UNIVERSIDADE FEDERAL DO PARANÁ
TIAGO LUIZ FERRAZZA

PARTIAL HOPF (CO)ACTIONS ON ALGEBRAS WITHOUT IDENTITY

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Orientador: Prof. Dr. Marcelo Muniz Silva Alves.

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All we have to decide is what to do with the time that is given us.

J.R.R. Tolkien, The Fellowship of the Ring.

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Resumo

Neste trabalho buscamos uma boa definição para (co)ações parciais de álgebras de Hopf em álgebras sem unidade, de forma a obter resultados análogos ao caso clássico, onde as álgebras são unitárias, como por exemplo a existência de uma globalização e de um contexto Morita, o estudo dos invariantes parciais e a teoria de Hopf-Galois parcial.

Palavras-chave: Álgebras de Hopf, ações parciais, coações parciais, representações parciais, globalização, equivalência de Morita, par combinado de Álgebras de Hopf.

Abstract

In this work we looked for a good definition for partial (co)actions of a Hopf algebra on an algebra without unity, in such way that we obtain results analogous to the classical case, where the algebras are unital, as for example the existence of a globalization and of a Morita context, the study of the partial invariants and the partial Hopf-Galois theory.

Keywords: Hopf algebras, partial actions, partial coactions, partial representations, globalization, Morita equivalence, matched pair of Hopf algebras.

Contents

Resumo Expandido

Nesse trabalho, buscamos generalizar os conceitos de ações e coações parciais de álgebras de Hopf em álgebras com unidade definido por Caenepeel e Jansen em [13] e depois desenvolvido por Alves, Batista, Dokuchaev, Paques e outros.

Durante todo o trabalho, consideraremos $\Bbbk$ um corpo. Além disso, nos referiremos à estruturas $\Bbbk$-lineares simplesmente como lineares ($\Bbbk$-álgebras como álgebras e assim por diante).

O estudo de ações parciais de álgebras de Hopf em álgebras unitárias surgiu do interesse de se generalizar o conceito de ações parciais de grupo em álgebras unitárias, saindo de um ambiente mais topológico para um puramente algébrico. Nesse trabalho apresentamos uma definição de ação parcial de uma álgebra de Hopf H em uma álgebra sem unidade A de tal forma que não só satisfaz condições suficientes para uma boa estrutura do produto smash $A\#H$, como também fornece uma relação estreita entre ações parcias da álgebra de Hopf $\Bbbk G$ e ações parciais do grupo G.

Definição 2.4. *Seja A uma álgebra associativa. Uma aplicação linear $\cdot : H \otimes A \longrightarrow A$, $h \otimes a \longmapsto h \cdot a$ é uma ação parcial se, para todo $a, b \in A$, $h, k \in H$,*

1. $1_H \cdot a = a$;
2. $h \cdot (a(k \cdot b)) = \sum (h_{(1)} \cdot a)(h_{(2)}k \cdot b)$.

Nesse caso, A é denominada uma H-módulo álgebra parcial. Diremos que a ação parcial é simétrica se, adicionalmente, $h \cdot ((k \cdot b)a) = \sum (h_{(1)}k \cdot b)(h_{(2)} \cdot a)$, para todo $a, b \in A$, $h, k \in H$.

Para o caso com unidade, Caenepeel e Jansen mostraram que existe uma bijeção entre ações parciais de $\Bbbk G$ e ações parciais de G onde os ideias envolvidos são gerados por idempotentes centrais. Já para o caso sem unidade, mostramos que isso vale mas substituindo os ideais gerados por idempotentes centrais por projeções que satisfazem algumas propriedades. Mais especificamente:

Proposição 2.29. *Seja A uma álgebra associativa que é idempotente, possui somente o zero como anulador à direita ou possui somente o zero como anulador à equerda. Então existe uma correspondência bijetiva entre $\Bbbk G$-ações parciais simétricas em A e G-ações parciais α em A que fornecem epimorfismos de álgebras $p_g : A \to D_g$, tais que $p_1 : A \to A$ é a identidade;*

1. $p_g^2 = p_g$;
2. $p_g p_k = p_k p_g$;
3. $p_g \alpha_k = \alpha_k p_{k^{-1}g} p_k^{-1}$,

para todo $g, k \in G$.

Definição 2.30. *Denominaremos tal família de morfismos de álgebras $\{p_g : A \to D_g\}$ de α-projeções.*

Outro fato importante é a relação que conseguimos entre H-módulo álgebras parciais e álgebras na categoria dos H-módulos parciais, que é mais geral que o caso estudado em [6], em que as álgebras em questão possuem unidade .

Teorema 2.24. *Seja H uma álgebra de Hopf com antípoda bijetiva. Então, quando são consideradas álgebras idempotentes ou que possuem somente o elemento zero como anulador à direita e à esquerda, existe uma correspondência bijetiva entre álgebras na categoria dos H-módulos parciais e H-módulo álgebras parciais simétricas.*

Assumimos então essa definição de ações parciais e mostramos que todas as ações parciais simétricas possuem globalização minimal (para álgebras com somente o zero como anulador à direita ou à esquerda) ou algo que se assemelha com uma globalização minimal (caso mais geral).

Definição 2.56. *Uma quasi-globalização da ação parcial $\cdot : H \otimes A \to A$ é um par (B, θ) tal que*

1. *B é uma H-módulo álgebra (sem unidade), com ação $\triangleright$;*
2. *$\theta : A \longrightarrow B$ é um monomorfismo de álgebras;*
3. *Para todo $a, b \in A$ e $h \in H$, temos que $\theta((h \cdot a)b) = (h \triangleright \theta(a))\theta(b)$ e $\theta(b(h \cdot a)) = \theta(b)(h \triangleright \theta(a))$*
4. *$B = H \triangleright \theta(A)$.*

Usamos a nomenclatura *quasi-globalização* porque ainda não conseguimos mostrar que a ação parcial considerada é a restrição da ação em B no ideal $\theta(A)$. Quando A é uma álgebra com somente o zero como anulador à direita ou à esquerda, como é o caso das álgebras com unidades locais, isso é possível.

Além disso, conseguimos mostrar que se H possui antípoda bijectiva, A é uma H-módulo álgebra parcial idempotente e B uma quasi-globalização, então existe um contexto de Morita estrito entre $\underline{A\#H}$ e $B\#H$, isso ainda mostra que existe uma equivalência entre a categoria dos $\underline{A\#H}$-módulos unitários e livre de torção e a categoria dos $B\#H$-módulos unitários e livres de torção, de acordo com a teoria de Morita para álgebras idempotentes desenvolvida por García e Simón em [19].

Teorema 2.81. *Seja H uma álgebra de Hopf com antípoda bijetiva, $A^2 = A$, A uma H-módulo álgebra parcial e (B, θ) uma quasi-globalização. Então existe um contexto de Morita estrito entre $\underline{A\#H}$ e $B\#H$.*

Nesse trabalho também apresentamos o conceito de equivalência de Morita entre ações parciais de álgebras de Hopf que generaliza a definição de equivalência de Morita de ações parciais de grupo apresentado por Abadie at al. em [1].

Definição 2.84. *Sejam A e B duas H-módulo álgebras parciais idempotentes com ações parciais $\cdot_A$ e $\cdot_B$, respectivamente. Diremos que $\cdot_A$ e $\cdot_B$ são ações parciais Morita equivalentes se*

1. *A é Morita equivalente à B, com contexto de Morita estrito $(A, B, {}_AM_B, {}_BN_A, \tau, \sigma)$, onde M e N são bimódulos unitários;*
2. *Existe uma ação parcial $\triangleright : H \otimes C \to C$, onde C é a álgebra do contexto de Morita $C = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, tal que $\triangleright$ restrita à $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ e $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$, é $\cdot_A$ e $\cdot_B$, respectivamente.*

Não só estendemos alguns dos resultados de [1] como também apresentamos uma relação entre equivalênca de Morita de ações parciais de kG e equivalência de Morita de ações parciais de G que satisfazem algumas propriedades.

Lema 2.96. *Sejam*

$$\alpha = \{\alpha_g : D_{g^{-1}} \to D_g\} \text{ e } \alpha' = \{\alpha'_g : D'_{g^{-1}} \to D'_g\}$$

ações parciais regulares de G nas álgebras idempotentes A e A', respectivamente, que são Morita equivalentes com ação parcial produto em C dada por $\theta = \{\theta_g : E_{g^{-1}} \to E_g\}$. Suponha que existam morfismos de álgebras $p_g : A \to D_g$, $p'_g : A' \to D'_g$ e $P_g : C \to E_g$ que são α-projeções, α'-projeções e θ-projeções, respectivamente, e que cada P_g restrito às cópias de A e A' é p_g e p'_g, respectivamente, i.e.

$$\begin{aligned} P_g\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} p_g(a) & 0 \\ 0 & 0 \end{pmatrix}, \\ P_g\begin{pmatrix} 0 & 0 \\ 0 & a' \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & p'_g(a') \end{pmatrix}. \end{aligned}$$

Então as ações parciais induzidas de $\Bbbk G$ em A e A' são também Morita equivalentes.

Lema 2.97. *Sejam A e A' $\Bbbk G$-módulo álgebras parciais idempotentes que posssuem ações parciais simétricas Morita equivalentes. Então as ações parciais induzidas de G em A e A' são Morita equivalentes.*

Nisso, também mostramos uma relação curiosa entre a definição de α-módulos, apresentada em [1], e a definição de (A, H)-módulos parciais, apresentada por Cavalheiro em [14] em um outro contexto.

De forma semelhante, sugerimos uma definição de coações parciais de uma álgebra de Hopf em uma álgebra sem unidade e relacionamos ações e coações parciais quando lidamos com uma álgebra de Hopf de dimensão finita. Mostramos também que sempre existe uma globalização de uma coação parcial simétrica, mas, o verdadeiro destaque nesse momento do estudo foi a identificação da globalização minimal de uma ação parcial via a globalização padrão da coação parcial induzida, quando trabalhamos com uma álgebra de Hopf de dimensão finita.

Proposição 3.25. *Sejam H uma álgebra de Hopf de dimensão finita, A uma álgebra associativa e $\cdot : H \otimes A \to A$ uma ação parcial simétrica. Seja $\rho : A \to A \otimes H^*$ a coação parcial induzida $\rho(a) = \sum_{i=1}^{n} h_i \cdot a \otimes h_i^*$, onde $\{h_i\}_{i=1}^{n}$ é a base de H considerada e $\{h_i^*\}_{i=1}^{n}$ sua base dual, e considere a ação $\rightharpoonup$: $H \otimes A \otimes H^* \to A \otimes H^*$ dada por $h \rightharpoonup a \otimes k^* = \sum k^*_{(2)}(h) a \otimes k^*_{(1)}$. Então $(H \rightharpoonup \rho(A), \rho)$ é a quasi-globalização minimal de $\cdot : H \otimes A \to A$.*

Já sabíamos, por Alves e Batista em [5], que podíamos construir a globalização de uma coação parcial via a globalização da ação parcial induzida, aqui fazemos o contrário, fornecemos uma ferramenta para identificar a globalização minimal de uma ação parcial como uma subálgebra de $A \otimes H^*$ em vez de uma subálgebra de $Hom(H, A)$.

Depois de já estabelecido os conceitos de ações e coações parciais em álgebras sem unidade e de apresentar resultados e argumentos que mostram que essas definições são coerentes, decidimos generalizar outro trabalho de Alves e Batista [7] que trata dos invariantes parciais e da teoria de Hopf-Galois parcial. Definimos os invariantes e coinvariantes parciais para álgebras sem unidade, e, sendo A uma H-módulo álgebra parcial, mostramos que existe um contexto de Morita entre a subálgebra dos invariantes parciais $A^{\underline{H}}$ e o produto smash parcial $\underline{A\#H}$, da mesma forma que acontecia para o caso com unidade.

Definição 4.1. *Seja A uma H-módulo álgebra parcial associativa. Definimos como a subálgebra dos invariantes parciais de A o subespaço*

$$A^{\underline{H}} = \{a \in A \,|\, h \cdot (ab) = a(h \cdot b) \text{ e } h \cdot (ba) = (h \cdot b)a, \, \forall b \in A, \, h \in H\}.$$

Teorema 4.6. $(\underline{A\#H}, A^{\underline{H}}, {}_{\underline{A\#H}}A_{A^{\underline{H}}}, {}_{A^{\underline{H}}}A_{\underline{A\#H}})$ *forma um contexto de Morita.*

Seguindo a mesma estrutura de [7], desenvolvemos a teoria de Hopf-Galois parcial e a usamos para tentar determinar quando o contexto de Morita citado é estrito.

Teorema 4.9. *Seja H uma álgebra de Hopf de dimensão finita, $0 \neq t \in \int_H^\ell$, A uma H-módulo álgebra parcial e $A^2 = A$. Suponha que a aplicação canônica $\beta : A \otimes_{A^{\underline{H}}} A \to \underline{A \otimes H^*}$ é sobrejetiva. Então*

1. *Para cada $c \in A$, existem $a_{c1}, \cdots, a_{ck}$ e $b_{c1}, \cdots, b_{ck}$ em A tais que $\phi_{ci} : A \to A^{\underline{H}}$ dada por $\phi_{ci}(a) = t \cdot (b_{ci}a)$ é um morfismo de $A^{\underline{H}}$-módulos à direita e $ca = \sum_{i=1}^k a_{ci}\phi_{ci}(a)$ para todo $a \in A$. Então A é um $A^{\underline{H}}$-módulo unitário à direita e para cada $c \in A$, todo subespaço cA é finitamente gerado como $A^{\underline{H}}$-módulo à direita (unitário);*

2. *Se $t_A(A \otimes_{A^{\underline{H}}} A) = \{x \in A \otimes_{A^{\underline{H}}} A; \, ax = 0, \, \forall a \in A\} = 0$, então β é bijetiva.*

Teorema 4.14. *Seja H uma álgebra de Hopf de dimensão finita com uma integral não nula t, A uma H-módulo álgebra parcial, $A^2 = A$ e $t_A(A \otimes_{A^{\underline{H}}} A) = 0$. As seguintes afirmações são equivalentes:*

1. *$A^{\underline{H}} \subset A$ é uma estensão H^*-Galois parcial;*

2. *$[\cdot, \cdot] : A \otimes_{A^{\underline{H}}} A \to \underline{A\#H}$ é sobrejetora.*

No último capítulo deste trabalho, consideramos um par combinado de álgebras de Hopf $(H, L, \triangleright, \triangleleft)$ e buscamos descobrir a compatibilidade entre as estruturas de H-módulo álgebra parcial e L-módulo álgebra parcial de uma álgebra sem unidade A, de forma que essas estruturas induzissem uma estrutura de $H \bowtie L$-módulo álgebra parcial em A, onde $H \bowtie L$ é o produto cruzado duplo (que é uma álgebra de Hopf) construído a partir do par combinado $(H, L, \triangleright, \triangleleft)$, de acordo com Majid em [20]. O que conseguimos foi responder essa pergunta para o caso onde uma das ações parciais é na verdade uma ação global.

Definição 5.5. *Seja $(H, L, \triangleright, \triangleleft)$ um par combinado de álgebras de Hopf. Seja A uma H-módulo álgebra com ação $\cdot_H$ e uma L-módulo álgebra parcial com ação parcial $\cdot_L$. Se existe uma ação parcial de $H \bowtie L$ tal que restrita à H e L recuperamos as ações parciais originais, diremos que $(\cdot_H, \cdot_L)$ é um par admissível de ações parciais do tipo 1.*

Definição 5.11. *Seja $(H, L, \triangleright, \triangleleft)$ um par combinado de álgebras de Hopf. Seja A uma H-módulo álgebra parcial com ação parcial $\cdot_H$ e uma L-módulo álgebra com ação $\cdot_L$. Se existe uma ação parcial de $H \bowtie L$ tal que restrita à H e L recuperamos as ações parciais originais, diremos que $(\cdot_H, \cdot_L)$ é um par admissível de ações parciais do tipo 2.*

Proposição 5.10. *Seja $(H, L, \triangleright, \triangleleft)$ um par combinado de álgebras de Hopf, A uma álgebra associativa que é uma H-módulo álgebra com ação $\cdot_H$ e uma L-módulo álgebra parcial com ação parcial $\cdot_L$. Se $r(A) = 0$ e H possui antípoda bijetiva, então $(\cdot_H, \cdot_L)$ é um par admissível de ações parciais do tipo 1 se e somente se a aplicação $(h \otimes x) \cdot a = h \cdot_H x \cdot_L a$ determina uma açaõ parcial de $H \bowtie L$ em A, e isso acontece se e somente se*

$$\sum (x_{(1)} \cdot_L b)(x_{(2)} \cdot_L h \cdot_H y \cdot_L a) \quad = \quad \sum (x_{(1)} \cdot_L b)((x_{(2)} \triangleright h_{(1)}) \cdot_H ((x_{(3)} \triangleleft h_{(2)})y) \cdot_L a),$$

para todo $a, b \in A$, $h \in H$, $x, y \in L$.

Proposição 5.16. *Seja A uma H-módulo álgebra parcial com ação parcial $\cdot_H$ e uma L-módulo álgebra com ação $\cdot_L$. Se $r(A) = 0$, então $(\cdot_H, \cdot_L)$ é um par admissível de ações parciais do tipo 2 se e somente se a aplicação linear*

$$h \otimes x \cdot a = \sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \cdot_L S_H(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \cdot_H a$$

determina uma ação parcial.

Seguindo essa mesma ideia, buscamos estudar o caso das representações parciais, que também conseguimos as compatibilidades para quando uma das representações parciais é na verdade uma representação.

Definição 5.29. *Suponha que π_H e π_L são representações parciais de H em uma álgebra unitária A tais que induzem uma representação parcial de $H \bowtie L$. Se π_H é uma representação, diremos que (π_H, π_L) é um par admissível de representações parciais do tipo 1. Se π_L é uma representação, diremos que (π_H, π_L) é um par admissível de representações parciais do tipo 2.*

Proposição 5.30. *Seja A uma álgebra unitária e $\pi_H : H \to A$ e $\pi_L : L \to A$ representações parciais. Então*

1. *(π_H, π_L) é um par admissível de representações parciais do tipo 1 se e somente se π_H é uma representação e*

$$\pi_L(x)\pi_H(h) = \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)}).$$

2. *(π_H, π_L) é um par admissível de representações parciais do tipo 2 se e somente se π_L é uma representação e*

$$\pi_L(x)\pi_H(h) = \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)}).$$

Em ambos os casos de ações e representações parciais, conseguimos estudar a situação estritamente parcial para a álgebra de Hopf $\Bbbk^G \# \Bbbk F$ no corpo $\Bbbk$. Mais ainda, para ações parciais, conseguimos determinar a compatibilidade para termos uma ação parcial de $\Bbbk^G \# \Bbbk F$ na álgebra $FMat_{\mathbb{N}}(\Bbbk)$, ou seja, as matrizes de ordem infinita com finitas entradas não nulas.

Definição 5.20. *Seja G um grupo finito e $i \mapsto gi$ uma ação de G em $\mathbb{N}$. Uma ação parcial caminhante é uma ação parcial de $\Bbbk G$ em $FMat_{\mathbb{N}}(\Bbbk)$ da forma $g \cdot E_{ij} = \alpha_{ij}(g)E_{(gi)(gj)}$, onde cada α_{ij} associa cada $g \in G$ a um escalar em $\Bbbk$.*

Teorema 5.22. *Sejam G, F grupos finitos, G abeliano, k um corpo tal que $char\,\Bbbk \nmid |G|$, $\triangleleft : G \times F \to G$ uma ação à direita dada por automorfismos de grupos e $A = Mat_{n\times n}(\Bbbk)$. Considere uma G-graduação parcial boa em A determinada por um subgrupo H de G, e uma ação parcial caminhante de $\Bbbk F$ em A determinada por um subgrupo L de F. Então existe uma ação parcial de $\Bbbk^G \# \Bbbk F$ em A que restrita à $\Bbbk^G$ e $\Bbbk F$ recuperamos as ações parciais originais, se e somente se $t_{ij}H \triangleleft x = t_{(x^{-1}i)(x^{-1}j)}H$. Em particular, H é invariante pela ação $\triangleleft|_{G\times L}$.*

Introduction

In this work we investigate partial actions of Hopf algebras on nonunital algebras. Our aim is to extend the main results of the theory of partial actions of Hopf algebras on unital algebras introduced by Caenepeel and Jansen [13] and later developed by Alves, Batista, Dokuchaev, Paques and others.

At first, we wanted to work exclusively with algebras with local units, but throughout the study we noticed that we could extend the definitions and results even more, in such way that the suggested definition provide a relation between symmetrical partial $\Bbbk G$-actions and a class of partial G-actions on a given associative algebras, as was already done for unital algebras. Hence, we started to work with associative algebras with trivial right annihilator, and we extended some of the fundamental results of the previous theory for this case, i.e., the Globalization Theorem, the construction of the partial smash product, the Morita context between $\underline{A\#H}$ and $B\#H$, where B is a globalization of the partial action of H on A, the definition of the partial invariants subalgebra and the partial Hopf Galois theory.

Here, we would like to highlight that in the most general case treated in this thesis, that of a partial action on an algebra with trivial right annihilator, it is not possible to recover the original partial action as an induced action from the globalization, in this case, we talk about a ”quasi-globalization”. But when we consider the ideal as an algebra with local units, we can define the partial action induced by the action.

We also developed the theory of Morita equivalence of partial Hopf actions that generalize the concept of Morita equivalence of partial group actions presented in [1]. We proved that Morita equivalence of partial $\Bbbk G$-actions is closely related to Morita equivalence of partial G-actions and, also, this theory provides an interesting relation of the definition of α-modules, defined in [1], and the definition of partial (A, H)-modules, defined in [14]

Alves and Batista proved that, under some assumptions, the globalization of the partial action induced by a partial coaction is also a globalization of the original partial coaction. We proved that, when the Hopf algebra is finite dimensional, the standard globalization of the partial coaction induced by a partial action is actually the minimal globalization of the original partial action.

Partial actions on categories are defined in [2] and every partial action on a linear category $\mathcal{C}$ induces a partial action on the “matrix algebra” $a(\mathcal{C})$, which is an algebra with local units. Actually, when we worked with algebras with local units, we found a especial class of partial actions that reverses this process: we call these categorizable, because they can be related to a partial action on a specific linear category naturally associated to the algebra and the chosen system of local units. Since we did not find any mention of partial coactions on categories in the literature, we develop a theory of such coactions in order to associate them to the categorizable partial coations.

Moreover, if we consider an algebra A with system of local units S, we prove that the category of the left unital modules of A is equivalent to the category of the left modules of the category $\mathcal{C}^S(A)$. Also, if $\mathcal{C}$ is a linear category, we proved that the category of the left

$\mathcal{C}$-modules and the category of the left unital $a(\mathcal{C})$-modules are equivalent.

Following the first results of Abadie, Dokuchaev, Exel and Simón, in [1], we defined the concept of Morita equivalence of partial actions, and with this we proved that if H is a Hopf algebra and A is a partial H-module algebra with system of local units S, then $\underline{A\#H}$ and $\underline{a(\mathcal{C}^S(A))\#H}$ are Morita equivalent.

We also worked with a particular example of partial coactions, which are the good partial G-gradings on $FMat_{\mathbb{N}}(\Bbbk)$, i.e., partial coactions of kG on $FMat_{\mathbb{N}}(\Bbbk)$. We described completely the good partial G-gradings on $FMat_{\mathbb{N}}(\Bbbk)$ using the classification of partial actions of $\Bbbk^G$ on Schurian categories, presented in [2].

Finally, we consider a matched pair of Hopf algebras $(H, L, \triangleright, \triangleleft)$ in the sense of Majid [20]. We wished to know when two partial actions of Hopf algebras H and L on an algebra induce a partial action of the double crossed product $H \bowtie L$ on A such that, when restricted to H and L, we recover the original partial actions. However, as we worked toward answering this question, we noticed that it is difficult to determine such partial action, when it exists. Because of this, we considered two subcases: the case where both actions are global and the case where one action is partial and the other is global.

For strictly partial actions, we consider the Hopf algebra $\Bbbk^G\#\Bbbk F$ associated to a specific matched pair of groups and the partial action is on the field $\Bbbk$ and on $Mat_{n\times n}(\Bbbk)$. An analogous question appears for partial representations of the double crossed product $H \bowtie L$, we proceed as in the case of actions, considering in the first place representations of H and L on an algebra A, and then considering a partial representation and a representation. We finish by considering partial representations of the Hopf algebra $\Bbbk^G\#\Bbbk F$ on the field $\Bbbk$ and verifying the relations between the induced partial actions of $\Bbbk^G$ and $\Bbbk F$ on $\Bbbk$.

Throughout this work, all the linear structures will be considered over a field $\Bbbk$; when we write module we mean left module, and by partial action we mean left partial action. For example, by algebra we mean $\Bbbk$-algebra.

Chapter 1

Algebras without identity

In this section, we will present some definitions that will be useful throughout this work. For this, we will always assume that A is an associative algebra.

First, we will recall the definition of an algebra with local units, which will let us to relate the concept of partial actions on categories with the concept of partial actions on this kind of algebras, that we will introduce later.

Definition 1.1. *The set $S = \{e_\lambda\}_{\lambda\in\Lambda}$ is called a system of local units of A if $e_\lambda^2 = e_\lambda$, for every $\lambda \in \Lambda$, and for every finite subset F of A, there exists $e_\alpha \in S$ such that $e_\alpha a = ae_\alpha = a$ for every $a \in F$. If A has a system of local units, it is called an algebra with local units.*

There is an equivalent definition for algebras with local units, that is: for every $a \in A$, there exist $e^2 = e \in A$ such that $ea = ae = a$. But, in this work, we will need a regularity on the system of local units, we need it to be directed, i.e., for every $e_\lambda, e_\alpha \in S$, there exist e_β such that $e_\beta e_\lambda = e_\lambda e_\beta = e_\lambda$ and $e_\alpha e_\beta = e_\beta e_\alpha = e_\alpha$.

Note that, in this case, every system of local units of an algebra A is a partially ordered set. Let $S = \{e_\lambda\}_{\lambda\in\Lambda}$ be a system of local units for A, we say that $\alpha \leq \beta \Leftrightarrow e_\alpha e_\beta = e_\beta e_\alpha = e_\alpha$. Sometimes we will write $e_\alpha \leq e_\beta$ instead of $\alpha \leq \beta$.

Another type of algebras is the s-unital ones, that includes the algebras with local units and is a particular class of the algebras with trivial left (and right) annihilator.

Definition 1.2. *A is called a left s-unital algebra if $a \in Aa$, for every $a \in A$, and a right s-unital algebra if $a \in aA$, for every $a \in A$. If A is both left and right s-unital, we just say that A is an s-unital algebra.*

Definition 1.3. *A is called idempotent if $A^2 = A$, i.e., every element of A is a finite sum of products of elements of A.*

Example 1.4. *If A is an algebra with local units, then A is idempotent and s-unital.*

When we consider modules over idempotent algebras, or even over algebras with local units, we lose the property of $1m = m$, which gives us the equality $AM = M$, where M is an A-module and A is a unital algebra. Hence we will present the definition of a unital module, which is an A-module M that satisfies $AM = M$, where A is an associative algebra.

Definition 1.5. *An A-module M is called an unital left A-module if $M = AM$, i.e., for every $m \in M$ there exist elements $a_1, \cdots, a_n \in A$ and $m_1, \cdots, m_n \in M$ such that $m = \sum_{i=1}^n a_i m_i$.*

Example 1.6. *If A is idempotent, A is a unital A-module.*

For some of the results related to partial actions that we will prove in this work, as for example the existence of a globalization, we will need to consider algebras with trivial left (and/or right) annihilator.

Definition 1.7. *The left annihilator of A is denoted by* $l(A) = \{a \in A \,|\, ab = 0, \,\forall b \in A\}$.

Definition 1.8. *The right annihilator of A is denoted by* $r(A) = \{a \in A \,|\, ba = 0, \,\forall b \in A\}$.

Definition 1.9. *Let M be a left A-module, its torsion submodule is* $t_A(M) = \{m \in M | am = 0, \,\forall a \in A\}$.

Example 1.10. *If A is a left s-unital algebra and M is a unital A-module, then* $t_A(M) = 0$.

There are two "models" of algebras with local units which will appear throughout this work: the algebra of "finite matrices" and the algebra associated to a linear category.

Let A be an associative algebra and I be a nonempty set. The $A - A$-bimodule of $I \times I$ matrices over A is the direct product $A^{I \times I}$, which we will denote by $Mat_I(A)$. As usual, we denote an element of $Mat_I(A)$ by (a_{ij}). The algebra of "finite matrices" $FMat_I(A)$ is the submodule of the matrices with finite support, i.e., those matrices (a_{ij}) where the number of pairs $(i, j) \in I \times I$ with $a_{ij} \neq 0$ is finite, endowed with the matrix product

$$(a_{ij})(b_{ij}) = (\sum_l a_{il} b_{lj}).$$

If A is a unital algebra then $FMat_I(A)$ is an algebra with local units. The idempotents

$$E_{kk} = (e_{ij})$$

where $e_{kk} = 1_A$ and $e_{ij} = 0$ if $(i, j) \neq (k, k)$ are mutually orthogonal and the set of finite sums of such (distinct) idempotents is a system of local units. If A itself is not unital but has local units then again $FMat_I(A)$ is an algebra with local units.

A second example comes from linear categories. Let $\mathcal{C}$ be a $\Bbbk$-linear category, i.e., a category where every set of morphisms has a structure of $\Bbbk$-vector space (or $\Bbbk$-module, if $\Bbbk$ is a commutative ring) and the composition of morphisms is a $\Bbbk$-bilinear map. The algebra $a(\mathcal{C})$ consists of elements of the form $({}_y f_x)_{x,y \in \mathcal{C}_0}$, where $\mathcal{C}_0$ is the set of objects of $\mathcal{C}$, with ${}_y f_x : x \to y$ and with finite ${}_y f_x \neq 0$, where multiplication is given by matrix multiplication and the composition of $\mathcal{C}$, i.e.,

$$({}_y f_x)({}_y g_x) = (\sum_z {}_y f_z \circ {}_z g_x).$$

Chapter 2

Partial actions

In this section, we will introduce the concept of partial action of a Hopf algebra H on an algebra without identity A in such way that when A is unital it is a partial action in the usual way. We begin by recalling the definition of a partial action on a unital algebra.

Definition 2.1 ([2],[5])**.** *Let H be a Hopf algebra and A an algebra with unit 1_A. A linear map $\cdot : H \otimes A \longrightarrow A$, $h \otimes a \mapsto h \cdot a$ is called a partial action if, for all $a, b \in A$, $h, g \in H$,*

1. $1_H \cdot a = a$;
2. $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$;
3. $h \cdot (g \cdot a) = \sum (h_{(1)} \cdot 1_A)(h_{(2)} g \cdot a)$.

Where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. In this case we say that A is a partial H-module algebra with unity. If also $h \cdot (g \cdot a) = \sum (h_{(1)} g \cdot a)(h_{(2)} \cdot 1_A)$, then we say that this partial action is symmetrical.

This definition not just satisfies the requisites to obtain a good structure of the smash product $A\#H$, as studied in [13], but also is closely related to partial G-actions on unital algebras, when we consider $H = kG$, as was also presented in [13].

Remark 2.2. *The smash product $A\#H$ is an algebra determined by: $A\#H = A \otimes H$ as a vector space and multiplication given by $(a \otimes h)(b \otimes k) = \sum a(h_{(1)} \cdot b) \otimes h_{(2)} k$.*

Remark 2.3. *In [13], Caenepeel and Janssen presented the following equivalences with respect to the axioms of a partial action on a unital algebra:*

- *Item $2)$ is a necessary and sufficient condition for the smash product $A\#H$ be an A-bimodule with actions*

$$b(a\#h)b' = \sum ba(h_{(1)} \cdot b')\#h_{(2)}.$$

- *Item $1)$ is a necessary and sufficient condition for the inclusion $i : A \longrightarrow A\#H$, $a \mapsto a\#1_H$, be a right A-linear map.*

- *Item $3)$ is a necessary and sufficient condition for the smash product $A\#H$ be an associative algebra;*

And in [4], Alves et al. proved that

- *the symmetrical property is a necessary and sufficient condition for $\varphi(A)$ be an ideal of $H \triangleright \varphi(A)$, where $\varphi : A \longrightarrow Hom(H, A)$, $\varphi(a)(h) = h \cdot a$, and $(h \triangleright \varphi)(a)(k) = \varphi(a)(kh)$.*

2.1 Partial actions on non unital algebras

Inspired by [2], [5] and, mainly, by the equivalences of Remark 2.3 presented in [13], we will suggest the following definition of partial action on associative algebras.

Definition 2.4. *Let A be an associative algebra. A linear map $\cdot : H \otimes A \longrightarrow A$, $h \otimes a \longmapsto h \cdot a$ will be called a partial action if, for all $a, b \in A$, $h, k \in H$,*

1. $1_H \cdot a = a$;
2. $h \cdot (a(k \cdot b)) = \sum (h_{(1)} \cdot a)(h_{(2)} k \cdot b)$.

In this case, A will be called a partial H-module algebra. We will say that the partial action is symmetrical if, additionally, $h \cdot ((k \cdot b)a) = \sum (h_{(1)} k \cdot b)(h_{(2)} \cdot a)$, for every $a, b \in A$, $h, k \in H$.

Note that if A is a partial H-module algebra we have that

$$\begin{aligned} h \cdot (ab) &= h \cdot (a(1_H \cdot b)) \\ &= \sum (h_{(1)} \cdot a)(h_{(2)} 1_H \cdot b) \\ &= \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b). \end{aligned}$$

Hence, if A is unital, the definitions of partial actions coincide.

Moreover, we will show later that this new concept is somehow related to the concept of partial group actions on nonunital algebras, and that the known relation between partial kG-actions and partial G-actions on unital algebras is a particular case of it.

Example 2.5. *Let A be any associative k-algebra, where $\Bbbk$ is a field, H a Hopf $\Bbbk$-algebra and $\lambda : H \to \Bbbk$ a linear map such that*

$$\begin{aligned} \lambda(1_H) &= 1 \\ \lambda(h)\lambda(g) &= \sum \lambda(h_{(1)})\lambda(h_{(2)} g) = \sum \lambda(h_{(1)} g)\lambda(h_{(2)}), \end{aligned}$$

for every $h, g \in H$. Then the map

$$\begin{aligned} \cdot : H \otimes A &\to A \\ h \otimes a &\mapsto h \cdot a = \lambda(h)a, \end{aligned}$$

defines a symmetric partial action.

Example 2.6. *Let A be any associative algebra, G be a finite group with $|G| = n$ and $\{p_g \,;\, g \in G\}$ be its dual basis in $(\Bbbk G)^*$. Suppose that $char \Bbbk \nmid n$, then, the mapping $p_g \cdot a = \frac{1}{n} a$ defines a symmetrical partial action of $(\Bbbk G)^*$ on A.*

Example 2.7. *Let $P = C_c(\mathbb{R})$ be the set of the continuous function with compact support. Note that P is an s-unital algebra, because for every $f \in P$ there exist $g \in P$ such that $supp\, f \subseteq g^{-1}(1)$, i.e., the support of f lies in the pre-image of 1 by g. Then, the mappings $(\overline{0} \cdot f)(x) = f(x)$ and $(\overline{1} \cdot f)(x) = f(-x)$ determine a (global) action of $k\mathbb{Z}_2$ on P.*

The following lemmas refer to Remark 2.3. For this, we will assume that H is a Hopf algebra, A is an associative algebra with $r(A) = 0$, $\cdot : H \otimes A \to A$ is a linear map given by $\cdot(h \otimes a) = h \cdot a$ and $A \# H$ its associated smash product.

Lemma 2.8. *$A\#H$ is an associative algebra if and only if $h \cdot (a(k \cdot b)) = \sum(h_{(1)} \cdot a)(h_{(2)}k \cdot b)$ for every $h, k \in H$, $a, b \in A$.*

Proof. In fact, $A\#H$ is an associative algebra if and only if for every $a, b, c \in A$, $h, k, l \in H$,

$$((c\#l)(a\#h))(b\#k) = (c\#l)((a\#h)(b\#k))$$
$$\Updownarrow$$
$$\sum c(l_{(1)} \cdot a)(l_{(2)}h_{(1)} \cdot b)\#l_{(3)}h_{(2)}k = \sum c(l_{(1)} \cdot (a(h_{(1)} \cdot b)))\#l_{(2)}h_{(2)}k$$
$$\Downarrow^{I\otimes\varepsilon,\, k=1_H}$$
$$\sum c(l_{(1)} \cdot a)(l_{(2)}h \cdot b) = c(l \cdot (a(h \cdot b)))$$
$$\Downarrow$$
$$c[\sum(l_{(1)} \cdot a)(l_{(2)}h \cdot b) - (l \cdot (a(h \cdot b)))] = 0.$$

Since $r(A) = 0$, we have the required equality. Conversely, if $h \cdot (a(k \cdot b)) = \sum(h_{(1)} \cdot a)(h_{(2)}k \cdot b)$, clearly $A\#H$ is associative. □

Lemma 2.9. *$A\#H$ is an A-bimodule with structure given by $b(a\#h)b' = \sum ba(h_{(1)} \cdot b')\#h_{(2)}$ if and only if $h \cdot ab = \sum(h_{(1)} \cdot a)(h_{(2)} \cdot b)$, for every $h \in H$, $a, b \in A$.*

Proof. Since A is associative, we only need to prove that $((c\#h)a)b = (c\#h)(ab)$, and this holds if and only if

$$\sum c(h_{(1)} \cdot a)(h_{(2)} \cdot b)\#h_{(3)} = \sum c(h_{(1)} \cdot ab)\#h_{(2)}$$
$$\Downarrow^{I\otimes\varepsilon}$$
$$\sum c(h_{(1)} \cdot a)(h_{(2)} \cdot b) = c(h \cdot ab)$$
$$\Downarrow^{r(A)=0}$$
$$\sum(h_{(1)} \cdot a)(h_{(2)} \cdot b) = h \cdot ab.$$

The converse is straightforward. □

Lemma 2.10. *$\iota : A \to A\#H$, $a \mapsto a\#1_H$, is a right A-linear morphism if and only if $1_H \cdot a = a$.*

Proof. We have that ι is a right A-linear morphism if and only if

$$ab\#1_H = (a\#1_H)b$$
$$\Updownarrow$$
$$ab\#1_H = \sum a((1_H)_{(1)} \cdot b)\#(1_H)_{(2)}$$
$$\Downarrow^{I\otimes\varepsilon}$$
$$ab = a(1_H \cdot b).$$

Since $r(A) = 0$, we have that $b = 1_H \cdot b$. The converse is straightforward. □

Based on these lemmas, we have that the suggested definition for partial Hopf algebras on nonunital algebras is good enough for us to continue trying to extend the classical results using it.

As for the case when we consider unital algebras, we have the following result.

Theorem 2.11. *Let H be a cocommutative Hopf algebra. If A and B are both (symmetrical) partial H-module algebras, then $A \otimes B$ is a (symmetrical) partial H-module algebra via*

$$h \cdot (a \otimes b) = \sum h_{(1)} \cdot a \otimes h_{(2)} \cdot b.$$

Proof. Clearly, the first axiom of partial actions holds. For the second one, let $a, b \in A$, $x, y \in B$, $h, k \in H$, then

$$\begin{aligned} h \cdot ((a \otimes x)(k \cdot b \otimes y)) &= \sum h_{(1)} \cdot (a(k_{(1)} \cdot b)) \otimes h_{(2)} \cdot (x(k_{(2)} \cdot y)) \\ &= \sum (h_{(1)} \cdot a)(h_{(2)}k_{(1)} \cdot b) \otimes (h_{(3)} \cdot x)(h_{(4)}k_{(2)} \cdot y) \\ &= \sum (h_{(1)} \cdot a)(h_{(3)}k_{(1)} \cdot b) \otimes (h_{(2)} \cdot x)(h_{(4)}k_{(2)} \cdot y) \\ &= \sum (h_{(1)} \cdot a \otimes h_{(2)} \cdot x)(h_{(3)}k_{(1)} \cdot b \otimes h_{(4)}k_{(2)} \cdot y) \\ &= \sum (h_{(1)} \cdot (a \otimes x))(h_{(2)}k \cdot (b \otimes y)). \end{aligned}$$

□

2.2 Partial H-module algebras and algebras in ${}_H\mathcal{M}^{par}$

In this section we will show that, when the antipode of H is bijective, there is a bijective correspondence between partial H-module algebras with symmetrical partial actions and (not necessary unital) algebras in the category of the partial H-modules. This result was proved by Alves at al. in [6] for the case of unital algebras. We will show that this holds for algebras A that have at least one of the following properties:

1. $A^2 = A$;
2. $r(A) = l(A) = 0$.

First, note that if A is any partial H-module algebra, then for all $x, y \in A$, $h \in H$, we have that

$$\sum h_{(1)} \cdot S(h_{(2)}) \cdot xy = \sum (h_{(1)} \cdot S(h_{(2)}) \cdot x)y, \tag{2.1}$$

because

$$\begin{aligned} \sum h_{(1)} \cdot S(h_{(2)}) \cdot xy &= \sum h_{(1)} \cdot [(S(h_{(3)}) \cdot x)(S(h_{(2)}) \cdot y)] \\ &= \sum (h_{(1)} \cdot S(h_{(4)}) \cdot x)(h_{(2)}S(h_{(3)}) \cdot y) \\ &= \sum (h_{(1)} \cdot S(h_{(2)}) \cdot x)y. \end{aligned}$$

Equation (2.1) means that the linear map $a \mapsto \sum h_{(1)} \cdot S(h_{(2)}) \cdot a$ is a right A-module map from A to A for every $h \in H$. Analogously, if the partial action is symmetrical, equation

$$\sum S(h_{(1)}) \cdot h_{(2)} \cdot xy = x(\sum S(h_{(1)}) \cdot h_{(2)} \cdot y) \tag{2.2}$$

means that the linear map $a \mapsto \sum S(h_{(1)}) \cdot h_{(2)} \cdot a$ is a left A-module map from A to A for every $h \in H$.

Definition 2.12 ([6]). *Let H be a Hopf algebra with antipode S and A a unital algebra. A linear map $\pi : H \to A$ is a partial representation of H on A if*

1. $\pi(1_H) = 1_A$;
2. $\sum \pi(h)\pi(k_{(1)})\pi(S(k_{(2)})) = \sum \pi(hk_{(1)})\pi(S(k_{(2)}))$;

3. $\sum \pi(k_{(1)})\pi(S(k_{(2)}))\pi(h) = \sum \pi(k_{(1)})\pi(S(k_{(2)})h)$;

4. $\sum \pi(h)\pi(S(k_{(1)}))\pi(k_{(2)}) = \sum \pi(hS(k_{(1)}))\pi(k_{(2)})$;

5. $\sum \pi(S(k_{(1)}))\pi(k_{(2)})\pi(h) = \sum \pi(S(k_{(1)}))\pi(k_{(2)}h)$,

for every $h, k \in H$.

Remark 2.13 ([6])**.** *If* H *is cocommutative, then the items in the definition of a partial representation coalesce into* 1), 2) *and* 5).

