operator L_h is neither globally γ^s–hypoelliptic nor globally solvable in the smooth sense, but it is globally γ^s–solvable, for any $s \geq 1$.

6.2 Low order perturbations

The next step for the study of low order perturbations is to consider the operator $L_q := X + q$, where $q \in \Gamma_{\{M_k\}}(G)$. The idea is to establish a connection between the global hypoellipticity and the global solvability in Komatsu sense of L_q and $L_{q_0} = X + q_0$, where q_0 is the average of q in G.

In [5], Bergamasco proved that the operator

$$L_q = \partial_t + a\partial_x + q,$$

where $a \in \mathbb{R}$ is an irrational non-Liouville number and $q \in C^\infty(\mathbb{T}^2)$, is globally hypoelliptic if and only if it is the operator $L_{q_0} = \partial_t + a\partial_x + q_0$, where $q_0 = \int_{\mathbb{T}^2} q(t,x)\, dxdt$. The key to make this connection is the fact that $L_q \circ e^{-Q} = e^{-Q} \circ L_{q_0}$, where $Q \in C^\infty(\mathbb{T}^2)$ satisfies $(\partial_t + a\partial_x)Q = q - q_0$. The existence of such Q is guaranteed by the global hypoellipticity of the operator $\partial_t + a\partial_x$.

For the study of the operator $L = X + q$, with $q \in \Gamma_{\{M_k\}}(G)$, we can not assume the global hypoellipticity of X in view of the Greenfield-Wallach's conjecture. Hence, we will assume as hypothesis that there exists $Q \in \Gamma_{\{M_k\}}(G)$ such that

$$XQ = q - q_0,$$

where $q_0 = \int_G q(x)\, dx$.

From Proposition 3.16, we have

$$L_q \circ e^{-Q} = e^{-Q} \circ L_{q_0}, \tag{6.14}$$

in $C^\infty(G)$ and $\mathcal{D}'(G)$. The aim of the next lemma is to extend this conjugation to Komatsu classes.

Lemma 6.9. *If $f \in \Gamma_{\{M_k\}}(G)$, then $e^f \in \Gamma_{\{M_k\}}(G)$.*

Proof. By the characterization of ultradifferentiable function of Roumieu type, there exist $C, h > 0$ such that

$$|\partial^\alpha f(x)| \leq Ch^{|\alpha|} M_{|\alpha|},$$

for all $\alpha \in \mathbb{N}_0^d$, $x \in G$.

Let $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| = p$. We have that

$$|\partial^\alpha e^{f(x)}| \leq |e^{f(x)}| h^p \sum_{k=1}^{p} C^k \left(\sum_{\lambda \in \Delta(p,k)} \binom{p}{\lambda} \frac{1}{r(\lambda)!} \prod_{j=1}^{k} M_{\lambda_j} \right),$$

where $\Delta(p,k) = \{\lambda \in \mathbb{N}^k; |\lambda| = p \text{ and } \lambda_1 \geq \cdots \geq \lambda_k \geq 1\}$ and $r(\lambda) \in \mathbb{N}_0^d$, where $r(\lambda)_j$ counts how many times j appears on λ. For example, $\lambda = (2,2,1,1) \in \Delta(6,4)$ and $r(\lambda) = (2,2,0,0,0,0)$. Since $\binom{p}{\lambda} = \frac{p!}{\lambda_1! \cdots \lambda_k!}$, by property (M.4) we obtain

$$\binom{p}{\lambda} \prod_{j=1}^{k} M_{\lambda_j} = p! \prod_{j=1}^{k} \frac{M_{\lambda_j}}{\lambda_j!} \leq p! \frac{M_{|\lambda|}}{|\lambda|!} = M_p$$

Then

$$|\partial^\alpha e^{f(x)}| \leq K h^p M_p \sum_{k=1}^{p} C^k \sum_{\lambda \in \Delta(p,k)} \frac{1}{r(\lambda)!}$$

We have that

$$\sum_{k=1}^{p} C^k \sum_{\lambda \in \Delta(p,k)} \frac{1}{r(\lambda)!} = \sum_{k=1}^{p} \binom{p-1}{k-1} \frac{C^k}{k!}.$$

Therefore, $\sum_{k=1}^{p} C^k \sum_{\lambda \in \Delta(p,k)} \frac{1}{r(\lambda)!} \leq 2^p e^C$ and we obtain

$$|\partial^\alpha e^{f(x)}| \leq K(2h)^p M_p,$$

which implies that $e^f \in \Gamma_{\{M_k\}}(G)$. □

Remark 6.10. *With a slight modification in the above proof it is possible that $e^f \in \Gamma_{(M_k)}(G)$ whenever $f \in \Gamma_{(M_k)}(G)$.*

From Lemma 6.9, we obtain that $e^Q v \in \Gamma_{\{M_k\}}(G)$, whenever $v \in \Gamma_{\{M_k\}}(G)$. Moreover, for $u \in \Gamma'_{\{M_k\}}(G)$, we also have $e^Q u \in \Gamma'_{\{M_k\}}(G)$. The equality (6.14) motivates us to define the global $\Gamma'_{\{M_k\}}$–solvability of L_q as:

Definition 6.11. *Let G be a compact Lie group, $X \in \mathfrak{g}$, and $Q \in \Gamma_{\{M_k\}}(G)$. We say that the operator*

$$L_q = X + q,$$

where $XQ = q - q_0$, $q_0 = \int_G q(x)\,dx$, is globally $\Gamma'_{\{M_k\}}$–solvable if $L_q(\mathcal{D}'(G)) = \mathcal{J}_q$, where

$$\mathcal{J}_q := \{v \in \Gamma'_{\{M_k\}}(G);\ e^Q v \in \mathcal{K}_{q_0}\}.$$

Proposition 6.12. *Let G be a compact Lie group and consider the operator $L = X + q$, where $X \in \mathfrak{g}$ and $q \in \Gamma_{\{M_k\}}(G)$. Assume that there exists $Q \in \Gamma_{\{M_k\}}(G)$ satisfying $XQ = q - q_0$, where $q_0 = \int_G q(x)\,dx$. The operator L_q is globally $\Gamma_{\{M_k\}}$–hypoelliptic if and only if L_{q_0} is globally $\Gamma_{\{M_k\}}$–hypoelliptic. Moreover, the operator L_q is globally $\Gamma_{\{M_k\}}$–solvable if and only if L_{q_0} is globally $\Gamma_{\{M_k\}}$–solvable.*

Proof. The proof is analogous to the demonstration of Theorem 3.7. □

Corollary 6.13. *If L_q is globally $\Gamma_{\{M_k\}}$–hypoelliptic, then L is globally $\Gamma_{\{M_k\}}$–solvable.*

Example 6.14. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Consider

$$L_q = \partial_t + \alpha X + q(t,x),$$

where $\alpha = \left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$ and $q(t,x) = \cos(t) + h(x) + \frac{1}{2}i$, where h is expressed in Euler's angles by

$$h(x(\phi,\theta,\psi)) = -\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\phi+\psi}{2}\right).$$

Notice that q is an analytic function, which implies that $q \in \gamma^s(\mathbb{T}^1 \times \mathbb{S}^3)$ for all $s \geq 1$. Let $Q(t,x) = \sin(t) + \frac{1}{\alpha}\mathrm{tr}(x)$, where tr is the trace function given in Euler's angles by

$$\mathrm{tr}(x(\phi,\theta,\psi)) = 2\cos\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\phi+\psi}{2}\right).$$

The vector field X is the operator ∂_ψ in Euler's angle and we obtain $X\mathrm{tr}(x) = h(x)$. Hence,

$$(\partial_t + \alpha X)Q(t,x) = q(t,x) - \tfrac{1}{2}i,$$

and by Proposition 6.12 it is enough to study the global properties of

$$L_{q_0} = \partial_t + \alpha X + \tfrac{1}{2}i.$$

In Example 5.22 we have seen that L_{q_0} is not globally γ^s–hypoelliptic but it is globally γ^s–solvable, for any $s \geq 1$. In addition, in Example 2.31 we have seen that the operator L_{q_0} is not globally solvable in the smooth sense. Therefore, the operator

$$L_q = \partial_t + \alpha X + \cos(t) + h(x) + \tfrac{1}{2}i$$

is not globally γ^s–hypoelliptic but it is globally γ^s–solvable, for any $s \geq 1$. Moreover, L_q is not globally solvable in the smooth sense. Similarly, we can conclude that

$$L_q = \partial_t + \sqrt{2}X + \cos(t) + h(x) + \tfrac{1}{2}i$$

is globally γ^s–hypoelliptic, for any $s \geq 1$, because in Example 5.22 we have seen that the operator $L_q = \partial_t + \sqrt{2}X + \frac{1}{2}i$ has this property.

Consider now the operator

$$L_{q_1} = \partial_t + \alpha X + q_1(t, x),$$

where $q_1(t, x) = \cos(t) + h(x) + \alpha i$. Analogously to the previous example, we have

$$(\partial_t + \alpha X)Q(t, x) = q(t, x) - \alpha i$$

and by Proposition 6.12, it is enough to study the operator

$$L_{q_{1_0}} = \partial_t + \alpha X + i\alpha.$$

This operator was already completely characterized in Examples 2.31 and 5.22. Hence, we conclude that

$$L_{q_1} = \partial_t + \alpha X + \cos(t) + h(x) + \alpha i$$

is not globally γ^s–hypoelliptic but it is globally γ^s–solvable, for any $s \geq 1$. Moreover, L_q is not globally solvable in the smooth sense.

Example 6.15. $G = \mathbb{S}^3 \times \mathbb{S}^3$

Let us analyze the same operator studied in Example 3.23. Consider

$$L = X_1 + \sqrt{2}X_2 + q(x_1, x_2),$$

where X_1 acts in the first variable, X_2 acts in the second variable, and $q : \mathbb{S}^3 \to \mathbb{C}$ is expressed in Euler's angles by

$$q(x_1, x_2) = p_1(x_1) + \sqrt{2}\, p_2(x_2) + \tfrac{1}{2}i,$$

where p_1 and p_2 are the projections of $\mathrm{SU}(2) \simeq \mathbb{S}^3$ given in Euler's angle by

$$p_1(x(\phi,\theta,\psi)) = \cos\left(\tfrac{\theta}{2}\right) e^{i(\phi+\psi)/2} \quad \text{and} \quad p_2(x(\phi,\theta,\psi)) = i \sin\left(\tfrac{\theta}{2}\right) e^{i(\phi-\psi)/2},$$

with $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$, $-2\pi \leq \psi < 2\pi$. Notice that q is an analytic function, so $q \in \gamma^s(\mathbb{T}^1 \times \mathbb{S}^3)$, for any $s \geq 1$. Moreover, the function $Q(x_1, x_2) = 2i(p_2(x_2) - p_1(x_1))$ satisfies

$$(X_1 + \sqrt{2}X_2)Q(x_1, x_2) = q(x_1, x_2) - \tfrac{1}{2}i.$$

The set $\mathcal{N}$ for the operator

$$L_0 = X_1 + \sqrt{2}X_2 + \tfrac{1}{2}i$$

has infinitely many elements (see Example 2.32), so L is not globally γ^s–hypoelliptic, for any $s \geq 1$. Since L_0 is globally solvable in the smooth sense, we conclude by Corollary 5.9 that L is globally γ^s–solvable, for any $s \geq 1$.

