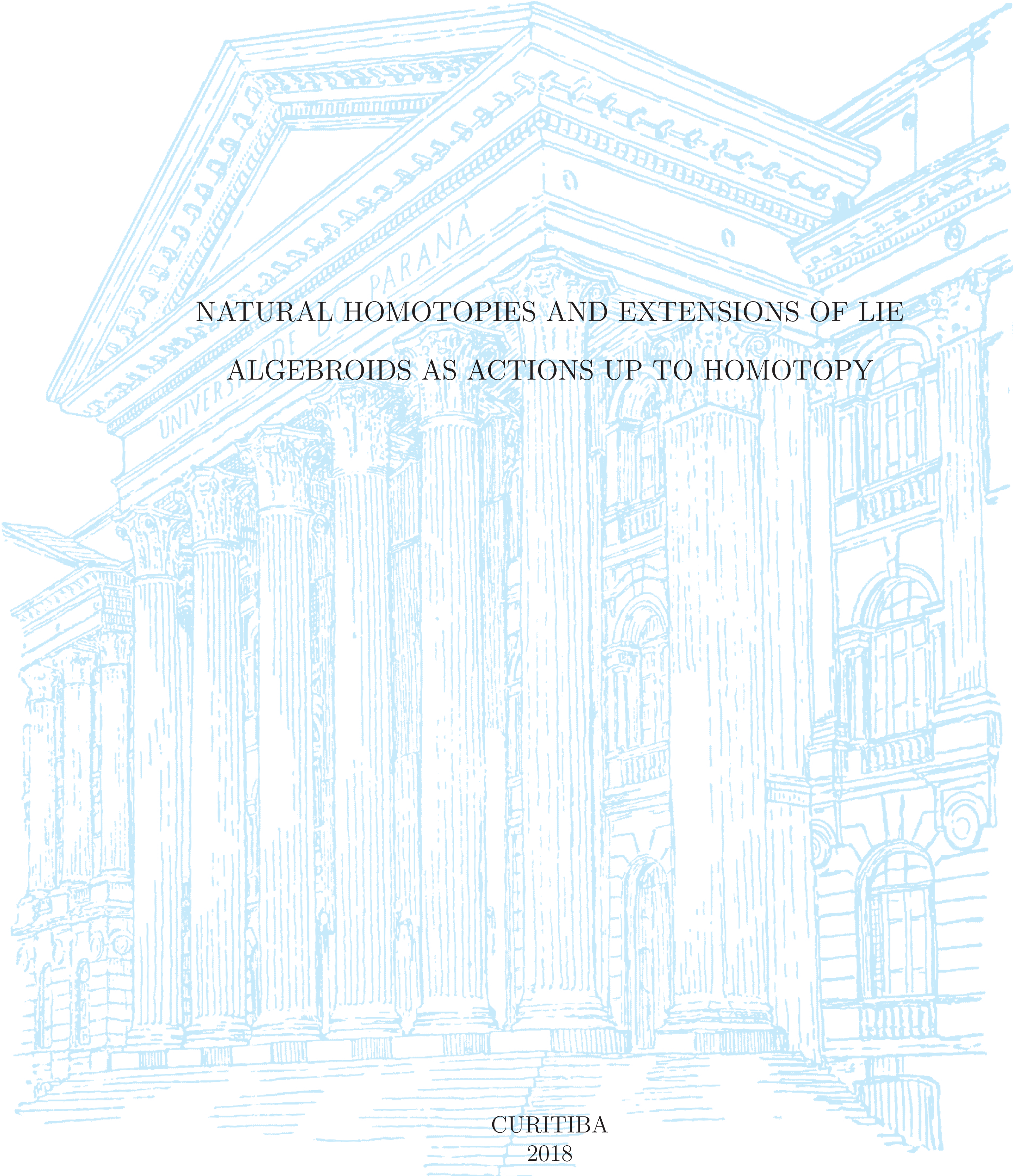
UNIVERSIDADE FEDERAL DO PARANÁ

DION ROSS PASIEVITCH BONI ALVES

NATURAL HOMOTOPIES AND EXTENSIONS OF LIE ALGEBROIDS AS ACTIONS UP TO HOMOTOPY



CURITIBA
2018

DION ROSS PASIEVITCH BONI ALVES

NATURAL HOMOTOPIES AND EXTENSIONS OF LIE ALGEBROIDS AS ACTIONS UP TO HOMOTOPY

Tese apresentada como requisito parcial à obtenção do grau de Doutor em Matemática, no Curso de Pós-Graduação em Matemática, Setor de Ciências Exatas, da Universidade Federal do Paraná.

Orientador: Prof. Dr. Olivier Brahic.

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PROGRAMA DE PÓS-GRADUAÇÃO MATEMÁTICA

ATA Nº025

ATA DE SESSÃO PÚBLICA DE DEFESA DE DOUTORADO PARA A OBTENÇÃO DO GRAU DE DOUTOR EM MATEMÁTICA

No dia trinta de agosto de dois mil e dezoito às 17:00 horas, na Sala de Seminários 300, Jayme Machado Cardoso, Departamento de Matemática – UFPR, Setor de Ciências Exatas – Centro Politécnico – Jardim das Américas, CEP: 81531-980, Curitiba – Paraná, foram instalados os trabalhos de arguição do doutorando **DION ROSS PASIEVITCH BONI ALVES** para a Defesa Pública de sua tese intitulada **Natural Homotopies and Extensions of Lie Algebroids as Actions up to Homotopy**. A Banca examinadora, designada pelo Colegiado do Programa de Pós-Graduação em MATEMÁTICA da Universidade Federal do Paraná, foi constituída pelos seguintes Membros: OLIVIER BRAHIC (UFPR), CARLOS EDUARDO DURAN FERNANDEZ (UFPR), PEDRO FREJLICH (UFRGS), CRISTIAN ANDRÉS ORTIZ GONZÁLES (IME-USP), HUDSON DO NASCIMENTO LIMA (UFPR). Dando início à sessão, a presidência passou a palavra ao discente, para que o mesmo expusesse seu trabalho aos presentes. Em seguida, a presidência passou a palavra a cada um dos Examinadores, para suas respectivas arguições. O aluno respondeu a cada um dos arguidores. A presidência retomou a palavra para suas condições finais. A Banca Examinadora, então, reuniu-se e, após a discussões de suas avaliações, decidiu-se pela aprovação do aluno. O doutorando foi convidado a ingressar novamente na sala, bem como os demais assistentes, após o que a presidência fez a leitura do Parecer da Banca Examinadora. A aprovação no rito de defesa deverá ser homologada pelo Colegiado do programa, mediante o atendimento de todas as indicações e correções solicitadas pela banca dentro dos prazos regimentais do programa. A outorga do título de doutor está condicionada ao atendimento de todos os requisitos e prazos determinados no regimento do Programa de Pós-Graduação. Nada mais havendo a tratar a presidência deu por encerrada a sessão, da qual, eu OLIVIER BRAHIC, lavrei a presente ata, que vai assinada por mim e pelos membros da Comissão Examinadora.

Curitiba, 30 de Agosto de 2018.

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Presidente da Banca Examinadora

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TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em MATEMÁTICA da Universidade Federal do Paraná foram convocados para realizar a arguição da tese de Doutorado de **DION ROSS PASIEVITCH BONI ALVES** intitulada: **Natural Homotopies and Extensions of Lie Algebroids as Actions up to Homotopy**, após terem inquirido o aluno e realizado a avaliação do trabalho, são de parecer pela sua aprovação no rito da defesa.

A outorga do título de doutor está sujeita à homologação pelo colegiado, ao atendimento de todas as indicações e correções solicitadas pela banca e ao pleno atendimento das demandas regimentais do Programa de Pós-Graduação.

Curitiba, 30 de Agosto de 2018.

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To my parents Inês and Sebastião.

RESUMO

Nesta tese introduzimos uma noção apropriada de homotopia entre morfismos de grupoides de Lie, obtemos o objeto infinitesimal correspondente e explicamos como integrá-lo. Depois, caracterizamos morfismos entre extensões de álgebroides de Lie por meio de conexões de Ehresmann. Em seguida, definimos ações à menos de homotopia, as quais são certas ações que generalizam representações à menos de homotopia de 2-termos, e mostramos que a categoria correspondente é equivalente à categoria de extensões de álgebroides de Lie. Então discutimos como uma ação à menos de homotopia se integra num 2-funtor estrito adequado, chamado de 2-funtor de holonomia. Discutimos também, brevemente, esta integração ao nível de morfismos.

Palavras-chave: Transformações naturais suaves; homotopias naturais; extensões de algebróides de Lie; ações à menos de homotopia.

ABSTRACT

In this thesis we introduce an appropriate notion of homotopy between morphisms of Lie groupoids. We obtain the corresponding infinitesimal object and we also explain how to integrate it. Next, we characterize morphisms between extensions of Lie algebroids by means of Ehresmann connections. Afterwards, we define actions up to homotopy, which are certain actions which generalize 2-term representations up to homotopy, and we show that the corresponding category is equivalent to the category of extensions of Lie algebroids. Then we discuss how an action up to homotopy integrates to a suitable strict 2-functor, called the holonomy 2-functor. We also discuss briefly this integration at the level of morphisms.

Keywords: Smooth natural transformations; natural homotopies; Lie algebroid extensions; actions up to homotopy.

Contents

Introduction

It is a classical fact that every finite dimensional Lie group induces a finite dimensional Lie algebra and, conversely, every finite dimensional Lie algebra is associated, up to a unique isomorphism, to a simply connected finite dimensional Lie group. Lie groups are the *global* objects corresponding to Lie algebras and Lie algebras are the *infinitesimal* objects corresponding to Lie groups. The process of passing from Lie groups to Lie algebras is called *differentiation* and the process of passing from Lie algebras to Lie groups is called *integration*. This has been a very fruitful theory and has been explored in a more general setting, where Lie groups are replaced by Lie groupoids and Lie algebras are replaced by Lie algebroids. It is well known that there is a *differentiation* procedure that allows us to pass from Lie groupoids to Lie algebroids. However, the *integration* procedure that would allow us to pass from Lie algebroids to Lie groupoids is not always possible. In other words, not every Lie algebroid "comes" from a Lie groupoid. We say that a given Lie algebroid A is *integrable* when there exists a Lie groupoid whose associated Lie algebroid is isomorphic to A. We then say that such Lie groupoid is an *integration* or *integrates* the given Lie algebroid. The problem of deciding whether a Lie algebroid is integrable or not, known as the *integrability problem*, was a longstanding one and it was completely solved in the seminal paper of Marius Crainic and Rui Loja Fernandes [17]. In that work, to a given Lie algebroid A is associated a source simply-connected topological groupoid $\mathcal{G}(A)$, called the *Weinstein groupoid* of A. The obstructions to the smoothness of $\mathcal{G}(A)$ are equivalent to the obstructions to the integrability of A, and whenever $\mathcal{G}(A)$ is a Lie groupoid it is the unique (up to isomorphism) source simply-connected Lie groupoid integrating A. This approach was inspired by the integration scheme for Lie algebras presented in [19].

Recently, much attention has been drawn to objects, even more general than Lie groupoids, called *VB-groupoids*. Roughly speaking, a VB-groupoid is a vector bundle object in the category of Lie groupoids: the total and the base spaces of the vector bundle are Lie groupoids and the vector bundle structure maps are required to define Lie groupoid morphisms. Again, there is a natural *differentiation* procedure that leads us from VB-groupoids to infinitesimal objects called *VB-algebroids*. Roughly speaking, *VB-algebroids* are vector bundle objects in the category of Lie algebroids: the total and the base spaces of the vector bundle are Lie algebroids and the vector bundle structure maps are required to define Lie algebroid morphisms. It turns out that it is not always possible to *integrate* VB-algebroids to VB-groupoids. The integration problem for VB-algebroids has been solved in [15] where the authors show that a VB-algebroid is integrable to a VB-groupoid if and only if the base algebroid is integrable and the spherical periods of certain underlying cohomology classes vanish identically.

VB-algebroids are intimately connected to algebraic objects called 2-term representations up to homotopy. Such representations, in a more general form, first appeared in the work [2] where the authors searched for a suitable definition of representation of a Lie algebroid. The connection with VB-algebroids was explained in [23] where the authors have shown that, after the choice of a linear splitting, a VB-algebroid $D \longrightarrow E$ over $A \longrightarrow M$ gives rise to a *2-term representation up to homotopy* of A on the graded vector bundle $\mathcal{E} := E \oplus C$ over M. This is a degree one operator $D : \Omega(A, \mathcal{E}) \longrightarrow \Omega(A, \mathcal{E})$ which squares to zero and satisfies a certain derivation rule. Conversely, any such operator induces a VB-algebroid structure on the *trivial double vector bundle* $A \oplus E \oplus C \longrightarrow E$ over $A \longrightarrow M$. Indeed, using this data one can produce an equivalence of categories as shown in [18].

In a recent pre-print [13], the authors proposed an integration scheme for 2-term representations up to homotopy via 2-categorical methods. In a slightly different form, the idea behind can be summarized as follows: starting from a 2-term representation up to homotopy of a Lie algebroid A on the graded vector bundle $E \oplus C$, they associated a strict 2-functor $\mathsf{Hol} : \mathcal{P}(A) \longrightarrow 2\text{-}\mathsf{Gau}(E \oplus C)$ where $\mathcal{P}(A)$ is a 2-categorical version of the Weinstein groupoid of A and where $2\text{-}\mathsf{Gau}(E \oplus C)$ is a gauge 2-groupoid associated to the *2-vector bundle* $\mathcal{E} := (E \oplus C \rightrightarrows E)$. A 2-vector bundle is simply a groupoid object in the category of vector bundles. The strict 2-functor Hol must be thought of as a higher action of $\mathcal{P}(A)$ on $\mathcal{E}$. Why is it appropriate to call this procedure *an integration*? Well, associated to this action there is a *semi-direct* product $\mathcal{P}(A) \ltimes_{\mathsf{Hol}} \mathcal{E}$ which is a strict 2-groupoid. It turns out that truncating this strict 2-groupoid at the level of 1-morphisms, *i.e.*, identifying two 1-morphisms when they are related via a 2-morphism, we recover the Weinstein groupoid of the corresponding VB-algebroid $A \oplus E \oplus C$. Briefly put, after the choice of a linear splitting, one can integrate the total space of VB-algebroids via strict 2-functors. This is purely categorical in contrast with the classical integration scheme described in [17].

In this work we would like to understand the previous scenario in a more general setting: we would like to replace VB-algebroids by the so called *extensions of Lie algebroids*. Roughly speaking, extensions of Lie algebroids are *fibration objects* in the category of Lie algebroids: the total and the base spaces of the Lie algebroid are fibrations and the projections are morphisms of fibrations. These are general enough to encompass VB-algebroids, extensions of Lie algebras, surjective submersions, infinitesimal actions, etc. Extensions of Lie algebroids should be thought of as non-linear versions of VB-algebroids. In this general setting we would like to answer:

Q1) What is the analogue of a 2-term representation up to homotopy?

Q2) What plays the role of the gauge 2-groupoid 2-$\mathsf{Gau}(E \oplus C)$ and what is the corresponding "integration" strict 2-functor Hol?

Q3) How does the theory behave at the level of morphisms?

Q4) Is this idealized integration scheme functorial?

The question Q3) has not even been answered in the context of [13] which we described above.

We believe we were able to give satisfactory answers to the first three questions. As to the last one, we are convinced the answer is positive, however, we did not touch upon it in the text, but we shall discuss it in a future work.

The text is organized into three parts which we briefly summarize below. More complete introductions are given in the respective parts.

Part I. Background

In this part we discuss the strictly essential material in order to make the text as self-contained as possible. We split it into two chapters for convenience otherwise we would end up with a strangely huge chapter. The material presented here, up to presentation, is more or less standard. Although we have omitted most proofs, appropriate references are always supplied.

Part II. Natural Homotopies

This part arose from an attempt to answer the third question we mentioned previously. It turns out the resulting theory is interesting on its own, and it is presented in an independent manner. However, we shall use the theory developed here in a subsequent part. Briefly put, in this part we discuss *global homotopies* and *infinitesimal homotopies.* We explain why a smooth natural transformation is not the appropriate notion of homotopy to be considered in the context of Lie groupoids and why it does not have an infinitesimal counterpart. The replacements for natural transformations are called *natural homotopies.* We develop a Lie theory for those objects.

Part III. Infinitesimal Actions up to Homotopies

In this part we discuss the category of extensions of Lie algebroids taking into account the possibility of choosing splittings. We define *actions up to homotopy* which are the analogue of 2-term representations up to homotopy when we replace VB-algebroids by extensions of Lie algebroids. We also show that the category of extensions of Lie algebroids and the category of actions up to homotopy are equivalent. Furthermore, we explain how it is possible to integrate actions up to homotopy by means of strict 2-functors. We also discuss this integration for morphisms. Finally, we point out some indications of future research.

Next, we emphasize the main contributions of this work:

- The discussion concerning morphisms of Lie algebroid extensions in the fourth chapter is original;
- The third and the fifth chapters are entirely new;
- The definition of the holonomy 2-functor in the context of Lie algebroid extensions is new but all the ingredients to its definition were already present in [12];
- The discussion concerning the integration of morphisms in the sixth chapter is also original.

Part I - Background

Chapter 1

Infinitesimal Background

In this chapter we gather the strictly essential material in order to make the text as self-contained as possible. Furthermore, we fix the notations we shall be adopting along the manuscript.

1.1 On Graded Objects

Along this text we shall deal constantly with some graded objects. For the sake of convenience and clarity we present the basic definitions.

Throughout this section we shall denote by R a commutative ring with identity. We assume the reader is familiar with the theory of modules over rings at the level of [5].

Definition 1.1.1. A *graded R-module* is a family of R-modules $M = (M_n)_{n\in\mathbb{Z}}$. An element $m \in M_n$ is said to be *homogeneous of degree $|m| = n$.*

We shall frequently define graded modules specifying only M_n for $n \geq 0$. In such cases, it must be understood $M_n = 0$ for every negative integer.

Definition 1.1.2. Let $M = (M_n)_{n\in\mathbb{Z}}$ and $N = (N_n)_{n\in\mathbb{Z}}$ be two graded R-modules. A *homogeneous R-linear map of degree p* is a family of R-linear maps $f = (f_n : M_n \longrightarrow N_{n+p})_{n\in\mathbb{Z}}$. If $p = 0$ we call such f a *morphism of graded R-modules.*

We shall denote $f = (f_n : M_n \longrightarrow N_{n+p})_{n\in\mathbb{Z}}$ simply by $f : M_n \longrightarrow N_{n+p}$.

Morphisms between graded R-modules compose naturally and such composition gives rise to the category of graded R-modules.

Definition 1.1.3. A *graded R-algebra* is graded R-module $A = (A_n)_{n\in\mathbb{Z}}$ together with R-bilinear products:

$$\begin{aligned} \cdot : A_p \times A_q &\longrightarrow A_{p+q} \\ (a, b) &\longmapsto a \cdot b. \end{aligned}$$

An *identity* for such graded R-algebra is an element $1 \in A_0$ such that:

$$1 \cdot a = a = a \cdot 1$$

for every $a \in A_p$ and every $p \in \mathbb{Z}$. Elements a of A_p are called *homogeneous of degree* $|a| = p$.

Whenever we write $a \in A$ it must be understood $a \in A_n$ for some $n \in \mathbb{Z}$. We define morphisms of graded R-algebras as expected:

Definition 1.1.4. Let A and B be two graded R-algebras and let $p \in \mathbb{Z}$. A *morphism of graded algebras from A to B* is a morphism of graded R-modules $f : A \longrightarrow B$ such that:

$$f(a \cdot b) = f(a) \cdot f(b)$$

for every $a, b \in A$.

If the graded R-algebras A and B have identities 1_A and 1_B, respectively, we shall require f to be compatible with them, in the sense that $f(1_A) = 1_B$. Morphisms of graded R-algebras compose naturally and such composition gives rise to the category of graded R-algebras.

Next we define the tensor product between two graded R-algebras. This will be important along the text.

Definition 1.1.5. The *tensor product* of two graded R-algebras A and B is the graded R-module:

$$(A \otimes_R B)_n := \bigoplus_{i+j=n} A_i \otimes_R B_j,$$

endowed with the product which extends:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (-1)^{|b_1||a_2|}(a_1 \cdot a_2) \otimes (b_1 \cdot b_2).$$

We notice the sign $(-1)^{|b_1||a_2|}$ in the above definition is not arbitrary, it comes from the Koszul's rule for signs which states that whenever two homogeneous elements of degrees i and j are intertwined the sign $(-1)^{ij}$ must appear. We shall adopt it consistently.

Next we introduce modules which are defined over graded algebras.

Definition 1.1.6. Let A be a graded R-algebra. A *graded left A-module* is a graded R-module M with R-bilinear products:

$$\begin{aligned} \cdot : A_m \times M_n &\longrightarrow M_{m+n} \\ (a, m) &\longmapsto a \cdot m, \end{aligned}$$

such that:

$$a_1 \cdot (a_2 \cdot m) = (a_1 \cdot a_2) \cdot m,$$

for every $a_1, a_2 \in A$ and $m \in M$.

In the case A has an identity 1_A we require that $1_A \cdot m = m$ for every $m \in M$. Analogously, one can define a *graded right A-module.*

Morphisms of graded left A-modules are defined as follows:

Definition 1.1.7. Let M and N be two graded left A-modules. A *morphism of graded left A-modules* is a morphism of graded R-modules $f : M \longrightarrow N$ such that:

$$f(a \cdot m) = a \cdot f(m)$$

for every $a \in A$ and $m \in M$.

We proceed to introduce the differential graded objects we shall be dealing with along the manuscript.

Definition 1.1.8. Let M be a graded R-module. A *differential* on M is a degree one R-linear map $\partial_M : M \longrightarrow M$ such that $\partial_M \circ \partial_M = 0$. The pair (M, ∂_M) is called a *differential graded R-module.*

Morphisms between DGMs are defined as expected:

Definition 1.1.9. Let (M, ∂_M) and (N, ∂_N) be two differential graded R-modules. A *morphism of differential graded R-modules* from (M, ∂_M) to (N, ∂_N) is a morphism of graded R-modules $f : M \longrightarrow N$ such that $\partial_N \circ f = f \circ \partial_M$.

One can also talk about differential graded R-algebras:

Definition 1.1.10. A *differential graded R-algebra* (or a *DGA over R* for short) is a pair (A, ∂_A) consisting of a graded R-algebra A together with a degree one R-linear map $\partial_A : A \longrightarrow A$ such that $\partial_A^2 = 0$ and:

$$\partial_A(a \cdot b) = \partial_A(a) \cdot b + (-1)^{|a|} a \cdot \partial_A(b).$$

One can define a category of DGAs over R introducing the following notion of morphism:

Definition 1.1.11. Let (A, ∂_A) and (B, ∂_B) be two DGAs over R. A *morphisms of DGAs over R from (A, ∂_A) to (B, ∂_B)* is a morphism of graded R-algebras $f : A \longrightarrow B$ such that $\partial_B \circ f = f \circ \partial_A$.

One more graded object we deal with is the following:

Definition 1.1.12. A *differential graded left-module over a differential graded R-algebra* (A, ∂_A) is a pair (M, ∂_M) consisting of a graded left-module over the graded R-algebra A together with a degree one R-linear map $\partial_M : M \longrightarrow M$ such that $\partial_M^2 = 0$ and:

$$\partial_M(a \cdot m) = \partial_A(a) \cdot m + (-1)^{|a|} a \cdot \partial_M(m),$$

for every $a \in A$ and every $m \in M$.

A category of such objects is obtained once we define:

Definition 1.1.13. Let (M, ∂_M) and (N, ∂_N) be two differential graded left-modules over the differential graded R-algebra (A, ∂_A). A *morphism from* (M, ∂_M) *to* (N, ∂_N) is a morphism of DG-modules modules $f : (M, \partial_M) \longrightarrow (N, \partial_N)$ which is also a morphism of modules over the graded algebra (A, ∂_A).

Further details concerning the objects we defined in this section can be found in [35].

1.2 Basics on Vector Bundles

Let us fix three vector bundles $p_A : A \longrightarrow M$, $p_E : E \longrightarrow M$ and $p_B : B \longrightarrow N$ and let $\Phi : A \longrightarrow B$ be a vector bundle morphism covering $\phi : M \longrightarrow N$.

Whenever $M = N$ and ϕ is the identity map of M, we call Φ a *strong vector bundle morphism.*

We denote by $\Gamma(A)$ or by $\Gamma(M, A)$ the $C^\infty(M)$-module of sections of A. The product of $a \in \Gamma(A)$ by $f \in C^\infty(M)$ is the section fa defined by

$$(fa)(p) := f(p)a(p)$$

for every $p \in M$. We shall write $\Omega(A)$ for the $C^\infty(M)$-graded algebra

$$\Omega(A) = \left\{\Omega^k(A)\right\}_{k \geq 0}$$

where

$$\Omega^k(A) := \begin{cases} \Gamma(\bigwedge^k A^*) & \text{if} \quad k > 0 \\ C^\infty(M) & \text{if} \quad k = 0 \end{cases}.$$

The product in this algebra is the *exterior product*

$$\wedge : \Omega^p(A) \otimes_{C^\infty(M)} \Omega^q(A) \longrightarrow \Omega^{p+q}(A),$$

which is defined as

$$(\varepsilon_1 \wedge \varepsilon_2)(a_1, \ldots, a_{p+q}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma)\varepsilon_1(a_{\sigma(1)}, \ldots, a_{\sigma(p)})\varepsilon_2(a_{\sigma(p+1)}, \ldots, a_{\sigma(p+q)}).$$

for every $\varepsilon_1 \in \Omega^p(A)$, $\varepsilon_2 \in \Omega^q(A)$ and $a_1, \ldots, a_{p+q} \in \Gamma(A)$, where $\mathsf{Sh}(p,q)$ denotes the set of those permutations σ of the first $p+q$ integers which satisfy $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+q)$.

Let us write $\Omega(A, E)$ for the graded left-module over the graded $C^\infty(M)$-algebra $\Omega(A)$ given by

$$\Omega(A, E) := \{\Omega^p(A, E)\}_{p \geq 0}$$

where

$$\Omega^p(A, E) := \Omega^p(A) \otimes_{C^\infty(M)} \Gamma(E),$$

for every integer $p \geq 0$. The module product is the map:

$$\cdot : \Omega^p(A) \otimes_{C^\infty(M)} \Omega^q(A, E) \longrightarrow \Omega^{p+q}(A, E),$$

defined by the recipe

$$(\alpha \cdot \varepsilon)(a_1, \ldots, a_{p+q}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma)\alpha(a_{\sigma(1)}, \ldots, a_{\sigma(p)})\varepsilon(a_{\sigma(p+1)}, \ldots, a_{\sigma(p+q)}). \tag{1.1}$$

for every $\alpha \in \Omega^p(A)$, $\varepsilon \in \Omega^q(A)$ and $a_1, \ldots, a_{p+q} \in \Gamma(A)$.

If $M = N$ then Φ induces a $C^\infty(M)$-linear map:

$$\begin{aligned} \Phi_* : \Gamma(A) &\longrightarrow \Gamma(B) \\ a &\longmapsto \Phi \circ a. \end{aligned}$$

In order to keep the notation simple, we shall also denote Φ_* by Φ. In general, whenever $M \neq N$, we do not have an induced map at the level of sections unless the base map ϕ is a diffeomorphism.

We shall write

$$\phi^* B := \{(b, m) \in B \times M : p_B(b) = \phi(m)\}.$$

This is a vector bundle over M, called the *pullback of B along ϕ*. Any section $a \in \Gamma(B)$ induces a section

$$\phi^* a := (a \circ \phi, \mathsf{id}_M) \in \Gamma(\phi^* B).$$

According to [24], the map

$$\begin{aligned} C^\infty(M) \otimes_{C^\infty(N)} \Gamma(B) &\longrightarrow \Gamma(\phi^* B) \\ f \otimes a &\longmapsto f\phi^* a, \end{aligned} \tag{1.2}$$

is an isomorphism of $C^\infty(M)$-modules.

The vector bundle morphism Φ induces a vector bundle morphism

$$(\Phi, p_A) : A \longrightarrow \phi^* B$$

and this, in turn, induces morphisms between the exterior powers

$$\bigwedge^k (\Phi, p_A)^* : \bigwedge^k (\phi^* B)^* \longrightarrow \bigwedge^k A^*$$

so that, passing to sections, one gets a $C^\infty(M)$-linear map

$$\bigwedge^k (\Phi, p_A)^* : \Omega^k(\phi^* B) \longrightarrow \Omega^k(A).$$

We define

$$\Phi^* : \Omega^k(B) \longrightarrow \Omega^k(A)$$

by the commutativity of the diagram

$$\begin{tikzcd} \Omega^k(B) \arrow[r, "\Phi^*"] \arrow[d, "1_{\mathbb{R}} \otimes \mathsf{id}"'] & \Omega^k(A) \\ C^\infty(M) \otimes_{C^\infty(N)} \Omega^k(B) \arrow[r, "\simeq"'] & \Omega^k(\phi^* B) \arrow[u, "\bigwedge^k (\Phi, p_A)^*"'] \end{tikzcd}$$

More explicitly, Φ^* is given by

$$\langle \Phi^* \varepsilon, a_1, \ldots, a_k \rangle := \langle \varepsilon \circ \phi, \Phi \circ a_1, \ldots, \Phi \circ a_k \rangle,$$

for every $\varepsilon \in \Omega^k(B)$ and $a_1, \ldots, a_k \in \Gamma(A)$. This is a morphism of graded algebras, that is

$$\Phi(\varepsilon_1 \wedge \varepsilon_2) = \Phi\varepsilon_1 \wedge \Phi\varepsilon_2,$$

for every $\varepsilon_1, \varepsilon_2 \in \Omega(A)$.

We shall constantly use the identifications of $C^\infty(M)$-modules

$$\Omega^k(A) \simeq \mathsf{Alt}^k_{C^\infty(M)}(\Gamma(A), C^\infty(M))$$

and

$$\Omega^k(A, E) \simeq \mathsf{Alt}^k_{C^\infty(M)}(\Gamma(A), \Gamma(E)),$$

between forms and vector valued forms and the respective $C^\infty(M)$-modules of multilinear and alternating maps.

We will write ϕ^* for the morphism of $\mathbb{R}$-algebras

$$\begin{aligned} \phi^* : C^\infty(N) &\longrightarrow C^\infty(M) \\ f &\longmapsto f \circ \phi. \end{aligned}$$

This allows us to realise every $C^\infty(M)$-module as a $C^\infty(N)$-module via restriction of scalars.

For $\varepsilon \in \Omega^k(B)$ and a section $b \in \Gamma(B)$ we shall also write

$$\phi^*\varepsilon := \varepsilon \circ \phi \quad \text{and} \quad \phi^* b := b \circ \phi.$$

We emphasize $\phi^* b$ should not be confused with $\phi^* b \in \Gamma(\phi^* B)$ we defined earlier. The context will make it clear what $\phi^* b$ means.

There is an important class of sections of a given vector bundle, namely, the projectable ones: a section $a \in \Gamma(A)$ is Φ-*projectable* if there exists $\Phi(a) \in \Gamma(B)$ such that

$$\Phi \circ a = \Phi(a) \circ \phi.$$

In this case, we shall also say that a *projects* on $\Phi(a)$. We shall make some remarks concerning projectable sections:

- There is no reason why $\Phi(a)$ should be uniquely determined by a, however, that is the case whenever $\phi^* : C^\infty(N) \longrightarrow C^\infty(M)$ is injective. In such case, there is a well defined $C^\infty(N)$-linear map:

  $$\begin{aligned} \Phi(-) : \Gamma_{\Phi\text{-proj}}(A) &\longrightarrow \Gamma(B) \\ a &\longmapsto \Phi(a) \end{aligned}$$

  where $\Gamma_{\Phi\text{-proj}}(A)$ is the $C^\infty(N)$-module of Φ-projectable sections of A.

- For a given $a \in \Gamma_{\Phi\text{-proj}}(A)$ we readily see that $\langle \Phi^*\varepsilon, a\rangle \in \phi^* C^\infty(N)$ for every $\varepsilon \in \Omega^1(B)$. Whenever ϕ is a surjective submersion the converse holds, that is, if $\langle \Phi^*\varepsilon, a\rangle \in \phi^* C^\infty(N)$ for every $\varepsilon \in \Omega^1(B)$ then a is Φ-projectable.

- If $[\cdot,\cdot]_A$ and $[\cdot,\cdot]_B$ are Lie brackets on $\Gamma(A)$ and $\Gamma(B)$, respectively, then $\Gamma_{\Phi\text{-proj}}(A)$

is a Lie subalgebra of $\Gamma(A)$ and:

$$\Phi \circ [a, b]_A = [\Phi(a), \Phi(b)]_B,$$

for every $a, b \in \Gamma_{\Phi\text{-proj}}(A)$.

The tangent bundle of a manifold M shall be denoted by TM and its $C^\infty(M)$-module of sections shall be written $\mathfrak{X}(M)$. For every $p \in M$ there exists $\varepsilon = \varepsilon(p) > 0$, an open set $U = U(p) \subset M$ around p and a smooth map

$$\begin{aligned} \Phi^X : U \times (-\varepsilon, \varepsilon) &\longrightarrow M \\ (u, t) &\longmapsto \Phi^X_t(u) \end{aligned}$$

called the *local flow* of X, such that:

- $\Phi^X_t(U)$ is an open set of M around p for every $t \in U$;
- $\Phi^X_t : U \longrightarrow \Phi^X_t(U)$ is a diffeomorphism for every $t \in (-\varepsilon, \varepsilon)$;
- The following identifies hold:

$$\Phi^X_{s+t} = \Phi_s \circ \Phi^X_t \quad \text{and} \quad \Phi_0 = \mathsf{id}_U.$$

In particular, $(\Phi^X_t)^{-1} = \Phi^X_{-t}$.

The *Lie derivative* along a vector field $X \in \mathfrak{X}(M)$ is the map

$$\mathcal{L}_X : C^\infty(M) \longrightarrow C^\infty(M),$$

defined via the flow of X as follows:

$$\mathcal{L}_X(f)(p) := \lim_{t \to 0} \frac{1}{t}[f(\Phi^X_t(p)) - f(p)],$$

for every $f \in C^\infty(M)$ and $p \in M$. It is a well known fact that the correspondence $X \longmapsto \mathcal{L}_X$ gives rise to an isomorphism of $C^\infty(M)$-modules between $\mathfrak{X}(M)$ and the $C^\infty(M)$-module $\mathsf{der}_{\mathbb{R}}(C^\infty(M))$ of *derivations* of the graded $\mathbb{R}$-algebra $C^\infty(M)$.

We must remark that for given a smooth map $f : M \longrightarrow N$ and a vector field $X \in \mathfrak{X}(M)$ which is *df*-projectable on $Y \in \mathfrak{X}(N)$, the flows of X and Y commute in the following sense:

$$f \circ \Phi^X_t = \Phi^Y_t \circ f,$$

for every $t \in I := [0, 1]$.

There is an obvious notion of time-dependent vector fields on a manifold. The local flow of a such a vector field X shall be denoted by $\Phi^X_{t,s}$ and it is defined analogously to the time-independent case.

For further details on smooth manifolds, vector bundles and flows we refer the reader to [6], [24], [27] and [30].

1.3 On Lie Algebroids

In this section we limit ourselves to presenting and clarifying those aspects concerning Lie algebroids we will need along the text.

Definition 1.3.1. A *Lie algebroid* over a manifold M is a vector bundle A over M together with a Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ and a morphism of vector bundles $\sharp_A : A \longrightarrow TM$, called the *anchor* of A, satisfying the Leibniz rule:

$$[a, fb]_A = f[a,b]_A + \mathcal{L}_{\sharp_A a}(f)b, \tag{1.3}$$

for every $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$.

Using the Jacobi identity for $[\cdot,\cdot]_A$ and (1.3), we can easily show the anchor $\sharp_A$ induces a morphism of Lie algebras:

$$\mathcal{L}_{\sharp_A(-)} : \Gamma(A) \longrightarrow \mathfrak{X}(M).$$

Next we present several interesting examples of Lie algebroids. Further examples and additional details can be found in [22], [33] and [36].

Example 1.3.2. A Lie algebroid over a point $\{*\}$ is equivalent to a Lie algebra.

Example 1.3.3. Any vector bundle A becomes a Lie algebroid once we take $\sharp_A$ and $[\cdot,\cdot]_A$ as the zero maps.

Example 1.3.4. The tangent bundle TM of a manifold M is a Lie algebroid. The Lie bracket on its space of sections is the commutator of vector fields and the anchor is the identity map. In particular, the Lie algebroid structure on the tangent bundle TI of the unit interval $I := [0,1]$ reads:

$$\left[f\frac{\partial}{\partial t}, g\frac{\partial}{\partial t}\right]_{TI} := (fg' - f'g)\frac{\partial}{\partial t} \quad \text{and} \quad \sharp_{TI}\left(f\frac{\partial}{\partial t}\right) := f\frac{\partial}{\partial t}.$$

for every $f, g \in C^\infty(I)$, where (I, t) stands for the standard coordinate chart of I. This will be the Lie algebroid structure we shall always consider on TI along the text.

Example 1.3.5. Lie algebroid structures on the trivial bundle $M \times \mathbb{R}$ are parametrized by vector fields on M. Given $X \in \mathfrak{X}(M)$, the corresponding Lie algebroid is denoted by A_X.

As a vector bundle, $A_X := M \times \mathbb{R}$. The Lie bracket of two sections $f, g \in \Gamma(A_X) \simeq C^\infty(M)$ is defined by:

$$[f, g] := f\mathcal{L}_X(g) - \mathcal{L}_X(f)g,$$

while the anchor is defined as $\sharp_{A_X}(f) := fX$ for every $f \in C^\infty(M)$.

Example 1.3.6. An *infinitesimal action* of a Lie algebra $\mathfrak{g}$ on a manifold M is a morphism of Lie algebras $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$. This defines a Lie algebroid structure on the trivial vector bundle $\mathfrak{g} \ltimes M := \mathfrak{g} \times M$ on M. The Lie bracket on $\Gamma(\mathfrak{g} \ltimes M) \simeq C^\infty(M) \otimes \mathfrak{g}$ is given by:

$$[f \otimes u, g \otimes v]_{\mathfrak{g}\ltimes M} := f\mathcal{L}_{\rho(u)}(g) \otimes v - g\mathcal{L}_{\rho(v)}(f) \otimes u + fg \otimes [u, v],$$

for every $f, g \in C^\infty(M)$ and $u, v \in \mathfrak{g}$. The anchor is defined, at the level of sections, by:

$$\sharp_{\mathfrak{g}\ltimes M}(f \otimes u) := f\rho(u),$$

for every $f \in C^\infty(M)$ and $u \in \mathfrak{g}$. The Lie algebroid $\mathfrak{g} \ltimes M$ is called the *action algebroid*.

Example 1.3.7. Any closed 2-form ω on a manifold M defines a Lie algebroid A_ω. As a vector bundle $A_\omega := TM \oplus (M \times \mathbb{R})$. The Lie bracket on $\Gamma(A_\omega) \simeq \mathfrak{X}(M) \oplus C^\infty(M)$ is defined by:

$$[X \oplus f, Y \oplus g] := [X, Y]_{TM} \oplus (\mathcal{L}_X(g) - \mathcal{L}_Y(f) + \omega(X, Y)),$$

for every $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ whereas the anchor is defined as

$$\sharp_{A_\omega}(X \oplus f) := X$$

for every $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

Example 1.3.8. Let G be a Lie group and let P be a G-principal bundle over M and define $A(P) := TP/G$. It turns out that $A(P)$ is a vector bundle over $M := P/G$. The sections of $A(P)$ correspond to G-invariant vector fields on P and therefore $\Gamma(A(P))$ has a canonical Lie bracket. With this bracket and with the anchor induced by the differential of the projection of P on M we get a Lie algebroid.

One can describe all the possible Lie algebroid structures on a given vector bundle via DGAs. The following characterization is due to Vaintrob [41].

Proposition 1.3.9. Let A be a vector bundle over M. There is a one-to-one correspondence between Lie algebroid structures on A and DGA-structures on the graded algebra $(\Omega(A), \wedge)$.

For a given Lie algebroid A over M, the corresponding DGA structure on the graded algebra $\Omega(A)$ will be denoted by d_A and it is defined by:

$$\begin{aligned}\langle d_A\varepsilon, a_1, \ldots, a_{p+1}\rangle := &\sum_{\sigma\in\mathsf{Sh}(1,p)} \mathsf{sgn}(\sigma)\mathcal{L}_{\sharp_A a_{\sigma(1)}}\langle\varepsilon, a_{\sigma(2)}, \ldots, a_{\sigma(p+1)}\rangle \\ &+ \sum_{\sigma\in\mathsf{Sh}(2,p-1)} \mathsf{sgn}(\sigma)\langle\varepsilon, [a_{\sigma(1)}, a_{\sigma(2)}], a_{\sigma(3)}, \ldots, a_{\sigma(p+1)}\rangle,\end{aligned}$$

for every $\varepsilon \in \Omega^p(A)$ and $a_1, \ldots, a_{p+1} \in \Gamma(A)$.

Example 1.3.10. Let $A = \mathfrak{g}$ be a Lie algebra, seen as a Lie algebroid over a point. Then $d_\mathfrak{g}$ coincides with the Chevalley-Eilenberg differential of $\mathfrak{g}$.

Example 1.3.11. For a given manifold M, the differential of the corresponding Lie algebroid TM coincides with De Rham differential of M.

Next, we discuss morphisms between Lie algebroids.

Definition 1.3.12. Let A and B two Lie algebroids over M and N, respectively, and let $\Phi : A \longrightarrow B$ be a morphism of vector bundles covering $\phi : M \longrightarrow N$. We say:

M1) Φ is *compatible with the anchors of A and B* if the diagram:

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d, "\sharp_A"'] & B \arrow[d, "\sharp_B"] \\ TM \arrow[r, "d\phi"'] & TN \end{tikzcd}$$

is commutative.

M2) Φ is *compatible with the Lie brackets $[\cdot,\cdot]_A$ and $[\cdot,\cdot]_B$* if for any pair of sections $a, b \in \Gamma(A)$ such that:

$$\Phi \circ a = \sum_i f_i \cdot \phi^* a_i \quad \text{and} \quad \Phi \circ b := \sum_j g_j \cdot \phi^* b_j$$

with $f_i, g_j \in C^\infty(M)$ and $a_i, b_j \in \Gamma(B)$, we have:

$$\begin{aligned}\Phi \circ [a, b]_A = &\sum_{i,j} (f_i g_j) \cdot \phi^*[a_i, b_j]_B \\ &+ \sum_j \mathcal{L}_{\sharp_A a}(g_j) \cdot \phi^* b_j - \sum_i \mathcal{L}_{\sharp_A b}(f_i) \cdot \phi^* a_i.\end{aligned}$$

The compatibility condition M2) was introduced by Higgins-Mackenzie in the paper [25] where the authors study the category of Lie algebroids. The compatibility between the Lie brackets is intricate because morphisms of vector bundles defined over distinct bases do not induce a map at the level of sections.

Definition 1.3.13. Let A and B be two Lie algebroids over M and N, respectively. A *Lie algebroid morphism from A to B* is a vector bundle morphism $\Phi : A \longrightarrow B$ which is compatible with the anchors and with the Lie brackets.

In [25], it was shown that Lie algebroids form a category with the above notion of morphism.

The condition of being a morphism of Lie algebroids can also be formulated in the context of DGAs.

Proposition 1.3.14. Let A and B be two Lie algebroids over M and N, respectively, and let $\Phi : A \longrightarrow B$ be a morphism of vector bundles covering $\phi : M \longrightarrow N$. Then Φ is a morphism of Lie algebroids if and only if the morphism of graded algebras $\Phi^* : (\Omega(B), \wedge) \longrightarrow (\Omega(A), \wedge)$ is a morphism of the corresponding DGAs, that is, $d_A \circ \Phi^* = \Phi^* \circ d_B$.

