UNIVERSIDADE FEDERAL DO PARANÁ

DANILO MICALI FUCCI

TRIPARTITE REALISM-BASED QUANTUM NONLOCALITY

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Dissertação apresentada ao Programa de Pós-graduação em Física do Setor de Ciências Exatas da Universidade Federal do Paraná, como parte dos requisitos necessários à obtenção do grau de Mestre em Física.

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RESUMO

Duas das características mais intrigantes da mecânica quântica são sua incompatibilidade com a hipótese da causalidade local e a indeterminação intrínseca das propriedades físicas num estado quântico, frequentemente referida como o problema do realismo. Foi recentemente proposto um critério operacional para realismo físico com uma medida de não-localidade quântica contextual subjacente que não apenas trata, mas também relaciona, ambos aspectos. Essa medida foi posteriormente generalizada para um quantificador de não-localidade para sistemas bipartidos independente de contexto, que apresenta diversas características contrastantes com a não-localidade de Bell. Apesar disso, um quantificador de não-localidade baseado em realismo para sistemas multipartidos ainda não existe. Neste trabalho, nós visamos iniciar este programa de pesquisa propondo uma medida de não-localidade genuinamente tripartida. Nós mostramos que ela se reduz a emaranhamento genuinamente tripartido para uma classe específica de estados puros tripartidos e que ela diagnostica como não-locais estados mistos que apresentam apenas correlações clássicas. Além disso, nós conduzimos um estudo de caso para estados GHZ e W ruidosos e investigamos as propriedades de monogamia da medida.

Palavras-chaves: realismo. localidade. emaranhamento. sistemas tripartidos.

ABSTRACT

Two of the most intriguing features of quantum mechanics are its incompatibility with the local causality hypothesis and the intrinsic indeterminacy of physical properties in a quantum state, often referred to as the realism problem. An operational criterion for physical realism was recently proposed with an underlying contextual measure for quantum nonlocality that not only addresses, but relates, both aspects. This measure was further generalized into a context-independent nonlocality quantifier for two-part quantum systems, shown to display several features that contrast with Bell nonlocality. However, a realism-based nonlocality quantifier for multipartite systems is still lacking. Here we aim at starting such research program by proposing a genuine tripartite realism-based nonlocality measure. We show that it reduces to genuine multipartite entanglement for a specific class of pure tripartite states and that it diagnoses as nonlocal mixed states that present only classical correlations. Moreover, we conduct a case study for noisy GHZ and W states and investigate the monogamy of the measure.

Key-words: realism. locality. entanglement. tripartite systems.

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1 INTRODUCTION

Almost one hundred years have passed since the term “quantum mechanics” started roaming around the corridors of the University of Göttingen. *Quantenmechanik*, in German, as it was originally created by a group of physicists composed, among others, by Max Born, Wolfgang Pauli and Werner Heisenberg. It was Born who first introduced the term in a paper published in 1924 [1]. This fundamental theory of physics, since then, had its mathematical body refined, incorporated conceptual elegance and dwelt victoriously into experimental grounds. Notwithstanding, its predictions are so departed from our intuitions, developed by evolution for the sake of our survival in a classical world, that the task of interpreting it is almost heroic. Picture, for example, the superposition phenomenon. If one is bold enough not to hide behind a pure mathematical abstraction and try to imagine the physical scenario itself, it will be clear that visualizing an object being in more than one place at the same time is not adequate. Our classical intuition is such that to visualize an object, what we really do is to visualize it by means of its physical properties, such as the position in space it occupies. But in such a scenario, it is not fair to the theory to say that the object is really “there”. The indefiniteness of its position lies in the superposition circumstance. Object’s physical properties indeterminacy impairs our ability to attribute “reality” to it. Still relying on our classical intuition, one could say that the object is real if the physical quantities assigned to its properties are well defined at all times, independently from any observer. If we assume, a priori, reality to a quantum state, in situations like the double-slit experiment, one would conclude that the particle interacts with itself and passes through both slits simultaneously, incurring, thus, to several physical inconsistencies. To make justice to quantum mechanics, the irreality phenomenon should be properly addressed.

Several takes on this issue were proposed in the literature. The first approach was made indirectly by Einstein, Podolsy and Rosen (EPR) in the classical paper “Can Quantum-Mechanical Description of Physical Reality be Considered Complete?” [2]. In this article, which will be thoroughly covered in section 2.4, the concept of “elements of reality” is crafted, based on the premise of full predictability without disturbance. It relies on the assumption of an objective reality that can be inferred via a physical theory. Bohr’s approach on the subject [3] is contrasting, since it emerges from an operational point of view. Being one of the founders of the Copenhagen interpretation of quantum mechanics, he states that it only makes sense to speak about elements of reality after the measurement procedure. In this sense, the quantum mechanics formalism only provides a symbolic representation of the physical system, being useful once it gives correct predictions for the experimental outcomes. In the context of the aforementioned paper, Einstein was proposing an ontic interpretation of quantum mechanics, with the wave function representing the physical state itself. In contrast, Bohr’s standpoint is for an epistemic view, where the wave function represents our knowledge about the underlying physical state.

Still in the 1935 EPR's paper, a thought experiment is proposed where the assumption of a set of premises together with their criterion of reality lead to a contradiction where it is possible to simultaneously know the quantities assigned to incompatible observables. Using a pair of entangled 1/2 spin particles (more about entanglement in section 3.2) and assuming locality (see 2.3.2), it is possible to infer the values assigned to the spin of one particle performing measurements on the other particle in a far away site. Depending on the direction chosen for the spin measurement, and assuming the measurement on one particle does not affect, upon the locality premise, the state assigned to the other, it is possible to assign simultaneous elements of reality to the spin in whichever direction. This development led the authors to state that quantum mechanics should be an incomplete theory, in the sense that there are elements of reality not captured by its mathematical formalism.

In 1964, motivated by discussions inaugurated in the aforementioned paper and by further developments of the theory, like the Bohmian Mechanics [4], Bell tried to supplement quantum mechanics with general local hidden variables and came to the conclusion that, holding the locality premise, such a procedure cannot recover the results predicted by the theory [5] (an in depth discussion about this work is given in section 3.1). In face of this fact, it is shown that the premise which should be left out is not the one of the completeness of quantum mechanics, but that quantum mechanics is a local theory. The incompatibility of quantum mechanics with the local causality hypothesis is known as the Bell theorem and is regarded as one of the most important results in physics, being, for example, referred by Henry Stapp as the “most profound discovery of science” [6]. The first experimental result that corroborated this theoretical result came forward eighteen years after the original paper was published [7], which has been since then recognized as the first quantum “loophole” experiment.

Even though Bell's original paper makes no reference to realism at all and his argument is quite clear, a lot of confusion regarding the terms realism and locality is present in still ongoing discussions. For example, the abstract of a Nature journal article, published by Hensen [8], states: “Bell proved that no theory of nature that obeys locality and realism can reproduce all the predictions of quantum theory: in any local-realist theory, the correlations between outcomes of measurements on distant particles satisfy an inequality that can be violated if the particles are entangled. Numerous Bell inequality tests have been reported”. Gisin addressed the issue in [9] arguing about the vagueness that usually underlies the term “realism”.

A sober development of the concept of realism and its connection with the nonlocality phenomenon was given by Bilobran and Angelo in 2015 [10], bringing some light on this issue. They provide an operational criterion for realism that allows us to quantify the reality, or its counterpart, “irreality”, that transcends, but includes, previously developed realism criteria. It is based on the premise that after a measurement is made, there is reality assigned to the observable measured, even if no one comes to know the outcome of the measurement. In contrast with EPR's elements of reality, it is also suitable for dealing with mixed states (see more about pure and mixed states in 2.1.2). Embedded in such a formalism, a quantifier for nonlocality is suggested that accounts for spacelike change events in the irreality of quantum states upon local operations in distant partitions of a bipartite system.

The theoretical framework by which Bilobran and Angelo's realism criterion is developed is that of quantum information theory. Such a field generalizes classical information theory, inaugurated by Claude Shannon in 1948 [11], to a quantum context and had its development tractioned since the publication of Rolf Laundauer's article in 1961 [12] where he proposes that information is physical. For a brief introduction, we can think of a physical system, like a coin, which can yield head or tail and ask what the informational content of such a system is. For someone who shares the knowledge of what a coin is, it suffices a binary piece of information, head or tail, to reproduce the state of the system. If the system is more complex, like a deck of cards, it is also possible, for someone who knows what a deck of cards is, to reproduce its state by giving a series of binary pieces of information: information *bits*. In physics, information is always stored, transmitted and processed by means of physical systems. Landauer showed that the process of erasing information implies in the increase of entropy and, after that, Bennett used this result to resolve the paradox of Maxwell's Demon [13]. The basic tools of classical and quantum information theory will be given in section 2.2.

Quantum information theory is also highly valuable in the task of quantifying entanglement. Since entanglement is a prerequisite for the performance of tasks like superdense coding and quantum teleportation [14], it is important to have appropriate tools to assess how much entanglement a specific physical state presents. Likewise, nonlocality can also be regarded as a kind of resource, demanding tools for its quantification. However, the quantification of nonlocality is still in the center of recent debates since it is a complicated task. This impairs us to a certain degree in the task of investigating the relationship between nonlocality and entanglement, where we come to face, for example, scenarios where states that are maximally entangled are not diagnosed as maximally Bell nonlocal, called "anomalies" [15–18].

The realism-based nonlocality developed by Bilobran and Angelo in [10] is context dependent, that is, it accounts for irreality changes given a specific setting of observables to be measured. This concept was further generalized afterwards by Gomes and Angelo and a realism-based nonlocality quantifier that is a function of the quantum system alone was crafted [19]. Such an approach for nonlocality is fundamentally different from Bell's take on the subject and it showed to provide a nonanomalous nonlocality measure for pure states. That is, it diagnoses maximally entangled pure states as maximally nonlocal, not incurring in the anomaly problem.

This quantifier, however, was thought of for bipartitions. Multipartite states are known to present a much richer phenomenology, and thus, a much more challenging ground for the analysis of its properties [20]. For example, states of this kind present different kinds of entanglement, distinct nonlocal behaviour and bounds that restrict the shareability of its quantum resources among the system partitions, known as monogamy. While the quantification of entanglement for bipartite states is reasonably well understood, there are still ongoing debates concerning the quantification of entanglement in multipartite mixed states [21]. Likewise, despite recent advances in this research, the development of nonlocality measures for the multipartite case is also tricky [22–27].

More specifically, proper analyses of multipartite states usually involve investigations of their bipartite cuts: ways to divide the system into two partitions. It could happen that a multipartite state that presents entanglement, for example, presents this feature only with respect to some of its possible cuts, but is not globally entangled. States whose entanglement shows up simultaneously with respect to all of its possible cuts are said to present genuine multipartite entanglement. Criteria for the crafting of genuinely multipartite resource measures are proposed in the literature, such as Bennett's postulates for multipartite correlations [28]. Two well known states that show genuine multipartite entanglement are the GHZ states, named after Greenberger, Horne and Zeilinger [29], and the W states, firstly characterized by Wolfgang Dür *et al.* [30]. The former is known to provide predictions that violate the local causality hypothesis without resorting to a statistical context, differently from the bipartite case, and presents an entanglement that is much more fragile against noise exposure and the discard of subsystems than the latter.

Here we aim to start a research program in the field of multipartite realism-based nonlocality. For such, we address specifically the tripartite case, once it provides many of the characteristic features of multipartite quantum systems but is also simple enough for us to deal with in operational grounds, conducting numerical analysis. We employ Bilobran and Angelo's realism criterion and extend their contextual nonlocality for a tripartite context. Then, we follow the procedure employed by Gomes and Angelo to make it context independent and extend the "genuine multipartite correlation" criterion proposed by Bennett, to craft a genuine realism-based nonlocality measure for tripartite systems, that is, a nonlocality quantifier that is only sensible for states whose nonlocality phenomenon manifests itself in all of its bipartite cuts. An analysis of its properties is conducted, highlighting its similarities and discrepancies with its bipartite analogue, and we employ this measure in a case study concerning tripartite states of interest, also adding white noise to them, in order to investigate the resilience of this measure. Finally, we investigate its monogamy properties, with both analytical and numerical methodologies.

The structure of this work is organized as follows: in chapter 2, a brief overview of the basis of quantum mechanics in its density operator formulation is provided, followed by an introduction of both classical and quantum information theory. Then, we characterize the terms causality and locality. This chapter is closed by an exposure of the EPR paradox, to bring forward some of the intriguing features presented by quantum mechanics. In chapter 3, we investigate the relationship between realism, causality and locality. We start by providing a resolution for the EPR paradox discussing Bell's theorem. We give a brief overview about entanglement after that and then Bilobran and Angelo's criterion of realism is introduced and discussed. The realism-based nonlocality is thus brought forward and most of its properties are presented, in order to provide a clear picture of its conception for a straightforward development of our central results in chapter 4. In this chapter, we start with a review of the multipartite scenario, together with a more precise elaboration of "genuine multipartite correlations" and monogamy. Our tripartite realism-based nonlocality is gradually introduced, and its properties are presented in a similar exposure to that of the bipartite case, to make it easy for the reader to compare each case. The work comes to an end by presenting a case study for noisy GHZ and W states and an investigation of the monogamous properties of the measure.

2 THEORETICAL FOUNDATIONS

2.1 QUANTUM MECHANICS FORMALISM

Quantum mechanics can be seen as a mathematical framework by which our physical theories are constructed [14]. It flaunts the highest standard among our current frameworks to understand Nature once it exhibits, for more than a century now, a flawless accordance between the theoretical landscape it provides and the experimental observations.

To provide a convenient overview of quantum mechanics structure, we will now offer a brief revision of its central pillars. We hope that the reader can benefit from a concise summary for a straightforward reading of this work. Our main source for this section is [14].

2.1.1 Postulates of quantum mechanics

In classical mechanics, a physical system is completely described by a set of coordinates that takes place in a phase space. For the most usual formulation of quantum mechanics, a complete description of a physical system is given by a *state vector*, a unit vector existing in the *state space*. The state space, in its turn, is a complex vector space equipped with an inner product, a Hilbert space.

The fundamental quantum mechanical system is the *qubit*. It can be shown that every possible finite-dimensional quantum system can be described in terms of qubits; it is the fundamental entity in quantum information theory and it will be largely employed in the course of this work. A qubit resides in a two-dimensional state space whose basis is given by two orthonormal vectors $|0\rangle$ and $|1\rangle$. A qubit can be written in a general form as

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,, \tag{2.1}$$

with $a, b \in \mathbb{C}$. Since $|\psi\rangle$ is a unit vector, $\langle\psi|\psi\rangle = 1$, where $\langle\psi|\psi\rangle$ is the inner product between $|\psi\rangle$ and $|\psi\rangle$. It follows that $|a|^2 + |b|^2 = 1$. The term qubit makes reference to a "quantum bit", once the vectors $|0\rangle$ and $|1\rangle$ are analogous to the values 0 and 1 that a bit can assume. However, in a qubit, these two states can exist in a superposition state.

We will not proceed to formulate quantum mechanics in terms of state vectors, however. Another formulation, more general, is provided by means of the *density operator* or *density matrix*, being more suitable for our purposes, since it accounts for cases where the quantum states that comprise a quantum system are not completely known. A more in depth discussion about the density operator and its connection with the state vector is given in the following subsection, but for now its definition suffices: a density operator ρ is a hermitian positive operator with trace one.

Quantum mechanics formalism can be stated by means of four postulates:

1. There is a *state space*, a Hilbert space, associated to any isolated physical system, whose complete description is given by a *density operator* ρ, that acts on the state space. If the system is in a quantum state ρ_i with probability p_i, it is described by the density operator $\sum_i p_i \rho_i$.

2. The time evolution of a closed quantum system is governed by a unitary transformation. That is, the relation between the description ρ of the system at a time t_1 and ρ' at t_2 is given by
$$\rho' = U \rho U^\dagger, \tag{2.2}$$
where U is an unitary operator dependent only on t_1 and t_2.

3. Quantum measurements are comprised by a set $\{M_m\}$ of measurement operators, *i.e.*, operators satisfying the completeness equation, $\sum_m M_m^\dagger M_m = \mathbb{1}$, that acts on the state space of the system where the measurement is performed. The index m stands for the possible measurement outcomes. If the state of the system immediately before the measurement is ρ, the probability for the outcome m to occur is
$$p(m) = \mathrm{Tr}\left(M_m^\dagger M_m \rho\right) \tag{2.3}$$
and the post-measurement state is
$$\frac{M_m \rho M_m^\dagger}{\mathrm{Tr}\left(M_m^\dagger M_m \rho\right)}. \tag{2.4}$$

4. The state space of a composite quantum system is given by the tensor product of its component system states. If the component systems have states $\rho_1, \rho_2, ..., \rho_n$, the joint state of the composite system is $\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n$.

2.1.2 Density operator

If a quantum system is in a state $|\psi_i\rangle$ with probability p_i, the density operator for the system is given by
$$\rho \equiv \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|. \tag{2.5}$$
In fact, it is a theorem that an operator ρ is the density operator associated with the ensemble $\{p_i, |\psi_i\rangle\}$ if and only if it is positive and with trace one, which coincides with the definition we gave before. This characterization makes explicit the reason why we said this formalism is useful for dealing with quantum systems whose composition is uncertain.

When the state that describes the system is fully known, say, if the system is in the state $|\psi\rangle$, the density operator takes the form $\rho = |\psi\rangle\langle\psi|$ and the system is said to be in a *pure state*. Otherwise, if there is uncertainty, the system is in a *mixed state*, that is, in a mixture of pure states. To determine whether it is the case for a given density operator ρ, there is a simple criterion: if $\mathrm{Tr}\,(\rho^2) = 1$, the state is pure, otherwise, that is, if $\mathrm{Tr}\,(\rho^2) < 1$, the state is mixed. When $\rho = \mathbb{1}/d$, where d is the dimension of the Hilbert space where ρ acts, the lack of knowledge about the system is maximum and ρ is said to be a completely mixed state.