6.3 The general case

We can use the results about perturbations of constant coefficient vector fields presented in Section 6.2 to study the operator L_{aq} defined on $G_1 \times G_2$ by

$$L_{aq} = X_1 + a(x_1)X_2 + q(x_1, x_2),$$

where $a \in \Gamma_{\{M_k\}}(G_1)$ is a real-valued ultradifferentiable function and $q \in \Gamma_{\{M_k\}}(G_1 \times G_2)$.

As discussed in Section 6.2, we will assume that there exists $Q \in \Gamma_{\{M_k\}}(G_1 \times G_2)$ such that

$$(X_1 + a(x_1)X_2)Q = q - q_0,$$

where q_0 is the average of q in $G_1 \times G_2$. We have that $e^Q \in \Gamma_{\{M_k\}}(G_1 \times G_2)$ and

$$e^Q \circ L_{aq} = L_{aq_0} \circ e^Q,$$

where $L_{aq_0} = X_1 + a(x_1)X_2 + q_0$. Now, we obtain

$$\Psi_a \circ L_{aq_0} = L_{a_0 q_0} \circ \Psi_a,$$

where $L_{a_0q_0} = X_1 + a_0X_2 + q_0$. Therefore,

$$\Psi_a \circ e^Q \circ L_{aq} = \Psi_a \circ L_{aq_0} \circ e^Q = L_{a_0q_0} \circ \Psi_a \circ e^Q.$$

The next result is a consequence of what was done previously.

Proposition 6.16. *The operator L_{aq} is globally $\Gamma_{\{M_k\}}$–hypoelliptic if and only if $L_{a_0q_0}$ is globally $\Gamma_{\{M_k\}}$–hypoelliptic. Similarly, the operator L_{aq} is globally $\Gamma_{\{M_k\}}$–solvable if and only if $L_{a_0q_0}$ is globally $\Gamma_{\{M_k\}}$–solvable.*

Example 6.17. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Consider

$$L_{aq} = \partial_t + a(t)X + q(t,x)$$

where $X \in \mathfrak{s}^2$, $a(t) = \sin(t) + \alpha$, and $q(t,x) = \cos(t) + (\sin(t) + \alpha)h(x) + \frac{1}{2}i$, where h is expressed in Euler's angle by

$$h(x(\phi,\theta,\psi)) = -\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\phi+\psi}{2}\right),$$

where $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$, $-2\pi \leq \psi < 2\pi$. Notice that q is an analytic function, which implies that $q \in \gamma^s(\mathbb{T}^1 \times \mathbb{S}^3)$ for all $s \geq 1$.

The vector field X is the operator ∂_ψ in Euler's angle and we have that $X\mathrm{tr}(x) = h(x)$, where the trace function tr is expressed in Euler's angle by

$$\mathrm{tr}(x(\phi,\theta,\psi)) = 2\cos\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\phi+\psi}{2}\right).$$

The function $Q(t,x) = \sin(t) + \mathrm{tr}(x)$ satisfies

$$(\partial_t + a(t)X)Q(t,x) = q(t,x) - \tfrac{1}{2}i.$$

By Proposition 6.16, the operator

$$L_{aq} = \partial_t + (\sin(t) + \alpha)X + \left\{\cos(t) + (\sin(t) + \alpha)h(x) + \tfrac{1}{2}i\right\}$$

is globally γ^s–hypoelliptic if and only if

$$L_{a_0q_0} = \partial_t + \alpha X + \tfrac{1}{2}i$$

is globally γ^s–hypoelliptic. In Example 5.22 we have seen that $L_{a_0q_0}$ is globally γ^s–hypoelliptic, for any $s \geq 1$. We conclude that L_{aq} is globally γ^s–hypoelliptic for any $s \geq 1$, which implies

that L_{aq} is also globally γ^s–solvable, for any $s \geq 1$. In addition, the operator L_{aq} is neither globally hypoelliptic nor globally solvable, because $L_{a_0q_0}$ has these properties.

Similarly, the operator

$$L_{aq} = \partial_t + (\sin(t) + \alpha)X + \{\cos(t) + (\sin(t) + \alpha)h(x) + \alpha i\}$$

is not globally γ^s–hypoelliptic but is globally γ^s–solvable because

$$L_{a_0q_0} = \partial_t + \alpha X + \alpha i$$

has these properties. Again, the operator L_{aq} is neither globally hypoelliptic nor globally solvable.

Example 6.18. $G = \mathbb{S}^3 \times \mathbb{S}^3$

Consider the operator

$$L_{hq} = X_1 + h(x_1)X_2 + q(x_1, x_2),$$

where q is given by:

$$q(x_1, x_2) = p_1(x_1) + h(x_1)p_2(x_2) + \tfrac{1}{2}i,$$

where p_1 and p_2 are the projections of $\mathrm{SU}(2) \simeq \mathbb{S}^3$ given in Euler's angle by

$$p_1(x(\phi, \theta, \psi)) = \cos\left(\tfrac{\theta}{2}\right) e^{i(\phi+\psi)/2} \quad \text{and} \quad p_2(x(\phi, \theta, \psi)) = i \sin\left(\tfrac{\theta}{2}\right) e^{i(\phi-\psi)/2},$$

where $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$, $-2\pi \leq \psi < 2\pi$. As in Example 3.23, the function $Q(x_1, x_2) = 2i(p_2(x_2) - p_1(x_1))$ satisfies

$$(X_1 + h(x_1)X_2)Q(x_1, x_2) = q(x_1, x_2) - \tfrac{1}{2}i.$$

Since Q is analytic, we have that $Q \in \gamma^s(\mathbb{S}^3 \times \mathbb{S}^3)$, for any $s \geq 1$. By Proposition 6.16, we can extract the global properties of L_{hq} from the operator

$$L_{h_0q_0} = X_1 + \alpha X_2 + \tfrac{1}{2}i.$$

We have seen in Example 5.23 that the operator $L_{h_0q_0}$ is globally γ^s–hypoelliptic for any $s \geq 1$, but is not globally solvable (in the smooth sense). By Proposition 6.16, the operator L_{hq} has these same properties.

Appendices

A Partial Fourier series

Let G_1 and G_2 be compact Lie groups, and set $G = G_1 \times G_2$. Consider the representations $\xi \in \mathrm{Hom}(G_1, \mathrm{Aut}(V_1))$ and $\eta \in \mathrm{Hom}(G_2, \mathrm{Aut}(V_2))$. The external tensor product representation $\xi \otimes \eta$ of G on $V_1 \otimes V_2$ is defined by

$$\begin{array}{rccl} \xi \otimes \eta: & G_1 \times G_2 & \to & \mathrm{Aut}(V_1 \otimes V_2) \\ & (x_1, x_2) & \mapsto & (\xi \otimes \eta)(x_1, x_2): \quad V_1 \otimes V_2 \to V_1 \otimes V_2 \\ & & & \qquad\qquad (v_1, v_2) \mapsto \xi(x_1)(v_1) \otimes \eta(x_2)(v_2) \end{array}$$

We point out that the external tensor product of unitary representation is also unitary. Moreover, if $\xi \in \mathrm{Hom}(G, \mathrm{U}(d_\xi))$ and $\eta \in \mathrm{Hom}(G, \mathrm{U}(d_\eta))$ are matrix unitary representations, then $\xi \otimes \eta \in \mathrm{Hom}(G, \mathrm{U}(d_\xi d_\eta))$ is also a matrix unitary representation and

$$\xi \otimes \eta(x_1, x_2) = \xi(x_1) \otimes \eta(x_2) \in \mathbb{C}^{d_\xi d_\eta \times d_\xi \times d_\eta},$$

where $\xi(x_1) \otimes \eta(x_2)$ is the Kronecker product of these matrices.

It is enough to study continuous irreducible unitary representations of G_1 and G_2 to obtain the elements of $\widehat{G}$, since for every $[\phi] \in \widehat{G}$, there exist $[\xi] \in \widehat{G_1}$ and $[\eta] \in \widehat{G_2}$ such that $\phi \sim \xi \otimes \eta$, that is, $[\phi] = [\xi \otimes \eta] \in \widehat{G}$ and $d_\phi = d_\xi \cdot d_\eta$. Moreover, $[\xi_1 \otimes \eta_1] = [\xi_2 \otimes \eta_2]$ if and only if $[\xi_1] = [\xi_2]$ and $[\eta_1] = [\eta_2]$. The proof of this fact can be found on [9] (Chapter II, Proposition 4.14). Therefore, the map $[\xi \otimes \eta] \mapsto ([\xi], [\eta])$ is a bijection from $\widehat{G}$ to $\widehat{G_1} \times \widehat{G_2}$.

It is easy to see that $\mathcal{L}_G = \mathcal{L}_{G_1} + \mathcal{L}_{G_2}$, so we have $\nu_{[\xi \otimes \eta]} = \nu_{[\xi]} + \nu_{[\eta]}$. Therefore we have

$$\frac{1}{2}(\langle \xi \rangle + \langle \eta \rangle) \leq \langle \xi \otimes \eta \rangle \leq \langle \xi \rangle + \langle \eta \rangle, \tag{A.1}$$

for all $[\xi] \in \widehat{G_1}$ and $[\eta] \in \widehat{G_2}$.

Let $f \in L^1(G)$ and $[\phi] \in \widehat{G}$. Let $[\xi] \in \widehat{G_1}$ and $[\eta] \in \widehat{G_2}$ such that $[\phi] = [\xi \otimes \eta]$. Notice that

$$\begin{aligned}\widehat{f}(\xi \otimes \eta) &= \int_G f(x)(\xi \otimes \eta)(x)^* \, dx \\ &= \int_{G_2}\int_{G_1} f(x_1,x_2)(\xi(x_1) \otimes \eta(x_2))^* \, dx_1 dx_2 \\ &= \int_{G_2}\int_{G_1} f(x_1,x_2)\xi(x_1)^* \otimes \eta(x_2)^* \, dx_1 dx_2.\end{aligned}$$

Thus $\widehat{f}(\xi \otimes \eta) \in \mathbb{C}^{d_\xi d_\eta \times d_\xi d_\eta}$ with elements

$$\begin{aligned}\widehat{f}(\xi \otimes \eta)_{ij} &= \int_{G_2}\int_{G_1} f(x_1,x_2)(\xi(x_1)^* \otimes \eta(x_2)^*)_{ij} \, dx_1 dx_2 \\ &= \int_{G_2}\int_{G_1} f(x_1,x_2)\overline{\xi(x_1)_{nm}}\,\overline{\eta(x_2)_{sr}} \, dx_1 dx_2\end{aligned}$$

where $1 \le m,n \le d_\xi$, $1 \le r,s \le d_\eta$ are given by

$$\begin{aligned} m &= \left\lfloor \tfrac{i-1}{d_\eta} \right\rfloor + 1, & r &= i - \left\lfloor \tfrac{i-1}{d_\eta} \right\rfloor d_\eta, \\ n &= \left\lfloor \tfrac{j-1}{d_\eta} \right\rfloor + 1, & s &= j - \left\lfloor \tfrac{j-1}{d_\eta} \right\rfloor d_\eta. \end{aligned}$$

Similarly for $u \in \mathcal{D}'(G)$, we have

$$\widehat{u}(\xi \otimes \eta)_{ij} = \left\langle u, \overline{(\xi \otimes \eta)_{ji}} \right\rangle = \left\langle u, \overline{\xi_{nm} \times \eta_{sr}} \right\rangle,$$

where $(\xi_{nm} \times \eta_{sr})(x_1,x_2) := \xi(x_1)_{nm}\eta(x_2)_{sr}$.

