UNIVERSIDADE FEDERAL DO PARANÁ

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MULTI-OBJECTIVE OPTIMAL CONTROL PROBLEMS WITH APPLICATION TO ENERGY MANAGEMENT SYSTEMS

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MULTI-OBJECTIVE OPTIMAL CONTROL PROBLEMS WITH APPLICATION TO ENERGY MANAGEMENT SYSTEMS

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RESUMO

Este trabalho investiga problemas de controle ótimo em tempo contínuo. Em horizonte de tempo finito, apresentamos uma nova abordagem ao analisar problemas com objetivos de diferentes naturezas, que precisam ser minimizados simultaneamente. Um objetivo está na forma clássica de Bolza e o outro é definido como uma função de máximo. Baseados na teoria de Hamilton-Jacobi-Bellman caracterizamos a fronteira de Pareto fraca e a fronteira de Pareto para tais problemas. Primeiramente, definimos um problema de controle ótimo auxiliar sem restrições de estado e mostramos que a fronteira de Pareto fraca é um subconjunto do conjunto de nível zero da função valor correspondente. Em seguida, com uma abordagem geométrica estabelecemos a caracterização da fronteira de Pareto. Alguns resultados numéricos são considerados para mostrar a relevância do nosso método. Os bons resultados obtidos para horizonte de tempo finito nos motivaram a investigar problemas de controle ótimo multiobjetivo com horizonte de tempo infinito. Com uma abordagem similar, caracterizamos a fronteira de Pareto para essa classe de problemas. Introduzimos um método, baseado no princípio da programação dinâmica, para reconstrução de trajetórias de problemas de controle ótimo com restrições de estado e horizonte de tempo infinito. A teoria é aplicada em sistemas de gestão de energia. Para problemas de energia simples, mas ainda representativos, que minimizam custo de geração e emissão de CO2, comparamos a habilidade de diferentes baterias como substituto para o mecanismo de deslocamento de demanda de ponta (load shaving). Com a resolução do problema multiobjetivo é possível obter uma relação entre a minimização dos custos de geração de energia e de emissão de gás carbônico das usinas termoelétricas consideradas no modelo.

Palavras-chave: Controle ótimo multi-objetivo; caracterização da fronteira de Pareto; abordagem de Hamilton-Jacobi-Bellman; baterias para armazenamento de energia; resposta à demanda.

ABSTRACT

In this work we investigate optimal control problems in continuous time. A novel theory is developed for finite horizon problems with two objectives of different nature that need to be minimized simultaneously. One objective is in the classical Bolza form and the other one is defined as a maximum function. Based on the Hamilton-Jacobi-Bellman framework we characterize the weak Pareto front and the Pareto front for such problems. First we define an auxiliary optimal control problem without state constraints and show that the weak Pareto front is a subset of the zero level set of the corresponding value function. Then with a geometrical approach we establish a characterization of the Pareto front. Some numerical examples are considered to show the interest of our proposal. The encouraging results obtained with the finite horizon motivated us to investigate infinite horizon multi-objective optimal control problems and characterize the corresponding Pareto front. Additionally, we introduce a method, based on the dynamical programming principle, to reconstruct optimal trajectories for infinite horizon control problems with state constraints. The theory is applied to energy management systems. We compare the ability of different batteries as a substitute of the load shaving mechanism in smoothing the load peaks, for simple, yet representative, power mix systems with two different objectives. The multi-objective approach makes it possible to obtain a compromise between the minimization of generation costs and the carbon emissions of the thermal power plants in the mix.

Keywords: Multi-Objective optimal control problems; Pareto front characterization; Hamilton-Jacobi-Bellman approach; energy management systems; battery energy storage systems; demand response.

Contents

Chapter 1

Introduction

1.1 General setting

Optimization is a very important field in Mathematics that studies problems with the goal of maximizing or minimizing some function, subject to constraints on the decision variables. An important branch in this area is the optimal control theory.

Optimal control considers systems evolving in time that are susceptible to change through external actions. For a system with such features, the aim is to control its evolution along a certain period of time in a manner that the pair of control-and-state is optimal with respect to some criterion, given in the objective function. The system dynamics can be formulated in discrete [Iso17, Ten01] or continuous time [BCD97, CHL91]. The decision variables are of two distinctive types:

- state variables, describing the evolution of the dynamical system under consideration by means of difference equations in discrete time or by means of a differential equation in continuous time;
- control variables, or external decisions, allowing for example to maintain the trajectories of the state variables within given bounds.

This work focuses on optimal control problems in continuous time. In this case, the objective function is defined in a functional space. As for the optimization horizon it can be finite or infinite. An example of finite horizon problem, presented in [ABZ13], considers a boat that navigates in a river from an initial position x. The goal is to reach an island with minimal fuel consumption at time T. The state variables are the coordinates of the boat while the control variables are the impulse of the motor and the wheel angle that changes the boat direction. The dynamics defines the speed of the boat as a function of the current drift and the control variables. Figure 1.1 illustrates the example including obstacles that the boat needs to avoid to attain the island; the river shores are natural bounds for the trajectory of the boat.

Infinite time horizon problems arise in Economy and Biology when imposing a final horizon is artificial. For instance, the pest control problem in [CHL91] studies the joint evolution of two species: a nuisance for humans and its predator. The population of these species are the state variables. The growth of both populations represents the dynamics that is intertwined forever, without any final time ending the relationship. The control variable is the rate at which a chemical is sprayed to poison the pest (which also kills the predator). To control the expansion of the nuisance, the work [CHL91] considers an objective function that adds the cost of spraying to the size of the nuisance population.

Photo Credits: *Mauricio Kuchla*

Figure 1.1: Illustration of the boat example.

These are two objectives of different nature that could be better treated with the multi-objective approach studied in this dissertation.

Since optimal control problems are stated in functional spaces, their numerical treatment requires some form of discretization. There are two well-known approaches in the literature: "discretize-and-optimize" or "optimize-and-discretize". This work falls into the second category, specifically, the so-called Hamilton-Jacobi-Bellman (HJB) framework [BCD97]. The idea is to consider the value function defined by the optimal value of the control problem, when seen as a function of the initial state and time of the dynamical system (in the boat example, the starting position of the boat, x). Under reasonable assumptions, the dynamical programming principle [Bel57] is satisfied and the value function can be characterized as the unique solution of a partial differential equation, called Hamilton-Jacobi-Bellman (HJB) equation [BCD97, ABIL13]. After solving this equation using a discretization method, an approximation of the value function is obtained for any initial state x. This feature is in sheer contrast with the discretize-and-optimize approach, that provides the optimal value for only one initial state. For the boat example, the HJB technique yields the minimal fuel consumption to reach the island from any starting position of the boat. Another remarkable by-product is the possibility of reconstructing optimal trajectories, see [ABDZ18, RV91] for finite and [BCD97] for infinite horizon problems. Reconstructing an optimal trajectory for the pest example can give an indication if the species will be extinct in the long term.

In general, when the set of trajectories is not closed, it is not possible to guarantee the existence of a minimizer for the optimal control problem. A well-known approach to deal with this issue is to consider a relaxed optimal control problem over a compactified set of trajectories; see for instance [AC84, FR99]. Under suitable assumptions, the value function of the original problem coincides with the value function of the relaxed problem, [BCD97, FR99].

Because of the non-linearity of the HJB equations, defining a solution in the classical sense is not possible when the value function is not sufficiently smooth. The concept of *viscosity* solution, based on sub and super-differentials and introduced by Crandall and Lions in [CL83], can be employed. If the problem includes state constraints, the value function not only lacks smoothness but can also be discontinuous. For the boat example, state constraints appear when modeling the obstacles in the river. The HJB characterization can still be derived in this setting, as long as certain controllability assumptions are satisfied. The most well-known are the inward and outward pointing conditions.

The former, introduced in [Son86], requires that each trajectory initiated on the boundary can be approximated by a sequence of trajectories lying in the interior of the set of state constraints. This strong assumption ensures Lipschitz continuity of the value function, that can then be characterized as the unique viscosity solution of an HJB equation.

The outward pointing condition, introduced in [FP00, FV00], ensures that the value function is the unique lower semi-continuous solution to an HJB equation. The condition states that each point in the boundary of the feasible set can be hit by an interior trajectory.

In general, showing satisfaction of such controllability conditions is not straightforward. In order to overcome this drawback, other characterizations of the value function can be derived as in [HWZ17, HZ15] that assume that the set of constraints is endowed with a stratified structure. See also [BFZ11].

In this work we focus on an alternative approach introduced in [ABZ13]. The technique describes the *epigraph* of a state constrained value function by means of an auxiliary control problem that has no state constraints. The corresponding value function is Lipschitz continuous and its level sets provide a mechanism to recover the original value function. Thanks to this ingenuous idea, a characterization via a unique viscosity solution of an HJB equation can be derived, without any controllability assumption.

As suggested by the pest example, multi-objective dynamical optimization is an area of great interest in applications. In this context, generally, it is not possible to minimize all the criteria simultaneously. For this reason, several solution concepts have been proposed in the literature. In the classical work "Cours d'Economie Politique" [Par96], the pioneering economist V. Pareto introduced the notion of efficient or Pareto solution. At a Pareto solution it is not possible to improve one criterion without worsening at least one of the other ones. The image of the set of all Pareto solutions by the objective function is called Pareto front. A larger set is given by weak Pareto solutions, at which it is not possible to improve all the objective functions simultaneously. The Pareto front is useful for practitioners for finding a trade-off between conflicting criteria.

Figure 1.2 shows the (weak) Pareto optimal set of the problem of minimizing x_1 and x_2 in the feasible set represented by the hexagon. Note that the weak Pareto optimal set contains the Pareto optimal set. In this case, the (weak) Pareto front coincides with the (weak) Pareto optimal set.

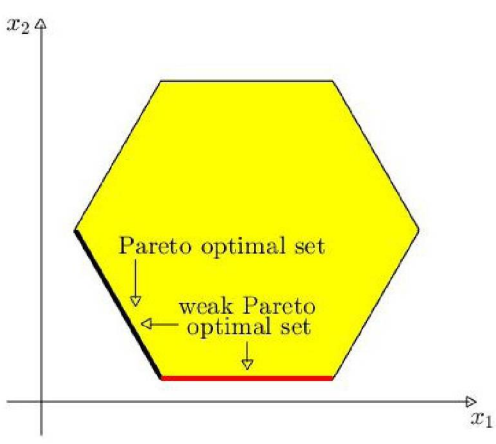


Figure 1.2: Pareto optimal set in black and weak Pareto optimal set in black and red

One of the most common approaches for solving multi-objective optimization problem is to relate it with a family of mono-objective optimization problems, in such a way that the solutions of the multi-objective problem can be obtained by solving a sequence of classical nonlinear programming problems. The most popular scalarization techniques are the weighted sum method and the weighted Chebyshev method [Jah11, Mie99]. For multi-objective optimal control problems several numerical algorithms based on scalarization techniques have been developed (see for instance [BK10, KM14, LHDv10] and the references therein).

Multi-objective optimal control problems have also been investigated within the HJB framework. A method that combines the HJB approach and the weighted sum method to find some points of the Pareto front was introduced in [MS03]. In [KV10], an approach based on HJB theory is investigated in the context of exit time problems. In that paper, the Pareto front is characterized by using the value function associated to an auxiliary control problem where one of the cost objectives is chosen as primary cost and the other objectives are transformed into auxiliary variables subject to state constraints. This idea was extended to a class of hybrid control problems, see [TCKV15]. In [Gui13] the set-valued function is characterized as a unique generalized solution of an HJB equation. In [DZ18], the idea of introducing an auxiliary problem to deal with mono-objective optimal control problems with state constraints [ABZ13], is extended to work with finite horizon multi-objective optimal control problems.

In practice, for some problems, it is difficult to calculate the sets of (weak) Pareto solutions and what it can be obtained is just an approximate set of solutions. The notion of approximate optimal solutions for multi-objective problems was introduced by [Lor84]. Several notions of ε-Pareto solutions can be considered, see [Whi86]. We discuss in this dissertation three of these concepts, one of them corresponds to the concept introduced in [Lor84] and the other ones have been considered in [EW07a, EW07b, GM04].

When developing the theory for multiobjective optimal control problems, the type of each criterion is of great importance. Chapter 3 in this dissertation presents a novel theory for finite horizon optimal control problems with objectives of different nature that need to be minimized simultaneously. Namely, in the vector objective function, one component can be an integral cost, called of Bolza cost in the literature, and another one a maximum running cost. Considering mixed objectives is important in applications. For the boat example the integral criterion is suitable for the fuel consumption. A second objective could be to maintain the boat near to one river shore, which is naturally defined as a max-type criterion. In the considered problem it is not possible to guarantee that the set of trajectories is closed, so we introduce a convexified (relaxed) problem where it is possible to guarantee that the set of trajectories is compact. This convexified set of trajectories is the closure of the set of trajectories of the original problem. Moreover we prove that if a feasible pair state-and-control $(\mathbf{y}, \mathbf{u})$ is a Pareto optimal solution for the convexified problem, then there exists an ε-Pareto optimal solution of the original problem that is in the neighborhood of $(\mathbf{y}, \mathbf{u})$. Following some ideas developped in [DZ18], we define an adequate auxiliary control problem and show that the Pareto front of the convexified problem is a subset of the zero level set of the corresponding value function. Moreover, with a geometrical approach we establish a characterization of the Pareto front.

In addition to the contributions in Chapter 3, this work contains new material regarding the topics below:

- the characterization of the (weak) Pareto front for infinite horizon problems with state constraints and objective functions of integral type. As shown in Chapter 4, extending the finite horizon results from [DZ18] is not straightforward;

- in Chapter 4, a method to reconstruct optimal trajectories for infinite horizon optimal control problems with state constraints is introduced;
- a study of energy management systems with two objectives, is given in Chapter 5, and briefly described below.

1.2 Energy management systems

The electrical grid must have the generation capacity to satisfy the demand of consumers at every time. However, the use of the energy varies daily and seasonally, and generating sufficient power at peak times, without any waste, is a challenging issue [LSN+14]. The increasing penetration of photovoltaic and wind power, which have an intermittent or uncertain nature, adds another layer of complexity to the problem. Demand-side management tools, more precisely load shaving, can be used to mitigate abrupt changes in the generation. Another tool to control the negative impact of the new technologies is to rely on a virtual entity representing a group of small generators coupled with a battery energy storage system (BESS).

Without storage devices, energy must be generated and consumed immediately. In these circumstances, to ensure that enough electricity is provided, generators must over produce. The BESS is a good alternative in this sense, because it has the capacity to store energy surplus when generation is larger than demand and supplies energy to the system when it lacks power. The BESS then functions as a “reservoir” that can be depleted when most needed. This storage is not free as some energy is lost when charging and discharging the battery.

The works [BBL+13, HMS+17] formulate the functioning of a BESS as a deterministic optimal control problem model with the objective of minimizing the energy production cost.

The Brazilian electricity generation matrix is “clean”. However, in recent years, the number of thermal power plants has grown. Due to the difficulty in building large hydroelectric power plants, it is necessary to study the environmental impact caused by the energy generation of thermal plants. We consider a model similar to [HMS+17], with the important difference that we extend the approach to the multi-objective case, [CKAN13]. Namely, in addition to the usual cost-minimization we incorporate an environmental concern, referred to minimizing fuel emissions.

We also compare the usage of different batteries instead of load shaving. For a simple, yet representative, power mix, coupled with batteries of different capacities and output, we investigate their ability in smoothing the load peaks, when compared with the load shaving mechanism. The multi-objective approach makes it possible to obtain a compromise between the minimization of generation costs and the carbon emissions of the thermal power plants in the mix. Moreover, the sensitivity analysis developed in Chapter 5.4.1 gives a systematic approach to quantify the impact of both conflicting objectives.

1.3 Objectives

The main objectives of this work are described below.

- To investigate multi-objective optimal control problems with and without state constraints. This includes infinite horizon control problems and also cost functions in different forms (Bolza and maximum running cost).

- To develop theoretical and numerical tools for optimal trajectories reconstruction of infinite horizon optimal control problems with state constraints.
- To study the Pareto front for the energy problem with load shaving or different batteries. The considered objectives are the simultaneous minimization of generation costs and fuel emissions. To determine to which extent a BESS can be used to smooth the peaks of demand in the energy problem.

1.4 Contributions

The main contributions of this dissertation are:

- to tackle, for the first time, a finite horizon bi-objective optimal control problem with objectives of different nature. By mean of the HJB approach it was possible to obtain (weak) ε-Pareto optimal solutions for this kind of problems when there is no guarantee of existence of (weak) Pareto optimal solutions. This part of the work constitutes the manuscript [CZ18], submitted to the Journal of Optimization Theory and Applications.
- To characterize the (weak) Pareto front for infinite horizon problems with state constraints and with all objective functions of integral type. Moreover to present a method based on the dynamical programming principle to reconstruct optimal trajectories for infinite horizon optimal control problems with state constraints. This part of the work constitutes a manuscript close to completion.
- To compare the ability of different batteries as a substitute of the load shaving mechanism in smoothing the load peaks, for simple, yet representative, power mix systems. The multi-objective approach makes it possible to obtain a compromise between the minimization of generation costs and the fuel emission of the thermal power plants in the mix. This part of the work constitutes a manuscript to be submitted to International Journal of Electrical Power and Energy Systems.

1.5 Organization

This work is organized as follows. Chapter 2 recalls background material on the HJB approach for optimal control problems. Some definitions and results in multi-objective optimization are also presented. Chapter 3 studies finite horizon optimal control problems with two costs of different nature, namely of integral and max-type. In Chapter 4 we characterize the (weak) Pareto front for infinite horizon optimal control problems with state constraints and introduce a method of reconstruction of trajectories using the dynamical programming principle. In Chapter 5 we analyze an energy management problem in multi-objective setting and compare the ability of different batteries as a substitute of the load shaving mechanism in smoothing the load peaks. Chapter 6 concludes the manuscript with final remarks and comments on future steps.

Chapter 2

Background for HJB approach and multi-objective optimization

Optimal control problems for ordinary differential equations consist of controlling the evolution of dynamical systems, along a certain period of time, in a manner that the pair of control-and-state is optimal with respect to some objective function. The form of this objective function is of great importance when developing the theory. It can be defined, for example, in a Bolza form [ABZ13, HZ15, CDL90, CDI84, Vin00] or in a max-type [ABDZ18, BI89].

Continuous optimal control problems have been studied extensively in the literature. Different approaches have been developed to characterize and compute the optimal solutions. In particular, the approach based on the Dynamic Programming principle consists on the analyses of the value function that associates, to every initial data, the optimal value of the control problem, see [BCD97] and the references therein. Under suitable conditions, such value function satisfies a dynamic programming principle and if the value function is differentiable, it was proved that it is a classical solution of a partial differential equation, called Hamilton-Jacobi-Bellman (HJB) equation [BCD97, ABIL13]. However, usually it is just possible to guarantee that the value function is continuous and, consequently, solutions to the HJB equations need to be understood in a weak sense. The most suitable framework to deal with these equations is the viscosity solution theory, based on sub and super-differentials, introduced by Crandall and Lions in [CL83]. This theory provides existence and uniqueness of viscosity solutions for a large class of nonlinear partial differential equations, including those that arise from optimal control problems. The value function can be characterized as the unique viscosity solution of an HJB equation [ABIL13] and contains all necessary information to reconstruct the optimal trajectories and control strategies [ABDZ18, BCD97, Ber89, RV91].

On the other hand, we are faced with problems that have more than one goal set by the decision maker in several areas, such as engineering, economics, administration, among others. It motivates the study of optimization problems with several goals and a great number of works are developed in this area [Cen77, CH83, Cho15, DZ18, Jah11, KM14, Mie99, San04].

Generally, it is not possible to minimize simultaneously all the objectives subject to some constraints, which leads to several concepts of optimal solution. The most well-known are the weak Pareto solution and the Pareto solution [Par96].

In this chapter we present some necessary definitions and results to the development of the work. First, we present the optimal control problem in finite and infinite horizon. We also introduce the concept of viscosity solution, necessary to characterize the value function of the optimal control problem as a unique viscosity solution of an HJB equation. An algorithm to obtain the optimal trajectory is presented. Finally we recall some concepts of solution and methods to solve multi-objective problems.

2.1 Unconstrained optimal control problems

Following [BCD97], for a given nonempty compact subset U of $\mathbb{R}^m$ $(m \geq 1)$ consider the set of admissible controls defined by:

$$\mathcal{U} = \{\mathbf{u} : [0, \infty[\mapsto \mathbb{R}^m \text{ measurable } \mathbf{u}(s) \in U \text{ a.e}\}.$$

Consider the controlled system:

$$\begin{cases} \dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)) \text{ a.e } s > 0, \\ \mathbf{y}(0) = x, \end{cases} \tag{2.1}$$

where $\mathbf{u} \in \mathcal{U}$, and $f : \mathbb{R}^n \times U \mapsto \mathbb{R}^n$ is continuous and is assumed to satisfy the following hypothesis:

$$\begin{cases} (i) \quad f \text{ is continuous on } \mathbb{R}^n \times U. \\ (ii) \text{ There exists } L_f \geq 0, \text{such that, for any } x,\ y \in \mathbb{R}^n \text{ and for all } u \in U : \\ \qquad |f(x,u) - f(y,u)| \leq L_f |x - y|. \end{cases} \tag{2.2}$$

where $|\cdot|$ is a norm on $\mathbb{R}^n$. By a solution to (2.1) we mean an absolutely continuous function $\mathbf{y}(\cdot)$ that satisfies

$$\mathbf{y}(s) = x + \int_0^s f(\mathbf{y}(\tau), \mathbf{u}(\tau))d\tau, \qquad \text{for all } s \geq 0. \tag{2.3}$$

Under assumption (2.2), for any $\mathbf{u} \in \mathcal{U}$ there exists an unique absolutely continuous trajectory satisfying (2.1), denoted by $\mathbf{y}_x^{\mathbf{u}}$, see for instance [BCD97, Thm. 3.5.5] or [CLSW98, Thm. 4.1.1].

Note that (2.2) implies that there exits $c_f > 0$ such that for all $x \in \mathbb{R}^n$

$$\max\{|f(x,u)| : u \in U\} \leq c_f(1 + |x|). \tag{2.4}$$

Lemma 2.1 *[BCD97, Lem. 3.5.1](Gronwall Inequality) If* $v \in L^1([t_0, t_1])$ *and* $h \in L^\infty([t_0, t_1])$ *satisfy, for some* $L \geq 0$,

$$v(t) \leq h(t) + L\int_{t_0}^t v(s)ds, \qquad \textit{for a.e. } t \in [t_0, t_1],$$

then

$$v(t) \leq h(t) + Le^{Lt}\int_{t_0}^t h(s)e^{-Ls}ds, \qquad \textit{for a.e. } t \in [t_0, t_1].$$

If, in addition, h *is non decreasing, then*

$$v(t) \leq h(t)e^{L(t-t_0)}, \qquad \textit{for a.e. } t \in [t_0, t_1].$$

By Gronwall's Lemma 2.1 and hypothesis (2.2) each solution of (2.1) satisfies the estimates presented in following proposition. (See [BCD97, Chapter III]).

Proposition 2.2 *Assume that* (2.2) *holds. Then for all* $x,\ x' \in \mathbb{R}^n$, $s > 0$ *and* $\mathbf{u} \in \mathcal{U}$

$$\begin{array}{lll} (a) & |\mathbf{y}_x^{\mathbf{u}}(s) - x| \leq c_f(1 + |x|)s & \forall s \geq 0; \\ (b) & |\dot{\mathbf{y}}_x^{\mathbf{u}}(s)| \leq c_f(1 + |x|)e^{c_f s} & \textit{for a.e } s > 0; \\ (c) & |\mathbf{y}_x^{\mathbf{u}}(s) - \mathbf{y}_{x'}^{\mathbf{u}}(s)| \leq e^{L_f s}|x - x'| & \forall s \geq 0,\ \forall x, x' \in \mathbb{R}^n; \\ (d) & 1 + |\mathbf{y}_x^{\mathbf{u}}(s)| \leq (1 + |x|)e^{c_f s} & \forall s \geq 0. \end{array}$$

2.1.1 Finite time horizon

Let $T > 0$ be a fixed final time. For $t \in [0, T]$ consider the following cost functional:

$$J(t, x, \mathbf{u}) = \int_0^t \ell(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s))ds + \varphi(\mathbf{y}_x^{\mathbf{u}}(t)), \tag{2.5}$$

where the distributed cost $\ell : \mathbb{R}^n \times U \to \mathbb{R}$ satisfies:

$$\begin{cases} (i) \quad \ell \text{ is continuous on } \mathbb{R}^n \times U \\ (ii) \text{ There exists } L_\ell \geq 0, \text{such that, for any } x,\ y \in \mathbb{R}^n \text{ and for all } u \in U : \\ \qquad |\ell(x, u) - \ell(y, u)| \leq L_\ell |x - y|, \end{cases} \tag{2.6}$$

and the final cost function satisfying:

$$\varphi : \mathbb{R}^n \to \mathbb{R} \text{ is Lipschitz continuous.} \tag{2.7}$$

Note that (2.6) implies that there exits $c_\ell > 0$ such that for all $x \in \mathbb{R}^n$ and $u \in U$

$$|\ell(x, u)| \leq c_\ell(1 + |x|). \tag{2.8}$$

This functional cost is said to be in the *Bolza* form. If the distributed cost function is null, the optimal control problem is said to be in the *Mayer* form, and if the final cost function is null, the problem is said to be in the *Lagrange* form. However, we can formulate each one of the problems in each of the other forms, as presented in [Cho15] and [Ten01]. Another important form is when the objective function is in a max-type, that is,

$$\max_{\theta \in [0,T]} \psi(\mathbf{y}_x^{\mathbf{u}}(\theta)),$$

where $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ is a Lipschitz continuous function. This kind of format was studied in [BI89] and in [ABDZ18] for unconstrained and state-constrained problems, respectively.

The optimal control problem consists in finding an admissible control $\mathbf{u} \in \mathcal{U}$ that minimizes the functional J. The value function for the finite horizon optimal control problem is

$$\vartheta(t, x) = \inf_{\mathbf{u} \in \mathcal{U}} J(t, x, \mathbf{u}), \tag{2.9}$$

We consider here a whole family of optimal control problems, all with the same dynamics (2.1) and cost functional (2.5). We are interested in how the minimum cost varies, depending on the initial position x. In the following theorem we present some known results from the literature of value functions.

Proposition 2.3 *Assume that* (2.2), (2.6) *and* (2.7) *hold.*

a) *[BCD97, Prop. 3.3.1] Then the value function ϑ is Lipschitz continuous that is, $\exists\ L_\vartheta > 0$ such that for all $t \leq s \leq T$ and $x,\ y \in \mathbb{R}^n$, the following holds:*

$$|\vartheta(t, x) - \vartheta(s, y)| \leq L_\vartheta (|x - y| + |t - s|).$$

b) *[BCD97, Prop. 3.3.2] (Dynamic Programming Principle) Let $t \in [0, T]$ and $x \in \mathbb{R}^n$ be given. Then, for all $x \in \mathbb{R}^n$ and $s \in [0, t]$, we have:*

$$\vartheta(t, x) = \inf_{\mathbf{u} \in \mathcal{U}} \left\{ \int_0^s \ell(\mathbf{y}_x^{\mathbf{u}}(\tau), \mathbf{u}(\tau))d\tau + \vartheta(t - s, \mathbf{y}_x^{\mathbf{u}}(s)) \right\}.$$

The Dynamic Programming Principle (DPP) allows to determine the value function at the point (t, x) by splitting the trajectories at time $t-s$ and starting with the position of the trajectory $\mathbf{y}_x^{\mathbf{u}}$ at time s

If the value function ϑ is differentiable, the DPP principle allow us to show that ϑ is a classical solution of the following HJB equation

$$\begin{cases} \partial_t \vartheta(t,x) + H(x, \nabla_x \vartheta(t,x)) = 0 & \text{in } [0,T] \times \mathbb{R}^n, \\ \vartheta(0,x) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{2.10}$$

where $\partial_t \vartheta(t,x)$ denotes the partial derivative with respect to t, $\nabla_x \vartheta(t,x)$ the gradient at point x and

$$H(x,p) = \sup_{u \in U} \{-f(x,u) \cdot p - \ell(x,u)\}. \tag{2.11}$$

By classical solution of (2.10) we mean: a function ϑ is said to be a classical solution of (2.10) over a domain if ϑ is differentiable over the entire domain and satisfies the above equation at every point. In general, the value function ϑ is only Lipschitz continuous, which means that ϑ cannot be a classical solution for (2.10). To deal with the value function lack of smoothness, the notion of viscosity solution, introduced in [CL83], and presented in next section, can be employed.

2.1.2 Viscosity solution

The theory of viscosity solutions introduced by Crandall and Lions in [CL83] allows continuous functions to be solution of a large class of partial differential equations of Hamilton-Jacobi (HJ) type, including the HJB equations that arise in optimal control. The theory provides results of existence and uniqueness of solution.

Consider the HJ equation

$$F(x, v(x), Dv(x)) = 0 \quad x \in \Omega, \tag{2.12}$$

where Ω is an open set of $\mathbb{R}^d$ and the Hamiltonian $F = F(x,r,p)$ is a continuous real valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^d$.

In what follows we denote by:

- $C(\Omega)$: the space of continuous functions $v : \Omega \to \mathbb{R}$.
- $C^1(\Omega)$: the subspace of $C(\Omega)$ of functions with continuous partial derivatives in Ω.

Definition 2.4 *[BCD97, Def. 2.1.1] A function $v \in C(\Omega)$ is a* ***viscosity sub-solution*** *of* (2.12) *if, for any $\overline{v} \in C^1(\Omega)$,*

$$F(x_0, v(x_0), D\overline{v}(x_0)) \leq 0$$

at any local maximum point $x_0 \in \Omega$ of $v - \overline{v}$. Similarly, $v \in C(\Omega)$ is a ***viscosity super-solution*** *of* (2.12) *if, for any $\overline{v} \in C^1(\Omega)$,*

$$F(x_1, v(x_1), D\overline{v}(x_1)) \geq 0$$

at any local minimum point $x_1 \in \Omega$ of $v - \overline{v}$. Finally v is a ***viscosity solution*** *of* (2.12) *if it is simultaneously a viscosity sub and super-solution.*

There are some equivalent definitions using super and sub-differentials. One example presented in [BCD97] associates to a function $v \in C(\Omega)$ and $x \in \Omega$ the sets

$$D^+v(x) := \left\{ p \in \mathbb{R}^n : \limsup_{y \to x, y \in \Omega} \frac{v(y) - v(x) - p \cdot (y-x)}{|x-y|} \leq 0 \right\}$$

$$D^-v(x) := \left\{ p \in \mathbb{R}^n : \liminf_{y \to x, y \in \Omega} \frac{v(y) - v(x) - p \cdot (y-x)}{|x-y|} \geq 0 \right\}$$

These sets are called, respectively, the *super-* and the *sub-differential* of v at x. In other words a vector $p \in \mathbb{R}^n$ is in the super-differential if and only if the hyperplane $y \mapsto v(x) + p \cdot (y-x)$ is tangent from above to the graph of v at the point x. Similarly a vector $p \in \mathbb{R}^n$ is in the sub-differential if and only if the hyperplane $y \mapsto v(x) + p \cdot (y-x)$ is tangent from below to the graph of v at point x.

Some properties of the sub- and super-differential are collected in next lemma.

Lemma 2.5 *[BCD97, Lem. 2.1.8] Let $v \in C(\Omega)$ and $x \in \Omega$. Then,*

a) if v is differentiable at x, then $\{Dv(x)\} = D^+v(x) = D^-v(x)$;

b) if for some x both $D^+v(x)$ and $D^-v(x)$ are nonempty, then

$$D^+v(x) = D^-v(x) = \{Dv(x)\};$$

c) the sets $A^+ = \{x \in \Omega : D^+v(x) \neq \emptyset\}$, $A^- = \{x \in \Omega : D^-v(x) \neq \emptyset\}$ are dense in Ω.

The next lemma provides a description of $D^+v(x)$, $D^-v(x)$ in terms of test functions.

Lemma 2.6 *[BCD97, Lem. 2.1.7] Let $v \in C(\Omega)$. Then,*

a) $p \in D^+v(x)$ if and only if there exists $\overline{v} \in C^1(\Omega)$ such that $D\overline{v}(x) = p$ and $v - \overline{v}$ has a local maximum at x;

b) $p \in D^-v(x)$ if and only if there exists $\overline{v} \in C^1(\Omega)$ such that $D\overline{v}(x) = p$ and $v - \overline{v}$ has a local minimum at x.

As a direct consequence of Lemma 2.6 the following new definition of viscosity solution turns out to be equivalent to the initial one.

Definition 2.7 *A function $v \in C(\Omega)$ is a viscosity subsolution of* (2.12) *in Ω if*

$$F(x, v(x), p) \leq 0 \quad \forall\, x \in \Omega,\ \forall\, p \in D^+v(x); \tag{2.13}$$

a viscosity supersolution of (2.12) *in Ω if*

$$F(x, v(x), p) \geq 0 \quad \forall\, x \in \Omega,\ \forall\, p \in D^-v(x). \tag{2.14}$$

As before, v will be called a viscosity solution of (2.12) *in Ω if (2.13) and (2.14) hold simultaneously.*

Remark 1 *Note that a function v is a classical solution of* (2.12) *if the following implication holds true:*

$$p = Dv(x) \quad \Longrightarrow \quad F(x, v(x), p) = 0.$$

This can be written in an equivalent way as:

$$p \in D^+v(x) \text{ and } p \in D^-v(x) \quad \Longrightarrow \quad F(x, v(x), p) \leq 0 \text{ and } F(x, v(x), p) \geq 0, \tag{2.15}$$

which implies, by Definition 2.7, that v is a viscosity solution of (2.12).

With this new notion of solution and under assumptions (2.2), (2.6), (2.7), we obtain that the value function ϑ is the unique viscosity solution of equation (2.10), see for example [BCD97, Thm. 3.3.7].

2.1.3 Infinite time horizon

Consider the cost functional J associated with the trajectories of the system (2.1) in the infinite horizon time:

$$J(x,\mathbf{u}) = \int_0^{+\infty} e^{-\lambda s}\ell(\mathbf{y}_x^{\mathbf{u}}(s),\mathbf{u}(s))ds, \tag{2.16}$$

where $\lambda > 0$ is a discount factor and the distributed cost $\ell : \mathbb{R}^n \times U \to \mathbb{R}$ satisfies (2.6).

The value function for infinite horizon optimal control problems is

$$\vartheta(x) = \inf_{\mathbf{u}\in\mathcal{U}} J(x,\mathbf{u}).$$

Note that in this case the value function ϑ depends only on the initial position x, not on time.

The next result states that the value function ϑ is Hölder continuous and satisfies a functional equation, called the Dynamic Programming Principle. Afterwards, we characterize the value function ϑ, as the unique viscosity solution of a partial differential equation.

Proposition 2.8

a) *[BCD97, Prop. 3.2.1] Assume that* (2.2) *and* (2.6) *hold. The value function ϑ is Hölder continuous, that is, there exists $L_\vartheta > 0$ depending on L_f and L_ℓ such that, for all $x,\ y \in \mathbb{R}^n$,*

$$|\vartheta(x) - \vartheta(y)| \le L_\vartheta |x-y|^\gamma,$$

where the exponent γ satisfies

$$\gamma = \begin{cases} 1 & \text{if } \lambda > L_f, \\ \text{any } \gamma < 1 & \text{if } \lambda = L_f, \\ \lambda/L_f & \text{if } \lambda < L_f. \end{cases}$$

b) *[BCD97, Prop. 3.2.5](Dynamic Programming Principle) Assume that* (2.2) *and* (2.6) *hold. Then, for all $x \in \mathbb{R}^n$ and $t > 0$,*

$$\vartheta(x) = \inf_{\mathbf{u}\in\mathcal{U}} \left\{ \int_0^t e^{-\lambda s}\ell(\mathbf{y}_x^{\mathbf{u}}(s),\mathbf{u}(s))ds + e^{-\lambda t}\vartheta(\mathbf{y}_x^{\mathbf{u}}(t)) \right\}.$$

Assuming that (2.2) and (2.6) hold and, moreover, that the cost functional ℓ and the dynamics f are bounded, in [BCD97] it is shown that the value function ϑ is the unique viscosity solution in the space of bounded uniformly continuous functions of the following HJB equation:

$$\lambda\vartheta + H(x,\nabla\vartheta(x)) = 0 \ \text{ in } \mathbb{R}^n, \tag{2.17}$$

where the Hamiltonian H is defined in (2.11) and $\nabla\vartheta(x)$ denotes the gradient of ϑ at the point x. The result was extended for unbounded uniformly continuous functions in [Ish84].

2.2 State constrained optimal control problems

In many applications the state trajectory is constrained, for example by some bound. In this section we present some results about state-constrained optimal control problems.

For $\mathcal{K} \subset \mathbb{R}^n$ nonempty and closed, consider the value function for the optimal control problem with $T > 0$

$$\vartheta(T, x) = \min_{\mathbf{u} \in \mathcal{U}} \left\{ \int_0^T e^{-\lambda s} \ell(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}(s)) ds \mid \mathbf{y}_x^{\mathbf{u}}(\theta) \in \mathcal{K}, \ \forall\, \theta \in (0, T) \right\}, \tag{2.18}$$

where $\mathbf{y}_x^{\mathbf{u}}$ denotes the absolutely continuous solution of (2.1). For the infinite horizon, it means when $T = +\infty$, consider $\lambda > 0$. For the finite horizon problem, when $T < +\infty$, consider $\lambda = 0$. For notational simplicity, the finite horizon problem is in Lagrange form, that is $\varphi \equiv 0$.

Note that, if $\mathcal{K} = \mathbb{R}^n$ we have an unconstrained control problem, as presented in Sections 2.1.1 and 2.1.3 for finite and infinite horizon, respectively. In this section we consider $\mathcal{K} \subsetneq \mathbb{R}^n$.

As presented in [HZ15], to show the existence of a minimizer $\mathbf{u} \in \mathcal{U}$, when dealing with a distributed cost in a state-constrained problem, it is usual to introduce an augmented dynamic. To this end, we define

$$\delta(x, u) = c_\ell(1 + |x|) - \ell(x, u), \quad \forall (x, u) \in \mathbb{R}^n \times U$$

Then, we consider the augmented dynamics $G : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n \times \mathbb{R}$ defined by

$$G(\tau, x) = \left\{ \left(\begin{array}{c} f(x, u) \\ e^{-\lambda \tau}(\ell(x, u) + r) \end{array} \right) \middle| \begin{array}{c} u \in U, \\ 0 \le r \le \delta(x, u) \end{array} \right\}, \quad \forall (\tau, x) \in \mathbb{R} \times \mathbb{R}^n.$$

Moreover, throughout this section, we will also assume that

$$G(\cdot) \text{ has convex images on a neighborhood of } [0, T) \times \mathcal{K}. \tag{2.19}$$

With this assumption the existence of a minimizer for the state-constrained optimal control problem (2.18), that is, when $t = +\infty$, can be shown.

Proposition 2.9 *[HZ15, Prop. 3.2] Suppose that* (2.2), (2.6) *and* (2.19) *hold. If* $\vartheta(x) \in \mathbb{R}$ *for some* $x \in \mathcal{K}$,

a) $T < +\infty$ *and* $\lambda = 0$ *then there exists a minimizer* $\mathbf{u} \in \mathcal{U}$ *of the finite horizon optimal control problem* (2.18).

b) $T = +\infty$ *and* $\lambda > c_f$ *then there exists a minimizer* $\mathbf{u} \in \mathcal{U}$ *of the infinite horizon optimal control problem* (2.18).

When the control problem is in presence of state constraints, Soner [Son86] associated a state-constrained HJB equation to the value function ϑ that, when $T < +\infty$, is given by

$$\begin{aligned} &\vartheta_t + H(x, \nabla_x \vartheta) = 0 \quad \forall x \in \mathcal{K}, \\ &\vartheta(0, x) = \varphi(x) \qquad\quad\ \forall x \in \mathcal{K}, \end{aligned} \tag{2.20}$$

and, when $T = +\infty$, is

$$\lambda\vartheta + H(x, \nabla_x \vartheta) = 0 \quad \forall x \in \mathcal{K}, \tag{2.21}$$

where H is defined by (2.11). In Soner's formulation the value function satisfies (2.20) or (2.21) in the state-constrained viscosity sense, that means that ϑ is a sub-solution on $\mathring{\mathcal{K}}$ and a super-solution on $\mathcal{K}$. The uniqueness of solution is more complicated to show. Indeed in absence of any controllability assumption on the behavior of the solution on the boundary, the state-constrained HJB equation (2.20) or (2.21), derived in finite and infinite horizon, respectively, can assume several solutions, in the constrained viscosity sense, see for example the discussion on [BFZ11, IK96].

One way to guarantee the uniqueness of solution is to require that the dynamics satisfy a special controllability assumption on the boundary of the set of state constraints. The most known are the "inward pointing condition" and the "outward pointing condition". The former, introduced in [Son86], requires that each trajectory initiated on the boundary can be approximated by a sequence of trajectories lying in the interior of the set of state constraints. This strong assumption ensures Lipschitz continuity of the value function, that can then be characterized as the unique viscosity solution of an HJB equation.

The outward pointing condition, introduced in [FP00, FV00], ensures that the value function is the unique lower semi-continuous solution to an HJB equation. The condition states that each point in the boundary of the feasible set can be reached by an interior trajectory.

Unfortunately, in many control problems, those controllability assumptions can not be satisfied. Some results in optimal control problems without state constraints can be seen, for instance in [BFZ11, HZ15, HWZ17]. An alternative way for dealing with the constrained case is introduced in [ABZ13], where the authors relate the epigraph of the value function ϑ with an auxiliary control problem without state constraints w:

$$w(T,x,z) = \min_{\mathbf{u}\in\mathcal{U}} \left(\int_0^T e^{-\lambda s}\ell(\mathbf{y}_x^{\mathbf{u}}(s),\mathbf{u}(s))ds - z \right) \bigvee \sup_{\theta\in(0,T]} (e^{-\lambda\theta} g(\mathbf{y}_x^{\mathbf{u}}(\theta))). \tag{2.22}$$

where $a \vee b = \max(a,b)$, $\lambda = 0$ when $T < +\infty$, $\lambda > 0$ when $T = +\infty$ and g is a Lipschitz continuous function satisfying

$$g(x) \leq 0 \Longleftrightarrow x \in K.$$

In this new problem there are no state constraints and the value function w can be characterized as a Lipschitz continuous viscosity solution of an HJB equation, without any controllability assumption. Moreover, under reasonable assumptions the epigraph of ϑ satisfies

$$Epi(\vartheta(\cdot)) = \{(x,z) \in \mathbb{R}^n \times \mathbb{R},\ w(T,x,z) \leq 0\}.$$

and we have

$$\vartheta(T,x) = \min\{z \in \mathbb{R},\ w(T,x,z) \leq 0\}.$$

In this work, based on [ABZ13, DZ18], we present in Chapter 4 a method to characterize the weak Pareto front and the Pareto front for the infinite horizon multi-objective optimal control problem with state constraints.

2.3 Synthesis of trajectory reconstruction

In this section we focus our attention on one important problem for applications: the reconstruction of an approximate optimal feedback control and the associated trajectory.

We present here the algorithm proposed in [RV91] for unconstrained finite horizon optimal control problems. This procedure applies the Dynamical Programming Principle to the function ϑ, and uses a piecewise constant control to obtain an optimal trajectory.

Algorithm 2.10

Let $\{t_0 = 0, t_1, \ldots, t_{N-1}, t_N = T\}$ *be a uniform partition of* $[0,T]$*. Let* $h_N = T/N$*. A process* $\{\mathbf{y}^N(\cdot), \mathbf{u}^N(\cdot)\}$ *is defined recursively on the intervals* $(t_0, t_1]$*,* $(t_1, t_2], \ldots, (t_{N-1}, t_N]$ *as follows. Set* $\mathbf{y}^N(t_0) = x$*.*

Step 1 *Select an optimal control value* $u_k^N \in U$ *at time* t_k^N *such that*

$$u_k^N \in \operatorname{argmin}_{u\in U} \vartheta\left(t_k, \mathbf{y}^N(t_k) + h_N f(t_k, \mathbf{y}^N(t_k), u)\right) + h_n \ell(t_k, \mathbf{y}^N(t_k), u)$$

Step 2 *Define* $\mathbf{u}^N(t) = u_k^N$ *a.e* $t \in (t_k, t_{k+1}]$*, and a new position* $\mathbf{y}^N(t)$ *on* $(t_k, t_{k+1}]$ *as solution to*

$$\dot{\mathbf{y}}(t) = f(t, \mathbf{y}(t), \mathbf{u}_k^N(t)) \ \ a.e\ t \in (t_k, t_{k+1}],$$

with initial condition $\mathbf{y}^N(t_k)$*.*

The following theorem shows the convergence of the Algorithm 2.10.

Theorem 2.11 *[RV91, Thm. 3.2] Suppose* (2.2)*,* (2.6)*,* (2.19) *and* (2.7) *hold. Let* $\{\mathbf{y}^n(\cdot), \mathbf{u}^n(\cdot)\}$ *be a sequence generated by Algorithm 2.10 for* $n \geq 1$*. Then* $\{\mathbf{y}^n(\cdot)\}$ *has cluster points with respect to the topology of uniform convergence, and corresponding to any cluster point* $\mathbf{y}(\cdot)$*, there exists a control* $\mathbf{u}(\cdot)$ *such that* $(\mathbf{y}(\cdot), \mathbf{u}(\cdot))$ *is an optimal process.*

A similar scheme was developed for finite horizon problems with one objective and with state constraints by [ABDZ18]. A result for infinite horizon without state constraints can be found in [BCD97]. Our interest is in the situation of infinite horizon with one objective and with state constraints. Moreover the results that we developed hold true also in case of multi-objective infinite horizon optimal control problems, as presented in Chapter 4.

2.4 Multi-objective optimization

In this section we are going to consider the following multi-objective optimization problem in a general context:

$$\begin{cases} \text{Minimize } \ f(x) = (f_1(x), \ldots, f_r(x)) \\ \text{subject to } x \in Q \end{cases} \tag{2.23}$$

where we have $r \geq 2$ objective functions $f_i : X \to \mathbb{R}$, a feasible nonempty set $Q \subset X$ and X is a Banach space.

In problem (2.23), the aim is to minimize all components of the objective function f at the same time. If there exists no conflict between the cost functions f_i, $i = 1, \ldots, r$, then a solution $x^* \in Q$ may exist such that:

$$f_i(x^*) = \min\{f_i(x),\ x \in Q\}, \quad i = 1, \ldots, r.$$

In this work, we assume that there is no single solution that minimizes all the objective functions simultaneously. This means that the objective functions are at least partly conflicting and several solution concepts may be associated with the problem (2.23). A predominant optimality notion for problem (2.23) is the one of Pareto optimality.

Definition 2.12 *A vector $x^* \in Q$ is a **weak Pareto solution** (or weak efficient solution) of* (2.23) *if there exist no $x \in Q$, $x \neq x^*$ such that $f_i(x) < f_i(x^*)$ for all $i = 1, \ldots, r$. Denote by $\mathcal{P}_w$ the set of all weak Pareto solutions of* (2.23)*.*

Definition 2.13 *A vector $x^* \in Q$ is a **Pareto solution** (or efficient solution) of* (2.23) *if there exist no $x \in Q$ such that $f(x) \neq f(x^*)$ and $f_i(x) \leq f_i(x^*)$ for all $i = 1, \ldots, r$. Denote by $\mathcal{P}$ the set of all Pareto solutions of* (2.23)*.*

The set of all vectors of objective values at the Pareto (resp. weak Pareto) minima is said to be the *Pareto front* (resp. *weak Pareto front*). More precisely, we have the following definition.

Definition 2.14 *We will call **Pareto front** $\mathcal{F}$ (respectively **weak Pareto front** $\mathcal{F}_w$) the image of the Pareto optimal solution set $\mathcal{P}$ (respectively of $\mathcal{P}_w$) by the objective application f:*

$$\mathcal{F} = \{f(x) = (f_1(x), \ldots, f_r(x)),\ x \in \mathcal{P}\}$$
$$\mathcal{F}_w = \{f(x) = (f_1(x), \ldots, f_r(x)),\ x \in \mathcal{P}_w\}$$

Besides, it is known [Lin76] that the (weak) Pareto front is contained in the boundary of the *attainable set* $\mathcal{Z}$, that is defined as:

$$\mathcal{Z} := \{f(x),\ x \in Q\} \subset \mathbb{R}^r.$$

It is immediate from the definitions that $\mathcal{P} \subset \mathcal{P}_w$. The reciprocal is false, as shown by Example 2.15.

Example 2.15 *Consider the problem from Moulin and Soulié [MS79]:*

$$\begin{cases} \text{Minimize } f(x) = (f_1(x), f_2(x)) \\ \text{subject to: } x \in Q \end{cases} \tag{2.24}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f_1(x) = -x_1$ e $f_2(x) = -x_2$, and $Q = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \cup [-1, 0] \times [0, 1] \subseteq \mathbb{R}^2$ (see Figure 2.1).

