Universidade Federal do Paraná
Programa de Pós-Graduação em Física
Dissertação de Mestrado

Matheus Fritsch Savi

Elements of Modern Physics Within Lightweight Reference Frames

Time Contraction and Quantumness Invariance

Dissertação apresentada ao Programa de Pós-Graduação em Física do Setor de Ciências Exatas da Universidade Federal do Paraná, como parte dos requisitos necessários à obtenção do grau de Mestre em Física.

Orientador: Prof. Dr. Renato M. Angelo

Curitiba - PR
2017

S267e Savi, Matheus Fritsch

Elements of Modern Physics Within Lightweight Reference Frames - Time Contraction and Quantumness Invariance / Matheus Fritsch Savi. – Curitiba, 2017.

95 f. : il. color. ; 30 cm.

Dissertação - Universidade Federal do Paraná, Setor de Ciências Exatas, Programa de Pós-Graduação em Física, 2017.

Orientador: Renato Moreira Angelo .

Bibliografia: p. 90-95.

1. Relatividade especial (Física). 2. Teoria quântica. 3. Mecânica quântica. I. Universidade Federal do Paraná. II.Angelo, Renato Moreira. III. Título.

CDD: 530.12

MINISTÉRIO DA EDUCAÇÃO
UNIVERSIDADE FEDERAL DO PARANÁ
PRÓ-REITORIA DE PESQUISA E PÓS-GRADUAÇÃO
Setor CIÊNCIAS DE CIÊNCIAS EXATAS
Programa de Pós-Graduação FÍSICA

TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em FÍSICA da Universidade Federal do Paraná foram convocados para realizar a arguição da Dissertação de Mestrado de **MATHEUS FRITSCH SAVI** intitulada: **Elements of Modern Physics Within Lightweight Reference Frames - Time Contraction and Quantumness Invariance**, após terem inquirido o aluno e realizado a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.

A outorga do título de mestre está sujeita à homologação pelo colegiado, ao atendimento de todas as indicações e correções solicitadas pela banca e ao pleno atendimento das demandas regimentais do Programa de Pós-Graduação.

Curitiba, 28 de Julho de 2017.

RENATO MOREIRA ANGELO
Presidente da Banca Examinadora (UFPR)

Alexandre Dias Ribeiro
ALEXANDRE DIAS RIBEIRO
Avaliador Interno (UFPR)

FABIANO MANOEL DE ANDRADE
Avaliador Externo (UEPG)

"To say that nothing is true is to realize that the foundations of society are fragile, and that we must be the shepherds of our own civilization. To say that everything is permitted is to understand that we are the architects of our own actions, and that we must live with their consequences, whether glorious or tragic."

– Ezio Auditore da Firenze

Agradecimentos

Meus sinceros agradecimentos volvem-se:

- Ao Prof. Renato, pela enorme disposição, interesse, confiança, carisma, altruísmo e dedicação comigo e com o trabalho. Devo dizer que quando eu estava próximo da desistência do curso de graduação por falta de rumo, foram suas aulas que motivaram-me a continuar.
- Aos meus pais que, em meio a tantos problemas e conflitos, nunca deixaram de me apoiar e nortear.
- À Kerllyn Cardoso, pela alegria, amor e segurança que emanam de sua existência e permeiam meus dias.
- Ao Leonardo Siqueira, meu grande amigo, pela disposição em revisar este manuscrito.
- Ao grupo de pesquisa da UFPR, organizado pelo Prof. Renato e Prof. Alexandre, com sábias e delicadas discussões.
- Ao grupo docente do Departamento de Física da UFPR, pelos ensinamentos sobre física, matemática e ciência.
- Aos membros da banca examinadora pelo aceitamento em avaliar e corroborar com este manuscrito.
- À fundação CAPES pelo suporte financeiro.

Abstract

In order to realize how the notion of reference frame is important, consider the following question: what would be the physics of a particle in an empty universe? What would its velocity, charge, temperature, or wave function be? What about its time evolution? Although it is one of the most basic and fundamental elements of a physical theory, the notion of reference frame is almost never rigorously defined. It is frequently confused with the notion of a coordinate system, an abstract mathematical entity devoid of physical significance. An experimental physicist, however, has the strong tendency to disapprove this notion. Therefore, at the end of the 20th century and in recent years, some works have investigated the physics described from the perspective of physical reference frames, of finite-mass and interacting ones. However, the accumulated knowledge hitherto about this topic, specially in Special Relativity and Quantum Mechanics, is incipient. In this work we intend to contribute to this context through two preliminary studies regarding simple models from Modern Physics. In the first part of this dissertation, we revisit the time dilation problem, where we attribute to the laboratory an arbitrary mass so as to participate in the conservation laws, for it is able to "feel" the emission of a photon; this consideration accentuates the usual formula by a dimensionless factor involving the Planck constant. In the second part, we approach a fundamental question, which is: is the total quantumness of quantum resources – coherences and correlations – invariant under change of quantum reference frames? Our study is realized in systems composed of few particles where, firstly, we construct and treat a classical model which is "analog" to the quantum case and only then we attend to the latter; we consider position and spin states described by distinct reference frames and evaluate how the resources behave in each frame. Despite of the simplicity of the approached models, we believe that our results are already able to reveal subtle and important aspects, deserving then a more profound investigation.

Keywords: special relativity, time contraction, lightweight reference frames, quantum reference frames, quantum correlations and coherence, relative spin.

Resumo

Para percebermos como a noção de referencial é importante, considere a seguinte questão: qual seria a física de uma partícula num universo vazio? Qual seria sua velocidade, carga, temperatura, ou função de onda? Como seria sua evolução temporal? Embora seja um dos elementos mais básicos e fundamentais de uma teoria física, a noção de referencial quase nunca é rigorosamente definida. Muitas vezes confunde-se com a noção de sistemas de coordenadas, uma entidade matemática abstrata e desprovida de substância física. Um físico experimental, no entanto, tem a firme tendência de desaprovar esta noção. Por conseguinte, no final do século 20 e em anos recentes, alguns trabalhos investigaram a física descrita da perspectiva de referenciais físicos, de massa finita e interagentes. Entretanto, o conhecimento acumulado até o momento sobre este tópico, especialmente nas áreas de Relatividade Especial e Mecânica Quântica, é incipiente. Neste trabalho pretendemos contribuir para este contexto através de dois estudos preliminares em modelos simples da Física Moderna. Na primeira parte desta dissertação, revisitamos o problema da dilatação do tempo, onde atribuímos ao laboratório uma massa arbitrária de modo a participar das leis de conservação, pois este é capaz de "sentir" a emissão de um fóton; esta consideração acentua a fórmula usual por um fator adimensional envolvendo a constante de Planck. Na segunda parte, abordamos uma questão fundamental, qual seja: é a quanticidade total de recursos quânticos – coerências e correlações – invariante sob troca de referenciais quânticos? Nosso estudo é realizado em sistemas de poucas partículas onde, primeiramente, construímos e tratamos um modelo clássico "análogo" ao caso quântico para então tratarmos este último; consideramos estados de posição e de spin descritos por referenciais distintos e avaliamos como os recursos comportam-se em cada referencial. Apesar da simplicidade dos modelos considerados, acreditamos que nossos resultados já sejam capazes de revelar aspectos sutis e importantes, merecendo então uma investigação mais aprofundada.

Palavras-chave: relatividade especial, contração do tempo, referenciais leves, referenciais quânticos, correlações quânticas e coerência, spin relativo.

CONTENTS

CHAPTER 1 INTRODUCTION

The following introduction is divided in two main parts: Special Relativity and Quantum Mechanics. We begin with a brief historical introduction about these theories in order to highlight the reason for which it is important to further consider the matter of *reference frames* in physics.

1.1 Absolute to Relational

1.1.1 System of Reference: From Mathematical to Physical

Physics is a relative science. That is, for a phenomenon to be measured and registered, it must be so relative to a *reference frame*. We can rephrase this statement by saying: there is no physics in a universe composed of only one single particle (or object, if the reader prefers). All physical quantities are measured relative to some previously determined quantity*. Let us say we have a single particle universe and such particle has a charge, mass, and temperature. Could we say if this particle has a negative or positive charge? Could we say if this particle is a heavy or a light one? If this particle is hot or cold? There is nothing else in the universe for us to compare it with. Say that its mass is 10 kg, then imagine that there are many other particles in the universe and their masses are all of 5 kg. These particles will then conclude that the 10 kg particle is heavy. If, on the other hand, the universe is composed of particles with masses 20 kg and they observe the 10 kg particle, they will conclude that such particle is light; a different scenario from the previous case. The concept of light and heavy can only be applied when in comparison to something†. Hence, we defend that a negative answer should be given to the previous questions.

When we are concerned with the motion of physical bodies, analogously, we need a reference frame for the same reason. Newton's laws of motion are valid only to a particular category of reference frames: *inertial*. An inertial reference frame is a frame in which it is at rest or is moving with constant

*Consider the following example: a pre-historical man wishes to measure some object's width and he does not have any kind of measure already defined for it. Being a smart man, he then defines a standard measure for width. Let us say his arm's width; calling it 1 arm. Using this definition, he is able to determine the width of everything else as a function of 1 arm. A tree's height equals 10.5 arm; his leg equals 1.7 arm, and so forth. His arm's width is the previously determined quantity for which other things may be measured and their widths are to be compared with.

†A more practical example. Say a object has mass equals to 100 kg. Is this object heavy or light? That depends, what is this object? If it is a car, it is light. If it is a pencil, it is heavy. That is the reasoning we are using, and it can be used to anything: velocity, volume, temperature, momentum, force, and so on.

linear velocity (no net force is acting upon it). But, does this reference frame really exists? The physics as we know it is described relative to such reference frame and, if the frame in question is not inertial, our laws must be altered to accommodate such fact. Take the Earth, for example; it is, approximately, an inertial reference frame, for, practically, it is not influenced by the observed bodies. There are cases, however, that is necessary to consider the Earth's non-inertiallity (*e.g.*, Coriolis force) and, in these cases, additional terms need to be included in the equations of motion. A strictly inertial reference frame may not exist; so how can we study physics if all our laws are written relative to an inertial frame? We arrive at the usual procedure adopted almost unconsciously: starting from an external and absolute reference frame, which is inertial and immaterial (usually called the *ether*), we write the equations of motion relative to it, and only then we move to the relative coordinates of the system[‡]. We will show an example that illustrates this fact to give an insight to the reader of how the concept of reference frame plays an important role in physics: the two-body problem, discussed next.

Consider an inertial (external) reference frame α. This problem consists of two mass points, m_1 and m_2, positioned at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, with the only force present that of an interaction potential $V(\mathbf{r}_2 - \mathbf{r}_1)$. For us to treat it relative to the external frame, we need to specify six quantities: three for the position vector $\mathbf{r}_1$ and three for the position vector $\mathbf{r}_2$. On the other hand, it is possible to treat it using another set of coordinates: the center of mass $\mathbf{R}$ and the relative coordinate between the particles $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, see Fig. 1.1. If we write the Lagrangian $\mathscr{L}$ with such central potential $V(r = |\mathbf{r}|)$, using the first set of coordinates, we get

$$\mathscr{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(r), \tag{1.1.1}$$

where the dot denotes a time derivative. The first two terms are simply the kinetic energy of both particles. Alternatively, if we write it using the second set, which contains the center of mass and relative orientation, we can rewrite the kinetic energy as

$$K = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + K', \qquad K' = \frac{1}{2}m_1\dot{\mathbf{r}}_1'^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2'^2, \tag{1.1.2}$$

with $\mathbf{r}_i'$ denoting the position vector of particle i relative to the center of mass, which are related to $\mathbf{r}$ through

$$\mathbf{r}_1' = -\frac{m_2}{m_1 + m_2}\mathbf{r}, \qquad \mathbf{r}_2' = \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{1.1.3}$$

Substituting this result in the Lagrangian, it follows that

$$\mathscr{L} = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r), \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}, \tag{1.1.4}$$

where μ is the reduced mass. The variable $\mathbf{R}$ is said to be cyclic, meaning that the center of mass is either at rest or moving with constant velocity (it can be explicitly calculated that $\ddot{\mathbf{R}} = \mathbf{0}$). Hence, none of the equations of motion for $\mathbf{r}$ will contain terms involving $\mathbf{R}$ or $\dot{\mathbf{R}}$ and, as a consequence, we can drop the first term in our calculations. Therefore, the dynamics of two moving bodies in a central potential is equivalent of only one "body" with mass μ. If we follow the same analysis using Newton's second law, we will find that the relative acceleration $\ddot{\mathbf{r}}$ will be given by

$$\ddot{\mathbf{r}} = \frac{m_2 + m_1}{m_1 m_2}m_1\ddot{\mathbf{r}}_1 = \frac{\mathbf{F}_{12}}{\mu}, \tag{1.1.5}$$

[‡]As the Earth can be treated with such properties, our laws are applicable to everyday systems.

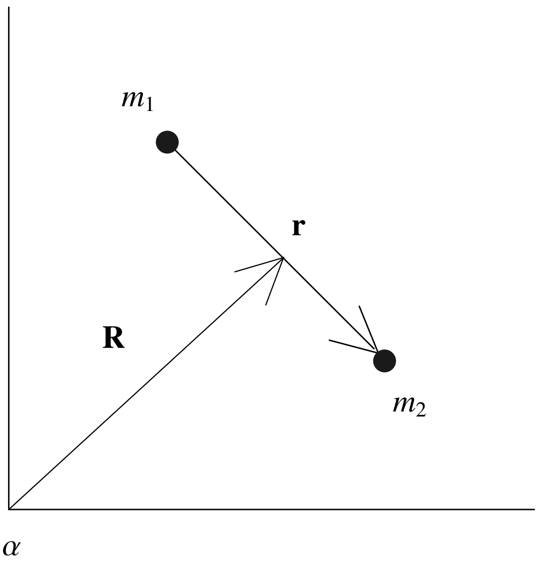

Figure 1.1: Illustration of the set of coordinates in α's viewpoint.

with $\mathbf{F}_{12}$ the force that particle 2 exerts on particle 1. This equation shows that particle 1 moves relative to particle 2 with a mass equals to μ. We recommend [1–3] for the interested reader for a thorough analysis on the two body problem and Newtonian mechanics.

It is standard procedure to adopt a reference frame to write down the equations of motion wherein Newton's laws are valid, but is this reference frame a mathematical entity for scientists to make calculations and nothing else or is it a physical one that really represents something and can interact with the system? In the calculations afore discussed, (1.1.1) represents the former and (1.1.4) together with (1.1.5) represents the latter. If we move to a reference frame in which one of the particles is at rest, we cannot treat this problem in the same way we did, because such frame would not be an inertial one and this method would not be valid. However, one can see that if any of the masses is much heavier than the other, the reduced mass approaches the lighter one and the second law becomes absolute; which means that the heavier particle would interpret, supposedly, the role of a reference frame, showing us that a massive body is a good approximation of an inertial frame.

The discussion above was to demonstrate that the concept of reference frame does not need to be always a mathematical one and that we are able to (or better, we must to) consider them as a physical entity, for experimental data can only be gathered and verified relative to some reference frame/observer. If our frames do not exist or they are only abstract, what is the point of measuring? As we have seen, the Earth can be treated as inertial and be our reference frame, but this capability relies on the Earth's mass being much greater than anything else on our planet. But, what if the mass of our frame is not so much greater than the rest of the system? Measurements performed by an observer in this frame would then disturb the system and may influence on the resulting outcome, as we shall see later. Before addressing this matter, we revisit the initial steps of special relativity to bring forth some notions that will be used afterwards.

1.1.2 Reference Frames Become an Important Role: The Birth of Relativity

The study of reference frames really became notorious when Maxwell realized that the speed of light, $c \simeq 3 \times 10^8$ m/s, is the speed at which electromagnetic waves propagate in vacuum, being always constant in all directions and regardless of the source's movement. This is in contradiction with Newtonian mechanics because of Galilean transformations (discussed in Chapter 2). Newton's equations of motion, when undergone these transformations, preserve their form and are valid in others reference frames: a principle called covariance[§]. However, Newton got disturbed when he realized that his equations were invalid in a great number of situations, like an accelerated frame or a rotating one. His equations are thus valid only in a limited class of frames: inertial ones. According to Newton, there must exist an absolute space and these frames (inertial) are those that remain at rest or in uniform motion relative to this (absolute) space. Furthermore, Maxwell's equations from electrodynamics do not satisfy Galileo covariance. Maxwell himself thought that electromagnetic waves needed a medium to propagate, at the time known as ether. The constant velocity c would then be relative to this medium and, hence, Galilean transformations would preserve Maxwell's equations only in reference frames that are at rest or in uniform motion relative to it.

In 1905, Einstein published his seminal paper introducing his special theory of relativity [4]. His original motive was to fix the uncertainties of electromagnetism, but he ended up extending and generalizing Newtonian mechanics as well. What triggered his thoughts was, at that time, the mere coincidence between *induced* electromotive force (emf) and *motional* electromotive force, which are both related to Faraday's law of induction. The explanation of Faraday's law is fundamentally important, for it has been shown that one does not need an absolute medium (absolute reference frame) to describe it. We will give a brief qualitative review here, but the reader is invited to seek [5–7] for a quantitative one. Consider a closed circuit C and a permanent magnet with magnetic field $\mathbf{B}$. It was observed by Faraday that an electric current is induced in the circuit whenever there is movement between the magnet and the circuit, *i.e.*, if one keeps the circuit steady (relative to some external observer) and thrusts the magnet into or out of the circuit or the opposite. Let us take the former case to begin our analysis: steady circuit and moving magnet. In this situation, we have a varying magnetic field $\mathbf{B} = \mathbf{B}(t)$ which induces an electric field which, in turn, induces an electric force that generates an induced electromotive force in the circuit (at that time, this was Faraday's law of induction). Thus, the emf's origin here is the changing flux due to $\mathbf{B}$'s dependence on time. Now, let us take the other case: steady magnet and moving circuit. As the circuit moves with a certain velocity (relative to the external observer and the magnet), the charges on the circuit experience a magnetic force proportional to this velocity and generates the motional emf. Hence, the emf's origin in this case is due to the magnetic force on charges in the circuit (known as flux rule). Because Faraday's law and the flux rule predicts the same emf, physicists thought as that as a mere coincidence. Einstein, however, thought that the induced electromotive force and the motional one were equal not because of a coincidence, but because they were supposed to be. From his 1905's paper, in his own words:

> *"It is known that Maxwell's electrodynamics — as usually understood at the present time — when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise — assuming equality of relative motion in the two cases discussed — to electric currents of the same path and intensity as those produced by the electric forces in the former case."*

[§]That is, they have the same form in every inertial reference frame. That is the principle behind Newtonian relativity (only applicable to mechanics). Later on, Einstein uses this principle and generalizes it to all physical theories (not only mechanics).

In Newtonian mechanics, thanks to the relativity principle of Galileo, there was not an absolute reference frame because no experiment could distinguish between a stationary frame and a uniformly moving one. In electromagnetism (Maxwell-Lorentz theory), however, there was an absolute reference frame for electromagnetic phenomena, like the magnet and the circuit: the ether. Under this point of view, the equality between them would be a coincidence. In Einstein's thoughts, if electromagnetic phenomena only depends on the relative motion between the observer and the system, then there is no need to have an absolute reference frame. Therefore, the ether should not exist. It is well known now that the many negative results from the experiments that sought to prove the ether's existence, like the Michelson-Morley experiment [8], only prove Einstein's ideas. Following these statements, the next paragraph of his 1905's paper reads:

> *"Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium" suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell's theory for stationary bodies. The introduction of a "luminiferous ether" will prove to be superfluous inasmuch as the view here to be developed will not require an "absolutely stationary space" provided with special properties, nor assign a velocity-vector to a point of the empty space in which electromagnetic processes take place. The theory to be developed is based — like all electrodynamics — on the kinematics of the rigid body, since the assertions of any such theory have to do with the relationships between rigid bodies (systems of co-ordinates), clocks, and electromagnetic processes. Insufficient consideration of this circumstance lies at the root of the difficulties which the electrodynamics of moving bodies at present encounters."*

The fact that mechanics and electromagnetism are connected is not so surprising. What do we use to measure? Light, space, and time: the foundations of mechanical and electromagnetic phenomena. Although Newtonian mechanics is a particular case of Einstein mechanics, the concepts that Einstein introduced in his theory makes a great difference from what Newton initially thought.

With the advance of the theory of relativity, the study of transformations between reference frames has become more and more frequent. As we have seen above, a frame of reference plays a crucial role when we write down our equations, and the possibility of considering the reference frame as an interacting part of the system might alter these equations. Moreover, this kind of possibility is not so away from us as it may seem.

We have analysed earlier that a massive body can be a good approximation for inertial reference frames, but what if this body is not heavy enough to be considered as such? Does it mean we cannot make physics with it? Certainly not. We can make our reference frame as light as we want as long as it obeys certain conservation laws, because it must respect the same physics as does everything else. Our step now is to see how this description affects a particular known situation in relativity, but, before that, we introduce the concept of *lightweight reference frames* to show how this framework can be used. Additionally, for more on relativity theory, we recommend [3, 9–11].

1.1.3 Lightweight Reference Frames

The study of relativity concerns itself when one wishes to change reference frames. For one to achieve such change, a mathematical description is necessary to go from one frame to another. Such descriptions are based on the notion that some physical phenomenon is happening somewhere in space and time and they are concerned in how two observers, in distinct reference frames, state about the phenomenon. What we aim to do is a little different from that. We do not have two observers in relative motion wanting to compare their notes after observing some event; instead, we propose that this event occurs inside some laboratory, which will be a reference frame nonetheless, and it influences the laboratory itself because we attribute to the latter a very small mass, so it can participate in the event. The event we want to study is the *time dilation* effect seen by two observers: one inside a laboratory and another outside of it, watching carefully. The approach we want to use, however, is somewhat different from the usual time dilation description.

Consider the two-body problem discussed earlier; Eq. (1.1.5) tells us that exists a relative acceleration between the bodies (each body can be interpreted as an observer) and that it is proportional to their masses. Let us elucidate this case with a practical example. Consider that Bob is standing on a ring of ice, close to a wall, while holding a ball, and Alice is out of the ring just watching. Let us assume that Bob is a very heavy person and he throws the ball towards the wall and it bounces back to him. As he is very heavy (much heavier than the ball), his acceleration is practically zero (for all practical purposes, we can say that it is strictly zero), so as to receive the ball on the same position he threw it. Alice will then conclude that Bob has not moved at all and, to close her observations, she would measure the time it took for the ball to leave Bob's hands and return to them. Finally, at the end of the experiment, they would compare their results: the time interval elapsed to each. Now, what would happen if Bob is not so much heavier than the ball? His acceleration will not be so close to zero anymore. In fact, it can have any value that respects the conservation of momentum between him and the ball. As a consequence, Alice will then ascribe to Bob an equation of motion just as she ascribes one to the ball, and Bob can be identified as a *lightweight reference frame* where physical laws may be applied to him as well. In the end, would his motion alter Alice's or even his measurements? As we will show later, we can give to Bob a mass so close to zero that even if he emits a photon he will receive a kickback in order to conserve momentum and quantum mechanics is necessary to treat the problem.

The mathematical description to go from one frame to another does not include the observer as a part of the system; it just asks what two observers in relative motion state about a physical event. Although there are very few research concerning lightweight reference frames, their study can be promising because physical laws must be valid for them as well (not in the inertial way, however), and new effects may be discovered by studying them. One of these effects will be discussed later and it involves some concepts of quantum mechanics, our next section.

1.2 Classical to Quantum

1.2.1 A Change of Concepts: Wave-Particle Duality

At the end of the 19th century, the atomic hypothesis was widely known, but not universally accepted. No one knew how to precisely describe an atom or even an electron, which had just been discovered at that time. Generally speaking, when the subject was molecules and atoms, physics could not handle it and a new theory was in need: quantum theory. Surprisingly, if we can say that, the rise of quantum mechanics started when the matter was thermal radiation or, more accurately, *black-body* radiation.