Definition A.1. *Let G_1 and G_2 be compact Lie groups and, set $G = G_1 \times G_2$. Let $f \in L^1(G)$ and $\xi \in Rep(G_1)$. The x_1–Fourier coefficient of f is defined by*

$$\widehat{f}(\xi, x_2) = \int_{G_1} f(x_1,x_2)\,\xi(x_1)^* \, dx_1 \in \mathbb{C}^{d_\xi \times d_\xi}, \quad x_2 \in G_2,$$

with components

$$\widehat{f}(\xi, x_2)_{mn} = \int_{G_1} f(x_1,x_2)\,\overline{\xi(x_1)_{nm}} \, dx_1, \quad 1 \le m,n \le d_\xi.$$

Similarly, for $\eta \in Rep(G_2)$, we define the x_2–Fourier coefficient of f as

$$\widehat{f}(x_1, \eta) = \int_{G_2} f(x_1,x_2)\,\eta(x_2)^* \, dx_2 \in \mathbb{C}^{d_\eta \times d_\eta}, \quad x_1 \in G_1,$$

with components

$$\widehat{f}(x_1, \eta)_{rs} = \int_{G_2} f(x_1,x_2)\,\overline{\eta(x_2)_{sr}} \, dx_1, \quad 1 \le r,s \le d_\eta.$$

By the definition, the function

$$\begin{aligned}\widehat{f}(\xi,\,\cdot\,)_{mn}:\;\; G_2 &\longrightarrow \mathbb{C}\\ x_2 &\longmapsto \widehat{f}(\xi,x_2)_{mn}\end{aligned}$$

belongs to $L^1(G_1)$ for all $\xi \in \mathit{Rep}(G_1)$, $1 \leq m,n \leq d_\xi$. Similarly, the function

$$\begin{aligned}\widehat{f}(\,\cdot\,,\eta)_{rs}:\;\; G_1 &\longrightarrow \mathbb{C}\\ x_1 &\longmapsto \widehat{f}(x_1,\eta)_{rs}\end{aligned}$$

belongs to $L^1(G_2)$ for all $\eta \in \mathit{Rep}(G_2)$, $1 \leq r,s \leq d_\eta$.

Let $\xi \in \mathrm{Rep}(G_1)$ and $\eta \in \mathrm{Rep}(G_2)$. Since $\widehat{f}(\xi,\,\cdot\,)_{mn} \in L^1(G_2)$ for all $1 \leq m,n \leq d_\xi$, we can take its Fourier coefficient:

$$\widehat{\widehat{f}}(\xi,\eta)_{mn} := \int_{G_2} \widehat{f}(\xi,x_2)_{mn}\eta(x_2)^*\,dx_2 \in \mathbb{C}^{d_\eta\times d_\eta}$$

with components

$$\begin{aligned}\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} &= \int_{G_2} \widehat{f}(\xi,x_2)_{mn}\overline{\eta(x_2)_{sr}}\,dx_2\\ &= \int_{G_2}\int_{G_1} f(x_1,x_2)\overline{\xi(x_1)_{nm}}\,\overline{\eta(x_2)_{sr}}\,dx_1dx_2,\end{aligned}$$

for $1 \leq r,s \leq d_\eta$. Similarly, since $\widehat{f}(\,\cdot\,,\eta)_{rs} \in L^1(G_1)$ for all $1 \leq r,s \leq d_\eta$, we can take its Fourier coefficient:

$$\widehat{\widehat{f}}(\xi,\eta)_{rs} := \int_{G_1} \widehat{f}(x_1,\eta)_{rs}\xi(x_1)^*\,dx_1 \in \mathbb{C}^{d_\xi\times d_\xi}$$

with components

$$\begin{aligned}\widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}} &= \int_{G_1} \widehat{f}(x_1,\eta)_{rs}\overline{\xi(x_1)_{nm}}\,dx_1\\ &= \int_{G_1}\int_{G_2} f(x_1,x_2)\overline{\xi(x_1)_{nm}}\,\overline{\eta(x_2)_{sr}}\,dx_2dx_1,\end{aligned}$$

for $1 \leq m,n \leq d_\xi$.

Notice that

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = \widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}} = \widehat{f}(\xi\otimes\eta)_{ij},$$

with

$$i = d_\eta(m-1)+r, \quad j = d_\eta(n-1)+s,$$

for all $1 \leq m,n \leq d_\xi$ and $1 \leq r,s \leq d_\eta$.

Definition A.2. *Let G_1 and G_2 be compact Lie groups, and set $G = G_1 \times G_2$. Let $u \in \mathcal{D}'(G)$, $\xi \in Rep(G_1)$ and $1 \leq m, n \leq d_\xi$. The mn-component of the x_1–Fourier coefficient of u is the linear function defined by*

$$\begin{aligned} \widehat{u}(\xi, \cdot)_{mn} : \ C^\infty(G_2) &\longrightarrow \mathbb{C} \\ \psi &\longmapsto \langle \widehat{u}(\xi, \cdot)_{mn}, \psi \rangle := \left\langle u, \overline{\xi_{nm}} \times \psi \right\rangle_G. \end{aligned}$$

In similar way, for $\eta \in Rep(G_2)$ and $1 \leq r, s \leq d_\eta$, we define the rs-component of the x_2–Fourier coefficient of u as

$$\begin{aligned} \widehat{u}(\,\cdot\,, \eta)_{rs} : \ C^\infty(G_1) &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \langle \widehat{u}(\,\cdot\,, \eta)_{rs}, \varphi \rangle := \langle u, \varphi \times \overline{\eta_{sr}} \rangle_G. \end{aligned}$$

By definition, $\widehat{u}(\xi, \cdot)_{mn} \in \mathcal{D}'(G_2)$ and $\widehat{u}(\,\cdot\,, \eta)_{rs} \in \mathcal{D}'(G_1)$ for all $\xi \in Rep(G_1)$, $\eta \in Rep(G_2)$, $1 \leq m, n \leq d_\xi$ and $1 \leq r, s \leq d_\eta$.

Let $\xi \in \mathrm{Rep}(G_1)$ and $\eta \in \mathrm{Rep}(G_2)$. Since $\widehat{u}(\xi, \cdot)_{mn} \in \mathcal{D}'(G_2)$ for all $1 \leq m, n \leq d_\xi$, we can take its Fourier coefficient:

$$\widehat{\widehat{u}}(\xi, \eta)_{mn} := \langle \widehat{u}(\xi, \cdot)_{mn}, \eta^* \rangle \in \mathbb{C}^{d_\eta \times d_\eta}$$

with components

$$\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}} = \langle \widehat{u}(\xi, \cdot)_{mn}, \overline{\eta_{sr}} \rangle = \left\langle u, \overline{\xi_{nm}} \times \overline{\eta_{sr}} \right\rangle_G = \left\langle u, \overline{\xi_{nm} \times \eta_{sr}} \right\rangle_G,$$

for all $1 \leq r, s \leq d_\eta$. Now, since $\widehat{u}(\,\cdot\,, \eta)_{rs} \in \mathcal{D}'(G_1)$ for all $1 \leq r, s \leq d_\eta$ we can take its Fourier coefficient:

$$\widehat{\widehat{u}}(\xi, \eta)_{rs} := \langle \widehat{u}(\,\cdot\,, \eta)_{rs}, \xi^* \rangle \in \mathbb{C}^{d_\xi \times d_\xi}$$

with components

$$\widehat{\widehat{u}}(\xi, \eta)_{rs_{mn}} = \left\langle \widehat{u}(\,\cdot\,, \eta)_{rs}, \overline{\xi_{mn}} \right\rangle = \left\langle u, \overline{\xi_{nm}} \times \overline{\eta_{sr}} \right\rangle_G = \left\langle u, \overline{\xi_{nm} \times \eta_{sr}} \right\rangle_G,$$

for all $1 \leq m, n \leq d_\xi$. Notice that

$$\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}} = \widehat{\widehat{u}}(\xi, \eta)_{rs_{mn}} = \widehat{u}(\xi \otimes \eta)_{ij},$$

with

$$i = d_\eta(m-1) + r, \quad j = d_\eta(n-1) + s,$$

for all $1 \leq m, n \leq d_\xi$ and $1 \leq r, s \leq d_\eta$.

Notice that

$$\|\widehat{u}(\xi\otimes\eta)\|_{\mathtt{HS}}^2=\sum_{i,j=1}^{d_\xi d_\eta}|\widehat{u}(\xi\otimes\eta)_{ij}|^2=\sum_{m,n=1}^{d_\xi}\sum_{r,s=1}^{d_\eta}\left|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}\right|^2=:\left\|\widehat{\widehat{u}}(\xi,\eta)\right\|_{\mathtt{HS}}^2,\tag{A.2}$$

for all $[\xi]\in\widehat{G_1}$ and $[\eta]\in\widehat{G_2}$ whenever $u\in L^1(G)$ or $u\in\mathcal{D}'(G)$.

It follows from (A.1) and (A.2) the following adaption of Theorem 1.27 to characterize smooth functions and distributions defined on a product of compact Lie groups:

Theorem A.3. *Let G_1 and G_2 be compact Lie groups, and set $G=G_1\times G_2$. The following three statements are equivalent:*

(i) $f\in C^\infty(G)$;

(ii) For every $N>0$, there exists $C_N>0$ such that

$$\|\widehat{\widehat{f}}(\xi,\eta)\|_{\mathtt{HS}}\leq C_N(\langle\xi\rangle+\langle\eta\rangle)^{-N},\quad\forall[\xi]\in\widehat{G_1},\ [\eta]\in\widehat{G_2};$$

(iii) For every $N>0$, there exists $C_N>0$ such that

$$\left|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}\right|\leq C_N(\langle\xi\rangle+\langle\eta\rangle)^{-N},\quad\forall[\xi]\in\widehat{G_1},\ [\eta]\in\widehat{G_2},\ 1\leq m,n\leq d_\xi,\ 1\leq r,s\leq d_\eta.$$

Moreover, the following three statements are equivalent:

(iv) $u\in\mathcal{D}'(G)$;

(v) There exist $C,\ N>0$ such that

$$\|\widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}}\leq C(\langle\xi\rangle+\langle\eta\rangle)^{N},\quad\forall[\xi]\in\widehat{G_1},\ [\eta]\in\widehat{G_2};$$

(vi) There exist $C,\ N>0$ such that

$$\left|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}\right|\leq C(\langle\xi\rangle+\langle\eta\rangle)^{N},\quad\forall[\xi]\in\widehat{G_1},\ [\eta]\in\widehat{G_2},\ 1\leq m,n\leq d_\xi,\ 1\leq r,s\leq d_\eta.$$

In the next results we will investigate when a sequence of partial Fourier coefficients can define a smooth function or a distribution.