- *The point $a_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ cannot decrease strictly f_1, without increasing strictly f_2. The point a_1 is a Pareto solution belonging to set $\mathcal{P}$, where $\mathcal{P}$ is the arc between $(1, 0)$ and $(0, 1)$ on the picture of the left. Respectively, the image of the point a_1 belong to the Pareto Front $\mathcal{F}$, where $\mathcal{F}$ is the arc between $(-1, 0)$ and $(0, -1)$ on the picture of the right, of Figure 2.1.*

- *The point $a_2 = (\frac{-1}{2}, 1)$ can decrease f_1 without increasing strictly f_2, (consider for example, the point $(0, 1)$), but cannot improve both criteria simultaneously. The point $a_2 \in \mathcal{P}_w$ however a_2 cannot belong to $\mathcal{P}$, with:*

  $$\mathcal{P}_w = \mathcal{P} \cup \{(\lambda, 1) : -1 \leq \lambda \leq 0\}.$$

  Respectively, the image of the point a_2 belongs to the weak Pareto front $\mathcal{F}_w$, however cannot belong to $\mathcal{F}$, with

  $$\mathcal{F}_w = \mathcal{F} \cup \{(\gamma, -1) : 0 \leq \gamma \leq 1\}.$$

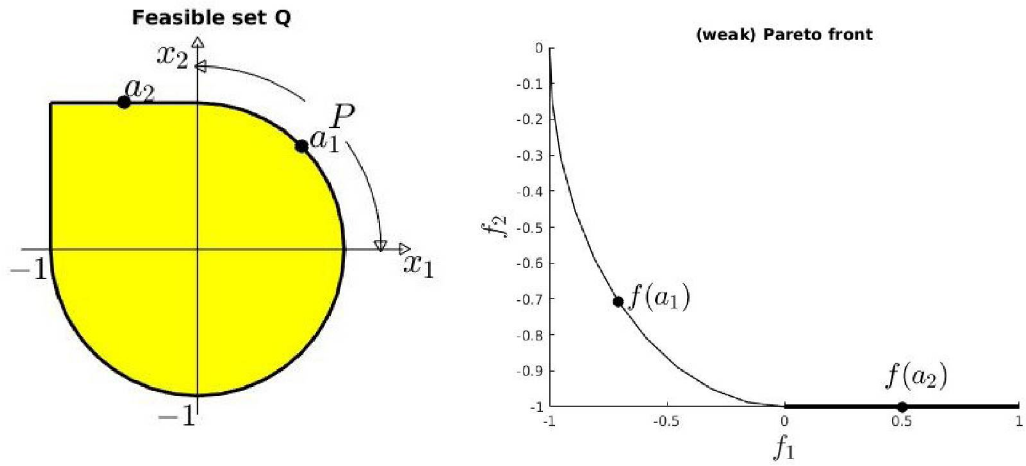


Figure 2.1: feasible set Q and (weak) Pareto front, respectively

2.4.1 Scalarization techniques

The term scalarization presents for a multi-objective problem a family of mono-objective optimization problems, in such a way that the solutions of the multi-objective problem can be obtained by solving a sequence of classical nonlinear programming problems. There is a large variety of methods for scalarizing a multi-objective optimization problem. There are some methods that generate just one Pareto optimal solution and others that try to generate all the set of (weak) Pareto optimal solutions. Our focus is on the second kind of methods, because it is important to represent all the (weak) Pareto front to show to the decision maker, who can select the desired compromise.

We will see two different scalarization methods: the weighted sum method and the weighted Chebyshev method. More details about those methods can be seen for instance in [Iso13, Jah11, Mie99].

Weighted sum method

Let $\Omega = \{\omega \in \mathbb{R}^r : \omega_i \geq 0,\ 1 \leq i \leq r \text{ and } \sum_{i=1}^{r} \omega_i = 1\}$. For each $\omega \in \Omega$, consider the following problem

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{r} \omega_i f_i(x) \\ \text{suject to:} & x \in Q. \end{array} \tag{2.25}$$

Some theoretical results about the weighted sum method are presented in the following theorem.

Theorem 2.16

a) *[Mie99, Thm. 3.1.1] If there exists $\omega \in \Omega$ such that $x^* \in Q$ is a solution of* (2.25)*, then x^* is a weak Pareto optimal solution of* (2.23).

b) *[Mie99, Thm. 3.1.2] If there exists $\omega \in \Omega$ with $\omega_i > 0$ for $i = 1, \dots, r$ such that $x^* \in Q$ is a solution of* (2.25)*, then x^* is a Pareto optimal solution of* (2.23).

c) *[Mie99, Thm. 3.1.3] If there exists $\omega \in \Omega$ such that the problem* (2.25) *has a unique solution $x^* \in Q$, then x^* is a Pareto optimal solution of* (2.23).

d) *[Mie99, Thm. 3.1.4] Let the multi-objective problem be convex. If $x^* \in Q$ is a Pareto optimal solution of* (2.23)*, then there exists a weighting vector $\omega \in \Omega$ such that x^* is a solution of* (2.25)*.*

The weighted sum method is a simple way to generate different Pareto optimal solutions and the Pareto optimality is guaranteed if the weighting coefficients are positive or the solution is unique. The weakness of the weighting method is that all the Pareto optimal points cannot be found if the problem is non-convex [KM14].

Weighted Chebyshev method

Before presenting the method we need the following definitions.

Definition 2.17 *The components f_i^* of the ideal objective vector $f^* \in \mathbb{R}^r$ are obtained by minimizing each of the objective functions individually subject to the constraints, that is, by solving*

$$f_i^* = \begin{cases} \textit{Minimize} & f_i(x) \\ \textit{suject to:} & x \in Q \end{cases} \tag{2.26}$$

for $i = 1, \dots, r$.

It is obvious that if the same x^* is the optimal solution for all the single objective problems it would be the solution of the multi-objective problem (2.23) and the Pareto optimal set would be reduced to it. This is not possible in general since there is some conflict among the objectives. But the ideal objective vector can be used as a lower bound of the Pareto optimal set for each objective function.

Definition 2.18 *An utopian objective vector $\beta^* \in \mathbb{R}^r$ is a vector which components are formed by $\beta_i^* = f_i^* - \varepsilon_i$ for all $i = 1, \dots, r$, where f_i^* is a component of the ideal objective vector and $\varepsilon_i > 0$.*

Consider Ω as in the weighted sum method. The weighted Chebyshev problem, for $\omega \in \Omega$, is of the form

$$\begin{array}{ll} \text{Minimize} & \max\limits_{i=1,\dots,r} \omega_i |f_i(x) - \beta_i^*|, \\ \text{suject to:} & x \in Q. \end{array} \tag{2.27}$$

Problem (2.27) is non-differentiable, but it can be solved in a differentiable form as long as the objective functions are differentiable and β^* is known globally. In this case, instead of problem (2.27), the problem

$$\begin{array}{ll} \text{Minimize} & \alpha \\ \text{suject to:} & x \in Q, \\ & \omega_i(f_i(x) - \beta_i^*) \le \alpha, \ i = 1, \dots, r. \end{array} \tag{2.28}$$

is solved, where both $x \in Q$ and $\alpha \in \mathbb{R}$, are variables.

In the following theorem we present some relations between the solutions of the multi-objective problem (2.23) and the solution of the scalar problem (2.27).

Theorem 2.19

a) *[Mie99, Thm. 3.4.2] The solution x^* of the weighted Chebyshev problem* (2.27) *is weak Pareto optimal if all weighting coefficients are positive.*

b) *[Mie99, Thm. 3.4.3] Weighted Chebyshev problem* (2.27) *has at least one Pareto optimal solution.*

c) *[Mie99, Cor. 3.4.4] If weighted Chebyshev problem* (2.27) *has a unique solution it is Pareto optimal.*

Convexity of the multi-objective optimization problem is needed in order to guarantee that every Pareto optimal solution can be found by the weighted sum method. On the other hand, the following theorem shows that every Pareto optimal solution can be found by the weighted Chebyshev method.

Theorem 2.20 *[Mie99, Thm. 3.4.5] Let $x^* \in Q$ be Pareto optimal. Then there exists a weighting vector $0 < \omega \in \Omega$, such that x^* is a solution of weighted Chebyshev problem* (2.27).

The method of weighted Chebyshev metric works for convex as well as non-convex problems, unlike the weighted sum method.

2.4.2 Algorithms to generate the Pareto front

Many methods have been developed for the numerical solution of multi-objective optimization problems. Many of these methods are only applicable to special problem classes. Some of them calculate just one Pareto solution or a part of the Pareto front. But in some applications it is important to find all the Pareto front, to observe all the possible solutions and after analyze what is the best one. Some algorithms to solve multi-objective problems are presented, for instance in [Arb97, DK11, Jah11, MKW16, Mie99, RW92].

For multi-objective optimal control problems several numerical algorithms based on scalarization techniques have been developed (see for instance [BK10, KM14, LHDv10] and the references therein). We present here the method presented in [KM14] that is based on the Chebyshev scalarization technique. The authors proposed the following algorithm that performs well on the numerical examples, but the important issue of showing convergence is not addressed in the paper. Although the Algorithm 2.21 is described for the bi-objective case it can be generalized to problems with more than two objectives.

Algorithm 2.21

Step 0.0: (Initialization) *Choose the utopia parameters, ε_1, $\varepsilon_2 > 0$.*
Set the number of discrete (approximating) points, $(N+1)$, in the Pareto front. Set $k=1$.

Step 0.1: (Boundary of the front)
(a) Find x^1 that solves Problem (2.26) *with $i=1$. Note that $f_1^* = f_1(x^1)$ and $\bar{f}_2 = f_2(x^1)$.*
Mark a boundary point in the Pareto front: $\bar{f}^N = (f_1^*, \bar{f}_2)$,
(b) Find x^2 that solves Problem (2.26) *with $i=2$. Note that $f_2^* = f_2(x^2)$ and $\bar{f}_1 = f_1(x^2)$.*
Mark a boundary point in the Pareto front: $\bar{f}^0 = (\bar{f}_1, f_2^*)$,

Step 0.2: (Utopia point) *Set $\beta^* = (\beta_1^*, \beta_2^*)$ with $\beta_i^* = f_i^* - \varepsilon_i$, $i = 1, 2$.*

Step 0.3: (Range of weights) *Set* $\omega_0 = \dfrac{(f_2^* - \beta_2^*)}{(\bar{f}_1 - \beta_1^*) + (f_2^* - \beta_2^*)}$ *and* $\omega_f = \dfrac{(\bar{f}_2 - \beta_2^*)}{(f_1^* - \beta_1^*) + (\bar{f}_2 - \beta_2^*)}$.
Set the increment $\Delta\omega = (\omega_f - \omega_0)/N$.

Step k.1: (Current weights) *Set $\omega = \omega_0 + k\Delta\omega$. Set $\omega_1 = \omega$ and $\omega_2 = 1 - \omega$.*

Step k.2: (A Pareto Minimun) *Find x^* that solves Problem* (2.28).
Assign a point in the Pareto front: $\bar{f}^k = (f(x^*), f_2(x^*))$.

Step k.3: (Stopping Criterion) *If $k = N$ then STOP. Otherwise, set k=k+1, and go to step k.1.*

Multi-objective optimal control problems have also been investigated within the HJB framework. A method that combines the HJB approach and the weighted sum method to find some points of the Pareto front was introduced in [MS03]. In [KV10], an approach based on HJB theory is investigated in the context of exit time problems. In that paper, the Pareto front is characterized by using the value function associated to an auxiliary control problem where one of the cost objectives is chosen as primary cost and the other objectives are transformed into auxiliary variables subject to state constraints. This idea was extended to a class of hybrid control problems, see [TCKV15]. In [Gui13] the set-valued function is characterized as a unique generalized solution of an HJB equation. In [DZ18], the idea of introducing an auxiliary problem to deal with mono-objective optimal control problems [ABZ13], is extended to work with multi-objective optimal control problems. Like in [KM14], the two criteria considered in [DZ18] must have the same nature, both of integral type.

Usually, the multi-objective control problems are investigated in the case when the cost functions are of the same nature (Bolza with free or fixed final time horizon). Our interest is to characterize the (weak) Pareto front for some kinds of multi-objective optimal control problems. Based on [DZ18], in Chapter 3 we present for the first time theory for tackling finite horizon optimal control problems with objectives of different nature. Namely, in the vector objective function, one component can be with a Bolza cost and another one as a maximum running cost. Moreover, the characterization of the (weak) Pareto front for a multi-objective infinite horizon optimal control problem with state constraints is presented in Chapter 4. Algorithm 2.21 is used at the test of the energy problem.

2.5 Concluding remarks

In this chapter we presented some important results about the HJB approach to solve optimal control problems with one objective and multi-objective theory for optimization problems in a general context. Those results are necessary for a better understanding of this thesis. In the following chapters we work with multi-objective control problems extending the HJB approach for this kind of problems.

Chapter 3

Contributions in finite horizon problems with different objectives

In this chapter, we consider finite horizon optimal control problems with two objective functions, of different nature, that need to be minimized simultaneously. Namely, in the vector objective function, one component is a Bolza cost and another one is a maximum running cost. So the considered bi-objective optimal control problem has the following form:

$$\inf_{(\mathbf{y},\mathbf{u})\in\mathbb{X}} \Big(\varphi(\mathbf{y}(T)) + \int_t^T \ell(\mathbf{y}(s),\mathbf{u}(s))ds,\ \max_{s\in[t,T]} \psi(\mathbf{y}(s))\Big),$$

for a given final cost $\varphi : \mathbb{R}^n \to \mathbb{R}$, a running cost $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and a given function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$, and where $\mathbb{X}$ is the feasible set, T is the final time horizon, $\mathbf{u} : [0,T] \to \mathbb{R}^m$ is the control variable and $\mathbf{y} : [0,T] \to \mathbb{R}^n$ is the state variable, solution of a differential system.

Several numerical algorithms based on scalarization techniques have been developed for multi-objective optimal control problems (see for instance [BK10, KM14, LHDv10] and the references therein). These problems have also been investigated within the HJB framework. A method that combines the HJB approach and the weighted sum method to find some points of the Pareto front was introduced in [MS03]. In [KV10], an approach based on HJB theory is investigated in the context of exit time problems. In that chapter, the Pareto front is characterized by using the value function associated to an auxiliary control problem where one of the cost objectives is chosen as primary cost and the other objectives are transformed into auxiliary variables subject to state constraints. This idea was extended to a class of hybrid control problems, see [TCKV15]. In [Gui13] the set-valued function is characterized as a unique generalized solution of an HJB equation. In [DZ18], the idea of introducing an auxiliary problem to deal with mono-objective optimal control problems [ABZ13], is extended to work with multi-objective optimal control problems.

In practice, for some problems, it is difficult to calculate the sets of (weak) Pareto solutions and what it could be obtained is just an approximate set of solutions. The notion of approximate optimal solutions for multi-objective problems was introduced by [Lor84]. Several notions of ε-Pareto solutions can be considered, see [Whi86]. We discuss in this chapter three of these concepts, one of them corresponds to the concept introduced in [Lor84] and the other ones have been considered in [EW07a, EW07b, GM04].

Usually, the multi-objective control problems are investigated in the case when the cost functions are of the same nature (Bolza with free or fixed final time horizon). In this work, we use a HJB approach to characterize the Pareto front for a finite horizon bi-objective optimal control problem with objectives of different nature. In the considered problem it is not possible to guarantee that the set of trajectories is closed, so we introduce a convexified (relaxed) problem where is possible to guarantee that the set of trajectories is compact. Moreover this convexified set of trajectories is the closure of the set of trajectories of the original problem. Moreover we prove that if a feasible pair $(\mathbf{y}, \mathbf{u})$ is a Pareto optimal solution for the convexified problem, then there exists an ε-Pareto optimal solution of the original problem that is in the neighborhood of $(\mathbf{y}, \mathbf{u})$. Following some ideas developed in [DZ18], we define an adequate auxiliary control problem and show that the Pareto front of the convexified problem is a subset of the zero level set of the corresponding value function. Moreover, with a geometrical approach we establish a characterization of the Pareto front.

With this idea it is also possible to characterize the (weak) Pareto front for bi-objective optimal control problem with objectives of different nature and with state constraints without any controllability assumption. Moreover, considering mixed objectives is important in applications and this is illustrated with some examples, where the method proposed in this chapter is applied to obtain the Pareto front.

3.1 Pareto optimality - General results

Consider the following bi-objective optimization problem:

$$\begin{cases} \text{Minimize} \;\; c(x) = (c_1(x), c_2(x)) \\ \text{subject to } x \in X, \end{cases} \tag{3.1}$$

where Y is a Banach space, $c_i : Y \to \mathbb{R}$ are continuous functions and $X \subset Y$ a feasible nonempty set.

In this chapter, we assume that there is no single solution that minimizes all the objective functions simultaneously. This means that the cost functions are at least partly conflicting and several solution concepts may be associated with the problem (3.1). The optimality notion for problem (3.1) is the one of Pareto optimality presented in Section 2.4.

Computing the Pareto fronts and the set of Pareto solutions is a challenging problem. In some cases, these sets may be empty, this is the case, for instance, when the feasible set is not closed. In such a context, it is natural to consider a set of *approximate* Pareto solutions. Different definitions for ε-solutions have been investigated in the literature, see [Lor84, Whi86]. In what follows, we will consider three of these concepts, where $|\cdot|$ is the maximum norm. Moreover we remember that $\mathcal{P}$ denotes the set of all Pareto solutions of (3.1).

Definition 3.1 (ε-Pareto solutions) *Let $\varepsilon \geq 0$, denote by $\varepsilon 1$ the vector in $\mathbb{R}^2$ such that all entries are equal ε. We define the following sets of ε-Pareto solutions:*

(i) $\mathcal{P}^{1,\varepsilon} = \{x \in X : \text{ there is no } y \in X \text{ such that}$

$$c(y) \leq c(x) - \varepsilon 1 \text{ and } c(y) \neq c(x) - \varepsilon 1\}.$$

(ii) $\mathcal{P}^{2,\varepsilon} = \{x \in X : \text{ there exists } y \in \mathcal{P} \text{ such that } |c(x) - c(y)| \leq \varepsilon\}.$

(iii) $\mathcal{P}^{3,\varepsilon} = \{x \in X : \text{ for any } y \in X, \text{ if } c(y) \neq c(x) \text{ and } c(y) \leq c(x) \text{ then}$

$$c(y) \geq c(x) - \varepsilon 1\}.$$

An ε-Pareto solution $x^* \in \mathcal{P}^{i,\varepsilon}$, $i = 1,2,3$, produces an ε-Pareto outcome $c(x^*)$ and the set of all ε-Pareto outcomes are denoted by $\mathcal{F}^{i,\varepsilon}$, $i = 1,2,3$. Note also that, if $\varepsilon_1 \leq \varepsilon_2$ then $\mathcal{F}^{i,\varepsilon_1} \subset \mathcal{F}^{i,\varepsilon_2}$, for $i = 1,2,3$. Moreover, if the feasible set X is a compact set, then the following relation between the ε-Pareto sets is established by [Whi86]:

$$\mathcal{F} \subset \mathcal{F}^{3,\varepsilon} \subseteq \mathcal{F}^{2,\varepsilon} \subseteq \mathcal{F}^{1,\varepsilon}. \tag{3.2}$$

Remark 3.2 *The reverse inclusions in* (3.2) *are not true. Consider the following bi-objective optimization problem:*

$$\begin{cases} \textit{Minimize } c(x) = (x_1, x_2) \\ \textit{subject to: } x \in X = [-1,1]^2 \setminus \{x \in [-1,0]^2 : x_1 + x_2 < -1\}. \end{cases} \tag{3.3}$$

The compact feasible set X is represented in Figure 3.1. Note that for all $\varepsilon > 0$ the point

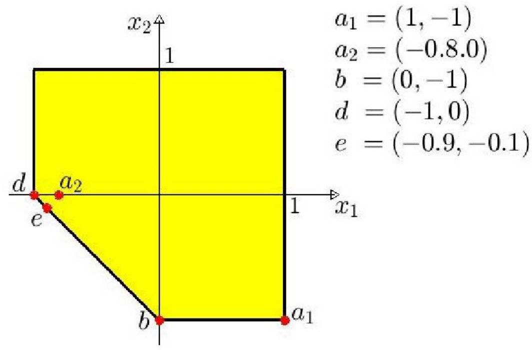


Figure 3.1: Feasible set X

$a_1 \in \mathcal{P}^{1,\varepsilon}$. Indeed, $c_2(a_1) - \varepsilon = -1 - \varepsilon$, so there is no $x \in X$ such that $c(x) \neq c(a_1) - \varepsilon$ and $c(x) \leq c(a_1) - \varepsilon$, therefore $a_1 \in \mathcal{P}^{1,\varepsilon}$. On the other hand, if $0 < \varepsilon < 1$ then $c_1 \notin \mathcal{P}^{3,\varepsilon}$ because $c(b) \neq c(a_1)$, $c(b) \leq c(a_1)$ and $c_1(b) = 0 < 1 - \varepsilon = c_1(a_1) - \varepsilon$.

Now, consider $\varepsilon = 0.1$. As the point e is a Pareto optimal solution, we obtain that $a_2 \in \mathcal{P}^{2,\varepsilon}$, but $a_2 \notin \mathcal{P}^{3,\varepsilon}$. In fact $c(d) \neq c(a_2)$, $c(d) \leq c(a_2)$ and $c_1(d) = -1 < -0.9 = c_1(a_2) - \varepsilon$, therefore $a_2 \notin \mathcal{P}^{3,\varepsilon}$.

Actually, in this simple example, the three concepts of ε-Pareto fronts give three different sets. These sets are represented in red in Figure 3.2, for $\varepsilon = 0.1$. In Figure 3.2a the black dashed line represents a segment that is not included in $\mathcal{F}^{1,\varepsilon}$.

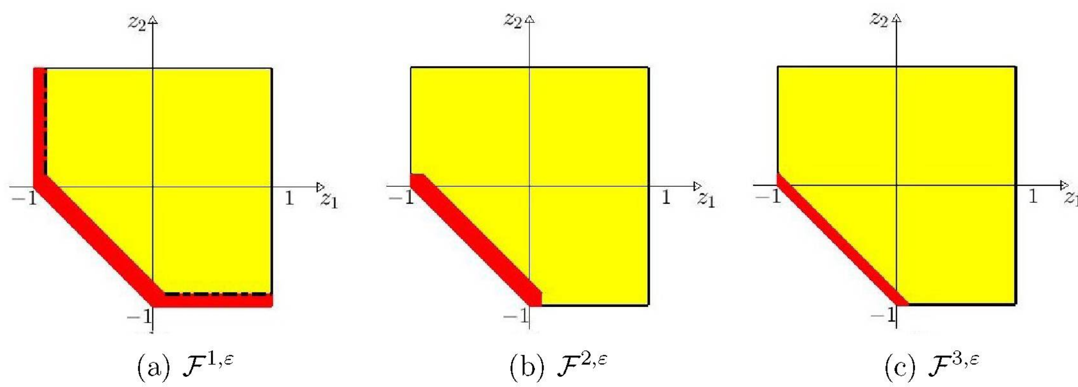
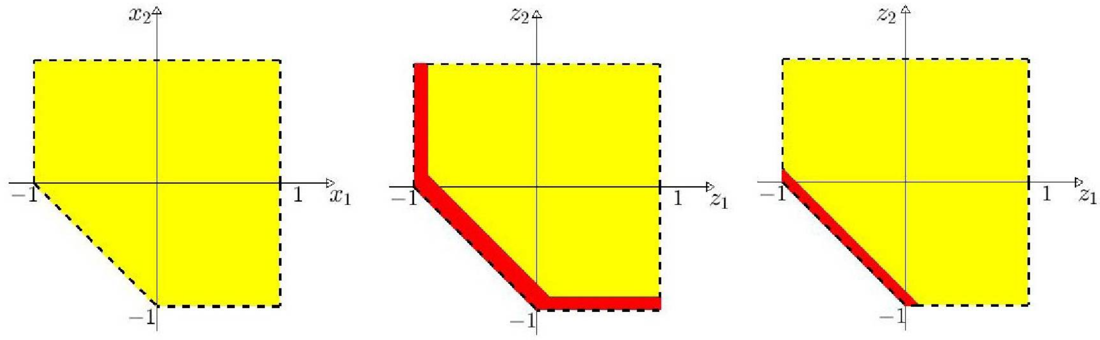


Figure 3.2: ε-Pareto fronts for Problem (3.3), $\varepsilon = 0.1$

Remark 3.3 *We stress on that the inclusions in (3.2) are true only when the feasible set X is a compact set. However if the feasible set is not closed, there is no guarantee of existence of (weak) Pareto solutions. Actually, the weak Pareto set can be empty. Consider, for example, the following problem:*

$$\begin{cases} \text{Minimize } c(x) = (x_1, x_2) \\ \text{subject to: } x \in X =]-1,1[^2 \backslash \{x \in (-1,0)^2 : x_1 + x_2 \leq -1\}. \end{cases} \tag{3.4}$$

In this case the feasible set is open, see Figure 3.3a, where the black dashed lines represents the boundary of X (this boundary is not included in X). For this example, the (weak) Pareto set and the the set $\mathcal{P}^{2,w}$ are empty. However the sets $\mathcal{P}^{1,\varepsilon}$ and $\mathcal{P}^{3,\varepsilon}$ are not empty and can be seen in red in Figures 3.3b and 3.3c, respectively.



(a) Open feasible set (b) $\mathcal{F}^{1,\varepsilon}$ (c) $\mathcal{F}^{3,\varepsilon}$

Figure 3.3: Feasible set and ε-Pareto fronts for Problem (3.4), $\varepsilon = 0.1$

Part of the concepts of approximate solutions, as defined in 3.1 can be also extended to the notion of weak ε-Pareto solutions (see in [Lor84]).

Definition 3.4 *[weak ε-Pareto solution] Let $\varepsilon \geq 0$. We define the following sets of weak ε-Pareto solutions:*

(i) $\mathcal{P}_w^{1,\varepsilon} = \{x \in X : \text{ there is no } y \in X \text{ such that } c(y) < c(x) - \varepsilon 1\}$.

(ii) $\mathcal{P}_w^{2,\varepsilon} = \{x \in X : \text{ there exists } y \in \mathcal{P}_w \text{ such that } |c(x) - c(y)| \leq \varepsilon\}$.

Motivated by the observations made on the example of Remark 3.3, we consider the problem of minimizing the objective functions over the closure of the feasible set X:

$$\begin{cases} \text{Minimize } \; c(x) = (c_1(x), c_2(x)) \\ \text{subject to } x \in \overline{X} \end{cases} \tag{3.5}$$

We denote by $\mathcal{P}^\#$ (resp. $\mathcal{P}_w^\#$) the set of Pareto (resp. weak Pareto) solutions of problem (3.5). We are interested in the link between the set $\mathcal{P}^\#$ (resp. $\mathcal{P}_w^\#$) and the sets of ε-Pareto (resp. weak ε-Pareto) solutions of the original problem (3.1).

Theorem 3.5 *Assume that the functions c_i are Lipschitz continuous, with Lipschitz constant L_i, $i = 1, 2$.*

(i) For any $x^* \in \mathcal{P}^\#$ and for any $\varepsilon > 0$, there exists $x_\varepsilon \in \mathcal{P}^{1,\varepsilon}$ such that

$$|x^* - x_\varepsilon| \leq \min_i(\varepsilon/L_i) \quad \text{and} \quad |c(x^*) - c(x_\varepsilon)| \leq \varepsilon.$$

(ii) For any $x^* \in \mathcal{P}_w^\#$ and for any $\varepsilon > 0$, there exists $x_\varepsilon \in \mathcal{P}_w^{1,\varepsilon}$ such that

$$|x^* - x_\varepsilon| \leq \min_i(\varepsilon/L_i) \quad \textit{and} \quad |c(x^*) - c(x_\varepsilon)| \leq \varepsilon.$$

(iii) Given $\varepsilon > 0$, for any $x^* \in \mathcal{P}^{3,\varepsilon}$ there exists $x_\varepsilon \in \mathcal{P}^\#$ such that $|c(x_\varepsilon) - c(x^*)| \leq 2\varepsilon$.

PROOF. (**i**) Let $x^* \in \mathcal{P}^\#$ and $\varepsilon > 0$. As $x^* \in \overline{X}$, there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} x_n = x^*$. Given $\varepsilon > 0$, define $\delta = \min_i(\varepsilon/L_i)$. So there exists x_N such that $|x^* - x_N| \leq \delta$. By the Lipschitz continuity of functions $c_i,\ i = 1, 2$ we obtain that

$$|c_i(x^*) - c_i(x_N)| \leq L_i|x^* - x_N| \leq L_i\delta \leq \varepsilon,$$

which means that $|c(x^*) - c(x_N)| \leq \varepsilon_i$. It remains to prove that $x_N \in \mathcal{P}^{1,\varepsilon}$. Note that, by definition of x^* there is no $x \in X$ such that $c_i(x) \leq c_i(x^*)$, for $i = 1, 2$. This implies that for any $x \in X$, we have:

$$c_1(x) > c_1(x^*) > c_1(x_N) - \varepsilon \quad \text{or} \quad c_2(x) > c_2(x^*) > c_2(x_N) - \varepsilon.$$

Therefore, there is no $x \in X$ such that $c_i(x) \leq c_i(x_N) - \varepsilon$ for $i = 1, 2$, which means that $x_N \in \mathcal{P}^{1,\varepsilon}$ and the assertion is now proved with $x_\varepsilon = x_N$.

(**ii**) The proof is similar to (i).

(**iii**) Let $x^* \in \mathcal{P}^{3,\varepsilon}$. Assume that there is no $x \in \mathcal{P}^\#$ such that $|c(x) - c(x^*)| < 2\varepsilon$. Then $x^* \notin \mathcal{P}^\#$, so there exists $y \in \mathcal{P}^\#$ such that $c(y) \neq c(x^*)$ and $c(y) \leq c(x^*)$. As $y \in \overline{X}$, there exists a sequence $\{y_n\} \subset X$ such that $\lim_{n\to\infty} y_n = y$. Choose y_N such that $c(y_N) \leq c(x^*)$ and $|c(y) - c(y_N)| \leq \varepsilon$. As $x^* \in \mathcal{P}^{3,\varepsilon}$, we must have $c(y_N) \geq c(x^*) - \varepsilon 1$. Hence $c(x^*) - \varepsilon 1 \leq c(y_N) \leq c(x^*)$, which means that $|c(y_N) - c(x^*)| < \varepsilon$. Then

$$|c(y) - c(x^*)| \leq |c(y) - c(y_N)| + |c(y_N) - c(x^*)| \leq 2\varepsilon,$$

what is a contradiction. □

Remark 3.6 *Assertion (i) of Theorem 3.5 amounts saying also that for any Pareto value $z^* \in \mathcal{F}^\#$ and its corresponding Pareto solution x^*, there exists an ε-Pareto solution $x_\varepsilon \in \mathcal{P}_\varepsilon^{1,\varepsilon}$ with corresponding value $z_\varepsilon = c(x_\varepsilon) \in \mathcal{F}^{1,\varepsilon}$ such that $|z^* - z_\varepsilon| \leq \varepsilon$ and $|x^* - x_\varepsilon| \leq$* $\min_i(\varepsilon/L_i)$.

3.2 Problem Statement

Let U be a given non-empty compact subset of $\mathbb{R}^m$ (for $m \geq 1$). A measurable function $\mathbf{u} : [0, +\infty[\to \mathbb{R}^m$ is said admissible if it satisfies $\mathbf{u}(s) \in U$ for almost every $s \geq 0$. The set of all admissible controls will be denoted by $\mathcal{U}$:

$$\mathcal{U} = \Big\{\mathbf{u} : [0, +\infty[\to \mathbb{R}^m \text{ measurable, } \mathbf{u}(s) \in U \text{ a.e.}\Big\}.$$

Let $T > 0$ be a fixed finite horizon, and consider the dynamical system:

$$\begin{cases} \dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)) & s \geq 0, \\ \mathbf{y}(t) = x. \end{cases} \tag{3.6}$$

The dynamics f satisfies the following hypothesis:

$$\begin{cases} (i) & \text{the function } f \text{ is continuous} \\ (ii) & \text{for any } R > 0, \exists L_f(R) > 0 \text{ such that. for every } u \in U: \\ & |f(x,u) - f(y,u)| \leq L_f(R)(|x-y|),\ \forall x, y \in \mathbb{R}^n \text{ with } |x| \leq R, |y| \leq R \\ (iii) & \exists\ c_f \geq 0, \text{such that for any } x \in \mathbb{R}^n \text{ we have}: \\ & \max\{|f(x,u)| :\ u \in U\} \leq c_f(1 + |x|). \end{cases} \tag{3.7}$$

By assumption (3.7), for any control input $u \in \mathcal{U}$, the system (3.6) admits a unique absolutely continuous solution $\mathbf{y}^{\mathbf{u}}_{t,x}$ in $W^{1,1}([t,T];\mathbb{R}^n)$. For every $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, we define the set, $\mathbb{X}_{t,x} \subset W^{1,1}([t,T];\mathbb{R}^n) \times \mathcal{U}$ as:

$$\mathbb{X}_{t,x} = \{(\mathbf{y}, \mathbf{u}) :\ \dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)),\ \text{for a.e. } s \in [t,T]; \quad \mathbf{y}(t) = x \text{ and } \mathbf{u} \in \mathcal{U}\}.$$

Let us introduce the running cost $\ell : \mathbb{R}^n \times U \to \mathbb{R}$ satisfying

$$\begin{cases} (i) & \text{the function } \ell \text{ is continuous} \\ (ii) & \text{for any } R > 0, \exists L_\ell(R) > 0 \text{ such that for every } u \in U: \\ & |\ell(x,u) - \ell(y,u)| \leq L_\ell(R)(|x-y|),\ \forall x, y \in \mathbb{R}^n \text{ with } |x| \leq R, |y| \leq R \\ (iii) & \exists\ c_\ell > 0 \text{ such that for any } x \in \mathbb{R}^n \text{ we have}: \\ & \max\{|\ell(x,u)|, u \in U\} \leq c_\ell(1 + |x|^{\lambda_\ell}), \end{cases} \tag{3.8}$$

and the final cost function $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\begin{cases} (i) & \text{For any } R > 0, \exists L_\varphi(R) > 0 \text{ such that }: \\ & |\varphi(x) - \varphi(y)| \leq L_\varphi(R)|x-y|\ \forall x, y \in \mathbb{R}^n \text{ with } |x| \leq R, |y| \leq R \\ (ii) & \exists\ c_\varphi > 0 \text{ such that for any } x \in \mathbb{R}^n \text{ we have}: \\ & \varphi(x) \leq c_\varphi(x)(1 + |x|^{\lambda_\varphi}) \text{ for every } x \in \mathbb{R}^n. \end{cases} \tag{3.9}$$

For $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, the objective function in Bolza form $\Phi(t,x;\cdot,\cdot)$: $W^{1,1}([t,T];\mathbb{R}^n) \times \mathcal{U} \mapsto \mathbb{R}$ is defined as

$$\Phi(t,x;\cdot,\cdot) : W^{1,1}([t,T];\mathbb{R}^n) \times \mathcal{U} \longrightarrow \mathbb{R}, \quad \Phi(t,x;\mathbf{y},\mathbf{u}) = \varphi(\mathbf{y}(T)) + \int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s))ds.$$

We are also interested by a second cost function that is measured all along the trajectory by:

$$\Psi(t,x;\cdot) : W^{1,1}([t,T];\mathbb{R}^n) \longrightarrow \mathbb{R}, \quad \Psi(t,x;\mathbf{y}) = \max_{s\in[t,T]} \psi(\mathbf{y}(s)),$$

where the function ψ satisfies:

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ is locally Lipschitz continuous.} \tag{3.10}$$

The bi-objective optimal control problem that will be investigated in this chapter is the following:

$$\begin{cases} \inf(\Phi(t,x;\mathbf{y},\mathbf{u}), \Psi(t,x;\mathbf{y})) \\ \text{s.t } (\mathbf{y},\mathbf{u}) \in \mathbb{X}_{t,x}. \end{cases} \tag{MOP}$$

The results presented in this chapter can be extended to the multi-objective case. As we work with objective functions of different nature we consider a bi-objective problem to simplify the notation. The characterization of the (weak) Pareto front for the multi-objective case will be similar with the characterization for the multi-objective infinite horizon optimal control problem considered in Chapter 4.

In what follows we will first consider a reformulation of the problem, and then introduce a scalarized control problem suitable for characterizing the ε-Pareto solutions of (MOP).

Define the set-valued function

$$G(x) = \left\{ \begin{pmatrix} f(x,u) \\ -\ell(x,u) - a \end{pmatrix}, \ 0 \le a \le \mathcal{A}(x,u), \ u \in U \right\},$$

where $\mathcal{A}(x,u) = c_\ell(1+|x|^{\lambda_\ell}) - \ell(x,u)$. Under assumptions (3.7) and (3.8) the function G is locally Lipschitz continuous in the sense that, for any $R > 0$, there exists $L_G(R) > 0$ such that:

$$G(x') \subset G(x) + |x-y| B(0, L_R) \qquad \forall x, x' \in \mathbb{R}^n \text{ with } |x| \le R, |x'| \le R.$$

We also define the following set of trajectories:

$$\mathcal{S}_{[t,T]}(x,0) = \{(\mathbf{y},\mathbf{z}) : \ (\dot{\mathbf{y}}(s), \dot{\mathbf{z}}(s))^\intercal \in G(\mathbf{y}(s)), \text{ for a.e. } s \in [t,T]; \quad (\mathbf{y}(t),\mathbf{z}(t)) = (x,0)\},$$

and the bi-objective optimal control problem:

$$\begin{cases} \inf \Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s)) \Big) \\ \text{s.t } (\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0). \end{cases} \tag{3.11}$$

The introduction of the dynamics G is a classical tool that is usually introduced to recast a cost in Bolza form into a cost in Mayer form. In this reformulation, the vector of state variables is increased by one more component.

Remark 3.7 *Let us stress on that the problem* (3.11) *is equivalent to problem* (MOP) *in the sense that every (weak) Pareto value of* (MOP) *(if it exists) corresponds to a (weak) Pareto value of* (3.11)*, and the reverse is true: every (weak) Pareto value of* (3.11) *corresponds to a (weak) Pareto solution of* (MOP)*.*

In fact, from the definition of the set of trajectories $\mathcal{S}_{[t,T]}(x,0)$*, every trajectory* $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0)$ *is associated to a control function* $\mathbf{u} \in \mathcal{U}$ *and* $\gamma \in \Gamma(\mathbf{y}(s), \mathbf{u}(s))$*, where*

$$\Gamma(x,u) = \{\gamma : [0,+\infty[\to \mathbb{R} \textit{ measurable}, \ \gamma(s) \in [0, \mathcal{A}(x,u)] \textit{ for a.e. } s\}.$$

Let $(\mathbf{y}^*, \mathbf{z}^*)$ *a Pareto optimal solution of* (3.11) *and* $\mathbf{u}^*$, γ^* *the respective controls. Then there exist no* $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0)$ *such that*

$$\Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s)) \Big) \le \Big(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s)) \Big),$$

$$\textit{and } \Big[\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) \quad \textit{or} \quad \max_{s\in[t,T]} \psi(\mathbf{y}(s)) < \max_{s\in[t,T]} \psi(\mathbf{y}^*(s)) \Big] \tag{3.12}$$

Note that

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) = \varphi(\mathbf{y}(T)) + \int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s)) ds + \int_t^T \gamma(s) ds.$$

As $0 \leq \gamma(s) \leq \mathcal{A}(\mathbf{y}(s), \mathbf{u}(s))$, for a.e. $s \geq 0$, and the second objective function of (3.11) *does not depend on control γ we obtain that for all $(\mathbf{y}^*, \mathbf{z}^*)$ that is a Pareto optimal solution of* (3.11) *the control function $\gamma^* \equiv 0$. Then by* (3.12)*, we obtain that $(\mathbf{y}^*, \mathbf{u}^*)$ is a Pareto optimal solution for problem* (MOP)*. With similar arguments it is possible to prove that if $(\mathbf{y}^*, \mathbf{z}^*)$ is a weak Pareto optimal solution of* (3.11) *with respective controls $\mathbf{u}^*$, and γ^* then $(\mathbf{y}^*, \mathbf{u}^*)$ is a weak Pareto optimal solution for problem* (MOP)*.*

Now consider $(\mathbf{y}^*, \mathbf{u}^*)$ a Pareto optimal solution of (MOP)*. Then there exist no $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t,T]}(x, 0)$ such that*

$$\begin{gathered}(\Phi(t, x; \mathbf{y}, \mathbf{u}), \Psi(t, x; \mathbf{y})) \leq (\Phi(t, x; \mathbf{y}^*, \mathbf{u}^*), \Psi(t, x; \mathbf{y}^*)) \\ \text{and } [\Phi(t, x; \mathbf{y}, \mathbf{u}) < \Phi(t, x; \mathbf{y}^*, \mathbf{u}^*) \quad \text{or} \quad \Psi(t, x; \mathbf{y}) < \Psi(t, x; \mathbf{y}^*)]\,.\end{gathered}$$

Define $\gamma^* \equiv 0$, then by definition of function Φ we obtain that there exist no $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t,T]}(x, 0)$ such that

$$\Phi(t, x; \mathbf{y}, \mathbf{u}) \leq \Phi(t, x; \mathbf{y}^*, \mathbf{u}^*) = \varphi(\mathbf{y}(T)) + \int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s)) + \gamma^*(s) ds = \varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T).$$

Therefore $(\mathbf{y}^*, \mathbf{z}^*)$ is a Pareto solution of (3.11)*, with associated controls $\mathbf{u}^*$ and γ^*. With similar arguments it is possible to prove that if $(\mathbf{y}^*, \mathbf{u}^*)$ is a weak Pareto optimal solution of* (MOP) *then $(\mathbf{y}^*, \mathbf{z}^*)$ is a weak Pareto optimal solution for problem* (3.11) *with respective controls $\mathbf{u}^*$, and $\gamma^* \equiv 0$.*

Moreover as $\gamma^* \equiv 0$ the (weak) Pareto front for problems (MOP) *and* (3.11) *coincides. Besides, for every $\varepsilon > 0$, problems* (MOP) *and* (3.11) *have the same (weak) ε-Pareto fronts.*

Without any additional assumption, the set of trajectories $\mathcal{S}_{[t,T]}(x, 0)$ is not necessarily closed, and the problem (3.11) might not have a solution. In this case, the (weak) Pareto fronts might be empty sets. One approach to obtain the closure of the set of trajectories $\mathcal{S}_{[t,T]}(x, z)$, is to introduce a convexified (relaxed) dynamical system [AC84, FR99], whose set of solutions is given by:

$$\mathcal{S}^{\#}_{[t,T]}(x, z) = \{(\mathbf{y}, \mathbf{z}) : (\dot{\mathbf{y}}(s), \dot{\mathbf{z}}(s))^\intercal \in \overline{co}\Big(G(\mathbf{y}(s))\Big), \text{ for a.e. } s \in [t, T]; (\mathbf{y}(t), \mathbf{z}(t)) = (x, z)\},$$

where for every subset $S \subset \mathbb{R}^n$, $\overline{co}(S)$ denotes the closed convex hull of S, that is the minimal convex set that contains S.

Under assumptions (3.7) and (3.8), following the same arguments of the proof of Filippov-Wazewski Theorem (see for instance [AC84], Theorem 10.4.3), the closure of $\mathcal{S}_{[t,T]}(x, z)$ in the space of continuous functions $C(t, T)$ is compact and equal to the set of solutions $\mathcal{S}^{\#}_{[t,T]}(x, z)$. So we introduce the following relaxed bi-objective optimal control problem

$$\begin{cases} \min \Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max\limits_{s \in [t,T]} \psi(\mathbf{y}(s))\Big) \\ \text{s.t } (\mathbf{y}, \mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x, 0). \end{cases} \tag{MORP}$$

For a fixed $(t, x) \in [0, T] \times \mathbb{R}^n$, we consider $\mathcal{P}^{i,\varepsilon}(t, x)$ and $\mathcal{P}^{i,\varepsilon}_w(t, x)$, $i = 1, 2, 3$, the sets of (weak) Pareto ε-solutions of problem (3.11) (according to the definitions 3.1 and 3.4). Besides, we denote by $\mathcal{P}^{\#}(t, x)$ and $\mathcal{P}^{\#}_w(t, x)$ the Pareto and the weak Pareto sets of problem (MORP), respectively. Moreover the Pareto front and the weak Pareto front of (MORP) by $\mathcal{F}^{\#}(t,x)$ and $\mathcal{F}^{\#}_w(t,x)$, respectively. The next proposition states the link between the optimal Pareto solution of the relaxed control problem (MORP) and the ε-Pareto solutions of the original problem (3.11).

Theorem 3.8 *Assume that* (3.7),(3.8),(3.9) *and* (3.10) *hold and let* $(t,x) \in [0,T] \times \mathbb{R}^n$.

(i) For any $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{\#}(t,x)$ *and for any* $\varepsilon > 0$, *define* $R = \|(\mathbf{y}^*, \mathbf{z}^*)\|_{L^\infty([0,T])} + \varepsilon$. *Then for* $\delta = \min(\varepsilon/(L_\varphi(R)+1), \varepsilon/(L_\psi(R)+1))$ *there exists* $(\mathbf{y},\mathbf{z}) \in \mathcal{P}^{1,\varepsilon}(t,x)$ *such that* $|(\mathbf{y}^*, \mathbf{z}^*) - (\mathbf{y},\mathbf{z})| \le \delta$ *and*

$$\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right)\right| \le \varepsilon.$$

(ii) For any $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{\#}_w(t,x)$ *and for any* $\varepsilon > 0$ *define* $R = \|(\mathbf{y}^*, \mathbf{z}^*)\|_{L^\infty([0,T])} + \varepsilon$, *then for* $\delta = \min(\varepsilon/(L_\varphi(R)+1), \varepsilon/(L_\psi(R)+1))$ *there exists* $(\mathbf{y},\mathbf{z}) \in \mathcal{P}^{1,\varepsilon}_w(t,x)$ *such that* $|(\mathbf{y}^*, \mathbf{z}^*) - (\mathbf{y},\mathbf{z})| \le \delta$ *and*

$$\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right)\right| \le \varepsilon.$$

(iii) Given $\varepsilon > 0$, *if* $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{3,\varepsilon}(t,x)$, *then there exists* $(\mathbf{y},\mathbf{z}) \in \mathcal{P}^{\#}(t,x)$ *such that*

$$\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right)\right| \le 2\varepsilon.$$

PROOF. (**i**) Let $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{\#}(t,x)$ and $\varepsilon > 0$. As $(\mathbf{y}^*, \mathbf{z}^*)$ is in the closure of $\mathcal{S}_{[t,T]}(x,0)$, there exists a sequence $\{(\mathbf{y}_n, \mathbf{z}_n)\} \subset \mathcal{S}_{[t,T]}(x,z)$ such that

$$\lim_{n\to\infty} (\mathbf{y}_n, \mathbf{z}_n) = (\mathbf{y}^*, \mathbf{z}^*).$$

Then, given $\varepsilon > 0$, for $R = \|(\mathbf{y}^*, \mathbf{z}^*)\|_{L^\infty([0,T])} + \varepsilon$ define $\delta = \min(\varepsilon/(L_\varphi(R)+1), \varepsilon/L_\psi(R))$, so there exists $(\mathbf{y}_N, \mathbf{z}_N)$ such that $|(\mathbf{y}^*, \mathbf{z}^*) - (\mathbf{y}_N, \mathbf{z}_N)| \le \delta$. Moreover, we have that $|(\mathbf{y}^*, \mathbf{z}^*)| \le R$ and $|(\mathbf{y}_N, \mathbf{z}_N)| \le R$. By the locally Lipschitz continuity of function φ and ψ, we obtain that

$$|\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) - \varphi(\mathbf{y}_N(T)) + \mathbf{z}_N(T)| \le (L_\varphi(R)+1)\delta \le \varepsilon,$$

$$\left|\max_{s\in[t,T]} \psi(\mathbf{y}^*(s)) - \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right| \le \max_{s\in[t,T]} |\psi(\mathbf{y}^*(s)) - \psi(\mathbf{y}_N(s))| \le L_\psi(R)\delta \le \varepsilon,$$

which means that

$$\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right)\right| \le \varepsilon.$$

It remains to prove that $(\mathbf{y}_N, \mathbf{z}_N) \in \mathcal{P}^{1,\varepsilon}(t,x)$. Note that, by definition of $(\mathbf{y}^*, \mathbf{z}^*)$ there is no $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0) \subset \mathcal{S}^{\#}_{[t,T]}(x,0)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right) \le \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right),$$

$$\text{and} \quad \left[\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) \quad \text{or} \quad \max_{s\in[t,T]} \psi(\mathbf{y}(s)) < \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right].$$

This means that for any $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0)$ at least for one of the following assertion holds:

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) > \varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) \geq \varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T) - \varepsilon, \text{ or}$$
$$\max_{s\in[t,T]} \psi(\mathbf{y}(s)) > \max_{s\in[t,T]} \psi(\mathbf{y}^*(s)) \geq \max_{s\in[t,T]} \psi(\mathbf{y}_N(s)) - \varepsilon.$$

Therefore, there is no $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right) \leq \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right) - (\varepsilon,\varepsilon),$$

$$\text{and } \left[\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T) - \varepsilon \text{ or } \max_{s\in[t,T]} \psi(\mathbf{y}(s)) < \max_{s\in[t,T]} \psi(\mathbf{y}_N(s)) - \varepsilon\right],$$

which means that $(\mathbf{y}_N,\mathbf{z}_N) \in \mathcal{P}^{1,\varepsilon}(t,x)$ and the assertion is now proved for $(\mathbf{y},\mathbf{z}) = (\mathbf{y}_N,\mathbf{z}_N)$.