Indeed, using the fact that Φ^* is a morphism of graded algebras and that $\Omega(B)$ is generated as a graded algebra by $\Omega^{\leq 1}(B)$ we have to verify the condition $d_A \circ \Phi^* = \Phi^* \circ d_B$ only in degrees zero and one. Furthermore, for degree zero the referred condition is equivalent to the compatibility with the anchor. A detailed proof of the previous proposition can be found in the work [11].

An important class of morphisms of Lie algebroids is given in the following:

Definition 1.3.15. Let A be a Lie algebroid over M. An *A-path* is a Lie algebroid morphism $TI \longrightarrow A$.

Any A-path has the form $a \cdot dt : TI \longrightarrow A$ where $a : I \longrightarrow A$ is a smooth map such that:

$$\sharp_A a(t) = \frac{d}{dt}\gamma(t),$$

where γ is the path on M induced via projection. We shall refer to γ as the *base path* of a. These morphisms play an important role in the theory of integration of Lie algebroids as we shall see further along.

Another important class of morphisms between Lie algebroids is given by:

Definition 1.3.16. Let A be a Lie algebroid over M. An *A-homotopy* is a Lie algebroid morphism $\sigma : TI \times TI \longrightarrow A$.

Considering the canonical coordinate chart $(I \times I, (t, s))$ one can readily check that any A-homotopy has the form

$$\sigma = a \cdot dt + b \cdot ds,$$

for smooth maps $a, b : I \times I \longrightarrow A$.

Using this definition one can introduce homotopies between A-paths:

Definition 1.3.17. Let $a_0 \cdot dt : TI \longrightarrow A$ and $a_1 \cdot dt : TI \longrightarrow A$ be two A-paths. We say *a_0 and a_1 are A-homotopic* if there exists an A-homotopy $\sigma = a \cdot dt + b \cdot ds : TI \times TI \longrightarrow A$ such that $a|_{I\times\{0\}} = a_0$ and $a|_{I\times\{1\}} = a_1$.

The above definition captures the right notion of homotopies in the world of Lie algebroids and they play an important role in the integration theory of Lie algebroids. See [22] for further details.

1.4 Derivations of Lie Algebroids

Derivations of Lie algebroids play an important role in this work. We start by dealing with the case of vector bundles. A general reference for this topic is [33].

Definition 1.4.1. Let E be a vector bundle over M. A *derivation of* E is a $\mathbb{R}$-linear map

$$D : \Gamma(E) \longrightarrow \Gamma(E)$$

for which there exists a vector field $X^D \in \mathfrak{X}(M)$ such that:

$$D(f \cdot e) = f \cdot De + \mathcal{L}_{X^D}(f) \cdot e,$$

for every $f \in C^\infty(M)$ and $e \in \Gamma(E)$. The vector field X^D is called the *symbol* of D.

We shall write $\mathsf{der}(E)$ for the the set of derivations of E. The $C^\infty(M)$-module structure on $\Gamma(E)$ induces a $C^\infty(M)$-module structure on $\mathsf{der}(E)$.

The symbol of a derivation is uniquely determined and the map:

$$\begin{aligned} (-)^D : \mathsf{der}(E) &\longrightarrow \mathfrak{X}(M) \\ D &\longmapsto X^D \end{aligned}$$

is a morphism of Lie algebras where $\mathsf{der}(E)$ is endowed with the Lie bracket given by the commutator

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1,$$

for every $D_1, D_2 \in \mathsf{der}(E)$. In general, the composite of two derivations may fail to be a derivation.

Derivations of a vector bundle are intimately related to linear vector fields as we shall explain now. If p denotes the projection of the vector bundle E, let us write $C^\infty_{\text{lin}}(E)$ for the $C^\infty(M)$-module of those smooth functions $f : E \longrightarrow \mathbb{R}$ such that $f|_{E_x} : E_x \longrightarrow \mathbb{R}$ are linear for every $x \in M$, where we write $E_x := p^{-1}(\{x\})$ for the fiber of E over $x \in M$. Notice $C^\infty_{\text{lin}}(E)$ is a $C^\infty(M)$-submodule of $C^\infty(E)$.

Definition 1.4.2. Let E be a vector bundle over M. A *linear vector field* on E is a vector field $X \in \mathfrak{X}(E)$ which preserves the submodule $C^\infty_{\text{lin}}(E)$, that is, $X(C^\infty_{\text{lin}}(E)) \subseteq C^\infty_{\text{lin}}(E)$.

We shall write $\mathfrak{X}_{\text{lin}}(E)$ for the set of linear vector fields of E. This is a $C^\infty(M)$-submodule of $\mathfrak{X}(E)$.

Linear functions on a vector bundle are intrinsically related to sections of the corresponding dual bundle. More precisely, there is an isomorphism of $C^\infty(M)$-modules

$$\begin{aligned} \ell : \Gamma(E^*) &\longrightarrow C^\infty_{\text{lin}}(E) \\ \varepsilon &\longmapsto \ell_\varepsilon := (e \longmapsto \varepsilon_{p(e)}(e)). \end{aligned} \tag{1.4}$$

The previous correspondence induces a 1-1 correspondence between derivations of $\Gamma(E^*)$ and linear vector fields on E. Given $X \in \mathfrak{X}_{\text{lin}}(E)$ the corresponding derivation

$$D^*_X : \Gamma(E^*) \longrightarrow \Gamma(E^*),$$

is characterized by the commutativity of the diagram

$$\begin{tikzcd} C^\infty_{\text{lin}}(E) \arrow[r, "\mathcal{L}_X"] & C^\infty_{\text{lin}}(E) \\ \Gamma(E^*) \arrow[u, "\ell"] \arrow[r, "D^*_X"'] & \Gamma(E^*) \arrow[u, "\ell"'] \end{tikzcd}$$

that is:

$$\ell_{D^*_X \varepsilon} = \mathcal{L}_X(\ell_\varepsilon),$$

for every $\varepsilon \in \Gamma(E^*)$. For details concerning this correspondence we refer the reader to [33].

We can also talk about derivations of Lie algebroids. In this case, we have to impose compatibility conditions with the bracket and the anchor.

Definition 1.4.3. Let A be a Lie algebroid over M. A *derivation* of A is a derivation D of the vector bundle A such that:

$$\begin{aligned} D([a,b]_A) &= [Da,b]_A + [a,Db]_A \\ \sharp D(a) &= [X^D, \sharp a]_{\mathfrak{X}(M)}, \end{aligned}$$

for every $a, b \in \Gamma(A)$.

Example 1.4.4. Let A be a Lie algebroid. Then $\mathsf{ad}_a := [a, -]_A$ is a derivation of A with symbol $\sharp_A a$. This follows from Jacobi identity and from the Leibniz rule.

Next we describe a general procedure in order to obtain the flow of a time-dependent derivation. This will play a relevant role in integration theory we propose in the last chapter. The basic reference is the appendix of [17].

Let D be a time-dependent derivation of a vector bundle $p_E : E \longrightarrow M$. The symbol of D is a time-dependent vector field X^D on M which has a flow $\Phi^{X^D}_{t,s} : U_M \longrightarrow \Phi^{X^D}_{t,s}(U_M)$. The *flow* of D is the unique family of smooth maps:

$$\Phi^D_{t,s} : U_E \longrightarrow \Phi^D_{t,s}(U_E)$$

defined on $U_E := p_E^{-1}(U_M)$ such that the diagram

$$\begin{tikzcd} U_E \arrow[r, "\Phi^D_{t,s}"] \arrow[d, "p_E"'] & \Phi^D_{t,s}(U_E) \arrow[d, "p_E"] \\ U_M \arrow[r, "\Phi^{X^D}_{t,s}"'] & \Phi^{X^D}_{t,s}(U_M) \end{tikzcd}$$

commutes and the following properties hold:

- $\Phi^D_{t,t} = \mathsf{id}$;
- $\Phi^D_{t,s} \circ \Phi^D_{s,u} = \Phi^D_{t,u}$;
- $\left.\dfrac{d}{dt}\right|_{t=s} (\Phi^D_{s,t} \circ a \circ \Phi^{X^D}_{t,s}) = D_s(a)$, for every $a \in \Gamma(A)$.

The flow $\Phi^D_{t,s}$ can be obtained as the flow of the linear vector field corresponding to D. For our purposes, the most important instance of the previous discussion is the following:

Example 1.4.5. Let A be a Lie algebroid and a a time-dependent section of A. Then $\mathsf{ad}_a := [a, -] : \Gamma(A) \longrightarrow \Gamma(A)$ is a time-dependent derivation whose symbol is $\sharp a$. The flow of ad_a will provide a family of isomorphisms:

$$\Phi^a_{t,s}|_{A_x} : A_x \longrightarrow A_{\Phi^{\sharp a}_{t,s}(x)}.$$

If the time-dependent vector field $\sharp_A a$ is complete then $\Phi^a_{s,t}|_{A_x}$ is defined for every s, t and for every $x \in M$.

1.5 Representations of Lie Algebroids

The theory of representations of Lie algebroids is based upon the notion of A-connection. A thorough study of A-connections can be found in [21].

Definition 1.5.1. Let A be a Lie algebroid over M and let E be a vector bundle over

M. An *A-connection on E* is a $\mathbb{R}$-linear map:

$$\begin{aligned} \nabla : \Gamma(A) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (a, s) &\longmapsto \nabla_a s, \end{aligned}$$

such that:

$$\begin{aligned} \nabla_{fa} s &= f \nabla_a s \\ \nabla_a(fs) &= f \nabla_a s + \mathcal{L}_{\sharp_A a}(f) s, \end{aligned}$$

for every $a \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$. The *curvature* of ∇ is the map:

$$\begin{aligned} R_\nabla : \Gamma(A) \times \Gamma(A) &\longrightarrow \mathsf{End}_{C^\infty(M)}(\Gamma(E)) \\ (a, b) &\longmapsto (s \longmapsto \nabla_a \nabla_b s - \nabla_b \nabla_a s - \nabla_{[a,b]} s). \end{aligned}$$

The connection is *flat* whenever $R_\nabla = 0$.

There are a few remarks to be made concerning A-connections:

- An A-connection $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$ can be equivalently defined as a $C^\infty(M)$-linear map:

  $$\begin{aligned} \nabla : \Gamma(A) &\longrightarrow \mathsf{der}(E) \\ a &\longmapsto (\nabla_a : s \longmapsto \nabla_a s) \end{aligned}$$

  such that:

  $$\nabla_a(fs) = f \nabla_a s + \mathcal{L}_{\sharp_A a}(f) s,$$

  for every $f \in C^\infty(M)$ and $s \in \Gamma(E)$;

- The curvature R_∇ defines a 2-form with values on $\mathsf{End}_{C^\infty(M)}(\Gamma(A))$, that is, $R_\nabla \in \Omega^2(A) \otimes \mathsf{End}_{C^\infty(M)}\Gamma(A)$;

- An A-connection ∇ is flat if and only if $\nabla : \Gamma(A) \longrightarrow \mathsf{der}(E)$ is a morphism of Lie algebras.

The theory of A-connections extends the classical theory of linear connections as follows:

Example 1.5.2. A TM-connection coincides with the classical definition of a linear connection on a vector bundle as defined, for instance, in [30].

Example 1.5.3. Let A be a Lie algebroid over M and let ∇ be a TM-connection on the vector bundle A. Then $\overline{\nabla}_a b := \nabla_{\sharp_A b} a + [a, b]_A$ defines an A-connection on A.

The next example is important in order to discuss morphisms between 2-term representations up to homotopy.

Exemplo 1.5.4. Let A be a Lie algebroid over M, E and C be two vector bundles over M. Any pair ∇^E and ∇^C of A-connections on E and on C induce a connection ∇^{Hom} on $\mathsf{Hom}(E,C)$ by

$$(\nabla_a T)(e) := \nabla_a^C(T(e)) - T(\nabla_a^E(e))$$

for every $a \in \Gamma(A)$, $e \in \Gamma(E)$ and $T \in \Gamma(\mathsf{Hom}(E,C))$.

By means of connections we can talk about representations of a Lie algebroid.

Definition 1.5.5. A *representation* of a Lie algebroid A over M consists of a vector bundle E over M together with a flat A-connection ∇ on E. We say that ∇ defines a *representation of A on E*.

Representations of Lie algebroids extend linear representations of Lie algebras as seen in the following:

Example 1.5.6. If $A = \mathfrak{g}$ is a Lie algebra and E is a vector space, that is, a vector bundle over a point, we recover the usual definition of a linear representation of $\mathfrak{g}$ on E.

We can interpret representations of a Lie algebroid in terms of differential graded modules (DG-modules for short). To explain that, we start by recalling that $\Omega(A,E)$ is a $\Omega(A)$-graded left-module with the product (1.1).

Proposition 1.5.7. Let A be a Lie algebroid over M and let E be a vector bundle over M. A representation of A on E is equivalent to a DG-module structure on the graded left $\Omega(A)$-module $\Omega(A,E)$, that is, a degree one differential

$$d : \Omega(A,E) \longrightarrow \Omega(A,E)$$

such that:

$$d(\alpha \cdot \varepsilon) = d_A\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot d\varepsilon, \tag{1.5}$$

for every $\alpha \in \Omega(A)$ and $\varepsilon \in \Omega(A,E)$.

Notice that the derivation property (1.5) tells us that the graded $\mathbb{R}$-module $(\Omega(A,E), d)$ becomes a differential graded left-module over the differential graded $\mathbb{R}$-algebra $(\Omega(A), d_A)$ as defined in section 1.1.

The DG-module structure corresponding to an A-connection ∇ on E will be denoted by $d_\nabla : \Omega(A, E) \longrightarrow \Omega(A, E)$ and it is defined as

$$\begin{aligned} d_\nabla \varepsilon(a_1, \ldots, a_{p+1}) = & \sum_{\sigma \in \mathsf{Sh}(1,p)} \mathsf{sgn}(\sigma) \nabla_{a_{\sigma(1)}} \langle \varepsilon, a_{\sigma(2)}, \ldots, a_{\sigma(p+1)} \rangle \\ & + \sum_{\sigma \in \mathsf{Sh}(2,p-1)} \mathsf{sgn}(\sigma) \langle \varepsilon, [a_{\sigma(1)}, a_{\sigma(2)}]_A, a_{\sigma(3)}, \ldots, a_{\sigma(p+1)} \rangle. \end{aligned} \tag{1.6}$$

for every $\varepsilon \in \Omega^p(A)$ and $a_1, \ldots, a_{p+q} \in \Gamma(A)$.

We can extend this to morphisms. Let us make clear what a morphism between two representations is:

Definition 1.5.8. Let (E, ∇) and (E', ∇') be two representations of a Lie algebroid A. A *morphism of representations from* (E, ∇) *to* (E', ∇') consists of a vector bundle morphism $\Phi : E \longrightarrow E'$ such that

$$\Phi_*(\nabla_a s) = \nabla'_a \Phi_*(s)$$

for every $a \in \Gamma(A)$ and $s \in \Gamma(E)$.

Let A, E and E' be vector bundles all over the same manifold M and let $\Phi : E \longrightarrow E'$ be a morphism of vector bundles. Then Φ induces a degree zero map

$$\Phi_* : \Omega(A, E) \longrightarrow \Omega(A, E')$$

given by:

$$\langle \Phi_* \varepsilon, a_1, \ldots, a_p \rangle := \Phi_*(\langle \varepsilon, a_1, \ldots, a_p \rangle),$$

for every $\varepsilon \in \Omega^p(A, E)$ and $a_1, \ldots, a_p \in \Gamma(E)$.

Proposition 1.5.9. Let A be a Lie algebroid and let (E, ∇) and (E', ∇') be two representations of A. A morphism of vector bundles $\Phi : E \longrightarrow E'$ is a morphism of representations $\Phi : (E, \nabla) \longrightarrow (E', \nabla')$ if and only if the induced map $\Phi_* : \Omega(A, E) \longrightarrow \Omega(A, E')$ is a morphism between the corresponding DG-modules, that is, Φ_* is $\Omega(A)$-linear and $d_{\nabla'} \circ \Phi_* = \Phi_* \circ d_\nabla$.

1.6 VB-Algebroids

In this section we recall the basic definitions concerning VB-algebroids. Further details can be found in [23] and [18].

Definition 1.6.1. A *double vector bundle* (DVB) is a commutative square of vector bundles

$$\begin{tikzcd} D \arrow[r, "p"] \arrow[d, "p_D"'] & A \arrow[d, "p_A"] \\ E \arrow[r, "p_E"'] & M \end{tikzcd} \tag{1.7}$$

such that p_D is a vector bundle morphism over p_A and $+_E : D \times_E D \longrightarrow D$ is a vector bundle morphism over $+ : A \times_M A \longrightarrow A$. We call A and E the *side bundles* and $C := \mathsf{Ker}(p) \cap \mathsf{Ker}(p_D)$ is the *core*.

The following basic example will be important:

Example 1.6.2. Let A, E and C be three vector bundles over a manifold M. Then:

$$\begin{tikzcd} A \oplus E \oplus C \arrow[r] \arrow[d] & A \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

is a double vector bundle whose core is the vector bundle C. This is called the *split double vector bundle*.

Given a double vector bundle (1.7), we shall write $0^A : M \longrightarrow A$, $0^E : M \longrightarrow E$, $0^D_E : E \longrightarrow D$ and $0^D_A : A \longrightarrow D$ for the various zero sections. In particular, the diagrams below commute:

$$\begin{tikzcd} D \arrow[d, "p_D"'] & A \arrow[l, "0^D_A"'] \arrow[d, "p_A"] \\ E & M \arrow[l, "0^E"] \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "p"] & A \\ E \arrow[u, "0^D_E"] \arrow[r, "p_E"'] & M \arrow[u, "0^A"'] \end{tikzcd} \tag{1.8}$$

Notice that the core of a double vector bundle has a natural structure of vector bundle over M and the corresponding projection will be denoted by $p_C : C \longrightarrow M$. The fiber of C over $m \in M$ is:

$$C_m = D_{0^A(m)} \cap D_{0^E(m)}.$$

In particular, the diagram below commutes:

$$\begin{tikzcd} & D \arrow[dl, "p_D"'] \arrow[dr, "p"] & \\ E & C \arrow[u, "\jmath_C"] \arrow[d, "p_C"] & A \\ & M \arrow[ul, "0^E"] \arrow[ur, "0^A"'] & \end{tikzcd} \tag{1.9}$$

where $\jmath_C : C \longrightarrow D$ stands for the inclusion map.

Taking (1.8) and (1.9) into account we obtain a well defined inclusion map

$$
\begin{aligned}
\jmath : \Gamma(M, C) &\longrightarrow \Gamma(E, D)\\
c &\longmapsto (e \longmapsto 0_E^D(e) +_A \jmath_C(c(p_E(e)))
\end{aligned}
$$

Definition 1.6.3. The image of the map $\jmath$ will be denoted $\Gamma_c(E, D)$ and its elements will be called *core sections*.

There is a another space of distinguished sections associated to any DVB:

Definition 1.6.4. A section $\chi \ni \Gamma(E, D)$ is *linear* if $\chi : E \longrightarrow D$ is a vector bundle morphism covering a section $a : M \longrightarrow A$. The space of linear sections is denoted by $\Gamma_l(E, D)$.

Every double vector bundle (1.7) induces a short exact sequence of vector bundles over E

$$
0 \longrightarrow p_E^* C \xrightarrow{\jmath} D \xrightarrow{(p, p_D)} p_E^* A \longrightarrow 0 \tag{1.10}
$$

where

$$
\jmath(e, c) := \jmath_C(c) +_A 0_E^D(e).
$$

Notice that there are canonical identifications of vector bundles:

$$
E \oplus C \simeq p_E^* C \quad \text{and} \quad A \oplus E \simeq p_E^* A.
$$

Definition 1.6.5. A *linear connection* on the double vector bundle (1.7) is a splitting of (1.10), that is, a strong vector bundles morphism $\sigma : p_E^* A \longrightarrow D$ such that $(p, p_D) \circ \sigma = \mathsf{id}_{p_E^* A}$.

Next we offer some alternative characterizations of linear connections.

Proposition 1.6.6. The following are equivalent:

(a) A linear connection $\sigma : A \oplus E \longrightarrow D$;

(b) A vector subbundle $H \subseteq D$ such that $D = A \oplus E \oplus H$;

(c) An isomorphism of double vector bundles $\Sigma : D \longrightarrow A \oplus E \oplus C$ over the identity maps on A, C and E;

(d) A $C^\infty(E)$-linear map $h : \Gamma(E) \longrightarrow \Gamma_{\mathrm{lin}}(E, D)$ such that $p \circ h(e) = e \circ p_E$.

We prove the above proposition in a more general setting later on.

Now we have all the ingredients to define a VB-algebroid:

Definition 1.6.7. A *VB-algebroid* is a double vector bundle:

$$\begin{tikzcd} D \arrow[r, "p"] \arrow[d, "p_D"'] & A \arrow[d, "p_A"] \\ E \arrow[r, "p_E"'] & M \end{tikzcd}$$

where D and A are Lie algebroids over E and M, respectively, and such that the anchor $\sharp_D : D \longrightarrow TE$ is a vector bundle morphism over $\sharp_A : A \longrightarrow TM$ and such that the following conditions hold:

1. $[\Gamma_l(E, D), \Gamma_l(E, D)]_D \subset \Gamma_l(E, D)$;
2. $[\Gamma_l(E, D), \Gamma_c(E, D)]_D \subset \Gamma_c(E, D)$;
3. $[\Gamma_c(E, D), \Gamma_c(E, D)]_D = 0$.

Our main interest relies on the relationship between VB-algebroids and 2-term representations up to homotopy which we shall discuss further on. For that, the following example will be important:

Example 1.6.8. Let $p : E \longrightarrow M$ be a vector bundle. Then the tangent bundle TE has two vector bundle structures. One as the tangent bundle of the manifold E and the second as a vector bundle over TM. The structure maps of $TE \longrightarrow TM$ are obtained differentiating the structure maps of $E \longrightarrow M$. We obtain a commutative diagram of vector bundles:

$$\begin{tikzcd} TE \arrow[r, "dp"] \arrow[d, "p"'] & TM \arrow[d, "\pi"] \\ E \arrow[r, "p"'] & M \end{tikzcd}$$

where $\pi : TM \longrightarrow M$ is the projection of the tangent bundle. This is a double vector bundle with core $E \longrightarrow M$. In fact, this is a VB-algebroid called the *tangent prolongation of* E. A linear splitting, in this case, is equivalent to a linear connection on E.

For further examples we refer the reader to [23].

Notice that a VB-algebroid induces a map

$$\begin{aligned} D^{E^*} : \Gamma_{\text{lin}}(D, E) \times \Gamma(E^*) &\longrightarrow \Gamma(E^*) \\ (X, \varepsilon) &\longmapsto D^{E^*}_X \varepsilon \end{aligned}$$

where $D^{E^*}_X \varepsilon$ is characterized, using the correspondence between linear vector fields and derivations, as follows:

$$\ell_{D^{E^*}_X \varepsilon} := \mathcal{L}_{\sharp_D X}(\ell_\varepsilon).$$

It also induces

$$\begin{aligned} D^C : \Gamma_{\text{lin}}(D, E) \times \Gamma(C) &\longrightarrow \Gamma(C) \\ (X, c) &\longmapsto [X, c]_D. \end{aligned}$$

Finally, since the anchor $\sharp_D$ defines a morphism of VB-algebroids from D to the prolonged tangent bundle TE, it induces a well defined morphism between the cores

$$\partial := \sharp_D|_C : C \longrightarrow E.$$

We will refer to D^{E^*}, D^C and ∂ as the *canonical operators* of the VB-algebroid.

1.7 Representations up to Homotopy of Lie Algebroids

The notion of representation of a Lie algebroid we gave in section 1.5 is too restrictive to be useful, due to the fact that we can not even define the adjoint representation due to the non-tensorial nature of the bracket. A more flexible notion of representation, called *representation up to homotopy*, was introduced by Abad-Crainic in [2]. Most of what we do in this work relies on that notion. In this section we explain briefly what they are.

Recall from proposition 1.5.7 that a representation (E, ∇) of a Lie algebroid A over M on a vector bundle E over M can be equivalently described as a degree 1 linear operator

$$d : \Omega(A, E) \longrightarrow \Omega(A, E)$$

which squares to zero and satisfies the derivation rule

$$d(\alpha \cdot \varepsilon) = d_A\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot d\varepsilon,$$

for every $\alpha \in \Omega(A)$ and $\varepsilon \in \Omega(A, E)$ homogeneous, where d_∇ is the differential corresponding to ∇. The idea behind the definition of a representation up to homotopy is to replace the vector bundle E by a graded vector bundle $\mathcal{E} = \{E_k\}_{k\in\mathbb{Z}}$ over M. In this case, $\Gamma(\mathcal{E}) = \{\Gamma(E_k)\}_{k\in\mathbb{N}}$ is naturally a $C^\infty(M)$-graded module and

$$\Omega(A, \mathcal{E}) := \Omega(A) \otimes_{C^\infty(M)} \Gamma(\mathcal{E})$$

is a $\Omega(A)$-graded left-module with the product given by

$$(\alpha \cdot \varepsilon)(a_1, \dots, a_{p+q}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma)\alpha(a_{\sigma(1)}, \dots, a_{\sigma(p)})\varepsilon(a_{\sigma(p+1)}, \dots, a_{\sigma(p+q)}),$$

defined for every $\alpha \in \Omega^p(A)$ and $\varepsilon \in \Omega^q(A, \mathcal{E})$.

In order to define a representation up to homotopy, we reproduce the characterization of representations in terms of DG-modules, as follows:

Definition 1.7.1. Let A be a Lie algebroid over a manifold M and $\mathcal{E}$ a graded vector bundle over M. A *representation up to homotopy* of A on $\mathcal{E}$ is a degree one $\mathbb{R}$-linear operator

$$D : \Omega(A, \mathcal{E}) \longrightarrow \Omega(A, \mathcal{E})$$

that satisfies $D^2 = 0$ and

$$D(\alpha \cdot \varepsilon) = d_A\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot D\varepsilon,$$

for every $\alpha \in \Omega(A)$ and $\varepsilon \in \Omega(A, \mathcal{E})$.

There is an evident notion of morphism between representations up to homotopy

Definition 1.7.2. Let D and D' be two representations up to homotopy of A on $\mathcal{E}$ and $\mathcal{E}'$, respectively. A *morphism of representations up to homotopy* from D to D' is a degree zero $\Omega(A)$-linear map

$$\Phi : \Omega(A, \mathcal{E}) \longrightarrow \Omega(A, \mathcal{E}')$$

which intertwines D and D', that is, $\Phi \circ D' = D \circ \Phi$.

This previous notion of morphism leads us to a category denoted by $\mathsf{Rep}^\infty(A)$.

We are mainly interested in the following particular case:

Definition 1.7.3. Let A be a Lie algebroid over M. A *2-term representation up to homotopy of* A is a representation up to homotopy of A on a 2-term graded vector bundle.

We shall write $\mathsf{Rep}^\infty_2(A)$ for the category of 2-term representations up to homotopy of A. The objects of this category consist of a 2-term graded vector bundle together with a representation up to homotopy of A on it. The morphisms are taken as in the definition 1.7.2.

The following characterization of 2-term representations up to homotopy can be found in [2]:

Theorem 1.7.4. There is a 1-1 correspondence between 2-term representations up to homotopy of A on the 2-term graded vector bundle $\mathcal{E} := E \oplus C$ and quadruples $(\partial, \nabla^E, \nabla^C, \omega)$ consisting of:

- A vector bundle morphism $\partial : C \longrightarrow E$;
- Two A-connections ∇^E and ∇^C on E and C, respectively, compatible with ∂ in the sense

$$\partial \circ \nabla^C = \nabla^E \circ \partial.$$

- A 2-form $\omega \in \Omega^2(A, \mathsf{Hom}(E, C))$, such that:

$$\partial \circ \omega = R_{\nabla^E}$$
$$\omega \circ \partial = R_{\nabla^C},$$

and which is *closed* in the sense

$$d_{\nabla^{\mathrm{Hom}}}\omega = 0,$$

where $d_{\nabla^{\mathrm{Hom}}}$ is the operator associated to the connection ∇^{Hom} induced by ∇^E and ∇^C on $\mathsf{Hom}(E, C)$, see example 1.5.4.

At the level of morphisms, the previous theorem extends as follows:

Theorem 1.7.5. Let $(\partial, \nabla^E, \nabla^C, \omega)$ and $(\partial', \nabla^{E'}, \nabla^{C'}, \omega')$ be two 2-term representations up to homotopy of A on $\mathcal{E} := E \oplus C$ and on $\mathcal{E}' := E' \oplus C'$, respectively. There is a 1-1 correspondence between morphisms of 2-term representations up to homotopy from the first to the latter and:

- Pairs $(\phi^{E',E}, \phi^{C',C})$ of vector bundle morphisms $\phi^{E',E} : E \longrightarrow E'$ and $\phi^{C',C} : C \longrightarrow C'$;
- Forms $\Theta \in \Omega^1(A, \mathsf{Hom}(E, C'))$;

subject to the following compatibility conditions:

$$\phi^{E',E} \circ \partial = \partial' \circ \phi^{C',C}$$
$$\nabla_a^{C'} \circ \phi^{C',C} - \phi^{C',C} \circ \nabla_a^C = \Theta(a) \circ \partial$$
$$\nabla_a^{E'} \circ \phi^{E',E} - \phi^{E',E} \circ \nabla_a^E = \partial' \circ \Theta(a)$$
$$\phi^{C',C} \circ \omega(a, b) - \omega'(a, b) \circ \phi^{E',E} = d_{\nabla^{\mathrm{Hom}}}\Theta(a, b),$$

where $d_{\nabla^{\mathrm{Hom}}}$ is the operator associated to the connection ∇^{Hom} induced by ∇^E and ∇^C on $\mathsf{Hom}(E, C)$, see example 1.5.4.

In the next section we explore a geometric interpretation of this kind of representation.

1.8 VB-Algebroids vs 2-Term Representations up to Homotopy

We now briefly explain the correspondence between VB-algebroids and 2-term representations up to homotopy.

Given a 2-term representation up to homotopy $(\nabla^E, \nabla^C, \partial, \omega)$ of A on $\mathcal{E} := E \oplus C$, we shall associate a VB-algebroid. The underlying DBV is the split double vector bundle

$$\begin{tikzcd} D := A \oplus E \oplus C \arrow[r] \arrow[d] & A \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

with core C. We have a canonical horizontal lift:

$$\begin{aligned} h : \Gamma(A) &\longrightarrow \Gamma_{\text{lin}}(E, D) \\ a &\longmapsto (a \circ p_E, \mathsf{id}_E, 0^C \circ p_E). \end{aligned}$$

The Lie algebroid structure on D is then defined as:

- The anchor $\sharp_D : D \longrightarrow TE$ acting on $h(a)$ is the linear vector field corresponding to the derivation

$$(\nabla_a^E)^* : \Gamma(E^*) \longrightarrow \Gamma(E^*),$$

  i.e., it is uniquely determined by the property:

$$\sharp_A(h(a))(\ell_\varepsilon) := \ell_{(\nabla_a^E)^*\varepsilon}$$

  for every $\varepsilon \in \Gamma(E^*)$. The action of the anchor $\sharp_D$ on a core section produces the vertical vector field

$$\sharp_D \jmath_C(c)) := \partial c^\uparrow,$$

  where:

$$\partial c^\uparrow(e) := \left.\frac{d}{dt}\right|_{t=0} (e + t\partial c(e)),$$

  for every $e \in E$.

- The Lie bracket on the space of sections of D is determined entirely by the following conditions:

$$\begin{aligned}
[\jmath_C(c_1), \jmath_C(c_2)] &= 0\\
[h(a), \jmath_C(c)] &= \nabla^C_a c\\
[h(a), h(b)] &:= h([a,b]) + \omega(a,b).
\end{aligned}$$

Reciprocally, given a VB-algebroid

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & A \arrow[d]\\
E \arrow[r] & M
\end{tikzcd}$$

with core C together with a horizontal lift $h : \Gamma(A) \longrightarrow \Gamma_{\mathrm{lin}}(E, D)$ let:

- $\partial : C \longrightarrow E$;
- $D^C : \Gamma_{\mathrm{lin}}(E, D) \times \Gamma(C) \longrightarrow \Gamma(C)$;
- $D^{E^*} : \Gamma_{\mathrm{lin}}(E, D) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)$;

be the associated canonical operators introduced in section 1.6. We define:

$$\begin{aligned}
\nabla^E : \Gamma(A) \times \Gamma(E) &\longrightarrow \Gamma(E)\\
(a, e) &\longmapsto D^{E^*}_{h(a)} e.
\end{aligned}$$

and:

$$\begin{aligned}
\nabla^C : \Gamma(A) \times \Gamma(C) &\longrightarrow \Gamma(C)\\
(a, c) &\longmapsto D^{C}_{h(a)} c,
\end{aligned}$$

Finally, we set:

$$\omega(a, b) := h([a, b]) - [h(a), h(b)].$$

Then, $(\nabla^E, \nabla^C, \partial, \omega)$ is a 2-term representation up to homotopy of A on the 2-term graded vector bundle $E \oplus C$.

According to [23] we can summarize the discussion in the following:

Theorem 1.8.1. There is a 1-1 correspondence between VB-algebroid structures on the trivial DVB $A \oplus E \oplus C$, or equivalently between VB-algebroids D with a linear splitting, and representations up to homotopy of A on $E \oplus C$.

Let us write $\mathsf{VB}(A)$ for the category of VB-algebroids with base A. The morphisms are evidently defined. Then, as shown in [18]:

Theorem 1.8.2. The categories $\mathsf{Rep}_2^{\infty}(A)$ and $\mathsf{VB}(A)$ are equivalent.

There is an increasing literature concerning double structures, VB-algebroids and representations up to homotopy see, for instance, [29], [31], [32] and [33].

Chapter 2

Global Background

In this chapter we review the structures of global nature we shall use along the manuscript. We start by recalling basic notions concerning Lie groupoids. Then, we deal with the basics about 2-groupoids. Finally, we describe an integration procedure of 2-term representations up to homotopy by means of strict 2-functors which motivated a large amount of this work.

2.1 Generalities on Lie Groupoids

Succinctly, a groupoid is a small category in which every morphism is invertible and a Lie groupoid is a groupoid in the category of manifolds. This is not entirely correct though because the category of manifolds does not have limits. The precise definition is as follows:

Definition 2.1.1. A *Lie groupoid*, denoted by $\mathcal{G} \rightrightarrows M$, consists of a diagram of smooth manifolds and smooth maps

$$\begin{tikzcd} \mathcal{G} \,{}_{\mathsf{s}}\!\times_{\mathsf{t}} \mathcal{G} \arrow[r, "\circ"] & \mathcal{G} \arrow[r, shift left, "\mathsf{s}"] \arrow[r, shift right, "\mathsf{t}"'] & M \arrow[r, "\mathsf{u}"] & \mathcal{G} \arrow[r, "\mathsf{i}"] & \mathcal{G} \end{tikzcd}$$

such that s and t are surjective submersions and the following diagrams commute:

$$\begin{tikzcd} & M \arrow[dl, "\mathsf{id}_M"'] \arrow[d, "\mathsf{u}"] \arrow[dr, "\mathsf{id}_M"] & \\ M & \mathcal{G} \arrow[l, "\mathsf{s}"'] \arrow[r, "\mathsf{t}"] & M \\ & \mathcal{G} \arrow[ul, "\mathsf{t}"] \arrow[u, "\mathsf{i}"] \arrow[ur, "\mathsf{s}"'] & \end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{G} \times (\mathcal{G} \times_M \mathcal{G}) \arrow[d, "\mathsf{id}_{\mathcal{G}} \times \circ"'] & \mathcal{G} \times_M \mathcal{G} \times_M \mathcal{G} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & (\mathcal{G} \times_M \mathcal{G}) \times \mathcal{G} \arrow[d, "\circ \times \mathsf{id}_{\mathcal{G}}"] \\
\mathcal{G} \times_M \mathcal{G} \arrow[r, "\circ"'] & \mathcal{G} \times_M \mathcal{G} & \mathcal{G} \times_M \mathcal{G} \arrow[l, "\circ"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{G} & \\
\mathcal{G} \arrow[ur, "\mathsf{id}_{\mathcal{G}}"] \arrow[r, "(\mathsf{id}_{\mathcal{G}}{,}\mathsf{us})"'] & \mathcal{G} \,{}_{\mathsf{s}}\!\times_{\mathsf{t}} \mathcal{G} \arrow[u, "\circ"] & \mathcal{G} \arrow[ul, "\mathsf{id}_{\mathcal{G}}"'] \arrow[l, "(\mathsf{ut}{,}\mathsf{id}_{\mathcal{G}})"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "(\mathsf{id}{,}\imath)"] \arrow[d, "t"'] & \mathcal{G} \,{}_{\mathsf{s}}\!\times_{\mathsf{t}} \mathcal{G} \arrow[d, "\circ"] & \mathcal{G} \arrow[l, "(\imath{,}\mathsf{id})"'] \arrow[d, "s"] \\
M \arrow[r, "u"'] & \mathcal{G} & M \arrow[l, "u"]
\end{tikzcd}$$

We call $\mathcal{G}$ and M the *space of arrows (morphisms)* and the *space of objects*, respectively. The elements of $\mathcal{G}$ and M are called *morphisms* and *objects*, respectively. The maps s, t, i and $\circ$ are the *structure maps*, s is the *source*, t is the *target*, u is the *unit section*, i is the *inversion* and $\circ$ is the *partial composition*. The morphism $1_x = \mathsf{u}(x)$ corresponding to $x \in M$ and the morphism $f^{-1} := \mathsf{i}(f)$ corresponding to $f \in \mathcal{G}$ are called the *identity* on x and the *inverse of* f, respectively. The elements of $\mathcal{G} \times_M \mathcal{G} = \{(f, g) : \mathsf{s}(f) = \mathsf{t}(g)\}$ are called *composable pairs*. We represent $f \in \mathcal{G}$ by $f : \mathsf{s}(f) \longrightarrow \mathsf{t}(g)$.

In general, we will not require $\mathcal{G}$ to be Hausdorff for this would be restrictive (see, for instance [22] for extra information). We require s and t to be surjective submersions in order to make the fibered product $\mathcal{G} \times_M \mathcal{G}$ into a manifold so that we could require the partial composition to be smooth. In fact, it suffices to suppose one of the maps s or t to be a surjective submersion. General references for Lie groupoids are [33] and [36].

Below we present some interesting examples of Lie groupoids.

Example 2.1.2. A Lie groupoid over $\{*\}$ is equivalent to a Lie group.

Example 2.1.3. Any manifold M defines a Lie groupoid $M \rightrightarrows M$, called the *trivial groupoid*, where $\mathsf{s} = \mathsf{t} = \mathsf{u} = \mathsf{i} = \mathsf{id}_M$. The partial composition is defined in the diagonal of M by $x \circ x := x$.

Example 2.1.4. Any manifold M defines a groupoid $M \times M \rightrightarrows M$, called the *pair groupoid*, where $\mathsf{s}(x, y) := y$ and $\mathsf{t}(x, y) := x$. The partial composition is defined by $(x, y) \circ (y, z) := (x, z)$. The unit section is given by $1_x := (x, x)$ whereas the inversion is defined as $(x, y)^{-1} := (y, x)$.

Example 2.1.5. Suppose $G \times M \longrightarrow M$, $(g, x) \longmapsto g \cdot x$, is a left action of a Lie group G on a manifold M. To this action we can associate a groupoid $G \ltimes M \rightrightarrows M$ with

$G \ltimes M := G \times M$, $\mathsf{s}(g,x) := x$, $\mathsf{t}(g,x) := g \cdot x$. The partial composition is defined by $(g, h\cdot x) \circ (h,x) := (gh, x)$. The unit section is $1_x := (e_G, x)$ where e_G is the identity of G. Finally, the inversion is given by $(g,x)^{-1} := (g^{-1}, g\cdot x)$. The Lie groupoid $G \ltimes M \rightrightarrows M$ is called the *action groupoid*.

Example 2.1.6. Let M be a manifold and let $\pi(M)$ be the set of relative homotopy classes of smooth paths on M with fixed end points. Then $\pi(M) \rightrightarrows M$ is a Lie groupoid with $\mathsf{s}([a]) := a(0)$, $\mathsf{t}([a]) := a(1)$. The partial composition is obtained by concatenation of paths $[a_1] \circ [a_0] := [a_1 * a_0]$, where

$$a_1 * a_0(t) := \begin{cases} a_0(2t) & \text{if} \quad t \in [0, 1/2] \\ a_1(2t-1) & \text{if} \quad t \in [1/2, 1] \end{cases} .$$

The unit section is $1_x := [0_x]$ where $0_x(t) := x$ is the constant path at x. The inversion is given by $[a]^{-1} := [\overline{a}]$ where $\overline{a}$ is the path defined by $\overline{a}(t) := a(1-t)$. The groupoid $\pi(M) \rightrightarrows M$ is called the *fundamental groupoid* of M.

Example 2.1.7. Let $\mathcal{U} := \{U_i\}_{i\in I}$ be an open cover of a manifold M. Then $\mathcal{G}_{\mathcal{U}} := \coprod_{(i,j)\in I\times I} U_i \cap U_j \rightrightarrows M$ is a Lie groupoid. Writing $x_i := (x,i) \in U_i \times \{i\}$ and $x_{ij} := (x,(i,j)) \in (U_i \cap U_j) \times \{(i,j)\}$ then $\mathsf{s}(x_{ij}) := x_j$ and $\mathsf{t}(x_{ij}) := x_i$. The partial composition is given by $x_{ij} \circ x_{jk} := x_{ik}$. The unit section is $1_{x_i} := x_{ii}$ and $x_{ij}^{-1} := x_{ji}$. This Lie groupoid is called *Čech groupoid* of the open cover $\mathcal{U}$.

Definition 2.1.8. Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be two Lie groupoids. A *Lie groupoid morphism* F from $\mathcal{G} \rightrightarrows M$ to $\mathcal{H} \rightrightarrows N$, denoted by $F : \mathcal{G} \longrightarrow \mathcal{H}$, consists of smooth maps $F_0 : M \longrightarrow N$ e $F_1 : \mathcal{G} \longrightarrow \mathcal{H}$ such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[d, "F_0"'] & \mathcal{G} \arrow[l, "\mathsf{s}"'] \arrow[r, "\mathsf{t}"] \arrow[d, "F_1"] & M \arrow[d, "F_0"] \\ N & \mathcal{H} \arrow[l, "\mathsf{s}"] \arrow[r, "\mathsf{t}"'] & N \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "\mathsf{u}"] \arrow[d, "F_0"'] & \mathcal{G} \arrow[d, "F_1"] \\ N \arrow[r, "\mathsf{u}"'] & \mathcal{H} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{G} \times_M \mathcal{G} \arrow[r, "F_1 \times F_1"] \arrow[d, "\circ"'] & \mathcal{H} \times_N \mathcal{H} \arrow[d, "\circ"] \\ \mathcal{H} \arrow[r, "F_1"'] & \mathcal{H} \end{tikzcd}$$

We shall refer to F_0 and F_1 as the *components* of F.

In the above definition we are abusing notations, by making no distinction between the structure maps of $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$. This, however, shall cause no harm and we shall follow this practice along the text. Besides, we shall usually omit the subscripts in the components F_0 and F_1 and shall write simply F.

We notice that, by its very definition, F preserves units and the partial composition and, consequently, it also preserves inversions.

Example 2.1.9. Consider a Lie groupoid $\mathcal{G} \rightrightarrows M$ with source s and target t. The map $(\mathsf{t}, \mathsf{s}) : \mathcal{G} \longrightarrow M \times M$ is a morphism of Lie groupoids.