Lemma 2.14. *Let* A *be a partial* H*-module algebra with symmetrical partial action and* H *a Hopf algebra. If* A *is either idempotent or,* $r(A) = l(A) = 0$ *and the antipode of* H *is bijective, then the linear map* $\pi : H \to End(A)$*, defined by* $\pi(h)(a) = h \cdot a$*, is a partial representation.*

Proof. Let us prove it first for idempotent algebras. Note that since for any element $a \in A$ there exist $b_1, \cdots, b_n, c_1, \cdots, c_n \in A$ such that $a = \sum_{i=1}^n b_i c_i$ and π is a linear map, then we only need to prove the axioms of partial representation for elements of the form $xy \in A$. Let $a, x, y \in A$ and $h, k \in H$, we have that $\pi(1_H)(a) = 1_H \cdot a = a$, then $\pi(1_H) = id_A$, and

$$\begin{aligned}
\sum \pi(h)\pi(k_{(1)})\pi(S(k_{(2)}))(xy) &= \sum h \cdot k_{(1)} \cdot S(k_{(2)}) \cdot (xy) \\
&\overset{(2.1)}{=} \sum h \cdot ((k_{(1)} \cdot S(k_{(2)}) \cdot x)y) \\
&= \sum (h_{(1)}k_{(1)} \cdot S(k_{(2)}) \cdot x)(h_{(2)} \cdot y) \\
&= \sum (h_{(1)}k_{(1)} \cdot S(k_{(4)}) \cdot x)(h_{(2)}k_{(2)}S(k_{(3)}) \cdot y) \\
&= \sum hk_{(1)} \cdot [(S(k_{(3)}) \cdot x)(S(k_{(2)}) \cdot y)] \\
&= \sum hk_{(1)} \cdot S(k_{(2)}) \cdot (xy) \\
&= \sum \pi(hk_{(1)})\pi(S(k_{(2)}))(xy).
\end{aligned}$$

This proves that item 2) holds. For the axiom 4) of partial representations, we use a similar calculation. For item 3), we have that

$$\begin{aligned}
\sum \pi(k_{(1)})\pi(S(k_{(2)}))\pi(h)(xy) &= \sum k_{(1)} \cdot S(k_{(2)}) \cdot h \cdot (xy) \\
&= \sum k_{(1)} \cdot S(k_{(2)}) \cdot (h_{(1)} \cdot x)(h_{(2)} \cdot y) \\
&= \sum k_{(1)} \cdot [(S(k_{(3)})h_{(1)} \cdot x)(S(k_{(2)}) \cdot h_{(2)} \cdot y)] \\
&= \sum (k_{(1)} \cdot S(k_{(2)})h_{(1)} \cdot x)(h_{(2)} \cdot y) \\
&= \sum k_{(1)} \cdot [(S(k_{(3)})h_{(1)} \cdot x)(S(k_{(2)})h_{(2)} \cdot y)] \\
&= \sum k_{(1)} \cdot S(k_{(2)})h \cdot (xy) \\
&= \sum \pi(k_{(1)})\pi(S(k_{(2)}h))(xy).
\end{aligned}$$

For the axiom 5), we use a similar calculation.

Now, we will assume that A is not necessarily idempotent, but that instead we have $r(A) = l(A) = 0$. Then, for every $a, x \in A$, $h, k \in H$, we have also that $\pi(1_H) = id_A$ and

$$\sum \pi(h)\pi(k_{(1)})\pi(S(k_{(2)}))(a)x = [h \cdot k_{(1)} \cdot S(k_{(2)}) \cdot a]x$$

$$
\begin{aligned}
&= \sum h_{(1)} \cdot [(k_{(1)} \cdot S(k_{(2)}) \cdot a)(S(h_{(2)}) \cdot x)] \\
&= \sum (h_{(1)}k_{(1)} \cdot S(k_{(3)}) \cdot a)(h_{(2)} \cdot S(h_{(3)}) \cdot x) \\
&= \sum h_{(1)}k_{(1)} \cdot [(S(k_{(3)}) \cdot a)(S(k_{(2)}) \cdot S(h_{(2)}) \cdot x)] \\
&= \sum h_{(1)}k_{(1)} \cdot S(k_{(2)}) \cdot (a(S(h_{(2)}) \cdot x)) \\
&= \sum h_{(1)}k_{(1)} \cdot [(S(k_{(3)}) \cdot a)(S(k_{(2)})S(h_{(2)}) \cdot x)] \\
&= \sum (hk_{(1)} \cdot S(k_{(2)}) \cdot a)x \\
&= \sum \pi(hk_{(1)})\pi(S(k_{(2)}))(a)x,
\end{aligned}
$$

since $l(A) = 0$ and this holds for all $x \in A$, we have that

$$\sum \pi(h)\pi(k_{(1)})\pi(S(k_{(2)})) = \sum \pi(hk_{(1)})\pi(S(k_{(2)})).$$

The proof for item $3)$ is entirely analogous. For item $4)$,

$$
\begin{aligned}
\sum x\pi(h)\pi(S(k_{(1)}))\pi(k_{(2)})(a) &= \sum x(h \cdot S(k_{(1)}) \cdot k_{(2)} \cdot a) \\
&= \sum h_{(2)} \cdot [(S^{-1}(h_{(1)}) \cdot x)(S(k_{(1)}) \cdot k_{(2)} \cdot a)] \\
&= \sum (h_{(2)} \cdot S^{-1}(h_{(1)}) \cdot x)(h_{(3)}S(k_{(1)}) \cdot k_{(2)} \cdot a) \\
&= \sum h_{(2)}S(k_{(1)}) \cdot [(k_{(2)} \cdot S^{-1}(h_{(1)}) \cdot x)(k_{(3)} \cdot a)] \\
&= \sum h_{(2)}S(k_{(1)}) \cdot [(k_{(2)}S^{-1}(h_{(1)}) \cdot x)(k_{(3)} \cdot a)] \\
&= \sum x(hS(k_{(1)}) \cdot k_{(2)} \cdot a) \\
&= \sum x\pi(hS(k_{(1)}))\pi(k_{(2)})(a),
\end{aligned}
$$

since $r(A) = 0$ and this holds for all $x \in A$, we have that $\sum \pi(h)\pi(S(k_{(1)}))\pi(k_{(2)}) = \sum \pi(hS(k_{(1)}))\pi(k_{(2)})$. The calculations to verify item $5)$ are similar, although in this case the bijectivity of the antipode of H is not needed. □

Corollary 2.15. *Let H be a cocommutative Hopf algebra and A a partial H-module algebra with symmetrical partial action. If $r(A) = 0$ or $l(A) = 0$, then the linear map $\pi : H \to End(A)$, defined by $\pi(h)(a) = h \cdot a$, is a partial representation.*

Proof. It is a direct consequence of Remark 2.13 and the previous lemma. □

This corollary shows that if G is a finite group, $H = kG$ and A is a partial H-module algebra with symmetrical partial action, then the mapping $\pi(g)(a) = g \cdot a$ defines a partial representation. And by [6], since

$$\sum \pi(h_{(1)})\pi(S(h_{(2)}))\pi(h_{(3)}) = \pi(h),$$

for every partial representation $\pi : H \to End(A)$, we have that

$$g \cdot g^{-1} \cdot g \cdot a = g \cdot a,$$

for every $a \in A$, $g \in G$.

Definition 2.16 ([6]). *Let H be a Hopf algebra. A partial H-module is a vector space M with a partial representation $\pi : H \to End(M)$.*

Motivated by the relation between H-modules and representations of H, Alves et al. defined in [6] an equivalent definition of partial H-module, which is a vector space M with a linear map $\bullet : H \otimes M \to M$ satisfying the following properties:

1. $1_H \bullet m = m$;
2. $\sum h \bullet (k_{(1)} \bullet (S(k_{(2)}) \bullet m)) = hk_{(1)} \bullet (S(k_{(2)}) \bullet m)$;
3. $\sum h_{(1)} \bullet (S(h_{(2)}) \bullet (k \bullet m)) = h_{(1)} \bullet (S(h_{(2)})k \bullet m)$;
4. $\sum h \bullet (S(k_{(1)}) \bullet (k_{(2)} \bullet m)) = hS(k_{(1)}) \bullet (k_{(2)} \bullet m)$;
5. $\sum S(h_{(1)}) \bullet (h_{(2)} \bullet (k \bullet m)) = S(h_{(1)}) \bullet h_{(2)} \bullet (k \bullet m)$.

In what follows we will use the latter definition.

Definition 2.17 ([6]). *Let H be a Hopf algebra. The algebra H_{par} is the quotient algebra $T(H)/I$, where $T(H)$ is the tensor algebra and I is the ideal generated by the elements:*

1. $\sum x \otimes y_{(1)} \otimes S(y_{(2)}) - xy_{(1)} \otimes S(y_{(2)})$;
2. $\sum x \otimes S(y_{(1)}) \otimes y_{(2)} - xS(y_{(1)}) \otimes y_{(2)}$;
3. $\sum S(x_{(1)}) \otimes x_{(2)} \otimes y - S(x_{(1)}) \otimes x_{(2)}y$;
4. $\sum x_{(1)} \otimes S(x_{(2)}) \otimes y - x_{(1)} \otimes S(x_{(2)})y$.

Recall from [6] that, when the Hopf algebra H has a bijective antipode, the category of the partial H-modules is isomorphic to the category of the H_{par}-modules. Also in [6], the authors proved that H_{par} is a Hopf algebroid, but to know this structure will be not necessary in this work.

Here, we will also denote the class of $x^1 \otimes \cdots \otimes x^n$ in $T(H)/I$ by $[x^1]\cdots[x^n]$, the category of partial H-modules by ${}_H\mathcal{M}^{par}$ and the base algebra of the Hopf algebroid H_{par} by

$$A = \{\epsilon_{h^1} \cdots \epsilon_{h^n};\ h^i \in H\},$$

where $\epsilon_h = [h_{(1)}][S(h_{(2)})]$. From [6] we know that every element of H_{par} is of the form

$$\epsilon_{k^1} \cdots \epsilon_{k^n}[h]$$

and there exists a partial action of H on A given by $h \cdot a = [h_{(1)}]a[S(h_{(2)})]$, and with this, Alves et al. proved that the linear map

$$\begin{array}{rcl} \underline{A\#H} & \to & H_{par} \\ (a\#h)1_A & \mapsto & a[h] \end{array}$$

is an algebra isomorphism.

With this, the structure on the category of partial H-modules obtained in the isomorphism ${}_H\mathcal{M}^{par} \cong {}_{H_{par}}\mathcal{M} \cong {}_{\underline{A\#H}}\mathcal{M}$, is given explicitly by:

$$h \bullet m = [h] \triangleright m = (1_A\#h) \triangleright m,$$

and, as we can see in [6], we have that for $a = \epsilon_{h^1} \cdots \epsilon_{h^n} \in A$, $m \in M$, $k \in H$

$$(a\#k) \triangleright m = \sum h^1_{(1)} \bullet (S(h^1_{(2)}) \bullet (\cdots (h^n_{(1)} \bullet (S(h^n_{(2)}) \bullet (k \bullet m)) \cdots)).$$

Also in [6], the authors highlight that every partial H-module M can be viewed as an A-bimodule by:

$$\begin{aligned} am &= s(a) \triangleright m = (a\#1_H) \triangleright m \\ ma &= t(a) \triangleright m = \sum (1_A\#h^n_{(2)})(1_A\#S^{-1}(h^n_{(1)})) \cdots (1_A\#h^1_{(2)})(1_A\#S^{-1}(h^1_{(1)})) \triangleright m. \end{aligned}$$

In the equation above, $a\#h = \sum a(h_{(1)} \cdot 1_A) \otimes h_{(2)} \in \underline{A\#H}$, and s is the morphism and t is the antimorphism of the Hopf algebroid structure of H_{par}.

Definition 2.18 ([14]). *Let A be a unital partial H-module algebra. A vector space M is a partial (A, H)-module if M is an A-module together with a linear map $H \otimes M \to M$, $h \otimes m \mapsto hm$ such that:*

1. $h(am) = \sum (h_{(1)} \cdot a)(h_{(2)}m)$;
2. $h(a(km)) = \sum (h_{(1)} \cdot a)((h_{(2)}k)m)$,

for every $h, k \in H$, $a \in A$, $m \in M$.

Since for every $h \in H$, we have that

$$h \cdot 1_A = \sum [h_{(1)}][S(h_{(2)})] = \epsilon_h$$

and, as we can see in [14], every $\underline{A\#H}$-module is a partial (A, H)-module, then we have that

$$h \bullet x = h \bullet (1_A x) = \sum \epsilon_{h_{(1)}}(h_{(2)} \bullet x), \text{ and} \tag{2.3}$$

$$h \bullet x = h \bullet (x 1_A) = \sum (h_{(1)} \bullet x)\epsilon_{h_{(2)}}. \tag{2.4}$$

Now we will present a definition of partial (A, H)-modules for when A does not have a unit. This is important to classify some $\underline{A\#H}$-modules.

Definition 2.19. *Let A be an associative partial H-module algebra. A vector space M is a partial (A, H)-module if M is an unital A-module together with a linear map $H \otimes M \to M$, $h \otimes m \mapsto hm$ such that:*

1. $h(am) = \sum (h_{(1)} \cdot a)(h_{(2)}m)$;
2. $h(a(km)) = \sum (h_{(1)} \cdot a)((h_{(2)}k)m)$,

for every $h, k \in H$, $a \in A$, $m \in M$.

In [14], the calculation was done for unital algebras, but even for a nonunital partial H-module algebra B, essentially the same argument shows that whenever M is a partial (B,H)-module, it is then a $\underline{B\#H}$-module with action

$$\sum a(h_{(1)} \cdot b)\#h_{(2)} \triangleright m = a(h \cdot (bm)),$$

and clearly if M is a unital B-module, then it will be a unital $\underline{B\#H}$-module.

But for the converse we have two problems: one of them is that even if M is a unital $\underline{B\#H}$-module, we don't know whether it is a unital B-module with the induced action. The other problem is that, even when M is a unital B-module, it is not clear how should one define a linear map $H \otimes M \to M$ such that, with the structure of B-module induced by the action of $\underline{B\#H}$, M turns out to be a partial (B, H)-module. If B is s-unital (Definition 1.2) this holds.

In fact, define $hm = \sum h_{(1)} \cdot x\#h_{(2)} \triangleright m$, where $x \in B$ is such that $xm = m$. Note that if also $m = ym$, there exists $z \in B$ such that $zy = y$ and $zx = x$, then $zm = m$ and

$$\begin{aligned}
\sum h_{(1)} \cdot x\#h_{(2)} \triangleright m &= \sum h_{(1)} \cdot zx\#h_{(2)} \triangleright m \\
&= \sum h_{(1)} \cdot z\#h_{(2)} \triangleright x\#1_H \triangleright m \\
&= \sum h_{(1)} \cdot z\#h_{(2)} \triangleright xm \\
&= \sum h_{(1)} \cdot z\#h_{(2)} \triangleright ym \\
&= \sum h_{(1)} \cdot z\#h_{(2)} \triangleright y\#1_H \triangleright m \\
&= \sum h_{(1)} \cdot zy\#h_{(2)} \triangleright m \\
&= \sum h_{(1)} \cdot y\#h_{(2)} \triangleright m,
\end{aligned}$$

hence $h \otimes m \mapsto h \cdot m$ is well defined. Now, let $a \in B$, $m \in M$, $h \in H$ and consider $x \in B$ such that $xa = a = ax$ and $xm = m$, then we have that

$$\begin{aligned}
h(am) &= \sum [h_{(1)} \cdot x\#h_{(2)}] \triangleright [a\#1_H] \triangleright m \\
&= \sum h_{(1)} \cdot xa\#h_{(2)} \triangleright m \\
&= \sum [h_{(1)} \cdot a\#1_H] \triangleright [h_{(2)} \cdot x\#h_{(3)}] \triangleright m \\
&= \sum (h_{(1)} \cdot a)(h_{(2)}m).
\end{aligned}$$

Analogously, we have that

$$h(a(km)) = \sum (h_{(1)} \cdot a)((h_{(2)}k)m),$$

hence M is a partial (B, H)-module.

Proposition 2.20. *Let H be a Hopf algebra and B be an s-unital partial H-module algebra.*

1. *If M is a unital $\underline{B\#H}$-module, then it is a (not necessarily unital) partial (B, H)-module;*
2. *If M is a partial (B, H)-module, then it is a unital $\underline{B\#H}$-module.*

Actually, for a more general class of algebras, if we assume that H has bijective antipode, we can show that if $t_B(M) = \{m \in M;\ am = 0,\ \forall a \in B\} = 0$, considering the action of B induced by the action of $\underline{B\#H}$, then M is a (not necessarily unital) partial (B, H)-module. In fact, for every $a, b, c \in B$, $m \in M$, we have that

$$\begin{aligned}
\sum a(h_{(2)} \cdot S^{-1}(h_{(1)}) \cdot b)\#h_{(3)} \triangleright cm &= \sum a(h_{(2)} \cdot S^{-1}(h_{(1)}) \cdot b)\#h_{(3)} \triangleright c\#1_H \triangleright m \\
&= \sum a(h_{(2)} \cdot ((S^{-1}(h_{(1)}) \cdot b)c))\#h_{(3)} \triangleright m \\
&= \sum ab(h_{(1)} \cdot c)\#h_{(2)} \triangleright m \\
&= ab\#1_H \triangleright \sum (h_{(1)} \cdot c)\#h_{(2)} \triangleright m.
\end{aligned}$$

Then, since every element of B is a finite sum of products in B and $t_B(M) = 0$, whenever $m = \sum_i a_i m_i = \sum_j b_j n_j \in M$, we have that $\sum (h_{(1)} \cdot a_i)\#h_{(2)} \triangleright m_i = \sum (h_{(1)} \cdot b_j)\#h_{(2)} \triangleright n_j$. Hence the linear map $H \otimes M \to M$ given by $h(am) = \sum h_{(1)} \cdot a\#h_{(2)} \triangleright m$ is well defined, and by a straightforward calculation we prove that M is a partial (B, H)-module.

Proposition 2.21. *Let H be a Hopf algebra with bijective antipode and B be any partial H-module algebra.*

1. *If M is a unital $\underline{B\#H}$-module such that $t_B(M) = \{m \in M;\ am = a\#1_H \triangleright m = 0,\ \forall a \in B\} = 0$, then it is a (not necessarily unital) partial (B, H)-module;*
2. *If M is a partial (B, H)-module, then it is a unital $\underline{B\#H}$-module.*

Lemma 2.22. *Let B be an (not necessarily unital) algebra in ${}_H\mathcal{M}^{par}$, where H is a Hopf algebra with bijective antipode. Then H acts partially in B with symmetrical partial action.*

Proof. In fact, for every $x, y \in B$, $h, k \in H$, since $H_{par} \simeq \underline{A\#H}$, we have that

$$1_H \bullet x = 1_A\#1_H \triangleright x = 1_{\underline{A\#H}} \triangleright x = x,$$

$$\begin{aligned}
h \bullet (x(k \bullet y)) &= 1_A\#h \triangleright (x(1_A\#k \triangleright y)) \\
&= \sum (1_A\#h_{(1)} \triangleright x)((1_A\#h_{(2)})(1_A\#k) \triangleright y) \\
&= \sum (h_{(1)} \bullet x)(((h_{(2)} \cdot 1_A)\#h_{(3)}k) \triangleright y) \\
&= \sum (h_{(1)} \bullet x)((\epsilon_{h_{(2)}}\#1_H)(1_A\#h_{(3)}k) \triangleright y) \\
&= \sum (h_{(1)} \bullet x)\epsilon_{h_{(2)}}((1_A\#h_{(3)}k) \triangleright y) \\
&= \sum (h_{(1)} \bullet x)\epsilon_{h_{(2)}}((h_{(3)}k) \bullet y) \\
&\overset{(2.3)}{=} \sum (h_{(1)} \bullet x)((h_{(2)}k) \bullet y),
\end{aligned}$$

and for the symmetrical case,

$$\begin{aligned}
h \bullet ((k \bullet x)y) &= \sum ((1_A\#h_{(1)})(1_A\#k) \triangleright x)(1_A\#h_{(2)} \triangleright y) \\
&= \sum ((h_{(1)} \cdot 1_A\#h_{(2)}k) \triangleright x)(h_{(3)} \bullet y) \\
&= \sum ((h_{(2)}S^{-1}(h_{(1)}) \cdot 1_A)(h_{(3)} \cdot 1_A)\#h_{(3)}k \triangleright x)(h_{(4)} \bullet y) \\
&= \sum (h_{(2)} \cdot S^{-1}(h_{(1)}) \cdot 1_A\#h_{(3)}k \triangleright x)(h_{(4)} \bullet y) \\
&= \sum ((1_A\#h_{(2)})(S^{-1}(h_{(1)}) \cdot 1_A\#k) \triangleright x)(h_{(3)} \bullet y) \\
&= \sum ((1_A\#h_{(4)})(S^{-1}(h_{(3)}) \cdot 1_A\#S^{-1}(h_{(2)})h_{(1)}k) \triangleright x)(h_{(5)} \bullet y) \\
&= \sum ([(1_A\#h_{(3)})(1_A\#S^{-1}(h_{(2)}))](1_A\#h_{(1)}k) \triangleright x)(h_{(4)} \bullet y) \\
&= \sum ([(1_A\#h_{(3)})(1_A\#S^{-1}(h_{(2)}))] \triangleright (1_A\#h_{(1)}k) \triangleright x)(h_{(4)} \bullet y) \\
&= \sum (1_A\#h_{(1)}k) \triangleright x)\epsilon_{h_{(2)}}(h_{(3)} \bullet y) \\
&= \sum (h_{(1)})k \bullet x)(h_{(2)} \bullet y).
\end{aligned}$$

□

Lemma 2.23. *Let B be a partial H-module algebra with symmetrical partial action. If B is idempotent or if $r(B) = l(B) = 0$ and H has bijective antipode, then B is an algebra in ${}_H\mathcal{M}^{par}$.*

Proof. We proved that a partial H-module algebra with these properties is an partial H-module and, since H acts partially in B, the multiplication of B is clearly a morphism of partial H-modules. Then, as the base algebra of the monoidal structure of ${}_H\mathcal{M}^{par}$ is A, i.e., the tensor of this category is over A, we only need to show that B is, in fact, an A-bimodule and the multiplication of B is balanced. First, we consider B as an A-bimodule with the structure presented in [6] given by

$$\begin{aligned} \epsilon_h x &= \sum h_{(1)} \bullet S(h_{(2)}) \bullet x \\ x\epsilon_h &= \sum h_{(2)} \bullet S^{-1}(h_{(1)}) \bullet x. \end{aligned}$$

We already saw that the linear map $x \mapsto \sum h_{(1)} \cdot S(h_{(2)}) \cdot x$ is a morphism of right B-modules and that the linear map $x \mapsto \sum h_{(2)} \cdot S^{-1}(h_{(1)}) \cdot x$ is a morphism of left B-modules. Hence, the multiplication of B is a morphism of A-bimodules. To show that the multiplication of B is also balanced, we have that

$$\begin{aligned} (x\epsilon_h)y &= (\sum h_{(2)} \bullet S^{-1}(h_{(1)}) \bullet x)y \\ &= \sum (h_{(2)} \bullet S^{-1}(h_{(1)}) \bullet x)(h_{(3)}S(h_{(4)}) \bullet y) \\ &= \sum (h_{(2)}S^{-1}(h_{(1)}) \bullet x)(h_{(3)} \bullet S(h_{(4)}) \bullet y) \\ &= \sum x(h_{(1)} \bullet S(h_{(2)}) \bullet y) \\ &= x(\epsilon_h y). \end{aligned}$$

□

These results lead us to the following theorem.

Theorem 2.24. *Let H be a Hopf algebra with bijective antipode. Then, when we consider only algebras that are idempotent or with trivial right and left annihilator, there exist a bijective correspondence between algebras in ${}_H\mathcal{M}^{par}$ and partial H-module algebras with symmetrical partial action.*

2.3 Partial G-actions and partial $\Bbbk G$-actions

In this section, we will prove that, under some conditions, there is a bijective correspondence between partial G-actions and partial $\Bbbk G$-actions when the algebra A does not have a unit. We already know, by [4] and [13], that this occurs when A is a unital algebra and the ideals of the partial G-action are generated by central idempotents, i.e., $D_g = Ae_g$ for every $g \in G$.

Definition 2.25 ([17])**.** *A partial action α of a group G on an algebra A consists of a family of two-sided ideals D_g in A, $g \in G$, and algebra isomorphisms $\alpha_g : D_{g^{-1}} \to D_g$, such that:*

1. *α_1 is the trivial isomorphism $A \to A$;*
2. *$\alpha_g(D_{g^{-1}} \cap D_h) \subseteq D_g \cap D_{gh}$;*
3. *$\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$, for any $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.*

The correspondence proved in [13] implies that given a symmetrical partial $\Bbbk G$-action on a unital algebra A, the ideals of the induced partial G-action are of the form $D_g = A(g \cdot 1_A)$.

But, for every $a \in A$, we have that $a(g \cdot 1_A) = (gg^{-1} \cdot a)(g \cdot 1_A) = g \cdot g^{-1} \cdot a$, which means that if we define the algebra maps

$$\begin{array}{rcl} \psi_g : A & \to & A \\ a & \mapsto & g \cdot g^{-1} \cdot a, \end{array}$$

then $D_g = \psi_g(A)$. Now we will assume that A is a (not necessarily unital) associative algebra.

Lemma 2.26. *Given a symmetric partial $\Bbbk G$-action on an associative algebra A, for every $g \in G$, ψ_g is an A-bimodule map. Hence $\psi_g(A)$ is an ideal of A.*

Proof. In fact, for every $g \in G$, $a, b, x \in A$, we have that

$$\begin{aligned} \psi_g(axb) &= g \cdot g^{-1} \cdot (axb) \\ &= (gg^{-1} \cdot a)(g \cdot g^{-1} \cdot (xb)) \\ &= a(g \cdot g^{-1} \cdot x)(gg^{-1} \cdot b) \\ &= a\psi_g(x)b. \end{aligned}$$

□

Also, every ψ_g is a projection whenever A is either idempotent or $r(A) = 0$ or $l(A) = 0$. In fact, if we assume that $A^2 = A$, every $x \in A$ is a sum $x = \sum_{i=1}^{n} y_i z_i$, with $y_i, z_i \in A$, then, since ψ_g is linear, we only need to check that the claim holds for products $ab \in A$. For every $a, b \in A$, we have that

$$\begin{aligned} \psi_g^2(ab) &= \psi_g(a\psi_g(b)) \\ &= \psi_g(a)\psi_g(b) \\ &= \psi_g(ab). \end{aligned}$$

Hence $\psi_g^2 = \psi_g$ whenever A is idempotent. And since

$$\begin{aligned} a\psi_g^2(b) &= \psi_g^2(ab) \\ &= \psi_g(ab) \\ &= a\psi_g(b), \end{aligned}$$

we have that $\psi_g^2 = \psi_g$ whenever $r(A) = 0$. For $l(A) = 0$ we use a similar calculation.

Remark 2.27. *Note that, when we consider symmetrical partial actions, for every two-sided ideal I of A, the algebra $g \cdot I$ is also a two-sided ideal of A. In fact, for every $x \in I$, $a, b \in A$,*

$$\begin{aligned} a(g \cdot x)b &= g \cdot ((g^{-1} \cdot a)x)b \\ &= g \cdot ((g^{-1} \cdot a)x(g^{-1} \cdot b)) \in g \cdot I. \end{aligned}$$

By Corollary 2.15, if A is a symmetrical partial $\Bbbk G$-module algebra and $A^2 = A$, $r(A) = 0$ or $l(A) = 0$, then $\pi : \Bbbk G \to End(A)$ defined by $\pi(g)(a) = g \cdot a$ is a partial representation of $\Bbbk G$, which is the same as a partial representation of G over $\Bbbk$, i.e.,

- $1_G \cdot a = a$;
- $g \cdot h \cdot h^{-1} \cdot a = gh \cdot h^{-1} \cdot a$;
- $g \cdot g^{-1} \cdot h \cdot a = g \cdot g^{-1}h \cdot a$,

for all $g, h \in G$, $a \in A$. Also, note that $\psi_g = \pi(g)\pi(g^{-1})$.

Now we will prove some equalities that will be useful later.

Lemma 2.28. *Let A be a partial $\Bbbk G$-module algebra with symmetrical partial action and ψ_g as before. If A is idempotent, $r(A) = 0$ or $l(A) = 0$, then:*

1. $g \cdot h \cdot h^{-1} \cdot a \in \psi_{gh}(A)$;
2. $g \cdot h \cdot a = gh \cdot h^{-1} \cdot h \cdot a$;
3. $\psi_g \psi_h = \psi_h \psi_g$;
4. $g \cdot g^{-1} \cdot h \cdot a = h \cdot h^{-1} g \cdot g^{-1} h \cdot a$,

for every $a \in A$, $g, h \in G$.

Proof. For item 1), we have that

$$\begin{aligned} g \cdot h \cdot h^{-1} \cdot a &= gh \cdot h^{-1} \cdot a \\ &= gh \cdot (gh)^{-1} \cdot gh \cdot h^{-1} \cdot a \\ &= \psi_{gh}(gh \cdot h^{-1} \cdot a) \in \psi_{gh}(A). \end{aligned}$$

For item 2),

$$\begin{aligned} gh \cdot h^{-1} \cdot h \cdot a &= g \cdot h \cdot h^{-1} \cdot h \cdot a \\ &= g \cdot h \cdot a. \end{aligned}$$

For the proof of items 3) and 4) we use similar calculations. □

Now we have all the tools to prove the following proposition.

Proposition 2.29. *Let A be an associative algebra such that $A^2 = A$, $r(A) = 0$ or $l(A) = 0$. Then, there is a bijective correspondence between symmetrical partial $\Bbbk G$-actions on A and partial G-actions $\alpha = \{\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G}\}$ on A endowed with algebra epimorphisms $p_g : A \to D_g$, such that:*

1. $p_1 : A \to A$ *is the identity of* A
2. $p_g^2 = p_g$;
3. $p_g p_h = p_h p_g$;
4. $p_g \alpha_k = \alpha_k p_{k^{-1}g}|_{D_{k^{-1}}}$.

Proof. Let A be a partial $\Bbbk G$-module algebra with symmetrical partial action. For each $g \in G$ consider $D_g = \psi_g(A)$, $p_g = \psi_g$ and define $\alpha_g : D_{g^{-1}} \to D_g$ by $\alpha_g(\psi_{g^{-1}}(a)) = g \cdot \psi_{g^{-1}}(a)$. We will prove that α is in fact a partial group action. First, note that $\psi_1(A) = A$ and $\alpha_1(\psi_1(a)) = a$. Now,

$$\begin{aligned} \psi_g(a) &= \psi_g^2(a) \\ &= g \cdot g^{-1} \cdot g \cdot g^{-1} \cdot a \\ &= \alpha_g(\psi_{g^{-1}}(g^{-1} \cdot a)), \end{aligned}$$

i.e., α_g is surjective. For the injectivity of α_g, assume that $\alpha_g(\psi_{g^{-1}}(a)) = \alpha_g(\psi_{g^{-1}}(b))$, then

$$g^{-1} \cdot g \cdot a = g^{-1} \cdot g \cdot g^{-1} \cdot g \cdot a$$

$$\begin{aligned} &= g^{-1} \cdot \alpha_g(\psi_{g^{-1}}(a)) \\ &= g^{-1} \cdot \alpha_g(\psi_{g^{-1}}(b)) \\ &= g^{-1} \cdot g \cdot g^{-1} \cdot g \cdot b \\ &= g^{-1} \cdot g \cdot b, \end{aligned}$$

i.e., $\psi_{g^{-1}}(a) = \psi_{g^{-1}}(b)$. Since all α_g are clearly algebra morphisms, we have that they are algebra isomorphisms.

Now, let us prove that $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$. In fact, for any $x \in D_{g^{-1}} \cap D_h$, there exist $a, b \in A$ such that $x = \psi_{g^{-1}}(a) = \psi_h(b)$, then on one side

$$\begin{aligned} \alpha_g(x) &= \alpha_g(\psi_{g^{-1}}(a)) \\ &= g \cdot g^{-1} \cdot g \cdot a \\ &= \psi_g(g \cdot a) \in D_g, \end{aligned}$$

and on the other side,

$$\begin{aligned} \alpha_g(x) &= \alpha_g(\psi_h(b)) \\ &= g \cdot h \cdot h^{-1} \cdot b \in \psi_{gh}(A) = D_{gh}. \end{aligned}$$

Hence $\alpha_g(D_{g^{-1}} \cap D_h) \subseteq D_g \cap D_{gh}$.

For the last axiom of the definition of partial group actions, let $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$, then there exists $a \in A$ such that $x = \psi_{h^{-1}}(a)$ and

$$\begin{aligned} \alpha_g(\alpha_h(x)) &= \alpha_g(\alpha_h(\psi_{h^{-1}}(a))) \\ &= g \cdot h \cdot h^{-1} \cdot h \cdot a \\ &= gh \cdot h^{-1} \cdot h \cdot a \\ &= \alpha_{gh}(x). \end{aligned}$$

For the converse of the proposition, suppose that α is a partial group action of G on A and that there exist algebra morphisms of A onto the ideals $p_g : A \to D_g$ that satisfy the items $1)$, $2)$, $3)$ and $4)$ of the proposition. Define $g \cdot a = \alpha_g(p_{g^{-1}}(a))$, then $1_G \cdot a = \alpha_1(p_1(a)) = a$ and

$$\begin{aligned} g \cdot (a(h \cdot b)) &= \alpha_g(p_{g^{-1}}(a\alpha_h(p_{h^{-1}}(b)))) \\ &= \alpha_g(p_{g^{-1}}(a))\alpha_g(p_{g^{-1}}(\alpha_h(p_{h^{-1}}(b)))) \\ &= \alpha_g(p_{g^{-1}}(a))\alpha_g(\alpha_h(p_{h^{-1}g^{-1}}p_{h^{-1}}(b))) \\ &= \alpha_g(p_{g^{-1}}(a))\alpha_{(gh)}(p_{h^{-1}g^{-1}}p_{h^{-1}}(b)) \\ &= \alpha_g(p_{g^{-1}}(a))\alpha_{(gh)}(p_{h^{-1}}p_{h^{-1}g^{-1}}(b)) \\ &= \alpha_g(p_{g^{-1}}(a))p_g(\alpha_{(gh)}(p_{h^{-1}g^{-1}}(b))) \\ &= p_g(\alpha_g(p_{g^{-1}}(a))\alpha_{(gh)}(p_{h^{-1}g^{-1}}(b))) \\ &= \alpha_g(p_{g^{-1}}(a))\alpha_{(gh)}(p_{h^{-1}g^{-1}}(b)) = (g \cdot a)(gh \cdot b). \end{aligned}$$

Analogously, $g \cdot ((h \cdot a)b) = (gh \cdot a)(g \cdot b)$. Hence A is a partial $\Bbbk G$-module algebra with symmetrical partial action. $\square$

Definition 2.30. *Let α be a partial action of the group G on the algebra A. A family of algebra morphisms $\{p_g : A \to D_g\}_{g \in G}$ that satisfies all the properties mentioned in the previous proposition will be called α-projections.*

Note that α-projections correspond, in the case of nonunital algebras, to the central idempotents that characterize, in the case of unital algebras, the partial G-actions that come from partial kG-actions. In fact, if A is unital and $D_g = A1_g$, we only need to consider the linear maps $p_g : A \to D_g$ defined by $p_g(a) = a1_g$. Clearly they are algebra maps and projections which compositions commute. For the last property, we have that

$$\begin{aligned}
\alpha_k p_{k^{-1}g} p_{k^{-1}}(a1_{k^{-1}}) &= \alpha_k(a1_{k^{-1}}1_{k^{-1}g}) \\
&= \alpha_k(a1_{k^{-1}})\alpha_k(1_{k^{-1}}1_{k^{-1}g}) \\
&= \alpha_k(a1_{k^{-1}})1_k 1_g \\
&= \alpha_k(a1_{k^{-1}})1_g \\
&= p_g\alpha_k(a1_{k^{-1}}).
\end{aligned}$$

Actually, the partial G-action induced by a partial $\Bbbk G$-action is a regular partial action, as we will prove later.

Definition 2.31 ([1]). *A regular partial G-action is a partial action such that for every $g_1, \cdots, g_n \in G$, we have that*

$$D_{g_1} \cap \cdots \cap D_{g_n} = D_{g_1} \cdot D_{g_2} \cdots D_{g_n}.$$

Let us show that if we consider only idempotent algebras, then the bijection of the previous proposition is between symmetrical partial kG-actions and regular partial G-actions. In fact, every element $a \in A$ can be viewed as $a = \sum_i a_{1i} \cdots a_{ni}$, for every $n \in \mathbb{N}$, then for every $g_1, \cdots g_k \in G$, we have that if $x \in D_{g_1} \cap \cdots \cap D_{g_k}$ there exist $a^1, \cdots, a^k \in A$ such that

$$x = p_{g_1}(a^1) = \cdots = p_{g_k}(a^k).$$

And since every p_{g_i} is a projection,

$$\begin{aligned}
x &= p_{g_1} \circ \cdots \circ p_{g_k}(a^1) \\
&= \sum_i p_{g_1} \circ \cdots \circ p_{g_k}(a^1_{1i} \cdots a^1_{ki}) \\
&= \sum_i (p_{g_1} \circ \cdots \circ p_{g_k}(a^1_{1i})) \cdots (p_{g_1} \circ \cdots \circ p_{g_k}(a^1_{ki})) \in D_{g_1} \cdot D_{g_2} \cdots D_{g_k},
\end{aligned}$$

because the projections commute. Hence $D_{g_1} \cap \cdots \cap D_{g_n} \subseteq D_{g_1} \cdot D_{g_2} \cdots D_{g_n}$, and the other inclusion is a consequence of every D_g be an ideal.

2.4 Partial actions on algebras with local units

Now that we have established a good definition for partial actions on associative algebras in general, we will restrict our study to algebras with local units and associate the obtained results to the known results about partial actions on categories. But first, we will present an equivalent definition of partial action.

Proposition 2.32. *Let H be a Hopf algebra, A an algebra with local units with system of local units $S = \{e_\lambda\}_{\lambda \in \Lambda}$, and $\cdot : H \otimes A \longrightarrow A$ a linear map. Then $\cdot$ is a partial action if and only if, for all $a, b \in A$, $h, k \in H$, we have:*

1. *$1_H \cdot a = a$;*

2. $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$;
3. $h \cdot (k \cdot a) = \sum (h_{(1)} \cdot e_\alpha)(h_{(2)}k \cdot a)$, *for every* $e_\alpha \in S$ *such that* $e_\alpha(k \cdot a) = k \cdot a$.

Additionally, this partial partial action is symmetrical if and only if

$$h \cdot (k \cdot a) = \sum (h_{(1)}k \cdot a)(h_{(2)} \cdot e_\beta)$$

for every $e_\beta \in S$ *such that* $(k \cdot a)e_\beta = k \cdot a$.

It is clear that if the linear map $\cdot : H \otimes A \to A$ is a partial action then (1), (2) and (3) hold. Conversely, assuming these properties, Axiom 2 of Definition 2.3 follows from the fact that if A is a partial H-module algebra with local units, for every $a, b \in A$, $h, k \in H$, there exists $e_\alpha \in S$ such that $be_\alpha = b$ and $e_\alpha(k \cdot a) = k \cdot a$, then:

$$\begin{aligned} h \cdot (b(k \cdot a)) &= \sum (h_{(1)} \cdot b)(h_{(2)} \cdot (k \cdot a)) \\ &= \sum (h_{(1)} \cdot b)(h_{(2)} \cdot e_\alpha)(h_{(3)}k \cdot a) \\ &= \sum (h_{(1)} \cdot be_\alpha)(h_{(2)}k \cdot a) \\ &= \sum (h_{(1)} \cdot b)(h_{(2)}k \cdot a). \end{aligned}$$

This calculation illustrates that if $\cdot : H \otimes A \longrightarrow A$ is a partial action for some s.l.u. (system of local units) S of A, then it will be a partial action for any other s.l.u. of A.

Now we will see a particular class of partial actions on algebras with local units, the *categorizable* ones. This terminology is motivated by the fact that given a categorizable partial action, then we can induce a partial action on an associated category.

We begin by recalling the construction of the algebra with local units associated to a linear category and the definition of partial H-module categories.

Definition 2.33. *Let $\mathcal{C}$ be a linear category. We will denote by $a(\mathcal{C})$ the algebra consisting of elements of the form $({}_yf_x)_{x,y\in\mathcal{C}_0}$, with finite ${}_yf_x \neq 0$, where the multiplication is given by matrix multiplication and the composition of $\mathcal{C}$.*

Definition 2.34 ([2])**.** *A partial action of H on a linear category $\mathcal{C}$ is a family of linear maps $\triangleright = \{\triangleright_{(x,y)} : H \otimes {}_y\mathcal{C}_x \longrightarrow {}_y\mathcal{C}_x\}_{x,y\in\mathcal{C}_0}$, such that, for every $x, y, z \in \mathcal{C}_0$, ${}_yf_x \in {}_y\mathcal{C}_x$ and ${}_zg_y \in {}_z\mathcal{C}_y$, we have that:*

1. $1_H \triangleright_{(x,y)} {}_yf_x = {}_yf_x$;
2. $h \triangleright_{(x,z)} ({}_zg_y \circ {}_yf_x) = \sum (h_{(1)} \triangleright_{(y,z)} {}_zg_y) \circ (h_{(2)} \triangleright_{(x,y)} {}_yf_x)$;
3. $h \triangleright_{(x,y)} (k \triangleright_{(x,y)} {}_yf_x) = \sum (h_{(1)} \triangleright_{(y,y)} {}_y1_y) \circ (h_{(2)}k \triangleright_{(x,y)} {}_yf_x)$;
4. *If additionally* $h \triangleright_{(x,y)} (k \triangleright_{(x,y)} {}_yf_x) = \sum (h_{(1)}k \triangleright_{(x,y)} {}_yf_x) \circ (h_{(2)} \triangleright_{(x,x)} {}_x1_x)$, $\triangleright$ *is called a symmetrical partial action.*

In this case, we say that $\mathcal{C}$ is a (symmetrical) partial H-module category.

Here, different from [2], we present the symmetrical condition as an additional property.