(ii) Let $(\mathbf{y}^*,\mathbf{z}^*) \in \mathcal{P}^{\#}_w(t,x)$ and $\varepsilon > 0$. Given $\varepsilon > 0$, for $R = \|(\mathbf{y}^*,\mathbf{z}^*)\|_{L^\infty([0,T])} + \varepsilon$ define $\delta = \max\left(\varepsilon/(L_\varphi(R)+1), \varepsilon/(L_\psi(R)+1)\right)$. With similar arguments as in item (i) we obtain $(\mathbf{y}_N,\mathbf{z}_N) \in \mathcal{S}_{[t,T]}(x,0)$ such that $|(\mathbf{y}^*,\mathbf{z}^*) - (\mathbf{y}_N,\mathbf{z}_N)| \leq \delta$ and

$$\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right)\right| \leq \varepsilon.$$

It remains to prove that $(\mathbf{y}_N,\mathbf{z}_N) \in \mathcal{P}^{1,\varepsilon}_w(t,x)$. Note that, by definition of $(\mathbf{y}^*,\mathbf{z}^*)$ there is no $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0) \subset \mathcal{S}^{\#}_{[t,T]}(x,0)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right) < \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right).$$

This implies that for any $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0)$ at least for one of the following assertion holds:

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) \geq \varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) \geq \varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T) - \varepsilon, \text{ or}$$
$$\max_{s\in[t,T]} \psi(\mathbf{y}(s)) \geq \max_{s\in[t,T]} \psi(\mathbf{y}^*(s)) \geq \max_{s\in[t,T]} \psi(\mathbf{y}_N(s)) - \varepsilon.$$

Therefore, there is no $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right) < \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right) - (\varepsilon,\varepsilon),$$

which means that $(\mathbf{y}_N,\mathbf{z}_N) \in \mathcal{P}^{1,\varepsilon}_w(t,x)$ and the assertion is now proved for $(\mathbf{y},\mathbf{z}) = (\mathbf{y}_N,\mathbf{z}_N)$.

(iii) Let $(\mathbf{y}^*,\mathbf{z}^*) \in \mathcal{P}^{3,\varepsilon}(t,x)$. Assume that there is no $(\mathbf{y},\mathbf{z}) \in \mathcal{P}^{\#}(t,x)$ such that

$$\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right)\right| \leq 2\varepsilon.$$

Then $(\mathbf{y}^*,\mathbf{z}^*) \notin \mathcal{P}^{\#}(t,x)$, so there exists $(\mathbf{y},\mathbf{z}) \in \mathcal{P}^{\#}(t,x)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right) \neq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right)$$

$$\text{and } \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right) \leq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right),$$

As $(\mathbf{y}, \mathbf{z})$ is in the closure of $\mathcal{S}_{[t,T]}(x, 0)$, there exists a sequence $\{(\mathbf{y}_n, \mathbf{z}_n)\} \subset \mathcal{S}_{[t,T]}(x, 0)$ such that $\lim_{n\to\infty}(\mathbf{y}_n, \mathbf{z}_n) = (\mathbf{y}, \mathbf{z})$. Choose $(\mathbf{y}_N, \mathbf{z}_N)$ such that

$$\left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right) \leq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right),$$

$$\text{and} \left|\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right)\right| \leq \varepsilon.$$

As $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{3,\varepsilon}(t, x)$, we must have

$$\left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right) \geq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - (\varepsilon, \varepsilon).$$

Hence

$$\begin{aligned}\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - (\varepsilon, \varepsilon) &\leq \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right) \\ &\leq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right),\end{aligned}$$

which means that

$$\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right)\right| \leq \varepsilon.$$

Then

$$\begin{aligned}&\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right)\right| \\ \leq &\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right)\right| \\ &+ \left|\left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s\in[t,T]} \psi(\mathbf{y}_N(s))\right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right)\right| \\ \leq &2\varepsilon.\end{aligned}$$

what is a contradiction. □

3.3 Auxiliary control problem

In this section we would like to characterize the (weak) Pareto fronts of the relaxed control problem (MORP). These fronts allow also to obtain the (weak) ε-Pareto fronts of the original problem (3.11) (see Theorem 3.8).

To characterize a Pareto front, the general idea consists of considering a family of scalarized optimal control problems whose optimal values correspond to Pareto values of the bi-objective problem. A predominant method for scalarization is based on the weighted sum problem where the cost function would take the following form: for $\alpha \in [0, 1]$ solve the control problem:

$$\min \left\{\alpha(\varphi(\mathbf{y}(T)) - \mathbf{z}(T)) + (1 - \alpha) \max_{s\in[t,T]} \psi(\mathbf{y}(s)) \mid (\mathbf{y}, \mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x, 0)\right\}.$$

As mentioned earlier, it is known that this weighted sum scalarization can characterize only a part of the Parto front and not the entire front. Another idea would be to consider a control problem where one of the cost functions is chosen as primary cost and the other one is transformed into an auxiliary variable subject to a state constraint:

$$\min\left\{\max_{s\in[t,T]}\psi(\mathbf{y}(s)) \mid (\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0) \quad \text{and } \varphi(\mathbf{y}(T))-\mathbf{z}(T))\leq z_1\right\}, \tag{3.13}$$

or

$$\min\left\{\varphi(\mathbf{y}(T))-\mathbf{z}(T)) \mid (\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0) \quad \text{and } \max_{s\in[t,T]}\psi(\mathbf{y}(s))\leq z_2\right\}. \tag{3.14}$$

Following the same ideas as in [KV10], one can show that all the Pareto values correspond to the optimal values of problem (3.13), when z_1 runs through $\mathbb{R}$. The same characterization holds if we use (3.14) and let z_2 runs through $\mathbb{R}$. However, it should be noticed that problems (3.13) and (3.14) are in presence of state constraints which make these problems difficult to analyse and to solve, see [Son86, HZ15] and the references therein.

In this section, we will follow some ideas introduced in [DZ18], and consider an auxiliary control problem and its value function $w:[0,T]\times\mathbb{R}^n\times\mathbb{R}^2\to\mathbb{R}$ defined as:

$$w(t,x,z_1,z_2)=\min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)}\left[\Big(\varphi(\mathbf{y}(T))-\mathbf{z}(T)-z_1\Big)\bigvee\max_{s\in[t,T]}(\psi(\mathbf{y}(s))-z_2)\right], \tag{3.15}$$

where the notation $a\bigvee b$ stands for $\max(a,b)$. Let us point out that the additional state components are very important to get a Dynamic Programming Principle for the value function w. Moreover, we note that under assumptions (3.7),(3.8),(3.9) and (3.10), there exists an admissible pair $(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)$ that minimizes the auxiliary control problem (3.15).

Let us remark that from the definition of $\mathcal{S}^{\#}_{[t,T]}(x,z)$ and from the definition of w, it follows that:

$$w(t,x,z_1,z_2)=\min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,z_1)}\left[\Big(\varphi(\mathbf{y}(T))-\mathbf{z}(T)\Big)\bigvee\max_{s\in[t,T]}(\psi(\mathbf{y}(s))-z_2)\right]. \tag{3.16}$$

The auxiliary control problem (3.15) satisfies the following Dynamic Programming Principle.

Proposition 3.9 (Dynamic Programming Principle) *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold. Then for all $h\geq 0$, such that $t+h<T$ and $(x,z)\in\mathbb{R}^n\times\mathbb{R}^2$, we have*

$$w(t,x,z_1,z_2) \;=\; \min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,t+h]}(x,z_1)}\left\{w\left(t+h,\mathbf{y}(t+h),\mathbf{z}(t+h),z_2\right)\bigvee\max_{s\in[t,t+h]}(\psi(\mathbf{y}(s))-z_2)\right\}.$$

PROOF. For any measurable control $\mathbf{u}\in\mathcal{U}$, we shall denote $\mathbf{u}_1$ the restriction of $\mathbf{u}$ on $[t,t+h]$ and $\mathbf{u}_2$ the restriction of $\mathbf{u}$ on $[t+h,T]$. Note that, for $t+h\leq s\leq T$ we have $\mathbf{y}^{\mathbf{u}}_{t,x}(s)=\mathbf{y}^{\mathbf{u}_2}_{t+h,\mathbf{y}^{\mathbf{u}_1}_{t,x}(t+h)}(s)$ and denote $\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}^{\mathbf{u}_1}_{t,x}}(s)=\mathbf{y}^{\mathbf{u}_2}_{t+h,\mathbf{y}^{\mathbf{u}_1}_{t,x}(t+h)}(s)$. So we obtain on the first hand:

$$\begin{aligned}\varphi(\mathbf{y}^{\mathbf{u}}_{t,x}(T))-\mathbf{z}^{\mathbf{u}}_{t,z_1}(T)&=\varphi(\mathbf{y}^{\mathbf{u}}_{t,x}(T))+\int_t^T\ell(\mathbf{y}^{\mathbf{u}}_{t,x}(s),\mathbf{u}(s))ds-z_1\\&=\varphi(\mathbf{y}^{\mathbf{u}}_{t,x}(T))+\int_{t+h}^T\ell(\mathbf{y}^{\mathbf{u}}_{t,x}(s),\mathbf{u}(s))ds+\int_t^{t+h}\ell(\mathbf{y}^{\mathbf{u}}_{t,x}(s),\mathbf{u}(s))ds-z_1\\&=\varphi(\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}^{\mathbf{u}_1}_{t,x}}(T))+\int_{t+h}^T\ell(\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}^{\mathbf{u}_1}_{t,x}}(s),\mathbf{u}_2(s))ds-\mathbf{z}^{\mathbf{u}_1}_{t,z_1}(t+h),\end{aligned} \tag{3.17}$$

and on the other hand

$$\max_{s\in[t,T]}\left(\psi\left(\mathbf{y}^{\mathbf{u}}_{t,x}(s)\right)-z_2\right)=\max_{s\in[t,t+h]}\left(\psi\left(\mathbf{y}^{\mathbf{u}_1}_{t,x}(s)\right)-z_2\right)\bigvee\max_{s\in[t+h,T]}\left(\psi\left(\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}^{\mathbf{u}_1}_{t,x}}(s)\right)-z_2\right). \quad (3.18)$$

Combining (3.17) and (3.18), taking measurable controls $\mathbf{u}_1 : [t,t+h]\to U$ and $\mathbf{u}_2 : [t+h,T]\to U$ we obtain

$$\begin{aligned}
w(t,x,z_1,z_2) &= \min_{\mathbf{u}_1}\min_{\mathbf{u}_2}\left(\varphi(\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}^{\mathbf{u}_1}_{t,x}}(T))+\int_{t+h}^{T}\ell(\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}^{\mathbf{u}_1}_{t,x}}(s),\mathbf{u}_2(s))ds-\mathbf{z}^{\mathbf{u}_1}_{t,z_1}(t+h)\right)\\
&\qquad\bigvee\max_{s\in[t+h,T]}\left(\psi\left(\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}^{\mathbf{u}_1}_{t,x}}(s)\right)-z_2\right)\bigvee\max_{s\in[t,t+h]}\left(\psi\left(\mathbf{y}^{\mathbf{u}_1}_{t,x}(s)\right)-z_2\right)\\
&=\min_{\mathbf{u}_1}w\left(t+h,\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}^{\mathbf{u}_1}_{t,x}}(t+h),\mathbf{z}^{\mathbf{u}_1}_{t,z_1}(t+h),z_2\right)\bigvee\max_{s\in[t,t+h]}\left(\psi\left(\mathbf{y}^{\mathbf{u}_1}_{t,x}(s)\right)-z_2\right)\\
&=\min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,t+h]}(x,z_1)}\left\{w\left(t+h,\mathbf{y}(t+h),\mathbf{z}(t+h),z_2\right)\bigvee\max_{s\in[t,t+h]}(\psi(\mathbf{y}(s))-z_2)\right\},
\end{aligned}$$

which is the desired result. □

Following [DZ18], the value function w satisfies the following property.

Proposition 3.10 *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold. The value function w is locally Lipschitz continuous on* $[0,T]\times\mathbb{R}^n\times\mathbb{R}^2$*.*

PROOF. Consider $R>0$, $t\in[0,T]$ and (x,z), $(x',z')\in\mathbb{R}^n\times\mathbb{R}^2$ such that

$$|\mathbf{y}^{\mathbf{u}}_{t,x}(s)|<R,\ |\mathbf{y}^{\mathbf{u}}_{t,x}(s)|<R,\ \text{for } s\in[t,T].$$

By using the definition of w and the simple inequalities:

$$\min(A)-\min(B)\le\max(A-B)\quad\text{and}\quad\max(A,B)-\max(C,D)\le\max(A-C,B-D),$$

we get

$$\begin{aligned}
|w(t,x,z_1,z_2)-w(t,x',z_1',z_2')| &= \Bigg|\min_{\mathbf{u}\in\mathcal{U}}\left[\left(\varphi(\mathbf{y}^{\mathbf{u}}_{t,x}(T))-\mathbf{z}^{\mathbf{u}}_{t,z_1}(T)\right)\bigvee\max_{s\in[t,T]}(\psi(\mathbf{y}^{\mathbf{u}}_{t,x}(s))-z_2)\right]\\
&\qquad-\min_{\mathbf{u}\in\mathcal{U}}\left[\left(\varphi(\mathbf{y}^{\mathbf{u}}_{t,x'}(T))-\mathbf{z}^{\mathbf{u}}_{t,z_1'}(T)\right)\bigvee\max_{s\in[t,T]}(\psi(\mathbf{y}^{\mathbf{u}}_{t,x'}(s))-z_2')\right]\Bigg|\\
&\le\Bigg|\max_{\mathbf{u}\in\mathcal{U}}\Big[\left(\varphi(\mathbf{y}^{\mathbf{u}}_{t,x}(T))-\mathbf{z}^{\mathbf{u}}_{t,z_1}(T)\right)\bigvee\max_{s\in[t,T]}(\psi(\mathbf{y}^{\mathbf{u}}_{t,x}(s))-z_2)\\
&\qquad-\left(\varphi(\mathbf{y}^{\mathbf{u}}_{t,x'}(T))-\mathbf{z}^{\mathbf{u}}_{t,z_1'}(T)\right)\bigvee\max_{s\in[t,T]}(\psi(\mathbf{y}^{\mathbf{u}}_{t,x'}(s))-z_2')\Big]\Bigg|\\
&\le\Bigg|\max_{\mathbf{u}\in\mathcal{U}}\Big[\left(\varphi(\mathbf{y}^{\mathbf{u}}_{t,x}(T))-\mathbf{z}^{\mathbf{u}}_{t,z_1}(T)-\varphi(\mathbf{y}^{\mathbf{u}}_{t,x'}(T))-\mathbf{z}^{\mathbf{u}}_{t,z_1'}(T)\right)\\
&\qquad\bigvee\max_{s\in[t,T]}(\psi(\mathbf{y}^{\mathbf{u}}_{t,x}(s))-\psi(\mathbf{y}^{\mathbf{u}}_{t,x'}(s))-z_2+z_2')\Big]\Bigg|\\
&\le\max_{\mathbf{u}\in\mathcal{U}}\Bigg(\int_t^T\left|\,\ell(\mathbf{y}^{\mathbf{u}}_{t,x}(s),\mathbf{u}(s))-\ell(\mathbf{y}^{\mathbf{u}}_{t,x'}(s),\mathbf{u}(s))\,\right|ds+|z_1'-z_1|\\
&\qquad|\varphi(\mathbf{y}^{\mathbf{u}}_{t,x}(T))-\varphi(\mathbf{y}^{\mathbf{u}}_{t,x'}(T))\bigvee\max_{s\in[t,T]}\left|\,(\psi(\mathbf{y}^{\mathbf{u}}_{t,x}(s))-\psi(\mathbf{y}^{\mathbf{u}}_{t,x'}(s))-z_2+z_2')\,\right|\\
&\le\max_{\mathbf{u}\in\mathcal{U}}\Bigg(L_\ell(R)\int_t^T|\mathbf{y}^{\mathbf{u}}_{t,x}(s)-\mathbf{y}^{\mathbf{u}}_{t,x'}(s)|ds+L_\varphi(R)|\mathbf{y}^{\mathbf{u}}_{t,x}(T)-\mathbf{y}^{\mathbf{u}}_{t,x'}(T)|+|z_1-z_1'|\\
&\qquad\bigvee L_\psi(R)\max_{s\in[t,T]}|\mathbf{y}^{\mathbf{u}}_{t,x}(s)-\mathbf{y}^{\mathbf{u}}_{t,x'}(s)|+|z_2-z_2'|\Bigg), \qquad (3.19)
\end{aligned}$$

where $L_\ell(R),\ L_\varphi(R),\ L_\psi(R)$ are the locally Lipschitz constant for $\ell,\ \varphi,\ \psi$, respectively. By Proposition 2.2(c) for $s \in [t,T]$,

$$\left| \mathbf{y}^{\mathbf{u}}_{t,x}(s) - \mathbf{y}^{\mathbf{u}}_{t,x'}(s) \right| \le e^{L_f(R)(s-t)}|x-x'| \le e^{L_f(R)T}|x-x'|,$$

where $L_f(R)$ is the locally Lipschitz constant of f. Then we conclude that:

$$\begin{aligned}
&|w(t,x,z) - w(t,x',z')| \\
&\le \max\Big((L_\ell(R)T + L_\varphi(R))e^{L_f(R)T}|x-x'| + |z_1 - z_1'|, L_\psi(R)e^{L_f(R)T}|x-x'| + |z_2 - z_2'| \Big) \\
&\le C\left(|x-x'| + |z_1 - z_1'| + |z_2 - z_2'|\right), \qquad (3.20)
\end{aligned}$$

where $C = \max\big((L_\ell(R)T + L_\varphi(R)e^{L_f(R)T}, L_\psi(R)e^{L_f(R)T}, 1\big)$. On the other hand, for $h > 0$ such that $t + h < T$, using that $w(t+h,x,z_1,z_2) \ge \psi(\mathbf{y}(t+h)) - z_2$, we deduce from Proposition 3.9 that

$$\begin{aligned}
&|w(t,x,z_1,z_2) - w(t+h,x,z_1.z_2)| \\
&= \Bigg| \min_{(\mathbf{y},\mathbf{z})\in S^{\#}_{[t,t+h]}(x,z_1)} \max\Big(w(t+h,\mathbf{y}(t+h),\mathbf{z}(t+h),z_2), \max_{s\in[t,t+h]}(\psi(\mathbf{y}(s)) - z_2) \Big) \\
&\qquad - \max(w(t+h,x,z_1,z_2), \psi(\mathbf{y}(t+h)) - z_2)| \\
&\le \max_{(\mathbf{y},\mathbf{z})\in S^{\#}_{[t,T]}(x,z_1)} \max\Big(C\left(|\mathbf{y}(t+h) - x| + |\mathbf{z}(t+h) - z_1|\right), L_\psi(R) \max_{s\in[t,t+h]} |\mathbf{y}(s) - \mathbf{y}(t+h)| \Big)
\end{aligned}$$

where we have used (3.20). Now using Proposition 2.2 (a) we obtain that

$$\begin{aligned}
|w(t,x,z_1,z_2) - w(t+h,x,z_1,z_2)| &\le \max\left(C\left(c_f(1+|x|) + c_\ell(1+|z_1|)\right)h, L_\psi(R)c_f(1+|x|)h\right) \\
&= \max\left(C\left(c_f(1+|x|) + c_\ell(1+|z_1|)\right), L_\psi(R)c_f(1+|x|)\right)h. \qquad (3.21)
\end{aligned}$$

Combining the inequalities (3.20) and (3.21), we get that exist some constant $\tilde{C} > 0$ such that, for $t,t' \in [0,T]$ and $(x,z),\ (x',z') \in \mathbb{R}^{n+2}$ such that $|\mathbf{y}^{\mathbf{u}}_{t,x}(s)| < R$, $|\mathbf{y}^{\mathbf{u}}_{t,x}(s)| < R$, for $s \in [t,T]$,

$$|w(t,x,z) - w(t',x',z')| \le \tilde{C}\left(|t-t'| + |x-x'| + |z-z'|\right),$$

that means that the value function w is locally Lipschitz continuous. □

Now we are going to characterize the value function w by a HJB equation. To prove the uniqueness of solution of the HJB equation, a comparison principle is necessary. Consider the HJ equation in presence of the obstacle term:

$$\begin{aligned}
&\min(-\partial_t v + H(x,Dv), v - g(x)) = 0 \quad \text{on } [0,T]\times\mathbb{R}^d, \\
&v(T,x) = v_0(x), \quad x \in \mathbb{R}^d, \qquad (3.22)
\end{aligned}$$

where $T > 0$, $g : \mathbb{R}^d \to \mathbb{R}$ and $H : \times\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are continuous and assume that H satisfies:

- there exists $c \ge 0$ such that, for all $x,\ p,\ q$ in $\mathbb{R}^d$,

$$|H(x,p) - H(x,q)| \le C(|x|+1)|p-q|; \qquad (3.23)$$

- for any $R > 0$, there exists a function $\omega_R : [0,\infty[\to [0,\infty[$, $\lim_{r\mapsto 0^+}\omega_R(r) = 0$ and

$$|H(x,p) - H(y,p)| \le \omega_R((1+|p|)|x-y|) \qquad (3.24)$$

for every $p \in \mathbb{R}^d$, $x,\ y \in B(0,R)$, where $B(0,R)$ denotes the open ball centered at 0 and of radius R.

Theorem 3.11 *[ABZ13, Theorem A.1.] Assume that* (3.23) *and* (3.24) *are satisfied. Let* $v,\ \overline{v}$ *be two continuous functions on* $[0,T]\times\mathbb{R}^d$*, and let* $g,\ h$ *be continuous on* $[0,T]\times\mathbb{R}^d$*. We assume that* v *(resp.* $\overline{v}$*) is a sub-solution (resp. super-solution) of* (3.22) *in* $[0,T]\times\mathbb{R}^d$*:*

$$\min(-\partial_t v + H(x,Dv), v-g) \le 0 \quad \text{in } [0,T]\times\mathbb{R}^d$$

$$\min(-\partial_t \overline{v} + H(x,D\overline{v}), \overline{v}-h) \ge 0 \quad \text{in } [0,T]\times\mathbb{R}^d$$

We denote $v_T(x) := v(T,x)$ *and* $\overline{v}_T(x) := \overline{v}(T,x)$*. Then for all* $t\in[0,T]$*,*

$$\max_{\mathbb{R}^d}(v(t,\cdot)-\overline{v}(t,\cdot)) \le \max\left(\max_{\mathbb{R}^d}(v_T-\overline{v}_T),\ \max_{(t,T)\mathbb{R}^d}(g-h)\right).$$

Using Theorem 3.11, we can characterize the value function of the auxiliary control problem (3.15) as a unique solution of HJB equation below.

Theorem 3.12 *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold. The function* w *is the unique viscosity solution of the following HJB equation:*

$$\min\Big(\partial_t w(t,x,z) + \mathcal{H}^{\#}(x,z,D_x w, D_z w), w(t,x,z)-(\psi(x)-z_2)\Big) = 0 \quad \text{for } t\in[0,T),\ x\in\mathbb{R}^n,\ z\in\mathbb{R}^2_;$$
$$w(T,x,z) = \Big(\varphi(x)-z_1\Big)\bigvee\Big(\psi(x)-z_2\Big) \quad \text{for } x\in\mathbb{R}^n, z\in\mathbb{R}^2, \tag{3.25}$$

where the function $\mathcal{H}^{\#}$ *is defined by:*

$$\mathcal{H}^{\#}(x,z,p,q) = \max_{(v_x,v_{z_1})\in\overline{co}(G(x))}\Big(-v_x\cdot p - v_{z_1}\cdot q_1\Big), \quad \forall p\in\mathbb{R}^N, \forall q=(q_1,q_2)\in\mathbb{R}^2.$$

PROOF. We first show that w is a solution of (3.25). The fact that w satisfies the final condition comes directly from the definition of w. Let us check the super-solution property of w. From the Proposition 3.9, we get that for any $0\le\tau\le T$

$$w(t,x,z_1,z_2) \ge \min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,t+h]}(x,z_1)} w\left(t+\tau,\mathbf{y}(t+\tau),\mathbf{z}(t+\tau),z_2\right). \tag{3.26}$$

Now for a function $\chi\in C^1([0,T]\times\mathbb{R}^{n+2})$ assume that (t,x,z_1,z_2) is a local minimum point of $w-\chi$, that is, for some $r>0$

$$w(t,x,z_1,z_2) - w(s,x',z_1',z_2') \le \chi(t,x,z_1,z_2) - \chi(s,x',z_1',z_2') \tag{3.27}$$

if $|(x,z_1,z_2)-(x',z_1',z_2')| < r,\ |t-s|<r$. For $\varepsilon>0$ and $0<h\le T$, by the equation (3.26), there exits $(\overline{\mathbf{y}},\overline{\mathbf{z}})\in\mathcal{S}_{[t,t+h]}$ (depending on ε) such that

$$w(t,x,z_1,z_2) \ge w(t+h,\overline{\mathbf{y}}(t+h),\overline{\mathbf{z}}(t+h),z_2) - h\varepsilon. \tag{3.28}$$

Then

$$\chi(t,x,z_1,z_2) - \chi(t+h,\overline{\mathbf{y}}(t+h),\overline{\mathbf{z}}(t+h),z_2) = -\int_t^{t+h}\frac{d}{ds}\chi(s,\overline{\mathbf{y}}(s),\overline{\mathbf{z}}(s),z_2)\,ds$$
$$= -\int_t^{t+h}[\partial_t\chi(s,\overline{\mathbf{y}}(s),\overline{\mathbf{z}}(s)) + D_x\chi(s,\overline{\mathbf{y}}(s),\overline{\mathbf{z}}(s),z_2)\,v_x(s) + D_{z_1}\chi(s,\overline{\mathbf{y}}(s),\overline{\mathbf{z}}(s),z_2)\,v_{z_1}(s)]\,ds$$
$$= -\int_t^{t+h}[\partial_t\chi(t,x,z_1,z_2) + D_x\chi(t,x,z_1,z_2)v_x + D_{z_1}\chi(t,x,z_1,z_2)v_{z_1}]\,ds + o(h), \tag{3.29}$$

where $(v_x(s), v_{z_1}(s)) \in \overline{co}(G(\mathbf{y}_x(s))$ and $(v_x, v_{z_1}) \in \overline{co}(G(x))$.

Then (3.27) with $(s, x', z_1', z_2') = (t+h, \overline{\mathbf{y}}(t+h), \overline{\mathbf{z}}(t+h), z_2)$ and (3.29) give

$$\begin{aligned} &-\int_t^{t+h} [\partial_t \chi(t,x,z_1,z_2) + D_x\chi(t,x,z_1,z_2)\ v_x + D_{z_1}\chi(t,x,z_1,z_2) v_{z_1}] ds + o(h) \\ &\geq w(t,x,z_1,z_2) - w(t+h, \overline{\mathbf{y}}(t+h), \overline{\mathbf{z}}(t+h), z_2), \end{aligned}$$

and using (3.28) we obtain

$$\begin{aligned} &\int_t^{t+h} \{-\partial_t \chi(t,x,z_1,z_2) - D_x\chi(t,x,z_1,z_2) v_x + D_{z_1}\chi(t,x,z_1,z_2) v_{z_1}\} ds + o(h) \\ &\qquad \geq w(t+h, \overline{\mathbf{y}}(t+h), \overline{\mathbf{z}}(t+h), z_2) - w(t+h, \overline{\mathbf{y}}(t+h), \overline{\mathbf{z}}(t+h), z_2) - h\varepsilon = -h\varepsilon. \end{aligned} \tag{3.30}$$

The term in brackets in the integral is estimated from above by

$$-\partial_t \chi(t,x,z) + \max_{(v_x, v_{z_1}) \in \overline{co}(G(x))} \{-D_x\chi(t,x,z_1,z_2) v_x + D_{z_1}\chi(t,x,z_1,z_2) v_{z_1}\}$$

so we can divide (3.30) by $h > 0$ and pass to the limit to get

$$-\partial_t \chi(t,x,z_1,z_2) + \max_{(v_x, v_{z_1}) \in \overline{co}(G(x))} \{-D_x\chi(t,x,z_1,z_2) v_x + D_{z_1}\chi(t,x,z_1,z_2) v_{z_1}\} \geq -\varepsilon$$

Since ε is arbitrary, we obtain that

$$-\partial_t \chi(t,x,z) + \mathcal{H}^{\#}(x,z,D_x v, D_z v) \geq 0 \tag{3.31}$$

Moreover, by the definition of w, for every $(t,x,z_1,z_2) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^2$, we have

$$w(t,x,z_1,z_2) \geq \min_{(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,z_1)} \max_{s \in [t,T]} (\psi(\mathbf{y}(s)) - z_2) \geq \psi(x) - z_2. \tag{3.32}$$

Combining (3.31) and (3.32), we get

$$\min\left(\partial_t w(t,x,z) + \mathcal{H}^{\#}(x,z,D_x w, D_z w), w(t,x,z) - (\psi(x) - z_2)\right) \geq 0, \tag{3.33}$$

for all $t \in [0,T]$, $x \in \mathbb{R}^n$, and $z \in \mathbb{R}^2$, in the viscosity sense, that is, w is a super-solution of (3.25).

Now we will prove that w is a sub-solution. Let $(t,x,z_1,z_2) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^2$. If $w(t,x,z_1,z_2) \leq \psi(x) - z_2$ then it is clear that w satisfies

$$\min\left(\partial_t w(t,x,z) + \mathcal{H}^{\#}(x,z,D_x w, D_z w), w(t,x,z) - (\psi(x) - z_2)\right) \leq 0.$$

Now assume that $w(t,x,z_1,z_2) > \psi(x) - z_2$. By continuity of ψ and w and assumptions (3.7) and (3.8), there exits some $h > 0$ such that for every $(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,z_1)$, we have $|(\mathbf{y},\mathbf{z}) - (x,z_1)| < h$ and $w(\theta, \mathbf{y}(\theta), \mathbf{z}(\theta), z_2) > \psi(\mathbf{y}(\theta)) - z_2$ for all $\theta \in [t, t+h]$. Hence, by using Proposition 3.9, we get that

$$w(t,x,z_1,z_2) = \min_{(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,t+h]}(x,z_1)} w(t+h, \mathbf{y}(t+h), \mathbf{z}(t+h), z_2) \tag{3.34}$$

for h small enough. Now let $\chi \in C^1([0,T] \times \mathbb{R}^{n+2})$ and (t,x,z_1,z_2) be a local maximum point of $w - \chi$, that is, for some $r > 0$,

$$w(t,x,z_1,z_2) - w(s,x',z_1',z_2') \geq \chi(t,x,z_1,z_2) - \chi(s,x',z_1',z_2')$$

if $|(x,z_1,z_2)-(x',z_1',z_2')|<r,\ |t-s|<r$. Fix an arbitrary $(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,t+h]}(x,z_1)$ For h small enough $(\mathbf{y}(t+h),\mathbf{z}(t+h))\in B((x,z_1),r)$ (by Proposition 2.2) and then

$$\chi(t,x,z_1,z_2)-\chi(t+h,\mathbf{y}(t+h),\mathbf{z}(t+h),z_2)\le w(t,x,z_1,z_2)-w(t+h,\mathbf{y}(t+h),\mathbf{z}(t+h),z_2)$$

for all $0\le h\le h_0$. By using the equation (3.34) we get

$$\begin{aligned}&\chi(t,x,z_1,z_2)-\chi(t+h,\mathbf{y}(t+h),\mathbf{z}(t+h),z_2)\\&\le w(t+,\mathbf{y}(t+h),\mathbf{z}(t+h),z_2)-w(t+h,\mathbf{y}(t+h),\mathbf{z}(t+h),z_2)=0\end{aligned}$$

therefore, dividing by $h>0$ and letting $h\mapsto 0$, we obtain, by differentiability of χ, and the continuity of $\mathbf{y},\ \mathbf{z},\ f$, and ℓ

$$-\partial_t\chi(t,x,z_1,z_2)-D_x\chi(t,x,z_1,z_2)\dot{\mathbf{y}}(0)-D_{z_1}\chi(t,x,z_1,z_2)\dot{\mathbf{z}}(0)\le 0,$$

Since $(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,t+h]}(x,z_1)$ is arbitrary we have

$$-\partial_t\chi(t,x,z_1,z_2)+\max_{(v_x,v_{z_1})\in\overline{co}(G(x))}(-D_x\chi(t,x,z_1,z_2)v_x+D_{z_1}\chi(t,x,z_1,z_2)v_{z_1})\le 0,$$

an therefore we have proved that w is a viscosity sub-solution of (3.25).

To prove the uniqueness note that, for every $(x,z)\in\mathbb{R}^{n+2}$, $(p,q),\ (p',q')\in\mathbb{R}^{n+2}$

$$\begin{aligned}|\mathcal{H}^{\#}(x,z,p,q)-\mathcal{H}^{\#}(x,z,p',q')|&\le\left|\max_{(v_x,v_{z_1})\in\overline{co}(G(x))}(v_x\cdot(p'-p)+v_{z_1}\cdot(q_1-q_1'))\right|\\&\le\left|\max_{(v_x,v_{z_1})\in\overline{co}(G(x))}v_x\cdot(p-p')\right|+\left|\max_{(v_x,v_{z_1})\in\overline{co}(G(x))}v_{z_1}(q_1-q_1')\right|\\&\le c_f(1+|x|)|p-p'|+c_\ell(1+|x|^{\lambda_\ell})|q_1-q_1'|\le C_1(|p-p'|+|q_1-q_1'|),\end{aligned}$$

for some constant $C_1>0$ and . Furthermore for every $(x,z),\ (x',z')\in B(0,R),\ R>0$, $(p,q)\in\mathbb{R}^{n+2}$

$$|\mathcal{H}^{\#}(x,z,p,q)-\mathcal{H}^{\#}(x',z',p,q)|\le C_2(|p|+|q_1|+1)(|x-x'|+|z_1-z_1'|),$$

for some constant $C_2>0$. Then we are in the hypothesis of Theorem 3.11, so for any sub-solution χ of the HJB equation (3.25) we have that

$$\sup_{\mathbb{R}^{n+2}}(\chi(t,\cdot)-w(t,\cdot))\le\max\left(\sup_{\mathbb{R}^{n+2}}(\chi(T,\cdot)-w(T,\cdot),\sup_{[t,T]\times\mathbb{R}^{n+2}}(\psi-\psi)\right)=0.$$

Hence $\chi(t,\cdot)\le w(t,\cdot)$ in $[t,T]\times\mathbb{R}^{n+2}$. On the other hand

$$\sup_{\mathbb{R}^{n+2}}(w(t,\cdot)-w_2(t,\cdot))\le 0,$$

for every super-solution w_2 of (3.25). Therefore w is the unique solution of the HJB equation (3.25). □

Another interesting property is that w is monotone decreasing with respect to the third argument as proved in the following proposition.

Proposition 3.13 *Assume that* (3.7),(3.8),(3.9) *and* (3.10) *hold and let* $(t,x)\in[0,T]\times\mathbb{R}^n$. *Then*

$$\forall z,z'\in\mathbb{R}^2,\ \Big(z\le z'\Rightarrow w(t,x,z)\ge w(t,x,z')\Big).$$

PROOF. Let $z, z' \in \mathbb{R}^2$ such that $z \leq z'$ and one admissible trajectory $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x, 0)$. Then

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z'_1 \leq \varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1 \quad \text{and} \quad \max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z'_2 \leq \max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2,$$

and then

$$(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z'_1) \bigvee \left(\max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z'_2 \right) \leq (\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee \left(\max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2 \right).$$

Taking the minimum over all $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x, 0)$ it follows from the last inequality that

$$w(t, x, z') \leq w(t, x, z).$$

□

Notice that from the definition of the Hamiltonian $\mathcal{H}^{\#}$, we have:

$$\begin{aligned} \mathcal{H}^{\#}(x, z, p, q) &= \sup_{(v_x, v_{z_1})\in G(x)} \Big(- v_x \cdot p - v_{z_1} \cdot q_1 \Big), \quad \forall p \in \mathbb{R}^N, \forall q = (q_1, q_2) \in \mathbb{R}^2, \\ &= \sup_{\substack{u\in U,\\ a\in[0,\mathcal{A}(x,u)]}} \left(-f(x, u) \cdot p + \ell(x, u) \cdot q_1 + a \cdot q_1 \right). \end{aligned}$$

If $q_1 \leq 0$, then we have:

$$\mathcal{H}^{\#}(x, z, p, q) = \sup_{u\in U} \left(-f(x, u) \cdot p + \ell(x, u) \cdot q_1 \right).$$

Since the value function w is decreasing with respect to the variable z, by using the DPP and using the classical viscosity arguments as in proof of Theorem 3.12, one can prove the following:

Theorem 3.14 *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold. The value function w is the unique viscosity solution to the following HJB equation:*

$$\begin{aligned} &\min\Big(\partial_t w(t, x, z) + \mathcal{H}(x, z, D_x w, D_z w), w(t, x, z) - (\psi(x) - z_2)\Big) = 0 \quad \text{for } t \in [0, T),\ x \in \mathbb{R}^n,\ z \in \mathbb{R}^2, \\ &w(T, x, z) = \Big(\varphi(x) - z_1\Big) \bigvee \Big(\psi(x) - z_2\Big) \quad \text{for } x \in \mathbb{R}^n, z \in \mathbb{R}^2, \end{aligned} \tag{3.35}$$

where the function $\mathcal{H}$ is defined by:

$$\mathcal{H}(x, z, p, q) = \sup_{u\in U} \left(-f(x, u) \cdot p + \ell(x, u) \cdot q_1 \right).$$

Remark 3.15 *If we consider the problem*

$$\widetilde{w}(t, x, z_1, z_2) = \min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}_{[t,T]}(x,0)} \left[\Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1\Big) \bigvee \max_{s\in[t,T]} (\psi(\mathbf{y}(s)) - z_2) \right]. \tag{3.36}$$

With similar arguments as in in proof of Theorem 3.12, we can prove that the function $\tilde{w}$ can be also characterized as the unique viscosity solution of the HJB equation (3.35) *By uniqueness of the solution for* (3.35)*, we conclude that the two function w and $\widetilde{w}$ are the same: $w = \widetilde{w}$. This result is not surprising as we know that the set $\mathcal{S}^{\#}_{[t,T]}(x, 0)$ is the closure of $\mathcal{S}_{[t,T]}(x, 0)$.*

3.4 Characterization of the Pareto fronts of the convexified bi-objective optimal control problem

In what follows we consider the single objective optimal control problems and the associated value functions:

$$\vartheta_1(t,x) = \min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)} \varphi(\mathbf{y}(T)) - \mathbf{z}(T), \qquad \vartheta_2(t,x) = \min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)} \max_{s\in[t,T]} \psi(\mathbf{y}(s)). \quad (3.37)$$

For every $t \in [0,T]$, $x \in \mathbb{R}^n$ and $i = 1,2$, we introduce the value:

$$z_i^*(t,x) = \inf\left\{\zeta \in \mathbb{R} \,\middle|\, \exists z \in \mathbb{R}^2 \text{ with } z_i = \zeta,\ w(t,x,z) \le 0\right\}. \quad (3.38)$$

Proposition 3.16 *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold and let* $(t,x) \in [0,T]\times\mathbb{R}^N$.

(i) *For every* $z \in \mathbb{R}^2$, *we have that* $w(t,x,z) \le 0$ *if and only if there exists* $(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)$ *such that:*

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) \le z_1, \quad \textit{and} \quad \max_{s\in[t,T]} \psi(\mathbf{y}(s)) \le z_2. \quad (3.39)$$

(ii) *Moreover, for* $i = 1,2$ *and every* $(t,x) \in [0,T]\times\mathbb{R}^n$, *we have* $z_i^*(t,x) = \vartheta_i(t,x)$.

PROOF. Assertion **(i)** follows directly from (3.15): Let $z \in \mathbb{R}^2$, then

$$w(t,x,z) \le 0 \Leftrightarrow \exists(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0) \text{ s.t. } \varphi(\mathbf{y}(T)) - \mathbf{z}(T) \le z_1 \text{ and } \max_{s\in[t,T]} \psi(\mathbf{y}(s)) \le z_2.$$

(ii) Let show that $\vartheta_i(t,x) \le z_i^*(t,x)$. By item (i) we have that for all $z \in \mathbb{R}^2$ such that $w(t,x,z) \le 0$

$$\exists(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0) \text{ s.t. } \varphi(\mathbf{y}(T)) - \mathbf{z}(T) \le z_1 \text{ and } \max_{s\in[t,T]} \psi(\mathbf{y}(s)) \le z_2.$$

Therefore $\vartheta_i(t,x) \le z_i$ for all $z \in \mathbb{R}^2$ such that $w(t,x,z) \le 0$ and then

$$\vartheta_i(t,x) \le \inf\{\gamma \in \mathbb{R} \mid\ z \in \mathbb{R}^2, w(t,x,z) \le 0 \text{ with } z_i = \gamma\} = z_i^*(t,x).$$

Let show now that $\vartheta_i(t,x) \ge z_i^*(t,x)$. Without loss of generality, we assume here that $i = 1$. The proof will be the same for $i = 2$. Assume that $\vartheta_1(t,x) < z_1^*(t,x)$. Then there exists $\delta \in \mathbb{R}$ such that $\vartheta_1(t,x) < \delta < z_1^*(t,x)$. The inequality $\vartheta_1(t,x) < \delta$ implies that there exists $(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)$ such that $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \delta$. Then for $z_2 = \max_{s\in[t,T]} \psi(\mathbf{y}(s))$ we have that,

$$w(t,x,\delta,z_2) \le (\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - \delta) \bigvee \left(\max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2\right) = 0,$$

which implies that $\delta \in \{\gamma \in \mathbb{R} \mid\ \exists z \in \mathbb{R}^2 \text{ with } z_1 = \gamma,\ w(t,x,z) \le 0\}$. But, we have chosen δ such that $\delta < z_1^*(t,x)$ which is impossible. □

Let us also denote

$$\beta^*(t,x) = (z_1^*(t,x), z_2^*(t,x)) \in \mathbb{R}^2. \quad (3.40)$$

It follows from the proposition 3.16 that β^* is the utopian point associated with the bi-objective control problem for a given $(t, x) \in [0, T] \times \mathbb{R}^n$. If this point is feasible (i.e., there is an admissible pair $(\mathbf{y}_x, \mathbf{z}_0)$ that realizes the minimum of both cost functions $\varphi(\mathbf{y}_x(T)) - \mathbf{z}_0(T)$ and $\max_{s\in[t,T]} \psi(\mathbf{y}_x(s))$, then the Pareto front is reduced to this point. In what follows it is assumed that the utopian point is not feasible. In this case, we have:

$$w(t, x, \beta^*(t, x)) > 0. \tag{3.41}$$

In the following theorem, we give the first link between the solutions of the bi-objective problem (MORP) and the function w.

Theorem 3.17 *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold and let* (t, x) *be in* $[0, T] \times \mathbb{R}^n$. *Then the weak Pareto front* $\mathcal{F}^{\#}_w(t, x)$ *for the bi-objective optimal control problem* (MORP) *with the initial condition* (t, x) *is a subset of the zero level set of the value function* $w(t, x, \cdot, \cdot)$*:*

$$\mathcal{F}^{\#}(t, x) \subset \mathcal{F}^{\#}_w(t, x) \subset \left\{ z \in \mathbb{R}^2 \;\middle|\; w(t, x, z) = 0 \right\}.$$

PROOF. Let $z \in \mathcal{F}^{\#}_w(t, x)$. Then there exists an admissible pair $(\overline{\mathbf{y}}, \overline{\mathbf{z}}) \in \mathcal{S}^{\#}_{[t,T]}(x, 0)$ such that

$$\varphi(\overline{\mathbf{y}}(T)) - \overline{\mathbf{z}}(T) = z_1, \quad \max_{s\in[t,T]} \psi(\overline{\mathbf{y}}(s)) = z_2$$

and there is no other admissible pair that dominates $(\mathbf{y}, \mathbf{z})$. This means that for any $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x, 0)$, one of the following assertions holds:

(i) $z_1 \le \varphi(\mathbf{y}(T)) - \mathbf{z}(T)$,

(ii) or $z_2 \le \max_{s\in[t,T]} \psi(\mathbf{y}(s))$.

We can easily check that in the two above cases, we have

$$(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee (\max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2) \ge 0.$$

Therefore,

$$\begin{aligned} w(t, x, z) &= \min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)} \left[(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee (\max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2) \right] \\ &= \left[(\varphi(\overline{\mathbf{y}}(T)) - \overline{\mathbf{z}}(T) - z_1) \bigvee (\max_{s\in[t,T]} \psi(\overline{\mathbf{y}}(s)) - z_2) \right] = 0. \end{aligned}$$

□

Let $x \in \mathbb{R}^n$ and $t > 0$. We define:

$$\begin{aligned} \overline{z_1}(t, x) &= \inf \left\{ \zeta \in \mathbb{R} \;\middle|\; w(t, x, \zeta, z_2^*(t, x)) = 0 \right\}, \\ \overline{z_2}(t, x) &= \inf \left\{ \zeta \in \mathbb{R} \;\middle|\; w(t, x, z_1^*(t, x), \zeta) = 0 \right\}. \end{aligned} \tag{3.42}$$

Denote by S_1 and S_2 the set of admissible pairs in $\mathcal{S}^{\#}_{[t,T]}(x, 0)$ that realize, respectively, the minimum of $\varphi(\mathbf{y}(T)) - \mathbf{z}(T)$ and $\max_{s\in[t,T]} \psi(\mathbf{y}(s))$.

Then, the values $\overline{z}_i(t,x)$ can be interpreted as:

$$\overline{z}_1(t,x) = \min_{(\mathbf{y},\mathbf{z})\in S_2} \varphi(\mathbf{y}(T)) - \mathbf{z}(T), \quad \overline{z}_2(t,x) = \min_{(\mathbf{y},\mathbf{z})\in S_1} \max_{s\in[t,T]} \psi(\mathbf{y}(s)).$$

The minimum is achieved, since the sets S_1 and S_2 are still compact.

Let us denote

$$\Omega = [z_1^*(t,x), \overline{z_1}(t,x)] \times [z_2^*(t,x), \overline{z_2}(t,x)]. \tag{3.43}$$

Theorem 3.18 *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold and let* (t,x) *be in* $[0,T]\times\mathbb{R}^n$. *The following assertions hold:*

(i) $\mathcal{F}^{\#}(t,x) \subset \mathcal{F}_w^{\#}(t,x) \cap \Omega \subset \{z \in \Omega \mid w(t,x,z) = 0\}$.

(ii) *Let* $z \in \Omega$ *such that* $w(t,x,z) = 0$. *If there exists an admissible pair* $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}^{\#}(x,0)$ *such that* $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) = z_1$ *and* $\max_{s\in[t,T]} \psi(\mathbf{y}(s)) = z_2$, *then* $z \in \mathcal{F}_w^{\#}(t,x)$.

PROOF. **(i)** By Theorem 3.17 we obtain immediately that

$$\mathcal{F}_w^{\#}(t,x) \cap \Omega \subset \{z \in \Omega \mid w(t,x,z) = 0\}.$$

Moreover $\mathcal{F}^{\#}(t,x) \subset \mathcal{F}_w^{\#}(t,x)$. It remains to prove that $\mathcal{F}^{\#}(t,x) \subset \Omega$.

Let $z = (z_1,z_2) \in \mathcal{F}^{\#}(t,x)$. By Proposition 3.16, $z_i^*(t,x) = \vartheta_i(t,x)$, for $i = 1,2$. Then for every $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}^{\#}(x,0)$ we have

$$z_1^*(t,x) \le \varphi(\mathbf{y}_1(T)) - \mathbf{z}_1(T), \qquad z_2^*(t,x) \le \max_{s\in[t,T]} \psi(\mathbf{y}(s)).$$

Therefore, $z_1 \ge z_1^*(t,x)$ and $z_2 \ge z_2^*(t,x)$.

Now, assume that $z_1 > \overline{z}_1(t,x)$. In this case, by definition of $\overline{z}_1(t,x)$, it would exists $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}^{\#}(x,0)$ such that

$$\varphi(\mathbf{y}_1(T)) - \mathbf{z}_1(T) \le \overline{z}_1(t,x) < z_1 \quad \text{and} \quad \max_{s\in[t,T]} \psi(\mathbf{y}(s)) \le z_2^*(t,x) \le z_2,$$

which contradicts the fact that $z \in \mathcal{F}^{\#}(t,x)$. We conclude that $z_1 \le \overline{z}_1(t,x)$. The same argument shows also that $z_2 \le \overline{z}_2(t,x)$, and then z belongs to Ω.