Example 2.1.10. Let G be a Lie group. The division map $\delta : G \times G \longrightarrow G$, $(g, h) \longmapsto gh^{-1}$ is a morphism of Lie groupoids from the pair groupoid $G \times G \rightrightarrows G$ to the groupoid $G \rightrightarrows \{*\}$.

Example 2.1.11. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $\mathcal{U} := \{U_i\}_{i \in I}$ be an open cover of M. A morphism $F : \mathcal{G}_{\mathcal{U}} \longrightarrow \mathcal{G}$ is equivalent to a family of smooth maps $\alpha_{ij} : U_i \cap U_j \longrightarrow M$ such that $\alpha_{ij}(x)\alpha_{jk}(x) = \alpha_{ik}(x)$ for every $x \in U_i \cap U_j \cap U_k$.

Finally, we define morphisms between morphisms of Lie groupoids.

Definition 2.1.12. Let $\mathcal{G}_M \rightrightarrows M$ and $\mathcal{G}_N \rightrightarrows N$ be two Lie groupoids and let $F_0, F_1 : \mathcal{G}_M \longrightarrow \mathcal{G}_N$ be two morphisms of Lie groupoids. A *natural transformation* $\eta : F_0 \Longrightarrow F_1$ is given by an assignment that associates to any object x in M an arrow η_x in $\mathcal{G}_N$, in such a way that for any arrow $g : x \longrightarrow y$ in $\mathcal{G}_M$, we have $F_1(a) \circ \eta_x = \eta_y \circ F_0(a)$ as illustrated below:

$$
\begin{tikzcd}
\mathcal{G}_M \arrow[rr, "F_0, F_1"] & & \mathcal{G}_N & \\
x \arrow[d, bend right, "a"'] & & F_0(x) \arrow[r, "\eta_x"] \arrow[d, bend right, "F_0(a)"'] & F_1(x) \arrow[d, bend right, "F_1(a)"'] \\
y & & F_0(y) \arrow[r, "\eta_y"'] & F_1(y)
\end{tikzcd}
$$

In the third chapter we give an alternative point of view for natural transformations between morphisms of Lie groupoids.

2.2 The Lie Functor

In this section we define a functor $\mathsf{Lie} : \mathsf{LieGrpd} \longrightarrow \mathsf{LieAlgd}$ called the *Lie functor*. This functor encodes the fact the Lie groupoids differentiate to Lie algebroids.

At the level of objects, for a given Lie groupoid $\mathcal{G} \rightrightarrows M$ the corresponding Lie algebroid $\mathsf{Lie}(\mathcal{G}) \longrightarrow M$ is obtained from the vector bundle

$$\mathsf{Lie}(\mathcal{G}) := \mathsf{u}^*\mathsf{Ker}(\mathsf{ds}) = \coprod_{x \in M} \mathsf{Ker}(\mathsf{ds}_{1_x}) = \coprod_{x \in M} T_{1_x}\mathsf{s}^{-1}(\{x\}),$$

where $\mathsf{u} : M \longrightarrow \mathcal{G}$ and $\mathsf{s} : \mathcal{G} \longrightarrow M$ are the respective unit map and the source map of $\mathcal{G} \rightrightarrows M$. Since the source map is a submersion, $\mathsf{Lie}(\mathcal{G})$ is indeed a vector bundle over M. Now, consider the $C^\infty(M)$-module of right invariant vector fields on $\mathcal{G}$:

$$\mathfrak{X}^r(\mathcal{G}) := \{X \in \Gamma(\mathsf{Ker}(\mathsf{ds})) : X_{hg} = (\mathsf{dR}_g)_h(X_h),\ \forall (h,g) \in \mathcal{G} \times_M \mathcal{G}\},$$

where R_g is the right multiplication by g, that is, the map:

$$\begin{aligned} \mathsf{R}_g : \mathbf{s}^{-1}(\mathbf{t}(g)) &\longrightarrow \mathbf{s}^{-1}(\mathbf{s}(g)) \\ h &\longmapsto hg. \end{aligned}$$

It is well-known that $\mathfrak{X}^r(\mathcal{G})$ is a Lie algebra when endowed with the usual Lie bracket of vector fields. A key observation is the fact that any $X \in \mathfrak{X}^r(\mathcal{G})$ satisfies

$$X_g = X_{\mathsf{u}(\mathsf{t}(g))g} = (\mathsf{dR}_g)_{\mathsf{u}(\mathsf{t}(g))}(X_{\mathsf{u}(\mathsf{t}(g))}),$$

for every $g \in \mathcal{G}$. This tells us that X is completely determined once we know its values on $\mathsf{u}(M)$. This makes the map

$$\begin{aligned} \Gamma(\mathsf{Lie}(\mathcal{G})) &\longrightarrow \mathfrak{X}^r(\mathcal{G}) \\ X &\longmapsto X^r \end{aligned}$$

where:

$$X^r(g) := (\mathsf{dR}_g)_{\mathsf{u}(\mathsf{t}(g))}(X_{\mathsf{t}(g))})$$

into an isomorphism of $C^\infty(M)$-modules. The proof of this fact can be found in [42]. This allows us to transport the Lie bracket from $\mathfrak{X}^r(\mathcal{G})$ to $\Gamma(\mathsf{Lie}(\mathcal{G}))$. More precisely, we define

$$[X, Y]_{\mathsf{Lie}(\mathcal{G})} := [X^r, Y^r] \circ \mathsf{u}.$$

The anchor map is:

$$\sharp_{\mathsf{Lie}(\mathcal{G})} := dt|_{\mathsf{Lie}(\mathcal{G})}.$$

Every Lie groupoid morphism $\Phi : \mathcal{G}_M \longrightarrow \mathcal{G}_N$ induces a Lie algebroid morphism between the corresponding Lie algebroids:

$$\mathsf{Lie}(\Phi) : \mathsf{Lie}(\mathcal{G}_M) \longrightarrow \mathsf{Lie}(\mathcal{G}_N),$$

which is given by:

$$\mathsf{Lie}(\Phi) = d\Phi|_{\mathsf{Lie}(\mathcal{G}_M)}.$$

A complete and thorough treatment of the contents of this section can be found, for instance, in [42] and [33].

2.3 Lie Theory

We saw in the previous section that a Lie groupoid gives rise to a Lie algebroid. Having this in mind, it is natural to ask whether every Lie algebroid can be obtained in this manner. In this section, we will see this is in general not the case. The presentation is inspired by the seminal papers [17] and [22].

Definition 2.3.1. A Lie algebroid A over M is *integrable* if it is isomorphic to the Lie algebroid of a Lie groupoid $\mathcal{G}$ over M. We then say that $\mathcal{G}$ *integrates* A or that $\mathcal{G}$ is an *integration* of A.

The following theorem, which is one of the main results in the seminal work [17], provides a generalization of the analogous result for Lie algebras.

Theorem 2.3.2 (Lie I)**.** Let A be an integrable Lie algebroid. Then, up to isomorphism, there exists a unique source-simply connected Lie groupoid integrating A.

Saying that a Lie groupoid is source-simply connected means that the fibers of its source map are simply connected.

In analogy to the case of finite dimensional Lie algebras, it was shown in [34] that:

Theorem 2.3.3 (Lie 2)**.** Let $F : A \longrightarrow B$ be a morphism of integrable Lie algebroids. Suppose $\mathcal{G}_A$ and $\mathcal{G}_B$ are *integrations* of A and B, respectively. If $\mathcal{G}_A$ is source-simply connected then there exists a unique morphism of Lie groupoids $\mathcal{F} : \mathcal{G} \longrightarrow \mathcal{H}$ integrating F.

As shown for instance in [19], every finite dimensional Lie algebra is integrable. This is called Lie's third theorem and this does not extend to Lie algebroids. The integrability problem for Lie algebroids was solved in [17] where the authors managed to obtain the precise obstructions to the integrability of a Lie algebroid. Below we discuss a procedure to find an integration of a given algebroid.

Inspired by the theory of Lie groups, as in [**?**], the starting point of the integration theory of Lie algebroids is the following:

Definition 2.3.4. Let A be a Lie algebroid. Two A-paths $a_0 \cdot dt$ and $a_1 \cdot dt$ are *A-homotopic* if there exists an A-homotopy $\sigma = a \cdot dt + b \cdot ds$ such that

$$a|_{I\times\{0\}} = a_0 \quad \text{and} \quad a|_{I\times\{1\}} = a_1.$$

The relation of "being A-homotopic" is denoted by $\sim_A$.

Let us write $P_1(A)$ for the space of A-paths. As noticed in [17], $P_1(A)$ is a Banach manifold. Using this data, the authors of [17] introduced the following groupoid:

Definition 2.3.5. Let A be a Lie algebroid over M. The *Weinstein groupoid* of A is

$$\mathcal{G}(A) := P_1(A)/\sim_A \rightrightarrows M$$

where $\sim_A$ is the relation of "being A-homotopic" and where the structural maps are given by:

- The source and the target maps $\mathsf{s}, \mathsf{t} : \mathcal{G}(A) \longrightarrow M$ are defined by

$$\mathsf{s}([a]) := \pi(a(0)) \quad \text{and} \quad \mathsf{t}([a]) := \pi(a(1));$$

- The composition is given by concatenation of A-paths:

$$a_1 * a_0(t) := \begin{cases} 2a_0(2t) & \text{if} \quad t \in [0, 1/2] \\ 2a_1(2t-1) & \text{if} \quad t \in [1/2, 1] \end{cases}.$$

- The unit section $\mathsf{u} : M \longrightarrow P(A)$ takes x to the class $[0_x]$ of the constant path at x;
- The inversion $\mathsf{i} : \mathcal{G}(A) \longrightarrow \mathcal{G}(A)$ associates a class $[a]$ to the class $[\overline{a}]$ where $\overline{a}(t) := -a(1-t)$.

There is a minor technical detail concerning smoothness of the concatenation of A-paths. A reparametrization of the A-paths is required in order to ensure smoothness. This however, presents no major problem since it is possible to perform such reparametrizations without changing A-homotopy classes. The interested reader may check [17] for further details.

A priori, the Weinstein groupoid of A is a topological groupoid. The smoothness of $\mathcal{G}(A)$ is related to the integrability of A. The following is one of the main results in [17]:

Theorem 2.3.6. Let A be a Lie algebroid over M. Then $\mathcal{G}(A)$ is a source-simply connected groupoid. Furthermore, whenever A is integrable, $\mathcal{G}(A)$ admits a smooth structure that turns it into the unique source-simply connected groupoid integrating A.

At the level of morphisms we have the following scenario:

Theorem 2.3.7. Let $\Phi : A \longrightarrow B$ be a Lie algebroid morphism. Then:

- The induced map

$$\begin{aligned} F : \mathcal{G}(A) &\longrightarrow \mathcal{G}(B) \\ [a] &\longmapsto [\Phi \circ a] \end{aligned}$$

  is a morphism between the corresponding topological groupoids.

- Whenever A and B are integrable $F : \mathcal{G}(A) \longrightarrow \mathcal{G}(B)$ is smooth and integrates Φ.

2.4 2-Categories

In this section we recall the basic definitions concerning strict 2-categories. Since all the categories we deal with in this work are strict we shall refer to them briefly as 2-categories. The presentation follows [38] very closely.

Definition 2.4.1. A *(small) 2-category* consists of the following data:

- a set of *objects*;
- for every pair of objects (X, Y) there is a set of *1-morphisms* whose elements are denoted by $f : X \longrightarrow Y$;
- for every pair of 1-morphisms (f, g), $f, g : X \longrightarrow Y$, there is a set of *2-morphisms* whose elements are written $\varphi : f \Rightarrow g$;

together with the following structure:

1) for every pair of 1-morphisms (f, g), $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, there is a 1-morphism $g \circ f : X \longrightarrow Y$ called the *composition* of f and g;

2) for every object X there is a 1-morphism $\mathsf{id}_X : X \longrightarrow X$ called the *identity 1-morphism of* X;

3) for every pair of 2-morphisms (φ, ψ), $\varphi : f \Rightarrow g$ and $\psi : g \Rightarrow h$ there is a 2-morphism $\psi \bullet \varphi : f \Rightarrow h$ called the *vertical composition* of φ and ψ;

4) for every 1-morphism f there is a 2-morphism $\mathsf{id}_f : f \Rightarrow f$ called the *identity 2-morphism of* f;

5) for every triple of objects (X, Y, Z) and 1-morphisms $f, f' : X \longrightarrow Y$ and $g, g' : Y \longrightarrow Z$ and for every pair of 2-morphisms, $\varphi : f \Rightarrow f'$ and $\psi : g \Rightarrow g'$ there is a 2-morphism $\psi \circ \varphi : g \circ f \Rightarrow g' \circ f'$ called the *horizontal composition* of φ and ψ.

This structure must satisfy the axioms:

i) the composition of 1-morphisms and the horizontal and vertical 2-morphisms are associative;

ii) the identity 1-morphisms are units with respect to the composition of 1-morphisms and the identity 2-morphisms are units with respect to the vertical composition, that is:

$$\varphi \bullet \mathrm{id}_f = \mathrm{id}_g \bullet \varphi$$

for every 2-morphism $\varphi : f \Rightarrow g$. The horizontal composition preserves the identity 2-morphisms in the following sense:

$$\mathrm{id}_g \circ \mathrm{id}_f = \mathrm{id}_{g \circ f}.$$

iii) the horizontal and the vertical compositions are compatible in the following sense

$$(\psi_1 \bullet \psi_2) \circ (\varphi_1 \bullet \varphi_2) = (\psi_1 \circ \varphi_1) \bullet (\psi_2 \circ \varphi_2)$$

whenever the composites are defined.

Rigorously, the above definition corresponds to the notion of *strict 2-category*. The general definition of a *2-category* requires weaker notions of the associativities and units. We shall omit the precise definition.

The prototype of a (strict) 2-category is that whose objects are categories, whose 1-morphisms are functors and whose 2-morphisms are natural transformations. The composition of 1-morphisms is the standard composition of functors whereas the vertical and horizontal compositions of 2-morphisms are the vertical and horizontal compositions of natural transformations.

Next we proceed to define the notion of a (strict) 2-groupoid.

Definition 2.4.2. A 1-morphism $f : X \longrightarrow Y$ in a 2-category is called a *strict 1-isomorphism* or *strictly invertible* if there exists another 1-morphism $g : Y \longrightarrow X$ such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$. A 2-category in which every 1-morphism is strictly invertible is called a *(strict) 2-groupoid*.

In general, we can also talk about non-strict 2-groupoid in which we require a weaker notion of invertibility. We shall omit the precise definition.

To relate two 2-categories we use the notion of (strict) 2-functor:

Definition 2.4.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. A *(strict) 2-functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an assignment:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} F(X) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow F(\alpha)"] & F(Y) \end{tikzcd}$$

such that:

(a) The vertical structure is respected in the following sense:

$$F(\psi \bullet \varphi) = F(\psi) \bullet F(\varphi) \quad \text{and} \quad F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$$

for any composable 2-morphisms φ and ψ and for any 1-morphism f;

(b) The composition of 1-morphisms is respected in the following sense:

$$F(g) \circ F(f) = F(g \circ f)$$

for any composable 1-morphisms f and g and the horizontal composition of 2-morphisms is respected in the following sense:

$$F(\psi) \circ F(\varphi) = F(\psi \circ \varphi)$$

for any composable 2-morphisms φ and ψ;

For general 2-categories there is a weaker notion of 2-functors.

In order to compare 2-functors, we use the notion of pseudo-natural transformation:

Definition 2.4.4. Let $\mathcal{C}$ and $\mathcal{D}$ be two (strict) 2-categories and let $F_1, F_2 : \mathcal{C} \longrightarrow \mathcal{D}$ be two (strict) 2-functors. A *pseudo-natural transformation* $\rho : F_1 \longrightarrow F_2$ is an assignment:

$$X \xrightarrow{f} Y \quad \overset{\rho}{\longmapsto} \quad \begin{tikzcd} F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "\rho_X"'] & F_1(Y) \arrow[d, "\rho_Y"] \arrow[dl, Rightarrow, "\rho_f"'] \\ F_2(X) \arrow[r, "F_2(f)"'] & F_2(Y) \end{tikzcd}$$

of a 2-isomorphism $\rho_f : \rho_Y \circ F_1(f) \Longrightarrow F_2(f) \circ \rho_X$ in $\mathcal{D}$ for every 1-morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ such that:

(a) The composition of 1-morphisms in $\mathcal{C}$ is respected in the following sense:

$$
\begin{tikzcd}
F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "\rho_X"'] & F_1(Y) \arrow[r, "F_1(g)"] \arrow[d, "\rho_Y"] \arrow[dl, phantom, "\rho_f \swArrow"] & F_1(Z) \arrow[d, "\rho_Z"] \arrow[dl, phantom, "\rho_g \swArrow"] \\
F_2(X) \arrow[r, "F_2(f)"'] & F_2(Y) \arrow[r, "F_2(g)"'] & F_2(Z)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_1(X) \arrow[r, "F_1(g\circ f)"] \arrow[d, "\rho_X"'] & F_1(Z) \arrow[d, "\rho_Z"] \arrow[dl, phantom, "\rho_{g\circ f} \swArrow"] \\
F_2(X) \arrow[r, "F_2(g\circ f)"'] & F_2(Z)
\end{tikzcd}
$$

(b) It is compatible with 2-morphisms:

$$
\begin{tikzcd}
F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "\rho_X"'] & F_1(Y) \arrow[d, "\rho_Y"] \arrow[dl, phantom, "\rho_f \swArrow"] \\
F_2(X) \arrow[r, "F_2(f)"] \arrow[r, bend right=50, "F_2(g)"'] \arrow[r, phantom, shift right=3ex, "\Downarrow F_2(\varphi)"] & F_2(Y)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_1(X) \arrow[r, bend left=50, "F_1(f)"] \arrow[r, phantom, shift left=3ex, "\Downarrow F_1(\varphi)"] \arrow[r, "F_1(g)"'] \arrow[d, "\rho_X"'] & F_1(Y) \arrow[d, "\rho_Y"] \arrow[dl, phantom, "\rho_g \swArrow"] \\
F_2(X) \arrow[r, "F_2(g)"'] & F_2(Y)
\end{tikzcd}
$$

In particular, $\rho(\mathrm{id}_X) = \mathrm{id}_{\rho_X}$ for every object X. Pseudonatural transformations $\rho_1 : F_1 \longrightarrow F_2$ and $\rho_2 : F_2 \longrightarrow F_3$ can naturally be composed to a pseudo-natural transformation $\rho_2 \circ \rho_1 : F_1 \longrightarrow F_3$.

Finally, we can compare two pseudo-natural transformations using modifications:

Definition 2.4.5. Let $F_1, F_2 : S \longrightarrow T$ be two strict 2-functors and let $\rho_1, \rho_2 : F_1 \longrightarrow F_2$ be pseudo-natural transformations. A *modification* $\mathcal{A} : \rho_1 \Longrightarrow \rho_2$ is an assignment

$$
X \stackrel{\mathcal{A}}{\longmapsto}
\begin{tikzcd}
F_1(X) \arrow[r, bend left=40, "\rho_1(X)"] \arrow[r, bend right=40, "\rho_2(X)"'] \arrow[r, phantom, "\Downarrow \mathcal{A}(X)"] & F_2(X)
\end{tikzcd}
$$

of a 2-morphism $\mathcal{A}(X)$ in T to any object X in S which satisfies

$$
\begin{tikzcd}
F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "\rho_1(X)"] \arrow[d, bend right=80, "\rho_2(X)"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow \mathcal{A}(X)"] & F_1(Y) \arrow[d, "\rho_1(Y)"] \arrow[dl, phantom, "\rho_1(f) \swArrow"] \\
F_2(X) \arrow[r, "F_2(f)"'] & F_2(Y)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "\rho_2(X)"'] & F_1(Y) \arrow[d, "\rho_2(Y)"'] \arrow[d, bend left=80, "\rho_1(X)"] \arrow[d, phantom, shift left=4ex, "\Leftarrow \mathcal{A}(Y)"] \arrow[dl, phantom, "\rho_2(f) \swArrow"] \\
F_2(X) \arrow[r, "F_2(f)"'] & F_2(Y)
\end{tikzcd}
$$

For two fixed strict 2-categories S and T, we recognize the following structures:

1) For two strict 2-functors $F_1, F_2 : S \longrightarrow T$, the pseudonatural transformations $\rho : F_1 \longrightarrow F_2$ together with modifications and vertical composition, form a category $\mathsf{Hom}(F_1, F_2)$.

2) Even more, strict 2-functors from S to T, together with pseudonatural transformations, modifications and the assignments $\circ$ and $\bullet$ defined above, form a strict 2-category $\mathsf{Func}(S, T)$.

2.5 The Weinstein 2-Groupoid

In this section we present the definition of the Weinstein 2-groupoid which first appeared in [13]. This will play an important role in the integration theory we discuss later on.

Definition 2.5.1. Let A be a Lie algebroid over M and let $a \cdot dt : TI \longrightarrow A$ be an A-path.

- The *source* of $a \cdot dt$ is $\mathbf{s}(a \cdot dt) := p_A(a(0))$;
- The *target* of $a \cdot dt$ is $\mathbf{t}(a \cdot dt) := p_A(a(1))$;
- The *inverse* of $a \cdot dt$ is $\overline{a}\ dt := -a(1-t) \cdot dt$;
- For $x \in M$, $1_x := 0_x \cdot dt$ denotes the *unit path* at x where 0_x is the neutral element of the fiber A_x.

Up to now, the unit and the inverse we just defined are simply formal. In order to make them legit we need to take the quotient by thin-homotopies which we shall explain below.

Definition 2.5.2. An A-path $a \cdot dt : TI \longrightarrow A$ is *flat in its boundaries* if for any $\theta \in \Gamma(A^*)$, the map $\langle \theta, a \rangle : I \longrightarrow \mathbb{R}$ vanishes for $t \in \{0, 1\}$ together with all its higher derivatives.

Definition 2.5.3. The *concatenation* of two A-paths $a \cdot dt, b\ dt : TI \longrightarrow A$, where $t(a\ dt) = s(b\ t)$, is defined to be:

$$(b * a)(t) \cdot dt := \begin{cases} 2a(2t) \cdot dt & \text{if} \quad t \in [0, 1/2] \\ 2b(2t-1) \cdot dt & \text{if} \quad t \in [1/2, 1] \end{cases}.$$

Obviously, $(b * a) \cdot dt$ is smooth whenever a and b are flat in their boundaries. In order to concatenate arbitrary A-paths we must replace them by their reparametrizations. For that, we consider a cutoff function $\tau : I \longrightarrow I$, which can be taken as the restriction of any increasing smooth function $\tau : \mathbb{R} \longrightarrow [0, 1]$ such that $\tau|_{(-\infty,0)} = \tau|_{(1,\infty)} = 0$ such that all its derivatives vanish at 0 and 1. Then, the reparametrization a^τ is the A-path given by $a^\tau(t) \cdot dt := a \circ d\tau : TI \longrightarrow A$ where $a^\tau(t) := \tau'(t)(a \circ \tau)(t)$.

Definition 2.5.4. An *A-homotopy* between two A-paths $a^0 \cdot dt$ and $a^1 \cdot dt$ is a morphism of Lie algebroids $\mathbf{h} := a \cdot dt + b \cdot ds : TI^2 \longrightarrow A$ such that $a^0 \cdot dt = a \cdot dt|_{s=0}$ and $a^1 \cdot dt = a \cdot dt|_{s=1}$ satisfying the boundary conditions $b|_{t=0,1} = 0$.

We denote by $s_v(\sigma) = a^0 \cdot dt$ and $t^v(\sigma) = a^1 \cdot dt$ the *source* and the *target* of $\mathbf{h}$ and we write $\mathbf{h} : a^0 \Rightarrow a^1$.

Definition 2.5.5. An A-homotopy $\mathbf{h} := a \cdot dt + b \cdot ds : TI^2 \longrightarrow A$ is *flat in its boundary* if for any smooth section $\eta \in \Gamma(A^*)$ the following conditions hold:

i) the map $\langle \eta, b\rangle : I \longrightarrow \mathbb{R}$ vanishes in $\{s = 0, 1\}$ as well as all its higher derivatives;

ii) the map $\langle \eta, a\rangle : I \longrightarrow \mathbb{R}$ vanishes in $\{t = 0, 1\}$ as well as all its higher derivatives.

Definition 2.5.6. Given two A-homotopies $\mathbf{h} := a \cdot dt + b \cdot dts$ and $\mathbf{h}' = a' \cdot dt + b' \cdot ds$ such that $t_v(\mathbf{h}) = s_v(\mathbf{h}')$, we define its *vertical concatenation* as:

$$(\mathbf{h}' \bullet_v \mathbf{h})(t, s) = \begin{cases} a(t, 2s) \cdot dt + 2b(t, 2s) \cdot ds & \text{if} \quad s \in [0, 1/2] \\ a'(t, 2s - 1) \cdot dt + 2b'(t, 2s - 1) \cdot ds & \text{if} \quad s \in [1/2, 1] \end{cases}.$$

The vertical concatenation is smooth whenever $\mathbf{h}$ and $\mathbf{h}'$ are flat in their boundaries. In order to smoothly concatenate arbitrary A-homotopies we must replace them by their respective reparametrizations. The reparametrization of an A-homotopy $\mathbf{h}$ is given by $\mathbf{h}^\tau := \mathbf{h} \circ d(\tau \times \tau)$, namely, $\mathbf{h}^\tau = \tau'(t)a(\tau(t), \tau(s)) \cdot dt + \tau'(s)b(\tau(t), \tau(s)) \cdot ds$ where τ is a cutoff function as we described above.

Definition 2.5.7. The *vertical inverse* of an A-homotopy $\mathbf{h} = a \cdot dt + b \cdot ds$ is given by

$$(\mathbf{h}^{-1_v})(t, s) = a(t, s) \cdot dt - b(t, 1 - s) \cdot ds.$$

The *vertical unit* at an A-path $a \cdot dt$ is the A-homotopy $1^v_{a \cdot dt} := a(t) \cdot dt + 0_{\gamma(t)} \cdot ds$.

Definition 2.5.8. The *horizontal source* s_H and the *horizontal target* t_H are defined as $s_H := s \circ s_v$ e $t_H := t \circ s_v$.

Definition 2.5.9. Given two A-homotopies $\mathbf{h} = a \cdot dt + b \cdot ds$ and $\mathbf{h}' := a' \cdot dt + b' \cdot ds$ such that $t_H(\mathbf{h}) = s_H(\mathbf{h}')$, we define their *horizontal concatenation* as:

$$(\mathbf{h}' \bullet_H \mathbf{h})(t, s) := \begin{cases} 2a(2t, s) \cdot dt + b(2t, s) \cdot ds & \text{if} \quad t \in [0, 1/2] \\ 2a'(2t - 1, s) \cdot dt + b'(2t - 1, s) \cdot ds & \text{if} \quad t \in [1/2, 1] \end{cases}.$$

The horizontal concatenation is smooth whenever $\mathbf{h}$ and $\mathbf{h}'$ are flat in their boundaries.

Definition 2.5.10. The *horizontal inverse* of an A-homotopy is defined as:

$$(\mathbf{h}^{-1_H})(t, s) := -a(1 - t, s) \cdot dt + b(1 - t, s) \cdot ds.$$

Given $x \in M$, the *horizontal unit* is the A-homotopy $1^H_x := 0_x \cdot dt + 0_x \cdot ds$ between the unit path 1_x and itself.

Definition 2.5.11. A 3-*homotopy* is any Lie algebroid morphism $H := H_1 \cdot dt + H_2 \cdot ds + H_3 \cdot du : TI^3 \longrightarrow A$ satisfying the boundary conditions $H_3 \cdot du|_{t=0,1} = 0 \cdot du$ and $H_3 \cdot du|_{s=0,1} = 0 \cdot du$.

In this case, H defines an A-homotopy between $\mathbf{h}_0 := (H_1 \cdot dt + H_2 \cdot ds)|_{u=0}$ and $\mathbf{h}_1 := (H_1 \cdot dt + H_2 \cdot ds)|_{u=1}$. Notice that, as a consequence of H being a Lie algebroid morphism, $\mathbf{h}_1$ is an A-homotopy if and only if $\mathbf{h}_0$ is.

Definition 2.5.12. Given a Lie algebroid A over M, the *Weinstein 2-groupoid of* A, denoted by 2-$\mathcal{P}(A)$, is the 2-groupoid whose:

- objects are the elements of M;
- 1-morphisms are thin-homotopy classes of A-paths;
- 2-morphisms are 3-homotopy classes of A-homotopies;

and whose compositions and unities are the ones we defined previously.

We denote by $P_1(A)$ the space of thin-homotopy classes of A-paths and by $P_2(A)$ the space of 3-homotopy classes of A-homotopies.

In [13], the authors proved the following interesting result:

Theorem 2.5.13. Let A be a Lie algebroid over M. The truncation at the level of 1-morphisms of the Weinstein 2-groupoid of A, namely, $P_1(A)/P_2(A)$ identifies with the Weinstein groupoid $\mathcal{G}(A)$ of A.

2.6 The Gauge 2-Groupoid of a 2-Vector Bundle

In this section we define a certain gauge 2-groupoid which is necessary in order to define representations of 2-groupoids. The presentation follows [13] very closely.

Definition 2.6.1. A *2-vector bundle* over a manifold M is a groupoid object in the category of vector bundles over M.

Namely, a 2-vector bundle over M is a category $E_1 \rightrightarrows E_0$ where both the space of objects E_0 and of arrows E_1 are vector bundles over M and where all structure maps (source, target, unit and composition) are vector bundle morphisms covering the identity.

Definition 2.6.2. A *2-vector bundle morphism* is a *linear functor*, that is, a functor which is a vector bundle map both at the level of objects and of arrows.

With the above notion of morphism, 2-vector bundles over M give rise to a category which will be denoted by 2-Vect_M.

The next example will be important:

Example 2.6.3. Let $\partial : C \longrightarrow E$ be a morphism of vector bundles over M. Then $E \oplus C \rightrightarrows E$ is a 2-vector bundle where the structure maps are:

$$\begin{aligned} s(e,c) &:= e \\ t(e,c) &:= e + \partial c \\ (e,c')(e+\partial c', c) &:= (e, c+c') \\ (e,c)^{-1} &:= (-e, c) \\ 1_e &:= (0,e). \end{aligned}$$

The following special case will be important:

Definition 2.6.4. A *2-vector space* is a 2-vector bundle over a point.

Given a 2-vector bundle $E_1 \rightrightarrows E_0$ over M then $(E_1)_x \rightrightarrows (E_0)_x$ is a 2-vector space. Therefore, we can think of 2-vector bundles over M as a family of 2-vector spaces parametrized smoothly by points of M. For a given 2-vector bundle $\mathcal{E} = (E_1 \rightrightarrows E_0)$ over M and $x \in M$ let us write $\mathcal{E}_x := ((E_1)_x \rightrightarrows (E_0)_x)$.

Every 2-vector bundle over M gives rise to a strict 2-groupoid over M called the *gauge 2-groupoid*.

Definition 2.6.5. Let $\mathcal{E} = (E_1 \rightrightarrows E_0)$ be a 2-vector bundle over M. The *gauge 2-groupoid of* $\mathcal{E}$, written 2-$\mathsf{Gau}(\mathcal{E})$ is the 2-groupoid whose:

- space of objects is M;
- space of 1-morphisms consists of invertible linear functors $F : \mathcal{E}_x \longrightarrow \mathcal{E}_y$ with x and y varying on M;
- space of 2-morphisms consists of linear natural transformations;
- The various compositions and units are the obvious ones.

Part II - Integration of Infinitesimal Natural Homotopies

Chapter 3

Natural Homotopies

Natural transformations play an important role in the theory of categories as well as in the theory of Lie groupoids. The notion of natural transformation between Lie groupoid morphisms is clear, it is an internal natural transformation in the category of smooth manifolds, that is, a natural transformation which is smooth as a map from the space of objects to the space of morphisms. Despite the fact that, recently, Lie groupoids have been the subject of intensive studies, the notion of natural transformation is hardly addressed, which partially motivated this part of the work.

Regarding natural transformations, there are two basic questions that arise in the context of Lie groupoids:

- Is there any infinitesimal counterpart of a (smooth) natural transformation in the context of Lie groupoids?
- What would the corresponding integration procedure be?

In this part of the work, we address both questions, to which, surprisingly, we could not find any explicit answer in the literature. Consequently, and although we have no doubt some of the present results might be known by a few experts in one form or another, we are convinced this gap was worth clarifying. Furthermore, we shall apply the theory developed in this part later in the text.

Our main result in this part is the aforementioned integration procedure:

Theorem 3.0.1. *Let A_M and A_N be Lie algebroids over smooth manifolds M and N, respectively, together with a smooth family $\Phi_t : A_M \longrightarrow A_N$ of Lie algebroid morphisms parametrized by $t \in [0,1]$.*

We denote by $\phi_t : M \longrightarrow N$ the underlying family of smooth maps, and by

$$\Phi_t^* : \Omega(A_N) \longrightarrow \Omega(A_M)$$

the smooth family of morphisms of cochains complexes induced by Φ_t.

Assume there exists a smooth family $\theta_t \in \Gamma(\phi_t^* A_N)$ of sections of A_N supported by ϕ_t that satisfies the following condition:

$$\frac{\partial}{\partial t}\Phi_t^* = d_{A_M} \circ \imath_{\theta_t}^{\Phi_t} + \imath_{\Theta_t} \circ d_{A_N}$$

where $\imath_{\theta_t}^{\Phi_t} : \Omega(A_N) \longrightarrow \Omega(A_M)$ is the degree -1 operator defined by:

$$\langle \imath_{\theta_t}^{\Phi_t}\varepsilon, a_1, \ldots, a_{k-1}\rangle := \langle \varepsilon \circ \phi_t, \theta_t, \Phi_t \circ a_1, \ldots, \Phi_t \circ a_{k-1}\rangle.$$

Then:

i) The morphisms Φ_0^* and Φ_1^* are homotopic as cochain maps;

ii) Whenever A_M and A_N are integrable Lie algebroids, the assignment:

$$\eta_m := [\theta(-, m) \cdot dt]_{A_M}$$

defines a smooth natural transformation $\eta : F_0 \Longrightarrow F_1$ where $F_0, F_1 : \mathcal{G}(A_M) \longrightarrow \mathcal{G}(A_N)$ denotes the Lie groupoid morphisms integrating Φ_0 and Φ_1, respectively.

Here, we denote by $[\theta(-, m)\cdot dt]_{A_M}$ the A_N-homotopy class of the A_N-path $\theta(-, m)\cdot dt : TI \longrightarrow A_N$.

This part is organized as follows:

- We start by discussing derivations of the graded algebra of a Lie algebroid;
- Next, we deal with basic time-dependent objects on Lie algebroids;
- We then discuss smooth natural transformations and explain how they can be seen as *discrete* homotopies;
- Afterwards, we explain why there is not an infinitesimal counterpart to a natural transformation and we introduce the main notion of this part, namely, that of natural homotopy. We then show how a natural homotopy is encoded in a family of natural transformations;
- We then proceed to explain how infinitesimal homotopies are the infinitesimal counterpart of natural homotopies;
- Subsequently, we prove the aforementioned integration procedures and discuss a few examples.

Regarding the existing literature, let us emphasize the following points. Before the notion of Lie algebroid even emerged, the relevance of the notion of homotopy between maps of DGAs was already pointed out by Sullivan in [40]. In the context of Lie algebroids, homotopies between Lie algebroid morphisms appear in the earlier work of Kubarski [28] on Chern-Weil homomorphism (see also [8]) then studied later on in connection with Poisson sigma-models [11]. However, the interpretation of such homotopies in terms of natural transformations does not appear in those works.

A generalization to L_∞-algebras with a strong categorical point of view can be found in [26]. The application of these results to Lie algebroids, however, is not straightforward, unless considering a Lie algebroid as an infinitesimal Lie algebra which turns out to be quite restrictive.

The presentation of this part is independent of the rest of the work. However, the theory presented here emerged from discussions in the context of extensions of Lie algebroids so there is really an underlying relationship with the rest of the text. It is even possible to see an infinitesimal homotopy as a morphism between suitable extensions of the tangent bundle TI. In the final part of this work we shall point out how to apply the integration procedure proven in this part can be applied in the context of extensions of Lie algebroids.

3.1 Derivations of Lie Algebroids

In this section we discuss derivation maps between forms of different Lie algebroids. This will be an important tool in later sections.

Definition 3.1.1. Given a smooth map $\phi : M \longrightarrow N$ and a vector bundle A_N over N, a *section of A_N supported by ϕ* is a section θ of the pullback bundle $\theta \in \Gamma(\phi^* A_N)$.

In particular, if $A_M = TM$ and $A_N = TN$ are tangent bundles, then θ is just a vector field supported by ϕ, that is, essentially, a vector tangent at ϕ in the space of smooth maps from M to N.

Definition 3.1.2. Let $\Phi : A_M \longrightarrow A_N$ be a vector bundle map covering $\phi : M \longrightarrow N$ and denote by $\Phi^* : \Omega(A_N) \longrightarrow \Omega(A_M)$ the induced map on forms.

A Φ^*-*derivation of degree k* is a degree k linear application

$$\nu : \Omega(A_N) \longrightarrow \Omega(A_M)$$

such that:

$$\nu(\alpha \wedge \beta) = \nu(\alpha) \wedge \Phi^*(\beta) + (-1)^{k|\alpha|}\Phi^*(\alpha) \wedge \nu(\beta), \tag{3.1}$$

for every $\alpha, \beta \in \Omega(A_N)$ homogeneous. In the case, $k = 0$, $A_M = A_N$ and $\Phi = \mathsf{id}_{A_N}$, we say that ν is a derivation of $\Omega(A_M)$.

Proposition 3.1.3. Under the assumptions and notations of definition 3.1.2, there is a one-to-one correspondence between:

- Φ^*-derivations of degree -1;
- sections $\theta \in \Gamma(\phi^* A_N)$ supported by ϕ.

Explicitly, to any section $\theta \in \Gamma(\phi^* A_N)$ supported by ϕ, the corresponding Φ^*-derivation $\nu := \imath^\Phi_\theta$ is given by:

$$\langle \nu\beta, a_1, \dots, a_{k-1}\rangle := \langle \beta \circ \phi, \theta \wedge \Phi \circ a_1 \wedge \dots \wedge \Phi \circ a_{k-1}\rangle \in C^\infty(M) \tag{3.2}$$

where $\beta \in \Omega^k(A_N)$ and $a_1, \dots, a_{k-1} \in \Gamma(A_M)$.

Proof. First, a direct computation shows $\nu = \imath^\Phi_\theta$ is indeed a Φ^*-derivation. To see that any Φ^*-derivation ν can be obtained this manner, one notices that, since $\Omega(A_N)$ is generated (as a graded algebra) by $\Omega^{\leq 1}(A_N)$, and because of the Φ^*-derivation condition (3.1), ν is entirely determined by its restriction to $\Omega^{\leq 1}(A_N)$. Since ν has degree -1, this restriction vanishes on $C^\infty(N)$ so it reduces to a map

$$\nu|_{\Omega^1(A_N)} : \Omega^1(A_N) \longrightarrow C^\infty(M)$$

which is $C^\infty(M)$-linear. Here $\Omega^1(A_N)$ is a $C^\infty(M)$-module by extension of scalars via the pullback map $\phi^* : C^\infty(N) \longrightarrow C^\infty(M)$. In other words, $\nu|_{\Omega^1(A_N)}$ can be seen as a $C^\infty(M)$-linear map $\nu|_{\Omega^1(A_N)} : \Gamma(\phi^* A^*_N) \longrightarrow C^\infty(M)$ and, by biduality, it corresponds to a section $\theta \in \Gamma(\phi^* A_N)$, which is precisely what equation (3.2) expresses for $k = 1$. This last argument also makes it clear that the assignment $\theta \longmapsto \imath^\Phi_\theta$ is injective. □

In fact, the degree -1 operator $\imath^\Phi_\theta : \Omega(A_N) \longrightarrow \Omega(A_M)$ defined in (3.2) can be obtained as a composition $\imath^\Phi_\theta = \Phi^* \circ \imath_\theta$ where $\imath_\theta$ denotes the interior product with θ, seen as the degree -1 map $\imath_\theta : \Omega(A_N) \longrightarrow \Omega(\phi^* A_N)$ and $\Phi^* : \Omega(\phi^* A_N) \longrightarrow \Omega(A_M)$ is the obvious map induced by Φ.

Later on, we will have to deal with computations involving both $\imath^\Phi_\theta = \Phi^* \circ \imath_\theta$ and Lie algebroid differentials. Since, the pullback bundle $\phi^* A_N$ does not come equipped with a Lie algebroid structure we will rather use the notation $\imath^\Phi_\theta$.

3.2 Lie Algebroid Maps and Time Dependence

Soon we will deal with various types of time-dependent geometric objects like sections and vector bundle maps. Below we clarify what is meant by smoothness of such objects.

Definition 3.2.1. A *smooth family of maps* $\phi_t : M \longrightarrow N$ parametrized by $t \in I := [0, 1]$ is a smooth map $\phi : M \times I \longrightarrow N$.

We define and use similar notions for smooth families of vector bundle maps, sections and so on. In particular, for forms, supported sections and derivations we shall use the following notations:

Definition 3.2.2. Given a vector bundle $A_M \longrightarrow M$, a *time-dependent k-form* on A_M is a section of the vector bundle $\bigwedge^k A_M^* \times I$ over $M \times I$.

We denote by

$$\Omega(A_M)^I := \Gamma(\bigwedge A_M^* \times I),$$

the space of time-dependent forms on A_M. We shall see any $\alpha \in \Omega(A_M)^I$ as a family of sections $\alpha_t \in \Omega(A_M)$ via the pullback

$$\alpha_t := (\imath_t^{A_M})^* \alpha$$

where $\alpha_t^{A_M} : A_M \longrightarrow A_M \times I$ denotes the obvious injection and $(\imath_t^{A_M})^*$ is the map induced on forms. Using these notations and the fact that $(\imath_t^{A_M})^* : \Omega(A_M)^I \longrightarrow \Omega(A_M)$ is a morphism of graded algebras, we have

$$(\alpha \wedge \beta)_t = \alpha_t \wedge \beta_t. \tag{3.3}$$

Definition 3.2.3. Given a smooth family of maps $\phi_t : M \longrightarrow N$ and a vector bundle A_N over N, a *smooth family θ_t of sections of A_N supported by ϕ_t* is, by definition, a section θ of the pullback $\theta \in \Gamma(\phi^* A_N)$ where ϕ is the corresponding smooth map $\phi : M \times I \longrightarrow N$.

For a time-dependent vector bundle map, the notion of time-dependent Φ^*-derivation is obtained as follows:

Definition 3.2.4. Consider $\Phi_t : A_M \longrightarrow A_N$ a time-dependent vector bundle map covering $\phi_t : M \longrightarrow N$ and denote by $\Phi^* : \Omega(A_N) \longrightarrow \Omega(A_M)^I$ the induced map on forms. A Φ^*-derivation of degree k is a degree k linear map

$$\nu : \Omega(A_N) \longrightarrow \Omega(A_M)^I$$

such that

$$\nu(\alpha \wedge \beta) = \nu(\alpha) \wedge \Phi^*(\beta) + (-1)^{k|\alpha|} \Phi^*(\alpha) \wedge \nu(\beta)$$

for every $\alpha, \beta \in \Omega(A_N)$ homogeneous.

Because of the equation (3.3), a Φ^*-derivation ν is the same as *smooth family* ν_t of Φ_t^*-derivations. Notice, however, that the terminology of a *smooth family* here is a bit misleading for both Φ^* and ν. Indeed, we view, for instance Φ^*, as a mere algebraic object, namely, a morphism of graded algebras $\Omega(A_N) \longrightarrow \Omega(A_M)^I$. There is no topology involved neither on $\Omega(A_N)$ nor on $\Omega(A_M)^I$. Rather, the algebraic properties of Φ^* are what ensure the smoothness of Φ. For instance, in degree 0, the only condition is to have a morphism of commutative algebras $\phi^* : C^\infty(N) \longrightarrow C^\infty(M \times I)$, then it is well known that such morphisms are always pullbacks of a *smooth* map $\phi : M \times I \longrightarrow N$ (see, for example [10]).