If radiation hits a body, it can enter its interior or be reflected by its surface. Depending on the proportions of absorbed and reflected radiation, such body receives a different name according to these proportions. For example, if all incident radiation is reflected by the body, without any loss, it is called white. If, on the other hand, all radiation is absorbed by it, it is called black; they are both a good idealization which greatly simplify the calculations regarding the energy density of radiation. The energy density, as a function of the body's temperature T, of the radiation emitted by a black-body was, at first, described by two separate and contradictory laws in classical physics: the Rayleigh-Jeans radiation law, which covered the range of long wavelength (low frequency) radiation, stating that the energy density was linear on the temperature; and the Wien's law, for short wavelength (high frequency), stating that it was proportional to $\exp(-1/T)$. Moreover, the former law predicted that the total amount of energy emitted, in all frequency ranges, would be infinite, which violates the conservation of energy principle[¶]. It was Max Planck who provided the correct explanation to the problem with his suggestion of *quantization* of energy, which states that for an electromagnetic wave of frequency ν, the only possible energies are integer multiples of the quantum $h\nu$, where $h \simeq 6.62 \times 10^{-34}$ J·s is a new constant, now called Planck constant. This idea has been formulated into Planck's radiation law and, together with the statistical development of the Boltzmann distribution, successfully explained the black-body radiation in all ranges of wavelength.

Alongside these problems, we had the matter of light, which has always been a subject of interest when concerning physical phenomena. In order to explain them, the corpuscle theory was developed by Newton and the wave theory by Huygens, almost simultaneously. Some properties of light, like rectilinear propagation and reflection, can be explained by both theories, but some, like interference and diffraction, can only be explained by the wave theory. So, after all, in which model does light fit?

Maxwell's electrodynamics, in 1864, which interprets light as an electromagnetic wave, seemed to confirm the wave theory, but the photoelectric effect, discovered by Heinrich Hertz in 1887 [12], could not be explained if light belonged in the wave theory, only in the corpuscular one. As a result of the latter, after Planck's hypothesis of quantization of energy, Einstein used this same hypothesis, in 1905, to extend Planck's explanation: light consists of a beam of particles, called photons, each possessing an energy $h\nu$; this step made it possible to explain very simply the photoelectric effect. Then, twenty years later, Compton confirmed the photon's existence with the so-called Compton effect. So, does light belong in the wave theory or in the corpuscle theory? Would it not be possible, in spite of everything, for light to belong in *both* theories?

Investigation on the nature of light, with the aid of many experiments, showed that depending on the kind of experiment performed, it must be described by the wave *or* particle theory. These results led to the following conclusion: the interaction of electromagnetic waves with matter occur by means of elementary indivisible processes, in which radiation appears to be composed of particles. Particle parameters, like energy E and momentum $\mathbf{p}$ of a photon, and wave parameters, like angular frequency $\omega = 2\pi\nu$ and wave vector $\mathbf{k}$, where ν is the wave's frequency and $|\mathbf{k}| = 2\pi/\lambda$ (λ its wavelength), are linked by the Planck-Einstein relations

$$\begin{cases} E = h\nu = \hbar\omega \\ \mathbf{p} = \hbar\mathbf{k}, \end{cases} \tag{1.2.1}$$

with $\hbar = h/2\pi$ the reduced Planck constant. It is important to notice that a complete interpretation of light phenomena can only be obtained by conserving both its wave *and* particle aspect.

This dual behaviour has received the name of wave-particle duality and it is well known now

[¶]This fact became known as the ultraviolet catastrophe, for when the frequencies reached the ultraviolet region of electromagnetic spectrum, the total emitted energy diverged greatly from experimental observations.

that material particles also demonstrate it[||], which is why classical mechanics and electrodynamics fail when trying to explain atomic phenomena. For instance, if we apply these classical theories to explain an electron that moves in a circular orbit around the nucleus, electrodynamics would tell us that the electron must lose its energy and fall into the nucleus; the consequence being that all atoms would be unstable, which is clearly not true. Another example, parallel to the discovery of photons, is the study of atomic emission and absorption spectra, which are composed by narrow lines and cannot be explained by classical physics. However, when one accepts the idea that atoms only emit or absorb photons with well-determined energy (or frequency), it can be interpreted very easily: the difference between the allowed energy values is directly proportional to an integer n of the quantum $h\nu$, not just any value. The existence of such discrete energy levels was confirmed by the Franck-Hertz experiment [13], and the wavelike aspect of matter by electron diffraction experiments (interference patterns could be obtained with material particles as electrons), like the one performed by Davisson and Germer [14].

Furthermore, the wavelike behaviour is responsible for many non-intuitive aspects regarding quantum mechanics. Most importantly, the intrinsic indeterminacy of the theory. The matter of how to interpret the wave aspect that describes particles and if a physical reality can be given to such wave was a subject to enormous discussion at that time (still is, actually). Max Born proposed a probabilistic interpretation to the wave that describes a particle with the calculation of the so-called *wave function* ψ associated with such particle, creating the term *guiding field* (which came up originally with Einstein's idea of "ghostfield"). The guiding field is a scalar function of the coordinates and time of the particle in question (or a system formed by many particles), where the particle's motion is determined by the laws of energy and momentum conservation, together with appropriated boundary conditions. The probability that the particle will follow a particular path is given by the intensity of the field, *i.e.*, its square modulus $|\psi|^2 = \psi^*\psi$. This interpretation was extended to the Copenhagen interpretation known nowadays, formulated by, among others, Niels Bohr and Werner Heisenberg, and it is the standard one. Notwithstanding, many interpretations and theories have emerged to describe the wave function and quantum mechanics; as some examples, we cite the Broglie-Bohm theory and the many-worlds interpretation.

Unlike particles, waves are not localized in space; meaning that one cannot ascribe to it a specific trajectory in which it "moves" or, in other words, we can tell, with uncertainty zero (theoretically), where the particle came from and where has it gone to, whereas for a wave this is not possible because it is extended (it is not located in a single point, but in a region) throughout all space. Newton's laws of motion describe how a particle moves when under the influence of a force by solving a differential equation given a set of boundary conditions. In consequence thereof, we see that the classical concept of trajectory must be replaced by another, known as *quantum state*, which must obey a wave equation.

The quantum state of a particle is characterized by a wave function $\psi(\mathbf{r}, t) = \langle \mathbf{r} | \psi(t) \rangle$, containing all the information that is possible to obtain about the particle. Once the problem is specified, we evaluate the wave function by solving the Schrödinger equation, which determines all possible eigenstates that the particle can reach. To each eigenstate, there is an associated eigenvalue, which is the physical object that can be measured. However, there is nothing in the theory that provides us how the outcome of a measurement occurs; only the probabilities that every eigenvalue has of being "selected". That is, until a measurement is made, we assume that the particle is in a superposition of all eigenstates (superposition principle) and, solely after the measurement, we are able to tell precisely, with probability 1, in which eigenstate the particle is. According to Greiner, in [15]:

[||] This matter was de Broglie's thesis in his doctorate, which, together with other physicists like Sommerfeld, gave birth to what is known as old quantum theory.

> *"Quantum mechanics is an indeterminate theory which states that there are physical measurements whose results are not definitely determined by the state of the systems prior to the measurement (at least, as far as it is in principle possible to observe that state). If, just before the measurement, the wave function of the system is not an eigenfunction of the operator whose observable is to be measured, then the result of the measurement is not definitely predictable; only the probability of the various possible results can be determined."*

In his book, several conceptual difficulties about quantum mechanics are briefly treated; like locality, measurement theory, hidden-variable theories, reality and many others. Posteriorly, when we talk about invariance on quantum mechanics, the concept of reality will be brought up. For further information on quantum mechanics, we recommend [15–21].

As we have seen in §1.1, particles can be regarded as reference frames, and, if such particles must be described by quantum theory, due to their wave nature coming from their very small mass or when their wavelength is comparable to the dimensions of the problem in question, these reference frames are to be identified as *quantum reference frames*. Before we discuss this matter, there is a pioneer subject involving the concepts of relativity and quantum mechanics that we shall address first: the relative states formulation, by Hugh Everett.

1.2.2 Relative States Formulation

Quantum mechanics concerns on the calculation of a system's wave function $\psi(\mathbf{r}, t)$, which gives us only information about the probabilities of the results of various possibles observations that can be made on the system. For a particle of mass m, this wave function obeys the Schrödinger equation

$$H\psi(\mathbf{r}, t) = \mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = E\psi(\mathbf{r}, t), \quad \text{with} \quad H = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) \tag{1.2.2}$$

the Hamiltonian operator, ∇^2 the Laplacian operator, $\mathrm{i} = \sqrt{-1}$ and E a real number (eigenvalue) associated with the total energy of the system. But, as we have seen, the measurement process is not included and, hence, not described by the Schrödinger equation. In other words, Eq. (1.2.2) does not tell us how the particle changes its state when measured; it only gives us the probabilities wherein such states can occur, their evolution in time, and their description (very precisely, actually). If the wave aspect really is real and prior to the measurement the particle is in a superposition state, does it mean that the particle is "dispersed" throughout the wave? If it is and we perform a measurement, the dispersed particle abruptly comes together in a single place almost immediately? Or is the theory lacking some details and the particle is never in a superposition state? Then, the probabilities for each state would just be some kind of measuring ignorance of our own. However, there are two kinds of ignorance we must consider. *Exempli gratia*, every measuring equipment, like a ruler, has an uncertainty due to the limitation in which the equipment is able to measure: if the smallest distance the ruler can measure is 1 mm, then its uncertainty is ± 0.5 mm; this kind of ignorance is due to the equipment's limitations. The ignorance regarding quantum mechanics is that, regardless of the equipment and the method utilized, it is generally impossible to tell what will be the outcome of a measurement with a probability equals to unity.

The Copenhagen interpretation of quantum mechanics states that, as the theory can only predict the probabilities of each result, there is no defined properties of physical systems prior to the measurement. The act of measurement then affects the system, causing it to "be" in a single state immediately after the measurement: feature known as the wave function *collapse*. Think of it this way: presume Adam is an observer in some reference frame and he will measure some observable A in some determined state which is in superposition $|\psi\rangle = c_1 |A_1\rangle + c_2 |A_2\rangle$ with eigenvalues a_1 and a_2, respectively and c_1, c_2 are complex numbers (called coefficients). He measured it and his result is, say, a_1. His explanation of the outcome result, a_1, is that, prior to being measured, the particle was in more than one state at the same time (the linear combination of $|A_1\rangle$ and $|A_2\rangle$) and, after the measurement, it has immediately collapsed into the eigenstate that is associated with the eigenvalue measured; in our case, $|A_1\rangle$. With only one observer, the collapse notion is rather understandable, but when one considers a situation that a quantum system is being observed by more than one observer, it becomes difficult to understand.

The description Everett introduced in his paper [22] has started the conciliation of quantum mechanics and relativity. Not in the sense of relative motion between two reference frames, but in the sense of changing from one observer to another; from one frame to another. Let us see how.

It is believed that the wave function contains all possible information about the (isolated) system which it represents. Thus, given the Hamiltonian, one is able to solve (1.2.2), evaluate the wave function $\psi(\mathbf{r}, t)$ and calculate the desired quantities. Everett then states that there are two processes, different and mutually excluding, in which the wave function can change. Although they were written using a distinct notation in the original paper, we simply adapt them to our present notation with no loss of generality. That being said, they are:

Process I: The discontinuous change brought about by the observation of a quantity with eigenstates $|\phi_1\rangle, |\phi_2\rangle, \ldots$, in which the state $|\psi\rangle$ will be changed to the state $|\phi_j\rangle$ with probability $\left|\langle\phi_j|\psi\rangle\right|^2$.

Process II: The continuous, deterministic change of state of the (isolated) system with time according to a wave equation $i\hbar\frac{\partial}{\partial t}|\psi\rangle = H(t)|\psi\rangle$, where $H(t)$ is a linear operator.

To exemplify them, imagine we have a system S, with wave function ψ^S, and two observers, A and B. While A, inside a closed room, performs measurements upon system S, the totality $(A + S)$ forms the system for observer B, who is outside the room, as depicted in Fig. 1.2. In such scenario, we are allowing B to assign a quantum state for system S and the observer A: ψ^{A+S}. Hence, if B does not interact with them, ψ^{A+S} will evolve according to Process II, even if A measures some observable in S. We are in possession of a paradox now, because B can never say that a discontinuity occurs, for he is using Process II, whilst A, using Process I, will say that ψ^S has collapsed into an eigenstate after a measurement. How can one of them see the collapse and the other cannot?

After presenting some alternatives to fix this problem, Everett concludes that Process I must be abandoned and the validity of wave mechanics is assumed for *all* physical systems, including observers and measuring apparatus. Observation processes must then be described by the state function of the system as a whole (observer and object-system), obeying, at all times, Process II.

He also states that correlations play an important role when concerning quantum mechanics. Consider the following case. An observer A is correlated with a system S and performs some measurements upon it. As A cannot describe himself, such correlation, between him and S, will never be available for consideration in his description (A's Hamiltonian describes the system formed by S) and he will say that, after the measurement, a collapse has occurred. Now, if an observer B watches A's measurements, as he is able to include the correlation between A and S, B will say nothing about collapse, for he has an additional degree of freedom available (B's Hamiltonian describes the system formed by $A + S$). So, does that mean that B is a privileged observer? That cannot be true. In fact, if A realizes that he is not including all the system in his description and introduces B into his equations (A's Hamiltonian now describes the system formed by $B + S$), he will no longer need to invoke any notion of collapse and both observers will concur on the results, as it must be.

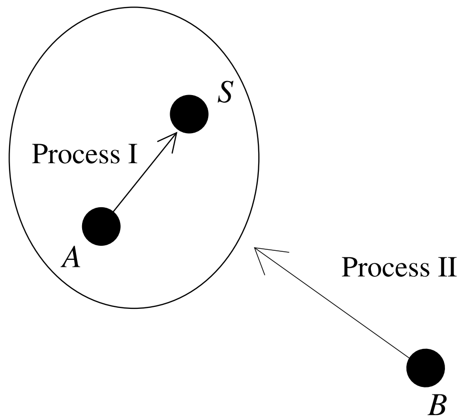

Figure 1.2: An observer A performs measurements in a system S inside a closed room. As a consequence of A's reduced description, which contains only the degrees of freedom of S, ψ^S evolves in time discontinuously, as prescribed by the Process I (collapse). Outside the room, an observer B analyses a larger system, $A + S$. B's Hamiltonian contains the degrees of freedom of both A and S, such that the wave function ψ^{A+S} evolves continuously, in accordance with the Process II (Schrödinger equation).

We see that the observer must also be included in one's description for the following reason. If an observer ascribes a state function to a system, why cannot the system (now observer) ascribes a state function to the observer (now observable)? In other words, why the observer's state should not be included in the description? One may argue that this reasoning is a senseless one because the system will not measure anything and hence it does not need a state function for the observer, which may be true. However, all reference frames must agree upon observations (despite of different explanations) because how can something happen only to a single observer and not to all? In order to achieve that, everything that is physical must be taken into consideration and, if we defend the notion that a reference frame *is* a physical entity, it must be accounted, as well; providing us a reason to allow quantum mechanics to describe a reference frame.

1.2.3 Quantum Reference Frames

The study of quantum reference frames started with Aharonov and Susskind, with their papers [23, 24], demonstrating that, theoretically, nothing prohibits the notion of reference frames as quantum objects. Later on, many works have been published [25–47] studying their properties and dynamics. As some examples, Ref. [27] demonstrates a formalism wherein the principle of equivalence (from relativity) is extended to quantum reference frames; Ref. [32] uses a large spin as a quantum mechanical gyroscope as a reference for which spin 1/2 particles' angular momenta can be measured; Ref. [43] analyses a simple thought experiment in which labelling one of the particles as reference frames correctly solves a presupposed paradox that emerges if this approach is not considered. Also, quantum reference frames are of great importance in information theory and superselection rules, in which Ref. [33] is a great guide.

As we go deeper into this subject, we see that it is not possible to consider quantum reference frames without the notion of *correlation*. We say that two particles are correlated when accessing the information of one particle gives to the observer information about the other, without even disturbing (measuring/interacting with) the latter; measuring only one of them is enough to know what the global state of the system is after the measurement (if the system is composed of two particles only)[**]. In

[**]Of course correlations also exist in classical physics, but, in the quantum one, they lead to exotic effects, as Bell's

fact, we many times use these correlations to measure because there are physical observables we cannot access directly. For example, we use an ammeter to measure the current flowing in an electric circuit; the intensity of the current is associated to where the pointer in the ammeter is located. To directly measure the current flowing in the circuit, we would have to have a powerful magnifying glass to zoom in into the wire and count how many electrons are passing through in a determined amount of time, but we cannot do that. As another example, we measure temperature using a thermometer wherein the liquid's height (usually, mercury) is elevated due to the pressure inflicted in it which, in turn, is due to the temperature of the material. The liquid's height is then associated with a given temperature. To directly measure it, we would have to measure the velocities and the masses of the particles that compose the material and calculate the thermal energy through their kinetic energy, but, again, we cannot do that[††]. When we measure with quantum pointers, we also do it indirectly, but, this time, through entanglement: the pointer gets entangled with the rest of the system and once the state of the former is known, we can infer the state of the latter as a result.

The entanglement was first deeply considered in the known EPR (Einstein, Podolsky and Rosen) paradox [48], where they also discuss the concept of *element of reality*. This work demands that, for every physical observable, there must exist an element of reality associated with it, and, if one is able to know the value of such observable, without perturbing the system in question, then the observable is said to be real. They argue that quantum mechanics is not a complete theory due to the fact that there are cases involving entangled states where one can predict the outcomes of incompatible observables in a far distant site without disturbing the subsystem in that site and yet the theory does not provide simultaneous elements of reality for both observables. The reader may associate these ideas with the relative states discussion; the contradiction between two observers while describing a measurement process.

Let us consider now the following puzzling scenario. Assume we are observers in the laboratory frame and we will perform the famous double-slit experiment with an electron. But, this time, the slit is light enough so that when the electron goes through, if it moves upwards (downwards) the slit must move downwards (upwards) to conserve momentum. Thus, if we are able to know whether the electron goes up or down, its state cannot be a superposition of up/down states: the electron's movement is entangled with the slit's, for if the latter goes up, we know the former went down. Therefore, we have a correlation between them and we will not see any fringes of interference on the screen. However, an observer that is attached with the slit cannot detect the slit's movement and will not have access to any correlations and, hence, the state of the electron would be a superposition and fringes should appear on the screen. Which observer is right then? This experiment is an example wherein quantum reference frames must take place to address the inconsistency, which will be done in the proceeding chapters.

To conclude the study of quantum reference frames, we want to investigate if there is some kind of *invariance* regarding coherences and correlations of quantum states. The concept of invariance is well established within the framework of classical mechanics and relativity: Galilean transformations leave invariant some quantities, like space and time intervals; Lorentz transformations leave invariant the notion of interval $\mathrm{d}s^2$ and events of the same type (spacelike and timelike). What about quantum mechanics? Is it really promising to describe quantum phenomena through the collapse, and to privilege some particular observer? If we want to find a theory that connects relativity and quantum mechanics, invariance must be inherent to such theory: something that all observers agree about.

We now conclude this chapter of introduction. In Chapter 2, a theoretical background in relativity and quantum mechanics will be given; all that is needed to accompany us to future chapters. In

non-locality.

[††]In fact, these measurements are not possible due to quantum principles.

Chapter 3 our discussion about lightweight reference frames and time dilation will take place; showing the unavoidable difference from the usual calculation and a connection to quantum mechanics through a dimensionless correction involving the Planck constant. In Chapter 4, we use some concepts of information theory to aid us in the evaluation of irreality (absence of reality) for some particular cases, involving position and momentum variables, and angular ones, like spin, with the intend to generalize it for any state. Finally, we close this work with the conclusions in Chapter 5 and an Appendix about "rigid-body motion" in special relativity.

CHAPTER 2

THEORETICAL BACKGROUND

In this chapter lie the necessary tools for the development of our main results in future chapters. We start with special relativity and, posteriorly, quantum mechanics. The topics treated in relativity are: **(i)** transformations of coordinates; **(ii)** kinematics and dynamics; **(iii)** mass-energy equivalence; and **(iv)** time dilation.

2.1 Relativity

2.1.1 Transformations of Coordinates

Galilean Transformation

We begin with Galilean transformation, a group of space-time coordinate transformation that leaves Newton's law of motion invariant, being the foundations of the principle of relativity in classical mechanics. Consider two inertial reference frames: $\mathbb{S}$, with coordinate system $\{x, y, z, t\}$ and origin O, and $\mathbb{S}'$ with coordinate system $\{x', y', z', t'\}$ and origin O'; where the latter moves with constant velocity $\mathbf{u}$ relative to the former and both origins coincide at $t = t' = 0$. The transformation that takes the vector $\mathbf{r}(t)$ with entries (x, y, z, t) of an event from reference $\mathbb{S}$ to the vector $\mathbf{r}'(t')$ with entries (x', y', z', t') of the same event, but now seen from $\mathbb{S}'$, known as Galilean transformation, is given by

$$\mathbf{r}'(t') = \mathbf{r}(t) - \mathbf{u}t, \qquad t' = t. \tag{2.1.1}$$

For simplicity, we will consider situations wherein the reference frames have aligned axes and their relative motion occurs in the x axis only, *i.e.*, $\mathbf{u} = u\hat{\mathbf{x}}$, as illustrated in Fig. 2.1. This simplifies the above relation to

$$\begin{cases} x' = x - ut, \\ y' = y, \\ z' = z, \\ t' = t. \end{cases} \tag{2.1.2}$$

The transformation for the velocity vector is easily obtained by direct differentiation with respect to time. Using $\mathrm{d}\mathbf{r}/\mathrm{d}t = \mathbf{v}$, firstly to (2.1.1), we get

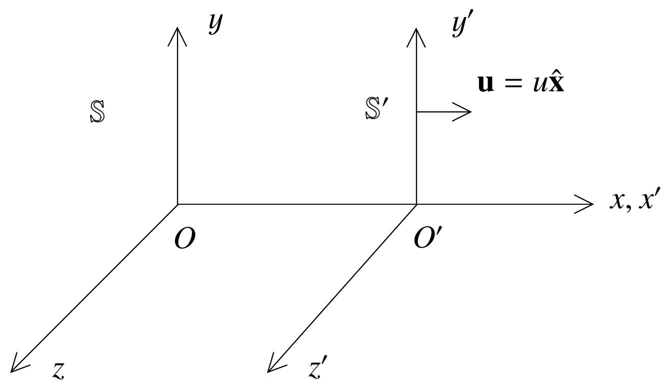

Figure 2.1: Reference frame $\mathbb{S}'$, with origin O', moves with constant velocity $\mathbf{u} = u\hat{\mathbf{x}}$ relative to another reference frame $\mathbb{S}$ with origin O. Both origins coincide at $t = t' = 0$.

$$\mathbf{v}'(t') = \mathbf{v}(t) - \mathbf{u}, \tag{2.1.3}$$

and, for (2.1.2)

$$\begin{cases} v'_x = v_x - u, \\ v'_y = v_y, \\ v'_z = v_z. \end{cases} \tag{2.1.4}$$

As far as kinematics is concerned, and since $\mathbb{S}$ moves with velocity $-\mathbf{u}$ relative to $\mathbb{S}'$, the inverse transformation is easily obtained by interchanging the primed and the unprimed variables and simultaneously replacing $\mathbf{u}$ by $-\mathbf{u}$. In other words, this means that both frames are equivalent.