(ii) Let $z \in \Omega$ such that $w(t,x,z) = 0$ and there exists an admissible pair $(\overline{\mathbf{y}},\overline{\mathbf{z}}) \in \mathcal{S}_{[t,T]}^{\#}(x,0)$ such that

$$\varphi(\overline{\mathbf{y}}(T)) - \overline{\mathbf{z}}(T) = z_1 \quad \text{and} \quad \max_{s\in[t,T]} \psi(\overline{\mathbf{y}}(s)) = z_2.$$

By definition of w

$$\min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}_{[t,T]}^{\#}(x,0)} (\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee \left(\max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2 \right) = 0.$$

That means, there exists no admissible par $(\mathbf{y},\mathbf{z})$ such that

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \varphi(\overline{\mathbf{y}}(T)) - \overline{\mathbf{z}}(T) = z_1 \quad \text{and} \quad \max_{s\in[t,T]} \psi(\mathbf{y}(s)) < \max_{s\in[t,T]} \psi(\overline{\mathbf{y}}(s)) = z_2.$$

Therefore, by definition of the weak Pareto optimal solution $z \in \mathcal{F}_w^{\#}(t,x)$. □

Remark 3.19 *As proved in [DZ18], outside of the set Ω, only some trivial parts of the weak Pareto front might exists, that is*

$$z \in \mathcal{F}_w^{\#}(t,x) \bigcap \Omega^C \;\Leftrightarrow\; z_1 = z_1^*(t,x) \text{ and } z_2 > \overline{z_2}(t,x), \text{ or } z_2 = z_2^*(t,x) \text{ and } z_1 > \overline{z_1}(t,x).$$

Now we are going to give a characterization of the Pareto optimal front of bi-objective optimal control problem (MORP) using the value function w. Assume that the hypothesis (3.7),(3.8),(3.9) and (3.10) hold and let (t,x) be in $[0,T] \times \mathbb{R}^n$. Let us introduce the operators on $\mathbb{R}^2$ defined by:

$$\pi_1(z) = z_1, \;\; \pi_2(z) = z_2. \tag{3.44}$$

Let denote

$$\Omega_1 = [z_1^*, \overline{z_1}[; \quad \Omega_2 = [z_2^*, \overline{z_2}[$$

and introduce the following extended functions

$$\eta_1 : [z_1^*, \overline{z_1}] \to [z_2^*, \overline{z_2}], \;\; \eta_1(\zeta_1) = \inf\{\gamma \mid w(t,x,\zeta_1,\gamma) \le 0\}, \tag{3.45a}$$
$$\eta_2 : [z_2^*, \overline{z_2}] \to [z_1^*, \overline{z_1}], \;\; \eta_2(\zeta_2) = \inf\{\gamma \mid w(t,x,\gamma,\zeta_2) \le 0\}. \tag{3.45b}$$

Proposition 3.20 *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold and let* (t,x) *be in* $[0,T] \times \mathbb{R}^n$*. Then for* $j = 1,2$ *the functions* $\eta_j(\cdot)$ *are decreasing:*

$$\forall \zeta, \zeta' \in [z_j^*, \overline{z_j}], \quad \Big(\zeta \le \zeta' \Rightarrow \eta_j(\zeta) \ge \eta_j(\zeta')\Big).$$

PROOF. Let assume that there exists $\xi \le \zeta$ such that

$$\eta_1(\xi) < \eta_1(\zeta).$$

It follows from definition (3.45) that $w(t,x,\zeta,\alpha) > 0$, for all $\alpha < \eta_1(\zeta)$.

Let us take $\alpha = \eta_1(\xi)$ and consider the point $z = (\zeta, \eta_1(\xi))$. It is clear that $w(t,x,z) > 0$. On the other hand, by Proposition 3.13 we obtain that

$$w(t,x,z) = w(t,x,\zeta,\eta_1(\xi)) \le w(t,x,\xi,\eta_1(\xi)) \le 0$$

because $\xi \le \zeta$. This is in contradiction with the fact that $w(t,x,z) > 0$ as established before. □

The following theorem gives a characterization of the Pareto front.

Theorem 3.21 *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold and let* (t,x) *be in* $[0,T] \times \mathbb{R}^n$.

(i) $\mathcal{F}^{\#}(t,x) = \Big\{(\zeta, \eta_1(\zeta)), \; \zeta \in dom(\eta_1)\Big\} \bigcap \Big\{(\eta_2(\zeta), \zeta), \; \zeta \in dom(\eta_2)\Big\}.$

(ii) *For any* $z \in \mathcal{F}^{\#}(t,x)$ *let a trajectory* $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)$ *be optimal for the auxiliary problem* (3.15)*. Then* $(\mathbf{y}, \mathbf{z})$ *is a Pareto optimal solution of* (MORP).

PROOF.

(**i**) We split the proof in two steps.

Step 1. Let us show that

$$\Big\{(\zeta,\eta_1(\zeta)),\ \zeta\in\mathrm{dom}(\eta_1)\Big\}\bigcap\Big\{(\eta_2(\zeta),\zeta),\ \zeta\in\mathrm{dom}(\eta_2)\Big\}\subset\mathcal{F}^{\#}(t,x).$$

Let $z\in\Big\{(\zeta,\eta_1(\zeta)),\ \zeta\in\Omega_1 \text{ and } \eta_1(\zeta)<+\infty\Big\}\bigcap\Big\{(\eta_2(\zeta),\zeta),\ \zeta\in\Omega_2 \text{ and } \eta_2(\zeta)<+\infty\Big\}$. First, we have to show that such a point is feasible, that is, there exists an admissible pair $(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)$ such that $\Big(\varphi(\mathbf{y}(T))-\mathbf{z}(T),\ \max_{s\in[t,T]}\psi(\mathbf{y}(s))\Big)=z$. By definition of the functions η_i, $i=1,2$, we have that $w(t,x,z)=0$. Then there exists at least one admissible pair $(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)$ such that

$$\Big((\varphi(\mathbf{y}(T))-\mathbf{z}(T))\bigvee(\max_{s\in[t,T]}\psi(\mathbf{y}(s))-z_2)\Big)\le 0\ \Leftrightarrow\ \varphi(\mathbf{y}(T))-\mathbf{z}(T)\le z_1 \text{ and } \max_{s\in[t,T]}\psi(\mathbf{y}(s))\le z_2 \tag{3.46}$$

Assume that z is not feasible. Then for any admissible pair $(\overline{\mathbf{y}},\overline{\mathbf{z}})\in\mathcal{S}^{\#}_{[t,T]}(x,0)$ satisfying (3.46), we have that

$$\varphi(\overline{\mathbf{y}}(T))-\overline{\mathbf{z}}(T)<z_1\quad\text{or}\quad\max_{s\in[t,T]}\psi(\overline{\mathbf{y}}(s))<z_2.$$

Let us recall that by choice of z we have that $z_i=\eta_i(\pi_i(z))$. Then, without loss of generality, take

$$\zeta=\varphi(\overline{\mathbf{y}}(T))-\overline{\mathbf{z}}(T),\quad\text{so}\quad w(t,x,\zeta,z_2)\le 0$$

with $\zeta<\eta_1(\pi_1(z))$ which is in contradiction with the definition of $\eta_1(\pi_1(z))$ (see (3.45)).

Now, let us show that $z=(z_1,z_2)$ is Pareto optimal. Assume that there exists $(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)$ such that

$$\varphi(\mathbf{y}(T))-\mathbf{z}(T)=\xi_1\le z_1\quad\text{and}\quad\max_{s\in[t,T]}\psi(\mathbf{y}(s))=\xi_2\le z_2,$$

with $(\xi_1,\xi_2)=\xi\neq z$. Consider, without loss of generality, that $\xi_1<z_1$, then $w(t,x,\xi_1,z_2)\le 0$. As $\xi_1<z_1$, by Proposition 3.13 we have that

$$w(t,x,\xi_1,z_2)\ge w(t,x,z_1,z_2)=0.$$

So we conclude that $w(t,x,\xi_1,z_2)=0$, with $\xi_1<z_1=\eta_1(\pi_1(z))$ which is a contradiction.

(**i**) **Step 2.** Let us show that

$$\mathcal{F}^{\#}(t,x)\subset\Big\{(\zeta,\eta_1(\zeta)),\ \zeta\in\mathrm{dom}(\eta_1)\Big\}\bigcap\Big\{(\eta_2(\zeta),\zeta),\ \zeta\in\mathrm{dom}(\eta_2)\Big\}.$$

Assume that $z\in\mathcal{F}^{\#}(t,x)$ and let $(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)$ be an admissible pair such that

$$z_1=\varphi(\mathbf{y}(T))-\mathbf{z}(T)\quad\text{and}\quad z_2=\max_{s\in[t,T]}\psi(\mathbf{y}(s)).$$

It follows from Theorem 3.17 that $w(t,x,z)=0$. Then it is obvious that $\eta_1(\pi_1(z))<+\infty$ and $\eta_2(\pi_2(z))<+\infty$. If

$$z\notin\Big\{(\zeta,\eta_1(\zeta)),\ \zeta\in\mathrm{dom}(\eta_1)\Big\}\bigcap\Big\{(\eta_2(\zeta),\zeta),\ \zeta\in\mathrm{dom}(\eta_2)\Big\}$$

then $\exists j$ such that $z_j \neq \eta_j(\pi_j(z))$. Consider, without loss of generality, that $z_1 \neq \eta_1(\pi_1(z))$. As $w(t,x,z) = 0$, we obtain that $z_1 > \eta_1(\pi_1(z))$. Consider

$$\xi = (\eta_1(\pi_1(z)), z_2).$$

By the definition of the function η_1 we have that $w(t,x,\xi) = 0$ and then there exists an admissible pair $(\overline{\mathbf{y}}, \overline{\mathbf{z}}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)$ such that

$$\varphi(\overline{\mathbf{y}}(T)) - \overline{\mathbf{z}}(T) \leq \xi_1 < z_1 \quad \text{and} \quad \max_{s\in[t,T]} \psi(\overline{\mathbf{y}}(s)) \leq \xi_2 = z_2,$$

what is in contradiction with the assumption that z is Pareto optimal.

(ii). Let $z \in \mathcal{F}^{\#}(t,x)$. Then $w(t,x,z) = 0$ and z is feasible. Take a trajectory $(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)$ that is optimal for the auxiliary control problem (3.15). Then, as it was be shown (see Proposition 3.16),

$$(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1 \bigvee \max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2) = 0 \Leftrightarrow (\varphi(\mathbf{y}(T)) - \mathbf{z}(T) \leq z_1, \text{ and } \max_{s\in[t,T]} \psi(\mathbf{y}(s)) \leq z_2.$$

If $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < z_1$ then $\xi = (\varphi(\mathbf{y}(T)) - \mathbf{z}(T), z_2)$ is a feasible vector that dominates z which is impossible. In the same manner, if $\max_{s\in[t,T]} \psi(\mathbf{y}(s)) < z_2$ then $\xi' = (z_1, \max_{s\in[t,T]} \psi(\mathbf{y}(s)))$ is a feasible vector that dominates z which is impossible. So, for any trajectory $(\mathbf{y},\mathbf{z})$ that is optimal for (3.15) we have that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right) = z$$

that means that the pair $(\mathbf{y},\mathbf{z})$ is Pareto optimal for (MORP).

□

3.5 ε-Pareto solutions of the original bi-objective optimal control problem

In this section we return to the original problem presented in this chapter. We prove that using the auxiliary value function w it is possible to obtain a region where (weak) ε-Pareto fronts are contained. Moreover (weak) ε-Pareto optimal solutions for problem MOP can be obtained by applying an algorithm of trajectory reconstruction to the auxiliary control problem (3.15).

Theorem 3.22 *Assume that* (3.7), (3.8), (3.9) *and* (3.10) *hold. Let* (t,x) *be in* $[0,T]\times\mathbb{R}^n$ *and* $\varepsilon > 0$.

(i) $\mathcal{F}^{1,\varepsilon}(t,x) \subset \mathcal{F}^{1,\varepsilon}_w(t,x) \subset \left\{ z \in \mathbb{R}^2 \;\middle|\; -\varepsilon \leq w(t,x,z) \leq 0 \right\}$.

(ii) *Let* $z_\varepsilon \in \left\{ z \in \mathbb{R}^2 \;\middle|\; -\varepsilon \leq w(t,x,z) \leq 0 \right\}$. *If there exists* $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{S}_{[t,T]}(x,0)$ *that is optimal for the auxiliary control problem* (3.15). *Then* $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{P}^{1,\varepsilon}_w(t,x)$ *of problem* (3.11).

(iii) *Let* $z_\varepsilon \in \left\{ z \in \mathbb{R}^2 \;\middle|\; -\varepsilon < w(t,x,z) \leq 0 \right\}$. *If there exists* $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{S}_{[t,T]}(x,0)$ *that is optimal for the auxiliary control problem* (3.15). *Then* $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{P}^{1,\varepsilon}(t,x)$ *of problem* (3.11).

PROOF. (**i**) Let $z \in \mathcal{F}_w^{1,\varepsilon}(t,x)$. Then there exists an admissible pair $(\overline{\mathbf{y}}, \overline{\mathbf{z}}) \in \mathcal{S}_{[t,T]}(x,0)$ such that

$$\varphi(\overline{\mathbf{y}}(T)) - \overline{\mathbf{z}}(T) = z_1, \quad \max_{s\in[t,T]} \psi(\overline{\mathbf{y}}(s)) = z_2,$$

then

$$\begin{aligned} w(t,x,z) &= \min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)} \Big[(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee (\max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2)\Big] \\ &\leq \Big[(\varphi(\overline{\mathbf{y}}(T)) - \overline{\mathbf{z}}(T) - z_1) \bigvee (\max_{s\in[t,T]} \psi(\overline{\mathbf{y}}(s)) - z_2)\Big] = 0. \end{aligned}$$

Moreover as $z \in \mathcal{F}_w^{1,\varepsilon}(t,x)$, then for any $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0)$, one of the following assertions holds:

(a) $z_1 - \varepsilon \leq \varphi(\mathbf{y}(T)) - \mathbf{z}(T)$,

(b) or $z_2 - \varepsilon \leq \max_{s\in[t,T]} \psi(\mathbf{y}(s))$.

It is possible to check that in the two above cases, we have

$$(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee (\max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2) \geq -\varepsilon.$$

As the $\mathcal{S}^{\#}_{[t,T]}(x,0)$ is the closure of the $\mathcal{S}_{[t,T]}(x,0)$, we can conclude that one of assertions above holds for any $(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)$. Therefore,

$$w(t,x,z) = \min_{(\mathbf{y},\mathbf{z})\in\mathcal{S}^{\#}_{[t,T]}(x,0)} \Big[(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee (\max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_2)\Big] \geq -\varepsilon.$$

(**ii**) Let $z_\varepsilon \in \Big\{z \in \mathbb{R}^2 \;\Big|\; -\varepsilon \leq \; w(t,x,z) \leq 0\Big\}$ and $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{S}_{[t,T]}(x,0)$ that is optimal for the auxiliary problem $w(t,x,z_\varepsilon)$. Then,

$$-\varepsilon \leq (\varphi(\mathbf{y}_\varepsilon(T)) - \mathbf{z}_\varepsilon(T) - z_{1,\varepsilon} \bigvee \max_{s\in[t,T]} \psi(\mathbf{y}_\varepsilon(s)) - z_{2,\varepsilon}.$$

Assume that there exists $(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0)$ such that

$$\Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\Big) < (z_{1,\varepsilon} - \varepsilon, z_{2,\varepsilon} - \varepsilon).$$

Therefore,

$$\Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_{1,\varepsilon} \bigvee \max_{s\in[t,T]} \psi(\mathbf{y}(s)) - z_{2,\varepsilon}\Big) < \Big(\varphi(\mathbf{y}_\varepsilon(T)) - \mathbf{z}_\varepsilon(T) - z_{1,\varepsilon} \bigvee \max_{s\in[t,T]} \psi(\mathbf{y}_\varepsilon(s)) - z_{2,\varepsilon}\Big)$$

which is impossible. So, for any trajectory $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon)$ that is optimal for the auxiliary control problem $w(t,x,z_\varepsilon)$ we have that the pair $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{P}_w^{1,\varepsilon}(t,x)$ of problem (3.11).

(**iii**) The proof can be obtained with similar arguments of (**ii**). □

3.6 State constrained optimal control problem

In many applications the optimal control problem is subject to state constraints, one example is the boat problem considered in section 3.8.3. In this section we study finite horizon optimal control problems with two different objective functions and with constraints on the state variables.

Let $\mathcal{K}_1 \subset \mathbb{R}^n$ nonempty and closed set of constraints and $\mathcal{K}_2 \subset \mathbb{R}^n$ a non empty closed target set. Consider the following bi-objective optimal control problem

$$\begin{cases} \inf & (\Phi(t,x;\mathbf{y},\mathbf{u}),\Psi(t,x;\mathbf{y})) \\ \\ \text{subject to} & \dot{\mathbf{y}}(s) = f(\mathbf{y}(s),\mathbf{u}(s)), \quad \text{a.e. } s \in [0,\infty[, \\ & \mathbf{y}(t) = x, \\ & \mathbf{y}(s) \in \mathcal{K}_1 \quad \forall s \in [t,T], \\ & \mathbf{y}(T) \in \mathcal{K}_2, \\ & \mathbf{u} \in \mathcal{U}. \end{cases} \tag{3.47}$$

Let $g_1 : \mathbb{R}^n \to \mathbb{R}$ a Lipschitz function such that

$$g_1(x) \leq 0 \Leftrightarrow x \in \mathcal{K}_1. \tag{3.48}$$

Since $\mathcal{K}_1$ is closed, such a function g_1 exists. Indeed, denote by $d_{\mathcal{K}_1}$ the signed distance to $\mathcal{K}_1$, where

$$d_{\mathcal{K}_1}(x) = \begin{cases} d(x,\partial\mathcal{K}_1) & \text{if } x \in \mathcal{K}_1^C \\ -d(x,\partial\mathcal{K}_1) & \text{if } x \in \mathcal{K}_1. \end{cases}$$

The function $g_1 \equiv d_{\mathcal{K}_1(\cdot)}$ is Lipschitz continuous and satisfies the statement (3.48).

Consider also $g_2 : \mathbb{R}^n \to \mathbb{R}$ a Lipschitz function such that

$$g_2(x) \leq 0 \Leftrightarrow x \in \mathcal{K}_2. \tag{3.49}$$

In this section we are going to assume that:

$$\text{For every } x \in \mathbb{R}^n, \left\{ \begin{pmatrix} f(x,u) \\ -\ell(x,u) - a \end{pmatrix}, u \in U,\ 0 \leq a \leq \mathcal{A}(x,u) \right\}, \text{ is a convex subset of } \mathbb{R}^n \times \mathbb{R}. \tag{3.50}$$

With this assumption and (3.7) and (3.8) the set of trajectories $\mathcal{S}_{[t,T]}(x,z_1)$ is a compact set in the space of continuous functions $C(t,T)$.

We associate with the optimal control problem (3.47) the following auxiliary control problem and its value function $\overline{w}$:

$$\overline{w}(t,x,z) = \min_{(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,z_1)} \left[\Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T)\Big) \bigvee \max_{s\in[t,T]} (\psi(\mathbf{y}(s)) - z_2) \bigvee \max_{s\in[t,T]} g_1(\mathbf{y}(s)) \bigvee g_2(\mathbf{y}(T)) \right]. \tag{3.51}$$

The value function of the auxiliary control problem (3.51) is locally Lipschitz continuous, satisfies the Dynamic Programming Principle given in the next theorem and can be characterize as a unique viscosity solution of a HJB equation. As the proof is similar to the proof presented for the unconstrained case, we are not going to rewrite it here.

Theorem 3.23 *Assume that* (3.7)*,* (3.8)*,* (3.9)*,* (3.10) *and* (3.50) *hold.*

(i) *The value function* $\overline{w}$ *is locally Lipschitz continuous on* $[0,T] \times \mathbb{R}^n \times \mathbb{R}^2$.

(ii) *For all $h \geq 0$ such that $t+h < T$ and $(x,z) \in \mathbb{R}^n \times \mathbb{R}^2$, we have*

$$\overline{w}(t,x,z) = \min_{(\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,t+h]}(x,z_1)} \left\{ \overline{w}\,(t+h, \mathbf{y}(t+h), \mathbf{z}(t+h), z_2) \bigvee \max_{s\in[t,t+h]} (\psi(y(s)) - z_2) \bigvee \max_{s\in[t,t+h]} g_1(y(s)) \right\}.$$

(iii) *The function $\overline{w}$ is the unique viscosity solution of the following HJB equation:*

$$\min\Big(\partial_t \overline{w}(t,x,z) + \mathcal{H}(x,z,D_x w, D_z w), \overline{w}(t,x,z) - (\psi(x) - z_2) \bigvee g_1(x)\Big) = 0, \ \forall t \in [0,T), \ (x,z) \in \mathbb{R}^{n+2}$$
$$\overline{w}(T,x,z) = (\varphi(x) - z_1) \bigvee (\psi(x) - z_2) \bigvee g_1(x) \bigvee g_2(x), \ \forall \ (x,z) \in \mathbb{R}^{n+2}. \quad (3.52)$$

where $\mathcal{H}$ is given by

$$\mathcal{H}(x,z,p,q) = \sup_{u\in U} \left(-f(x,u)\cdot p + \ell(x,u)\cdot q_1\right).$$

The characterization of the weak Pareto front and the Pareto front for the bi-objective optimal control problem (3.47) can be obtained considering the function $\overline{w}$ in Theorems 3.18 and 3.21, respectively.

3.7 Reconstruction of the Pareto optimal trajectories

Theorems 3.18 and 3.21 provide the characterization of the weak Pareto front and the Pareto front, respectively, of the convexified problem (MORP) in the case without state constraints and of the problem (3.47) with state constraints. Another important concern is to reconstruct an optimal trajectory corresponding to a given Pareto optimal solution.

Once the auxiliary value function w is known by Theorem 3.21 we have a characterization of the Pareto front $\mathcal{F}^{\#}(t,x)$. Now, let z be an optimal Pareto solution. Then a corresponding Pareto trajectory can be obtained by using the value function w. Indeed, by applying an algorithm of trajectory reconstruction to the function w on $[t,T]$ with the initial conditions (x,z) as the algorithm presented in [ABDZ18], we get an approximation of the optimal trajectory for $w(t,x,z)$. Now, by Theorem 3.21, item (ii), if the trajectory is optimal for the auxiliary problem (3.15), then it is a Pareto optimal trajectory of (MORP).

In the case of weak Pareto solutions, by Theorem 3.18

$$\mathcal{F}_w^{\#}(t,x) \cap \Omega \subset \big\{ z \in \Omega \mid w(t,x,z) = 0 \big\}.$$

And if there exists an admissible pair $(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)$ such that $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) = z_1$ and $\max\limits_{s\in[t,T]} \psi(\mathbf{y}(s)) = z_2$, then $z \in \mathcal{F}_w^{\#}(t,x)$. So we can take $z \in \Omega$ such that $w(t,x,z) = 0$, apply an algorithm of trajectory reconstruction to the function w on $[t,T]$ with the initial (x,z) as in [ABDZ18], and see if such a trajectory exists. If there is a trajectory in such conditions this is an approximation of a weak Pareto trajectory of (MORP).

Now consider $\varepsilon > 0$ and let $z_\varepsilon \in \mathbb{R}^2$, such that $-\varepsilon \leq w(t,x,z_\varepsilon) \leq 0$. By applying an algorithm of trajectory reconstruction to the function w on $[t,T]$ with the initial conditions (x,z_ε) we get an approximation of the optimal trajectory for $w(t,x,z_\varepsilon)$. Now by Theorem 3.22, if there exists $(\mathbf{y}_\varepsilon,\mathbf{z}_\varepsilon) \in \mathcal{S}_{[t,T]}(x,0)$ that is optimal for the auxiliary control problem (3.15). Then $(\mathbf{y}_\varepsilon,\mathbf{z}_\varepsilon) \in \mathcal{P}_w^{1,\varepsilon}(t,x)$ of problem (3.11). Moreover if we consider that $-\varepsilon < w(t,x,z_\varepsilon) \leq 0$ and it is possible to obtain $(\mathbf{y}_\varepsilon,\mathbf{z}_\varepsilon) \in \mathcal{S}_{[t,T]}(x,0)$ that is optimal for the auxiliary control problem (3.15). Then $(\mathbf{y}_\varepsilon,\mathbf{z}_\varepsilon) \in \mathcal{P}^{1,\varepsilon}(t,x)$ of problem (3.11).

3.8 Numerical examples

In this section we present some examples where the method proposed in this chapter was applied to construct the weak Pareto front and the Pareto front. In all examples the HJB equation was solved by a finite difference method implement at C++ HJB-solver "ROC-HJ" [ROC].

3.8.1 A problem of business strategy

Denote by $\mathbf{y}(s)$ the quantity of steel produced by an industry, at time s. At every moment, such production can either be reinvested to expand the productive capacity or sold. The initial productive capacity is $x > 0$; such capacity grows as the reinvestment rate. Let the function $\mathbf{u} : [0,T] \to [0,1]$, where $\mathbf{u}(s)$ is the fraction of the output at time s that should be reinvested. The objective is to maximize the total sales. As $\mathbf{u}(s)$ is the fraction of the output $\mathbf{y}(s)$ that we reinvest, then $(1-\mathbf{u}(s))\mathbf{y}(s)$ is the part of $\mathbf{y}(s)$ that we sell by a a price P at time s, that is constant over the time horizon. So the first objective is

$$\max \int_0^T P(1-\mathbf{u}(s))\mathbf{y}(s)ds = \min \int_0^T P(\mathbf{u}(s)-1)\mathbf{y}(s)ds.$$

However, consider that the owner also wants the production to be around a quantity c, so the second objective can be modeled as

$$\min \max_{s\in[0,T]} |\mathbf{y}(s)-c|\,.$$

Hence the bi-objective problem is

$$\begin{cases} \text{minimize} & \left(\int_t^T P(\mathbf{u}(s)-1)\mathbf{y}(s)ds,\ \max_{s\in[0,T]} |\mathbf{y}(s)-c|\right) \\ \text{subject to} & \dot{\mathbf{y}}(s) = \mathbf{y}(s)\mathbf{u}(s) \\ & \mathbf{y}(t) = x_0 \\ & 0 \le \mathbf{u}(s) \le 1. \end{cases} \tag{3.53}$$

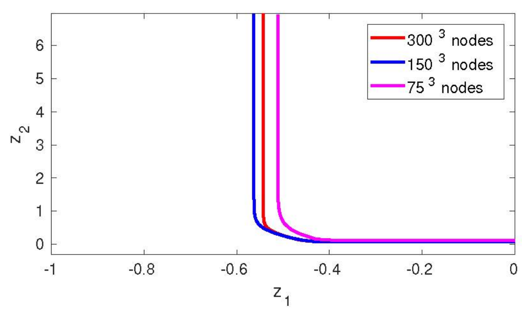
For numerical simulation we considered $P = 0.5$, $c = 0.5$ and $T = 2$. Note that the image set of the augmented dynamics is convex. So, by Remark 3.15 the auxiliary value function w is the unique viscosity solution to the following HJB equation:

$$\min\Big(\partial_t w(t,x,z) + \mathcal{H}(x, D_x w, D_z w), w(t,x,z) - (|x-c| - z_2)\Big) = 0, \qquad \text{for } t\in[0,T), x\in\mathbb{R}, z\in\mathbb{R}^2$$
$$w(T,x,z) = -z_1 \bigvee (|x-c|-z_2) \qquad \text{for } x\in\mathbb{R}, z\in\mathbb{R}^2.$$

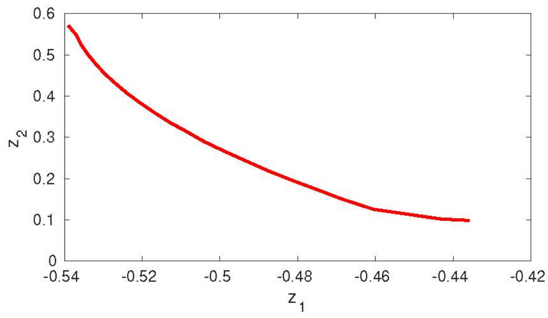
where the Hamiltonian is given by

$$\mathcal{H}(x,p,q) = \max(-xp + Pxq_1, 0) - Pxq_1, \quad \forall x,p\in\mathbb{R}, q\in\mathbb{R}^2.$$

The HJB equation is solved on a x-grid of 300^3 nodes on the domain $[0,e^2]\times[-e^2,0]\times[0,7]$. For this simple example, we use an explicit Euler scheme in time and Lax-Friedrich discretization in space (we refer to [ABDZ18, OS91, BFZ10] for more details). This discretization is known to be stable and convergent under an adequate interplay between the mesh size of the x-grid and the time step.



(a) Approximation of the 0-level set on different grids.



(b) Zoom on $\mathcal{F}(0, x_0)$. Computation on a grid of 300^3 nodes.

Figure 3.4: Business strategy problem: analysis of the 0-level set of $w(0, x_0, \cdot)$ for $x_0 = 0.4$

Now, we fix the initial conditions as follows: $x_0 = 0.4$ and $t = 0$. Figure 3.4a presents the 0-set-level of the value function $w(0, x_0, \cdot)$ computed on three different x-grids (with respectively 75^3, 150^3 and 300^3 grid points). This test confirms the (already known) stability of the numerical scheme.

On another hand, formula (3.38) allows to give an approximation of the utopian point associated with the bi-objective control problem:

$$z_1^*(t, x_0) = \inf \left\{ \zeta_1 \in \mathbb{R} \,\middle|\, \exists z_2 \in \mathbb{R} \text{ s.t. } \; w(t, x_0, \zeta_1, z_2) \leq 0 \right\} \cong -0.539,$$

$$z_2^*(t, x_0) = \inf \left\{ \zeta_2 \in \mathbb{R} \,\middle|\, \exists z_1 \in \mathbb{R} \text{ s.t. } \; w(t, x_0, z_1, \zeta_2) \leq 0 \right\} \cong 0.0985.$$

Following (3.42), we get also an approximation of the upper bounds of Ω:

$$\overline{z_1}(t, x_0) = \inf \left\{ \zeta \in \mathbb{R} \,\middle|\, w(t, x_0, \zeta, z_2^*(t, x_0)) = 0 \right\} \cong -0.4359,$$
$$\overline{z_2}(t, x_0) = \inf \left\{ \zeta \in \mathbb{R} \,\middle|\, w(t, x_0, z_1^*(t, x), \zeta) = 0 \right\} \cong 0.5711.$$

These values lead to an approximation of the set

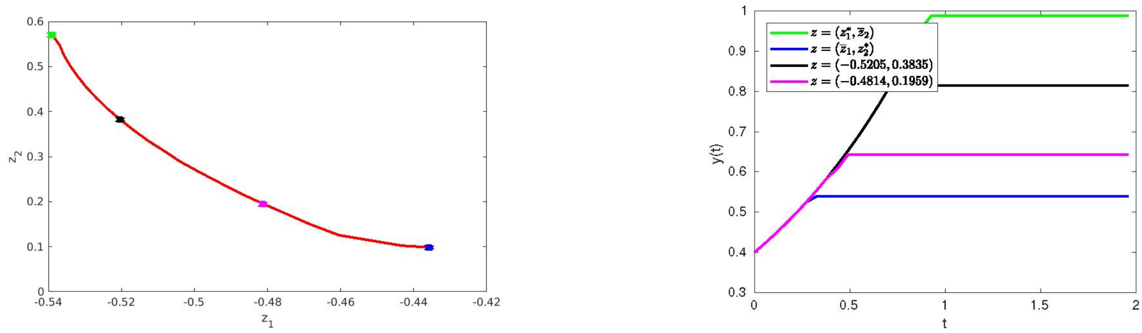
$$\Omega := [z_1^*(t, x_0), \overline{z_1}(t, x_0)] \times [z_2^*(t, x_0), \overline{z_2}(t, x_0)].$$

Figure 3.4b shows the intersection of 0-level set of $w(0, x_0, \cdot)$ with the set Ω. In our case this intersection is equal to the Pareto front. The possible points in the weak Pareto front are just trivial points and are outside the set Ω.

The value function w is also useful to reconstruct Pareto optimal trajectories. We chose four point z_i, $i = 1, 2, 3, 4$ at the Pareto front, that are represented by points with different colors in Figure 3.5a. We compute the optimal trajectories for $w(t, x_0, z_i), i = 1, 2, 3, 4$, using the reconstruction algorithm from [ABDZ18]. Figure 3.5b shows these optimal trajectories for the initial states (x_0, z_i) (in figure 3.5b, the color used for each trajectory refers to the corresponding Pareto value z_i in figure 3.5a). Let us point out that these trajectories are Pareto optimal for the optimal control problem (3.53).

3.8.2 Pest control problem

Consider a Lotka-Volterra model describing the interaction between two species (predator-prey model): a prey, that is a nuisance for humans, and a predator. Both



(a) Points z_i for initial condition of computed trajectories

(b) Optimal trajectories for the initial states (x, z_i)

Figure 3.5: Business strategy: optimal trajectories as Pareto optimal solutions

species grow according to the following system

$$\begin{aligned} \dot{\mathbf{y}}_1(s) &= \mathbf{y}_1(s) - \mathbf{y}_1(s)\mathbf{y}_2(s) \\ \dot{\mathbf{y}}_2(s) &= -\mathbf{y}_2(s) + \mathbf{y}_1(s)\mathbf{y}_2(s) \end{aligned}$$

where $\mathbf{y}_1$ represents the pest population, $\mathbf{y}_2$ the predator. We can act on this model by spraying a chemical to poison the pest (the poison may also kill a part of predator population). The dynamical system becomes

$$\begin{aligned} \dot{\mathbf{y}}_1(s) &= \mathbf{y}_1(s) - \mathbf{y}_1(s)\mathbf{y}_2(s) - \mathbf{y}_1(s)c_{\mathbf{y}_1}\mathbf{u}(s) \\ \dot{\mathbf{y}}_2(s) &= -\mathbf{y}_2(s) + \mathbf{y}_1(s)\mathbf{y}_2(s) - \mathbf{y}_2(s)c_{\mathbf{y}_2}\mathbf{u}(s) \end{aligned}$$

where the constants $c_{\mathbf{y}_1}$, $c_{\mathbf{y}_2}$ represent the rate of each population that will be killed by the poison. As in [SBD+06], the value of the constants $c_{\mathbf{y}_1}$, $c_{\mathbf{y}_2}$ are set to 0.4 and 0.2 respectively. The control function $\mathbf{u}(s)$ is restricted to take values of either 0 or 1. The goal is to keep nuisance expansion under control by minimizing the maximum difference of certain proportion of both species along the time horizon $T - t$:

$$\max_{s\in[t,T]} 0.25(\mathbf{y}_1(s) - K\mathbf{y}_2(s))^2$$

and also minimize the cost of spraying the chemical:

$$\int_t^T P\mathbf{u}(s)ds,$$

where $K = 0.7$, $P = 0.3$ and $T = 10$. So, the optimal control problem has two different objective functions. Since the control is allowed to take only the values 0 and 1, the set of trajectories is not compact. In the relaxed control problem, the control input may take values in the interval $[0, 1]$. As described in the previous sections, we introduce an auxiliary control problem whose value function w is solution of the following HJB equation on $[0, T] \times \mathbb{R}^4$:

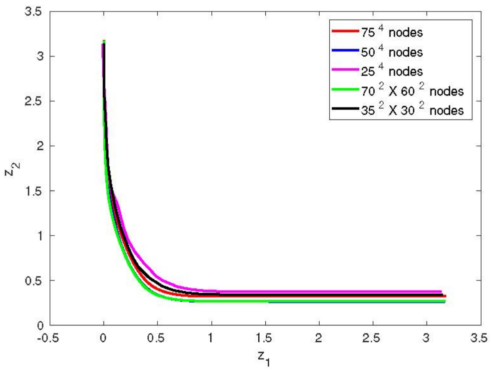
$$\min\Big(\partial_t w(t,x,z) + \mathcal{H}(x, D_x w, D_z w), w(t,x,z) - (0.25(x_1 - Kx_2)^2 - z_2)\Big) = 0, \quad \text{for } t \in [0,T), x, z \in \mathbb{R}^2$$
$$w(T,x,z) = -z_1 \bigvee (0.25(x_1 - Kx_2)^2 - z_2) \qquad \text{for } x \in \mathbb{R}^2, z \in \mathbb{R}^2.$$

where the Hamiltonian is given by

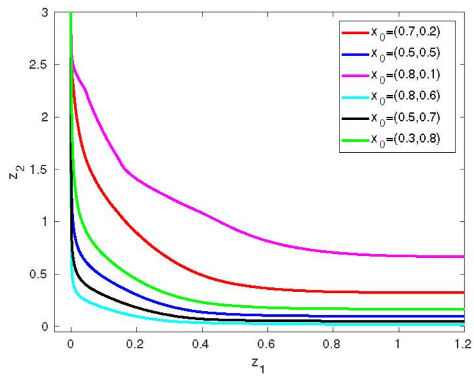
$$\mathcal{H}(x,p,q) = -(x_1 - x_1x_2)p_1 + (x_2 - x_1x_2)p_2 + \max\left(c_{\mathbf{y}_1}x_1p_1 + c_{\mathbf{y}_2}x_2p_2 + Pq_1, 0\right), \quad \forall x,p,q \in \mathbb{R}^2.$$

The HJB equation is solved on a x-grid of 75^4 nodes on the domain $[0, 3.5] \times [0, 3.5] \times [0, 3] \times [0, 3]$. As in the previous example, we use an explicit Euler scheme in time and Lax-Friedrich discretization in space.

First, we fix the initial conditions as follows: $x_0 = (0.7, 0.2)$ and $t = 0$. Figure 3.6a presents the 0-set-level of the value function $w(0, x_0, \cdot)$ computed on five different x-grids. This numerical test confirms the (already known) stability of the numerical scheme. One important feature of the HJB approach to solve bi-objective optimal control problems is the possibility of obtain the Pareto front for different initial states with the same auxiliary value function w. Figure 3.6b shows the Pareto front for different initial states x_0.



(a) 0-level set of $w(0, x_0, \cdot)$ computed on different grids



(b) Pareto front for different initial states x_0

Figure 3.6: Pest control problem: 0-level sets for different initial states and on different computational grids

From now on, we fix $t = 0$ and $x_0 = (0.7, 0.2)$ and we consider the approximation of the value function w obtained on x-grid of on a x-grid of 75^4 nodes. From formula (3.38), we get an approximation of the utopian point associated with the bi-objective control problem:

$$\begin{aligned} z_1^*(0, x_0) &= \inf \left\{ \zeta_1 \in \mathbb{R} \,\middle|\, \exists z_2 \in \mathbb{R} \text{ s.t. } \; w(0, x_0, \zeta_1, z_2) \leq 0 \right\} \cong 0, \\ z_2^*(0, x_0) &= \inf \left\{ \zeta_2 \in \mathbb{R} \,\middle|\, \exists z_1 \in \mathbb{R} \text{ s.t. } \; w(0, x_0, z_1, \zeta_2) \leq 0 \right\} \cong 0.3245. \end{aligned}$$

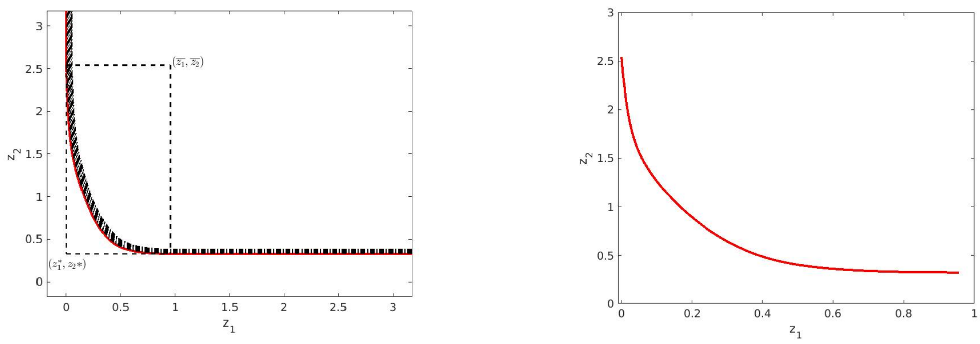
Following (3.42), we get also an approximation of the upper bounds of Ω:

$$\begin{aligned} \overline{z_1}(0, x_0) = \inf \left\{ \zeta \in \mathbb{R} \,\middle|\, w(0, x_0, \zeta, z_2^*(0, x_0)) = 0 \right\} \cong 0.9560, \\ \overline{z_2}(0, x_0) = \inf \left\{ \zeta \in \mathbb{R} \,\middle|\, w(0, x_0, z_1^*(0, x_0), \zeta) = 0 \right\} \cong 2.5427. \end{aligned}$$

These values lead to an approximation of the set

$$\Omega := [z_1^*(0, x_0), \overline{z_1}(0, x_0)] \times [z_2^*(0, x_0), \overline{z_2}(0, x_0)].$$

Figure 3.7a shows the 0-level set of the value function $w(0, x_0, \cdot)$ in red that contains the (weak) Pareto front $\mathcal{F}^{\#}(0, x_0) \subset \mathcal{F}_x^{\#}(0, x_0)$. Moreover, the black region in figure 3.7a represents a region where it is possible to obtain points in $\mathcal{F}_w^{1;\varepsilon}(0, x_0)$, for $\varepsilon = 0.05$. In this figure the set Ω is represented by a box delimited by black dashed lines. Figure 3.7b shows the intersection of 0-level set of $w(0, x_0, \cdot)$ with the set Ω. In our case this intersection is



(a) The 0-level set of $w(0, x_0, \cdot)$ (in red). The set $\{z \mid w(0, x_0, z) \leq -\varepsilon\}$ for $\varepsilon = 0.05$ (in black)

(b) zoom on $\mathcal{F}^{\#}(0, x_0)$

Figure 3.7: Pest control problem: analysis of the negative level set of $w(0, x_0, \cdot)$, for $x_0 = (0.7, 0.2)$

equal to the Pareto front $\mathcal{F}^{\#}(0, x_0)$. The possible points in the weak Pareto front are just trivial points and are outside the set Ω.

The value function w is also useful to reconstruct Pareto optimal trajectories. With the algorithm of trajectory reconstruction from [ABDZ18] we reconstruct an optimal trajectory for $w(0, x_0, z_1^*(0, x_0), \overline{z_2}(0, x_0))$ (resp. $w(0, x_0, \overline{z_1}(0, x_0), z_2^*(0, x_0)))$ that is also optimal trajectory of the scalar problems $\vartheta_1(0, x_0) = z_1^*(0, x_0)$ (resp. $\vartheta_2(0, x_0) = z_2^*(0, x_0)$). This trajectory and the associated optimal control is represented in green (resp. in blue) in Figure 3.8.

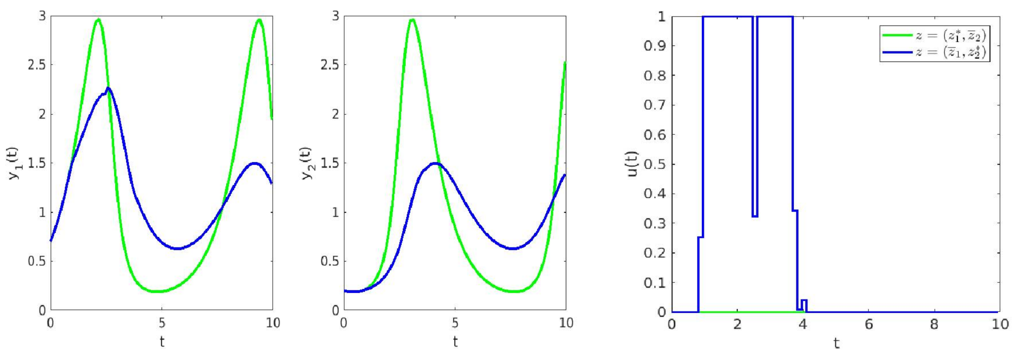


Figure 3.8: Pest control problem: optimal trajectories, and corresponding optimal control laws, associated to the Pareto values $(z_1^\star(0, x_0), \overline{z}_2(0, x_0))$ (in green) and $(\overline{z}_1(0, x_0)), z_2^\star(0, x_0))$ (in blue)

As can be seen in figure 3.8 the optimal control law for $\vartheta_1(0, x_0)$ is identically 0 (i.e., $\mathbf{u} \equiv 0$), which means that not only the point $(z_1^*(0, x_0), \overline{z_2}(0, x_0))$ belongs to the Pareto front of the relaxed problem $\mathcal{F}^{\#}(0, x_0)$, but it is also optimal Pareto for the original (non-relaxed) control problem $(z_1^*(0, x_0), \overline{z_2}(0, x_0)) \in \mathcal{F}(0, x_0)$.

The case of the Pareto value $(\overline{z_1}(0, x_0), z_2^*(0, x_0)) \in \mathcal{F}^{\#}(0, x_0)$ is different, because the optimal control law seems to take other values than just 0 and 1. So, we cannot guarantee that $(\overline{z_1}(0, x_0), z_2^*(0, x_0))$ belongs to $\mathcal{F}(0, x_0)$. However, by theorem 3.8, there exists a ε-Pareto solution in a neighborhood of $(\overline{z_1}(0, x_0), z_2^*(0, x_0))$, where the control takes values in the set $\{0, 1\}$.

We choose $\varepsilon = 0.01$ and consider the point $(\overline{z_1}(0, x_0) + \varepsilon, z_2^*(0, x_0) + \varepsilon)$ that is in the black region of Figure 3.7a. In figure 3.9, we display in blue the relaxed optimal

Pareto trajectory-control corresponding to $(\overline{z_1}(0,x_0), z_2^*(0,x_0)) \in \mathcal{F}^\#(0,x_0)$ and in red the optimal trajectory-control corresponding to the ε-Pareto value $(\overline{z_1}(0,x_0)+\varepsilon, z_2^*(0,x_0)+\varepsilon)$. As can be seen in this example, the two trajectories are very close to each other, while the structure of the controls laws are different.

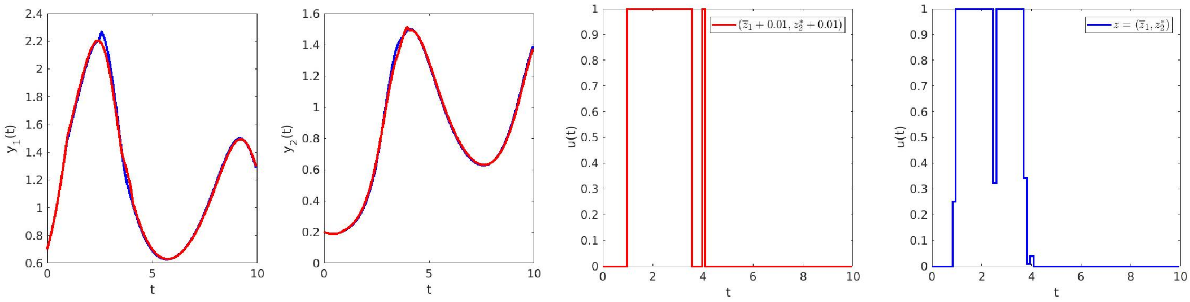
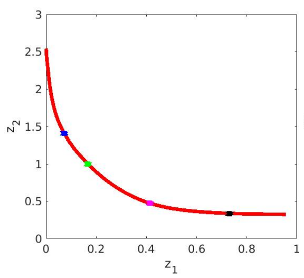
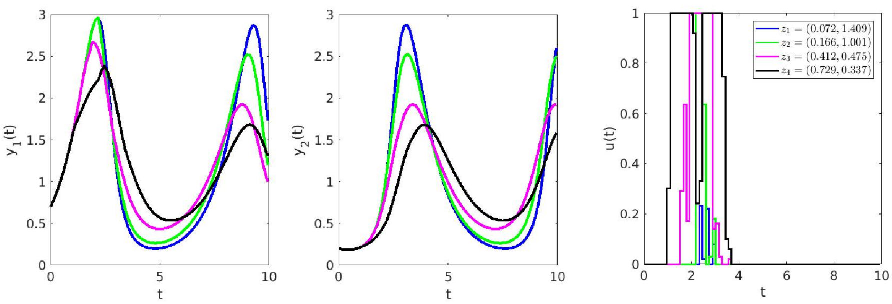


Figure 3.9: Pest control problem: in blue: relaxed optimal solution corresponding to the relaxed Pareto value $(\overline{z_1}(0,x_0), z_2^*(0,x_0))$. In red: ε-solution to the original problem corresponding to the ε-Pareto value $(\overline{z_1}(0,x_0)+\varepsilon, z_2^*(0,x_0)+\varepsilon)$ with $\varepsilon = 0.01$

To analyze more examples of optimal Pareto trajectories, we choose four Pareto values z_i, $i = 1,2,3,4$, that are represented by points with diffrent colors in figure 3.10a. We compute the optimal trajectories for $w(0,x_0,z_i), i = 1,2,3,4$. Figure 3.10 shows these optimal trajectories for the initial states (x_0, z_i), the color used for each trajectory refers to the corresponding Pareto value z_i in figure 3.10.



(a) Points z_i for initial conditions of computed trajectories



(b) Pareto optimal trajectories and controls

Figure 3.10: Pest control problem: optimal Pareto solutions in $\mathcal{P}^\#(0,x_0)$

We observe that the optimal pairs represented in Figure 3.10 are in $\mathcal{P}^\#(0,x_0)$ but are not in $\mathcal{P}(0,x_0)$, because the respective control functions take values different from 0 and 1. So here again, ε-Pareto values can be taken in the neighborhood of $\mathcal{F}^\#(0,x_0)$.