Similar considerations hold for ν. Let us spell out the correspondence $\nu \leftrightarrow \nu_t$ in order to avoid confusions. Given ν we obtain a smooth family ν_t setting $\nu_t := (\imath_t^{A_M})^* \circ \nu$. Reciprocally, given a family of applications $\nu_t : \Omega(A_N) \longrightarrow \Omega(A_M)$, we set:

$$\langle \nu(\alpha_N)_{(m,t)}, a_1, \ldots, a_k \rangle := \langle \nu_t(\alpha_N)_m, a_1, \ldots, a_k \rangle$$

for every $\alpha_N \in \Omega^k(A_N)$ and $a_1, \ldots, a_k \in (A_M \times I)_{(m,t)}$. Here, we are identifying $(A_M \times I)_{(m,t)}$ with $(A_M)_m$. Then, by definition, the family ν_t is *smooth* if ν *takes values* in $\Omega(A_M)^I$, the space of smooth sections of $\bigwedge A_M^* \times I \longrightarrow M \times I$.

Assume now that $A_M \longrightarrow M$ is a Lie algebroid. Then, there is on $A_M \times I$ an obvious structure of a Lie algebroid over $M \times I$ such that for every $t \in I$, the injections $\imath_t^{A_M} : A_M \hookrightarrow A_M \times I$ define Lie subalgebroids. The corresponding differential on $\Omega(A_M)^I$ is characterized by

$$(d_{A_M \times I}\alpha)_t = d_{A_M}\alpha_t, \tag{3.4}$$

for any time-dependent forms $\alpha, \beta \in \Omega(A_M)^I$ and $t \in I$.

Notice that $d_{A_M \times I}$ is $C^\infty(I)$-linear, and the obvious operator:

$$\frac{\partial}{\partial t} : \Omega(A_M)^I \longrightarrow \Omega(A_M)^I$$

is a degree 0 derivation of $(\Omega(A_M)^I, \wedge)$ that commutes with $d_{A_M \times I}$. More explicitly:

$$\Big\langle \frac{\partial}{\partial t}\alpha_t, a_t \Big\rangle := \frac{\partial}{\partial t}\Big|_{t=0} \Big\langle \alpha_t, a_t \Big\rangle - \Big\langle \alpha_t, \frac{\partial}{\partial t}\Big|_{t=0} \alpha_t \Big\rangle,$$

for every time-dependent one-form α_t and every time-dependent section a_t.

Likewise, time-dependent forms can be integrated with respect to the time variable, yielding a degree zero operator

$$\begin{aligned} \int_0^1 : \Omega(A_M)^I &\longrightarrow \Omega(A_M) \\ \alpha &\longmapsto \int_0^1 \alpha_t \; dt. \end{aligned}$$

The easiest way to define this is by duality as follows:

$$\Big\langle \int_0^1 \alpha_t \; dt, a_1, \ldots, a_n \Big\rangle := \int_0^1 \langle \alpha_t, a_1, \ldots, a_n \rangle \; dt,$$

for every $\alpha \in \Omega^n(A_M)$ and $a_i \in \Gamma(A_M)$, $i = 1, \ldots, n$. Here, on the right hand side, one just integrates a function on $M \times I$ with respect to the time variable. This makes it easy to check that such an integration commutes with the Lie algebroid differentials, namely, we have:

$$\int_0^1 (d_{A_M \times I}\alpha)_t \, dt = d_{A_M} \int_0^1 \alpha_t \, dt.$$

The proof of this fact for the case of tangent bundles can be found in [9].

Lemma 3.2.5. There is a one-to-one correspondence between smooth families of vector bundle maps $\Phi_t : A_M \longrightarrow A_N$ and morphisms of graded algebras

$$\Phi^* : (\Omega(A_N), \wedge) \longrightarrow (\Omega(A_M)^I, \wedge).$$

Furthermore, $\Phi_t : A_M \longrightarrow A_N$ defines a smooth family of Lie algebroid morphisms if and only if the induced application $\Phi^* : (\Omega(A_N), \wedge, d_{A_N}) \longrightarrow (\Omega(A_M)^I, \wedge, d_{A_M \times I})$ is a morphism of DGAs.

Proof. The first statement essentially follows from the definitions: a *smooth family* of vector bundle maps $\Phi_t : A_M \longrightarrow A_N$ is, by definition, a vector bundle map $\Phi : A_M \times I \longrightarrow A_N$, which is well-known [41] to be the same as a morphism of graded algebras $\Phi^* : \Omega(A_N) \longrightarrow \Omega(A_M)^I$. The second statement is a direct consequence of (3.3) and (3.4). □

The Proposition 3.1.3 has a time-dependent counterpart, whose proof is obtained simply replacing M with $M \times I$ and following the definitions:

Proposition 3.2.6. Let $\Phi_t : A_M \longrightarrow A_N$ be a smooth family of vector bundle maps covering $\phi_t : M \longrightarrow N$ and denote by $\Phi^* : \Omega(A_N) \longrightarrow \Omega(A_M)^I$ the induced map on forms.

There is a one-to-one correspondence between:

- degree -1 Φ^*-derivations;
- smooth families of time-dependent sections $\theta_t \in \Gamma(\phi_t^* A_N)$ supported by ϕ_t.

More precisely, to any section $\theta \in \Gamma(\phi^* A_N)$ supported by ϕ, one associates the Φ^*-derivation $\nu := \imath_\theta^\Phi$ defined by:

$$\langle \nu\beta, a_1, \ldots, a_{k-1} \rangle := \langle \beta \circ \phi, \theta, \Phi \circ a_1, \ldots, \Phi \circ a_{k-1} \rangle \in C^\infty(M \times I) \tag{3.5}$$

for every $\beta \in \Omega^k(A_N)$ and $a_1, \ldots, a_{k-1} \in \Gamma(A_M)$. Furthermore, any Φ^*-derivation is of this form.

3.3 Natural Transformations as Discrete Homotopies

Consider two abstract (small) groupoids $\mathcal{C}$ and $\mathcal{D}$, together with functors

$$F_0, F_1 : \mathcal{C} \longrightarrow \mathcal{D}$$

that shall be fixed for the remainder of this section. Since we shall largely discuss the notion of natural transformation, let us recall the basic definition.

Definition 3.3.1. A *natural transformation* $\eta : F_0 \Longrightarrow F_1$ is given by an assignment that associates to any object x in $\mathcal{C}$ an arrow η_x in $\mathcal{D}$, in such a way that for any arrow $a : x \longrightarrow y$ in $\mathcal{C}$, we have $F_1(a) \circ \eta_x = \eta_y \circ F_0(a)$ as illustrated below:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "{F_0, F_1}"] & & \mathcal{D} & \\
x \arrow[d, bend right, "a"'] & & F_0(x) \arrow[r, "\eta_x"] \arrow[d, bend right, "F_0(a)"'] & F_1(x) \arrow[d, bend right, "F_1(a)"'] \\
y & & F_0(y) \arrow[r, "\eta_y"'] & F_1(y)
\end{tikzcd}$$

We also introduce the following basic tool, that will be useful in order to understand natural transformations.

Definition 3.3.2. We call *discrete interval* the pair groupoid over $\{0, 1\}$ and denote it by $\mathbb{I}$.

More precisely, the discrete interval is the groupoid

$$\mathbb{I} := \{0 \rightleftarrows 1\}$$

with $\{0, 1\}$ as space of objects and whose only non-identical morphisms are given by a single arrow $\tau_{1,0} : 0 \longrightarrow 1$ together with its inverse $\tau_{0,1} : 1 \longrightarrow 0$.

In the theory of categories, the discrete interval plays a similar role as the usual interval $I := [0, 1]$ in basic homotopy theory as we now explain. Although, we are only interested in groupoids, the following discussion can be easily adapted to categories.

One may consider the product groupoid $\mathcal{C} \times \mathbb{I}$ and notice that it contains two copies of $\mathcal{C}$. More precisely, there are two obvious embeddings

$$\imath_0, \imath_1 : \mathcal{C} \hookrightarrow \mathcal{C} \times \mathbb{I},$$

which send an object x in $\mathcal{C}$ to $(x, 0)$, respectively, to $(x, 1)$ and a morphism a in $\mathcal{C}$ to $a \times \mathsf{id}_0$, respectively, to $a \times \mathsf{id}_1$, where id_0 and id_1 are the identities in $\mathbb{I}$.

Proposition 3.3.3. Given two functors $F_0, F_1 : \mathcal{C} \longrightarrow \mathcal{D}$, there is a one-to-one correspondence between:

- natural transformations $\eta : F_0 \Longrightarrow F_1$;
- functors $F : \mathcal{C} \times \mathbb{I} \longrightarrow \mathcal{D}$ such that $F_0 = F \circ \imath_0$ and $F_1 = F \circ \imath_1$.

Proof. Given $\eta : F_0 \longrightarrow F_1$, the restrictions $F|_{\imath_0(\mathcal{C})}$ and $F|_{\imath_1(\mathcal{C})}$ being imposed, all we need in order to define F is to specify the images of the morphisms $\mathsf{id}_x \times \tau : (x, 0) \longrightarrow (x, 1)$ and $\mathsf{id}_x \times \tau^{-1} : (x, 1) \longrightarrow (x, 0)$. This is easily done setting

$$F(\mathsf{id}_x \times \tau_{1,0}) = \eta_x$$
$$F(\mathsf{id}_x \times \tau_{0,1}) = \eta_x^{-1}.$$

The conditions for η to be a natural transformation imply that F is indeed a functor.

Conversely, given a functor $F : \mathcal{C} \times \mathbb{I} \longrightarrow \mathcal{D}$, the above formulas define a natural transformation $\eta : F_0 \Longrightarrow F_1$.

The functor F, and how it relates to η can be pictured as follows:

$$\mathcal{C} \times \mathbb{I} \xrightarrow{\quad F \quad} \mathcal{D}$$

$$\begin{tikzcd}
(x,0) \arrow[r, "\mathsf{id}_x \times \tau_{0,1}"] \arrow[d, bend right, "a \times \mathsf{id}_0"'] & (x,1) \arrow[d, bend right, "a \times \mathsf{id}_1"] & & F_0(x) \arrow[r, "\eta_x"] \arrow[d, bend right, "F_0(a)"'] & F_1(x) \arrow[d, bend right, "F_1(a)"] \\
(y,0) \arrow[r, "\mathsf{id}_y \times \tau_{0,1}"] & (y,1) & & F_0(y) \arrow[r, "\eta_y"] & F_1(y)
\end{tikzcd}$$

Notice that the diagram on the left hand side always commutes because of the multiplication on the product $\mathcal{C} \times \mathbb{I}$. By functoriality, the diagram of the right hand side is also commutative. This explains the commutativity condition in the definition 3.3.1 of a natural transformation. □

Although Definition (3.3.1) is the one usually appearing in textbooks, one may argue that the description obtained via proposition (3.3.3) is more natural from the categorical point of view, in the sense that it spells out the notion of natural transformation purely in terms of functors. It will turn out to be our starting point in order to understand natural transformations in the smooth setting.

The idea of seeing natural transformations as discrete homotopies between functors is not original, see, for instance, the comments in [16] and [39].

3.4 Smooth Natural Transformations and Lie Groupoid Morphisms

In this section, we consider two Lie groupoids $\mathcal{G}_M \rightrightarrows M$ and $\mathcal{G}_N \rightrightarrows N$, together with Lie groupoid morphisms $F_0, F_1 : \mathcal{G}_M \rightrightarrows \mathcal{G}_N$. We denote by $\phi_0 : M \longrightarrow N$ and $\phi_1 : M \longrightarrow N$ the corresponding smooth maps induced on the bases.

There is no subtlety in the definition of a natural transformation between Lie groupoid morphisms. We just need to adapt the definition to the differential geometric setting.

Definition 3.4.1. A *(smooth) natural transformation* $\eta : F_0 \Longrightarrow F_1$ is a smooth map $\eta : M \longrightarrow \mathcal{G}_N$ such that $F_1(a) \circ \eta_x = \eta_y \circ F_0(a)$ for any $a \in \mathcal{G}_M$.

One may notice that the discrete interval $\mathbb{I} \rightrightarrows \{0, 1\}$ endowed with the discrete topology has a natural Lie groupoid structure. Furthermore, proposition (3.3.3) immediately generalizes to Lie groupoids as follows.

Proposition 3.4.2. Given two Lie groupoid morphisms $F_0, F_1 : \mathcal{G}_M \longrightarrow \mathcal{G}_N$, there is a one-to-one correspondence between:

- Natural transformations $\eta : F_0 \Longrightarrow F_1$;
- Lie groupoid morphisms $F : \mathcal{G}_M \times \mathbb{I} \longrightarrow \mathcal{G}_N$ such that $F_0 = F \circ \imath_0$ and $F_1 = F \circ \imath_1$.

The above proposition makes it clear that there is no useful infinitesimal counterpart to a natural transformation. Let us spell this out: in the theory of Lie groupoids, the *infinitesimal counterpart* is obtained by applying the Lie functor (see section 2.2). By doing so to a Lie groupoid morphism of the form $F : \mathcal{G}_M \times \mathbb{I} \longrightarrow \mathcal{G}_N$ as in proposition (3.4.2), we obtain a Lie algebroid morphism

$$\mathsf{Lie}(F) : \mathsf{Lie}(\mathcal{G}_M \times \mathbb{I}) \longrightarrow \mathsf{Lie}(\mathcal{G}_N).$$

However, it is easily seen that $\mathsf{Lie}(\mathcal{G}_M \times \mathbb{I})$ can be canonically identified with the disjoint union of two copies of $A_M := \mathsf{Lie}(\mathcal{G}_M)$, namely, we have:

$$\mathsf{Lie}(\mathcal{G}_M \times \mathbb{I}) = A_M \coprod A_M.$$

This is due to the fact that the space of arrows in $\mathbb{I}$ comes with a discrete topology. As a consequence, $\mathcal{G}_M \times \mathbb{I}$ cannot have connected source-fibers, even if $\mathcal{G}_M$ does. Hence, when we apply the Lie functor to F, all information about the natural transformation η is lost. In other words, from this point of view there is no infinitesimal counterpart to a single natural transformation.

3.5 Natural Homotopies

The discussion of the previous section seems to make hopeless the existence of an infinitesimal object that would integrate to a natural transformation. However, as suggested by proposition 3.3.3, the discrete interval $\mathbb{I}$ is a mere algebraic model for homotopies between functors. Yet, in the context of Lie groupoids, there is another candidate to play this role, namely the pair groupoid $\mathbf{I} := (I \times I \rightrightarrows I)$ over the interval $I := [0,1]$. We shall denote by $\tau_{t_1,t_0} : t_0 \longrightarrow t_1$ the arrows in $\mathbf{I}$.

Notice that $\mathbf{I}$ comes equipped with a natural Lie groupoid structure. Furthermore, given a Lie groupoid $\mathcal{G}_M \rightrightarrows M$, we now have a whole family of embeddings parametrized by $t \in I$, as follows:

$$\imath_t : \mathcal{G}_M \hookrightarrow \mathcal{G}_M \times \mathbf{I}.$$

Notice there is also an obvious embedding $\mathbb{I} \hookrightarrow \mathbf{I}$, so that we can identify $\imath_t$ for $t = 0, 1$ with the embeddings $\mathcal{G}_M \times \mathbb{I}$ defined in the previous section.

With this smooth model of an interval at hand, the proposition (3.3.3) suggests that we adapt our approach as follows:

Definition 3.5.1. Given two Lie groupoid morphisms $F_0, F_1 : \mathcal{G}_M \longrightarrow \mathcal{G}_N$, a *natural homotopy* $F : F_0 \Longrightarrow F_1$ is a Lie groupoid morphism

$$F : \mathcal{G}_M \times \mathbf{I} \longrightarrow \mathcal{G}_N$$

such that $F_0 = F \circ \imath_0$ and $F_1 = F \circ \imath_1$.

Associated to a natural homotopy, rather than a single natural transformation, we now recover a whole smooth family of them.

Proposition 3.5.2. Given two Lie groupoid morphisms $F_0, F_1 : \mathcal{G}_M \longrightarrow \mathcal{G}_N$, there is one-to-one correspondence between:

- natural homotopies $F : \mathcal{G}_M \times \mathbf{I} \longrightarrow \mathcal{G}_N$ such that $F_0 = F \circ \imath_0$ and $F_1 = F \circ \imath_1$;
- smooth families of natural transformations $\eta_t : F_0 \longrightarrow F_t$.

Proof. The proof is analogue to that of proposition (3.3.3). Given $F : \mathcal{G}_M \times \mathbf{I} \longrightarrow \mathcal{G}_N$ we define a family of functors $F_t := F \circ \imath_t : \mathcal{G}_M \longrightarrow \mathcal{G}_N$ and an application $\eta : M \times \mathbf{I} \longrightarrow \mathcal{G}_N$ as follows

$$\eta_{(x,t)} := F(\mathsf{id}_x \times \tau_{t,0}).$$

As illustrated below, we have:

$$\mathcal{G}_M \times \mathbf{I} \xrightarrow{\quad F \quad} \mathcal{G}_N$$

$$\begin{tikzcd}
(x,0) \arrow[r, "\mathsf{id}_x \times \tau_{0,t}"] \arrow[d, bend right, "a \times \mathsf{id}_0"'] & (x,t) \arrow[d, bend right, "a \times \mathsf{id}_t"] & F_0(x) \arrow[r, "\eta^t_x"] \arrow[d, bend right, "F_0(a)"'] & F_t(x) \arrow[d, bend right, "F_t(a)"] \\
(y,0) \arrow[r, "\mathsf{id}_x \times \tau_{0,t}"] & (y,t) & F_0(y) \arrow[r, "\eta^t_y"] & F_t(y)
\end{tikzcd}$$

The diagram on the left hand side commutes as a consequence of the definition of the multiplication in the product $\mathcal{G}_M \times \mathbf{I}$; therefore, the one on the right hand side commutes by functoriality of F. It follows that $\eta_t : F_0 \Longrightarrow F_t$ is a natural transformation for all $t \in I$. Furthermore, η is clearly smooth as a map $\eta : M \times \mathbf{I} \longrightarrow \mathcal{G}_N$.

Reciprocally, given a smooth family of natural transformations $\eta_t : F_0 \longrightarrow F_t$ we define F as follows:

$$F(\mathsf{id}_x \times \tau_{t_2,t_1}) := \eta_{(x,t_2)} \circ \eta^{-1}_{(x,t_1)}$$

for every $x \in M$. It is then straightforward to check that F defines a Lie groupoid morphism $\mathcal{G}_M \times \mathbf{I} \longrightarrow \mathcal{G}_N$ as a consequence of η_t being a natural transformation for all $t \in I$. □

3.6 Infinitesimal Counterpart of a Natural Homotopy

The Lie algebroid of the Lie groupoid $\mathbf{I} \rightrightarrows I$ is given by the tangent bundle:

$$\mathsf{Lie}(\mathbf{I}) := (TI \longrightarrow I).$$

Having proposition (3.5.2) in mind, there is an obvious infinitesimal counterpart for a natural homotopy, obtained by applying the **Lie** functor to definition (3.5.1) as follows:

Definition 3.6.1. Given two Lie algebroid morphisms $\Phi_0, \Phi_1 : A_M \longrightarrow A_N$, a *natural homotopy* $\Psi : \Phi_0 \Longrightarrow \Phi_1$ is a Lie algebroid morphism

$$\Psi : A_M \times TI \longrightarrow A_N$$

such that $\Phi_0 = \Psi \circ \imath_0$ and $\Phi_1 = \Psi \circ \imath_1$, where $\imath_0, \imath_1 : A_M \hookrightarrow A_M \times TI$ denote the obvious injections.

Example 3.6.2. An A-path $a \cdot dt : TI \longrightarrow A$ between $x := p_A(a(0))$ and $y = p_A(a(1))$ can be seen as a natural homotopy between the Lie algebroid morphisms $0_x : 0_* \longrightarrow A$ and $0_y : 0_* \longrightarrow A$. Here, 0_* denotes the trivial Lie algebroid over a point, and 0_x, 0_y are the obvious Lie algebroid maps associated with x and y.

Below, we shall look at natural homotopies between Lie algebroid morphisms in more detail. Part of the discussion is based on [11], though our approach is slightly different, avoiding graphs.

Along the text, we shall abuse notations, in particular ∂t may denote indistinctly the obvious section of either the vector bundle $TI \longrightarrow I$ or $A_M \times TI \longrightarrow M \times I$.

Similarly, we shall not distinguish $dt \in \Omega^1(TI)$ from its pullback along the projection $A_M \times TI \longrightarrow TI$.

Lemma 3.6.3. Any vector bundle map $\Psi : A_M \times TI \longrightarrow A_N$ can be written uniquely in the form

$$\Psi := \Phi + \theta \cdot dt \tag{3.6}$$

where $\Phi_t : A_M \longrightarrow A_N$ is a smooth family of vector bundle maps covering $\phi_t : M \longrightarrow N$ and $\theta_t \in \Gamma(\phi_t^* A_N)$ a smooth family of sections of A_N supported by ϕ_t. Here $\phi : M \times I \longrightarrow N$ is the map underlying Ψ.

Reciprocally, any pair (Φ_t, θ_t) where $\Phi_t : A_M \longrightarrow A_N$ is a vector bundle map and $\theta_t \in \Gamma(\phi_t^* A_N)$ is a smooth family of sections supported by the base map $\phi_t : M \longrightarrow N$ of Φ_t, induces by (3.6) a vector bundle map $\Psi : A_M \times TI \longrightarrow A_N$.

Proof. The vector bundle $A_M \times TI \longrightarrow M \times I$ is obtained as the Whitney sum of vector bundles over $M \times I$ as follows:

$$A_M \times TI := (A_M \times I) \oplus (M \times TI). \tag{3.7}$$

Furthermore, a vector bundle map $A_M \times TI \longrightarrow A_N$ covering a smooth map $\phi : M \times I \longrightarrow N$ is the same as a vector bundle map $A_M \times TI \longrightarrow \phi^* A_N$ covering the identity of $M \times I$. Decomposing Ψ into each factor in (3.7) the result easily follows. It suffices to take $\phi := \Psi|_{A_M \times I}$ and $\theta := \Psi \circ \partial t$ so that $\Psi|_{M \times TI} = \theta \cdot dt$. The converse statement should be obvious now. □

As pointed out in Proposition 1.3.14, Lie algebroid morphisms can be described in terms of morphisms of DGAs. The aim of the discussion below is to use this approach to translate the conditions for $\Psi = \Phi + \theta \cdot dt$ to define a morphism of Lie algebroids into conditions on Φ and θ.

Let us start by fixing a smooth family of vector bundle maps $\Phi_t : A_M \longrightarrow A_N$ as in section 3.2. Recall that we denote by $\phi_t : M \longrightarrow N$ the underlying family of smooth maps and by $\Phi^* : \Omega(A_N) \longrightarrow \Omega(A_M)^I$ the morphisms of graded algebras induced on forms. Also, recall that:

$$\Omega(A_N)^I := \Gamma(\bigwedge A_N^* \times I),$$

where $A_N \times I$ is seen as a vector bundle over $N \times I$.

Proposition 3.6.4. Let $\Phi_t : A_M \longrightarrow A_N$ be a smooth family of vector bundle maps covering $\phi_t : M \longrightarrow N$ and suppose $\theta_t \in \Gamma(\phi_t^* A_N)$ is a smooth family of sections of A_N supported by ϕ_t.

Then $\Psi = \Phi + \theta \cdot dt : A_M \times TI \longrightarrow A_N$ defines a Lie algebroid morphism if and only if the following conditions hold for any $t \in I$:

i) $\Phi_t : A_M \longrightarrow A_N$ is a Lie algebroid morphism;

ii) The Φ_t^*-derivation $\imath_{\theta_t}^{\Phi_t}$ as in proposition 3.1.3 satisfies

$$\frac{\partial}{\partial t}\Phi_t^* = d_{A_M} \circ \imath_{\theta_t}^{\Phi_t} + \imath_{\theta_t}^{\Phi_t} \circ d_{A_N}.$$

Proof. Any k-form $\eta \in \Omega^k(A_M \times TI)$ decomposes uniquely as $\eta = \alpha + \beta \wedge dt$ where $\alpha \in \Omega^k(A_M)^I$ and $\beta \in \Omega^{k-1}(A_M)^I$. In other words, there is a canonical isomorphism

$$\Omega^k(A_M \times TI) = \Omega^k(A_M)^I \oplus \Omega^{k-1}(A_M)^I \wedge dt. \tag{3.8}$$

Using this decomposition, one checks that both $d_{A_M\times TI}$ and Ψ^* can be decomposed in the following way:

$$d_{A_M\times TI} = d_{A_M\times I} + (-1)^{|\cdot|}\frac{\partial}{\partial t}(\cdot) \wedge dt \tag{3.9}$$

$$\Psi^* = \Phi^* - (-1)^{|\cdot|}\imath_\theta^\Phi \wedge dt. \tag{3.10}$$

More precisely, for any $\alpha \in \Omega^k(A_M)^I$ and $\beta \in \Omega^{k-1}(A_M)^I$ we have:

$$d_{A_M\times TI}(\alpha + \beta \wedge dt) = d_{A_M\times I}(\alpha) + \left((-1)^k\frac{\partial}{\partial t}\alpha + d_{A_M\times I}\beta\right) \wedge dt,$$

and for any $\alpha_N \in \Omega^l(A_N)$ we have:

$$\Psi^*(\alpha_N) = \Phi^*(\alpha_N) - (-1)^l \imath_\theta^\Phi(\alpha_N) \wedge dt.$$

Here, $d_{A_M\times I}$ is the obvious Lie algebroid differential on $A_M \times I$. Furthermore, the operator $\imath_\theta^\Phi$ is given by the obvious time-dependent version (3.5) of the formula describing $\imath_\theta^\Phi$ as a Φ^*-derivation.

Substituting the equations (3.9) and (3.10) in the condition $d_{A_M\times TI}\circ\Psi^* = \Psi^*\circ d_{A_N}$ for Ψ to define a Lie algebroid morphism and then taking into account the uniqueness of the decomposition of a form in (3.8), we deduce that Ψ is a Lie algebroid morphism if and only if:

$$d_{A_M\times I} \circ \Phi^* = \Phi \circ d_{A_N}$$

$$\frac{\partial}{\partial t}\Phi^* = d_{A_M\times I} \circ \imath_\theta^\Phi + \imath_\theta^\Phi \circ d_{A_N}.$$

One concludes by noticing that both equations hold if and only if they hold timewise. For the first equation, this follows more precisely from lemma (3.2.5) and (3.4). Similarly, for the second equation one can see $\nu := \imath_\theta^\Phi$ as a smooth family $\nu_t : \Omega(A_N) \longrightarrow \Omega(A_M)$ of Φ_t^*-derivations of degree k to which, by Proposition (3.5), we associate a family $\theta_t \in \Gamma(\phi_t^* A_M)$

supported by ϕ_t. $\square$

In the terminology of [11], Proposition 3.6.4 says that a natural homotopy is a homotopy of Lie algebroid morphisms by *gauge transformations*. Note, however, that the precise relation between gauge transformations and smooth natural transformations is not established in [11], neither its time-dependent version made explicit.

The theorem below, which essentially rephrases Proposition 3.6.4, summarizes the discussion of this section:

Theorem 3.6.5. Let $\Phi_i : A_M \longrightarrow A_N$ be two Lie algebroid morphisms ($i = 0, 1$) and denote by $\Phi_0^*, \Phi_1^* : (\Omega(A_N), \wedge, d_{A_N}) \longrightarrow (\Omega(A_M), \wedge, d_{A_M})$ the corresponding morphisms of DGAs.

Then Φ_0 and Φ_1 are naturally homotopic if and only if there exists:

- a morphism of DGAs $\Phi^* : (\Omega(A_N), \wedge, d_{A_N}) \longrightarrow (\Omega(A_M)^I, \wedge, d_{A_M \times I})$ extending Φ_0^* and Φ_1^*;
- a Φ^*-derivation $\nu : \Omega(A_N) \longrightarrow \Omega(A_M)^I$ of degree -1 such that:

$$\frac{\partial}{\partial t}\Phi_t^* = d_{A_M \times I} \circ \nu + \nu \circ d_{A_N}.$$

3.7 Integration of Natural Homotopies

We now have enough tools in hand in order to prove the main result of this part, which is to provide an integration procedure in order to obtain smooth natural transformations.

Theorem 3.7.1. Let A_M and A_N be Lie algebroids over smooth manifolds M and N, respectively, together with a smooth family $\Phi_t : A_M \longrightarrow A_N$ of Lie algebroid morphisms parametrized by $t \in [0, 1]$.

We denote by $\phi_t : M \longrightarrow N$ the underlying family of smooth maps and by

$$\Phi_t^* : \Omega(A_N) \longrightarrow \Omega(A_M)$$

the smooth family of morphisms of DGAs induced by Φ_t.

Assume that there exists a smooth family $\theta_t \in \Gamma(\phi_t^* A_N)$ of sections of A_N supported by ϕ_t that satisfies the following condition

$$\frac{\partial}{\partial t}\Phi_t^* = d_{A_M} \circ \imath_{\theta_t}^{\Phi_t} + \imath_{\theta_t}^{\Phi_t} \circ d_{A_N} \tag{3.11}$$

where $\imath_{\theta_t}^{\Phi_t} : \Omega(A_N) \longrightarrow \Omega(A_M)$ is the family of degree -1 Φ_t^*-derivations defined as

$$\langle \imath_{\theta_t}^{\Phi_t} \alpha_N, v_1, \ldots, v_{p-1} \rangle := \langle \alpha_N \circ \phi_t, \theta_t(m) \wedge \Phi_t(v_1), \ldots, \Phi_t(v_{p-1}) \rangle$$

for any $v_1, \ldots, v_{p-1} \in (A_M)_m$.

Then the following assertions hold:

i) The morphisms Φ_0^* and Φ_1^* are cochain homotopic;

ii) Whenever A_M and A_N are integrable Lie algebroids, the assignment:

$$M \ni m \longmapsto \eta_m := [\theta(m)\ dt]_{A_N} \in \mathcal{G}(A_N) \tag{3.12}$$

defines a smooth natural transformation $\eta : F_0 \Longrightarrow F_1$, where

$$F_0, F_1 : \mathcal{G}(A_M) \longrightarrow \mathcal{G}(A_N)$$

denote the Lie groupoid morphisms integrating Φ_0 and Φ_1, respectively.

Here, we denoted by $[\theta(m)\ dt]_{A_N}$ the A_N-homotopy class of the A_N-path $\theta(m)\ dt : TI \longrightarrow A_N$.

Proof. The proof of the first assertion is similar to the homotopy invariance of smooth maps in the De Rham cohomology (see, for instance [30]). More precisely, given $\alpha \in \Omega(A_N)$, we check using the condition (3.11) that

$$\begin{aligned}
\Phi_1^*\alpha - \Phi_0^*\alpha &= \int_0^1 \frac{\partial}{\partial t}\Phi_t^*(\alpha)\ dt \\
&= d_{A_M}\int_0^1 \imath_{\theta_t}^{\Phi_t}(\alpha)\ dt - \int_0^1 \imath_{\theta_t}^{\Phi_t}(d_{A_N}\alpha)\ dt
\end{aligned}$$

so that the degree -1 operator $\Theta : \Omega(A_N) \longrightarrow \Omega(A_M)$ given by

$$\Theta(\alpha) := \int_0^1 \imath_{\theta_t}^{\Phi_t}(\alpha)\ dt$$

defines indeed a cochain homotopy between Φ_0^* and Φ_1^*. Here we used the fact that integration with respect to time parameter commutes with the Lie algebroid differentials.

In order to prove the second assertion, we use lemma 3.6.3 according to which $\Psi := \Phi + \theta \cdot dt$ defines a Lie algebroid morphism $\Psi : A_M \times TI \longrightarrow A_N$ such that $\Phi_0 = \Psi \circ \imath_0$ and $\Phi_1 = \Psi \circ \imath_1$. Then, applying the integration functor to Ψ, we obtain a Lie groupoid morphism

$$F : \mathcal{G}(A_M) \times \mathbf{I} \longrightarrow \mathcal{G}(A_N)$$

such that $F_0 = F \circ \imath_0$ and $F_1 = F \circ \imath_1$. Therefore, by proposition 3.5.2, there exist natural transformations $\eta_t : F_0 \Longrightarrow F_t$, in particular for $t = 1$.

Finally, the explicit formula for the natural transformation η is a consequence of the formula for η given in the proof of proposition 3.5.2. Namely, the element $\mathsf{id}_x \times \tau_{1,0}$, as a homotopy class of $A_M \times TI$-path, is represented by the following path:

$$\begin{aligned}(0^A_x \times \partial t)\; dt : TI &\longrightarrow A_M \times TI\\ \partial t &\longmapsto (0^{A_M}_x, \partial t).\end{aligned}$$

By the definition of the integration functor, the image of $\mathsf{id}_x \times \tau_{1,0}$ by F is given by the homotopy class of the A_N-path $\Psi \circ (0^{A_M}_x \times \partial t)\; dt$. Then using the formula $\Psi = \Phi + \theta \cdot dt$, we successively obtain:

$$\begin{aligned}\eta_x &= F(\mathsf{id}_x \times \tau_{1,0})\\ &= F([(0^{A_M}_x \times \partial t)\; dt]_{A_M})\\ &= [\Psi \circ (0^{A_M}_x \times \partial t)\; dt]_{A_N}\\ &= [\theta(x, -)\; dt]_{A_N},\end{aligned}$$

which completes the proof. $\square$

The usual notion of homotopy between smooth maps is of course recovered from our results as follows:

Proposition 3.7.2. Let M and N be smooth manifolds and $\phi_i : M \longrightarrow N$ $(i = 0, 1)$ be smooth maps. Then smooth homotopies $\psi : M \times I \longrightarrow N$ are the same as natural homotopies $\Psi : TM \times TI \longrightarrow TN$.

Proof. This follows from the well known fact that a vector bundle map $\Psi : TM \times TI \longrightarrow TN$ covering $\phi : M \times I \longrightarrow N$ is a Lie algebroid morphism if and only if $\Psi = d\psi$. $\square$

In that case, our main theorem reads:

Proposition 3.7.3. Given smooth manifolds M, N and a smooth homotopy $\phi : M \times I \longrightarrow N$ between $\phi_0 : M \longrightarrow N$ and $\phi_1 : M \longrightarrow N$, the map $\eta : M \longrightarrow \pi(N)$ given by

$$\eta_m := [\gamma^\phi_m],$$

where $[\gamma^\phi_m]$ denote the homotopy class of the path $\phi^\phi_m : I \longrightarrow N$, $t \longmapsto \phi_t(m)$, defines a smooth natural transformation:

$$\begin{tikzcd} \Pi(M) \arrow[r, bend left, "\Phi_0"] \arrow[r, bend right, "\Phi_1"'] \arrow[r, phantom, "\eta\Uparrow"] & \Pi(N) \end{tikzcd}$$

Here $\pi(M) \rightrightarrows M$ and $\pi(N) \rightrightarrows N$ denote the fundamental groupoids of M and N, respectively, and $\Phi_0, \Phi_1 : \pi(M) \longrightarrow \pi(N)$ are the Lie groupoid morphisms induced by ϕ_0 and ϕ_1, respectively.

Proof. Given a smooth homotopy $\phi : M \times I \longrightarrow N$ between $\phi_0 : M \longrightarrow N$ and $\phi_1 : M \longrightarrow N$ we obtain a smooth family of Lie algebroid morphisms $\Phi_t := d\phi_t : TM \longrightarrow TN$ that integrate to a family of Lie groupoid morphisms:

$$F_t : \pi(M) \longrightarrow \pi(N).$$

The differential $\Psi := d\phi : TM \times TI \longrightarrow TN$ defines a natural homotopy between Φ_0 and Φ_1. Furthermore, Φ is clearly of the form

$$\Phi_{x,t} := d\phi_t + \frac{\partial \phi}{\partial t}\, dt.$$

In other words, in that case, we have:

$$\theta_m := \frac{\partial \phi}{\partial t}.$$

The result follows from our main theorem 3.7.1. □

One may wonder what happens if we replace the fundamental groupoids $\pi(M)$ and $\pi(N)$ by the pair groupoids $M \times M \rightrightarrows M$ and $N \times N \rightrightarrows N$ that also integrate TM and TN, respectively. However, by doing so, the proposition 3.7.3 becomes void. Indeed, for any Lie groupoid $\mathcal{G} \rightrightarrows M$ and any couple of Lie groupoid morphisms $F_0, F_1 : \mathcal{G} \longrightarrow N \times N$, there exists a unique natural isomorphism $\eta : F_0 \Longrightarrow F_1$ (it is given by $\eta_x := (F_0(x), F_1(x)) \in N \times N$, for any $x \in M$).

3.8 Deformation Retracts of Lie Algebroids

As an application of our theory of natural homotopies we discuss deformation retracts of Lie algebroids. In the sequel we consider a Lie algebroid $A_M \longrightarrow M$ together with a *Lie subalgebroid* $A_R \longrightarrow R$. Recall that this means we have an injective Lie algebroid morphism

$$\imath_{A_R} : A_R \hookrightarrow A_M,$$

such that the corresponding base map $\imath_R : R \hookrightarrow M$ is an embedding.

Definition 3.8.1. Given a Lie subalgebroid $\imath_{A_R} : A_R \hookrightarrow A_M$, a *(weak) deformation retraction* from A_M to A_R is a natural homotopy between $\Phi_0 := \mathsf{id}_{A_M}$ and a retraction from A_M to A_R.

More precisely, a deformation retraction is given by a Lie algebroid morphism $\Psi = \Phi + \theta \cdot dt : A_M \times TI \longrightarrow A_M$ such that:

i) for $t = 0$ we have $\Phi_0 = \mathsf{id}_{A_M}$;

ii) for $t = 1$ the map $\Phi_1 : A_M \longrightarrow A_M$ factors through a Lie algebroid morphism

$\check{\Phi}_1 : A_M \longrightarrow A_R$ which is a retraction, namely:

$$\begin{aligned} \imath_{A_R} \circ \check{\Phi}_1 &= \Phi_1 \\ \check{\Phi}_1 \circ \imath_{A_R} &= \mathsf{id}_{A_R}, \end{aligned}$$

as illustrated below:

$$\begin{tikzcd} A_M \arrow[r, "\Phi_1"] \arrow[dr, "\check{\Phi}_1"] & A_M \\ A_R \arrow[u, hook, "\imath_{A_R}"] \arrow[r, "\mathsf{id}_{A_R}"] & A_R \arrow[u, hook, "\imath_{A_R}"'] \end{tikzcd}$$

We say A_R is a *deformation retract* of A_M if there exists a deformation retraction from A_M to A_R.

Proposition 3.8.2. A Lie algebra $\mathfrak{g}$ admits no deformation retract except from itself. More precisely, if $\mathfrak{g}_R \subset \mathfrak{g}$ is a deformation retract of $\mathfrak{g}$, then $\mathfrak{g}$ and $\mathfrak{g}_R$ are isomorphic as Lie algebras.

Proof. If $A_M = \mathfrak{g}$ is a lie algebra then Φ_1 is necessarily invertible. $\square$

We conclude this section with an interesting theorem which discuss deformation retracts of integrable Lie algebroids.

Theorem 3.8.3. Let A_M be an integrable Lie algebroid and $\Psi = \Phi + \theta \cdot dt : A_M \times TI \longrightarrow A_M$ be a deformation retraction of A_M to A_R. Then the following assertions hold:

i) The Lie algebroid A_R is integrable;

ii) $\mathcal{G}(A_M)$ identifies, as a groupoid with the pullback groupoid $\phi_1^! \mathcal{G}(A_T)$ where ϕ_1 denotes the restriction of the base map of Ψ to $M \times \{1\}$, seen as a map $\phi_1 : M \longrightarrow R$.

iii) If, furthermore, the base map $\check{\phi}_1 : M \longrightarrow R$ is transversal to the characteristic foliation of A_R, that is, if:

$$\mathsf{Im}((d\check{\phi}_1)_m) + \mathsf{Im}(\sharp_{A_R})_{\check{\phi}_1(m)} = T_{\check{\phi}_1(m)} R$$

for any $m \in M$, then the identification of item ii) is a Lie groupoid isomorphism.

Proof. Item i) is a well known result from Moerdijk and Mrcun [37], stating that any Lie subalgebroid of an integrable Lie algebroid is integrable. In order to prove ii), one first checks that, for any Lie algebroid morphism $\check{\Phi}_1 : A_M \longrightarrow A_R$ covering $\check{\phi}_1 : M \longrightarrow R$, the map

$$\begin{aligned} K : \mathcal{G}(A_M) &\longrightarrow \check{\phi}_1^{\,!} \mathcal{G}(A_R) \\ [a_M\ dt] &\longmapsto (t([a_M\ dt]), \check{F}_1([a_M\ dt]), s([a_M\ dt])), \end{aligned}$$

always defines a (topological) groupoid morphism. Here $[a_M\ dt]$ denotes the A_M-homotopy class of any A_M-path $a_M\ dt : TI \longrightarrow A_M$ and $\check{F}_1 : \mathcal{G}(A_M) \longrightarrow \mathcal{G}(A_R)$ is the Lie groupoid morphism integrating $\check{\Phi}_1 : A_M \longrightarrow A_R$.

Whenever there is a deformation retraction $\Psi = \Phi + \theta \cdot dt$ as in definition (3.8.1), then K admits the following map as an inverse:

$$\begin{aligned} L : \check{\phi}_1^{\,!}\mathcal{G}(A_R) &\longrightarrow \mathcal{G}(A_M) \\ (y, [a_R\ dt], x) &\longmapsto \eta_y \cdot [\imath_{A_R}(a_R)\ dt] \cdot (\eta_x)^{-1}. \end{aligned}$$

Here, $\eta_m := [\theta(m)\ dt] \in \mathcal{G}(A_M)$ is the natural transformation as in (3.12).

The transversality condition in *iii*) ensures that both the pullback groupoid $\check{\phi}_1^{\,!}\mathcal{G}(A_N)$ is a Lie groupoid, then the isomorphisms K and L above are automatically smooth (see, for instance, [25] for pullbacks of Lie groupoids and of Lie algebroids). $\square$

Part III - Infinitesimal Actions up to Homotopy

Chapter 4

Extensions of Lie Brackets

In this chapter we deal with extensions of Lie algebroids. These are fibrations within the category of Lie algebroids. By a fibration we mean a (not-necessarily locally trivial) surjective submersion between manifolds. The concept of extension is general enough to encompass surjective submersions, extensions of Lie algebras, transitive Lie algebroids, actions of Lie algebroids and VB-algebroids. We pay particular attention to Ehresmann connections for they play a fundamental role in the theory. Most of the exposition is based upon [12]. Our main contribution in this chapter is the section concerning morphisms between extensions of Lie algebroids.

4.1 Extensions of Lie Algebroids

In this section we deal with Lie algebroid extensions which is one of the main building block of our work. We make sure to present several examples that should make it clear why Lie algebroid extensions are worth to be studied.

Definition 4.1.1. An *extension* of a Lie algebroid A_B is a surjective Lie algebroid morphism $\pi : A_E \longrightarrow A_B$ which covers a surjective submersion $\pi_0 : E \longrightarrow B$. A_E will be called the *total space*, A_B the *base* and $\mathcal{K} := \mathsf{Ker}(\pi)$ the *kernel*.

Under the previous conditions, the kernel $\mathcal{K}$ is a sub-vector bundle of A_E. Furthermore, as proved in [12], $\mathcal{K}$ is a Lie subalgebroid of A_E by means of the inclusion map $\jmath : \mathcal{K} \longrightarrow E$. Thus, we obtain a short exact sequence of Lie algebroids over different bases:

$$\mathcal{K} \xhookrightarrow{\jmath} A_E \xrightarrow{\pi} A_B \ .$$

We will also refer to A_E as an *extension of* A_B *by* $\mathcal{K}$. However, it must be emphasized that the injection $\mathcal{K} \subset A_E$ does not determine π nor π_0 so one should not think of A_B as a quotient of A_E by $\mathcal{K}$.