If a resulting force $\mathbf{F}$ is imposed on a particle of mass m in frame $\mathbb{S}$, the particle will obtain an acceleration which obeys Newton's second law, $m\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2 = \mathbf{F}$ and, from (2.1.1), it follows that

$$\frac{\mathrm{d}^2\mathbf{r}'}{\mathrm{d}t'^2} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}, \tag{2.1.5}$$

where we see that the acceleration in both frames is the same. If, further, we add the assumption of classical physics that the mass of a body is a constant, independent of its motion relative to an observer, we have $m' = m$ and

$$\mathbf{F}' = m'\frac{\mathrm{d}^2\mathbf{r}'}{\mathrm{d}t'^2} = \mathbf{F}, \tag{2.1.6}$$

showing us that Newton's second law is valid in all inertial systems: the principle of Newtonian relativity.

If we calculate the distance between two events in frame $\mathbb{S}$ and $\mathbb{S}'$, occurring at (x, y, z, t) and (x', y', z', t'), respectively, we will find that

$$x'_2 - x'_1 = x_2 - ut_2 - (x_1 - ut_1) = x_2 - x_1 - u(t_2 - t_1), \tag{2.1.7}$$

which, if the events are simultaneous, $t_2 = t_1 = t'_1 = t'_2$, yields

$$x'_2 - x'_1 = x_2 - x_1. \tag{2.1.8}$$

They need to be simultaneous if the distance $x_2 - x_1$ is to be interpreted as a length measurement. If such is the case, we have that length is a conserved quantity when changing inertial frames with (2.1.2).

Through the set of equations that form the Galilean transformation and Newton's second law, we can conclude that classical mechanics implies that *length*, *time interval* and *mass* are invariant quantities. However, this formalism is only applicable within the limit of low velocities $u \ll c$, in such a way that we are in need of a new one that covers all ranges of velocities.

Lorentz Transformation

The new transformation of coordinates can be derived using the homogeneity assumption, which states that all points in space and time are equivalent, meaning that the results of a measurement of a length or time interval of a specific event should not depend on where or when the interval happens in our reference frame, and the postulates of special relativity theory:

Postulate I: *The laws of physics are the same in all inertial systems. No preferred inertial system exists.* (The Principle of Relativity.)

Postulate II: *The speed of light in free space has the same value c in all inertial systems.* (The Principle of the Constancy of the Speed of Light.)

Consider the same set of reference frames depicted in Fig. 2.1. The transformation that relates an event in the $\mathbb{S}$ frame, characterized by space-time coordinates (x, y, z, t) to the $\mathbb{S}'$ frame, where the same event is recorded by the coordinates (x', y', z', t'), is given by the Lorentz transformation*

$$\begin{cases} x' = \gamma_u(x - ut), \\ y' = y, \\ z' = z, \\ t' = \gamma_u \left(t - \dfrac{ux}{c^2}\right), \end{cases} \tag{2.1.9}$$

with $\gamma_u = \left(1 - \beta_u^2\right)^{-1/2}$ and $\beta_u = |\mathbf{u}|/c$ the so-called velocity factor. One can find that, in the regime where $u \ll c$, we have $\gamma_u \simeq 1$ and we obtain the Galilean transformation (2.1.2), which agrees with experimental data: Newtonian mechanics is a good description of nature in the low velocities limit.

Transformation (2.1.9) was first introduced by Lorentz in his electron theory, but his motivation and interpretation were different from that of relativity. The velocity u appearing in the velocity factor was the speed relative to the absolute frame: the ether. The derivation of this transformation using the relativity principle is due to Einstein.

The Lorentz transformation has an important feature: the invariance of the *interval*†, $\mathrm{d}s^2$, defined by

$$\mathrm{d}s^2 \equiv c^2\mathrm{d}t^2 - \mathrm{d}\mathbf{r}^2 = c^2\mathrm{d}t^2 - \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{2.1.10}$$

If $\mathrm{d}s'^2$ is the interval measured by $\mathbb{S}'$, we have

*This transformation can also be written in vector form, but this is not the form we will use throughout the rest of this work. For the interested reader, they are

$$\begin{cases} \mathbf{r}' = \mathbf{r} + \mathbf{u}\left[(\mathbf{r}\cdot\mathbf{u})\,(\gamma_u - 1)\,/u^2 - \gamma_u t\right], \\ t' = \gamma_u\left[t - (\mathbf{u}\cdot\mathbf{r})/c^2\right], \end{cases}$$

with $\mathbf{r}\cdot\mathbf{u}$ the scalar product.

†Some authors call this invariant property as *proper time* $\mathrm{d}\tau$, for the difference between them is just a constant of proportionality c. If, furthermore, one uses natural units $c = 1$, one gets that $\mathrm{d}s = \mathrm{d}\tau$.

$$ds'^2 = c^2 dt'^2 - d\mathbf{r}'^2 = ds^2. \tag{2.1.11}$$

Depending if $ds^2 > 0$, $ds^2 = 0$, or $ds^2 < 0$, we have, respectively: timelike, lightlike, and spacelike events. As (2.1.9) preserves the interval for inertial reference frames, if an event is, say, timelike, in one inertial frame, this event will also be timelike in any other inertial frame. Therefore, these quantities are also invariant under a Lorentz transformation.

Consequences of the Lorentz Transformation: Length Contraction and Time Dilation

The Lorentz transformation entails some interesting consequences about our notion of length and time; mainly due to the fact that an event that is simultaneous in some inertial frame $\mathbb{S}$, will not be simultaneous in another reference frame $\mathbb{S}'$ which is moving with velocity $\mathbf{u}$ relative to the former. This matter regards the discussion about simultaneity of events, not considered here.

Length Contraction: Imagine a rod lying at rest along the x' axis in $\mathbb{S}'$ frame and assume that its ends are at x'_2 and x'_1. Its length, in $\mathbb{S}'$, is simply the difference between its ends: $L' = x'_2 - x'_1$. Let us see how this length behaves when seen by $\mathbb{S}$ using (2.1.9). From the first equation of the set, we get

$$x'_2 = \frac{x_2 - ut_2}{\sqrt{1-\beta_u^2}}, \qquad x'_1 = \frac{x_1 - ut_1}{\sqrt{1-\beta_u^2}}. \tag{2.1.12}$$

Implying that

$$x'_2 - x'_1 = \frac{(x_2 - x_1) - u(t_2 - t_1)}{\sqrt{1-\beta_u^2}}. \tag{2.1.13}$$

So, the rod's length in $\mathbb{S}$ is the distance between its ends points, x_2 and x_1, measured at the same time, $t_2 = t_1$, in that frame; which is

$$L = x_2 - x_1 = L' \sqrt{1-\beta_u^2}, \tag{2.1.14}$$

being shorter than L'. We can rephrase this result as follows: a body's length is measured to be greater when it is at rest relative to the observer. When it moves with a velocity u relative to the observer, its measured length, L, is contracted in the direction of movement by the factor $\sqrt{1-\beta_u^2}$, whereas its dimensions perpendicular to the direction of motion are unaffected, for the corresponding spatial transformations are $y' = y$ and $z' = z$. This phenomenon, known as *length contraction*, has been experimentally verified a number of times [49–60].

Time Dilation: Consider a clock at rest at the position x' in $\mathbb{S}'$. An observer in this same frame, in possession of such clock, measures a time interval $\Delta t' = t'_2 - t'_1$ elapsed between two events. Using the last equation of the set (2.1.9), an observer, in $\mathbb{S}$, records the instants associated to each event as

$$t_1 = \frac{t'_1 + (u/c^2)x'}{\sqrt{1-\beta_u^2}}, \qquad t_2 = \frac{t'_2 + (u/c^2)x'}{\sqrt{1-\beta_u^2}}. \tag{2.1.15}$$

The clock in $\mathbb{S}'$ is at a fixed location x', but the times t_1 and t_2 are read at two different places, whereas a clock is positioned at each place in $\mathbb{S}$. Assuming they are synchronized, the observer in $\mathbb{S}$ can state that the time interval that has passed to him for the event is

$$\Delta t = t_2 - t_1 = \frac{\Delta t'}{\sqrt{1-\beta_u^2}}, \tag{2.1.16}$$

being greater than $\Delta t'$. In words, this result is: a clock is measured to go at its fastest rate when it is at rest relative to the observer. When it moves with a velocity u relative to the observer, its time flow is measured to have slowed down by a factor $\sqrt{1-\beta_u^2}$. This phenomenon is known as *time dilation*, also verified experimentally many times in history [61–69].

In relativity, there are some terms worth mentioning. The frame in which the observed body is at rest is called *proper frame*. The length of such body in such frame is then called *proper length*; and the time interval recorded by a clock attached to the observed body is called *proper time*. The latter can be thought of as being the time interval between two events occurring at the same place in $\mathbb{S}'$ or the time interval measured by a single clock at one place. Thus, an improper time interval would be the one measured with two different clocks at two different places.

2.1.2 Relativistic Kinematics and Dynamics

We will review some important concepts, that will be used later, regarding kinematics and dynamics of a moving particle as a consequence of the Lorentz transformation. Basically, Newtonian mechanics is inconsistent with special relativity because it is invariant under the Galilean transformation, not the Lorentz one. Thereupon, it has to be adapted in order to respect invariance under a Lorentz transformation.

Addition of Velocities

The transformation of velocities for (2.1.9) can be obtained in a similar fashion as the Galilean case. Suppose a particle moves with velocity $\mathbf{v}' = \mathrm{d}\mathbf{r}'/\mathrm{d}t' = (v_x', v_y', v_z')$ in the $\mathbb{S}'$ frame while this one moves with velocity $\mathbf{u} = u\hat{\mathbf{x}}$ relative to the $\mathbb{S}$ frame. The particle's velocity in the $\mathbb{S}$ frame is $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t = (v_x, v_y, v_z)$. The transformation then reads

$$v_x = \frac{v_x' + u}{1 + uv_x'/c^2}, \qquad v_y = \frac{v_y'}{\gamma_u\left(1 + uv_x'/c^2\right)}, \qquad v_z = \frac{v_z'}{\gamma_u\left(1 + uv_x'/c^2\right)}. \tag{2.1.17}$$

As a special case, consider that $\mathbf{v}' = c\hat{\mathbf{x}}'$, that is, the "particle" is now a light pulse moving in the x' axis in $\mathbb{S}'$. Through the set above, with $(v_x', v_y', v_z') = (c, 0, 0)$, we get

$$v_x = \frac{c+u}{1 + cu/c^2} = \frac{c+u}{c(c+u)}c^2 = c, \tag{2.1.18}$$

being independent of u. This means that, if a light pulse travels with velocity c in one inertial frame, all inertial frames, regardless of their velocities relative to the former, will state that the light pulse is also travelling at c, which had to be, since this is an assumption for obtaining the Lorentz transformation.

Relativistic Linear Momentum

The momentum $\mathbf{p}$ of a moving particle with velocity $\mathbf{v}$ also needs to be redefined in order to have its conservation law in collisions invariant under Lorentz transformations. Such transformation does not leave length and time measurements invariant when going from one reference to another; it may be that the measured mass of a body also depends on its motion relative to the observer, differing from the classical assumption. It is known now that this is indeed true, so the relativistic momentum of a particle with mass m moving with velocity $\mathbf{v}$ is such that

$$\mathbf{p} = \gamma_v m\mathbf{v} = m(v)\mathbf{v}, \qquad m(v) = \frac{m_0}{\sqrt{1-\beta_v^2}}, \tag{2.1.19}$$

where $m(0) = m_0$ is the particle's proper mass or rest mass, being identical to the one assigned to a particle in Newtonian mechanics. $m(v)$ may now be labelled as the relativistic mass of the particle.

As $\sum_i \mathbf{p}_i = \sum_i m_i \mathrm{d}\mathbf{r}_i/\mathrm{d}t = \sum_i m_i \mathbf{v}_i$ (total linear momentum) is a conserved quantity in Newtonian mechanics, when no external forces are affecting the system, we have that, if (2.1.19) denotes the linear momentum of the jth particle that composes the system, the sum $\sum_j \mathbf{p}_j = \gamma_{v_j} m_j \mathbf{v}_j$ is our conserved quantity in relativistic mechanics (under the same conditions), being the one which must be used if one wishes to correctly describe collisions and dynamics within the framework of relativity. Furthermore, as $\mathbf{p}$ is a vector, its conservation implies that each component p_k must be conserved. Explicitly writing (2.1.19) in components, given that $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$ and $v = |\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} = v_x^2 + v_y^2 + v_z^2$, where the same analysis applies to $\mathbf{p}$, we get that

$$p_k = \frac{m_0 v_k}{\sqrt{1-\beta_v^2}}, \qquad \text{with } k = x, y, z, \tag{2.1.20}$$

must be conserved for $\mathbf{p}$ to be as well. In addition, this way we emphasize that the magnitude of v appear on the denominator, through $\beta_v = v/c$, whilst the components appear on the numerator.

Relativistic Force and Energy

In classical mechanics, the work done by a resulting force $\mathbf{F}$, which acts upon a particle, is equal to the variation of the kinetic energy K of the particle. This force, according to Newton's second law, is

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \tag{2.1.21}$$

but this time, (2.1.19) must be used to properly carry the Lorentz transformation. When $\mathbf{F}$ is zero, we have that $\mathbf{p}$ must be a constant.

As in classical mechanics, the rate of work W done by this force is defined by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{F}\cdot\mathbf{v}, \tag{2.1.22}$$

where $\mathbf{v}$ is the particle's velocity. Moreover, the kinetic energy of a particle is such that

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}t}. \tag{2.1.23}$$

Using (2.1.19), (2.1.21) and (2.1.22), we get

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{v}\cdot\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{m_0\mathbf{v}}{\sqrt{1-\beta_v^2}}\right), \tag{2.1.24}$$

which can be evaluated to

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0c^2}{\sqrt{1-\beta_v^2}}\right). \tag{2.1.25}$$

Introducing it in (2.1.23) and integrating, we obtain

$$K = \frac{m_0c^2}{\sqrt{1-\beta_v^2}} + C, \tag{2.1.26}$$

with C a constant of integration. Since the kinetic energy can be taken as zero when $\mathbf{v} = \mathbf{0}$, it follows that $C = -m_0c^2$; thus

$$K = m_0c^2\left(\frac{1}{\sqrt{1-\beta_v^2}} - 1\right). \tag{2.1.27}$$

If we now take $E = mc^2$ as the total relativistic energy of the particle, we can express the foregone equation as

$$E = m_0c^2 + K, \tag{2.1.28}$$

where m_0c^2 is known as the particle's rest energy. In the classical limit, $v \ll c$, it can be found that $K \approx \frac{1}{2}m_0v^2$, the classical kinetic energy.

We see that all deviations between classical and relativistic mechanics are at least of second order in v/c, explaining why some earlier theories, like the electron theory developed by Lorentz, which were based on classical mechanics, were able to explain all effects of first order. Furthermore, this is the reason why the Michelson-Morley experiment was the decisive one about the ether theory, for its results could measure terms in second order, $(v/c)^2$.

To close this part, we can rewrite (2.1.28), noting that, if p_i and E are the momentum components and energy of the particle, respectively, in frame $\mathbb{S}$ and the same applies to p'_i and E' to $\mathbb{S}'$, p_x can be given by

$$p_x = \frac{m_0v_x}{\sqrt{1-\beta_v^2}} = \frac{m_0(v'_x + u)}{\sqrt{1-u'^2/c^2}\sqrt{1-v'^2/c^2}}, \tag{2.1.29}$$

or still,

$$p_x = \frac{p'_x + uE'/c^2}{\sqrt{1-\beta_v^2}}. \tag{2.1.30}$$

In the same way, we have

$$p_y = p'_y, \qquad p_z = p'_z, \qquad E = \frac{E' + up'_x}{\sqrt{1-\beta_u^2}}. \tag{2.1.31}$$

When comparing both with the Lorenz transformation (2.1.9), we see a certain similarity, meaning that the four quantities p_x, p_y, p_z and E/c^2 are transformed in the same way as the space-time coordinates (x, y, z, t); leading us to believe that, as $\mathrm{d}s^2 = \mathrm{d}s'^2$ (which is an invariant quantity), being both a function of (x, y, z, t) and (x', y', z', t'), respectively, there may be an invariant quantity for $(p_x, p_y, p_z, E/c^2)$ and $(p'_x, p'_y, p'_z, E'/c^2)$ as well. Indeed, there is: in analogy with (2.1.10), we arrive at

$$p^2 - \frac{E^2}{c^2} = p'^2 - \frac{E'^2}{c^2}, \tag{2.1.32}$$

and, through (2.1.19) and (2.1.28), we see that the invariant quantity is $-m_0c^2$; which means that, in any inertial reference frame,

$$E = \sqrt{p^2c^2 + m_0^2c^4}. \tag{2.1.33}$$

Mass-Energy Equivalence

The fact that energy is directly proportional to a body's mass, $E = mc^2$, is one of the most important, if not *the* most, result from Einstein's theory of relativity. We shall give the reader an example of what this equation says before continuing to our next section.

For such, consider an inelastic collision of two bodies, A and B, whose rest masses are m_0, each with energy K, colliding with one another to form a body with mass M_0, as seen by an observer in $\mathbb{S}'$ frame. Before the collision, these bodies have velocities oppositely directed and along the x' axis, *i.e.*, $\mathbf{v}'_A = -v'_A\hat{\mathbf{x}}' = -v'\hat{\mathbf{x}}'$ and $\mathbf{v}'_B = v'_B\hat{\mathbf{x}}' = v'\hat{\mathbf{x}}'$. After the collision, the combined body C is at rest in $\mathbb{S}'$, as required by conservation of momentum. In frame $\mathbb{S}$, which moves relatively to $\mathbb{S}'$ with speed $\mathbf{u} = -\mathbf{v}' = -v'\hat{\mathbf{x}}'$, the body C will have a velocity $\mathbf{v}_C = v_C\hat{\mathbf{x}} = u\hat{\mathbf{x}}$; A will be stationary before the collision, and B will have a speed $\mathbf{v}_B = v_B\hat{\mathbf{x}}$, as depicted in Figs. 2.2 and 2.3.

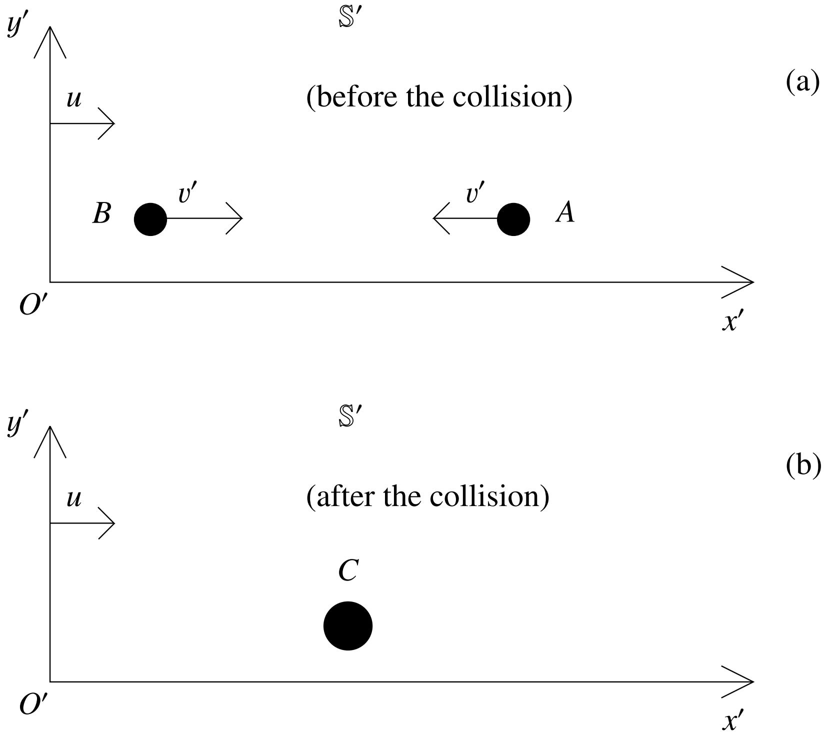

Figure 2.2: A particular inelastic collision as viewed by an observer in $\mathbb{S}'$ (a) before the collision and (b) after the collision.

The velocity of B in the $\mathbb{S}$ frame can be obtained through (2.1.17) as

$$v_B = \frac{v' + v'}{1 + v'^2/c^2} = \frac{2v'}{1 + v'^2/c^2}. \tag{2.1.34}$$

So, the relativistic mass of B, in $\mathbb{S}$, is

$$m_B = \frac{m_0(1 + v'^2/c^2)}{(1 - v'^2/c^2)}. \tag{2.1.35}$$

Hence, conservation of momentum in this frame gives us

$$\frac{m_0}{\sqrt{1 - v_B^2/c^2}} v_B = \frac{M_0}{\sqrt{1 - u^2/c^2}} u; \tag{2.1.36}$$

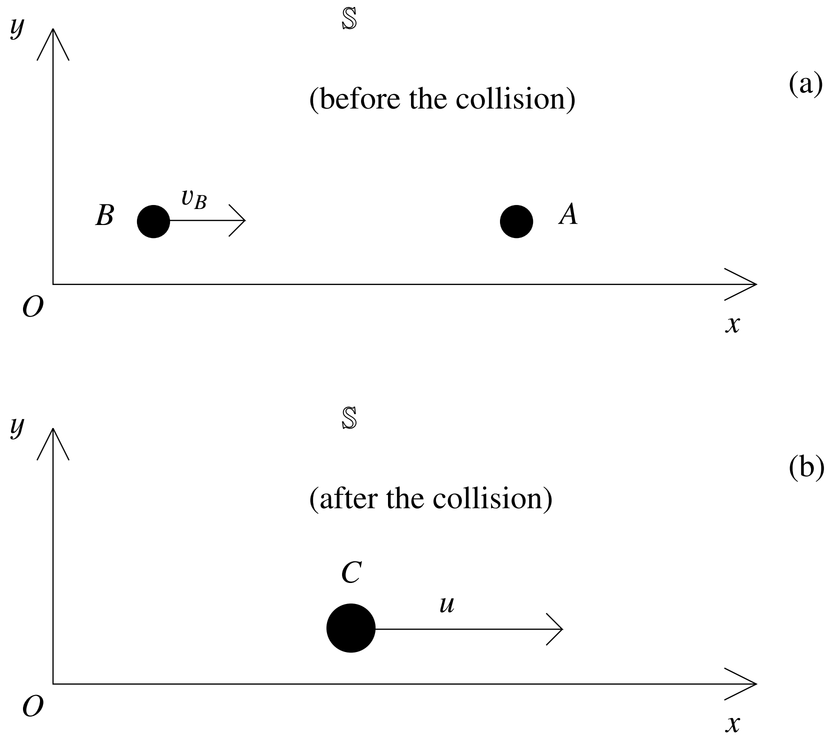

Figure 2.3: The same collision as in the previous figure; now viewed by an observer in $\mathbb{S}$ (a) before the collision and (b) after the collision.

with $u = v'$ and v_B given above, it yields

$$M_0 = \frac{2m_0}{\sqrt{1 - v'^2/c^2}}, \tag{2.1.37}$$

being greater than the sum of the combined bodies by an amount of

$$M_0 - 2m_0 = 2m_0\left(\frac{1}{\sqrt{1 - v'^2/c^2}} - 1\right). \tag{2.1.38}$$

If we look at the total kinetic energy, before the collision, in $\mathbb{S}'$, we have

$$K_A + K_B = 2K = 2m_0c^2\left(\frac{1}{\sqrt{1 - v'^2/c^2}} - 1\right), \tag{2.1.39}$$

and, since C is at rest in this frame, the final kinetic energy is zero. So, where did the energy $2K$ go to? Comparing $M_0 - 2m_0$ with $2K$, we see that $K_A + K_B = (M_0 - 2m_0)c^2$, showing us that the increase in mass, after the collision, is equal to the kinetic energy before the collision. Thus, the rest mass must also be included in the conservation of energy principle, being equivalent to energy. Although the kinetic energy is not conserved, the total energy is; the latter includes the former plus the rest mass energy. Hence, we are led to the following conclusion: conservation of total energy implies conservation of relativistic mass; this feature has been corroborated throughout the years and in many different approaches [70–80].