First, note that if we consider a (symmetrical) partial H-module category $\mathcal{C}$ which partial action is determined by the family of linear maps $\triangleright = \{\triangleright_{(x,y)} : H \otimes {}_y\mathcal{C}_y \to {}_y\mathcal{C}_y\}$, we can induce a linear map $\cdot : H \otimes a(\mathcal{C}) \to a(\mathcal{C})$ given by $h \cdot ({}_yf_x)_{x,y} = (h \triangleright_{(x,y)} {}_yf_x)_{x,y}$ that, actually, defines an S-categorizable partial action of H on $a(\mathcal{C})$, where each local unit of S is a finite sum of the elements E_{xx} given by ${}_x1_x$ in the position xx, and zero otherwise. In fact, clearly $1 \cdot ({}_yf_x)_{x,y} = ({}_yf_x)_{x,y}$ and

$$h \cdot ({}_yf_x)_{x,y}({}_yg_x)_{x,y} = \sum (h_{(1)} \cdot ({}_yf_x)_{x,y})(h_{(2)} \cdot ({}_yg_x)_{x,y}).$$

For the third axiom of partial action, note that

$$\begin{aligned} h \cdot k \cdot ({}_yf_x)_{x,y} &= (h \triangleright_{(x,y)} k \triangleright_{(x,y)} {}_yf_x)_{x,y} \\ &= (h \triangleright_{(y,y)} ({}_y1_y(k \triangleright_{(x,y)} {}_yf_x)))_{x,y} \\ &= \sum ((h_{(1)} \triangleright_{(y,y)} {}_y1_y)(h_{(2)}k \triangleright_{(x,y)} {}_yf_x))_{x,y} \\ &= \sum (h_{(1)} \cdot \sum_{y;\, {}_yf_x \neq 0} E_{yy})(h_{(2)}k \cdot ({}_yf_x)_{x,y}), \end{aligned}$$

i.e., there exists at least one local unit E in S such that $E(k \cdot ({}_yf_x)_{x,y}) = k \cdot ({}_yf_x)_{x,y}$ and $h \cdot k \cdot ({}_yf_x)_{x,y} = (h_{(1)} \cdot E)(h_{(2)}k \cdot ({}_yf_x)_{x,y})$. Analogously, if the original partial action is symmetrical, there exists at least one local unit F in S such that $(k \cdot ({}_yf_x)_{x,y})F = k \cdot ({}_yf_x)_{x,y}$ and $h \cdot k \cdot ({}_yf_x)_{x,y} = \sum (h_{(1)}k \cdot ({}_yf_x)_{x,y})(h_{(2)} \cdot F)$.

Now, note that whenever $X, Y \in S$ are such that $X \leq Y$, in this case, we have that $Y - X \in S$. Moreover, given a Hopf algebra H and an algebra A with s.l.u. $S = \{e_\lambda\}_{\lambda \in \Lambda}$, for every linear map $\cdot : H \otimes A \to A$, where $H \cdot e_\lambda \subseteq e_\lambda A e_\lambda$, and $e_\beta - e_\alpha \in S$ whenever $e_\alpha \leq e_\beta$, we have that $\sum (h_{(1)} \cdot e_\lambda)(h_{(2)}k \cdot a) = \sum (h_{(1)} \cdot e_\gamma)(h_{(2)}k \cdot a)$ for every $e_\lambda, e_\gamma \in S$ such that $e_\lambda a = a = e_\gamma a$. In fact, let e_α, e_β be local units such that $e_\alpha a = a = e_\beta a$, then there exists a local unit e_γ such that $e_\alpha, e_\beta \leq e_\gamma$. Hence

$$\begin{aligned} \sum (h_{(1)} \cdot e_\gamma)(h_{(2)}k \cdot a) &= \sum (h_{(1)} \cdot (e_\gamma - e_\alpha + e_\alpha))(h_{(2)}k \cdot a) \\ &= \sum (h_{(1)} \cdot (e_\gamma - e_\alpha))(h_{(2)}k \cdot a) + (h_{(1)} \cdot e_\alpha)(h_{(2)}k \cdot a) \\ &= \sum (h_{(1)} \cdot (e_\gamma - e_\alpha))(e_\gamma - e_\alpha)(e_\alpha)(h_{(2)}k \cdot a) + (h_{(1)} \cdot e_\alpha)(h_{(2)}k \cdot a) \\ &= \sum (h_{(1)} \cdot e_\alpha)(h_{(2)}k \cdot a). \end{aligned}$$

Analogously, $\sum (h_{(1)} \cdot e_\gamma)(h_{(2)}k \cdot a) = \sum (h_{(1)} \cdot e_\beta)(h_{(2)}k \cdot a)$.

This is useful when one wants to verify if some linear map $\cdot : H \otimes A \longrightarrow A$ is a (symmetrical) partial action or not, because we will only need to calculate the third axiom of partial actions for one local unit that satisfies the required property.

Consequently, one can conclude that the linear map $\cdot : H \otimes a(\mathcal{C}) \to a(\mathcal{C})$ mentioned before is, in fact, a (symmetrical) partial action.

Conversely, given a (symmetrical) partial action $\cdot : H \otimes A \longrightarrow A$, if there exist some s.l.u. $S = \{e_\lambda\}_{\lambda \in \Lambda}$ such that $h \cdot e_\alpha \in e_\alpha A e_\alpha$, for all $\alpha \in \Lambda$, $h \in H$, we have that

$$\begin{aligned} e_\alpha(h \cdot e_\alpha a e_\beta)e_\beta &= \sum e_\alpha(h_{(1)} \cdot e_\alpha)(h_{(2)} \cdot e_\alpha a e_\beta)(h_{(3)} \cdot e_\beta)e_\beta \\ &= \sum (h_{(1)} \cdot e_\alpha)(h_{(2)} \cdot e_\alpha a e_\beta)(h_{(3)} \cdot e_\beta) \\ &= h \cdot e_\alpha a e_\beta, \end{aligned}$$

for all $e_\alpha, e_\beta \in S$, i.e., $H \cdot e_\alpha A e_\beta \subseteq e_\alpha A e_\beta$. Hence, we can induce a (symmetrical) partial action of H on the category $\mathcal{C}^S(A)$, where the set of objects is Λ, the set of morphisms from α to β is given by $\mathcal{C}^S(A)(\alpha, \beta) = {}_\beta\mathcal{C}^S(A)_\alpha = e_\beta A e_\alpha$ and the composition is the multiplication of A.

Definition 2.35. *If a linear map $\cdot : H \otimes A \longrightarrow A$ is a (symmetrical) partial action and there exist some s.l.u. $S = \{e_\lambda\}_{\lambda\in\Lambda}$ such that $H \cdot e_\alpha \subseteq e_\alpha A e_\alpha$, for every $e_\alpha \in S$, we will say that this partial action is an S-categorizable (symmetrical) partial action.*

Example 2.36. *To see that there exist partial actions that are not categorizable, consider the algebra $A = FMat_{\mathbb{Z}}(\Bbbk)$ with system of local units $S = \{\sum_i^{finite} E_{ii}\}$ and the (non trivial) action of $k\mathbb{Z}$ on $\mathbb{Z}$ given $n \triangleright i = n + i$. Then, we have the induced (global) action of $k\mathbb{Z}$ on A given by $n \cdot E_{ij} = E_{(i+n)(j+n)}$, that is clearly not S-categorizable.*

2.5 Morita equivalence between A and $a(\mathcal{C}^S(A))$.

In this section we will show that every algebra A with s.l.u. S is Morita equivalent to the algebra $a(\mathcal{C}^S(A))$. In order to do this, we will prove that the category of the unital A-modules is equivalent to the category of $\mathcal{C}^S(A)$-modules, and then we will show that for every linear category $\mathcal{C}$, the category of the $\mathcal{C}$-modules is equivalent to the category of the unital $a(\mathcal{C})$-modules. But first, let us recall the definition of modules of a linear category.

Definition 2.37. *Let $\mathcal{C}$ be a linear category. A $\mathcal{C}$-module $\mathcal{M}$ is a functor from $\mathcal{C}$ to the category of vector spaces* **Vec** *over the same field of the linear category structure. In other words, for every object $x \in \mathcal{C}_0$, a vector space ${}_x\mathcal{M}$ is determined, and for every linear map $f : x \to y$ in $\mathcal{C}_1$, a linear transformation $f : {}_x\mathcal{M} \to {}_y\mathcal{M}$ is determined, such that:*

1. ${}_x 1_x\, {}_x m = {}_x m$;
2. ${}_y g_z (\, {}_z f_x\, {}_x m) = ({}_y g_z \circ {}_z f_x)\, {}_x m$.

We know that given an algebra with local units A and a fixed s.l.u. $S = \{e_\lambda\}_{\lambda\in\Lambda}$, every A-module M can be seen as a $\mathcal{C}^S(A)$-module, namely $\mathcal{M} = \{\, {}_\lambda\mathcal{M}\}_{\lambda\in\Lambda}$ where ${}_\lambda\mathcal{M} = e_\lambda M$ and the actions in every $e_\lambda M$ are given by the original action of A on M.

Now, to construct an A-module arising from an $\mathcal{C}^S(A)$-module, note that given a $\mathcal{C}^S(A)$-module $\mathcal{M} = \{\, {}_\lambda\mathcal{M}\}_{\lambda\in\Lambda}$, for every $e_\lambda \leq e_\alpha$, we have that $e_\lambda \in e_\alpha A e_\lambda \cap e_\lambda A e_\alpha \cap e_\lambda A e_\lambda$, hence $e_\lambda \cdot_{(\lambda,\lambda)}\, {}_\lambda m = e_\lambda \cdot_{(\alpha,\lambda)} e_\lambda \cdot_{(\lambda,\alpha)}\, {}_\lambda m$ for every ${}_\lambda \in {}_\lambda\mathcal{M}$. Since $e_\lambda \cdot_{(\lambda,\lambda)} \square = id_{{}_\lambda\mathcal{M}}$, we have that $I_{(\lambda,\alpha)} = e_\lambda \cdot_{(\lambda,\alpha)} \square$ is an injective linear map. This calculation is illustrated by the following diagram.

$$\begin{tikzcd} {}_\lambda M \arrow[loop left, "e_\lambda\cdot_{\lambda,\lambda}"] \arrow[r, shift left, "e_\lambda\cdot_{(\lambda,\alpha)}"] & {}_\alpha M \arrow[l, shift left, "e_\lambda\cdot_{(\alpha,\lambda)}"] \end{tikzcd}$$

Lemma 2.38. $\{\{\, {}_\lambda\mathcal{M}\}_{\lambda\in\Lambda}, \{I_{(\lambda,\alpha)} : {}_\lambda\mathcal{M} \to {}_\alpha\mathcal{M}\}_{\lambda\leq\alpha\in\Lambda}\}$ *forms a direct system of vector spaces.*

Proof. Follows directly from the fact that $e_\gamma e_\alpha = e_\alpha$ whenever $e_\alpha \leq e_\gamma$. $\square$

It is well known that every direct system of vector spaces $\{\iota_{ij} : M_i \to M_j\}$ has a (unique) limit, i.e., there exists a vector space M with inclusions $\iota_i : M_i \to M$ such that for every vector space N with inclusions $\nu_i : M_i \to N$ there exists a unique linear transformation $\theta : M \to N$ such that $\theta \circ \iota_i = \nu_i$ for every i.

To describe the limit of the direct system of the previous lemma, consider $T \subset \oplus\, {}_\lambda\mathcal{M}$ the set of finite sums of elements of the form ${}_\lambda m - I_{(\lambda,\alpha)}(\, {}_\lambda m)$ with $e_\lambda \leq e_\alpha$. Hence, by Proposition 2.6.8 of [26], we have that $\varinjlim\, {}_\lambda\mathcal{M} = \oplus\, {}_\lambda\mathcal{M}/T$.

Note that we can identify ${}_\lambda M$ as $\overline{{}_\lambda M} \subset M$ by ${}_\lambda m \mapsto \overline{{}_\lambda m}$, the equivalence class of ${}_\lambda m$ in M. This identification is well defined because $I_{(\lambda,\alpha)}$ is injective.

Now, consider $M = \varinjlim {}_\lambda\mathcal{M}$ and define

$$\begin{array}{rcl} \bullet : A \otimes M & \longrightarrow & M \\ a \otimes \overline{{}_\lambda m} & \mapsto & \overline{e_\beta a e_\alpha \cdot_{(\alpha,\beta)} I_{(\lambda,\alpha)}\, {}_\lambda m}, \end{array}$$

where $ae_\alpha = a = e_\beta a$ and $e_\lambda \leq e_\alpha$. In other words, $a \bullet \overline{{}_\lambda m} = \overline{a \cdot_{(\alpha,\beta)} e_\lambda \cdot_{(\lambda,\alpha)}\, {}_\lambda m} = \overline{ae_\lambda \cdot_{(\lambda,\beta)}\, {}_\lambda m}$

Proposition 2.39. *M is an A-module via $\bullet$.*

Proof. Since the proof that it determines an action is straightforward, we will prove that $\bullet$ is well defined. Because of the description $a \bullet \overline{{}_\lambda m} = \overline{ae_\lambda \cdot_{(\lambda,\beta)}\, {}_\lambda m}$, that does not depend on the choice of e_α, we will show that $\overline{ae_\lambda \cdot_{(\lambda,\beta)}\, {}_\lambda m} = \overline{ae_\lambda \cdot_{(\lambda,\gamma)}\, {}_\lambda m}$ whenever $e_\beta \leq e_\gamma$, in fact

$$\begin{array}{rcl} \overline{ae_\lambda \cdot_{(\lambda,\beta)}\, {}_\lambda m} & = & \overline{I_{(\beta,\gamma)}(ae_\lambda \cdot_{(\lambda,\beta)}\, {}_\lambda m)} \\ & = & \overline{e_\gamma \cdot_{(\beta,\gamma)} ae_\lambda \cdot_{(\lambda,\beta)}\, {}_\lambda m} \\ & = & \overline{e_\gamma a e_\lambda \cdot_{(\lambda,\gamma)}\, {}_\lambda m} \\ & = & \overline{ae_\lambda \cdot_{(\lambda,\gamma)}\, {}_\lambda m}. \end{array}$$

Finally, if we chose e_α and e_β such that $e_\beta a = e_\alpha a = a$, there always exist $e_\gamma \geq e_\alpha, e_\beta$, and we are done. $\square$

Theorem 2.40. *Let A be an algebra with s.l.u. S. Then the category of the unital A-modules is equivalent to the category of $\mathcal{C}^S(A)$-modules.*

Proof. First, note that if $\overline{{}_\lambda m} = \overline{{}_\alpha m} \in G(\mathcal{M})$, there exist $\gamma \in \Lambda$ such that $I^{\mathcal{M}}_{(\lambda,\gamma)}({}_\lambda m) = I^{\mathcal{M}}_{(\alpha,\gamma)}({}_\alpha m)$. Let $f : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathcal{C}^S(A) - Mod$, then

$$\begin{array}{rcl} I^{\mathcal{N}}_{(\lambda,\gamma)}(f_\lambda({}_\lambda m)) & = & e_\lambda \cdot_{(\lambda,\gamma)} f_\lambda({}_\lambda m) \\ & = & f_\lambda(e_\lambda \cdot_{(\lambda,\gamma)}\, {}_\lambda m) \\ & = & f_\gamma(I^{\mathcal{M}}_{(\lambda,\gamma)}({}_\lambda m)) \\ & = & f_\gamma(I^{\mathcal{M}}_{(\alpha,\gamma)}({}_\alpha m)) \\ & = & I^{\mathcal{N}}_{(\alpha,\gamma)}(f_\alpha({}_\alpha m)), \end{array}$$

i.e., $\overline{f_\lambda({}_\lambda m)} = \overline{f_\alpha({}_\alpha m)}$ and G is well defined. Now, let $A - Mod$ be the category of all unital A-modules. We have the following functors

$$\begin{array}{rcl} F : A - Mod & \longrightarrow & \mathcal{C}^S(A) - Mod \\ (M, \cdot) & \mapsto & \{e_\lambda M\}_{\lambda \in \Lambda} \\ f : M \to N & \mapsto & \{F(f)_\lambda = f|_{e_\lambda M}\} \end{array}$$

$$\begin{array}{rcl} G : \mathcal{C}^S(A) - Mod & \longrightarrow & A - Mod \\ \mathcal{M} & \mapsto & (\varinjlim \mathcal{M}_\lambda, \bullet) \\ f = \{f_\lambda : {}_\lambda\mathcal{M} \to {}_\lambda\mathcal{N}\} & \mapsto & G(f)(\overline{{}_\lambda m}) = \overline{f_\lambda({}_\lambda m)} \end{array}$$

Since $\varinjlim e_\lambda M = M$ whenever $M \in A - Mod$ and the A-module structure of M coincides with that given by the functor G, we have that $GF = Id_{A-Mod}$ in the objects of the categories, and is easy to see that $GF = Id_{A-Mod}$ in the morphisms too. Now, consider the linear map

$$\varphi_\lambda : {}_\lambda\mathcal{M} \longrightarrow e_\lambda \bullet G(\mathcal{M})$$

$$ {}_\lambda m \quad \mapsto \quad \overline{{}_\lambda m}. $$

Since $\overline{{}_\lambda m} = \overline{e_\lambda \cdot_{(\lambda,\lambda)}\ {}_\lambda m} = e_\lambda \bullet \overline{{}_\lambda m}$, we have that φ is well defined. Now, suppose that there exist ${}_\lambda m,\ {}_\lambda n \in\ {}_\lambda\mathcal{M}$ such that $\overline{{}_\lambda m} = \overline{{}_\lambda n}$. Then, there exist $e_\lambda \leq e_\beta$ such that $I_{(\lambda,\beta)}({}_\lambda m) = I_{(\lambda,\beta)}({}_\lambda n)$, since $I_{(\lambda,\beta)}$ is injective, we conclude that ${}_\lambda m =\ {}_\lambda n$. Finally, note that for every $\lambda, \beta \in \Lambda$ there exist $\gamma \in \Lambda$ such that $e_\lambda \leq e_\gamma$ and $e_\beta \leq e_\gamma$, then

$$
\begin{aligned}
e_\lambda \bullet \overline{{}_\beta m} &= \overline{e_\lambda e_\gamma \cdot_{(\gamma,\lambda)} I_{(\beta,\gamma)}({}_\beta m)} \\
&= \varphi_\lambda(e_\lambda e_\gamma \cdot_{(\gamma,\lambda)} I_{(\beta,\gamma)}({}_\beta m)),
\end{aligned}
$$

since $e_\lambda e_\gamma \cdot_{(\gamma,\lambda)} I_{(\beta,\gamma)}({}_\beta m) \in\ {}_\lambda\mathcal{M}$, we have that φ_λ is surjective and, consequently, each φ_λ is an isomorphism of vector spaces. Moreover, as

$$
\begin{aligned}
e_\beta a e_\lambda \bullet \varphi_\lambda({}_\lambda m) &= e_\beta a e_\lambda \bullet \overline{{}_\lambda m} \\
&= \overline{e_\beta a e_\lambda \cdot_{(\lambda,\beta)}\ {}_\lambda m} \\
&= \varphi_\beta(e_\beta a e_\lambda \cdot_{(\lambda,\beta)}\ {}_\lambda m),
\end{aligned}
$$

we have that the following diagrams commute

$$
\begin{tikzcd}
{}_\lambda\mathcal{M} \arrow[r, "e_\beta a e_\lambda"] \arrow[d, "\varphi_\lambda"'] & {}_\beta\mathcal{M} \arrow[d, "\varphi_\beta"] \\
e_\lambda \bullet G(\mathcal{M}) \arrow[r, "e_\beta a e_\lambda"'] & e_\beta \bullet G(\mathcal{M})
\end{tikzcd}
$$

i.e., since every φ_λ is a linear isomorphism, $\varphi = \{\varphi_\lambda\}_{\lambda\in\Lambda}$ is a $\mathcal{C}^S(A)$-module isomorphism. Hence $FG \cong Id_{\mathcal{C}^S(A)-Mod}$ when restricted to the objects. Finally, we have that

$$ FG(f)(\varphi_\lambda^{\mathcal{M}}({}_\lambda m)) = G(f)(\varphi_\lambda^{\mathcal{M}}({}_\lambda m)) = \varphi_\lambda^{\mathcal{N}}(f_\lambda({}_\lambda m)). $$

Then φ is actually a natural transformation, hence a natural isomorphism. □

Theorem 2.41. *Let $\mathcal{C}$ be a linear category. Then the category of the $\mathcal{C}$-modules and the category of the unital $a(\mathcal{C})$-modules are equivalent.*

Proof. In fact, consider the functors

$$
\begin{aligned}
F : \mathcal{C} - Mod &\rightarrow a(\mathcal{C}) - Mod \\
M = \{{}_x M\}_{x\in\mathcal{C}_0} &\mapsto \bigoplus_{x\in\mathcal{C}_0} {}_x M \\
\theta : M \to N &\mapsto F(\theta)(({}_x m)_{x\in\mathcal{C}_0}) = (\theta_x({}_x m))_{x\in\mathcal{C}_0}
\end{aligned}
$$

and

$$
\begin{aligned}
G : a(\mathcal{C}) - Mod &\rightarrow \mathcal{C} - Mod \\
M &\mapsto \{e_x M\}_{x\in\mathcal{C}_0} \\
\alpha : M \to N &\mapsto G(\alpha)_x = \alpha,
\end{aligned}
$$

where e_x denotes the matrix with ${}_x 1_x$ in the (x,x) position and 0 otherwise. Note that $S = \{\sum_{i=1}^n e_{x_i}\}$ is a system of local units for $a(\mathcal{C})$. Now, we have that

$$ FG(M) = \bigoplus_x {}_x G(M) = \bigoplus_x e_x M = M $$

and

$$ {}_yGF(\{{}_xM\}) = e_y(\bigoplus_x {}_xM) = {}_yM, $$

then $FG = Id_{a(\mathcal{C})-Mod}$ and $GF = Id_{\mathcal{C}-Mod}$ in the objects. For the morphisms, we have that

$$ \begin{aligned} (FG(\alpha))(m) &= (FG(\alpha))(\sum_{i=1}^{n} e_{x_i}m) \\ &= \sum_{i=1}^{n} \alpha(e_{x_i}m) \\ &= \sum_{i=1}^{n} e_{x_i}\alpha(m) \\ &= \alpha(m), \end{aligned} $$

where $e = \sum_{i=1}^{n} e_{x_i}$ is such that $em = m$. And

$$ (GF(\theta))_y({}_ym) = F(\theta)({}_ym) = \theta_y({}_ym). $$

□

Now we can enunciate the desired result.

Corollary 2.42. *Let A be an algebra with s.l.u. S. Then A and $a(\mathcal{C}^S(A))$ are Morita equivalent.*

By [10], every algebra with local units is Morita equivalent to some algebra with enough idempotents, i.e., to some algebra R such that $R = \bigoplus_e eR = \bigoplus_e Re$, where $\{e\}$ is a family of orthogonal idempotents in R. The previous corollary prove the same thing and give a better description of such algebra with enough idempotents. Also, despite of the notation as subset of a system of local units, nothing require that the local units must be different. So, if we consider a unital algebra A with a s.l.u. $S = \{e_1, \cdots, e_n\}$ where every $e_i = 1_A$, this corollary provides the well known Morita equivalence between A and $Mat_{n\times n}(A)$.

We can push this result even further: in [18] the authors prove that given two algebras with enough idempotents A and B, there exists an infinite set of indexes X such that $FMat_X(A) \cong FMat_X(B)$ as algebras. As a consequence, we obtain the following result.

Corollary 2.43. *Let A be an algebra with s.l.u. $S = \{e_\lambda\}_{\lambda\in\Lambda}$ and B an algebra with s.l.u. $T = \{f_i\}_{i\in\Gamma}$. If A and B are Morita equivalent, then there exist an infinite set of indices X whose cardinality is greater than or equal to those of Λ and Γ such that $FMat_X(a(\mathcal{C}^S(A))) \cong FMat_X(a(\mathcal{C}^T(B)))$.*

2.6 S-categorizable partial actions.

In Section 4 we saw that when A is a partial H-module algebra with local units with an S-categorizable (symmetrical) partial action, we can induce a (symmetrical) partial action on the category $\mathcal{C}^S(A)$. Here, we will highlight some peculiarities coming from the concept of categorizable partial actions: we will study the connections of partial actions on $\mathcal{C}^S(A)$ and S-categorizable partial actions on A.

Given a partial H-module structure on $\mathcal{C}^S(A)$, namely $\triangleright$, suppose that every inclusion $i_{(\lambda,\lambda)}^{(\gamma,\gamma)} : e_\lambda Ae_\lambda \longrightarrow e_\gamma Ae_\gamma$ is a morphism of partial actions, where $e_\lambda, e_\gamma \in S$ with $e_\lambda \leq e_\gamma$, i.e.,

$h \triangleright_{(\gamma,\gamma)} e_\gamma e_\lambda a e_\lambda e_\gamma = i^{(\gamma,\gamma)}_{(\lambda,\lambda)}(h \triangleright_{(\lambda,\lambda)} e_\lambda a e_\lambda)$, for every $h \in H$, $a \in A$. Then we can define a linear map

$$\begin{array}{rcl} \star : H \otimes A & \longrightarrow & A \\ h \otimes a & \mapsto & h \triangleright_{(\lambda,\lambda)} a, \end{array}$$

where $e_\lambda \in S$ is such that $e_\lambda a = a = a e_\lambda$. Note that if $e_\gamma \in S$ is such that $e_\gamma a = a = a e_\gamma$, there exist $e_\beta \in S$ such that $e_\lambda \leq e_\beta$ and $e_\gamma \leq e_\beta$, and

$$\begin{array}{rcl} h \triangleright_{(\lambda,\lambda)} a = h \triangleright_{(\lambda,\lambda)} e_\lambda a e_\lambda & = & i^{(\beta,\beta)}_{(\lambda,\lambda)}(h \triangleright_{(\lambda,\lambda)} e_\lambda a e_\lambda) \\ & = & h \triangleright_{(\beta,\beta)} e_\beta e_\lambda a e_\lambda e_\beta \\ & = & h \triangleright_{(\beta,\beta)} a. \end{array}$$

Analogously, $h \triangleright_{(\gamma,\gamma)} a = h \triangleright_{(\beta,\beta)} a$, hence $\star$ is well defined. Now, we have that

1. $1_H \star a = 1_H \triangleright_{(\lambda,\lambda)} a = a$, where $e_\lambda \in S$ is such that $e_\lambda a = a = a e_\lambda$;

2. Take $e_\beta \in S$ such that $e_\beta a = a = a e_\beta$ and $e_\beta b = b = b e_\beta$, then

$$\begin{array}{rcl} h \star (ab) & = & h \triangleright_{(\beta,\beta)} (ab) \\ & = & \sum (h_{(1)} \triangleright_{(\beta,\beta)} a)(h_{(2)} \triangleright_{(\beta,\beta)} b) \\ & = & \sum (h_{(1)} \star a)(h_{(2)} \star b). \end{array}$$

3. Take $e_\lambda \in S$ such that $e_\lambda a = a = a e_\lambda$, then

$$\begin{array}{rcl} h \star (g \star a) & = & h \star (g \triangleright_{(\lambda,\lambda)} a) \\ & = & h \triangleright_{(\lambda,\lambda)} (g \triangleright_{(\lambda,\lambda)} a) \\ & = & \sum (h_{(1)} \triangleright_{(\lambda,\lambda)} e_\lambda)(h_{(2)} g \triangleright_{(\lambda,\lambda)} a) \\ & = & \sum (h_{(1)} \star e_\lambda)(h_{(2)} g \star a). \end{array}$$

Hence $\star$ is a partial action, actually an S-categorizable partial action. Additionally, if $\triangleright$ is symmetrical, the induced partial action $\star$ will be symmetrical.

Note that the inclusions $i^{(\gamma,\gamma)}_{(\lambda,\lambda)} : e_\lambda A e_\lambda \longrightarrow e_\gamma A e_\gamma$ are always morphisms of partial actions when the partial action in $\mathcal{C}^S(A)$ is induced by an S-categorizable partial action on A, and, moreover, the partial action $\star$ defined as before is the original S-categorizable partial action on A. However, if the inclusions $i^{(\gamma,\gamma)}_{(\lambda,\lambda)} : e_\lambda A e_\lambda \longrightarrow e_\gamma A e_\gamma$ are morphisms of partial actions, then the partial action $\star$ defined as before, induces a new partial action on $\mathcal{C}^S(A)$ that is possibly different from the first one. To recover the original partial action, we must have that, whenever $e_\lambda \leq e_\gamma$ and $e_\alpha \leq e_\beta$, every inclusion $i^{(\gamma,\beta)}_{(\lambda,\alpha)} : e_\alpha A e_\lambda \longrightarrow e_\beta A e_\gamma$ preserves partial actions, i.e., $h \triangleright_{(\beta,\gamma)} e_\gamma e_\lambda a e_\alpha e_\beta = i^{(\gamma,\beta)}_{(\lambda,\alpha)}(h \triangleright_{(\alpha,\lambda)} e_\lambda a e_\alpha)$, for every $h \in H$, $a \in A$.

Example 2.44. *Let k be a field and consider the algebra $A = ke + kf$ where $e^2 = e$, $f^2 = f$ and $ef = fe = 0$, which is an algebra with s.l.u. $S = \{1_A, e\}$. Let $H = k\mathbb{Z}$. Note that $\mathcal{C}^S(A)$ is an H-module category with H-module structure on each morphism space given by*

$$\begin{array}{rcl} \triangleright_y : H \otimes {}_yA_y & \longrightarrow & {}_yA_y \\ i \otimes e & \mapsto & i \triangleright_y e = e, \end{array}$$

$$\begin{array}{rcl} i \otimes f & \mapsto & i \triangleright_y f = f, \\ \triangleright_x : H \otimes {}_xA_x & \longrightarrow & {}_xA_x \\ i \otimes e & \mapsto & i \triangleright_x e = e, \\ \triangleright_{(x,y)} : H \otimes {}_yA_x & \longrightarrow & {}_yA_x \\ i \otimes e & \mapsto & i \triangleright_{(x,y)} e = 2^i e, \\ \triangleright_{(y,x)} : H \otimes {}_xA_y & \longrightarrow & {}_xA_y \\ i \otimes e & \mapsto & i \triangleright_{(y,x)} e = 2^{-i} e, \end{array}$$

for all $i \in \mathbb{Z}$, where the objects y and x correspond to the local units 1_A and e, respectively. To show that $\mathcal{C}^S(A)$ is indeed an H-module category (hence a partial H-module category), we must verify the conditions where e is considered as a morphism in ${}_xA_y$ and ${}_yA_x$, because when e is located in ${}_xA_x$ or in ${}_yA_y$, where the action is trivial, the equalities are easily verified. So, since $\Delta_H(i) = i \otimes i$, we have

- $(i \triangleright_{(y,x)} e)(i \triangleright_{(x,y)} e) = 2^{-i}2^i e = e = i \triangleright_x e$;
- $(i \triangleright_{(x,y)} e)(i \triangleright_{(y,x)} e) = 2^i 2^{-i} e = e = i \triangleright_y e$.

Then $\mathcal{C}^S(A)$ is an H-module category, as defined in [2] and the induced partial action $\star$ on A is the trivial action, which makes A an H-module algebra, but the partial action on $\mathcal{C}^S(A)$ induced by the trivial action on A does not coincide with $\triangleright$.

Remark 2.45. *If every inclusion $i_{(\lambda,\lambda)}^{(\gamma,\gamma)} : e_\lambda A e_\lambda \longrightarrow e_\gamma A e_\gamma$ is a morphism of partial actions for every pair of local units $e_\lambda \leq e_\gamma$ that lies in a subset T of S that is also a s.l.u. for A, we can define a partial action on A in the same way as $\star$, but this partial action will be T-categorizable, not necessarily S-categorizable.*

Definition 2.46 ([3]). *Let H be a Hopf algebra, A an algebra with s.l.u. $S = \{e_\lambda\}_{\lambda \in \Lambda}$ and $\triangleright : H \otimes A \longrightarrow A$ a linear map. We say that $\triangleright$ measures A if for every $h \in H$, $a, b \in A$, we have:*

1. $h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$;
2. $h \triangleright e_\lambda = \varepsilon_H(h) e_\lambda$, *for every* $e_\lambda \in S$.

Proposition 2.47. *Let A be an algebra with s.l.u. $S = \{e_\lambda\}_{\lambda \in \Lambda}$ and H a Hopf algebra. Suppose that the family of linear maps $\triangleright = \{\triangleright_{(\lambda,\gamma)} : H \otimes e_\gamma A e_\lambda \longrightarrow e_\gamma A e_\lambda\}$ determine a partial action on the category $\mathcal{C}^S(A)$ such that $h \triangleright_{(\lambda,\gamma)} e_\alpha = \varepsilon_H(h) e_\alpha$, for every $h \in H$ and every $e_\alpha, e_\lambda, e_\gamma \in S$ such that $e_\alpha \in e_\gamma A e_\lambda$, i.e., independently of the location of the local units, H acts trivially on them. Note that this is more than just say that $\mathcal{C}^S(A)$ is an H-module category. Then, $\triangleright$ comes from a partial action that measures A.*

Proof. Suppose that $e_\lambda \leq e_\gamma$ and $e_\alpha \leq e_\beta$ and consider the inclusion $\iota : e_\alpha A e_\lambda \longrightarrow e_\beta A e_\gamma$. We will show that $h \triangleright_{(\lambda,\alpha)} a = h \triangleright_{(\gamma,\beta)} a$, for every $h \in H$ and $a \in e_\alpha A e_\lambda$, i.e., every inclusion ι of this type preserve the partial action. In fact, take $a \in e_\alpha A e_\lambda$ and let $e_\theta \in S$ be such that $e_\lambda, e_\gamma, e_\alpha, e_\beta \leq e_\theta$, then $a \in e_\theta A e_\theta$, $e_\lambda \in e_\theta A e_\lambda$, $e_\alpha \in e_\alpha A e_\theta$ and

$$\begin{aligned} h \triangleright_\theta a &= \sum (h_{(1)} \triangleright_{(\alpha,\theta)} e_\alpha)(h_{(2)} \triangleright_{(\lambda,\alpha)} a)(h_{(3)} \triangleright_{(\theta,\lambda)} e_\lambda) \\ &= \sum \varepsilon(h_{(1)}) e_\alpha (h_{(2)} \triangleright_{(\lambda,\alpha)} a) \varepsilon(h_{(3)}) e_\lambda \\ &= e_\alpha (h \triangleright_{(\lambda,\alpha)} a) e_\lambda \end{aligned}$$

$$= \quad h \triangleright_{(\lambda,\alpha)} a.$$

Analogously, we prove that $h \triangleright_\theta a = h \triangleright_{(\gamma,\beta)} a$. Hence, this partial action induces a partial action on A such that $h \cdot e_\alpha = \varepsilon_H(h) e_\alpha$ for every $h \in H$ and $e_\alpha \in S$, and by the third property of partial action, we have $h \cdot (k \cdot a) = hk \cdot a$, for every $h, k \in H$, $a \in A$. $\square$

2.7 Partial G-gradings on $FMat_{\mathbb{N}}(\Bbbk)$

In this section, we will use our knowledge about partial actions on algebras with local units obtained until now to extend some results about partial gradings of matrices for partial gradings of the algebra with local units $FMat_{\mathbb{N}}(\Bbbk)$.

In [6], the authors proved that if G is a finite group and $\Bbbk$ is a field such that its characteristic does not divide $|G|$, then there is a bijective correspondence between subgroups of G and partial G-gradings of $\Bbbk$. When we consider the algebra of the matrices of infinite order but with finitely many nonzero entries in the field k, which is an algebra with local units, we can show that this results still holds.

Definition 2.48 ([6])**.** *Let G be a finite group and $\Bbbk^G$ its dual group algebra. Denote by $p_g \in \Bbbk^G$ the morphism such that $p_g(h) = \delta_{g,h}$, then $\{p_g\}_{g \in G}$ determines a canonical basis for $\Bbbk^G$ as a vector space. A partial G-grading of a unital algebra A is a linear map $\cdot : \Bbbk^G \otimes A \longrightarrow A$ that is a symmetrical partial action, in other words, this linear map satisfies the following conditions, for every $a, b \in A$, $g, t \in G$:*

1. $\sum_{g \in G} p_g \cdot a = a$;
2. $p_g \cdot (ab) = \sum_{l \in G} (p_{gl^{-1}} \cdot a)(p_l \cdot b)$;
3. $p_g \cdot (p_t \cdot a) = (p_{gt^{-1}} \cdot 1_A)(p_t \cdot a) = (p_t \cdot a)(p_{t^{-1}g} \cdot 1_A)$.

Definition 2.49. *Let A be an algebra with s.l.u. $S = \{e_\lambda\}_{\lambda \in \Lambda}$ and G a finite group. We will say that A admits a partial G-grading if it has a partial $\Bbbk^G$-module algebra structure given by a symmetrical partial action. In other words, there exist a linear map $\cdot : \Bbbk^G \otimes A \longrightarrow A$ such that for every $a, b \in A$, $g, t \in G$:*

1. $\sum_{g \in G} p_g \cdot a = a$;
2. $p_g \cdot (ab) = \sum_{l \in G} (p_{gl^{-1}} \cdot a)(p_l \cdot b)$;
3. $p_g \cdot (p_t \cdot a) = (p_{gt^{-1}} \cdot e_\lambda)(p_t \cdot a) = (p_t \cdot a)(p_{t^{-1}g} \cdot e_\alpha)$, *for every $e_\lambda, e_\alpha \in S$ such that* $e_\lambda(p_t \cdot a) = (p_t \cdot a) = (p_t \cdot a)e_\alpha$.

In [15], the authors called a good G-grading of $M_n(\Bbbk)$ a grading where every matrix unit E_{ij} is homogeneous, and they proved that there is a bijective correspondence between good G-gradings of $M_n(\Bbbk)$ and elements of G^{n-1}.

Now, note that if we consider the algebra with local units $FMat_{\mathbb{N}}(\Bbbk)$ and if $(g_i)_{i \in \mathbb{N}}$ is a sequence of elements of a finite group G, then, as in [6], we have a G-grading of $FMat_{\mathbb{N}}(\Bbbk)$ given by the formula

$$deg(E_{ij}) = g_i g_j^{-1}.$$

Definition 2.50. *A good G-grading of $FMat_{\mathbb{N}}(\Bbbk)$ is a G-grading where every matrix unit E_{ij} is homogeneous.*

Proposition 2.51. *There is a bijective correspondence between good G-gradings of $FMat_{\mathbb{N}}(\Bbbk)$ and sequences of elements of G where $g_1 = 1_G$.*

Sketch of the proof. First, note that the sequences $(g_1, g_2, \ldots)$ and $(1_G, g_2g_1^{-1}, g_3g_1^{-1}, \ldots)$ provide the same good G-grading on $FMat_{\mathbb{N}}(\Bbbk)$. Conversely, every good G-grading on $FMat_{\mathbb{N}}(\Bbbk)$ provides a family $\{h_i\}_{i\in\mathbb{N}}$ of elements of G such that $deg\, E_{ii+1} = h_i$, and the others degrees are given by

$$\begin{aligned} deg\, E_{ij} &= h_i h_{i+1} \cdots h_{j-1} \\ deg\, E_{ji} &= h_{j-1}^{-1} h_{j-2}^{-1} \cdots h_i^{-1}, \end{aligned}$$

for every $1 \leq i < j \leq n$, and we assume $g_1 = 1_G$ and $g_i = h_i^{-1} h_{i-1}^{-1} \cdots h_1^{-1}$. □

In [6], the authors defined a *good partial G-grading* as a partial G-grading of $M_n(\Bbbk)$ where every matrix unit E_{ij} is an eigenvector for all operators p_g.□.

Definition 2.52. *Consider the algebra $FMat_{\mathbb{N}}(\Bbbk)$ and let G be a finite group. We will say that a partial G-grading of $FMat_{\mathbb{N}}(\Bbbk)$ is good if every $E_{i,j}$ is a eigenvector for all operators p_g.*□.

Example 2.53. *Suppose that the group G is abelian, take $A = FMat_{\mathbb{N}}(\Bbbk)$ endowed with a G-grading given by a fixed sequence $(g_i)_{i\in\mathbb{N}}$ as before, and assume that the characteristic of $\Bbbk$ does not divide $|G|$. Then, we can tensor A by a partial G-grading of $\Bbbk$ given by a subgroup H of G, in order to obtain a partial G-grading on $\Bbbk \otimes A \cong A$, as was done in [6]. This partial grading is given by*

$$p_g \cdot E_{ij} = \frac{1}{|H|} \delta_{gH, g_i g_j^{-1} H} E_{ij}.$$

Proposition 2.54. *Let G be a finite abelian group and $\Bbbk$ a field such that $char(\Bbbk) \nmid |G|$. Then, there is a bijective correspondence between good partial G-gradings of $FMat_{\mathbb{N}}(\Bbbk)$ and good X-gradings of $FMat_{\mathbb{N}}(\Bbbk)$, where X is a quotient of G.*

Proof. According to the proof of Corollary 6.7 of [6], we have that every subalgebra $\Bbbk E_{ii}$ is a partial G-graded subalgebra of $FMat_{\mathbb{N}}(\Bbbk)$, and the partial grading of every kE_{ii} is given by the same subgroup H of G. This is possible because every good partial G-grading on $FMat_{\mathbb{N}}(\Bbbk)$ induce a good partial G-grading on each algebra $Mat_{n\times n}(\Bbbk)$, that are naturally included in $FMat_{\mathbb{N}}(\Bbbk)$. Usin the same idea, it is not hard to see that the formula

$$q_{gH} \triangleright M = |H| p_g \cdot M$$

defines a good G/H-grading of $FMat_{\mathbb{N}}(\Bbbk)$, where $M \in FMat_{\mathbb{N}}(\Bbbk)$ and $\{q_{gH}\}_{gH\in G/H}$ is the canonical basis of $\Bbbk^{G/H}$. □

Basically, in the proof of the previous proposition, we use indirectly that the family of the finite matrices forms a direct system which direct limit is $FMat_{\mathbb{N}}(\Bbbk)$.

2.8 Globalization of a partial action

In [4] and [5], Alves and Batista proved that every partial Hopf action on a unital algebra has an enveloping action (globalization), actually, there exists a minimal globalization, that is unique up to isomorphism.

In this section we will show that, in fact, every partial Hopf action on an algebra with trivial right (or left) annihilator is, actually, a restriction of an action as in the case of partial actions on algebras with unit. For associative algebras in general, this is not necessarily possible, but we will show that there exists almost a globalization.

Definition 2.55 ([4],[5]). *Let $\cdot : H \otimes A \longrightarrow A$ be a symmetrical partial action, where A is a unital algebra. A pair (B, θ) is called an enveloping action, or globalization, for the partial action $\cdot$ if:*

1. *B is an H-module algebra (without unit), with action $\triangleright$;*
2. *$\theta : A \longrightarrow B$ is a monomorphism of algebra;*
3. *$\theta(A)$ is an ideal of B;*
4. *$\theta(h \cdot a) = \theta(1_A)(h \triangleright \theta(a)) = (h \triangleright \theta(a))\theta(1_A)$, $\forall a \in A$, $h \in H$;*
5. *$B = H \triangleright \theta(A)$.*

Item $4)$ means that θ is a morphism of partial actions, because the mapping

$$h \rightharpoonup \theta(a) = \theta(1_A)(h \triangleright \theta(a))$$

determines a partial action on $\theta(A)$.

2.8.1 Globalization theorem for non unital algebras

Let A be any associative algebra that is a partial H-module algebra with symmetrical partial action $\cdot : H \otimes A \to A$.