Theorem A.4. *Let G_1 and G_2 be compact Lie groups, $G=G_1\times G_2$, and let $\{\widehat{f}(\,\cdot\,,\eta)_{rs}\}$ be a sequence of functions on G_1. Define*

$$f(x_1,x_2):=\sum_{[\eta]\in\widehat{G_2}}d_\eta\sum_{r,s=1}^{d_\eta}\widehat{f}(x_1,\eta)_{rs}\eta_{sr}(x_1).$$

Then $f \in C^\infty(G)$ if and only if $\widehat{f}(\,\cdot\,,\eta)_{rs} \in C^\infty(G_1)$, for all $[\eta] \in \widehat{G_2}$, $1 \le r,s \le d_\eta$ and for every $\beta \in \mathbb{N}_0^n$ and $\ell > 0$ there exist $C_{\beta\ell} > 0$ such that

$$\left|\partial^\beta \widehat{f}(x_1,\eta)_{rs}\right| \le C_{\beta\ell}\langle\eta\rangle^{-\ell}, \quad \forall x_1 \in G_1,\ [\eta] \in \widehat{G},\ 1 \le r,s \le d_\eta.$$

Proof. ($\Longleftarrow$) It is sufficient to consider $N \in \mathbb{N}$ in Theorem A.3 to conclude that $f \in C^\infty(G)$. Recall that $-\nu_{[\xi]}$ is the eigenvalue of the Laplacian operator $\mathcal{L}_{G_1}$ associated to the eigenfunctions $\{\xi_{mn},\ 1 \le m,n \le d_\xi\}$, and we have

$$\widehat{\mathcal{L}_{G_1} g}(\xi)_{mn} = \left\langle \mathcal{L}_{G_1} g, \overline{\xi_{nm}} \right\rangle = \left\langle g, \mathcal{L}_{G_1}\overline{\xi_{nm}} \right\rangle = -\nu_{[\xi]}\left\langle g, \overline{\xi_{nm}} \right\rangle = -\nu_{[\xi]}\widehat{g}(\xi)_{mn},$$

for all $g \in C^\infty(G_1)$, $[\xi] \in \widehat{G_1}$, and $1 \le m,n \le d_\xi$. In particular, for $N \in \mathbb{N}$, we obtain

$$\begin{aligned}
\nu_{[\xi]}^N |\,\widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}}| &= \left|\widehat{\mathcal{L}_{G_1}^N \widehat{f}}(\xi,\eta)_{rs_{mn}}\right| \\
&= \left|\int_{G_1} \mathcal{L}_{G_1}^N \widehat{f}(x_1,\eta)_{rs}\overline{\xi(x_1)_{nm}}\,dx_1\right| \\
&\le \int_{G_1} |\mathcal{L}_{G_1}^N \widehat{f}(x_1,\eta)_{rs}||\xi(x_1)_{nm}|\,dx_1 \\
&\le \left(\int_{G_1} |\mathcal{L}_{G_1}^N \widehat{f}(x_1,\eta)_{rs}|^2\,dx_1\right)^{1/2}\left(\int_G |\xi(x_1)_{nm}|^2\,dx_1\right)^{1/2} \\
&\le \frac{1}{\sqrt{d_\xi}} \sum_{|\beta|=2N} \max_{x_1\in G_1} |\partial^\beta \widehat{f}(x_1,\eta)_{rs}|
\end{aligned}$$

By Proposition 1.24, there exists $C > 0$ such that $\langle\xi\rangle \le C\nu_{[\xi]}$ for all non-trivial $[\xi] \in \widehat{G_1}$. Thus for all $\ell = N$ we have

$$|\,\widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}}| \le C_N\langle\xi\rangle^{-N}\langle\eta\rangle^{-N} \le C_N 2^N(\langle\xi\rangle + \langle\eta\rangle)^{-N}.$$

Therefore $f \in C^\infty(G)$.

($\Longrightarrow$) Let $E_2 := (I - \mathcal{L}_{G_2})^{\frac{1}{2}}$. Since $f \in C^\infty(G)$, for all $\beta \in N_0^n$ and $N \in \mathbb{N}_0$ we have $\partial^\beta E_2^N f \in C^\infty(G)$ and then, by the compactness of G, there exists $C_{\beta N} \ge 0$ such that

$$|\partial^\beta E_2^N f(x_1,x_2)| \le C_{\beta N}, \qquad \forall (x_1,x_2) \in G_1 \times G_2. \tag{A.3}$$

Fix $\eta \in \mathrm{Rep}(G_2)$, $1 \leq r, s \leq d_\eta$. We already know that $\widehat{f}(\,\cdot\,,\eta)_{rs} \in C^\infty(G_1)$. Moreover

$$\begin{aligned}
|\langle\eta\rangle^N \partial^\beta \widehat{f}(x_1,\eta)_{rs}| &= |\partial^\beta \widehat{E_2^N f}(x_1,\eta)_{rs}| \\
&= \left|\partial^\beta \int_{G_2} E_2^N f(x_1,x_2)\overline{\eta(x_2)_{sr}}\,dx_2\right| \\
&\leq \int_{G_2} |\partial^\beta E_2^N f(x_1,x_2)||\overline{\eta(x_2)_{sr}}|\,dx_2 \\
&\leq \left(\int_{G_2} |\partial^\beta E_2^N f(x_1,x_2)|^2\,dx_2\right)^{\frac{1}{2}} \left(\int_{G_2} |\eta(x_2)_{sr}|^2\,dx_2\right)^{\frac{1}{2}} \\
&\overset{(A.3)}{\leq} \tfrac{1}{\sqrt{d_\eta}} C_{\beta N}.
\end{aligned}$$

Therefore,

$$|\partial^\beta \widehat{f}(x_1,\eta)_{rs}| \leq C_{\beta N}\langle\eta\rangle^{-N},$$

for all $x_1 \in G_1$, $[\eta] \in \widehat{G_2}$, $1 \leq r, s \leq d_\eta$. □

Theorem A.5. *Let G_1 and G_2 be compact Lie groups, set $G = G_1 \times G_2$, and let $\{\widehat{u}(\,\cdot\,,\eta)_{rs}\}$ be a sequence of distributions on G_1. Define*

$$u = \sum_{[\eta]\in\widehat{G}} d_\eta \sum_{r,s=1}^{d_\eta} \widehat{u}(\,\cdot\,,\eta)_{rs}\eta_{sr}$$

Then $u \in \mathcal{D}'(G)$ if and only if there exist $K \in \mathbb{N}$ and $C > 0$ such that

$$\left|\langle\widehat{u}(\cdot,\eta)_{rs},\varphi\rangle\right| \leq C\, p_K(\varphi)\langle\eta\rangle^K, \tag{A.4}$$

for all $\varphi \in C^\infty(G_1)$ and $[\eta] \in \widehat{G}$, where $p_K(\varphi) := \sum_{|\beta|\leq K} \|\partial^\beta\varphi\|_{L^\infty(G_1)}$.

Proof. ($\Longleftarrow$) Take $\varphi = \overline{\xi_{nm}}$, $[\xi] \in \widehat{G_1}$, $1 \leq m, n \leq d_\xi$. Let $\beta \in \mathbb{N}_0^n$, $|\beta| \leq K$, with K as in (A.4). Since the symbol of ∂^β at $x_1 \in G_1$ and $\xi \in \mathit{Rep}(G_1)$ is given by

$$\sigma_{\partial^\beta}(x_1,\xi) = \xi(x_1)^*(\partial^\beta\xi)(x_1),$$

we have

$$\begin{aligned}
|\partial^\beta \xi_{nm}(x_1)| &= \left|\sum_{i=1}^{d_\xi} \xi_{ni}(x)\sigma_{\partial^\beta}(\xi)_{im}\right| \\
&\leq \sum_{i=1}^{d_\xi} |\xi_{ni}(x)||\sigma_{\partial^\beta}(\xi)_{im}| \\
&\leq \left(\sum_{i=1}^{d_\xi} |\xi_{ni}(x)^2|\right)^{1/2} \left(\sum_{i=1}^{d_\xi} |\sigma_{\partial^\beta}(\xi)_{im}|^2\right)^{1/2}
\end{aligned}$$

Let $M \in \mathbb{Z}$ satisfying $M \geq \frac{\dim G_1}{2}$. By (1.6) we have

$$\left(\sum_{i=1}^{d_\xi} |\xi_{ni}(x)^2|\right)^{1/2} \leq \left(\sum_{i=1}^{d_\xi} \|\xi_{ni}\|^2_{L^\infty(G_1)}\right)^{1/2} \leq C_M \sqrt{d_\xi} \langle \xi \rangle^M$$

and, by Proposition 1.25, there exists $C > 0$ such that

$$d_\xi \leq C \langle \xi \rangle^M.$$

Moreover, notice that

$$\left(\sum_{i=1}^{d_\xi} |\sigma_{\partial^\beta}(\xi)_{im}|^2\right)^{1/2} \leq \|\sigma_{\partial^\beta}(\xi)\|_{\mathtt{HS}} \leq \sqrt{d_\xi} \|\sigma_{\partial^\beta}(\xi)\|_{op} \leq \sqrt{d_\xi} C_0^{|\beta|} \langle \xi \rangle^{|\beta|},$$

where the last inequalities come from (1.10) and (1.11). Hence

$$\begin{aligned} |\partial^\beta \xi_{nm}(x_1)| &\leq C \langle \xi \rangle^M \sqrt{d_\xi} \|\sigma_{\partial^\beta}(\xi)\|_{\mathtt{HS}} \\ &\leq C \langle \xi \rangle^M d_\xi \|\sigma_{\partial^\beta}(\xi)\|_{op} \\ &\leq C C_0^{|\beta|} \langle \xi \rangle^{2M+|\beta|}. \end{aligned}$$

Then

$$p_K(\overline{\xi_{nm}}) = p_K(\xi_{nm}) \leq C \langle \xi \rangle^{2M+K}.$$

Hence

$$\begin{aligned} \left|\widehat{\widehat{u}}(\xi,\eta)_{rs_{mn}}\right| = \left|\langle \widehat{u}(\cdot,\eta)_{rs}, \overline{\xi_{nm}} \rangle\right| &\leq C\, p_K(\overline{\xi_{nm}}) \langle \eta \rangle^K \\ &\leq C \langle \xi \rangle^{2M+K} \langle \eta \rangle^{2M+K} \\ &\leq C(\langle \xi \rangle + \langle \eta \rangle)^{2(2M+K)}. \end{aligned}$$

Therefore $u \in \mathcal{D}'(G)$.