Probably the most useful resource the density operator formalism renders us is a descriptive tool for subsystems of a composite quantum system, the *reduced density operator*. For the bipartite case, a system comprised of two partitions $\mathcal{A}$ and $\mathcal{B}$ is described by $\rho_{\mathcal{AB}}$. If we discard a partition, we are left with the reduced density operator. Supposing we discard $\mathcal{B}$, the remaining state, $\rho_{\mathcal{A}}$, is given by

$$\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho_{\mathcal{AB}}), \tag{2.6}$$

where $\mathrm{Tr}_{\mathcal{B}}$ is the partial trace with respect to the partition $\mathcal{B}$, defined by

$$\mathrm{Tr}_{\mathcal{B}}(|a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|) \equiv |a_i\rangle\langle a_j| \langle b_j|b_i\rangle\,, \tag{2.7}$$

with $|a_i\rangle$, $|a_j\rangle$ being any two vectors in the partition $\mathcal{A}$ and $|b_i\rangle$, $|b_j\rangle$ in $\mathcal{B}$. Similar considerations hold if we want to discard the subsystem $\mathcal{A}$ and an extension for a multipartite case is direct.

It is clear that a statistical nature is embedded in the formulation of quantum mechanics, given that, for example, the predictions for measurement outcomes are given in terms of probabilities. But it is also possible to take into account, through the density operator formalism, the subjective ignorance one could have about the system state. All this uncertainty, that arises both from ontological and epistemic contexts, suggests a fertile ground for an informational approach of quantum mechanics.

2.2 CLASSICAL AND QUANTUM INFORMATION

It is possible to treat a quantum state with regards to the information it contains. And for quantifying it, a series of tools were developed in the context of what we call “quantum information theory”. To lay down its basic principles, we start by introducing basic notions of classical information theory and then we generalize it for a quantum context.

2.2.1 Shannon entropy

So, first of all, what is information? Does it relate to *Truth*? Is it some kind of substance? A quality? A correspondence? Surprisingly, to convey information a significance that suffices for us to deal with it in operational terms, it is enough to say that information is what resolves uncertainty. Embedded in this rationale, uncertainty is suggested as a counterpart for information. Along this train of thought, one could say that for the knowing of something that is highly uncertain, the amount of information acquired is higher than for the knowing of something that is less uncertain. However, if one is to quantify information, this task is now dependent upon the quantification of uncertainty. And such a quantification, in the context of information theory, is given by information entropy.

The concept of information entropy is written in the foundations of information theory, as it comes to light in the inaugural article of the field published in 1948 by Claude Shannon: "*A Mathematical Theory of Communication*" [11] (renamed in 1949 to "*The Mathematical Theory of Communication*", due to a recognition of its generality). Information entropy or Shannon entropy is defined in a probabilistic context such that, given a random variable X, it quantifies how much information we obtain, on average, when we assess the value of X. Equivalently, we could say that it quantifies how much uncertainty there is about X before the knowing of its value. That is, it quantifies the information after the knowing, or the uncertainty before the knowing [14]. For a greater appreciation of this concept, instead of just throwing it right away, let us arrive gradually to it by different paths.

For a purely mathematical approach to derive information entropy, following the steps given in an exercise found in [14], we see that the form of such a tool can be molded if we require it to meet a series of reasonable assumptions. That is, if we look for a function H of a probabilistic variable X, $H(X)$, such that:

1. $H(X)$ is a function solely of the probabilities assigned to X, so that if it is $p \in [0, 1]$, $H(X) = H(p)$;
2. H is a smooth function;
3. If X consists of a set of independent non-zero probabilities, say, $p, q \in]0, 1]$, $H(pq) = H(p) + H(q)$,

one can show that $H(p) = k \log p$ for some constant k. As a consequence, for the knowing of a random event pertaining to a set of mutually exclusive events with assigned probabilities $p_1, p_2, ..., p_n$, the average information gain is given by $k \sum_i p_i \log p_i$, which is the Shannon entropy up to a constant factor.

It is also possible to follow a more down to earth route by deriving information entropy through the pursuit of a specific goal within a concrete example. For such, we will use the arguments presented in [31]. Still on the paper [11], Shannon accomplished two major goals by determining:

1. How much a message can be compacted, that is, how much redundancy there is in the information.
2. At what rate a message can be reliably communicated over a noisy channel, that is, how much redundancy is necessary to be introduced in order to protect the content of the message when it is transmitted over a noisy channel.

So, inspired by 1, let us consider a message and try to find a way to compress it.

A message can be thought of as a string of k letters from an alphabet $\{a_1, a_2, ..., a_k\}$. Suppose that the showing up of each letter is an independent statistical event with probability assigned, a priori, like $p(a_x)$, with $\sum_{x=1}^{k} p(a_x) = 1$. First, for simplicity, let the alphabet be binary, that is, an alphabet with only $\{a_1, a_2\}$, with $a_1 = 0$ and $a_2 = 1$, such that $p(a_1) = p$ and $p(a_2) = 1 - p$. Now, we ask: for a long message, with n letters, $n \gg 1$, is it possible to compact it to a smaller string while keeping its information?

Since n is large, the incidence of 0's and 1's in the original message string is given by pn and $(1-p)n$, respectively. The number of possible strings of this kind is of the order of $\binom{n}{np}$. Employing the Stirling approximation, $\log n! \cong n \log n - n$, we have:

$$\log \binom{n}{np} = \log \left(\frac{n!}{(np)![n(1-p)]!} \right) \tag{2.8}$$

$$\cong n \log n - n - [np \log np - np + n(1-p) \log n(1-p) - n(1-p)] \tag{2.9}$$

$$= nH(p), \tag{2.10}$$

where $H(p) = -p \log p - (1-p) \log(1-p)$. Therefore, for base 2 log (since we are dealing with bits), the order of such strings is of $2^{nH(p)}$. For the original message, one could think that we need n bits to store all of its information. However, we could just employ a code block that allocates a positive integer for each of the $2^{nH(p)}$ typical strings, taking $nH(p)$ bits to write each number, and convey the original message by the number of the correspondent string. Since $0 \leq H(p) \leq 1$ with $H(p) = 1$ only when $p = 1/2$, we can reduce the amount of resources necessary for the storage if the letters are not equiprobable, compressing the message without loss of information.

The generalization for a non binary alphabet is straightforward. For an alphabet of k letters, where the probability correspondent to the letter x is $p_x \equiv p(x)$, in a large message of n letters, the occurrence of x is np_x and the order of possible strings is

$$\frac{n!}{\prod_x (np_x)!} \cong 2^{-nH(X)}, \tag{2.11}$$

where

$$H(X) \equiv -\sum_x p_x \log p_x \tag{2.12}$$

is the entropy of information, or Shannon entropy, for an ensemble $X = \{x, p_x\}$. For such, we know that the average information carried by a letter in this message is $H(X)$, being thus able to quantify the minimum amount of resources needed for the storage of the message.

To give a clear picture of this procedure, we will present a simple example found in [14]. Suppose an information source provides one out of four possible signs, α, β, γ, and δ. To store a sign produced by one use of the source, without compression, it would require two bits. However, if it is known that the probabilities for the source to produce each sign are, respectively, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{8}$, we could make use of this bias to assign less bits for the storage of the more probable sign and more bits for the less probable ones to compress the source. One possible arrangement is to encode α as 0, β as 10, γ as 110, and δ as 111. For such, the average length of the compressed string by use of the source is $\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 = \frac{7}{4}$ bits. Interestingly, this result coincides with the entropy of the source: $H(X) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{8}\log\frac{1}{8} - \frac{1}{8}\log\frac{1}{8} = \frac{7}{4}$. Henceforth, any attempts of further compression would result in a irretrievable loss of information.

From here on, we will use the log in the entropy, equation 2.12, in the basis e. Also, to avoid any indeterminacy when $p_x = 0$, we use the convention $0 \log 0 \equiv 0$. $H(X)$ is non-negative and, for d possible outcomes, it assumes its maximal value $\log d$ when all outcomes are equiprobable.

Proceeding, for a deeper incursion into the classical information theory, some more measures relying upon the information entropy will be introduced, as well as some of their more important properties. These tools will be useful later in this work either directly or as foundations for the development of further concepts. To the end of this subsection, our main reference will still be [14].

- *Joint entropy*: the total uncertainty assigned to a pair of random variables X and Y is given by:
$$H(X,Y) \equiv -\sum_{x,y} p(x,y) \log p(x,y). \tag{2.13}$$
This definition can be extended for any set of random variables.

- *Conditional entropy*: after the knowing of Y, that is, in possession of the information $H(Y)$, the remaining uncertainty about the pair (X,Y) reduces to
$$H(X|Y) \equiv H(X,Y) - H(Y), \tag{2.14}$$
that is the entropy of X conditional on knowing Y.

- *Mutual information*: the common information to X and Y is expressed via
$$I(X;Y) \equiv H(X) + H(Y) - H(X,Y). \tag{2.15}$$
While summing $H(X)$ and $H(Y)$ the information that is common to both X and Y is counted twice and the information that is not common is counted only once. To remain only with the common, or mutual, information of X and Y, we subtract $H(X,Y)$. It is possible to relate the mutual information and the conditional entropy by inserting 2.15 into 2.14, arriving to the expression $I(X;Y) = H(X) - H(X|Y)$.

For a better grasp of the relationship and behaviour of these quantities, we present a diagram (figure 1) and list some properties regarding them. These properties are enlisted originally in [14] together with their proofs.

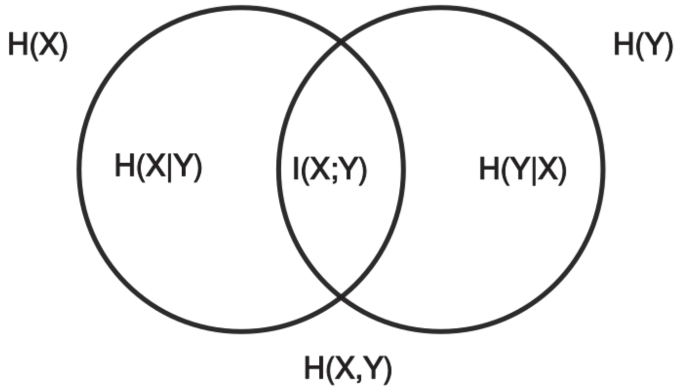

Figure 1 – Different measures of entropy and their relationships in a Venn diagram. Original figure found at [32].

1. $H(X,Y)$ and $I(X;Y)$ are invariant under the permutation of X and Y. For the mutual information, this is made clear by the fact that the information we acquire about Y by learning X is the same we acquire about X by learning Y.

2. $H(X,Y) \geq H(X)$, with equality if and only if $Y = f(X)$, that is, if Y is a function of X.

3. $H(Y|X) \geq 0$, hence $I(X;Y) \leq H(Y)$. Equality holds if and only if $Y = f(X)$.

4. $H(Y|X) \leq H(Y)$, hence $I(X;Y) \geq 0$. Equality holds if and only if X and Y are independent, since, for such a case, the knowing of X does not implies in the learning about Y and vice versa. Another way of seeing these inequalities is that the knowing of X cannot reduce our information about Y and vice versa.

5. $H(X|Y,Z) \geq H(X|Y)$, since the information we have about X given that we already know Y and Z is lesser than the information we have about X when we only know about Y.

6. *Subadditivity*: $H(X,Y) \leq H(X) + H(Y)$. Equality holds if and only if X and Y are independent.

7. *Strong subadditivity*: $H(X,Y,Z) + H(Y) \leq H(X,Y) + H(Y,Z)$. Equality holds if and only if $X \to Y \to Z$ is a Markov chain.

One last, but not least important, tool: if we want to measure the proximity of two probability distributions $p(x)$ and $q(x)$, defined over the same index set x, we can employ the entropy-like measure *relative entropy*,

$$H(p(x)||q(x)) \equiv \sum_x p(x) \log \frac{p(x)}{q(x)} = -H(X) - \sum_x p(x) \log q(x). \tag{2.16}$$

We assume that $-0 \log 0 \equiv 0$ and $-p(x) \log 0 \equiv +\infty$ if $p(x) > 0$. This quantity is non-negative, $H(p(x)||q(x)) \geq 0$, with equality holding if and only if $p(x) = q(x)$ for all x.

With these concepts in mind, we are now ready to proceed and see how information theory and quantum mechanics together bring forward quantum information theory.

2.2.2 Von Neumann entropy

Most of the content in this subsection is based on the expositions on [14], [31], and [33]. Then, to give a feeling of how information theory can be applied to quantum mechanics, suppose that a source gives a message of n letters chosen from measurements on a quantum states ensemble. That is, we have a density operator ρ comprised by a set of quantum states ρ_x occurring with probabilities p_x, $\rho = \sum_x p_x \rho_x$, and each letter corresponds to a measurement of an observable in this state. The letters a, for instance, correspond to the outcomes assigned to the measurement of the observable A, with probabilities $p(a) = \mathrm{Tr}\left(|a\rangle\langle a|\,\rho\right)$. We introduce now the von Neumann entropy, named after Jon von Neumann, that determines the entropy of a quantum state ρ:

$$S(\rho) \equiv -\mathrm{Tr}\left(\rho \ln \rho\right). \tag{2.17}$$

If we diagonalize ρ, writing it in a orthonormal basis $|a\rangle$, remaining with $\rho = \sum_a \lambda_a |a\rangle\langle a|$, the von Neumann entropy takes the form

$$S(\rho) = -\sum_a \lambda_a \log \lambda_a, \tag{2.18}$$

which is identical to the Shannon entropy $H(A)$ for the ensemble $A = \{a, \lambda_a\}$.

If it is possible to describe the message source in terms of a density operator comprised by a sum of orthonormal states, the quantum source behaves like a classical one. However, this is not always possible. In this case, the quantum source cannot be seen as a classical one, since pure non-orthonormal states cannot be totally distinguished [14]. This is a fundamental difference between quantum and classical information. While uncertainty in a classical scenario can be understood as an ignorance about underlying definite properties of the system, in a quantum scenario, Nature is intrinsically uncertain. One can think about the difference between a coin toss, where the outcome can be determined by a detailed knowledge about the system's dynamics, and a qubit, a state in superposition, where the dynamics involved is intrinsically probabilistic.

Von Neumann entropy can thus be seen as a generalization of Shannon entropy. While the quantum source behaves classically, von Neumann entropy quantifies the minimum amount of bits required, per letter, to store its information, or even more, the maximum amount of information per letter, in bits, we can acquire about the source given the optimal set of measurements. For when it behaves like a quantum source, von Neumann entropy quantifies the information of each letter in terms of qubits. Later we will see that it is also useful for quantifying entanglement in pure states.

Just like in the classical case, other measures of entropy can be defined regarding composed quantum systems.

- *Quantum joint entropy*: the entropy assigned for a composite system with partitions $\mathcal{A}$ and $\mathcal{B}$ is given by
$$S(\rho_{\mathcal{AB}}) \equiv -\rho_{\mathcal{AB}} \log \rho_{\mathcal{AB}}, \tag{2.19}$$
where $\rho_{\mathcal{AB}}$ is the density matrix of the system $\mathcal{AB}$.

- *Quantum conditional entropy*: is defined as
$$S(\rho_{\mathcal{A}}|\rho_{\mathcal{B}}) = S(\rho_{\mathcal{AB}}) - S(\rho_{\mathcal{B}}). \tag{2.20}$$

- *Quantum mutual information*: is given by
$$S(\mathcal{A} : \mathcal{B}) \equiv S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}) - S(\rho_{\mathcal{AB}}) \tag{2.21}$$
$$= S(\rho_{\mathcal{A}}) - S(\rho_{\mathcal{A}}|\rho_{\mathcal{B}}) = S(\rho_{\mathcal{B}}) - S(\rho_{\mathcal{B}}|\rho_{\mathcal{A}}). \tag{2.22}$$

We list now some of the most important properties regarding von Neumann entropy. Most of them will be very useful in the development of our main results. Others are evoked for the sake of a more complete exposition. Their proofs can be found in [14].

1. *Non-negativity*: the entropy is greater than or equal to zero, $S(\rho) \geq 0$, vanishing if and only if ρ is pure, that is, our knowledge about the state is maximal.

2. *Maximum value*: the maximum value the entropy assumes is $\log d$, where d is the dimension of the Hilbert space where ρ acts, $S(\rho) \leq \log d$, with saturation if and only if ρ is a completely mixed state, $\rho = \mathbb{1}/d$, since, for such, our ignorance about the state is maximal.

3. *Invariance*: the entropy is invariant by unitary transformations, $S(U\rho U^\dagger)$, where U is an unitary operator.

4. The entropy increases under projective measurements. Let A_a be a complete set of orthogonal projectors that defines the observable $A = \sum_a a A_a$. If A is measured, the post measurement state is $\Phi_A(\rho) = \sum_a A_a \rho A_a$ and we have:
$$S(\Phi_A(\rho)) \geq S(\rho), \tag{2.23}$$
with equality if and only if $\Phi_A(\rho) = \rho$.

5. For a pure composite system $\rho_{\mathcal{AB}}$, it follows that $S(\rho_{\mathcal{A}}) = S(\rho_{\mathcal{B}})$.

6. The entropy of a product state, a tensor product of states, is equal to the sum of the entropies of the states. For a tensor product of two states, it reads: $S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$.

7. For probabilities p_i, $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$, and states ρ_i that have support on orthogonal subspaces, it follows:

$$S\left(\sum_i p_i \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i) \tag{2.24}$$

8. *Joint entropy theorem*: for p_i probabilities, with orthogonal states $|i\rangle$ in subspace $\mathcal{A}$ and any set of density operators ρ_i in $\mathcal{B}$, we have

$$S\left(\sum_i p_i |i\rangle \langle i| \otimes \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i). \tag{2.25}$$

9. *Concavity of entropy*: our uncertainty about a mixture of states ρ_i is bigger than our average uncertainty about the same states:

$$S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S(\rho_i). \tag{2.26}$$

10. *Subaditivity*: for two quantum states in $\mathcal{A}$ and $\mathcal{B}$,

$$S(\rho_{\mathcal{AB}}) \leq S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}), \tag{2.27}$$

with equality if and only if $\rho_{\mathcal{AB}} = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$, where $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho_{\mathcal{AB}})$ and similarly for $\rho_{\mathcal{B}}$.

11. *Araki-Lieb inequality*: a lower bound for the previous inequality is given by

$$S(\rho_{\mathcal{AB}}) \geq |S(\rho_{\mathcal{A}}) - S(\rho_{\mathcal{B}})|. \tag{2.28}$$

12. *Strong subaditivity*: for a tripartite quantum state $\mathcal{ABC}$,

$$S(\rho_{\mathcal{ABC}}) + S(\rho_{\mathcal{B}}) \leq S(\rho_{\mathcal{AB}}) + S(\rho_{\mathcal{BC}}). \tag{2.29}$$

While it is always true that for random variables X and Y, $H(X, Y) \geq H(X)$, another disparity between the quantum and classical case can be spotted given that, in general, $S(\rho_{\mathcal{AB}}) \ngeq S(\rho_{\mathcal{A}})$. Take for example the case where $\rho_{\mathcal{AB}} = |\psi\rangle\langle\psi|$ where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Given that $\rho_{\mathcal{AB}}$ is a pure state, $S(\mathcal{A}, \mathcal{B})$ is zero. However, since the density operator describing $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho_{\mathcal{AB}})$ is $\mathbb{1}/2$, $S(\rho_{\mathcal{A}}) = \log 2$.