Definition 2.56. *A quasi-globalization for the partial action $\cdot$ is a pair (B, θ) such that*

1. *B is an H-module algebra (without unit), with action $\triangleright$;*
2. *$\theta : A \longrightarrow B$ is a monomorphism of algebra;*
3. *For every $a, b \in A$ and $h \in H$, we have $\theta((h \cdot a)b) = (h \triangleright \theta(a))\theta(b)$ and $\theta(b(h \cdot a)) = \theta(b)(h \triangleright \theta(a))$*
4. *$B = H \triangleright \theta(A)$.*

Note that if (B, θ) is a quasi-globalization for $\cdot$, from item $3)$ and $4)$, we have that $\theta(A)$ is an ideal of B, as can be seen in [4].

We call (B, θ) a quasi-globalization because we cannot say that the partial action on A comes from a restriction of the action of B on $\theta(A)$. However, we shall see later that this is true when $r(A) = 0$.

Here, we can see that if A is a partial H-module algebra with unit and (B, θ) is a quasi-globalization, then (B, θ) is actually a globalization in the sense of Definition 2.55. Conversely, if (B, θ) is a globalization, then it is a quasi-globalization.

Definition 2.57. *Let $\cdot : H \otimes A \longrightarrow A$ be a symmetrical partial action. A quasi-globalization (B, θ) will be called minimal if $\sum_i kh_i \cdot a_i = 0$, for all $k \in H$, implies $\sum_i h_i \triangleright \theta(a_i) = 0$.*

In [4] the authors defined minimal globalizations in the following way:

Definition 2.58. *Let H be a Hopf algebra and A a unital partial H-module algebra. A globalization (B, θ) is minimal if $\theta(1_A)M = 0$ implies that $M = 0$, for every H-submodule M of B.*

Since $\theta(A)$ is an ideal of B, we can consider the mapping $\Pi_0 : B \to A$ given by $\Pi_0(x) = \theta(1_A)x$ that is a projection of B on $\theta(A)$. Then, the previous definition is equivalent to say that the kernel of Π does not contain any nonzero H-submodule of B.

Note that, if A is any partial H-module algebra such that $r(A) = 0$ and (B, θ) is a quasi-globalization, then the linear map $\Pi : B \to \theta(A)$, given by $\Pi(h \triangleright \theta(a)) = \theta(h \cdot a)$ (that is equivalent to Π_0 when A is unital), is well defined because if $\sum_i h_i \triangleright \theta(a_i) = \sum_j k_j \triangleright \theta(b_j)$, then for every $c \in A$ we have that

$$\begin{aligned} \theta(c\sum_i h_i \cdot a_i) &= \theta(c)\sum_i h_i \triangleright \theta(a_i) \\ &= \theta(c)\sum_j k_j \triangleright \theta(b_j) \\ &= \theta(c\sum_j k_j \cdot b_j), \end{aligned}$$

and since $r(A) = 0$ and θ is a momonorphism, then $\sum_i h_i \cdot a_i = \sum_j k_j \cdot b_j$.

Proposition 2.59. *Let H be a Hopf algebra and A a (not necessarily unital) partial H-module algebra. If $r(A) = 0$, then a quasi-globalization (B, θ) is minimal if and only if the kernel of Π does not contain any nonzero H-submodule of B.*

Proof. Suppose that the quasi-globalization (B, θ) is minimal and let M be an H-submodule of B such that $\Pi(M) = 0$. Then for every $\sum_i h_i \triangleright \theta(a_i) \in M$, we have that $\sum_i kh_i \triangleright \theta(a_i) \in M$ for every $k \in H$, wich means that $\theta(\sum_i kh_i \cdot a_i) = 0$. Since θ is injective and the quasi-globalization is minimal, we conclude that $\sum_i h_i \triangleright \theta(a_i) = 0$, hence $M = 0$. Let us now assume that the kernel of Π does not contain any nonzero H-submodule of B and suppose that $\sum_i kh_i \cdot a_) = 0$ for every $k \in H$. Then the H-submodule M of B generated by $\sum_i h_i \triangleright \theta(a_i)$ is contained in the kernel of Π, but since $ker\Pi$ does not contain any nonzero H-submodule of B, we conclude that $M = 0$, hence $\sum_i h_i \triangleright \theta(a_i) = 0$. □

Remember that if $\cdot : H \otimes A \longrightarrow A$ is a linear map, we can induce a linear map $\varphi : A \longrightarrow Hom(H, A)$ given by $\varphi(a)(h) = h \cdot a$, and $Hom(H, A)$ is an algebra with the convolution product and a left H-module algebra with action $\triangleright$ given by $(k \triangleright f)(h) = f(hk)$. Note that $H \triangleright \varphi(A)$ is the smaller H-submodule algebra of $Hom(H, A)$ that contains $\varphi(A)$.

Lemma 2.60. *Let $\cdot : H \otimes A \longrightarrow A$ be a partial action and suppose that $r(A) = 0$. Then $\varphi(A)$ is an ideal of $B = H \triangleright \varphi(A)$ if and only if the partial action is symmetrical.*

Proof. First, take $k, h \in H$, $a, b \in A$. Then

$$\begin{aligned} (\varphi(a)(h \triangleright \varphi(b)))(k) &= \sum (k_{(1)} \cdot a)(k_{(2)}h \cdot b) \\ &= k \cdot (a(h \cdot b)) = \varphi(a(h \cdot b))(k), \end{aligned}$$

hence $\varphi(A)$ is a right ideal of B. Analogously, if the partial action is symmetrical, we prove that $\varphi(A)$ is a left ideal of B. Now, suppose that $\varphi(A)$ is a left ideal of B. Then, for $k \in H$, $a \in A$, for every $b \in A$, we have that $(k \triangleright \varphi(a))\varphi(b) \in \varphi(A)$. Hence, for every $c \in A$,

$$\begin{aligned} \varphi(c)\varphi((k \cdot a)b)(h) &= \sum (h_{(1)} \cdot c)(h_{(2)} \cdot (k \cdot a))(h_{(3)} \cdot b) \\ &= \sum (h_{(1)} \cdot (c(k \cdot a)))(h_{(2)} \cdot b) \\ &= \sum (h_{(1)} \cdot c)(h_{(2)}k \cdot a)(h_{(3)} \cdot b) \\ &= \varphi(c)(k \triangleright \varphi(a))\varphi(b)(h). \end{aligned}$$

Now, since $r(A) = 0$ and φ is a monomorphism of algebras, we have that $r(\varphi(A)) = 0$. Then, as the previous calculation holds for every $c \in A$, we have that

$$\varphi((k \cdot a)b)(h) = (k \triangleright \varphi(a))\varphi(b)(h),$$

i.e., the partial action is symmetrical. □

Proposition 2.61. *Let* $\cdot : H \otimes A \longrightarrow A$ *be a symmetrical partial action. Then, the pair* (B, φ)*, where* $B = H \triangleright \varphi(A)$*, is a minimal quasi-globalization for* $\cdot$.

Proof. In fact,

1. B is an H-module algebra with action $\triangleright$;
2. $\varphi : A \longrightarrow B$ is a monomorphism of algebras because if $\varphi(a) = \varphi(b)$ then $a = 1_H \cdot a = \varphi(a)(1_H) = \varphi(b)(1_H) = 1_H \cdot b = b$;
3. $\varphi(A)$ is an ideal of B by the previous lemma.
4. $B = H \triangleright \varphi(A)$ by hypothesis;
5. Suppose that for every $k \in H$ we have $\sum_i kh_i \cdot a_i = 0$. Then $0 = \sum_i kh_i \cdot a_i = (\sum_i h_i \triangleright \varphi(a_i))(k)$, for every $k \in H$, hence $\sum_i h_i \triangleright \varphi(a_i) = 0$.

□

Definition 2.62. (B, φ) *will be called the standard quasi-globalization.*

We already know that when we consider only unital algebras, there exists only one minimal globalization, up to isomorphism. The next theorem shows that this also holds for some associative algebras.

Theorem 2.63. *Let* H *be a Hopf algebra and* A *a partial* H*-module algebra with symmetrical partial action. If* $r(A) = 0$ *or* $l(A) = 0$*, there exist only one minimal quasi-globalization of the partial action on* A *up to isomorphism.*

Proof. In fact, we will prove that every minimal quasi-globalization is isomorphic to the standard quasi-globalization. Let (B', θ) be a quasi-globalization and (B, φ) denote the standard quasi-globalization. Consider the linear map

$$\begin{array}{rcl} \Phi : B' & \to & B \\ \displaystyle\sum_{i=1}^{n} h_i \triangleright' \theta(a_i) & \mapsto & \displaystyle\sum_{i=1}^{n} h_i \triangleright \varphi(a_i). \end{array}$$

First we will prove that Φ is well defined. For this, suppose that $x = \sum_{i=1}^{n} h_i \triangleright' \theta(a_i) = 0$, then for every $c \in A$ and $k \in H$,

$$\begin{aligned} 0 &= \theta(c)(k \triangleright' x) \\ &= \theta(c)(\sum_{i=1}^{n} kh_i \triangleright' \theta(a_i)) \\ &= \theta(c \sum_{i=1}^{n} kh_i \cdot a_i). \end{aligned}$$

Since θ is injective, we have that $\sum_{i=1}^n c(kh_i \cdot a_i) = 0$ for every $c \in A$, $k \in H$. Then, if we suppose that $r(A) = 0$, $\sum_{i=1}^n kh_i \cdot a_i = 0$, for every $k \in H$, and since the standard quasi-globalization is minimal, we have that $\sum_{i=1}^n h_i \triangleright \varphi(a_i) = 0$. If $l(A) = 0$ we use a similar calculation. By construction, Φ is a morphism of H-modules, and the proof that Φ is a morphism of (non unital) algebras is the same as in [4], where is used essentially the equation

$$(h \triangleright \theta(a))(k \triangleright \theta(b)) = \sum h_{(1)} \triangleright \theta(a(S(h_{(2)})k \cdot b)),$$

that is a consequence of axiom 3) of the definition of quasi-globalization. Finally, in the same way we can construct a morphism of H-module algebras from B to B' that, since (B', θ) is minimal, is well defined and is mutual inverse with Φ. □

Corollary 2.64. *Under the assumptions of the previous theorem, if (B', θ) is a quasi-globalization, then there exists an epimorphism of H-module algebras from B' to a minimal quasi-globalization.*

Now we will show that, whenever $r(A) = 0$ or $l(A) = 0$, any quasi-globalization of a partial action $\cdot : H \otimes A \to A$ is actually a globalization, i.e., the partial action is induced by the action of the quasi-globalization.

Proposition 2.65. *Let B be an H-module algebra with action $\triangleright$ and A an ideal of B. If there exists a projection $\pi : H \triangleright A \to A$ such that:*

1. $\pi(H \triangleright (A)) = A$;
2. $\pi(h \triangleright (a\pi(k \triangleright b))) = \sum \pi(h_{(1)} \triangleright a))\pi(h_{(2)}k \triangleright b)$,

for every $h, k \in H$ and $a, b \in A$, then the linear map $h \rightharpoonup a = \pi(h \triangleright a)$ defines a partial action of H on A.

Proof. In fact, we have that $1_H \rightharpoonup a = \pi(1_H \triangleright a) = \pi(a) = a$ and

$$\begin{aligned} h \rightharpoonup (a(k \rightharpoonup b)) &= \pi(h \triangleright (a\pi(k \triangleright b))) \\ &= \sum \pi(h_{(1)} \triangleright a))\pi(h_{(2)}k \triangleright b) \\ &= \sum (h_{(1)} \rightharpoonup a)(h_{(2)}k \rightharpoonup b), \end{aligned}$$

for every $h, k \in H$ and $a, b \in A$. □

Proposition 2.66. *Let A be a partial H-module algebra with symmetrical partial action given by $\cdot : H \otimes A \to A$ and (B, θ) a quasi-globalization where B is an H-module algebra with action $\triangleright$. If $r(A) = 0$ or $l(A) = 0$, then (B, θ) is a globalization.*

Proof. We will assume that $r(A) = 0$, since the case where $l(A) = 0$ folows from an analogous argument. Then, as (B, θ) is a quasi-globalization, we have that

$$\theta(x)(h \triangleright \theta(a)) = \theta(x)\theta(h \cdot a),$$

wich implies that, if $\sum_i h_i \triangleright \theta(a_i) = \sum_j k_j \triangleright \theta(b_j)$,

$$\begin{aligned} \sum_i \theta(x)\theta(h_i \cdot a_i) &= \sum_i \theta(x)(h_i \triangleright \theta(a_i)) \\ &= \sum_j \theta(x)(k_j \triangleright \theta(b_j)) \end{aligned}$$

$$= \sum_j \theta(x)\theta(k_j \cdot b_j).$$

Hence, since $r(A) = 0$ and $\theta : A \to B$ is a monomorphism, $\sum_i \theta(h_i \cdot a_i) = \sum_j \theta(k_j \cdot b_j)$. With this, we have that the linear map $\pi : B \to A$ defined by $\pi(h \triangleright \theta(a)) = \theta(h \cdot a)$ is well defined and determines a projection with the properties of the previous proposition, then

$$\begin{aligned} h \rightharpoonup \theta(a) &= \pi(h \triangleright \theta(a)) \\ &= \theta(h \cdot a), \end{aligned}$$

wich means that θ is a morphism of partial actions. □

Corollary 2.67. *Every partial action on an associative algebra A, with $r(A) = 0$ or $l(A) = 0$, has a (minimal) globalization.*

2.8.2 Globalization theorem for algebras with local units

Since every algebra with local units A satisfies $r(A) = l(A) = 0$, according to the previous subsection we already know that every symmetrical partial action on an algebra with l.u. has a globalization. Here we will present an equivalent definition of globalization considering algebras with l.u. that will be useful for the relation between globalizations of categorizable partial actions and globalizations of the induced partial action on the category associated to the considered system of local units.

But first, we will follow the idea of Proposition 2.64 to provide a sufficient condition to determine if we can induce a partial action on an ideal with local units of an H-module algebra.

Definition 2.68. *Let A be an algebra with local units and S be a s.l.u. of A. A subfamily $T \subseteq S$ which is also a s.l.u. of A, will be called a subsystem of local units of S.*

Proposition 2.69. *Let B be an H-module algebra with action $\triangleright$ and A be an ideal of B. If A has a s.l.u. $S = \{e_\lambda\}_{\lambda \in \Lambda}$ and for every $h \in H$, $a \in A$ there exist subsystems of local units $L(h, a)$ and $R(h, a)$ of S such that $e(h \triangleright a) = (h \triangleright a)f$, for all $e \in L(h, a)$, $f \in R(h, a)$, then the linear map $h \rightharpoonup a = e(h \triangleright a)$, where $e \in L(h, a)$, determines a partial action of H on A.*

Proof. Note that if we take $e \in L(h, a)$, we can add every local unit $x \geq e$ to $L(h, a)$ without problem. In fact, for every local unit $x \geq e$, since $x(h \triangleright a) \in A$, A is an ideal of B and $R(h, a)$ is a subsystem of local units, there exist $f \in R(h, a)$ such that $x(h \triangleright a) = x(h \triangleright a)f = xe(h \triangleright a) = e(h \triangleright a)$. Then for every $h_1, \cdots, h_n \in H$ and $a_1, \cdots, a_n \in A$ we can construct a family $\{L_i = L(h_i, a_i)\}_{i=1}^n$ whose intersection $L = \cap_i L_i$ is also a subsystem of local unit, hence, in fact, we can consider $L_i = L$ for every $i = 1, \cdots, n$. Analogously, we can consider $R(h_i, a_i) = R$, for some subsystem of local units R, for every $i = 1, \cdots, n$. This observation is important to verify that the linear map $\pi : H \triangleright A \to A$ given by $\pi(h \triangleright a) = e(h \triangleright a)$, where $e \in L(h, a)$, is well defined. Moreover, we have that

$$\begin{aligned} \pi(h \triangleright (a\pi(k \triangleright b))) &= e_1(h \triangleright (ae_2(k \triangleright b))) \\ &= \sum e_1(h_{(1)} \triangleright a)(h_{(2)}k \triangleright b) \\ &= \sum e_1(h_{(1)} \triangleright a)e_3(h_{(2)}k \triangleright b) \\ &= \sum \pi(h_{(1)} \triangleright a)\pi(h_{(2)}k \triangleright b). \end{aligned}$$

Here, we choose $e_2 \in L(k, b)$ such that $ae_2 = a$, e_1 as consequence of the intersections above, and e_3 as consequence of the intersections and because A is an ideal of B. □

Definition 2.70. *Let $\cdot : H \otimes A \longrightarrow A$ be a symmetrical partial action where A is an algebra with s.l.u. $S = \{e_\lambda\}_{\lambda\in\Lambda}$. Then the pair (B, θ) is a globalization for $\cdot$ if and only if:*

1. *B is an H-module algebra (without unit), with action $\triangleright$;*
2. *$\theta : A \longrightarrow B$ is an algebra monomorphism;*
3. *$\theta(A)$ is an ideal of B;*
4. *$\theta(e_\alpha)(h \triangleright \theta(a)) = (h \triangleright \theta(a))\theta(e_\beta)$, for every $e_\alpha, e_\beta \in S$, $a \in A$, such that $e_\alpha(h \cdot a) = h\cdot a = (h\cdot a)e_\beta$, and $\theta(h\cdot a) = \theta(e_\alpha)(h\triangleright\theta(a))$, for every $e_\alpha \in S$ such that $e_\alpha(h \cdot a) = h\cdot a$;*
5. *$B = H \triangleright \theta(A)$.*

Note that when A has local units and (B, θ) is a globalization, we have that

$$\begin{aligned}\theta(h \cdot a) &= \theta(e)(h \triangleright \theta(a))\\ &= (h \triangleright \theta(a))\theta(f),\end{aligned}$$

for every $e, f \in A$ such that $e(h \cdot a) = h \cdot a = (h \cdot a)f$. Then, in this case the subsystems considered are $L(h, a) = \{e_\alpha \in S;\ e_\alpha(h\cdot a) = h\cdot a\}$ and $R(h, a) = \{e_\alpha \in S;\ (h\cdot a)e_\alpha = h\cdot a\}$, hence item 4 means that θ is a morphism of partial actions.

Definition 2.71. *Let $\cdot : H \otimes A \longrightarrow A$ be a symmetrical partial action and let $\theta : A \to B$ be a globalization. The globalization (B, θ) will be called minimal if $\sum_i kh_i \cdot a_i = 0$, for all $k \in H$, implies $\sum_i h_i \triangleright \theta(a_i) = 0$.*

Proposition 2.72. *Let $\cdot : H \otimes A \longrightarrow A$ be a symmetrical partial action and $S = \{e_\lambda\}_{\lambda\in\Lambda}$ an s.l.u. for A. Then, the pair (B, φ), where $B = H \triangleright \varphi(A)$, is a minimal globalization for $\cdot$.*

For the next example, we will use the results and definitions presented until now to find the minimal globalization of a good partial G-graduation on $FMat_{\mathbb{N}}(k)$. For this, when we consider the matrix algebra $FMat_{\mathbb{N}}(X)$, where X is an associative algebra, we will write $(\alpha)E_{ij}$ to denote that the element $\alpha \in X$ is on the entry ij of the matrix.

Example 2.73. *Let G be a finite abelian group such that $char(k) \nmid |G|$, $H = (kG)^*$ and $A = FMat_{\mathbb{N}}(k)$. Suppose that the map $\cdot : H \otimes A \to A$ determines a good partial G-grading on A. Then, by [2] and [6], we have that there exists a subgroup L of G and a family $\{t_{ij}\}_{i,j\in\mathbb{N}} \subset G$ such that $t_{ik}t_{kj}L = t_{ij}L$ and $p_g \cdot E_{ij} = \delta_{gL,t_{ij}L}\frac{1}{|L|}E_{ij}$. Consider B the subalgebra of $FMat_{\mathbb{N}}(kG)$ described by: $B = \bigoplus_{i,j\in\mathbb{N}} B_{ij}$ where $B_{ij} = \bigoplus_{g\in t_{ij}L} kg$. Define*

$$\begin{aligned}\theta : A &\to B\\ E_{ij} &\mapsto (\delta_{(x,y),(i,j)}\frac{1}{|L|}\sum_{g\in t_{ij}L} g)_{x,y\in\mathbb{N}} = (\frac{1}{|L|}\sum_{g\in t_{ij}L} g)E_{ij}.\end{aligned}$$

Note that the natural action of H in kG given by $p_g \rightharpoonup h = \delta_{g,h}h$ induces an structure of H-module algebra on $FMat_{\mathbb{N}}(kG)$ and B turns out to be an H-submodule algebra of $FMat_{\mathbb{N}}(kG)$. This action of H on B is given explicitly by

$$p_g \triangleright (\sum_{h\in t_{ij}L} \alpha_h h)E_{ij} = (\delta_{gL,t_{ij}L}\alpha_g g)E_{ij}.$$

Then (B, θ) is a minimal globalization for the good partial G-grading on A. In fact, it is not hard to show that B is an H-module algebra with action $\triangleright$, $B = H \triangleright \theta(A)$ and θ is a *monomorphism of algebras. To prove that $\theta(A)$ is an ideal of B, note that for every $g \in G$, $i, j, k \in \mathbb{N}$, we have*

$$\begin{aligned}\theta(E_{ik})(p_g \triangleright \theta(E_{kj})) &= \delta_{gL,t_{kj}L}\frac{1}{|L|}\theta(E_{ij}),\\ (p_g \triangleright \theta(E_{ik}))\theta(E_{kj}) &= \delta_{gL,t_{ik}L}\frac{1}{|L|}\theta(E_{ij}).\end{aligned}$$

Consequently,

$$\begin{aligned}\theta(p_g \cdot E_{ij}) &= \delta_{gL,t_{ij}L}\frac{1}{|L|}\theta(E_{ij})\\ &= \theta(E_{ij})(p_g \triangleright \theta(E_{ij}))\\ &= (p_g \triangleright \theta(E_{ij}))\theta(E_{jj}).\end{aligned}$$

Hence $\theta(p_g \cdot X) = \theta(E)(p_g \triangleright \theta(X)) = (p_g \triangleright \theta(X))\theta(F)$, for every finite sum E, F of $E_{ii}'s$ such that $EX = X = XF$. Finally, suppose that $\sum_i kh_i \cdot a_i = 0$ for every $k \in H$, with $h_i = \sum_{g\in G}\alpha^i_g p_g$ and $a_i = \sum_{j,l\in\mathbb{N}} a^i_{jl}E_{jl}$, where almost every $a^i_{jl} = 0$. Then, for every $g \in G$, choosing $k = p_g$, we have that

$$\begin{aligned}0 = \sum_i \alpha^i_g p_g \cdot a_i &= \sum_{i,j,l}\alpha^i_g p_g \cdot a^i_{jl}E_{jl}\\ &= \sum_{i,j,l}\alpha^i_g \delta_{gL,t_{jl}L}a^i_{jl}\frac{1}{|L|}E_{jl},\end{aligned}$$

then for every $j, l \in \mathbb{N}$, we have that

$$\sum_i \alpha^i_g \delta_{gL,t_{jl}L}a^i_{jl} = 0.$$

Hence,

$$\begin{aligned}\sum_i h_i \triangleright \theta(a_i) &= \sum_{i,g}\alpha^i_g p_g \triangleright (\sum_{j,l}\sum_{h\in t_{jl}L}\frac{1}{|L|}a^i_{jl}h)E_{jl}\\ &= \sum_{i,j,l,g}(\alpha^i_g \frac{1}{|L|}a^i_{jl}\delta_{gL,t_{jl}L}g)E_{jl} = 0,\end{aligned}$$

then (B, θ) is the minimal globalization of the partial action $\cdot : H \otimes A \to A$.

Example 2.74. *Let G be a finite group with $|G| = n$, $A = FMat_{\mathbb{N}}(k)$ and $H = (kG)^*$. Consider the partial H-module algebra structure on A given by $p_g \cdot a = \frac{1}{n}a$. Then (B, θ), where $B = FMat_{\mathbb{N}}(kG)$ and*

$$\begin{aligned}\theta : A &\to B\\ E_{ij} &\mapsto \left(\delta_{(x,y),(i,j)}\frac{1}{n}\sum_{g\in G} g\right)_{x,y} = \left(\frac{1}{n}\sum_{g\in G} g\right)E_{ij},\end{aligned}$$

is a minimal globalization, where the action of H on B is given by $p_h \triangleright (\sum_{g\in G}\alpha_g g)E_{ij} = \alpha_h hE_{ij}$. In fact, this is just the previous example where $L = G$.

Now, we will make some observations about globalizations of S-categorizable partial actions and globalizations of partial actions on categories.

Definition 2.75 ([2]). *Before we proceed, we will present some definitions:*

1. *A linear semicategory is almost a category, i.e., satisfies all axioms needed to be a category, except the existence of a identity for every object.*

2. *A $\mathcal{C}_0$-semicategory is a semicategory which class of objects is $\mathcal{C}_0$.*

3. *A $\mathcal{C}_0$-semifunctor is a functor between $\mathcal{C}_0$-semicategories that is the identity on the objects.*

Definition 2.76 ([2]). *Let $\mathcal{C}$ be a linear category. An ideal of $\mathcal{C}$ is a $\mathcal{C}_0$-subsemicategory ℓ of $\mathcal{C}$ such that ${}_zf_x \circ {}_xl_y \circ {}_yg_w \in {}_z\ell_w$ for every ${}_xl_y \in {}_x\ell_y$, ${}_zf_x \in {}_z\mathcal{C}_x$, ${}_yg_w \in {}_y\mathcal{C}_w$.*

Definition 2.77 ([2]). *A central idempotent in a linear category $\mathcal{C}$ is an idempotent natural transformation e of the identity functor $Id_{\mathcal{C}}$ to itself, i.e., it is a collection $e = \{{}_xe_x\}_{x\in\mathcal{C}_0}$, where each ${}_xe_x \in {}_x\mathcal{C}_x$ is an idempotent endomorphism such that*

$$ {}_ye_y \circ {}_yf_x = {}_yf_x \circ {}_xe_x. $$

Given a central idempotent e, the ideal ℓ of $\mathcal{C}$ generated by e is given by

$$ {}_y\ell_x = {}_ye_y\, {}_y\mathcal{C}_x\, {}_xe_x = {}_ye_y\, {}_y\mathcal{C}_x = {}_y\mathcal{C}_x\, {}_xe_x. $$

The definition of a central idempotent of a category was motivated by the well known fact that if B is an H-module algebra with action $\triangleright$ and A is the ideal of B generated by a central idempotent e, i.e., $A = Be$, then the mapping $g \cdot a = (h \triangleright a)e$ determines a partial action of H on A. Moreover, if $\mathcal{C}$ is a semicategory with finite number of objects, then $B = a(\mathcal{C})$ is a nonunital algebra, and a central idempotent of this algebra must be of the form $\sum_{x\in\mathcal{C}_0} e_{xx}$ such that $e_{xx} \circ {}_xf_y = {}_xf_y \circ e_{yy}$ for every $x, y \in \mathcal{C}_0$, i.e., $e = \{e_{xx}\}$ determines a central idempotent as in the definition above. The definition is written in such way that it works even for categories with infinite number of objects.

Definition 2.78 ([2]). *Let $\mathcal{C}$ be a partial H-module category. A globalization of the partial action is a pair $(\mathcal{B}, F)$ where*

1. *$\mathcal{B}$ is an H-module semicategory over $\mathcal{C}_0$, with action $\triangleright$;*

2. *$F : \mathcal{C} \longrightarrow \mathcal{B}$ is a faithful $\mathcal{C}_0$-semifunctor and $F(\mathcal{C})$ is the ideal of $\mathcal{B}$ generated by the central idempotent $e = \{F({}_x1_x)\}_{x\in\mathcal{C}_0}$;*

3. *$\mathcal{B} = H \triangleright F(\mathcal{C})$;*

4. *F intertwines the partial action on $\mathcal{C}$ and the induced partial action on $F(\mathcal{C})$, i.e., for every ${}_yf_x \in {}_y\mathcal{C}_x$ we have*

$$ F(h \cdot {}_yf_x) = F({}_y1_y)(h \triangleright F({}_yf_x)) = (h \triangleright F({}_yf_x))F({}_x1_x). $$

Proposition 2.79. *Let $\cdot : H \otimes A \longrightarrow A$ be an S-categorizable symmetrical partial action and (B, θ) be an enveloping action. Then the partial action of H on $\mathcal{C}^S(A)$ induced by $\cdot$ has a globalization $(\mathcal{B}, F)$ given by:*

- $\mathcal{B}_0 = \mathcal{C}^S(A)_0$
  $\mathcal{B}(\alpha, \beta) = {}_\beta\mathcal{B}_\alpha = H \triangleright \theta({}_\beta\mathcal{C}^S(A)_\alpha) = H \triangleright \theta(e_\beta A e_\alpha)$;

- $F : \mathcal{C}^S(A) \longrightarrow \mathcal{B}$ *is given by* $F_0(\alpha) = \alpha$ *and* $F_1(e_\beta a e_\alpha) = \theta(e_\beta a e_\alpha)$.

Proof. In fact,

1. $\mathcal{B}$ is an H-module semicategory over $\mathcal{C}^S(A)_0$, with action $\blacktriangleright$ induced by $\triangleright$;

2. F is a faithful functor, because θ is monomorphism, $e = \{\theta(e_\alpha)\}_{e_\alpha \in S}$ is a central idempotent and $F(\mathcal{C}^S(A))$ is an ideal of $\mathcal{B}$ generated by e;

3. $\mathcal{B} = H \triangleright F(\mathcal{C}^S(A))$;

4. $F(h \cdot e_\beta a e_\alpha) = \theta(h \cdot e_\beta a e_\alpha) = \theta(e_\beta)(h \triangleright \theta(e_\beta a e_\alpha)) = F(e_\beta)(h \triangleright F(e_\beta a e_\alpha))$ and $F(h \cdot e_\beta a e_\alpha) = \theta(h \cdot e_\beta a e_\alpha) = (h \triangleright \theta(e_\beta a e_\alpha))\theta(e_\alpha) = (h \triangleright F(e_\beta a e_\alpha))F(e_\alpha)$.

□

Conversely, if the partial action $\triangleright : H \otimes \mathcal{C} \longrightarrow \mathcal{C}$ has a globalization $(\mathcal{B}, F)$, then the partial action $\cdot : H \otimes a(\mathcal{C}) \longrightarrow a(\mathcal{C})$ induced by $\triangleright$ has a globalization given by the pair $(a(\mathcal{B}), \theta)$, where $\theta(({}_y f_x)_{x,y}) = (F({}_y f_x))_{x,y}$ and the action on $a(\mathcal{B})$ is induced by the action on $\mathcal{B}$.

2.9 Morita context

Recall that if A is a partial H-module algebra with unit, then the smash product $A\#H$ is the algebra defined by: $A\#H = A \otimes H$ as vector space, and the product is given by

$$(a \otimes h)(b \otimes k) = \sum a(h_{(1)} \cdot b) \otimes h_{(2)}k.$$

This structure is defined in [13], where Caenepeel and Jansen noticed that $A\#H$ may not have unit, but the subalgebra $\underline{A\#H} = (A\#H)(1_A\#1_H)$,that is called the partial smash product, has unit $1_A\#1_H$. Also in [13], as we recalled in the beginning of this chapter, the authors proved that $A\#H$ is an A-bimodule, and here we highlight that $\underline{A\#H}$ is, in fact, the unital part of $A\#H$ as an A-bimodule. Note that $\underline{A\#H}$ is generated by the elements $\underline{a\#h} = \sum a(h_{(1)} \cdot 1_A) \otimes h_{(2)}$ and $\sum a(h_{(1)} \cdot 1_A)\#h_{(2)} = \underline{a\#h}$.

In [5], Alves and Batista also proved that there exists a strict Morita context between $\underline{A\#H}$ and $B\#H$, whenever (B, θ) is a globalization for the symmetrical partial action on A and H has bijective antipode.

For nonunital partial H-module algebras, we will construct the smash product in the same way: $A\#H = A \otimes H$ as vector space, and the product is given by

$$(a \otimes h)(b \otimes k) = \sum a(h_{(1)} \cdot b) \otimes h_{(2)}k.$$

Now, consider the vector space $\underline{A\#H} = (A\#H)(A\#1_H)$, which corresponds to the partial smash product when A has unit, and bacause of that, we will also call it partial smash product. Note that $\underline{A\#H}$ is also the unital sub A-bimodule of $A\#H$, and it is generated by the elements $\sum a(h_{(1)} \cdot \overline{b)\#h_{(2)}}$.

Lemma 2.80. *Let A be a partial H-module algebra and $\theta : A \to B$ be a quasi-globalization. Then, there is an algebra monomorphism from $\underline{A\#H}$ into $B\#H$.*

Proof. Consider the linear map $\Phi : \underline{A\#H} \longrightarrow B\#H$, given by

$$\Phi(\sum a(h_{(1)} \cdot b)\#h_{(2)}) = \sum \theta(a)(h_{(1)} \triangleright \theta(b))\#h_{(2)}.$$

The calculation is the same as in [4]: Φ is a well defined algebra morphism because of the properties of θ, and Φ is injective because θ is injective. □

Theorem 2.81. *Let H be a Hopf algebra with bijective antipode, $A^2 = A$, A a partial H-module algebra and (B, θ) a quasi-globalization. Then, there exists a strict Morita context between $\underline{A\#H}$ and $B\#H$.*

Proof. Consider the vector spaces

$$\begin{aligned} M &= \Phi(A\#H) = \{\sum_i \theta(a_i)\#h_i;\ a_i \in A,\ h_i \in H\} \\ N &= \{\sum_{i,(h_i)} (h_i)_{(1)} \triangleright \theta(a_i)\#(h_i)_{(2)};\ a_i \in A,\ h_i \in H\}, \end{aligned}$$

i.e., N is the subspace of $B\#H$ generated by the elements $\sum h_{(1)} \triangleright \theta(a)\#h_{(2)}$. Since M and N are subspaces of $B\#H$ and $\underline{A\#H}$ can be seen as a subspace of $B\#H$, by the previous lemma, it is not hard to show that M is an $\underline{A\#H} - B\#H$-bimodule and N is an $B\#H - \underline{A\#H}$-bimodule. And this structures are the same structures of bimodules presented in [4].

For the rest of the Morita context, define the maps

$$\begin{aligned} \tau &: M \otimes_{B\#H} N \longrightarrow \underline{A\#H} \cong \Phi(\underline{A\#H}) \subseteq B\#H \\ \sigma &: N \otimes_{\underline{A\#H}} M \longrightarrow B\#H, \end{aligned}$$

both given by the usual multiplication. This is possible because M, N and $\underline{A\#H}$ are seen as subspaces of $B\#H$, as mentioned before. Since the multiplication on $B\#H$ is associative, we have that both τ and σ are bimodule morphisms, then we only need to prove that they are surjective.

First, we have that $MN \subseteq \Phi(\underline{A\#H})$ because for every $a, b \in A$ and $h, k \in H$ we have

$$\begin{aligned} (\theta(a)\#h)(\sum k_{(1)} \triangleright \theta(b)\#k_{(2)}) &= \sum \theta(a)(h_{(1)}k_{(1)} \triangleright \theta(b))\#h_{(2)}k_{(2)} \\ &= \sum \theta(a)((hk)_{(1)} \triangleright \theta(b))\#(hk)_{(2)} \ \in \ \Phi(\underline{A\#H}). \end{aligned}$$

Since $\sum \theta(a)(h_{(1)} \triangleright \theta(b))\#h_{(2)} = (\theta(a)\#h)(\theta(b)\#1_H)$ and $\theta(b)\#1_H$ lies in N, we have that $MN = \Phi(\underline{A\#H})$. Finally, as $NM \subseteq B\#H$, $h \triangleright \theta(a)\#k$ is a generator of $B\#H$ as vector space and

$$h \triangleright \theta(a)\#k = \sum (h_{(1)} \triangleright \theta(a_1)\#h_{(2)})(\theta(a_2)\#S(h_{(3)})k) \ \in \ NM,$$

where $a = \sum_i a_{1i}a_{2i} = \sum a_1 a_2$ (A is idempotent), we have that $NM = B\#H$. $\square$

Since A is idempotent, we have that both $\underline{A\#H}$ and $B\#H$ are also idempotent, and by [19] we have the following result.

Theorem 2.82. *Under the hypothesis of the previous theorem, we have that the category of the unital and torsionfree left $B\#H$-modules ($B\#H$-mod) and the category of the unital and torsionfree left $\underline{A\#H}$-modules ($\underline{A\#H}$-mod) are equivalent.*

Corollary 2.83. *Under the hypothesis of the previous theorem, if A is a unital algebra, even if $B\#H$ does not have unit, there still exist an equivalence between the categories of $\underline{A\#H}$-modules and the category of the unital and torsionfree left $B\#H$-modules.*

The fact that partial Hopf actions is a generalization of partial group actions, give this theorem greater importance.

2.10 Morita equivalence of partial actions

In this section we will prove that given a partial H-module algebra with local units A with S-globalizable partial action, we have that $\underline{A\#H}$ is Morita equivalent to $\underline{a(\mathcal{C}^S(A))\#H}$, where the partial action on $a(\mathcal{C}^S(A))$ is induced by the S-categorizable partial action on A. In order to do this, we will generalize some results presented in [1].

We recall from [19] that two idempotent rings are Morita equivalent, i.e., its categories of unital and torsionfree modules are equivalent, if and only if there exists a strict Morita context where the modules are unital.

Definition 2.84. *Let A and B be idempotent partial H-module algebras with partial actions $\cdot_A$ and $\cdot_B$, respectively. We will say that $\cdot_A$ and $\cdot_B$ are Morita equivalent partial actions if*

1. *A is Morita equivalent to B, with strict Morita context $(A, B, {}_AM_B, {}_BN_A, \tau, \sigma)$, where M and N are unital bimodules;*

2. *There exists a partial action $\triangleright : H \otimes C \to C$, where C is the context algebra $C = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, such that its restriction to $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ coincides with $\cdot_A$ and $\cdot_B$, respectively.*

We recall that the multiplication of matrices in C comes from the bimodule structures on M and N and from the maps $\tau : M \otimes_B N \to A$ and $\sigma : N \otimes_A M \to B$ of the Morita context. Throughout this subsection, we will denote $\tau(m, n) = (mn)$ and $\sigma(n, m) = (nm)$, for every $m \in M, n \in N$.

Note that if $\cdot_A$ and $\cdot_B$ are Morita equivalent, then the partial action $\triangleright$ on C provides a linear map $\varphi_M : H \otimes M \to M$, $\varphi_M(h \otimes m) = hm$. In fact, since M is a unital left A-module, for any $m \in M$, we have that $m = \sum_i a_i m_i$, with $a_i \in A$, $m_i \in M$ and

$$\begin{aligned} h \triangleright \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} &= \sum_i h \triangleright \begin{pmatrix} a_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix} \\ &= \sum_i \begin{pmatrix} h_{(1)} \cdot_A a_i & 0 \\ 0 & 0 \end{pmatrix} h_{(2)} \triangleright \begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} A & M \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Analogously, since M is a unital right B-module, we have that

$$h \triangleright \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & M \\ 0 & B \end{pmatrix},$$

hence

$$h \triangleright \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & M \\ 0 & B \end{pmatrix} \cap \begin{pmatrix} A & M \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix},$$

and therefore, we define hm by the equation

$$h \triangleright \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & hm \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}.$$

In the same way, the partial action on C provides a linear map $\varphi_N : H \otimes N \to N$, $\varphi_N(h \otimes n) = hn$.

Note also that the linear map φ_M satisfies the following properties:

1. $1_H m = m$;
2. $h(am) = \sum (h_{(1)} \cdot_A a)(h_{(2)} m)$;
3. $h(mb) = \sum (h_{(1)} m)(h_{(2)} \cdot_B b)$;
4. $h(a(km)) = \sum (h_{(1)} \cdot_A a)((h_{(2)} k) m)$;
5. $h(m(k \cdot_B b)) = \sum (h_{(1)} m)(h_{(2)} k \cdot_B b)$,

for every $h, k \in H$, $m \in M$. In fact, for item $2)$, we have that

$$\begin{aligned}
\begin{pmatrix} 0 & h(am) \\ 0 & 0 \end{pmatrix} &= h \triangleright \begin{pmatrix} 0 & am \\ 0 & 0 \end{pmatrix} \\
&= h \triangleright \left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \right) \\
&= \sum \left(h_{(1)} \triangleright \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) \left(h_{(2)} \triangleright \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \right) \\
&= \sum \begin{pmatrix} h_{(1)} \cdot_A a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & h_{(2)} m \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sum (h_{(1)} \cdot_A a)(h_{(2)} m) \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

for every $a \in A$, $m \in M$, $h \in H$. The other items are proved using similar arguments.

Analogously, φ_N satisfies similar properties, and additionally,

1. $h \cdot_A (mn) = \sum (h_{(1)} m)(h_{(2)} n)$;
2. $h \cdot_B (nm) = \sum (h_{(1)} n)(h_{(2)} m)$;
3. $h \cdot_A (m(kn)) = \sum (h_{(1)} m)((h_{(2)} k) n)$;
4. $h \cdot_B (n(km)) = \sum (h_{(1)} n)((h_{(2)} k) m)$,

for every $h, k \in H$, $m \in M$, $n \in N$. In fact, for item $1)$, we have that

$$\begin{aligned}
\begin{pmatrix} h \cdot_A (mn) & 0 \\ 0 & 0 \end{pmatrix} &= h \triangleright \begin{pmatrix} (mn) & 0 \\ 0 & 0 \end{pmatrix} \\
&= h \triangleright \left(\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \right) \\
&= \sum \left(h_{(1)} \triangleright \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \right) \left(h_{(2)} \triangleright \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \right) \\
&= \sum \begin{pmatrix} 0 & h_{(1)} m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ h_{(2)} n & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sum ((h_{(1)} \cdot_A a)(h_{(2)} m)) & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

for every $m \in M$, $n \in N$, $h \in H$. The other items are proved using similar arguments.

Example 2.85. *If A is an idempotent partial H-module algebra with symmetrical partial action $\cdot_A$, then A is a symmetrical partial (A, H)-bimodule with usual A-bimodule structure.*

In the definition of Morita equivalent partial actions, item 2) can be replaced by the existence of compatible partial $(A,H)-(B,H)$ and $(B,H)-(A,H)$-bimodule structures on M and N, respectively.

Proposition 2.86. *Let A and B be idempotent partial H-module algebras with partial actions $\cdot_A$ and $\cdot_B$, respectively, and assume that item 1) of Definition 2.77 holds. The following are equivalent:*

1. *There exist a partial action $\triangleright : H\otimes C \to C$, where C is the context algebra $C = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, such that its restriction to $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ coincides with $\cdot_A$ and $\cdot_B$, respectively.*
2. *M has a partial $(A,H)-(B,H)$-bimodule structure and N has a partial $(B,H)-(A,H)$-bimodule structure such that:*
   - *$h\cdot_A (m(kn)) = \sum (h_{(1)}m)((h_{(2)}k)n)$;*
   - *$h\cdot_B (n(km)) = \sum (h_{(1)}n)((h_{(2)}k)m)$,*

   for every $h\in H$, $m\in M$, $n\in N$.