($\Longrightarrow$) Since $u \in \mathcal{D}'(G)$, then there exist $C > 0$ and $K \in \mathbb{N}$ such that

$$\left|\widehat{\widehat{u}}(\xi,\eta)_{rs_{mn}}\right| \leq C(\langle \xi \rangle + \langle \eta \rangle)^K, \tag{A.5}$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq r, s \leq d_\eta$, and $1 \leq m, n \leq d_\xi$ and

$$u = \sum_{[\xi] \in \widehat{G_1}} \sum_{[\eta] \in \widehat{G}} d_\xi d_\eta \sum_{m,n=1}^{d_\xi} \sum_{r,s=1}^{d_\eta} \widehat{\widehat{u}}(\xi,\eta)_{rs_{mn}} \xi_{nm} \eta_{sr}.$$

For $\varphi \in C^\infty(G_1)$ we have

$$|\langle \widehat{u}(\cdot,\eta)_{rs}, \varphi\rangle| = |(u, \varphi \times \overline{\eta_{sr}})| = \left| \sum_{[\xi]\in\widehat{G_1}} \sum_{[\eta]\in\widehat{G}} d_\xi d_\eta \sum_{m,n=1}^{d_\xi} \sum_{k,\ell=1}^{d_\eta} \widehat{\widehat{u}}(\xi,\eta)_{k\ell_{mn}} \langle \xi_{nm}, \varphi\rangle_{G_1} \langle \eta_{\ell k}, \overline{\eta_{sr}}\rangle_{G_2} \right|$$

Notice that $\langle \eta_{\ell k}, \overline{\eta_{sr}}\rangle_{G_2} = \frac{1}{d_\eta}\delta_{\ell s}\delta_{kr}$, since the set $\mathcal{B}$ is orthonormal (see (1.1)). Moreover, $\widehat{\varphi}(\overline{\xi})_{mn} = \langle \xi_{nm}, \varphi\rangle_{G_1}$. So

$$\begin{aligned} |\langle \widehat{u}(\cdot,\eta)_{rs}, \varphi\rangle| &= \left| \sum_{[\xi]\in\widehat{G_1}} d_\xi \sum_{m,n=1}^{d_\xi} \widehat{\widehat{u}}(\xi,\eta)_{rs_{mn}} \widehat{\varphi}(\overline{\xi})_{mn} \right| \\ &\leq \sum_{[\xi]\in\widehat{G_1}} d_\xi \sum_{m,n=1}^{d_\xi} \left| \widehat{\widehat{u}}(\xi,\eta)_{rs_{mn}} \right| \left| \widehat{\varphi}(\overline{\xi})_{mn} \right| \\ &\leq C \sum_{[\xi]\in\widehat{G_1}} d_\xi \sum_{m,n=1}^{d_\xi} \left(\langle\xi\rangle + \langle\eta\rangle\right)^K \left| \widehat{\varphi}(\overline{\xi})_{mn} \right|, \end{aligned}$$

where the last inequality comes from (A.5). Notice that for all $K \in \mathbb{N}$ it holds $(\langle\xi\rangle + \langle\eta\rangle)^K \leq 2^K \langle\xi\rangle^K \langle\eta\rangle^K$. In addition, we have

$$\sum_{m,n=1}^{d_\xi} |\widehat{\varphi}(\xi)_{mn}| \leq \left(d_\xi^2 \sum_{m,n=1}^{d_\xi} |\widehat{\varphi}(\xi)_{mn}|^2 \right)^{\frac{1}{2}} = d_\xi \|\widehat{\varphi}(\xi)\|_{\mathtt{HS}}.$$

Since $\langle\xi\rangle = \langle\overline{\xi}\rangle$ and the summation is over all $\widehat{G_1}$, we have

$$\sum_{[\xi]\in\widehat{G_1}} d_\xi \langle\xi\rangle^K \sum_{m,n=1}^{d_\xi} |\widehat{\varphi}(\overline{\xi})_{mn}| = \sum_{[\xi]\in\widehat{G_1}} d_{\overline{\xi}} \langle\overline{\xi}\rangle^K \sum_{m,n=1}^{d_\xi} |\widehat{\varphi}(\xi)_{mn}| = \sum_{[\xi]\in\widehat{G_1}} d_\xi \langle\xi\rangle^K \sum_{m,n=1}^{d_\xi} |\widehat{\varphi}(\xi)_{mn}|.$$

Thus

$$\begin{aligned} |\langle \widehat{u}(\cdot,\eta)_{rs}, \varphi\rangle| &\leq C\langle\eta\rangle^K \sum_{[\xi]\in\widehat{G_1}} d_\xi \langle\xi\rangle^K \sum_{m,n=1}^{d_\xi} |\widehat{\varphi}(\overline{\xi})_{mn}| \\ &\leq C\langle\eta\rangle^K \sum_{[\xi]\in\widehat{G_1}} d_\xi^2 \langle\xi\rangle^K \|\widehat{\varphi}(\xi)\|_{\mathtt{HS}} \end{aligned}$$

The series $\sum_{[\xi]\in\widehat{G_1}} d_\xi^2 \langle\xi\rangle^{-2t}$ converges if and only if $t > \frac{\dim G_1}{2}$, which implies that there exists $C > 0$ such that $d_\xi \leq C\langle\xi\rangle^{\dim G_1}$, for all $[\xi] \in \widehat{G_1}$. Hence,

$$
\begin{aligned}
|\langle \widehat{u}(\cdot,\eta)_{rs},\varphi\rangle| &= C\langle\eta\rangle^{K}\sum_{[\xi]\in\widehat{G_1}}\left(d_\xi\langle\xi\rangle^{-\dim G_1}\right)\left(d_\xi\langle\xi\rangle^{K+\dim G_1}\|\widehat{\varphi}(\xi)\|_{\mathtt{HS}}\right)\\
&\leq C\langle\eta\rangle^{K}\left(\sum_{[\xi]\in\widehat{G_1}} d_\xi^2\langle\xi\rangle^{-2\dim G_1}\right)^{\frac{1}{2}}\left(\sum_{[\xi]\in\widehat{G_1}} d_\xi^2\langle\xi\rangle^{2(K+\dim G_1)}\|\widehat{\varphi}(\xi)\|_{\mathtt{HS}}^2\right)^{\frac{1}{2}}\\
&\leq C\langle\eta\rangle^{K}\left(\sum_{[\xi]\in\widehat{G_1}} d_\xi\langle\xi\rangle^{2(K+2\dim G_1)}\|\widehat{\varphi}(\xi)\|_{\mathtt{HS}}^2\right)^{\frac{1}{2}}
\end{aligned}
$$

Let $L\in\mathbb{N}_0$ such that $K+2\dim G_1\leq 2L$. So

$$
\begin{aligned}
|\langle \widehat{u}(\cdot,\eta)_{rs},\varphi\rangle| &\leq C\langle\eta\rangle^{K}\left(\sum_{[\xi]\in\widehat{G_1}} d_\xi\langle\xi\rangle^{4L}\|\widehat{\varphi}(\xi)\|_{\mathtt{HS}}^2\right)^{\frac{1}{2}}\\
&= C\langle\eta\rangle^{K}\left(\sum_{[\xi]\in\widehat{G_1}} d_\xi\|\widehat{E_1^{2L}\varphi}(\xi)\|_{\mathtt{HS}}^2\right)^{\frac{1}{2}}\\
&= C\langle\eta\rangle^{K}\|E_1^{2L}\varphi\|_{L^2(G_1)}\\
&\leq C\|E_1^{2L}\varphi\|_{L^2(G_1)}\langle\eta\rangle^{2L},
\end{aligned}
$$

where $E_1=(I-\mathcal{L}_{G_1})^{\frac{1}{2}}$, and the last equality comes from the Plancherel formula (1.3). Notice that

$$\|E_1^{2L}\varphi\|_{L^2(G_1)}\leq\|E_1^{2L}\varphi\|_{L^\infty(G_1)}=\|(I-\mathcal{L}_{G_1})^{L}\varphi\|_{L^\infty(G_1)}\leq Cp_{2L}(\phi).$$

Therefore,

$$|\langle \widehat{u}(\cdot,\eta)_{rs},\varphi\rangle|\leq Cp_{2L}(\phi)\langle\eta\rangle^{2L}.$$

□

Now we will present the characterization of ultradifferentiable functions and ultradistributions in Komatsu classes of both Roumieu and Beurling types through the analysis of the behavior of their partial Fourier series. First, as in the smooth case, we have the following characterization of ultradifferentiable functions and ultradistributions:

Theorem A.6. *Let G_1 and G_2 be compact Lie groups, and set $G=G_1\times G_2$. The following three statements are equivalent:*

(i) $f\in\Gamma_{\{M_k\}}(G)$;

(ii) There exist $C, N > 0$ such that

$$\| \widehat{\widehat{f}}(\xi,\eta)\|_{\mathtt{HS}} \leq C \exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\}, \quad \forall [\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2};$$

(iii) There exist $C, N > 0$ such that

$$\left| \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} \right| \leq C \exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\},$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m,n \leq d_\xi,\ 1 \leq r,s \leq d_\eta$.

Moreover, the following three statements are equivalent:

(iv) $u \in \Gamma'_{\{M_k\}}(G)$;

(v) For every $N > 0$, there exists $C_N > 0$ such that

$$\| \widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}} \leq C_N \exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\}, \quad \forall [\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2};$$

(vi) For every $N > 0$, there exists $C_N > 0$ such that

$$\left| \widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} \right| \leq C_N \exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\},$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m,n \leq d_\xi,\ 1 \leq r,s \leq d_\eta$.

Theorem A.7. *Let G_1 and G_2 be compact Lie groups, and set $G = G_1 \times G_2$. The following three statements are equivalent:*

(i) $f \in \Gamma_{(M_k)}(G)$;

(ii) For every $N > 0$, there exists $C_N > 0$ such that

$$\| \widehat{\widehat{f}}(\xi,\eta)\|_{\mathtt{HS}} \leq C_N \exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\}, \quad \forall [\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2};$$

(iii) For every $N > 0$, there exists $C_N > 0$ such that

$$\left| \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} \right| \leq C_N \exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\},$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m,n \leq d_\xi,\ 1 \leq r,s \leq d_\eta$.

Moreover, the following three statements are equivalent:

(iv) $u \in \Gamma'_{(M_k)}(G)$;

(v) There exist $C, N > 0$ such that

$$\|\widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}} \leq C\exp\{M(N(\langle\xi\rangle+\langle\eta\rangle))\}, \quad \forall[\xi]\in\widehat{G_1},\ [\eta]\in\widehat{G_2};$$

(vi) There exist $C, N > 0$ such that

$$\left|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}\right| \leq C\exp\{M(N(\langle\xi\rangle+\langle\eta\rangle))\},$$

for all $[\xi]\in\widehat{G_1}$, $[\eta]\in\widehat{G_2}$, $1\leq m,n\leq d_\xi$, $1\leq r,s\leq d_\eta$.