We will denote by $\sharp_E : A_E \longrightarrow TE$, $\sharp_B : A_B \longrightarrow TB$, $\sharp_{\mathcal{K}} : \mathcal{K} \longrightarrow TE$ and $[\cdot,\cdot]_{A_E}$, $[\cdot,\cdot]_{A_B}$ and $[\cdot,\cdot]_{\mathcal{K}}$ the respective anchors and brackets.

It is straightforward that the characteristic foliation of $\mathcal{K}$ lies on $\mathsf{Vert} := \mathsf{Ker}(dp)$, that is, $\sharp_{\mathcal{K}}(\mathcal{K}) \subset \mathsf{Vert}$. Therefore, the restriction

$$\mathcal{K}|_{E_x} = \{(k, p_{\mathcal{K}}(k)) \in \mathcal{K} \times E : \pi_0(p_{\mathcal{K}}(k)) = x\},$$

of the vector bundle $\mathcal{K}$ to the fiber $E_x := p^{-1}(\{x\})$ over $x \in B$ is a Lie algebroid over E_x. Therefore, $\mathcal{K}$ can be thought of as a family of Lie algebroids parametrized by points of B.

In the literature, extensions are also called *fibrations*, see for instance [33]. For an anternative definition of extension we refer the reader to [33].

Afterwards, we will illustrate Definition 4.1.1 with a wide range of examples. Most of the following examples were taken from [12] and additional ones can be found therein.

Example 4.1.2 (Lie Algebroids)**.** Any Lie algebroid A_E, by means of the identity morphism, defines an extension whose total space coincides with the base. The kernel of this extension is the zero vector bundle 0_E over E.

Example 4.1.3 (Lie Algebras)**.** Let $A_E := \mathfrak{e}$ and $A_B := \mathfrak{b}$ be two Lie algebras. An extension with total space $\mathfrak{e}$ and base $\mathfrak{k}$ is, simply, a surjective Lie algebra morphism $\pi : \mathfrak{e} \longrightarrow \mathfrak{b}$, that is, an extension of Lie algebras. The kernel $\mathfrak{k}$ of this extension is a Lie ideal of $\mathfrak{e}$.

Example 4.1.4 (Submersions)**.** A wide range of examples can be obtained considering surjective submersions between manifolds. In fact, any surjective submersion $\pi_0 : E \longrightarrow B$ defines an extension of TB by means of its differential $d\pi_0 : TE \longrightarrow TB$. The kernel is the vertical subbundle $\mathsf{VE} := \mathsf{Ker}(d\pi_0)$ of TE.

Example 4.1.5 (Infinitesimal Actions)**.** Consider an infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold M and let $\mathfrak{g} \ltimes M$ be the corresponding action algebroid. The projection onto the first component realizes $A_E := \mathfrak{g} \ltimes M$ as an extension of $A_B := \mathfrak{g}$ whose kernel is $\mathcal{K} = \{0\} \times M$. In fact, we can define an infinitesimal action as an extension of a Lie algebra with trivial kernel. More generally, suppose the Lie algebroid A_B acts on a surjective submersion $\pi_0 : E \longrightarrow B$ and denote by $A_E := A_B \ltimes E$ the associated action algebroid. As in the previous example, the projection onto the first component defines an extension of A_B whose kernel is $\mathcal{K} = E \times \{0\}$. Conversely, an extension of an algebroid A_B with vanishing kernel $\mathcal{K}$ is naturally associated with an action of A_B on E.

Example 4.1.6 (Pullback Lie Algebroids)**.** The pullback of a Lie algebroid A_B by a submersion $\pi_0 : E \longrightarrow B$ was introduced in [25] as

$$p^! A_B := \{(X, \alpha) \in TE \times A_B : d\pi_0(X) = \sharp_B(\alpha)\}.$$

This is a Lie algebroid over TE, and the projection onto the second factor defines an extension of A_B whose kernel is $\mathcal{K} = \mathsf{Vert}$.

Example 4.1.7 (Bundle of Lie Algebras)**.** Any extension $\pi : A_E \longrightarrow A'_E$, where A_E and A'_E are Lie algebroids over the same base, induces a vanishing anchor on $\mathcal{K}$, so the kernel is a bundle of Lie algebras (see [33] for the notion of bundle of Lie algebras).

Example 4.1.8 (Fibered Lie Algebroids)**.** An extension of $A_B = \{0\} \times B$ necessarily coincides with the kernel $\mathcal{K}$ which stands as a fibered Lie algebroid over $\pi_0 : E \longrightarrow B$ (see [12] for the notion of fibered Lie algebroid).

Example 4.1.9 (Transitive Lie Algebroids)**.** A transitive Lie algebroid $A \longrightarrow B$ fits into an extension of $A_B := TB$. The projection π is given by the anchor $\pi = \sharp_A$. Then $\mathcal{K} = \mathsf{Ker}(\sharp_A)$ is a bundle of Lie algebras. A regular Lie algebroid $A \longrightarrow B$ with foliation $F \subset TB$ can be treated similarly as an extension of F by a bundle of Lie algebras.

Example 4.1.10 (Extensions of TI)**.** Let I be the unit interval $[0, 1]$ and A_B be a Lie algebroid. Endow TI with the Lie algebroid structure given in example 1.3.4. The product Lie algebroid $A_E := A_B \times TI$ defines an extension of TI by means of the projection onto the second factor. The kernel is $\mathcal{K} = A_B \times I$. This extension is related to natural homotopies as we shall explain later on.

Example 4.1.11 (VB-Algebroids)**.** Any VB-algebroid $A_E \longrightarrow E$ over $A_B \longrightarrow B$ defines an extension of A_B. By [13], the core $C \longrightarrow B$ identifies naturally with $E \oplus C \longrightarrow B$. It is instructive to think of a VB-algebroid as a linear version of an extension of Lie algebroids.

Next we present a rather elementary proposition whose content, although elementary, has not yet been made explicit in the literature. Besides, it will be useful in a later section of this work.

Proposition 4.1.12. Let A_B be a Lie algebroid, A_E be a vector bundle and $\pi : A_E \longrightarrow A_B$ be a surjective morphism of vector bundles covering a surjective submersion. There is a one to one correspondence between Lie algebroid structures on A_E which turn $\pi : A_E \longrightarrow A_B$ into an extension and DGA-structures on the graded algebra $\Omega(A_E)$ which satisfy the derivation rule:

$$d(\alpha \cdot \varepsilon) = d_B\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot d\varepsilon, \tag{4.1}$$

for every $\alpha \in \Omega(A_B)$ and $\varepsilon \in \Omega(A_E)$ homogeneous. Here, $\cdot$ denotes the graded left-module structure over $\Omega(B)$ induced by the morphism of graded algebras $\pi^* : \Omega(B) \longrightarrow \Omega(E)$.

Proof. First, since π is a morphism of vector bundles $\pi^* : \Omega(A_B) \longrightarrow \Omega(A_E)$ is a morphism of graded algebras. A Lie algebroid structure on A_E which turn $\pi : A_E \longrightarrow A_B$ into an extension is equivalent to a DGA-structure d on the graded algebra $\Omega(A_E)$ such that:

$$d \circ \pi^* = d_E \circ \pi^*. \tag{4.2}$$

If this holds, then for any $\alpha \in \Omega(A_B)$ and $\varepsilon \in \Omega(A_E)$ homogeneous we find

$$\begin{aligned} d(\alpha \cdot \varepsilon) &= d(\pi^*\alpha \wedge \varepsilon) \\ &= d(\pi^*\alpha) \wedge \varepsilon + (-1)^{|\pi^*(\alpha)|}\pi^*\alpha \wedge d\varepsilon \\ &= \pi^*(d_B\alpha) \wedge \varepsilon + (-1)^{|\alpha|}\pi^*\alpha \cdot d\varepsilon \\ &= d_B\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot d\varepsilon. \end{aligned}$$

On the other hand, given d and applying (4.1) with $\varepsilon = 1_{\mathbb{R}}$ we get:

$$d(\pi^*\alpha) = d(\alpha \cdot 1_{\mathbb{R}}) = d_B\alpha \cdot 1_{\mathbb{R}} = \pi^*(d_B\alpha),$$

forall $\alpha \in \Omega(A_B)$ homogeneous. So, d defines a Lie algebroid structure on A_E with respect to which π becomes a morphism of Lie algebroids and therefore an extension. $\square$

Briefly put, the above proposition tells us that once A_E and π are fixed, π defines an extension of A_B if and only if $\Omega(A_E)$ has a DGA-structure which also turns it into a DG-module over $\Omega(A_B)$ with respect to the module structure induced by π^*.

Next, we define the category of extensions of a given algebroid.

Definition 4.1.13. Let $\pi : A_E \longrightarrow A_B$ and $\pi' : A_{E'} \longrightarrow A_{B'}$ be extensions covering π_0 and π_0', respectively. A *morphism of extensions* from π to π' consists of two morphisms of Lie algebroids $\Phi_E : A_E \longrightarrow A_{E'}$ covering ϕ_E and $\Phi_B : A_B \longrightarrow A_{B'}$ covering ϕ_B such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[rr, "\phi_E"] \arrow[dd, "\pi_0"'] & & E' \arrow[dd, "\pi_0'"'] & \\
& A_E \arrow[ul, "p_{A_E}"] \arrow[rr, "\Phi_E"] \arrow[dd, "\pi"'] & & A_{E'} \arrow[ul, "p_{A_{E'}}"'] \arrow[dd, "\pi'"] \\
B \arrow[rr, "\phi_B"] & & B' & \\
& A_B \arrow[ul, "p_{A_B}"] \arrow[rr, "\Phi_B"] & & A_B' \arrow[ul, "p_{A_{B'}}"']
\end{tikzcd}$$

We shall make some remarks:

- The condition $\pi' \circ \Phi_E = \Phi_B \circ \pi$ implies $\Phi_E(\mathcal{K}) \subseteq \mathcal{K}'$ and, since $\mathcal{K}$ and $\mathcal{K}'$ are Lie subalgebroids, we have an induced morphism of Lie algebroids $\Phi_E|_{\mathcal{K}} : \mathcal{K} \longrightarrow \mathcal{K}'$ covering ϕ_E;
- We shall deal mainly with the case where $A_B = A_{B'}$ and we shall denote the corresponding category of extensions of A_B by $\mathsf{Ext}(A_B)$.

In Proposition 4.1.12 we have shown that extensions can be be parametrized by certain DGA-structures. Next we put morphisms of extensions into that context:

Proposition 4.1.14. Let A_B be a Lie algebroid, A_E and $A_{E'}$ be two vector bundles and let $\pi : A_E \longrightarrow A_B$ and $\pi' : A_{E'} \longrightarrow A_B$ be two surjective morphisms of vector bundles covering surjective submersions.

Consider A_E and $A_{E'}$ endowed with the unique Lie algebroid structures which turn π and π' into extensions of Lie algebroids, as in Proposition 4.1.12. There is a 1-1 correspondence between morphisms of extensions $\Phi : A_E \longrightarrow A_{E'}$ and morphisms of DGAs $\Phi^* : \Omega(A_{E'}) \longrightarrow \Omega(A_E)$ which are $\Omega(A_B)$-linear, that is:

$$\Phi(\alpha \cdot \varepsilon) = \alpha \cdot \Phi(\varepsilon)$$

for every $\alpha \in \Omega(A_B)$ and $\varepsilon \in \Omega(A_{E'})$.

Proof. From Proposition 1.3.14 we already know that morphisms of Lie algebroids $\Phi : A_E \longrightarrow A_{E'}$ are in 1-1 correspondence with morphisms of DGAs $\Phi^* : \Omega(A_{E'}) \longrightarrow \Omega(A_E)$. We contend Φ is a morphism of extensions if and only if Φ^* is $\Omega(A_B)$-linear. First, notice:

$$\pi' \circ \Phi = \pi \Leftrightarrow \Phi^* \circ (\pi')^* = \pi^*.$$

If $\pi' \circ \Phi = \pi$ then for every $\alpha \in \Omega(A_B)$ and $\varepsilon \in \Omega(A_{E'})$ we find:

$$\begin{aligned}
\Phi^*(\alpha \cdot \varepsilon) &= \Phi^*((\pi')^*\alpha \wedge \varepsilon) \\
&= \Phi^*((\pi')^*\alpha) \wedge \Phi^*(\varepsilon) \\
&= \pi^*\alpha \wedge \Phi^*(\varepsilon) \\
&= \alpha \cdot \Phi(\varepsilon).
\end{aligned}$$

On the other hand, supposing Φ^* is $\Omega(A_B)$-linear then:

$$\begin{aligned}
\Phi^*((\pi')^*\alpha) &= \Phi^*(\alpha \cdot 1_{\mathbb{R}}) \\
&= \alpha \cdot \Phi^*(1_{\mathbb{R}}) \\
&= \alpha \cdot 1_{\mathbb{R}} \\
&= (\pi^*\alpha) \cdot 1_{\mathbb{R}} \\
&= \pi^*(\alpha),
\end{aligned}$$

for every $\alpha \in \Omega(A_B)$. □

We shall study morphism of extensions thoroughly by means of Ehresmann connections in a later section.

4.2 Ehresmann Connections

In this section we introduce a notion of connection [20] for extensions of Lie algebroids. These connections correspond to the linear splittings found in the theory of

VB-algebroids. As we shall see, the choice of an Ehresmann connection sheds some light on the structural components of extensions and of their morphisms. We follow a different but equivalent approach to that presented on [12].

We start noticing that to any Lie algebroid extension $\mathcal{K} \hookrightarrow A_E \overset{\pi}{\twoheadrightarrow} A_B$ we can associate a short exact sequence of vector bundles:

$$\mathcal{K} \overset{\jmath}{\hookrightarrow} A_E \xrightarrow{(\pi, p_{A_E})} p^* A_B \ , \tag{4.3}$$

where p_{A_E} is the projection of A_E. We will refer to this sequence as the *Ehresmann sequence* of the extension.

Definition 4.2.1. An *Ehresmann connection* on a Lie algebroid extension $A_E \xrightarrow{\pi} A_B$ covering $\pi_0 : E \longrightarrow B$ is a section of the corresponding Ehresmann sequence, that is, a strong morphism of vector bundles $\sigma : \pi_0^* A_B \longrightarrow A_E$ such that $(\pi, p_{A_E}) \circ \pi = \mathsf{id}_{\pi_0^* A_B}$.

The condition $(\pi, p_{A_E}) \circ \sigma = \mathsf{id}_{\pi_0^* A_B}$ is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} E & \pi_0^* A_B \arrow[l, "\mathrm{pr}_E"'] \arrow[r, "\mathrm{pr}_{A_B}"] \arrow[d, "\sigma"] & A_B \\ & A_E \arrow[ul, "p_{A_E}"] \arrow[ur, "\pi"'] & \end{tikzcd} \tag{4.4}$$

where pr_E and pr_{A_B} stand for the projections from $\pi_0^* A_B$ on E and B, respectively.

In general, the Ehresmann sequence does not split in the category of Lie algebroids. However, using a partition of unit argument, it is possible to show it always splits in the category of vector bundles:

Proposition 4.2.2. There exist Ehresmann connections for any Lie algebroid extension.

Ehresmann connections can come in several flavours as we show below.

Proposition 4.2.3. Let $\mathcal{K} \hookrightarrow A_E \xrightarrow{\pi} A_B$ be a Lie algebroid extension covering $\pi_0 : E \longrightarrow B$. The following are equivalent:

(i) Ehresmann connections $\sigma : \pi_0^* A_B \longrightarrow A_E$ on $\mathcal{K} \hookrightarrow A_E \xrightarrow{\pi} A_B$;

(ii) Smooth subbundles $\mathcal{H} \subseteq A_E$ such that $A_E = \mathcal{H} \oplus \mathcal{K}$. We call such $\mathcal{H}$ a *horizontal subbundle of* A_E;

(iii) Strong vector bundle isomorphisms $\Sigma : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow A_E$ such that the following diagram commutes:

$$\begin{tikzcd} \pi_0^* A_B \oplus \mathcal{K} \arrow[rr, "\Sigma"] \arrow[dr, "\mathrm{pr}_{A_B}"'] & & A_E \arrow[dl, "\pi"] \\ & A_B & \end{tikzcd} \tag{4.5}$$

We call such Σ a *decomposition of* A_E;

(iv) $C^\infty(N)$-linear maps $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$ such that the diagram commutes:

$$\begin{tikzcd} E \arrow[r, "h(a)"] \arrow[d, "\pi_0"'] & A_E \arrow[d, "\pi"] \\ B \arrow[r, "a"'] & A_B \end{tikzcd} \tag{4.6}$$

for every section $a \in \Gamma(A_B)$. We call such h a *horizontal lift.*

Proof. *(i)* $\Leftrightarrow$ *(ii)* Given σ we take $\mathcal{H} := \mathsf{Im}(\sigma)$. On the other hand, given $\mathcal{H}$ such that $A_E = \mathcal{H} \oplus \mathcal{K}$ then $(\pi, p_{A_E})|_{\mathcal{H}} : \mathcal{H} \longrightarrow \pi_0^* A_B$ is a strong vector bundle isomorphism. We take

$$\sigma := \jmath \circ (\pi, p_{A_E})|_{\mathcal{H}}^{-1} : \pi_0^* A_B \longrightarrow A_E,$$

where $\jmath$ is the inclusion map of $\mathcal{H}$ on A_E. Then:

$$\begin{aligned} (\pi, p_{A_E}) \circ \sigma &= (\pi, p_{A_E}) \circ \jmath \circ (\pi, p_{A_E})|_{\mathcal{H}}^{-1} \\ &= (\pi, p_{A_E})|_{\mathcal{H}} \circ (\pi, p_{A_E})|_{\mathcal{H}}^{-1} \\ &= \mathsf{id}_{\pi_0^* A_B}. \end{aligned}$$

(i) $\Leftrightarrow$ *(iii)* Given σ, define:

$$\begin{aligned} \Sigma : \pi_0^* A_B \oplus \mathcal{K} &\longrightarrow A_E \\ (a, e) \oplus k &\longmapsto \sigma(a, e) +_{A_E} k. \end{aligned}$$

This is a strong isomorphism of vector bundles with inverse:

$$\Sigma^{-1}(a) := (\pi(a), p_{A_E}(a)) \oplus (a -_{A_E} \sigma(\pi(a), p_{A_E}(a)).$$

That $a -_{A_E} \sigma(\pi(a), p_{A_E}(a))$ belongs to $\mathcal{K}$ is a consequence of (4.4). The commutativity of (4.5) follows from (4.4). On the other hand, given Σ we set, of course, $\sigma := \Sigma|_{\pi_0^* A_B}$. Then $\pi \circ \sigma = \mathrm{pr}_{A_B}$ and since Σ is a morphism of vector bundles it follows $p_{A_E} \circ \sigma = \mathrm{pr}_E$. Therefore, $(\pi, p_{A_E}) \circ \sigma = \mathsf{id}_{\pi_0^* A_B}$.

(i) $\Leftrightarrow$ *(iv)* Given σ, we define:

$$\begin{aligned} h : \Gamma(A_B) &\longrightarrow \Gamma(A_E) \\ a &\longmapsto \sigma \circ \pi_0^*(a), \end{aligned}$$

where $\pi_0^*(a) = (a \circ \pi_0, \mathsf{id}_E)$ is the induced section on $\pi_0^* A_B$. This map is $C^\infty(B)$-linear because the following diagram commutes:

$$\begin{tikzcd}
\Gamma(A_B) \arrow[rr, "h"] \arrow[d, "1_{\mathbb{R}}\otimes\mathsf{id}"'] & & \Gamma(A_E) \\
C^\infty(E) \otimes_{C^\infty(B)} \Gamma(A_B) \arrow[rr, "\simeq"'] & & \Gamma(p^* A_B) \arrow[u, "\sigma_*"']
\end{tikzcd}$$

where the bottom line stands for the isomorphism $f \otimes a \longmapsto f\pi_0^*(a)$ and where σ_* is the map induced at the level of sections by σ. Furthermore, by (4.4) we find:

$$\begin{aligned}
\pi \circ h(a) &= \pi \circ \sigma \circ (a \circ \pi_0, \mathsf{id}_E) \\
&= \mathrm{pr}_{A_B} \circ (a \circ \pi_0, \mathsf{id}_E) \\
&= a \circ \pi_0,
\end{aligned}$$

for every $a \in \Gamma(A_B)$. On the other hand, given h we define $\sigma : \pi_0^* A_B \longrightarrow A_E$ at the level of sections by:

$$\sigma(\pi_0^* a) := h(a),$$

for every $a \in \Gamma(A_B)$ and we extend it by $C^\infty(E)$-linearity. This is in fact an Ehresmann connection because:

$$\pi \circ \sigma(a \circ \pi_0, \mathsf{id}_E) = \pi \circ h(a) = a \circ \pi_0,$$

for every $a \in \Gamma(A_B)$, and therefore $\pi \circ \sigma = \mathrm{pr}_{A_B}$. Also, since $h(a)$ is a section of A_E we get:

$$\mathrm{pr}_{A_E} \circ \sigma(a \circ \pi_0, \mathsf{id}_E) = \mathrm{pr}_{A_E} \circ h(a) = \mathsf{id}_E,$$

for every $a \in \Gamma(A_B)$, and therefore $p_{A_E} \circ \sigma = \mathrm{pr}_E$.

$\square$

There are several interesting points concerning Ehresmann connections we would like to emphasize:

Remarks 4.2.4.

(i) The choice of an Ehresmann connection is not canonical;

(ii) Given a horizontal lift $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$, we readily see the corresponding decomposition $\Sigma^h : \pi_0^* B \oplus \mathcal{K} \longrightarrow A_E$ is given at the level of sections by:

$$\Sigma^h(\pi_0^* a \oplus k) = h(a) + k.$$

In particular, as noticed in [12], any section of A_E can be written as a finite sum whose terms have the form $f \cdot h(a) + k$ with $a \in \Gamma(A_B)$ and $f \in C^\infty(E)$. As usual, there is a unique Lie algebroid structure on $\pi_0^* A_B \oplus \mathcal{K}$ which turns Σ^h into a Lie algebroid isomorphism. We will come back to this in the next section.

(iii) The property $\pi \circ h(a) = a \circ \pi_0$ means the section $h(a)$ is π-projectable on a. Indeed, $h(a)$ can be characterized as the only section of the corresponding horizontal subbundle with this property.

(iv) In general, a horizontal lift $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$ is not a morphism of Lie algebras. In fact, this is the case if and only if the associated horizontal subbundle is a Lie subalgebroid of A_E.

(v) Morphisms of extensions, generally, do not preserve horizontal subbundles.

(vi) We can also define an Ehresmann connection as a retraction of the Ehresmann sequence, that is, a strong vector bundle morphism $\Theta : A_E \longrightarrow \mathcal{K}$ such that $\Theta \circ \jmath = \mathsf{id}_{\mathcal{K}}$. This can be seen as a form on A_E with values on $\mathcal{K}$ which restricts to the identity on $\mathcal{K}$. We however do not use this approach in the text.

Analogously to the theory of VB-algebroids, in which linear splittings provide a nice description of the structure of such objects, Ehresmann connections play that role in the context of extensions as we shall see in the next section.

4.3 The Structure of an Extension

Let $\mathcal{K} \hookrightarrow A_E \xrightarrow{\pi} A_E$ be an extension. In the previous section, we noticed that the choice of a horizontal lift $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$ induces a Lie algebroid structure on $\pi_0^* A_B \oplus \mathcal{K}$ such that the associated decomposition

$$\Sigma^h : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow A_E$$

is an isomorphism of Lie algebroids. In order to better understand the induced Lie algebroid structure on $\pi_0^* A_B \oplus \mathcal{K}$ and hence that of A_E we start proving the following:

Lemma 4.3.1. Let $\mathcal{K} \hookrightarrow A_E \xrightarrow{\pi} A_E$ be an extension of Lie algebroids covering $\pi_0 : E \longrightarrow B$ and consider a horizontal lift $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$. Then:

(a) There is a well defined $C^\infty(B)$-linear map $\nabla^h : \Gamma(A_B) \longrightarrow \mathsf{der}(\mathcal{K})$ given by:

$$\nabla^h_a k := [h(a), k]_{A_E}.$$

for every $a \in \Gamma(A_B)$ and $k \in \Gamma(\mathcal{K})$. Furthermore, ∇^h_a has symbol $X^{\nabla^h_a} = \sharp_E h(a)$ which is $d\pi_0$-projectable on $\sharp_B a$ or, equivalently,

$$\mathcal{L}_{\sharp_E h(a)}(\pi_0^* f) = \pi_0^* \mathcal{L}_{\sharp_B a}(f),$$

for every $a \in \Gamma(A_B)$ and $f \in C^\infty(B)$.

(b) The map $\omega^h : \Gamma(A_B) \times \Gamma(A_B) \longrightarrow \Gamma(\mathcal{K})$ defined by:

$$\omega^h(a,b) := [h(a), h(b)]_{A_E} - h([a,b]_{A_B}),$$

for every $a, b \in \Gamma(A_B)$, is a well defined $C^\infty(B)$-bilinear map, that is, $\omega^h \in \Omega^2(B, \mathcal{K})$.

(c) The maps ∇^h e ω^h are compatible in the following sense:

$$\begin{aligned}\nabla^h_{[a,b]_{A_B}} - [\nabla^h_a, \nabla^h_b]_{\mathsf{der}(\mathcal{K})} &= [\omega^h(a,b), -]_{\mathcal{K}} \\ \oint_{a,b,c} \nabla^h_a \omega^h(b,c) + \omega^h(a, [b,c]_{A_B})) &= 0,\end{aligned}$$

where $\oint_{a,b,c}$ denotes the cyclic sum over $a, b, c \in \Gamma(A_B)$.

Proof. (a) Since $\pi \circ h(a) = a \circ \pi_0$ and $\pi \circ k = 0 \circ \pi_0$, we have:

$$\pi(\nabla^h_a k) = \pi([h(a), k]_{A_E}) = [a, 0]_{A_B} \circ \pi_0 = 0 \circ \pi_0,$$

hence ∇_a sends $\Gamma(\mathcal{K})$ into itself. The $C^\infty(B)$-linearity can be checked as follows:

$$\begin{aligned}\nabla^h_{f_B \cdot a} k &= [h(f_B), k]_{A_E} \\ &= [\pi_0^*(f_B) \cdot h(a), k] \\ &= \pi_0^*(f_B)[h(a), k]_{A_E} - \mathcal{L}_{\sharp_E k}(\pi_0^*(f_B)) \cdot h(a) \\ &= \pi_0^*(f_B) \nabla^h_a k,\end{aligned}$$

where we used $\sharp_{\mathcal{K}}(\mathcal{K}) \subseteq \mathsf{Ker}(d\pi_0)$ to get the last equality. Now, the equality

$$\nabla^h_a(f \cdot k) = f \cdot \nabla^h_a k + \mathcal{L}_{\sharp_E a}(f) \cdot k,$$

tells us $X^{\nabla^h_a} = \sharp_E h(a)$. This is $d\pi_0$-projectable because:

$$d\pi_0 \circ X^{\nabla^h_a} = d\pi_0 \circ \sharp_E h(a) = \sharp_B a \circ \pi_0.$$

Since $\sharp_{\mathcal{K}} = \sharp_E|_{\mathcal{K}}$ we find:

$$\sharp^{\mathcal{K}} \nabla^h_a k = \sharp_E [h(a), k]_{A_E} = [\sharp_E h(a), \sharp_E k]_{A_E} = [X^{\nabla^h_a}, \sharp_{\mathcal{K}} k]_{TE}.$$

Finally, from the Jacobi identity, ∇^h_a is derivation of the bracket of $\Gamma(\mathcal{K})$.

(b) Using that π is a morphism of Lie algebroids and using projectability we find:

$$\begin{aligned}
\pi \circ \omega^h(a,b) &= \pi \circ [h(a),h(b)]_{A_E} - \pi \circ h([a,b]_{A_B}) \\
&= [a,b]_{A_B} \circ \pi_0 - [a,b]_{A_B} \circ \pi_0 \\
&= 0 \circ \pi_0.
\end{aligned}$$

and therefore ω^h is well defined. The skew-symmetry of ω^h is straightforward. Let us show ω^h is $C^\infty(B)$-bilinear. In fact, using that h is $C^\infty(B)$-linear we find:

$$\begin{aligned}
\omega^h(a,fb) &= h([a,fb]_{A_B}) - [h(a),h(fb)]_{A_E} \\
&= h(f[a,b]_{A_B} + \mathcal{L}_{\sharp_B a}(f)b) - [h(a),fh(b)]_{A_E} \\
&= \pi_0^* fh([a,b]_{A_B}) + \pi_0^*\mathcal{L}_{\sharp_B a}(f)h(b) - \pi_0^* f[h(a),h(b)]_E \\
&- \mathcal{L}_{\sharp_E h(a)}(\pi_0^* f)h(b) \\
&= f\omega^h(a,b) + \pi_0^*\mathcal{L}_{\sharp_B a}(f)h(b) - \mathcal{L}_{\sharp_E h(a)}(\pi_0^* f)h(b) \\
&= f\omega^h(a,b) + \pi_0^*\mathcal{L}_{\sharp_B a}(f)h(b) - \pi_0^*\mathcal{L}_{\sharp_B a}(f)h(b) \\
&= f\omega^h(a,b).
\end{aligned}$$

(c) If fact, given $a,b \in \Gamma(A_B)$ e $\kappa \in \Gamma(\mathcal{K})$ we have:

$$\begin{aligned}
(\nabla^h_{[a,b]_{A_B}} - [\nabla^h_a,\nabla^h_b]_{\mathsf{der}(\mathcal{K})})(k) &= [h([a,b]_{A_B}),k]_{A_E} - [h(a),[h(b),k]_{A_E}]_{A_E} \\
&\quad +[h(b),[h(a),k]_{A_E}]_{A_E} \\
&= [h([a,b]_{A_B}),k]_{A_E} - [h(a),[h(b),k]_{A_E}]_{A_E} \\
&\quad -[h(b),[k,h(a)]_{A_E}]_{A_E} \\
&= [h([a,b]_{A_B}),k]_{A_E} + [k,[h(a),h(b)]_{A_E}]_{A_E} \\
&= [h([a,b]_{A_B}),k]_{A_E} - [[h(a),h(b)]_{A_E},k]_{A_E} \\
&= [h([a,b]_{A_B}) - [h(a),h(b)]_{A_E},k]_{A_E} \\
&= [\omega^h(a,b),k]_{A_E}.
\end{aligned}$$

Finally, notice that for any $a,b,c \in \Gamma(A_B)$:

$$\begin{aligned}
\nabla^h_a\omega^h(b,c) + \omega^h(a,[b,c]_{A_B}) &= [h(a),\omega^h(b,c)]_{A_E} + h([a,[b,c]_{A_B}]_{A_B}) \\
&- [h(a),h([b,c]_{A_B})]_{A_E} \\
&= [h(a),h([b,c]_{A_B})]_{A_E} - [h(a),[h(b),h(c)]_{A_E}]_{A_E} + h([a,[b,c]_{A_B}]_{A_B}) \\
&- [h(a),h([b,c]_{A_B})]_{A_E} \\
&= h([a,[b,c]_{A_B}]_{A_B}) - [h(a),[h(b),h(c)]_{A_E}]_{A_E}.
\end{aligned}$$

Taking the cyclic sum, using the linearity of h and the Jacobi property of $[\cdot,\cdot]_{A_E}$ and

$[\cdot,\cdot]_{A_B}$ we find:

$$\oint \nabla^h_a \omega^h(b,c) + \omega^h(a,[b,c]_{A_B}) = 0.$$

$\square$

Using this we prove:

Theorem 4.3.2. Let $\mathcal{K} \hookrightarrow A_E \xrightarrow{\pi} A_B$ be an extension of Lie algebroids covering $\pi_0 : E \longrightarrow B$ and consider a horizontal lift $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$. Then:

(a) The unique Lie algebroid structure on $\pi_0^* A_B \oplus \mathcal{K}$ which turns Σ^h into an isomorphism of Lie algebroids is determined by:

$$\begin{aligned}
[k_1, k_2] &= [k_1, k_2]_{\mathcal{K}} \\
[\pi_0^* a, k] &= \nabla^h_a k \\
[\pi_0^* a, \pi_0^* b] &= \pi_0^*([a,b]_{A_B}) \oplus \omega(a,b) \\
\sharp \pi_0^*(a) &= X^{\nabla^h_a} \\
\sharp k &= \sharp_{\mathcal{K}} k.
\end{aligned}$$

In particular, the Lie algebroid structure on A_E is determined by the conditions:

$$\begin{aligned}
[k_1, k_2]_{A_E} &= [k_1, k_2]_{\mathcal{K}} \\
[h(a), k]_{A_E} &= \nabla^h_a k \\
[h(a), h(b)]_{A_E} &= h([a,b]_{A_B}) + \omega^h(a,b) \\
\sharp^{A_E} h(a) &= X^{\nabla^h_a} \\
\sharp^{A_E} h(a) &= \sharp_{\mathcal{K}} k.
\end{aligned}$$

(b) Endowing $\pi_0^* A_B \oplus \mathcal{K}$ with the previous Lie algebroid structure, the projection onto the first factor $\mathrm{pr}_B : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow A_B$ becomes an extension of A_B by $\mathcal{K}$ which comes equipped with a canonical horizontal lift $h' : \Gamma(A_B) \longrightarrow \Gamma(\pi_0^* A_B \oplus \mathcal{K})$ such that the following diagram commutes:

$$\begin{tikzcd}
\Gamma(\pi_0^* A_B \oplus \mathcal{K}) \arrow[rr, "\Sigma^h"] & & \Gamma(A_E) \\
& \Gamma(A_B) \arrow[ul, "h'"] \arrow[ur, "h"'] &
\end{tikzcd}$$

(c) Σ^h is an isomorphism of extensions.

Proof.

(a) In fact, for every $a_1, a_2 \in \Gamma(A_B)$ and $k_1, k_2 \in \Gamma(\mathcal{K})$ we find:

$$\begin{aligned}
[\pi_0^* a_1 \oplus k_1, \pi_0^* a_2 \oplus k_2] &= (\Sigma^h)^{-1}([\Sigma^h(\pi_0^* a_1 \oplus k_1), \Sigma^h(\pi_0^* b \oplus k_2)]) \\
&= (\Sigma^h)^{-1}([h(a_1) + k_1, h(a_2) + k_2]) \\
&= (\Sigma^h)^{-1}([h(a_1), h(a_2)] + [h(a_1), k_2] - [h(a_2), k_1)] + [k_1, k_2]) \\
&= (\Sigma^h)^{-1}(h([a_1, a_2]_B) + \omega^h(a_1, a_2) + \nabla^h_{a_1} k_2 - \nabla^h_{a_2} k_1 + [k_1, k_2]) \\
&= \pi_0^*([a_1, a_2]) \oplus \omega^h(a_1, a_2) + \nabla^h_{a_1} k_2 - \nabla^h_{a_2} \kappa_1 + [k_1, k_2]
\end{aligned}$$

whereas the anchor is:

$$\sharp(\pi_0^* a \oplus k) = \sharp_E \Sigma^h(\pi_0^* a \oplus k) = \sharp_E(h(a) + k),$$

for every $a \in \Gamma(A_B)$ and $k \in \Gamma(\mathcal{K})$.

(b) We already know Σ^h is an isomorphism of Lie algebroids. By proposition 4.2.3, $\mathrm{pr}_{A_B} = \pi \circ \Sigma$ so that pr_{A_B} is a composition of Lie algebroid morphisms and therefore is itself a Lie algebroid morphism. The horizontal lift h' is simply $h'(a) = \pi_0^*(a)$. Thus:

$$\Sigma^h h'(a) = \Sigma^h(\pi_0^*(a)) = h(a),$$

as we had already noticed.

(c) In fact, Σ^h is an isomorphism of Lie algebroids which satisfies $\mathrm{pr}_{A_B} = \pi \circ \Sigma^h$.

□

Notice that by the first part of theorem 4.3.2 the Lie algebroid structure is encoded in that of A_B, $\mathcal{K}$ and in the pair (∇^h, ω^h). We can get rid of the splitting h now:

Proposition 4.3.3. Let A_B and $\mathcal{K}$ be two Lie algebroids and $\pi_0 : E \longrightarrow B$ be a surjective submersion. Suppose $\sharp_{\mathcal{K}}(\mathcal{K}) \subseteq \mathsf{Ker}(d\pi_0)$. Given any pair (∇, ω) consisting of a $C^\infty(B)$-linear map $\nabla : \Gamma(A_B) \longrightarrow \mathsf{der}(\mathcal{K})$ and a $C^\infty(B)$-bilinear map $\nabla : \Gamma(A_B) \times \Gamma(A_B) \longrightarrow \Gamma(\mathcal{K})$ such that:

$$\begin{aligned}
d\pi_0 \circ X^{\nabla_a} &= \sharp^{A_B} a \circ \pi_0 \\
\nabla_{[a,b]} - [\nabla_a, \nabla_b]_{\mathsf{der}(\mathcal{K})} &= [\omega(a,b), -]_{\mathcal{K}} \\
\oint \nabla_a \omega(b,c) + \omega([a,b],c)) &= 0,
\end{aligned}$$

for every $a, b, c \in \Gamma(A_B)$, then $\pi_0^* A_B \oplus \mathcal{K}$ is a Lie algebroid with:

$$\begin{array}{rcl}
[k_1, k_2] & = & [k_1, k_2]_{\mathcal{K}} \\
[\pi_0^* a, k] & = & \nabla_a k \\
[\pi_0^* a, \pi_0^* b] & = & \pi_0^*([a, b]_{A_B}) \oplus \omega(a, b) \\
\sharp \pi_0^*(a) & = & X^{\nabla_a} \\
\sharp k & = & \sharp_{\mathcal{K}} k.
\end{array}$$

defined for every $a, b \in \Gamma(A_B)$ and $k, k_1, k_2 \in \Gamma(\mathcal{K})$. Furthermore, $\mathrm{pr}_{A_B} : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow A_B$ is an extension whose kernel is $\mathcal{K}$ and the inclusion $\Gamma(\pi_0^* A_B) \longrightarrow \Gamma(\pi_0^* A_B \oplus \mathcal{K})$ is a horizontal lift which induces the pair (∇, ω).

Proof. Immediate from the previous discussions. □

We must think of the pair (∇, ω) as a kind of action of A_B on the family of Lie algebroids $\mathcal{K} \longrightarrow E \xrightarrow{\pi_0} B$. The semi-direct product corresponding to this action is $\pi_0^* A_B \oplus \mathcal{K}$ which, in fact, defines an extension of A_B via the projection onto the first factor. Conversely:

Corollary 4.3.4. Let A_B be a Lie algebroid, $\pi_0 : E \longrightarrow B$ be a surjective submersion and $\mathcal{K}$ be a Lie algebroid such that $\sharp_{\mathcal{K}}(\mathcal{K}) \subseteq \mathsf{Ker}(d\pi_0)$. There are 1-1 correspondences between:

(a) Extensions of A_B by $\mathcal{K}$ together with a horizontal lift;

(b) Pairs (∇, ω) consisting of $\nabla \in \Omega^1_B(A_B, \mathsf{der}(\mathcal{K}))$, $\omega \in \Omega^2_B(A_B, \mathcal{K})$ such that:

$$\begin{aligned}
X^{\nabla_a}(\pi_0^* f_B) &= \pi_0^* \mathcal{L}_{\sharp_B a}(f_B) \\
\nabla_{[a,b]} - [\nabla_a, \nabla_b] &= [\omega(a, b), -]_{\mathcal{K}} \\
\oint \nabla_a \omega(b, c) + \omega([a, b], c) &= 0,
\end{aligned} \tag{4.7}$$

for every $a, b, c \in \Gamma(A_B)$ and $f_B \in C^\infty(B)$;

(c) Lie algebroid structures on the vector bundle $\pi_0^* A_B \oplus \mathcal{K}$ which turn the projection onto the first factor $\mathrm{pr}_B : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow A_B$ into an extension.

Proof. $(a) \Rightarrow (b)$ Suppose $\mathcal{K} \hookrightarrow A_E \overset{\pi}{\twoheadrightarrow} A_B$ is an extension together with a horizontal lift $h \in \Omega^1_B(A_B, A_E)$. We take $(\nabla, \omega) := (\nabla^h, \omega^h)$ as in lemma 4.3.1 and we are done.

$(b) \Rightarrow (c)$ Given a pair (∇, ω) we endow $\pi_0^* A_B \oplus \mathcal{K}$ with the following Lie algebroid structure:

$$\begin{aligned}
[k_1, k_2] &= [k_1, k_2]_{\mathcal{K}} \\
[\pi_0^* a, k] &= \nabla_a k \\
[\pi_0^* a, \pi_0^* b] &= \pi_0^*([a, b]_{A_B}) \oplus \omega(a, b) \\
\sharp \pi_0^*(a) &= X^{\nabla_a} \\
\sharp k &= \sharp_{\mathcal{K}} k.
\end{aligned}$$

We already pointed out in 4.3.3 that this defines an extension.

$(c) \Rightarrow (a)$ We take the extension $\mathcal{K} \hookrightarrow \pi_0^* A_B \oplus \mathcal{K} \xrightarrow{\mathrm{pr}_B} A_B$ together with the inclusion $h : \Gamma(\pi_0^* A_B) \longrightarrow \Gamma(\pi_0^* A_B \oplus \mathcal{K})$. □

4.4 On Morphisms of Extensions: Geometric Viewpoint

In this section we use horizontal lifts to study morphisms of extensions. The discussion presented in this section and on the next one is original.

Lemma 4.4.1. Let $\mathcal{K} \hookrightarrow A_E \xrightarrow{\pi} A_B$ and $\mathcal{K}' \hookrightarrow A_{E'} \xrightarrow{\pi'} A_B$ be extensions of Lie algebroids covering $\pi_0 : E \longrightarrow B$ and $\pi_0' : E' \longrightarrow B$, respectively. Given horizontal lifts $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$ and $h' : \Gamma(A_B) \longrightarrow \Gamma(A_E')$, there exists a 1-1 correspondence between:

- Morphisms of vector bundles $\Phi_{\mathrm{tot}} : A_E \longrightarrow A_{E'}$ covering $\phi : E \longrightarrow E'$, such that $\pi' \circ \Phi_{\mathrm{tot}} = \pi$ and $\pi_0' \circ \phi = \pi_0$;
- Pairs (Φ, Θ) consisting of vector bundle morphisms $\Phi : \mathcal{K} \longrightarrow \mathcal{K}'$ and $\Theta : \pi_0^* A_B \longrightarrow \phi^* \mathcal{K}'$.

At the level of sections, the correspondence is characterized by:

$$\Phi_{\mathrm{tot}}(h(a) + k) = h'(a) \circ \phi + \mathrm{pr}_{\mathcal{K}'} \Theta(\pi_0^* a) + \Phi(k) \tag{4.8}$$

for every $a \in \Gamma(A_B)$ and $k \in \Gamma(\mathcal{K})$.