Proof. The proof of 1) $\Rightarrow$ 2) is in the previous calculations. For 2) $\Rightarrow$ 1), one can consider the linear map $\triangleright : H\otimes C\to C$ given by:

$$h\triangleright \begin{pmatrix} a & m \\ n & x \end{pmatrix} = \begin{pmatrix} h\cdot_A a & hm \\ hn & h\cdot_B x \end{pmatrix},$$

and it defines a partial action of H on C. □

Note that if the hypothesis are satisfied, then we have that $h\cdot_A (mn) = \sum (h_{(1)}m)(h_{(2)}n)$ and $h\cdot_B (nm) = \sum (h_{(1)}n)(h_{(2)}m)$.

Proposition 2.87. *Morita equivalence of partial actions is an equivalence relation.*

Proof. Let A, A', $A"$ be idempotent partial H-module algebras with partial actions $\cdot$, $\cdot_1$, $\cdot_2$, respectively. Suppose that $\cdot$ is Morita equivalent to $\cdot_1$ and $\cdot_1$ is Morita equivalent to $\cdot_2$ with strict Morita contexts (A,A',M,M') and $(A',A",L',L")$, respectively. Now, consider the linear maps

$$\begin{array}{rcl} \tau : (M\otimes_{A'} L')\otimes_{A"} (L"\otimes_{A'} M') & \to & A \\ m\otimes l'\otimes l"\otimes m' & \mapsto & (m((l'l")m')), \end{array}$$

and

$$\begin{array}{rcl} \sigma : (L"\otimes_{A'} M')\otimes_{A} (M\otimes_{A'} L') & \to & A" \\ l"\otimes m'\otimes m\otimes l' & \mapsto & (l"((m'm)l')). \end{array}$$

Since these maps are induced by morphisms of strict Morita contexts, when we consider the natural $(A,A")$-bimodule structure of $M\otimes_{A'} L'$ and the natural $(A",A)$-bimodule structure of $L"\otimes_{A'} M'$, we have the strict Morita context $(A,A",M\otimes_{A'} L', L"\otimes_{A'} M',\tau,\sigma)$. Finally, defining

$$h\triangleright \begin{pmatrix} a & m\otimes l' \\ l"\otimes m' & a" \end{pmatrix} = \sum \begin{pmatrix} h\cdot a & h_{(1)}m\otimes h_{(2)}l' \\ h_{(1)}l"\otimes h_{(2)}m' & h\cdot_2 a" \end{pmatrix},$$

we have that the partial actions $\cdot$ and $\cdot_2$ are Morita equivalent. Since clearly Morita equivalence of partial actions is reflexive and symmetric, it is an equivalence relation. □

Theorem 2.88. *Let A and B be idempotent partial H-module algebras with partial actions $\cdot_A$ and $\cdot_B$, respec.. If $\cdot_A$ is Morita equivalent to $\cdot_B$, then $\underline{A\#H}$ is Morita equivalent to $\underline{B\#H}$.*

Proof. Suppose that the Morita equivalence of A and B is given by the strict Morita context $(A, B, {}_AM_B, {}_BN_A)$. Note that $M \otimes H$ is an $A\#H - B\#H$-bimodule with structure given by:

$$\begin{aligned}(m \otimes h)(x \otimes k) &= \sum m(h_{(1)} \cdot_B x) \otimes h_{(2)}k \\ (a \otimes h)(m \otimes k) &= \sum a(h_{(1)}m) \otimes h_{(2)}k.\end{aligned}$$

Analogously $N \otimes H$ is an $B\#H - A\#H$-bimodule. Now, since $\underline{A\#H}$ and $\underline{B\#H}$ are subalgebras of $A\#H$ and $B\#H$, respectively, we have that $M \otimes H$ is also an $\underline{A\#H} - \underline{B\#H}$-bimodule and $N \otimes H$ is also an $\underline{B\#H} - \underline{A\#H}$-bimodule. Since

$$\begin{aligned}(M \otimes H)(B\#1_H)(\underline{B\#H}) &= (M \otimes H)(B\#1_H)(B\#H)(B\#1_H) \\ &\subseteq (M \otimes H)(B^2\#H)(B\#1_H) \\ &\subseteq (M \otimes H)(B\#1_H),\end{aligned}$$

because $(M \otimes H)(B\#H) \subseteq M \otimes H$, we have that $(M \otimes H)(B\#1_H)$ is an $\underline{A\#H} - \underline{B\#H}$-sub-bimodule of $M \otimes H$, and this sub bimodule structure is given by

$$\begin{aligned}\sum a(h_{(1)} \cdot_A b)\#h_{(2)} \blacktriangleright \sum m(k_{(1)} \cdot_B z) \otimes k_{(2)} &= \sum a(h_{(1)}(bm))(h_{(2)}k_{(1)} \cdot_B z) \otimes h_{(3)}k_{(2)}, \\ \sum m(k_{(1)} \cdot_B z) \otimes k_{(2)} \blacktriangleleft \sum x(h_{(1)} \cdot_B y)\#h_{(2)} &= \sum m(k_{(1)} \cdot_B zx)(k_{(2)}h_{(1)} \cdot_B y) \otimes k_{(3)}h_{(2)},\end{aligned}$$

for every $h, k \in H$, $a, b \in A$, $x, y \in B$, $m \in M$.

Analogously, $(N \otimes H)(A\#H)$ is an $\underline{B\#H} - \underline{A\#H}$-sub-bimodule of $N \otimes H$ with structure given by

$$\begin{aligned}\sum x(h_{(1)} \cdot_B y)\#h_{(2)} \blacktriangleright \sum n(k_{(1)} \cdot_A c) \otimes k_{(2)} &= \sum x(h_{(1)}(yn))(h_{(2)}k_{(1)} \cdot_A c) \otimes h_{(3)}k_{(2)}, \\ \sum n(k_{(1)} \cdot_A c) \otimes k_{(2)} \blacktriangleleft \sum a(h_{(1)} \cdot_A b)\#h_{(2)} &= \sum n(k_{(1)} \cdot_A ca)(k_{(2)}h_{(1)} \cdot_A b) \otimes k_{(3)}h_{(2)},\end{aligned}$$

for every $h, k \in H$, $a, b, c \in A$, $x, y, z \in B$, $n \in N$.

Since M and N are unital bimodules, then $(M \otimes H)(B\#1_H)$ and $(N \otimes H)(A\#1_H)$ are also unital bimodules. Finally, consider the linear maps

$$\begin{aligned}\tau : (M \otimes H)B \otimes_{\underline{B\#H}} (N \otimes H)A &\to \underline{A\#H} \\ (\sum m(h_{(1)} \cdot_B x) \otimes h_{(2)}) \otimes (\sum n(k_{(1)} \cdot_A a) \otimes k_{(2)}) &\mapsto \sum (m(h_{(1)}(xn)))(h_{(2)}k_{(1)} \cdot_A a)\#h_{(3)}k_{(2)},\end{aligned}$$

and

$$\begin{aligned}\sigma : (N \otimes H) \otimes_{\underline{A\#H}} (M \otimes H) &\to \underline{B\#H} \\ (\sum n(k_{(1)} \cdot_A a) \otimes k_{(2)}) \otimes (\sum m(h_{(1)} \cdot_B x) &\mapsto \sum (n(h_{(1)}(am)))(h_{(2)}k_{(1)} \cdot_B x)\#h_{(3)}k_{(2)},\end{aligned}$$

that are well defined because M and N are unital, and are also surjective because the Morita context (A, B, M, N) is strict. Clearly τ and σ are morphisms of bimodules, and they are balanced by construction. Hence $\underline{A\#H}$ is Morita equivalent to $\underline{B\#H}$. $\square$

Consider A a partial H-module algebra with s.l.u. $S = \{e_\lambda\}_{\lambda \in \Lambda}$ with symmetrical S-categorizable partial action. To prove that $\underline{A\#H}$ and $\underline{a(\mathcal{C}^S(A))\#H}$ are Morita equivalent, first we will use the strict Morita context $(A, a(\mathcal{C}^S(A)), \oplus_\lambda Ae_\lambda, \oplus_\lambda e_\lambda A, \tau, \sigma)$, where the elements of $\oplus_\lambda Ae_\lambda$ are seen as row matrices, the elements of $\oplus_\lambda e_\lambda A$ are seen as column matrices, the A-module structures are the usual and the $a(\mathcal{C}^S(A))$-module structures and the morphisms τ and σ are given by matrix multiplication. Actually, this Morita context comes from the Morita equivalence of A and $a(\mathcal{C}^S(A))$ of Section 2.5.

Corollary 2.89. *Let A be a partial H-module algebra with s.l.u. $S = \{e_\lambda\}_{\lambda\in\Lambda}$ with symmetrical S-categorizable partial action and consider the induced symmetrical partial action on $a(\mathcal{C}^S(A))$. Then $\underline{A\#H}$ and $\underline{a(\mathcal{C}^S(A))\#H}$ are Morita equivalent.*

Proof. We only need to consider the linear map

$$h \triangleright \begin{pmatrix} a & (a^\lambda)_\lambda \\ (a_\lambda)_\lambda & (a(\lambda,\alpha))_{\lambda,\alpha} \end{pmatrix} = \begin{pmatrix} h\cdot a & (h\cdot a^\lambda)_\lambda \\ (h\cdot a_\lambda)_\lambda & (h\cdot a(\lambda,\alpha))_{\lambda,\alpha} \end{pmatrix},$$

where $a^\lambda \in Ae_\lambda$, $a_\lambda \in e_\lambda A$ and $a(\lambda,\alpha)\in e_\alpha A e_\lambda$. Since the partial action on A is symmetrical S-categorical, this map is well defined and is straightforward that it determines a partial action on $C = \begin{pmatrix} A & \oplus_\lambda Ae_\lambda \\ \oplus_\lambda e_\lambda A & a(\mathcal{C}^S(A)) \end{pmatrix}$. □

In [2], the authors proved that given a partial H-module k-category $\mathcal{C}$, we have that $\underline{a(\mathcal{C})\#H}$ and $a(\underline{\mathcal{C}\#H})$ are isomorphic, where the partial action on $a(\mathcal{C})$ is induced by the partial action on $\mathcal{C}$. Then, by Theorem 2.41 we have the following result.

Corollary 2.90. *Let A be a partial H-module algebra with s.l.u. S with symmetrical S-categorizable partial action. Then the category of the unital $\underline{A\#H}$ modules is equivalent to the category of the $\underline{\mathcal{C}^S(A)\#H}$ modules.*

In [1], it was proved that every partial group action is Morita equivalent to a globalizable partial action. Here we will show that this holds even for partial Hopf actions: we will present some Morita equivalent partial actions proving that every symmetrical partial action is Morita equivalent to a globalizable partial action. We saw in Section 2.8 that every symmetrical partial action on an associative algebra A has a quasi-globalization, and if $r(A) = 0$ this quasi-globalization is actually a globalization.

Let A be an associative algebra and consider $B = A/r(A)$ and $C = A/l(A)$. In [19], García and Símon highlighted the fact that A, B and C are Morita equivalent algebras. Explicitly, denoting the equivalence class of $a\in A$ in B and in C by $[a]_r$ and $[a]_l$, respectively, we have that the mappings $a[b]_r = ab$ and $[a]_r b = [ab]_r$ provide a structure of right B-module in A and a structure of right A-module in B.

Hence, considering the natural structures of left A-module of A and of left B-module of B, we have that $(A, B, {}_AA_B, {}_BB_A)$ determine a strict Morita context where the bimodules morphisms of the context are given by the same mappings of the module structures.

Analogously, $(C, A, {}_AC_C, {}_CA_A)$ determine a strict Morita context between C and A.

For the partial actions we will assume that A is a partial H-module algebra with symmetrical partial action. Note that if H has bijective antipode, then for every $x\in r(A)$, $a\in A$ and $h\in H$, we have that

$$\begin{aligned} a(h\cdot x) &= \sum (h_{(2)}S^{-1}(h_{(1)})\cdot a)(h_{(3)}\cdot x) \\ &= \sum h_{(2)}\cdot ((S^{-1}(h_{(1)})\cdot a)x) \\ &= 0, \end{aligned}$$

i.e., $h\cdot x\in r(A)$. Hence we can define a partial H-module algebra structure in B given by $h\cdot [a]_r = [h\cdot a]_r$.

For the algebra C, even if H does not have bijective antipode, for every $x\in l(A)$, $a\in A$ and $h\in H$, we have that

$$(h\cdot x)a = \sum (h_{(1)}\cdot x)(h_{(2)}S(h_{(3)})\cdot a)$$

$$\begin{aligned} &= \sum(h_{(1)} \cdot (x(S(h_{(2)}) \cdot a)) \\ &= 0, \end{aligned}$$

i.e., $h \cdot x \in l(A)$. Hence we can define a partial H-module algebra structure in C given by $h \cdot [a]_l = [h \cdot a]_l$.

Clearly these partial actions are symmetrical and put the bimodules ${}_AA_B$ and ${}_BB_A$ (analogously ${}_AC_C$ and ${}_CA_A$) in the hypothesis of Proposition 2.84, i.e., the symmetrical partial action of H on A is Morita equivalent to the induced symmetrical partial action of H on C and to the induced (if H has bijective antipode) partial action on B.

Remember that if A is idempotent, then $r(A/r(A)) = 0 = l(A/l(A))$, hence $l(C) = 0$ and by Section 2.8, every symmetrical partial action on C is globalizable.

Theorem 2.91. *Every symmetrical partial Hopf action on an idempotent algebra is Morita equivalent to a globalizable partial action.*

Considering the results presented in [1], the authors constructed a canonical Morita globalization for a regular partial action and proved that whenever two regular partial G-actions are Morita equivalent, the global actions of its canonical Morita globalizations are also Morita equivalent. Here, we will prove that a similar result holds for partial H-actions.

Proposition 2.92. *Let A and B be two symmetrical partial H-module algebras with Morita equivalent partial actions. Then the global actions of its standard quasi-globalizations are also Morita equivalent.*

Proof. In fact, consider the strict Morita context $(A, B, {}_AM_B, {}_BN_A, \tau, \sigma)$ from the Morita equivalence of the partial H-actions on A and on B. We will denote the two partial actions on A and B by $\cdot$, the elements of A by $a, b, \ldots$, the elements of B by $x, y, \ldots$ and the mappings of the Morita context by $\tau(m, n) = (mn)$ and $\sigma(n, m) = (nm)$, for every $m \in M$, $n \in N$. We will also denote the standard quasi-globalizations by $A' = H \triangleright \varphi_A(A)$ and $B' = H \triangleright \varphi_B(B)$ (note that here we will use the notation $\triangleright$ to represent the actions of both quasi-globalizations, because it won't cause any trouble in the calculation).

We know that the Morita equivalence of the partial actions yields two linear maps $H \otimes M \to M$ and $H \otimes N \to N$, then we can define linear maps $\varphi_M : M \to Hom(H, M)$ and $\varphi_N : N \to Hom(H, N)$ by $\varphi_M(m)(h) = hm$ and $\varphi_N(n)(h) = hn$, respectively, for all $m \in M, n \in N, h \in H$.

Also, as for partial H-module algebras, the vector spaces $Hom(H, M)$ and $Hom(H, N)$ are H-module algebras with H-module structures given by $(k \triangleright f)(h) = f(hk)$ (again, the same notation for the action), particularly, $(k \triangleright \varphi_M(m))(h) = (hk)m$.

Now, we will consider the vector spaces $M' = H \triangleright \varphi_M(M)$ and $N' = H \triangleright \varphi_N(N)$ and prove that M' is an (A', B')-bimodule and N' is an (B', A')-bimodule. In fact, define

$$(h \triangleright \varphi_A(a) \blacktriangleright k \triangleright \varphi_M(m))(l) = \sum(h \triangleright \varphi_A(a)(l_{(1)}))(k \triangleright \varphi_M(m)(l_{(2)})) = \sum(l_{(1)}h \cdot a)((l_{(2)}k)m).$$

Note that if $\sum h^i \triangleright \varphi_A(a^i) = 0$, then $\sum lh^i \cdot a = 0$ for every $l \in H$, hence $\sum h^i \triangleright \varphi_A(a^i) \blacktriangleright k \triangleright \varphi_M(m) = 0$. Also, we have that

$$\begin{aligned} (h \triangleright \varphi_A(a) \blacktriangleright k \triangleright \varphi_M(m))(l) &= \sum(l_{(1)}h \cdot a)((l_{(2)}k)m) \\ &= \sum lh_{(1)}(a((S(h_{(2)})k)m)) \\ &= (\sum h_{(1)} \triangleright \varphi_M(a((S(h_{(2)})k)m)))(l), \end{aligned}$$

i.e., $h \triangleright \varphi_A(a) \blacktriangleright k \triangleright \varphi_M(m) \in M'$, and we proved that $\blacktriangleright$ is well defined. Moreover, since $\blacktriangleright$ works like the "convolution product" together with the A-module structure of M, it follows that $\blacktriangleright$ is actually an action. Now, instead of using the notations φ_A, φ_B, φ_M and φ_N we will only write φ. We can also use the previous calculation to show that the mappings

$$\begin{aligned} \tau'(h \triangleright \varphi(m), k \triangleright \varphi(n))(l) &= \sum \tau((l_{(1)}h)m, (l_{(2)}k)n), \\ \sigma'(k \triangleright \varphi(n), h \triangleright \varphi(m))(l) &= \sum \sigma((l_{(2)}k)n, (l_{(1)}h)m), \end{aligned}$$

define linear maps

$$\begin{aligned} \tau' : M' \otimes N' &\to A' \\ \sigma' : N' \otimes M' &\to B', \end{aligned}$$

that are well defined balanced bimodule maps. Additionally, they are surjective because τ and σ are surjective and M and N are unital bimodules. In other words, $(A', B', M', N', \tau', \sigma')$ is a strict Morita context. Finally, with the considered H-actions on M' and N', we can easily see that the H-actions on A' and on B' are Morita equivalent. □

Actually, in [1], Abadie et al. proved that the global actions of the globalizations of two Morita equivalent partial group actions are also Morita equivalent. Until now, we proved that the actions of the minimal quasi-globalizations of two Morita equivalent partial actions are also Morita equivalent, and we know from [4] that for the case of unital algebras, every globalization of a partial kG-action is minimal. Here we will show that even if we consider nonunital algebras, under some assumptions, this result also holds.

Lemma 2.93. *Let A be an associative partial kG-module algebra, where G is a finite group, and let (B, θ) be a globalization. Then:*

1. *if A is unital, then B has local units;*
2. *if A has local units, then B has local units;*
3. *if A is s-unital, then B is s-unital;*
4. *if A i idempotent, then B is idempotent;*
5. *if $r(A) = 0$, then $r(B) = 0$ if and only if the globalization is minimal.*

Proof. We begin by proving item 3), because the proof of item 1) and 2) is analogous. Assume that A is s-unital, then for every $h \triangleright \theta(a) \in B$, there exist $x \in A$ such that $xa = ax = a$, hence

$$\begin{aligned} (h \triangleright \theta(x))(h \triangleright \theta(a)) &= h \triangleright \theta(a) \\ &= (h \triangleright \theta(a))(h \triangleright \theta(x)). \end{aligned}$$

We already proved in Section 2.9 that item 4) holds for any Hopf algebra. Now, suppose that $r(A) = 0$ and that $\sum_i kh_i \cdot a_i = 0$ for every $k \in G$. Then, for every $l \triangleright \theta(b) \in B$, we have that

$$(l \triangleright \theta(b))(\sum_i h_i \triangleright \theta(a_i)) = \sum_i l \triangleright \theta(b(l^{-1}h_i \cdot a_i)) = 0,$$

i.e., $B(\sum_i h_i \triangleright \theta(a_i)) = 0$, and if $r(B) = 0$, then $\sum_i h_i \triangleright \theta(a_i) = 0$ and we conclude that the globalization is minimal. Conversely, if the globalization is minimal and $B(\sum_i h_i \triangleright \theta(a_i)) = 0$, then for every $l \triangleright \theta(b) \in B$ we have that $\sum_i l \triangleright \theta(b(l^{-1}h_i \cdot a_i)) = 0$. Hence

$$0 = l^{-1} \triangleright \sum_i l \triangleright \theta(b(l^{-1}h_i \cdot a_i))$$

$$= \sum_i \theta(b(l^{-1}h_i \cdot a_i)),$$

and since this holds for every $l \in G$, $b \in A$, $r(A) = 0$ and θ is injetive, then $\sum_i l^{-1}h_i \cdot a_i = 0$ for every $l \in G$, and since the globalization is minimal, we must have that $\sum_i h_i \triangleright \theta(a_i) = 0$, i.e., $r(B) = 0$. $\square$

Corollary 2.94. *Let A be an associative partial kG-module algebra, where G is a finite group, and let (B, θ) be a globalization. Then, if A is at least s-unital, the globalization is minimal.*

Now, we will associate the definitions of Morita equivalence of partial kG-actions and of Morita equivalence of partial G-actions, as was done for partial actions.

Definition 2.95 ([1]). *Let*

$$\alpha = \{\alpha_g : D_{g^{-1}} \to D_g\} \text{ and } \alpha' = \{\alpha'_g : D'_{g^{-1}} \to D'_g\}$$

be regular partial actions of G on algebras A and A', respectively. We say that α is Morita equivalent to α' if:

1. *A is Morita equivalent to A', with strict Morita context $(A, A', {}_AM_{A'}, {}_{A'}M'_A, \tau, \sigma)$, where M and M' are unital bimodules such that $M'D_gM = D'_g$ for any $g \in G$;*
2. *There exists a product partial action $\theta = \{\theta_g : E_{g^{-1}} \to E_g\}$ of G on C, where C is the context algebra $C = \begin{pmatrix} A & M \\ M' & A' \end{pmatrix}$, such that θ restricted to $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}$, is α and α', respectively.*

Recall from [1] that "θ restrict to $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ is α" means that, for every $g \in G$

$$E_g \cap \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D_g & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\theta_g \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_g(a) & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall a \in D_{g^{-1}}.$$

Remember that whenever A is idempotent and α is a regular partial action of G on an idempotent algebra A and there exist α-projections $p_g : A \to D_g$, then the linear map $\cdot : kG \otimes A \to A$ defined by $g \cdot a = \alpha_g(p_{g^{-1}}(a))$ is a symmetrical partial action (Proposition 2.29).

Lemma 2.96. *Let*

$$\alpha = \{\alpha_g : D_{g^{-1}} \to D_g\} \text{ and } \alpha' = \{\alpha'_g : D'_{g^{-1}} \to D'_g\}$$

be Morita equivalent regular partial actions of G on algebras A and A', respectively, with product partial action on C given by $\theta = \{\theta_g : E_{g^{-1}} \to E_g\}$. Suppose that there exist algebra morphisms $p_g : A \to D_g$, $p'_g : A' \to D'_g$ and $P_g : C \to E_g$ that are α-projections, α'-projections and θ-projections, respectively, and that every P_g restricted to the copy of A and A' is p_g and p'_g, respectively, i.e.

$$P_g \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p_g(a) & 0 \\ 0 & 0 \end{pmatrix},$$

$$P_g \begin{pmatrix} 0 & 0 \\ 0 & a' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & p'_g(a') \end{pmatrix}.$$

Then the induced partial actions of kG on A and A' are also Morita equivalent.

Proof. We only need to prove that the induced partial action of kG on the context algebra restricted to $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}$ are, respectively, the induced partial actions of kG on A and A'. In fact, in the first case, we have that

$$\begin{aligned} g \cdot \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} &= \theta_g P_{g^{-1}} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \\ &= \theta_g \begin{pmatrix} p_{g^{-1}}(a) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_g(p_{g^{-1}}(a)) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} g \cdot a & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Analogously, the induced partial action of kG on the context algebra restricted to $\begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}$ is the induced partial action of kG on A'. □

Lemma 2.97. *Let A and A' be idempotent partial kG-module algebras with Morita equivalent symmetrical partial actions. Then the induced partial actions of G on A and A' are Morita equivalent.*

Proof. Since A is idempotent, we know that the induced partial action of G on C is regular, hence by [1] it is a product partial action.

Now, consider the Morita context (A, A', M, M') given by the Morita equivalence of the symmetrical partial actions of kG on A and on A'. In order to show that the restriction properties holds, note that

$$g \cdot g^{-1} \cdot M = g \cdot g^{-1} \cdot AM = (g \cdot g^{-1} \cdot A)M = M(g \cdot g^{-1} \cdot A'),$$

$$g \cdot g^{-1} \cdot M' = g \cdot g^{-1} \cdot A'M' = (g \cdot g^{-1} \cdot A')M' = M'(g \cdot g^{-1} \cdot A),$$

$D_g = g \cdot g^{-1} \cdot A$ and $D'_g = g \cdot g^{-1} \cdot A'$. Then, since MD'_gM' is generated by elements of the form $m(g \cdot g^{-1} \cdot a)m'$, we have that

$$\begin{aligned} m(g \cdot g^{-1} \cdot a)m' &= (g \cdot (g^{-1}m)(g^{-1} \cdot a))m' \\ &= g \cdot (g^{-1}m)(g^{-1} \cdot a)(g^{-1}m') \\ &= g \cdot g^{-1} \cdot (\tau(m, am')) \in D_g. \end{aligned}$$

Moreover, for every $g \cdot g^{-1} \cdot a \in D_g$, there exist $m_i's \in M$ and $n_i's \in M'$ such that $a = \sum_i \tau(m_i, n_i)$, and as M is a unital right A'-module, we have that every m_i is of the form $m_i = \sum_j m_{ij} a'_{ij}$, hence

$$\begin{aligned} g \cdot g^{-1} \cdot a &= \sum_{i,j} g \cdot g^{-1} \cdot \tau(m_{ij} b_{ij}, n_i) \\ &= \sum_{i,j} \tau(g(g^{-1} m_{ij})(g \cdot g^{-1} \cdot b_{ij}), g(g^{-1} n_i)) \in MD'_gM'. \end{aligned}$$

Then $MD'_gM' = D_g$ and analogously $M'D_gM = D'_g$. Hence

$$E_g = C(g \cdot g^{-1} \cdot C)$$

$$
\begin{aligned}
&= \begin{pmatrix} A & M \\ M' & A' \end{pmatrix} \begin{pmatrix} g \cdot g^{-1} \cdot A & g \cdot g^{-1} \cdot M \\ g \cdot g^{-1} \cdot M' & g \cdot g^{-1} \cdot A' \end{pmatrix} \\
&= \begin{pmatrix} A & M \\ M' & A' \end{pmatrix} \begin{pmatrix} D_g & D_g M \\ D'_g M' & D'_g \end{pmatrix} \\
&= \begin{pmatrix} D_g & D_g M \\ D'_g M' & D'_g \end{pmatrix},
\end{aligned}
$$

then

$$
\begin{aligned}
E_g \cap \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} D_g & 0 \\ 0 & 0 \end{pmatrix}, \\
E_g \cap \begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & D'_g \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
\theta_g \begin{pmatrix} g^{-1} \cdot g \cdot a & 0 \\ 0 & 0 \end{pmatrix} &= g \cdot g^{-1} \cdot g \cdot \begin{pmatrix} g^{-1} \cdot g \cdot a & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} g \cdot g^{-1} \cdot g \cdot g^{-1} \cdot g \cdot a & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} g \cdot g^{-1} \cdot a & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \alpha_g(g^{-1} \cdot g \cdot a) & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Hence the restriction of the induced partial action of G on C to $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ is the induced partial action of G on A. Analogously, the induced partial action of G on C restrict to $\begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}$ is the induced partial action of G on A'. □

Another interesting fact, is the relation of the concept of α-modules, presented in [1], and the concept of partial (A, kG)-modules, presented in [14].

Definition 2.98 ([1]). *Let $\alpha = \{\alpha_g : D_{g^{-1}} \to D_g\}$ be a regular partial action of G on an algebra A. By a left α-module we mean a left unital A-module M, together with a family of linear isomorphisms $\gamma_g : D_{g^{-1}}M \to D_gM$, such that the following properties are satisfied for every $g, h \in G$:*

1. *γ_1 is the natural isomorphism $M \to M$;*
2. *$\gamma_g \circ \gamma_h(m) = \gamma_{gh}$, for all $m \in D_{h^{-1}}D_{(gh)^{-1}}M$;*
3. *$\gamma_g(am) = \alpha_g(a)\gamma_g(m)$, for all $a \in D_{g^{-1}}, m \in D_{g^{-1}}M$.*

Definition 2.99. *Let M be a left α-module with structure given by $\gamma_g : D_{g^{-1}}M \to D_gM$, where α is a partial action of G on an algebra A, with α-projections $\{p_g\}_{g\in G}$. We call γ-projections a family of algebra epimorphisms $q_g : M \to D_gM$, such that:*

1. *$q_1 : M \to M$ is the identity;*
2. *$q_g^2 = q_g$;*

3. $q_g q_h = q_h q_g$;
4. $q_g \gamma_k = \gamma_k q_{k^{-1}g} q_k^{-1}$;
5. $q_g(am) = p_g(a) q_g(m)$.

Proposition 2.100. *Let $\alpha = \{\alpha_g : D_{g^{-1}} \to D_g\}$ be a regular partial action of G on an algebra A and M be a left α-module. Suppose that α has α-projections $\{p_g\}_{g\in G}$ and γ has γ-projections $\{q_g\}_{g\in G}$, then M is a symmetrical partial (A, kG)-module.*

Proof. We only need to define the linear map $kG \otimes M \to M$ by $g \otimes m \mapsto (gm = \gamma_g(q_{g^{-1}}(m)))$. The calculation is similar to the case for partial actions. □

Proposition 2.101. *Let M be a partial (A, kG)-module where A is a symmetrical partial kG-module algebra. Then M is a left α-module, where α is the induced partial action of G on A.*

Proof. We define $\gamma_g(m) = gm$, for every $m \in D_{g^{-1}}M$, where $D_g = g \cdot g^{-1} \cdot A$. In order to show that this define an α-module structure on M, we use calculations similar to the case of partial actions. Note also that

$$g(g^{-1}(am)) = (g \cdot g^{-1} \cdot a)m,$$

hence γ has γ-projections $q_g(m) = g(g^{-1}m)$. □

Chapter 3

Partial coactions

Now that partial actions on algebras are well understood, we will define the concept of partial coactions on an algebra in such way that when it has unit, this is a partial coaction, in the sense presented in [5].

Definition 3.1 ([5])**.** *Let H be a Hopf algebra and A an algebra with unit. A linear map $\rho : A \longrightarrow A \otimes H$ is called a partial coaction if the following holds:*

1. $(I \otimes \varepsilon_H)\rho(a) = a$;
2. $\rho(ab) = \rho(a)\rho(b)$;
3. $(\rho \otimes I)\rho(a) = (\rho(1_A) \otimes 1_H)(I \otimes \Delta)\rho(a)$,

for every $a, b \in A$. In this case, A is called a partial H-comodule algebra. If, additionally, $(\rho \otimes I)\rho(a) = [(I \otimes \Delta_H)\rho(a)](\rho(1_A) \otimes 1_H)$ for every $a \in A$, then ρ is called a symmetrical partial coaction.

We will use the notation $\rho(a) = \sum a^{(0)} \otimes a^{(1)}$. Moreover, we can show that if H is finite dimensional, the linear map $\rightharpoonup$: $H^* \otimes A \longrightarrow A$, that sends $f \otimes a$ to $f \rightharpoonup a = \sum a^{(0)} f(a^{(1)})$, is a partial action if and only if ρ is a partial coaction.

3.1 Partial coactions on non unital algebras

Definition 3.2. *Let H be a Hopf algebra. A will be called a partial H-comodule algebra if there exists a linear map $\rho : A \longrightarrow A \otimes H$ that satisfies, for every $a, b \in A$,*

1. $(I \otimes \varepsilon_H)\rho(a) = a$;
2. $(\rho \otimes I)((b \otimes 1_H)\rho(a)) = (\rho(b) \otimes 1_H)(I \otimes \Delta_H)\rho(a)$;

If, additionally, $(\rho \otimes I)(\rho(a)(b \otimes 1_H)) = [(I \otimes \Delta_H)\rho(a)](\rho(b) \otimes 1_H)$ for every $a, b \in A$, ρ will be called a symmetrical partial coaction.

Note that if A is a partial H-comodule algebra, then we have that for every $a, b \in A$,

$$\begin{array}{ccc}
\sum (ab^{(0)})^{(0)} \otimes (ab^{(0)})^{(1)} \otimes b^{(1)} & = & \sum a^{(0)} b^{(0)} \otimes a^{(1)} (b^{(1)})_{(1)} \otimes (b^{(1)})_{(2)} \\
& \Downarrow^{I \otimes I \otimes \varepsilon} & \\
\sum (ab^{(0)})^{(0)} (ab^{(0)})^{(1)} \varepsilon(b^{(1)}) & = & \sum a^{(0)} b^{(0)} \otimes a^{(1)} (b^{(1)})_{(1)} \varepsilon((b^{(1)})_{(2)}) \\
& \Downarrow &
\end{array}$$

$$\sum (ab)^{(0)} \otimes (ab)^{(1)} \quad = \quad \sum a^{(0)}b^{(0)} \otimes a^{(1)}b^{(1)},$$

i.e., $\rho(ab) = \rho(a)\rho(b)$. Which means that if A is a partial H-comodule algebra and A has unit, then A is a partial H-comodule algebra in the usual sense.

Proposition 3.3. *Let H be a finite dimensional Hopf algebra.*

1. *If A is a partial H-comodule algebra with (symmetrical) partial coaction ρ, then A is a partial H^*-module algebra with (symmetrical) partial action $\rightharpoonup$: $H^* \otimes A \longrightarrow A$ given by $f \rightharpoonup a = \sum a^{(0)} f(a^{(1)})$;*

2. *If A is a partial H-module algebra with (symmetrical) partial action $\cdot$, then A is a partial H^*-comodule algebra with (symmetrical) partial coaction $\rho : A \longrightarrow A \otimes H^*$ given by $\rho(a) = \sum_i d_i \cdot a \otimes p_i$, where $dim\, H = n$, $\{d_i\}_{i=1}^n$ is a basis for H and $\{p_i\}_{i=1}^n$ is its dual basis in H^*.*

Theorem 3.4. *Let H be a commutative Hopf algebra. If A and B are both (symmetrical) partial H-comodule algebras, then $A \otimes B$ is a (symmetrical) partial H-comodule algebra via $\rho(a \otimes b) = \sum a^{(0)} \otimes b^{(0)} \otimes a^{(1)}b^{(1)}$.*

Proof. We ask for H to be commutative because, for every $a, b \in A$, $x, y \in B$,

$$\begin{aligned} \rho(a \otimes x)\rho(b \otimes y) &= \sum [a^{(0)} \otimes x^{(0)} \otimes a^{(1)}x^{(1)}][b^{(0)} \otimes y^{(0)} \otimes b^{(1)}y^{(1)}] \\ &= \sum a^{(0)}b^{(0)} \otimes x^{(0)}y^{(0)} \otimes a^{(1)}x^{(1)}b^{(1)}y^{(1)}, \end{aligned}$$

and

$$\begin{aligned} \rho(ab \otimes xy) &= \sum (ab)^{(0)} \otimes (xy)^{(0)} \otimes (ab)^{(1)}(xy)^{(1)} \\ &= \sum a^{(0)}b^{(0)} \otimes x^{(0)}y^{(0)} \otimes a^{(1)}b^{(1)}x^{(1)}y^{(1)}, \end{aligned}$$

hence if H is commutative, then $\rho(a \otimes x)\rho(b \otimes y) = \rho(ab \otimes xy)$. □

3.2 Partial coaction on algebras with local units

In this subsection, we will work with algebras with local units. For this, we will present an equivalent definition of partial coaction.

Proposition 3.5. *Let H be a Hopf algebra, A an algebra with local units and $S = \{e_\lambda\}_{\lambda \in \Lambda}$ an s.l.u. for A. The linear map $\rho : A \longrightarrow A \otimes H$ is a partial coaction if and only if, for every $a, b \in A$,*

1. *$(I \otimes \varepsilon_H)\rho(a) = a$;*

2. *$\rho(ab) = \rho(a)\rho(b)$;*

3. *$(\rho \otimes I)\rho(a) = (\rho(e_\alpha) \otimes 1_H)(I \otimes \Delta_H)\rho(a)$, for every $e_\alpha \in S$ such that $(e_\alpha \otimes 1_H)\rho(a) = \rho(a)$.*

The partial coaction ρ is symmetrical if and only if, additionally,

$$(\rho \otimes I)\rho(a) = [(I \otimes \Delta_H)\rho(a)](\rho(e_\beta) \otimes 1_H)$$

for every $e_\beta \in S$ such that $\rho(a)(e_\beta \otimes 1_H) = \rho(a)$.

Note that, as for partial actions, if A is a partial H-comodule algebra with local units, satisfying the conditions $1) - 3)$ for some s.l.u. S, then the same conditions holds for every s.l.u. T of A.

Definition 3.6. *If A is a partial H-comodule algebra with local units with (symmetrical) partial coaction ρ, and for some s.l.u. $S = \{e_\lambda\}_{\lambda\in\Lambda}$ we have that $\rho(e_\alpha) \in e_\alpha A e_\alpha \otimes H$, then, as for partial actions, we have that $\rho(e_\alpha A e_\beta) \subseteq e_\alpha A e_\beta \otimes H$. In this case, ρ will be called a (symmetrical) S-categorizable partial coaction.*

Since we do not find any mention of partial Hopf comodule categories in the literature, we will introduce them in the following section.

3.3 Partial comodule categories

Before we suggest the definition of a partial coaction on a linear category, we will remind the definition of coaction.

Definition 3.7 ([3],[23],[24]). *Let H be a Hopf algebra. A linear category $\mathcal{C}$ is an H-comodule category if there exist a family of linear maps $\rho = \{\rho_{(x,y)} : {}_y\mathcal{C}_x \to {}_y\mathcal{C}_x \otimes H\}_{x,y\in\mathcal{C}_0}$ such that, for every ${}_yf_x \in {}_y\mathcal{C}_x$, ${}_zg_y \in {}_z\mathcal{C}_y$,*

1. $\rho_{(x,z)}(gf) = \rho_{(y,z)}(g)\rho_{(x,y)}(f)$;
2. $\rho_{(x,x)}({}_x1_x) = {}_x1_x \otimes 1_H$;
3. $(I \otimes \Delta)\rho_{(x,y)} = (\rho_{(x,y)} \otimes I)\rho_{(x,y)}$;
4. $(I \otimes \varepsilon)\rho_{(x,y)} = \rho_{(x,y)}$.

Example 3.8. *Consider $A = Mat_{n\times n}(k)$ with a good G-gradding, where G is a finite group. Then, the linear category $\mathcal{C}$ with n objects and ${}_i\mathcal{C}_j = k$ is an H-comodule category with coaction $\rho_{(i,j)}(\alpha) = \alpha \otimes g$ iff $E_{ij} \in A_g$.*

3.3.1 Partial coactions

Definition 3.9. *Let H be a Hopf algebra. A linear category $\mathcal{C}$ is a partial H-comodule category if there exist a family of linear maps $\rho = \{\rho_{(x,y)} : {}_y\mathcal{C}_x \to {}_y\mathcal{C}_x \otimes H\}_{x,y\in\mathcal{C}_0}$ such that, for every ${}_yf_x \in {}_y\mathcal{C}_x$, ${}_zg_y \in {}_z\mathcal{C}_y$,*

1. $(I \otimes \varepsilon)\rho_{(x,y)} = \rho_{(x,y)}$
2. $\rho_{(x,z)}(gf) = \rho_{(y,z)}(g)\rho_{(x,y)}(f)$;
3. $(\rho_{(x,y)} \otimes I)\rho_{(x,y)}(f) = (\rho_{(y,y)}({}_y1_y) \otimes 1_H)(I \otimes \Delta)\rho_{(x,y)}(f)$.

In this case, ρ will be called a partial coaction. The partial coaction ρ will be called symmetrical if, additionally, $(\rho_{(x,y)} \otimes I)\rho_{(x,y)}(f) = (I \otimes \Delta)\rho_{(x,y)}(f)[\rho_{(x,x)}({}_x1_x) \otimes 1_H]$.

Example 3.10. *Every H-comodule category is a partial H-comodule category.*

Example 3.11. *If A is a (symmetrical) partial H-comodule unital algebra, then the unitary category associated with A is a (symmetrical) partial H-comodule category.*

As for partial actions, if A is a partial H-comodule algebra with local units with (symmetrical) S-categorizable partial coaction, then the linear category $\mathcal{C}^S(A)$ is a partial H-comodule category. Conversely, if $\mathcal{C}$ is a (symmetrical) partial H-comodule category, then $a(\mathcal{C})$ is a (symmetrical) partial H-comodule algebra with local units.

As for algebras, we have the following results that we won't prove, because they follow from calculations analogous to the case for unital algebras.

Proposition 3.12. *Let H be a finite dimensional Hopf algebra.*

1. *If $\mathcal{C}$ is a partial H-module category, then $\mathcal{C}$ is a partial H^*-comodule category;*
2. *If $\mathcal{C}$ is a partial H-comodule category, then $\mathcal{C}$ is a partial H^*-module category.*

Theorem 3.13. *Let H be a commutative Hopf algebra. If $\mathcal{A}$ and $\mathcal{B}$ are both (symmetrical) partial H-comodule categories, then $\mathcal{A} \otimes \mathcal{B}$ is a (symmetrical) partial H-module category.*

Corollary 3.14. *Let H be a commutative Hopf algebra. If A and B are (symmetrical) partial H-comodule unital algebras, then $A \otimes B$ is a (symmetrical) partial H-comodule unital algebra.*

3.3.2 Globalization

In this subsection we will prove that there always exist a globalization of a symmetrical partial coaction on a linear category.

Definition 3.15. *Let H be a Hopf algebra. A linear semicategory $\mathcal{H}$ is an H-comodule semicategory if there exist a family of linear maps $\rho = \{\rho_{(x,y)} : {}_y\mathcal{H}_x \to {}_y\mathcal{H}_x \otimes H\}_{x,y \in \mathcal{H}_0}$ such that, for every ${}_yf_x \in {}_y\mathcal{H}_x$, ${}_zg_y \in {}_z\mathcal{H}_y$,*

1. *$\rho_{(x,z)}(gf) = \rho_{(y,z)}(g)\rho_{(x,y)}(f)$;*
2. *$(I \otimes \Delta)\rho_{(x,y)} = (\rho_{(x,y)} \otimes I)\rho_{(x,y)}$;*
3. *$(I \otimes \varepsilon)\rho_{(x,y)}(f) = f$.*

Proposition 3.16. *Let $\mathcal{H}$ be an H-comodule semicategory with coaction γ. Suppose $\mathcal{C}$ is a subcategory that is an ideal of $\mathcal{H}$, then $\mathcal{C}$ is a symmetrical partial H-comodule category with partial coaction ρ given by*

$$\begin{aligned}\rho_{(x,y)}(f) &= (1_{{}_y\mathcal{C}_y} \otimes 1_H)\gamma_{(x,y)}(f) \\ &= \gamma_{(x,y)}(f)(1_{{}_x\mathcal{C}_x} \otimes 1_H),\end{aligned}$$

for every $f \in {}_y\mathcal{C}_x$.

Following the idea presented in [5], we will prove that every partial coaction on a linear category has a globalization.