As a final remark, if kinetic energy is regarded as external energy, then the rest mass energy may be regarded as an internal one; with its nature coming from molecular motion, which changes when heat energy is absorbed or given by the body, or intermolecular potential energy, which changes when chemical reactions occur. Additionally, we can cite still atomic potential energy, changing when an atom absorbs or emits radiation, or nuclear potential energy, changed by nuclear reactions. For example, given one atomic mass unit, 1 u, being approximately equal to 1.66×10^{-27} kg, the rest mass of a proton is 1.00731 u and that of a neutron is 1.00867 u. A deuteron is known to consist of a neutron and a proton and has a rest mass of 2.01360 u; being less than the combined rest masses (proton plus neutron) by an amount of 0.00238 u. This quantity is equivalent to 3.57×10^{-13} J, which is equal to 2.22 MeV (mega electron-volts, the standard unit for energy in atomic and nuclear physics). This "missing energy" of 2.22 MeV comes from the internal energy of the deuteron.

2.1.3 Time Dilation

Although Lorentz transformations were invented to account for the invariance of the speed of light, the change from Galilean relativity to special relativity brought some kinematic consequences for material objects moving at speeds close to c. One of them will be at the focus of our analysis in Chapter 3: *time dilation*. Time dilation can be derived through many possibilities, but we shall study the one that will allow us to perform adaptations for the next step.

Consider the usual case of Fig. 2.1, but this time, let us assume we have an observer on the $\mathbb{S}'$ frame and that this frame is a moving train with constant velocity $\mathbf{u}$ relative to the $\mathbb{S}$ frame that is an observer on the ground. The observer on the train then prepares an experiment: he puts a light emitter on the floor, call it point A, and one mirror on the top of the train, point B, facing downwards. Points A and B are separated by a distance L, as shown in Fig. 2.4. Then, a single photon is emitted from point A towards point B (first event) and is reflected from B to A again (second event). The observer on the train measures the total time interval the photon travels during these events, which can easily be evaluated to

$$\Delta t' = \frac{2L}{c}. \tag{2.1.40}$$

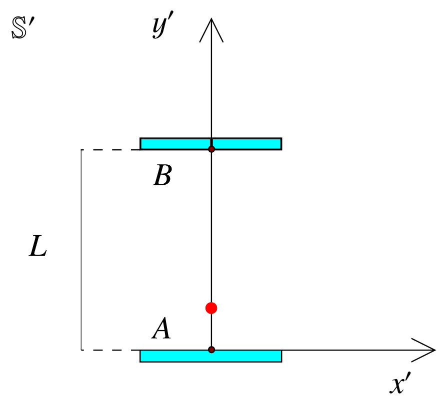

Figure 2.4: Representation of the emitted photon from the laser located in point A in the $\mathbb{S}'$ frame, which moves with velocity $\mathbf{u} = u\hat{\mathbf{x}}$ relative to $\mathbb{S}$. The photon (red spot) travels towards the mirror located in B and is reflected to A. The points are stationary for the $\mathbb{S}'$ observer and are a distance L apart.

We will analyse now the same sequence of events from the observer in the $\mathbb{S}$ frame. As the train moves with velocity $\mathbf{u}$ relative to this observer, the distance travelled by the train while the photon is going from A to B is $\frac{1}{2}u\Delta t$, which is the same from B to A, so that the total distance is $u\Delta t$. The photon, however, travels a distance $\frac{1}{2}c\Delta t$ while going up and another $\frac{1}{2}c\Delta t$ while going down, see Fig. 2.5.

From the triangle $A\hat{O}B$, we see that the distance travelled by the photon in $\mathbb{S}$ is greater than $\mathbb{S}'$. But, we know, through Einstein's postulates, that light travels with the same velocity c in both frames,

$\mathbb{S}$ and $\mathbb{S}'$. As the photon travels a greater distance in $\mathbb{S}$ with the same velocity, it takes more time to reach the mirror and return. Therefore, the total time interval has to be greater in $\mathbb{S}$ as well.

Using the rectangular triangle $A\hat{O}B$, one has that

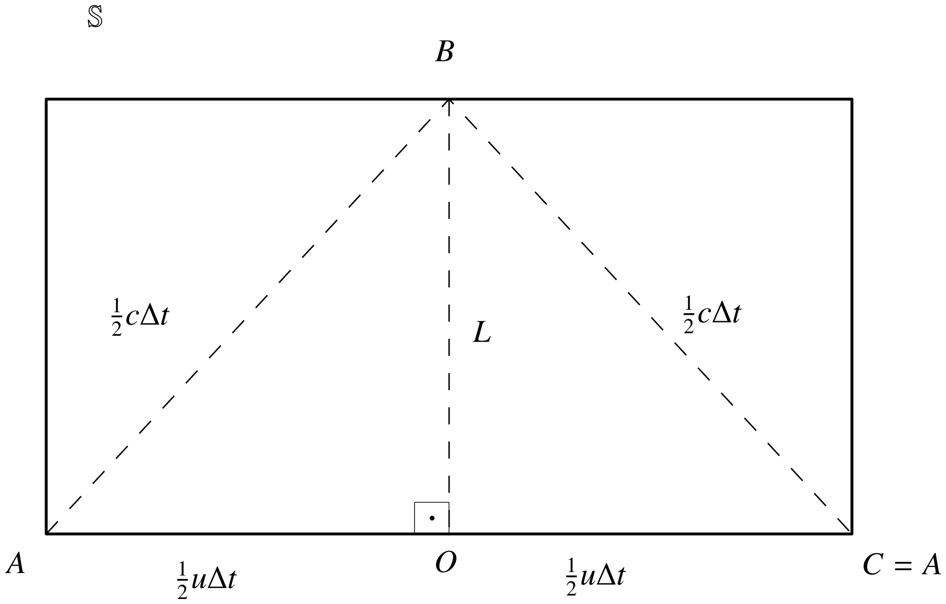

Figure 2.5: Representation of the emitted photon from the laser located in point A in the $\mathbb{S}$ frame. The movement of the photon is now bidimensional due to the train's movement on the x axis.

$$\left(\frac{1}{2}c\Delta t\right)^2 = L^2 + \left(\frac{1}{2}u\Delta t\right)^2, \tag{2.1.41}$$

and, with L given by (2.1.40), after rearranging terms, we arrive at

$$\Delta t^2 \left(1 - \beta_u^2\right) = \Delta t'^2. \tag{2.1.42}$$

Hence,

$$\Delta t = \gamma_u \Delta t', \tag{2.1.43}$$

which is equal to (2.1.16), as it must be.

Equation (2.1.43) is the usual one for time dilation, but it can also be written as a *time contraction* one

$$\Delta t' = \Delta t/\gamma_u, \tag{2.1.44}$$

giving us $\Delta t'$ as a function of Δt; meaning that, for a given time interval in $\mathbb{S}$ frame, $\mathbb{S}'$ frame will measure a shorter interval.

Although this method gives us the usual equation for time dilation, we see that when the photon is emitted from the laser in A and the mirror in B, it is supposed, due to their heavy masses, that all momentum is carried by the photon; for A and B do not acquire any velocity in the vertical direction. Our proposal is to assume that A and B can have their masses so close to zero (as an electron, say) that their variations of momentum, due to the photon's emission and absorption, are not negligible anymore and must be taken into account in the conservation laws.

2.2 Quantum Mechanics

This section is intended to provide the fundamental tools of our approach to quantum mechanics. They will be presented using the Dirac braket notation, due to their convenience and usefulness. We begin with the postulates of quantum mechanics in order to analyse how the concepts of probabilities are applied within the quantum framework, and the time evolution of a quantum state to prepare our discussion about the comparison between the description of a physical system through quantum and classical theory. Afterwards, we will dive into information theory to study quantifying measures of entropy, entanglement, coherence and quantum correlations for our final matter: irreality measures using the Bilobran-Angelo quantifier, which shall be introduced later.

2.2.1 Postulates of Quantum Mechanics

In classical mechanics, we know that the motion of any physical system is uniquely determined if the interaction potential, which may be a function of position, velocity, or even time, between the particles that compose such system is given and the boundary conditions of the problem have been set. For this to happen, one introduces, for the ith particle, generalized coordinates $q_i(t)$ whose derivatives with respect to time, $\dot{q}_i(t)$, are the generalized velocities (the index i may refer to the degrees of freedom of the system, as well). With these quantities specified, we are able to evaluate, at any given instant, the position and the velocity of the ith particle of the system with the Lagrangian formalism, *i.e.*, given the Lagrangian $\mathscr{L}(q_i, \dot{q}_i, t)$, the conjugate momentum p_i of each of the generalized coordinates q_i is

$$p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i}. \tag{2.2.1}$$

The variables q_i and p_i are known as the dynamical variables of the system. Once they have been calculated, all physical quantities can be expressed in terms of these variables. For example: the total energy of the system, given by the Hamiltonian function $\mathscr{H}$ which is found through the Legendre transformation

$$\mathscr{H}(q_i, p_i, t) = \sum_i^N \dot{q}_i p_i - \mathscr{L}, \tag{2.2.2}$$

with N the number of particles (or degrees of freedom) of the system. The motion of the system can yet be studied through the Hamilton equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial \mathscr{H}}{\partial p_i}, \tag{2.2.3a}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial \mathscr{H}}{\partial q_i}. \tag{2.2.3b}$$

Thus, the classical description of a system with N degrees of freedom is achieved through the following steps: **(i)** the state of a system at a fixed time t_0 is defined by specifying N generalized coordinates $q_i(t_0)$ and their N conjugate momenta $p_i(t_0)$; **(ii)** the value, at a given time, of the various physical quantities is completely determined when the state of the system at this time is known, since it enables one to predict with certainty the result of any measurement; **(iii)** the time evolution of the state of the system is given by the Hamilton equations; since these are first order differential equations, their solution $\{q_i(t), p_i(t)\}$ is unique if the value of these functions at a given time t_0 is fixed $\{q_i(t_0), p_i(t_0)\}$ (boundary conditions). The state of the system is known for all times t if its initial state is known. This may be interpreted as a deterministic notion: as the positions and momenta of all particles in the universe are defined (even though we do not know them), the future will be defined, as well; there is no such thing as "free will" in classical physics.

Now, what about quantum mechanics? The description of a system is based on the postulates upon which quantum mechanics has been formulated. They answer us: **(i)** how is the state of a quantum system at a given time described mathematically? **(ii)** Given this state, how can we predict the results of the measurement of various physical quantities? **(iii)** How can the state of the system at an arbitrary time t be found when the state at time t_0 is known?

To this end, we first introduce the notion of Hilbert space, also known as state space, denoted by $\mathcal{H}$, which is an extension of the usual Euclidean space, for the former comprises complex vectors of infinite dimension. Then, to each particle is assigned a quantum state represented by $|\psi\rangle$, which belongs to a ket space $\mathcal{E}$, where $\mathcal{E} \subset \mathcal{H}$, spanned by eigenstates of a given observable. The ket space may be discrete or continuous, with finite or infinite dimensionality. Therefore, the quantum state of a particle at a fixed time t_0 is characterized by a ket in the state space:

First Postulate: At a fixed time t_0, the state of a physical system is defined by specifying a normalized ket $|\psi(t_0)\rangle$ belonging to the state space $\mathcal{H}$.

As $\mathcal{H}$ is a linear vector space, the first postulate implies the superposition principle: a linear combination of vectors is also a vector. Given **A**, **B** and $\mathbf{C} \in \mathbb{R}^n$, $\mathbf{A} = c_1\mathbf{B} + c_2\mathbf{C}$, with $c_1, c_2 \in \mathbb{R}$ (that is, n-dimensional vectors with real coefficients). This means that the state of a system can be described by a superposition of states: given $|\alpha\rangle$, $|\beta\rangle$ and $|\gamma\rangle \in \mathcal{H}$, $|\alpha\rangle = a\,|\beta\rangle + b\,|\gamma\rangle$, with $a, b \in \mathbb{C}$ (that is, infinite-dimensional vectors, in general, with complex coefficients).

We now turn to the second postulate, which is the description of physical quantities, and the third, related to measurements of observables.

Second Postulate: Every measurable physical quantity $\mathscr{A}$ is described by an Hermitian operator A acting in $\mathcal{H}$; this operator is an observable.
Third Postulate: The only possible result of the measurement of a physical quantity $\mathscr{A}$ is one of the eigenvalues of the corresponding operator A.

Consider now a system described by a ket $|\psi\rangle$, normalized $\langle\psi|\psi\rangle = 1$, and we want to predict the outcome of a measurement of the observable A. As we know, the spectrum of A can be discrete or continuous, degenerate or nondegenerate. To each case, we separate the fourth postulate.

Fourth Postulate (discrete and nondegenerate spectrum)**:** When the physical quantity $\mathscr{A}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $P(a_n)$ of obtaining the nondegenerate eigenvalue a_n of the corresponding observable A is:

$$P(a_n) = |\langle\alpha_n|\psi\rangle|^2 \,, \tag{2.2.4}$$

where $|\alpha_n\rangle$ is the normalized eigenvector of A associated with the eigenvalue a_n.

For continuous and nondegenerate spectrum of A, with corresponding eigenvectors $|a_\mu\rangle$, $A\,|a_\mu\rangle = a_\mu\,|a_\mu\rangle$, where they form a continuous basis in $\mathcal{E}$, in which $|\psi\rangle$ can be expanded, $|\psi\rangle = \int \mathrm{d}\mu\, c(\mu)\,|a_\mu\rangle$, we have that the probability density (because the eigenvalues form a continuous) is $\mathrm{d}P(\mu) = p(\mu)\mathrm{d}\mu$, with $p(\mu) = \left|\langle a_\mu|\psi\rangle\right|^2$ and:

Fourth Postulate (continuous and nondegenerate spectrum)**:** When the physical quantity $\mathscr{A}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $\mathrm{d}P(\mu)$ of obtaining a result included in μ and $\mu + \mathrm{d}\mu$ is equal to

$$\mathrm{d}P(\mu) = \left|\langle a_\mu|\psi\rangle\right|^2 \mathrm{d}\mu \tag{2.2.5}$$

where $|a_\mu\rangle$ is the eigenvector corresponding to the eigenvalue μ of the observable A associated with $\mathscr{A}$.

As we will not encounter a degenerate case in this work, it will not be treated here.

With the fourth postulate stated, we are able to state the fifth, which is due to the state of the system after a measurement.

Fifth Postulate (Collapse): If the measurement of the physical quantity $\mathscr{A}$ on the system in the state $|\psi\rangle$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection, $\dfrac{P(a_n)\,|\psi\rangle}{\sqrt{\langle\psi|P(a_n)|\psi\rangle}}$, of $|\psi\rangle$ onto the eigensubspace associated with a_n.

Therefore, the state of the system immediately after a measurement is always an eigenvector of the observable in question. Finally, we state the last postulate, concerning the time evolution of a state.

Sixth Postulate: The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle = H(t)\,|\psi(t)\rangle\,, \tag{2.2.6}$$

where $H(t)$ is the observable associated with the total energy of the system.

H is called the Hamiltonian *operator* of the system, as it is obtained from the classical Hamiltonian *function* through the quantization rules.

As we can see from the postulates, quantum mechanics is a non-deterministic theory relying only on probabilities. But, what exactly is this probability associated with in quantum mechanics? For us to answer that, we have to discuss the concept of *ensemble*, which will take us to the path of the density operator formalism.

2.2.2 Density Operator Formalism

Ensemble

The probabilistic interpretation of quantum mechanics only works if thought as a collection of identically prepared states, all characterized by the same ket $|\psi\rangle$, such that the experiments and, thus, the measurements, are performed in each individual state that forms the collection of states. That is, when we say we have the state $|\psi\rangle$ and calculate that some given probability is 50%, empirically, it means that we have a great number of laboratories performing the same experiment, all with the same state $|\psi\rangle$, and, at the end of the day, they compare their results so as to conclude that certain eigenvalue, associated with the 50% probability, had been registered 50% of the times. This idealized collection of infinitely many states prepared under identical conditions receives the name of ensemble.

The concept of ensemble is also a very familiar one in statistical mechanics. As mentioned in Pathria's book [81]:

"It may, therefore, make sense if we consider, at a single instant of time, a rather large number of systems − all being some sort of "mental copies" of the given system − which are characterized by the same macrostate as the original system, but are, naturally enough, in all sorts of microstates. Then, under ordinary circumstances, we may expect that the average behaviour of any system in this collection, which we call an ensemble, would be identical to the time-averaged behaviour of the given system."

This ensemble can be a pure or a mixed one. A pure one can be regarded as if all members of the ensemble were characterized by a single and common ket that describes the system. A mixed ensemble, however, can be regarded as if the members were characterized by different probability weights, as if 70% were in $|\psi\rangle$ and the remaining 30% in $|\varphi\rangle$, and they do not need to be orthogonal. Consider a spin 1/2 system as an example. A pure ensemble would be as if all members had the same spin in a definite orientation alone, the positive z axis, say. A mixed one would be something as if 70% had the spin in the positive z axis and 30% in the negative x axis. Roughly speaking, a mixed ensemble can be viewed as a mixture of pure ensembles; hence the name. Additionally, there is the concept of completely random ensemble, which would be as if the probability weights of each state were equal, 50% and 50%.

A ket is only able to describe a pure ensemble. In order to describe the others, we need a new formalism: the density operator. This formalism is a powerful one, for it can describe all kinds of ensembles and it is a generalization of the usual ket state.

Density Operator

The formalism of the density operator was presented by J. von Neumann in 1927 [82], being capable to quantitatively describes physical situations with mixed as well as pure ensembles. If we define w_i as probability weights, they must satisfy the normalization condition

$$\sum_i w_i = 1. \tag{2.2.7}$$

However, the number of terms in the sum need not coincide with the dimensionality of the ket space.

Suppose now we want to evaluate the expectation value of some observable A, with eigenkets $|\alpha_k\rangle$ and corresponding eigenvalues a_k, on a mixed ensemble, taken with respect to some state $|\varphi_i\rangle$. The answer is given by the ensemble average of A, defined by

$$\begin{aligned} \langle A\rangle &\equiv \sum_i w_i \langle\varphi_i|A|\varphi_i\rangle \\ &= \sum_i \sum_j w_i \left|\langle\alpha_j|\varphi_i\rangle\right|^2 a_j, \end{aligned} \tag{2.2.8}$$

with $\langle\varphi_i|A|\varphi_i\rangle$ the usual expression for the expectation value of A. The probabilistic concept enter twice in the expression above: first in $\left|\langle\alpha_j|\varphi_i\rangle\right|^2$, which tells us the probability for the state $|\alpha_j\rangle$ to be found in $|\varphi_i\rangle$; and again in the probability factor w_i for finding in the ensemble a quantum state described by $|\alpha_j\rangle$.

If we write it using a more general basis $\{|\beta_m\rangle\}$, we get

$$
\begin{aligned}
\langle A \rangle &= \sum_i w_i \sum_m \sum_n \langle \varphi_i | \beta_m \rangle \langle \beta_m | A | \beta_n \rangle \langle \beta_n | \varphi_i \rangle \\
&= \sum_m \sum_n \left(\sum_i w_i \langle \beta_n | \varphi_i \rangle \langle \varphi_i | \beta_m \rangle \right) \langle \beta_m | A | \beta_n \rangle ,
\end{aligned} \tag{2.2.9}
$$

which motivates us to define the density operator ρ as

$$
\rho \equiv \sum_i w_i | \varphi_i \rangle \langle \varphi_i | , \tag{2.2.10}
$$

with corresponding matrix elements,

$$
\langle \beta_n | \rho | \beta_m \rangle = \sum_i w_i \langle \beta_n | \varphi_i \rangle \langle \varphi_i | \beta_m \rangle , \tag{2.2.11}
$$

defining the density matrix. Returning this expression into (2.2.9), we see that

$$
\begin{aligned}
\langle A \rangle &= \sum_{m,n} \langle \beta_n | \rho | \beta_m \rangle \langle \beta_m | A | \beta_n \rangle \\
&= \mathrm{Tr}(\rho A),
\end{aligned} \tag{2.2.12}
$$

where $\mathrm{Tr}(X)$ gives us the trace of matrix X. As the trace is independent of representation, the foregone relation can be used with any convenient basis. Additionally, the probability for obtaining the eigenvalue a_j, given that $A |\alpha_j\rangle = a_j |\alpha_j\rangle$, is

$$
\langle \alpha_j | \rho | \alpha_j \rangle = \sum_i w_i \left| \langle \alpha_j | \varphi_i \rangle \right|^2 = \sum_i w_i P_i(a_j), \tag{2.2.13}
$$

where the same reasoning about the probabilities in the expectation value case can be applied here.

Before continuing, we list some properties of the density operator below.

- Normalization: $\mathrm{Tr}\,\rho = 1$;
- Hermiticity: $\rho_{ij} = \rho^*_{ji}$, which is the same as $\rho = \rho^\dagger$;
- Positivity: $\rho \geq 0$;
- Purity: $\mathrm{Tr}\,\rho^2 \leq 1$, where the equality holds only for a pure state $\rho = |\varphi\rangle \langle \varphi|$.

Time Evolution of the Density Operator

Let us see how the density matrix evolves as a function of time. We start with the time-dependent Schrödinger equation and its complex conjugate

$$
\mathrm{i}\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle , \tag{2.2.14a}
$$

$$
-\mathrm{i}\hbar \frac{\partial}{\partial t} \langle \psi | = \langle \psi | H, \tag{2.2.14b}
$$

differentiate the density matrix of a mixed state (2.2.10) with respect to time, multiply it by $\mathrm{i}\hbar$ and combine with the set above to yield

$$
\begin{aligned}
\mathrm{i}\hbar\frac{\partial\rho}{\partial t} &= \mathrm{i}\hbar\sum_i w_i\left(|\dot{\psi}_i\rangle\langle\psi_i| + |\psi_i\rangle\langle\dot{\psi}_i|\right) \\
&= \sum_i w_i\left(H\rho_i - \rho_i H\right) \\
&= [H,\rho],
\end{aligned} \tag{2.2.15}
$$

where $\rho_i = |\psi_i\rangle\langle\psi_i|$. This equation is known as the von Neumann equation, being the quantum mechanical analog of the classical Liouville equation.

Particularly, we can solve it for the case where H is not time-dependent considering the time evolution operator

$$
U(t,t_0) = \exp\left[-\frac{\mathrm{i}}{\hbar}H(t-t_0)\right], \tag{2.2.16}
$$

with $t > t_0$, which allows us to write

$$
|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle \tag{2.2.17}
$$

in order to obtain

$$
\rho(t) = U\rho(t_0)U^\dagger. \tag{2.2.18}
$$

In addition, we can prove that, as $UU^\dagger = U^\dagger U = \mathbb{1}$, $\mathrm{Tr}\,\rho^2(t)$ is not time dependent:

$$
\mathrm{Tr}\,\rho^2(t) = \mathrm{Tr}\left(U\rho(t_0)U^\dagger U\rho(t_0)U^\dagger\right) = \mathrm{Tr}\left(\rho^2(t_0)U^\dagger U\right) = \mathrm{Tr}\,\rho^2(t_0), \tag{2.2.19}
$$

meaning that the purity of a state given ρ is not altered while ρ evolves in time.

2.2.3 Information Theory

Information theory is one of the few theories that we can state, with certainty, where it has begun: Shannon's paper [83], in 1948, which has introduced a quantitative model of communication as a statistical process. His motivation was to approach a fundamental problem in communication which, in his words, *"[...] is that of reproducing at one point, either exactly or approximately, a message selected at another point."*. Thus, information theory studies quantification, storage, and communication of information in communicating systems. The reader is invited to seek [84] for a detailed study on information theory.

For an example of the quantifying measures, consider the following. You have the task of uncovering a certain word that was written in English and you somehow obtain the information that the letter "e" is in the word; that is good. However, "e" is the most common letter in English words and you have not narrowed your options very much, so the information you have gained about the word, by knowing that the letter "e" is in it, was not much. If, on the other hand, you discover that the letter "j" is present in the word, which is the least common in English, your options have been narrowed much further than that of the "e" case. In other words, you have gained much more information on the latter case than on the former. Shannon has provided a quantifying measure of this difference.