3.8.3 Boat example

Based on [ABZ13] consider the classical Zermelo type problem. A boat with coordinates $\mathbf{y}(s) = (\mathbf{y}_1(s), \mathbf{y}_2(s))$ navigates in a canal $\mathbb{R} \times [-2.5, 2.5]$, starting from $\mathbf{y}(0) = (x_1, x_2)$, and wants to reach an island, with minimal fuel consumption. The dynamics is given by

$$\dot{\mathbf{y}}_1(s) = \mathbf{u}_2(s)cos(\mathbf{u}_1(s)) + \beta - \gamma\mathbf{y}_2^2(s)$$
$$\dot{\mathbf{y}}_2(s) = \mathbf{u}_2(s)sin(\mathbf{u}_1(s)),$$

where $\mathbf{u}_1 \in [0, 2\pi]$ is the first control (angle), $\mathbf{u}_2 \in [0,1]$ is a second control (the speed of the boat), and $\beta - \gamma \mathbf{y}_2^2(s)$ is the current drift (along the $\mathbf{y}_1$-axis). We shall choose the parameters $\beta = 0.25$, $\gamma = 0.04$, so in this case the current drift is in the same direction that the boat navigates and is zero near to the shores and increases when is approximating of the middle of the river. The fuel consumption is proportional to

$$\int_0^T 0.5\mathbf{u}_2(s)ds.$$

A second objective is to maintain the boat near to one shore

$$\max_{0\le s\le T}(\theta - \mathbf{y}_2(s)). \tag{3.54}$$

where we considered $\theta = 2.5$. Consider the set of constraints given by

$$\mathcal{K}_1 = \{x \in \mathbb{R}^2,\ g_1(x) \le 0\},$$

where

$$g_1(x) = \max\left(r_a - \max(|x_1 - a_1|, \frac{1}{2}|x_2 - a_2|), r_b - \max(|x_1 - b_1|, \frac{1}{2}|x_2 - b_2|)\right)$$

and where $r_a = 0.5$, $a = (-3.5, 1)$ and $r_b = 0.25$, $b = (-1.5, 0.25)$.

The target $\mathcal{K}_2 \equiv B(c, r_0)$ with $r_0 = 0.5$ and $c = (0, 0.2)$ is represented by a function g_2 defined by

$$g_2(x) = ||x - c|| - r_0.$$

The optimal control problem is

$$\left\{\begin{array}{l} \inf \quad \left(\int_0^T 0.5\mathbf{u}_2(s)ds, \max\limits_{0\le s\le T} \theta - \mathbf{y}_2(s)\right), \\ \\ \quad\quad \dot{\mathbf{y}}_1(s) = \mathbf{u}_2(s)cos(\mathbf{u}_1(s)) + \beta - \gamma\mathbf{y}_2^2(s), \\ \quad\quad \dot{\mathbf{y}}_2(s) = \mathbf{u}_2(s)sin(\mathbf{u}_1(s)), \\ \quad\quad \mathbf{y}(0) = (x_1, x_2), \\ \quad\quad \mathbf{y}(s) \in \mathcal{K}_1, \forall s \in [0, T], \\ \quad\quad \mathbf{y}(T) \in \mathcal{K}_2, \\ \quad\quad \mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{U}, \end{array}\right. \tag{3.55}$$

where $\mathcal{U}$ is the set of measurable controls $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) : [0, T] \to [0, 2\pi] \times [0, 1]$.

Therefore it is a bi-objective problem with different cost functions and with state constraints. So we are in the situation of the Section 3.6, and we introduce an auxiliary control problem whose value function w is solution of the following HJB equation on $[0, T] \times \mathbb{R}^4$:

$$\min\Big(\partial_t w(t, x, z) + \mathcal{H}(x, D_x w, D_z w), w(t, x, z) - \max\big(\theta - x_2 - z_2, g_1(x)\big)\Big) = 0, \quad \text{for } t \in [0, T), x, z \in \mathbb{R}^2$$

$$w(T, x, z_1, z_2) = -z_1 \bigvee \theta - x_2 - z_2 \bigvee g_1(x) \bigvee g_2(x), \quad x \in \mathbb{R}^2,\ z \in \mathbb{R}^2.$$

where the Hamiltonian is given by

$$\mathcal{H}(x, p, q) = \max(0, \sqrt{p_1^2 + p_2^2} + 0.5q_1) - (\beta - \gamma x_2^2)p_1, \quad \forall x, p, q \in \mathbb{R}^2.$$

The HJB equation is solved on a x-grid of $70^2 \times 50^2$ nodes on the domain $[-6,1] \times [-2.5, 2.5] \times [0,3] \times [0,5]$. As in the previous examples, we use an explicit Euler scheme in time and Lax-Friedrich discretization in space.

We fixed the initial position of the boat equal $x_0 = (-5.5, 1)$. From formula (3.38), we get an approximation of the utopian point associated with the bi-objective control problem:

$$\begin{aligned} z_1^*(0,x_0) &= \inf\left\{\zeta_1 \in \mathbb{R} \,\middle|\, \exists z_2 \in \mathbb{R} \text{ s.t. } \; w(0,x_0,\zeta_1,z_2) \le 0\right\} \cong 2.071, \\ z_2^*(0,x_0) &= \inf\left\{\zeta_2 \in \mathbb{R} \,\middle|\, \exists z_1 \in \mathbb{R} \text{ s.t. } \; w(0,x_0,z_1,\zeta_2) \le 0\right\} \cong 1.786. \end{aligned}$$

Following (3.42), we get also an approximation of the upper bounds of Ω:

$$\begin{aligned} \overline{z_1}(0,x_0) &= \inf\left\{\zeta \in \mathbb{R} \,\middle|\, w(0,x_0,\zeta,z_2^*(0,x_0)) = 0\right\} \cong 2.88, \\ \overline{z_2}(0,x_0) &= \inf\left\{\zeta \in \mathbb{R} \,\middle|\, w(0,x_0,z_1^*(0,x_0),\zeta) = 0\right\} \cong 3.1. \end{aligned}$$

These values lead to an approximation of the set

$$\Omega := [z_1^*(0,x_0), \overline{z_1}(0,x_0)] \times [z_2^*(0,x_0), \overline{z_2}(0,x_0)].$$

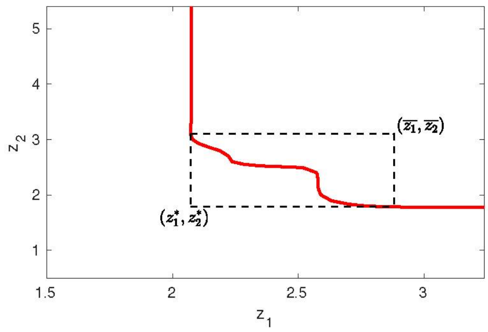
Figure 3.11a shows the 0-level set of the value function $w(0,x_0,\cdot)$ in red that contains the (weak) Pareto front $\mathcal{F}(0,x_0) \subset \mathcal{F}_x(0,x_0)$. In this figure the set Ω is represented by a box delimited by black dashed lines. In this example the intersection of the zero level set of the value function $w(0,x_o,\cdot)$ it is not equal to the Pareto front of the bi-objective problem (3.55). The Pareto front can be obtained by using Theorem (3.21). First the functions η_1 and η_2 can be derived by using (3.45):

$$\eta_1 : [z_1^*, \overline{z_1}] \to [z_2^*, \overline{z_2}], \quad \eta_1(\zeta_1) = \inf\{\gamma \mid w(t,x,\zeta_1,\gamma) \le 0\},$$

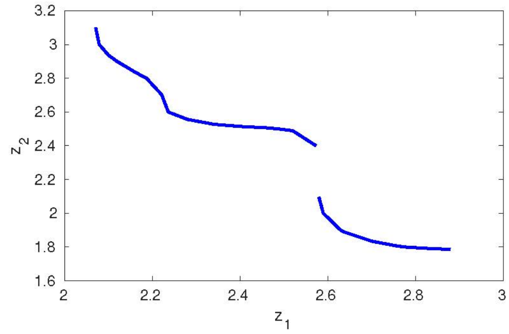
$$\eta_2 : [z_2^*, \overline{z_2}] \to [z_1^*, \overline{z_1}], \quad \eta_2(\zeta_2) = \inf\{\gamma \mid w(t,x,\gamma,\zeta_2) \le 0\}.$$

The graphs of these functions are shown on Figures (3.11b) (blue line) and (3.11c) (green line), respectively. By theorem (3.21), the Pareto front for the initial condition x_0 is obtained as intersection of two graphs. The resulted front is shown on Figure (3.11d). In this example, the Pareto front is non convex and discontinuous.

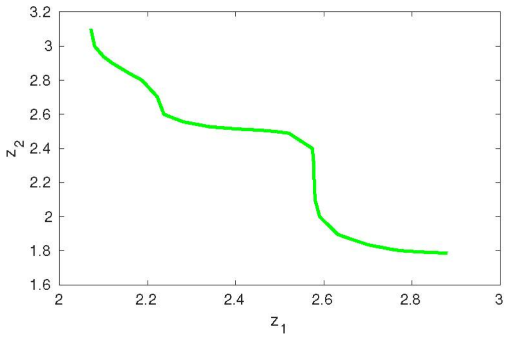
The value function w is also useful to reconstruct Pareto optimal trajectories. We chose three points z_i, $i = 1,2,3$ at the Pareto front, that are represented by points with different colors in Figure 3.12a. We compute the optimal trajectories for $w(t,x_0,z_i), i = 1,2,3$, using the reconstruction algorithm from [ABDZ18]. Figure 3.12b shows these optimal trajectories for the initial states (x_0, z_i) (in figure 3.12b, the color used for each trajectory refers to the corresponding Pareto value z_i in figure 3.12a). Furthermore the obstacles in the river, that is the state constraints are represented in red, and the island, that is the target set is represented in yellow. Let us point out that these trajectories are Pareto optimal for the optimal control problem (3.55). The trajectory in green is optimal for $w(0,x_0,z_1^*(0,x_0),\overline{z_2}(0,x_0))$ that is also optimal trajectory of the scalar problems $\vartheta_1(0,x_0) = z_1^*(0,x_0)$. Moreover the trajectory in blue is optimal for $w(0,x_0,\overline{z_1}(0,x_0),z_2^*(0,x_0)))$ that is also optimal trajectory of the scalar problems $\vartheta_2(0,x_0) = z_2^*(0,x_0))$.



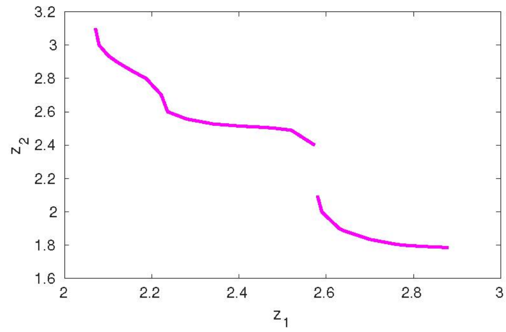
(a) 0-level set of $w(0, x_0, \cdot)$



(b) Graph of function $\eta_1(z_1)$

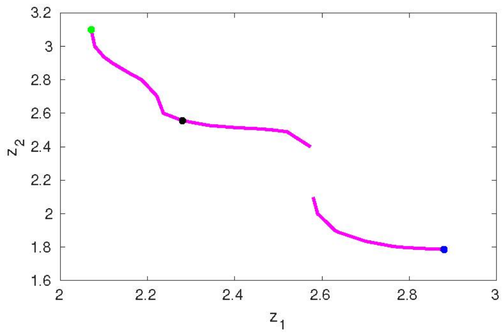


(c) Graph of function $\eta_2(z_2)$

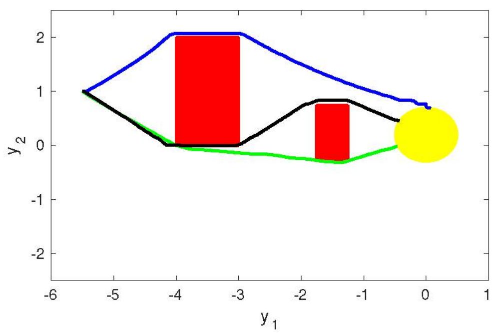


(d) Pareto front $\mathcal{F}(0, x_0)$ equal the intersection of graphs of $\eta_1(z_1)$ and $\eta_2(z_2)$

Figure 3.11: Boat example: Analysis of the 0-level set of the value function w at $t = 0$ and $x_0 = (-5.5, 1)$.



(a) Points z_i for initial condition of computed trajectories



(b) Optimal trajectories for the initial states (x, z_i)

Figure 3.12: Boat example: optimal trajectories as Pareto optimal solutions

3.9 Concluding remarks

In this chapter we have investigated a bi-objective optimal control problem with cost functions of different nature. We considered the situation where the set of trajectories is not compact and the set of Pareto solutions may be empty. We have studied the relation of the Pareto front corresponding to the relaxed (convexified) bi-objective problem with the original one. More precisely, we proved that for any x^* is a (weak) Pareto optimal solution of the relaxed problem and for any $\varepsilon > 0$ there exist a (weak) ε-Pareto solution for the original problem that is in a neighborhood of x^*. Moreover, the distance between the Pareto optimal value achieved by x^* and the optimal value achieved by x is small and is of order ε.

We gave a characterization of the Pareto front of the relaxed bi-objective problem and we show that it is contained in a 0-level set of the value function w associated to an adequate auxiliary control problem. We gave also a characterization of the ε-Pareto front of the original problem by using the same value function w.

We have tested the relevance of our approach on some bi-objective optimal control problems. The numerical simulations confirm all the theoretical results. However, it should be noticed that for a control problem with two state variables and two cost functions, the method requires to solve a HJB equation in dimension 4. To get an approximation of the Pareto front in a reasonable time, the dimension of state should remain less than 4 or 5. We stress on that this thesis addresses mainly some theoretical questions, the numerical aspects should be investigated further.

The interesting results obtained in this chapter motivated us to investigate the infinite horizon multi-objective optimal control problems. The study of this kind of problems is done in the next chapter.

Chapter 4

Contributions in multi-objective infinite horizon problems

In this chapter we study multi-objective infinite horizon optimal control problems with state constraints and with all objective functions of integral type. Infinite horizon problems are important in applications when imposing a final time is artificial, for instance in some problems that arise in economics and biology. Infinite horizon control problems have been extensively studied in the literature. An overview on optimality conditions of infinite optimal control problems in the form of Pontryagin principles can be found in [CHL91, Hal79]. Such control problems have also been investigated within the HJB approach, see for instance [BCD97, IK96].

Numerical methods to calculate the value function of optimal control problems have also been investigated. The central idea of the numerical methods is the discretization of the continuum that makes the problem finite, and therefore, enables its solution through computers. A well-known technique is the Semi-Lagrangian method [FF14, FG99]. This scheme is obtained in two steps. First, by discretizing in time the dynamic programming principle, which provides an interesting interpretation of the approximations in terms of a discrete representation formula for the value function. The second step consists on a space discretization and it results in a finite dimensional problem that can be solved.

We investigated multi-objective infinite horizon control problems based on the idea proposed in [DZ18] and in Chapter 3 of this thesis, both in finite horizon case. First we introduced an auxiliary optimal control problem, free of state constraints. To compute numerically the value function of such problem we introduced a semi-Lagrangian scheme. Afterwards, with similar arguments as in the finite horizon case we show that the weak Pareto front is contained in the zero level set of the value function of this auxiliary control problem. Moreover, a more detailed characterization of the Pareto front for the multi-objective infinite horizon control problem was presented.

However, after obtaining the (weak) Pareto front for the multi-objective control problem
another question that arises is: can we obtain the (weak) Pareto optimal trajectories? Reconstruction algorithms for finite horizon control problems were proposed for instance in [ABDZ18] and [RV91]. Some results can be found in [BCD97, Appendix A], for infinite horizon control problems without state constraints and with state constraints using some controllability assumption on the set of state constraints. However, there are no results in the literature for infinite horizon optimal control problems with state constraints and without assuming any controllability assumption on the set of constraints.

So another purpose of this chapter is the reconstruction of optimal trajectories for infinite horizon optimal control problems with state constraints. The method proposed in this chapter consider the dynamic programming principle for the value function of this auxiliary problem to generate (weak) Pareto optimal trajectories for the multi-objective control problem without any controllability assumption. This procedure is also a new result to generate optimal trajectories for infinite horizon optimal control problems with one objective where we can apply the method to the auxiliary control problem (2.22) introduced in [ABZ13].

4.1 Problem formulation

For a given nonempty compact subset U of $\mathbb{R}^m$ ($m \geq 1$) consider the set of admissible controls defined by:

$$\mathcal{U} = \{\mathbf{u} : [0, \infty[\to \mathbb{R}^m \text{ measurable, } \mathbf{u}(t) \in U \text{ a.e}\}.$$

Consider the controlled system:

$$\begin{cases} \dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)) \text{ a.e } s \in [0, \infty[, \\ \mathbf{y}(0) = x, \end{cases} \tag{4.1}$$

where $\mathbf{u} \in \mathcal{U}$, and $f : \mathbb{R}^n \times U \mapsto \mathbb{R}^n$ satisfies:

$$\begin{cases} (i) \ \exists\ L_f \geq 0, \text{such that for any } x,\ y \in \mathbb{R}^n \text{ and for all } u \in U : \\ \qquad |f(x,u) - f(y,u)| \leq L_f|x - y|, \end{cases} \tag{4.2}$$

Note that (4.2) implies that there exits $c_f > 0$ such that for all $x \in \mathbb{R}^n$

$$\max\{|f(x,u)| : u \in U\} \leq c_f(1 + |x|). \tag{4.3}$$

It is known that under assumption (4.2), for any $\mathbf{u} \in \mathcal{U}$ there exists a unique absolutely continuous trajectory $\mathbf{y} = \mathbf{y}_x^{\mathbf{u}}$ satisfying (4.1). The set of all absolutely continuous solution of (4.1) on $[0, \infty[$, starting from the position x will be denoted as:

$$\mathbb{X}_x = \{(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) :\ \mathbf{y}_x^{\mathbf{u}} \text{ satisfies (4.1) for } \mathbf{u} \in \mathcal{U}\}.$$

Let $\mathcal{K} \subset \mathbb{R}^n$ be a nonempty closed set of constraints. The set of all admissible trajectories starting from the position x will be denoted by

$$\mathbb{X}_x^{\mathcal{K}} = \{(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x :\ \mathbf{y}_x^{\mathbf{u}}(s) \in \mathcal{K},\ \forall\ s \in [0, \infty[\}.$$

Clearly, in view of (4.2), the set $\mathbb{X}_x$ is nonempty, while the set $\mathbb{X}_x^{\mathcal{K}}$ may be empty if there is no admissible control input that keeps the trajectory in the set $\mathcal{K}$.

Now consider the r-dimensional running cost $\ell : \mathbb{R}^n \times U \to \mathbb{R}^r$, satisfying

$$\begin{cases} (i) \ \ \exists\ L_i > 0,\ i = 1, \ldots, r \text{ such that, for any } x,\ y \in \mathbb{R}^n \text{ and } u \in U : \\ \qquad |\ell_i(x,u) - \ell_i(y,u)| \leq L_i|x - y| \ \forall\ x, y \ \in\ \mathbb{R}^n,\ i = 1, \ldots, r. \\ (ii) \ \exists\ M_i > 0,\ i = 1, \ldots, r \text{ such that, for any } x \in \mathbb{R}^n \text{ and } u \in U : \\ \qquad |\ell_i(x,u)| \leq M_i,\ i = 1, \ldots, r. \end{cases} \tag{4.4}$$

For $x \in \mathbb{R}^n$, the objective functions $J_i(x; \cdot, \cdot) : W^{1,1}[0, \infty[\times \mathcal{U} \to \mathbb{R}$ are defined, for $i = 1, \ldots, r$, as:

$$J_i(x; \mathbf{y}, \mathbf{u}) = \int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}(s), \mathbf{u}(s)) ds. \tag{4.5}$$

Note here that we are considering that the discount factor $e^{\lambda t}$ is the same for both objectives, this fact is important to be able to obtain a Dynamic Programming Principle for the considered optimal control problem. Moreover, in order to prove some results we are going to need the following hypothesis on the discount factor λ

$$\lambda > L_f + c_f \tag{4.6}$$

We introduce also the $\mathbb{R}^r$ valued objective application $J(x;\cdot,\cdot) : W^{1,1}[0,\infty[\times U \to \mathbb{R}^r$, defined by:

$$J(x;\mathbf{y},\mathbf{u}) = (J_1(x;\mathbf{y},\mathbf{u}),\dots,J_r(x;\mathbf{y},\mathbf{u})).$$

Now, consider the multi-objective optimal control problem:

$$\begin{cases} \inf \quad J(x;\mathbf{y}_x^{\mathbf{u}},\mathbf{u}) \\ \\ \text{s.t.} \quad (\mathbf{y}_x^{\mathbf{u}},\mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}. \end{cases} \tag{4.7}$$

If there exists a control $\mathbf{u} \in \mathcal{U}$ at which all objectives attain its optimum, no special methods are needed. To avoid such trivial cases we assume that there does not exist a single control that minimizes all objective functions.

4.2 Auxiliary control problem

In what follows we also consider the single objective optimal control problems and the associated value functions:

$$\vartheta_i(x) = \inf_{(\mathbf{y}_x^{\mathbf{u}},\mathbf{u})\in\mathbb{X}_x^{\mathcal{K}}} J_i(x;\mathbf{y}_x^{\mathbf{u}},\mathbf{u}), \quad i=1,\dots,r. \tag{4.8}$$

In this section, we introduce a scalar control problem whose value function will provide a characterization of the (weak) Pareto front and will also provide all the single value functions $\vartheta_i(x)$, for $i=1,\dots,r$. For this, define an augmented state vector $(x,z) \in \mathbb{R}^n \times \mathbb{R}^r$ and the following augmented dynamical system for $s \geq 0$

$$\begin{cases} \dot{\mathbf{y}}(s) = f(\mathbf{y}(s),\mathbf{u}(s)) \\ \dot{\mathbf{z}}(s) = \lambda\mathbf{z}(s) - \ell(\mathbf{y}(s),\mathbf{u}(s)) \\ \mathbf{y}(0) = x, \ \mathbf{z}(0) = z. \end{cases} \tag{4.9}$$

By assumptions (4.2) and (4.4) for every $\mathbf{u} \in \mathcal{U}$ there exists a unique absolutely continuous trajectory $(\mathbf{y}_x^{\mathbf{u}},\mathbf{z}_z^{\mathbf{u}})$ satisfying (4.9). Moreover, note that

$$\mathbf{z}_{z_i}^{\mathbf{u}}(s) = e^{\lambda s}z_i - \int_0^s e^{\lambda(s-\tau)}\ell_i(\mathbf{y}_x^{\mathbf{u}}(\tau),\mathbf{u}(\tau))d\tau, \text{ for } i=1,\dots,r$$

.

In what follows we are going to assume that:

$$\forall\, \tau \in \mathbb{R},\ x \in \mathbb{R}^n,\ \left\{ \begin{pmatrix} f(x,u) \\ e^{-\lambda\tau}\ell(x,u)+\eta \end{pmatrix}, u \in U,\ 0 \leq \eta \leq M_\ell - \ell(x,u) \right\}, \tag{4.10}$$

is a convex subset of $\mathbb{R}^n \times \mathbb{R}^r$, where M_ℓ is the vector formed by the bounds M_i of the functions ℓ_i according to assumption (4.4).

With the purpose of characterizing the weak Pareto front and the Pareto front for the infinite horizon multi-objective optimal control problem (4.7), consider $g : \mathbb{R}^n \to \mathbb{R}$ a Lipschitz and bounded function such that

$$g(x) \leq 0 \Leftrightarrow x \in \mathcal{K}. \tag{4.11}$$

Remark 4.1 *Since $\mathcal{K}$ is closed, such a function g exists. Indeed, let $\eta_1, \eta_2 > 0$ and denote by $d_{\mathcal{K}}$ the signed distance to $\mathcal{K}$, where*

$$d_{\mathcal{K}}(x) = \begin{cases} d(x, \partial\mathcal{K}) & \text{if } x \in \mathcal{K}^C \\ -d(x, \partial\mathcal{K}) & \text{if } x \in \mathcal{K}. \end{cases}$$

The function $g(x) = \min\left(\max\left(-\eta_1, d_{\mathcal{K}}(x)\right), \eta_2\right)$ satisfies the statement (4.11)*, is Lipschitz continuous and $|g(x)| \leq \max\{\eta_1, \eta_2\}$, for all $x \in \mathbb{R}^n$.*

In the sequel we denote by L_g the Lipschitz constant of g and by M_g the positive constant such that $|g(x)| \leq M_g$, for all $x \in \mathbb{R}^n$.

We associate with the multi-objective optimal control problem (4.7) the following augmented control problem and its value function w:

$$w(x,z) = \inf_{\mathbf{u}\in\mathcal{U}} \left\{ \left[\bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s)) ds - z_i \right) \right] \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}}(\theta)\right) \right\}. \quad (4.12)$$

Under assumptions (4.2), (4.4) and (4.10), the value function $w(x,z)$ has a minimizer $\mathbf{u} \in \mathcal{U}$. Following [DZ18] the value function w satisfies the following property.

Proposition 4.2 *Assume that* (4.2), (4.4) *and* (4.6) *hold, then the value function w is Lipschitz continuous.*

PROOF. Consider (x,z), $(x',z') \in \mathbb{R}^n \times \mathbb{R}^r$. By using the definition of w and the simple inequalities:

$$\inf(A) - \inf(B) \leq \sup(A-B) \quad \text{and} \quad \max(A,B) - \max(C,D) \leq \max(A-C, B-D),$$

we get

$$\begin{aligned}
|w(x,z) - w(x',z)| &= \Bigg| \inf_{\mathbf{u}\in\mathcal{U}} \left\{ \bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s)) ds - z_i \right) \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g(\mathbf{y}_x^{\mathbf{u}}(\theta)) \right\} \\
&\quad - \inf_{\mathbf{u}\in\mathcal{U}} \left\{ \bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_{x'}^{\mathbf{u}}(s), \mathbf{u}(s)) ds - z_i' \right) \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g(\mathbf{y}_{x'}^{\mathbf{u}}(\theta)) \right\} \Bigg| \\
&\leq \Bigg| \sup_{\mathbf{u}\in\mathcal{U}} \Bigg\{ \left(\bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s)) ds - z_i \right) \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g(\mathbf{y}_x^{\mathbf{u}}(\theta)) \right) \\
&\quad - \left(\bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_{x'}^{\mathbf{u}}(s), \mathbf{u}(s)) ds - z_i' \right) \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g(\mathbf{y}_{x'}^{\mathbf{u}}(\theta)) \right) \Bigg\} \Bigg| \\
&\leq \sup_{\mathbf{u}\in\mathcal{U}} \Bigg\{ \bigvee_{i=1}^{r} \left| \int_0^\infty e^{-\lambda s} \left[\ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s)) - \ell_i(\mathbf{y}_{x'}^{\mathbf{u}}(s), \mathbf{u}(s)) \right] ds + z_i' - z_i \right| \qquad (4.13) \\
&\qquad \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} \left| g(\mathbf{y}_x^{\mathbf{u}}(\theta)) - g(\mathbf{y}_{x'}^{\mathbf{u}}(\theta)) \right| \Bigg\}.
\end{aligned}$$

For $T > 0$, define

$$\Delta_i = \int_0^T e^{-\lambda s} \left| \ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s)) - \ell_i(\mathbf{y}_{x'}^{\mathbf{u}}(s), \mathbf{u}(s)) \right| ds, \text{ for } i = 1, \dots, r,$$

$$\widetilde{\Delta}_i = \int_T^\infty e^{-\lambda s} \left| \ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s)) - \ell_i(\mathbf{y}_{x'}^{\mathbf{u}}(s), \mathbf{u}(s)) \right| ds, \text{ for } i = 1, \dots, r.$$

From (4.4) and Proposition(2.2) (c) we get, for $i = 1, \dots, r$,

$$\Delta_i \leq L_i \int_0^T e^{-\lambda s} \left|\mathbf{y}_x^{\mathbf{u}}(s) - \mathbf{y}_{x'}^{\mathbf{u}}(s)\right| ds \leq L_i \int_0^T |x-x'| e^{(L_f-\lambda)s} = L_i \frac{e^{(L_f-\lambda)T}-1}{L_f-\lambda}|x-x'|,$$

and

$$\widetilde{\Delta}_i \leq 2M_i \int_T^\infty e^{-\lambda s} ds = \frac{2M_i}{\lambda} e^{-\lambda T}.$$

Now choose T such that $e^{-T} \leq |x-x'|^{\frac{1}{\lambda}}$ and using the fact that $\lambda > L_f$ (assumption (4.6)), we get

$$\begin{aligned} \Delta_i &\leq \frac{L_i}{L_f-\lambda}|x-x'|(|x-x'|^{\frac{-L_f}{\lambda}+1} - 1) = \frac{L_i}{L_f-\lambda}|x-x'|^{2-\frac{L_i}{\lambda}} + \frac{L_f}{\lambda - L_f}|x-x'| \\ &\leq \frac{L_i}{\lambda-L_f}|x-x'|, \ i = 1, \dots, r, \end{aligned}$$

and

$$\widetilde{\Delta}_i \leq \frac{2M_i}{\lambda}|x-x'|, \ i = 1, \dots, r.$$

Therefore from (4.13)

$$\begin{aligned} &|w(x,z) - w(x',z')| \\ &\leq \sup_{\mathbf{u}\in\mathcal{U}} \left\{ \bigvee_{i=1}^r \left[\left(\frac{L_i}{\lambda - L_f} + \frac{2M_i}{\lambda} \right) |x-x'| + |z_i' - z_i| \right] \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} L_g \left|\mathbf{y}_x^{\mathbf{u}}(\theta) - \mathbf{y}_{x'}^{\mathbf{u}}(\theta)\right| \right\} \\ &\leq \max \left\{ \bigvee_{i=1}^r C_i(|x-x'| + |z'-z|), L_g|x-x'| \right\} \leq K(|x-x'| + |z-z'|), \end{aligned}$$

for some $K > 0$. □

The Dynamic Programming Principle for the multi-objective infinite horizon optimal control problem with state constraints is proved in the following theorem.

Proposition 4.3 (Dynamic Programming Principle) *Assume that* (4.2) *and* (4.4) *hold. Then, for all* $h \geq 0$ *and* $(x,z) \in \mathbb{R}^n \times \mathbb{R}^r$, *we have*

$$w(x,z) \quad = \quad \inf_{\mathbf{u}\in\mathcal{U}} \left\{ e^{-\lambda h} w(\mathbf{y}_x^{\mathbf{u}}(h), \mathbf{z}_z^{\mathbf{u}}(h)) \bigvee \max_{\theta\in[0,h]} e^{-\lambda\theta} g(\mathbf{y}_x^{\mathbf{u}}(\theta)) \right\}. \tag{4.14}$$

PROOF. For any measurable control $\mathbf{u} \in \mathcal{U}$, we shall denote $\mathbf{u}_1$ the restriction of $\mathbf{u}$ on $[0,h]$ and $\mathbf{u}_2$ the measurable control of $\mathcal{U}$ such that $\mathbf{u}_2(t) = \mathbf{u}(t+h)$ a.e. $t \geq 0$. Using $\mathbf{y}_x^{\mathbf{u}}(s+h) = \mathbf{y}_{\mathbf{y}_x^{\mathbf{u}_1}(h)}^{\mathbf{u}_2}(s)$ we obtain on the first hand, for $i = 1, \dots, r$,:

$$\begin{aligned} \int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s)) ds - z_i &= \int_h^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s)) ds + \int_0^h e^{-\lambda s} \ell_i(y_x^{\mathbf{u}}(s), \mathbf{u}(s)) ds - z_i \\ &= e^{-\lambda h} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}}(s+h), \mathbf{u}(s+h)) ds - \mathbf{z}_{z_i}^{\mathbf{u}_1}(h) \right) \\ &= e^{-\lambda h} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_{\mathbf{y}_x^{\mathbf{u}_1}(h)}^{\mathbf{u}_2}(s), \mathbf{u}_2(s)) ds - \mathbf{z}_{z_i}^{\mathbf{u}_1}(h) \right) \end{aligned} \tag{4.15}$$

and on the other hand

$$\sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}}(\theta)\right) = \sup_{\theta\in[0,\infty[} e^{-\lambda(\theta+h)} g\left(\mathbf{y}_x^{\mathbf{u}}(\theta)\right) \bigvee \max_{\theta\in[0,h]} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}_1}(\theta)\right)$$

$$= \left(e^{-\lambda h} \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g\left(\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}_x^{\mathbf{u}_1}(h)}(\theta)\right)\right) \bigvee \max_{\theta\in[0,h]} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}_1}(\theta)\right). \quad (4.16)$$

Combining (4.15) and (4.16), taking measurable controls $\mathbf{u}_1 : [0,h] \to U$ and $\mathbf{u}_2 \in \mathcal{U}$ we obtain

$$w(x,z) = \inf_{\mathbf{u}_1} \inf_{\mathbf{u}_2} \left[\bigvee_{i=1}^{r} e^{-\lambda h}\left(\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}_x^{\mathbf{u}_1}(h)}(s), \mathbf{u}_2(s))ds - \mathbf{z}_{z_i}^{\mathbf{u}_1}(h)\right)\right]$$

$$\bigvee \left(e^{-\lambda h} \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g\left(\mathbf{y}^{\mathbf{u}_2}_{\mathbf{y}_x^{\mathbf{u}_1}(h)}(\theta)\right)\right) \bigvee \max_{\theta\in[0,h]} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}_1}(\theta)\right)$$

$$= \inf_{\mathbf{u}_1} \left(e^{-\lambda h} w\left(\mathbf{y}_x^{\mathbf{u}_1}(h), \mathbf{z}_z^{\mathbf{u}_1}\right)\right) \bigvee \max_{\theta\in[0,h]} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}_1}(\theta)\right),$$

which is the desired result. □

Another interesting property is that w is monotone with respect to the third argument as proved in the following proposition.

Proposition 4.4 *Assume that* (4.2), (4.4) *hold and let* $x \in \mathbb{R}^n$. *Then*

$$\forall z, z' \in \mathbb{R}^r, \left(z \le z' \Rightarrow w(x,z) \ge w(x,z')\right).$$

PROOF. Let $z, z' \in \mathbb{R}^r$ such that $z \le z'$ and one admissible control $\mathbf{u} \in \mathcal{U}$. Then for all $i = 1, \ldots, r$ we have that

$$J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) - z_i' \le J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) - z_i,$$

and then

$$\left[\bigvee_{i=1}^{r} \left(J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) - z_i'\right)\right] \le \left[\bigvee_{i=1}^{r} \left(J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) - z_i\right)\right].$$

Taking the minimum over all $\mathbf{u} \in \mathcal{U}$ it follows from the last inequality that

$$w(x,z') \le w(x,z).$$

□

Remark 4.5 *It is possible to prove that the value function* w *satisfies the following HJB equation*

$$\min\left(\lambda w(x,z) + \mathcal{H}(x,z,D_x w, D_z w),\ w(x,z) - g(x)\right) = 0 \quad \textit{for } x \in \mathbb{R}^n,\ z \in \mathbb{R}^r \quad (4.17)$$

where the Hamiltonian $\mathcal{H}$ *is defined by:*

$$\mathcal{H}(x,z,p,q) = \sup_{u\in U}\left(-f(x,u)p - \sum_{i=1}^{r}(\lambda z_i - \ell_i(x,u))q_i\right).$$

However, the value function w is not the unique solution of (4.17)*. Usually numerical approximations are analyzed (convergence and error estimate) under the assumption that a comparison principle holds. In the following section we propose a numerical scheme for solving* (4.12) *and we prove the convergence results by using only the Dynamic Programming Principle* (4.14)*.*

4.2.1 Semi-Lagrangian scheme for the auxiliary optimal control problem

Based on [FF14], in order to obtain a Semi-Lagrangian scheme for the auxiliary optimal control problem (4.12) we first consider a time discretization. For this, we fix a time step $h = \Delta t$ and set $t_j = jh$ ($j \in \mathbb{N}$). The simplest way to discretize (4.1) is by using the explicit Euler scheme, which corresponds to the following discrete dynamical system:

$$\begin{cases} y_{j+1} = y_j + hf(y_j, u_j), & j \in \mathbb{N} \\ y_0 = x, \end{cases} \tag{4.18}$$

Here, the sequence of vectors $u_j \in U$ has the role of a discrete control. We denote by $u^n = \{u_j\}$ the sequence as a whole, and by U^n the set of all sequences $\{u_j\} \subset U$. Moreover we associate this sequence with the continuous piecewise constant control defined by

$$\mathcal{U}^h = \{\mathbf{u} \in \mathcal{U} : \mathbf{u}(s) = u_j,\ s \in [t_j, t_{j+1}),\ j \in \mathbb{N}\}.$$

Moreover we are going to denote by $y^n = \{y_j^u\}$ the sequence given by (4.18).

Remark 4.6 *It is also possible to consider higher-order time discretization by using, for example, Runge-Kutta type scheme in* (4.18)*. Some references in this subject are [Cun00, FF94].*

Given the time discretization (4.18) for the controlled system, the corresponding discrete version of the cost function J_i, $i = 1, \ldots, r$, (4.5) may be obtained by a rectangle quadrature

$$J_i^h(x; y^n, u^n) = h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y_j^u, u_j). \tag{4.19}$$

With the discrete dynamics and cost functional, (4.18) and (4.19), respectively, we can define the discrete value function

$$w^h(x,z) = \inf_{u^n \in U^n} \left\{ \left[\bigvee_{i=1}^{r} \left(h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y_j^u, u_j) - z_i \right) \right] \bigvee \sup_{0 \le j \le \infty} e^{-\lambda t_j} g(y_j^u) \right\}. \tag{4.20}$$

Adapting the arguments of the continuous case, it is possible to prove a discrete Dynamic Programming Principle for the optimal control problem (4.20).

Proposition 4.7 *[Discrete Dynamic Programming Principle (DDPP)] Assume that* (4.2)*,* (4.4) *and* (4.6) *hold. Then w^h is Lipschitz continuous. Furthermore for $h > 0$, $(x, z) \in \mathbb{R}^n \times \mathbb{R}^r$ and for any integer k,*

$$w^h(x,z) \;=\; \inf_{u^n \in U^n} \left\{ e^{-\lambda t_k} w^h(y_k, z_k) \bigvee \max_{0 \le j \le k-1} e^{-\lambda t_j} g(y_j) \right\}, \tag{4.21}$$

where $z_{i_k} = e^{\lambda t_k} \left(z_i - h \sum_{j=0}^{k-1} e^{-\lambda t_j} \ell_i(y_j, u_j) \right)$ *for* $i = 1, \ldots, r$.

PROOF. Consider (x,z), $(x',z') \in \mathbb{R}^n \times \mathbb{R}^r$. By similar arguments as in the proof of Proposition 4.2 we obtain that

$$|w^h(x,z) - w^h(x',z)| \leq \sup_{u^n \in U^n} \left\{ \bigvee_{i=1}^{r} \left| h \sum_{j=0}^{\infty} e^{-\lambda t_j} \left[\ell_i(y^u_{j_x}, u_j) - \ell_i(y^u_{j_{x'}}, u_j) \right] + z'_i - z_i \right| \right.$$
$$\left. \bigvee \sup_{0 \leq j \leq \infty} e^{-\lambda t_j} \left| g(y_{j_x}) - g(y_{j_{x'}}) \right| \right\}.$$

For $i = 1, \ldots, r$, we have that

$$\begin{aligned}
\Delta_i &= h \sum_{j=0}^{\infty} e^{-\lambda t_j} \left| \ell_i(y^u_{j_x}, u_j) - \ell_i(y^u_{j_{x'}}, u_j) \right| \\
&\leq L_i h \sum_{j=0}^{\infty} e^{-\lambda t_j} \left| y_{j_x} - y_{j_{x'}} \right| \\
&\leq L_i h |x - x'| \sum_{j=0}^{\infty} e^{(L_f - \lambda) t_j},
\end{aligned}$$

as by assumption (4.6) $\lambda > L_f$ then $\sum_{j=0}^{\infty} e^{(L_f-\lambda)t_j}$ is convergent. Moreover

$$\sup_{0 \leq j \leq \infty} e^{-\lambda t_j} \left| g(y_{j_x}) - g(y_{j_{x'}}) \right| \leq L_g |x - x'|$$

Therefore there exist $K > 0$ such that

$$|w^h(x,z) - w^h(x',z')| \leq K(|x - x'| + |z - z'|),$$

that means that the function w^h is Lipschitz continuous.

Now to prove the DDPP, define $z_{i_k} = e^{\lambda t_k} \left(z_i - h \sum_{j=0}^{k-1} e^{-\lambda t_j} \ell_i(y_j, u_j) \right)$, then for $i = 1, \ldots, r$, we get

$$\begin{aligned}
h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y^u_{j_x}, u_j) - z_i &= h \sum_{j=k}^{\infty} e^{-\lambda t_j} \ell_i(y^u_{j_x}, u_j) + h \sum_{j=0}^{k-1} e^{-\lambda t_j} \ell_i(y^u_{j_x}, u_j) - z_i \\
&= h \sum_{j=0}^{\infty} e^{-\lambda (t_j + t_k)} \ell_i(y^u_{j_{y_k}}, u_j) - e^{-\lambda t_k} z_{i_k} \\
&= e^{\lambda t_k} \left(h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y^u_{j_{y_k}}, u_j) - z_{i_k} \right) \qquad (4.22)
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\sup_{0 \leq j \leq \infty} e^{-\lambda t_j} g(y^u_{j_x}) &= \sup_{0 \leq j \leq \infty} e^{-\lambda (t_j + t_k)} g(y^u_{j_{y_k}}) \bigvee \max_{0 \leq j \leq k-1} e^{-\lambda t_j} g(y^u_{j_x}) \\
&= e^{-\lambda t_k} \sup_{0 \leq j \leq \infty} e^{-\lambda t_j} g(y^u_{j_{y_k}}) \bigvee \max_{0 \leq j \leq k-1} e^{-\lambda t_j} g(y^u_{j_x}). \qquad (4.23)
\end{aligned}$$

Combining (4.22) and (4.23) we obtain

$$w(x,z) = \inf_{u^n \in U^n} \left[\bigvee_{i=1}^{r} e^{\lambda t_k} \left(h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y^u_{j_{y_k}}, u_j) - z_{i_k} \right) \right]$$
$$\bigvee e^{-\lambda t_k} \sup_{0 \le j \le \infty} e^{-\lambda t_j} g(y^u_{j_{y_k}}) \bigvee \max_{0 \le j \le k-1} e^{-\lambda t_j} g(y^u_{j_x})$$
$$= \inf_{u^n \in U^n} \left\{ e^{-\lambda t_k} w^h(y_{k_x}, z_k) \bigvee \max_{0 \le j \le k-1} e^{-\lambda t_j} g(y_{j_x}) \right\},$$

which is the desired result. □

The following result gives us a convergence rate concerning the time discretization.

Theorem 4.8 *Assume that* (4.2)*,* (4.4) *and* (4.10) *hold and let* w^h *be defined as in* (4.20) *and* w *be the value function defined in* (4.12)*. Then*

$$||w - w^h||_\infty \le Ch, \quad \textit{for some constant } C > 0 \textit{ and } h \in]0,1[.$$

PROOF. **Step 1.** Let $(x,z) \in \mathbb{R} \times \mathbb{R}^r$, $h \in]0,1[$. By the assumptions there exist $u^* = \{u^*_j\} \in U^n$ and a associate discrete trajectory $\{y^*_j\}$ that are optimal for the discrete optimal control problem $w^h(x,z)$. Consider a control $\mathbf{u}^*_h(s) \in \mathcal{U}^h$ such that $\mathbf{u}^*_h(s) = u^*_j$ for $s \in [t_j, t_{j+1}[,\ j \in \mathbb{N}$. Then by definition of w, (4.12), and w^h, (4.20), we obtain that

$$w(x,z) - w^h(x,z) \le \left[\bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}^{\mathbf{u}^*_h}_x(s), \mathbf{u}^*_h(s)) ds - z_i \right) \right] \bigvee \sup_{\theta \in [0,\infty[} e^{-\lambda \theta} g\left(\mathbf{y}^{\mathbf{u}^*_h}_x(\theta) \right)$$
$$- \left[\bigvee_{i=1}^{r} \left(h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y^*_j, u^*_j) - z_i \right) \right] \bigvee \sup_{0 \le j \le \infty} e^{-\lambda t_j} g(y^*_j).$$
$$\le \left[\bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}^{\mathbf{u}^*_h}_x(s), \mathbf{u}^*_h(s)) ds - h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y^*_j, u^*_j) \right) \right]$$
$$\bigvee \left[\sup_{\theta \in [0,\infty[} e^{-\lambda \theta} g\left(\mathbf{y}^{\mathbf{u}^*_h}_x(\theta) \right) - \sup_{0 \le j \le \infty} e^{-\lambda t_j} g(y^*_j) \right] \tag{4.24}$$

Step 1.a Based on [BCD97], we claim that, for $i = 1, \ldots, r$ exist $K_i > 0$ such that

$$\left| \int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}^{\mathbf{u}^*_h}_x(s), \mathbf{u}^*_h(s)) ds - h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y^*_j, u^*_j) \right| \le K_i h. \tag{4.25}$$

In the following $\lfloor s/h \rfloor$ denotes the largest integer which is less than or equal to s/h. Define $\tilde{\mathbf{y}}^{\mathbf{u}^*_h}_x(s) = y^*_j$, for $s \in [t - j, t_{j+1}[$ and observe that $\tilde{\mathbf{y}}^{\mathbf{u}^*_h}_x$ can be expressed as

$$\tilde{\mathbf{y}}^{\mathbf{u}^*_h}_x(s) = x + \int_0^{\lfloor s/h \rfloor h} f(\tilde{\mathbf{y}}^{\mathbf{u}^*_h}_x(\tau), \mathbf{u}^*_h(\tau)) d\tau, \quad s \ge 0.$$

Then

$$|\mathbf{y}^{\mathbf{u}^*_h}_x(s) - \tilde{\mathbf{y}}^{\mathbf{u}^*_h}_x(s)| \le \int_0^{\lfloor s/h \rfloor h} |f(\mathbf{y}^{\mathbf{u}^*_h}_x(\tau), \mathbf{u}^*_h(\tau)) - f(\tilde{\mathbf{y}}^{\mathbf{u}^*_h}_x(\tau), \mathbf{u}^*_h(\tau))| d\tau + \int_{\lfloor s/h \rfloor h}^{s} |f(\mathbf{y}^{\mathbf{u}^*_h}_x(\tau), \mathbf{u}^*_h(\tau))| d\tau.$$

By the Lipschitz continuity of f

$$\begin{aligned}|\mathbf{y}_x^{\mathbf{u}_h^*}(s) - \tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}(s)| &\le L_f \int_0^{\lfloor s/h \rfloor h} |\mathbf{y}_x^{\mathbf{u}_h^*}(\tau) - \tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}(\tau)| d\tau + \int_{\lfloor s/h \rfloor h}^{s} c_f(1+|x|)e^{c_f \tau} d\tau \\ &\le L_f \int_0^{\lfloor s/h \rfloor h} |\mathbf{y}_x^{\mathbf{u}_h^*}(\tau) - \tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}(\tau)| d\tau + c_f(1+|x|)e^{c_f s} h.\end{aligned}$$

Hence, by Gronwall's Lemma (Lemma 2.1)

$$|\mathbf{y}_x^{\mathbf{u}^*}(s) - \tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}(s)| \le c_f(1+|x|)e^{c_f s} h e^{L_f s}. \tag{4.26}$$

Now, for $i = 1, \ldots, r$ we have that

$$\left| \int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}_h^*}(s), \mathbf{u}_h^*(s)) ds - h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y_j^*, u_j^*) \right| \le X_i + Y_i,$$

with

$$\begin{aligned}X_i &= \int_0^\infty e^{-\lambda s} \left| \ell_i(\mathbf{y}_x^{\mathbf{u}_h^*}(s), \mathbf{u}_h^*(s)) - \ell_i(\tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}(s), \mathbf{u}_h^*(s)) \right| ds \\ Y_i &= \left| \int_0^\infty e^{-\lambda s} \ell_i(\tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}(s), \mathbf{u}_h^*(s)) ds - h \sum_{j=0}^{\infty} e^{-\lambda t_j} \ell_i(y_j^*, u_j^*) \right|,\end{aligned}$$

for $i = 1, \ldots, r$. Let us estimate the $X_i's$ and $Y_i's$, $(i = 1, \ldots, r)$. From (4.4) and (4.26), we obtain, for $i = 1, \ldots, r$

$$\begin{aligned}X_i &\le L_i \int_0^\infty e^{-\lambda s} \left| \mathbf{y}_x^{\mathbf{u}^*}(s) - \tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}(s) \right| ds \le L_i c_f (1+|x|) h \int_0^\infty e^{(c_f + L_f - \lambda)s} ds \\ &\le \frac{L_i c_f (1+|x|)}{(\lambda - c_f - L_f)} h,\end{aligned}$$

where we used the assumption that $\lambda > c_f + L_f$. From definition of $\tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}$ and $\mathbf{u}_h^*$ and (4.4) we have that

$$Y_i \le \int_0^\infty |\ell_i(\tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}(s), \mathbf{u}_h^*(s))| e^{-\lambda s} - e^{-\lambda \lfloor s/h \rfloor h} ds \le M_i \int_0^\infty e^{-\lambda s} - e^{-\lambda \lfloor s/h \rfloor h} ds,$$

for $i = 1, \ldots, r$. Now , for $h < 1$ we have that $s - 1 < s - h \le \lfloor s/h \rfloor h \le s$, then from the mean value theorem we obtain that

$$Y_i \le M_i \int_0^\infty e^{-\lambda(s-1)} (\lambda(s - \lfloor s/h \rfloor h)) ds \le M_i e^{\lambda} h.$$

Therefore (4.25) is proved with $K_i = \dfrac{L_i c_f (1+|x|)}{(\lambda - c_f - L_f)} + M_i e^{\lambda}$.