Definition 3.17. *Let H be a Hopf algebra and $\mathcal{C}$ a partial H-comodule category with symmetrical partial coaction ρ. A globalization, or enveloping coaction, of ρ is a pair $(\mathcal{B}, F)$, where*

1. *$\mathcal{B}$ is an H-comodule semicategory, with coaction γ;*
2. *$F : \mathcal{C} \to \mathcal{B}$ is a faithful $\mathcal{C}_0$-semifunctor and $F(\mathcal{C})$ is the ideal of $\mathcal{B}$ generated by the central idempotent $e = \{F({}_x1_x)\}_{x \in \mathcal{C}_0}$;*

3. *$\mathcal{B}$ is generated by $F(\mathcal{C})$ as H-comodule semicategory, i.e., for every $x, y \in \mathcal{C}_0 = \mathcal{B}_0$, the set ${}_y\mathcal{B}_x$ is generated as vector space by monomials $w^k \cdots w^1$, where $w^i = h_i^* \rightharpoonup f^i = (I \otimes h^*)\gamma(f^i)$, with $h_i^* \in H^*$, $f^i \in {}_{x^{i+1}}\mathcal{C}_{x^i}$ and $x^1 = x$, $x^k = y$.*

4. *F intertwines the partial coaction on $\mathcal{C}$ and the induced partial coaction on $F(\mathcal{C})$, i.e., for every $f \in {}_y\mathcal{C}_x$, we have that $(F \otimes I)\rho_{(x,y)}(f) = (F({}_y1_y) \otimes 1_H)\gamma_{(x,y)}(F(f)) = \gamma_{(x,y)}(F(f))(F({}_x1_x) \otimes 1_H)$.*

Let $\mathcal{C}$ be a partial H-comodule category with symmetrical partial coaction ρ.

Consider the category $\mathcal{A} = \mathcal{C} \otimes H$, where ${}_y\mathcal{A}_x = {}_y\mathcal{C}_x \otimes H$. Note that $\mathcal{A}$ is a H-comodule category with coaction $\delta_{(x,y)} = I_{{}_y\mathcal{C}_x} \otimes \Delta$, for every $x, y \in \mathcal{C}_0$. Let $\mathcal{B}$ be the subsemicategory generated by $\rho(\mathcal{C})$, where ρ is seen as a $\mathcal{C}_0$-semifunctor. In other words, for every $x, y \in \mathcal{C}_0$, the set ${}_y\mathcal{B}_x$ is generated as vector space by monomials $w^k \cdots w^1$, where $w^i = h_i^* \rightharpoonup f^i$, with $h_i^* \in H^*$, $f^i \in {}_{x^{i+1}}\mathcal{C}_{x^i}$ and $x^1 = x$, $x^k = y$. Note that, in this case, $\rightharpoonup$ is given by

$$\begin{aligned} h^* \rightharpoonup \rho_{(x,y)}(f) &= (I_{{}_y\mathcal{C}_x} \otimes h^*)\delta_{x,y}(\rho_{(x,y)}(f)) \\ &= (I_{{}_y\mathcal{C}_x} \otimes h^*)(I_{{}_y\mathcal{C}_x} \otimes \Delta)\rho_{(x,y)}(f) \\ &= \sum f^{(0)} \otimes (f^{(1)})_{(1)} h^*((f^{(1)})_{(2)}). \end{aligned}$$

Theorem 3.18. *The pair $(\mathcal{B}, \rho)$, as above, is a globalization of the symmetrical partial coaction ρ.*

Proof. Since the other axioms of globalization hold, by hypothesis, and ρ is a faithful functor because of the property $1)$ of partial coaction, we only need to verify that $\rho(\mathcal{C})$ is the ideal of $\mathcal{B}$ generated by the central idempotent $e = \{\rho_{(x,x)}({}_x1_x)\}_{x \in \mathcal{C}_0}$. In fact, let $w = w^k \cdots w^1$ be a monomial in ${}_y\mathcal{B}_x$. First, we must prove that $\rho_{(y,y)}({}_y1_y)w$ and $w\rho_{(x,x)}({}_x1_x)$ lie in $\rho_{(x,y)}({}_y\mathcal{C}_x)$. In fact, we will prove that $\rho_{(y,z)}(f)w \in \rho_{(x,z)}({}_z\mathcal{C}_x)$ for every $f \in {}_z\mathcal{C}_y$ and every $z \in \mathcal{C}_0$. Let $h^* \in H^*$, $g \in {}_y\mathcal{C}_x$ and $f \in {}_z\mathcal{C}_y$, then

$$\begin{aligned} \rho_{(y,z)}(f)(h^* \rightharpoonup g) &= \sum f^{(0)} g^{(0)} \otimes f^{(1)} (g^{(1)})_{(1)} h^*((g^{(1)})_{(2)}) \\ &= \sum f^{(0)} {}_y1_y^{(0)} g^{(0)} \otimes f^{(1)} {}_y1_y^{(1)} (g^{(1)})_{(1)} h^*((g^{(1)})_{(2)}) \\ &= \sum f^{(0)} g^{(0)(0)} \otimes f^{(1)} g^{(0)(1)} h^*(g^{(1)}) \\ &= \sum (fg^{(0)})^{(0)} \otimes (fg^{(0)})^{(1)} h^*(g^{(1)}) \\ &= \sum \rho_{(x,z)}(fg^{(0)} h^*(g^{(1)})) \in \rho_{(x,z)}({}_z\mathcal{C}_x). \end{aligned}$$

The fact that $w\rho_{(z,x)}(f) \in \rho_{(z,y)}({}_y\mathcal{C}_z)$ for every $f \in {}_x\mathcal{C}_z$ and every $z \in \mathcal{C}_0$, is proved analogously using the fact that ρ is symmetrical. Hence, particularly, $\rho_{(y,y)}({}_y1_y)(h^* \rightharpoonup g) = \rho_{(x,y)}(g^{(0)} h^*(g^{(1)})) = (h^* \rightharpoonup g)\rho_{(x,x)}({}_x1_x)$, which means that $\rho(\mathcal{C})$ is the ideal of $\mathcal{B}$ generated by the central idempotent $e = \{\rho_{(x,x)}({}_x1_x)\}_{x \in \mathcal{C}_0}$. $\square$

3.4 Globalization of a partial coaction

In [5], Alves and Batista proved that every partial coaction on a unital algebra has an enveloping coaction. In this section we will show that this statement also holds for partial coactions on a more general class of algebras.

Definition 3.19 ([5]). *Let H be a Hopf algebra and A a partial H-comodule unital algebra with coaction ρ. A pair (B, θ) is called an enveloping coaction, or globalization, for ρ if:*

1. *B is an H-comodule algebra with coaction γ;*
2. *$\theta : A \longrightarrow B$ is a monomorphism of algebras;*
3. *$\theta(A)$ is an ideal of B;*
4. *$(\theta \otimes I)\rho(a) = (\theta(1_A) \otimes 1_H)\gamma(\theta(a))$, for all $a \in A$;*
5. *B is generated by $\theta(A)$ as an H-comodule.*

The item 4) means that θ is a morphism of partial coactions, where $\theta(A)$ has the induced partial coaction $\overline{\gamma}(\theta(a)) = (\theta(1_A) \otimes 1_H)\gamma(\theta(a))$.

3.4.1 Quasi-globalization

Definition 3.20. *Let A be a partial H-comodule algebra with symmetrical partial coaction ρ. A pair (B, θ) will be called a quasi-globalization for the partial coaction ρ if,*

1. *B is an H-comodule algebra with coaction γ;*
2. *$\theta : A \longrightarrow B$ is a monomorphism of algebras;*
3. *$\theta(A)$ is an ideal of B;*
4. *$(\theta \otimes I)((b \otimes 1_H)\rho(a)) = (\theta(b) \otimes 1_H)\gamma(\theta(a))$, $(\theta \otimes I)(\rho(a)(b \otimes 1_H)) = \gamma(\theta(a))(\theta(b) \otimes 1_H)$, for every $a, b \in A$;*
5. *B is generated by $\theta(A)$ as H-comodule.*

Note that when A has unit, (B, θ) is a quasi-globalization if and only if is a globalization in the classical sense.

As in the classical case, $A \otimes H$ has the trivial comodule structure given by $\delta = I \otimes \Delta$ and let B be the subcomodule algebra of $A \otimes H$ generated by $\rho(A)$, i.e., B is generated as algebra by $H^* \rightharpoonup \rho(A)$, where $h^* \rightharpoonup \rho(a) = (I \otimes h^*)\delta(\rho(a))$ and

$$\delta(\rho(a)) = (I_{A \otimes H} \otimes \Delta)(\sum a^{(0)} \otimes a^{(1)}) = \sum a^{(0)} \otimes (a^{(1)})_{(1)} \otimes (a^{(1)})_{(2)},$$

in other words, $h^* \rightharpoonup \rho(a) = \sum a^{(0)} \otimes (a^{(1)})_{(1)} h^*((a^{(1)})_{(2)})$.

Theorem 3.21. *Let $\rho : A \to A \otimes H$ be a symmetrical partial coaction. Then (B, ρ) is a quasi-globalization for ρ.*

Proof. As item 1) and 5) are hypothesis, let us begin by proving item 2). Suppose that $\rho(a) = \rho(b)$, since $a = (I \otimes \varepsilon_H)\rho(a) = (I \otimes \varepsilon_H)\rho(b) = b$ and $\rho(xy) = \rho(x)\rho(y)$ for every $x, y \in A$, we have that ρ is a monomorphism of algebras. Now, take $a, b \in A$ and $h^* \in H^*$, then

$$\begin{array}{rcl} (\rho \otimes I)((b \otimes 1_H)\rho(a)) & = & (\rho(b) \otimes 1_H)(I \otimes \Delta)\rho(a) \\ & \Downarrow & \\ (I \otimes I \otimes h^*)(\rho \otimes I)((b \otimes 1_H)\rho(a)) & = & \rho(b)(I \otimes I \otimes h^*)(I \otimes \Delta)\rho(a) \\ & \Downarrow & \end{array}$$

$$\begin{aligned}\rho(\sum ba^{(0)}h^*(a^{(1)})) &= \sum a^{(0)(0)} \otimes a^{(0)(1)}h^*(a^{(1)}) \\ &= \sum b^{(0)}a^{(0)} \otimes b^{(1)}(a^{(1)})_{(1)}h^*((a^{(1)})_{(2)}) \\ &= \rho(b)(\sum a^{(0)} \otimes (a^{(1)})_{(1)}h^*((a^{(1)})_{(2)})) \\ &= \rho(b)(h^* \rightharpoonup \rho(a)) \in \rho(A).\end{aligned}$$

Hence $\rho(A)$ is a right ideal of B. To show that $\rho(A)$ is also a left ideal, we use analogous calculation and the hypothesis that ρ is symmetrical. Finally, item $4)$ is an immediate consequence of ρ be a symmetrical partial action. □

3.4.2 Globalization of a partial coaction on an algebra with local units

As for partial actions, in this section we will prove that every partial coaction on an algebra with local units has a globalization, i.e., is a restriction of a global action to an ideal.

Remark 3.22. *Let B be an H-comodule algebra with coaction ρ, A an ideal with local units of B, and let $S = \{e_\lambda\}_{\lambda\in\Lambda}$ be an s.l.u. for A. If $(e_\alpha \otimes 1_H)\rho(a) = \rho(a)(e_\beta \otimes 1_H)$ for every $a \in A$ and $\alpha, \beta \in \Lambda$ such that $e_\alpha a = a = ae_\beta$, then the linear map $\overline{\rho}(a) = (e_\alpha \otimes 1_H)\rho(a)$, where $a \in A$ and $e_\alpha a = a$, is a symmetrical partial coaction.*

Definition 3.23. *Let H be a Hopf algebra, A a partial H-comodule algebra with coaction ρ and $S = \{e_\lambda\}_{\lambda\in\Lambda}$ an s.l.u. for A. The pair (B, θ) is an enveloping coaction, or globalization, for the partial coaction ρ if*

1. *B is an H-comodule algebra with coaction γ;*
2. *$\theta : A \longrightarrow B$ is a monomorphism of algebras;*
3. *$\theta(A)$ is an ideal of B;*
4. *$(\theta(e_\alpha) \otimes 1_H)\gamma(\theta(a)) = \gamma(\theta(a))(\theta(e_\beta) \otimes 1_H)$ for every pair of local units $e_\alpha, e_\beta \in S$ such that $(e_\alpha \otimes 1_H)\rho(a) = \rho(a) = \rho(a)(e_\beta \otimes 1_H)$, and*

$$(\theta \otimes I)\rho(a) = (\theta(e_\alpha) \otimes 1_H)\gamma(\theta(a)),$$

for every $e_\alpha \in S$ such that $(e_\alpha \otimes 1_H)\rho(a) = \rho(a)$;

5. *B is generated by $\theta(A)$ as H-comodule.*

Item $4)$ means that θ is a morphism of partial coactions.

Note that (B, θ) is a quasi-globalization for a partial coaction on an algebra with local units, then (B, θ) is actually a globalization.

Theorem 3.24. *Let H be a Hopf algebra, A a partial H-comodule algebra with local units with symmetrical partial coaction ρ, and let $S = \{e_\lambda\}_{\lambda\in\Lambda}$ be an s.l.u. for A. Then (B, ρ) is an enveloping coaction for ρ, where B is the subcomodule algebra of $A \otimes H$ generated by $\rho(A)$.*

Proof. It's a consequence of the Theorem 3.21. □

3.5 The minimal globalization of a partial action by a finite dimensional Hopf algebra

In [5] there is a description of the globalization of a coaction via the globalization of the induced partial action. Here, we will prove that when H is finite dimensional, the minimal globalization of a partial action coincides with a globalization of the induced partial coaction.

Proposition 3.25. *Let H be a finite dimensional Hopf algebra, A an associative algebra and $\cdot : H \otimes A \to A$ a symmetrical partial action. Let $\rho : A \to A \otimes H^*$ be the induced partial coaction $\rho(a) = \sum_{i=1}^{n} h_i \cdot a \otimes h_i^*$, where $\{h_i\}_{i=1}^{n}$ is the basis of H and $\{h_i^*\}_{i=1}^{n}$ its dual basis, and consider the action $\rightharpoonup$: $H \otimes A \otimes H^* \to A \otimes H^*$ given by $h \rightharpoonup a \otimes k^* = \sum k^*_{(2)}(h)a \otimes k^*_{(1)}$. Then, $(H \rightharpoonup \rho(A), \rho)$ is the minimal quasi-globalization of $\cdot : H \otimes A \to A$.*

Proof. First, let us prove that $B = H \rightharpoonup \rho(A)$ is an H-module algebra. For this, we only need to prove that

$$(h_k \rightharpoonup \rho(a))(h_l \rightharpoonup \rho(b)) = [\sum_{i=1}^{n}(h_i^*)_{(2)}(h_k)h_i \cdot a \otimes (h_i^*)_{(1)}][\sum_{j=1}^{n}(h_j^*)_{(2)}(h_l)h_j \cdot b \otimes (h_j^*)_{(1)}] \in B,$$

for every $k, l \in \{1, 2, \cdots, n\}$, $a, b \in A$. In fact, for any $h \in H$,

$$\begin{aligned}
&(I \otimes h)[\sum_{i=1}^{n}(h_i^*)_{(2)}(h_k)h_i \cdot a \otimes (h_i^*)_{(1)}][\sum_{j=1}^{n}(h_j^*)_{(2)}(h_l)h_j \cdot b \otimes (h_j^*)_{(1)}]\\
&= (I \otimes h)[\sum_{i,j}(h_i^*)_{(2)}(h_k)(h_j^*)_{(2)}(h_l)(h_i \cdot a)(h_j \cdot b) \otimes (h_i^*)_{(1)}(h_j^*)_{(1)}]\\
&= \sum_{i,j,(h)}(h_i^*)_{(2)}(h_k)(h_j^*)_{(2)}(h_l)(h_i \cdot a)(h_j \cdot b) \otimes (h_i^*)_{(1)}(h_{(1)})(h_j^*)_{(1)}(h_{(2)})\\
&= \sum_{i,j,(h)}(h_i^*)(h_{(1)}h_k)(h_j^*)(h_{(2)}h_l)(h_i \cdot a)(h_j \cdot b)\\
&= \sum(h_{(1)}h_k \cdot a)(h_{(2)}h_l \cdot b)\\
&= \sum(h_{(1)}(h_k)_{(1)} \cdot a)(h_{(2)}(h_k)_{(2)}S((h_k)_{(3)})h_l \cdot b)\\
&= \sum(h(h_k)_{(1)} \cdot a(S((h_k)_{(2)})h_l \cdot b))\\
&= \sum_i h_i^*(h(h_k)_{(1)})h_i \cdot a(S((h_k)_{(2)})h_l \cdot b))\\
&= \sum_i(h_i^*)_{(1)}(h)(h_i^*)_{(2)}((h_k)_{(1)})h_i \cdot a(S((h_k)_{(2)})h_l \cdot b))\\
&= (I \otimes h)(\sum_i(h_i^*)_{(2)}((h_k)_{(1)})h_i \cdot a(S((h_k)_{(2)})h_l \cdot b)) \otimes (h_i^*)_{(1)})\\
&= (I \otimes h)(\sum(h_k)_{(1)} \rightharpoonup \rho(a(S((h_k)_{(2)})h_l \cdot b))).
\end{aligned}$$

Hence

$$[\sum_{i=1}^{n}(h_i^*)_{(2)}(h_k)h_i{\cdot}a{\otimes}(h_i^*)_{(1)}][\sum_{j=1}^{n}(h_j^*)_{(2)}(h_l)h_j{\cdot}b{\otimes}(h_j^*)_{(1)}] = \sum(h_k)_{(1)} \rightharpoonup \rho(a(S((h_k)_{(2)})h_l{\cdot}b)) \in B.$$

The others requisites but the minimality, follow from

$$\rho(a(k \cdot b)) \quad = \quad \rho(a)(k \rightharpoonup \rho(b))$$

$$\rho((k \cdot b)a) \quad = \quad (k \rightharpoonup \rho(b))\rho(a).$$

Finally, for the minimality, suppose that $\sum_i kh^i \cdot a^i = 0$ for all $k \in H$. Then, for every $k \in H$

$$\begin{aligned}
(I \otimes k)(\sum_i h^i \rightharpoonup \rho(a^i)) &= (I \otimes k)(\sum_{i,j} h^i \rightharpoonup h_j \cdot a^i \otimes h_j^*) \\
&= (I \otimes k)(\sum_{i,j} (h_j^*)_{(2)}(h^i)h_j \cdot a^i \otimes (h_j^*)_{(1)}) \\
&= \sum_{i,j} h_j^*(kh^i)h_j \cdot a^i \\
&= \sum_i kh^i \cdot a^i = 0,
\end{aligned}$$

hence $\sum_i h^i \rightharpoonup \rho(a^i) = 0$. $\square$

This proposition can be illustrated by the following commutative diagram

$$\begin{tikzcd}
H \otimes A \otimes H^* \arrow[rr, "Id_H \otimes \theta"] \arrow[d, "\rightharpoonup"'] & & H \otimes Hom(H, A) \arrow[d, "\triangleright"] \\
A \otimes H^* \arrow[rr, "\theta"] & & Hom(H, A) \\
& A \arrow[ul, "\rho"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

where θ is the natural isomorphism.

Example 3.26. *Applying this result on Example 2.72, we have that*

$$\begin{aligned}
\rho(E_{ij}) &= \sum_{g \in G} p_g \cdot E_{ij} \otimes g \\
&= \sum_{g \in G} \frac{1}{|L|} \delta_{gt_{ij}L, t_{ij}L} E_{ij} \otimes g \\
&= \sum_{g \in t_{ij}L} \frac{1}{|L|} E_{ij} \otimes g,
\end{aligned}$$

and by the algebra isomorphism $FMat_{\mathbb{N}}(k) \otimes kG \simeq FMat_{\mathbb{N}}(kG)$, we may write

$$\rho(E_{ij}) = (\sum_{g \in t_{ij}L} \frac{1}{|L|} g)E_{ij} \in FMat_{\mathbb{N}}(kG).$$

Example 3.27. *Consider $A = k$ and G a finite group. The minimal globalization of a partial action $\cdot : kG \otimes k \to k$ given by $g \cdot 1 = \delta_{gH,H}$ where H is a subgroup of G, is the pair (k^H, θ), where $\theta(1) = \sum_{h \in H} p_h$.*

Example 3.28. *Let G be a finite group and A a partially G-graded associative algebra, i.e., A is a partial k^G-module algebra with symmetrical partial action. Note that for every $a \in A$, we have*

$$\rho(a) = \sum_{g \in G} p_g \cdot a \otimes g,$$

and

$$p_h \rightharpoonup \rho(a) = p_h \cdot a \otimes h.$$

Then the minimal globalization of the original partial action is given by an algebra that is generated as vector space by the elements $p_g \cdot a \otimes g$, with $g \in G$ and $a \in A$. Note also that

$$\begin{aligned} p_{gh} \cdot (a(p_h \cdot b)) &= \sum_{t \in G} (p_{ght^{-1}} \cdot a)(p_t p_h \cdot b) \\ &= (p_g \cdot a)(p_h \cdot b), \end{aligned}$$

then we can identify the minimal globalization as the G-graded algebra $B = \oplus_{g \in G} B_g$, where $B_g = p_g \cdot A$, and the monomorphism of algebras $\theta : A \to B$ is given by $\theta(a) = \sum_{g \in G} p_g \cdot a$.

Note that if we consider a partial G-grading on the algebra with local units $A = FMat_{\mathbb{N}}(k)$, then its minimal globalization is $B = \oplus_{g \in G} p_g \cdot FMat_{\mathbb{N}}(k)$, which can be identified with a subalgebra of $FMat_{\mathbb{N}}(kG)$, even if the partial grading is not good.

Proposition 3.29. *The minimal globalization of any partial G-grading of $FMat_{\mathbb{N}}(k)$ is a subalgebra of $FMat_{\mathbb{N}}(kG)$.*

Chapter 4

Partial Hopf actions and the partial invariant subalgebra

The goal of this section is to extend the result presented in [7] by Alves and Batista, where they proved that under some conditions, there exist a strict Morita context between the subalgebra of the partial invariants $A^{\underline{H}}$ and the partial smash product $\underline{A\#H}$.

4.1 Partial invariants

Let $\cdot : H \otimes A \to A$ be a partial action. In [7], the authors considered a unital partial H-module algebra A and assumed that for every $h \in H$, the elements $h \cdot 1_A$ lied in the center of A. With this, they defined the set of invariants of the partial action as the subalgebra

$$A^{\underline{H}} = \{a \in A;\ h \cdot a = a(h \cdot 1_A),\ \forall h \in H\}.$$

Definition 4.1. *Let A be an associative partial H-module algebra. We define the partial invariant subalgebra of A the subspace*

$$A^{\underline{H}} = \{a \in A \,|\, h \cdot (ab) = a(h \cdot b) \text{ and } h \cdot (ba) = (h \cdot b)a,\ \forall b \in A,\ h \in H\}.$$

Different from the unital case where $A^{\underline{H}}$ is a unital (sub)algebra, because $1_A \in A^{\underline{H}}$, when A does not have unit, we can say nothing about the structure of $A^{\underline{H}}$. In fact, even if A has a system of local units S and the partial action of H on A is S-categorizable, we only know that every $e \in S$ is in $(eAe)^{\underline{H}}$, because

$$h \cdot a = e(h \cdot a) = (h \cdot a)e$$

for every $a \in eAe$, but e is not necessarily in $A^{\underline{H}}$, i.e., we don't know if $A^{\underline{H}}$ also has local units. Actually, $A^{\underline{H}}$ may be neither an idempotent algebra.

Now, considering $\rho : A \to A \otimes H$ a partial coaction, when A has unit, Alves and Batista defined in [7] de set of coinvariants of the partial coaction as

$$A^{\underline{coH}} = \{a \in A;\ \rho(a) = a\rho(1_A),\ \forall h \in H\},$$

which is also a subalgebra of A.

Definition 4.2. *Let A be a partial H-comodule algebra. We define the partial coinvariant subalgebra of A the subspace*

$$A^{\underline{coH}} = \{a \in A \,|\, \rho(ab) = a\rho(b) \text{ and } \rho(ba) = \rho(b)(a \otimes 1_H),\ \forall b \in A\}.$$

4.2 Morita context

As in [7], we will consider H a finite dimensional Hopf algebra and the partial action on the associative algebra A will be considered symmetrical.

As H is finite dimensional, there exist a nonzero left integral $t \in H$ and we can define the partial trace map $\hat{t} : A \to A$, $\hat{t}(a) = t \cdot a$.

Lemma 4.3. *$\hat{t}$ is an $A^{\underline{H}}$-module map from A to $A^{\underline{H}}$.*

We will show that, under some assumptions, there exist a Morita context between the partial invariant subalgebra $A^{\underline{H}}$ and the partial smash product algebra $\underline{A\#H}$.

First, since the Hopf algebra H is finite dimensional, we have that the antipode S_H is bijective and the subspace generated by the left integrals in H, denoted by $\int_H^\ell$, is unidimensional. Then, if $t \in \int_H^\ell$, we have that $th \in \int_H^\ell$ for every $h \in H$, hence $th = \xi(h)t$ for some scalar $\xi(h)$, and this define an algebra morphism $\xi : H \to k$.

Lemma 4.4. *Given $c \in A$ and $\sum a(h_{(1)} \cdot b)\#h_{(2)} \in \underline{A\#H}$, the maps*

$$(\sum a(h_{(1)} \cdot b)\#h_{(2)}) \triangleright c = a(h \cdot (bc))$$

and

$$c \triangleleft (\sum a(h_{(1)} \cdot b)\#h_{(2)}) = \sum \xi(h_{(2)})(S_H^{-1}(h_{(1)}) \cdot ca)b$$

define left and right $\underline{A\#H}$-module structures on A. Moreover, considering the canonical left and right $A^{\underline{H}}$-module structures on A, we have that A is both an $(A^{\underline{H}}, \underline{A\#H})$-bimodule and $(\underline{A\#H}, A^{\underline{H}})$-bimodule.

Proof. First, note that since every associative partial H-module algebra A has a partial (A, H)-module structure that induce an action of $\underline{A\#H}$, which is the first mapping, we have that it is well defined. Now, take $\sum a(h_{(1)} \cdot x)\#h_{(2)}, \sum b(k_{(1)} \cdot y)\#k_{(2)} \in \underline{A\#H}$ and $c \in A$. Then, for the left $\underline{A\#H}$-module structure, we have

$$\begin{aligned}
((\sum a(h_{(1)} \cdot x)\#h_{(2)})(\sum b(k_{(1)} \cdot y)\#k_{(2)})) \triangleright c &= (\sum a(h_{(1)} \cdot (xb(k_{(1)} \cdot y))\#h_{(2)}k_{(2)}) \triangleright c \\
&= (\sum a(h_{(1)} \cdot (xb))(h_{(2)}k_{(1)} \cdot y)\#h_{(3)}k_{(2)}) \triangleright c \\
&= (\sum a(h_{(1)} \cdot (xb))((h_{(2)}k)_{(1)} \cdot y)\#(h_{(2)}k)_{(2)}) \triangleright c \\
&= \sum a(h_{(1)} \cdot (xb))(h_{(2)}k \cdot (yc)) \\
&= a(h \cdot (xb(k \cdot (yc)))) \\
&= (\sum a(h_{(1)} \cdot x)\#h_{(2)}) \triangleright (b(k \cdot (yc)) \\
&= (\sum a(h_{(1)} \cdot x)\#h_{(2)}) \triangleright ((\sum b(k_{(1)} \cdot y)\#k_{(2)})) \triangleright c).
\end{aligned}$$

And for the right $\underline{A\#H}$-module structure, we have

$$\begin{aligned}
(c \triangleleft (\sum a(h_{(1)} \cdot x)\#h_{(2)})) \triangleleft (\sum b(k_{(1)} \cdot y)\#k_{(2)}) = \\
&= (\sum \xi(h_{(2)})(S^{-1}(h_{(1)}) \cdot ca)x) \triangleleft (\sum b(k_{(1)} \cdot y)\#k_{(2)}) \\
&= \sum \xi(h_{(2)})\xi(k_{(2)})[S^{-1}(k_{(1)}) \cdot ((S^{-1}(h_{(1)}) \cdot ca)xb)]y
\end{aligned}$$

$$
\begin{aligned}
&=\sum \xi(h_{(2)}k_{(3)})[(S^{-1}(h_{(1)}k_{(2)})\cdot ca)(S^{-1}(k_{(1)})\cdot xb)]y\\
&=\sum \xi(h_{(3)}k_{(3)})[(S^{-1}(h_{(2)}k_{(2)})\cdot ca)(S^{-1}(k_{(1)})\varepsilon_H(h_{(1)})\cdot xb)]y\\
&=\sum \xi(h_{(4)}k_{(3)})[(S^{-1}(h_{(3)}k_{(2)})\cdot ca)(S^{-1}(k_{(1)})S^{-1}(h_{(2)})h_{(1)}\cdot xb)]y\\
&=\sum \xi(h_{(4)}k_{(3)})[(S^{-1}(h_{(3)}k_{(2)})\cdot ca)(S^{-1}(h_{(2)}k_{(1)})h_{(1)}\cdot xb)]y\\
&=\sum \xi(h_{(3)}k_{(2)})[S^{-1}(h_{(2)}k_{(1)})\cdot ca(h_{(1)}\cdot xb)]y\\
&=\sum \xi((h_{(2)}k)_{(2)})[S^{-1}((h_{(2)}k)_{(1)})\cdot ca(h_{(1)}\cdot xb)]y\\
&=c\triangleleft(\sum a(h_{(1)}\cdot xb)((h_{(2)}k)_{(1)}\cdot y)\#(h_{(2)}k)_{(2)}\\
&=c\triangleleft[(\sum a(h_{(1)}\cdot x)\#h_{(2)}))(\sum b(k_{(1)}\cdot y)\#k_{(2)})].
\end{aligned}
$$

Now, consider $\sum c(h_{(1)}\cdot x)\#h_{(2)} \in \underline{A\#H}$, $a \in A$ and $b \in A^{\underline{H}}$. Then

$$
\begin{aligned}
(ba)\triangleleft(\sum c(h_{(1)}\cdot x)\#h_{(2)}) &= \sum \xi(h_{(2)})(S^{-1}(h_{(1)})\cdot bac)x\\
&= \sum \xi(h_{(2)})b(S^{-1}(h_{(1)})\cdot ac)x\\
&= b\sum \xi(h_{(2)})(S^{-1}(h_{(1)})\cdot ac)x\\
&= b(a\triangleleft(\sum c(h_{(1)}\cdot x)\#h_{(2)})).
\end{aligned}
$$

On the other side,

$$
\begin{aligned}
((\sum c(h_{(1)}\cdot x)\#h_{(2)})\triangleright a)b &= c(h\cdot xa)b\\
&= c(h\cdot xab)\\
&= (\sum c(h_{(1)}\cdot x)\#h_{(2)})\triangleright(ab).
\end{aligned}
$$

□

For the Morita context we will use the same maps defined in [7], which are

$$
\begin{aligned}
[\cdot,\cdot]: A\otimes_{A^{\underline{H}}} A &\longrightarrow \underline{A\#H}\\
a\otimes b &\mapsto [a,b]=\sum a(t_{(1)}\cdot b)\#t_{(2)}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\cdot,\cdot\rangle: A\otimes_{\underline{A\#H}} A &\longrightarrow A^{\underline{H}}\\
a\otimes b &\mapsto \langle a,b\rangle=\hat{t}(ab)=t\cdot ab.
\end{aligned}
$$

Lemma 4.5. *The maps $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$ are well-defined.*

Proof. Take $a, b \in A$ and $c \in A^{\underline{H}}$, then

$$
\begin{aligned}
[a,cb] &= \sum a(t_{(1)}\cdot cb)\#t_{(2)}\\
&= \sum ac(t_{(1)}\cdot b)\#t_{(2)}\\
&= [ac,b].
\end{aligned}
$$

Now, take $a, b \in A$ and $\sum c(h_{(1)}\cdot x)\#h_{(2)} \in \underline{A\#H}$, then

$$
\langle a\triangleleft(\sum c(h_{(1)}\cdot x)\#h_{(2)}),b\rangle = \langle\sum \xi(h_{(2)})(S^{-1}(h_{(1)})\cdot ac)x,b\rangle
$$

$$
\begin{aligned}
&= t\cdot[\sum \xi(h_{(2)})(S^{-1}(h_{(1)})\cdot ac)xb]\\
&= \sum \xi(h_{(2)})t\cdot((S^{-1}(h_{(1)})\cdot ac)xb)\\
&= \sum \xi(h_{(2)})(t_{(1)}S^{-1}(h_{(1)})\cdot ac)(t_{(2)}\cdot xb)\\
&= \sum \xi(h_{(3)})(t_{(1)}S^{-1}(h_{(2)})\cdot ac)(t_{(2)}\varepsilon_H(h_{(1)})\cdot xb)\\
&= \sum \xi(h_{(4)})(t_{(1)}S^{-1}(h_{(3)})\cdot ac)(t_{(2)}S^{-1}(h_{(2)})h_{(1)}\cdot xb)\\
&= \sum \xi(h_{(3)})(tS^{-1}(h_{(2)})\cdot (ac(h_{(1)}\cdot xb)))\\
&= \sum(\xi(h_{(3)})tS^{-1}(h_{(2)})\cdot (ac(h_{(1)}\cdot xb)))\\
&= \sum th_{(3)}S^{-1}(h_{(2)})\cdot (ac(h_{(1)}\cdot xb))\\
&= t\cdot(ac(h\cdot xb))\\
&= \langle a, c(h\cdot xb)\rangle\\
&= \langle a, (\sum c(h_{(1)}\cdot x)\#h_{(2)})\triangleright b\rangle.
\end{aligned}
$$

□

Theorem 4.6. *$(\underline{A\#H}, A^{\underline{H}}, {}_{\underline{A\#H}}A_{A^{\underline{H}}}, {}_{A^{\underline{H}}}A_{\underline{A\#H}}, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$ is a Morita context.*

Proof. We must check that both $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$ are bimodule maps. First, since $\hat{t}$ is an $A^{\underline{H}}$-bimodule map, we have that $\langle\cdot,\cdot\rangle$ is an $A^{\underline{H}}$-bimodule map. Now, take $a, b\in A$ and $\sum c(h_{(1)}\cdot x)\#h_{(2)}\in \underline{A\#H}$, then

$$
\begin{aligned}
[(\sum c(h_{(1)}\cdot x)\#h_{(2)})\triangleright a, b] &= [c(h\cdot xa), b]\\
&= \sum c(h\cdot xa)(t_{(1)}\cdot b)\#t_{(2)}\\
&= \sum c(h_{(1)}\varepsilon_H(h_{(2)})\cdot xa)(t_{(1)}\cdot b)\#t_{(2)}\\
&= \sum c(h_{(1)}\cdot xa)(\varepsilon_H(h_{(2)})t_{(1)}\cdot b)\#t_{(2)}\\
&= \sum c(h_{(1)}\cdot xa)((\varepsilon_H(h_{(2)})t)_{(1)}\cdot b)\#(\varepsilon_H(h_{(2)})t)_{(2)}\\
&= \sum c(h_{(1)}\cdot xa)((h_{(2)}t)_{(1)}\cdot b)\#(h_{(3)}t)_{(2)}\\
&= \sum c(h_{(1)}\cdot xa)(h_{(2)}t_{(1)}\cdot b)\#h_{(2)}t_{(2)}\\
&= (\sum c(h_{(1)}\cdot x)\#h_{(2)})(\sum a(t_{(1)}\cdot b)\#t_{(2)})\\
&= (\sum c(h_{(1)}\cdot x)\#h_{(2)})[a, b].
\end{aligned}
$$

For the right hand side

$$
\begin{aligned}
[a, b\triangleleft(\sum c(h_{(1)}\cdot x)\#h_{(2)})] &= [a, \sum \xi(h_{(2)})(S_H^{-1}(h_{(1)})\cdot bc)x]\\
&= \sum a(t_{(1)}\cdot(\xi(h_{(2)})S_H^{-1}(h_{(1)})\cdot bc)x))\#t_{(2)}\\
&= \sum a((\xi(h_{(2)})t)_{(1)}\cdot(S_H^{-1}(h_{(1)})\cdot bc)x))\#(\xi(h_{(2)})t)_{(2)}\\
&= \sum a((th_{(2)})_{(1)}\cdot(S_H^{-1}(h_{(1)})\cdot bc)x))\#(th_{(2)})_{(2)}\\
&= \sum a(t_{(1)}h_{(2)}\cdot(S_H^{-1}(h_{(1)})\cdot bc)x))\#t_{(2)}h_{(3)}\\
&= \sum a(t_{(1)}h_{(2)}S_H^{-1}(h_{(1)})\cdot bc)(t_{(2)}h_{(3)}\cdot x)\#t_{(3)}h_{(4)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum a(t_{(1)} \cdot bc)(t_{(2)}h_{(1)} \cdot x)\#t_{(3)}h_{(2)} \\
&= \sum a(t_{(1)} \cdot bc)(t_{(2)} \cdot (h_{(1)} \cdot x))\#t_{(3)}h_{(2)} \\
&= (\sum a(t_{(1)} \cdot b)\#t_{(2)})(\sum c(h_{(1)} \cdot x)\#h_{(2)}) \\
&= [a,b](\sum c(h_{(1)} \cdot x)\#h_{(2)}).
\end{aligned}
$$

Finally, we must check the associativity of the brackets, i.e., $[a,b] \triangleright c = a\langle b,c\rangle$ and $a \triangleleft [b,c] = \langle a,b\rangle c$. For this, take $a,b,c \in A$, then

$$[a,b] \triangleright c = (\sum a(t_{(1)} \cdot b)\#t_{(2)}) \triangleright c = a(t \cdot bc) = a\langle b,c\rangle,$$

and

$$
\begin{aligned}
a \triangleleft [b,c] &= a \triangleleft (\sum b(t_{(1)} \cdot c)\#t_{(2)}) \\
&= \sum \xi(t_{(2)})(S_H^{-1}(t_{(1)}) \cdot ab)c \\
&= (t \cdot ab)c \\
&= \langle a,b\rangle c,
\end{aligned}
$$

here we used that $t = \sum \xi(t_{(2)})S_H^{-1}(t_{(1)})$, as we can see in [16]. $\square$

4.3 Partial Hopf Galois theory

In this section we will adapt the Partial Hopf Galois theory for algebras without identity in order to present some conditions for the Morita context $(\underline{A\#H}, A^{\underline{H}}, A, A, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$ be strict. First, in the case where A is a unital algebra, given a partial coaction $\rho : A \to A \otimes H$, we can define an A-bimodule structure on $A \otimes H$: the left A-module structure is given by the multiplication of A and the right A-module structure is given by $(a \otimes h)b = \sum ab^{(0)} \otimes hb^{(1)}$, where $\rho(b) = \sum b^{(0)} \otimes b^{(1)}$. In [7], the authors considered the sub A-bimodule of $A \otimes H$ defined by

$$\underline{A \otimes H} = \{\sum a1_A^{(0)} \otimes h1_A^{(1)};\ a \in A,\ h \in H\}.$$

Following the same construction, when A does not have unit, the tensor product $A \otimes H$ still is an A-bimodule and we consider the unital right submodule of $A \otimes H$

$$\underline{A \otimes H} = (A \otimes H)A.$$

Note that when $A^2 = A$, $A \otimes H$ is unital as left A-module.

Definition 4.7. *Let $\rho : A \to A \otimes H$ be a partial coaction. The extension $A^{\underline{coH}} \subset A$ will be called partial H-Hopf Galois if the canonical map $\beta : A \otimes_{A^{\underline{coH}}} A \to \underline{A \otimes H}$, given by $\beta(a \otimes b) = \sum ab^{(0)} \otimes b^{(1)}$, is a surjective A-bimodule map.*

Analogously to the case where A is unital, as can be seen in [7], we have the following result.

Lemma 4.8. *If H is a finite dimensional Hopf algebra, A is a partial H-module algebra and $\rho : A \to A \otimes H^*$ is the induced partial H^*-comodule structure on A, then $A^{\underline{H}} = A^{\underline{coH^*}}$.*

This last lemma says that when H is finite dimensional, we may consider on a partial H-module algebra A the induced structure of partial H^*-comodule algebra and, since $A^{\underline{H}} = A^{\underline{coH^*}}$, we consider the map $\beta : A \otimes_{A^{\underline{H}}} A \to \underline{A \otimes H^*}$.

Theorem 4.9. *Let H be a finite dimensional Hopf algebra, $0 \neq t \in \int_H^\ell$, A a partial H-module algebra and $A^2 = A$. Suppose that the canonical map $\beta : A \otimes_{A^{\underline{H}}} A \to \underline{A \otimes H^*}$ is surjective. Then*

1. *For every $c \in A$, there exists $a_{c1}, \cdots, a_{ck}$ and $b_{c1}, \cdots, b_{ck}$ in A such that $\phi_{ci} : A \to A^{\underline{H}}$ given by $\phi_{ci}(a) = t \cdot (b_{ci}a)$ is a right $A^{\underline{H}}$-module map and $ca = \sum_{i=1}^k a_{ci}\phi_{ci}(a)$ for every $a \in A$. Hence A is a right unital $A^{\underline{H}}$-module and for every $c \in A$, every subspace cA is finitely generated as right (unital) $A^{\underline{H}}$-module;*

2. *If $t_A(A \otimes_{A^{\underline{H}}} A) = \{x \in A \otimes_{A^{\underline{H}}} A;\ ax = 0,\ \forall a \in A\} = 0$, then β is bijective.*

Proof. For item $1)$, we will consider the isomorphism

$$\begin{array}{rcl} \theta : H^* & \to & H \\ f & \mapsto & \theta(f) = t \leftharpoonup f = \sum f(t_{(1)})t_{(2)}, \end{array}$$

presented in [16]. Then there exists $T \in H^*$ such that $1_H = \sum T(t_{(1)})t_{(2)}$. For a fixed $c \in A$, as β is surjective, there exist $a_{c1}, \cdots, a_{ck}$ and $b_{c1}, \cdots, b_{ck}$ in A such that

$$c^{(0)} \otimes Tc^{(1)} = \beta(\sum_{i=1}^k a_{ci} \otimes b_{ci}) = \sum_{i=1}^k a_{ci}b_{ci}^{(0)} \otimes b_{ci}^{(1)}.$$

Now, take $a \in A$, then

$$\begin{array}{rcl} ca = 1_H \cdot ca & = & (t \leftharpoonup T) \cdot ca \\ & = & \sum c^{(0)}a^{(0)}(c^{(1)}a^{(1)}(t \leftharpoonup T)) \\ & = & \sum c^{(0)}a^{(0)}T(t_{(1)})(c^{(1)}a^{(1)})(t_{(2)}) \\ & = & \sum c^{(0)}a^{(0)}T(t_{(1)})c^{(1)}(t_{(2)})a^{(1)}(t_{(3)}) \\ & = & \sum c^{(0)}a^{(0)}(Tc^{(1)})(t_{(1)})a^{(1)}(t_{(2)}) \\ & = & \sum_{i=1}^k a_{ci}b_{ci}^{(0)}a^{(0)}b_{ci}^{(1)}(t_{(1)})a^{(1)}(t_{(2)}) \\ & = & \sum_{i=1}^k a_{ci}b_{ci}^{(0)}a^{(0)}(b_{ci}^{(1)}a^{(1)})(t) \\ & = & \sum_{i=1}^k a_{ci}(t \cdot b_{ci}a) \\ & = & \sum_{i=1}^k a_{ci}\phi_{ci}(a) \end{array}$$

For item $2)$, consider the maps

$$\beta' : A \otimes_{A^{\underline{H}}} A \longrightarrow A \otimes H^*$$

$$a \otimes b \quad \mapsto \quad \sum a^{(0)} b \otimes a^{(1)}$$

and

$$\begin{aligned} \varphi : A \otimes H^* \quad &\longrightarrow \quad A \otimes H^* \\ a \otimes h^* \quad &\mapsto \quad \sum a^{(0)} \otimes a^{(1)} S_{H^*}(h^*). \end{aligned}$$

Note that $\beta' = \varphi\beta$. Then, if $\sum_{j=1}^m a_j \otimes b_j \in ker\beta$, we have $0 = \varphi\beta(\sum_{j=1}^m a_j \otimes b_j) = \sum_j a_j^{(0)} b_j \otimes a_j^{(1)}$. Hence, for every $c \in A$, we have

$$\begin{aligned} \sum_{j=1}^m ca_j \otimes b_j \quad &= \quad \sum_{j,i} a_{ci}\phi_{ci}(a_j) \otimes b_j \\ &= \quad \sum_i a_{ci} \otimes (\sum_j \phi_{ci}(a_j) b_j) \\ &= \quad \sum_i a_{ci} \otimes (\sum_j (t \cdot b_{ci} a_j) b_j) \\ &= \quad \sum_i a_{ci} \otimes (\sum_j b_{ci}^{(0)} a_j^{(0)} (b_{ci}^{(1)} a_j^{(1)})(t) b_j) = 0, \end{aligned}$$

since $t_A(A \otimes_{A\underline{H}} A) = 0$, we have that $\sum_{j=1}^k a_j \otimes b_j = 0$, hence β is injective. $\square$

For the next corollary, we will need the following definition that appears in [10].