Practically, all these properties can be derived from a single concept: entropy; which we address now. First, we take a look at the classical definitions and, then, we move to quantum information theory.

Shannon Entropy

In classical statistical thermodynamics, the entropy S_{cl} is defined as

$$S_{\mathrm{cl}} = k_B \log \Omega, \tag{2.2.20}$$

with k_B the Boltzmann constant and Ω the number of accessible microstates of the system. The logarithm can be expressed in any desired basis; commonly, it is the $\ln = \log_e$, $\log_2$ or $\log_{10}$. If all microstates are equally probable, then the probability[‡] p of each one is $p = 1/\Omega$ and (2.2.20) can be written as

$$S_{\mathrm{cl}} = -k_B \log p. \tag{2.2.21}$$

If, on the other hand, the probabilities are not equal, one has that the average entropy, considering that the probability for the ith state is p_i, is given by

$$S_{\mathrm{cl}} = -k_B \sum_i p_i \log p_i. \tag{2.2.22}$$

Now, consider the following situation. A message is a string of letters from an alphabet of k letters represented by the set $\{a_1, a_2, \ldots, a_k\}$. Suppose the letters are statistically independent and that each letter a_i occurs with a probability p_i, with the constraint that

$$\sum_{i=1}^{k} p_i = 1. \tag{2.2.23}$$

As an example, we cite the binary alphabet, where our set would be represented by $\{0, 1\}$. Hence, if 1 occurs with probability p, 0 occurs with probability $(1 - p)$.

If we address long messages, containing n letters, with $n \gg 1$, we may ask: is it possible to compress this set of letters to a shorter version containing, essentially, the same information?

Well, for large n, we have that typical strings will contain, in the binary case, np 1's and $n(1 - p)$ 0's. The number of distinct strings is of order of the binomial coefficient of n and np, which, using Stirling's approximation (valid for $n \gg 1$) $\log n! \approx n \log n - n$, we get

$$\log \binom{n}{np} = \log \left(\frac{n!}{(np)!\,[n(1-p)]!} \right) \simeq nH(p), \tag{2.2.24}$$

where

$$H(p) = -\left[p \log p + (1 - p) \log(1 - p)\right] \tag{2.2.25}$$

is the entropy function. Thus, the number of typical strings is of order $2^{nH(p)}$ (it is very convenient to use base 2 in the log function when concerning bits).

This result, which is due to Shannon, tells us that all the information carried by a sequence of n bits can be assigned into a chosen block code of positive integers associated to each of the typical strings, and such block code has about $2^{nH(p)}$ letters. Furthermore, for any $p \neq 1/2$, the message is shortened by the block code, since $H(1/2) = 1$ and $0 < H(p) < 1$ otherwise.

We are able to generalize the formalism for the case of any number of letters, where letter x occurs with probability $p(x)$. In a string of n letters, x typically occurs about $np(x)$ times, and the number of typical strings is of order

‡This label is not to be confused with the momentum one, as it is only a matter of standard notation. In cases where it may cause confusion, a comment will appear.

$$\frac{n!}{\prod_x [np(x)]!} \simeq 2^{nH(X)}, \tag{2.2.26}$$

with

$$H(X) = -\sum_x p(x) \log p(x) \tag{2.2.27}$$

the Shannon entropy of the ensemble $X = \{x, p(x)\}$.

Through the notion of block code presented earlier, which assigns an integer to each typical sequence, the information in a string of n letters can be compressed to $nH(X)$ bits. In this sense, a letter x chosen from the ensemble carries, on average, $H(X)$ bits of information.

Mutual Information

Suppose we have the pair (X, Y) of two ensembles X and Y. If we know some values of Y, we obtain some information $H(Y)$ about the pair. The conditional entropy of X, given we know something of Y, is a measure of how ignorant, on average, we still are with respect to X after knowing Y, and it is defined as

$$H(X|Y) = -\sum_{x,y} p(x, y) \log p(x, y) - H(Y) = H(X, Y) - H(Y), \tag{2.2.28}$$

where $H(X, Y)$, known as joint entropy, measures the total ignorance about the pair (X, Y). We may interpret this quantity as the number of additional bits per letter needed to specify both x and y once the latter is known.

The Shannon entropy $H(X)$ quantifies how much information is transmitted, on average, by a letter taken from the ensemble X, for it tells us how many bits are required to encode it. The mutual information quantifies the correlation of two messages from two different ensembles, X and Y, by telling us how much about X we gain when learning something from Y or, in other words, by quantifying how much the number of bits per letter needed to specify X is reduced when Y is known. Thus,

$$\begin{aligned} I(X : Y) &= H(X) + H(Y) - H(X, Y), \\ &= H(X) - H(X|Y), \\ &= H(Y) - H(Y|X), \end{aligned} \tag{2.2.29}$$

being symmetric when X and Y are interchanged, since we find out as much about X when learning Y as about Y when learning X. $I(X : Y)$ is called the mutual information of X and Y.

As a property, $I(X : Y)$ is nonnegative, because learning something from some ensemble can never reduce the knowledge of the other. If yet X and Y are uncorrelated, meaning that $p(x, y) = p(x)p(y)$, we get $I(X : Y) = 0$. Naturally, as we cannot know anything about X by learning about Y if they are not correlated.

Having these quantifiers, definitions and properties in mind, we are able to proceed to quantum information theory and analyse how they adapt to quantum states.

Quantum Information Theory

Quantum information theory (QIT) is the subject concerning the elements of classical information theory through quantum concepts. What we need now is to generalize classical elements to quantum theory.

Imagine then a source that prepares messages of k letters $\{a_1, a_2, \ldots, a_k\}$ where each one is chosen from an ensemble of quantum states ρ_k with probability p_k. The probability of any outcome of any measurement of a letter chosen from this ensemble, if the observer has no knowledge about which letter was prepared, can be completely characterized by the density operator

$$\rho = \sum_k p_k \rho_k. \tag{2.2.30}$$

The von Neumann entropy $S(\rho)$ is an extension of (2.2.22) and is defined as

$$S(\rho) = -\operatorname{Tr}(\rho \log \rho). \tag{2.2.31}$$

By choosing an orthonormal basis $\{|\varphi_i\rangle\}$ which diagonalizes ρ, such as

$$\rho = \sum_i p_i |\varphi_i\rangle \langle\varphi_i|, \tag{2.2.32}$$

and the property $\langle\varphi_j| \log\left(\sum_i p_i |\varphi_i\rangle\langle\varphi_i|\right) |\varphi_k\rangle = \sum_i \log p_i \langle\varphi_j|\varphi_i\rangle \langle\varphi_i|\varphi_k\rangle$, which is a direct consequence of ρ being diagonal in the $\{|\varphi_i\rangle\}$ basis, (2.2.31) takes the form

$$\begin{aligned}
S(\rho) &= -\operatorname{Tr}\left[\sum_i p_i |\varphi_i\rangle\langle\varphi_i| \log\left(\sum_j p_j |\varphi_j\rangle\langle\varphi_j|\right)\right] \\
&= -\sum_k \sum_i p_i \langle\varphi_k|\varphi_i\rangle \left(\sum_j \log p_j\right) \langle\varphi_i|\varphi_j\rangle \langle\varphi_j|\varphi_k\rangle \\
&= -\sum_{i,j,k} \delta_{ki}\delta_{ij}\delta_{jk} p_i \log p_j \\
&= -\sum_i p_i \log p_i,
\end{aligned} \tag{2.2.33}$$

which is the same as the Shannon entropy. This is not surprising because we can imagine the classical message symbols to be replaced by quantum states. Since the latter are orthogonal, they can be distinguished with certainty, as can the classical symbols, and hence there is no physical difference between the two situations.

Von Neumann's entropy is a very meaningful quantity to obtain because it plays many roles: **(i)** it quantifies the quantum information content per letter of the ensemble, which is the minimum number of qubits (quantum bits) per letter necessary to encode it; **(ii)** it also quantifies the classical information content, which is the maximum amount of information per letter (in bits) that is available to obtain; **(iii)** and yet, it quantifies entanglement of a bipartite pure state. Moreover, nonorthogonal pure states cannot be distinguished from the orthogonal ones because von Neumann's entropy is the same for both; something that has no classical analog.

We now state some mathematical properties of $S(\rho)$. As their proofs will not be given in this volume, the reader may refer to [85]. They are:

- Purity: $S(\rho) = 0$ if and only if ρ is pure;
- Invariance: The entropy is unchanged under unitary transformations $S(U\rho U^\dagger) = S(\rho)$, with $UU^\dagger = U^\dagger U = \mathbb{1}$, which can be seen straightforwardly since $S(\rho)$ depends only on ρ's eigenvalues;

- Maximum: if ρ has u nonvanishing eigenvalues, then $S(\rho) \leq \log u$, where the equality holds when all nonzero eigenvalues are equal. Meaning that the entropy is maximized when the quantum state is randomly chosen;

- Concavity: if $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ and $\sum_i \lambda_i = 1$, then $S(\lambda_1\rho_1 + \ldots + \lambda_n\rho_n) \geq \lambda_1 S(\rho_1) + \ldots + \lambda_n S(\rho_n)$. In words, von Neumann's entropy is larger the more ignorant we are about how the state was prepared. This property is a consequence of the convexity of the logarithm function;

- Entropy of measurement: Suppose that, in a state ρ, we measure the observable $A = \sum_x a_x |\alpha_x\rangle\langle\alpha_x|$ so that the outcome a_x occurs with probability $p_x = \langle\alpha_x|\rho|\alpha_x\rangle$. Then, the Shannon entropy of the ensemble of measurement outcomes $X = \{a_x, p_x\}$ satisfies $H(X) \geq S(\rho)$, where the equality holds when $[A, \rho] = 0$. Mathematically, this is the statement that $S(\rho)$ increases if we replace all off-diagonal matrix elements of ρ by zero, in any basis. Physically, it says that the randomness of the measurement outcome is minimized if we choose to measure an observable that commutes with the density matrix;

- Entropy of preparation: if a pure state is drawn randomly from the ensemble $\{|\varphi_x\rangle, p_x\}$, so that the density matrix is $\rho = \sum_x p_x |\varphi_x\rangle\langle\varphi_x|$, then $H(X) \geq S(\rho)$, with equality if the states of the signals $|\varphi_x\rangle$ are mutually orthogonal. This statement indicates that distinguishability is lost when we mix nonorthogonal pure states;

- Subadditivity: consider a bipartite system AB in the state ρ. Then, $S(\rho) \leq S(\rho_A) + S(\rho_B)$, where $\rho_A = \mathrm{Tr}_B\,\rho$ is the partial trace over B subspace, with equality for $\rho = \rho_A \otimes \rho_B$. Thus, entropy is additive for uncorrelated systems, but otherwise the entropy of the whole is less than the sum of its parts. This property is analogous to $H(X,Y) \leq H(X) + H(Y)$, of Shannon entropy; it holds because some of the information in XY (or AB) is encoded in the correlations between X and Y (A and B);

- Triangle inequality (Araki-Lieb inequality): for a bipartite system, $S(\rho) \geq |S(\rho_A) - S(\rho_B)|$. This inequality contrasts sharply with the analogous property of the Shannon entropy, which says that $H(X,Y) \geq H(Z)$, with $Z = X, Y$. The Shannon entropy of a classical bipartite system exceeds the Shannon entropy for either part, meaning that there is more information in the whole system than in part of it.

Quantum Discord

We have seen that the concepts of entropy and mutual information enable us to quantify correlations, classical and quantum. In the quantum framework, there is also another quantifier: quantum discord, which was introduced by Ollivier and Zuerk in [86] and Henderson and Vedral in [87]. It measures correlations that can also be present in certain mixed separable states and it is based on quantum mutual information. More precisely, quantum discord is the difference between the total mutual information of the subsystems and the mutual information that can be extracted by local measurements. Further, in the case of pure states, the quantum discord measures the entropy of entanglement.

If we apply von Neumann's entropy for a bipartite system consisting of two qubits, A and B, ρ, and the local entropies as $S(\rho_A) = S(\mathrm{Tr}_B\,\rho)$ and similarly for $S(\rho_B)$, we calculate the quantum mutual information between A and B as

$$I_{A:B}(\rho) = S(\rho_A) + S(\rho_B) - S(\rho). \tag{2.2.34}$$

If $\rho = \rho_A \otimes \rho_B$, which means that the subsystems are completely independent, the sum of the information contents of the subsystems, $S(\rho_A) + S(\rho_B)$, is equal to the information content $S(\rho)$. However, if they are correlated, a measurement upon one subsystem also contain information about the other and the total information is smaller than the sum of the two subsystems. Therefore, $I_{A:B}(\rho)$ measures the total correlation between them.

Moreover, with the set of projective operators, $\prod_j^A = |\alpha_j\rangle\langle\alpha_j|$ (with $A|\alpha_j\rangle = a_j|\alpha_j\rangle$), on subsystem A, we define the quantum conditional entropy $S(\rho_B|A)$ as

$$S(\rho_B|A) = \min \sum_{j \in \mathcal{E}_A} S\left(\rho_{B|\prod_j^A}\right), \tag{2.2.35}$$

where the sum in j runs over the dimensionality of the ket space spanned by A's eigenvectors: $\mathcal{E}_A = \text{span}\{|\alpha_j\rangle\}$[§]. We need to specify the set of projective operators that will be measured because there will be an ambiguity otherwise, for the resulting state after the measurement ($\rho_{B|\prod_j^A}$) relies on what observable is measured. As every measurement disturbs the system being measured, we choose the one that will minimize the disturbance, hence the "min" on the right-hand side of the equation. To summarize, $S(\rho_B|A)$ gives the entropy for the state ρ_B given that a projective measurement has been performed on ρ_A.

Through these quantities, we calculate the difference

$$J_A(\rho) = S(\rho_B) - S(\rho_B|A), \tag{2.2.36}$$

which specifies the information gained about B as a result of a measurement on some set of observables on A. (For a classical system, as there is no ambiguity concerning measurements, we have that $I(\rho) = J(\rho)$.) Hence, we are able to define the quantum discord as

$$D_A(\rho) = I_{A:B}(\rho) - J_A(\rho) = S(\rho_A) - S(\rho) + S(\rho_B|A). \tag{2.2.37}$$

If $D_A(\rho) \neq 0$, it means that measurements upon subsystem A disturb subsystem B, which can happen even if A and B are not entangled. Equation (2.2.37) can then be interpreted as the difference between the total mutual information and the mutual information that can be extracted by local measurements.

Entanglement

The concept of entangled states can be applied to any number N, finite or infinite, of composite systems, with each subsystem lying in a Hilbert space $\mathcal{H}_i$, and total Hilbert space given by their tensor product $\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$. However, as we will only treat bipartite systems, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, in this work, we shall only consider entanglement in this situation. Hereafter, we will denote $\mathcal{D}(\mathcal{H})$ the set of operators $\mathcal{O}$ that satisfy the properties of positivity, hermiticity, and normalization, which act on $\mathcal{H}$.

The definition of entanglement is given by a negative:

Definition 2.2.1. Entanglement
A state $\rho \in \mathcal{D}(\mathcal{H})$ is said to be *separable* if it can be written as the convex sum

$$\rho = \sum_i w_i \rho_A^i \otimes \rho_B^i, \qquad w_i \geq 0, \qquad \sum_i w_i = 1. \tag{2.2.38}$$

Otherwise, the state of the system is said to be *entangled*.

[§]For example, if we are dealing with a spin 1/2 system, and $A = \sigma_z$, with $\sigma_z|\pm\rangle = \pm|\pm\rangle$, our set of projectors are $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ and $\mathcal{E}_A = \{|+\rangle, |-\rangle\}$; thus, $\dim(\mathcal{E}_A) = 2$.

For the particular case concerning pure states, the definition above reduces to

Definition 2.2.2. Entanglement of Pure States
A state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is said to be separable if

$$|\psi\rangle = |\phi\rangle_A \otimes |\varphi\rangle_B \,, \qquad |\phi\rangle_A \in \mathcal{H}_A, \quad |\varphi\rangle_B \in \mathcal{H}_B. \tag{2.2.39}$$

Otherwise, the state is said to be entangled.

These definitions may be rather simple, one just has to attempt a factorization of the global state to see if it is possible to write it as a product of two separable states. However, this step is not always so easily obtainable; maybe because the state is really not separable or the separation is not so clear. As a result, there came to be a necessity to find quantifiers of entanglement without having to explicitly separate the state. Although there are many of them, we will consider only the one which we will use: entropy of entanglement, which holds only for pure states. If the reader is interested in more details about entanglement and its quantifiers, we recommend [88, 89].

The entropy of entanglement $E(|\psi\rangle)$ is given by von Neumann's entropy of the reduced state. It is one of the simplest entanglement measures. It vanishes for a reduced pure state, $E = 0$, and it is maximum for a completely mixed reduced state, $E = \log d$, where $\dim(\mathcal{H}) = d$. $E(\rho)$ is defined for bipartite pure states as the von Neumann's entropy of one of the reduced states:

$$E(\rho) = S\,(\rho_A) = S\,(\rho_B)\,. \tag{2.2.40}$$

If ρ is a product state, $\rho = \rho_A \otimes \rho_B$, such as, $|\psi\rangle = |\alpha_i\beta_j\rangle$, each reduced state is a pure one and the entropy vanishes. However, if the state is maximally entangled, such as $|\psi\rangle = 1/\sqrt{2}\left(|\alpha_i\beta_j\rangle + |\alpha_k\beta_l\rangle\right)$, the subsystems are completely mixed, $\rho_A = \rho_B = \frac{1}{2}\mathbb{1}$, and the entropy is maximum.

2.2.4 Irreality Measure

Through the postulates of quantum mechanics and the EPR paper [48], we have seen that quantum theory has a great problem of telling us what is happening, exactly, with a state $|\psi\rangle$ before a measurement takes place and during its process. The situation is even more shocking when entanglement is present in the state. Due to this causes, EPR has claimed that quantum theory is incomplete, for, among other reasons, it cannot enable one to predict with certainty what will be the outcome of a measurement and, as a consequence, an element of physical reality cannot be, in general, assigned to the particle's state before it has been measured.

To this end, a measure of irreality (absence of reality) may be taken to quantify such element, in the same way that the gain/loss of information is quantified by Shannon's/von Neumann's entropy. We use the Bilobran-Angelo measure defined in [90] to do it, for it is an extension of EPR's criterion: the latter imputes an element of reality only for observables' eigenstates (pure state), whilst the former is able to describe reality when concerning mixed states, as well.

Consider an experimental procedure that prepares a physical state for a multipartite system. A task is defined which consists of determining, via state tomography, the most complete description for this preparation. Thus, we get to know that, every time the procedure runs, the quantum-mechanical description for the system will be ρ, see Fig. 2.6(a). Then, we are exposed to a different scheme, presented in Fig. 2.6(b). Again, we are asked to propose a complete description for the system state, given the same preparation and tomography, but now a measurement of an observable $\mathcal{O}_1 = \sum_k o_{1k}\mathrm{O}_{1k}$, with $\mathrm{O}_{1k} = |o_{1k}\rangle\langle o_{1k}|$ acting on $\mathcal{H}_1$, is secretly performed by an agent between the preparation and the tomography, in every run of the procedure. Quantum theory predicts that the system will be in the state

$O_{1k} \otimes \rho_{2|o_{1k}}$ with probability p_k after the measurement is performed, where $\rho_{2|o_{1k}} = \mathrm{Tr}_1 (O_{1k}\rho O_{1k}) / p_k$ is the state of the rest of the system given the outcome o_{1k} and $p_k = \mathrm{Tr} (O_{1k}\rho O_{1k})$. As the observable $\mathcal{O}_1 \in \mathcal{H}_1$ has been measured, we know the state has been projected into one of the eigenstates $O_{1k}\rho O_{1k}$ that belongs to the eigenspace of $\mathcal{O}_1$. But, as we do not know exactly into which of them, we have to sum over them all. Hence, the best description one can give to the system, after the measurement, is

$$\Phi_{\mathcal{O}_1} (\rho) = \sum_k O_{1k}\rho O_{1k} = \sum_k p_k O_{1k} \otimes \rho_{2|o_{1k}}. \tag{2.2.41}$$

According to EPR's criterion, the agent is certain that the observable is real after each measurement is made. It follows then that the probability p_k reflects only our subjective ignorance about the actual value of $\mathcal{O}_1$.

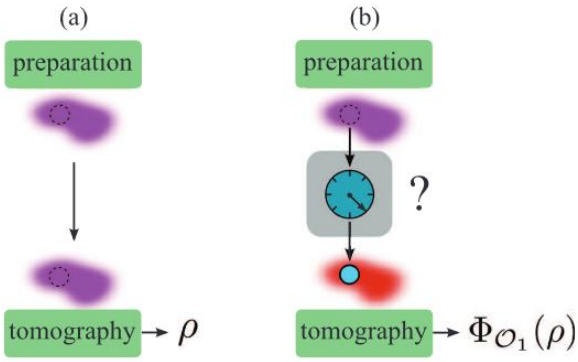

Figure 2.6: (a) A preparation ρ is determined by state tomography. (b) An observable $\mathcal{O}_1$ is secretly measured after the preparation, so that it is surely real before the tomography, which then predicts the state $\Phi_{\mathcal{O}_1} (\rho)$. If $\Phi_{\mathcal{O}_1} (\rho) = \rho$, then the secret measurement has just revealed a pre-existing element of reality. Image taken from [90].

If the situation is such that $\Phi_{\mathcal{O}_1} (\rho) = \rho$, the agent can conclude that an element of reality for $\mathcal{O}_1$ was implied by the very preparation. In this case, the agent's measurements did not create reality, but only revealed a pre-existing one; suggesting us the following definition:

Definition 2.2.3. Element of Reality
An observable $\mathcal{O}_1 = \sum_k o_{1k} O_{1k}$, with projectors $O_{1k} = |o_{1k}\rangle \langle o_{1k}|$ acting on $\mathcal{H}_1$, is real for a preparation $\rho \in \bigotimes_{i=1}^N \mathcal{H}_i$ if and only if

$$\Phi_{\mathcal{O}_1} (\rho) = \rho. \tag{2.2.42}$$

It agrees with EPR's about the reality of $\mathcal{O}_1$ when the preparation is an eigenstate of this observable, *i.e.*, $\rho = O_{1k}$, and, furthermore, it also predicts an element of reality for a mixture of eigenstates, $\rho = \sum_k p_k O_{1k}$, as in this case

$$\begin{aligned} \Phi_{\mathcal{O}_1} (\rho) &= \sum_k O_{1k} \left(\sum_j p_j O_{1j} \right) O_{1k}, \\ &= \sum_k |o_{1k}\rangle \sum_j p_j \delta_{kj} \delta_{jk} \langle o_{1k}| , \\ &= \rho. \end{aligned} \tag{2.2.43}$$

Another point to be noticed is that this formalism already incorporates the fact that a measurement preserves a pre-existing reality, meaning that

$$\begin{aligned}\Phi_{\mathcal{O}_1\mathcal{O}_1}(\rho) &= \Phi_{\mathcal{O}_1}\left(\Phi_{\mathcal{O}_1}(\rho)\right),\\ &= \sum_k \mathrm{O}_{1k}\left(\sum_j \mathrm{O}_{1j}\rho\mathrm{O}_{1j}\right)\mathrm{O}_{1k},\\ &= \sum_k |o_{1k}\rangle \sum_j \delta_{kj}\langle o_{1j}|\rho|o_{1j}\rangle\,\delta_{jk}\langle o_{1k}|,\\ &= \sum_k \mathrm{O}_{1k}\rho\mathrm{O}_{1k},\\ &= \Phi_{\mathcal{O}_1}(\rho).\end{aligned} \tag{2.2.44}$$

Regarding this criterion, it induces us a measure of by how much a given state ρ is far from a state with $\mathcal{O}_1$ real. That is, if $\Phi_{\mathcal{O}_1}(\rho)$ stands for the state ρ with $\mathcal{O}_1$ real and, in general, ρ does not, we define the irreality of the observable $\mathcal{O}_1$ given the preparation $\rho \in \mathcal{H}$ as the entropic distance

$$\Im(\mathcal{O}_1|\rho) \equiv S\left(\Phi_{\mathcal{O}_1}(\rho)\right) - S(\rho), \tag{2.2.45}$$

where $S(\rho)$ is von Neumann's entropy of the state ρ. As projective measurements can never reduce the entropy, we have that $\Im(\mathcal{O}_1|\rho) \geq 0$, where the equality holds if, and only if, $\Phi_{\mathcal{O}_1}(\rho) = \rho$. As a qualitative example (the calculations will be performed in Chapter 4), consider the state $\rho = |+\rangle\langle+|$. We get $\Im(\sigma_z|\rho) = 0$, *i.e.*, the operator σ_z has a well-defined reality: if a measurement is performed, it will certainly yield the result $|+\rangle$. However, if we are concerned with the operators σ_x or σ_y, we obtain $\Im(\sigma_x|\rho) = \Im(\sigma_y|\rho) = \ln 2 > 0$; that is, σ_x and σ_y do not have a well-defined reality, since $|+\rangle = (|+\rangle_x + |-\rangle_x)/\sqrt{2} = (|+\rangle_y + \mathrm{i}\,|-\rangle_y)/\sqrt{2}$.