Step 1.b Now, we claim that

$$\left| \sup_{\theta \in [0,\infty[} e^{-\lambda \theta} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(\theta)\right) - \sup_{0 \le j \le \infty} e^{-\lambda t_j} g(y_j^*) \right| \le K_0 h. \tag{4.27}$$

In fact, by (4.26), we obtain

$$\left|\sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(\theta)\right) - \sup_{0\le j\le\infty} e^{-\lambda t_j} g(y_j^*)\right| = \left|\sup_{0\le j\le\infty}\max_{\theta\in[t_j,t_{j+1}]} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(\theta)\right) - \sup_{0\le j\le\infty} e^{-\lambda t_j} g(y_j^*)\right|$$
$$\le \left|\sup_{0\le j\le\infty}\max_{\theta\in[t_j,t_{j+1}]} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(\theta)\right) - \sup_{0\le j\le\infty} e^{-\lambda t_j} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(t_j)\right)\right| + \left|\sup_{0\le j\le\infty} e^{-\lambda t_j} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(t_j)\right) - \sup_{0\le j\le\infty} e^{-\lambda t_j} g(y_j^*)\right|$$
$$\le \sup_{0\le j\le\infty}\left|\max_{\theta\in[t_j,t_{j+1}]} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(\theta)\right) - e^{-\lambda t_j} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(t_j)\right)\right| + \sup_{0\le j\le\infty} e^{-\lambda t_j}\left|g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(t_j)\right) - g(y_j^*)\right|$$
$$\le \sup_{0\le j\le\infty}\left|\max_{\theta\in[t_j,t_{j+1}]}\left[e^{-\lambda(\theta-t_j)} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(\theta)\right) + e^{-\lambda t_j}\left(g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(\theta)\right) - g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(t_j)\right)\right)\right]\right| + \sup_{0\le j\le\infty} L_g e^{-\lambda t_j}\left|\mathbf{y}_x^{\mathbf{u}_h^*}(t_j) - y_j^*\right|$$
$$\le M_g h + L_g \sup_{0\le j\le\infty}\max_{\theta\in[t_j,t_{j+1}]} e^{-\lambda t_j}|\mathbf{y}_x^{\mathbf{u}_h^*}(\theta) - \mathbf{y}_x^{\mathbf{u}_h^*}(t_j)| + \sup_{0\le j\le\infty} L_g e^{-\lambda t_j}\left|\mathbf{y}_x^{\mathbf{u}_h^*}(t_j) - y_j^*\right|$$
$$\le M_g h + L_g c_f(1+|x|)e^{c_f}h \sup_{0\le j\le\infty} e^{(c_f-\lambda)t_j} + L_g c_f(1+|x|)h \sup_{0\le j\le\infty} e^{(c_f+L_f-\lambda)t_j}|.$$

Then, by the assumption that $\lambda > c_f + L_f$

$$\left|\sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}_h^*}(\theta)\right) - \sup_{0\le j\le\infty} e^{-\lambda t_j} g(y_j^*)\right| \le (M_g + L_g c_f(1+|x|)e^{c_f} + L_g c_f(1+|x|))h, \tag{4.28}$$

so (4.27) is proved with $K_0 = M_g + L_g c_f(1+|x|)e^{c_f} + L_g c_f(1+|x|)$.

Substituting the previous estimates (4.25) and (4.27) in (4.24) we obtain that

$$w(x,z) - w^h(x,z) \le \bigvee_{i=0}^{r} K_i h. \tag{4.29}$$

Step 2. To perform the reverse estimate, Let $(x,z)\in\mathbb{R}\times\mathbb{R}^r$, $h\in]0,1[$ and consider $\mathbf{u}^*\in\mathcal{U}$ that is a minimizer for $w(x,z)$. Then

$$\begin{aligned} w(x,z) &= \bigvee_{i=1}^{r}\left(\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}_x^{\mathbf{u}^*}(s),\mathbf{u}^*(s))ds - z_i\right)\bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}^*}(\theta)\right) \\ &\ge \bigvee_{i=1}^{r}\left(\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}_x^{\mathbf{u}^*}(s),\mathbf{u}^*(s))ds - z_i\right)\bigvee\left(\sup_{0\le j\le\infty} e^{-\lambda t_j} g\left(\mathbf{y}_x^{\mathbf{u}^*}(t_j)\right)\right) \end{aligned} \tag{4.30}$$

Moreover, by Hypothesis (4.10), according with [AC84, Thm. 0.5.3], for $j\in\mathbb{N}$, there exists $\overline{u}\in U$ such that

$$\int_{t_j}^{t_{t_{j+1}}} f(\overline{y}_j,\mathbf{u}^*(s))ds = hf(\overline{y}_j,\overline{u}_j), \tag{4.31}$$

where $\overline{y}_j$ is given by (4.18) with the controls $\overline{u}_j$. Define $\overline{\mathbf{y}}_x(s) = \overline{y}_j$, for $s\in[t_j,t_{j+1}[$. Observe that by (4.31), $\overline{\mathbf{y}}_x$ can be expressed as

$$\begin{aligned} \overline{\mathbf{y}}_x(s) &= x + h\sum_{j=0}^{\lfloor s/h\rfloor} f(\overline{y}_j,\overline{u}_j) \\ &= x + \int_0^{(\lfloor s/h\rfloor+1)h} f(\overline{y}_j,\mathbf{u}^*(\tau))d\tau \\ &= x + \int_0^{(\lfloor s/h\rfloor+1)h} f(\overline{\mathbf{y}}_x(\tau),\mathbf{u}^*(\tau))d\tau, \quad s\ge 0. \end{aligned}$$

Then, by Lipschitz continuity of f

$$|\mathbf{y}_x^{\mathbf{u}^*}(s)-\overline{\mathbf{y}}_x(s)| \le \int_0^s |f(\mathbf{y}_x^{\mathbf{u}^*}(\tau),\mathbf{u}^*(\tau))-f(\overline{\mathbf{y}}_x(\tau),\mathbf{u}^*(\tau))|d\tau + \int_s^{(\lfloor s/h\rfloor+1)h} |f(\overline{\mathbf{y}}_x(\tau),\mathbf{u}^*(\tau))|d\tau$$
$$\le L_f\int_0^s |\mathbf{y}_x^{\mathbf{u}^*}(\tau)-\overline{\mathbf{y}}_x(\tau)|d\tau + c_f(1+|x|)e^{c_f(s+1)}h.$$

Hence, by Gronwall's Lemma (Lemma 2.1)

$$|\mathbf{y}_x^{\mathbf{u}^*}(s)-\overline{\mathbf{y}}_x(s)| \le c_f(1+|x|)e^{c_f(s+1)}he^{L_f s}. \tag{4.32}$$

Now from definition of w^h and (4.30) we obtain that

$$w^h(x,z)-w(x,z) \le \left[\bigvee_{i=1}^r\left(h\sum_{j=0}^\infty e^{-\lambda t_j}\ell_i(\overline{y}_j,\overline{u}_j)-z_i\right)\right]\bigvee \sup_{0\le j\le\infty} e^{-\lambda t_j}g(\overline{y}_j)$$
$$-\bigvee_{i=1}^r\left(\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}_x^{\mathbf{u}^*}(s),\mathbf{u}^*(s))ds - z_i\right)\bigvee\left(\sup_{0\le j\le\infty} e^{-\lambda t_j}g\left(\mathbf{y}_x^{\mathbf{u}^*}(t_j)\right)\right)$$
$$\le\bigvee_{i=1}^r\left(h\sum_{j=0}^\infty e^{-\lambda t_j}\ell_i(\overline{y}_j,\overline{u}_j)-\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}_x^{\mathbf{u}^*}(s),\mathbf{u}^*(s))ds\right) \tag{4.33}$$
$$\bigvee\sup_{0\le j\le\infty} e^{-\lambda t_j}\left(g(\overline{y}_j)-g\left(\mathbf{y}_x^{\mathbf{u}^*}(t_j)\right)\right)$$

Step 2.a For $1=\dots,r$ we have that

$$\left|h\sum_{j=0}^\infty e^{-\lambda t_j}\ell_i(\overline{y}_j,\overline{u}_j)-\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}_x^{\mathbf{u}^*}(s),\mathbf{u}^*(s))ds\right| \le \overline{K}_i h, \tag{4.34}$$

for $\overline{K}_i$ positive constants for $i=1,\dots,r$. The proof follows with similar arguments as in Step 1.a, so for $i=1,\dots,r$

$$\left|h\sum_{j=0}^\infty e^{-\lambda t_j}\ell_i(\overline{y}_j,\overline{u}_j)-\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}_x^{\mathbf{u}^*}(s),\mathbf{u}^*(s))ds\right| \le \overline{X}_i+\overline{Y}_i,$$

with

$$\overline{X}_i = \int_0^\infty e^{-\lambda s}\left|\ell_i(\mathbf{y}_x^{\mathbf{u}^*}(s),\mathbf{u}^*(s))-\ell_i(\overline{\mathbf{y}}_x(s),\mathbf{u}^*(s))\right|ds$$
$$\overline{Y}_i = \left|\int_0^\infty e^{-\lambda s}\ell_i(\overline{\mathbf{y}}_x(s),\mathbf{u}^*(s))ds - h\sum_{j=0}^\infty e^{-\lambda t_j}\ell_i(\overline{y}_j,\overline{u}_j)\right|,$$

So $Y_i \le M_i e^\lambda h$. and assuming that $\lambda > c_f+L_f$ we obtain $X_i \le \dfrac{L_i c_f e^{c_f}(1+|x|)}{(\lambda-c_f-L_f)}h$, therefore (4.34) is proved with $\overline{K}_i = \dfrac{L_i c_f e^{c_f}(1+|x|)}{(\lambda-c_f-L_f)}+M_i e^\lambda$.

Step 2.b By (4.32), we have that

$$\left|\sup_{0\le j\le\infty} e^{-\lambda t_j}\left(g(\overline{y}_j)-g\left(\mathbf{y}_x^{\mathbf{u}^*}(t_j)\right)\right)\right| \le \sup_{0\le j\le\infty} e^{-\lambda t_j}L_g|\overline{y}_j-\mathbf{y}_x^{\mathbf{u}^*}(t_j)|$$
$$=\sup_{0\le j\le\infty} e^{-\lambda t_j}L_g|\overline{\mathbf{y}}_x(t_j)-\mathbf{y}_x^{\mathbf{u}^*}(t_j)|$$
$$\le L_g c_f(1+|x|)e^{c_f}h\sup_{0\le j\le\infty} e^{(c_f+L_f-\lambda)t_j}.$$

As $\lambda > c_f + L_f$ we obtain that

$$\left| \sup_{0 \le j \le \infty} e^{-\lambda t_j} \left(g(\overline{y}_j) - g\left(\mathbf{y}_x^{\mathbf{u}^*}(t_j) \right) \right) \right| \le L_g c_f (1+|x|) e^{c_f} h = \overline{K}_0 h, \tag{4.35}$$

where $\overline{K}_0 = L_g c_f (1+|x|) e^{c_f}$.

Substituting the previous estimates (4.34) and (4.35) in (4.33) we obtain that

$$w^h(x,z) - w(x,z) \le \bigvee_{i=0}^{r} \overline{K}_i h. \tag{4.36}$$

Therefore, as $(x,z) \in \mathbb{R}^n \times \mathbb{R}^r$ are arbitrary, from (4.29) and (4.36) we obtain that

$$||w - w^h||_\infty \le Ch, \text{ for } h \in]0,1[,$$

where $C = \max\{\bigvee_{i=0}^{r} K_i h, \bigvee_{i=0}^{r} \overline{K}_i h\}$. □

Remark 4.9 *The result of Theorem 4.8 can also be obtained by assuming that the function f is bounded and $\lambda > L_f$ instead of $\lambda > L_f + c_f$. In this case the estimate*

$$|\mathbf{y}_x^{\mathbf{u}^*}(s) - \tilde{\mathbf{y}}_x^{\mathbf{u}_h^*}(s)| \le M_f h e^{L_f s},$$

holds true instead of (4.26) *and* (4.32)*. Following the same steps of the prove of Theorem 4.8 and considering the fact that $\lambda > L_f$, Hypothesis* (4.6)*, it is possible to obtain $C > 0$ such that*

$$||w - w^h||_\infty \le Ch, \text{ for } h \in]0,1[.$$

The full discretization (time and space) for (4.20) is performed in a standard way by considering, for $\Delta = (h, \Delta x, \Delta z)$, a regular grid

$$\mathcal{G} = \{(x_k, z_i) \mid x_k = (k_1 \Delta x_1, \ldots, k_n \Delta x_n) \text{ and } z_i = (i_1 \Delta z_1, \ldots, i_r \Delta z_r), \quad (k,i) \in \mathbb{Z}^n \times \mathbb{Z}^r\}.$$

So the fully discrete scheme is the following

$$w_{k,i}^\Delta = \min_{u \in U} \left\{ e^{-\lambda h} [w^\Delta] \left(x_k + h f(x_k, u), e^{\lambda h} \left(z_i - h\ell(x_k, u) \right) \right) \bigvee g(x_k) \right\}, \tag{4.37}$$

where $[w^\Delta]$ is an interpolation function of $w_{k,i}^\Delta$, $k \in \mathbb{Z}$, $i \in \mathbb{Z}$.

We observe that starting from a node (x_k, z_i) from the grid $\mathcal{G}$, the point of the approximate trajectory $x_k + hf(x_k,u)$ and $e^{\lambda h}(z_i - h\ell(x_k,u)$ may not be a point of $\mathcal{G}$. It is for this reason that we calculate

$$[w^\Delta] \left(x_k + h f(x_k, u), e^{\lambda h} \left(z_i - h\ell(x_k, u) \right) \right)$$

by a bilinear interpolation. For this consider the barycentric coefficients $\mu_j^{k,i}(u)$ such that:

$$0 \le \mu_j^{k,i}(u) \le 1, \quad \sum_j \mu_j^{k,i}(u) = 1,$$

$$\sum_j \mu_j^{k,i}(u) x_j = x_k + h f(x_k, u) \text{ and } \sum_j \mu_j^{k,i}(u) z_j = e^{\lambda h} \left(z_i - h\ell(x_k, u) \right).$$

So one possible interpolation for $[w^\Delta] \left(x_k + h f(x_k, u), e^{\lambda h} \left(z_i - h\ell(x_k, u) \right) \right)$ is given by

$$[w^\Delta] \left(x_k + h f(x_k, u), e^{\lambda h} \left(z_i - h\ell(x_k, u) \right) \right) = \sum_j \mu_j^{k,i}(u) w_{k,i}^\Delta.$$

Theorem 4.10 *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_4})$ hold and let w^Δ defined as in (4.37) and w the value function defined in (4.12). Then there exist $C_1, C_2 > 0$ such that

$$||w - w^\Delta||_\infty \le C_1 h + C_2 \frac{|(\Delta x, \Delta z)|}{h}, \quad \text{for } h \in]0,1[.$$

PROOF. In order to estimate the space discretization error for the fully discrete scheme (4.37), consider a control u^* that is optimal for w^h and denoting $\overline{x}_k = x_k + hf(x_k, u^*)$ and $\overline{z}_i = e^{\lambda h}(z_i - h\ell(x_k, u))$ we obtain, by the discrete Dynamic Programming Principle, Proposition (4.7), the one-side estimate

$$\begin{aligned} w^\Delta_{k,i} - w^h(x_k, z_i) &\le e^{-\lambda h}[w^\Delta](\overline{x}_k, \overline{z}_i) \bigvee g(x_k) - e^{-\lambda h} w^h(\overline{x}_k, \overline{z}_i) \bigvee g(x_k) \\ &\le e^{-\lambda h}\Big|[w^\Delta](\overline{x}_k, \overline{z}_i) - [w^h](\overline{x}_k, \overline{z}_i)\Big| + e^{-\lambda h}\Big|[w^h](\overline{x}_k, \overline{z}_i) - w^h(\overline{x}_k, \overline{z}_i)\Big| \\ &\le e^{-\lambda h}||w^\Delta - w^h||_\infty + \mathcal{O}(|(\Delta x, \Delta z)|), \end{aligned}$$

where $[w^h]$ is an interpolation function of w^h, and by Lipschitz continuity of w^h, we have bounded its interpolation error with $\mathcal{O}(\Delta x, \Delta z)$. With similar arguments it is possible to obtain the opposite bound and passing to the ∞-norm, we obtain

$$||w^\Delta - w^h||_\infty \le C_2 \frac{|(\Delta x, \Delta z)|}{h}, \text{ for some } C_2 > 0. \tag{4.38}$$

Combining time and discretization error estimates (Theorem 4.8 and equation (4.38)), we have the complete estimate:

$$||w - w^\Delta||_\infty \le C_1 h + C_2 \frac{|(\Delta x, \Delta z)|}{h}, \quad \text{for } h \in]0,1[\text{ and } C_1,\ C_2 > 0.$$

□

4.3 Characterization of the Pareto fronts

In order to characterize the weak Pareto front for the multi-objective optimal control problem (4.7), for $x \in \mathbb{R}^n$, consider the negative level set of $w(x, \cdot)$:

$$Z(x) = \{z \in \mathbb{R}^r |\ w(x,z) \le 0\}.$$

By Proposition 4.2, the set $Z(x)$ is a closed set of $\mathbb{R}^r$. However, one can notice that this set may be empty whenever $\mathbb{X}^{\mathcal{K}}_x = \emptyset$, that is, when there is no admissible trajectory that remains in the set $\mathcal{K}$. In this case the weak Pareto front is also empty. In particular, for $x \in \mathcal{K}^c$, $Z(x) = \emptyset$.

For every $x \in \mathbb{R}^n$ and $i = 1, \dots, r$, we introduce also the value:

$$z^*_i(x) := \inf \Big\{ \zeta \in \mathbb{R} \ \Big| \exists z \in \mathbb{R}^r \text{ with } z_i = \zeta,\ w(x,z) \le 0 \Big\}. \tag{4.39}$$

With this notation, $z^*_i = +\infty$ if $Z(x) = \emptyset$. Moreover, if $Z(x) \neq \emptyset$ we can remark that

$$Z(x) \subset \prod_{i=1}^r [z^*_i(x), +\infty[. \tag{4.40}$$

The following proposition establishes a link between the negative level set of w, $Z(x)$ and the optimal control problems defined in (4.8).

Proposition 4.11 *Assume that (4.2), (4.4), (4.6) and (4.10) hold. Let $x \in \mathcal{K}$ such that $Z(x) \neq \emptyset$. Then $\vartheta_i(x) < +\infty$, for $i = 1, \ldots, r$ and the following relations hold:*

(i) *For every $z \in \mathbb{R}^r$, we have that $z \in Z(x)$ if and only if there exists $\mathbf{u} \in \mathcal{U}$ such that:*

$$\mathbf{y}_x^{\mathbf{u}}(s) \in \mathcal{K} \ \forall s \in [0, \infty[\quad \textit{and} \quad J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \leq z_i, \ \textit{for } i = 1, \ldots, r. \tag{4.41}$$

(ii) *Moreover,*

$$z_i^*(x) = \vartheta_i(x), \ \textit{for } i = 1, \ldots, r. \tag{4.42}$$

PROOF. **(i)** From the definition of the value function w in (4.12) we have:

$$z \in Z(x) \qquad \Leftrightarrow \quad \exists\ \mathbf{u} \in \mathcal{U}, \text{ s.t. } \forall\ s \in [0, \infty[, \ \mathbf{y}_x^{\mathbf{u}}(s) \in \mathcal{K},$$
$$J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) = \int_0^\infty e^{-\lambda s} \ell_i(y_x^{\mathbf{u}}(s), \mathbf{u}(s)) ds \leq z_i \text{ for } i = 1, \ldots, r$$

(ii) Let $i \in \{1, \ldots, r\}$. It is clear from the characterization of the set $Z(x)$ that if $Z(x) \neq \emptyset$ then $\mathbb{X}_x^{\mathcal{K}} \neq \emptyset$ and so $\vartheta_i(x) < +\infty$.

Let show that $\vartheta_i(x) \leq z_i^*(x)$. By item **(i)** we have that for all $z \in Z(x)$

$$\exists (\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}} \text{ s.t. } J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \leq z_i \text{ for } i = 1, \ldots, r.$$

Therefore $\vartheta_i(x) \leq z_i$ for all $z \in Z(x)$ and then

$$\vartheta_i(x) \leq \inf\{\zeta \in \mathbb{R} \mid \ z \in Z(x) \text{ with } z_i = \zeta\} = z_i^*(x).$$

Let show now that $\vartheta_i(x) \geq z_i^*(x)$. Without loss of generality, we assume here that $i = 1$. The proof will be the same for $i = 2, \ldots, r$. Assume that $\vartheta_1(x) < z_1^*(x)$. Then there exists $\delta \in \mathbb{R}$ such that $\vartheta_1(x) < \delta < z_1^*(x)$. The inequality $\vartheta_1(x) < \delta$ implies that $\exists\ (\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ such that $J_1(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) < \delta$. Then for any $z_i \geq J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u})$, $i = 1, \ldots, r$ we have that,

$$w(x, \delta, z_2, \ldots, z_r) = (J_1(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) - \delta) \bigvee \left[\bigvee_{i=2}^{r} (J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) - z_i) \right] \bigvee \sup_{\theta \in [0, \infty[} e^{-\lambda\theta} g(\mathbf{y}_x^{\mathbf{u}}(\theta)) \leq 0.$$

And then $w(x, \delta, z_2, \ldots, z_r) \leq 0$ which implies that

$$\delta \in \{\gamma \in \mathbb{R} \mid \ \exists z \in \mathbb{R}^r \text{ with } z_1 = \gamma, \ w(t, x, z) \leq 0\}.$$

But, we have chosen δ such that $\delta < z_1^*(x)$ which is impossible. □

We can show that the weak Pareto front of the problem (4.7) is a subset of the zero level set of the value function w.

Theorem 4.12 *Assume that (4.2),(4.4),(4.6),(4.10) hold. Let x such that $Z(x) \neq \emptyset$. Then the weak Pareto front $\mathcal{F}_w(x)$ for the multi-objective optimal control problem (4.7) with the initial condition x is a subset of the zero level set of the value function $w(x, \cdot)$:*

$$\mathcal{F}(x) \subset \mathcal{F}_w(x) \subset \{z \in \mathbb{R}^r \mid \ w(x, z) = 0\}.$$

PROOF. Let $z \in \mathcal{F}_w(x)$. Therefore there exist an admissible pair $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ such that $J(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) = z$ and $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u})$ is a weak Pareto solution of (4.7). By definition of w, we get that $w(x,z) \leq 0$. Assume that $w(x,z) < 0$. Then by Proposition 4.11 there exist $(\overline{\mathbf{y}}_x^{\mathbf{u}}, \overline{\mathbf{u}}) \in \mathbb{X}_x^{\mathcal{K}}$ such that

$$J_i(x; \overline{\mathbf{y}}_x^{\mathbf{u}}, \overline{\mathbf{u}}) < z_i = J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}), \quad \text{for all} \quad i = 1, \ldots, r,$$

which is in contradiction with the weak optimality of $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u})$. We conclude that

$$w(x,z) = 0.$$

□

Let $x \in \mathcal{K}$ such that $Z(x) \neq \emptyset$. For $i = 1, \ldots, r$ define

$$\overline{z}_i(x) = \inf\{\zeta \in \mathbb{R} | \ \big((z_1^*(x), \ldots, z_{i-1}^*(x), \zeta, z_{i+1}^*(x), \ldots, z_r^*(x)\big) \in Z(x)\}.$$

By definition, $z_i^*(x) \leq \overline{z}_i(x) \leq +\infty$, for all $i = 1, \ldots, r$. We consider the closed set of $\overline{\mathbb{R}}^r$

$$\Omega = \prod_{i=1}^{r} [z_i^*(x), \overline{z}_i(x)]. \tag{4.43}$$

Theorem 4.13 *Assume that* (4.2), (4.4), (4.6) *and* (4.10) *hold and let* $x \in \mathcal{K}$ *be such that* $Z(x) \neq \emptyset$. *The following assertions hold:*

(i) $\mathcal{F}(x) \subset \mathcal{F}_w(x) \cap \Omega \subset \big\{z \in \Omega \mid w(x,z) = 0\big\}$.

(ii) *Let* $z \in \Omega$ *such that* $w(x,z) = 0$. *If there exists a admissible pair* $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ *such that* $J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) = z_i$, *for* $i = 1, \ldots, r$ *then* $z \in \mathcal{F}_w(x)$.

PROOF. **(i)** By Theorem 4.12 we obtain immediately that

$$\mathcal{F}_w(x) \cap \Omega \subset \{z \in \Omega \mid w(x,z) = 0\}.$$

Moreover $\mathcal{F}(x) \subset \mathcal{F}_w(x)$. It remains to prove that $\mathcal{F}(x) \subset \Omega$.

Let $z \in \mathcal{F}(x)$. By (4.40) we obtain that $z_i \geq z_i^*(x)$ for $i = 1, \ldots, r$. To prove that $z \leq \overline{z}(x)$, assume that there exist $j \in \{1, \ldots, r\}$ such that $z_j > \overline{z}_j(x)$ and consider

$$\tilde{z} = (z_1^*(x), \ldots, z_{j-1}^*(x), \overline{z}_j(x), z_{j+1}^*(x), \ldots, z_r^*(x)).$$

By definition of $z_i^*(x)$, $\overline{z}(x)$ and w, it comes that $w(x, \tilde{z}) = 0$ and there exist $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ such that $J(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \leq \tilde{z} \leq z$ with $\tilde{z}_j < z_j$, which is impossible because $z \in \mathcal{F}(x)$. Henceforth $z \leq \overline{z}(x)$.

(ii) Let $z \in \Omega$ such that $w(x,z) = 0$ and there exists an admissible pair $(\overline{\mathbf{y}}_x^{\overline{\mathbf{u}}}, \overline{\mathbf{u}}) \in \mathbb{X}_x^{\mathcal{K}}$ such that

$$J_i(x; \overline{\mathbf{y}}_x^{\overline{\mathbf{u}}}, \overline{\mathbf{u}}) = z_i \quad \text{for} \quad i = 1, \ldots, r.$$

By definition of w

$$\inf_{\mathbf{u} \in \mathcal{U}} \left[\bigvee_{i=1}^{r} \left(J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) - z_i \right) \right] \bigvee \sup_{\theta \in [0, \infty[} e^{-\lambda\theta} g\left(\mathbf{y}_x^{\mathbf{u}}(\theta)\right) = 0.$$

That means, there exists no admissible par $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ such that

$$J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) < J_i(x; \overline{\mathbf{y}}_x^{\overline{\mathbf{u}}}, \overline{\mathbf{u}}) = z_i \quad \text{for all} \quad i = 1, \dots, r.$$

Therefore, by definition of the weak Pareto optimal solution $z \in \mathcal{F}_w(x)$. □

In order to obtain a more precise characterization of the Pareto front we introduce the operators Π_k, for $k = 1, \dots, r$:

$$\Pi_k : \mathbb{R}^r \to \mathbb{R}^{r-1}, \quad \Pi_k : z \mapsto \Pi_k(z) = (z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_r). \tag{4.44}$$

Let $x \in \mathcal{K}$ be such that $Z(x) \neq \emptyset$. For $k = 1, \dots, r$ denote by Ω_k the subset of $\mathbb{R}^{r-1}$ given by

$$\Omega_k = \Pi_k(\Omega) = \prod_{i=1,\ i \neq k}^{r} [z_i^*(x), \overline{z}_i(x)[,$$

and introduce the extended functions $\Gamma_i : \Omega_i \to [z_i^*(x), \overline{z}_i(x)[\cup \{+\infty\}$ defined by

$$\Gamma_i(x) = \inf\{\gamma \mid w(x, z_1, \dots, z_{i-1}, \gamma, z_{i+1}, \dots, z_r) \leq 0\}, \tag{4.45}$$

with $z = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_r) \in \Omega_i$.

Proposition 4.14 *Assume that* (4.2)*,* (4.4)*,* (4.6) *and* (4.10) *hold and let $x \in \mathcal{K}$ be such that $Z(x) \neq \emptyset$. Then, for all $i = 1, \dots, r$, the functions $\Gamma_i(\cdot)$ are decreasing:*

$$\forall \zeta, \zeta' \in \Omega_i, \quad \Big(\zeta \leq \zeta' \Rightarrow \Gamma_i(\zeta) \geq \Gamma_i(\zeta')\Big).$$

PROOF. Let $i \in \{1, \dots, r\}$ and assume that there exists $\xi \leq \zeta$ such that

$$\Gamma_i(\xi) < \Gamma_i(\zeta).$$

It follows from definition (4.45) that $w(x, \zeta_1, \dots, \zeta_{i-1}, \alpha, \zeta_{i+1}, \dots, \zeta_r) > 0$, for all $\alpha < \Gamma_i(\zeta)$.

Let us take $\alpha = \Gamma_i(\xi)$ and consider the point $z = (\zeta_1, \dots, \zeta_{i-1}, \Gamma_i(\xi), \zeta_{i+1}, \dots, \zeta_r)$. It is clear that $w(x, z) > 0$. On the other hand, by Proposition 4.4 we obtain that

$$w(x, z) = w(x, \zeta_1, \dots, \zeta_{i-1}, \Gamma_i(\xi), \zeta_{i+1}, \dots, \zeta_r) \leq w(x, \xi_1, \dots, \xi_{i-1}, \Gamma_i(\xi), \xi_{i+1}, \dots, \xi_r) \leq 0$$

because $\xi \leq \zeta$. This is in contradiction with the fact that $w(t, x, z) > 0$ as established before. □

Now, for $i = 1, \dots, r$ we use the notation $\operatorname{Gr}\Gamma_i$ for the set

$$\operatorname{Gr}\Gamma_i = \{(z_1, \dots, z_{i-1}, \Gamma_i(z), z_{i+1}, \dots, z_r) \ \mid \ z = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_r) \in \Omega_i\}. \tag{4.46}$$

The next theorem gives a characterization of the Pareto front.

Theorem 4.15 *Assume that* (4.2)*,*(4.4)*,*(4.6)*,*(4.10) *hold and let x be such that $Z(x) \neq \emptyset$. Then*

$$\mathcal{F}(x) = \bigcap_{i=1}^{r} \operatorname{Gr}\Gamma_i.$$

PROOF. We split the proof on two steps.

Step 1 Let us show that $\bigcap_{i=1}^r \operatorname{Gr}\Gamma_i \subset \mathcal{F}(x)$. Let $z \in \bigcap_{i=1}^r \operatorname{Gr}\Gamma_i$. First, we have to show that such a point is feasible, that is, there exists an admissible pair $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ such that $J(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) = z$. By definition of the functions Γ_i, $i = 1, \ldots, r$, we have that $w(x, z) = 0$. Then there exists at least one admissible pair $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ such that

$$\bigvee_{i=1}^{r} (J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) - z_i) \leq 0 \quad \Leftrightarrow \quad J_i(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \leq z_i, \quad i = 1, \ldots, r. \tag{4.47}$$

Assume that z is not feasible. Then for any admissible pair $(\overline{\mathbf{y}}_x^{\overline{\mathbf{u}}}, \overline{\mathbf{u}}) \in \mathbb{X}_x^{\mathcal{K}}$ satisfying (4.47), there exist $j \in \{1, \ldots, r\}$ such that $J_j(x; \overline{\mathbf{y}}_x^{\overline{\mathbf{u}}}, \overline{\mathbf{u}}) < z_j$. Let us recall that by choice of z we have that $z_i = \Gamma_i(\Pi_i(z))$. Then by taken $\zeta = J_j(x; \overline{\mathbf{y}}_x^{\overline{\mathbf{u}}}, \overline{\mathbf{u}})$ we obtain that $w(x, z_1, \ldots, z_{j-1}, \zeta, z_{j_1}, \ldots, z_r) \leq 0$ with $\zeta < \Gamma_j(\Pi_j(z))$ which is in contradiction with the definition of $\Gamma_j(\Pi_j(z))$ (see (4.45)).

Now, let us show that z is Pareto optimal for problem (4.7). Assume that there exists $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ such that

$$J(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) = \xi,$$

with $\xi \leq z$ and $\xi \neq z$. Then $w(x, \xi) \leq 0$. As $\xi \leq z$ and $w(x, \cdot)$ is decreasing (Proposition 4.4) we have that $w(x, \xi) \geq w(x, z) = 0$. So we can conclude that $w(x, \xi) = 0$. Now recall that

$$\xi \leq z \text{ and } \xi \neq z \quad \Leftrightarrow \quad \xi_i \leq z_i, \ \forall i = 1, \ldots, r \text{ and } \exists\, j \in \{1, \ldots, r\} \text{ s.t. } \xi_j < z_j.$$

Then $0 = w(x, z) \geq w(x, z_1, \ldots, z_{j-1}, \xi_j, z_{j+1}, \ldots, z_r)$ with $\xi_j < z_1 = \Gamma_j(\Pi_j(z))$ which is a contradiction with the definition of Γ_j.

Step 2 Let us show that

$$\mathcal{F}(x) \subset \bigcap_{i=1}^{r} \operatorname{Gr}\Gamma_i.$$

Assume that $z \in \mathcal{F}(x)$ and let $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ be an admissible pair such that

$$J(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) = z.$$

It follows from Theorem 4.12 that $w(x, z) = 0$. Then it is obvious that $\Gamma_i(\Pi_i(z)) < +\infty$, for $i = 1, \ldots, r$. If $z \notin \bigcap_{i=1}^r \operatorname{Gr}\Gamma_i$ then $\exists j \in \{1, \ldots, r\}$ such that $z_j \neq \Gamma_j(\Pi_j(z))$. As $w(x, z) = 0$, we obtain that $z > \Gamma_j(\Pi_j(z))$. Consider

$$\xi = (z_1, \ldots, z_{j-1}, \Gamma_j(\Pi_j(z)), z_{j+1}, \ldots, z_r).$$

By definition of the function Γ_j we have that $w(x, \xi) = 0$ and then there exists an admissible pair $(\overline{\mathbf{y}}_x^{\overline{\mathbf{u}}}, \overline{\mathbf{u}}) \in \mathbb{X}_x^{\mathcal{K}}$ such that

$$J(x; \overline{\mathbf{y}}_x^{\overline{\mathbf{u}}}, \overline{\mathbf{u}}) \leq \xi \leq z = J(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) = z.$$

As $\xi \neq z$ it is in contradiction with the assumption that z is Pareto optimal value of (4.7).
□

It is possible to obtain an equivalent characterization of the Pareto front that needs only one of the functions Γ_i. This result is presented in the following Theorem.

Theorem 4.16 *Assume that* (4.2),(4.4),(4.6),(4.10) *hold and let x be such that $Z(x) \neq \emptyset$. Then for any $i = 1, \dots, r$, the Pareto front can be characterized by the following: $z \in \mathcal{F}(x)$ if and only if*

$$z_i = \Gamma_i(\Pi_i(z)) \quad \textit{and} \quad \forall\, \zeta \in \Omega_i,\ (\zeta \leq \Pi_i(z) \textit{ and } \zeta \neq \Pi_i(z)) \ \Rightarrow\ \Gamma_i(\zeta) > \Gamma_i(\Pi_i(z)). \tag{4.48}$$

PROOF. Consider $z \in \mathbb{R}^r$ satisfying (4.48). It follows from theorem 4.15 that $z \in \mathcal{F}(x)$ if and only if $z_i = \Gamma_i(\Pi_i(z))$, for all $i = 1, \dots, r$. Assume that $z_j \neq \Gamma_j(\Pi_j(z))$ for some $j \in \{1, \dots, r\}$. For $i \neq j$, as $z_i = \Gamma_i(\Pi_i(z))$ we have that $w(x, z) = 0$ and then $\Gamma_j(\Pi_j(z)) < +\infty$ and $z_j > \Gamma_j(\Pi_j(z))$. Consider

$$\xi = (z_1, \dots, z_{j-1}, \Gamma_j(\Pi_j(z)), z_{j+1}, \dots, z_r) \in \Omega.$$

By definition of $\Gamma_j(\cdot)$, $w(x, \xi) = 0$. By construction $\Pi_i(\xi) \leq \Pi_i(z)$ and $\Pi_i(\xi) \neq \Pi_i(z)$. Then under assumption (4.48) $\Gamma_i(\Pi_i(\xi)) > \Gamma_i(\Pi_i(z)) = z_i = \xi_i$. So we have that $\xi_i < \Gamma_i(\Pi_i(\xi))$ then $w(x, \xi) > 0$, which is a contradiction.

Now assume that $z \in \mathcal{F}(x)$ and let $(\mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ be an admissible pair such that $J(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u}) = z$. Fix $j \in \{1, \dots, r\}$. It follows from Theorem 4.15 that $z_j = \Gamma_j(\Pi_j(z))$. Let us show that the second part of (4.48) holds. assume that rhere exist $\zeta\Omega_j$ such that $\zeta \leq \Pi_j(z)$, $\zeta \neq \Pi_j(z)$ and $\Gamma_i(\zeta) \leq \Gamma_i(\Pi_i(z)) = z_j$. Consider

$$\xi = (\zeta_1, \dots, \zeta_{j-1}, \Gamma_j(\zeta), \zeta_{j+1}, \dots, \zeta_r).$$

It is clear that $\xi \leq z$, $\xi \neq z$ and $w(x, \xi) = 0$. Then there exist an admissible pair $(\bar{\mathbf{y}}_x^{\mathbf{u}}, \mathbf{u}) \in \mathbb{X}_x^{\mathcal{K}}$ such that $J(x; \bar{\mathbf{y}}_x^{\mathbf{u}}, \mathbf{u}) \leq \xi \leq z = J(x; \mathbf{y}_x^{\mathbf{u}}, \mathbf{u})$, which is in contradiction with the fact that z is a Pareto optimal value of (4.7). □

4.4 Reconstruction of optimal trajectories

The purpose of this section is the reconstruction of optimal trajectories for infinite horizon optimal control problems with state constraints. It is important to recall here that in previous sections we characterized the weak Pareto front and the Pareto front for multi-objective infinite horizon control problems using an auxiliary optimal control problem. In this section we will use the dynamical programming principle for the value function of this auxiliary problem to generate optimal trajectories. This procedure is also a new result to generate optimal trajectories for the infinite horizon optimal control problems with one objective and with state constraints.

The following algorithm is a procedure for calculating a trajectory with a piecewise constant control function for the multi-objective infinite horizon optimal control. The main idea of this approach is using the Dynamical Programing Principal for the auxiliary control problem w defined in (4.12).

Algorithm 4.17

Fix $(x, z) \in \mathbb{R}^n \times \mathbb{R}^r$. *For* $h > 0$, *consider for* $k = 0, 1, \dots$ *a partition* $s_k = kh$ *of* $[0, \infty[$.
Set $y_0^h = x$, $z_0^h = z$. *For* $k = 0, 1, \dots,$

Step 1 *Compute an optimal control value $u_k^h \in U$ such that*

$$u_k^h \in \operatorname*{argmin}_{u\in U} \left\{ e^{-\lambda h} w\left(y_k^h + hf(y_k^h, u), z_k^h(u)\right) \bigvee e^{-\lambda kh} g(y_k^h) \right\},$$

where $z_k^h(u) = e^{\lambda h}[z_k - h\ell(y_k^h, u)]$.

Step 2 *Compute a new state position (y_{k+1}^h, z_{k+1}^h)*

$$\begin{aligned} y_{k+1}^h &= y_k^h + hf(y_k^h, u_k^h) \\ z_{k+1}^h &= e^{\lambda h}[z_k^h - h\ell(y_k^h, u_k^h)]. \end{aligned}$$

Associate to this sequence of controls, a piecewise constant control $\mathbf{u}^h(s) = u_k^h$ on $s \in [kh, (k+1)h)$, and an approximate trajectory $\mathbf{y}^h$ such that

$$\begin{aligned} &\dot{\mathbf{y}}^h(s) = f(y_k^h, u_k^h) \qquad a.e\ s \in [kh, (k+1)h) \\ &\mathbf{y}^h(kh) = y_k^h. \end{aligned}$$

Note that in step 1 the value of u_k^h can also be defined as a minimizer of

$$u_k^h \in \operatorname*{argmin}_{u\in U} w\left(y_k^h + hf(y_k^h, u), z_k^h(u)\right)$$

since this will imply in turn to be a minimizer of

$$u_k^h \in \operatorname*{argmin}_{u\in U} \left\{ e^{-\lambda h} w\left(y_k^h + hf(y_k^h, u), z_k^h(u)\right) \bigvee e^{-\lambda kh} g(y_k^h) \right\}.$$

The next theorem ensures the convergence of the sequence $(\mathbf{y}^h)_{h>0}$ to an optimal trajectory for the function w. The precise statement is given in the next theorem and the proof is based on some arguments used in [ABDZ18] and [RV91].

Theorem 4.18 *Assume that* (4.2),(4.4),(4.6),(4.10) *hold. Let $(x, z) \in \mathbb{R}^n \times \mathbb{R}^r$, $\{y_k^h\}$, $\{z_k^h\}$ be the sequences generated by Algorithm 4.17.*

(i) The approximate trajectories $\{y_k^h\}$ are minimizing sequences in the following sense:

$$w(x, z) = \lim_{h\to 0} \bigvee_{i=1}^{r} \left(h \sum_{k=0}^{\infty} e^{-\lambda kh} \ell_i(y_k^h, u_k^h) - z_i \right) \bigvee \left(\bigvee_{k=0}^{\infty} e^{-\lambda kh} g(y_k^h) \right).$$

(ii) Moreover, the family $(\mathbf{y}^h)_{h>0}$ admits cluster points, for the $L^1([0, +\infty[; e^{-\lambda t}dt)$ norm, when $h \to 0$. For any cluster point $\bar{y}$, we have $\bar{\mathbf{y}} \in \mathbb{X}_x$ and that $\bar{\mathbf{y}}$ is an optimal trajectory for $w(x, z)$.

PROOF. **(i)** The proof will be divided in several steps. Let $(x, z) \in \mathbb{R}^n \times \mathbb{R}^r$ and consider, for every $h > 0$ the discrete trajectory $y^h = (y_0^h, y_1^h, \ldots)$, $z^h = (z_0^h, z_1^h, \ldots)$ and the discrete control $u^h = (u_0^h, u_1^h, \ldots)$ given by Algorithm 4.17. In the sequel of the proof, and for simplicity of notation, we shall denote y_k (respectively z_k, u_k) instead of y_k^h (respectively z_k^h, u_k^h).

Step 1. Let us first establish that there exist $K_0, K > 0$ such that

$$w(y_0, z_0) \geq e^{-\lambda h} w(y_1, z_1) \bigvee g(y_0) - (e^{-\lambda h} K_0 + K) h^2.$$

From the Dynamical Programing Principle for w (recall that $y_0 = x$ and $z_0 = z$):

$$\begin{aligned} w(y_0, z_0) &= \inf_{\mathbf{u} \in \mathcal{U}} \left\{ e^{-\lambda h} w\left(\mathbf{y}^{\mathbf{u}}_{y_0}(h), \mathbf{z}^{\mathbf{u}}_{z_0}(h)\right) \bigvee \max_{\theta \in [0,h]} e^{-\lambda \theta} g(\mathbf{y}^{\mathbf{u}}_{y_0}(\theta)) \right\} \\ &\geq \inf_{\mathbf{u} \in \mathcal{U}} \left\{ e^{-\lambda h} w\left(\mathbf{y}^{\mathbf{u}}_{y_0}(h), \mathbf{z}^{\mathbf{u}}_{z_0}(h)\right) \bigvee g(x) \right\}. \end{aligned} \tag{4.49}$$

Consider $\mathbf{u}_0^* \in \mathcal{U}$ a minimizer of the term (4.49). By using the convexity of the set $G(0, y_0)$ (Assumption (4.10)), according with [AC84, Thm. 0.5.3], there exists $u_0^* \in U$ such that

$$\int_0^h f(y_0, \mathbf{u}_0^*(s)) ds = h f(y_0, u_0^*) \tag{4.50}$$

$$\int_0^h \ell_i(y_0, \mathbf{u}_0^*(s)) ds = h \ell_i(y_0, u_0^*), \quad \text{for } i = 1, 2. \tag{4.51}$$

Now consider the trajectory $\mathbf{y}^{\mathbf{u}_0^*}_{y_0}$ solution of (4.1) corresponding to the control $\mathbf{u}_0^*$ and starting from y_0. From (4.50) and Proposition 2.2(a) we obtain that

$$\begin{aligned} |\mathbf{y}^{\mathbf{u}_0^*}_{y_0}(h) - y_0 - h f(y_0, u_0^*)| &= \left| y_0 + \int_0^h f(\mathbf{y}^{\mathbf{u}_0^*}_{y_0}(s), \mathbf{u}_0^*(s)) ds - y_0 - \int_0^h f(y_0, \mathbf{u}_0^*(s)) ds \right| \\ &\leq \int_0^h |f(\mathbf{y}^{\mathbf{u}_0^*}_{y_0}(s), \mathbf{u}_0^*(s)) - f(y_0, \mathbf{u}_0^*(s))| ds \\ &\leq \int_0^h L_f |\mathbf{y}^{\mathbf{u}_0^*}_{y_0}(s) - y_0| \leq L_f c_f (1 + |x|) h^2 = K_0 h^2, \end{aligned}$$

where $K_0 = L_f c_f (1 + |x|)$. And by the definition of

$$\mathbf{z}^{\mathbf{u}}_{z_i}(h) = e^{\lambda h} z_i - \int_0^h e^{\lambda(h-s)} \ell_i(\mathbf{y}^{\mathbf{u}}_x(s), \mathbf{u}(s)) ds,$$

for $i = 1, \ldots, r$ and (4.51), we have that

$$\begin{aligned} \left| \mathbf{z}^{\mathbf{u}_0^*}_{z_{i_0}}(h) - e^{\lambda h}[z_{i_0} - h \ell_i(y_0, u_0^*)] \right| &= e^{\lambda h} \left| \int_0^h \ell_i(y_0, \mathbf{u}_0^*(s)) ds - \int_0^h e^{-\lambda s} \ell_i(\mathbf{y}^{\mathbf{u}_0^*}_{y_0}(s), \mathbf{u}_0^*(s)) ds \right| \\ &\leq e^{\lambda h} \int_0^h C_i \left(\left| y_0 - \mathbf{y}^{\mathbf{u}_0^*}_{y_0}(s) \right| + s \right) ds \\ &\leq e^{\lambda h} C_i \left(c_f (1 + |x|) + 1/2 \right) h^2 = e^{\lambda h} K_i h^2, \end{aligned}$$

where $K_i = C_i \left(c_f (1 + |x|) + 1/2 \right)$ and C_i is the Lipschitz constant of function $e^{-\lambda s} \ell_i(x, u)$.