Theorem A.8. *Let G_1 and G_2 be compact Lie groups, set $G = G_1\times G_2$, and let $f\in C^\infty(G)$. Then $f\in\Gamma_{\{M_k\}}(G)$ if and only if $\widehat{f}(\,\cdot\,,\eta)_{rs}\in\Gamma_{\{M_k\}}(G_1)$ for every $[\eta]\in\widehat{G_2}$, $1\leq r,s\leq d_\eta$ and there exist $h, C, \varepsilon > 0$ such that*

$$\max_{x_1\in G_1}|\partial^\alpha\widehat{f}(x_1,\eta)_{rs}| \leq Ch^{|\alpha|}M_{|\alpha|}\exp\{-M(\varepsilon\langle\eta\rangle)\}, \tag{A.6}$$

for all $[\eta]\in\widehat{G_2}$, $1\leq r,s\leq d_\eta$ and $\alpha\in\mathbb{N}_0^{d_1}$.

Proof. ($\Longleftarrow$) Let $\alpha\in\mathbb{N}_0$. Recall that $-\nu_{[\xi]}$ is the eigenvalue of the Laplacian operator $\mathcal{L}_{G_1}$ associated to the eigenfunctions $\{\xi_{mn},\ 1\leq m,n\leq d_\xi\}$. By (1.7), we obtain

$$\begin{aligned}
\nu_{[\xi]}^\alpha|\widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}}| &= \left|\widehat{\mathcal{L}_{G_1}^\alpha f}(\xi,\eta)_{rs_{mn}}\right| \\
&= \left|\int_{G_1}\mathcal{L}_{G_1}^\alpha\widehat{f}(x_1,\eta)_{rs}\overline{\xi(x_1)_{nm}}\,dx_1\right| \\
&\leq \int_{G_1}|\mathcal{L}_{G_1}^\alpha\widehat{f}(x_1,\eta)_{rs}||\xi(x_1)_{nm}|\,dx_1 \\
&\leq \left(\int_{G_1}|\mathcal{L}_{G_1}^\alpha\widehat{f}(x_1,\eta)_{rs}|^2\,dx_1\right)^{1/2}\left(\int_G|\xi(x_1)_{nm}|^2\,dx_1\right)^{1/2}.
\end{aligned}$$

Notice that, by (1.1), we have $\|\xi_{nm}\|_{L^2(G_1)}\leq 1$, for all $[\xi]\in\widehat{G_1}$. Moreover, we can write $\mathcal{L}_{G_1}^\alpha$ as a sum of d_1^α derivatives of order 2α, where $d_1 = \dim G_1$. So, by (A.6), we obtain

$$\nu_{[\xi]}^\alpha|\widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}}| \leq Cd_1^\alpha h^{2\alpha}M_{2\alpha}\exp\{-M(\varepsilon\langle\eta\rangle)\}.$$

By Proposition 1.24, there exists $C>0$ such that $\langle\xi\rangle^2\leq C\nu_{[\xi]}$, for all non-trivial representation. By the property (M.2) of the sequence $\{M_k\}$, we have $M_{2\alpha}\leq AH^{2\alpha}M_\alpha^2$. Thus

$$|\widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}}| \leq C(\sqrt{d_1}hH)^{2\alpha}\langle\xi\rangle^{-2\alpha}M_\alpha^2\exp\{-M(\varepsilon\langle\eta\rangle)\}, \quad \forall\alpha\in\mathbb{N}_0$$

Hence,

$$\begin{aligned}|\widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}}| &\leq C\left(\inf_{\alpha\in\mathbb{N}_0}\frac{M_\alpha}{(\langle\xi\rangle(\sqrt{d_1}hH)^{-1})^\alpha}\right)^2\exp\{-M(\varepsilon\langle\eta\rangle)\\ &= C\exp\{-2M((\sqrt{d_1}hH)^{-1}\langle\xi\rangle)\}\exp\{-M(\varepsilon\langle\eta\rangle)\}\\ &\leq C\exp\{-M((\sqrt{d_1}hH)^{-1}\langle\xi\rangle)\}\exp\{-M(\varepsilon\langle\eta\rangle)\}\end{aligned}$$

Set $2N=\min\{(\sqrt{d_1}hH)^{-1},\varepsilon\}$. In this way, we get

$$|\widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}}|\leq C\exp\{-M(2N\langle\xi\rangle)\}\exp\{-M(2N\langle\eta\rangle)\}$$

and by Proposition 1.34,

$$|\widehat{\widehat{f}}(\xi,\eta)_{rs_{mn}}|\leq C\exp\{-M(N(\langle\xi\rangle+\langle\eta\rangle))\},$$

for all $[\xi]\in\widehat{G_1}$ non-trivial, $[\eta]\in\widehat{G_2}$. It is easy to see that we can also obtain this inequality for the trivial representation of G_2 from the hypothesis. Therefore $f\in\Gamma_{\{M_k\}}(G)$.

($\Longrightarrow$) We can characterize the elements of $\Gamma_{\{M_k\}}(G)$ as follows (Theorem 2.3 of [12]): $\varphi\in\Gamma_{\{M_k\}}(G)$ if and only if there exist $C,h>0$ such that

$$\max_{(x_1,x_2)\in G}|\partial_1^\alpha\partial_2^\beta\varphi(x_1,x_2)|\leq Ch^{|\alpha|+|\beta|}M_{|\alpha|+|\beta|},$$

for all $\alpha\in\mathbb{N}_0^{d_1},\beta\in\mathbb{N}_0^{d_2}$.

For $f\in\Gamma_{\{M_k\}}(G)$ we have

$$\begin{aligned}\nu_{[\eta]}^\beta|\partial_1^\alpha\widehat{f}(x_1,\eta)_{rs}| &= |\partial_1^\alpha\widehat{\mathcal{L}_{G_2}^\beta f}(x_1,\eta)_{rs}|\\ &\leq\int_{G_2}|\partial_1^\alpha\mathcal{L}_{G_2}^\beta f(x_1,x_2)||\overline{\eta(x_2)_{sr}}|\,dx_2\\ &\leq\left(\int_{G_2}|\partial_1^\alpha\mathcal{L}_{G_2}^\beta f(x_1,x_2)|^2\,dx_2\right)^{1/2}\left(\int_{G_2}|\eta(x_2)_{sr}|^2\,dx_2\right)^{1/2}\\ &\leq\frac{1}{\sqrt{d_\eta}}\sum_{|\gamma|=2\beta}\max_{(x_1,x_2)\in G}|\partial_1^\alpha\partial_2^\gamma f(x_1,x_2)|\\ &\leq Cd_2^\beta h^{|\alpha|+2\beta}M_{|\alpha|+2\beta},\end{aligned}$$

where $d_2=\dim G_2$. Thus, when $[\eta]$ is not trivial we obtain

$$\begin{aligned}|\partial_1^\alpha\widehat{f}(x_1,\eta)_{rs}| &\leq Cd_2^\beta h^{|\alpha|+2\beta}M_{|\alpha|+2\beta}\langle\eta\rangle^{-2\beta}\\ &\leq Ch^{|\alpha|+2\beta}AH^{|\alpha|+2\beta}M_{|\alpha|}h^{2\beta}d_2^\beta M_{2\beta}\langle\eta\rangle^{-2\beta}\\ &\leq C(hH)^{|\alpha|}M_{|\alpha|}h^{2\beta}d_2^\beta H^{4\beta}M_\beta^2\langle\eta\rangle^{-2\beta}\\ &\leq C(hH)^{|\alpha|}M_{\alpha|}\exp\{-2M((\sqrt{d_2}hH^2)^{-1}\langle\eta\rangle)\}\\ &\leq C(hH)^{|\alpha|}M_{\alpha|}\exp\{-M((\sqrt{d_2}hH^2)^{-1}\langle\eta\rangle)\}.\end{aligned}$$

Put $h' = hH$ and $\varepsilon = (\sqrt{d_2}hH^2)^{-1}$ to obtain

$$\max_{x_1 \in G_1} |\partial_1^\alpha \widehat{f}(x_1, \eta)_{rs}| \leq C h'^{|\alpha|} M_{|\alpha|} \exp\{-M(\varepsilon\langle\eta\rangle)\},$$

for all non-trivial $[\eta] \in \widehat{G_2},\ 1 \leq r, s \leq d_\eta,\ \alpha \in \mathbb{N}_0^n$.

For $[\eta] = [\mathbb{1}_{G_2}]$ we have

$$\begin{aligned} |\partial_1^\alpha \widehat{f}(x_1, \mathbb{1}_{G_2})| &= \left| \int_{G_2} \partial_1^\alpha f(x_1, x_2)\, dx_2 \right| \\ &\leq |\partial_1^\alpha f(x_1, x_2)| \\ &\leq C h^{|\alpha|} M_{|\alpha|}. \end{aligned}$$

In this way, adjusting C if necessary, we obtain

$$|\partial_1^\alpha \widehat{f}(x_1, \mathbb{1}_{G_2})| \leq C h^{|\alpha|} M_{|\alpha|} \exp\{-M(\varepsilon\langle\mathbb{1}_{G_2}\rangle)\}.$$

□

In the next results, we will be concerned about estimates involving only non-trivial representations since the trivial case is treated similarly as in the proof of Theorem A.8.

Theorem A.9. *Let G_1 and G_2 be compact Lie groups, set $G = G_1 \times G_2$, and let $f \in C^\infty(G)$. Then $f \in \Gamma_{(M_k)}(G)$ if and only if $\widehat{f}(\,\cdot\,, \eta)_{rs} \in \Gamma_{(M_k)}(G_1)$ for every $[\eta] \in \widehat{G_2}$, $1 \leq r, s \leq d_\eta$ and for all $h > 0$ and $\varepsilon > 0$ there exists $C_{h\varepsilon} > 0$ such that*

$$\max_{x_1 \in G_1} |\partial^\alpha \widehat{f}(x_1, \eta)_{rs}| \leq C_{h\varepsilon} h^{|\alpha|} M_{|\alpha|} \exp\{-M(\varepsilon\langle\eta\rangle)\},$$

for all $[\eta] \in \widehat{G_2},\ 1 \leq r, s \leq d_\eta$ and $\alpha \in \mathbb{N}_0^{d_1}$.

Proof. ($\Longleftarrow$) By the proof of Theorem A.8, we have

$$|\,\widehat{\widehat{f}}(\xi, \eta)_{rs_{mn}}| \leq C_{h\varepsilon} \exp\{-M((\sqrt{d_1}hH)^{-1}\langle\xi\rangle)\} \exp\{-M(\varepsilon\langle\eta\rangle)\}.$$

Given $N > 0$, choose $h = \dfrac{1}{2\sqrt{d_1}NH}$ and $\varepsilon = 2N$. So

$$\begin{aligned} |\,\widehat{\widehat{f}}(\xi, \eta)_{rs_{mn}}| &\leq C_N \exp\{-M(2N\langle\xi\rangle)\} \exp\{-M(2N\langle\eta\rangle)\} \\ &\leq C_N \exp\{-M(N(\langle\xi\rangle + \langle\eta\rangle))\}. \end{aligned}$$

Therefore $f \in \Gamma_{(M_k)}(G)$.