There is yet another way to express (2.2.45). Considering that $\Phi_{\mathcal{O}_1}(\rho_2) = \rho_2$, where $\rho_2 = \mathrm{Tr}_1\,\rho$ (this is not a restriction to bipartite systems: one can think of a system formed by many particles and the partial trace as a sectioning between one particle, and the rest of the system), we can add and subtract terms to $\Im$ in the following manner:

$$\begin{aligned}\Im(\mathcal{O}_1|\rho) = S\left(\Phi_{\mathcal{O}_1}(\rho)\right) - S(\rho) + \Big[S\left(\Phi_{\mathcal{O}_1}(\rho_1)\right) - S\left(\Phi_{\mathcal{O}_1}(\rho_1)\right)\Big] +&\\ \Big[S(\rho_1) - S(\rho_1)\Big] + \Big[S(\rho_2) - S\left(\Phi_{\mathcal{O}_1}(\rho_2)\right)\Big]&.\end{aligned} \tag{2.2.46}$$

After rearranging them, we get

$$\Im(\mathcal{O}_1|\rho) = \Im(\mathcal{O}_1|\rho_1) + D_{[\mathcal{O}_1]}(\rho), \tag{2.2.47}$$

where $D_{[\mathcal{O}_1]}(\rho) = I_{1:2}(\rho) - I_{1:2}\left(\Phi_{\mathcal{O}_1}(\rho)\right)$ is a discordlike measure written in terms of the mutual information $I_{1:2}(\rho) = S(\mathrm{Tr}_2\,\rho) + S(\mathrm{Tr}_1\,\rho) - S(\rho)$. The term $\Im(\mathcal{O}_1|\rho_1)$ can be seen as a measure of local irreality, as it quantifies the irreality of $\mathcal{O}_1$ given the local state $\rho_1 = \mathrm{Tr}_2\,\rho$. The total irreality thus assimilates the "local quantumness" (which is due to coherence, superposition, waviness), referring essentially to coherent superposition within a local space, plus the "global quantumness", which arises from correlations between subspaces. Recalling the state $\rho = |+\rangle\langle+|$, we obtained $\Im(\sigma_x|\rho) = \Im(\sigma_y|\rho) = \ln 2$; the contribution for the corresponding irrealities, in both cases, is due to coherent superposition. These quantities have been used by other authors to quantify waviness and coherence in [91, 92], for example.

CHAPTER 3

TIME CONTRACTION WITHIN LIGHTWEIGHT REFERENCE FRAMES

3.1 Preliminary Concepts

We now work our way through one of the main parts of this dissertation: how does the usual formula for time contraction, (2.1.44), behaves when we consider the train, which is a laboratory wherein lies the $\mathbb{S}'$ frame and an observer with clocks and rulers, to have a mass so small as to be sensitive enough in order to receive a kickback upon a photon's emission?

Before proceeding, we will adapt our notation a little: all quantities will have a subscript referring to which reference frame this quantity is being measured. For example, $\Delta t'$, the time interval from $\mathbb{S}'$ frame, will now be denoted by $\Delta t_{\mathbb{S}'}$. The reason for this adaptation will become clear shortly.

Consider the same situation treated in §2.1.3, where light is emitted from the floor of the train, reflects in the roof, and is absorbed at the same point in the floor. An observer within the train measures, using a single clock, a time interval $\Delta t_{\mathbb{S}'}$ between the two events (emission and absorption in the floor), which occur at the same position in his inertial reference frame $\mathbb{S}'$. An observer in an external inertial reference frame $\mathbb{S}$ measures, using two synchronized clocks placed in different locations, a time interval $\Delta t_{\mathbb{S}}$. According to the laws of special relativity, these time intervals are related by the usual formula, obtained via Lorentz transformation,

$$\Delta t_{\mathbb{S}'} = \Delta t_{\mathbb{S}}/\gamma_u, \qquad \gamma_u = \frac{1}{\sqrt{1-\beta_u^2}}, \qquad \beta_u = \frac{u}{c}, \tag{3.1.1}$$

with u the velocity of $\mathbb{S}'$ relative to $\mathbb{S}$.

Consider now a sort of "microscopic elastic version" of this problem in which the light beam is replaced with a single photon and the *rigid train** with two very light plates, which can move independently. In our model, the upper plate will be a mirror and the lower plate will play the role of moving reference frame. The motivation behind this scheme is to understand how the formula (3.1.1) changes in a regime wherein the moving system is allowed to get kickbacks upon emission and absorption of light, as would do a quantum particle. Although, on the one hand, we may suspect that any

*The need of this replacement lies on the fact that the definition of rigid-body in special relativity is not clearly stated and accepted. Therefore, we leave this discussion, along with some works that have already been done on this matter, and our contributions, in the Appendix.

eventual correction must be negligible, on the other, conservation laws ensure that it is fundamentally unavoidable.

Even though we conceive the lower plate as a quantum particle, with tiny mass, we will not assign to it a ket state, for it can be thought of as being prepared in minimal uncertainty wavepackets with large mean momentum and, as a further simplification, we may assume that the wavepackets remain significantly localized during the experiment, so that we can apply Ehrenfest's theorem† to the dynamics and thus effectively treat the quantum systems as classical particles. Nevertheless, even with such approximation, we still expect to get some insight on the sort of phenomenon we should meet from the perspective of relativistic quantum particles. After all, be localized as a classical particle or delocalized as a quantum wave, any finite-mass system is compulsorily submitted to kickbacks deriving from the conservation laws.

3.2 Model

Let us consider the framework illustrated in Fig. 3.1. Two parallel plates, each of mass M, move with velocities $\mathbf{V}_{\mathbb{S}}^{\mathfrak{U}} = \mathbf{V}_{\mathbb{S}}^{\mathfrak{L}} = \mathbf{u}_{\mathbb{S}} = u\hat{\mathbf{x}}_{\mathbb{S}} = (u_x, u_y) = (u, 0)$ relative to an inertial reference frame $\mathbb{S}$, where $\hat{\mathbf{x}}_{\mathbb{S}}$ is a unit vector associated with the cartesian coordinate system $\{x, y\}_{\mathbb{S}}$ that defines $\mathbb{S}$. As the problem is two-dimensional, the z axis will not be explicitly considered. The superscripts $\mathfrak{L}$ and $\mathfrak{U}$ refer to points located at the *lower* and *upper* plates, respectively. These indexes are also used to name the plates themselves. Rigidly attached to point $\mathfrak{L}$ of the lower plate is the origin of a cartesian system $\{x, y\}_{\mathbb{S}'}$, which then defines the moving reference frame $\mathbb{S}'$. For future convenience, we also consider an auxiliary reference frame $\mathbb{A}$, equipped with a cartesian system $\{x, y\}_{\mathbb{A}}$, that moves with constant velocity $u\hat{\mathbf{x}}_{\mathbb{S}}$ relative to $\mathbb{S}$ and is perfectly aligned with $\{x, y\}_{\mathbb{S}'}$. Hence, the initial velocities of the lower plate and the mirror relative to $\mathbb{A}$ are $\mathbf{V}_{\mathbb{A}}^{\mathfrak{L}} = \mathbf{V}_{\mathbb{A}}^{\mathfrak{U}} = (0, 0)$. The velocity of the lower plate relative to its own coordinate system is $\mathbf{V}_{\mathbb{S}'}^{\mathfrak{L}} = (0, 0)$ and the velocity of the mirror relative to the lower plate is $\mathbf{V}_{\mathbb{S}'}^{\mathfrak{U}} = (0, 0)$.

When a photon is emitted from point $\mathfrak{L}$ and moves towards point $\mathfrak{U}$, the lower plate gets a kickback and starts to move along the $y_{\mathbb{S},\mathbb{A}}$ axes. Notice that from the perspective of $\mathbb{S}$, the motion of $\mathbb{S}'$ is two-dimensional, whilst for $\mathbb{A}$ is one-dimensional. This is the reason why $\mathbb{A}$ is useful. From now on, besides considering the velocities of the plates $\mathbf{V}_{\mathbb{F}}^{\mathfrak{L},\mathfrak{U}}$ relative to a given frame $\mathbb{F}$, with $\mathbb{F} = \mathbb{S}, \mathbb{S}', \mathbb{A}$, we also look at the photon's velocity $\mathbf{v}_{\mathbb{F}}$ relative to $\mathbb{F}$.

At this moment, it is worth noticing that the kickback imposed on $\mathbb{S}'$ by the photon's emission makes it turn into a non-inertial reference frame only for an insignificant lapse of time. This is so because the photon is quite a peculiar entity that cannot be accelerated; either it does not exist (before the emission) or it exists and moves with speed c (after the emission). As a consequence, we have to admit that the velocity of $\mathbb{S}'$ relative to $\mathbb{S}$ changes from a given constant vector to another constant vector *instantaneously*. It then follows that $\mathbb{S}'$ is effectively inertial during all the relevant time intervals, so that we can safely apply the Lorentz transformations.

†This theorem shows us that

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle x\rangle = \left\langle -\frac{\mathrm{d}}{\mathrm{d}x}V(x)\right\rangle,$$

which is a quantum analog for Newton's second law. If the expectation values above were calculated with sufficiently localized states, then it will be true that

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle x\rangle = -\frac{\mathrm{d}}{\mathrm{d}\langle x\rangle}V(\langle x\rangle),$$

identical to Newton's second law. Ehrenfest's original paper, in German, and a translation to Portuguese, can be found, respectively, in [93, 94].

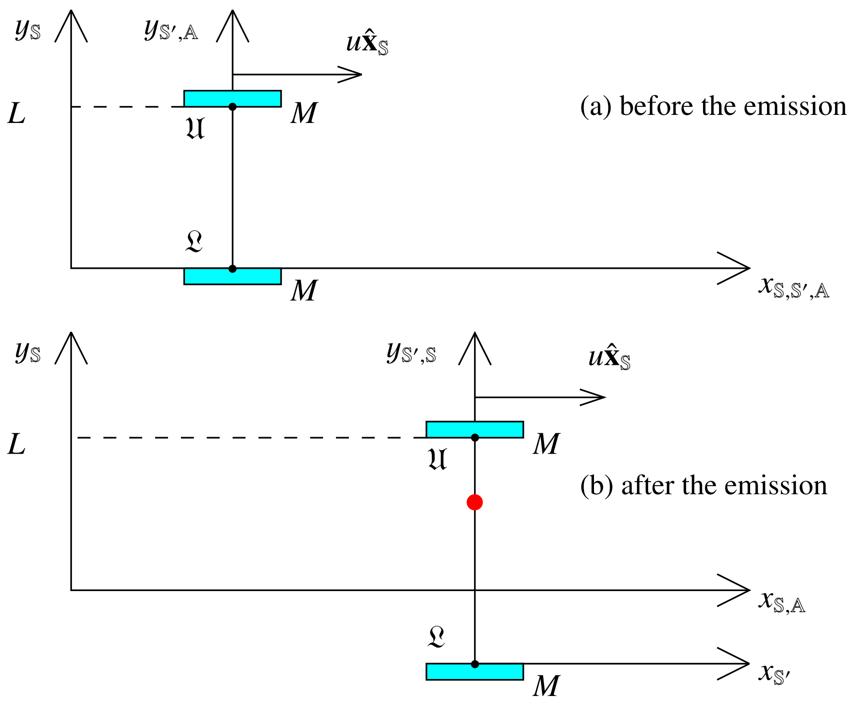

Figure 3.1: (a) Two plates of mass M move independently with velocity $u\hat{\mathbf{x}}_{\mathbb{S}}$ relative to an external inertial reference frame $\mathbb{S}$. The upper plate is a mirror and the lower one, which is equipped with a source of photons and a cartesian system $\{xy\}_{\mathbb{S}'}$, assumes the role of moving reference frame $\mathbb{S}'$. An auxiliary reference frame $\mathbb{A}$ moves with velocity $u\hat{\mathbf{x}}_{\mathbb{S}}$ relative to $\mathbb{S}$. (b) After a photon (red spot) is emitted from the point $\mathfrak{L}$ at the lower plate towards the point $\mathfrak{U}$ at the upper mirrored plate, $\mathbb{S}'$ starts to move along the $y_{\mathbb{S},\mathbb{A}}$ axes.

To determine the velocity acquired by $\mathbb{S}'$ due to the emission of the photon, we apply the relativistic conservation laws from the perspective of the inertial reference frame $\mathbb{A}$. Energy and momentum conservation laws imply, respectively, that

$$Mc^2 = \frac{M_{\mathfrak{L}\mathbb{A}}c^2}{\sqrt{1 - V_{\mathfrak{L}\mathbb{A}}^2/c^2}} + h\nu_{\mathbb{A}} \qquad \text{and} \qquad \frac{h\nu_{\mathbb{A}}}{c} = \frac{M_{\mathfrak{L}\mathbb{A}}V_{\mathfrak{L}\mathbb{A}}}{\sqrt{1 - V_{\mathfrak{L}\mathbb{A}}^2/c^2}}, \tag{3.2.1}$$

where $M_{\mathfrak{L}\mathbb{A}}$ (M) is the rest mass of the lower plate after (before) the photon's emission, h is the Planck constant, $\nu_{\mathbb{A}}$ is the photon's frequency relative to $\mathbb{A}$, and $-V_{\mathfrak{L}\mathbb{A}}\hat{\mathbf{y}}_{\mathbb{A}}$ is the lower plate's velocity relative to $\mathbb{A}$. Solving the set above for the mass and velocity of the lower plate yields

$$\frac{V_{\mathfrak{L}\mathbb{A}}}{c} = \frac{\epsilon}{1-\epsilon} \qquad \text{and} \qquad \frac{M_{\mathfrak{L}\mathbb{A}}}{M} = \sqrt{1-2\epsilon}, \qquad \epsilon \equiv \frac{h\nu_{\mathbb{A}}}{Mc^2}. \tag{3.2.2}$$

Because the photon always carries a non-zero energy and $V_{\mathfrak{L}\mathbb{A}} < c$, the parameter ϵ has to be bounded as $0 < \epsilon < 1/2$. In ordinary instances involving low energy photons and heavy plates, one has that $\epsilon \ll 1/2$. In this regime, it follows that $Mc^2 \simeq M_{\mathfrak{L}\mathbb{A}}c^2 + h\nu_{\mathbb{A}}$, which is an expression of the mass-energy conservation expected for decay processes in nonrelativistic regime [46, 95]. Throughout this work, however, we maintain ϵ arbitrary in the range $(0, 1/2)$.

In order to link the results of distinct inertial reference frames, we use the Lorentz transformation. Let $\mathbb{F}'$ be a reference frame moving with constant velocity $v\hat{\mathbf{x}}_{\mathbb{F}}$ relative to $\mathbb{F}$, another inertial frame. In this instance, the Lorentz transformations (2.1.9) can be written as

$$
\begin{pmatrix} \Delta x_{\mathbb{F}'} \\ \Delta y_{\mathbb{F}'} \\ \Delta t_{\mathbb{F}'} \end{pmatrix} = \begin{pmatrix} \gamma_v & 0 & -v\gamma_v \\ 0 & 1 & 0 \\ -v\gamma_v/c^2 & 0 & \gamma_v \end{pmatrix} \begin{pmatrix} \Delta x_{\mathbb{F}} \\ \Delta y_{\mathbb{F}} \\ \Delta t_{\mathbb{F}} \end{pmatrix}. \tag{3.2.3}
$$

For the inverse transformation, we have

$$
\begin{pmatrix} \Delta x_{\mathbb{F}} \\ \Delta y_{\mathbb{F}} \\ \Delta t_{\mathbb{F}} \end{pmatrix} = \begin{pmatrix} \gamma_v & 0 & v\gamma_v \\ 0 & 1 & 0 \\ v\gamma_v/c^2 & 0 & \gamma_v \end{pmatrix} \begin{pmatrix} \Delta x_{\mathbb{F}'} \\ \Delta y_{\mathbb{F}'} \\ \Delta t_{\mathbb{F}'} \end{pmatrix}, \tag{3.2.4}
$$

where caution must be taken if the frame $\mathbb{F}'$ is moving in another axis relative to $\mathbb{F}$; in this case, the transformations must be accordingly adapted. We are now able to compute the time elapsed since the photon's emission at point $\mathfrak{L}$ until its absorption at this same point. For convenience, we divide the kinematics in two parts: the photon's rise and the photon's descent.

Photon's Rise ($\mathfrak{L} \to \mathfrak{U}$)

We first calculate the time interval $\Delta t_{\mathbb{A}}^{\mathfrak{L}\,\mathfrak{U}}$ referring to the photon's rise from $\mathfrak{L}$ to $\mathfrak{U}$ from the perspective of $\mathbb{A}$. In this case, the events to be considered are: **(i)** the photon's emission at $\mathfrak{L}$, located in the lower plate, and **(ii)** the photon's absorption at $\mathfrak{U}$, which is a point located in the upper mirrored plate. From the discussion above, one has that the velocity of the lower plate after the emission of the photon is $\mathbf{V}_{\mathbb{A}}^{\mathfrak{L}} = (0, -V_{\mathfrak{L}\mathbb{A}})$, whereas the velocity of the mirror is $\mathbf{V}_{\mathbb{A}}^{\mathfrak{U}} = (0, 0)$. As the speed of the photon is the same in all reference frames, we have $\mathbf{v}_{\mathbb{A}} = (0, c)$.

Concerning space-time intervals, for the events in question, it is clear that $\Delta x_{\mathbb{A}}^{\mathfrak{L}\,\mathfrak{U}} = 0, \Delta y_{\mathbb{A}}^{\mathfrak{L}\,\mathfrak{U}} = L$, and $\Delta t_{\mathbb{A}}^{\mathfrak{L}\,\mathfrak{U}} = L/c$. Then, we can apply the transformations (3.2.3) and (3.2.4), with pertinent adaptations (for the relative motion is now occurring in the y direction), to obtain the frame conversions $\mathbb{A} \to \mathbb{S}'$ and $\mathbb{S}' \to \mathbb{S}$. The results can be expressed as

$$
\begin{aligned}
\Delta x_{\mathbb{S}'}^{\mathfrak{L}\,\mathfrak{U}} &= 0, & \Delta y_{\mathbb{S}'}^{\mathfrak{L}\,\mathfrak{U}} &= \frac{\Delta y_{\mathbb{A}}^{\mathfrak{L}\,\mathfrak{U}}}{(1-2\epsilon)^{1/2}}, \\
\Delta t_{\mathbb{S}'}^{\mathfrak{L}\,\mathfrak{U}} &= \frac{\Delta t_{\mathbb{A}}^{\mathfrak{L}\,\mathfrak{U}}}{(1-2\epsilon)^{1/2}}, & \Delta x_{\mathbb{S}}^{\mathfrak{L}\,\mathfrak{U}} &= \frac{\gamma_u\, u\, \Delta t_{\mathbb{S}'}^{\mathfrak{L}\,\mathfrak{U}}}{(1-2\epsilon)^{-1/2}}, \\
\Delta y_{\mathbb{S}}^{\mathfrak{L}\,\mathfrak{U}} &= \frac{\Delta y_{\mathbb{S}'}^{\mathfrak{L}\,\mathfrak{U}}}{(1-2\epsilon)^{-1/2}}, & \Delta t_{\mathbb{S}}^{\mathfrak{L}\,\mathfrak{U}} &= \frac{\gamma_u\, \Delta t_{\mathbb{S}'}^{\mathfrak{L}\,\mathfrak{U}}}{(1-2\epsilon)^{-1/2}}.
\end{aligned} \tag{3.2.5}
$$

The last relation above shows that the usual dilation factor γ_u is reduced by a recoil factor $\sqrt{1-2\epsilon}$. In particular, no dilation will occur when $h\nu_{\mathbb{A}} = Mu^2/2$ since, in this case, $\gamma_u \sqrt{1-2\epsilon} = 1$. Of course, this regime cannot be reached when low energy photons and heavy plates are involved.

Photon's Descent ($\mathfrak{U} \to \mathfrak{L}$)

Before considering the descent of the photon from $\mathfrak{U}$ to $\mathfrak{L}$, we need to look at its scattering by the mirror. From $\mathbb{A}$'s perspective, the energy-momentum conservation in the *absorption process*, at $\mathfrak{U}$, demands that

$$
h\nu_{\mathbb{A}} + Mc^2 = \frac{M_{\mathfrak{U}\mathbb{A}}c^2}{\sqrt{1 - V_{\mathfrak{U}\mathbb{A}}^2/c^2}} \qquad \text{and} \qquad \frac{h\nu_{\mathbb{A}}}{c} = \frac{M_{\mathfrak{U}\mathbb{A}} V_{\mathfrak{U}\mathbb{A}}}{\sqrt{1 - V_{\mathfrak{U}\mathbb{A}}^2/c^2}}, \tag{3.2.6}
$$

whose solution is similar to the previous case:

$$\frac{V_{\mathfrak{U}\mathbb{A}}}{c} = \frac{\epsilon}{1+\epsilon} \qquad \text{and} \qquad \frac{M_{\mathfrak{U}\mathbb{A}}}{M} = \sqrt{1+2\epsilon}, \tag{3.2.7}$$

with ϵ defined by (3.2.2). Here, $V_{\mathfrak{U}\mathbb{A}}\hat{\mathbf{y}}_{\mathbb{A}}$ and $M_{\mathfrak{U}\mathbb{A}}$ denote, respectively, the velocity and the rest mass of the mirror after the photon's absorption. From a quantum mechanical viewpoint, the photon is absorbed by one atom of the mirror and is posteriorly emitted with a different frequency (as we will discuss later). The time elapsed between these two events — the lifetime of the corresponding electronic emission — is denoted here by $\tau_{\mathbb{A}}$.