Finally, we introduce a very important entropy-like measure, analogue to the classical relative entropy, the *quantum relative* entropy. For density operators ρ and σ, the quantum relative entropy of ρ to σ is

$$S(\rho||\sigma) \equiv \mathrm{Tr}\,(\rho \log \rho) - \mathrm{Tr}\,(\rho \log \sigma). \tag{2.30}$$

If there is non-trivial intersection between the kernel of σ and the support of ρ, this quantity is defined as $+\infty$, otherwise it is finite. The non-negativity of the quantum relative entropy is given by a theorem named *Klein's inequality*:

$$S(\rho||\sigma) \geq 0 \tag{2.31}$$

with equality if and only if $\rho = \sigma$.

2.3 CAUSALITY AND LOCALITY

We are almost ready to explore some of the most strange features of quantum mechanics. But first, we need to clarify some concepts. Throughout this work we often refer to the terms causality and locality. Here, we aim at giving a more precise definition for these concepts, even though it is not our intention to dwell into extensive metaphysical matters. For our purposes, it will suffice to embody these terms with enough content just for them to have a clear physical meaning, that is, to be expressed in the same language with which we built our physical theories.

2.3.1 Causality

Causality is the relationship between cause and effect. In a probabilistic context, to assert that the event A necessarily causes the event B implies a deterministic causal relation between the events, that is, given A, B must follow. For a situation of this kind, let $P(B|A)$ be the probability of B to occur given that A already happened. This quantity must be equal to unity. Furthermore, since

$$P(B|A) = \frac{P(A, B)}{P(A)}, \tag{2.32}$$

where $P(A, B)$ and $P(A)$ are the probabilities of occurrence of $A \wedge B$ and A, respectively, it's clear that $P(A, B) = P(A)$. For the case where A and B share no causal relation and no correlation with each other, in general, $P(A, B) = P(A)P(B)$.

Suppose that we have two coins $\mathcal{A}$ and $\mathcal{B}$. Tossing the coin $\mathcal{A}$ gives us the set of equiprobable results $A \in \{h, t\}$ and $\mathcal{B}$ gives $B \in \{h, t\}$. There is no relation between the outcomes of each coin, that is, $P(A|B) = P(A)$ and $P(B|A) = P(B)$. Then, by 2.32, it is clear that by tossing both coins, the probability for us to get the result $h_{\mathcal{A}} \wedge h_{\mathcal{B}}$, that is $P(h_{\mathcal{A}}, h_{\mathcal{B}})$, is given by

$$P(h_{\mathcal{A}}, h_{\mathcal{B}}) = P(h_{\mathcal{A}})P(h_{\mathcal{B}}) = \frac{1}{4}. \tag{2.33}$$

Now, suppose that each coin has a strong magnetic moment embedded in such a way that every time the coins are tossed close to each other the results are the same, both heads or both tails. There is a physical phenomenon that implies a deterministic causal connection between the position of each coin. In this case, we have

$$P(h_{\mathcal{A}}, h_{\mathcal{B}}) = P(h_{\mathcal{A}}) = P(h_{\mathcal{B}}) = \frac{1}{2}. \tag{2.34}$$

However, it's also possible for us to have a situation in which the magnetization of each coin is not strong enough to establish determinism, where the outcome of the tossing of a coin causes influence over the outcome of the other, but does not determines it completely, such as for $0.5 < P(h_{\mathcal{A}}|h_{\mathcal{B}}) < 1$. In this case, even though we do not have a deterministic effect, there is still causation. For a setting like that, the causal relation relies on the fact that it's not possible to factorize the amount $P(h_{\mathcal{A}}, h_{\mathcal{B}})$ as a product of solely $P(h_{\mathcal{A}})$ and $P(h_{\mathcal{B}})$.

2.3.2 Locality

Locality is the assumption that an object can only be influenced by its immediate surroundings. A local physical theory, that is, a physical theory which implements the principle of locality, is incompatible with "action at a distance" phenomena. Special and general relativity as well as quantum field theory are examples of such local physical theories. It is the violation of the assumption of locality that is behind Einstein's appeal to ridicule in his famous "spooky action at a distance", that we will discuss in the next section.

Let us now turn to the framework of special relativity in order to see how a local physical theory imposes constraints over causal phenomena. Together with the principle of locality, special relativity also imposes that no influence, that is, a causal relation of some kind, can propagate with speed greater than c, the speed of light in vacuum. Explicitly, if an event A and another event B happens in two distinct locations separated by a distance d and aparted in time by an amount T such as $d > cT$, there are no means by which one event can exert influence over the other. Another way to frame this is by implementing the spacetime interval,

$$\Delta s = \sqrt{(c\Delta t)^2 - (\Delta x)^2}, \tag{2.35}$$

where Δt is the time separation between the events and Δx is the spatial separation. If the spacetime interval for the events A and B is negative, $\Delta s^2 < 0$, the separation is said to be spacelike, there can be no causal relation between the events. One event resides outside the light cone of the other.

Turning back to the example of the magnetized coins, for the sake of the argument, let us suppose that the magnetization is strong enough so that the influence of one coin over the other could be significant even for very large distances. Suppose also that the two coins are taken apart to two very distant sites separated by a constant distance Δx. The coins are tossed almost simultaneously, so that Δt is small. For a situation like that, $\Delta s^2 < 0$, which implies that no causal relation can be established between each outcome and thus the probability for us to obtain an outcome $h_A \wedge h_B$ is $P(h_A, h_B) = 1/4$.

2.4 EPR PARADOX

The EPR paradox is a thought experiment where the assumption of some premises together with the formalism of quantum mechanics lead through inductive reasoning to a paradox, in the sense of a contradiction. It was introduced in a 1935 paper by Einstein, Podolsky and Rosen (EPR) entitled "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" [2] and the interpretation given for the occurrence of such a paradox is that the description of physical reality implied by quantum mechanics is incomplete, in such a way that a new and more fundamental physical theory should be devised.

The motivation for this argument emerged from two major reservations Einstein had with quantum theory. The first one is that quantum mechanics appeared to be departed from what, in his opinion, is a fundamental task of a physical theory: the capability to provide knowledge about aspects of nature that are independent of an observer and his observations. And the second one was the apparently intrinsic statistical nature of the theory in such a way that, unlike classical statistical mechanics, where the indeterminacy arises from subjective ignorance about fine details, quantum mechanics would render a fundamentally indeterministic description of reality. So, EPR proceeded by establishing some premises for what a proper physical theory should provide.

They start by arguing about the necessity of distinguishing the objective reality from the physical concepts by which a theory operates, intended to correspond to objective reality itself providing a way for it to be represented. Then, they state that such a theory is successful if the answer for the two following questions is positive:

1. "Is the theory correct?"
2. "Is the description given by the theory complete?"

The first question refers to the accuracy by which the theory predicts experimental results. For quantum mechanics, the answer is yes. The central point in the paper, however, is that quantum mechanics fails to meet the requirement given by the second question. For a clear understanding of what exactly the second question is, two definitions are given.

- *Condition of completeness*: "every element of the physical reality must have a counterpart in the physical theory."

Such elements of physical reality, they propose, cannot be assessed by a priori philosophical reasoning, but by experimental procedures. The definition for element of physical reality, or just element of reality, is thus enunciated.

- *Element of reality*: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity."

To derive the incompleteness of quantum mechanics they pose the following proposition. Quantum mechanics predicts that if two observables A and B are such that they do not commute, the knowing of one quantity precludes the knowing of the other. Say, the knowing of position precludes the knowing of momentum. Then, if we know the position of a particle, there is no element of reality corresponding to momentum. There are, thus, two alternatives: (1) the description of the physical reality given by the wave function is incomplete; (2) physical quantities assigned to incompatible observables cannot have simultaneous reality. The two alternatives are mutually exclusive, if it is possible for incompatible observables to have simultaneous reality, the description given by the wave function would be complete. And, if it is complete, it should be possible to know the values corresponding to the incompatible observables.

Quantum mechanics formalism presumes that (1) is false and, therefore, (2) should be true. In a thought experiment, though, EPR seem to show that quantum mechanics also implies that if (1) is false, (2) should also be false, arriving, thus, to a paradox. The resolution comes with the conclusion that (1) should be true, and, hence, that quantum mechanics is incomplete. For simplicity, the version of the thought experiment we will present is the one proposed by Bohm in [34], since it deals with discrete variables, opposed to the original argument that deals with continuous ones.

Alice is at her laboratory and, far away, Bob is in his laboratory. Each one possesses a qubit from a pair described by a singlet state like

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|z+;z-\rangle - |z-;z+\rangle) \tag{2.36}$$

$$= \frac{1}{\sqrt{2}}(|x+;x-\rangle - |x-;x+\rangle), \tag{2.37}$$

where $|z+\rangle$ ($|z-\rangle$) is the eigenstate of the operator σ_z with eigenvalue +1 (-1) and $|x+\rangle$ ($|x-\rangle$) the eigenstate of σ_x and eigenvalue +1 (-1). Assuming that (1) is false, such a description should be complete. Alice and Bob can perform measurements over their qubits, but the arrangement of the laboratories is such that the measurement events are spacelike separated. Alice proceeds to measure the spin of her qubit. If a measurement is made with respect to the σ_z observable, giving a +1 (-1) outcome, the post measurement state reads $|z+;z-\rangle$ ($|z-;z+\rangle$), rendering Bob's qubit uniquely determined. Similarly, if the measurement is made with respect to σ_x, the qubits of both Alice and Bob are given uniquely by eigenstates of σ_x. That is, depending on the choice of measurement made by Alice, Bob's qubit can be described by two distinct wave functions. Now, the argument provided by EPR states that, since the measurement events are spacelike distant, there is no causal connection between Alice and Bob's laboratories, in such a way that it is possible to ascribe simultaneously two distinct wave functions to Bob's qubit. Thus, for Bob's qubit, there are elements of reality corresponding simultaneously to both σ_z and σ_x. Since σ_z and σ_x are incompatible, they concluded that (2) is false too.

This thought experiment motivated heated debates that were dragged throughout decades. Is quantum mechanics indeed an incomplete theory? In the next chapter we will find that it is not. Beyond that, we will see that the flaw behind EPR's reasoning is an underlying a priori assumption that quantum mechanics should be a local physical theory.

3 QUANTUM NONLOCALITY

With our basic tools laid out and with some of the intriguing features displayed by quantum mechanics exposed, we are now in position to start an investigation on the role played by causality, locality and realism and the interaction of such concepts in this theoretical framework. An operational measure of realism and an embedded measure of nonlocality will be introduced, paving ways for the development of the central part of this work, in the upcoming chapter.

3.1 BELL'S THEOREM

In 1964, Bell published an article entitled "On the Einstein Podolsky Rosen Paradox" [5] where, aiming to shed some light over the EPR's paradox, a very important and conceptually rich result was found, later coming to be known as the "Bell's theorem". It proves that the predictions made by quantum mechanics are incompatible with a local causality hypothesis.

Such a conclusion was made possible through the proof that quantum mechanics is incompatible with a local hidden variable theory, since the original argument laid out on that EPR's paper pointed in the direction that the assumption of quantum mechanics to be a local physical theory implied in the theory's incompleteness, which demanded its supplementation via the introduction of new variables still unrecognized, the "hidden variables". Such variables would provide information regarding the preparation of the physical state as well as ascertain the definiteness of its physical properties at all times.

Here we will outline the argument not in its original form, but on the lines of a more recent development presented in [35] since it is clearer and employs the "CHSH inequality", a lemma of Bell's theorem named after John Clauser, Michael Horne, Abner Shimony, and Richard Holt who introduced it in 1969 in [36], which is largely employed in experiments crafted to verify the validity of such theorem.

Alice and Bob, who reside in two distant sites, receive each one a system that previously interacted with each other. Alice performs a series of measurements of a chosen observable A obtaining an outcome a, while Bob does the same for B obtaining an outcome b. The outcomes obtained by the measurements of the chosen observables can vary for each iteration. It is possible, thus, after a large number of measurements, to obtain the probability distribution $p(a, b|A, B)$ associated with the outcomes.

For an experiment of this kind where the states being measured are a pair of spin 1/2 particles and the chosen observables are their spins along some direction, Alice and Bob see that, in general,

$$p(a, b|A, B) \neq p(a|A)p(b|B). \tag{3.1}$$

This points to the fact that the statistical profile that governs the outcomes obtained by Alice and Bob are not independent of each other.

This result, a priori, does not necessarily point to a nonlocal phenomenon, since it is possible that this profile of statistical distribution occurs due to some dependency relation that was established when the systems interacted with each other. Turning this consideration into a hypothesis, its implementation is made possible describing a set of past factors that have a causal relation with the outcomes obtained in each site by means of some variable λ. Thus, we can say the outcomes a obtained by Alice are due only to the choice of an observable A and a set of past factors described by λ, therefore being independent of the choice of an observable B and outcomes b obtained by Bob in a distant site. Similar considerations are valid for the dependencies of the outcomes b. Hence, this assumption allows us to factorize the probabilities for a and b:

$$p(a, b|A, B, \lambda) = p(a|A, \lambda)p(b|B, \lambda). \tag{3.2}$$

To make this assumption even more general, we take into account that the variables λ does not need to be fixed for each iteration of the experiment since, even though the parameters set for each measurement performed are held fixed, λ accounts for unknown physical quantities that are not necessarily controllable. This way, it's reasonable to infer that the possible values λ assumes are characterized by a probability distribution $p(\lambda)$, which allows us to restate 3.2 as

$$p(a, b|A, B, \lambda) = \int_\lambda p(\lambda)p(a|A, \lambda)p(b|B, \lambda)d\lambda. \tag{3.3}$$

Brunner et al. argue in [35] that this statement does not make any assumption of determinism. However, in [37] Cavalcanti and Wiseman claim that the assumption of determinism is implied by Bell's theorem, in such a way that 3.3 is often referred as the "local causality hypothesis".

Now, we show that the predictions made by quantum mechanics are incompatible with 3.3. More specifically, the probability profile $p(a, b|A, B)$ accessed via Bell experiments involving entangled particles are not subject to a decomposition of the kind 3.3. In fact, this constitutes a mathematical theorem. So, for simplicity, let us consider an experiment where A and B are restricted to a set of only two possible choices, $A, B \in \{1, 0\}$, with associated outcomes +1 and −1, $a, b \in \{-1, +1\}$. The expectation value of the product ab is given by $\langle a_A b_B\rangle = \sum_{a,b} abp(a, b|A, B)$. We now introduce a function of the probabilities $p(a, b|A, B)$: $S = \langle a_0 b_0\rangle + \langle a_0 b_1\rangle + \langle a_1 b_0\rangle - \langle a_1 b_1\rangle$. These probabilities satisfy the local causality hypothesis if and only if they also satisfy the CHSH inequality:

$$S = \langle a_0 b_0\rangle + \langle a_0 b_1\rangle + \langle a_1 b_0\rangle - \langle a_1 b_1\rangle \leq 2. \tag{3.4}$$

To see this, let us assume 3.3, then we have $\langle a_A b_B\rangle = \int_\lambda p(\lambda)\langle a_A\rangle_\lambda \langle b_B\rangle_\lambda d\lambda$, where $\langle a_A\rangle_\lambda = \sum_a ap(a|A, \lambda)$ and $\langle b_B\rangle_\lambda = \sum_b bp(b|B, \lambda)$, with values in $[-1, 1]$. S now takes the form $S = \int_\lambda p(\lambda)S_\lambda d\lambda$, with $S_\lambda = \langle a_0\rangle_\lambda(\langle b_0\rangle_\lambda + \langle b_1\rangle_\lambda) + \langle a_1\rangle_\lambda(\langle b_0\rangle_\lambda - \langle b_1\rangle_\lambda)$. Because $\langle a_0\rangle_\lambda, \langle a_1\rangle_\lambda \in [-1, 1]$ and assuming, without loss of generality, that $\langle b_0\rangle_\lambda \geq \langle b_1\rangle_\lambda$, we know that $S_\lambda \leq |\langle b_0\rangle_\lambda + \langle b_1\rangle_\lambda| + |\langle b_0\rangle_\lambda - \langle b_1\rangle_\lambda| = 2\langle b_0\rangle_\lambda \leq 2$. Hence, $S \leq 2\int_\lambda p(\lambda)d\lambda = 2$.

Finally, let us consider now an experiment comprising a state of two qubits, one to be measured by Alice and the other by Bob, prepared in a singlet state like $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Here, $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z with associated eigenvalues +1 and −1 respectively. The choice of measurements to be performed is upon the spins of the qubits along some direction, for Alice $A = \vec{A} \cdot \vec{\sigma}$ and for Bob $B = \vec{B} \cdot \vec{\sigma}$, where $\vec{\sigma}$ is the Pauli vector, $(\sigma_x, \sigma_y, \sigma_z)$. Quantum mechanics predicts that, for such a case, $\langle a_A, b_B\rangle = -\vec{A} \cdot \vec{B}$. Let the two possible choices of A correspond to two orthogonal vectors $\hat{e}_1$ and $\hat{e}_2$ and for B the pair $-\frac{1}{\sqrt{2}}(\hat{e}_1 + \hat{e}_2)$ and $\frac{1}{\sqrt{2}}(\hat{e}_1 - \hat{e}_2)$. Such an arrangement gives us $\langle a_0 b_0\rangle = \langle a_0 b_1\rangle = \langle a_1 b_0\rangle = \frac{1}{\sqrt{2}}$ and $\langle a_1 b_1\rangle = -\frac{1}{\sqrt{2}}$ such that

$$S = 2\sqrt{2} > 2, \tag{3.5}$$

violating the CHSH inequality 3.4. This implies that quantum mechanics is incompatible with a local hidden variable theory and, therefore, incompatible with the local causality hypothesis 3.3. Such a theoretical assessment came to be experimentally verified many times through different methodologies, see [8, 38–43].