Proof. In fact, it suffices to take Φ as the restriction $\Phi_{\mathrm{tot}}|_{\mathcal{K}}$ and:

$$\Theta(\pi_0^* a) := (\Phi_{\mathrm{tot}} \circ h(a) - h'(a) \circ \phi, \mathsf{id}).$$

Notice $\Theta : \pi_0^* A_B \longrightarrow \phi^*\mathcal{K}'$ is well defined, since:

$$\begin{aligned}\pi' \circ \Theta(\pi_0^* a) &= \pi' \circ \Phi_{\mathrm{tot}} \circ h(a) - \pi' \circ h'(a) \circ \phi\\ &= \pi \circ h(a) - a \circ \pi_0' \circ \phi\\ &= a \circ \pi_0 - a \circ \pi_0\\ &= 0,\end{aligned}$$

for every $a \in \Gamma(A_B)$. Besides, the induced map $\Theta : \Gamma(\pi_0^* A_B) \longrightarrow \Gamma(\phi^*\mathcal{K}')$ is $C^\infty(E)$-linear and therefore, Θ defines a vector bundle morphism. $\square$

Using the previous notation we give a standard characterization:

Proposition 4.4.2. There are 1-1 correspondences between:

- Vector bundle morphisms $\Theta : \pi_0^* A_B \longrightarrow \mathcal{K}'$ covering $\phi : E \longrightarrow E'$;
- Strong vector bundle morphisms $\Theta : \pi_0^* A_B \longrightarrow \phi^*\mathcal{K}'$;
- Forms $\Theta \in \Omega^1(A_B, \phi^*\mathcal{K}')$;
- Maps $\Theta : \Gamma(A_B) \longrightarrow C^\infty_\phi(E, \mathcal{K}')$ which are $C^\infty(B)$-linear, where we write

$$C^\infty_\phi(E, \mathcal{K}') := \{f \in C^\infty(E, \mathcal{K}') : p_{\mathcal{K}'} \circ f = \phi\}.$$

We shall abuse the notations and we shall make no distinction among any of the above forms of the morphism Θ. Using the last formulation

$$\Theta(a) = \Phi_{\mathrm{tot}} \circ h(a) - h'(a) \circ \phi,$$

so that Θ measures the failure of Φ_{tot} to preserve horizontal lifts.

In terms of the corresponding decompositions $\Sigma^h : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow A_E$ and $\Sigma^{h'} : \pi_0'^* A_B \oplus \mathcal{K} \longrightarrow A_{E'}$, we see that $\Phi : \mathcal{K} \longrightarrow \mathcal{K}'$ and $\Theta : \pi_0^* A_B \longrightarrow \mathcal{K}'$ are characterized via the commutativity of the diagram:

$$\begin{tikzcd} \pi_0^* A_B \oplus \mathcal{K} \arrow[r, "\begin{pmatrix} \phi\times\mathrm{id} & 0 \\ \Theta & \Phi \end{pmatrix}"] \arrow[d, "\Sigma^h"'] & \pi_0'^* A_B \oplus \mathcal{K}' \\ A_E \arrow[r, "\Phi_{\mathrm{tot}}"'] & A_{E'} \arrow[u, "\Sigma^{h'}"'] \end{tikzcd}$$

In other words, if we denote by $\Phi^{h,h'}_{\mathrm{tot}} : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow \pi_0'^* A_B \oplus \mathcal{K}'$ the transported map Φ_{tot} via the isomorphisms Σ^h and $\Sigma^{h'}$ then:

$$\Phi^{h,h'}_{\mathrm{tot}}(a_B, e, k) = (a_B, \phi(e), \Theta(a_B, e) + \Phi(k)),$$

for every $(a_B, e, k) \in \pi_0^* A_B \oplus \mathcal{K}$. Taking this into account, it is evident one should be able to translate the conditions in order to Φ_{tot} be a morphism of Lie algebroids into conditions on the pair (Φ, Θ). The main problem is that the pullback bundle $\phi^*\mathcal{K}'$ does not come equipped with a Lie algebroid structure hence we will have to go through the dual level. This is done on the next section.

4.5 On Morphisms of Extensions: Dual Viewpoint

The starting point is the fact, already noticed in [12], that there is an isomorphism of $C^\infty(E)$-graded algebras:

$$\Omega(A_B) \otimes_{C^\infty(B)} \Omega(\mathcal{K}) \simeq \Omega(\pi_0^* A_B \oplus \mathcal{K}).$$

Explicitly, in degree k, this isomorphism is defined as:

$$\begin{aligned} \Sigma : \Omega^k(\pi_0^* A_B \oplus \mathcal{K}) &\longrightarrow \bigoplus_{p+q=k} \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}) \\ \varepsilon &\longmapsto \sum_{p+q=k} \varepsilon^{p,q} \end{aligned} \tag{4.9}$$

where $\varepsilon^{p,q} \in \Omega^p(A_B) \otimes \Omega^q(\mathcal{K})$ is given by:

$$\langle \varepsilon^{p,q}(a_1, \dots, a_p), k_1, \dots, k_q \rangle := \langle \varepsilon, h(a_1), \dots, h(a_p), \dots, k_q \rangle,$$

for every $a_1, \dots, a_p \in \Gamma(A_B)$ and $k_1, \dots, k_q \in \Gamma(\mathcal{K})$. Using this isomorphism, [12] decomposed the differential in $\Omega(\pi_0^* A_B \oplus \mathcal{K})$ as:

Theorem 4.5.1. The differential:

$$d : \bigoplus_{p+q=k} \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}) \longrightarrow \bigoplus_{r+s=k+1} \Omega^r(A_B) \otimes \Omega^s(\mathcal{K})$$

decomposes as follows:

$$d = \begin{pmatrix} d_{0,1}^{0,p+q} & 0 & 0 & \dots & 0 & 0 \\ d_{1,0}^{0,p+q} & d_{0,1}^{1,p+q-1} & 0 & \dots & 0 & 0 \\ d_{2,-1}^{0,p+q} & d_{1,0}^{1,p+q-1} & d_{0,1}^{2,p+q-2} & \dots & 0 & 0 \\ 0 & d_{2,-1}^{1,p+q-1} & d_{1,0}^{2,p+q-2} & \dots & 0 & 0 \\ 0 & 0 & d_{2,-1}^{2,p+q-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{0,1}^{p+q-1,1} & 0 \\ 0 & 0 & 0 & \dots & d_{1,0}^{p+q-1,1} & d_{0,1}^{p+q,0} \\ 0 & 0 & 0 & \dots & d_{2,-1}^{p+q-1,1} & d_{1,0}^{p+q,0} \end{pmatrix}$$

where:

- $d_{0,1}^{p,q} : \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}_E) \longrightarrow \Omega^p(A_B) \otimes \Omega^{q+1}(\mathcal{K})$ is the extension of the differential of $\mathcal{K}$, defined by:

$$d_{0,1}^{p,q}\varepsilon(a_1, \ldots, a_p) := (-1)^p d_{\mathcal{K}}(\varepsilon(a_1, \ldots, a_p)),$$

- $d_{1,0}^{p,q} : \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}_E) \longrightarrow \Omega^{p+1}(A_B) \otimes \Omega^q(\mathcal{K})$ is the map induced by ∇:

$$\begin{aligned} d_{1,0}^{p,q}\varepsilon(a_1, \ldots, a_{p+1}) := &\sum_{j=1}^{p+1} (-1)^{j+1} \nabla_{a_j}(\langle \varepsilon, a_1, \ldots, \widehat{a_j}, \ldots, a_{p+1}\rangle) \\ &+ \sum_{i<j} (-1)^{i+j} \varepsilon([a_i, a_j], a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_{p+2}). \end{aligned}$$

- $d_{2,-1}^{p,q} : \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}_E) \longrightarrow \Omega^{p+2}(A_B) \otimes \Omega^{q-1}(\mathcal{K}_E)$ is the map induced by ω:

$$d_{2,-1}^{p,q}\varepsilon(a_1, \ldots, a_{p+2}) := \sum_{i<j} (-1)^{i+j} \imath_{\omega(a_i,a_j)}(\varepsilon(a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_{p+2})).$$

In order to keep the notation simple we shall denote $d_{0,1}$, $d_{1,0}$ and $d_{2,-1}$, regardless the bidegree, respectively by $d_{\mathcal{K}}$, d_{∇} and d_{ω}. Succinctly, we may write:

$$d = d_{\mathcal{K}} + d_{\nabla} + d_{\omega}.$$

In order to relate Φ_{tot} to the pair (Φ, Θ) we use the isomorphism (4.9) to decompose the induced map on forms

$$\Phi_{\text{tot}}^* : \Omega(\pi_0'^* A_B \oplus \mathcal{K}') \longrightarrow \Omega(\pi_0^* A_B \oplus \mathcal{K}).$$

This is done in the following:

Theorem 4.5.2. The map:

$$\Phi_{\text{tot}}^* : \bigoplus_{p+q=k} \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}') \longrightarrow \bigoplus_{r+s=k} \Omega^r(A_B) \otimes \Omega^s(\mathcal{K})$$

decomposes as follows:

$$\Phi_{\text{tot}}^* = \begin{pmatrix} (\imath_{\Theta}^{\Phi})^{0,p+q}_{0,0} & 0 & \cdots & 0 & 0 \\ (\imath_{\Theta}^{\Phi})^{0,p+q}_{1,-1} & (\imath_{\Theta}^{\Phi})^{1,p+q-1}_{0,0} & \cdots & 0 & 0 \\ (\imath_{\Theta}^{\Phi})^{0,p+q}_{2,-2} & (\imath_{\Theta}^{\Phi})^{1,p+q-1}_{1,-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (\imath_{\Theta}^{\Phi})^{0,p+q}_{p+q-1,-(p+q-1)} & (\imath_{\Theta}^{\Phi})^{1,p+q-1}_{p+q-2,-(p+q_2)} & \cdots & (\imath_{\Theta}^{\Phi})^{p+q-1,1}_{0,0} & 0 \\ (\imath_{\Theta}^{\Phi})^{0,p+q}_{p+q,-(p+q)} & (\imath_{\Theta}^{\Phi})^{1,p+q-1}_{p+q-1,-(p+q-1)} & \cdots & (\imath_{\Theta}^{\Phi})^{p+q-1,1}_{1,-1} & (\imath_{\Theta}^{\Phi})^{p+q,0}_{0,0} \end{pmatrix} \qquad (4.10)$$

where, for every $l \geq 0$, the map:

$$(\imath^{\Phi}_{\Theta})^{p,q}_{l,-l} : \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}') \longrightarrow \Omega^{p+l}(A_B) \otimes \Omega^{q-l}(\mathcal{K})$$

is defined by:

$$\begin{aligned}\big\langle (\imath^{\Phi}_{\Theta})^{p,q}_{l,-l}\varepsilon(a_1, \dots, a_{p+l}), k_1, \dots, k_{q-l}\big\rangle \\ = (-1)^p \sum_{\sigma \in \mathsf{Sh}(p,l)} \mathsf{sgn}(\sigma)\big\langle \varepsilon^{p,q}(a_{\sigma(l+1)}, \dots, a_{\sigma(p+l)}), \Theta(a_{\sigma(1)}), \dots \\ \dots, \Theta(a_{\sigma(l)}), \Phi \circ k_1, \dots, \Phi \circ k_{q-l}\big\rangle. \quad (4.11)\end{aligned}$$

Proof. For a given $\varepsilon^{p,q} \in \Omega^p(A_B) \otimes \Omega^q(\mathcal{K})$ let us write

$$\varepsilon^{p,q}_{\Sigma} := \Sigma^{-1}(\varepsilon^{p,q}) \in \Omega^{p+q}(\pi_0^* A_B \oplus \mathcal{K}),$$

where Σ is the isomorphism (4.9). Then:

$$\varepsilon^{p,q} = \sum_{r+s=k} (\varepsilon^{p,q}_{\Sigma})^{r,s} \Rightarrow (\varepsilon^{p,q}_{\Sigma})^{r,s} = \begin{cases} \varepsilon^{p,q} & \text{if} \quad (r,s) = (p,q) \\ 0 & \text{if} \quad (r,s) \neq (p,q) \end{cases}.$$

The map $\Phi^*_{\text{tot}} : \bigoplus_{p+q=k} \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}') \longrightarrow \bigoplus_{p'+q'=k} \Omega^{p'}(A_B) \otimes \Omega^{q'}(\mathcal{K})$ decomposes as follows:

$$\Phi^*_{\text{tot}}\left(\sum_{p+q=k} \varepsilon^{p,q}\right) = \sum_{p+q=k} \sum_{r+s=k} (\Phi^*_{\text{tot}} \varepsilon^{p,q}_{\Sigma})^{r,s}.$$

We are left to study $(\Phi^*_{\text{tot}} \varepsilon^{p,q}_{\Sigma})^{r,s}$ with $r + s = k = p + q$. For the sake of brevity let us write:

$$(\mathsf{LHS}) := \langle (\Phi^*_{\text{tot}} \varepsilon^{p,q}_{\Sigma})^{r,s}(a_1, \dots, a_r), k_1, \dots, k_s \rangle.$$

Then:

$$\begin{aligned}
(\mathsf{LHS}) &= \left\langle \Phi^*_{\mathrm{tot}}\varepsilon^{p,q}_{\Sigma}, h(a_1)_\wedge \ldots {}_\wedge h(a_r), k_{1\wedge} \ldots {}_\wedge k_s \right\rangle \\
&= \left\langle \varepsilon^{p,q}_{\Sigma}, h'(a_1) + \Theta(a_1), \ldots, h'(a_r) + \Theta(\alpha_r), \Phi(k_1), \ldots, \Phi(k_s) \right\rangle \\
&= \sum_{l=0}^{r} \sum_{\sigma \in \mathsf{Sh}(l,r-l)} \mathsf{sgn}(\sigma) \left\langle \varepsilon^{p,q}_{\Sigma}, h(a_{\sigma(l+1)}), \ldots, h(a_{\sigma(r)}), \Theta(a_{\sigma(1)}), \ldots \right. \\
&\qquad\qquad \left. \ldots, \Theta(a_{\sigma(l)}), \Phi \circ k_1, \ldots, \Phi \circ k_s \right\rangle \\
&= \sum_{l=0}^{r} \sum_{\sigma \in \mathsf{Sh}(l,r-l)} \mathsf{sgn}(\sigma) \left\langle (\varepsilon^{p,q}_{\Sigma})^{r-l,s+l}(a_{\sigma(l+1)}, \ldots, a_{\sigma(r)}), \Theta(a_{\sigma(1)}), \ldots \right. \\
&\qquad\qquad \left. \ldots, \Theta(a_{\sigma(l)}), \Phi \circ k_1, \ldots, \Phi \circ k_s \right\rangle
\end{aligned}$$

But:

$$(\varepsilon^{p,q}_{\Sigma})^{r-l,s+l} = \begin{cases} \varepsilon^{p,q} & \text{if} \quad (p,q) = (r-l, s+l) \\ 0 & \text{if} \quad (p,q) \neq (r-l, s+l) \end{cases}.$$

Consequently:

$$\begin{aligned}
(\mathsf{LHS}) &= \sum_{\sigma \in \mathsf{Sh}(r-p,p)} \mathsf{sgn}(\sigma) \left\langle \varepsilon^{p,q}(a_{\sigma(r-p+1)}, \ldots, a_{\sigma(r)}), \Theta(a_{\sigma(1)}), \ldots \right. \\
&\qquad\qquad \left. \ldots, \Theta(a_{\sigma(q-s)}), \Phi \circ k_1, \ldots, \Phi \circ k_s \right\rangle \\
&= (-1)^p \sum_{\sigma \in \mathsf{Sh}(p,r-p)} \mathsf{sgn}(\sigma) \left\langle \varepsilon^{p,q}(a_{\sigma(r-p+1)}, \ldots, a_{\sigma(r)}, \Theta(a_{\sigma(1)}), \ldots \right. \\
&\qquad\qquad \left. \ldots, \Theta(a_{\sigma(q-s)}), \Phi \circ k_1, \ldots, \Phi \circ k_s \right\rangle \\
&= \left\langle (\imath^{\Phi}_{\Theta})^{p,q}_{r-p,q-s} \varepsilon(a_1, \ldots, a_r), k_1, \ldots, k_s \right\rangle.
\end{aligned}$$

from what we deduce:

$$(\Phi^*_{\mathrm{tot}}\varepsilon^{p,q}_{\Sigma})^{r,s} = (\imath^{\Phi}_{\Theta})^{p,q}_{r-p,q-s},$$

and therefore:

$$\Phi^*_{\mathrm{tot}} = \sum_{p+q=k} \sum_{r+s=k} (\imath^{\Phi}_{\Theta})^{p,q}_{r-p,q-s},$$

which is another way of writing 4.10.

□

Before we proceed, let us write:

$$\Phi^*_{\mathcal{K}} := (\imath^{\Phi}_{\Theta})_{0,0} \quad \text{and} \quad \imath^{\Phi}_{\Theta \underbrace{\wedge \ldots \wedge}_{p} \Theta} := (\imath^{\Phi}_{\Theta})_{p,-p}$$

With these notations, $\Phi^*_{\rm tot}$ may be written as:

$$\Phi^*_{\rm tot} = \Phi^*_{\mathcal{K}} + \imath^{\Phi}_{\Theta} + \imath^{\Phi}_{\Theta\wedge\Theta} + \dots .$$

As we shall see, only $\Phi^*_{\mathcal{K}}$, $\imath^{\Phi}_{\Theta}$ and $\imath^{\Phi}_{\Theta\wedge\Theta}$ will be relevant in the study of $\Phi_{\rm tot}$ and for that reason we explicit their actions below. The first map:

$$\Phi^*_{\mathcal{K}} : \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}') \longrightarrow \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}),$$

is given by:

$$\langle \Phi^*_{\mathcal{K}} \varepsilon(a_1, \dots, a_p), k_1, \dots, k_q\rangle := (-1)^p \Phi^*(\langle \varepsilon(a_1, \dots, a_p), k_1, \dots, k_q\rangle,$$

whereas the second map:

$$\imath^{\Phi}_{\Theta} : \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}') \longrightarrow \Omega^{p+1}(A_B) \otimes \Omega^{q-1}(\mathcal{K})$$

is defined by:

$$\begin{aligned}&\big\langle \imath^{\Phi}_{\Theta}\varepsilon(a_1, \dots, a_{p+1}), k_1, \dots, k_{q-1}\big\rangle \\ &\qquad := (-1)^p \sum_{j=1}^{p+1} (-1)^j \big\langle \varepsilon, a_1, \dots, \widehat{a_j}, \dots, a_{p+1}, \Theta(a_j), \Phi\circ k_1, \dots, \Phi \circ k_{q-1}\big\rangle.\end{aligned}$$

and finally, the third map:

$$\imath^{\Phi}_{\Theta\wedge\Theta} : \Omega^p(A_B) \otimes \Omega^q(\mathcal{K}') \longrightarrow \Omega^{p+2}(A_B) \otimes \Omega^{q-2}(\mathcal{K})$$

is obtained as:

$$\begin{aligned}&\big\langle \imath^{\Phi}_{\Theta\wedge\Theta}\varepsilon(a_1, \dots, a_{p+2}), k_1, \dots, k_{q-1}\big\rangle \\ &\qquad\qquad = (-1)^p \sum_{i<j} (-1)^{i+j} \big\langle \varepsilon(a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_{p+2}), \\ &\qquad\qquad\qquad\qquad \Theta(a_i), \Theta(a_j), \Phi \circ k_1, \dots, \Phi\circ k_{q-2}\big\rangle.\end{aligned}$$

Finally, we can prove the main result of this section:

Theorem 4.5.3. The map $\Phi^*_{\rm tot} : \Omega(\pi'^*_0 A_B \oplus \mathcal{K}') \longrightarrow \Omega(\pi^*_0 A_B \oplus \mathcal{K})$ is a morphism of DGAs if and only if:

$$d_{\mathcal{K}} \circ \Phi^*_{\mathcal{K}} = \Phi^*_{\mathcal{K}} \circ d_{\mathcal{K}'} \tag{M1}$$

$$d_{\nabla} \circ \Phi^*_{\mathcal{K}} - \Phi^*_{\mathcal{K}} \circ d_{\nabla'} = \imath^{\Phi}_{\Theta} \circ d_{\mathcal{K}'} - d_{\mathcal{K}} \circ \imath^{\Phi}_{\Theta} \tag{M2}$$

$$d_{\omega} \circ \Phi^*_{\mathcal{K}} - \Phi^*_{\mathcal{K}} \circ d_{\omega'} = \imath^{\Phi}_{\Theta} \circ d_{\nabla'} - d_{\nabla} \circ \imath^{\Phi}_{\Theta} + \imath^{\Phi}_{\Theta\wedge\Theta} \circ d_{\mathcal{K}'} - d_{\mathcal{K}} \circ \imath^{\Phi}_{\Theta\wedge\Theta}. \tag{M3}$$

Proof. Let us write d_{tot} and d'_{tot} for the respective differentials on $\Omega(\pi_0^* A_B \oplus \mathcal{K})$ and $\Omega(\pi_0'^* A_B \oplus \mathcal{K}')$. Recall Φ^*_{tot} is a morphism of DGAs if and only if

$$d_{\text{tot}} \circ \Phi^*_{\text{tot}} = \Phi^*_{\text{tot}} \circ d'_{\text{tot}}. \tag{4.12}$$

This has to be verified only in total degree 0 and total degree 1. By theorem 4.5.1, d_{tot} and d'_{tot} decomposes as

$$d_{\text{tot}} = d_{\mathcal{K}} + d_{\nabla} + d_{\omega} \quad \text{and} \quad d'_{\text{tot}} = d_{\mathcal{K}'} + d_{\nabla'} + d_{\omega'},$$

whereas, by theorem 4.5.2, Φ^*_{tot} decomposes as

$$\Phi^*_{\text{tot}} = \Phi^*_{\mathcal{K}} + \imath^{\Phi}_{\Theta} + \imath^{\Phi}_{\Theta\wedge\Theta} + \dots .$$

Writing down the commutativity condition (4.12) using these decompositions and comparing degrees we find the stated equalities (M1) – (M3). □

Chapter 5

Infinitesimal Actions up to Homotopy

Since extensions of Lie algebroids are a non-linear version of VB-groupoids it is natural to look for a non-linear version of representations up to homotopy. In this chapter we undertake this purpose.

5.1 Actions up to Homotopy of Lie Algebroids

Let us start fixing a surjective submersion $\pi_0 : E \longrightarrow B$, A_B a Lie algebroid over B and $\mathcal{K}$ a vector bundle over E. Define:

$$\Omega_B(A_B, \Omega(\mathcal{K})) := \Omega_B(A_B) \otimes_{C^\infty(B)} \Omega(\mathcal{K}),$$

where the $C^\infty(B)$-module structure on $\Omega(\mathcal{K})$ is that induced by the morphism of $\mathbb{R}$-algebras $\pi_0^* : C^\infty(B) \longrightarrow C^\infty(E)$. By means of this morphism of algebras we can consider any $C^\infty(E)$-module as a $C^\infty(B)$-module and will do it without further mention.

Notice $\Omega_B(A_B, \Omega(\mathcal{K}))$ is a $C^\infty(B)$-graded algebra with the product:

$$(\varepsilon_1 \wedge \varepsilon_2)(a_1, \dots, a_{p+q}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma)\varepsilon_1(a_{\sigma(1)}, \dots, a_{\sigma(p)}) \wedge \varepsilon_2(a_{\sigma(p+1)}, \dots, a_{\sigma(p+q)})$$

defined for every $\varepsilon_1 \in \Omega^p_B(A_B, \Omega^r(\mathcal{K}))$, $\varepsilon_2 \in \Omega^q_B(A_B, \Omega^s(\mathcal{K}))$ and every $a_1, \dots, a_{p+q} \in \Gamma(A_B)$. The wedge product $\wedge$ on the right hand side is happening between forms on $\Omega(\mathcal{K})$.

Besides, $\Omega_B(A_B, \Omega(\mathcal{K}))$ is also a graded left $\Omega(A_B)$-module by restriction of scalars, that is, via the product

$$(\alpha \cdot \varepsilon)(a_1, \dots, a_{p+q}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma)\, \underbrace{\alpha(a_{\sigma(1)}, \dots, a_{\sigma(p)})}_{\in C^\infty(B)}\, \varepsilon(a_{\sigma(p+1)}, \dots, a_{\sigma(p+q)})$$

defined for every $\alpha \in \Omega^p_B(A_B)$, $\varepsilon \in \Omega^q_B(A_B, \Omega^r(\mathcal{K}))$ and every $a_1, \ldots, a_{p+q} \in \Gamma(A_B)$.

Now we are ready to define the main notion of this chapter.

Definition 5.1.1. Let $\pi_0 : E \longrightarrow B$ be a surjective submersion, A_B be a Lie algebroid over B and $\mathcal{K}$ be a vector bundle over E. An *action up to homotopy of A_B on $\mathcal{K}$ along π_0* is a degree one $\mathbb{R}$-linear operator

$$D : \Omega_B(A_B, \Omega(\mathcal{K})) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$$

such that $D^2 = 0$ and

$$D(\alpha \cdot \varepsilon) = d_{A_B}\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot D\varepsilon \tag{5.1}$$

$$D(\varepsilon_1 \wedge \varepsilon_2) = D\varepsilon_1 \wedge \varepsilon_2 + (-1)^{|\varepsilon_1|}\varepsilon_1 \wedge D\varepsilon_2 \tag{5.2}$$

for every $\alpha \in \Omega(A_B)$ and $\varepsilon, \varepsilon_1, \varepsilon_2 \in \Omega_B(A_B, \Omega(\mathcal{K}))$ homogeneous.

We do not provide any examples since we shall prove later on all those actions arise from extensions of Lie algebroids.

Whenever it is not relevant to mention the map π_0 we will omit it and we shall refer to D simply as an *action up to homotopy*

We have an evident notion of morphism between actions up to homotopy.

Definition 5.1.2. Let D be an action up to homotopy of A_B on $\mathcal{K}$ along $\pi_0 : E \longrightarrow B$ and D' be an action up to homotopy of A_B on $\mathcal{K}'$ along $\pi'_0 : E' \longrightarrow B$. A *morphism of actions up to homotopy from D to D'* is a degree zero $\mathbb{R}$-linear operator

$$\Phi : \Omega_B(A_B, \Omega(\mathcal{K}')) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$$

such that $\Phi \circ D' = D \circ \Phi$ and

$$\Phi(\alpha \cdot \varepsilon) = \alpha \cdot \Phi\varepsilon \tag{5.3}$$

$$\Phi(\varepsilon_1 \wedge \varepsilon_2) = \Phi\varepsilon_1 \wedge \Phi\varepsilon_2 \tag{5.4}$$

for every $\alpha \in \Omega(A_B)$ and $\varepsilon, \varepsilon, \varepsilon_1 \in \Omega_B(A_B, \Omega(\mathcal{K}))$ homogeneous.

The conditions 5.3 and 5.4 tell us Φ is a $\Omega(A_B)$-linear map which respects the products.

It is clear that morphisms of actions up to homotopy compose in an associative manner and this composition has an evident unit and therefore we have a category of actions up to homotopy of A_B which will be denoted by $\mathsf{Act}_\infty(A_B)$. The objects of this category are triples $(\mathcal{K}, \pi_0, D)$ consisting of a vector bundle $\mathcal{K} \longrightarrow E$, a surjective submersion $\pi_0 : E \longrightarrow B$ and an action up to homotopy D of A_B on $\mathcal{K}$ along π_0.

The goal of this chapter is to study actions up to homotopy thoroughly and to explain its relationship with extensions of Lie algebroids and with representations up to homotopy.

5.2 The Structural Theorems

In this section we decompose actions up to homotopy and their morphisms into homogeneous components in order to gain insight into them. In order to do that, we need some algebraic machinery. First, let us write $\mathsf{End}_B(\Omega(\mathcal{K}))$ for the set of graded endomorphisms $T : \Omega(\mathcal{K}) \longrightarrow \Omega(\mathcal{K})$ which are *projectable along* π_0^* in the sense:

$$T(\pi_0^* C^\infty(B)) \subset \pi_0^* C^\infty(B).$$

We shall write:

$$\Omega_B(A_B, \mathsf{End}_B(\Omega(\mathcal{K}))) := \Omega_B(A_B) \otimes_{C^\infty(B)} \mathsf{End}_B(\Omega(\mathcal{K})).$$

This is an associative $\mathbb{R}$-algebra with the product

$$T_1 \bullet T_2(a_1, \ldots, a_{p+q}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma) T_1(a_{\sigma(1)}, \ldots, a_{\sigma(p)}) \circ T_2(a_{\sigma(p+1)}, \ldots, a_{\sigma(p+q)})$$

defined for every $T_1 \in \Omega_B^p(A_B, \mathsf{End}^r(\Omega(\mathcal{K})))$, $T_2 \in \Omega_B^q(A_B, \mathsf{End}^s(\Omega(\mathcal{K})))$ and $a_1, \ldots, a_{p+q} \in \Gamma(A_B)$. In particular, we can make it into a $\mathbb{R}$-graded Lie algebra with the graded commutator

$$[T_1, T_2] := T_1 \bullet T_2 - (-1)^{|T_1||T_2|} T_2 \bullet T_1.$$

Furthermore, $\Omega_B(A_B, \Omega(\mathcal{K}))$ becomes a graded left $\Omega_B(A_B, \mathsf{End}(\Omega(\mathcal{K})))$-module with the product

$$(T \bullet \varepsilon)(a_1, \ldots, a_{p+q}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma) T(a_{\sigma(1)}, \ldots, a_{\sigma(p)})(\varepsilon(a_{\sigma(p+1)}, \ldots, a_{\sigma(p+q)})) \qquad (5.5)$$

defined for every $T \in \Omega_B^p(A_B, \mathsf{End}^r \Omega(\mathcal{K}))$, $\varepsilon \in \Omega_B^q(A_B, \Omega^s(\mathcal{K}))$ and $a_1, \ldots, a_{p+q} \in \Gamma(A_B)$. In particular,

$$T_1 \bullet (T_2 \bullet \varepsilon) = (T_1 \bullet T_2) \bullet \varepsilon$$

for every $T_1, T_2 \in \Omega_B(A_B, \mathsf{End}\ \Omega(\mathcal{K}))$ and $\varepsilon \in \Omega_B(A_B, \Omega(\mathcal{K}))$ homogeneous. This will be important later on.

We shall soon see an action up to homotopy consists of some $\Omega(A_B)$-linear pieces and only one piece which is not $\Omega(A_B)$-linear. In what follows lets us write $\mathsf{der}_B \Omega(\mathcal{K})$ for the graded Lie algebra where:

- $\mathsf{der}^r_B\Omega(\mathcal{K})$ consists of those $C^\infty(B)$-linear derivations of the graded algebra $\Omega(\mathcal{K})$ for every $r \neq 0$;
- $\mathsf{der}^0_B\Omega(\mathcal{K})$ consists of those derivations D of the graded algebra $\Omega(\mathcal{K})$ which satisfy $D(\pi_0^* C^\infty(B)) \subset \pi_0^* C^\infty(B)$.

In order to deal with the linear pieces let us write $\mathsf{der}_B\Omega(\mathcal{K})$ for the graded Lie algebra of $C^\infty(B)$-linear derivations of the graded algebra $\Omega(\mathcal{K})$. Then, we shall make use of the following:

Lemma 5.2.1. Let $T \in \Omega^p_B(A_B, \mathsf{der}^r_B\Omega(\mathcal{K}))$. The left-multiplication map

$$\begin{aligned} L_T : \Omega^q_B(A_B, \Omega^s(\mathcal{K})) &\longrightarrow \Omega^{p+q}_B(A_B, \Omega^{r+s}(\mathcal{K})) \\ \varepsilon &\longmapsto T \bullet \varepsilon \end{aligned}$$

defines a 1-1 correspondence between

- Elements $T \in \Omega^p_B(A_B, \mathsf{der}^r_B\Omega(\mathcal{K}))$;
- $\Omega(A_B)$-linear derivations of the graded algebra $\Omega_B(A_B, \Omega(\mathcal{K}))$ which rise the bidegree by (p, r).

Proof. By the very definition, L_T is well defined and produces a $\Omega(A_B)$-linear derivation of $\Omega_B(A_B, \Omega(\mathcal{K}))$ which rises the bidegree by (p, r). On the other hand, given a degree one $\Omega(A_B)$-linear derivation Υ which rises the bidegree by (p, r), we define:

$$T(a_1, \dots, a_p)(\varepsilon) := \Upsilon\varepsilon(a_1, \dots, a_p),$$

for every $\varepsilon \in \Omega^l(\mathcal{K}) \simeq \Omega^0_B(A_B, \Omega^l(\mathcal{K}))$. From the properties of Υ it follows $T \in \Omega^p_B(A_B, \mathsf{der}^r_B\,\Omega(\mathcal{K}))$. Furthermore, L_T coincides with Υ on $\Omega(\mathcal{K})$ and therefore $L_T = \Upsilon$ since $\Omega_B(A_B, \Omega(\mathcal{K}))$ is generated as a graded-module by $\Omega(A_B)$ and L_T is $\Omega(A_B)$-linear. □

As to the non-linear piece we shall need the following:

Lemma 5.2.2. There is a 1-1 correspondence between degree one $\mathbb{R}$-linear operators

$$\partial : \Omega_B(A_B, \Omega(\mathcal{K})) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$$

which rises the bidegree by $(1, 0)$ and satisfy

$$\partial(\alpha \cdot \varepsilon) = d_{A_B}\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot \partial\varepsilon \tag{5.1}$$

$$\partial(\varepsilon_1 \wedge \varepsilon_2) = \partial\varepsilon_1 \wedge \varepsilon_2 + (-1)^{|\alpha|}\alpha \cdot \partial\varepsilon_2 \tag{5.2}$$

for every $\alpha \in \Omega(A_B)$, $\varepsilon, \varepsilon_1, \varepsilon_2 \in \Omega_B(A_B, \Omega(\mathcal{K}))$ homogeneous and covariant derivative operators $\nabla^* \in \Omega^1_B(A_B, \mathsf{der}^0_{\mathbb{R}}\Omega(\mathcal{K}))$ such that

$$\nabla^*_a(\pi_0^* f_B) = \pi_0^* \mathcal{L}_{\sharp_{A_B} a}(f_B) \tag{5.3}$$

for every $a \in \Gamma(A_B)$ and $f_B \in C^\infty(B)$.

Proof. Given ∇^* we define

$$\partial_{\nabla^*}\varepsilon(a_1,\ldots,a_{p+1}) := \sum_{\sigma\in\mathsf{Sh}(1,p)} \mathsf{sgn}(\sigma)\nabla^*_{a_{\sigma(1)}}\varepsilon(a_{\sigma(2)},\ldots,a_{\sigma(p+1)}) \tag{5.4}$$
$$+ \sum_{\sigma\in\mathsf{Sh}(2,p-1)} \mathsf{sgn}(\sigma)\varepsilon([a_{\sigma(1)},a_{\sigma(2)}],a_{\sigma(3)},\ldots,a_{\sigma(p+1)}) \tag{5.5}$$

for every $\varepsilon \in \Omega^p_B(A_B,\Omega^q(\mathcal{K}))$ and $a_1,\ldots,a_{p+1} \in \Gamma(A_B)$. Since ∇^* takes its values on degree zero derivation of $\Omega(\mathcal{K})$ we readily see it defines a derivation of the graded algebra $\Omega_B(A_B,\Omega(\mathcal{K}))$ and by the very definition it rises the bidegree by $(1,0)$. The condition (5.3) ensures (5.1) holds. On the other hand, given ∂ we define $\nabla^* \in \Omega^1_B(A_B,\mathsf{der}^0\Omega(\mathcal{K}))$ by

$$\langle\nabla^*_a\varepsilon,k_1,\ldots,k_q\rangle := \langle\partial\varepsilon(a),k_1,\ldots,k_q\rangle$$

for every $a \in \Gamma(A_B)$ and every $\varepsilon \in \Omega^q(\mathcal{K}) \simeq \Omega^0_B(A_B,\Omega^q(\mathcal{K}))$. Then (5.2) ensures ∇^* takes its values on derivations of $\mathcal{K}$ and since ∂ rises the bidegree by $(1,0)$, this derivation has degree zero. Finally, (5.1) readily implies (5.3). □

Notice we can break the operator ∂_{∇^*} given in (5.4) into two pieces

$$\partial_{\nabla^*} = L_{\nabla^*} + \partial_{[\cdot,\cdot]}$$

where

$$L_{\nabla^*}\varepsilon(a_1,\ldots,a_{p+1}) := \sum_{\sigma\in\mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma)\nabla_{a_{\sigma(1)}}\varepsilon(a_{\sigma(2)},\ldots,a_{\sigma(p+1)})$$
$$\partial_{[\cdot,\cdot]}\varepsilon(a_1,\ldots,a_{p+1}) := \sum_{\sigma\in\mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma)\varepsilon([a_{\sigma(1)},a_{\sigma(2)}],a_{\sigma(3)},\ldots,a_{\sigma(p+1)}),$$

for every $\varepsilon \in \Omega^p_B(A_B,\Omega^q(\mathcal{K}))$ and $a_1,\ldots,a_{p+1} \in \Gamma(A_B)$. Some remarks are required:

- L_{∇^*} does not map $\Omega_B(A_B,\Omega(\mathcal{K}))$ into itself once ∇^* does not take its values on $C^\infty(B)$-linear derivations of $\Omega(\mathcal{K})$;
- $\partial_{[\cdot,\cdot]}$ is clearly $\Omega(A_B)$-linear, rises the bidegree by $(1,0)$ and satisfies $\partial^2_{[\cdot,\cdot]} = 0$.

Notice we can define a similar operator $\partial_{[\cdot,\cdot]}$ on $\Omega_B(A_B,\mathsf{End}\Omega(\mathcal{K}))$ which rises the bidegree by $(1,0)$ setting

$$\partial_{[\cdot,\cdot]}(T)(a_1,\ldots,a_{p+1}) := \sum_{\sigma\in\mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma)T([a_{\sigma(1)},a_{\sigma(2)}],a_{\sigma(3)},\ldots,a_{\sigma(p+1)})$$

for every $T \in \Omega^p_B(A_B,\mathsf{End}^q\Omega(\mathcal{K}))$ and $a_1,\ldots,a_{p+1} \in \Gamma(A_B)$. This operator is compatible with the $\Omega_B(A_B,\mathsf{End}\Omega(\mathcal{K}))$-graded left-module structure (5.5) on $\Omega_B(A_B,\Omega(\mathcal{K}))$ in the following sense

$$\partial_{[\cdot,\cdot]}(T \bullet \varepsilon) = \partial_{[\cdot,\cdot]}(T) \bullet \varepsilon + (-1)^{|T|} T \bullet \partial_{[\cdot,\cdot]}(\varepsilon)$$

for every $T \in \Omega_B(A_B, \mathsf{End}\Omega(\mathcal{K}))$ and $\varepsilon \in \Omega_B(A_B, \Omega(\mathcal{K}))$ homogeneous. This derivation property will be important in what follows.

Before we proceed, we have to do the following important:

Remark 5.2.3. Let A be a Lie algebroid and let $\Omega(A)$ be its differential graded algebra of forms. The multiplication in this algebra is given by the wedge product. Since $\Omega(A)$ is generated as a graded algebra by forms of degree 0 and 1, any derivation of degree inferior to -1 of $\Omega(A)$ must vanish.

Now we are ready to prove the first main theorem of this section.

Theorem 5.2.4. There is a 1-1 correspondence between actions up to homotopy

$$D : \Omega_B(A_B, \Omega(\mathcal{K})) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$$

and triples $(\partial_{\mathcal{K}}, \nabla^*, \partial_\omega)$ where

- $\partial_{\mathcal{K}} \in \Omega^0_B(A_B, \mathsf{der}^1_B \Omega(\mathcal{K}))$;
- $\nabla^* \in \Omega^1_B(A_B, \mathsf{der}^0 \Omega(\mathcal{K}))$ is such that $\nabla^*_a(\pi_0^* f_B) = \pi_0^* \mathcal{L}_{\sharp_{A_B} a}(f_B)$;
- $\partial_\omega \in \Omega^2_B(A_B, \mathsf{der}^{-1}_B \Omega(\mathcal{K}))$;

satisfying the compatibility conditions

$$\begin{aligned}
\frac{1}{2}[\partial_{\mathcal{K}}, \partial_{\mathcal{K}}] &= 0\\
[\nabla^*, \partial_{\mathcal{K}}] &= 0\\
\frac{1}{2}[\nabla^*, \nabla^*] + [\partial_{\mathcal{K}}, \partial_\omega] + \partial_{[\cdot,\cdot]}(\nabla^*) &= 0\\
[\nabla^*, \partial_\omega] + \partial_{[\cdot,\cdot]}(\partial_\omega) &= 0\\
\frac{1}{2}[\partial_\omega, \partial_\omega] &= 0.
\end{aligned}$$

The correspondence is characterized by:

$$D\varepsilon = \partial_{\mathcal{K}} \bullet \varepsilon + \partial_{\nabla^*}(\varepsilon) + \partial_\omega \bullet \varepsilon = \partial_{\mathcal{K}} \bullet \varepsilon + \nabla^* \bullet \varepsilon + \partial_{[\cdot,\cdot]}(\varepsilon) + \partial_\omega \bullet \varepsilon.$$

Proof. An action up to homotopy is an operator:

$$D : \bigoplus_{p+q=k} \Omega^p_B(A_B, \Omega^q(\mathcal{K})) \longrightarrow \bigoplus_{p+q=k+1} \Omega^p(A_B, \Omega^q(\mathcal{K})),$$

and therefore it is determined by its restrictions on $\Omega_B^p(A_B, \Omega^q(\mathcal{K}))$ as:

$$D = D_0 + D_1 + D_2 + \ldots,$$

where D_j rises the bidegree by $(j, 1-j)$, that is:

$$D_j : \Omega_B^p(A_B, \Omega^q(\mathcal{K})) \longrightarrow \Omega_B^{p+j,q+1-j}(A_B, \Omega^q(\mathcal{K})).$$

Then by (5.1):

$$\begin{aligned}
D_0(\alpha \cdot \varepsilon) + D_1(\alpha \cdot \varepsilon) + D_2(\alpha \cdot \varepsilon) + \ldots &= (D_0 + D_1 + D_2 + \ldots)(\alpha \cdot \varepsilon) \\
&= D(\alpha \cdot \varepsilon) \\
&= d_B\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot D\varepsilon \\
&= d_B\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot (D_0\varepsilon + D_1\varepsilon + D_2\varepsilon + \ldots) \\
&= d_B\alpha \cdot \varepsilon + (-1)^{|\alpha|}(\alpha \cdot D_0\varepsilon + \alpha \cdot D_1\varepsilon + \alpha \cdot D_2\varepsilon + \ldots) \\
&= d_B\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot D_0\varepsilon + (-1)^{|\alpha|}\alpha \cdot D_1\varepsilon + (-1)^{|\alpha|}\alpha \cdot D_2\varepsilon + \ldots
\end{aligned}$$

Comparing degrees we find

$$D_j(\alpha \cdot \varepsilon) = (-1)^{|\alpha|}\alpha \cdot D(\varepsilon) \tag{5.6}$$

for every $j \neq 1$ and:

$$D_1(\alpha \cdot \varepsilon) = d_B\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot D_1\varepsilon.$$

To sum up, (5.6) together with (5.2) implies:

- D_0 is a $\Omega(A_B)$-linear derivation of $\Omega_B(A_B, \Omega(\mathcal{K}))$ which rises the bidegree by $(0,1)$, hence it corresponds to $d_{\mathcal{K}} \in \Omega_B^0(A_B, \Omega_B^1(\mathcal{K}))$;
- D_1 is a degree one derivation of $\Omega_B(A_B, \Omega(\mathcal{K}))$ which rises the bidegree by $(1,0)$ and satisfy (5.6), hence, by lemma 5.4, it corresponds to $\nabla^* \in \Omega_B^1(A_B, \mathsf{der}^0\ \Omega(\mathcal{K}))$ such that $\nabla_a^*(\pi_0^* f_B) = \pi_0^* \mathcal{L}_{\sharp_{A_B} a}(f_B)$ for every $a \in \Gamma(A_B)$ and $f_B \in C^\infty(B)$;
- D_2 is a $\Omega(A_B)$-linear derivation of $\Omega_B(A_B, \Omega(\mathcal{K}))$ which rises de bidegree by $(2,-1)$, hence it corresponds to $d_\omega \in \Omega_B^2(A_B, \mathsf{der}_B^{-1}(\Omega(\mathcal{K}))$;
- For $j \geq 3$ we have $D_j \in \Omega_B^j(A_B, \mathsf{der}_B^{1-j}(\Omega(\mathcal{K})))$ and since $1-j \leq -2$ we have $D_j = 0$ once $\mathsf{der}_B^r(\Omega(\mathcal{K})) = 0$ for $r \leq -2$ as noticed in remark 5.2.3.