During a time interval that comprises the photon's rise and the atomic lifetime, the lower plate moves downwards a distance $V_{\mathfrak{L}\mathbb{A}}\left(\Delta t_{\mathbb{A}}^{\mathfrak{L}\,\mathfrak{U}} + \tau_{\mathbb{A}}\right)$. By its turn, the mirror moves upwards a distance $V_{\mathfrak{U}\mathbb{A}}\tau_{\mathbb{A}}$. From this moment on, the two events to be considered are: **(i)** the emission of the photon at $\mathfrak{U}$ and **(ii)** its absorption at $\mathfrak{L}$. The time equations for the photon and the lower plate can be respectively written as $y_{\mathbb{A}}(t_{\mathbb{A}}) = (L + V_{\mathfrak{U}\mathbb{A}}\tau_{\mathbb{A}}) - ct_{\mathbb{A}}$ and $Y_{\mathbb{A}}^{\mathfrak{L}}(t_{\mathbb{A}}) = -V_{\mathfrak{L}\mathbb{A}}(\Delta t_{\mathbb{A}}^{\mathfrak{L}\,\mathfrak{U}} + \tau_{\mathbb{A}} + t_{\mathbb{A}})$, where $t_{\mathbb{A}}$ is the time elapsed since the photon's emission at $\mathfrak{U}$. Equating these expressions, and noting that $t_{\mathbb{A}} = \Delta t_{\mathbb{A}}^{\mathfrak{U}\,\mathfrak{L}}$, we can easily determine the time elapsed between the two events as

$$\Delta t_{\mathbb{A}}^{\mathfrak{U}\,\mathfrak{L}} = \frac{L/c}{(1-2\epsilon)} + \frac{2\epsilon\tau_{\mathbb{A}}}{(1+\epsilon)(1-2\epsilon)}. \tag{3.2.8}$$

Now, using the Lorentz transformations and the above results, we can compute the total time interval from the emission at $\mathfrak{L}$ until the absorption at this same point in all reference frames:

$$\begin{aligned}
\Delta t_{\mathbb{A}} &= \Delta t_{\mathbb{A}}^{\mathfrak{L}\,\mathfrak{U}} + \tau_{\mathbb{A}} + \Delta t_{\mathbb{A}}^{\mathfrak{U}\,\mathfrak{L}} = \frac{2L}{c}\left(\frac{1-\epsilon}{1-2\epsilon}\right) + \tau_{\mathbb{A}}\left(\frac{1+\epsilon-2\epsilon^2}{1-\epsilon-2\epsilon^2}\right),\\
\Delta t_{\mathbb{S}} &= \gamma_u \Delta t_{\mathbb{A}},\\
\Delta t_{\mathbb{S}'} &= \gamma_{-V_{\mathfrak{L}\mathbb{A}}}\left[\Delta t_{\mathbb{A}} - \frac{(-V_{\mathfrak{L}\mathbb{A}})\,\Delta y_{\mathbb{A}}}{c^2}\right] = \Delta t_{\mathbb{A}}/\gamma_{V_{\mathfrak{L}\mathbb{A}}},
\end{aligned} \tag{3.2.9}$$

where we have used $\Delta y_{\mathbb{A}} = -V_{\mathfrak{L}\mathbb{A}}\Delta t_{\mathbb{A}}$ to derive the last equality. To conclude, we write

$$\Delta t_{\mathbb{S}'} = \frac{\Delta t_{\mathbb{S}}}{\gamma_u \gamma_{V_{\mathfrak{L}\mathbb{A}}}} = \left(\frac{\sqrt{1-2\epsilon}}{1-\epsilon}\right)\Delta t_{\mathbb{S}}/\gamma_u. \tag{3.2.10}$$

It is interesting to notice that this relation does not depend on any information regarding the photon's scattering in the mirror; such information is encoded only in $\Delta t_{\mathbb{A}}$. In addition, the first equality above gives an intuitive relation: $\Delta t_{\mathbb{S}'}$ connects with $\Delta t_{\mathbb{S}}$ through Lorentz factors referring to both the horizontal motion and the vertical motion (due to the recoil) of $\mathbb{S}'$ relative to $\mathbb{S}$. Also, we can note that this result does not depend on the relative velocity between the plates and, consequently, no alteration regarding this result should be expected when considering a rigid-body. In fact, perhaps the only alteration would be through the mass dependence with ϵ, which would contain the mass of the laboratory as a whole and not just the plate's. The correction term, ϵ, arose due to the conservation laws concerning the photon-laboratory interaction, and not to its internal structure.

Relativistic Doppler Effect

So far, our results have been expressed in terms of $\nu_{\mathbb{A}}$, which is the frequency observed from the auxiliary frame $\mathbb{A}$ during the photon's travel. Now, we want to abandon $\mathbb{A}$ and rewrite our results in terms of $\nu_{\mathbb{S}'}$, which is the frequency measured in $\mathbb{S}'$. To this end, we apply the longitudinal relativistic

Doppler effect and we need to know about the relative motion between the source and the detector. It is clear that when the photon is rising, the "source" at $\mathfrak{L}$ separates with speed $V_{\mathfrak{L}\mathbb{A}}$ from the "detector", which is fixed say at the origin $y_{\mathbb{A}} = 0$ of $\mathbb{A}$. Then, it follows that the frequency values during the photon's rise and its descent (as the source is also moving away from the detector), as seen by $\mathbb{A}$ and $\mathbb{S}'$, are related through

$$\nu_{\mathbb{A}} = \nu_{\mathbb{S}'} \sqrt{\frac{1 - V_{\mathfrak{L}\mathbb{A}}/c}{1 + V_{\mathfrak{L}\mathbb{A}}/c}} = \nu_{\mathbb{S}'} \sqrt{1 - 2\epsilon}. \tag{3.2.11}$$

However, this is still not the complete solution to the problem, for the right-hand side term in the last equality depends on $\nu_{\mathbb{A}}$ through the relation $\epsilon = h\nu_{\mathbb{A}}/Mc^2$. Using it, we are able to solve (3.2.11) for $\nu_{\mathbb{A}}$ to obtain

$$\nu_{\mathbb{A}} = \nu_{\mathbb{S}'} \left(\sqrt{1 + \varepsilon^2} - \varepsilon \right), \qquad \varepsilon \equiv \frac{h\nu_{\mathbb{S}'}}{Mc^2}, \tag{3.2.12}$$

where ε has substituted ϵ in the role of "significant dimensionless parameter".

Finally, we are in position to finish our calculations. Through (3.2.11) and (3.2.12), we learn how to express $\sqrt{1 - 2\epsilon}$ as a function of ε. With that, we come back to (3.2.10) to derive, after some algebraic manipulations, our final result:

$$\Delta t_{\mathbb{S}'} = \frac{\Delta t_{\mathbb{S}}/\gamma_u}{\sqrt{1 + \varepsilon^2}}. \tag{3.2.13}$$

In comparing it with (3.1.1), which is obtained in the usual context of rigid infinite-mass laboratory, one can readily regard $\sqrt{1 + \varepsilon^2}$ as a correction factor deriving from considering a nonrigid finite-mass laboratory. Indeed, (3.2.13) reduces to (3.1.1) whenever $Mc^2 \gg h\nu_{\mathbb{S}'}$. It is worth noticing that ε will be small even in extreme scenarios, as for example when the laboratory is thought of as being formed by only two hydrogen atoms (*e.g.*, a H_2 molecule), one representing the lower plate and the other the mirror. Taking $M \simeq 1.0$ u for the mass of each plate gives $Mc^2 \simeq 930$ MeV. If we consider the highest-energy photon that an hydrogen atom could eventually emit, we can estimate that $h\nu_{\mathbb{S}'} \simeq 14$ eV. It follows that $\varepsilon \simeq 1.5 \times 10^{-8}$, which makes ε^2 negligible in (3.2.13).

Frequency Change Upon Reflection in a Light Moving Mirror

A quick remark is opportune here with respect to the phenomenon of frequency alteration upon reflection in a light moving mirror. Consider a mirror plate of mass M moving with velocity $v\hat{\mathbf{y}}_{\mathbb{S}}$ relative to an inertial reference frame $\mathbb{S}$. The plane of the mirror is always perpendicular to $\hat{\mathbf{y}}_{\mathbb{S}}$. A photon with velocity $c\hat{\mathbf{y}}_{\mathbb{S}}$ and frequency ν_i impinges on the mirror and reflects with velocity $-c\hat{\mathbf{y}}_{\mathbb{S}}$ and frequency ν_r. After the photon's reflection, the mirror moves with speed v'. This situation is illustrated in Fig. 3.2. The conservation laws for this scenario,

$$h\nu_i + \frac{Mc^2}{\sqrt{1 - \beta_v^2}} = \frac{Mc^2}{\sqrt{1 - \beta_{v'}^2}} + h\nu_r \tag{3.2.14a}$$

and

$$\frac{h\nu_i}{c} + \frac{Mv}{\sqrt{1 - \beta_v^2}} = \frac{Mv'}{\sqrt{1 - \beta_{v'}^2}} - \frac{h\nu_r}{c}, \tag{3.2.14b}$$

require that the frequency change $\nu_i \to \nu_r$ upon reflection be described, to the $\mathbb{S}$ perspective, as

$$\nu_r = \nu_i \left(\frac{1-\beta_v}{1+\beta_v}\right)\Gamma, \qquad \Gamma \equiv \left(1 + 2\epsilon\sqrt{\frac{1-\beta_v}{1+\beta_v}}\right)^{-1}, \tag{3.2.15}$$

remembering that $\beta_v = v/c$, and $\epsilon = h\nu_i/Mc^2$. For infinite-mass mirrors, $\epsilon \to 0$ and $\Gamma \to 1$, in which case we recover the usual formula for light reflection in a moving mirror [96]. Notice that even if $v = 0$, a correction $\Gamma = (1+2\epsilon)^{-1}$ will be required due to the lightness of the mirror. Of course, this correction can be implemented in the problem under consideration in this work, if required. In our approach, however, this was not necessary, since the results have been exhibited in terms of the initial frequency of the photon, the one defined in the very first emission at the lower plate.

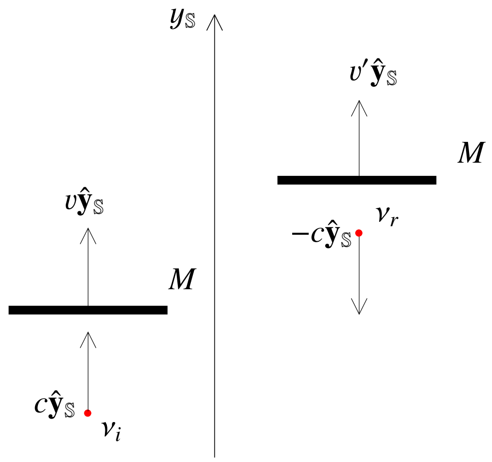

Figure 3.2: A mirror of arbitrary mass M moving with velocity $v\hat{\mathbf{y}}_{\mathbb{S}}$, relative to an inertial reference frame $\mathbb{S}$, absorbs a photon with frequency ν_i, and posteriorly reflects it with frequency ν_r. As a consequence, the mirror's velocity is now $v'\hat{\mathbf{y}}_{\mathbb{S}}$. The lightness of the mirror implies a frequency alteration, even if $v = 0$.

The results of this chapter can be found in: M. F. Savi and R. M. Angelo, "Time Contraction Within Lightweight Reference Frames", *Braz. J. Phys* **47**, 333 (2017).

3.3 A Final Remark: Rigid-Body in Relativity

The "older" version of this model consisted in a laboratory as a whole, with each wall and mirror attached to each other, composing a rigid-body; not as two independent and unattached plates. However, a difficulty arose as to how a rigid-body must be treated within the framework of relativity and, furthermore, how a rigid-body is even defined in such framework or if really exists some notion of proper length in relativity. Due to these conceptual difficulties, we have abandoned such model and searched for a new one, which is the one presented in this volume.

Not so surprisingly, our final result (3.2.13) is the same in both models (it is much simpler to solve it using the laboratory as a rigid-body, actually). The interested reader is invited to seek the Appendix, where a brief discussion about the difficulty associated with rigid-bodies in special relativity is made.

CHAPTER 4

QUANTUMNESS INVARIANCE

This chapter consists of the second main part of this work. For convenience, we separate it in two parts: variables regarding linear degrees of freedom, and variables regarding angular degrees of freedom.

We have seen that special relativity and classical mechanics have some invariant quantities when one changes from a given inertial reference frame to another, like the concept of interval, $\mathrm{d}s^2 = \mathrm{d}s'^2$, for the former, and the distance, $\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r}'$, for the latter. The invariance in these cases are both concerning the kinematic of the system. If we attend to the same analysis for the operators in quantum mechanics, we would possibly see some similarity with the former cases. However, this is not enough, for we need to analyse how the states may change under a change of reference frame and if it is necessary to include such frames, in the quantum description, or if remaining with the classical one is sufficient. The works about quantum reference frames presented earlier give us the possibility to assign to a reference frame a state vector, answering the first question. What remains to be answered is: is there an invariant quantity under a change of reference frames in quantum mechanics, like the interval $\mathrm{d}s^2$ in special relativity? And, in the positive case, what would it be?

In light of the paradox regarding the floating-slit experiment, we want to investigate a possible invariance concerning the *correlations* exhibited by the quantum states. In order to achieve that, we shall consider some transformations of states, between two inertial reference frames, within systems composed of few particles. We have demonstrated in §2.2.3 that one is able to use quantities like mutual information and discordlike measure to quantify correlations between two ensembles. A little further, we have seen that our definition of irreality, (2.2.45), can be written as a function of such discord measure, as in (2.2.47). In consequence thereof, one may wonder whether the notion of realism is reference frame independent, since it is written as a contribution of two terms: local coherence and quantum correlations; both elements arising from correlations between entities. That is, must two quantum observers, in two distinct reference frames, agree upon the realism of a quantum system under observation?

Furthermore, the literature is very well equipped with works involving positional relativity, as an example of such works, we cite [43, 45]. But, what about the orientational one, like angular momentum or spin? With all that has been said so far, would we not be able to treat the latter in a relative manner?

To answer all these questions, we start with a given system prepared in some state ρ, described by an observer in some inertial reference frame* S and through a transformation of coordinates, yet to be

*Note that despite using the same letter to denote the reference frame, it is represented differently. This is so because determined, we compare how the amount of correlations is given in these two frames.

4.1 Variables with Linear Degrees of Freedom

4.1.1 Classical Boosts

Consider a particle in the following generic state,

$$|\psi\rangle = \int dx\, |x\rangle \langle x|\psi\rangle = \int dx\, \psi(x)\, |x\rangle\,, \tag{4.1.1}$$

described by some observer in an inertial reference frame S, written in the position basis $\{|x\rangle\}$, with wave function $\langle x|\psi\rangle = \psi(x)$. From the generator of translation $\hat{G}_\xi = e^{i\xi\hat{p}/\hbar}, \xi \in \mathbb{R}$, satisfying

$$\hat{G}_\xi\, |p_i\rangle = e^{i\xi\hat{p}/\hbar}\, |p_i\rangle = e^{i\xi p_i/\hbar}\, |p_i\rangle\,, \tag{4.1.2}$$

we have

$$\hat{G}_\xi\, |x\rangle = |x - \xi\rangle\,, \tag{4.1.3}$$

which is just the shift operator for a displacement $-\xi$. Applying it onto $|\psi\rangle$ yields

$$|\psi_\xi\rangle = \hat{G}_\xi\, |\psi\rangle = \int dx\, \psi(x)\hat{G}_\xi\, |x\rangle = \int dx\, \psi(x)\, |x - \xi\rangle = \int dx\, \psi(x + \xi)\, |x\rangle\,, \tag{4.1.4}$$

with wave function $\langle x|\psi_\xi\rangle = \psi(x + \xi)$. This procedure is known as *Galilean boost*; before explaining it, let us continue a little further. If one conceives the wave function as a Gaussian function g with center at x_0 and width δ, *i.e.*, $\psi(x) = g_\delta(x - x_0)$, then it is clear that $\langle x|\psi_\xi\rangle = \psi(x+\xi) = g_\delta\,(x - (x_0 - \xi))$ represents the same Gaussian, but with center at $x_0 - \xi$. Also, by direct calculation, we see that

$$\langle\psi_\xi|\hat{x}|\psi_\xi\rangle = \langle\psi|\hat{x}|\psi\rangle - \xi. \tag{4.1.5}$$

The interpretation is the following. The application (4.1.4) can be viewed as a physical action on the vector state; the wave function is shifted in space by $-\xi$ and the mean position of the particle changes accordingly (this is what we call an *active picture*, which will be discussed in §4.2.3). This view is, however, indistinguishable from that according to which nothing has happened with the system from S's perspective, but the description has changed to the perspective of a reference frame S′ which is displaced relatively to S by a distance ξ. By rewriting the above equation as $\langle\hat{x}\rangle' = \langle\hat{x}\rangle - \xi$, we can make direct contact with the usual Galilean scenarios involving an original inertial reference frame S, a moving one S′, and $\xi = ut$, with u the speed of S′ relative to S. In this connection, $|\psi_\xi\rangle$ is to be interpreted as the state vector as seen from S′.

We can rephrase this formalism in terms of a *passive picture*. In the active one, the application $\hat{G}_\xi\, |\psi\rangle = |\psi_\xi\rangle$ produces a vector state in the same vector space, *i.e.*, $\hat{G}_\xi : \mathcal{H} \to \mathcal{H}$, as we often have when discussing dynamical evolutions of a system within a laboratory. However, the Galilean boost is to be understood here as a mere change of perspective, an abstract operation towards the physical description made by another reference, not as a physical interaction with the system. To mathematically state the Galilean boost, we then proceed as follows. Let us rewrite the preparation (4.1.1) as

the frames treated in the quantum framework are not, generally, in relative motion. Hence, hereafter, in order to distinguish this feature from relativity, we represent them in this fashion to avoid ambiguity.

$$|\psi\rangle_{\mathrm{S}} = \int \mathrm{d}x\, \psi(x)\, |x\rangle_{\mathrm{S}}\,. \tag{4.1.6}$$

Here, $|\psi\rangle_{\mathrm{S}} \in \mathcal{H}_{\mathrm{S}}$ is the quantum state as seen by S, which adopts a Hilbert space $\mathcal{H}_{\mathrm{S}}$. Now, let $|x-\xi\rangle_{\mathrm{S}'}$ be the corresponding object in a Hilbert space $\mathcal{H}_{\mathrm{S}'}$ used by frame S′. We then conceive a mapping that establishes the bijective correspondence $|x\rangle_{\mathrm{S}} \leftrightarrow |x-\xi\rangle_{\mathrm{S}'}$ between the elements of each vector space. In other words, we take $\hat{G}_\xi : \mathcal{H}_{\mathrm{S}} \to \mathcal{H}_{\mathrm{S}'}$. Its meaning appears when it acts on the state, via the prescription $\hat{x}\,|x_i\rangle_{\mathrm{S}} = x_i\,|x_i\rangle_{\mathrm{S}}$ and $\hat{x}\,|x_i-\xi\rangle_{\mathrm{S}'} = (x_i-\xi)\,|x_i-\xi\rangle_{\mathrm{S}'}$.

Consider now the same procedure, but with a two particle system prepared in the following generic state relative to S:

$$|\psi\rangle_{\mathrm{S}} = \int \mathrm{d}x_1\mathrm{d}x_2\, \psi(x_1,x_2)\, |x_1\rangle_{\mathrm{S}}\, |x_2\rangle_{\mathrm{S}}\,. \tag{4.1.7}$$

If $\psi(x_1,x_2) = \phi(x_1)\varphi(x_2)$, then $|\psi\rangle$ is a product state with no entanglement. Otherwise, it is entangled. Now, take the linear operator $\hat{\mathscr{G}}_\xi = \hat{G}_\xi \otimes \hat{G}_\xi = \mathrm{e}^{\mathrm{i}\xi(\hat{p}_1+\hat{p}_2)/\hbar}$ such that $\hat{\mathscr{G}}_\xi : \mathcal{H}^{(1)}_{\mathrm{S}} \otimes \mathcal{H}^{(2)}_{\mathrm{S}} \to \mathcal{H}^{(1)}_{\mathrm{S}'} \otimes \mathcal{H}^{(2)}_{\mathrm{S}'}$. It then follows that

$$|\psi\rangle_{\mathrm{S}'} = \hat{\mathscr{G}}_\xi\, |\psi\rangle_{\mathrm{S}} = \hat{G}_\xi \otimes \hat{G}_\xi\, |\psi\rangle_{\mathrm{S}} = \int \mathrm{d}x_1\mathrm{d}x_2\, \psi\,(x_1+\xi, x_2+\xi)\, |x_1\rangle_{\mathrm{S}'}\, |x_2\rangle_{\mathrm{S}'}\,. \tag{4.1.8}$$

Because $\hat{\mathscr{G}}_\xi$ is a *local unitary transformation*, it cannot change the amount of quantum correlations in $|\psi\rangle_{\mathrm{S}}$, *i.e.*, $|\psi\rangle_{\mathrm{S}'}$ and $|\psi\rangle_{\mathrm{S}}$ have the same amount of entanglement (the observers in S and S′ see the same entanglement). In other words, we can say that correlations are invariant under classical boosts. We see that from the last expression above: the application of $\hat{\mathscr{G}}_\xi$ onto $|\psi\rangle_{\mathrm{S}}$, yielding $|\psi\rangle_{\mathrm{S}'}$, only shifts the corresponding wave functions in space by $-\xi$; it does not alter them (does not remove or generate correlations).

This formalism is not only applicable to Galilean boosts, where $\xi = ut$. If we consider a simple translation, $\xi = d$, or still $\xi = at^2/2$, in which case the frame S′ would be accelerated relative to S, the conclusion, that both observers register the same amount of correlations, remains unaltered. This happens as long as the reference frames are treated classically, *i.e.*, with position and momentum well defined at all instant of times.

4.1.2 Observer-Observable Symmetry Principle

In a universe composed of two physical systems, A and B, we can make good meaningful relational physics. Of course, in this framework, only one degree of freedom exists, namely, the one referring to the physics of one of the systems relative to the other. Now, we should notice that in such scenario both systems have the right to play the role of observer, in the sense that we can speak of the physics of B relative to A and vice-versa. For instance, if we say that $|x\rangle^{\mathrm{B}}_{\mathrm{A}}$ gives the position of B relative to A, then it is perfectly legitimate to say that $|-x\rangle^{\mathrm{A}}_{\mathrm{B}}$ yields the position of A relative to B. We call this legitimacy *OO symmetry* and postulate it as a principle of relativity (or relationality), be it classical or quantum.

A Classical Argument

For example, consider the following situation regarding classical mechanics. A system of two particles, A and B, moving in one dimension inside an inertial laboratory on Earth, as depicted in Fig. 4.1, under the only influence of a potential $V(x_{\mathrm{B}} - x_{\mathrm{A}})$, leads us to the corresponding Lagrangian

$$\mathscr{L} = \frac{1}{2}m_{\rm A}\dot{x}_{\rm A}^2 + \frac{1}{2}m_{\rm B}\dot{x}_{\rm B}^2 - V(x_{\rm B} - x_{\rm A}), \tag{4.1.9}$$

where the generalized momenta are

$$p_{\rm S} = \frac{\partial \mathscr{L}}{\partial \dot{x}_{\rm S}} = m_{\rm S}\dot{x}_{\rm S}, \qquad ({\rm S} = {\rm A}, {\rm B}). \tag{4.1.10}$$

The Hamiltonian function $\mathscr{H}$ takes the form

$$\mathscr{H} = \sum_{\rm S} \dot{x}_{\rm S} p_{\rm S} - \mathscr{L} = \frac{p_{\rm A}^2}{2m_{\rm A}} + \frac{p_{\rm B}^2}{2m_{\rm B}} + V(x_{\rm B} - x_{\rm A}). \tag{4.1.11}$$

Also, by manipulating the Hamilton equations, we are able to obtain

$$m_{\rm A}\ddot{x}_{\rm A} = \frac{\partial}{\partial(x_{\rm B} - x_{\rm A})}V(x_{\rm B} - x_{\rm A}) \qquad m_{\rm B}\ddot{x}_{\rm B} = -\frac{\partial}{\partial(x_{\rm B} - x_{\rm A})}V(x_{\rm B} - x_{\rm A}). \tag{4.1.12}$$

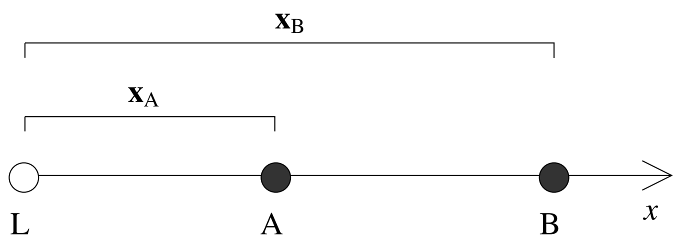

Figure 4.1: Illustration of the positions of A and B relative to L. Respectively, they are: $\mathbf{x}_{\rm A} = x_{\rm A}\hat{\mathbf{x}}$ and $\mathbf{x}_{\rm B} = x_{\rm B}\hat{\mathbf{x}}$. We depict L, A, and B differently, for A and B are to be treated as bodies and L is a point of reference. Considering the viewpoint of an observer in the external reference frame, he also assigns a position vector to L: $\mathbf{x}_{\rm L} = x_{\rm L}\hat{\mathbf{x}} = \mathbf{0}$. The relative positions are then $\mathbf{x}_{\rm A} - \mathbf{x}_{\rm L}$ and $\mathbf{x}_{\rm B} - \mathbf{x}_{\rm L}$.