This procedure allows us to tell if a quantum state can display nonlocal behaviour. But it is also possible to quantify the nonlocality of such a state by making use of measures of Bell nonlocality. Several of them were developed, being suitable for both pure and mixed states. The crafting of such measures involves features like the maximization of Bell inequalities violation [44], the amount of information needed for the simulation of quantum correlations via classical communication (see more about classical communication in section 3.2) [45–49], resilience to noise [50, 51], and the amount of settings for a given state that manifests nonlocal behaviour [52].

3.2 ENTANGLEMENT

In the previous section, a situation where the CHSH inequality violation happened was presented by means of introducing what we called an "entangled state", suggesting that entanglement and nonlocality are closely related. In a nutshell, if a quantum system composed by various parts is such that it is impossible to describe the state of one of its parts independently of the total system, there is entanglement. An extensive review about the subject can be found in [21].

Entanglement is a kind of correlation that typically arises when two particles interact with each other. It is not possible, however, to generate quantum entanglement or other quantum correlations via "LOCC", local operations and classical communication. To clarify, local operation is an operation that is locally performed over one part of a quantum composite system, such as measurements, appending auxiliary systems or "tracing out" subsystems; and classical communication is a process of broadcast of information via classical medium, such as in the case where Alice tells Bob something about her qubit via a text message (transmitted via classical internet).

Such restrictions on the possibilities to create entanglement and the usefulness of such a characteristic in tasks like quantum teleportation [14] justifies entanglement to be seen as a quantum resource (for more about resource theories, see [53]). It is thus clear the necessity to diagnose and quantify entanglement, so we will discuss it in a little bit more depth. In spite of the fact that such a phenomenon can be spotted in multipartite quantum systems, in this chapter we will focus solely on bipartite quantum states, pure and mixed.

3.2.1 Entanglement in pure states

A bipartite pure state is said to be entangled if it is impossible to put it in the form $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. For, if it is possible, a measurement performed in the partition $\mathcal{A}$ of the system, say, $(A \otimes \mathbb{1}_{\mathcal{B}})$, with A acting on $\mathcal{H}_{\mathcal{A}}$, renders the partition $\mathcal{B}$ untouched, establishing an independence between the parts of the system. After all, the measurement of A is independent of the partition $\mathcal{B}$ and a further measure on $\mathcal{B}$ is independent of the outcome of A. Hence, it is possible to describe the state of one of the parts of the system without needing to resort to the state of the total system, pointing to no entanglement.

For a product state $\rho_{pro} = |\Psi\rangle \langle\Psi|$, we have $\rho_{pro} = |\psi_1\rangle \langle\psi_1| \otimes |\psi_2\rangle \langle\psi_2|$. The probabilities $p(a, b|A, B)$ are given by $\text{Tr}\,(A_a \otimes B_B \rho_{pro}) = p(a|A)p(b|B)$, making it clear that a product state of the kind $|\Psi\rangle$ suits the factorization required in 3.2 for the local causality hypothesis to be met. An entangled pure state, thus, a nonproduct state, will necessarily violate such a hypothesis. In fact, for pure states, Bell nonlocality occurs if and only if entanglement occurs [54, 55].

Even though the definition is quite easy to grasp, the task of identifying whether a particular quantum state can be written like $|\Psi\rangle$ is not trivial. If it is possible to write a particular pure state as a product state, we are sure that such a state is not entangled. However, if one cannot do that, either the state is entangled or a possible decomposition could not be identified. To avoid this issue, different methodologies to identify whether a state is entangled other than trying to put it as a product state were developed, the measures of entropy. Such measures need to meet a list of requirements, such as the one presented in [56], that applies for both pure and mixed states and are able to distinguish between more or less entangled states. In the case of pure states, there are plenty of tools suitable for this task. One of them is the entropy of entanglement.

- *Entropy of entanglement*: for a pure state ρ in $\mathcal{H}$, with a bipartition given by $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, the entanglement between A and B is given by $E(\rho) = S(\rho_{\mathcal{A}})$, where $\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}}(\rho)$ and $S(\rho_{\mathcal{A}}) = S(\rho_{\mathcal{B}})$.

There is another useful tool to access the entanglement properties of a state, the Schmidt decomposition. For a pure bipartite state $|\psi\rangle$ there exist orthonormal states $|i_{\mathcal{A}}\rangle$ for the partition $\mathcal{A}$ and $|i_{\mathcal{B}}\rangle$ for $\mathcal{B}$ such that,

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |i_{\mathcal{A}}\rangle\, |i_{\mathcal{B}}\rangle \,, \tag{3.6}$$

with real non-negative coefficients λ_i such that $\sum_i \lambda_i = 1$. The number of coefficients λ_i is called the Schmidt number. It is a necessary and sufficient condition for the state $|\psi\rangle$ to be product that its Schmidt number is 1, and, therefore, $\lambda = 1$. The state is diagnosed as entangled when the Schmidt number is greater than 1 and it is maximally entangled if all λ_i are equal. To better picture this, we give the maximally entangled bipartite states for qubits, the Bell states:

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle); \quad |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{3.7}$$

These states present Schmidt number 2 and all of their coefficients are equal.

3.2.2 Entanglement in mixed states

Just as for the pure states case, the definition of entanglement for mixed states is given through a negative. If, for a bipartite mixed state ρ, there are no local states $\rho_i^{\mathcal{A}}$ and $\rho_i^{\mathcal{B}}$ and no non-negative values p_i such that ρ can be written as a convex sum $\rho_{sep} = \sum_i p_i \rho_i^{\mathcal{A}} \otimes \rho_i^{\mathcal{B}}$, a separable state, ρ is said to be entangled. Now, in contrast with the case for pure states, a nonentangled mixed state can exhibit correlations, since the outcome of a local measurement performed in $\mathcal{A}$ and the outcome of one in $\mathcal{B}$ can be correlated in such a way that $\mathrm{Tr}\,(A \otimes B\rho_{sep}) \neq \mathrm{Tr}\,(A \otimes \mathbb{1}_{\mathcal{B}}\rho_{sep})\mathrm{Tr}\,(\mathbb{1}_{\mathcal{A}} \otimes B\rho_{sep})$. However, such a correlation is not quantum by nature, it is classical, since it is given by the classical probabilities p_i.

The absence of quantum correlations in a state of the kind ρ_{sep} can be readily appreciated in the non violation of the local causality hypothesis, given that $p(a, b|A, B) = \mathrm{Tr}\,(A \otimes B\rho_{sep})$ admits a factorization, being, thus, Bell local. Bell nonlocal mixed states, in turn, imply in entanglement. However, the converse was shown not to be true [35], entailing that the class of entangled states forms a superset of the Bell nonlocal states.

The quantification of entanglement for mixed states has shown to be a much more complicated task than for the pure state case. We dispose of mechanisms for such given quantum states of low dimension, but the quantifiers devised for higher dimensional states are still unsatisfactory. An example of a quantifier of entanglement that is suited for both pure and mixed states is the concurrence. Here, we will present it for the case of two qubits, composing a bipartite state:

- *Concurrence*: for a bipartite two qubit state ρ, the concurrence is given by $C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where the λ_is are the eigenvalues in decreasing order of $\rho\tilde{\rho}$, with $\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$, ρ^* the complex conjugate of ρ and σ_y the Pauli operator.

In an essay published by Gisin in [57] some open questions are enumerated regarding Bell nonlocality. One of them is: "why are almost all known Bell inequalities for more than 2 outcomes maximally violated by states that are not maximally entangled?" Indeed, there are situations where states are diagnosed as maximally entangled, but when the amount of Bell nonlocality assigned for such states is quantified, it is found that it is not maximal [15–18]. This is often called "anomaly of nonlocality" and may be a reflex of the difficulty to measure Bell nonlocality, since different methods for such can diagnose the same state as anomalous or not. There is another way, however, to attack this problem, devising nonlocality quantifiers that do not rely on Bell inequalities. Following this route, we are about to present a quantifier of nonlocality that has shown to be nonanomalous, reduces to entanglement for maximally entangled states, and is backed upon the presumption that nonlocal phenomena are closely related to violations of realism. So, now, we discuss realism and how to measure it.

3.3 REALISM

Realism is a term often used without a rigorous definition in physics' literature, so it is prudent to give a little bit more room for developing the criterion of realism that will be employed in this work.

It is common that, when someone says "realism" in a physics paper, what underlies this word is a hypothesis that there is always a definite value assigned to every physical quantity in such a way that a measurement only reveals it [9]. A seminal idea for the implementation of this concept can be found in [58], where Einstein, Podolsky and Rosen pose a definition for what they called "elements of reality" (section 2.4).

By EPR's criterion, it is straightforward for us to see that there are elements of reality corresponding to physical quantities like the electric charge of a macroscopic object, but the predictions made by quantum mechanics point towards situations where elements of reality are absent.

Consider for example a particle with spin 1/2. Upon a measurement of its z spin's component, we have obtained the outcome $\hbar/2$. For the present state, we know that further measurements in the z axis would give the same outcome and leave the particle state untouched. Thus, it is possible to predict the outcome with certainty without disturbing the system. We say then that there is an element of reality corresponding to the z component of the particle spin. However, what can we say about the x component of its spin? Quantum mechanics predictions ascertain that the outcome of such measurement cannot be predicted with certainty. Even more, such a measurement would imply in a change, and thus a disturbance, on the system state. In this case, we can say that there are no elements of reality corresponding to this physical quantity, the x component of the particle spin.

However, the direct application of this criterion for the cases where there is a subjective ignorance about the outcome of a measurement points to an absence of elements of reality. Suppose that Alice measured the z component of a spin $1/2$ particle but prevented Bob to know the outcome of the measurement. The best description Bob can ascribe to the post-measurement state is $\rho = (|0\rangle\langle 0| + |1\rangle\langle 1|)1/2$, that is, an ensemble of states with elements of reality corresponding to the spin z component. Even though if Bob had access to the system and submitted it to a new measurement it would not imply in a change in the system state, he cannot predict the measurement outcome with certainty. But it is not reasonable to say that there is a lack of elements of reality in this case due to Bob's ignorance.

Further proposals of realism criteria manage to avoid such inconsistencies, as the one given by A. Zeilinger et al. in [59]: "the assumption that measurement outcomes are well defined prior to and independent of the measurements". Here, we opt for the one proposed by Bilobran and Angelo in [10] since it is suitable for dealing with mixed states, avoiding the aforementioned problem, and, furthermore, it is operational and quantitative.

3.3.1 Bilobran and Angelo's criterion of realism

A fundamental assumption that underlies this criterion is that after a projective measurement of a discrete spectrum observable $A = \sum_a aA_a$ over a quantum state, even if the measurement outcome is not revealed, there is an element of reality corresponding to A. Following this rationale, a protocol is devised allowing for the quantification of the notion of "irreality" in a generic quantum state ρ on $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ [10].

Suppose a given source prepares a very large number of copies of a generic quantum state for a bipartite system. Upon submission of every copy to state tomography, we come to know that the most complete possible description of such preparation is given by $\rho \in \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. Suppose now that a given source prepares a very large number of copies of the same quantum state, but this time, before the submission of the copies to the procedure of state tomography, every copy is intercepted by an agent that always performs projective measurements of the operator $A = \sum_a aA_a$ acting over the space $\mathcal{H}_\mathcal{A}$ but never reveals its outcome. The best description we can come out with for such states is given by

$$\sum_a (A_a \otimes \mathbb{1}_\mathcal{B})\, \rho\, (A_a \otimes \mathbb{1}_\mathcal{B}) = \sum_a p_a A_a \otimes \rho_{\mathcal{B}|a} =: \Phi_A(\rho), \tag{3.8}$$

where $p_a = \mathrm{Tr}\,(A_a \otimes \mathbb{1}_\mathcal{B}\, \rho)$ is the probability for the agent to obtain the outcome a and $\rho_{\mathcal{B}|a} = \mathrm{Tr}_\mathcal{A}(A_a \otimes \mathbb{1}_\mathcal{B}\, \rho)/p_a$ is the state that describes the part of the system left untouched, figure 2. By the assumption that we took, it is clear that the post measurement state, $\Phi_A(\rho)$, has an element of reality corresponding to A. Because of that, the authors proceed to call the state $\Phi_A(\rho)$ an *A-reality state*. Now, the criterion is condensed to the following condition: *the observable A is real for the state ρ if and only if:*

$$\rho = \Phi_A(\rho). \tag{3.9}$$

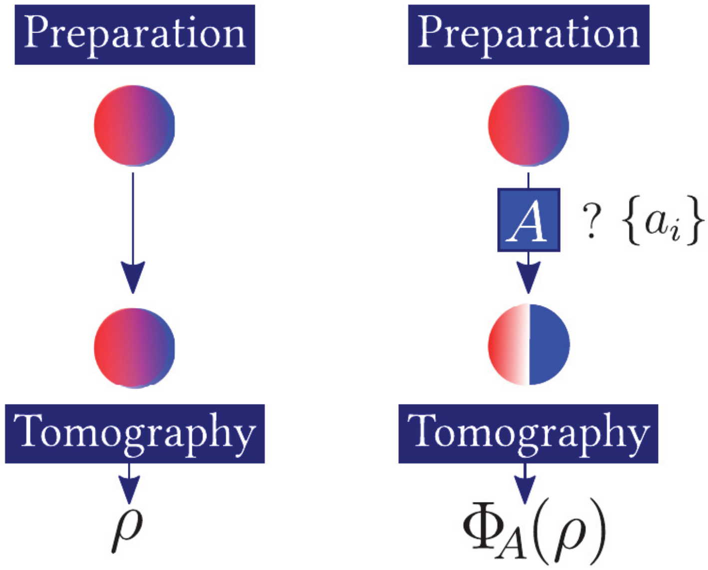

Figure 2 – On the left, a source prepares infinitely many copies of a quantum state that are sent to quantum tomography procedure that obtains the description ρ. On the right, a source prepares infinitely many copies of the same state, but this time, every copy is intercepted by an agent that measures, every time, the same observable A. The states are thus sent to a quantum tomography procedure that now obtains the description $\Phi_A(\rho)$. Figure took from [60]

Since this criterion will be employed from here on, it is useful to lay down some mathematical properties of the Φ_A map. Φ_A is a completely positive trace-preserving unital map. Let us unpack this a little bit. Since $\rho = \rho_{\mathcal{AB}}$, that is, ρ is a density operator that represents a joint system of $\mathcal{A}$ and $\mathcal{B}$, the property of Φ_A being completely positive guarantees that its action over the part $\mathcal{A}$ alone will still result in a valid density operator up to a normalization. To put this property more formally, if Φ_A acts only on the part $\mathcal{A}$ and I is the identity map acting over $\mathcal{B}$, the map $\Phi_A \otimes I$ will necessarily take positive operators to positive operators [14]. It is trace-preserving since, given that a fundamental property of density operators is that $\operatorname{Tr} \rho = 1$, it always maps a density operator into another one. And finally, its unital property reflects the fact that $\Phi_A(I) = I$, since it takes a completely mixed state into a completely mixed state.

To see this criterion in action, let us take a $1/2$ spin particle that after a measurement of the z component of its spin is left in the state $\rho = |0\rangle\langle 0|$. EPR's criterion would point to an element of reality corresponding to the z component of its spin, the observable S_z. It is straightforward to notice that the task of the unrevealed measurement of the S_z observable would leave the system state unaltered, that is, $\Phi_{S_z}(\rho) = \rho$. Then, BA's criterion of realism is in accordance to EPR's in this case. In contraposition, if the state under scrutiny is $\rho = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$, the same procedure would leave us with $\Phi_{S_z}(\rho) = \rho$, determining that the state ρ is an S_z-reality state according to BA's criterion in opposition to EPR's and, thus, showing a conceptual advantage of the former over the latter.

Indeed, due to the fact that the measurement of the observable A is projective, and thus $A_a A_{a'} = A_a \delta_{aa'}$, we can verify that further submissions of the system to the protocol of unrevealed measurement leaves the state untouched, that is,

$$\Phi_A(\Phi_A(\rho)) = \sum_a \sum_{a'} (A_a \otimes \mathbb{1}_{\mathcal{B}})(A_{a'} \otimes \mathbb{1}_{\mathcal{B}})\, \rho\, (A_{a'} \otimes \mathbb{1}_{\mathcal{B}})(A_a \otimes \mathbb{1}_{\mathcal{B}}) \tag{3.10}$$

$$= \sum_a (A_a \otimes \mathbb{1}_{\mathcal{B}})\, \rho\, (A_a \otimes \mathbb{1}_{\mathcal{B}}) = \Phi_A(\rho) \tag{3.11}$$

and, therefore, does not change its reality status.

Another insightful way of looking to the quantity $\Phi_A(\rho)$ is brought forward by the authors by making use of the Stinespring theorem [61]. A direct application of the theorem leads us to the conclusion that $\Phi_A(\rho)$ can be expressed as

$$\Phi_A(\rho) = \mathrm{Tr}_{\mathcal{X}}[U(\rho \otimes |x_0\rangle\langle x_0|)U^\dagger], \tag{3.12}$$

with U being an unitary operator acting over $\mathcal{H} \otimes \mathcal{H}_{\mathcal{X}}$ and $|x_0\rangle \in \mathcal{H}_{\mathcal{X}}$. To put it plainly, a quantum operation of the kind Φ can be viewed as if we coupled the system ρ to an ancillary state $|x_0\rangle\langle x_0|$ acting on a space $\mathcal{H}_{\mathcal{X}}$, submitted it to a unitary evolution U acting over both spaces and then discarded the ancillary system. If we regard the ancillary system as an informer, that is, a physical system whose degrees of freedom allow for the storage of information about ρ and is discarded afterwards, 3.12 points to the emergence of reality upon the dynamic generation of correlation between the system ρ and such an informer. In the context of the unrevealed measurement protocol, the informer is the secret agent that interacts with the system by means of the measurement, storing its information and being discarded, once it never returns to interact with the system and does not reveal the outcome assessed.

Now, we show a result concerning this quantity which will be useful later on in this work. Given two orthonormal bases in $\mathcal{H}_{\mathcal{A}}$, $\{|a_i\rangle\}$ and $\{|a_i'\rangle\}$, these are called "mutually unbiased bases", MUB, if it is true that $|\langle a_i|a_i'\rangle|^2 = \frac{1}{d_{\mathcal{A}}}$, where $d_{\mathcal{A}}$ is the dimension of $\mathcal{H}_{\mathcal{A}}$. After all, a measurement performed in one basis leaves all the possible outcomes on the other basis equiprobable and, thus, its information inaccessible by such a measurement. Using the first and second bases, we can define a pair of "maximally unbiased", MU, observables, A and A', respectively. In the case where we consider a state $\rho = \Phi_A(\rho)$ and submit it to an unrevealed measurement protocol of an observable A' that is maximally incompatible to A, the result is the following state:

$$\Phi_{A'}(\Phi_A(\rho)) = \frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}} \otimes \rho_{\mathcal{B}}, \tag{3.13}$$

with $d_{\mathcal{A}} = \dim \mathcal{H}_{\mathcal{A}}$, and $\rho_{\mathcal{B}}$ being the reduced state of the part $\mathcal{B}$, $\mathrm{Tr}_{\mathcal{A}}(\rho) = \rho_{\mathcal{B}}$.