($\Longrightarrow$) We can characterize the elements of $\Gamma_{\{M_k\}}(G)$ as follows (see [12]): $\varphi \in \Gamma_{\{M_k\}}(G)$ if and only if for all $h > 0$ there exists $C_h > 0$ such that

$$\max_{(x_1,x_2)\in G} |\partial_1^\alpha \partial_2^\beta \varphi(x_1,x_2)| \leq C h^{|\alpha|+|\beta|} M_{|\alpha|+|\beta|},$$

for all $\alpha \in \mathbb{N}_0^{d_1}, \beta \in \mathbb{N}_0^{d_2}$. Let $f \in \Gamma_{(M_k)}$. In the proof of Theorem A.8 we have obtained

$$|\partial_1^\alpha \widehat{f}(x_1,\eta)_{rs}| \leq C_h (hH)^{|\alpha|} M_{|\alpha|} \exp\{-M((\sqrt{n}hH^2)^{-1}\langle\eta\rangle)\}$$

Given $\ell, \varepsilon > 0$. If $\ell\varepsilon < (\sqrt{n}H)^{-1}$, take $h = \ell H^{-1}$. In this case,

$$\begin{aligned} |\partial_1^\alpha \widehat{f}(x_1,\eta)_{rs}| &\leq C_{\ell\varepsilon} \ell^{|\alpha|} M_{|\alpha|} \exp\{-M((\sqrt{n}\ell H)^{-1}\langle\eta\rangle)\} \\ &\leq C_{\ell\varepsilon} \ell^{|\alpha|} M_{|\alpha|} \exp\{-M((\varepsilon\langle\eta\rangle)\} \end{aligned}$$

If $\ell\varepsilon \geq (\sqrt{n}H)^{-1}$, take $h = (\sqrt{n}\varepsilon H^2)^{-1}$. So

$$\begin{aligned} |\partial_1^\alpha \widehat{f}(x_1,\eta)_{rs}| &\leq C_{\ell\varepsilon} (\sqrt{n}\varepsilon H^2)^{-|\alpha|} M_{|\alpha|} \exp\{-M((\varepsilon\langle\eta\rangle)\} \\ &\leq C_{\ell\varepsilon} \ell^{|\alpha|} M_{|\alpha|} \exp\{-M((\varepsilon\langle\eta\rangle)\} \end{aligned}$$

□

Theorem A.10. *Let G_1 and G_2 be compact Lie groups, and set $G = G_1 \times G_2$. Then $u \in \Gamma'_{\{M_k\}}(G)$ if and only if for all $\varepsilon, h > 0$ there exists $C_{h\varepsilon} > 0$ such that*

$$|\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \varphi\rangle| \leq C_{h\varepsilon} \|\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\}, \quad \forall \varphi \in \Gamma_{M_k}(G_1),$$

where $\|\varphi\|_h := \sup_{\alpha,x_1} \big|\partial^\alpha \varphi(x_1)\big| h^{-|\alpha|} M_{|\alpha|}^{-1}$.

Proof. ($\Longleftarrow$) Let $\varphi = \overline{\xi_{nm}}$. We have

$$|\partial^\beta \xi_{nm}(x_1)| \leq C C_0^{|\beta|} \langle\xi\rangle^{p+|\beta|},$$

where p is any natural number satisfying $p \geq \frac{\dim G}{2}$ (see [11]). Then

$$\begin{aligned} |\big\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \overline{\xi_{nm}}\big\rangle| &\leq C_{h\varepsilon} \|\overline{\xi_{nm}}\|_h \exp\{M(\varepsilon\langle\eta\rangle)\} \\ &= C_{h\varepsilon} \sup_{\alpha,x_1} |\partial^\alpha \overline{\xi_{nm}}(x_1) h^{-|\alpha|} M_{|\alpha|}^{-1}| \exp\{M(\varepsilon\langle\eta\rangle)\} \\ &\leq C_{h\varepsilon} \langle\xi\rangle^p \sup_{\alpha} |C_0^{|\alpha|} \langle\xi\rangle^{|\alpha|} h^{-|\alpha|} M_{|\alpha|}^{-1}| \exp\{M(\varepsilon\langle\eta\rangle)\} \\ &= C_{h\varepsilon} \langle\xi\rangle^p \exp\{M(h^{-1}C_0\langle\xi\rangle)\} \exp\{M(\varepsilon\langle\eta\rangle)\} \end{aligned}$$

By Proposition 1.35, we have

$$\langle\xi\rangle^p \exp\{M(h^{-1}C_0\langle\xi\rangle)\} \leq A(h^{-1}C_0)^{-p}M_p \exp\{M(Hh^{-1}C_0\langle\xi\rangle)\}.$$

By Proposition 1.34, we obtain

$$|\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \overline{\xi_{nm}}\rangle| \leq C_{h\varepsilon} \exp\{M(H(Hh^{-1}C_0\langle\xi\rangle + \varepsilon\langle\eta\rangle))\}.$$

Given $N > 0$, choose $h = \frac{H^2C_0}{N}$ and $\varepsilon = \frac{N}{H}$. In this way,

$$|\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C_N \exp\{M(N(\langle\xi\rangle + \langle\eta\rangle))\},$$

which implies that $u \in \Gamma'_{\{M_k\}}(G)$.

($\Longrightarrow$) Since $u \in \Gamma'_{\{M_k\}}(G)$, for every $\ell > 0$, there exists $C_\ell > 0$ such that

$$|\langle u, \psi\rangle| \leq C_\ell \sup_{\alpha,\beta} \ell^{|\alpha|+|\beta|} M^{-1}_{|\alpha|+|\beta|} ||\partial_1^\alpha \partial_2^\beta \psi||_{L^\infty(G)},$$

for all $\psi \in \Gamma_{\{M_k\}}(G)$. Given $\varphi \in \Gamma_{\{M_k\}}(G_1)$, take $\psi = \varphi \times \overline{\eta_{sr}}$. Then

$$\begin{aligned}|\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \varphi\rangle| &= |\langle u, \varphi \times \overline{\eta_{sr}}\rangle| \\ &\leq C_\ell \sup_{\alpha,\beta} \ell^{|\alpha|+|\beta|} M^{-1}_{|\alpha|+|\beta|} \sup_{x_1} |\partial_1^\alpha \varphi(x_1)| \sup_{x_2} |\partial_2^\beta \overline{\eta_{sr}(x_2)}|\end{aligned}$$

Similar to what was done above, we have

$$\sup_{\beta,x_2} |\partial_2^\beta \overline{\eta_{sr}(x_2)} \ell^{|\beta|} M^{-1}_{|\beta|}| \leq C_\ell \exp\{M(H\ell C_0\langle\eta\rangle)\}.$$

By the property $M_{|\alpha|}M_{|\beta|} \leq M_{|\alpha|+|\beta|}$ we obtain

$$|\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \varphi\rangle| \leq C_\ell \sup_{\alpha,x_1} |\partial_1^\alpha \varphi(x_1) \ell^{|\alpha|} M^{-1}_{|\alpha|}| \exp\{M(H\ell C_0\langle\eta\rangle)\}.$$

Given $h, \varepsilon > 0$. If $\varepsilon h \leq C_0 H$, take $\ell = \frac{\varepsilon}{C_0 H}$. Thus $\ell \leq h^{-1}$ and

$$|\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \varphi\rangle| \leq C_{h\varepsilon} \|\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\}.$$

On the other hand, if $\varepsilon h > C_0 H$, take $\ell = h^{-1}$. Thus $H\ell C_0 < \varepsilon$ and

$$|\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \varphi\rangle| \leq C_{h\varepsilon} \|\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\}.$$

□

Theorem A.11. *Let G_1 and G_2 be compact Lie groups, and set $G = G_1 \times G_2$. Then $u \in \Gamma'_{(M_k)}(G)$ if and only if there exist $\varepsilon, h, C > 0$ such that*

$$|\langle \widehat{u}(\,\cdot\,,\eta)_{rs}, \varphi\rangle| \leq C\|\varphi\|_h \exp\{M(\varepsilon\langle\eta\rangle)\}, \quad \forall \varphi \in \Gamma_{(M_k)}(G_1).$$

The proof of this theorem is analogous to the Roumieu case and it will be omitted.

B Auxiliary results

Lemma B.1. *Let $\lambda \in \mathbb{C}$ and consider the equation*

$$\frac{d}{dt}u(t) + \lambda u(t) = f(t), \tag{B.1}$$

where $f \in C^\infty(\mathbb{T}^1)$.

If $\lambda \notin i\mathbb{Z}$ then the equation (B.1) has a unique solution that can be expressed by

$$u(t) = \frac{1}{1 - e^{-2\pi\lambda}} \int_0^{2\pi} e^{-\lambda s} f(t-s)\, ds, \tag{B.2}$$

or equivalently,

$$u(t) = \frac{1}{e^{2\pi\lambda} - 1} \int_0^{2\pi} e^{\lambda r} f(t+r)\, dr. \tag{B.3}$$

If $\lambda \in i\mathbb{Z}$ and $\int_0^{2\pi} e^{\lambda s} f(s)\, ds = 0$ then we have that

$$u(t) = e^{-\lambda t} \int_0^t e^{\lambda s} f(s)\, ds \tag{B.4}$$

is a solution of the equation (B.1).

Proof. Notice that the function u defined in (B.2), (B.3), and (B.4) is a smooth function on $\mathbb{T}^1$. Let us prove now that u defined in (B.2) is a solution of (B.1). Notice that

$$\begin{aligned}
\frac{d}{dt}u(t) &= \left(1 - e^{-2\pi\lambda}\right)^{-1} \int_0^{2\pi} e^{-\lambda s} \frac{d}{dt} f(t-s) ds \\
&= -\left(1 - e^{-2\pi\lambda}\right)^{-1} \int_0^{2\pi} e^{-\lambda s} \frac{d}{ds} f(t-s) ds \\
&= -\left(1 - e^{-2\pi\lambda}\right)^{-1} \left(e^{-\lambda s} f(t-s)\right)\Big|_{s=0}^{2\pi} - \int_0^{2\pi} f(t-s)(-\lambda) e^{-\lambda s} ds \\
&= -\left(1 - e^{-2\pi\lambda}\right)^{-1} \left(e^{-2\pi\lambda} f(t-2\pi) - f(t)\right) + \lambda \int_0^{2\pi} e^{-\lambda s} f(t-s) ds \\
&= -\left(1 - e^{-2\pi\lambda}\right)^{-1} f(t) \left(e^{-2\pi\lambda} - 1\right) - \lambda \left(1 - e^{-2\pi\lambda}\right)^{-1} \int_0^{2\pi} e^{-\lambda s} f(t-s) ds \\
&= f(t) - \lambda u(t)
\end{aligned}$$

Analogously we prove that u defined on (4.4) is a solution of (B.1). Finally, using the expression on (B.4) we obtain

$$\frac{d}{dt}u(t) = -\lambda e^{-\lambda t} \int_0^t e^{\lambda s} f(s) ds + e^{-\lambda t} e^{\lambda t} f(t) = -\lambda u(t) + f(t).$$

Therefore, the functions defined on (B.2), (B.3), and (B.4) are solutions of (B.1).