Definition 4.10 ([22]). *Let I be a partially ordered set and R a ring. A direct system of R-modules over I is a family $\{M_i\}_{i \in I}$ of R-modules together with a family of morphisms*

$$\psi_i^j : M_i \to M_j$$

for $i \le j$ such that $\psi_i^i = id_{M_i}$ $\psi_i^j \psi_j^k = \psi_i^k$ whenever $i \le j \le k$.

Definition 4.11 ([10]). *Let A be a ring with local units. A unital A-module P is a locally projective module if there exist a direct system $(P_i)_{i \in I}$ of finitely generated projective direct summands of P together with projections $\psi_i : P \to P_i$ such that ψ_i factors through ψ_j whenever $i \le j$, and such that $\varinjlim P_i = P$.*

Example 4.12. *If A is an algebra with local units, then A_A is a locally projective right A-module, because for every local unit e_α of A, the unital right A-module $e_\alpha A$ is a finitely generated projective A-module. In fact, note that for every local unit e_α of A, we have that $A = e_\alpha A \oplus (1 - e_\alpha)A$, where*

$$(1 - e_\alpha)A = \{a \in A; e_\alpha a = 0\}.$$

Then, we have that every $e_\alpha A$ is a finitely generated right A-module that is a direct summand. To show that every $e_\alpha A$ is projective (as right A-module), note that every right A-module map $g : e_\alpha A \to N$ is determined by $g(e_\alpha)$. Then, given a surjective right A-module map $f : M \to N$, we determine the right A-module map $h : e_\alpha A \to M$ by $h(e_\alpha) = m'$, where m' is chosen in M such that $f(m') = g(e_\alpha)$. Hence, the following diagram commutes:

$$\begin{tikzcd} & e_\alpha A \arrow[dl, "h"'] \arrow[d, "g"] & \\ M \arrow[r, "f"'] & N \arrow[r] & 0. \end{tikzcd}$$

Corollary 4.13. *Let H be a finite dimensional Hopf algebra, $0 \neq t \in \int_H^{\ell}$, and let A be a partial H-module algebra with s.l.u. $S = \{e_\lambda\}_{\lambda\in\Lambda}$, such that the canonical map $\beta : A \otimes_{A^{\underline{H}}} A \to \underline{A \otimes H^*}$ is surjective. Then*

1. *For every $e_\lambda \in S$, there exist $a_{\lambda 1}, \cdots, a_{\lambda n}$ and $b_{\lambda 1}, \cdots, b_{\lambda n}$ in A such that $\phi_{\lambda i} : A \to A^{\underline{H}}$ given by $\phi_{\lambda i}(a) = t \cdot (b_{\lambda i} a)$ is a right $A^{\underline{H}}$-module map and $a = \sum_{i=1}^n a_{\lambda i}\phi_{\lambda i}(a)$ for every $a \in e_\lambda A$. Then every subspace $e_\lambda A$ is a finitely generated projective right $A^{\underline{H}}$-module, hence A is a locally projective right unital $A^{\underline{H}}$-module;*

2. *β is bijective.*

Proof. The fact that, in this case, every subspace $e_\lambda A$ is a projective right $A^{\underline{H}}$-module, follows from [22], because we provide a projective basis for every $e_\lambda A$. □

Theorem 4.14. *Let H be a finite dimensional Hopf algebra with a nonzero integral t, A a partial H-module algebra and $A^2 = A$. The following statements are equivalent:*

1. *$A^{\underline{H}} \subset A$ is a partial H^*-Galois extension;*

2. *$[\cdot,\cdot] : A \otimes_{A^{\underline{H}}} A \to \underline{A\#H}$ is surjective.*

Proof. Let $\theta : H^* \to H$ be the H-module isomorphism $f \mapsto (t \leftharpoonup f = \sum f(t_{(1)})t_{(2)})$. Then, we can easily prove that $[a,b] = (I \otimes \theta)\beta(a \otimes b)$. Hence, β is surjective if and only if $[\cdot,\cdot]$ is surjective. □

Example 4.15. *Consider the s-unital algebra $P = C_0(\mathbb{R})$ with the $k\mathbb{Z}_2$-action as in Example 4.6. Is easy to see that $P^{k\mathbb{Z}_2} = \{f \in P \mid f(x) = f(-x), \forall x \in \mathbb{R}\}$. First, we will write $+$ for the addition in $k\mathbb{Z}_2$ as vector space and the multiplication for the operation of the group, then $t = \overline{0} + \overline{1}$ is a left integral of $k\mathbb{Z}_2$. Since $t \cdot f = 2f$ for every $f \in P^{k\mathbb{Z}_2}$, we have that the map $\widehat{t}$ is surjective. Now, note that for every $f \in P$, we have that $f \otimes p_{\overline{1}} \in \underline{A \otimes k^{\mathbb{Z}_2}}$. Define $g'(x) = g(-x)$, for every $g \in P$, and suppose that the canonical map $\beta : P \otimes_{P^{k\mathbb{Z}_2}} P \to \underline{A \otimes k^{\mathbb{Z}_2}}$ is surjective, then there exist $f_1, \cdots, f_n, g_1, \cdots, g_n \in P$ such that $f \otimes p_{\overline{1}} = \beta(\sum_{i=1}^n f_i \otimes g_i) = \sum_{i=1}^n f_i g_i \otimes p_{\overline{0}} + \sum_{i=1}^n f_i g_i' \otimes p_{\overline{1}}$, i.e., for every $x \in \mathbb{R}$, $\sum_{i=1}^n f_i(x)g_i(x) = 0$ and $\sum_{i=1}^n f_i(x)g_i(-x) = f(x)$, then we must have $f(0) = 0$. Hence β is not surjective.*

Example 4.16. *Let G be a finite abelian group and set $A = FMat_{\mathbb{N}}(k)$. Assume that the partial action $\cdot : kG \otimes A \to A$ is a walker partial action (see Definition 5.20) given by $g \cdot E_{ij} = \delta_{gH,H}\alpha_{ij}E_{ij}$ (see Proposition 5.21), where H is a subgroup of G. Then $\underline{A \otimes k^G}$ is generated by the elements*

$$\begin{aligned}
(E_{i(gk)} \otimes p_g)E_{kj} &= \sum_{h\in G}(E_{i(gk)} \otimes p_g)(h \cdot E_{kj} \otimes p_h) \\
&= \sum_{h\in H}(E_{i(gk)} \otimes p_g)(\alpha_{kj}(h)E_{(hk)(hj)} \otimes p_h) \\
&= \sum_{h\in H}(\alpha_{kj}(h)E_{i(gk)}E_{(hk)(hj)} \otimes p_g p_h) \\
&= \delta_{gH,H}\alpha_{kj}(g)E_{(gk)(gj)} \otimes p_g,
\end{aligned}$$

and $\beta : A \otimes_{A^{\underline{kG}}} A \to \underline{A \otimes k^G}$ is given by

$$\beta(E_{ij} \otimes E_{kl}) = \sum_{g\in G} E_{ij}(g \cdot E_{kl}) \otimes p_g$$

$$= \sum_{g \in H} \alpha_{kl}(g)\delta_{j,gk} E_{i(gl)} \otimes p_g.$$

Hence,

$$\begin{aligned} \beta(E_{(gk)(gk)} \otimes E_{kj}) &= \sum_{h \in H} \alpha_{kj}(h)\delta_{gk,hk} E_{(gk)(hj)} \otimes p_h \\ &= \alpha_{kj}(g) E_{(gk)(gj)} \otimes p_g, \end{aligned}$$

for every $g \in H$. Therefore, the extension $A^{\underline{kG}} \subseteq A$ is partial k^G-Hopf Galois.

Chapter 5

On partial actions and partial representations of $H \bowtie L$

By Majid [20], whenever we have a matched pair of Hopf algebras (H, L), we can construct a new Hopf algebra $H \bowtie L$. In this section, we will study the relations of the actions, partial actions, representations and partial representations of $H \bowtie L$ and the actions, partial actions, representations and partial representations of H and L.

5.1 Actions of $H \bowtie L$

First we will investigate when an H-action and an L-action on an associative algebra A induce an $H \bowtie L$-action on A.

We begin by recalling the definition of a matched pair of Hopf algebras and of the double crossed product associated to a matched pair, according to Majid.

Remark 5.1. *A left (right) H-module coalgebra is a coalgebra C in the category of the left (right) H-modules.*

Definition 5.2 ([20])**.** *Let H, L be Hopf algebras. The pair $(H, L, \triangleright, \triangleleft)$ is a right-left matched pair if L is a right H-module coalgebra with action $\triangleleft$ and H is a left L-module coalgebra with action $\triangleright$ such that:*

1. $(xy) \triangleleft h = \sum (x \triangleleft (y_{(1)} \triangleright h_{(1)}))(y_{(2)} \triangleleft h_{(2)})$;

2. $x \triangleright (gh) = \sum (x_{(1)} \triangleright g_{(1)})((x_{(2)} \triangleleft g_{(2)}) \triangleright h)$;

3. $x \triangleright 1_H = \varepsilon_L(x) 1_H$;

4. $1_L \triangleleft h = \varepsilon_H(h) 1_L$;

5. $\sum x_{(1)} \triangleleft h_{(1)} \otimes x_{(2)} \triangleright h_{(2)} = \sum x_{(2)} \triangleleft h_{(2)} \otimes x_{(1)} \triangleright h_{(1)}$,

for every $g, h \in H$, $x, y \in L$.

As we can see in [20], given a right-left matched pair $(H, L, \triangleright, \triangleleft)$, we have that the double crossed product $H \bowtie L$ built on the vector space $H \otimes L$ is a Hopf algebra with multiplication

$$(g \otimes x)(h \otimes y) = \sum (g(x_{(1)} \triangleright h_{(1)})) \otimes ((x_{(2)} \triangleleft h_{(2)})y)$$

with $1_H \otimes 1_L$ as unit, and coalgebra structure given by the tensor product of coalgebras. The antipode is given by

$$S_{H \bowtie L}(h \otimes x) = (1_H \otimes S_L(x))(S_H(h) \otimes 1_L).$$

Note that the canonical inclusions of H and L in $H \bowtie L$ are morphisms of Hopf algebras, then every action of $H \bowtie L$ on an associative algebra A induce actions of H and L on A.

From now on, we will say that $(H, L, \triangleright, \triangleleft)$ is a matched pair of Hopf algebras, instead of right-left matched pair.

Definition 5.3. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras and A an associative algebra. Suppose that A is an H-module algebra with action $\cdot_H$ and an L-module algebra with action $\cdot_L$, if there exist an action of $H \bowtie L$ on A such that the induced actions of H and L on A coincide with $\cdot_H$ and $\cdot_L$, respectively, we will say the $(\cdot_H, \cdot_L)$ is an admissible pair of actions.*

Proposition 5.4. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras and A an associative algebra. Suppose that A is an H-module algebra with action $\cdot_H$ and an L-module algebra with action $\cdot_L$. Then $(\cdot_H, \cdot_L)$ is an admissible pair of actions if and only if $x \cdot_L h \cdot_H a = \sum (x_{(1)} \triangleright h_{(1)}) \cdot_H (x_{(2)} \triangleleft h_{(2)}) \cdot_L a$, for every $x \in L$, $h \in H$.*

Proof. If $(\cdot_H, \cdot_L)$ is an admissible pair of actions then for every $x \in L$, $h \in H$, we have that

$$\begin{aligned}
x \cdot_L h \cdot_H a &= (1_H \otimes x) \cdot (h \otimes 1_L) \cdot a \\
&= \sum (x_{(1)} \triangleright h_{(1)} \otimes x_{(2)} \triangleleft h_{(2)}) \cdot a \\
&= \sum ((x_{(1)} \triangleright h_{(1)} \otimes 1_L)(1_H \otimes x_{(2)} \triangleleft h_{(2)})) \cdot a \\
&= \sum (x_{(1)} \triangleright h_{(1)} \otimes 1_L) \cdot (1_H \otimes x_{(2)} \triangleleft h_{(2)}) \cdot a \\
&= \sum (x_{(1)} \triangleright h_{(1)}) \cdot_H (x_{(2)} \triangleleft h_{(2)}) \cdot_L a.
\end{aligned}$$

Now, suppose that this equality holds and define $(h \otimes x) \cdot a = h \cdot_H x \cdot_L a$, then clearly $(1_H \otimes 1_L) \cdot a = a$ and

$$\begin{aligned}
(g \otimes x) \cdot (h \otimes y) \cdot a &= g \cdot_H x \cdot_L h \cdot_H y \cdot_L a \\
&= \sum g \cdot_H (x_{(1)} \triangleright h_{(1)}) \cdot_H (x_{(2)} \triangleleft h_{(2)}) \cdot_L y \cdot_L a \\
&= \sum (g(x_{(1)} \triangleright h_{(1)}) \otimes (x_{(2)} \triangleleft h_{(2)})y) \cdot a \\
&= \sum ((g \otimes x)(h \otimes y)) \cdot a,
\end{aligned}$$

for every $g, h \in H$, $x, y \in L$. □

5.2 Partial actions of $H \bowtie L$

Now we will investigate some (symmetrical) partial actions of $H \bowtie L$, on some classes of associative algebras, induced by partial actions of H and L when one of them is actually a global action. We don't consider partial actions of $H \bowtie L$ induced by strictly partial actions of H and L, because in this case we don't even know how the induced partial actions of $H \bowtie L$ would work.

This study is motivated by the fact that every partial action of $H \bowtie L$ induces actions of H and L.

Definition 5.5. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. Let A be an H-module algebra with action $\cdot_H$ and a partial L-module algebra with partial action $\cdot_L$. If there exist a partial action of $H \bowtie L$ such that its restriction to H and L yield the original partial actions, we will say that $(\cdot_H, \cdot_L)$ is an admissible pair of partial actions of type* 1.

Note that if we consider partial actions on an associative algebra A with $r(A) = 0$ and suppose that H and L have bijective antipode, if both partial actions are global actions, then the induced partial action of $H \bowtie L$ on A must be also global. In fact, if the global action of H and L on A induce a partial action of $H \bowtie L$, namely $\cdot$, we have that

$$\begin{aligned}
b(x \cdot_L h \cdot_H a) &= \sum (x_{(2)} S_L^{-1}(x_{(1)}) \cdot_L b)(x_{(3)} \cdot_L h \cdot_H a) \\
&= \sum x_{(2)} \cdot_L ((S_L^{-1}(x_{(1)}) \cdot_L b)(h \cdot_H a)) \\
&= \sum (1_H \otimes x_{(2)}) \cdot ((1_H \otimes S_L^{-1}(x_{(1)})) \cdot b)((h \otimes 1_L) \cdot a) \\
&= \sum (1_H \otimes x_{(2)}) \cdot (1_H \otimes S_L^{-1}(x_{(1)})) \cdot b)((1_H \otimes x_{(3)})(h \otimes 1_L) \cdot a) \\
&= \sum (x_{(2)} \cdot_L S_L^{-1}(x_{(1)}) \cdot_L b)((1_H \otimes x_{(3)})(h \otimes 1_L) \cdot a) \\
&= \sum \varepsilon_L(x_{(1)}) b((1_H \otimes x_{(2)})(h \otimes 1_L) \cdot a) \\
&= b(1_H \otimes x)(h \otimes 1_L) \cdot a \\
&= \sum b(x_{(1)} \triangleright h_{(1)} \otimes x_{(2)} \triangleleft h_{(2)}) \cdot a,
\end{aligned}$$

for every $a, b \in A$, $x \in L$, $h \in H$. Analogously, we also have that

$$\begin{aligned}
b(h \cdot_H x \cdot_L a) &= \sum (h_{(2)} S_H^{-1}(h_{(1)}) \cdot_H b)(h_{(3)} \cdot_H x \cdot_L a) \\
&= \sum h_{(2)} \cdot_H [(S_H^{-1}(h_{(1)}) \cdot_H b)(x \cdot_L a)] \\
&= \sum h_{(2)} \otimes 1_L \cdot [(S_H^{-1}(h_{(1)}) \otimes 1_L \cdot b)(1_H \otimes x \cdot a)] \\
&= b(h \otimes x \cdot a),
\end{aligned}$$

then we must have that

$$\begin{aligned}
b(x \cdot_L h \cdot_H a) &= \sum b((x_{(1)} \triangleright h_{(1)} \otimes x_{(2)} \triangleleft h_{(2)}) \cdot a) \\
&= \sum b((x_{(1)} \triangleright h_{(1)}) \cdot_H (x_{(2)} \triangleleft h_{(2)}) \cdot_L a),
\end{aligned}$$

and since $r(A) = 0$,

$$(x \cdot_L h \cdot_H a) = \sum (x_{(1)} \triangleright h_{(1)} \otimes x_{(2)} \triangleleft h_{(2)}) \cdot a,$$

that is the necessary and sufficient condition for $\cdot$ be a global action, as proved previously.

By a similar calculation, we can prove the same fact if we consider that all partial actions involved are symmetrical and if we assume that $l(A) = 0$, but in this case H and L don't need to have bijective antipodes.

Now we will present some properties of S_H and S_L when $(H, L, \triangleright, \triangleleft)$ is a matched pair of Hopf algebras, that will be useful later.

Lemma 5.6. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. Then*

- $S_H(x \triangleright h) = (x \triangleleft h_{(1)}) \triangleright S_H(h_{(2)})$*;*
- $S_L(x \triangleleft h) = S_L(x_{(1)}) \triangleleft (x_{(2)} \triangleright h)$*.*

Proof. We will prove only the second equation, because the first one is proved analogously. Since L is an H-module coalgebra, we have that $\varepsilon_L(x \triangleleft h) = \varepsilon_L(x)\varepsilon_H(h)$, then

$$\begin{aligned}
\sum S_L(x_{(1)}) \triangleleft (x_{(2)} \triangleright h) &= \sum S_L(x_{(1)}) \triangleleft (x_{(2)} \triangleright h_{(1)})\varepsilon_L(x_{(3)})\varepsilon_H(h_{(2)}) \\
&= \sum S_L(x_{(1)}) \triangleleft (x_{(2)} \triangleright h_{(1)})\varepsilon_L(x_{(3)} \triangleleft h_{(2)}) \\
&= \sum S_L(x_{(1)}) \triangleleft (x_{(2)} \triangleright h_{(1)})(x_{(3)} \triangleleft h_{(2)})_{(1)}S_L((x_{(3)} \triangleleft h_{(2)})_{(2)}) \\
&= \sum S_L(x_{(1)}) \triangleleft (x_{(2)} \triangleright h_{(1)})(x_{(3)} \triangleleft h_{(2)})S_L(x_{(4)} \triangleleft h_{(3)}) \\
&= \sum (S_L(x_{(1)})x_{(2)} \triangleleft h_{(1)})S_L(x_{(3)} \triangleleft h_{(2)}) \\
&= \sum \varepsilon_L(x_{(1)})\varepsilon_H(h_{(1)})S_L(x_{(2)} \triangleleft h_{(2)}) \\
&= S_L(x \triangleleft h).
\end{aligned}$$

Note that we use the properties $1)$ and $4)$ of matched pair of Hopf algebras. □

In a similar way, we can prove the following result.

Lemma 5.7. *Let H and L be Hopf algebras where H is an L-module bialgebra. Then S_H is a morphism of L-modules.*

Lemma 5.8. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras, A an associative algebra that is an H-module algebra with action $\cdot_H$ and a partial L-module algebra with partial action $\cdot_L$. Suppose that $(\cdot_H, \cdot_L)$ is an admissible pair of partial actions of type 1. If $r(A) = 0$ and H has bijective antipode , or $l(A) = 0$ and all partial actions involved are symmetrical, then*

1. $(h \otimes x) \cdot a = h \cdot_H x \cdot_L a$*;*
2. $x \cdot_L (b(h \cdot_H a)) = \sum (x_{(1)} \cdot_L b)((x_{(2)} \triangleright h_{(1)} \otimes x_{(3)} \triangleleft h_{(2)}) \cdot a)$*.*

Proof. Item $1)$ follows from previous calculation and item $2)$ is an immediate consequence of the induced partial action $\cdot$ of $H \bowtie L$ on A. □

Observation 5.9. *If we assume that A has unit, then H don't need to have bijective antipode on the previous lemma to prove the first equality, because we can use $h \cdot_H 1_A = \varepsilon_H(h)1_A$.*

Proposition 5.10. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras, A an associative algebra that is an H-module algebra with action $\cdot_H$ and a partial L-module algebra with partial action $\cdot_L$. If $r(A) = 0$ and H has bijective antipode, then $(\cdot_H, \cdot_L)$ is an admissible pair of partial actions of type 1 if and only if the map $(h \otimes x) \cdot a = h \cdot_h x \cdot_L a$ determines a partial action of $H \bowtie L$ on A, and this holds if and only if*

$$\sum (x_{(1)} \cdot_L b)(x_{(2)} \cdot_L h \cdot_H y \cdot_L a) = \sum (x_{(1)} \cdot_L b)((x_{(2)} \triangleright h_{(1)}) \cdot_H ((x_{(3)} \triangleleft h_{(2)})y) \cdot_L a),$$

for every $a, b \in A$, $h \in H$, $x, y \in L$.

Proof. Note that clearly the map $(h \otimes x) \cdot a = h \cdot_h x \cdot_L a$ satisfies the first property of partial actions, then it determine a partial action if and only if the second property holds, i.e.,

$$(g \otimes x) \cdot (b((h \otimes y) \cdot a)) = \sum ((g_{(1)} \otimes x_{(1)}) \cdot b)((g_{(2)}(x_{(2)} \triangleright h_{(1)}) \otimes (x_{(3)} \triangleleft h_{(2)})y) \cdot a),$$

and if this holds, choosing $g = 1_H$, we have that

$$x \cdot_L (b(h \cdot_H y \cdot_L a)) \quad = \quad \sum (x_{(1)} \cdot_L b)((x_{(2)} \triangleright h_{(1)}) \cdot_H (x_{(3)} \triangleleft h_{(2)})y) \cdot_L a)$$

Conversely, if

$$\sum (x_{(1)} \cdot_L b)(x_{(2)} \cdot_L h \cdot_H y \cdot_L a) \quad = \quad \sum (x_{(1)} \cdot_L b)((x_{(2)} \triangleright h_{(1)}) \cdot_H ((x_{(3)} \triangleleft h_{(2)})y) \cdot_L a),$$

then the map $(h \otimes x) \cdot a = h \cdot_h x \cdot_L a$ determine a partial action if and only if

$$\begin{aligned}
(g \otimes x) \cdot (b((h \otimes y) \cdot a)) &= g \cdot_H x \cdot_L (b(h \cdot_H y \cdot_L a)) \\
&= g \cdot_H (\sum (x_{(1)} \cdot_L b)((x_{(2)} \triangleright h_{(1)}) \cdot_H ((x_{(3)} \triangleleft h_{(2)})y) \cdot_L a)) \\
&= \sum (g_{(1)} \cdot_H x_{(1)} \cdot_L b)(g_{(2)} \cdot_H (x_{(2)} \triangleright h_{(1)}) \cdot_H ((x_{(3)} \triangleleft h_{(2)})y) \cdot_L a) \\
&= \sum (g_{(1)} \cdot_H x_{(1)} \cdot_L b)(g_{(2)}(x_{(2)} \triangleright h_{(1)}) \cdot_H ((x_{(3)} \triangleleft h_{(2)})y) \cdot_L a) \\
&= \sum ((g_{(1)} \otimes x_{(1)}) \cdot b)((g_{(2)}(x_{(2)} \triangleright h_{(1)}) \otimes (x_{(3)} \triangleleft h_{(2)})y) \cdot a).
\end{aligned}$$

□

Now that the necessary and sufficient conditions for a global action of H and a partial action of L determine a partial action of $H \bowtie L$ on a specific class of associative algebras are determined, we will now consider the case when the action of H is partial and the action of L is global.

Definition 5.11. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. Let A be a partial H-module algebra with partial action $\cdot_H$ and an L-module algebra with action $\cdot_L$. If there exist a partial action of $H \bowtie L$ such that restrict to H and L we recover the original partial actions, we will say that $(\cdot_H, \cdot_L)$ is an admissible pair of partial actions of type 2.*

As for the first case, if $(\cdot_H, \cdot_L)$ is an admissible pair of partial actions of type 2, we want to describe the partial action of $H \bowtie L$ using $\cdot_H$ and $\cdot_L$. Here, we will assume that both H and L have bijective antipode.

Lemma 5.12. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras, then*

$$\sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \triangleright S_H(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) = \varepsilon_L(x)h.$$

Proof. Here we will use the second equality of the Lemma 5.12.

$$\begin{aligned}
&\sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \triangleright S(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \\
= \ &\sum (x_{(3)} \triangleleft (S_L^{-1}(x_{(2)}) \triangleright S^{-1}(h_{(2)}))) \triangleright S(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \\
= \ &\sum S_H(x_{(2)} \triangleright S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h)) \\
= \ &\varepsilon_L(x)h.
\end{aligned}$$

□

Lemma 5.13. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras, then*

$$\sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \triangleleft S_H(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) = \varepsilon_H(h)x.$$

Proof. Here we will use the first equality of the Lemma 5.12.

$$\begin{aligned}
& \sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \triangleleft S_H(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \\
= & \sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(3)})) \triangleleft (S_L^{-1}(x_{(1)}) \triangleleft S_H^{-1}(h_{(2)}) \triangleright h_{(1)}) \\
= & \sum S_L(S^{-1}(x) \triangleleft S^{-1}(h_{(2)}) \triangleleft h_{(1)}) \\
= & \ \varepsilon_H(h)x.
\end{aligned}$$

□

Lemma 5.14. *Let A be a partial H-module algebra with partial action $\cdot_H$ and an L-module algebra with action $\cdot_L$. If $(\cdot_H, \cdot_L)$ is an admissible pair of partial actions of type 2, then we have that*

$$x \cdot_L h \cdot_H a = \sum((x_{(1)} \triangleright h_{(1)}) \otimes (x_{(2)} \triangleleft h_{(2)})) \cdot a,$$

if and only if,

$$\sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \cdot_L S_H(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \cdot_H a = (h \otimes x) \cdot a.$$

Proof. Suppose that the first equality holds, then

$$\begin{aligned}
& \sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \cdot_L S(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \cdot_H a \\
= & \sum [S_L(S_L^{-1}(x_{(3)}) \triangleleft S_H^{-1}(h_{(3)})) \triangleright S(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \otimes S_L(S_L^{-1}(x_{(4)}) \triangleleft S_H^{-1}(h_{(4)})) \triangleleft S(S_L^{-1}(x_{(2)}) \triangleright S_H^{-1}(h_{(2)}))] \cdot a \\
= & \sum [S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \triangleright S(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \otimes S_L(S_L^{-1}(x_{(4)}) \triangleleft S_H^{-1}(h_{(4)})) \triangleleft S(S_L^{-1}(x_{(3)}) \triangleright S_H^{-1}(h_{(3)}))] \cdot a \\
= & \sum \varepsilon_L(x_{(1)})h_{(1)} \otimes \varepsilon_H(h_{(2)})x_{(2)} \cdot a \\
= & \ h \otimes x \cdot a.
\end{aligned}$$

To prove the converse, we begin with $\sum(x_{(1)} \triangleright h_{(1)})(x_{(2)} \triangleleft h_{(2)}) \cdot a$, apply the hypothesis and use the properties of matched pair of Hopf algebras and Lemma 5.13. □

Lemma 5.15. *Let A be a partial H-module algebra with partial action $\cdot_H$ and an L-module algebra with action $\cdot_L$. If $(\cdot_H, \cdot_L)$ is an admissible pair of partial actions of type 2 and $r(A) = 0$, then*

$$\sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \cdot_L S_H(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \cdot_H a = (h \otimes x) \cdot a.$$

Proof. Note that

$$\begin{aligned}
b(x \cdot_L h \cdot_H a) &= \sum(x_{(1)}S_L^{-1}(x_{(2)}) \cdot_L b)(x_{(3)} \cdot_L h \cdot_H a) \\
&= \sum(x_{(1)} \cdot_L S_L^{-1}(x_{(2)}) \cdot_L b)(x_{(3)} \cdot_L h \cdot_H a) \\
&= \sum((1_H \otimes x_{(1)}) \cdot (1_H \otimes S_L^{-1}(x_{(2)}) \cdot b)((1_H \otimes x_{(3)}) \cdot (h \otimes 1_L) \cdot a) \\
&= \sum(1_H \otimes x_{(1)}) \cdot ((1_H \otimes S_L^{-1}(x_{(2)}) \cdot b)((h \otimes 1_L) \cdot a)) \\
&= \sum((1_H \otimes x_{(1)}) \cdot (1_H \otimes S_L^{-1}(x_{(2)}) \cdot b)((1_H \otimes x_{(3)})(h \otimes 1_L) \cdot a) \\
&= \sum((1_H \otimes x_{(1)}S_L^{-1}(x_{(2)}) \cdot b)((1_H \otimes x_{(3)})(h \otimes 1_L) \cdot a)
\end{aligned}$$

$$= \quad b((1_H \otimes x)(h \otimes 1_L) \cdot a).$$

Since $r(A) = 0$, we have that $x \cdot_L h \cdot_H a = (1_H \otimes x)(h \otimes 1_L) \cdot a$ and then we use the previous lemma. $\square$

Note that as for the case of admissible partial actions of type 1, if we assume that A has unit, then L don't need to have bijective antipode.

Proposition 5.16. *Let A be a partial H-module algebra with partial action $\cdot_H$ and an L-module algebra with action $\cdot_L$. If $r(A) = 0$, then $(\cdot_H, \cdot_L)$ is an admissible pair of partial actions of type 2 if and only if the map*

$$h \otimes x \cdot a = \sum S_L(S_L^{-1}(x_{(2)}) \triangleleft S_H^{-1}(h_{(2)})) \cdot_L S_H(S_L^{-1}(x_{(1)}) \triangleright S_H^{-1}(h_{(1)})) \cdot_H a$$

determines a partial action.

When we work with symmetrical partial actions of $H \bowtie L$ on an associative algebra A, then the partial actions of H and L must also be symmetrical. Hence we can define the concept of admissible pair of symmetrical partial actions of type 1 (respect. type 2), and to determine how the induced symmetrical partial action of $H \bowtie L$ would work, we may ask for $l(A) = 0$ and the antipode of H (respect. of L) don't need to be bijective, because we can use

$$\begin{aligned}
(h \cdot_H x \cdot_L a)b &= \sum (h_{(1)} \cdot_H x \cdot_L a)(h_{(2)} S_{(h_{(3)})} b) \\
&= \sum h_{(1)} \cdot_H ((x \cdot_L a)(S_H(h_{(2)}) \cdot_H b)) \\
&= \sum ((h_{(1)} \otimes x) \cdot a)(h_{(2)} S_{(h_{(3)})} b) \\
&= ((h \otimes x) \cdot a)b,
\end{aligned}$$

and since $l(A) = 0$, we have that $(h \otimes x) \cdot a = h \cdot_H x \cdot_L a$. But when we have to verify if the map $(h \otimes x) \cdot a = h \cdot_H x \cdot_L a$ determine a symmetrical partial action of $H \bowtie L$, we have one more equation to calculate.

5.3 Partial actions of $\Bbbk^G \# \Bbbk F$

First of all, note that given two Hopf algebras H and L, if H is a left L-module bialgebra and we consider the trivial action of H on L, then (H, L) is a matched pair of Hopf algebras and its associated double crossed product $H \bowtie L$ coincides with the smash product $H \# L$.

Now, we want to understand when two partial actions of the Hopf algebras H and L on an algebra A induce a partial action of $H \# L$ on A. Throughout the study, we notice that is not easy even to determine how the partial action of $H \# L$ would work. Then, first we will work with a Hopf algebra associated to a particular matched pair of groups. For a more general study about Hopf algebras associated to matched pair of groups, see [25] and [9].

Let G, F be finite groups, $\Bbbk$ a field such that the order of G is not divisible by $char\, \Bbbk$ and $\triangleleft : G \times F \to G$ a right action by automorphisms of groups. Then we have a left action of kF on $\Bbbk^G$ given by $x \triangleright p_g = p_{g \triangleleft x^{-1}}$ and the induced smash product $R = \Bbbk^G \# \Bbbk F$. We have that R is the double crossed product associated with the matched pair of Hopf algebras $(\Bbbk^G, \Bbbk F)$ and it is a Hopf algebra with structure given by:

$$\begin{aligned}
\Delta_R(p_g x) &= \sum_{t \in G} p_t x \otimes p_{t^{-1}g} x \\
\varepsilon(p_g x) &= \delta_{g,e}
\end{aligned}$$

$$S(p_g x) \;=\; p_{g^{-1}\triangleleft x}x^{-1},$$

where we write $p_g \otimes x = p_g x$.

We can easily see that if A is an R-module algebra, then A is also a $\Bbbk^G$-module algebra and a $\Bbbk F$-module algebra.

The question is: if A is a partial $\Bbbk^G$-module algebra and a partial $\Bbbk F$-module algebra, is there exist a partial R-module algebra structure on A, such that its restrictions to $\Bbbk^G$ and $\Bbbk F$ recover the original partial actions?

Example 5.17. *Take $G = S_3$, $H = \langle(1,2)\rangle$, $F = \langle(1,2,3)\rangle$ and $A = \Bbbk$. Consider $\triangleleft$ the conjugation action of F on G and the partial actions of $\Bbbk^G$ and $\Bbbk F$ on $\Bbbk$ given by*

$$\begin{aligned} p_g \cdot_G 1 &= \frac{1}{2}\delta_{gH,H},\ \forall g \in G \\ x \cdot_F 1 &= 1,\quad \forall x \in F. \end{aligned}$$

If there exists a R-module algebra structure on A such that its restrictions to $\Bbbk^G$ and $\Bbbk F$ recover the original partial actions, then

$$\begin{aligned} \frac{1}{2} &= (132) \cdot_F p_{(12)} \cdot_G 1 \\ &= (132) \cdot_R p_{(12)} \cdot_R 1 \\ &= ((132) \cdot_R 1)((132)p_{(12)} \cdot_R 1) \\ &= ((132) \cdot_F 1)(p_{(12)\triangleleft(123)}(132) \cdot_R 1) \\ &= p_{(12)\triangleleft(123)}(132) \cdot_R 1 \\ &= p_{(23)}(132) \cdot_R 1. \end{aligned}$$

note that, in the general case,

$$p_g x \cdot_R p_h y \cdot_R 1 \;=\; (p_{g(h^{-1}\triangleleft x^{-1})}x \cdot_R 1)(p_{h\triangleleft x^{-1}}xy \cdot_R 1).$$

Then, taking $x = y = (132)$ and $g = h = (23)$, we have

$$\begin{aligned} \frac{1}{4} &= p_{(23)}(132) \cdot_R p_{(23)}(132) \cdot_R 1 \\ &= (p_{(23)((23)\triangleleft(123))}(132) \cdot_R 1)(p_{(23)\triangleleft(123)}(132)(132) \cdot_R 1) \\ &= (p_{(123)}(132) \cdot_R 1)(p_{(132)}(123) \cdot_R 1). \end{aligned}$$

And since

$$\begin{aligned} p_g x \cdot_R 1 &= (x \cdot_R 1)(p_g x \cdot_R 1) \\ &= x \cdot_R p_{g\triangleleft x} \cdot_R 1 \\ &= p_{g\triangleleft x} \cdot_R 1, \end{aligned}$$

whenever $x \cdot_R 1 = 1$, we have that

$$p_{(123)}(132) \cdot_R 1 \;=\; p_{(123)} \cdot_G 1 = 0.$$

Hence, there is no such partial R-module algebra structure on $\Bbbk$.

The following lemma is presented as an example in [8].

Lemma 5.18. *Let G be a finite group. Then the partial $\Bbbk G$-module algebra structures of $\Bbbk$ are in bijective correspondence with the subgroups of G.*

We already know that every partial action of $\Bbbk^G$ on $\Bbbk$ ($char\,\Bbbk \nmid |G|$) is of the form $p_g \cdot 1 = \frac{1}{|H|}\delta_{gH,H}$, where H is a subgroup of G.

Theorem 5.19. *Let G, F be finite groups, $\Bbbk$ a field such that $char\,\Bbbk \nmid |G|$ and $\triangleleft : G \times F \to G$ a right action by automorphisms of groups. Consider the partial actions*

$$\begin{aligned} \cdot_G : \Bbbk^G \otimes \Bbbk &\to \Bbbk \\ p_g \otimes 1 &\mapsto \frac{1}{|H|}\delta_{gH,H}, \\ \cdot_F : \Bbbk F \otimes \Bbbk &\to \Bbbk \\ x \otimes 1 &\mapsto \delta_{xL,L}, \end{aligned}$$

where H is a subgroup of G, and L is a subgroup of F. Then there exist a partial R-module algebra structure on $\Bbbk$, that its restrictions to $\Bbbk^G$ and $\Bbbk F$ recover the partial actions $\cdot_G$ and $\cdot_F$, if and only if, H is invariant by the action $\triangleleft|_{G\times L}$.

Proof. Suppose that there exist such partial R-module algebra structure on k. Then we must have

$$\begin{aligned} \frac{1}{|H|}\delta_{(g\triangleleft x)H,H}\delta_{xL,L} &= x \cdot_F p_{g\triangleleft x} \cdot_G 1 \\ &= x \cdot_R p_{g\triangleleft x} \cdot_R 1 \\ &= (x \cdot_R 1)(x p_{g\triangleleft x} \cdot_R 1) \\ &= (x \cdot_F 1)(p_g x \cdot_R 1) \\ &= \delta_{xL,L} p_g x \cdot_R 1. \end{aligned}$$

Hence, if $x \in L$, we must have that $p_g x \cdot_R 1 = \frac{1}{|H|}\delta_{(g\triangleleft x)H,H}$. Then, for every $g, h \in G$ and $x, y \in L$, we have that

$$\begin{aligned} \frac{1}{|H|^2}\delta_{(g\triangleleft x)H,H}\delta_{(h\triangleleft y)H,H} &= p_g x \cdot_R p_h y \cdot_R 1 \\ &= (p_{g(h^{-1}\triangleleft x^{-1})} x \cdot_R 1)(p_{h\triangleleft x^{-1}} xy \cdot_R 1) \\ &= \frac{1}{|H|^2}\delta_{((g\triangleleft x)h^{-1})H,H}\delta_{(h\triangleleft y)H,H}. \end{aligned}$$

Hence, $g \triangleleft x, h \triangleleft y \in H$ for any $x, y \in L$ if and only if $g \triangleleft x, h \triangleleft y, h \in H$ for any $x, y \in L$. Conversely, if H is invariant by the action $\triangleleft$ restrict to L, the map

$$p_g x \cdot_R 1 = \frac{1}{|H|}\delta_{(g\triangleleft x)H,H}\delta_{xL,L}$$

determines a partial action of R on $\Bbbk$. □

Note that in the example $5, 17$, H is not invariant by the action of F, because

$$(12) \triangleleft (123) = (132)(12)(123) = (13)$$

which is not an element of H.

We will consider now the same problem for a class of partial actions on matrix algebras.

Definition 5.20. *Let $A = Mat_{n\times n}(\Bbbk)$ and G a finite group. A symmetrical partial action $\cdot : \Bbbk G \otimes A \to A$ will be called a walker partial action if $g \cdot E_{ij} = \alpha_{ij}(g)E_{(gi)(gj)}$, where $\alpha_{ij}(g)$ is a scalar that depends of i, j and g, and $E_{(gi)(gj)}$ is determined by an action of G on the index set $\{1, \ldots, n\}$.*

Proposition 5.21. *Let $A = Mat_{n\times n}(\Bbbk)$ and G a finite group. The walker partial $\Bbbk G$-actions on A are in bijective correspondence with triples $(H, \tau, \{\alpha_{i,i+1}\}_{i=1}^{n-1})$, where H is a subgroup of G, τ is an action of G on the index set $\{1, \ldots, n\}$ and each $\alpha_{i,i+1} : H \to \Bbbk^\times$ is a linear map such that*

$$\alpha_{(hi)(hj)}(g)\alpha_{ij}(h) \;=\; \alpha_{ij}(gh),$$

for every $g, h \in H$. Particularly, if τ is the trivial action, then every α_{ij} is a group morphism.