3.3.2 Irreality

In the same work, BA propose a way to quantify reality or its counterpart, irreality. Employing von Neumann's entropy, a measure of the degree of violation of the realism criterion

3.9 laid above takes the form:

$$\mathfrak{I}_A(\rho) := S(\Phi_A(\rho)) - S(\rho). \tag{3.14}$$

$\mathfrak{I}_A(\rho)$ stands for the irreality of the observable A given the state ρ, after all, what its definition implements is an entropic difference between ρ and its A-reality counterpart, $\Phi_A(\rho)$.

Even more than a difference of entropies, $\mathfrak{I}_A(\rho)$ is equivalent to the entropic distance between the state ρ and $\Phi_A(\rho)$ itself:

$$\mathfrak{I}_A(\rho) = S(\rho||\Phi_A(\rho)). \tag{3.15}$$

Let us prove this. Putting the right-hand side of the above equation explicitly, we have

$$S(\rho||\Phi_A(\rho)) = \mathrm{Tr}\,(\rho \log \rho) - \mathrm{Tr}\,(\rho \log \Phi_A(\rho)). \tag{3.16}$$

A comparison between 3.14 and 3.16 leads us to the conclusion that 3.15 follows if it is true that

$$\mathrm{Tr}\,(\rho \log \Phi_A(\rho)) = \mathrm{Tr}\,(\Phi_A(\rho) \log \Phi_A(\rho)). \tag{3.17}$$

In order to show this, we remember that $\sum_a A_a = \mathbb{1}_{\mathcal{A}}$, $A_a^2 = A_a$, and that the trace is cyclical. So, we begin by writing

$$\mathrm{Tr}\,(\rho \log \Phi_A(\rho)) = \mathrm{Tr}\,(\underbrace{\sum_a (A_a \otimes \mathbb{1}_{\mathcal{B}})}_{\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}} \rho \log \Phi_A(\rho)) \tag{3.18}$$

$$= \mathrm{Tr}\,(\sum_a (A_a \otimes \mathbb{1}_{\mathcal{B}}) \rho \log \Phi_A(\rho)\, \underbrace{(A_a \otimes \mathbb{1}_{\mathcal{B}})}_{\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}}). \tag{3.19}$$

Now, since

$$\Phi_A(\rho)(A_a \otimes \mathbb{1}_{\mathcal{B}}) = (A_a \otimes \mathbb{1}_{\mathcal{B}})\rho(A_a \otimes \mathbb{1}_{\mathcal{B}}) \tag{3.20}$$

$$= (A_a \otimes \mathbb{1}_{\mathcal{B}})\Phi_A(\rho), \tag{3.21}$$

we see that $(A_a \otimes \mathbb{1}_{\mathcal{B}})$ commutes with $\Phi_A(\rho)$, and thus, $(A_a \otimes \mathbb{1}_{\mathcal{B}})$ commutes with $\log \Phi_A(\rho)$. With that in mind, we come to

$$\mathrm{Tr}\,(\rho \log \Phi_A(\rho)) = \mathrm{Tr}\,(\sum_a (A_a \otimes \mathbb{1}_{\mathcal{B}}) \rho \log \Phi_A(\rho)(A_a \otimes \mathbb{1}_{\mathcal{B}})) \tag{3.22}$$

$$= \mathrm{Tr}\,(\sum_a (A_a \otimes \mathbb{1}_{\mathcal{B}}) \rho (A_a \otimes \mathbb{1}_{\mathcal{B}}) \log \Phi_A(\rho)) \tag{3.23}$$

$$= \mathrm{Tr}\,(\Phi_A(\rho) \log \Phi_A(\rho)), \tag{3.24}$$

which proves 3.17, proving, thus, 3.15. This result is shown in [10] and a similar demonstration to the one we settled here can be found at [14].

The equivalence 3.15 and the non-negativity of the relative entropy allow us to derive an important property:

$$\mathfrak{I}_A(\rho) \geq 0, \tag{3.25}$$

with equality holding if and only if $\Phi_A(\rho) = \rho$. This assures us that the only state of maximum reality of an observable for a state is, indeed, $\Phi_A(\rho)$. Now, if we consider 3.15 together with the monotonicity of the relative entropy, we see that $\mathfrak{I}_A(\rho)$ is nonincreasing under completely positive trace-preserving maps, that is, in the context of quantum information, quantum channels. To put it explicitly, if $\mathcal{E}(\rho)$ is a quantum channel, then

$$\mathfrak{I}_A(\rho) = S(\rho||\Phi_A(\rho)) \geq S(\mathcal{E}(\rho)||\Phi_A(\mathcal{E}(\rho))) = \mathfrak{I}_A(\mathcal{E}(\rho)). \tag{3.26}$$

In [62], Freire and Angelo were able to show that this measure of irreality can be extended to the context of continuous variables and also derived a kind of uncertainty relation:

$$\mathfrak{I}_A(\rho) + \mathfrak{I}_{A'}(\rho) \geq \left(\rho||\frac{\mathbb{1}_{\mathcal{A}}}{d_{\mathcal{A}}} \otimes \rho_{\mathcal{B}}\right), \tag{3.27}$$

where A and A' are arbitrary observables acting on $\mathcal{H}_{\mathcal{A}}$. This relation shows that it is not possible to expect two observables to have simultaneous reality in general.

3.4 REALISM BASED NONLOCALITY

First of all, let us put down an example presented on [10] to see how the phenomenon of nonlocality and irreality are closely related. Suppose that two 1/2 spin particles share a singlet state, $\rho = |\psi\rangle\langle\psi|$, with $|\psi\rangle = \frac{1}{\sqrt{2}}(|z+;z-\rangle - |z-;z+\rangle)$. A physical preparation of this kind constrains the total spin $\mathfrak{s}_z$ of the system in accordance to $\mathfrak{s}_z = \mathfrak{s}_z^{\mathcal{A}} + \mathfrak{s}_z^{\mathcal{B}} = 0$. One could argue that the total spin of the system has a definite value such that $\Phi_{S_z}(\rho) = \rho$, whereby we diagnose the observable S_z as real for this preparation. Even though we separate each particle by a distance as big as we want, the constraint still holds and, consequently, the reality status of S_z holds as well. However, by the same criterion, the individual spin of the particles cannot be regarded as real for this preparation. Now, if a measurement of the particle $\mathcal{A}$ spin is performed and, thus, the observable $S_z^{\mathcal{A}}$ becomes real, the constraint obliges the particle $\mathcal{B}$'s spin, $S_z^{\mathcal{B}}$, to also become real. This phenomenon cannot be regarded as something locally causal, since the disturbance of the particle $\mathcal{A}$ in a remote site caused the emergence of reality in $\mathcal{B}$. Backed by this relation, we proceed to examine the nonlocality.

For a preparation comprising a bipartite state where we allocate each part of the system in two distant sites, Bilobran and Angelo argue that if a projective measurement of an observable B, $B = \sum_b bB_b$ for projectors B_b, acting on $\mathcal{B}$ does not change the irreality status of an observable A on $\mathcal{A}$, there can be no nonlocal phenomena arising. A locality hypothesis were thus crafted synthesising this proposition that is formally expressed as:

$$\mathfrak{I}_A(\rho) = \mathfrak{I}_A(\Phi_B(\rho)), \tag{3.28}$$

where $\Phi_B(\rho) = (\mathbb{1}_{\mathcal{A}} \otimes B_b)\rho(\mathbb{1}_{\mathcal{A}} \otimes B_b)$.

3.4.1 Contextual realism based nonlocality

Still on the same work, in a similar way to that as the irreality measure was presented, BA proposed a quantification for context nonlocality based on realism that is a measure of the degree of violation of the locality hypothesis 3.28,

$$\eta_{A|B}(\rho) := \mathfrak{I}_A(\rho) - \mathfrak{I}_A(\Phi_B(\rho)). \tag{3.29}$$

This quantity is called *contextual realism-based nonlocality*, since it accounts for the degree of change in the irreality of an observable A upon the measurement of an observable B for a preparation ρ, being, thus, defined over a context given by the pair $\{A, B\}$.

$\eta_{A|B}(\rho)$ is invariant under permutation of indices. This can be clearly seen if we write 3.29 in terms of von Neumann entropy:

$$\eta_{A|B}(\rho) = S(\Phi_A(\rho)) + S(\Phi_B(\rho)) - S(\Phi_{A,B}(\rho)) - S(\rho), \tag{3.30}$$

where $S(\Phi_{A,B}(\rho)) = S(\Phi_A(\Phi_B(\rho)))$ and recall that, since A acts on $\mathcal{A}$ and B acts on $\mathcal{B}$, A and B commutes, then $S(\Phi_{A,B}(\rho)) = S(\Phi_{B,A}(\rho))$.

The non-negativity of $\eta_{A|B}(\rho)$,

$$\eta_{A|B}(\rho) \geq 0, \tag{3.31}$$

arises naturally due to the fact that the irreality is nonincreasing under completely positive trace-preserving maps, such as Φ_B. To put it explicitly, 3.29 shows that if

$$\mathfrak{I}_A(\rho) \geq \mathfrak{I}_A(\Phi_B(\rho)), \tag{3.32}$$

3.31 follows. To show that this is true, we take advantage of the fact that the relative entropy is also nonincreasing under completely positive trace-preserving maps, concluding that

$$\begin{aligned} \mathfrak{I}_A(\rho) = S(\rho||\Phi_A(\rho)) \geq& S(\Phi_B(\rho)||\Phi_A(\Phi_B(\rho))) \\ =& S(\Phi_B(\rho)||\Phi_B(\Phi_A(\rho))) = \mathfrak{I}_A(\Phi_B(\rho)). \end{aligned} \tag{3.33}$$

The saturation of the inequality 3.31 is assured for fully uncorrelated states, $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$, and for states of A-reality, $\rho = \Phi_A(\rho)$, B-reality, $\rho = \Phi_B(\rho)$, and A, B-reality, $\rho = \Phi_{A,B}(\rho)$. For fully uncorrelated states:

$$\begin{aligned} \eta_{A|B}(\rho) &= S(\Phi_A(\rho)) + S(\Phi_B(\rho)) - S(\Phi_{A,B}(\rho)) - S(\rho) &(3.34)\\ &= S(\Phi_A(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}})) + S(\Phi_B(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}})) - S(\Phi_{A,B}(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}})) - S(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) &(3.35)\\ &= S(\Phi_A(\rho_{\mathcal{A}})) + S(\rho_{\mathcal{B}})) + S(\rho_{\mathcal{A}}) + S(\Phi_B(\rho_{\mathcal{B}})) \\ &\quad - S(\Phi_A(\rho_{\mathcal{A}})) - S(\Phi_B(\rho_{\mathcal{B}})) - S(\rho_{\mathcal{A}}) - S(\rho_{\mathcal{B}}) &(3.36)\\ &= 0. &(3.37) \end{aligned}$$

For states of A-reality, it is clear that $\mathfrak{I}_A(\Phi_A(\rho)) = 0$ and, from 3.32 and 3.31, it follows that $\eta_{A|B}(\Phi_A(\rho)) = 0$. The same is true for B-reality states, since $\eta_{A|B}(\rho)$ is invariant under permutation of indices. A similar argument also holds valid for the case of A, B-reality, since $\mathfrak{I}_A(\Phi_{A,B}(\rho)) = 0$.

3.4.2 Bipartite realism-based nonlocality

Later on, Gomes and Angelo [19] employed the contextual realism based nonlocality to come out with a nonlocality quantifier for bipartite states that relies solely upon a generic preparation ρ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, a measure of nonlocality that is independent of context. This was made possible by taking a maximization over all pairs of observables A and B acting on $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$ respectively, leading thus to the *bipartite realism based nonlocality*:

$$\mathcal{N}_2(\rho) := \max_{\{A,B\}} \eta_{A|B}(\rho). \tag{3.38}$$

There are two major motivations behind this definition. The first one is that it implements the intuition that a preparation ρ that is subject to greater changes to its irreality status upon a projective measurement in a distant site should be diagnosed as more nonlocal. The second one is that if $\mathcal{N}_2(\rho) > 0$, we rest assured that there is at last one context $\{A, B\}$ where nonlocality will manifest itself, that is, it is guaranteed the existence of a scenario where Bob measurements can change reality in Alice's site.

Due to the non-negativity of $\eta_{A|B}(\rho)$, $\mathcal{N}_2(\rho)$ is also non-negative,

$$\mathcal{N}_2(\rho) \geq 0. \tag{3.39}$$

The condition for which equality holds is given by $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$. Interestingly though, if we consider separable states such that entanglement is absent but there can be classical correlations, as the "classical-classical state", $\rho_{cc} = \sum_i p_i A' \otimes B'$, where $A' = \sum_i a_i' A_i'$ and $B' = \sum_i b_i' B_i'$, $\mathcal{N}_2(\rho_{cc})$ is not expected to vanish. To see this, let us choose a context $\{A, B\}$ for which A and A' as well as B and B' form pairs of maximally unbiased observables, and evaluate $\eta_{A|B}(\rho_{cc})$:

$$\eta_{A|B}(\rho_{cc}) = S(\Phi_A(\rho_{cc})) + S(\Phi_B(\rho_{cc})) - S(\Phi_{A,B}(\rho_{cc})) - S(\rho_{cc}). \tag{3.40}$$

To proceed with this task, we are going to employ the joint entropy theorem 2.25 together with the fact that, for a pair A and A' of MU observables, $\Phi_A(A') = \mathbb{1}_{\mathcal{A}}/d_{\mathcal{A}}$. Now, analysing term by term of the equation 3.40, starting from the last one:

$$S(\rho_{cc}) = S\left(\sum_i p_i A_i' \otimes B_i'\right) \tag{3.41}$$

$$= H(\{p_i\}) + \sum_i p_i S(A_i' \otimes B_i') \tag{3.42}$$

where $H(\{p_i\})$ is the Shanon entropy of the probability distribution p_i. Since $A_i' \otimes B_i'$ is a pure state, $S(A_i' \otimes B_i') = 0$. So,

$$S(\rho_{cc}) = H(\{p_i\}). \tag{3.43}$$

Turning ourselves now to $S(\Phi_A(\rho_{cc}))$,

$$S(\Phi_A(\rho_{cc})) = S\left(\sum_i p_i \Phi_A(A_i') \otimes B_i'\right) \tag{3.44}$$

$$= S\left(\sum_i p_i \frac{\mathbb{1}_{\mathcal{A}'}}{d_{\mathcal{A}'}} \otimes B_i'\right) \tag{3.45}$$

$$= S\left(\frac{\mathbb{1}_{\mathcal{A}'}}{d_{\mathcal{A}'}}\right) + S\left(\sum_i p_i B_i'\right) \tag{3.46}$$

$$= \log d_{\mathcal{A}'} + H(\{p_i\}) + \sum_i p_i S(B_i'). \tag{3.47}$$

B_i' is also a pure state, thus $S(B_i') = 0$. We remain with

$$S(\Phi_A(\rho_{cc})) = \log d_{\mathcal{A}'} + H(\{p_i\}). \tag{3.48}$$

Employing an analogous procedure to $S(\Phi_B(\rho_{cc}))$ we come to the conclusion that

$$S(\Phi_B(\rho_{cc})) = \log d_{\mathcal{B}'} + H(\{p_i\}). \tag{3.49}$$

For the last term, we have

$$S(\Phi_{A,B}(\rho_{cc})) = S\left(\sum_i pi \Phi_A(A_i') \otimes \Phi_B(B_i')\right) \tag{3.50}$$

$$= S\left(\sum_i p_i \frac{\mathbb{1}_{\mathcal{A}'}}{d_{\mathcal{A}'}} \otimes \frac{\mathbb{1}_{\mathcal{B}'}}{d_{\mathcal{B}'}}\right) \tag{3.51}$$

$$= S\left(\frac{\mathbb{1}_{\mathcal{A}'}}{d_{\mathcal{A}'}} \otimes \frac{\mathbb{1}_{\mathcal{B}'}}{d_{\mathcal{B}'}}\right) \tag{3.52}$$

$$= S\left(\frac{\mathbb{1}_{\mathcal{A}'}}{d_{\mathcal{A}'}}\right) + S\left(\frac{\mathbb{1}_{\mathcal{B}'}}{d_{\mathcal{B}'}}\right) \tag{3.53}$$

$$= \log d_{\mathcal{A}'} + \log d_{\mathcal{B}'}. \tag{3.54}$$

By substituting 3.43, 3.48, 3.49, and 3.54 in 3.40,

$$\eta_{A|B}(\rho_{cc}) = \log d_{\mathcal{A}'} + H(\{p_i\}) + \log d_{\mathcal{B}'} + H(\{p_i\}) - H(\{p_i\}) - \log d_{\mathcal{A}'} - \log d_{\mathcal{B}'} \tag{3.55}$$

and, finally:

$$\eta_{A|B}(\rho_{cc}) = H(\{p_i\}). \tag{3.56}$$

With this result in hands, it's straightforward to claim that

$$\mathcal{N}_2(\rho_{cc}) > 0, \tag{3.57}$$

which points to the fact that $\mathcal{N}_2(\rho)$ is not only sensitive to quantum correlations such as entanglement, but also to nonlocal effects that arise due to the incompatibility of observables.