Finally, squaring $D = D_0 + D_1 + D_2$ and comparing degrees we get the identities:

$$\begin{aligned} D_0 \circ D_0 &= 0 \\ D_1 \circ D_0 + D_0 \circ D_1 &= 0 \\ D_1 \circ D_1 + D_0 \circ D_2 + D_2 \circ D_0 &= 0 \\ D_1 \circ D_2 + D_2 \circ D_1 &= 0 \\ D_2 \circ D_2 &= 0. \end{aligned}$$

We show next these equations are equivalent to those we stated. In fact, since $D_0(\varepsilon) = \partial_{\mathcal{K}} \bullet \varepsilon$ one finds

$$\begin{aligned} D_0(D_0(\varepsilon)) &= \partial_{\mathcal{K}} \bullet (\partial_{\mathcal{K}} \bullet \varepsilon) \\ &= (\partial_{\mathcal{K}} \bullet \partial_{\mathcal{K}}) \bullet \varepsilon \\ &= \frac{1}{2}[\partial_{\mathcal{K}}, \partial_{\mathcal{K}}] \bullet \varepsilon. \end{aligned}$$

Since $D_1(\varepsilon) = \nabla^* \bullet \varepsilon + \partial_{[\cdot,\cdot]}(\varepsilon)$ and $\partial_{[\cdot,\cdot]}(\partial_{\mathcal{K}}) = 0$ we get:

$$\begin{aligned} &D_1(D_0(\varepsilon)) + D_0(D_1(\varepsilon)) = D_1(\partial_{\mathcal{K}} \bullet \varepsilon) + D_0(\nabla^* \varepsilon + \partial_{[\cdot,\cdot]}(\varepsilon)) \\ &\quad = \nabla^* \bullet (\partial_{\mathcal{K}} \bullet \varepsilon) + \partial_{[\cdot,\cdot]}(\partial_{\mathcal{K}} \bullet \varepsilon) + \partial_{\mathcal{K}} \bullet (\nabla^* \varepsilon + \partial_{[\cdot,\cdot]}(\varepsilon)) \\ &\quad = (\nabla^* \bullet \partial_{\mathcal{K}}) \bullet \varepsilon + \partial_{[\cdot,\cdot]}(\partial_{\mathcal{K}}) \bullet \varepsilon - \partial_{\mathcal{K}} \bullet \partial_{[\cdot,\cdot]}(\varepsilon) + \partial_{\mathcal{K}} \bullet (\nabla^* \bullet \varepsilon) + \partial_{\mathcal{K}} \bullet \partial_{[\cdot,\cdot]}\varepsilon \\ &\quad = (\nabla^* \bullet \partial_{\mathcal{K}}) \bullet \varepsilon + (\partial_{\mathcal{K}} \bullet \nabla^*) \bullet \varepsilon \\ &\quad = [\nabla^*, \partial_{\mathcal{K}}] \bullet \varepsilon. \end{aligned}$$

Since $D_2(\varepsilon) = \partial_\omega \bullet \varepsilon$ and $\partial^2_{[\cdot,\cdot]} = 0$ we get:

$$\begin{aligned} &D_1(D_1(\varepsilon)) + D_0(D_2(\varepsilon)) + D_2(D_0(\varepsilon))) \\ &\quad = D_1(\nabla^* \bullet \varepsilon + \partial_{[\cdot,\cdot]}(\varepsilon)) + D_0(\partial_\omega \varepsilon) + D_2(\partial_{\mathcal{K}} \bullet \varepsilon) \\ &\quad = \nabla^* \bullet (\nabla^* \bullet \varepsilon + \partial_{[\cdot,\cdot]}(\varepsilon)) + \partial_{[\cdot,\cdot]}(\nabla^* \bullet \varepsilon + \partial_{[\cdot,\cdot]}(\varepsilon)) \\ &\qquad\qquad + \partial_{\mathcal{K}} \bullet (\partial_\omega \bullet \varepsilon) + \partial_\omega \bullet (\partial_{\mathcal{K}} \bullet \varepsilon) \\ &\quad = (\nabla^* \bullet \nabla^*) \bullet \varepsilon + \nabla^* \bullet \partial_{[\cdot,\cdot]}(\varepsilon) + \partial_{[\cdot,\cdot]}(\nabla^*) \bullet \varepsilon - \nabla^* \bullet \partial_{[\cdot,\cdot]}(\varepsilon) \\ &\qquad\qquad + (\partial_{\mathcal{K}} \bullet \partial_\omega) \bullet \varepsilon + (\partial_\omega \bullet \partial_{\mathcal{K}}) \bullet \varepsilon \\ &\quad = \frac{1}{2}[\nabla^*, \nabla^*] \bullet \varepsilon + [\partial_{\mathcal{K}}, \partial_\omega] \bullet \varepsilon + \partial_{[\cdot,\cdot]}(\nabla^*) \bullet \varepsilon. \end{aligned}$$

Next:

$$\begin{aligned}D_1(D_2(\varepsilon)) + D_2(D_1(\varepsilon)) &= D_1(\partial_\omega \bullet \varepsilon) + D_2(\nabla^*\varepsilon + \partial_{[\cdot,\cdot]}(\varepsilon)) \\ &= \nabla^* \bullet (\partial_\omega \bullet \varepsilon) + \partial_{[\cdot,\cdot]}(\partial_\omega \bullet \varepsilon) + \partial_\omega \bullet (\nabla^*\varepsilon) + \partial_\omega \bullet \partial_{[\cdot,\cdot]}(\varepsilon) \\ &= (\nabla^* \bullet \partial_\omega) \bullet \varepsilon + \partial_{[\cdot,\cdot]}(\partial_\omega) \bullet \varepsilon - \partial_\omega \bullet \partial_{[\cdot,\cdot]}(\varepsilon) + (\partial_\omega \bullet \nabla^*) \bullet \varepsilon + \partial_\omega \bullet \partial_{[\cdot,\cdot]}(\varepsilon) \\ &= [\nabla^*, \partial_\omega] \bullet \varepsilon + \partial_{[\cdot,\cdot]}(\partial_\omega) \bullet \varepsilon.\end{aligned}$$

Finally

$$\begin{aligned}D_2(D_2(\varepsilon)) &= D_2(\partial_\omega \bullet \varepsilon) \\ &= \partial_\omega \bullet (\partial_\omega \bullet \varepsilon) \\ &= (\partial_\omega \bullet \partial_\omega) \bullet \varepsilon \\ &= \frac{1}{2}[\partial_\omega, \partial_\omega] \bullet \varepsilon.\end{aligned}$$

□

5.3 Morphisms: Dual Picture

Next, we study morphisms between actions up to homotopy. Again, we shall need to introduce some tools to handle them. Although the $C^\infty(B)$-module of $C^\infty(B)$-linear graded algebra morphisms:

$$\mathsf{Hom}_B(\Omega(\mathcal{K}'), \Omega(\mathcal{K})) := \mathsf{Hom}_{C^\infty(B)}(\Omega(\mathcal{K}'), \Omega(\mathcal{K})),$$

is not a graded algebra, we still can define a $C^\infty(B)$-bilinear map

$$\bullet : \Omega_B(A_B, \mathsf{Hom}_B(\Omega(\mathcal{K}'), \Omega(\mathcal{K}))) \otimes_{C^\infty(B)} \Omega_B(A_B, \Omega(\mathcal{K}')) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$$

by setting

$$(T \bullet \varepsilon)(a_1, \dots, a_{p+q}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma) T(a_{\sigma(1)}, \dots, a_{\sigma(p)})(\varepsilon(a_{\sigma(p+1)}, \dots, a_{\sigma(p+q)}),$$

for every $T \in \Omega^p_B(A_B, \mathsf{Hom}^r_B(\Omega(\mathcal{K}'), \Omega(\mathcal{K})))$ and $\varepsilon \in \Omega^q_B(A_B, \Omega^s(\mathcal{K}'))$.

Next we state the analogue of lemma 5.2.1 for morphisms.

Lemma 5.3.1. Let $\Phi \in \Omega^p_B(A_B, \mathsf{Hom}^r_B(\Omega(\mathcal{K}'), \Omega(\mathcal{K})))$. The left-multiplication

$$\begin{aligned}L_\Phi : \Omega^q_B(A_B, \Omega^s(\mathcal{K}')) &\longrightarrow \Omega^{p+q}_B(A_B, \Omega^{r+s}(\mathcal{K})) \\ \varepsilon &\longmapsto \Phi \bullet \varepsilon\end{aligned}$$

defines a 1-1 correspondence between

- Elements $\Phi \in \Omega^p_B(A_B, \mathsf{Hom}^r_B(\Omega(\mathcal{K}'), \Omega(\mathcal{K})))$;
- Degree zero $\Omega(A_B)$-linear operators from $\Omega_B(A_B, \Omega(\mathcal{K}'))$ to $\Omega_B(A_B, \Omega(\mathcal{K}))$ which increase the bidegree by (p, r).

Proof. The proof is analogue to that of lemma 5.2.1 and therefore we omit it. □

We can also define a $\mathbb{R}$-bilinear pairing

$$\bullet : \Omega_B(A_B, \mathsf{End}\ \Omega(\mathcal{K})) \otimes_{\mathbb{R}} \Omega_B(A_B, \mathsf{Hom}(\Omega(\mathcal{K}'), \Omega(\mathcal{K}))) \longrightarrow \Omega_B(A_B, \mathsf{Hom}(\Omega(\mathcal{K}'), \Omega(\mathcal{K}))),$$

setting

$$(\Phi_1 \bullet \Phi_2)(a_1, \ldots, a_{p+q}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma) \Phi_1(a_{\sigma(1)}, \ldots, a_{\sigma(p)}) \circ \Phi_2(a_{\sigma(p+1)}, \ldots, a_{\sigma(p+q)}),$$

for every $\Phi_1 \in \Omega^p_B(A_B, \mathsf{End}^r\ \Omega(\mathcal{K}))$ and $\varepsilon \in \Omega^q_B(A_B, \Omega^s(\mathcal{K}))$. Analogously, one can define a $\mathbb{R}$-bilinear pairing

$$\bullet : \Omega_B(A_B, \mathsf{Hom}(\Omega(\mathcal{K}'), \Omega(\mathcal{K})) \otimes_{\mathbb{R}} \Omega_B(A_B, \mathsf{End}\ \Omega(\mathcal{K}')) \longrightarrow \Omega_B(A_B, \mathsf{Hom}(\Omega(\mathcal{K}'), \Omega(\mathcal{K}))).$$

These define graded modules structures on $\Omega_B(A_B, \mathsf{Hom}(\Omega(\mathcal{K}'), \Omega(\mathcal{K}))$.

To finish the algebraic *intermezzo*, we define

$$\partial_{[\cdot,\cdot]}(\Phi)(a_1, \ldots, a_{p+1}) := \sum_{\sigma \in \mathsf{Sh}(p,q)} \Phi([a_{\sigma(1)}, a_{\sigma(2)}], a_{\sigma(3)}, \ldots, a_{\sigma(p+1)}),$$

for every $\Phi \in \Omega^p_B(A_B, \mathsf{Hom}^r(\Omega(\mathcal{K}'), \Omega(\mathcal{K}))$ and $a_1, \ldots, a_{p+1} \in \Gamma(A_B)$. This increases the bidegree by $(1, 0)$ and satisfies the compatibility condition:

$$\partial_{[\cdot,\cdot]}(\Phi \bullet \varepsilon) = \partial_{[\cdot,\cdot]}(\Phi) \bullet \varepsilon + (-1)^{|\Phi|} \Phi \bullet \partial_{[\cdot,\cdot]}(\varepsilon)$$

for every $\Phi \in \Omega_B(A_B, \mathsf{Hom}(\Omega(\mathcal{K}'), \Omega(\mathcal{K})))$ and $\varepsilon \in \Omega_B(A_B, \Omega(\mathcal{K}))$ homogeneous.

Finally, we get to the second main result of this chapter:

Theorem 5.3.2. There is a 1-1 correspondence between morphisms of actions up to homotopy

$$\Phi : \Omega_B(A_B, \Omega(\mathcal{K}')) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$$

from $D = \partial_{\mathcal{K}} + \partial_{\nabla^*} + \partial_\omega$ to $D' = \partial_{\mathcal{K}'} + \partial_{\nabla'^*} + \partial_{\omega'}$ and sequences $\{\Phi_j\}_{j \geq 0}$ of elements

$$\Phi_j \in \Omega^j_B(A_B, \mathsf{Hom}^{-j}_B(\Omega(\mathcal{K}'), \Omega(\mathcal{K}))$$

satisfying:

$$\Phi_k \bullet (\varepsilon_1 \wedge \varepsilon_2) = \sum_{i+j=k} \Phi_i \bullet \varepsilon \wedge \Phi_j \bullet \varepsilon$$

and

$$\sum_{i+j \geq k+1} \Phi_i \bullet \varepsilon \wedge \Phi_2 \bullet \varepsilon_2 = 0,$$

for every $k \geq 0$ and for every $\varepsilon_1, \varepsilon_2 \in \Omega_B(A_B, \Omega(\mathcal{K}))$, and which are compatible with D and D' in the sense

$$\begin{aligned}\partial_\omega \bullet \Phi_{k-2} + \nabla^* \bullet \Phi_{k-1} + \partial_{\mathcal{K}} \bullet \Phi_k + \partial_{[\cdot,\cdot]}(\Phi_{k-1})& \\ = \Phi_{k-2} \bullet \partial_{\omega'} + \Phi_{k-1} \bullet \nabla'^* + \Phi_k \bullet \partial_{\mathcal{K}'}&\end{aligned} \tag{5.7}$$

for every $k \geq 0$ where we interpret Φ_{-2} and Φ_{-1} as zero.

The correspondence is characterized by

$$\Phi(\varepsilon) = \Phi_0 \bullet \varepsilon + \Phi_1 \bullet \varepsilon + \Phi_2 \bullet \varepsilon + \ldots,$$

for every $\varepsilon \in \Omega_B(A_B, \Omega(\mathcal{K}))$ homogeneous.

Proof. A morphism of actions up to homotopy is an operator

$$\Phi : \bigoplus_{p+q=k} \Omega^p_B(A_B, \Omega^q(\mathcal{K})) \longrightarrow \bigoplus_{p+q=k} \Omega^p_B(A_B, \Omega^q(\mathcal{K}))$$

hence it is determined by its restriction to $\Omega^p_B(A_B, \Omega^q(\mathcal{K}))$ with $p+q=k$, which decomposes as

$$\Phi = \ldots + \Phi_{-2} + \Phi_{-1} + \Phi_0 + \Phi_1 + \Phi_2 + \ldots$$

where each Φ_j rises the bidegree by $(j, -j)$. Then:

- Since Φ is $\Omega(A_B)$-linear it follows immediately each Φ_j defines a degree zero $\Omega(A_B)$-linear operator from $\Omega_B(A_B, \Omega(\mathcal{K}'))$ to $\Omega_B(A_B, \Omega(\mathcal{K}))$ which rises the bidegree by $(j, -j)$. Therefore we can realise each Φ_j as the left multiplication by $\Phi_j \in \Omega^j_B(A_B, \mathsf{Hom}^{-j}_B(\Omega(\mathcal{K}')$ In particular, $\Phi_j = 0$ for every $j < 0$ and therefore:

$$\Phi(\varepsilon) = \Phi_0 \bullet \varepsilon + \Phi_1 \bullet \varepsilon + \Phi_2 \bullet \varepsilon + \ldots,$$

  for every $\varepsilon \in \Omega_B(A_B, \Omega(\mathcal{K}))$ homogeneous.

- Using the previous decomposition, it follows that:

$$\begin{aligned}\Phi(\varepsilon_1 \wedge \varepsilon_2) &= \left(\sum_j \Phi_j\right)(\varepsilon_1 \wedge \varepsilon_2)\\ &= \sum_j \Phi_j \bullet (\varepsilon_1 \wedge \varepsilon_2),\end{aligned}$$

whereas

$$\begin{aligned}\Phi(\varepsilon_1) \wedge \Phi(\varepsilon_2) &= \left(\sum_i \Phi_i\right) \bullet \varepsilon_1 \wedge \left(\sum_j \Phi_j\right) \bullet \varepsilon_2\\ &= \left(\sum_i \Phi_i \bullet \varepsilon_1\right) \wedge \left(\sum_j \Phi_j \bullet \varepsilon_2\right)\\ &= \sum_{i,j} \Phi_i \bullet \varepsilon_1 \wedge \Phi_j \bullet \varepsilon_2.\end{aligned}$$

Looking at the homogeneous components we find that

$$\Phi(\varepsilon_1 \wedge \varepsilon_2) = \Phi\varepsilon_1 \wedge \Phi\varepsilon_2$$

holds, if and only if:

$$\Phi_k \bullet (\varepsilon_1 \wedge \varepsilon_2) = \sum_{i+j=k} \Phi_i \bullet \varepsilon_1 \wedge \Phi_2 \bullet \varepsilon_2$$

and:

$$\sum_{i+j\geq k+1} \Phi_i \bullet \varepsilon_1 \wedge \Phi_2 \bullet \varepsilon_2 = 0,$$

for every $k \geq 0$.

- As to the compatibility conditions lets us write temporarily $\Upsilon_0 = \partial_{\mathcal{K}}$, $\Upsilon_1 = L_{\nabla^*}$ and $\Upsilon_2 = \partial_\omega$ and analogously for D'. Then we have:

$$D(\Phi(\varepsilon)) = \sum_{j,i} (\Upsilon_j \bullet \Phi_i) \bullet \varepsilon + \sum_i \partial_{[\cdot,\cdot]}(\Phi_i) \bullet \varepsilon + \sum_i \Phi_i \bullet \partial_{[\cdot,\cdot]}(\varepsilon).$$

On the other hand:

$$\Phi(D'(\varepsilon)) = \sum_{i,j} \Phi_i \bullet \Upsilon'_j + \sum_i \Phi_i \bullet \partial_{[\cdot,\cdot]}(\varepsilon).$$

Therefore, $\Phi \circ D = D' \circ \Phi$ if and only if

$$\sum_{i,j} \Upsilon_j \bullet \Phi_i + \sum_i \partial_{[\cdot,\cdot]}(\Phi_i) = \sum_{i,k} \Phi_i \bullet \Upsilon'_k.$$

Looking at the homogeneous components one finds for every $k \geq 0$

$$\sum_{i+j=k} \Upsilon_j \bullet \Phi_i + \partial_{[\cdot,\cdot]}(\Phi_{k-1}) = \sum_{i+j=k} \Phi_i \bullet \Upsilon'_j.$$

This gives the compatibility conditions (5.7).

□

Let us write down the compatibility conditions (5.7) for low values of k:

- For $k = 0$ the condition becomes

$$\partial_{\mathcal{K}} \bullet \Phi_0 = \Phi_0 \bullet \partial_{\mathcal{K}'}.$$

- For $k = 1$, taking into account $\partial_{[\cdot,\cdot]}(\Phi_0) = 0$, the condition becomes

$$\nabla^* \bullet \Phi_0 + \partial_{\mathcal{K}} \bullet \Phi_1 = \Phi_1 \bullet \partial_{\mathcal{K}'} + \Phi_0 \bullet \nabla'^*.$$

- For $k = 2$, one finds

$$\begin{aligned} &\partial_\omega \bullet \Phi_0 + \nabla^* \bullet \Phi_1 + \partial_{\mathcal{K}} \bullet \Phi_2 + \partial_{[\cdot,\cdot]}(\Phi_1) \\ &\qquad = \Phi_2 \bullet \partial_{\mathcal{K}'} + \Phi_1 \bullet \nabla'^* + \Phi_0 \bullet \partial_{\omega'}. \end{aligned}$$

5.4 Extensions of Lie Algebroids as Actions up to Homotopy

In this section we prove the category of actions up to homotopy of A_B is equivalent to the category of extensions of A_B. In order to do that we need the following:

Lemma 5.4.1. Let $\mathcal{K} \hookrightarrow A_E \overset{\pi}{\twoheadrightarrow} A_B$ be an extension covering $\pi_0 : E \longrightarrow B$ and let $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$ be a horizontal lift. The induced Lie algebroid isomorphism

$$\Sigma^h : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow A_E$$

induces an isomorphism of graded algebras and of $\Omega(A_B)$-graded left modules

$$(\Sigma^h)^* : \Omega(A_E) \longrightarrow \Omega(\pi_0^* A_B \oplus \mathcal{K})$$

where the $\Omega(A_B)$-graded left-module structure on $\Omega(\pi_0^*A_B \oplus \mathcal{K})$ is induced by the morphism of graded algebras $\mathrm{pr}_{A_B}^*$. In particular, $\Omega(A_E) \simeq \Omega_B(A_B, \Omega(\mathcal{K}))$.

Proof. We know $\Sigma^h : \pi_0^*A_B \oplus \mathcal{K} \longrightarrow A_E$ is an isomorphism of Lie algebroids. Hence:

$$\left(\Sigma^h\right)^* : \Omega(A_E) \longrightarrow \Omega(\pi_0^*A_B \oplus \mathcal{K}),$$

is an isomorphism of graded algebras. We are left to check $\left(\Sigma^h\right)^*$ preserves the $\Omega(A_B)$-graded left-module structure. In fact, let $\alpha \in \Omega^p(A_B)$. Then:

$$\begin{aligned}
\langle \left(\Sigma^h\right)^* \pi^*\alpha_B, \pi_0^*a_1 \oplus k_1, \ldots, \pi_0^*a_p \oplus k_p\rangle &= \langle \pi^*\alpha_B, h(a_1) + k_1, \ldots, h(a_p) + k_p\rangle \\
&= \langle \pi_0^*\alpha_B, \pi(h(a_1) + k_1), \ldots, \pi(h(a_p) + k_p)\rangle \\
&= \langle \pi_0^*\alpha_B, a_1 \circ \pi_0, \ldots, a_p \circ \pi_0\rangle \\
&= \langle \mathrm{pr}_{A_B}^*\alpha_B, \pi_0^*a_1 \oplus k_1, \ldots, \pi_0^*a_p \oplus k_p\rangle.
\end{aligned}$$

Therefore $\left(\Sigma^h\right)^* (\pi^*\alpha_B) = \mathrm{pr}_{A_B}^*\alpha_B$. The last claim follows by following the sequence of canonical isomorphisms:

$$\begin{aligned}
\Omega(\pi_0^*A_B \oplus \mathcal{K}) &\simeq \Omega(p^*A_B) \otimes_{C^\infty(E)} \Omega(\mathcal{K}) \\
&\simeq \Omega(A_B) \otimes_{C^\infty(B)} C^\infty(E) \otimes_{C^\infty(E)} \Omega(\mathcal{K}) \\
&\simeq \Omega_B(A_B, \Omega(\mathcal{K})).
\end{aligned}$$

The explicit isomorphism $\Xi^{-1} : \Omega_B(A_B, \Omega(\mathcal{K})) \longrightarrow \Omega(\pi_0^*A_B \oplus \mathcal{K})$ sends $\varepsilon \in \Omega_B^p(A_B, \Omega^q(\mathcal{K}))$ to:

$$\Xi\varepsilon(\pi_0^*a_1 \oplus k_1, \ldots, \pi_0^*a_{p+q} \oplus k_{p+q}) = \sum_{\sigma \in \mathsf{Sh}(p,q)} \mathsf{sgn}(\sigma)\varepsilon(a_{\sigma(1)}, \ldots, a_{\sigma(p)})(k_{\sigma(p+1)}, \ldots, k_{\sigma(p+q)}).$$

The inverse has components:

$$\langle (\Xi\varepsilon)^{p,q} (a_1, \ldots, a_p), k_1, \ldots, k_q\rangle := \langle \varepsilon, \pi_0^*a_1, \ldots, \pi_0^*a_p, k_1, \ldots, k_q\rangle,$$

for every $\varepsilon \in \Omega^k(\pi_0^*A_B \oplus \mathcal{K})$. □

Using this lemma we prove the main result of this section, namely:

Theorem 5.4.2. There is an equivalence of categories between $\mathsf{Act}_\infty(A_B)$ and $\mathsf{Ext}(A_B)$.

Proof. Let us define functors

$$F : \mathsf{Ext}(A_B) \longrightarrow \mathsf{Act}_\infty(A_B) \quad \text{e} \quad G : \mathsf{Act}_\infty(A_B) \longrightarrow \mathsf{Ext}(A_B)$$

which will provide the desired equivalence.

- The functor F, on objects, is defined as follows: For each extension $\mathcal{K} \hookrightarrow A_E \overset{\pi}{\twoheadrightarrow} A_B$ we fix a horizontal lift $h : \Gamma(A_B) \longrightarrow \Gamma(A_B)$ (which is possible due to the axiom of choice). The functor F then associates an action up to homotopy of A_B on $\mathcal{K}$ along π_0 transporting the differential of A_E as follows:

$$\begin{tikzcd}
\Omega(A_E) \arrow[r, "\Sigma^h"] \arrow[d, "d_{A_E}"'] & \Omega(p^*A_B \oplus \mathcal{K}) \arrow[r, "\Xi"] \arrow[d, "d_{p^*A_B\oplus\mathcal{K}}"'] & \Omega_B(A_B, \Omega(\mathcal{K})) \arrow[d, "D"] \\
\Omega^\bullet(A_E) \arrow[r, "\Sigma^h"'] & \Omega(p^*A_B \oplus \mathcal{K}) \arrow[r, "\Xi"'] & \Omega_B(A_B, \Omega(\mathcal{K})
\end{tikzcd}$$

that is, we set:

$$D := (\Xi \circ \Sigma^h) d_{A_E} (\Xi \circ \Sigma^h)^{-1}$$

Then:

$$\begin{aligned}
D^2 &= (\Xi \circ \Sigma^h) d_{A_E} (\Xi \circ \Sigma^h)^{-1} \circ (\Xi \circ \Sigma^h) d_{A_E} (\Xi \circ \Sigma^h)^{-1} \\
&= (\Xi \circ \Sigma^h) d_{A_E}^2 (\Xi \circ \Sigma^h)^{-1} \\
&= 0.
\end{aligned}$$

Furthermore, since $\Xi \circ \Sigma^h$ is an isomorphism of graded algebras one gets

$$\begin{aligned}
D(\varepsilon_1 \wedge \varepsilon_2) &= (\Xi \circ \Sigma^h) d_{A_E} \big((\Xi \circ \Sigma^h)\varepsilon_1 \wedge (\Xi \circ \Sigma^h)\varepsilon_2 \big) \\
&= (\Xi \circ \Sigma^h) d_{A_E} (\Xi \circ \Sigma^h)\varepsilon_1 \wedge (\Xi \circ \Sigma^h)(\Xi \circ \Sigma^h)^{-1}\varepsilon_2 \\
&+ (-1)^{|\varepsilon_1|} (\Xi \circ \Sigma^h)(\Xi \circ \Sigma^h)^{-1}\varepsilon_1 \wedge (\Xi \circ \Sigma^h) d_{A_E} (\Xi \circ \Sigma^h)^{-1}\varepsilon_2 \\
&= D\varepsilon_1 \wedge \varepsilon_2 + (-1)^{|\varepsilon_1|}\varepsilon_1 \wedge D\varepsilon_2.
\end{aligned}$$

Finally, writing $\jmath^* : \Omega(A_B) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$ for

$$\langle \jmath^*\varepsilon, a_1, \ldots, a_p \rangle := \pi_0^* \langle \varepsilon, a_1, \ldots, a_p \rangle,$$

the $\Omega(A_B)$-module structure on $\Omega_B(A_B, \Omega(\mathcal{K}))$ reads

$$\alpha_B \cdot \varepsilon = \jmath^*\alpha_B \wedge \varepsilon.$$

Then, using that Σ and Ξ are isomorphism of $\Omega(A_B)$-graded left-modules we find:

$$(\Xi \circ \Sigma^h)^{-1}(\jmath^*\alpha_B) = \pi^*\alpha_B$$

and

$$\left(\Xi \circ \Sigma^h\right)(\pi^* d_{A_B}\alpha_B) = \mathrm{pr}_B^* d_{A_B}\alpha_B$$

hence, using $d_{A_E} \circ \pi^* = \pi^* \circ d_{A_B}$ we get

$$\begin{aligned} D(\jmath^*\alpha_B) &= (\Xi \circ \Sigma^h) d_{A_E} (\Xi \circ \Sigma^h)^{-1}(\jmath^*\alpha_B) \\ &= (\Xi \circ \Sigma^h) d_{A_E}(\pi^*\alpha_B) \\ &= (\Xi \circ \Sigma^h)\pi^*(d_{A_B}\alpha_B) \\ &= \mathrm{pr}_B^*(d_{A_B}\alpha_B) \end{aligned}$$

so that:

$$\begin{aligned} D(\alpha_B \cdot \varepsilon) &= D(\jmath^*\alpha_B \wedge \varepsilon) \\ &= D(\jmath^*\alpha_B) \wedge \varepsilon + (-1)^{|\alpha_B|}\jmath^*\alpha_B \wedge D\varepsilon \\ &= \mathrm{pr}_B^*(d_{A_B}\alpha_B) \wedge \varepsilon + (-1)^{|\alpha_B|}\jmath^*\alpha_B \wedge D\varepsilon \\ &= d_{A_B}\alpha_B \cdot \varepsilon + (-1)^{|\alpha_B|}\alpha_B \cdot D\varepsilon. \end{aligned}$$

This proves that D is indeed an action up to homotopy.

- The functor F, at the level of morphisms, is defined as follows: Given a morphism of extensions:

$$\begin{tikzcd} \mathcal{K} \arrow[r] \arrow[d] & A_E \arrow[r] \arrow[d, "\Phi_{\mathrm{ext}}"] & A_B \arrow[d] \\ \mathcal{K}' \arrow[r] & A_{E'} \arrow[r] & A_B \end{tikzcd}$$

  we define the corresponding morphism of actions up to homotopy

$$\Phi : \Omega_B(A_B, \Omega(\mathcal{K}')) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$$

  transporting the morphism $\Phi_{\mathrm{ext}}^* : \Omega(A_{E'}) \longrightarrow \Omega(A_E)$ as follows:

$$\Phi = (\Xi \circ \Sigma^{h'}) \circ \Phi_{\mathrm{ext}}^* \circ (\Xi \circ \Sigma^h)^{-1}$$

  where $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$ and $h' : \Gamma(A_B) \longrightarrow \Gamma(A_{E'})$ are the horizontal lifts we choose in order to obtain the corresponding action up to homotopy. Now a straightforward argument shows Φ is indeed a morphism of actions up to homotopy.

Next, let us define the functor $G : \mathsf{Act}_\infty(A_B) \longrightarrow \mathsf{Ext}(A_B)$.

- At the level of objects, G is defined as follows: Given D be an action up to homotopy of A_B on $\mathcal{K}$ along π_0, we define a Lie algebroid structure on $\pi_0^* A_B \oplus \mathcal{K}$ defining

$$d := \Xi^{-1} \circ D \circ \Xi.$$

By a similar argument we used above we show easily $d^2 = 0$ and

$$d(\varepsilon_1 \wedge \varepsilon_2) = d\varepsilon_1 \wedge \varepsilon_2 + (-1)^{|\varepsilon_1|}\varepsilon_1 \wedge d\varepsilon_2.$$

The vector bundle morphism $\mathrm{pr}_{A_B} : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow A_B$ is a submersion covering the surjective submersion π_0 and it is a Lie algebroid morphism since

$$\begin{aligned} D(\jmath^* \alpha_B) &= D(\alpha_B \cdot 1_{\mathbb{R}}) \\ &= d_{A_B}\alpha_B \cdot 1_{\mathbb{R}} \\ &= \jmath^* d_{A_B}\alpha_B, \end{aligned}$$

implies

$$\begin{aligned} d(\mathrm{pr}_{A_B}^* \alpha_B) &= \Xi^{-1}(D(\Xi(\mathrm{pr}_{A_B}^* \alpha_B)) \\ &= \Xi^{-1} D(\jmath_B^* \alpha_B) \\ &= \Xi^{-1}(\jmath^* d_{A_B}\alpha_B) \\ &= \mathrm{pr}_{A_B}^*(d_{A_B}\alpha_B). \end{aligned}$$

Therefore, $\mathrm{pr}_{A_B} : \pi_0^* A_B \oplus \mathcal{K} \longrightarrow A_B$ is an extension of A_B with kernel $\mathcal{K}$.

- At the level of morphisms, G is defined as follows: Given a morphism of actions up to homotopy $\Phi : \Omega_B(A_B, \Omega(\mathcal{K}')) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$ we set:

$$\Phi_{\mathrm{ext}}^* := \Xi^{-1} \circ \Phi \circ \Xi',$$

where $\Xi : \Omega(\pi_0^* A_B \oplus \mathcal{K}) \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}))$ and $\Xi' : \Omega(\pi_0'^* A_B \oplus \mathcal{K}') \longrightarrow \Omega_B(A_B, \Omega(\mathcal{K}'))$ are the canonical isomorphisms. Now it is straightforward to check this defines a morphisms of extensions.

Finally, by the very construction, G is fully faithful. It is also dense, for any extension $\mathcal{K} \hookrightarrow A_E \overset{\pi_0}{\twoheadrightarrow} A_E$ after the choice of a splitting h becomes isomorphic to $\pi_0^* A_B \oplus_{(D,\omega)} \mathcal{K}$ which is precisely the image by G of the transported differential of A_E. □

The proof of the previous theorem is conceptually clear, however, not very practical since we do not provide very explicit formulas. In the next section we adopt a more concrete approach.

5.5 Practical Considerations

It is interesting to write down explicitly how to pass from extensions to actions up to homotopy and vice-versa. In this section we summarize the previous discussions in order to have a more concrete view of the theory.

Let us consider an extension $\mathcal{K} \hookrightarrow A_E \overset{\pi}{\twoheadrightarrow} A_B$. As explained in section 4.3, after the choice of a horizontal lift, the structure of this extension is fully encoded on $\mathcal{K}$ and on a pair (∇, ω) consisting of $\nabla \in \Omega^1(A_B, \mathsf{der}(\mathcal{K}))$ and of $\omega \in \Omega^2(A_B, \mathcal{K})$ satisfying compatibility conditions (4.7). The corresponding action up to homotopy is given by:

$$D = D_0 + D_1 + D_2,$$

where:

$$\begin{aligned}\langle D_0\varepsilon(a_1,\dots,a_p),k_1,\dots,k_{q+1}\rangle\\ &= \sum_{j=1}^{q+1}\mathcal{L}_{\sharp^{\mathcal{K}}k_j}(\langle\varepsilon(a_1,\dots,a_p),k_1,\dots,\widehat{k_j},\dots,k_{q+1}\rangle)\\ &+\sum_{i<j}\langle\varepsilon(a_1,\dots,a_p),[k_i,k_j],\dots,\widehat{k_i},\dots,\widehat{k_j},\dots,k_{q+1}\rangle.\end{aligned}$$

whereas

$$\begin{aligned}\langle D_1\varepsilon(a_1,\dots,a_{p+1}),k_1,\dots,k_q\rangle\\ &=\sum_{j=1}^{p+1}(-1)^{j+1}X^{\nabla_{a_j}}(\langle\varepsilon(a_1,\dots,\widehat{a_j},\dots,a_{p+1},k_1,\dots,k_q\rangle)\\ &-\sum_{j=1}^{p+1}(-1)^{j+1}\sum_{i=1}^{q}\langle\varepsilon(a_1,\dots,\widehat{a_j},\dots,a_p),k_1,\dots,\nabla_{a_j}k_i,\dots,k_q\rangle\\ &+\sum_{i<j}(-1)^{i+j}\langle\varepsilon([a_i,a_j],a_1,\dots,\widehat{a_i},\dots,\widehat{a_j},\dots,a_{p+1}),k_1,\dots,k_q\rangle.\end{aligned}$$

and

$$\begin{aligned}\langle D_2\varepsilon(a_1,\dots,a_{p+2}),k_1,\dots,k_{q-1}\rangle\\ &=\sum_{i<j}(-1)^{i+j}\langle\varepsilon(a_1,\dots,\widehat{a_i},\dots,\widehat{a_j},\dots,a_{p+2}),\omega(a_i,a_j),k_1,\dots,k_{q-1}\rangle.\end{aligned}$$

There are some interesting points we would like to emphasize:

- The above formulas are precisely those corresponding to $\partial_{\mathcal{K}}\bullet\varepsilon$, $\partial_{\nabla}(\varepsilon)$ and $\partial_\omega\bullet\varepsilon$ we got in the previous sections;
- The operator D_0 is nothing but the natural extension of the differential of $\mathcal{K}$ to all $\Omega_B(A_B,\Omega(\mathcal{K}))$.
- For every $a\in\Gamma(A_B)$ we have an induced degree 0 derivation:

$$\nabla^*_a:\Omega(\mathcal{K})\longrightarrow\Omega(\mathcal{K})$$

which is given by:

$$\langle \nabla_a^* \varepsilon, k_1, \ldots, k_q \rangle := X^{\nabla_a} \langle \varepsilon, k_1, \ldots, k_q \rangle - \sum_{j=1}^{q} \langle \varepsilon, k_1, \ldots, \nabla_a k_j, \ldots, k_q \rangle,$$

for every $\varepsilon \in \Omega^q(\mathcal{K})$ and $k_1, \ldots, k_q$. This entirely determines D_1.

- For every pair of sections $a, b \in \Gamma(A_B)$ we have an induced degree -1 derivation:

$$\imath_{\omega(a,b)} : \Omega(\mathcal{K}) \longrightarrow \Omega(\mathcal{K})$$

given by:

$$\langle \imath_{\omega(a,b)} \varepsilon, k_1, \ldots, k_{q-1} \rangle := \langle \varepsilon, \omega(a,b), k_1, \ldots, k_{q-1} \rangle.$$

for every $\varepsilon \in \Omega^q(\mathcal{K})$ and $k_1, \ldots, k_{q-1}$. This extends via anti-symmetrization to D_2.

On the other hand, given an action up to homotopy $D = D_0 + D_1 + D_2$ of A_B on $\mathcal{K}$ along $\pi_0 : E \longrightarrow B$ the corresponding extension is obtained as follows:

- The Lie algebroid structure on $\mathcal{K}$ is obtained via the degree 1 differential:

$$d : \Omega(\mathcal{K}) \longrightarrow \Omega(\mathcal{K})$$

given by:

$$\langle d\varepsilon, k_1, \ldots, k_{q+1} \rangle := \langle D_0 \varepsilon, k_1, \ldots, k_{q+1} \rangle$$

for every $\varepsilon \in \Omega^q(\mathcal{K}) \simeq \Omega^0_B(A_B, \Omega^q(\mathcal{K}))$ and $k_1, \ldots, k_{q-1}$. The $\Omega(A_B)$-linearity of D_0 implies d is $C^\infty(B)$-linear and this implies $\sharp_{\mathcal{K}}(\mathcal{K}) \subset \mathsf{Ker}(d\pi_0)$. In fact, for any $f_B \in C^\infty(B)$ we have:

$$\begin{aligned} 0 &= \pi_0^* f_B \langle d(1_{\mathbb{R}}), k \rangle \\ &= \langle d(\pi_0^* f_B), k \rangle \\ &= \mathcal{L}_{\sharp_{\mathcal{K}} k}(\pi_0^* f_B) \\ &= df_B \circ d\pi_0 \circ \sharp_{\mathcal{K}} k. \end{aligned}$$

- The 1-form $\nabla \in \Omega(A_B, \mathsf{der}_B(\mathcal{K}))$ is uniquely determined by the fact:

$$\langle \varepsilon, \nabla_a k \rangle = D_1(\langle \varepsilon, k \rangle)(a) - \langle D_1 \varepsilon(a), k \rangle.$$

for every $\varepsilon \in \Omega^1(\mathcal{K})$, $a \in \Gamma(A_B)$ and $k \in \Gamma(\mathcal{K})$. Notice the symbol of ∇ is simply the restriction of D_1 to functions.

- Finally, $\omega \in \Omega^2(A_B, \mathcal{K})$ is uniquely determined by:

$$\langle \varepsilon, \omega(a,b) \rangle := D_2\varepsilon(a,b),$$

for every $\varepsilon \in \Omega^1(\mathcal{K}) \simeq \Omega^0_B(A_B, \Omega^1(\mathcal{K}))$ and $a, b \in \Gamma(A_B)$.

5.6 Actions up to Homotopy vs Representations up to Homotopy

In this section we discuss how actions up to homotopy relate to 2-term representations up to homotopy. To explain that, let us start discussing two ways of seeing a representations of a Lie algebroid. In order to to that, let us fix a Lie algebroid A over M and V a vector bundle over M.

Linear representations of A on V can be dealt with:

- *An Algebraic Approach.* As degree one $\mathbb{R}$-linear operators:

$$D : \Omega(A) \otimes \Gamma(V) \longrightarrow \Omega(A) \otimes \Gamma(V)$$

such that $D^2 = 0$ and:

$$D(\alpha \cdot \varepsilon) = d_A\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot D\varepsilon$$

for every $\alpha \in \Omega(A)$ and $\varepsilon \in \Omega(A, E)$ homogeneous.

- *A Geometric Approach.* As *semi-direct products*, that is, as Lie algebroid structures on the vector bundle $A \oplus V \longrightarrow M$ which turn the projection $A \oplus V \longrightarrow A$ into a morphism of Lie algebroids.

The algebraic approach was taken in [2] as the starting point to defining representations up to homotopy. The idea was simply to replace the vector bundle V by a graded vector bundle over M. Notice this replacement is purely algebraic and the result has a *covariant nature* since every morphism of graded vector bundles from V to V' induces a morphism of representations up to homotopy $\Phi : \Omega(A, V) \longrightarrow \Omega(A, V')$.

The main difference between our approach of actions up to homotopy and the usual approach of representations up to homotopy is that we take the geometric approach as a starting point. Let us clear this out. In the geometric approach, the representations of A on V are parametrized by Lie algebroid structures on $A \oplus V$ which turn the projection $A \oplus V \longrightarrow A$ into an extension of Lie algebroids. Such structures, as we have pointed out in proposition 4.1.12, are equivalent to degree one $\mathbb{R}$-linear derivations:

$$D : \Omega(A) \otimes \Omega(V) \longrightarrow \Omega(A) \otimes \Omega(V)$$

which are compatible with the evident $\Omega(A)$-module structure on $\Omega(A) \otimes \Omega(V)$ in the sense:

$$D(\alpha \cdot \varepsilon) = d_A\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha \cdot D\varepsilon$$

for every $\alpha \in \Omega(A)$ and $\varepsilon \in \Omega(A) \otimes \Omega(V)$ homogeneous. Notice that this approach has a *contravariant nature* since every morphism of (graded) vector bundles from V to V' induces a morphism $\Phi : \Omega(A, V') \longrightarrow \Omega(A, V)$.

In order to pass from the algebraic point of view to the geometric one and *vice-versa* it suffices to notice that representations of A on V are equivalent to representations of A on V^*. Anyway, from the geometer's point of view, we may contend that the geometric approach is more natural than the algebraic one.

Chapter 6

Integration of Actions up to Homotopy

In this final chapter we discuss the basic building blocks of an idealized integration theory for actions up to homotopy via 2-categorical methods. In light of our previous discussions, this integration theory should also provide a way to integrate extensions. The motivation for this chapter comes from [13] where the authors undertake the purpose of integrating 2-term representations up to homotopy via 2-functors. In the first section we recall their integration theory. In the subsequent section, to an action up to homotopy we associate a certain kind of holonomy. The definition of holonomy we give is not entirely new, it is merely an abstraction of the respective definition found in the work [12] where the author defines holonomy in the context of extensions of Lie algebroids. Subsequently, we present how such holonomy gives rise to a strict 2-functor, which should be thought of as a higher action of a Weinstein 2-groupoid in a certain gauge 2-groupoid. Afterwards, we discuss morphism of actions up to homotopy in this 2-categorical context. Finally, we indicate how the theory can be developed further in a future research.