Proof. Let $\cdot : \Bbbk G \otimes A \to A$ be a walker partial action and denote

$$H_{ij} = \{g \in G \,|\, g \cdot E_{ij} \neq 0\} = \{g \in G|\alpha_{ij}(g) \neq 0\}.$$

Since

$$\begin{aligned} g \cdot E_{ii} &= (g \cdot E_{ij})(g \cdot E_{ji}),\\ g \cdot E_{ij} &= (g \cdot E_{ii})(g \cdot E_{ij}),\\ g \cdot E_{ij} &= (g \cdot E_{ij})(g \cdot E_{jj}),\\ g \cdot E_{jj} &= (g \cdot E_{ji})(g \cdot E_{ij}), \end{aligned}$$

we have that

$$\begin{aligned} \alpha_{ii}(g)E_{(gi)(gi)} &= \alpha_{ij}(g)\alpha_{ji}(g)E_{(gi)(gi)},\\ \alpha_{ij}(g)E_{(gi)(gj)} &= \alpha_{ii}(g)\alpha_{ij}(g)E_{(gi)(gj)},\\ \alpha_{ij}(g)E_{(gi)(gj)} &= \alpha_{ij}(g)\alpha_{jj}(g)E_{(gi)(gi)},\\ \alpha_{jj}(g)E_{(gj)(gj)} &= \alpha_{ji}(g)\alpha_{ij}(g)E_{(gj)(gj)}, \end{aligned}$$

for every $i, j \in \{1, \cdots, n\}$, $g \in G$. Hence,

$$\begin{aligned} H_{ii} &\subseteq H_{ij} \cap H_{ji},\\ H_{ij} &\subseteq H_{ii},\\ H_{ij} &\subseteq H_{jj}. \end{aligned}$$

i.e., $H_{ij} = H_{ii} = H_{jj} = H$ for some subset H of G. Now, since

$$\begin{aligned} g \cdot h \cdot E_{ij} &= (g \cdot E_{(hi)(hi)})(gh \cdot E_{ij})\\ &= (gh \cdot E_{ij})(g \cdot E_{(hj)(hj)}), \end{aligned}$$

for every $g, h \in G$, we have that

$$\begin{aligned} \alpha_{(hi)(hj)}(g)\alpha_{ij}(h) &= \alpha_{(hi)(hi)}(g)\alpha_{ij}(gh)\\ &= \alpha_{ij}(gh)\alpha_{(hj)(hj)}(g), \end{aligned}$$

i.e., $H.H \subseteq H$. And, since clearly $1_G \in H$ and $g \cdot g^{-1} \cdot E_{ij} = (g \cdot E_{(g^{-1}i)(g^{-1}i)})E_{ij}$, we have that if $g \in H$, then $g^{-1} \in H$, i.e., H must be a subgroup of G. Moreover, by the equations above, we have that $\alpha_{ii}(g) = \alpha_{ii}(g)^2$, hence $\alpha_{ii}(g) = 0$ or 1 and $\alpha_{ii}(g) = 1$ only when $g \in H$. Additionally, $\alpha_{ij}(g)\alpha_{ji}(g) = \delta_{gH,H}$, i.e., whenever $g \in H$, $\alpha_{ji}(g) = \alpha_{ij}(g)^{-1}$, and since $E_{ij} = E_{(i)(i+1)} \cdots E_{(j-1)(j)}$, we have that

$$\alpha_{ij}(g) = \alpha_{(i)(i+1)}(g) \cdots \alpha_{(j-1)(j)}(g),$$

for every $g \in G$. Hence, the considered walker partial action $\cdot : kG \otimes A \to A$ is determined by a triple $(H, \tau, \{\alpha_{(i)(i+1)}\}_{i=1}^n)$ with the properties of the proposition. Conversely, given a triple $(H, \tau, \{\alpha_{(i)(i+1)}\}_{i=1}^n)$ on the hypothesis of this proposition, the linear map $\cdot : \Bbbk G \otimes A \to A$ given by

$$g \cdot E_{ij} = \delta_{gH,H}\alpha_{ij}(g)E_{(gi)(gi)},$$

where $\alpha_{ij}(g) = \alpha_{(i)(i+1)}(g) \cdots \alpha_{(j-1)(j)}(g)$ if $i < j$ and $\alpha_{ij}(g) = \alpha_{ji}(g)^{-1}$ if $i > j$. determines a walker partial action. $\square$

Theorem 5.22. *Let G, F be finite groups, G abelian, $\Bbbk$ a field such that $char\,\Bbbk \nmid |G|$, $\triangleleft : G \times F \to G$ a right action by automorphisms of groups and $A = Mat_{n\times n}(\Bbbk)$. Consider a good partial G-grading on A determined by a subgroup H of G and a family $\{t_{ij}\}_{i,j=1}^n \subset G$, and a walker partial action of $\Bbbk F$ on A determined by the triple $(L, \tau, \{\alpha_{(i)(i+1)}\}_{i=1}^n)$. Then, there exists a symmetrical partial action of $R = \Bbbk^G \# \Bbbk F$ on A such that its restrictions to $\Bbbk^G$ and $\Bbbk F$ recover the original partial actions, if and only if $t_{ij}H \triangleleft x \subseteq t_{(x^{-1}i)(x^{-1}j)}H$ for every $x \in L$. In this case, H is invariant by the action $\triangleleft|_{G\times L}$.*

Proof. Let $\cdot : R \otimes A \to A$ be a symmetrical partial action that its restrictions to $\Bbbk^G$ and $\Bbbk F$ recover the original partial actions. Then, on one side

$$x \cdot p_{g\triangleleft x} \cdot E_{ij} = \frac{1}{|H|}\delta_{xL,L}\delta_{(g\triangleleft x)H,t_{ij}H}\alpha_{ij}(g)E_{(gi)(gj)},$$

and on the other side

$$\begin{aligned} x \cdot p_{g\triangleleft x} \cdot E_{ij} &= (x \cdot E_{ii})(p_g x \cdot E_{ij}) \\ &= \frac{1}{|H|}\delta_{xL,L} p_g x \cdot E_{ij}. \end{aligned}$$

Hence, whenever $x \in L$, we have that

$$p_g x \cdot E_{ij} = \frac{1}{|H|}\delta_{(g\triangleleft x)H,t_{ij}H}\alpha_{ij}(g)E_{(gi)(gj)}.$$

Then, for every $x, y \in L$, we have that, on one side

$$p_g x \cdot p_h y \cdot E_{ij} = \frac{1}{|H|^2}\delta_{(g\triangleleft x)H,t_{(yi)(yj)}H}\delta_{(h\triangleleft y)H,t_{ij}H}\alpha_{(yi)(yj)}(x)\alpha_{ij}(y)E_{(xyi)(xyj)},$$

and on the other side,

$$\begin{aligned} p_g x \cdot p_h y \cdot E_{ij} &= \sum_{t\in G}(p_t x \cdot E_{(yi)(yi)})(p_{t^{-1}g} x p_h y \cdot E_{ij}) \\ &= \sum_{t\in G}(p_t x \cdot E_{(yi)(yi)})(p_{t^{-1}g} p_{h\triangleleft x^{-1}} xy \cdot E_{ij}) \\ &= \sum_{t\in G}(p_t x \cdot E_{(yi)(yi)})(\delta_{t^{-1}g,h\triangleleft x^{-1}} p_{h\triangleleft x^{-1}} xy \cdot E_{ij}) \\ &= (p_{g(h^{-1}\triangleleft x^{-1})} x \cdot E_{(yi)(yi)})(p_{h\triangleleft x^{-1}} xy \cdot E_{ij}) \end{aligned}$$

$$= \frac{1}{|H|^2}\delta_{((g\triangleleft x)h^{-1})H,H}\delta_{(h\triangleleft y)H,t_{ij}H}\alpha_{ij}(xy)E_{(xyi)(xyi)}.$$

Hence, $g\triangleleft x \in t_{(yi)(yj)}H$ and $h\triangleleft y \in t_{ij}H$ if and only if $(g\triangleleft x)h^{-1} \in H$ and $h\triangleleft y \in t_{ij}H$. Particularly, if $y \in L$, for every $h\triangleleft y \in t_{ij}H$, since $t_{(yi)(yj)} \in t_{(yi)(yj)}H$, we have that $t_{(yi)(yj)}h^{-1} \in H$, i.e., $h \in t_{(yi)(yj)}H$. Hence, if $h \in t_{ij}H$, then $h\triangleleft y\triangleleft y^{-1} \in t_{ij}H$ and $t_{(y^{-1}i)(y^{-1}j)} \in t_{(y^{-1}i)(y^{-1}j)}H$, consequently, $h \triangleleft y \in t_{(y^{-1}i)(y^{-1}j)}H$. Therefore $t_{ij}H \triangleleft y \subseteq t_{(y^{-1}i)(y^{-1}j)}H$, for every $y \in L$, $i,j = 1,\cdots,n$. Conversely, suppose that $t_{ij}H \triangleleft x \subseteq t_{(x^{-1}i)(x^{-1}j)}H$ for every $x \in L$. Then, the linear map $\cdot : R \otimes A \to A$ given by

$$pgx \cdot E_{ij} = \frac{1}{|H|^2}\delta_{xL,L}\delta_{(g\triangleleft x)H,t_{ij}H}\alpha_{ij}(x)E_{(xi)(xj)},$$

determines a symmetrical partial action. In fact, clearly $1_R \cdot E_{ij} = E_{ij}$, and

$$p_g x \cdot p_h y \cdot E_{ij} = \frac{1}{|H|^2}\delta_{xL,L}\delta_{yL,L}\delta_{(g\triangleleft x)H,t_{(yi)(yj)}H}\delta_{(h\triangleleft y)H,t_{ij}H}\alpha_{(yi)(yj)}(x)\alpha_{ij}(y)E_{(xyi)(xyj)},$$

and on the other side,

$$\begin{aligned}
\sum_{t\in G}(p_t x \cdot E_{(yi)(yi)})(p_{t^{-1}g}xp_h y \cdot E_{ij}) &= \sum_{t\in G}(p_t x \cdot E_{(yi)(yi)})(p_{t^{-1}g}p_{h\triangleleft x^{-1}}xy \cdot E_{ij})\\
&= \sum_{t\in G}(p_t x \cdot E_{(yi)(yi)})(\delta_{t^{-1}g,h\triangleleft x^{-1}}p_{h\triangleleft x^{-1}}xy \cdot E_{ij})\\
&= (p_{g(h^{-1}\triangleleft x^{-1})}x \cdot E_{(yi)(yi)})(p_{h\triangleleft x^{-1}}xy \cdot E_{ij})\\
&= \frac{1}{|H|^2}\delta_{xL,L}\delta_{(xy)L,L}\delta_{((g\triangleleft x)h^{-1})H,H}\delta_{(h\triangleleft y)H,t_{ij}H}\alpha_{ij}(xy)E_{(xyi)(xyi)}.
\end{aligned}$$

Since L is a subgroup of F, it follows that $\delta_{xL,L}\delta_{yL,L} = \delta_{xL,L}\delta_{(xy)L,L}$, and we already have that $\alpha_{(yi)(yj)}(x)\alpha_{ij}(y) = \alpha_{ij}(xy)$, whenever $x, y \in L$. Finally,

$$\begin{array}{rcl}
(g \triangleleft x)h^{-1} \in H & \text{and} & h \triangleleft y \in t_{ij}H\\
& \Updownarrow & \\
(g \triangleleft x)h^{-1} \in H & \text{and} & h \triangleleft y \in t_{ij}H \text{ and } h \in t_{(yi)(yj)}H\\
& \Updownarrow & \\
g \triangleleft x \in t_{(yi)(yj)}H & \text{and} & h \triangleleft y \in t_{ij}H \text{ and } h \in t_{(yi)(yj)}H\\
& \Updownarrow & \\
g \triangleleft x \in t_{(yi)(yj)}H & \text{and} & h \triangleleft y \in t_{ij}H,
\end{array}$$

whenever $y \in L$. In other words, $\delta_{((g\triangleleft x)h^{-1})H,H}\delta_{(h\triangleleft y)H,t_{ij}H} = \delta_{(g\triangleleft x)H,t_{(yi)(yj)}H}\delta_{(h\triangleleft y)H,t_{ij}H}$, whenever $y \in L$. Therefore,

$$p_g x \cdot p_h y \cdot E_{ij} = \sum_{t\in G}(p_t x \cdot E_{(yi)(yi)})(p_{t^{-1}g}xp_h y \cdot E_{ij}).$$

For the symmetrical property, we use a similar argument. $\square$

Note that Theorem 5.19 is actually a consequence of the Theorem 5.22

Remark 5.23. *Consider the assumptions of the Theorem 5.19, with $A = \Bbbk$. Then, we have that the globalization of $\cdot_G$ is the pair $(\Bbbk H, \theta_1)$ where $\theta_1(1) = \sum_{h\in H}\frac{1}{|H|}h$, and the globalization of $\cdot_F$ is the pair $(\Bbbk^L, \theta_2)$ where $\theta_2(1) = \sum_{x\in L}p_x$. Conversely, the globalization of $\cdot_R$ is (B, ρ) where $\rho(1) = \sum_{x\in L, g\in H}\frac{1}{|H|}gp_x$, i.e., $B = \Bbbk H \otimes \Bbbk^L$, and the structure of R-module of B is given by $p_g x \rightharpoonup hp_y = hp_{yx^{-1}}\delta_{g,h\triangleleft yx^{-1}}$.*

5.4 Representations of $H \bowtie L$

Now we will investigate when a representation of H and a representation of L induce a representation of $H \bowtie L$, as was done for global action. For this, we will always assume that $(H, L, \triangleright, \triangleleft)$ is a matched pair of Hopf algebras.

Definition 5.24. *A representation of a Hopf algebra H on a unital algebra A is a linear map $\pi : H \to A$ such that $\pi(hk) = \pi(h)\pi(k)$ and $\pi(1_H) = 1_A$.*

Definition 5.25. *Let A be an algebra with unit and suppose that $\pi_H : H \to A$ and $\pi_L : L \to A$ are representations. If there exist a representation $\pi : H \bowtie L \to A$ that restrict to H and L we recover π_H and π_L, respectively, i.e., that π_H and π_L induce π, we will say that (π_H, π_L) is an admissible pair of representations.*

Note that if (π_H, π_L) is an admissible pair of representations, then we must have that

$$\begin{aligned} \pi(h \otimes x) &= \pi(h \otimes 1_L)\pi(1_H \otimes x) \\ &= \pi_H(h)\pi_L(x), \end{aligned}$$

and

$$\begin{aligned} \pi_L(x)\pi_H(h) &= \pi(1_H \otimes x)\pi(h \otimes 1_L) \\ &= \sum \pi((x_{(1)} \triangleright h_{(1)}) \otimes (x_{(2)} \triangleleft h_{(2)})) \\ &= \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)}). \end{aligned}$$

Proposition 5.26. *Let A be an algebra with unit and $\pi_H : H \to A$ and $\pi_L : L \to A$ representations. Then (π_H, π_L) is an admissible pair of representations if and only if*

$$\pi_L(x)\pi_H(h) = \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)}).$$

Proof. The proof of the necessary condition is straightforward. For the sufficient condition, define $\pi : H \bowtie L \to A$ by $\pi(h \otimes x) = \pi_H(h)\pi_L(x)$ and assume that $\pi_L(x)\pi_H(h) = \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)})$. Since both π_H and π_L are representations, we have that π is also a representation. □

5.5 Partial representations of $H \bowtie L$

As for partial actions, if $\pi : H \bowtie L \to A$ is a partial representation, then π restricted to H and L results in partial representations, but for the converse, if we have partial representations of H and L and ask for them to induce a partial representation of $H \bowtie L$, we can't even determine how this new partial representation would work. Then we will assume that one of the original partial representations is actually a representation.

Definition 5.27 ([6])**.** *Let H be a Hopf algebra with antipode S and A a unital algebra. A linear map $\pi : H \to A$ is a partial representation of H on B if*

1. $\pi(1_H) = 1_A$;
2. $\sum \pi(h)\pi(k_{(1)})\pi(S(k_{(2)})) = \sum \pi(hk_{(1)})\pi(S(k_{(2)}))$;
3. $\sum \pi(k_{(1)})\pi(S(k_{(2)}))\pi(h) = \sum \pi(k_{(1)})\pi(S(k_{(2)})h)$;

4. $\sum \pi(h)\pi(S(k_{(1)}))\pi(k_{(2)}) = \sum \pi(hS(k_{(1)}))\pi(k_{(2)})$;
5. $\sum \pi(S(k_{(1)}))\pi(k_{(2)})\pi(h) = \sum \pi(S(k_{(1)}))\pi(k_{(2)}h)$,

for every $h, k \in H$.

Remark 5.28 ([6]). *If H is cocommutative, then the items in the definition of a partial representation coalesce into* 1), 2) *and* 5).

From now on, we will denote by $\pi_H : H \to A$ and $\pi_L : L \to A$ partial representations.

Definition 5.29. *Suppose that π_H and π_L induce a partial representation of $H \bowtie L$. If π_H is a representation, we will say that (π_H, π_L) is an admissible pair of partial representations of type* 1*. If π_L is a representation, we will say that (π_H, π_L) is an admissible pair of partial representations of type* 2*.*

In contrast to the case of partial actions, something truly remarkable happens when we consider the two types of admissible pair of partial representations.

Let (π_H, π_L) be an admissible pair of partial representations of type 1 and $\pi : H \bowtie L \to A$ the induced partial representation. Then we have that $\pi(1_H \otimes 1_L) = id_A$ and

$$\begin{aligned}
\pi(h \otimes x) &= \pi(1_H \otimes 1_L)\pi(h \otimes x) \\
&= \sum \pi(h_{(1)}S_H(h_{(2)}) \otimes 1_L)\pi(h_{(3)} \otimes x) \\
&= \sum \pi_H(h_{(1)}S_H(h_{(2)}))\pi(h_{(3)} \otimes x) \\
&= \sum \pi_H(h_{(1)})\pi_H(S_H(h_{(2)}))\pi(h_{(3)} \otimes x) \\
&= \sum \pi(h_{(1)} \otimes 1_L)\pi(S_H(h_{(2)}) \otimes 1_L)\pi(h_{(3)} \otimes x) \\
&= \sum \pi(h_{(1)} \otimes 1_L)\pi(S_H(h_{(2)}) \otimes 1_L)\pi(h_{(3)} \otimes 1_L)\pi(1_H \otimes x) \\
&= \pi(h \otimes 1_L)\pi(1_H \otimes x) \\
&= \pi_H(h)\pi_L(x),
\end{aligned}$$

and

$$\begin{aligned}
\pi_L(x)\pi_H(h) &= \pi(1_H \otimes x)\pi(h \otimes 1_L) \\
&= \sum \pi(1_H \otimes x)\pi(h_{(1)} \otimes 1_L)\pi(S_H(h_{(2)}) \otimes 1_L)\pi(h_{(3)} \otimes 1_L) \\
&= \sum \pi((1_H \otimes x)(h_{(1)} \otimes 1_L))\pi(S_H(h_{(2)}) \otimes 1_L)\pi(h_{(3)} \otimes 1_L) \\
&= \sum \pi((1_H \otimes x)(h_{(1)} \otimes 1_L))\pi_H(S_H(h_{(2)}))\pi_H(h_{(3)}) \\
&= \sum \pi((1_H \otimes x)(h_{(1)} \otimes 1_L))\pi_H(S_H(h_{(2)})h_{(3)}) \\
&= \pi((1_H \otimes x)(h \otimes 1_L)) \\
&= \sum \pi((x_{(1)} \triangleright h_{(1)}) \otimes (x_{(2)} \triangleleft h_{(2)})) \\
&= \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)}).
\end{aligned}$$

Now, let (π_H, π_L) be an admissible pair of partial representations of type 2 and consider $\pi : H \bowtie L \to A$ the induced partial representation. Then we have that

$$\pi(h \otimes x) = \sum \pi(h \otimes x_{(1)})\pi(1_H \otimes x_{(2)}S_L(x_{(3)}))$$

$$
\begin{aligned}
&= \sum \pi(h \otimes x_{(1)})\pi_H(x_{(2)}S_L(x_{(3)})) \\
&= \sum \pi(h \otimes x_{(1)})\pi_H(x_{(2)})\pi_L(S_L(x_{(3)})) \\
&= \sum \pi(h \otimes x_{(1)})\pi(1_H \otimes x_{(2)})\pi_L(1_H \otimes S_L(x_{(3)})) \\
&= \sum \pi(h \otimes 1_L)\pi(1_H \otimes x_{(1)})\pi(1_H \otimes x_{(2)})\pi_L(1_H \otimes S_L(x_{(3)})) \\
&= \pi(h \otimes 1_L)\pi(1_H \otimes x) \\
&= \pi_H(h)\pi_L(x),
\end{aligned}
$$

and

$$
\begin{aligned}
\pi_L(x)\pi_H(h) &= \sum \pi_L(x_{(1)}S_L(x_{(2)}))\pi_L(x_{(3)})\pi_H(h) \\
&= \sum \pi_L(x_{(1)})\pi_L(S_L(x_{(2)}))\pi_L(x_{(3)})\pi_H(h) \\
&= \sum \pi(1_H \otimes x_{(1)})\pi(1_H \otimes S_L(x_{(2)}))\pi(1_H \otimes x_{(3)})\pi(h \otimes 1_L) \\
&= \sum \pi(1_H \otimes x_{(1)})\pi(1_H \otimes S_L(x_{(2)}))\pi((1_H \otimes x_{(3)})(h \otimes 1_L)) \\
&= \pi((1_H \otimes x)(h \otimes 1_L)) \\
&= \sum \pi((x_{(1)} \triangleright h_{(1)}) \otimes (x_{(2)} \triangleleft h_{(2)})) \\
&= \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)}).
\end{aligned}
$$

I.e., the same equations are necessary conditions for (π_H, π_L) be either an admissible pair of partial representations of type 1 or of type 2.

Proposition 5.30. *Let A be a unital algebra and $\pi_H : H \to A$ and $\pi_L : L \to A$ partial representations. Then*

1. *(π_H, π_L) is an admissible pair of partial representations of type 1 if and only if π_H is a representation and*

$$\pi_L(x)\pi_H(h) = \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)}).$$

2. *(π_H, π_L) is an admissible pair of partial representations of type 2 if and only if π_L is a representation and*

$$\pi_L(x)\pi_H(h) = \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)}).$$

Proof. In both cases, the necessary condition is already proved. For the sufficient condition, we will prove the second axiom of partial representation for the first item, because the other axioms and the axioms of the second item are proved analogously. For item 1), suppose that π_H is a representation, $\pi_L(x)\pi_H(h) = \sum \pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)})$ and define $\pi : H \bowtie L \to A$ by $\pi(h \otimes x) = \pi_H(h)\pi_L(x)$, then the axiom 2) of partial representation is

$$\sum \pi(g \otimes x)\pi(h_{(1)} \otimes y_{(1)})\pi(S(h_{(2)} \otimes y_{(2)})) = \sum \pi((g \otimes x)(h_{(1)} \otimes y_{(1)}))\pi(S(h_{(2)} \otimes y_{(2)})),$$

And this equality holds if and only if

$$
\begin{aligned}
&\sum \pi_H(g)\pi_L(x)\pi_H(h_{(1)})\pi_L(y_{(1)})\pi_L(S_L(y_{(2)}))\pi_H(S_H(h_{(2)})) = \\
&\sum \pi_H(g(x_{(1)} \triangleright h_{(1)}))\pi_L((x_{(2)} \triangleleft h_{(2)})y_{(1)}))\pi_L(S_L(y_{(2)}))\pi_H(S_H(h_{(3)}))
\end{aligned}
$$

and this holds if and only if

$$\sum \pi_H(g)\pi_H(x_{(1)} \triangleright h_{(1)})\pi_L(x_{(2)} \triangleleft h_{(2)})\pi_L(y_{(1)})\pi_L(S_L(y_{(2)}))\pi_H(S_H(h_{(3)})) =$$
$$\sum \pi_H(g(x_{(1)} \triangleright h_{(1)}))\pi_L((x_{(2)} \triangleleft h_{(2)})y_{(1)}))\pi_L(S_L(y_{(2)}))\pi_H(S_H(h_{(3)})).$$

Here we use that

$$\begin{aligned}
\pi(S(h \otimes y)) &= \pi((1_H \otimes S_L(y))(S_H(h) \otimes 1_L)) \\
&= \sum \pi((S_L(y_{(2)}) \triangleright S_H(h_{(2)})) \otimes (S_L(y_{(1)}) \triangleleft S_H(h_{(1)}))) \\
&= \sum \pi_H(S_L(y_{(2)}) \triangleright S_H(h_{(2)}))\pi_L(S_L(y_{(1)}) \triangleleft S_H(h_{(1)})) \\
&= \pi_L(S_L(y))\pi_H(S_H(h)).
\end{aligned}$$

□

5.6 Two subcategories of the category of the partial $H \bowtie L$-modules.

In the previous section we fully describe the partial representations of $H \bowtie L$ induced by a pair of partial representations where one of them are actually a representation. In this section, we will show that the subcategory of the partial representations mentioned before, is isomorphic to a category of modules over some algebra.

Definition 5.31. *Let H be a Hopf algebra. A partial H-module is a vector space M with a partial representation $\pi : H \to End(M)$.*

Definition 5.32. *Let H be a Hopf algebra and M and N two partial H-modules with partial representations π_1 and π_2, respectively. A morphism of partial H-modules frome M to N is a linear map $f : M \to N$ such that $f \circ \pi_1(h) = \pi_2(h) \circ f$, for every $h \in H$.*

Since H is an L-module, there exists a linear morphism (actually an algebra morphism) $\varphi : L \to End(H)$, and this morphism induces an algebra morphism $\overline{\varphi} : T(L) \to End(H)$, i.e., H is an $T(L)$-module algebra with action $(x^1 \otimes \cdots \otimes x^n) \blacktriangleright h = x^1 \triangleright (x^2 \triangleright (\cdots (x^n \triangleright h) \cdots)) = (x^1 \cdots x^n) \triangleright h$.

Remember that, by [6], the Hopf algebroid L_{par} is isomorphic to $T(L)/I$, where I is the ideal generated by elements of the form:

- $\sum x \otimes y_{(1)} \otimes S_L(y_{(2)}) - xy_{(1)} \otimes S_L(y_{(2)})$;
- $\sum x \otimes S_L(y_{(1)}) \otimes y_{(2)} - xS_L(y_{(1)}) \otimes y_{(2)}$;
- $\sum S_L(x_{(1)}) \otimes x_{(2)} \otimes y - S_L(x_{(1)}) \otimes x_{(2)}y$;
- $\sum x_{(1)} \otimes S_L(x_{(2)}) \otimes y - x_{(1)} \otimes S_L(x_{(2)})y$.

As in [6], we will denote by $[x^1] \cdots [x^n]$ the class of $x^1 \otimes \cdots \otimes x^n$.

Remark 5.33. *Also in [6], the authors proved that the category of the partial L-modules is isomorphic to the category of the L_{par}-modules.*

Note that the action $\blacktriangleright$ factors through the ideal I, then we actually have an action of L_{par} on H, that we will also denote by $\blacktriangleright$.

Recall from the properties of a matched pair of Hopf algebras that for every $x, y, x \in L$, $h \in H$,

$$\begin{aligned}(x(yz)) \triangleleft h &= \sum (x \triangleleft (y_{(1)}z_{(1)} \triangleright h_{(1)}))(y_{(2)}z_{(2)} \triangleleft h_{(2)}) \\ &= \sum (x \triangleleft (y_{(1)}z_{(1)} \triangleright h_{(1)}))(y_{(2)} \triangleleft (z_{(2)} \triangleright h_{(2)})(z_{(3)} \triangleleft h_{(3)}).\end{aligned}$$

Now, consider the linear map $\blacktriangleleft$: $L_{par} \otimes H \to L_{par}$ given by

$$[x^1] \cdots [x^n] \blacktriangleleft h = \sum [x^1 \triangleleft (x^2_{(1)} \cdots x^n_{(1)} \triangleright h_{(1)})] \cdot [x^{n-1}_{(n-1)} \triangleleft (x^n_{(n-1)} \triangleright h_{(n-1)})][x^n_{(n)} \triangleleft h_{(n)}].$$

Note that it's actually an action, because it's induced by the action $\triangleleft$ and we can see the similarities with $x^1 \cdots x^n \triangleleft h$. We only need to verify that this action is well defined, i.e., that $X \blacktriangleleft h = 0$ for every $X \in I$ and $h \in H$. For this, we will need the following lemma.

Lemma 5.34. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. Then*

- $S_L(x) \triangleleft h = \sum S_L(x_{(1)} \triangleleft (S_L(x_{(2)}) \triangleright h))$;
- $x \triangleright S_H(h) = \sum S_H((x \triangleleft S_H(h_{(1)})) \triangleright h_{(2)})$.

Proof. We will only prove the first item, because the second is proved analogously. Then, by Lemma 5.5, we have that

$$\begin{aligned}\sum S_L(x_{(1)} \triangleleft (S_L(x_{(2)}) \triangleright h)) &= \sum S_L(x_{(1)}) \triangleleft (x_{(2)} \triangleright S_L(x_{(3)}) \triangleright h) \\ &= S_L(x) \triangleleft h.\end{aligned}$$

□

Now, to prove that the action $\blacktriangleleft$ is well defined, we will verify that every element of H acting on the generators of I is zero. In fact,

$$\begin{aligned}
& \sum x \otimes y_{(1)} \otimes S_L(y_{(2)}) \blacktriangleleft h \\
= & \sum x \triangleleft (y_{(1)}S_L(y_{(5)}) \triangleright h_{(1)}) \otimes y_{(2)} \triangleleft (S_L(y_{(4)}) \triangleright h_{(2)}) \otimes S_L(y_{(3)}) \triangleleft h_{(3)} \\
= & \sum x \triangleleft (y_{(1)}S_L(y_{(6)}) \triangleright h_{(1)}) \otimes y_{(2)} \triangleleft (S_L(y_{(5)}) \triangleright h_{(2)}) \otimes S_L(y_{(3)} \triangleleft (S_L(y_{(4)} \triangleright h_{(3)}))) \\
= & \sum x \triangleleft (y_{(1)}S_L(y_{(5)}) \triangleright h_{(1)}) \otimes (y_{(2)} \triangleleft (S_L(y_{(4)}) \triangleright h_{(2)}))_{(1)} \otimes S_L((y_{(2)} \triangleleft (S_L(y_{(4)} \triangleright h_{(2)})))_{(2)}) \\
= & \sum (x \triangleleft (y_{(1)}S_L(y_{(5)}) \triangleright h_{(1)}))(y_{(2)} \triangleleft (S_L(y_{(4)}) \triangleright h_{(2)}))_{(1)} \otimes S_L((y_{(2)} \triangleleft (S_L(y_{(4)} \triangleright h_{(2)})))_{(2)}) \\
= & \sum (x \triangleleft (y_{(1)}S_L(y_{(6)}) \triangleright h_{(1)}))(y_{(2)} \triangleleft (S_L(y_{(5)}) \triangleright h_{(2)})) \otimes S_L(y_{(3)} \triangleleft (S_L(y_{(4)} \triangleright h_{(3)}))) \\
= & \sum (x \triangleleft (y_{(1)}S_L(y_{(5)}) \triangleright h_{(1)}))(y_{(2)} \triangleleft (S_L(y_{(4)}) \triangleright h_{(2)})) \otimes S_L(y_{(3)}) \triangleleft h_{(3)} \\
= & \sum xy_{(1)} \triangleleft (S_L(y_{(3)}) \triangleright h_{(1)}) \otimes S_L(y_{(2)}) \triangleleft h_{(2)} \\
= & \sum xy_{(1)} \otimes S_L(y_{(2)}) \blacktriangleleft h.
\end{aligned}$$

For the other generators of I, we use similar calculations.

With this, we define the algebra $H \boxtimes^* L_{par}$, that is $H \otimes L_{par}$ as a vector space and the product is given by

$$(h \otimes [x^1] \cdots [x^n])(k \otimes [y^1] \cdots [y^m]) = \sum h([x^1_{(1)}] \cdots [x^n_{(1)}] \blacktriangleright k_{(1)}) \otimes ([x^1_{(2)}] \cdots [x^n_{(2)}] \blacktriangleleft k_{(2)})[y^1] \cdots [y^m].$$

Analogously, we can define ${H_{par}}^* \boxtimes L$

We will denote by $\mathcal{P}$ the category of the partial $H \bowtie L$-modules in which the partial representations are induced by an admissible pair of partial actions of type 1, and $\mathcal{Q}$ the category of the partial $H \bowtie L$-modules in which the partial representations are induced by an admissible pair of partial actions of type 2.

Proposition 5.35. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. Then $\mathcal{P} \cong {}_{H\boxtimes^* L_{par}}\mathcal{M}$.*

Proof. Consider the functors

$$\begin{array}{rcl} F : \mathcal{P} & \to & {}_{H\boxtimes^* L_{par}}\mathcal{M} \\ (M, \pi) & \mapsto & (M, \bullet), \end{array}$$

where $(h \otimes [x^1] \cdots [x^n]) \bullet m = \pi(h) \circ \pi(x^1) \circ \cdots \circ \pi(x^n)(m)$, and

$$\begin{array}{rcl} G : {}_{H\boxtimes^* L_{par}}\mathcal{M} & \to & \mathcal{P} \\ (M, \cdot) & \mapsto & (M, \pi_0), \end{array}$$

where $\pi_0(h \otimes x)(m) = h \cdot [x] \cdot m$. Note that, given $(M, \pi) \in \mathcal{P}$,

$$\begin{array}{rcl} \pi_0(h \otimes x)(m) & = & h \bullet [x] \bullet m \\ & = & \pi(h)\pi(x)(m) \\ & = & \pi(h \otimes x)(m), \end{array}$$

then $FG \simeq Id_{\mathcal{P}}$ in the objects. Additionally, given $(M, \cdot) \in {}_{H\boxtimes^* L_{par}}\mathcal{M}$, we have that

$$\begin{array}{rcl} h \otimes [x^1] \cdots [x^n] \bullet m & = & \pi_0(h)\pi_0(x^1) \cdots \pi_0(x^n)(m) \\ & = & h \cdot [x^1] \cdots [x^n] \cdot m \\ & = & (h \otimes [x^1] \cdots [x^n]) \cdot m, \end{array}$$

hence $GF \simeq Id_{{}_{H\boxtimes^* L_{par}}\mathcal{M}}$ in the objects. For the morphisms, consider (M, π_1) and (N, π_2) in $\mathcal{P}$ and f a morphism of partial $H \bowtie L$-modules from M to N, then

$$\begin{array}{rcl} f(h \otimes [x^1] \cdots [x^n] \bullet_M m) & = & f \circ \pi_1(h) \circ \pi_1(x^1) \cdots \pi_1(x^n)(m) \\ & = & \pi_2(h) \circ \pi_2(x^1) \cdots \pi_2(x^n) \circ f(m) \\ & = & h \otimes [x^1] \cdots [x^n] \bullet_N f(m). \end{array}$$

Then $F(f)$ is a morphism in ${}_{H\boxtimes^* L_{par}}\mathcal{M}$. Analogously, G is well defined in the morphisms and $FG \simeq Id_{\mathcal{P}}$ and $GF \simeq Id_{H\boxtimes^* L_{par}\mathcal{M}}$ in the morphisms. □

And by an analogous argument, we have the following result.

Proposition 5.36. *Let $(H, L, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. Then $\mathcal{Q} \cong {}_{{H_{par}}^* \boxtimes L}\mathcal{M}$.*

5.7 Partial representation of $\Bbbk^G \# kF$ on $\Bbbk$

In this section we will consider the Hopf algebra structure of $\Bbbk^G \# \Bbbk F$ given by an action by automorphisms of groups $\triangleleft : G \times F \to G$, as in the previous section.

Here, we will study when two partial representations on the field k, one of $\Bbbk^G$ and one of $\Bbbk F$, allow a partial representation of $R = \Bbbk^G \# \Bbbk F$ on $\Bbbk$ such that we can recover the original partial representations by restricting to $\Bbbk^G$ and $\Bbbk F$.

But first we will describe all partial representations of $\Bbbk^G$ on the field $\Bbbk$.

Lemma 5.37. *Let G be a finite group and X a subset of G. The following are equivalent:*

1. *If for some $g \in X$ and $h \in G$ we have that $gh \in X$, then $th \in X$ for every $t \in X$;*

2. *If for some $g \in X$ and $h \in G$ we have that $hg \in X$, then $ht \in X$ for every $t \in X$.*

Proof. $1) \Rightarrow 2)$ Note that for every $s, t \in X$ we have that $ss^{-1}t = t \in X$, then $rs^{-1}t \in X$ for every $r, s, t \in X$. Suppose that $hg \in X$ for some $g \in X$, then $hg(g^{-1}t) = ht \in X$ for every $t \in X$. Analogously $2) \Rightarrow 1)$. □

Lemma 5.38. *Let G be a finite group and X a subset of G. Then, X satisfies one of the equivalent items of the previous lemma if and only if $X = xH$, where $x \in G$ and H is a subgroup of G.*

Proof. Suppose that X satisfies $1)$ and consider $H = \{h \in G \,|\, th \in X \, \forall t \in X\}$. Note that if $h \in H$, then $th \in X$ and $(th)h^{-1} = t \in X$, hence $h^{-1} \in H$. Also, if $g, h \in H$, then $tg \in X$ and $tgh \in X$ for every $t \in X$, hence $gh \in H$. Consequently, H is a subgroup of G. We also have that $tH \subseteq X$ for every $t \in X$, and $x^{-1}y \in H$ for every $x, y \in X$, because $xx^{-1}y = y \in X$ and X satisfies $1)$. Then, $x = t(t^{-1}x) \in tH$ for every $x \in X$, hence $X = tH$. Conversely, if $X = xH$ where $x \in G$ and H is a subgroup of G, then $(xh)k \in X$ for some $h \in H$ if and only if $k \in H$, then $xtk \in X$ for every $t \in H$. □

Theorem 5.39. *Let G be a finite group and $\Bbbk$ a field such that its characteristic does not divide $|G|$. Then, there exist a bijective correspondence between partial representations of $\Bbbk^G$ on $\Bbbk$ and all left cosets of all subgroups of G.*

Proof. Suppose that $\pi : \Bbbk^G \to \Bbbk$ is a partial representation and let $X = \{g \in G; \pi(p_g) \neq 0\}$. Then, by the property $2)$ of partial representations, we have that for every $g \in X$ and $h \in G$,

$$\sum_{t \in X} \pi(p_t)\pi(p_{th}) = \pi(p_{gh}),$$

particularly for $h = 1_G$, follows that $\pi(p_g) = \sum_{t \in X} \pi(p_t)^2$, for every $g \in X$, hence $\pi(p_g) = \pi(p_h)$ for every $g, h \in X$. Since $\sum_{g \in X} \pi(p_g) = 1$, we have that $\pi(p_g) = \frac{1}{\#X}[g \in X]$, where $\#X$ denote the number of elements of X and $[g \in X] = 1$ if $g \in X$, and 0 otherwise. Now, also by the property $2)$ of partial representations, we have that, for every $g \in X$ and $h \in G$,

$$\pi(p_{gh}) \;=\; \sum_{t \in X} \frac{1}{(\#X)^2}\pi(p_{th}),$$

which means that if $gh \in X$, then $th \in X$ for every $t \in X$. Hence $X = xH$, for some subgroup H of G and $x \in G$. Analogously, by properties $3)$ and $4)$ of partial representations, follows that If for $g \in X$ and $h \in G$ we have that $hg \in X$, then $ht, h^{-1}t \in X$ for every $t \in X$. Conversely, consider $X = xH$ where $x \in G$ and H is a subgroup of G. Then the linear map $\pi(p_g) = \frac{1}{\#X}[g \in X]$ define a partial representation. In fact, the property $1)$ of partial representation is straightfore. For the property $2)$, we must have that for every $g, h \in G$,

$$\sum_{t \in X} \frac{1}{n^3}[g \in X][th \in X] = \frac{1}{n^2}[g \in X][gh \in X].$$

Then, for every $g \in X$, we must have that

$$\sum_{t \in X} \frac{1}{n^3}[th \in X] = \frac{1}{n^2}[gh \in X].$$

But this happens, because if $gh \in X$, we have that $th \in X$, for every $t \in X$. Analogously, the properties 3), 4) and 5) are satisfied. $\square$

Note that every partial representation π_F of $\Bbbk F$ on $\Bbbk$ is actually a extension of a representation of a subgroup L of F, i.e., $\pi_F(x) = \delta_{xL,L}$.

Theorem 5.40. *Let $\Bbbk$ be a field such that char $\Bbbk \nmid |G|$, $\pi_G : \Bbbk^G \to \Bbbk$ and $\pi_F : \Bbbk F \to \Bbbk$ partial representations, where π_F is given by a representation of a subgroup L of F and π_G is determined by the left coset xH as in Theorem 5.39. Then, there exist a partial representation $\pi : R \to \Bbbk$ such that we can recover π_G and π_F if and only if xH is invariant by the action $\triangleleft|_{xH\times L}$.*

Proof. Suppose that there exist such partial representation $\pi : R \to \Bbbk$. Then, we must have that, for all $g \in G$ and $x \in F$,

$$\begin{aligned}\pi_G(p_g)\pi_F(x)\pi_F(x^{-1}) &= \pi_G(p_g x)\pi_F(x^{-1});\\ \pi_F(x^{-1})\pi_F(x)\pi_G(p_g) &= \pi_F(x^{-1})\pi_G(xp_g) = \pi_F(x^{-1})\pi_G(p_{g\triangleleft x^{-1}}x),\end{aligned}$$

i.e., when $x \in L$, we must have that $\pi_G(p_g x) = \pi_G(p_g)\pi_F(x)$ and $\pi_G(p_g) = \pi_G(p_{g\triangleleft x^{-1}})$. Then we must have that xH is invariant by the action $\triangleleft|_{xH\times L}$. Conversely, if xH is invariant by the action $\triangleleft|_{xH\times L}$, define the map $\pi(p_g x) = \pi_G(p_g)\pi_F(x)$. The property 1) of partial representations is straightfore. For the property 2), for every $g, k \in G$, $x, y \in L$, in one side we must have that

$$\begin{aligned}\sum_{t\in G}\pi(p_g x)\pi(p_t y)\pi(p_{h^{-1}t\triangleleft x}x^{-1}) &= \sum_{t\in G}\pi_G(p_g)\pi_G(p_t)\pi_G(p_{h^{-1}t\triangleleft x})\pi_F(x)\pi_F(y)\pi_F(x^{-1})\\ &= \sum_{t\in G}\pi_G(p_g)\pi_G(p_t)\pi_G(p_{h^{-1}t\triangleleft x})\\ &= \sum_{t\in G}\pi_G(p_g)\pi_G(p_t)\pi_G(p_{h^{-1}t})\\ &= \pi_G(p_g)\pi_G(p_{h^{-1}g}),\end{aligned}$$

and on the other side

$$\begin{aligned}\sum_{t\in G}\pi(p_g x)\pi(p_t y)\pi(p_{h^{-1}t\triangleleft x}x^{-1}) &= \sum_{t\in G}\pi(p_g x p_t y)\pi(p_{h^{-1}t\triangleleft x}x^{-1})\\ &= \sum_{t\in G}\pi(p_g p_{t\triangleleft x^{-1}} xy)\pi(p_{h^{-1}t\triangleleft x}x^{-1})\\ &= \pi(p_g xy)\pi(p_{h^{-1}(g\triangleleft x)\triangleleft x}x^{-1})\\ &= \pi_G(p_g)\pi_F(xy)\pi_G(p_{h^{-1}(g\triangleleft x)\triangleleft x})\pi_F(x^{-1})\\ &= \pi_G(p_g)\pi_G(p_{h^{-1}(g\triangleleft x)\triangleleft x})\\ &= \pi_G(p_g)\pi_G(p_{h^{-1}(g\triangleleft x)}),\end{aligned}$$

then, whenever $\pi_G(p_g) \neq 0$, we must have $\pi_G(p_{hg}) = \pi_G(p_{ht})$ for every $t = g \triangleleft x$, with $x \in L$. But this is satisfied because xH is invariant by the action $\triangleleft|_{xH\times L}$ and $X = xH$ satisfies the properties of the Lemma 5.37. Analogously, the properties 3), 4) and 5) of partial representations holds. $\square$

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