One of the most important features of the bipartite realism-based nonlocality is that it reduces to entanglement for bipartite states that are maximally entangled, being, thus, nonanomalous. A bipartite state that is maximally entangled can be generally written in the form

$$|\psi\rangle = \sum_{i=1}^{d} \frac{|i\rangle\,|i\rangle}{\sqrt{d}}. \tag{3.58}$$

For $\rho = |\psi\rangle\,\langle\psi|$, we now aim to prove that

$$\mathcal{N}_2(\rho) = E(\rho), \tag{3.59}$$

where $E(\rho)$ is the entanglement entropy of ρ. Once again, we start by expressing $\eta_{A|B}(\rho)$ in terms of the von Neumann entropy:

$$\eta_{A|B}(\rho) = S(\Phi_A(\rho)) + S(\Phi_B(\rho)) - S(\Phi_{A,B}(\rho)) - S(\rho). \tag{3.60}$$

$S(\rho) = 0$ because ρ is a pure state and, as expressed by the non-negativity of irreality, 3.25, $S(\Phi_R(\rho)) \geq S(\rho)$ and $S(\Phi_{A,B}(\rho)) \geq S(\Phi_R(\rho))$, with R standing for either A or B. Such consideration allows us to write

$$2S(\Phi_{A,B}(\rho)) \geq S(\Phi_A(\rho)) + S(\Phi_B(\rho)). \tag{3.61}$$

Dividing 3.61 by 2 and comparing it with 3.60, we come to the conclusion that

$$\eta_{A|B}(\rho) \leq \frac{1}{2}[S(\Phi_A(\rho)) + S(\Phi_B(\rho))], \tag{3.62}$$

with equality holding in the case where $S(\Phi_{A,B}(\rho)) = S(\Phi_A(\rho)) = S(\Phi_B(\rho))$. Such a case is expected to occur if A and B are the Schmidt observables associated to the Schmidt decomposition of $|\psi\rangle$. That is, for $|\psi\rangle = \sum_i \sqrt{\xi_i}\,|\alpha_i\rangle\,|\beta_i\rangle$, we have the associated Schmidt observables $\alpha = \sum_i \alpha_i\,|\alpha_i\rangle\,\langle\alpha_i|$ and $\beta = \sum_i \beta_i\,|\beta_i\rangle\,\langle\beta_i|$ such that

$$\Phi_\alpha(\rho) = \sum_i (|\alpha_i\rangle\,\langle\alpha_i| \otimes \mathbb{1}_{\mathcal{B}}) \left(\sum_j \sqrt{\xi_j}\,|\alpha_j\rangle\,|\beta_j\rangle\right)\left(\sum_{j'} \sqrt{\xi_{j'}}\,\langle\alpha_{j'}|\,\langle\beta_{j'}|\right)(|\alpha_i\rangle\,\langle\alpha_i| \otimes \mathbb{1}_{\mathcal{B}}) \tag{3.63}$$

$$= \sum_i (|\alpha_i\rangle\,\langle\alpha_i| \otimes \mathbb{1}_{\mathcal{B}}) \left(\sum_j \sqrt{\xi_j}\,|\alpha_j\rangle\,|\beta_j\rangle\right)\left(\sum_{j'} \delta_{i,j'}\sqrt{\xi_{j'}}\,\langle\alpha_i|\,\langle\beta_{j'}|\right) \tag{3.64}$$

$$= \sum_i (|\alpha_i\rangle\,\langle\alpha_i| \otimes \mathbb{1}_{\mathcal{B}}) \left(\sum_j \sqrt{\xi_j}\,|\alpha_j\rangle\,|\beta_j\rangle\right)\left(\sqrt{\xi_i}\,\langle\alpha_i|\,\langle\beta_i|\right) \tag{3.65}$$

$$= \sum_i \left(\sum_j \delta_{i,j}\sqrt{\xi_j}\,|\alpha_i\rangle\,|\beta_j\rangle\right)\left(\sqrt{\xi_i}\,\langle\alpha_i|\,\langle\beta_i|\right) \tag{3.66}$$

$$= \sum_i \left(\sqrt{\xi_i}\,|\alpha_i\rangle\,|\beta_i\rangle\right)\left(\sqrt{\xi_i}\,\langle\alpha_i|\,\langle\beta_i|\right) \tag{3.67}$$

$$= \sum_i \xi_i\,|\alpha_i\rangle\,\langle\alpha_i| \otimes |\beta_i\rangle\,\langle\beta_i|\,. \tag{3.68}$$

Using the joint entropy theorem, we see that $S(\Phi_\alpha(\rho)) = H(\{\xi_i\})$. It is easy to verify that by employing similar procedures, $\Phi_\alpha(\rho) = \Phi_\beta(\rho) = \Phi_{\alpha,\beta}(\rho)$, implying that $S(\Phi_\alpha(\rho)) = S(\Phi_\beta(\rho)) = S(\Phi_{\alpha,\beta}(\rho)) = H(\{\xi_i\})$. Using this information, the saturation of 3.62 takes the form

$$\eta_{\alpha|\beta}(\rho) = H(\{\xi_i\}), \tag{3.69}$$

hence,

$$\mathcal{N}_2(\rho) = H(\{\xi_i\}). \tag{3.70}$$

Now, if we evaluate the partial traces of ρ, we see that $\mathrm{Tr}_{\mathcal{A}}(\rho) = \sum_i \xi_i |\beta_i\rangle\langle\beta_i|$ and $\mathrm{Tr}_{\mathcal{B}}(\rho) = \sum_i \xi_i |\alpha_i\rangle\langle\alpha_i|$; using again the joint entropy, we can evaluate the entropy of entanglement of ρ, $E(\rho) = S(\mathrm{Tr}_{\mathcal{A}}(\rho)) = S(\mathrm{Tr}_{\mathcal{B}}(\rho))$,

$$E(\rho) = H(\{\xi_i\}). \tag{3.71}$$

3.71 and 3.70 lead to 3.59, completing the proof.

Such concepts of realism and nonlocality provide us a new lens by which quantum phenomenon can be seen. For example, a thought experiment designed by Lucien Hardy in [63] is able to show Bell's nonlocality without resorting to inequalities. In it, an underlying assumption of realism implies a violation of causal locality. However, as shown in [64], it is possible to reassess this experiment by making use of irreality and realism-based nonlocality in such a way that, at the expense of the realism assumption, local causality can be restored. So, to further broaden our horizons, we will now expand the realism-based nonlocality to beyond the bipartite realm.

4 TRIPARTITE REALISM-BASED NONLOCALITY

Stepping upon the solid foundations rendered to us in [10] and [19], we were able to propose an extension for the bipartite realism-based nonlocality quantifier to craft a genuine tripartite realism-based nonlocality quantifier. It is an original work whose results were published in [65]. In this chapter we will first present an overview on multipartite quantum systems and then explore our quantifier, examining its properties, its behaviour when applied to some important tripartite states, and its monogamy. Also, in order to avoid notational chaos, we will refer to the bi and tripartite realism-based nonlocality quantifiers simply as "realism-based nonlocality", distinguishing it from Bell nonlocality.

4.1 MULTIPARTITE SYSTEMS AND MONOGAMY

In the previous chapter, we explored nonclassical aspects displayed by bipartite quantum states. Multipartite quantum states, those that act onto a space $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \ldots \otimes \mathcal{H}_{\mathcal{N}}$, are known to exhibit a much greater complexity. To investigate resources such as entanglement, new questions are brought forth due to the necessity of analysing, for example, the multiple partition cuts that are available for a state. One of them is how one resource is shared among different cuts of such a system. As a first step into the multipartite realm, we will focus mostly on tripartite systems, with states acting on $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{C}$, since they gives a nice balance between the complexity aforementioned and an operational convenience.

4.1.1 Multipartite entanglement

In the same way we defined for the bipartite case, a pure multipartite quantum state is entangled if it is not suitable to be written as a product state, $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$. For a mixed state, it is entangled if one cannot put it like $\rho_{sep} = \sum_i p_i \rho_i^{\mathcal{A}} \otimes \rho_i^{\mathcal{B}} \otimes \ldots \otimes \rho_i^{\mathcal{N}}$. It is clear that the definition of entanglement from the bipartite to the multipartite case is just a simple extension, but there is a huge qualitative jump from one case to another, since the phenomenology of entanglement already in pure states is much richer. To make this clear, let us bring forward a comparison between the bipartite and tripartite states for cases comprising just qubits.

Inspired by the arguments presented in [20], let us represent the states in a bipartite product basis as the corners of a square labeling them as bitstrings, figure 3 (a). So, the corner 01 represents the state $|01\rangle$. Two directly connected corners are separated by a bit flip operation. For such an arrangement, any superposition of states represented in adjacent vertices are separable. However, for equally weighted superpositions of states in the two diagonals of the square, we arrive to the maximally entangled states for the bipartite case, the Bell states.

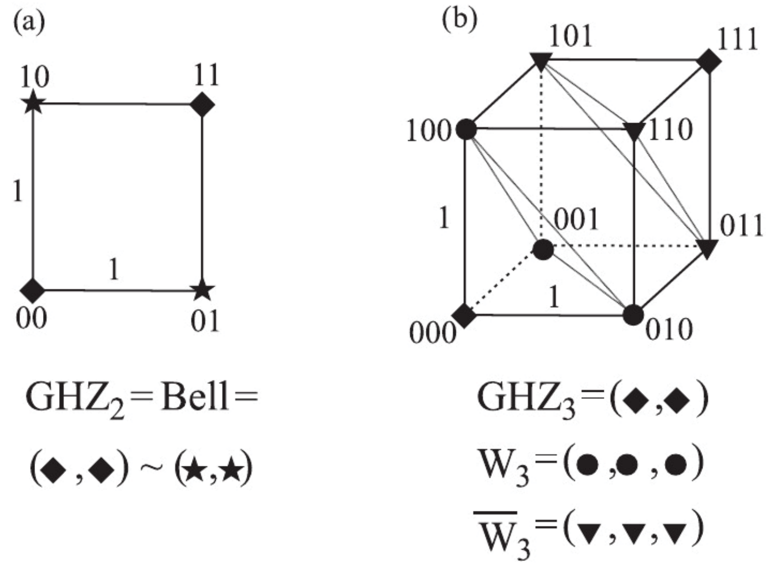

Figure 3 – Geometric representation for pure states of systems with (a) two qubits and (b) three qubits. In the square, balanced superpositions of states along of the two diagonals are the Bell states. For the cube, a balanced superposition of the state along the long diagonal is the state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and the states $|W_3\rangle$ and $|\overline{W}_3\rangle$ are formed by balanced superpositions of the states along the two parallel triangles. Original figure in [20]

Generalising this picture for the tripartite case, we arrive thus to a cube, figure 3 (b), with vertices labelled as bitstrings that represent a tripartite product state in the same fashion, with, for example, 001 corresponding to $|001\rangle$. Once again, setting the length of each edge as one, the distance corresponding to the pathway along the edges to make from one vertex to another corresponds to the number of bit flips that takes one state to another. Balanced superpositions of states separated by a distance of 2 are called biseparable states, since they present entanglement in two partitions but are a product with another. For example, taking the vertices 000 and 110, we see that

$$|\psi_{\mathcal{AB}|C}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) = |\Phi^+\rangle \otimes |0\rangle\,. \tag{4.1}$$

For a same weight superposition of states corresponding to two maximally distant corners, with distance 3, we get the analogues for the the Bell states, the tripartite GHZ states, named after Greenberger, Horne and Zeilinger, who first studied it in 1989 [29]. One of them is:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{4.2}$$

Such a state is nonproduct in every bipartition and as a consequence it presents what we call genuine tripartite entanglement (see more in 4.1.2). Another class of genuine tripartite entangled states are the $|W\rangle$ states. They are found by superposing with equal coefficients three states distancing by 2, forming triangles in the cube. One of such states is:

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{4.3}$$

Another state of the same nature is:

$$|\overline{W}\rangle = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle). \tag{4.4}$$

The entanglement of the GHZ states are of global nature and, therefore, quite fragile, as we can see by tracing out a part of the system and remaining with a fully mixed separable state. For $\rho_{GHZ} = |GHZ\rangle\langle GHZ|$,

$$\mathrm{Tr}_{\mathcal{R}}(\rho_{GHZ}) = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|), \tag{4.5}$$

with $\mathcal{R} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$. It contrasts with the entanglement of the states $|W\rangle$, which are very robust, since tracing out a subsystem leads to a bipartite entangled mixed state:

$$\mathrm{Tr}_{\mathcal{R}}(\rho_W) = \frac{1}{3}(|00\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|), \tag{4.6}$$

where $\rho_W = |W\rangle\langle W|$.

Extremely nonlocal effects are assigned to states of the kind $|GHZ\rangle$, since, for such states, quantum mechanics is known to make non-statistical predictions that conflict with the local causality hypothesis given a wise choice of observables to be measured. It was found to happen when the choice of observables A, B, and C, acting respectively over $\mathcal{H}_{\mathcal{A}}$, $\mathcal{H}_{\mathcal{B}}$, and $\mathcal{H}_{\mathcal{C}}$, are $(\sigma_x, \sigma_x, \sigma_x)$, $(\sigma_y, \sigma_y, \sigma_x)$, $(\sigma_y, \sigma_x, \sigma_y)$, or $(\sigma_x, \sigma_y, \sigma_y)$ [66]. This contrasts with the bipartite case, where the violation of the same hypothesis can be spotted in a context involving the mean value of outcomes and, thus, a statistical context.

A series of difficulties can arise when trying to quantify or, at least, qualify the entanglement of tripartite states. An example is the impossibility to readily generalize the Schmidt decomposition to multipartite systems. For tripartite systems, a quantum state admits a Schmidt decomposition if and only if the tracing out of any of the system partitions gives a fully separable, thus disentangled, state [67]. A first step we provide is the confection of what is called genuine tripartite entanglement.

4.1.2 Genuine multipartite correlations

To better restrict what exactly one refers to when talking about genuine multipartite correlations, Bennett et al. propose a series of postulates that should be satisfied by a measure or indicator of correlations of this kind [28]. In principle, it should be suitable for correlations in general, like genuine multipartite entanglement or genuine multipartite classical correlations. The postulates are:

- *Postulate 1*: “If an n-partite state does not have genuine n-partite correlations and one adds a party in a product state, then the resulting $n+1$-partite state does not have genuine n-partite correlations.”

- *Postulate 2*: "If an n-partite state does not have genuine n-partite correlations, then local operations and unanimous postselection (which mathematically correspond to the operation $\Lambda_1 \otimes \Lambda_2 \otimes ... \otimes \Lambda_n$, where n is the number of parties and each Λ_i is a trace nonincreasing operation acting on the ith party's subsystem) cannot generate genuine n-partite correlations."

- *Postulate 3*: "If an n-partite state does not have genuine n-partite correlations, then if one party splits his subsystem into two parts, keeping one part for himself and using the other to create a new $n+1$-st subsystem, then the resulting $n+1$-partite state does not have genuine $n+1$-partite correlations."

A definition of genuine multipartite correlation is thus provided, proven to satisfy all of the three postulates given above:

- *Definition*: "A state of n particles has genuine n-partite correlations if it is nonproduct in every bipartite cut."

Such an approach, as shown by Ma et al. in [68], is able to produce a genuine multipartite entanglement measure for pure states. Restricting ourselves to the tripartite case, it gives:

$$E_3(\rho) := \min\left\{E_{\mathcal{A}|\mathcal{B}\mathcal{C}}(\rho), E_{\mathcal{B}|\mathcal{A}\mathcal{C}}(\rho), E_{\mathcal{C}|\mathcal{A}\mathcal{B}}(\rho)\right\}, \tag{4.7}$$

being $E_{\mathcal{A}|\mathcal{B}\mathcal{C}}(\rho) = S(\rho_{\mathcal{A}})$, with $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\mathrm{Tr}_{\mathcal{C}}(\rho)) = \mathrm{Tr}_{\mathcal{C}}(\mathrm{Tr}_{\mathcal{B}}(\rho))$ the entropy of entanglement with respect to the partition $\mathcal{A}|\mathcal{B}\mathcal{C}$ of the state ρ. Similar interpretations are given for the other two terms. Now, given that an amount of entanglement is encoded in a tripartite quantum state, we ask: how is this entanglement distributed among the system parts?

4.1.3 Monogamy

Monogamy is a property of resource measures that imposes a restriction in the amount of shareability such resource has among a multipartite system parts. For a tripartite system, the monogamy relation of a resource Q with respect to a partition cut $\mathcal{A}|\mathcal{B}\mathcal{C}$ is stated through the inequality

$$Q_{\mathcal{A}|\mathcal{B}\mathcal{C}} \geq Q_{\mathcal{A}|\mathcal{B}} + Q_{\mathcal{A}|\mathcal{C}}. \tag{4.8}$$

Here, $Q_{\mathcal{A}|\mathcal{B}\mathcal{C}}$ stands for the amount of the related resource measured in the bipartite cut $\mathcal{A}|\mathcal{B}\mathcal{C}$, $Q_{\mathcal{A}|\mathcal{B}}$ for the cut $\mathcal{A}|\mathcal{B}$ of the reduced state $\rho_{\mathcal{A}\mathcal{B}}$ and similarly for $Q_{\mathcal{A}|\mathcal{C}}$. If the measure of the resource Q with respect to the parts $\mathcal{A}$ and $\mathcal{B}$ of a system is maximal, 4.8 imposes that the amount of Q for $\mathcal{A}$ and $\mathcal{C}$ will be zero.

Interestingly, while classical correlations can be freely shared among the parts of a system, being, for example, simultaneously maximal for both $\mathcal{A}|\mathcal{B}$ and $\mathcal{A}|\mathcal{C}$ in a tripartite system, the same is not true for quantum correlations. If the entanglement of a system of the same kind is maximal between $\mathcal{A}|\mathcal{B}$, there should be no entanglement for $\mathcal{A}|\mathcal{C}$.

Such a non-classical feature plays a central role in tasks involving quantum systems as it lies, for example, at the heart of quantum cryptography processes [69], but it is known that there are several quantum correlation measures that can violate the monogamy relation. The entanglement of formation, which accounts for the amount of entanglement necessary for the preparation of a specific bipartite quantum state, is a case of a quantum correlation that does not respect, in general, a monogamy relation [70].

Notwithstanding, it was shown that, given a quantum correlation that violates the monogamy relation for a given tripartite quantum state, it is always possible to define a monotonically increasing function of that measure that, for the same state, is monogamous [71]. More specifically, there exists $\alpha, \beta \in \mathbb{R}_{>0}$ such that $\alpha \geq \beta$, that, for a correlation measure C, it holds

$$C^{\alpha}_{\mathcal{A}|\mathcal{BC}}(\rho_{\mathcal{ABC}}) \geq C^{\alpha}_{\mathcal{A}|\mathcal{B}}(\rho_{\mathcal{AB}}) + C^{\alpha}_{\mathcal{A}|\mathcal{C}}(\rho_{\mathcal{AC}}). \tag{4.9}$$

For instance, if we take the entanglement of formation to the power $\sqrt{2}$, that is, for $\beta = \sqrt{2}$, we arrive to a function of the entanglement of formation that satisfies the monogamy relation [72].

4.2 GENUINE TRIPARTITE NONLOCALITY

Just like for the bipartite case, several methodologies to quantify Bell nonlocality in tripartite states where developed [22–27]. However, the research on realism-based nonlocality for such states is still incipient. Now we take the first step in this direction.