Those estimates along with (4.49), and by using the Lipschitz continuity of w, yields to

$$\begin{aligned} w(x, z) &\geq e^{-\lambda h} w(\mathbf{y}^{\mathbf{u}_0^*}_{y_0}(h), \mathbf{z}^{\mathbf{u}_0^*}_{,z_0}(h)) \bigvee g(y_0) \\ &\geq \left[e^{-\lambda h} w\left(y_0 + h f(y_0, u_0^*), e^{\lambda h}[z_0 - h \ell(y_0, u_0^*)] \right) - (e^{-\lambda h} K_0 + K) h^2 \right] \bigvee g(y_0), \end{aligned}$$

where $K = \sum_{i=1}^{r} K_i$. Now notice that for $c \geq 0$

$$(a-c) \vee b \geq a \vee b - c. \tag{4.52}$$

Then by the definition of the minimizer u_0 on Algorithm 4.17 we finally obtain

$$w(x,z) \geq e^{-\lambda h} w\left(y_0 + hf(y_0,u_0), e^{\lambda h}[z_0 - h\ell(y_0,u_0)]\right) \bigvee g(y_0) - (e^{-\lambda h}K_0 + K)h^2.$$

Knowing that $y_1 = y_0 + hf(y_0,u_0)$ and $z_1 = e^{\lambda h}[z_0 - h\ell(y_0,u_0)]$ we finally get the desired result:

$$w(y_0,z_0,) \geq e^{-\lambda h} w(y_1,z_1) \bigvee g(y_0) - (e^{-\lambda h}K_0 + K)h^2.$$

With exactly the same arguments, for al $k = 0,1,\ldots,$ we obtain

$$w(y_k,z_k) \geq e^{-\lambda h} w(y_{k+1},z_{k+1}) \bigvee g(y_k) - (e^{-\lambda h}K_0 + K)h^2. \tag{4.53}$$

Step 2. From (4.53) we get

$$\begin{aligned} w(y_0,z_0) &\geq e^{-\lambda h} w(y_1,z_1) \bigvee g(y_0) - (e^{-\lambda h}K_0 + K)h^2 \\ &= e^{-\lambda h}\left(e^{-\lambda h} w(y_2,z_2) \bigvee g(y_1) - (e^{-\lambda h}K_0 + K)h^2\right) \bigvee g(y_0) - (e^{-\lambda h}K_0 + K)h^2. \end{aligned}$$

Now using (4.52), we obtain that

$$\begin{aligned} w(x,z_1,z_2) &\geq (e^{-\lambda 2h}) w(y_2,z_2) \bigvee e^{-\lambda h} g(y_1) \bigvee g(y_0) - 2(e^{-\lambda h}K_0 + K)h^2 \\ &= e^{-\lambda 2h} w(y_2,z_2) \bigvee \left(\bigvee_{k=0}^{1} e^{-\lambda kh} g(y_k)\right) - \sum_{i=1}^{2}\left(e^{-\lambda ih}K_0 + e^{-\lambda(i-1)h}K\right)h^2. \end{aligned}$$

By induction we finally get that for all $n \in \mathbb{N}$

$$w(x,z) \geq e^{-\lambda nh} w(y_n,z_n) \bigvee \left(\bigvee_{k=0}^{n-1} e^{-\lambda kh} g(y_k)\right) - \sum_{i=1}^{n}\left(e^{-\lambda ih}K_0 + e^{-\lambda(i-1)h}K\right)h^2.$$

Consider $\mathbf{u}^* \in \mathcal{U}$ a minimizer $w(y_n,z_n)$ and using the definition of w, we get that

$$\begin{aligned} w(x,z) \geq & e^{-\lambda nh}\left[\bigvee_{i=1}^{r}\left(\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}_{y_n}^{\mathbf{u}^*}(s),\mathbf{u}^*(s))ds - z_{i_n}\right) \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g(\mathbf{y}_{y_n}^{\mathbf{u}^*}(\theta))\right] \\ & \bigvee \left(\bigvee_{k=0}^{n-1} e^{-\lambda kh} g(y_k)\right) - \sum_{i=1}^{n}\left(e^{-\lambda ih}K_0 + e^{-\lambda(i-1)h}K\right)h^2. \end{aligned}$$

Now using the definition of z_{i_n}, for $i = 1,\ldots,r$ we obtain

$$\begin{aligned} w(x,z) \geq & e^{-\lambda nh}\left[\bigvee_{i=1}^{r}\left(\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}_{y_n}^{\mathbf{u}^*}(s),\mathbf{u}^*(s))ds - e^{\lambda nh}[z_i - h\sum_{k=0}^{n-1} e^{-\lambda hk}\ell_i(y_k,u_k)]\right)\right. \\ & \left.\bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g(\mathbf{y}_{y_n}^{\mathbf{u}^*}(\theta))\right] \bigvee \left(\bigvee_{k=0}^{n-1} e^{-\lambda kh} g(y_k)\right) - \sum_{i=1}^{n}\left(e^{-\lambda ih}K_0 + e^{-\lambda(i-1)h}K\right)h^2 \\ = & \bigvee_{i=1}^{r}\left(e^{-\lambda nh}\int_0^\infty e^{-\lambda s}\ell_i(\mathbf{y}_{y_n}^{\mathbf{u}^*}(s),\mathbf{u}^*(s))ds - z_i + h\sum_{k=0}^{n-1} e^{-\lambda kh}\ell_i(y_k,u_k)\right) \\ & \bigvee e^{-\lambda nh} \sup_{\theta\in(0,\infty)} e^{-\lambda\theta} g(\mathbf{y}_{y_n}^{\mathbf{u}^*}(\theta)) \bigvee \left(\bigvee_{k=0}^{n-1} e^{-\lambda kh} g(y_k)\right) - \sum_{i=1}^{n}\left(e^{-\lambda ih}K_0 + e^{-\lambda(i-1)h}K\right)h^2. \end{aligned}$$

For all $N \in \mathbb{R}$, consider $n = \lfloor N/h^2 \rfloor$ the largest integer which is less than or equal to N/h^2. So when $h \to 0$, $nh \to \infty$ and $n \to \infty$. Therefore by passing the limit when $h \to 0$ it follows that:

$$w(x,z) \geq \limsup_{h\to 0} \bigvee_{i=1}^{r} \left(h \sum_{k=0}^{\infty} e^{-\lambda kh} \ell_i(y_k, u_k) - z_i \right) \bigvee \left(\bigvee_{k=0}^{\infty} e^{-\lambda kh} g(y_k) \right). \tag{4.54}$$

Step 3. On the other hand, define $\mathcal{U}_h$ the subset of $\mathcal{U}$ consisting of all controls which take constant values on each interval $[kh,\ (k+1)h),\ k = 0, 1, \ldots$. That is

$$\mathcal{U}_h = \{\mathbf{u} \in \mathcal{U} :\ \mathbf{u}(s) \equiv \mathbf{u}(kh),\ s \in [kh, (k+1)h),\ k \in \mathbb{N}\}.$$

Then by the definition of w

$$\begin{aligned} w(x,z) &= \min_{\mathbf{u}\in\mathcal{U}} \left[\bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s))ds - z_i \right) \right] \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g(\mathbf{y}_x^{\mathbf{u}}(\theta)) \\ &\leq \min_{\mathbf{u}\in\mathcal{U}_h} \left[\bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}}(s), \mathbf{u}(s))ds - z_i \right) \right] \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g(\mathbf{y}_x^{\mathbf{u}}(\theta)) \\ &\leq \bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}^h}(s), \mathbf{u}^h(s))ds - z_i \right) \bigvee \sup_{\theta\in[0,\infty[} e^{-\lambda\theta} g(\mathbf{y}_x^{\mathbf{u}^h}(\theta)), \end{aligned} \tag{4.55}$$

where $\mathbf{u}^h$ is the control generated by Algorithm 4.17.

With similar arguments as in the proof of step 1.a of Theorem 4.8, we can prove that for $i = 1, \ldots, r$, there exist $K_i > 0$ such that Based on [BCD97], we claim that, for $i = 1, 2$

$$\left| \int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_x^{\mathbf{u}^h}(s), \mathbf{u}^h(s))ds - h \sum_{k=0}^{\infty} e^{-\lambda kh} \ell_i(y_k, u_k) \right| \leq K_i h. \tag{4.56}$$

Moreover with similar arguments as in step 1.b of the proof of Theorem 4.8, there exist $K_0 > 0$ such that

$$\left| \sup_{\theta\in(0,\infty)} e^{-\lambda\theta} g(\mathbf{y}_x^{\mathbf{u}^h}(\theta)) - \bigvee_{k=0}^{\infty} e^{-\lambda kh} g(y_k) \right| \leq K_0 h. \tag{4.57}$$

The estimates (4.56) and (4.57) along with (4.55) yield to:

$$\begin{aligned} w(x,z) &\leq \bigvee_{i=1}^{r} \left(h \sum_{k=0}^{\infty} e^{-\lambda kh} \ell_i(y_k, u_k) - z_i \right) \bigvee \left(\bigvee_{k=0}^{\infty} e^{-\lambda kh} g(y_k) \right) + \sum_{i=0}^{r} K_i h \\ &\leq \liminf_{h\to 0} \bigvee_{i=1}^{r} \left(h \sum_{k=0}^{\infty} e^{-\lambda kh} \ell_i(y_k, \alpha_k) - z_i \right) \bigvee \left(\bigvee_{k=0}^{\infty} e^{-\lambda kh} g(y_k) \right). \end{aligned} \tag{4.58}$$

Hence, from (4.54) and (4.58), the right-hand side term has a limit and

$$w(x,z) = \lim_{h\to 0} \bigvee_{i=1}^{r} \left(h \sum_{k=0}^{\infty} e^{-\lambda kh} \ell_i(y_k, u_k) - z_i \right) \bigvee \left(\bigvee_{k=0}^{\infty} e^{-\lambda kh} g(y_k) \right).$$

This concludes the desired result of item (i).

(ii) For all $h > 0$ and $k = 0, 1, \ldots$, the function $\mathbf{y}^h(\cdot)$ satisfies

$$\dot{\mathbf{y}}^h(s) = f(\mathbf{y}^h(s), \mathbf{u}^h(s)), \qquad \text{a.e } s \in [0, \infty[.$$

Define the measure $d\mu = e^{-\lambda t} dt$ and let $L^1 = L^1([0, +\infty[; d\mu)$ be the Banach space of (the class of equivalence of) integrable real-valued functions on $[0, +\infty[$ for the measure $d\mu$. Consequently, we denote by $W^{1,1}$ the Sobolev space functions $\mathbf{y} : [0, +\infty[\to \mathbb{R}^n$ for which $|\mathbf{y}| \in L^1$ and whose weak derivative $\dot{\mathbf{y}}$ also verifies $|\dot{\mathbf{y}}| \in L^1$.

Let $\omega : [0, +\infty) \to \mathbb{R}$ be given by $\omega(t) := c_f(1 + |x|)e^{c_f t}$ for any $t \geq 0$. As $\lambda > c_f$, Proposition 2.2(b) implies that $\omega(\cdot)$ is a positive function in L^1 which dominates $|\dot{\mathbf{y}}^h|$. Moreover by Proposition 2.2(d) the sequence $\{\mathbf{y}^h(s)\}$ is relatively compact for any $s \geq 0$, hence the hypothesis of theorem [[AC84],Theorem 0.3.4] are satisfied and so, there exist a function $\bar{\mathbf{y}} \in W^{1,1}$ and a sub-sequence, still denoted by $\{\mathbf{y}^h\}$, such that

$$\mathbf{y}^h \text{ converges uniformly to } \bar{\mathbf{y}} \text{ on compact subsets of } [0, +\infty),$$

$$\dot{\mathbf{y}}^h \text{ converges weakly to } \dot{\bar{\mathbf{y}}} \text{ in } L^1([0, +\infty), \mathbb{R}^n, d\mu).$$

In addition by (4.2), f is Lipschitz continuous with closed images and by (4.10) it has convex images. So the Convergence Theorem [AC84, Thm. 1.4.1] implies that $(\dot{\bar{\mathbf{y}}}^h(s)) \in f(\bar{\mathbf{y}}^h(s), \mathbf{u}^h(s))$ for almost every $s \geq 0$.

The optimality of $\bar{\mathbf{y}}$ follows from item (i). □

A similar algorithm has been introduced in [RV91] for the case of finite horizon optimal control problems without state constraints. The result was extended for the case with state constraints in [ABDZ18]. In both cases the convergence result has been proved. We extended the result for the optimal control problems in infinite horizon and with state constraints.

4.4.1 Reconstruction using an approximate value function

Sometimes the exact value function of the control problem is not available. We just have an approximation of the function. In this section we prove a result that is an extension of the convergence of trajectories for piecewise constant control with the algorithm 4.17 using the approximation of the value function.

Consider for each $h > 0$, a function w^h being an approximation of the value function w, such that, for every $R > 0$ we have:

$$|w_h(x, z) - w(x, z)| \leq E_R h, \quad \text{ for } |x| \leq R, \ |z| \leq R, \tag{4.59}$$

The function w_h could be a numerical approximation obtained by the Semi-Lagrangian scheme proposed in Section 4.2.1.

Algorithm 4.19

Fix $(x, z) \in \mathbb{R}^n \times \mathbb{R}^r$. For $h > 0$, consider for $k = 0, 1, \ldots$ a partition $s_k = kh$ of $[0, \infty[$.
Set $y_0^h = x$, $z_0^h = z$. For $k = 0, 1, \ldots,$

Step 1 *Compute an optimal control value $u_k^h \in U$ such that*

$$u_k^h \in \underset{u\in U}{\operatorname{argmin}} \left\{ e^{-\lambda h} w_h\left(y_k^h + hf(y_k^h,u), z_k^h(u)\right) \bigvee e^{-\lambda kh} g(y_k^h) \right\},$$

where $z_k^h(u) = e^{\lambda h}[z_k - h\ell(y_k^h, u)]$.

Step 2 *Compute a new state position* (y_{k+1}^h, z_{k+1}^h)

$$\begin{aligned} y_{k+1}^h &= y_k^h + hf(y_k^h, u_k^h) \\ z_{k+1}^h &= e^{\lambda h}[z_k - h\ell(y_k^h, u_k^h)]. \end{aligned}$$

Associate to this sequence of controls, a piecewise constant control $\mathbf{u}^h(s) = u_k^h$ *on* $s \in [kh, (k+1)h)$*, and an approximate trajectory* $\mathbf{y}^h$ *such that*

$$\begin{aligned} \dot{\mathbf{y}}^h(s) &= f(y_k^h, u_k^h) \qquad a.e\ s \in [kh, (k+1)h) \\ \mathbf{y}^h(kh) &= y_k^h. \end{aligned}$$

The following result is an extension of the convergence Theorem 4.18, using an approximation for the value function w.

Theorem 4.20 *Assume that* (4.2),(4.4),(4.6),(4.10) *hold. Let* $(x,z) \in \mathbb{R}^n \times \mathbb{R}^r$, $\{y_k^h\}$, $\{z_k^h\}$ *be the sequences generated by Algorithm 4.17.*

(i) The approximate trajectories $\{y_k^h\}$ *are minimizing sequences in the following sense:*

$$w(x,z) = \lim_{h\to 0} \bigvee_{i=1}^{r} \left(h \sum_{k=0}^{\infty} e^{-\lambda kh} \ell_i(y_k^h, u_k^h) - z_i \right) \bigvee \left(\bigvee_{k=0}^{\infty} e^{-\lambda kh} g(y_k^h) \right).$$

(ii) Moreover, the family $(\mathbf{y}^h)_{h>0}$ *admits cluster points, for the* $L^1([0,+\infty[; e^{-\lambda t}dt)$ *norm, when* $h \to 0$*. For any such cluster point* $\bar{\mathbf{y}}$*, we have* $\bar{\mathbf{y}} \in \mathbb{X}_x$ *and that* $\bar{\mathbf{y}}$ *is an optimal trajectory for* $w(x,z)$*.*

PROOF. The proof of this theorem is similar with the proof of Theorem 4.18. So we are going to present just the different parts.

With same arguments used at the proof of Step 1 of Theorem 4.18, we obtain that

$$\begin{aligned} w(x,z) \geq & e^{-\lambda h} w\left(y_0 + hf(y_0,u_0^*), e^{\lambda h}[z_0 - h\ell(y_0,u_0^*)]\right) \\ & \bigvee g(y_0) - (e^{-\lambda h}K_0 + K)h^2, \end{aligned}$$

where $K = \sum_{i=1}^r K_i$. Now using the fact that $||w - w_h|| \leq E_R h$, we get

$$\begin{aligned} w_h(x,z) \geq & w(x,z) - E_R h \\ \geq & e^{-\lambda h} w\left(y_0 + hf(y_0,u_0^*), e^{\lambda h}[z_0 - h\ell(y_0,u_0^*)]\right) \\ & \bigvee g(y_0) - (e^{-\lambda h}K_0 + K)h^2 - E_R h \\ \geq & e^{-\lambda h}\left(w^h\left(y_0 + hf(y_0,u_0^*), e^{\lambda h}[z_0 - h\ell(y_0,u_0^*)]\right) - E_R h\right) \\ & \bigvee g(y_0) - (e^{-\lambda h}K_0 + K)h^2 - E_R h \\ \geq & e^{-\lambda h} w^h\left(y_0 + hf(y_0,u_0^*), e^{\lambda h}[z_0 - h\ell(y_0,u_0^*)]\right) \\ & \bigvee g(y_0) - (e^{-\lambda h}K_0 + K)h^2 - (1 + e^{-\lambda h})E_R h. \end{aligned}$$

Then by the definition of the minimizer u_0 on Algorithm 4.19 we finally obtain

$$w^h(y_0,z_0) \geq e^{-\lambda h} w^h(y_1,z_1) \bigvee g(y_0) - (e^{-\lambda h}K_0 + K)h^2 - (1+e^{-\lambda h})E_R h. \tag{4.60}$$

With exactly the same arguments as in Step 2 of the proof of Theorem 4.18 and using the fact that $||w - w^h|| \leq E_R h$, we get

$$\begin{aligned} w^h(x,z) \geq & e^{-\lambda nh} w^h(y_n,z_n) \bigvee \left(\bigvee_{k=0}^{n-1} e^{-\lambda kh} g(y_k) \right) - \sum_{i=1}^{n} \left(\varepsilon_1^i h^2 + \varepsilon_2^i E_R h \right), \\ \geq & e^{-\lambda nh} w(y_n,z_n) \bigvee \left(\bigvee_{k=0}^{n-1} e^{-\lambda kh} g(y_k) \right) - \sum_{i=1}^{n} \left(\varepsilon_1^i h^2 + \varepsilon_2^i E_R h \right) - e^{-\lambda nh} E_R h, \end{aligned}$$

where $\varepsilon_1^i = e^{-\lambda ih}K_0 + e^{-\lambda(i-1)h}K$ and $\varepsilon_2^i = e^{-\lambda(i-1)h}(1+e^{-\lambda h})$.
Consider $\mathbf{u}^* \in \mathcal{U}$ a minimizer $w(y_n,z_n)$ and using the definition of w, we get that

$$\begin{aligned} w_h(x,z) \geq & e^{-\lambda nh} \left[\bigvee_{i=1}^{r} \left(\int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_{y_n}^{\mathbf{u}^*}(s), \mathbf{u}^*(s)) ds - z_{i_n} \right) \bigvee \sup_{\theta \in [0,\infty[} e^{-\lambda \theta} g(\mathbf{y}_{y_n}^{\mathbf{u}^*}(\theta)) \right] \\ & \bigvee \left(\bigvee_{k=0}^{n-1} e^{-\lambda kh} g(y_k) \right) - \sum_{i=1}^{n} \left(\varepsilon_1 h^2 + \varepsilon_2 E_R h \right) - e^{-\lambda nh} E_R h. \end{aligned}$$

Now using the definition of z_{i_n}, for $i = 1, \ldots, r$ we obtain

$$\begin{aligned} w_h(x,z) \geq & \bigvee_{i=1}^{r} \left(e^{-\lambda nh} \int_0^\infty e^{-\lambda s} \ell_i(\mathbf{y}_{y_n}^{\mathbf{u}^*}(s), \mathbf{u}^*(s)) ds - z_i + h \sum_{k=0}^{n-1} e^{-\lambda kh} \ell_i(y_k,u_k) \right) \\ & \bigvee e^{-\lambda nh} \sup_{\theta \in [0,\infty[} e^{-\lambda \theta} g(\mathbf{y}_{y_n}^{\mathbf{u}^*}(\theta)) \bigvee \left(\bigvee_{k=0}^{n-1} e^{-\lambda kh} g(y_k) \right) - \sum_{i=1}^{n} \left(\varepsilon_1 h^2 + \varepsilon_2 E_R h \right) - e^{-\lambda nh} E_R h. \end{aligned}$$

For all $N \in \mathbb{R}$, consider $n = \lfloor N/h^2 \rfloor$ the largest integer which is less than or equal to N/h^2. So when $h \to 0$, $nh \to \infty$ and $n \to \infty$. Therefore by passing the limit when $h \to 0$ it follows that:

$$w(x,z) \geq \limsup_{h \to 0} \bigvee_{i=1}^{r} \left(h \sum_{k=0}^{\infty} e^{-\lambda kh} \ell_i(y_k,u_k) - z_i \right) \bigvee \left(\bigvee_{k=0}^{\infty} e^{-\lambda kh} g(y_k) \right). \tag{4.61}$$

The other side is similar with Step 3 of proof of Theorem 4.18. □

We point out that the advantage of the Algorithm 4.19 is that the exact value function is not necessary. The reconstruction of the trajectory can be computed when an approximation of the value function is available, that is the case when the value function is obtained by numerical approximation, for example by the Semi-Lagrangian scheme proposed in Section 4.2.1. Similar result was introduced for finite horizon optimal control problem in [ABDZ18].

4.4.2 Reconstruction with a penalization of control variation

In the algorithm of this section we consider a trajectory procedure with a perturbation term in the definition of the optimal control value. This perturbation takes the form of a penalization term on the variation of the control with respect to the previously computed control values.

As suggested in [ABDZ18] for the reconstruction of trajectories for finite horizon optimal control problem with a maximum running cost, the penalization by control variation aims to prove an improvement on the performance of the algorithms 4.17 and 4.19. This method allows to select the control for which the value is close to the last value of the control.

For this end, for every $k \geq 1$, we introduce a function $q_k : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^+$ that represents a penalization term for the control value. For instance, if $\mathbb{U}_k = (u_0, u_1, \ldots, u_{k-1})$ is a vector in $\mathbb{R}^k$, we can choose

$$q_k(u, \mathbb{U}_k) = ||u - u_{k-1}||, \quad \text{or} \quad q_k(u, \mathbb{U}_k) = ||u - \frac{1}{p}\sum_{i=1}^{p} u_{k-i}|| \quad \text{for some } p \geq 1. \tag{4.62}$$

Algorithm 4.21

Let $(\lambda_h)_{h>0}$ be a family of positive constants. Fix $(x, z) \in \mathbb{R}^n \times \mathbb{R}^r$. For $h > 0$, consider for $k = 0, 1, \ldots$ a partition $s_k = kh$ of $[0, \infty)$.
Set $y_0^h = x$, $z_0^h = z$. For $k = 0, 1, \ldots,$

Step 1 *Compute an optimal control value $u_k^h \in U$ such that*

$$u_k^h \in \operatorname*{argmin}_{u \in U} \left\{ e^{-\lambda h} w_h \left(y_k^h + hf(y_k^h, u), z_k^h(u) \right) \bigvee e^{-\lambda k} g(y_k^h) + \lambda_h q_k(u, \mathbb{U}_k) \right\},$$

where $z_k^h(u) = e^{\lambda h}[z_k - h\ell(y_k^h, u)]$.

Step 2 *Compute a new state position (y_{k+1}^h, z_{k+1}^h)*

$$\begin{aligned} y_{k+1}^h &= y_k^h + hf(y_k^h, u_k^h) \\ z_{k+1}^h &= e^{\lambda h}[z_k - h\ell(y_k^h, u_k^h)]. \end{aligned}$$

Associate to this sequence of controls, a piecewise constant control $\mathbf{u}^h(s) = u_k^h$ on $s \in [kh, (k+1)h)$, and an approximate trajectory $\mathbf{y}^h$ such that

$$\begin{aligned} \dot{\mathbf{y}}^h(s) &= f(y_k^h, u_k^h) \qquad a.e\ s \in [kh, (k+1)h) \\ \mathbf{y}^h(kh) &= y_k^h. \end{aligned}$$

Theorem 4.22 *Assume that* (4.2),(4.4),(4.6),(4.10) *hold. Assume also that the penalization term is bounded: there exists $M_q > 0$ such that $|q_k(u, \mathbb{U})| \leq M_q$ for every $u \in U$, every $\mathbb{U} \in \mathbb{U}_k$, and*

$$\frac{\lambda_h}{h^2} \to 0 \qquad \text{as } h \to 0.$$

Let $(x, z) \in \mathbb{R}^n \times \mathbb{R}^r$, $\{y_k^h\}$, $\{z_k^h\}$ be the sequences generated by Algorithm 4.17.

(i) The approximate trajectories $\{y_k^h\}$ are minimizing sequences in the following sense:

$$w(x, z) = \lim_{h \to 0} \bigvee_{i=1}^{r} \left(h \sum_{k=0}^{\infty} e^{-\lambda kh} \ell_i(y_k^h, u_k^h) - z_i \right) \bigvee \left(\bigvee_{k=0}^{\infty} e^{-\lambda kh} g(y_k^h) \right).$$

(ii) Moreover, the family $\{\mathbf{y}^h\}_{h>0}$ *admits cluster points, for the* $L^1([0,+\infty); e^{-\lambda t}dt)$ *norm, when* $h \to 0$*. For any such cluster point* $\bar{\mathbf{y}}$*, we have* $\bar{\mathbf{y}} \in \mathbb{X}_x$ *and that* $\bar{\mathbf{y}}$ *is an optimal trajectory for* $w(x,z)$*.*

PROOF. The arguments of the proof are similar to the ones used in the proof of Theorems 4.18 and 4.20. The only change is that instead of the estimate (4.60), we get now:

$$w_h(y_n, z_n) \geq e^{-\lambda h} w_h(y_{n+1}, z_{n+1}) \bigvee g(y_0) - (e^{-\lambda h} K_0 + K) h^2 - (1 + e^{-\lambda h}) E_R h - M_q \lambda_h. \quad (4.63)$$

The rest of the proof remains unchanged. □

4.5 Concluding remarks

We have investigated multi-objective optimal control problems in infinite horizon and with state constraints. First we defined an auxiliary optimal control problem without state constraints and propose a semi-Lagrangian scheme to compute the value function of this auxiliary problem. Then we show that the weak Pareto front is a subset of the zero level set of the corresponding value function. Moreover, we established a more detailed characterization of the Pareto front for the multi-objective infinite horizon optimal control problem.

After obtaining the (weak) Pareto front, another important question is the reconstruction of (weak) Pareto optimal trajectories. In finite horizon, there are some algorithms of trajectory reconstruction that can be employed, but in infinite horizon, without assuming any controllability assumption, there are no results in the literature. So we introduced a method, based on the dynamical programming principle, to reconstruct optimal trajectories for infinite horizon control problems with state constraints and with no controllability assumption on the set of constraints. The convergence of the method when the exact value function is not available, that means, when the value function is obtained by some numerical approximation method, was also proved. Our method extends for infinite horizon optimal control problems with state constraints the results for finite horizon problems without and with state constraints introduced in [RV91] and [ABDZ18], respectively.

Chapter 5

Application to energy management systems

Intermittent sources of energy represent a challenge for electrical networks, particularly regarding demand satisfaction at peak times. Demand-side management tools, such as load shaving, can be used to mitigate abrupt changes in the generation. Another tool to control the negative impact of the new technologies is to rely on the so-called Energy Management Systems (EMS). Typically, this virtual entity represents a group of small generators, for example a photo-voltaic power plant, a wind turbine, a diesel generator, etc; coupled with a battery energy storage system (BESS).

The model, developed in collaboration with Wim van Ackooij from EDF, France is similar to [HMS+17]; see also [BBL+13] with the important difference that we extend the approach to the multi-objective case. Namely, in addition to the usual generation cost minimization we incorporate an environmental concern, referred to minimizing fuel emissions.

The question that arises is: to which extent can a BESS be used as a replacement for load shaving? For three instances representing typical configurations in Brazil, France and Germany, we analyze if appending batteries smooths peaks in demand, similarly to the load shaving mechanism.

5.1 An overview of battery energy storage systems

Energy problems deal with finding minimum cost production schedules that meet the customer load and satisfy all the operational and technological rules of the considered assets that can be used to generate electricity. But another important concern is the environmental impact. How much fuel is burnt as a result of the thermal generation? So based on [CKAN13], we incorporate in the model a second objective of minimizing the carbon emission of the thermal units.

Solar and wind generation are intermittent sources of energy. Without storage devices, energy must be generated and consumed immediately. The BESS is a good alternative in this sense, because it has the capacity to store energy surpluses when generation is larger than demand and supplies energy to the system when it lacks power and can have a wide range of system sizes and layouts to accommodate the requirements specific to each energy storage project [Ltd17]. This storage is not free as some energy is lost when charging and discharging the battery.

Nowadays batteries are employed as storage devices all over the world, see for example [Cic17]. In [RZVV13] the authors discuss the use of the battery to eliminate the peaks and valleys in the load profile and conclude that the method can effectively reduce the losses and demand of the system, moreover the batteries improve the voltage profile at peak hours. A multi-objective study is done is [XS12], where the objectives considered are the reliability and economy by choosing optimal energy storage parameters subject to the constraints for a specific distribution system. The use of the battery from the point of the view of the customer is presented in [TD16], considering that the price to buy energy in off-peak hours is less than the price in on-peak hours.

We are interested in comparing the ability of different batteries as a substitute of the load shaving mechanism in smoothing the load peaks. The multi-objective approach makes it possible to obtain a compromise between the minimization of generation costs and the fuel emission of the thermal power plants in the mix.

5.2 Description of the set of assets

In this section we describe the set of thermal, hydro, solar and wind plants used to generate electricity. We also present the load shaving mechanism and the BESS models as well as both objectives of the bi-objective energy management problem. The models for thermal units will be those of economic dispatch (for a complete account of trends in unit-commitment we refer to [TvFL15, vDF$^+$18]). The models for thermal units are simplified, i.e., those of economic dispatch, since our interest is in computing exactly the efficient frontier of the bi-objective problem. The used algorithm requires solving iteratively many sub-problems, which would be (excessively) time consuming if they were mixed integer linear, non-convex, or large scale optimization problems. Nonetheless, the framework is in principle compatible. Let us also refer to [Td17] for a deep discussion on cascaded reservoir management.

Within our model, the time horizon is assumed to consist of T homogenously spaced time steps of h hours each. The (customers') demand/load at time step t will be denoted by L^t. The energy balance equation will ensure that load is met by generation of the set of available assets. Since the latter sets change, so does this balance equation.

5.2.1 Thermal units

A thermal unit j can be used to produce electricity, which involves the choice of an output level p_j^t in megawatts (MW) at a given proportional cost c_j in e.g., €/ MWh. Although we will not explicitly consider this, convex quadratic costs could also be immediately incorporated in the model. Altogether, when considering N thermal units, the total cost of generation is:

$$\sum_{t=1}^{T} \sum_{j=1}^{N} c_j p_j^t h.$$

Based on [CKAN13], we also consider the carbon emissions of thermal unit j, denoted by f_j, that results when generating power from thermal units. The objective is to minimize the worst case CO_2 emissions (tonnes/MWh), over time. Therefore, this objective function is:

$$\max_{t=1,\dots,T} \left(\frac{\sum_{j=1}^{N} f_j p_j^t}{L^t} \right),$$

The model of this second objective as a maximum allows to have a better control oft he fuel emission during the time horizon, while still allowing for a certain amount of CO_2

emission over the planning period.

Thermal units have some technological constraints, the output level p_j^t needs to be less or equal than some bound p_j^{max}, that represents the maximum power capacity of generation of thermal unit j:

$$0 \leq p_j^t \leq p_j^{max}, \quad t = 1, \ldots, T.$$

Furthermore, thermal units are subject to ramping constraints, limiting the variation of power between two consecutive time steps. This ramping rate θ_j for $j = 1, \ldots, N$, is considered constant and links adjacent moments in time. With an initial power level p_j^0, for $j = 1, \ldots, N$, assumed to be given, this yields the following constraints:

$$-\theta_j h \leq p_j^t - p_j^{t-1} \leq \theta_j h, \quad t = 1, \ldots, T.$$

5.2.2 Hydropower plants with cascaded reservoirs

The considered hydro power plants are distributed along a hydro valley. This is a set K of reservoirs connected through turbines that can be used to generate electricity. We assume that each reservoir k, disposes of a set of uphill reservoirs $A(k)$ and a set of downhill reservoirs $B(k)$. Associated with each connection between reservoirs m and k is an uphill and downhill flow delay $D_{m,k}^{up}$, $D_{m,k}^{dn}$ as well as a turbining station. Moreover each reservoir disposes of natural inflows I_k^t (m^3/h) as well as two volumetric bounds V_k^{min}, V_k^{max}. For reservoirs m, k we associate with the arc (m, k) the set of turbines $\mathcal{T}(m, k)$. With each turbine $i \in \cup \mathcal{T}(m, k)$ we associate a flow rate q_i^t (m^3/h) and turbining efficiency ρ_i^t (MWh/m^3).

Since water is naturally free, and not attributing it a value would lead to premature depletion of this resource, we attribute it a value c_{w_k} in (€$/m^3$), for the water of each reservoir k. We refer to [vHMZ14] for an extensive discussion on how this value of water can be computed. Denote by V_k^t the water level of each reservoir at time t. The initial volume V_k^0 is known in advance, so we define $z_k^0 = V_k^0 - V_k^{min}$ indicating the amount of water available for generating power at the initial time. We also introduce variables $z_k^T = V_k^T - V_k^{min}$ indicating for each reservoir k the amount of water available for generating power at the final time. Then, the value of the final water level to be minimized in the cascaded reservoir system is equal to:

$$\sum_{k=1}^{K} c_{w_k}(z_k^0 - z_k^T) = \sum_{k=1}^{K} c_{w_k}(V_k^0 - V_k^T).$$

The valuation induced by $\sum_{k=1}^{K} W_k z_k^0$ is in fact a constant and can be omitted.

The water balance equation and further constraints are:

$$V_k^t = V_k^{t-1} + I_k^t h + \sum_{m \in A(k)} \sum_{i \in \mathcal{T}(m,k)} q_i^{t-D_{m,k}^{dn}} h - \sum_{m \in B(k)} \sum_{i \in \mathcal{T}(k,m)} q_i^{t-D_{k,m}^{up}} h,$$
$$\text{for } t = 1, \ldots, T, \ k = 1, \ldots, K$$

$$V_k^{min} \leq V_k^t \leq V_k^{max} \qquad t = 1, \ldots, T, \ k = 1, \ldots, K$$

$$z_k^T = V_k^t - V_k^0 \qquad k = 1, \ldots, K$$

$$0 \leq z_k^T \leq V_k^{max} - V_k^{min} \quad k = 1, \ldots, K$$

$$q_i^{min} \leq q_i^t \leq q_i^{max} \qquad \forall i \in \bigcup_{(m,k)} \mathcal{T}(m, k), \ t = 1, \ldots, T.$$

The amount of generated power at time t is

$$\sum_{(m,k)} \sum_{i \in \mathcal{T}(m,k)} \rho_i^t q_i^t.$$

5.2.3 Solar, wind and run-of-the-river plants

The solar panel, wind turbine and the run-of-the-river hydro plants produce electricity without any additional cost, but the generation pattern cannot be controlled. In this, simplified, model, we consider the data to be deterministic. Hence, U^t in MW at time t is the sum of solar, wind and run-of-the-river plants. As this generation is deterministic the value U^t is removed from the demand.

5.2.4 First demand management tool: load shaving

Load shaving, or more generally demand side management, is related to some incentive to displace a portion of the electrical consumption from one moment in time to another. The electrical uses could for instance be: charging an electrical vehicle, or using a dish-washer. The idea is to program a shift in time of such an electrical use, under some constraints. The advantage of disposing of such a tool stems from the idea of shifting electrical use away from peaking hours where it is produced with highly costly means to off-peak hours where the same energy can be produced using cheaper technologies. Although the literature discusses, seemingly, different forms of such mechanisms which go by names such as: peak shaving, valley filling, load shifting etc..., they are essentially and mathematically the same mechanism resulting from moving load, potentially beyond the borders of the model. Still one could argue that some of these shifts are not of zero sum as total energy consumption is considered. Indeed one can think of rebound effects (requiring additional energy to heat a room when temperature has fallen too much). Typically incorporating such features requires the use of binary variables. Again for reasons outlined above, we consider a simpler model which has energy neutral shift:

$$\begin{aligned} & -\gamma \leq \delta^t \leq v^t \leq \gamma, && t = 1, \ldots, T, \\ & v^t \geq 0 && t = 1, \ldots, T, \\ & \sum_0^T \delta^t h = 0, \ \sum_0^T v^t h \leq \gamma, && \end{aligned}$$

where δ^t is the displaced energy at time step t, v^t is the maximum power that can be displaced at each period and γ is a bound for the power, in megawatts, that can be shifted along the planning horizon.

5.2.5 Second demand management tool: battery energy storage system

The BESS was modeled based in [CKAN13, HMS+17]. The aim is to analyze the impact of the BESS in the reduction of the cost and the CO_2 emission. The BESS can store energy for later use, but has limited capacity and power. Moreover the storage is not free as some energy is lost when charging and discharging the battery.

Given a initial state y^0, the state of the charge y of the BESS evolves according to the discrete dynamics

$$y^{t+1} = y^t \frac{h}{Q_B}(b_I^t \rho_I - \frac{b_O^t}{\rho_O}), \quad \text{for all } t = 1, \ldots, T, \tag{5.1}$$

where Q_B is the maximum capacity of the BESS, $b_I^t,\ b_O^t > 0$ are the input and output power of the BESS, and $\rho_I,\ \rho_O \in [0,1]$ are the efficiency ratios for the charge and discharge processes, assumed constant.

For physical reasons, the system is subject to the following constraints at every time step $t = 1, \dots, T$

$$y^t \in [\mathrm{c}^{min}, \mathrm{c}^{max}], \quad b_I^t \in [0, b_I^{max}] \quad \text{and} \quad b_O^t \in [0, b_O^{max}].$$

Since batteries typically have a short storage cycle and a cost of operation is not necessarily attributed to it, one has to think of how to handle day to day operation. A possibly rather natural constraint is to assume that the cycle balances off over the considered time frame. This is translated into the constraint that the final storage level should be at least as favourable as the initial one:

$$y^T \geq y^0.$$

5.3 Mathematical formulation

5.3.1 Case 1: Energy problem without management system

In the base case of the generation energy problem, there is no energy management system. Hence the generation of thermal, hydro power and solar plants need to satisfy the demand, i.e., the last constraint of problem (5.2). In this problem the first objective concerns the minimization of the cost generation of thermal units and the "cost" of the water. The second objective is to minimize the maximum carbon emissions of thermal units during the time horizon. The resulting problem is as follows:

$$\begin{cases} \min \quad \left(\displaystyle\sum_{j=1}^{N} \sum_{t=1}^{T} c_j^t p_j^t h + \sum_{k=1}^{K} c_{w_k} (z_k^0 - z_k^T),\ \max_{t=1,\dots,T} \left(\frac{\sum_{j=1}^{N} f_j p_j^t}{L^t} \right) \right) \\ \text{s. t.} \quad p_j^t \in P_j, \quad j = 1, \dots, N,\ t = 1, \dots, T \\ \qquad z_k^T, q_i^t \in H, \quad i \in \bigcup_{(m,k)} \mathcal{T}(m,k),\ t = 1, \dots, T \\ \qquad \displaystyle\sum_{j=1}^{N} p_j^t + \sum_{(m,k)} \sum_{i \in \mathcal{T}(m,k)} \rho_i^t q_i^t = L^t - U^t,\ t = 1, \dots, T \end{cases} \tag{5.2}$$

where P_j is the set of technological constraints of the thermal units, described in section 5.2.1 and H is the set of Hydropower technological constraints described in section 5.2.2.

5.3.2 Case 2: Energy problem with load shaving

When we consider the energy problem with load shaving, the amount of energy displaced at each time must be added to the power balance equation. Moreover we add to this problem the constraints of the load-shaving model presented in section 5.2.4.

$$\begin{cases} \min & \left(\sum_{j=1}^{N} \sum_{t=1}^{T} c_j^t p_j^t h + \sum_{k=1}^{K} c_{w_k} (z_k^0 - z_k^T), \max_{t=1,\ldots,T} \left(\frac{\sum_{j=1}^{N} f_j p_j^t}{L^t} \right) \right) \\ \text{s. t.} & p_j^t \in P_j, \quad j = 1,\ldots,N, \ t = 1,\ldots,T \\ & z_k^T, q_i^t \in H, \quad i \in \bigcup_{(m,k)} \mathcal{T}(m,k), \ t = 1,\ldots,T \\ & -\gamma \leq \delta^t \leq v^t \leq \gamma, \quad t = 1,\ldots,T, \\ & v^t \geq 0 \quad t = 1,\ldots,T, \\ & \sum_{t=1}^{T} \delta^t = 0, \ \sum_{t=1}^{T} v^t \leq \gamma, \\ & \sum_{j=1}^{N} p_j^t + \sum_{(m,k)} \sum_{i \in \mathcal{T}(m,k)} \rho_i^t q_i^t + \delta^t = L^t - U^t, \ t = 1,\ldots,T \end{cases} \tag{5.3}$$

5.3.3 Case 3: Energy problem with battery energy storage system

The inclusion of a BESS within the energy problem requires to add the amount of energy input or output of the battery at each time in the power balance equation. Moreover, in that case, taking into account the demand and the power generation devices, we obtain that b_O and b_I can be written as nonlinear functions, meaning that if the generation is bigger than the demand the energy will be included in the battery and if the generation is less than the demand the energy of the battery will be used to satisfy the demand. So for $t = 1,\ldots,T$, we have that

$$b_O^t = -\min\Big(0, \sum_{j=1}^{N} p_j^t + \sum_{(m,k)} \sum_{i \in \mathcal{T}(m,k)} \rho_i^t q_i^t - L^t + U^t\Big),$$
$$b_I^t = \max\Big(0, \sum_{j=1}^{N} p_j^t + \sum_{(m,k)} \sum_{i \in \mathcal{T}(m,k)} \rho_i^t q_i^t - L^t + U^t\Big).$$

Those constraints can be written as, for $t = 1,\ldots,T$,

$$b_I^t \geq 0 \text{ and } b_I^t \geq \sum_{j=1}^{N} p_j^t + \sum_{(m,k)} \sum_{i \in \mathcal{T}(m,k)} \rho_i^t q_i^t - L^t + U^t,$$
$$b_O^t \geq 0 \text{ and } -b_O^t \leq \sum_{j=1}^{N} p_j^t + \sum_{(m,k)} \sum_{i \in \mathcal{T}(m,k)} \rho_i^t q_i^t - L^t + U^T.$$

In this configuration it is necessary to add the constraints of the BESS as described in section 2.2.5. So we obtain the following multi-objective energy problem:

$$\left\{\begin{array}{ll} \min & \left(\sum_{j=1}^{N}\sum_{t=1}^{T} c_j^t p_j^t h + \sum_{k=1}^{K} c_{w_k}(z_k^0 - z_k^T),\ \max_{t=1,\dots,T}\left(\frac{\sum_{j=1}^{N} f_j p_j^t}{L^t}\right)\right) \\ \text{s. t.} & p_j^t \in P_j, \quad j=1,\dots,N,\ t=1,\dots,T \\ & z_k^T, q_i^t \in H, \quad i \in \bigcup_{(m,k)} \mathcal{T}(m,k),\ t=1,\dots,T \\ & y^{t+1} = y^t + \frac{h}{Q_B}\left(b_I^t \rho_I - \frac{b_O^t}{\rho_O}\right),\ t=0,1,\dots,T-1 \\ & y^T \geq y^0, y^{min} \leq y^t \leq y^{max},\ t=1,\dots,T \\ & 0 \leq b_O^t \leq b_O^{max}, 0 \leq b_I^t \leq b_I^{max},\ t=1,\dots,T \\ & b_O^t = -b_O^t \leq \sum_{j=1}^{N} p_j^t + \sum_{(m,k)} \sum_{i\in\mathcal{T}(m,k)} \rho_i^t q_i^t - L^t + U^t,\ t=1,\dots,T \\ & b_I^t \geq \sum_{j=1}^{N} p_j^t + \sum_{(m,k)} \sum_{i\in\mathcal{T}(m,k)} \rho_i^t q_i^t - L^t + U^t,\ t=1,\dots,T \\ & \sum_{j=1}^{N} p_j^t + \sum_{(m,k)} \sum_{i\in\mathcal{T}(m,k)} \rho_i^t q_i^t + b_O^t - b_I^t = L^t - U^t,\ t=1,\dots,T \end{array}\right. \tag{5.4}$$

5.4 Chebyshev scalarization method

A known approach for solving multi-objective problems is the scalarization techique. The term "scalarization" refers to techniques defining a family of mono-objective optimization problems, in such a way that the solutions of the multi-objective problem can be obtained by solving classic nonlinear programming problems. There is a large variety of methods for scalarizing a multi-objective optimization problem, some of them presented in Section 2.4.1.

In this work we use the Chebyshev approach to obtain the Pareto front, see Section 2.4.1. There are also methods that compute just one (weak) Pareto optimal solutions, but in this work we look for methods to compute the entire Pareto front. Indeed if we could content ourselves with computing a single (weak) Pareto solution, then implicitly we are capable of finding a desirable trade-off between both. Then in essence the problem is a regular mono-objective optimization problem. It is our assumption that this is usually not the case. Hence, it is important that all possible solutions are shown to the decision maker, who can select the desired compromise.

Let us first briefly provide an abstract form for these problems. To this end, denote by x the vector of variables and by X the feasible set for problems (5.2), (5.3) and (5.4). For example, in (5.2) the decision variable is $x = (p_j^t, q_i^t, z_k^T)$, for $t = 1,\dots,T$, $j = 1,\dots,N$, $k = 1,\dots,K$ and $i \in \bigcup_{(m,k)} \mathcal{T}(m,k)$. Moreover denote the first objective function by $J_1(x)$ and the second objective function by $J_2(x) = \max_{t=1,\dots,T} J_2^t(x)$. Then the problems (5.2), (5.3) and (5.4) can be rewritten, in abstract form, as:

$$\left\{\begin{array}{l} \text{Minimize } \ J(x) := (J_1(x), J_2(x)) \\ \text{subject to } x \in X. \end{array}\right. \tag{5.5}$$

We note here that for the results presented in this section J_1 and J_2 can be arbitrary. The Chebyshev scalarization method applied to (5.5) can be written as follows.

$$\begin{cases} \min & \max \left\{ \omega(J_1(x) - \beta_1^*), (1-\omega)(J_2(x) - \beta_2^*) \right\} \\ \\ \text{s.t.} & x \in X \end{cases} \tag{5.6}$$

where *zomega* $\in [0,1]$ is a weight and $\beta_i^*,\ i = 1, 2$, are the respective utopian objective values, defined shortly hereafter. A utopian objective vector β^* associated with a bi-objective problem consists of components $\beta_i^* = z_i^* - \varepsilon_i$, where $\varepsilon_i > 0$ for $i = 1, 2$ and z_i^* is the optimal value obtained by minimizing the objective function $J_i,\ i = 1, 2$ individually subject to the constraints.

For $\omega \in [0, 1]$, problem (5.6) can be written as

$$\begin{cases} \min & \alpha \\ \\ \text{subject to} & x \in X \text{ and} \\ & \omega\left(J_1(x) - \beta_1^*\right) \le \alpha, \\ & (1-\omega)\left(J_2(x) - \beta_2^*\right) \le \alpha, \end{cases}$$

or equivalently

$$\begin{cases} \min & \alpha \\ \\ \text{subject to} & x \in X \text{ and} \\ & \omega\left(J_1(x) - \beta_1^*\right) \le \alpha, \\ & (1-\omega)\left(J_2^t(x) - \beta_2^*\right) \le \alpha,\ t = 1, \dots, T. \end{cases} \tag{5.7}$$

The scalarized problem (5.7) is a linear optimization problem and can be solved by methods developed for linear optimization. To generate the Pareto front for problems (5.2), (5.3), (5.4) we considered the Algorithm proposed in [DK11] and presented in Section 2.4.2, in which the scalarized problem (5.7) is solved for a large number of parameters ω. This algorithm guides the user through appropriate choices for ω.

In Section 5.5 we compare some results of those three different approaches of the energy problem considering data that represent the French, Brazilian and German systems. As the scalarized problem (5.7) is linear, in the different approaches of the energy problem, we solved the mono-objective problems using the solver *linprog* of matlab.

5.4.1 Sensitivity analysis

In this section we study the variation of one objective function when we change the other. This sensitivity analysis is an additional information that can be provided to the decision maker as useful side information. We show moreover that by solving (5.7) it can be immediately constructed from the available information. In particular, when given a single element of the Pareto front, this sensitivity information can be used to locally approximate the set of Pareto optimal solutions. For the purpose of analysis, let us introduce the perturbation function $v : \mathbb{R} \to \bar{\mathbb{R}}$ defined as:

$$v(r) = \min_{x \in X} \left\{ J_2(x) \mid J_1(x) = r \right\}. \tag{5.8}$$

Moreover, let $N_X(x^*)$ denote the normal cone (of convex analysis) to the set X at the point x^*, that is

$$N_X(x^*) = \{ g \in \mathbb{R}^n \mid \ g^\top (y - x^*) \le 0,\ y \in X \}.$$

Since J_1 is affine and is polyhedral, the optimality conditions for (5.8) ensure there exist a multiplier $\pi_r \in \mathbb{R}$ and a normal element $\nu \in N_X(x^*)$ such that

$$0 \in \partial J_2(x^*) + \pi_r \nabla J_1(x^*) + \nu. \tag{5.9}$$

It is known that $-\pi_r \in \partial v(r)$, [HUL93, Chapter VII, Thm 3.3.2]. So the multiplier in problem (5.8) gives the rate of change of the optimal value J_2 with respect to variations in the right hand side r, which corresponds to values of J_1.