The expressions (B.2) and (B.3) are actually equivalents. Indeed, we have

$$u(t) = \frac{1}{1 - e^{-2\pi\lambda}} \int_0^{2\pi} e^{-\lambda s} f(t - s)\, ds \tag{B.2}$$

and make the substitution $s \mapsto -r + 2\pi$. Hence, $s = 0$ implies that $r = 2\pi$ and $s = 2\pi$ implies that $r = 0$. Moreover, $ds = -dr$. So

$$\frac{1}{1 - e^{-2\pi\lambda}} \int_0^{2\pi} e^{-\lambda s} f(t - s)\, ds = -\frac{1}{1 - e^{-2\pi\lambda}} \int_{2\pi}^{0} e^{\lambda(r-2\pi)} f(t + r - 2\pi)\, dr.$$

Since f is 2π–periodic, we obtain

$$-\frac{1}{1 - e^{-2\pi\lambda}} \int_{2\pi}^{0} e^{\lambda(r-2\pi)} f(t + r - 2\pi)\, dr = \frac{e^{-2\pi\lambda}}{1 - e^{-2\pi\lambda}} \int_0^{2\pi} e^{\lambda r} f(t + r)\, dr.$$

Now, we have

$$\frac{e^{-2\pi\lambda}}{1 - e^{-2\pi\lambda}} = \frac{1}{e^{2\pi\lambda} - 1}.$$

Therefore,

$$u(t) = \frac{1}{e^{2\pi\lambda} - 1} \int_0^{2\pi} e^{\lambda r} f(t + r)\, dr. \tag{B.3}$$

Let us prove now that the equation (B.1) has a unique solution when $\lambda \notin i\mathbb{Z}$. Assume that $u_1, u_2 \in C^\infty(\mathbb{T}^1)$ are solutions of (B.1). For $u = u_1 - u_2$ we obtain

$$\frac{d}{dt} u(t) + \lambda u(t) = 0,$$

or equivalently,

$$\frac{d}{dt} \left(e^{\lambda t} u(t) \right) = 0,$$

which implies that $u(t) = ce^{-\lambda t}$, for some $c \in \mathbb{C}$. By the fact that u is 2π–periodic, we have

$$u(t) = u(t + 2\pi) = ce^{-\lambda(t+2\pi)} = ce^{-\lambda t} e^{-\lambda 2\pi} = u(t) e^{-\lambda 2\pi},$$

for all $t \in [0, 2\pi]$. Since $\lambda \notin i\mathbb{Z}$, we have $e^{-\lambda 2\pi} \neq 1$ and we conclude that $u \equiv 0$, that is, $u_1 = u_2$.

To conclude the proof let us see how to obtain the expressions (B.2), (B.3), and (B.4).

If $\lambda \in i\mathbb{Z}$ we have that the function $t \mapsto e^{\lambda t}$ is well-defined on $\mathbb{T}^1$ and so we can write (B.1) as

$$\frac{d}{dt} \left(e^{\lambda t} u(t) \right) = e^{\lambda t} f(t).$$

Hence,

$$u(t) = e^{-\lambda t} \int_0^t e^{\lambda s} f(s) ds. \tag{B.4}$$

Since u must be 2π–periodic, we have $u(2\pi) = u(0) = 0$, that is,

$$\int_0^{2\pi} e^{\lambda s} f(s) ds = 0.$$

Assume now that $\lambda \notin i\mathbb{Z}$ and notice that $E \in \mathcal{D}'(\mathbb{T}^1)$ defined by $E = \left(1 - e^{-2\pi\lambda}\right)^{-1} e^{-\lambda t}$ is a fundamental solution of the operator $\frac{d}{dt} + \lambda$. Hence, a solution of (B.1) can be expressed as

$$\begin{aligned} u(t) = (E * f)(t) &= \int_0^{2\pi} \left(1 - e^{-2\pi\lambda}\right)^{-1} e^{-\lambda s} f(t-s) ds \\ &= \frac{1}{1 - e^{-2\pi\lambda}} \int_0^{2\pi} e^{-\lambda s} f(t-s) ds \end{aligned} \tag{B.2}$$

□

Lemma B.2. *Are equivalent:*

1. *There exist $C, M > 0$ such that*

$$|k + c_0 \mu_r(\eta) - iq| \geq C(|k| + \langle \eta \rangle)^{-M},$$

for all $k \in \mathbb{Z}$, $[\eta] \in \widehat{G}$, $1 \leq r \leq d_\eta$, whenever $k + c_0\mu_r(\eta) - iq \neq 0$.

2. *There exist $C, M > 0$ such that*

$$\left|1 - e^{\pm 2\pi i (c_0 \mu_r(\eta) - iq)}\right| \geq C \langle \eta \rangle^{-M}, \tag{B.5}$$

for all $[\eta] \in \widehat{G}$, $1 \leq r \leq d_\eta$, whenever $c_0\mu_r(\eta) - iq \notin \mathbb{Z}$.

Proof. Assume that 2. does not hold, so for all $j \in \mathbb{N}$ there exist $[\eta_j] \in \widehat{G}$ and $1 \leq r_j \leq d_{\eta_j}$ such that

$$0 < \left|1 - e^{\pm 2\pi i \left(c_0 \mu_{r_j}(\eta_j) - iq\right)}\right| < \tfrac{1}{j} \langle \eta_j \rangle^{-j}$$

Setting $c_0 = a_0 + ib_0$, with $a_0, b_0 \in \mathbb{R}$, we have that $|\mathrm{Re}(q) - b_0 \mu_{r_j}(\eta_j)| \to 0$ and there exists a sequence of integers $\{k_j\}$ such that $|k_j + a_0 \mu_{r_j}(\eta_j) + \mathrm{Im}(q)| \to 0$, when $j \to \infty$. Hence, by the Mean Value Theorem we have

$$\left|1 - e^{\pm 2\pi i \left(c_0 \mu_{r_j}(\eta_j) - iq\right)}\right| \geq \left|1 - e^{\pm 2\pi \left(\mathrm{Re}(q) - \mu_{r_j}(\eta_j) b_0\right)}\right| \geq e^{-1} 2\pi \left|\mathrm{Re}(q) - \mu_{r_j}(\eta_j) b_0\right|$$

and

$$\left|\sin\left(2\pi\left(k_j + \mu_{r_j}(\eta_j) a_0 + \mathrm{Im}(q)\right)\right)\right| \geq \pi \left|k_j + \mu_{r_j}(\eta_j) a_0 + \mathrm{Im}(q)\right|,$$

for j sufficiently large. Thus,

$$\begin{aligned}\pi\left|k_j+\mu_{r_j}(\eta_j)a_0+\mathrm{Im}(q)\right| &\leq \left|\sin\left(2\pi\left(k_j+\mu_{r_j}(\eta_j)a_0+\mathrm{Im}(q)\right)\right)\right| \\ &\leq 2e^{2\pi\left(\mathrm{Re}(q)-\mu_{r_j}(\eta_j)b_0\right)}\left|\sin\left(2\pi\left(k_j+\mu_{r_j}(\eta_j)a_0+\mathrm{Im}(q)\right)\right)\right| \\ &= 2\left|\mathrm{Im}\left(1-e^{\pm 2\pi i\left(\mu_{r_j}(\eta_j)c_0-iq\right)}\right)\right| \\ &\leq 2\left|1-e^{\pm 2\pi i\left(\mu_{r_j}(\eta_j)c_0-iq\right)}\right|.\end{aligned}$$

We conclude that for j sufficiently large there exists $C>0$ such that

$$\begin{aligned}0<\left|k_j+\mu_{r_j}(\eta_j)c_0-iq\right| &\leq \left|\mathrm{Re}(q)-\mu_{r_j}(\eta_j)b_0\right|+\left|k_j+\mu_{r_j}(\eta_j)a_0+\mathrm{Im}(q)\right| \\ &\leq C\left|1-e^{\pm 2\pi i\left(\mu_{r_j}(\eta_j)c_0-iq\right)}\right| \\ &\leq \frac{C}{j}\langle\eta_j\rangle^{-j},\end{aligned}$$

which implies that 1. is not satisfied.

Conversely, assume now that 1. is not valid, so for all $j\in\mathbb{N}$ there exist $k_j\in\mathbb{Z}$, $[\eta_j]\in\widehat{G}$ and $1\leq r_j\leq d_{\eta_j}$ such that

$$0<|k_j+\mu_{r_j}(\eta_j)c_0-iq|<\tfrac{1}{j}(|k_j|+\langle\eta_j\rangle)^{-j}.$$

In particular, we have $|k_j+\mu_{r_j}(\eta_j)-iq|\to 0$ and $|\,\mathrm{Re}\,(q)-\mu_{r_j}(\eta_j)|\to 0$ when $j\to\infty$. The ideas to verify that 2. does not hold is similar to the previous case and so the details is omitted. For j sufficiently large, by Mean Value Theorem we obtain a constant $C>0$ such that

$$\begin{aligned}\left|1-e^{\pm 2\pi i\left(\mu_{r_j}(\eta_j)c_0-iq\right)}\right| &\leq \left|1-e^{\pm 2\pi\left(\mathrm{Re}(q)-\mu_{r_j}(\eta_j)b_0\right)}\cos\left(2\pi\left(\mu_{r_j}(\eta_j)a_0+\mathrm{Im}(q)\right)\right)\right| \\ &+\left|e^{\pm 2\pi\left(\mathrm{Re}(q)-\mu_{r_j}(\eta_j)b_0\right)}\right|\cdot\left|\sin\left(2\pi\left(\mu_{r_j}(\eta_j)a_0+\mathrm{Im}(q)\right)\right)\right| \\ &\leq \left|1-\cos\left(2\pi\left(k_j+\mu_{r_j}(\eta_j)a_0+\mathrm{Im}(q)\right)\right)\right|+\left|1-e^{\pm 2\pi\left(\mathrm{Re}(q)-\mu_{r_j}(\eta_j)b_0\right)}\right| \\ &+e^{\pm 2\pi\left(\mathrm{Re}(q)-\mu_{r_j}(\eta_j)b_0\right)}\left|\sin\left(2\pi\left(k_j+\mu_{r_j}(\eta_j)a_0+\mathrm{Im}(q)\right)\right)\right| \\ &\leq C\left(\left|k_j+\mu_{r_j}(\eta_j)a_0+\mathrm{Im}(q)\right|+\left|\mathrm{Re}(q)-\mu_{r_j}(\eta_j)b_0\right|\right) \\ &\leq 2C|k_j+\mu_{r_j}(\eta_j)c_0-iq| \\ &\leq \tfrac{2C}{j}(|k_j|+\langle\eta_j\rangle)^{-j} \\ &\leq \tfrac{2C}{j}\langle\eta_j\rangle^{-j},\end{aligned}$$

and so Condition 2. is not satisfied. □

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