To finally introduce the genuine tripartite realism-base nonlocality quantifier, two other measures were crafted. In this section, since each quantifier relies upon the preceding one, we will present them in their natural order.

4.2.1 Contextual bipartite nonlocality for tripartite states

Given a tripartite preparation $\rho \in \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{C}}$ whose parts were split and sent to three distant laboratories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, a quantifier was devised to account for the change in the irreality of the observable A on $\mathcal{H}_{\mathcal{A}}$ when local measurements of B on $\mathcal{H}_{\mathcal{B}}$ and C on $\mathcal{H}_{\mathcal{C}}$, with $C = \sum_c cC_c$, are performed. It is

$$\eta_{A|B,C}(\rho) \coloneqq \mathfrak{I}_A(\rho) - \mathfrak{I}_A(\Phi_{B,C}(\rho)), \tag{4.10}$$

applying for whatever permutation of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. For $\eta_{A|B,C}(\rho)$, due to the commutation of A, B, and C, we have $\eta_{A|B,C}(\rho) = \eta_{A|C,B}(\rho)$ and analogous results for the other permutations.

Since the quantifiers we will introduce are natural extensions of the ones devised for bipartite states, it will be clear that their properties as well as their respective demonstrations are analogous to the ones we developed in the previous chapter. Such is the case for the non-negativity of $\eta_{A|B,C}(\rho)$,

$$\eta_{A|B,C}(\rho) \geq 0, \tag{4.11}$$

since irreality is non increasing under completely positive trace-preserving maps. The saturation of this inequality is reached when $\rho = \Phi_A(\rho)$, after all, given 4.11, if $\mathfrak{I}_A(\rho) = 0$, it follows that $\mathfrak{I}_A(\Phi_{B,C}(\rho))$ is also zero. Expressing $\eta_{A|B,C}(\rho)$ in terms of the von Neumann entropy,

$$\eta_{A|B,C}(\rho) = S(\Phi_A(\rho)) + S(\Phi_{B,C}(\rho)) - S(\Phi_{A,B,C}(\rho)) - S(\rho), \tag{4.12}$$

it is clear that $\eta_{A|B,C}$ remains unchanged under the permutation of A and B, C, which implies that the saturation is also reached in the case where $\rho = \Phi_{B,C}(\rho)$. A third case for the saturation happens when we are dealing with full reality states, $\bigotimes_{\mathcal{R}=\mathcal{A},\mathcal{B},C} \frac{\mathbb{1}_\mathcal{R}}{d_\mathcal{R}} \equiv \frac{\mathbb{1}}{d}$, where $d = d_\mathcal{A} d_\mathcal{B} d_C$. To see this, it suffices to remember that the state of full reality is simultaneously a state of A-reality and B, C-reality, suiting the two previous conditions for saturation of $\eta_{A|B,C}(\rho)$. The fourth and last case where $\eta_{A|B,C}(\rho)$ vanishes is for uncorrelated states of the kind $\rho = \rho_\mathcal{A} \otimes \rho_{\mathcal{B}C}$ (which includes the fully uncorrelated state $\rho = \rho_\mathcal{A} \otimes \rho_\mathcal{B} \otimes \rho_C$). For such,

$$\begin{aligned}
\eta_{A|B,C}(\rho_\mathcal{A} \otimes \rho_{\mathcal{B}C}) &= S(\Phi_A(\rho_\mathcal{A} \otimes \rho_{\mathcal{B}C})) + S(\Phi_{B,C}(\rho_\mathcal{A} \otimes \rho_{\mathcal{B}C})) \\
&\qquad - S(\Phi_{A,B,C}(\rho_\mathcal{A} \otimes \rho_{\mathcal{B}C})) - S(\rho_\mathcal{A} \otimes \rho_{\mathcal{B}C}) && (4.13)\\
&= S(\Phi_A(\rho_\mathcal{A})) + S(\rho_{\mathcal{B}C}) + S(\rho_\mathcal{A}) + S(\Phi_{B,C}(\rho_{\mathcal{B}C})) \\
&\qquad - S(\Phi_A(\rho_\mathcal{A})) - S(\Phi_{B,C}(\rho_{\mathcal{B}C})) - S(\rho_\mathcal{A}) - S(\rho_{\mathcal{B}C}) && (4.14)\\
&= 0. && (4.15)
\end{aligned}$$

It is reasonable to expect that $\eta_{A|B,C}(\rho)$ reduces to $\eta_{A|B}(\rho)$ if the state we are dealing with has its C partition fully uncorrelated with the rest of the system, that is, for $\rho = \rho_{\mathcal{AB}} \otimes \rho_C$. Fortunately, such is the case, as we can easily verify:

$$\begin{aligned}
\eta_{A|B,C}(\rho_{\mathcal{AB}} \otimes \rho_C) &= S(\Phi_A(\rho_{\mathcal{AB}} \otimes \rho_C)) + S(\Phi_{B,C}(\rho_{\mathcal{AB}} \otimes \rho_C)) \\
&\qquad - S(\Phi_{A,B,C}(\rho_{\mathcal{AB}} \otimes \rho_C)) - S(\rho_{\mathcal{AB}} \otimes \rho_C) && (4.16)\\
&= S(\Phi_A(\rho_{\mathcal{AB}})) + S(\rho_C) + S(\Phi_B(\rho_{\mathcal{AB}})) + S(\Phi_C(\rho_C)) \\
&\qquad - S(\Phi_{A,B}(\rho_{\mathcal{AB}})) - S(\Phi_C(\rho_C)) - S(\rho_{\mathcal{AB}}) - S(\rho_C) && (4.17)\\
&= S(\Phi_A(\rho_{\mathcal{AB}})) + S(\Phi_B(\rho_{\mathcal{AB}}) - S(\Phi_{A,B}(\rho_{\mathcal{AB}})) - S(\rho_{\mathcal{AB}}) && (4.18)\\
&= \eta_{A|B}(\rho_{\mathcal{AB}}). && (4.19)
\end{aligned}$$

Similar considerations applies to the case where $\mathcal{B}$ is uncorrelated.

4.2.2 Bipartite nonlocality for tripartite states

Proceeding along the lines Gomes and Angelo followed to come up with $\mathcal{N}_2(\rho)$, we were able to develop a quantifier of nonlocality $\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho)$ that is independent of a context

$\{A, B, C\}$. $\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho)$ accounts for the amount of nonlocality for a given state that is associated with the change in the realism in part $\mathcal{A}$ when projective measurements are performed in $\mathcal{B}$ and C and it is constructed by running a maximization over all possible trios of observables A, B, and C in $\eta_{A|B,C}(\rho)$:

$$\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho) := \max_{\{A,B,C\}} \eta_{A|B,C}(\rho). \tag{4.20}$$

The non-negativity of $\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho)$,

$$\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho) \geq 0, \tag{4.21}$$

follows from the non-negativity of $\eta_{A|B,C}(\rho)$. Just as $\mathcal{N}_2(\rho) > 0$ implies the existence of at least one context in which the nonlocal changes in the irreality of observables manifests, if $\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho) > 0$, there is at least one context $\{A, B, C\}$ in which the measurement of B and C in distant sites implies a change of the irreality of A.

We saw that there are four possibilities for $\eta_{A|B,C}(\rho) = 0$. Two of them, for A-reality and B, C-reality states, are dependent of context. For example, if we choose for an A-reality state a context that includes an A' observable that acts on the same space and is incompatible with A, $\eta_{A'|B,C}(\rho)$ will not, in general, be zero. The same argument applies for a B, C-reality state. We are left, thus, with two possibilities for which $\eta_{A|B,C}(\rho)$ reaches zero and are independent of a specific setting of observables, being, therefore, the conditions for which $\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho)$ vanishes. The first one is for the full reality state, where $\rho = \frac{\mathbb{1}}{d}$, since it will be a state of reality for A and B, C whichever the context. And the second is for states of the kind $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}C}$, for it is also zero for whichever context, as shown in the previous subsection.

Another property of $\eta_{A|B,C}(\rho)$ that is independent of context and, consequently, carries on for $\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho)$ is its reduction to $\eta_{A|B}(\rho)$ when dealing with states of the kind $\rho = \rho_{\mathcal{A}\mathcal{B}} \otimes \rho_C$. For such a case, we see that

$$\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho) = \mathcal{N}_2(\rho). \tag{4.22}$$

That is, for states where the partition C is uncorrelated, the bipartite realism-based nonlocality for tripartite states reduces to the bipartite realism-based nonlocality.

In the same way that $\mathcal{N}_2(\rho)$ is sensitive for irreality changes in states that are devoid of quantum correlations, we will show how $\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho)$ presents a similar behavior for tripartite quantum states. For such, let us take the "classical-classical-classical" state $\rho_{ccc} = \sum_i p_i A'_i \otimes B'_i \otimes C'_i$. Just as its bipartite counterpart, there is no quantum correlation to be spotted in such a state. But for a context $\{A', B', C'\}$ which is maximally incompatible with $\{A, B, C\}$, we will see that $\eta_{A'|B'C'}(\rho_{ccc}) > 0$. The procedure, once again, will be similar to the one employed for the analysis in the bipartite case. We start by expressing $\eta_{A'|B'C'}(\rho_{ccc})$ in terms of the von Neumann entropy,

$$\eta_{A|B,C}(\rho_{ccc}) = S(\Phi_A(\rho_{ccc})) + S(\Phi_{B,C}(\rho_{ccc})) - S(\Phi_{A,B,C}(\rho_{ccc})) - S(\rho_{ccc}), \tag{4.23}$$

and by analysing term by term with the aid of the joint entropy theorem. The last one gives us

$$S(\rho_{ccc}) = S\left(\sum_i p_i A_i' \otimes B_i' \otimes C_i'\right) \tag{4.24}$$

$$= H(\{p_i\}) + \sum_i p_i S(B_i' \otimes C_i') \tag{4.25}$$

and, since $B_i' \otimes C_i'$ is a pure state, $S(B_i' \otimes C_i') = 0$ and we are left only with

$$S(\rho_{ccc}) = H(\{p_i\}). \tag{4.26}$$

Moving on,

$$S(\Phi_A(\rho_{ccc})) = S\left(\sum_i p_i \Phi_A(A_i') \otimes B_i' \otimes C_i'\right) \tag{4.27}$$

$$= S\left(\sum_i p_i \frac{\mathbb{1}_{\mathcal{A}'}}{d_{\mathcal{A}'}} \otimes B_i' \otimes C_i'\right) \tag{4.28}$$

$$= S\left(\frac{\mathbb{1}_{\mathcal{A}'}}{d_{\mathcal{A}'}}\right) + S\left(\sum_i p_i B_i' \otimes C_i'\right) \tag{4.29}$$

$$= \log d_{\mathcal{A}'} + H(\{p_i\}) + \sum_i p_i S(C_i') \tag{4.30}$$

$$= \log d_{\mathcal{A}'} + H(\{p_i\}). \tag{4.31}$$

Also,

$$S(\Phi_{B,C}(\rho_{ccc})) = S\left(\sum_i p_i A_i' \otimes \Phi_B(B_i') \otimes \Phi_C(C_i')\right) \tag{4.32}$$

$$= S\left(\sum_i p_i A_i' \otimes \frac{\mathbb{1}_{\mathcal{B}'}}{d_{\mathcal{B}'}} \otimes \frac{\mathbb{1}_{C'}}{d_{C'}}\right) \tag{4.33}$$

$$= S\left(\sum_i p_i A_i'\right) + S\left(\frac{\mathbb{1}_{\mathcal{B}'}}{d_{\mathcal{B}'}}\right) + S\left(\frac{\mathbb{1}_{C'}}{d_{C'}}\right) \tag{4.34}$$

$$= H(\{p_i\}) + \log d_{\mathcal{B}'} + \log d_{C'}. \tag{4.35}$$

The remaining term gives us

$$S(\Phi_{A,B,C}(\rho_{ccc})) = S\left(\sum_i p_i \Phi_A(A_i') \otimes \Phi_B(B_i') \otimes \Phi_C(C_i')\right) \tag{4.36}$$

$$= S\left(\frac{\mathbb{1}_{\mathcal{A}'}}{d_{\mathcal{A}'}}\right) + S\left(\frac{\mathbb{1}_{\mathcal{B}'}}{d_{\mathcal{B}'}}\right) + S\left(\frac{\mathbb{1}_{C'}}{d_{C'}}\right) \tag{4.37}$$

$$= \log d_{\mathcal{A}'} + \log d_{\mathcal{B}'} + \log d_{C'}. \tag{4.38}$$

Inserting 4.26, 4.31, 4.35, and 4.38 in 4.23, we see that

$$\eta_{A|B,C}(\rho_{ccc}) = H(\{p_i\}), \tag{4.39}$$

which guarantees that $\mathcal{N}_{\mathcal{A}|\mathcal{B}C}(\rho) > 0$.

4.2.3 Genuine tripartite nonlocality

Even though $\mathcal{N}_{\mathcal{A}|\mathcal{BC}}(\rho)$ cannot be strictly regarded as a measure of quantum correlations, since, as we just saw, it is sensitive to the incompatibility of observables in states that lack quantum correlations, we still find inspiration in Bennett et al.'s proposal for genuine tripartite quantum correlation measures (subsection 4.1.2) to construct a measure for genuine tripartite nonlocality.

The reasoning behind it is that a state ρ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{C}}$ that presents at least one of its three bipartite cuts whose irreality is insensitive to projective measurements applied to its parts is diagnosed as having zero genuine tripartite nonlocality. For such, this quantifier is constructed like:

$$\mathcal{N}_3(\rho) := \min\left\{\mathcal{N}_{\mathcal{A}|\mathcal{BC}}(\rho), \mathcal{N}_{\mathcal{B}|\mathcal{AC}}(\rho), \mathcal{N}_{\mathcal{C}|\mathcal{AB}}(\rho)\right\}. \tag{4.40}$$

The non-negativity of $\mathcal{N}_3(\rho)$ follows as a consequence of the non-negativity of the quantifiers of the kind $\mathcal{N}_{\mathcal{A}|\mathcal{BC}}(\rho)$. It is zero for states of full reality, $\mathbb{1}/d$, since for states of this kind the nonlocality associated with all of its three bipartite cuts is zero, and for states of the kind $\rho = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{BC}}$ as well as for states with permuted indices, because for such a case the irreality correspondent to at least one of its bipartite cuts is assured to be zero.

We found that, just as $\mathcal{N}_2(\rho)$ reduces to entanglement for pure states that are maximally entangled, $\mathcal{N}_3(\rho)$ reduces to genuine tripartite entanglement for a particular class of tripartite entangled pure states, those which admit a Schmidt decomposition, $\zeta = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_i \sqrt{\xi_i}\,|\alpha_i\rangle|\beta_i\rangle|\gamma_i\rangle$. We will now prove this. We start with

$$\eta_{A|B,C}(\zeta) = S(\Phi_A(\zeta)) + S(\Phi_{B,C}(\zeta)) - S(\Phi_{A,B,C}(\zeta)) - S(\zeta), \tag{4.41}$$

and notice that, since ζ is pure, the last term is zero. Since $S(\Phi_R(\zeta)) \geq S(\zeta)$, $S(\Phi_{R,Q}(\zeta)) \geq S(\Phi_R(\zeta))$, and $S(\Phi_{A,B,C}(\zeta)) \geq S(\Phi_{R,Q}(\zeta))$, with $R, Q \in \{A, B, C\}$, it is true that

$$2S(\Phi_{A,B,C}(\zeta)) \geq S(\Phi_A(\zeta)) + S(\Phi_{B,C}(\zeta)), \tag{4.42}$$

which saturates in the case where $S(\Phi_{A,B,C}(\zeta)) = S(\Phi_{B,C}(\zeta)) = S(\Phi_A(\zeta))$. The comparison of 4.42 divided by 2 and 4.41 leads us to

$$\eta_{A|B,C}(\zeta) \leq \frac{1}{2}[S(\Phi_A(\zeta)) + S(\Phi_{B,C}(\zeta))]. \tag{4.43}$$

The condition for equality in this expression gives us the maximization of $\eta_{A|B,C}(\zeta)$ and, therefore, $\mathcal{N}_{\mathcal{A}|\mathcal{BC}}(\zeta)$. This condition is met with the choice of the context $\{A, B, C\}$ as the Schmidt observables defined by $|\psi\rangle$, $A = \alpha = \sum_i \alpha_i |\alpha_i\rangle\langle\alpha_i|$, $B = \beta = \sum_i \beta_i |\beta_i\rangle\langle\beta_i|$, and $C = \gamma = \sum_i \gamma_i |\gamma_i\rangle\langle\gamma_i|$, since, for such, $S(\Phi_{\alpha,\beta,\gamma}(\zeta)) = S(\Phi_{\beta,\gamma}(\zeta)) = S(\Phi_\alpha(\zeta)) = \sum_i \xi_i |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \otimes |\gamma\rangle\langle\gamma| = H(\{\xi_i\})$. Due to the symmetry of the state $|\psi\rangle$, we expect to obtain the same results from the evaluation of nonlocality respective to the other bipartitions, hence, $\mathcal{N}_3(\zeta) = H(\{\xi_i\})$.

Now, the entropy of entanglement of the bipartite cut $\mathcal{A}|\mathcal{BC}$ is $E_{\mathcal{A}|\mathcal{BC}}(\zeta) = S(\mathrm{Tr}_{\mathcal{A}}\zeta) = H(\{\xi_i\})$ and, again, by the symmetry of the state, the same result comes from the other bipartitions, leading, finally, to

$$\mathcal{N}_3(\zeta) = E_3(\zeta). \tag{4.44}$$

We verified, then, that for states of the kind $|\psi\rangle$, $\mathcal{N}_3(\rho)$ is a nonanomalous measure with respect to genuine tripartite entanglement according to 4.7.

4.3 CASE STUDY FOR NOISY GHZ AND W STATES

To further investigate the behaviour of the tipartite realism-based nonlocality, numerical computations were implemented for states of interest. We used *Mathematica* to evaluate $\mathcal{N}_3(\rho)$ for the noisy three-qubit states:

$$\rho_{\mathfrak{n}}^{\chi} := \mathfrak{n}\,\frac{\mathbb{1}}{8} + (1-\mathfrak{n})\,|\Psi_\chi\rangle\,\langle\Psi_\chi|\,, \tag{4.45}$$

with the noise $\mathfrak{n} \in [0,1]$, and $\chi \in \{GHZ, W\}$, in such a way that

$$|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right), \tag{4.46}$$

$$|\Psi_{W}\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right). \tag{4.47}$$

The convex sum in 4.45 allows us to introduce white noise into a pure state in a controllable way by varying the parameter $\mathfrak{n}$. For $\mathfrak{n} = 0$, we get a pure state and for $\mathfrak{n} = 1$ we are left with a mixed state. With such approach, we were able to assess how $\mathcal{N}_3(\rho)$ behaves with gradual increments in noise.