6.1 The Holonomy 2-Representation

In this section we discuss briefly the integration of 2-term representations up to homotopy via strict 2-functors which first appeared in [13]. We start recalling one of the main results in [13]:

Theorem 6.1.1. Let $(\nabla = (\nabla^E, \nabla^C), \omega, \partial)$ be a 2-term representation of a Lie algebroid over M on the graded vector bundle $E \oplus C \longrightarrow M$ and let $\mathcal{E} := (E \oplus C \rightrightarrows E)$ be the 2-vector bundle corresponding to $\partial : C \longrightarrow E$. There is a strict 2-functor:

$$\mathsf{Hol} : \mathsf{P}(A) \longrightarrow \text{2-}\mathsf{Gau}(\mathcal{E})$$

defined by the assignment:

- for each A-path $a \cdot dt : TI \longrightarrow A$ from x to y, $\mathsf{Hol}(a)$ is the linear functor:

$$\begin{tikzcd} E_x \oplus C_x \arrow[r, "(\mathsf{Hol}^E(a),\mathsf{Hol}^C(a))"] \arrow[d, shift left] \arrow[d, shift right] & E_y \oplus C_y \arrow[d, shift left] \arrow[d, shift right] \\ E_x \arrow[r, "\mathsf{Hol}(a)"'] & E_y \end{tikzcd}$$

where Hol^E and Hol^C denote the holonomy of ∇^E and ∇^C, respectively.

- for each A-homotopy $\sigma = a \cdot dt + b \cdot dt : TI^2 \longrightarrow A$ the assignment

$$\mathsf{Hol}(\sigma) : E_x \longrightarrow E_y \oplus C_y$$

is defined by:

$$\mathsf{hol}(\sigma) := \left(\mathsf{Hol}^E(a^1) - \mathsf{Hol}^E(a_0), \int_0^1 \int_0^1 \mathsf{Hol}^C_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s(t)} \circ \mathsf{Hol}_{a^s_{t,0}}(-) \; ds \; dt \right).$$

The previous theorem leads to the following:

Definition 6.1.2. The 2-functor $\mathsf{Hol} : \mathcal{P}(A) \longrightarrow 2\text{-}\mathsf{Gau}(\mathcal{E})$ is called the *holonomy 2-representation* associated to the 2-term representation up to homotopy $(\nabla^E, \nabla^C, \omega, \partial)$ of A on $E \oplus C$.

There are a few points to be emphasized:

- The definitions of the Weinstein 2-groupoid $\mathcal{P}(A)$ and of the gauge 2-groupoid $2\text{-}\mathsf{Gau}(\mathcal{E})$ are discussed in section 2.5;
- In the work [13], they do not use $2\text{-}\mathsf{Gau}(\mathcal{E})$ as we defined. Instead, their definition of this gauge 2-groupoid is based upon 2-term cochain complexes of vector bundles. However, our approach is equivalent since 2-term cochain complexes of vector bundles are equivalent to 2-vector bundles;
- The 2-functor $\mathsf{Hol} : \mathcal{P}(A) \longrightarrow 2\text{-}\mathsf{Gau}(\mathcal{E})$ can be seen as a kind of linear action of $\mathcal{P}(A)$ on $\mathcal{E}$.

Associated to this representation there is a transformation 2-groupoid:

Definition 6.1.3. The *transformation 2-groupoid* associated to the holonomy 2-functor $\mathsf{Hol} : \mathsf{P}(A) \longrightarrow 2\text{-}\mathsf{Gau}(\mathcal{E})$ is the strict 2-groupoid $\mathcal{P}(A) \ltimes_{\mathsf{Hol}} \mathcal{E}$ defined as follows:

- *objects:* the space of objects of $\mathcal{P}(A) \ltimes_{\mathsf{Hol}} \mathcal{E}$ is given by E;
- *1-morphisms:* the space of 1-morphisms of $\mathcal{P}(A) \ltimes_{\mathsf{Hol}} \mathcal{E}$ is:

$$P_1(A) \ltimes \mathcal{E} := t^*C \oplus s^*E$$

where $s, t : P_1(A) \longrightarrow M$ are the source and target maps of 1-morphisms of the Weinstein 2-groupoid. The source, target, and inverse maps of 1-morphisms are given as follows:

$$\begin{aligned}
\tilde{s}(c, a, e) &= e \\
\tilde{t}(c, a, e) &= \mathsf{Hol}_a^E(e) + \partial c \\
(c, a, e)^{-1}) &= (-\mathsf{Hol}_{a^{-1}}^C(c), a^{-1}, \mathsf{Hol}_a^E(e) + \partial c),
\end{aligned}$$

two 1-morphisms (c_1, a_1, e_1) and (c_0, a_0, e_0) are composable provided $e_1 = \mathsf{Hol}_{a_0}^E(e_0) + \partial c_0$ and their multiplication is given by:

$$(c_1, a_1, e_1) \cdot (c_0, a_0, e_0) = (c_1 + \mathsf{Hol}_{a_1}^C(c_0), a_1 \cdot a_0, e_0),$$

- *2-morphisms:* the space of 2-morphisms in $\mathcal{P}(A) \ltimes_{\mathsf{Hol}} \mathcal{E}$ is given by:

  $$P_2(A) \ltimes_{\mathsf{Hol}} \mathcal{E} := t_H^* C \oplus s_H^* E$$

  where $s_H, t_H : P_2(A) \longrightarrow M$ are the horizontal source and target maps of the Weinstein 2-groupoid;

- *vertical structure maps:* the vertical source and target maps $\tilde{s}_V, \tilde{t}_V : P_2(A) \ltimes_{\mathsf{Hol}} \mathcal{E} \longrightarrow P_1(A) \ltimes_{\mathsf{Hol}} \mathcal{E}$ are given by:

  $$\begin{aligned}
  \tilde{s}_V(c, \sigma, e) &:= (c, s_V(\sigma), e) \\
  \tilde{t}_V(c, \sigma, e) &:= (c - \mathsf{Hol}(\sigma)_e, t_V(\sigma), e).
  \end{aligned}$$

  Two 2-morphisms (c_2, σ_2, e_2) and (c_1, σ_1, e_1) are composable vertically provided $s_V(\sigma_2) = t_V(\sigma_1) \in P_1(A)$ and $c_2 = c_1 + \mathsf{Hol}(\sigma_1)(e_1)$, and their vertical multiplication is given by:

  $$(c_2, \sigma_2, e_2) \bullet_V (c_1, \sigma_1, e_1) = (c_1, \sigma_2 \bullet_V \sigma_1, e_1).$$

  The vertical inverse of a 2-morphism (c, σ, e) is defined by the following formula:

  $$(c, \sigma, e)^{-1_V} := (c - \mathsf{Hol}(\sigma)_e, \sigma^{-1_V}, e).$$

- *horizontal structure maps:* the horizontal source and target maps are given by:

  $$\begin{aligned}
  \tilde{s}_H(c, \sigma, e) &:= e \\
  \tilde{t}_H(c, \sigma, e) &= \mathsf{Hol}_{t_V(\sigma)}(e) + \partial c.
  \end{aligned}$$

The horizontal composition and horizontal inverse are given by:

$$(c,\sigma,e)\bullet_H(b,\tau,f) := (c+\mathsf{Hol}^C_{s_V(\sigma)}(b),\sigma\bullet_H\tau,f)$$
$$(c,\sigma,e)^{-1_H} := (\mathsf{Hol}^C_{s_V(\sigma)}(c),\sigma^{-1_H},e).$$

Then, according to [13]:

Theorem 6.1.4. Let A be a Lie algebroid over M. Then $\mathcal{P}(A)\ltimes_{\mathsf{Hol}}\mathcal{E}$ is a strict 2-groupoid.

The main theorem obtained in [13] can be stated as follows:

Theorem 6.1.5. Let $A\longrightarrow M$ be a Lie algebroid $(\nabla^E,\nabla^C,\omega,\partial)$ be a 2-term representation up to homotopy of A on $E\oplus C\longrightarrow M$ and denote by D the corresponding VB-algebroid. The Weinstein groupoid $\mathcal{G}(D)$ of D identifies with the 1-truncation of the transformation 2-groupoid $\mathcal{P}(A)\ltimes_{\mathsf{Hol}}\mathcal{E}$ associated to the holonomy 2-representation.

For the proof of this important theorem as well as for further discussions and examples we refer the reader to the original work [13].

In the subsequent sections we try to put the previous discussions in the context of our work.

6.2 Complete Horizontal Lifts and Holonomy

In this section we discuss how it is possible to define a parallel transport notion in the context of extensions of Lie algebroids. This will play a relevant role later on.

Let $a\cdot dt : TI\longrightarrow A_B$ be an A_B-path whose base path is $\gamma : I\longrightarrow B$. Choose a time-dependent section $a_t\in\Gamma(A_B)$ extending $a(t)$, that is, such that

$$a_t(\gamma(t)) = a(t),$$

for every $t\in I$. This is always possible, it suffices to use a partition of unity. It is even possible to choose a_t with compact support. Then $h(a)$ is a time-dependent section of A_E and $\nabla^h_{a_t} := [h(a_t),-]_{A_E}$ is a time-dependent derivation of A_E whose symbol is the time-dependent vector field $\sharp_{A_E}h(a_t)$ on E. By section 1.4, the time-dependent derivation $\nabla^h_{a_t}$ has a flow $\Phi^{\nabla^h_{a_t}} : \mathcal{K}\longrightarrow\mathcal{K}$ covering the flow $\Phi^{\sharp_{A_E}h(a_t)} : E\longrightarrow E$. Since $\sharp_{A_E}h(a_t)$ is $d\pi_0$-projectable on $\sharp a_t$ the flows of $\sharp_{A_E}h(a_t)$ and $\sharp_{A_B}a_t$ commute, hence, for every $k\in\mathcal{K}$ such that $\pi_0(p_{\mathcal{K}}(k)) = x$, one finds:

$$\pi_0 p_{\mathcal{K}}(\Phi^{\nabla^h_{a_t}}_{t,s}(k)) = \pi_0\Phi^{\sharp_{A_E}h(\alpha_t)}_{t,s}(p_{\mathcal{K}}(k)) = \Phi^{\sharp\alpha}_{t,s}(\pi_0(p_{\mathcal{K}}(k)) = \Phi^{\sharp\alpha}_{t,s}(x).$$

This shows there is an well defined map:

$$\Phi^{h(a)}_{t,s}(x) := \Phi^{\nabla^h_{a_t}}_{t,s}|_{\mathcal{K}|_{E_x}} : \mathcal{K}|_{E_x}\longrightarrow\mathcal{K}|_{E_{\Phi^{\sharp a}_{t,s}(x)}}. \tag{6.1}$$

A priori, this is not defined for every x and every $s, t \in I$. However, this is the case whenever the vector field $\sharp_{A_B} a_t$ is complete for every $t \in I$. This motivates the following:

Definition 6.2.1. Let $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$ be a horizontal lift associated to an extension $\mathcal{K} \hookrightarrow A_E \overset{\pi}{\twoheadrightarrow} A_B$. We say h is *complete* if the vector field $\sharp_{A_E} h(a)$ on E is complete for every $a \in \Gamma(A_B)$ such that $\sharp_{A_B} a$ is complete.

Therefore, for complete horizontal lifts one get an isomorphism of Lie algebroids:

$$\Phi^{h(a)}_{1,0}(\gamma(0)) : \mathcal{K}|_{E_{\gamma(0)}} \longrightarrow \mathcal{K}|_{E_{\gamma(1)}},$$

which, by integration, defines a Lie groupoid isomorphism:

$$\mathsf{Hol}^h_a : \mathcal{G}(\mathcal{K}|_{E_{\gamma(0)}}) \longrightarrow \mathcal{G}(\mathcal{K}|_{E_{\gamma(1)}})$$

which covers

$$\mathsf{hol}^h_a := \Phi^{\sharp a}_{0,1} : E_{\gamma(0)} \longrightarrow E_{\gamma(1)}.$$

The above construction does not depend on how we extend the A_B-path a into a time-dependent section.

Definition 6.2.2. Let $\mathcal{K} \hookrightarrow A_E \overset{\pi}{\twoheadrightarrow} A_B$ be an extension of Lie algebroids, $h : \Gamma(A_B) \longrightarrow \Gamma(A_E)$ be a complete horizontal lift and $a \cdot dt : TI \longrightarrow A_B$ an A_B-path over γ. We call $\mathsf{Hol}^h_a : \mathcal{G}(\mathcal{K}|_{E_{\gamma(0)}}) \longrightarrow \mathcal{G}(\mathcal{K}|_{E_{\gamma(1)}})$ the *holonomy of h along a.*

As noticed in [12], the following functorial properties hold

$$\mathsf{Hol}^h_{a_0 * a_1} = \mathsf{Hol}^h_{a_0} \circ \mathsf{Hol}^h_{a_1} \quad \text{and} \quad \mathsf{Hol}^h_{a_x} = \mathsf{id},$$

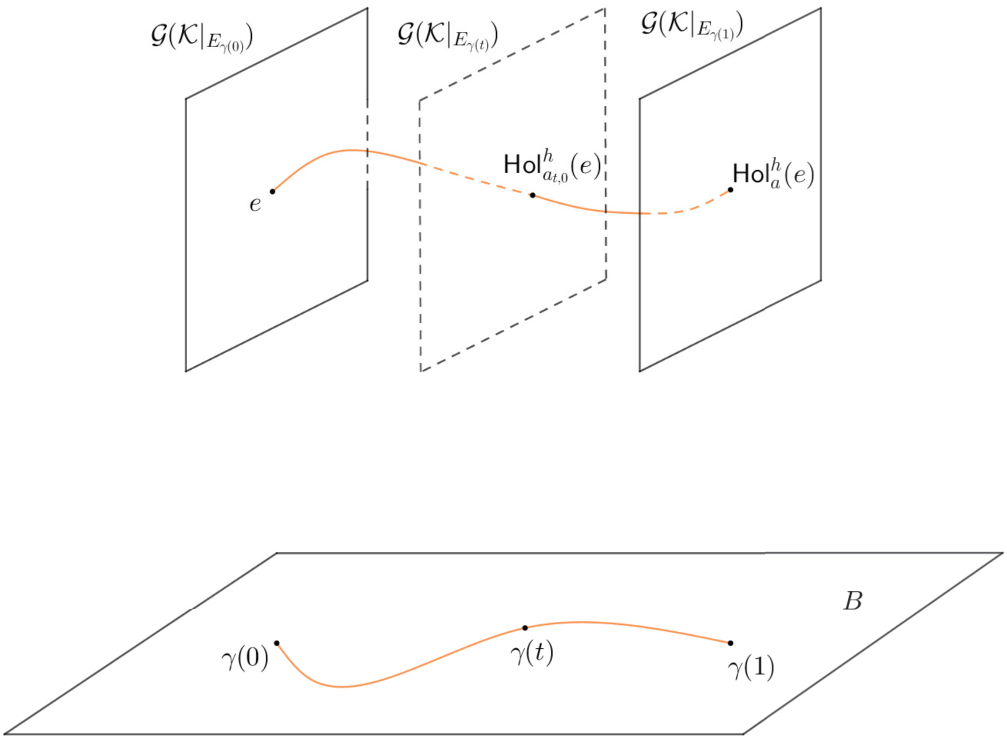
for any A_B-paths $a_0 \cdot dt, a_1 \cdot dt : TI \longrightarrow A_B$ where $0_x \cdot dt : TI \longrightarrow A_B$ stands for the constant path over x. From this it follows

$$(\mathsf{Hol}^h_a)^{-1} = \mathsf{Hol}^h_{\overline{a}},$$

where $\overline{a}\ dt : TI \longrightarrow A_B$ is the A_B-path obtained from $\overline{a}(t) := -a(1-t)$.

It is very illustrative to have in mind the following picture of Hol^h_a: In [12], the author used this in order to get a description of the Weinstein-groupoid of A_E in terms of A_B and $\mathcal{K}$.

The previous discussion was based upon the choice of an Ehresmann connection. But the relevant data is the fact that ∇^h is a $C^\infty(B)$-linear map which takes values on derivations of the kernel $\mathcal{K}$ whose symbol projects on the anchor of A_B. So, we can get



rid of the splitting h and define the holonomy for any $\nabla \in \Omega^1(A_B, \mathsf{der}(\mathcal{K}))$ which satisfies

$$X^{\nabla_a}(\pi_0^* f_B) = \pi_0^* \mathcal{L}_{\sharp_{A_B} a}(f_B),$$

for every $a \in \Gamma(A_B)$ and every $f_B \in C^\infty(B)$, and such that X^{∇_a} is a complete vector field for every $a \in \Gamma(A_B)$ such that $\sharp_{A_B} a$ is complete.

6.3 The Holonomy 2-Functor

In this section, we define, in the context of actions up to homotopy, the analogue of the holonomy 2-functor we discussed in the first section of this chapter.

From now on, whenever we consider an action up to homotopy (∇, ω) we shall suppose X^{∇_a} is complete for every $a \in \Gamma(A_B)$ such that $\sharp_{A_B} a$ is complete.

Let $\pi_0 : E \longrightarrow B$ be a surjective submersion and suppose $\mathcal{K} \longrightarrow E$ is a Lie algebroid such that $\sharp_{\mathcal{K}}(\mathcal{K}) \subset \mathsf{Ker}(d\pi_0)$. The restrictions of $\mathcal{K} \longrightarrow E$ to the fibres $E_x := \pi_0^{-1}(x) \subset E$ are Lie algebroids. Notice that:

$$\mathcal{K}|_{E_x} = \{(k, p_{\mathcal{K}}(k)) : k \in \mathcal{K}, \pi_0(p_{\mathcal{K}}(k)) = x\}.$$

Therefore, we can think of $\mathcal{K}$ as a family of Lie algebroids parametrized by points of B. Integrating each $\mathcal{K}|_{E_x}$ we get a family of, *a priori*, topological groupoids $\mathcal{G}(\mathcal{K}|_{E_x}) \rightrightarrows E_x$.

We shall denote by 2-$\mathsf{Gau}((\mathcal{G}(\mathcal{K}), \pi_0))$ the strict 2-groupoid whose:

- *objects* are the points of B;
- *1-morphisms* from x to y are invertible functors from $\mathcal{G}(\mathcal{K}|_{E_x})$ to $\mathcal{G}(\mathcal{K}|_{E_y})$;
- *2-morphisms* are natural transformations;
- the various compositions and the units are the evident.

We shall not discuss smoothness in this context. This probably must be dealt with by means of *diffeologies*. Next we explain how we can put actions up to homotopy in the 2-categorical context.

Let (∇, ω) be an action up to homotopy of a Lie algebroid A_B on $\mathcal{K}_E \longrightarrow E$ along $\pi_0 : E \longrightarrow B$. Then there exists a strict 2-functor:

$$\mathsf{Hol}^{\nabla,\omega} : \mathcal{P}(A_B) \longrightarrow 2\text{-}\mathsf{Gau}\big((\mathcal{G}(\mathcal{K}), \pi_0)\big)$$

defined as follows:

- at the level of objects $\mathsf{Hol}^{\nabla,\omega}$ restricts to the identity of B;
- to an A_B-path $a \cdot dt : x \longrightarrow y$:

  $$\mathsf{Hol}^{\nabla,\omega}(a) := \mathsf{Hol}^{\nabla}_a : \mathcal{G}(\mathcal{K}|_{E_x}) \longrightarrow \mathcal{G}(\mathcal{K}|_{E_y})$$

  where Hol^{∇}_a is as defined in the previous section;
- Let $a^0 \cdot dt : x \longrightarrow y$ and $a^1 \cdot dt : x \longrightarrow y$ be two A_B-paths and suppose $\sigma = a \cdot dt + b \cdot ds : a^0 \cdot dt \Longrightarrow a^1 \cdot dt$ is an A_B-homotopy. Then

  $$\mathsf{Hol}^{\nabla,\omega}(\sigma) : E_x \longrightarrow \mathcal{G}(\mathcal{K}|_{E_y})$$

  is induced by the assignment:

  $$e \longmapsto \left[\int_0^1 \mathsf{Hol}^{\nabla}_{a^s_{1,t}} \circ \omega(a,b)_{\gamma^s_t} \circ \mathsf{Hol}^{\nabla}_{a^s_{t,0}}(e)\ dt\right]_{\mathcal{K}}.$$

Some remarks are required:

- The fact that $\mathsf{Hol}^{\nabla,\omega}$ takes its values on 2-$\mathsf{Gau}\big((\mathcal{G}(\mathcal{K}), \pi_0)\big)$ is a direct consequence from the construction of the holonomy along A-paths;
- The difficult part to verify that $\mathsf{Hol}^{\nabla,\omega}$ is indeed a functor is that $\mathsf{Hol}^{\nabla,\omega}$ does not depend on thin-homotopy classes of A-paths;

- It is not very difficult to verify that $\mathsf{Hol}^{\nabla,\omega}$ preserves the various compositions and identities.

We shall think of $\mathsf{Hol}^{\nabla,\omega}$ as "higher" action of the Weinstein 2-grupoid $\mathcal{P}(A_B)$ on $\mathcal{G}(\mathcal{K}) \rightrightarrows E$.

6.4 Integrating Morphisms

In the previous section we explained how an action up to homotopy (∇, ω) of A_B on $\mathcal{K} \longrightarrow E$ along $\pi_0 : E \longrightarrow B$ induces an action of $\mathcal{P}(A_B)$ on $\mathcal{G}(\mathcal{K}) \rightrightarrows E$ along $\pi_0 : E \longrightarrow B$. In this section we discuss this at the level of morphisms.

Strangely as it may seem at first, the following theorem was the motivating point for the development of the theory of natural homotopies we dealt with in a previous part.

Theorem 6.4.1. Let (∇, ω) and (∇', ω') be two actions up to homotopy of a Lie algebroid on $\mathcal{K}$ and $\mathcal{K}'$ along $\pi_0 : E \longrightarrow B$ and $\pi_0' : E' \longrightarrow B$, respectively. Let $(\Phi, \theta) : (\nabla, \omega) \longrightarrow (\nabla', \omega')$ be a morphism of actions up to homotopy. Then:

i) For any $x \in B$, Φ induces, by restriction, a morphism of Lie algebroids

$$\Phi_x : \mathcal{K}|_{E_x} \longrightarrow \mathcal{K}'|_{E'_x}.$$

ii) For any A_B-path $a : x \longrightarrow y$, θ induces a natural isomorphism:

$$\mathsf{Hol}^{\theta}_a : F_y \circ \mathsf{Hol}^{\nabla}_a \Longrightarrow \mathsf{Hol}^{\nabla'}_a \circ F_x$$

given by:

$$E_x \ni e \longmapsto \left[\mathsf{Hol}^{\nabla'}_{a_{t,0}} \circ \theta(a_t) \circ \mathsf{Hol}^{\nabla}_{\gamma_{1,t}}(e) \cdot dt\right]_{\mathcal{K}'|_{E'_y}} \in \mathcal{G}(\mathcal{K}'|_{E'_y}), \tag{6.2}$$

where:

- For each $x \in B$

$$F_x : \mathcal{G}(\mathcal{K}|_{E_x}) \longrightarrow \mathcal{G}(\mathcal{K}'|_{E'_x})$$

denotes the morphism of Lie groupoids integrating Φ_x;

- For each A_B-path

$$\left[\mathsf{Hol}^{\nabla'}_{a_{t,0}} \circ \theta(a_t) \circ \mathsf{Hol}^{\nabla}_{\gamma_{1,t}}(e) \cdot dt\right]_{\mathcal{K}'|_{E'_y}},$$

denotes the $\mathcal{K}'|_{E'_y}$-homotopy class of the $\mathcal{K}'$-path

$$\mathsf{Hol}^{\nabla'}_{a_{t,0}} \circ \theta(a_t) \circ \mathsf{Hol}^{\nabla}_{\gamma_{1,t}}(e) \cdot dt.$$

This is illustrated in the following diagram:

$$x \xrightarrow{a\ dt} y \qquad \xmapsto{\mathrm{Hol}^{(\Theta,\Phi)}} \qquad \begin{tikzcd} \mathcal{G}(\mathcal{K}_{E_x}) \arrow[rr, "\mathsf{Hol}^{\nabla}_a"] \arrow[d, "F_x"'] & & \mathcal{G}(\mathcal{K}_{E_y}) \arrow[d, "F_y"] \arrow[dll, Rightarrow, "\mathsf{Hol}^{\Theta}_a"'] \\ \mathcal{G}(\mathcal{K}_{E'_x}) \arrow[rr, "\mathsf{Hol}^{\nabla'}_a"'] & & \mathcal{G}(\mathcal{K}_{E'_y}) \end{tikzcd}$$

Proof. In order to obtain the natural isomorphism

$$\mathsf{Hol}^{\theta}_a : F_y \circ \mathsf{Hol}^{\nabla}_a \Longrightarrow \mathsf{Hol}^{\nabla'}_a \circ F_x$$

we are going to apply theorem 3.7.1. With this goal in mind, let us consider the one-parameter family of Lie algebroid morphisms defined by:

$$\Psi_t := \mathsf{Hol}^{\nabla'}_{a_{1,t}} \circ \Phi_{\gamma(t)} \circ \mathsf{Hol}^{\nabla}_{a_{t,0}} : \mathcal{K}|_{E_x} \longrightarrow \mathcal{K}'|_{E'_y},$$

This family is obtained via the composition illustrated below:

$$\begin{tikzcd} \mathcal{K}|_{E_x} \arrow[r, "\mathsf{Hol}^{\nabla}_{a_{t,0}}"] & \mathcal{K}|_{E_{\gamma(t)}} \arrow[d, "\Phi_{\gamma(t)}"] & \\ & \mathcal{K}'|_{E'_{\gamma(t)}} \arrow[r, "\mathsf{Hol}^{\nabla'}_{a_{1,t}}"'] & \mathcal{K}'|_{E'_y} \end{tikzcd}$$

Notice that

$$\Psi_0 = \mathsf{Hol}^{\nabla'}_a \circ \Phi_x \quad \text{and} \quad \Psi_1 = \Phi_y \circ \mathsf{Hol}^{\nabla}_a.$$

All we are left to do is to verify that Ψ_0 and Ψ_1 are naturally homotopic and that the natural homotopy will be given by (6.2). To show Ψ_0 and Ψ_1 are indeed naturally homotopic let us consider the one-parameter family of smooth maps:

$$\Theta(a_t) := \mathsf{Hol}^{\nabla'}_{a_{t,0}} \circ \theta(a_t) \circ \mathsf{hol}^{\nabla}_{\gamma_{1,t}} : E_x \longrightarrow \mathcal{K}'|_{E'_y}.$$

A straightforward computation shows that the degree -1 induced map on forms:

$$\imath^{\Psi_s}_{\Theta(a_s)} : \Omega(\mathcal{K}'|_{E'_y}) \longrightarrow \Omega(\mathcal{K}|_{E_x})$$

is given by:

$$\imath^{\Psi_s}_{\Theta(a_s)} = \mathsf{Hol}^{\nabla^*}_{a_{s,0}} \circ \imath^{\Phi}_{\theta(a_s)} \circ \mathsf{Hol}^{\nabla'*}_{a_{1,s}}, \tag{6.3}$$

for every $s \in I$. We proceed to show the vector bundle morphism

$$\Psi + \Theta \cdot dt : \mathcal{K}'|_{E_x} \times TI \longrightarrow \mathcal{K}'|_{E'_y}$$

is a natural homotopy from Ψ_0 to Ψ_1. Since Ψ_t is a Lie algebroid morphism for every $t \in I$, by proposition 3.6.4, we are left to check the compatbility structure equation:

$$\left.\frac{d}{dt}\right|_{t=s} \Psi^*_t = d_{\mathcal{K}} \circ \imath^{\Psi_t}_{\Theta(a_s)} + \imath^{\Psi_s}_{\Theta(a_s)} \circ d_{\mathcal{K}'}.$$

In fact, since (Φ, θ) is a morphism of actions up to homotopy, we know that:

$$\nabla^*_a \circ \Phi^* - \Phi^* \circ \nabla'^*_a = d_{\mathcal{K}} \circ \imath^{\Phi}_{\theta(a)} + \imath^{\Phi}_{\theta(a)} \circ d_{\mathcal{K}'} \tag{6.4}$$

for every $a \in \Gamma(A_B)$. Combining (6.3) and (6.4), we compute:

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=s} \Psi^*_t &= \left.\frac{d}{dt}\right|_{t=s} \mathsf{Hol}^*_{a_{t,0}} \circ \Phi \circ \mathsf{Hol}'^*_{a_{1,t}} \\
&= \left.\frac{d}{dt}\right|_{t=s} \left[\mathsf{Hol}^*_{a_{s,0}} \circ \mathsf{Hol}^*_{t,s} \circ \Phi^* \circ \mathsf{Hol}'^*_{a_{s,t}} \circ \mathsf{Hol}'^*_{1,s}\right] \\
&= \mathsf{Hol}^*_{a_{s,0}} \circ \left[\left.\frac{d}{dt}\right|_{t=s} \mathsf{Hol}'^*_{a_{t,s}} \circ \Phi^* \circ \mathsf{Hol}'^*_{a_{s,t}}\right] \circ \mathsf{Hol}'^*_{a_{1,s}} \\
&= \mathsf{Hol}^*_{a_{s,0}} \circ \left[\left.\frac{d}{dt}\right|_{t=s} \mathsf{Hol}^*_{a_{t,s}} \circ \Phi^* \circ \mathsf{Hol}'^*_{s,s} \right. \\
&\qquad\qquad \left. + \left.\frac{d}{dt}\right|_{t=s} \mathsf{Hol}^*_{a_{s,s}} \circ \Phi^* \circ \mathsf{Hol}'^*_{a_{s,t}}\right] \circ \mathsf{Hol}'^*_{a_{1,s}} \\
&= \mathsf{Hol}^*_{a_{s,0}} \circ \left[\left(\left.\frac{d}{dt}\right|_{t=s} \mathsf{Hol}^*_{a_{t,s}}\right) \circ \Phi^* + \Phi^* \circ \left(\left.\frac{d}{dt}\right|_{t=s} \mathsf{Hol}'^*_{a_{s,t}}\right)\right] \circ \mathsf{Hol}'^*_{a_{1,s}} \\
&= \mathsf{Hol}^*_{a_{s,0}} \circ \left(\nabla^*_{a_s} \circ \Phi^* - \Phi^* \circ \nabla'^*_{a_s}\right) \circ \mathsf{Hol}'^*_{a_{1,s}} \\
&= \mathsf{Hol}^*_{a_{s,0}} \circ \left[d_{\mathcal{K}} \circ \imath^{\Phi}_{\theta(a_s)} + \imath^{\Phi}_{\theta(a_s)} \circ d_{\mathcal{K}'}\right] \circ \mathsf{Hol}'^*_{a_{1,s}} \\
&= d_{\mathcal{K}} \circ \left[\mathsf{Hol}^*_{a_{s,0}} \circ \imath^{\Phi}_{\theta(a_s)} \circ \mathsf{Hol}'^*_{a_{1,s}} + \mathsf{Hol}^*_{a_{s,0}} \circ \imath_{\theta(a_s)} \circ \mathsf{Hol}'^*_{a_{1,s}}\right] \circ d_{\mathcal{K}'} \\
&= d_{\mathcal{K}} \circ \imath^{\Psi_s}_{\Theta(a_s)} + \imath^{\Psi_s}_{\Theta(a_s)} \circ d_{\mathcal{K}'}.
\end{aligned}$$

Applying theorem 3.7.1 we obtain that

$$E_x \ni e \longmapsto \left[\mathsf{Hol}^{\nabla'}_{a_{t,0}} \circ \theta(a_t) \circ \mathsf{Hol}^{\nabla}_{\gamma_{1,t}}(e) \cdot dt\right]_{\mathcal{K}'|_{E'_y}} \in \mathcal{G}(\mathcal{K}'|_{E'_y}),$$

defines a natural isomorphism from the integration of Ψ_0, which is $\mathsf{Hol}^{\nabla'}_a \circ F_x$, to the integration of Ψ_1, which is $F_y \circ \mathsf{Hol}_a^{\nabla}$. $\square$

6.5 Future Research: A Functorial Integration

Along the development of this work some natural questions arose and we discuss them in this final part. In fact, we already have some answers in an heuristic form and we would like to present them in the same spirit. Most of the discussions are informal but we believe they are coherent enough to be included here.

First, we discuss how the tools we developed in the previous parts can be used to produce a functorial scheme for integrating extensions of Lie algebroids. Afterwards, we point how it is possible to discuss natural homotopies within the category of extensions of Lie algebroids and, of course, in the category of VB-algebroids. Next, we point out some applications which may be a consequence of our work.

In the previous part we showed how an action up to homotopy (∇, ω) of a Lie algebroid A_B on $\mathcal{K} \longrightarrow E$ along a surjective submersion $\pi_0 : E \longrightarrow B$ gives rise to a strict 2-functor:

$$\mathsf{Hol}^{\nabla,\omega} : \mathcal{P}(A_B) \longrightarrow \text{2-}\mathsf{Gau}((\mathcal{G}(\mathcal{K}), \pi_0)).$$

The first step in our future research is to obtain the description of the corresponding action 2-groupoid $\mathcal{P}(A_B) \ltimes_{\mathsf{Hol}^{\nabla,\omega}} \text{2-}\mathsf{Gau}((\mathcal{G}(\mathcal{K}), \pi_0))$ and to show that its truncation identifies with the Weinstein groupoid integrating the Lie algebroid $\pi_0^* A_B \oplus_{\nabla,\omega} \mathcal{K}$.

Also, we already pointed out how a morphism (Φ, θ) of actions up to homotopy allows us to compare different holonomies. A natural question is, are those constructions functorial? The answer to this question seems to be true and we hope to show it soon. More precisely, we believe we can show the following:

- Let $(\Phi, \theta) : (\nabla, \omega) \longrightarrow (\nabla', \omega')$ be a morphism of actions up to homotopy of A_B on $\mathcal{K}$ along $\pi_0 : E \longrightarrow B$ and on $\mathcal{K}'$ along $\pi_0' : E' \longrightarrow B$, respectively. Then the assignment:

$$x \xrightarrow{a\ dt} y \qquad \xmapsto{\ \mathrm{hol}^{(\Theta,\Phi)}\ } \qquad \begin{tikzcd} \mathcal{G}(\mathcal{K}_{E_x}) \arrow[r, "\mathsf{Hol}_a^{\nabla}"] \arrow[d, "F_x"'] & \mathcal{G}(\mathcal{K}_{E_y}) \arrow[d, "F_y"] \arrow[dl, Rightarrow, "\mathsf{Hol}_a^{\Theta}"'] \\ \mathcal{G}(\mathcal{K}_{E'_x}) \arrow[r, "\mathsf{Hol}_a^{\nabla'}"'] & \mathcal{G}(\mathcal{K}_{E'_y}) \end{tikzcd}$$

  where F_x and F_y denote the groupoid morphism integrating Φ_x and Φ_y, respectively, and $\mathsf{Hol}_a^{\theta} : E_x \longrightarrow \mathcal{G}(\mathcal{K}_{E'_y})$ is given by:

$$\mathsf{Hol}_a^{\theta} := \left[s \longmapsto \mathsf{Hol}_{a_{t,1}}^{\nabla'} \circ \theta(a) \circ \mathsf{hol}_{a_{t,0}}^{\nabla} \right],$$

induces a pseudo-natural transformation:

$$\mathsf{Hol}^{\Phi,\theta} : \mathsf{Hol}^{\nabla,\omega} \Longrightarrow \mathsf{Hol}^{\nabla',\omega'},$$

that is, a morphism in $\mathsf{Func}(\mathcal{P}(A_B), \mathsf{2Grpd})$ from $\mathsf{Hol}^{\nabla,\omega}$ to $\mathsf{Hol}^{\nabla',\omega'}$.

The difficult part in the previous statement is to show that the assignment is well defined, that is, it does not depend on the thin-homotopy class of the A_B-paths. Once the above is proven, it is more or less easily to show:

$$\mathsf{Hol} : \mathsf{Act}_2^{\infty}(A_B) \longrightarrow \mathsf{Funct}(\mathcal{P}(A_B), 2\text{-}\mathsf{Grpd}),$$

is a functor where 2-Grpd is the category whose:

- *objects* are groupoids;
- *1-morphisms* are invertible functors;
- *2-morphisms* are natural transformations;
- compositions and identities are the usual ones.

Of course, we also intend to investigate examples and develop applications for the theory discussed.

Bibliography

[1] Abad, C. A., Crainic, M.. *Representations up to homotopy and Bott's spectral sequence for Lie groupoids*. Advances in Mathematics, Elsevier Science, vol. 248, 2013, 416-.

[2] Abad, C. A., Crainic, M.. *Representations up to homotopy of Lie algebroids*. Journal für die reine und angewandte Mathematik (Crelles Journal), vol. 2012, issue 663, 2012, 91–126.

[3] Abad, C. A., Schätz, F.. *Deformations of Lie brackets and representations up to homotopy*. Indagationes Mathematicae, vol. 22, issues 1-2, 2011, 27-54.

[4] Abad, C. A., Schätz, F.. *The A_∞ de Rham theorem and integration of representations up to homotopy*, Int. Math. Res. Not, vol. 2013, issue 16, 2013, 3790-3855.

[5] Assem, I. *Algèbres et Modules*. Les Presses de I'Universite d'Ottawa, Masson, 1997

[6] Baum, H.. *Eichfeldtheorie: Eine Einführung in die Differentialgeometrie auf Faserbündeln*. Springer Spektrum, 2014.

[7] Baez, J. C., Crans, A. S.. *Higher-Dimensional Algebra VI: Lie 2-Algebras*. Preprint arXiv:math/0307263v6, 2010.

[8] Balcerzak, B., Kubarski, J., Walas, W. *Endomorphisms of the Lie algebroids $TM \times \mathfrak{g}$ up to homotopy*. Universitatis Iagellonicae Acta Mathematica, Faisciculus XL, vol. 2002, 215-223.

[9] Berger, M., Gostiaux, B. *Differential Geometry: Manifolds, Curves, and Surfaces*. Graduate texts in Mathematics, 1987.

[10] Bkouche, R. *Idéaux mous d'un anneau commutatif: Applications aux anneaux de fonctions*. C. R. Acad. Sci. Paris, vol. 1965, issue 260, 1965, 6496-6498.

[11] Bojowald, M., Kotov, Strobl, T. *Lie algebroid morphisms, Poisson sigma-models and off-shelf closed gauge symmetries*. J. Geom. Phys., vol. 54, issue 4, 2005, 400-426.

[12] Brahic, O.. *Extensions of Lie brackets*. Journal of Geometry and Physics, Elsevier Science, vol. 60, issue 2, 2010, 352-374.

[13] Brahic, O., Ortiz, C. *Integration of 2-term representations up to homotopy via 2-functors*. Preprint arxiv:1608.00664v1, 2016.

[14] Brahic, O., Zhu, C. *Lie algebroid fibrations*. Adv. Math. 226, vol. 226, issue 4, 2011, 3105-3135.

[15] Brahic, O., Cabrera, A., Ortiz, C. *Obstruction to the integrability of VB-algebroids*. Preprint arXiv:1403.1990v2, 2014

[16] Brown, R. *Topology and groupoids*. Booksurge LLc., S. Carolina, 2016.

[17] Crainic, M., Fernandes, R. L.. *Integrability of Lie Brackets*. Annals of Mathematics, John Hopkins University Press, vol. 157, issue 2, 2003, 575-620.

[18] Drummond, T., Jotz, M., Ortiz, C.. *VB-algebroid morphisms and representations up to homotopy*. Differential Geometry and its Applications, Elsevier Science, vol. 40, 2015, 332-357.

[19] Duistermaat, J. J, Kolk, J. A. C.. *Lie Groups*. Universitext, Springer Verlag, 1947.

[20] Ehresmann, C. *Les connexions infinitésimales dans un espace fibrée différentiable*. Colloque de topologie de Bruxelles, vol. 1, 1950, 153-168.

[21] Fernandes, R. L.. *Lie Algebroids, Holonomy and Characteristic Classes*. Advances in mathematics, Elsevier Science, vol. 170, issue 1, 2002, 119-179.

[22] Fernandes, R. L., Crainic, M.. *Lectures on Integrability of Lie Brackets*. Geometry and Topology Monographs, vol. 17, 2011, 1–107.

[23] Gracia-Saz, A., Mehta, R. A.. *Lie Algebroid Structures on Double Vector Bundles and Representation Theory of Lie Algebroids*. Advances in Mathematics, Elsevier Science, volume 223, issue 4, 2010, 1236-1275.

[24] Greub, W., Halperin, S., Vanstone, R.. *Connections, curvature and cohomology*, vol. 1. Pure and applied mathematics - A series of monographs and textbooks, Academic Press, 1972.

[25] Higgins, P. J., Mackenzie, K.. *Algebraic constructions in the category of lie algebroids*. Journal of Algebra, Elsevier Science, vol. 129, 1990, 194-230.

[26] Hisham, S., Schreiber, Stasheff, J. *L_∞-algebra connections and applications to string and Chern-Simons n-transport*. Quantum Field Theory: Competitive Models, Birkhäuser Basel, 2009, pp 303-424,.

[27] Knapp, K.. *Vektorbündel: Vom Möbius-Bündel bis zum J-Homomorphismus*. Springer Spektrum, 2013.

[28] Kubarski, J. *The Chern-Weil homomorphism of regular Lie algebroids*. Publ., Dép. Math. Université de Lyon, vol. 1, 1991, 1-69.

[29] Konieczna, K., Urbański, P.. *Double vector bundles and duality*. Archivum Mathematicum, Department of Mathematics, Faculty of Science of Masaryk University, Brno, volume 35, issue 1, 1999, 59-95.

[30] Lee, J. *Introduction to smooth manifolds* (2003). Springer Graduate Texts in Mathematics 218.

[31] Mackenzie, K.. *Double Lie Algebroids and Second-order Geometry, I*. Advances in Mathematics, Elsevier Science, volume 94, issue 2, 1992, 180-239.

[32] Mackenzie, K.. *Double Lie Algebroids and Second-order Geometry, II*. Advances in Mathematics, Elsevier Science, volume 154, issue 1, 2000, 46-75.

[33] Mackenzie, K.. *General Theory of Lie Groupoids and Lie Algebroids*. London Mathematical Society Lecture Note Series, Cambridge University Press, 2005.

[34] Mackenzie, K., Xu, P.. *Integration of Lie bialgebroids*. Topology, volume 39, issue 3, 2000, 445-467.

[35] MacLane. *Homology*. Grundlehren der mathematischen Wissenschaften, Springer, 1994.

[36] Moerdijk, I., Mrcun, J.. *Introduction to foliations and Lie groupoids*. Cambridge studies in advanced mathematics 91, Cambridge University Press, 2003.

[37] Moerdijk, I., Mrcun, J.. *On integrability of infinitesimal actions*. Amer. J. Math, volume 124, issue 3, 2002, 567-593 124.

[38] Schreiber, U., Waldorf, K. *Smooth functors vs. differential forms*. Homology Homotopy and Applications, International Press, volume 13, issue 1, 2011, p. 143-203.

[39] Segal, G. *Classifying spaces and spectral sequences*. Publications Mathématiques de L'IHÉS, volume, 34, 1968, p. 105-112.

[40] Sullivan, D.. *Infinitesimal computations in Topology*. Publications Mathématiques de l'IHÈS, vol. 47, 1997, p. 269-331.

[41] Vaintrob, A. Y.. *Lie Algebroids and homological vector fields*. Russian mathematical surveys, Turpion Limited, vol. 52, issue 2, 1997, 428-429.

[42] Valdés, J. N., Villalón, A. F. T, Alarcón, J. A. V. *Elementos de la Teoría de Grupoides y Algebroides*. Servicio de Publicaciones de la Universidad de Cádiz, 2006.