Of course it is less interesting to have to solve another optimization problem (5.8) to find π_r. We will show below that this is not necessary and π_r can be immediately retrieved from data available as a result of solving the Chebyshev scalarization. We recall that the latter problem is:

$$\begin{cases} \min & \alpha \\ \\ \text{subject to} & x \in X \\ & \omega\left(J_1(x) - \beta_1^*\right) \leq \alpha, \\ & (1-\omega)\left(J_2^t(x) - \beta_2^*\right) \leq \alpha, \ t = 1, \ldots, T. \end{cases} \tag{5.10}$$

Theorem 5.1 *For $\omega \in (0,1)$ let $(x_\omega^*, \alpha_\omega^*), \nu \in N_X(x_\omega^*)$ be an optimal solution of problem (5.10) with Lagrange multiplier $\mu_1^* \notin \{0,1\}$ associated with the constraint $\omega\left(J_1(x) - \beta_1^*\right) \leq \alpha$. Then x_ω^* satisfies (5.9) with $r = \beta_1^* + \dfrac{\alpha_\omega^*}{\omega}, \pi_r = \dfrac{\mu_1^* \omega}{(1-\mu_1^*)(1-\omega)}$ and ν.*

PROOF. For $\omega \in (0,1)$, let (x^*, α^*) be a optimal solution of problem (5.10). Then there exist $\mu_1^*, \ \mu_{2,t}^* \geq 0, \ t = 1, \ldots, T$ and $\nu \in N_X(x^*)$, such that

$$\begin{pmatrix} 1 - \mu_1^* - \sum_{t=1}^T \mu_{2,t}^* \\ \omega \mu_1^* \nabla J_1(x^*) + (1-\omega) \sum_{i=1}^T \mu_{2,t}^* \nabla J_2^t(x^*) + \nu \end{pmatrix} = 0, \tag{5.11}$$

$$\mu_1^* \big(\omega\left(J_1(x^*) - \beta_1^*\right) - \alpha^*\big) = 0, \tag{5.12}$$

$$\mu_{2,t}^* \big((1-\omega)\left(J_2^t(x^*) - \beta_2^*\right) - \alpha^*\big) = 0, \quad \text{for t=1,...,T} \tag{5.13}$$

By equation (5.12) we obtain that $\mu_1^* = 0$ or $\omega\left(J_1(x^*) - \beta_1^*\right) - \alpha^* = 0$. Assuming that $\mu_1^* \neq 0$, then $J_1(x^*) = \beta_1^* - \dfrac{\alpha^*}{\omega}$.

Now note that

$$\partial J_2(x^*) = conv\{\nabla J_2^t(x^*) \mid t = 1, ..., T \text{ s.t. } J_2^t(x^*) = J_2(x^*)\}.$$

By (5.13) we obtain that $J_2^t(x^*) = \beta_2^* - \dfrac{\alpha^*}{1-\omega}$ if and only if $\mu_{2,t}^* \neq 0$. By (5.11), we have $1 - \mu_1^* = \sum_{t=1}^T \mu_{2,t}^*$. This together with the assumption that $1 - \mu_1^* \neq 0$, implies that there must exist some $t \in \{1, \ldots, T\}$ with $\mu_{2,t}^* \neq 0$ (and actually is equivalent with it). We may thus conclude that $\displaystyle\sum_{t=1}^T \frac{\mu_{2,t}^*}{\sum_{t=1}^T \mu_{2,t}^*} = 1$ and consequently:

$$\sum_{i=1}^T \frac{\mu_{2,t}^*}{\sum_{t=1}^T \mu_{2,t}^*} \nabla J_2^t(x^*) \in \partial J_2(x^*).$$

Altogether we have thus shown that

$$\sum_{t=1}^{T} \frac{\mu_{2,t}^*}{\sum_{t=1}^{T} \mu_{2,t}^*} \nabla J_2^t(x^*) + \frac{\omega \mu_1^*}{(1-\mu_1^*)(1-\omega)} \nabla J_1(x^*) + \nu = 0,$$

which means that x^* satisfies (5.9) with the indicated identification of parameters. □

5.5 Numerical results

In this section we compare the results when considering the various energy problems: (5.2)-(5.4) on a set of three stylized systems representing the French, German and Brazilian cases respectively. The German system is thermal dominated, the Brazilian is hydro dominated and French system has a hydro thermal mix.

As the scalarized problem (5.7) is linear, in the different approaches of the energy problem, we solved the mono-objective problems using the solver *linprog* of MaTLaB.

The carbon emission rate f in kilogram per megawatt-hour (Kg/MWh) is considered constant over time. The considered values relate to each type of technology and are given in Table 5.1. We have used the same values for all test cases.

Thermal units	CO2 emission (Kg/MWh)
Nuclear	0
Lignite	1140
Coal	940
Gas	570
Mineral oil	770
Biomass	770
Combustion turbine	1200

Table 5.1: Carbon emission by type of thermal unit

5.5.1 Hydro - Thermal system : French case

In this problem we consider the time horizon T to span 48 hours with time step size h of 2 hours. The demand and solar generation for the whole period can be seen in Figure 5.1.

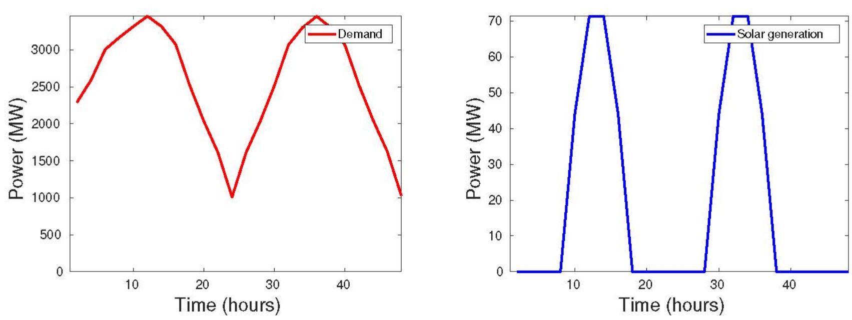


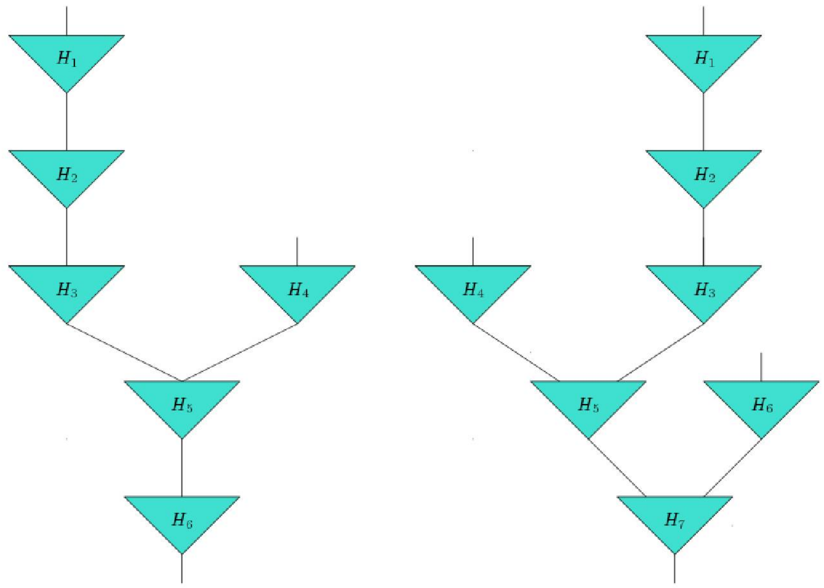
Figure 5.1: Demand and solar generation of a cold sunny winter day, respectively, for the French system.

In this system we have 9 thermal units of 4 different types: 3 nuclear (T_1-T_3), 2 coal (T_4-T_5), 3 gas (T_6-T_8) and 1 combustion turbine (T_9). The data used for each unit is given in Table 5.2 below. The cost c_j, in euro per megawatt-hour (€/MWh) is constant.

Thermal units	p^0	p^{max}	θ	c_j
T_1	750	900	100	30
T_2	750	900	100	35
T_3	750	900	100	37
T_4	75	300	30	45
T_5	75	300	30	55
T_6	0	200	20	60
T_7	0	200	20	100
T_8	0	200	20	110
T_9	0	150	10	150

Table 5.2: Thermal unit data

We also consider two typical hydro valleys called Ain and Isère whose hydraulic configuration is given in Figure 5.2. The data related to the hydro plants of Ain and Isère valleys are shown in Tables 5.3 and 5.4, respectively. All turbines of each power plant are considered identical, but each plant has a varying number of turbines. The volumes are given in hm^3, the maximum flow rate of each turbine q^{max} in m^3/s and the cost of the water c_w in euros.



(a) Ain valley (b) Isere valley

Figure 5.2: French hydro valleys.

The maximum power that we can displace with load shaving is considered equal to 100 MW. Regarding the battery, we consider three different batteries which differ in the maximum capacity of the battery Q_B and the maximum input and output power, b_I^{max} and b_O^{max}, respectively. The data is provided in Table 5.5. In the table, battery 3 corresponds to the largest battery in the market [Ric18]. Table 5.6 reports common further data for all three batteries.

Plant	Turbines	V^{min}	V^{max}	V^0	q^{max}	c_w
H_1	4	150	417.80	277.50	72.5	40
H_2	2	0.96	1.5	1.011	100	35
H_3	2	33.27	35.12	34.38	120	30
H_4	2	0.88	4.38	1.84	15	33
H_5	3	10.81	13.59	12.9	60	28
H_6	3	17.6	18.2	18.1	90	25

Table 5.3: Hydro plants data of Ain valley

Plant	Turbines	V^{min}	V^{max}	V^0	q^{max}	c_w
H_1	3	4	9.78	5	11	35
H_2	3	3	223.83	133.43	16	40
H_3	4	0	0.1	0.05	10	30
H_4	2	0	0.1	0.05	12	33
H_5	3	0.1	0.6	0.3	3.3	28
H_6	4	0.1	1.32	0.5	47.7	35
H_7	4	0.09	0.3	0.1	25	25

Table 5.4: Hydro plants data of Isère valley

	Q_B	b_I^{max}	b_O^{max}
Battery 1	117 MWh	13.2 MW	40 MW
Battery 2	234 MWh	26.4 MW	80 MW
Battery 3	400 MWh	100 MW	100 MW

Table 5.5: Battery data for French system

c^{min}	0.1
c^{max}	1
ρ_I	0.95
ρ_O	0.95
y^0	0.5

Table 5.6: Battery data equal for all models

Figure 5.3 shows the Pareto front for the French bi-objective energy problem. The number of points at the discretization is equal 100. By examining the Pareto front, the option that reduces the most both the cost and the carbon emission is battery 3, that is the largest battery. Load shaving is an efficient mechanism, comparable to battery 1. The black dots in Figure 5.3 show the optimal value of each multi-objective problem for $\omega = 0.5$, that is, when both objectives have the same weight in the scalarization.

Table 5.7, shows the final value of the two objectives when ω equal 1, 0, and 0.5, respectively. Recall from equation (5.7) that the parameter ω defines the scalarization; specifically, when $\omega = 1$ (respectively $\omega = 0$) only the generation cost (respectively the maximum CO_2 emission) is considered, while $\omega = 0.5$ assigns the same weight for both objectives. For each value of ω, a first row reports the generation cost and the maximum CO_2 emission for a problem without EMS, case 1. The gains in both objectives, absolute and percentage magnitudes, are displayed subsequently, for the model (5.3) with load shaving, and for the model (5.4), using the 3 battery configurations of Table 5.5.

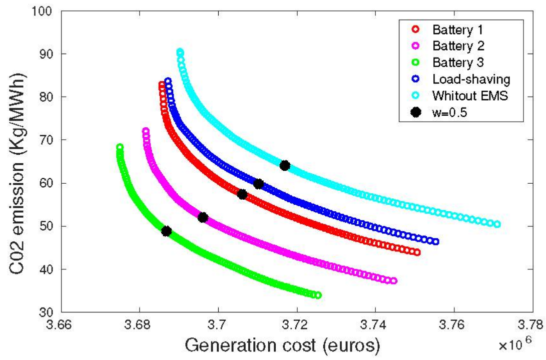


Figure 5.3: Pareto front for French system

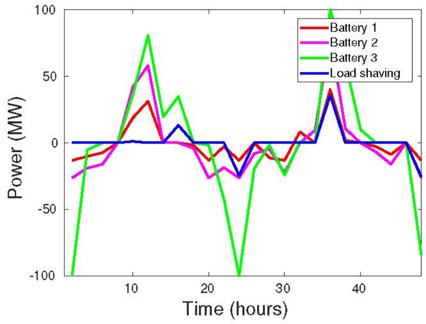
		Cost (euro)	% difference	CO_2 (Kg/MWh)	% difference
$\omega = 1$	Without EMS	3.690.352,00	-	90,41	-
	Load shaving	-3.100,00	-0.08 %	-6.82	-7.55 %
	Battery 1	-4.581,00	- 0.12 %	-7.14	-7.89 %
	Battery 2	-8.720,00	-0.24 %	-17.23	-19.05 %
	Battery 3	-15.320,00	-0.41 %	-22.16	-24.51 %
$\omega = 0$	Without EMS	3.770.995,00	-	50.38	-
	Load shaving	-15.632,00	-0.41%	-4.06	-8.07 %
	Battery 1	-20.363,00	-0.54 %	-6.54	-12.97%
	Battery 2	-26.335,00	-0.70%	-13.15	-26.10 %
	Battery 3	-45.581,00	-1.21 %	-16.49	-32.74 %
$\omega = 0.5$	Without EMS	3.716.891,00	-	64.05	-
	Load shaving	-6.684,00	-0.18%	-4.22	-6.59 %
	Battery 1	-10.828,00	-0.29 %	-6.64	-10.37%
	Battery 2	-20.841,00	-0.56%	-12.01	-18.76 %
	Battery 3	-30.140,00	-0.81%	-15.19	-23.72 %

Table 5.7: Generation cost and CO_2 emission French system

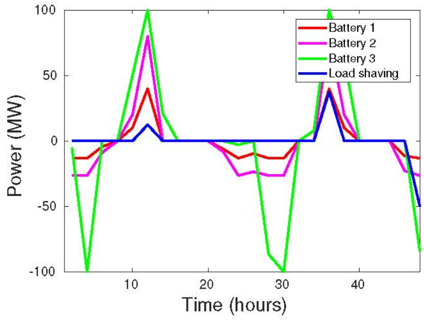
On Table 5.7, results with $\omega = 1$ and $\omega = 0$ correspond, respectively, to a "purely operational" and a "fully green" decision maker. In particular, considering only the minimization of emissions, Table 5.7 shows that the generation cost when considering an EMS also reduces the cost with respect to the base configuration (without EMS). As expected, in this case the CO_2 emissions decrease with the use of some resource of energy management system. Notably, in the opposite situation reported for $\omega = 1$ (CO_2 emission is not taken into account in the objective), using an EMS contributes to reducing the fuel emissions. When the same weight is assigned to both objectives, i.e., $\omega = 0.5$, we also obtain a decrease in both the generation cost and the maximum CO_2 emission with respect to the base configuration. However, when compared to the results for $\omega = 1$, we notice a better control over the total emissions.

On Figure 5.4 the batteries' power profiles and load shavings are shown for different values of the parameter ω. Positive values of the battery power profile mean energy discharged, the output, and negative values energy stored, input. We can see that the battery is used at peak times and the same happens with the load shaving. So we can conclude that both mechanism have similar behaviors and are good for smoothing the load peaks, consequently contributing to reduce the generation cost and the CO_2 emission.

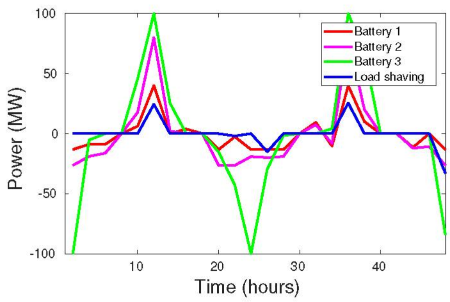
The sensitivity of the second objective with respect to the the generation cost,



(a) $\omega = 1$



(b) $\omega = 0$



(c) $\omega = 0.5$

Figure 5.4: Battery profile and load shaving for different values of ω (French system)

given by Theorem 5.1, is presented in Figure 5.5. This sensitivity is negative, as the CO_2 decreases when the generation cost increases, and has a similar behavior for all configurations of the problem. Moreover, it is possible to see that the rate of change decreases when the generation cost gets closer to the value obtained when minimizing just the second objective function.

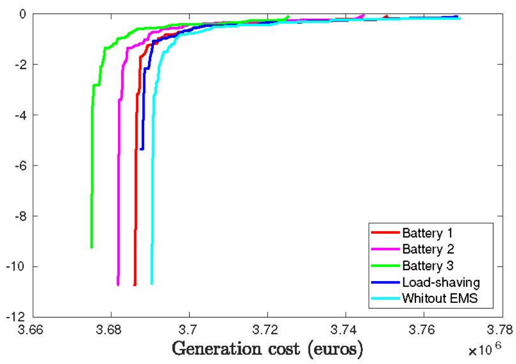


Figure 5.5: Sensitivity of the CO_2 emission with respect to the generation cost for the French system

5.5.2 Thermal dominated system : German case

In this model we consider six thermal units, a run-of-the-river hydro power plant, a solar and a wind plant that are described below. The time horizon and time step are picked identical to those for the French case.

The data used for each thermal unit is given in Table 5.8 below. The cost c_j in euro per megawatt-hour (€/MWh) is considered constant over time. Each of thermal unit is of a different type: nuclear, brown coal (lignite), hard coal, gas, mineral oil and biomass.

Thermal units	p^{max}	θ	c_j
1	9516	1057	35
2	19989.3	1999	45
3	22699.6	2270	55
4	24434.7	2443	90
5	2614.7	175	150
6	7320.1	732	40

Table 5.8: German's thermal unit data

The maximum power that we can displace with load shaving is equal to 2000 MW, the base data of the batteries is as shown in Table 5.6 with maximum capacity, output and input power as in Table 5.9. We note that the data for the load shaving and for the capacity, output and input of the battery are 20 times larger than in the French system, but the demand here is also around 20 times larger than in the French system.Moreover, we run the model for two different days, a winter and a spring days. The demand, initial power of thermal units (p^0), run-of-the-river, solar and wind plants of those days will be presented in next sections.

	(Q_B)	(b_I)	(b_O)
Battery 1	2340 MWh	264 MW	800 MW
Battery 2	4680 MWh	328 MW	1600 MW
Battery 3	8000 MWh	2000 MW	2000 MW

Table 5.9: Battery data for German system

Data and results of a winter day

The initial power of this day can be seen in Table 5.10. Moreover, the demand and wind and run-of-the-river generation for the whole period can be seen in Figure 5.6.

Thermal units	1	2	3	4	5	6
p^0	9329	14551	1946	1626	191	4489

Table 5.10: Initial power thermal units for a winter German day

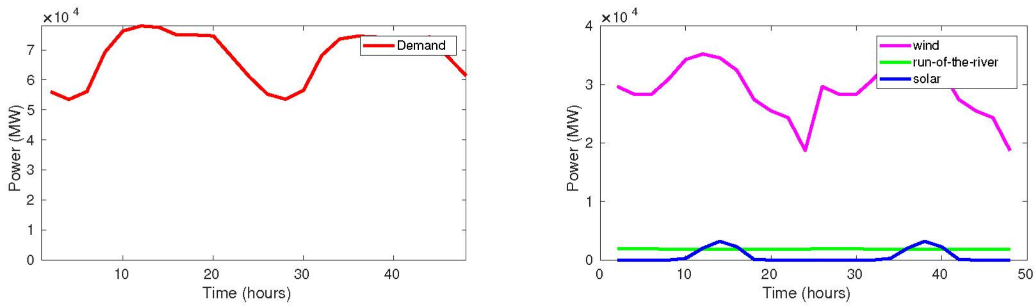


Figure 5.6: Demand and solar, wind and run-of-the-river generation, respectively for a winter German day.

Figure 5.7 shows the Pareto front for the German bi-objective energy problem for a winter day. The number of points at the discretization is considered equal 200. As in the French case, the option that reduces the most both the cost and the carbon emission is battery 3, that is the largest battery. Load shaving is an efficient mechanism, comparable to the smallest storage battery 1.

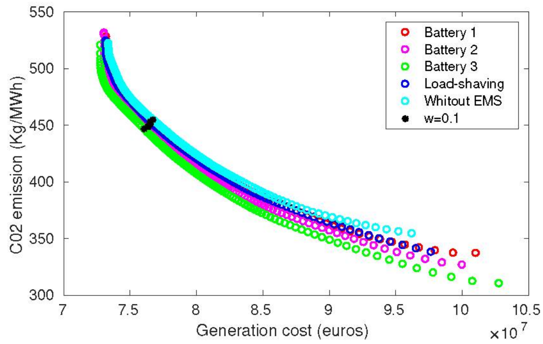


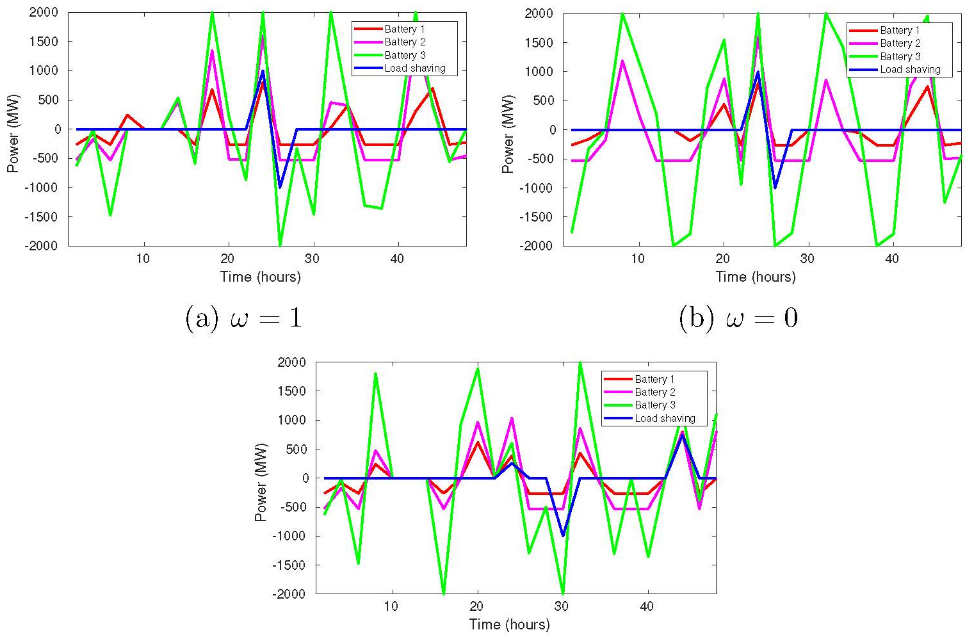
Figure 5.7: Pareto front for a winter German day

Table 5.11 show the final value of the two objectives when the parameter ω is equal 1, 0, and 0.1, respectively. Recall from equation (5.7) that the parameter ω defines the scalarization; specifically, when $\omega = 1$ (respectively $\omega = 0$) only the generation cost (respectively the maximum CO_2 emission) is considered. For each value of ω, a first row reports the generation cost and the maximum CO_2 emission beyond the target for a problem without EMS. The gains in both objectives, absolute and percentage magnitudes, are displayed subsequently, for the model (5.3) with load shaving, and for the model (5.4), using the three battery configurations in Table 5.9.

		Cost (euro)	% difference	CO_2 (Kg/MWh)	% difference
$\omega = 1$	Without EMS	73.360.494,00	-	524.44	-
	Load shaving	-19.310,00	-0.26 %	-0.0	-0.0 %
	Battery 1	-153.543,00	- 0.21 %	+3.45	+0.666 %
	Battery 2	-306.995,00	-0.42 %	+7.0	+1.3328 %
	Battery 3	-589.802,00	-0.80 %	-0.0	-0.0 %
$\omega = 0$	Without EMS	96.209.093,00	-	354.39	-
	Load shaving	+2.770.810,00	+2.88%	-18.42	-5.20 %
	Battery 1	+4.819.470,00	+5.00 %	-17.33	-4.89%
	Battery 2	+3.786.984,00	+3.94%	-27.72	-7.82 %
	Battery 3	+1.927.188,00	+2.00 %	-35.49	-10.0 %
$\omega = 0.1$	Without EMS	76.720.938,00	-	454.90	-
	Load shaving	-203.992,00	-0.26%	-73.80	-16.22 %
	Battery 1	-188.498,00	-0.24 %	-77.36	-17.00%
	Battery 2	-358.562,00	-0.47%	-81.20	-17.85 %
	Battery 3	-657.167,00	-0.86%	-85.68	-18.83 %

Table 5.11: Generation cost and CO_2 emission for a winter German day

On Figure 5.8 the batteries power profile and load shaving are shown for different values of the parameter ω. Positive values of the battery power profile mean energy discharged, the output, and negative values energy stored, input. In this case we see that



(a) $\omega = 1$ (b) $\omega = 0$

(c) $\omega = 0.1$

Figure 5.8: Battery profile and load shaving for different values of ω (winter German day)

the battery is not used just at peak times as the load-shaving, but several times during the day.

The sensitivity of the second objective with respect to the generation cost, Figure 5.9, as in the French system, has a similar behavior for all configurations of the problem and the rate of change decreases when the generation cost gets closer to the value obtained when minimizing just the second objective function. Note that the sensitivity is smaller than in the French case, this probably happens because the variation on the generation cost is larger than in the French case. So in the German case it is necessary to increase more the generation cost than in the French case to have reductions in the CO_2 emission.

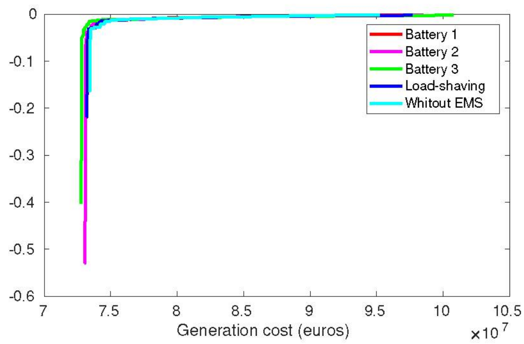


Figure 5.9: Sensitivity of the CO_2 emission with respect to the generation cost for a winter German day

Data and results of a spring day

The initial power of this day can be seen in Table 5.12. Moreover, the demand and wind and run-of-the-river generation for the whole period can be seen in Figure 5.10.

Thermal units	1	2	3	4	5	6
p^0	7762	15946	14681	1497	64	3939

Table 5.12: Initial power thermal units of a spring German day

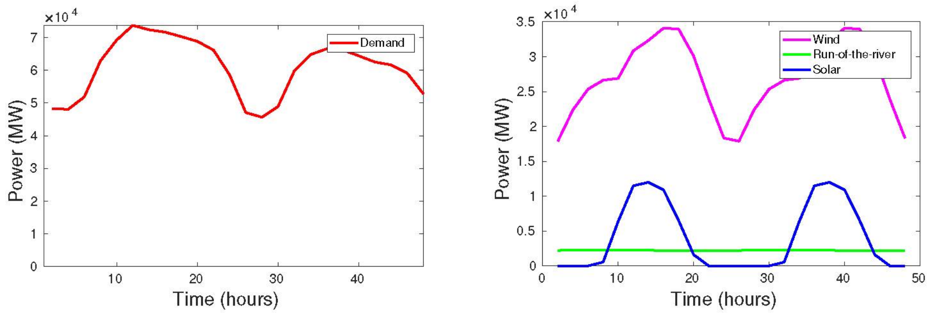


Figure 5.10: Demand and solar, wind and run-of-the-river generation, respectively of a spring German day

Figure 5.11 shows the Pareto front for the German bi-objective energy problem with data for a spring. The number of points at the discretization is also considered equal 200. As in the french problem, the option that reduces the most both the cost and the carbon emission is battery 3, that is the largest battery. Moreover the load shaving is an efficient mechanism, comparable to battery 1.

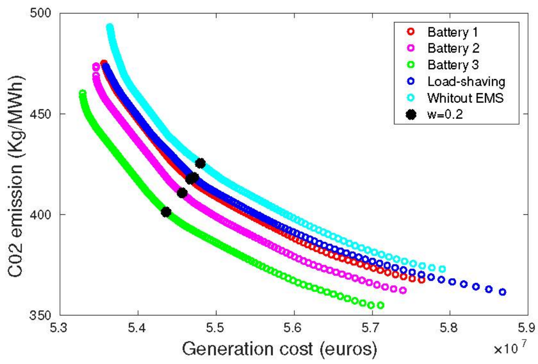


Figure 5.11: Pareto front for a spring German day

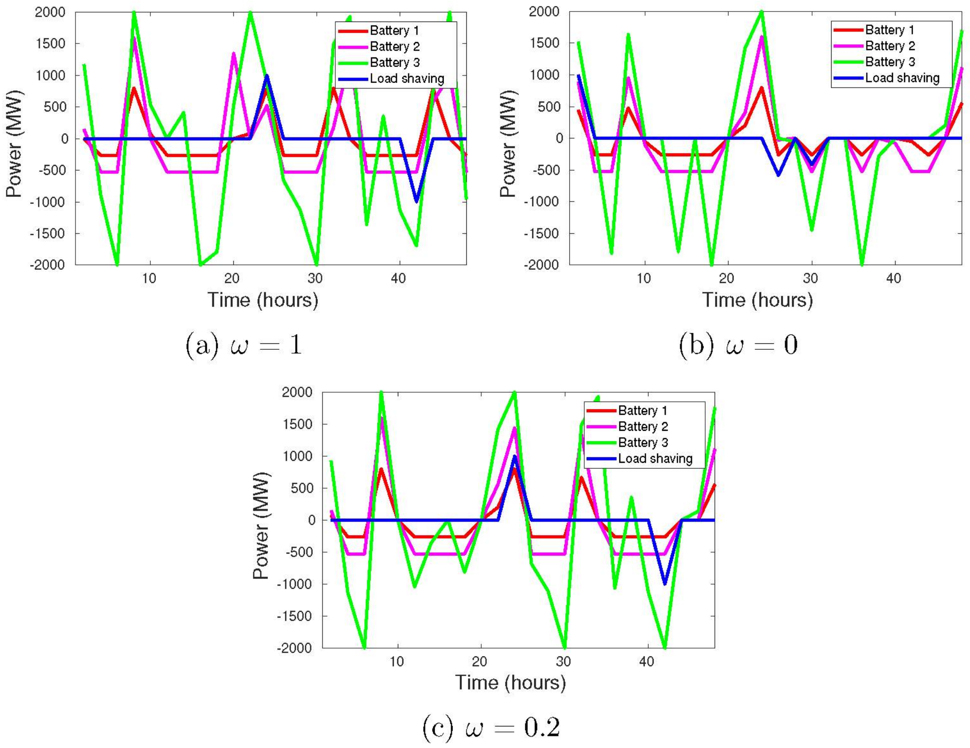
Table 5.13 show the final value of the two objectives when the parameter ω is equal 1, 0 and 0.2, respectively. The Pareto front for the different configurations of the energy problem are more spaced in spring day, Figure (5.11), than in the winter day (5.7). It happens because the difference between the lower and upper values for the objectives is smaller in the spring day. However comparing Tables 5.13 and 5.11, for $\omega = 1$, it is possible to see that the percentage of reduction in the cost generation is larger in the winter day than in the spring day. The same happens when comparing the percentage of the CO_2 emission on Tables 5.13 and 5.11, for $\omega = 0$.

On Figure 5.12 the batteries power profile and load shaving are shown for different values of the parameter ω. As in the winter day of the German system the batteries are used several times during the day, not just at peak times as the load-shaving.

The sensitivity of the second objective with respect to the generation cost, Figure 5.13, as in the previous cases, has a similar behavior for all configurations of the problem and the rate of change decreases when the generation cost gets closer to the value obtained

		Cost	%	CO_2	%
$\omega = 1$	Without EMS	53.640.087,00	-	492.94	-
	Load shaving	-58.865,00	-0.11 %	-18.60	-3.77 %
	Battery 1	-98.132,00	- 0.18 %	-15.32	-3.11 %
	Battery 2	-176.792,00	-0.33 %	-10.18	-2.06 %
	Battery 3	-348.998,00	-0.65 %	-15.69	-3.18 %
$\omega = 0$	Without EMS	57.903.467,00	-	372.77	-
	Load shaving	+975.380,00	+1.68%	-11.88	-3.17 %
	Battery 1	-264.926,00	-0.46 %	-5.25	-1.41%
	Battery 2	-504.849,00	-0.87%	-10.50	-2.81 %
	Battery 3	-789.857,00	-1.36 %	-17.94	-4.81 %
$\omega = 0.2$	Without EMS	54.798.647,00	-	425.42	-
	Load shaving	-89.702,00	-0.16%	-6.82	-1.68 %
	Battery 1	-130.522,00	-0.24 %	-7.76	-1.82%
	Battery 2	-237.123,00	-0.43%	-14.69	-3.45 %
	Battery 3	-441.068,00	-0.80%	-24.10	-5.67 %

Table 5.13: Generation cost and CO_2 emission for a spring German day



(a) $\omega = 1$

(b) $\omega = 0$

(c) $\omega = 0.2$

Figure 5.12: Battery profile and load shaving for different values of ω (spring German day)

when minimizing just the second objective function. Note that the sensitivity is smaller than in the French case, this probably happens because the variation on the generation cost is larger than in the French case. So in the German case it is necessary to increase more the generation cost than in the French case to have reductions in the CO_2 emission.

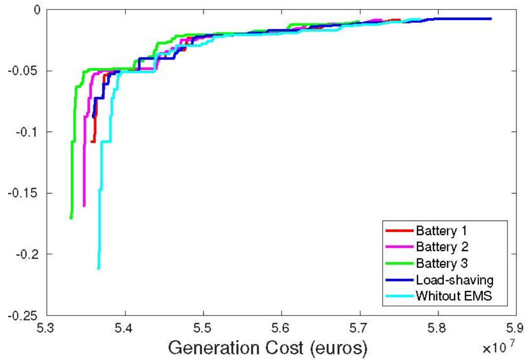


Figure 5.13: Sensitivity of the CO_2 emission with respect to the generation cost for a spring German day

5.5.3 Hydro dominated system : Brazilian case

The Brazilian system represents schematically the main subsystems in the country, each one synthesized in a bus. Each bus has its own demand to meet, but it is interconnected with some other buses and can exchange energy with them. The representation of the transmission system is an important modeling issue since the capacity of transmission lines may change the energy generation. Based on [Scu16], in this work a classical DC (direct current) model is employed, which considers the following premises:

- all bus voltages are considered fixed and equal to 1 p.u.;
- line resistences are negligible compared to line reactances;
- angle (θ) differences between buses are small, so that $\sin(\theta) \approx \theta$ in radians.

With these assumptions, the power flow in each transmission line can be modelled by linear constraints [TOC+99]. The limits of the transmission lines and the power balance equation, thus become:

$$-FL_l^{max} \leq \sum_{b=1}^{B} \Gamma_{lb}\Big(\sum_{j\in T_b} p_j^t + \sum_{i\in H_b} \rho_i^t q_i^t + W_b^t - L_b^t\Big) \leq FL_l^{max}$$

$$\sum_{b=1}^{B}\Big(\sum_{j\in T_b} p_j^t + \sum_{i\in H_b} \rho_i^t q_i^t + W_b^t\Big) = \sum_{b=1}^{B} L_b^t,$$

for $t = 1, \ldots, T$, where B is the number of buses, T_b is the set of thermal units connected to bus b, H_b is the set of hydro plants connected to bus b, $\rho_i q_i$ is the output power of hydro plant i, W_b^t represents the wind energy generated by bus b at time step t, L_b^t is the load demand of bus b at time step t, FL_l^{max} is the maximum power flow capacity of the transmission line l (MW) and Γ_{lb} is the power transfer distribution factor (PTDF) of transmission line l due to the injection of active power at bus b. Moreover, Γ_{lb} is a parameter that depends on the reactance of the transmission lines and represents the sensitivity of each power injection at bus b over line l. This parameter is obtained through manipulations on equations of power flow, power balance on buses and the incidence matrix as described in [VDBDD14].

The power system considered in this section is represented in Figure 5.14. The transmission system has 5 buses and 6 lines, whose transmission capacity is 1.000 MW and reactance 0, 0249 pu, with the exception of line $2-5$, which has a capacity of 2.000 MW and reactance 0, 0124 pu. The demand to be met is located in four load centers ($L_1 - L_4$). Bus 5 is considered to be the slack bus, i.e., it does not have a demand to meet. Moreover the system is composed of seven thermal units ($T_1 - T_7$), three hydro plants ($H_1 - H_3$) and two wind units ($W_1 - W_2$). We consider the time horizon T to span 24 hours and with time step size h equal to 1 hour.

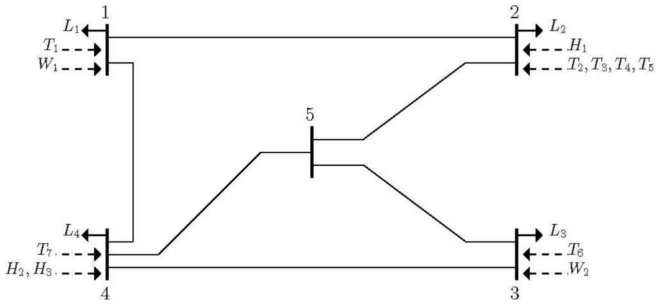


Figure 5.14: Diagrammatic representation of the Brazilian system

The demand for each bus and the wind generation of units W_1 and W_2 are given in Figure 5.15.

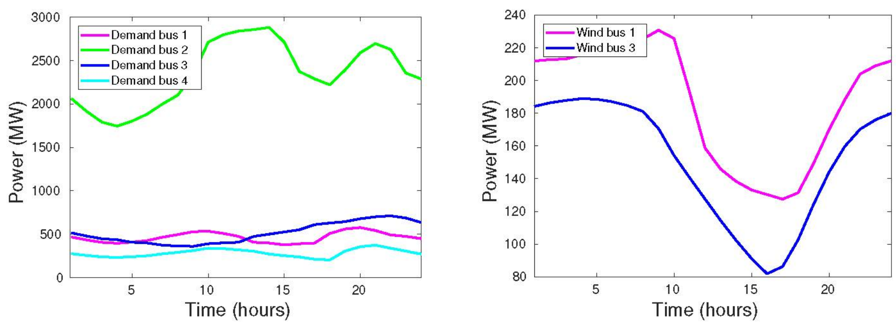


Figure 5.15: Demand and wind generation, respectively for the Brazilian system

The thermal unit data is given in Table 5.14 below of three different types: 2 gas ($T_1 - T_2$), 3 mineral oil ($T_3 - T_5$), 1 coal (T_6) and 1 biomass (T_7).

Thermal units	Bus	p^0	p^{max}	θ	c_j
T_1	1	0	455	75	15
T_2	2	0	160	70	20
T_3	2	0	80	30	50
T_4	2	0	55	15	60
T_5	2	0	55	15	80
T_6	3	200	455	100	35
T_7	4	0	130	40	25

Table 5.14: Thermal unit data for the Brazilian system

The data related to the hydro plants are shown in Table 5.15. The first power plant, H_1, consists of three, while $H2$ and $H3$ consist of five and four identical units, respectively. The volumes are given in hm^3, the maximum flow rate of each turbine in h^3/s and the cost of the water c_w in Brazilian Real (R\$). Reservoir H_3 is downstream with respect to

H_2 with travel time equal to one hour. Reservoir H_1 is uphill with respect to H_2 and there is no water delay between both reservoirs.

Hydro plant	Bus	V^{min}	V^{max}	V^0	q^{max}	c_w
H_1	2	1974	5776	2000	401	50
H_2	4	2283	3340	2500	312	50
H_3	4	4300	5100	4500	393	50

Table 5.15: Hydro plants data Brazilian system

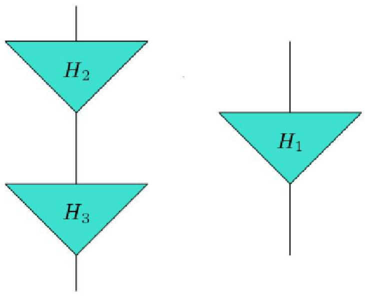


Figure 5.16: Brazilian hydraulic configuration

The load shaving and the battery energy storage system where connected to bus 4, that is the bus with the biggest capacity of power generation. Moreover the data of those mechanism where considered as in Frech system (Tables (5.6) and (5.5)).

Results

Figure 5.17 shows the Pareto front for the bi-objective energy problem considering the number of points at the discretization equal 100. The black dots in Figure 5.17 show the optimal value of each multi-objective problem for $w = 0.9$.

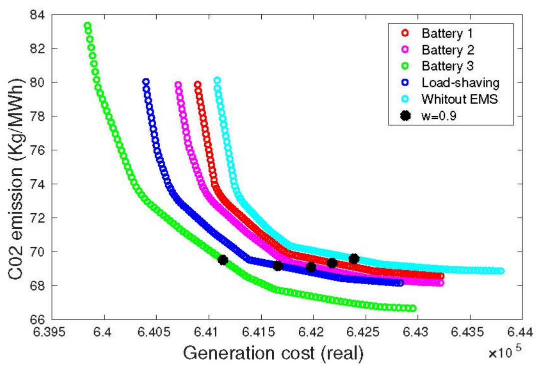


Figure 5.17: Pareto front for the Brazilian system

By examining the Pareto front, the option that reduces the most both the cost and the carbon emission is battery 3, that is the largest battery. Load shaving is a mechanism better than batteries 1 and 2 in this configuration, while in the previous configurations it was comparable with battery 1. Since in this case, we consider the transmission system that includes more constraints to the problem. The energy that can be transferred between the buses, needs to satisfy the limits of the lines. Hence, it is necessary to dispose of a larger battery to obtain better results than the load-shaving (which by nature can be diffuse).

Table 5.16 shows the final value of the two objectives when ω equal 1, 0, and 0.9, respectively. The gains in both objectives, absolute and percentage magnitudes, are displayed subsequently, for the model (5.3) with load shaving, and for the model (5.4), using the 3 battery configurations in Table 5.5.

		Cost	%	CO_2	%
$\omega = 1$	Without EMS	641.083,00	-	80.10	-
	Load shaving	-682,00	-0.11 %	-0.08	-0.1 %
	Battery 1	-187,00	- 0.03 %	-0.25	-0.31 %
	Battery 2	-374,00	-0.06 %	-0.27	-0.34 %
	Battery 3	-1.241,00	-0.19 %	+3.23	+4.03 %
$\omega = 0$	Without EMS	643.795,00	-	68.85	-
	Load shaving	-960,00	-0.15%	-0.70	-1.03 %
	Battery 1	-571,00	-0.09 %	-0.30	-0.44%
	Battery 2	-575,00	-0.09%	-0.70	-1.03 %
	Battery 3	-840,00	-0.13 %	-2.21	-3.21 %
$\omega = 0.9$	Without EMS	642.389,00	-	69.75	-
	Load shaving	-729,00	-0.11%	-0.41	-0.59 %
	Battery 1	-209,00	-0.03 %	-0.20	-0.34%
	Battery 2	-411,00	-0.06%	-0.48	-0.68 %
	Battery 3	-1.255,00	-0.19%	-0.06	-0.08 %

Table 5.16: Generation cost and CO_2 emission for Brazilian system

The sensitivity of the second objective with respect to the generation cost, given by Theorem 5.1 is presented in Figure 5.18. The rate of change decreases when the generation cost gets closer to the value obtained when minimizing just the second objective function, as obtained for French and German systems. Moreover, in this case, the sensitivity is larger than in the previous cases, meaning that it is possible to reduce the CO_2 emission with a smaller cost.

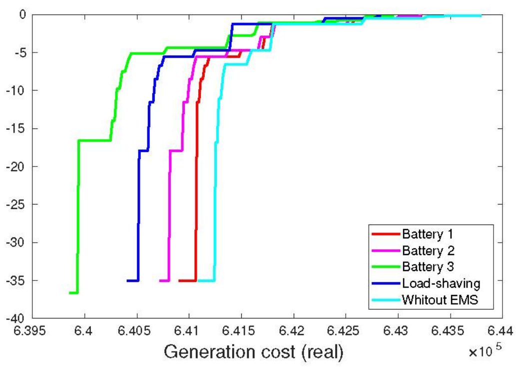


Figure 5.18: Sensitivity of the CO_2 emission with respect to the generation cost for the Brazilian system

5.6 Concluding remarks

In this work we studied the effect of different batteries, and a load shaving mechanism on bi-objective energy management problems. The first objective consists in controlling cost, whereas the second consists of controlling carbon emissions. The study was carried out on simple, yet representative, mixes for power systems: thermal dominated, hydro dominated, hydro-thermal system. Both mechanisms, load shaving and battery provide a reduction in generation cost and carbon emissions. Moreover, if the battery is sufficiently large, the results are better than with load shaving. The size of the required battery was shown to depend on modelling transmission system constraints. In our study demand side management was shown to provide good results comparable to the case with a battery. When recalling the fact that the former, does not, in principle, require the installation of physical material, demand side management may proof quite valuable.

The provided multi-objective approach can be extended to consider larger and more realistic models. We have also given a sensitivity analysis which provides useful information to the decision maker as to the compromise between conflicting objectives.

This preliminary study shows the interest of the approach. We plan to study extensions to a stochastic setting, and a continuous time model that can be investigated with the method presented in Chapter 3.

Chapter 6

Conclusions and future work

In this work we studied the multi-objective optimal control problems in finite and infinite horizon. More precisely, in Chapter 3 we investigated for the first time bi-objective optimal control problems with cost functions of different nature. We considered the situation where the set of trajectories is not compact and the set of Pareto solutions may be empty. We have studied the relation of the Pareto front corresponding to the relaxed (convexified) bi-objective problem with the original one. With the HJB approach to an adequate auxiliary control problem we gave a characterization of the Pareto front of the relaxed bi-objective problem. Moreover, we gave also a characterization of the ε-Pareto front of the original problem by using the same auxiliary value function. The presented numerical examples confirm that the method is good, especially considering that after solving the derived HJB equation, we can easily obtain the Pareto fronts for different initial states.

However, it should be noticed that for a control problem with two state variables and two cost functions, the method requires to solve a HJB equation in dimension 4. To get an approximation of the Pareto front in a reasonable time, the dimension of state should remain less than 4 or 5. We stress on that this thesis addresses mainly some theoretical questions, the numerical aspects should be investigated further.

The interesting results obtained for finite horizon motivated us to investigate infinite horizon multi-objective optimal control problems. We note that the extension of this results to the infinite horizon case was not trivial. Considering an adequate auxiliary control problem, in Chapter 4, we could characterize the Pareto fronts for this kind of problem. The issue here was that the value function of the auxiliary optimal control problem could be characterized as a solution of an HJB equation. However, the value function is not the unique solution of the HJB equation. Usually, numerical approximations are analyzed (convergence and error estimates) under the assumption that a comparison principle holds. In this work we proposed a numerical scheme to obtain the value of the auxiliary control problem and we proved the convergence results by using only the Dynamical Programming Principle.

Furthermore, in Section 4.4, we introduced a method for reconstruction of trajectories for infinite horizon optimal control problems with state constraints, based on the dynamical programming principle. The method can also be employed if the exact value function is not available, that means, when the value function is obtained by some numerical approximation method, as the Semi-Lagrangian method proposed in Section 4.2.1. Our method extends to infinite horizon optimal control problems with state constraints the results for finite horizon problems without and with state constraints introduced in [RV91] and [ABDZ18], respectively. A future step in this subject is solving numerical examples with the Semi-Lagrangian scheme proposed in this thesis and to reconstruct optimal trajectories.

Chapter 5 was devoted to the investigation of energy management systems with two objectives. We compared the ability of different batteries as a substitute of the load shaving mechanism in smoothing the load peaks, for a simple, yet representative, power mix systems representing typical configurations in Brazil, France and Germany. In addition to the usual generation cost minimization we incorporated an environmental concern, referred to minimizing fuel emissions. We concluded that both mechanisms, load shaving and a battery, have a positive effect on demand response. We observed a reduction in generation cost and carbon emission. Moreover, if the battery is sufficiently large, the results were better than load shaving.

The energy management systems can also be modeled in a continuous time. In an ongoing work we are considering the HJB approach proposed to solve finite multi-objective problems in Chapter 3 to solve the model. When solving the discrete energy problems (5.2), (5.3) and (5.4), we could conclude that both mechanisms, load shaving and the use of an battery, have a positive effect on demand response. This feature is promising regarding the optimal control problem, where it would be desirable to eliminate load shaving (state) variables. In view of the presented results, we expect that an optimal control model dealing only with a battery would induce a demand response mechanism similar to load shaving, provided the considered batteries are sufficiently large. We are interested also in analyzing the use of the batteries for primary frequency response (PFR). The dynamics of the system frequency can be governed by the first-order swing equation [TTS16, WLHL16]:

$$2H\frac{\partial \Delta f_r(t)}{\partial t} + D\Delta f_r(t) = \sum_{j=1}^{N} \Delta p_j(t) + \Delta b_I(t) - \Delta p_L(t), \tag{6.1}$$

where $\Delta f_r(t)$ is the frequency deviation, H [MWs/Hz] is the system inertia after generation loss which refers to the ability of the system to resist a frequency change following a contingency,D [1/Hz] represents the load-damping rate, Δp_j and Δb_I denotes respectively the increased power outputs from the thermal units and battery storage unit following the generation loss Δp_L [MW].

The problem of generating energy such that demand is satisfied at minimum operating cost over a daily or weekly time horizon usually can be divided in discrete periods of one hour and the PFR is necessary during the first few seconds following a severe disturbance such as the sudden loss of a generator or a massive increase in the load, when there is no more sun, for example. To include that constraint in the problem, a finer discretization needs to be done locally, at critical times. We are currently developing a model in continuous time that we plan to solve with the HJB approach.

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