The implementation of $\eta_{A|B,C}(\rho)$ as well for permuted indices is straightforward. The spin operator A is given by $A = \hat{a}\cdot\vec{\sigma}$ with $\hat{a} = (\sin\theta_a\cos\varphi_a, \sin\theta_a\sin\varphi_a, \cos\theta_a)$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Here, θ_a is the polar angle, $\theta_a \in [0,\pi]$, and φ_a the azimuthal one, $\varphi_a \in [0, 2\pi]$. Similar expressions are assigned for B and C and their respective parameters.

The task of maximization involved in the evaluation of $\mathcal{N}_{\mathcal{A}|\mathcal{BC}}$, however, turns to be a hard computational problem. Fortunately, the quantum states we worked with are invariant under permutation of subsystems, and, due to this symmetry, $\mathcal{N}_3 = \mathcal{N}_{\mathcal{A}|\mathcal{BC}} = \mathcal{N}_{\mathcal{B}|\mathcal{AC}} = \mathcal{N}_{\mathcal{C}|\mathcal{AB}}$. Because of that, the computational cost was cut to a third. To accomplish the maximization, two different strategies were employed.

For the first strategy, we defined a grid by letting the angles $\theta_{a,b,c}$ and $\varphi_{a,b,c}$ vary over their domains by steps of $\pi/8$. For example, we set $\theta_{a,b,c} = \varphi_{a,b,c} = 0$ and evaluated $\eta_{A|B,C}(\rho)$ for this setup. Then, for the next iteration, we hold $\theta_{a,b,c} = \varphi_{a,b} = 0$ and incremented φ_c by $\pi/8$, getting a new value for $\eta_{A|B,C}(\rho)$. If the new value obtained is bigger than the previous one, we keep it. After a few iterations, when φ reached the value of 2π, the value zero is assigned for this variable and the next one, say, θ_c, is thus incremented by $\pi/8$, repeating again the cycle for

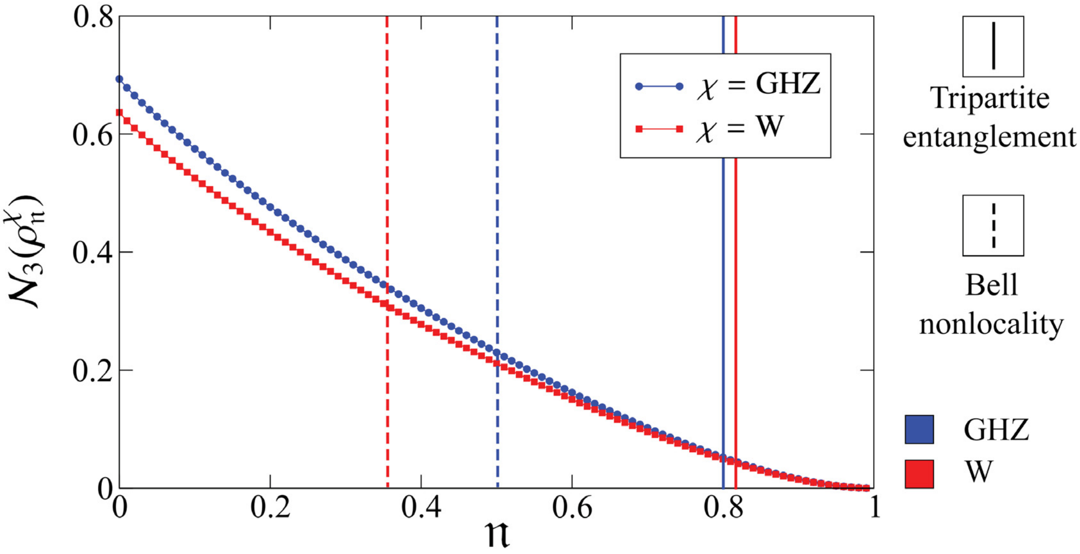

Figure 4 – Genuine tripartite nonlocality $\mathcal{N}_3(\rho_{\mathfrak{n}}^{\chi})$ for the noisy three-qubit states 4.46 as a function of the noise quantity $\mathfrak{n}$. The blue circles represents the noisy GHZ state ($\chi = GHZ$) and the red squares the noisy W state ($\chi = W$). Sudden deaths for tripartite entanglement and Bell nonlocality are indicated by the vertical full lines and dashed lines, respectively. Genuine tripartite nonlocality is a monotonically decreasing function of the noise.

φ_c. After its completion, this procedure yields a total of 2 985 984 distinct settings for $\{A, B, C\}$. Such a strategy was thought of because, given the nature of the states $|GHZ\rangle$ and $|W\rangle$, we wanted to make sure to cover specific settings that could generate maxima, like those where the spin operators fall along $\hat{x}$, $\hat{y}$ and $\hat{z}$, as well as other interesting directions, like $(1/\sqrt{2})(\hat{x} + \hat{z})$.

In the second one, the angles that define the spin operators were randomly generated. In this fashion, the maximization was performed on a set of 10^6 settings $\{A, B, C\}$. This was done for two main reasons. We wanted to verify if such an approach would involve a computational time compatible with the first approach, what turned out to be the case. The necessary time for the completion of the process was of approximately one third of the first method's time, which makes sense, given that, for this evaluation, the number of settings was roughly a third of the previous one. Also, we wished to see if this second approach would provide the same results obtained through the first one, and it did. This strategy has thus shown to be reliable, since it reasonably covers the specific settings for which maxima are found and it is more suitable for general states, once it does not depend on specific states symmetries. Even more, it gave further evidence that the results we obtained are statistically well founded.

For pure states, $\mathfrak{n} = 0$, remarkable settings were found where the contextual nonlocality is maximal. For the GHZ state, $\eta_{A|B,C}(\rho)$ reaches its maximum value, $\ln 2$, for when A is σ_z and at least one of the observable B and C is equal. It is also maximal for when (A, B, C) is $(\sigma_x, \sigma_x, \sigma_x)$, $(\sigma_y, \sigma_y, \sigma_x)$, $(\sigma_y, \sigma_x, \sigma_y)$, or $(\sigma_x, \sigma_y, \sigma_y)$, the set of observables known to provide predictions that are conflicting with the local causality hypothesis (section 4.1.1). For the W state, the maximum value the contextual nonlocality assumes is equal to 0.6364 and it is assessed for the case where

$A = B = C = \sigma_z$.

Several values for $\mathcal{N}_3$ were obtained by varying the noise parameter $\mathfrak{n}$ from 0 to 1 by increments of 0.01 for $\chi \in \{GHZ, W\}$ and the results are shown in the figure 4. When the "grid strategy" was employed for the evaluation, the required computation time for we to obtain each curve was of approximately one week.

It is noteworthy that the genuine tripartite nonlocality has shown to be a monotonically decreasing function of the noise parameter $\mathfrak{n}$, vanishing strictly for when this parameter is 1, the scenario for which the state is completely mixed. This puts in evidence the fact that $\mathcal{N}_3$ is highly resilient to noise. In fact, its resilience is far greater than those of several other measures of nonclassicality, known to vanish abruptly when higher levels of noise are introduced. For when $\chi = GHZ$, abrupt vanishings are found for tripartite entanglement for $\mathfrak{n} \geq 4/5$ [30, 73], Bell nonlocality, from two up to five measurements per site, for $\mathfrak{n} > 1/2$ [74], and steering for $\mathfrak{n} \gtrsim 0.225$ [75]. To $\chi = W$, entanglement vanishes abruptly for $\mathfrak{n} \geq 0.8220$ [76], Bell nonlocality, with two (three) measurements per site, for $\mathfrak{n} > 0.3558$ (0.3952) [74], and steering for $\mathfrak{n} \gtrsim 0.1634$ [75]. This property is compatible with the noise resilience found for $\mathcal{N}_2$ in [19].

4.4 TRIPARTITE NONLOCALITY MONOGAMY

For our last incursion, we investigate the monogamy properties of the genuine tripartite nonlocality. In fact, we want to assess by which extent the relation

$$\mathcal{N}_3^{\alpha}(\rho_{\mathcal{ABC}}) \geq \mathcal{N}_2^{\alpha}(\rho_{\mathcal{AB}}) + \mathcal{N}_2^{\alpha}(\rho_{\mathcal{AC}}), \tag{4.48}$$

with $\alpha \in \mathbb{R}_{>0}$, holds. We introduced the parameter α, constructing a monotonically increasing function of $\mathcal{N}_3$, in order to provide a broader context for which a monogamy relation can be found. This procedure was found to be successful for quantum correlation measures (see subsection 4.1.3). For such, both analytical and numerical approaches were employed.

We start by showing that 4.48 is not valid in general. For the GHZ pure state, the reduced states are given by $\rho_{\mathcal{AB}} = \rho_{\mathcal{AC}} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) \equiv \rho_{cc}$. It was shown in the subsection 3.4.2 that, for such states, $\mathcal{N}_2(\rho_{\mathcal{AB}}) = \mathcal{N}_2(\rho_{\mathcal{AB}}) = \ln 2$. Given that, in this case, $\mathcal{N}_2(\rho_{\mathcal{ABC}}) = \ln 2$, for whichever α, 4.48 is always violated. Therefore, $\mathcal{N}_3$ is not, neither can be deformed to, a monogamic measure for all states. Interestingly, the monogamy properties presented by the resource $\mathcal{N}_3$ are contrasting with those presented by quantum correlation measures, since it is always possible to define monotonically increasing functions of those measures that are monogamous in general.

Notwithstanding, numerical simulations obtained using *Mathematica* have shown that violations of monogamy for $\rho_{\mathfrak{n}}^{\chi}$, $\chi \in \{GHZ, W\}$ are relatively sparse in the parameter space. This was shown by means of calculating the following amount:

$$\delta\mathcal{N}_3^{\alpha}(\rho_{\mathcal{ABC}}) \coloneqq \mathcal{N}_3^{\alpha}(\rho_{\mathcal{ABC}}) - \left[\mathcal{N}_2^{\alpha}(\rho_{\mathcal{AB}}) + \mathcal{N}_2^{\alpha}(\rho_{\mathcal{AC}})\right], \tag{4.49}$$

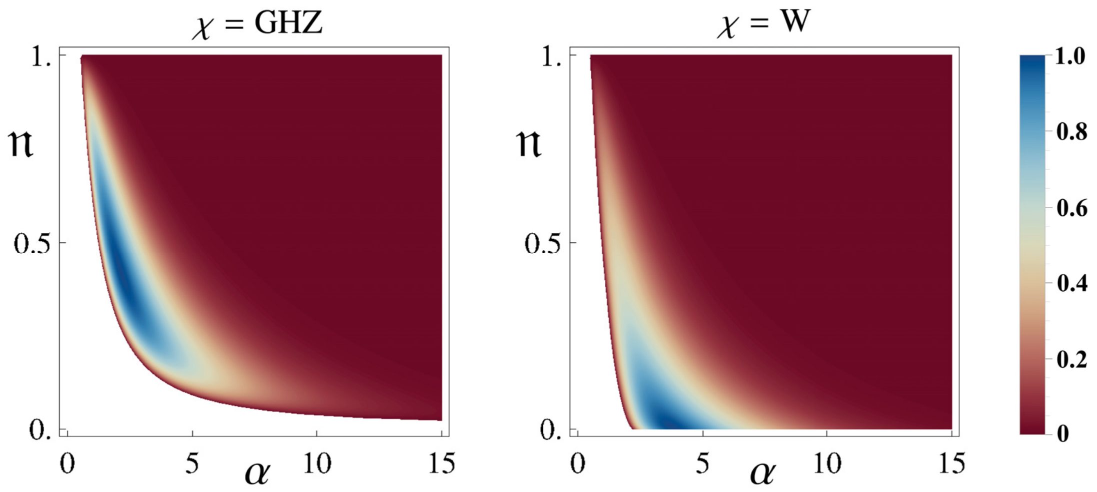

Figure 5 – Contour plots for the normalized monogamy witness $\delta N_3^\alpha(\rho_{\mathfrak{n}}^\chi)/N_\chi$ for the states $\rho_{\mathfrak{n}}^\chi$ as a function of the parameter α and the noise amount $\mathfrak{n}$. $N_{GHZ} \cong 0.019997$ and $N_W \cong 0.073296$. Monogamy holds where $\delta N_3^\alpha \geq 0$ (colored regions). $\mathcal{N}_3^\alpha$ is not monogamous for the GHZ pure state, blank region in the left panel for $\mathfrak{n} = 0$ and $\forall \alpha > 0$.

where $\rho_{\mathcal{ABC}} = \rho_{\mathfrak{n}}^\chi$. Comparing equations 4.48 and 4.49, wee see that $\delta\mathcal{N}_3^\alpha$ witnesses monogamy for the measure $\mathcal{N}_3^\alpha$ whenever it is positive.

The implementation of equation 4.49 was made by using our previous results for $\mathcal{N}_3(\rho_{\mathfrak{n}}^\chi)$ with $\chi \in \{GHZ, W\}$ and the calculations of $\mathcal{N}_2(\rho_{\mathfrak{n}}^\chi)$ were done in a similar fashion, employing the randomly generated spin operators strategy to perform the maximization involved. For this, the computation time is greatly reduced, because the dimension of the operators involved in the task is lower.

The results, shown in figure 5, point to the absence of monogamy for small values of $\{\mathfrak{n}, \alpha\}$ and to its presence for when those values are large. The result of absence of monogamy for the GHZ pure state we derived analytically was confirmed by the numerical simulation, $\delta N_3^\alpha(\rho_{\mathfrak{n}=0}^{GHZ}) < 0$. We were also able to verify that the genuine tripartite nonlocality for the pure W state, $N_3^\alpha(\rho_{\mathfrak{n}=0}^W)$, is not monogamous when $\alpha = 0$, but it is when $\alpha \gtrsim 2.1641$, in such a way that $\delta N_3^\alpha(\rho_{\mathfrak{n}=0}^W)$ peaks when $\alpha \cong 3.8372$.

5 CONCLUSION

The idea that Nature displays nonlocal phenomena can be so counter-intuitive that at first it was dismissed by EPR in [2] as a flaw in the construction of our physical theories. When Bell confirmed that the hypothesis of local causality is indeed incompatible with quantum mechanics [5] and had his analytical assessment confirmed by experimental results [7], foundational researches in nonlocality became a necessity. Perhaps an even stranger feature is that of the indeterminacy of the physical properties of quantum objects, known as irrealism. The plethora of realism definitions, frequently mutually excluding, and the confusion that surrounds the interplay between realism and nonlocality demand clarifications.

In this work we explored the connection between realism and nonlocality by means of the framework provided by Bilobran and Angelo [10], where an operational criterion for realism was devised, together with a contextual measure for nonlocality. We described how to quantify the nonlocality assigned to a bipartite quantum system with the bipartite realism-based nonlocality constructed by Gomes and Angelo [19]. Although based on premises rather different from those underlying Bell nonlocality, such measure has shown to diagnose nonlocality to states that are known to manifest nonlocal behaviour, like entangled states.

However, these quantities were not conceived for dealing with multipartite quantum systems. As we have shown, the multipartite realm is much wilder, comprising, for example, different kinds of entanglement and bounds for the shareability of resources among the system parts, a mechanism called monogamy. To start the research program in multipartite realism-based quantum nonlocality, we proceeded by trying to craft a tripartite realism-based nonlocality measure, laying down the first stone of this pathway.

We succeeded at doing it. In this entrepreneurship, three measures were crafted. The first one, the contextual bipartite nonlocality for tripartite states was developed by slightly adjusting the contextual nonlocality of Bilobran and Angelo for a tripartite system. We showed that this quantity is non-negative, likewise its bipartite counterpart, and we determined the conditions for which it vanishes. For tripartite systems that are a product state with respect to one of its partitions, this measure was shown to reduce to the bipartite contextual nonlocality.

Afterwards, in the same way that Gomes and Angelo conceived the bipartite realism-based nonlocality, independent of context, we employed the maximization over all sets of possible observables in the tripartite contextual measure to arrive to a bipartite nonlocality measure for tripartite states that is also independent of context. That is, it accounts for the nonlocality assigned to bipartitions of a tripartite system. It was shown to be non-negative and the conditions for it to be zero were exposed. Just like for the contextual measure case, when the tripartite system under scrutiny is product with respect to one of its partitions, this measure also reduces to the bipartite nonlocality. We have also shown that this measure is able to capture irreality changes in states that lack quantum correlations, being sensitive to the incompatibility of observables.

The tripartite realism-based nonlocality was thus introduced. For such, even though this measure cannot be strictly regarded as a quantum correlation measure, but more as a quantum resource measure, Bennett's postulates for genuine multipartite correlations [28] still provided a source of inspiration for us to conceive a genuine tripartite nonlocality quantifier. As such, it only manifests itself in tripartite states where all of its bipartite cuts display nonlocal behaviour. It was also shown that this quantity reduces to genuine tripartite entanglement for maximally entangled tripartite pure states that admit a Schmidt decomposition, like the GHZ state.

Numerical investigations were conducted in order to assess the behaviour of these amounts when dealing with states of interest: noisy GHZ and W states. First of all, in this task, we were able to verify that the computational implementation of these measures is feasible. We thus found that the tripartite realism-based nonlocality reaches its maximum value for the pure GHZ state, that it is lower for the pure W state and presents a monotonically decreasing behaviour as white noise is introduced in these systems, vanishing only when the states are in a complete mixture. This highlighted the high resilience of this measure, contrasting with several other quantum resources, which present sudden death.

Our last result concerns the monogamy of the measure. We provided an analytical result showing that the tripartite realism-based nonlocality is not, neither can be deformed into, a generally monogamous measure. However, we were able to numerically show that for the noisy GHZ and W states, violations of monogamy are relatively rare in the parameter space.

These results were published in [65] and we showed that the realism-based nonlocality concept is adequate for dealing with tripartite systems. Further generalizations of the tripartite to n-partite systems are now in reach. With the tripartite realism-based nonlocality stated, with its properties delineated and with its behaviour exposed for important states, we believe that we provided a robust start for the multipartite realism-based nonlocality research program.

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