Now, consider the transformation of coordinates:

$$x_{\rm A} \to q_1 \equiv x_{\rm L} - x_{\rm A} = -x_{\rm A} \qquad \text{and} \qquad x_{\rm B} \to q_2 \equiv x_{\rm B} - x_{\rm A}. \tag{4.1.13}$$

We see that q_2 denotes particle's B position relative to A, whereas q_1 can be interpreted as the position of the laboratory's origin relative to A (we are thus implementing the OO symmetry mentioned above)†. Through the inverse transformation, $x_{\rm A} = -q_1$ and $x_{\rm B} = q_2 - q_1$, we are able to rewrite the Lagrangian as

$$\mathscr{L} = \frac{1}{2}m_{\rm A}\dot{q}_1^2 + \frac{1}{2}m_{\rm B}\left(\dot{q}_2 - \dot{q}_1\right)^2 - V(q_2). \tag{4.1.14}$$

The new conjugate momenta, π_1 and π_2, will then be

$$\pi_1 = \frac{\partial \mathscr{L}}{\partial \dot{q}_1} = M\dot{q}_1 - m_{\rm B}\dot{q}_2 \qquad \text{and} \qquad \pi_2 = \frac{\partial \mathscr{L}}{\partial \dot{q}_2} = m_{\rm B}\left(\dot{q}_1 - \dot{q}_2\right), \tag{4.1.15}$$

with $M = m_A + m_B$ the total mass of the system. Solving the set for the generalized velocities, we obtain

†Record the procedure in §1.1: consider an absolute reference frame where it assigns the position $x_{\rm U}$ to U = L, A, and B; further, we yet consider that $x_{\rm L}$ lies in the origin of this frame. Certainly, the most important description is the relative one: A and B relative to L ($x_{\rm A} - x_{\rm L}$ and $x_{\rm B} - x_{\rm L}$) and vice-versa ($x_{\rm L} - x_{\rm A}$ and $x_{\rm L} - x_{\rm B}$). However, these reference frames are under the influence of a resulting force and, hence, are not inertial, meaning that Newton's laws are invalid for them.

$$\dot{q}_1 = \frac{\pi_1 + \pi_2}{m_A} \qquad \text{and} \qquad \dot{q}_2 = \frac{\pi_1}{m_A} + \frac{\pi_2}{\mu}, \tag{4.1.16}$$

where $\mu = m_A m_B / M$ is the reduced mass. The new Hamiltonian results in

$$\mathscr{H} = \sum_i \dot{q}_i \pi_i - \mathscr{L} = \frac{\pi_1^2}{2m_A} + \frac{\pi_2^2}{2\mu} + \frac{\pi_1 \pi_2}{m_A} + V(q_2), \tag{4.1.17}$$

which carries a term involving the coupling of momenta; note that the original Hamiltonian was carrying a coupling of positions. From Hamilton equations, one gets that

$$m_A \ddot{q}_1 = -\frac{\partial}{\partial q_2} V(q_2) \qquad \text{and} \qquad \mu \ddot{q}_2 = -\frac{\partial}{\partial q_2} V(q_2). \tag{4.1.18}$$

The second equation, giving the physics from B relative to A, is exactly the expected one, the same as (1.1.5). The first equation, giving the physics from the laboratory L relative to A, looks very reasonable. To L, particle A, of mass m_A, moves under the influence of a force $\partial V(q_2)/\partial q_2$ inflicted by B. By symmetry, it is legitimate to say, from the results above, that for A, the laboratory L, with effective mass m_A, moves under the influence of a force $-\partial V(q_2)/\partial q_2$ which arises from an external potential $V(q_2)$ and a gauge potential $\pi_1 \pi_2 / m_A$. Further, it can yet be shown that $\pi_1 = -(p_A + p_B)$ and $\pi_2 = p_B$. Therefore, although q_2 implicates the description of a relative position, π_2 maintains its original meaning: B's momentum relative to L. Hence, the new Hamiltonian is "hybrid"; one cannot identify in it a portion of relative energy only.

A Quantum Argument

We can perform the same analysis through quantum states. But, before we proceed, let us ratify the notation we shall use. A general ket vector will have two labels: one superscript and one subscript, as $|\varphi\rangle^X_Y$. We read it in the following way: $|\varphi\rangle^X_Y$ is the state $|\varphi\rangle$ of object X relative to frame Y. We have already used it in the example above, as in $|x\rangle^A_B$, which denotes the state $|x\rangle$ of particle A relative to B. Now, in cases where there is no subscript, it will be implicit that the frame, upon which the state is relative to, is the same as the frame on the left-hand side of the equation, unless specified otherwise. For example,

$$|\varphi\rangle_L = |\phi_1\rangle^A |\phi_2\rangle^B = |\phi_1 \phi_2\rangle^{AB} \tag{4.1.19}$$

means that the state $|\phi_1\rangle$ and $|\phi_2\rangle$ from particle A and B, respectively, are relative to L (laboratory). If, on the other hand, we have something like

$$|\varphi\rangle_L = |\phi_1\rangle^B_A |\phi_2\rangle^B, \tag{4.1.20}$$

it means that the state $|\phi_1\rangle^B_A$ is from particle B relative to frame A, whilst $|\phi_2\rangle^B$ is from particle B relative to frame L (as specified in the state on the left-hand side $|\varphi\rangle_L$). With that clarified, we may proceed.

Consider the following two particle superposition state prepared within a laboratory L on Earth:

$$|\psi\rangle_L \propto \left[|x\rangle^A |y\rangle^B + |x + \delta_x\rangle^A |y + \delta_y\rangle^B \right]; \tag{4.1.21}$$

where $|u\rangle^R$ denotes a narrow wavepacket for object R centred at u. This state represents an entangled superposition of both particles, A and B, relative to L; meaning that neither particle's position is defined (that is, with their corresponding irrealities greater than zero).

We use the "proportional to" sign because the position space is a continuous one and a state like the aforementioned is not normalizable if we treat $|u\rangle^{\mathrm{R}}$ as an eigenstate of position, where we would have $\langle u|u'\rangle = \delta(u - u')$. As $|u\rangle^{\mathrm{R}}$ denotes a wavepacket, the state (4.1.21), and the others to follow, are normalizable if we specify the details of each ket; but, as this is not our intention, we maintain the proportionality sign. Therefore, the subsequent "calculations" will be only through direct inspection (a qualitative analysis) of the states. When we attend to spins variables, which belong to a discrete space, then we will be able to quantify our results. However, by any means the ensuing discussion should be seen as insufficient, for the difficulty to quantify the irreality in this present case is a mathematical one, which is due to the projection operator in continuous spectra. Nevertheless, it is possible to *discretize* a continuous space in order to treat it as discrete one, such that, a priori, this procedure can be pursued; see [97], as an example.

Let us use the irreality measures introduced in §2.2.4, in particular (2.2.47) (where we can see the contributions: local coherence and quantum correlations), to analyse it in the current frame, L, so as to repeat it in other reference frames. As (4.1.21) is an entangled state, we immediately obtain a non-zero value for the discordlike measure, giving us a non-zero irreality for the position of A and B.

We now want to change our frame of reference relative to which the system is described. For us to achieve such deed, we repeat the procedure in the previous section: consider an inertial reference frame (an external one), labelled by α, where an observer in this frame assigns the states $|z\rangle^{\mathrm{L}}$, $|x\rangle^{\mathrm{A}}$ and $|y\rangle^{\mathrm{B}}$ to L, A, and B, respectively. As a result, (4.1.21) becomes

$$|\Psi\rangle_\alpha \propto |0\rangle^{\mathrm{L}} \left[|x\rangle^{\mathrm{A}} |y\rangle^{\mathrm{B}} + |x + \delta_x\rangle^{\mathrm{A}} |y + \delta_y\rangle^{\mathrm{B}} \right], \tag{4.1.22}$$

where we state that L's wavepacket is centred at the origin of the external frame, for convenience. This description enables us to change to the frames of A and B. For the former, we perform the map

$$|z\rangle^{\mathrm{L}}_\alpha |x\rangle^{\mathrm{A}}_\alpha |y\rangle^{\mathrm{B}}_\alpha \qquad \mapsto \qquad |z\rangle^{\mathrm{L}}_\alpha |z - x\rangle^{\mathrm{L}}_{\mathrm{A}} |y - x\rangle^{\mathrm{B}}_{\mathrm{A}}, \tag{4.1.23}$$

i.e., from a set that represents the position of L, A, and B, relative to α, we move to a set that represents the position of L relative to α, the difference from the position of L and A, yielding L's position relative to A, and the difference from the position of B and A, yielding B's position relative to A. As an example, the state $|0\rangle^{\mathrm{L}}_\alpha |1\rangle^{\mathrm{A}}_\alpha |2\rangle^{\mathrm{B}}_\alpha$ is transformed into $|0\rangle^{\mathrm{L}}_\alpha |{-1}\rangle^{\mathrm{L}}_{\mathrm{A}} |1\rangle^{\mathrm{B}}_{\mathrm{A}}$. Therefore, (4.1.22), relative to A, is given by

$$|\Psi\rangle_{\mathrm{A}} \propto |0\rangle^{\mathrm{L}}_\alpha \left[|{-x}\rangle^{\mathrm{L}} |y - x\rangle^{\mathrm{B}} + |{-x} - \delta_x\rangle^{\mathrm{L}} |y - x + \delta_y - \delta_x\rangle^{\mathrm{B}} \right], \tag{4.1.24}$$

where it also denotes an entangled superposition of the wavepackets, but this time from particle B and the laboratory L. As before, the conclusion is the same: a non-zero value for the discordlike measure, meaning that the irrealities for the position of L and B are also non-zero.

Analogously, for B's frame, we perform

$$|z\rangle^{\mathrm{L}}_\alpha |x\rangle^{\mathrm{A}}_\alpha |y\rangle^{\mathrm{B}}_\alpha \qquad \mapsto \qquad |z\rangle^{\mathrm{L}}_\alpha |z - y\rangle^{\mathrm{L}}_{\mathrm{B}} |x - y\rangle^{\mathrm{A}}_{\mathrm{B}}, \tag{4.1.25}$$

which for our previous example yields $|0\rangle^{\mathrm{L}}_\alpha |1\rangle^{\mathrm{A}}_\alpha |2\rangle^{\mathrm{B}}_\alpha \mapsto |0\rangle^{\mathrm{L}}_\alpha |{-2}\rangle^{\mathrm{L}}_{\mathrm{B}} |{-1}\rangle^{\mathrm{A}}_{\mathrm{B}}$; the state relative to B is then

$$|\Psi\rangle_{\mathrm{B}} \propto |0\rangle^{\mathrm{L}}_\alpha \left[|{-y}\rangle^{\mathrm{L}} |x - y\rangle^{\mathrm{A}} + |{-y} - \delta_y\rangle^{\mathrm{L}} |x - y + \delta_x - \delta_y\rangle^{\mathrm{A}} \right], \tag{4.1.26}$$

denoting an entangled superposition of the wavepackets of particle A and the laboratory L. Once more, the discordlike measure is non-zero and the irrealities for the position of A and L are non-zero, as well. Moreover, it is possible to see that the introduction of α does not change the relative physics between L, A, and B.

We are able to see some kind of pattern here: each object's position is indefinite in every frame's description. Let us look at some particular cases before drawing further conclusions, and, henceforth, we will ignore the states relative to α, since it is irrelevant to the relative viewpoint.

- If $\delta_x = \delta_y = 0$, we have:

$$\begin{aligned}
|\psi\rangle_{\mathrm{L}} &\propto |x\rangle^{\mathrm{A}} |y\rangle^{\mathrm{B}} ,\\
|\psi\rangle_{\mathrm{A}} &\propto |-x\rangle^{\mathrm{L}} |y - x\rangle^{\mathrm{B}} ,\\
|\psi\rangle_{\mathrm{B}} &\propto |-y\rangle^{\mathrm{L}} |x - y\rangle^{\mathrm{A}} ,
\end{aligned} \tag{4.1.27}$$

  where the similarity between them remains; the position of all objects, L, A, and B, are well defined. As none of them are entangled, there is no correlations and, as such, the discordlike measure is zero.

- If $\delta_x = \delta_y = \delta$:

$$\begin{aligned}
|\psi\rangle_{\mathrm{L}} &\propto \left(|x\rangle^{\mathrm{A}} |y\rangle^{\mathrm{B}} + |x + \delta\rangle^{\mathrm{A}} |y + \delta\rangle^{\mathrm{B}} \right),\\
|\psi\rangle_{\mathrm{A}} &\propto \left(|-x\rangle^{\mathrm{L}} + |-x - \delta\rangle^{\mathrm{L}} \right) |y - x\rangle^{\mathrm{B}} ,\\
|\psi\rangle_{\mathrm{B}} &\propto \left(|-y\rangle^{\mathrm{L}} + |-y - \delta\rangle^{\mathrm{L}} \right) |x - y\rangle^{\mathrm{A}} .
\end{aligned} \tag{4.1.28}$$

  Although their form is not the same, we see that: **(i)** for L, the positions of A and B are not defined, since they are in superposition. The contribution for the irrealities, from A and B, come from quantum correlations; **(ii)** for A, the position of L is in superposition (agreeing with L in the sense that A is in superposition relative to it), but the relative positive of B is well defined (not in a superposition, for it has been factorized), which makes sense because the displacement from their original positions (x for A and y for B) is the same, δ, (this relative position, between A and B, is also the same for L, but L is describing the state from A and B relative to itself and not relative to each other hence, there is no factorized state). The contribution of L's irreality comes from local coherence; **(iii)** for B, the position of L is in superposition (agreeing with A and L) and the relative position of A is also well defined (agreeing with A). The contribution of L's irreality also comes from local coherence.

- If $\delta_x = 0$, we obtain:

$$\begin{aligned}
|\psi\rangle_{\mathrm{L}} &\propto \left(|y\rangle^{\mathrm{B}} + |y + \delta_y\rangle^{\mathrm{B}} \right) |x\rangle^{\mathrm{A}} ,\\
|\psi\rangle_{\mathrm{A}} &\propto \left(|y - x\rangle^{\mathrm{B}} + |y - x + \delta_y\rangle^{\mathrm{B}} \right) |-x\rangle^{\mathrm{L}} ,\\
|\psi\rangle_{\mathrm{B}} &\propto \left(|-y\rangle^{\mathrm{L}} |x - y\rangle^{\mathrm{A}} + |-y - \delta_y\rangle^{\mathrm{L}} |x - y - \delta_y\rangle^{\mathrm{A}} \right).
\end{aligned} \tag{4.1.29}$$

  Then: **(i)** for L, we see that A's position is well defined and B is in a superposition state. B's irreality comes from local coherence; **(ii)** for A, L's position is well defined (agreeing with L) and B is in superposition, too (agreeing with L). B's irreality also comes from local coherence; **(iii)** for B, the position of L and A are not defined, but they are entangled. Hence, their irrealities are due to quantum correlations.

We can conclude from the study above that it seems possible to support *invariance* for the quantity

$$\mathscr{Q}_{\mathrm{S}} = \Im\left(X_{\mathrm{U}}|\rho_{\mathrm{U}}\right) + \Im\left(X_{\mathrm{V}}|\rho_{\mathrm{V}}\right) + \mathfrak{D}_{[X_{\mathrm{U}},X_{\mathrm{V}}]}(\rho_{\mathrm{S}}), \tag{4.1.30}$$

where U, V, and S can assume $\{\mathrm{A},\mathrm{B},\mathrm{L}\}$ with $\mathrm{U} \neq \mathrm{V} \neq \mathrm{S}$, X_{U} denotes the position operator of system U relative to S (similarly for X_{V}), $\rho_{\mathrm{U}} = \mathrm{Tr}_{\mathrm{VS}}\,\rho$ is a local state (similarly for ρ_{V}), and $\mathfrak{D}_{[\mathcal{O}_1,\mathcal{O}_2]}(\rho) = I_{1:2}\left(\rho\right) - I_{1:2}\left(\Phi_{\mathcal{O}_2}\Phi_{\mathcal{O}_1}\left(\rho\right)\right)$ is the symmetric quantum discord. Since the local irreality quantifies local coherence and the quantum discord quantifies quantum correlations, $\mathscr{Q}_{\mathrm{S}}$ can be interpreted as the total *quantumness* of the state ρ involving partitions US and VS. For example:

- for (4.1.21), we have the contribution from quantum correlations and no local coherence. This applies for L, A, and B. Hence, $\mathscr{Q}_{\mathrm{L}} = \mathscr{Q}_{\mathrm{A}} = \mathscr{Q}_{\mathrm{B}}$;
- for $\delta_x = \delta_y = 0$, we have null quantumness for all frames, giving us $\mathscr{Q}_{\mathrm{L}} = \mathscr{Q}_{\mathrm{A}} = \mathscr{Q}_{\mathrm{B}} = 0$;
- for $\delta_x = \delta_y = \delta$, $\mathscr{Q}_{\mathrm{L}}$ receives a contribution from quantum correlations, whereas $\mathscr{Q}_{\mathrm{A}} = \mathscr{Q}_{\mathrm{B}}$ receive from local coherence, yielding $\mathscr{Q}_{\mathrm{L}} = \mathscr{Q}_{\mathrm{A}} = \mathscr{Q}_{\mathrm{B}}$;
- for $\delta_x = 0$, L and A measures local coherence whilst B measure quantum correlations, in a way that $\mathscr{Q}_{\mathrm{L}} = \mathscr{Q}_{\mathrm{A}} = \mathscr{Q}_{\mathrm{B}}$.

Albeit we lack the mathematical proof of (4.1.30), we show that, in all cases covered throughout this volume, $\mathscr{Q}_{\mathrm{R}}$ remains invariant for $\mathrm{R} = \{\mathrm{L},\mathrm{A},\mathrm{B}\}$.

Floating-Slit Experiment: Solution

We are now in position to solve the emergent paradox about the experiment mentioned in §1.2.3: the floating-slit experiment, which was presented by N. Bohr in [98]. Further, although their concern and approach were different from ours, this paradox is also discussed in [90].

Consider a particle P and a light floating-slit F that precedes a double-slit system DS, as illustrated in Fig. 4.2. After interacting with the first slit (the floating one), the particle P moves towards DS, which is rigidly attached to the laboratory. Momentum conservation implies that, in order for P to move towards the upper (lower) slit, F has to move downwards (upwards). If m and M denote the masses of P and F, respectively, then the correlation generated in this experiment, as seen from an external observer E, can be described by the state

$$|\Psi\rangle_{\mathrm{E}} \propto \left[|v\rangle^{\mathrm{P}}\left|-\frac{mv}{M}\right\rangle^{\mathrm{F}} + |-v\rangle^{\mathrm{P}}\left|\frac{mv}{M}\right\rangle^{\mathrm{F}}\right], \tag{4.1.31}$$

where v (mv/M) is the speed of P (F). As the momentum variable is a continuous one as well, we maintain the considerations of the previous section. We see that it is an entangled state, for if one sees the slit going upwards, one knows that P went downwards; this correlation being the reasoning behind this scheme. One can demonstrate that $\Im\left(v|\rho_{\mathrm{P}}\right)$ ($\rho_{\mathrm{P}} = \mathrm{Tr}_{\mathrm{F}}\,|\Psi\rangle\,\langle\Psi|$) is a monotonically increasing function of the parameter $b \equiv \left|\left\langle \frac{mv}{M}\right|-\frac{mv}{M}\right\rangle\right|$ and that $\Im\left(v|\rho_{\mathrm{P}}\right) = 0$ only if $b = 0$. This shows that the velocity v of P given ρ_{P} will be real only if the motion of S can be unambiguously identified, *i.e.*, if the slit can properly play the role of an informer, in which case no interference pattern will be seen. Clearly, the reality of the velocity can be adjusted by the ratio m/M, whose value is previously chosen by the observer. When $m \ll M$ (a nearly fixed slit), momentum conservation will not be able to reveal the path of the particle, so the velocity will be maximally indefinite and interference fringes will appear.

Now, consider an observer attached to the floating-slit, in frame F, and that $b = 0$. Since this observer moves along with the slit, he is unable to see the slit's motion and, hence, is unable to see any correlations. Thus, according to him, the state of the particle is a superposition and interference will appear, even in the regime $b = 0$; contradicting E's point of view. However, if this observer, in frame F, includes the external observer, which lies in frame E, the state that the former ascribes to the system will take the form

$$|\Psi\rangle_{\mathrm{F}} \propto \left[|v\rangle^{\mathrm{P}} \left| \frac{mv}{M} \right\rangle^{\mathrm{E}} + |-v\rangle^{\mathrm{P}} \left| -\frac{mv}{M} \right\rangle^{\mathrm{E}} \right], \tag{4.1.32}$$

where now v (mv/M) denotes the velocity of the particle (external observer) relative to the observer that is attached with the slit. If he notices that the external observer went up, he can tell that the particle went up as well, for if the former went up, it is because the slit (along with him) went down, which is due to the particle going up!

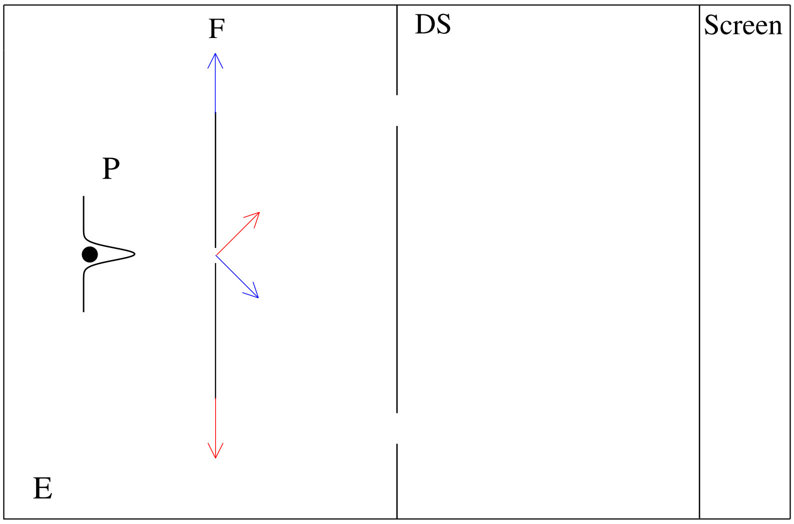

Figure 4.2: A particle P, represented by the gaussian wave packet, moves towards a floating-slit F. If P goes upwards (downwards), represented by the red (blue) arrow, F must go downwards (upwards) in order to conserve momentum in the vertical direction. F then provides to an observer in E "which-way" information about P and, as a result, after crossing with the double-slit DS, no fringes will appear on the screen.

Therefore, the particle's state is entangled with that of the external observer and no interference fringes will appear to F as well; now agreeing with E. By including the external observer in the descriptions, there is no paradox at all. This experiment has been recently performed in [99, 100] where the authors study this reasoning from the laboratory reference frame. Moreover, we can show that $\mathscr{Q}_{\mathrm{E}} = \mathscr{Q}_{\mathrm{F}}$, where the contribution comes from quantum correlations in both cases and, once again, the quantumness is preserved when changing frames.

4.2 Variables with Angular Degrees of Freedom

4.2.1 Relative Angular Momentum

Now that the reader is acquainted with our procedure of relative orientation when concerning position and momentum variables, we shall attend to the case of relative orientation concerning angular variables, like angular momentum and spin.

What one understands as "relative position" is very clear and familiar: given an arbitrary inertial reference frame, the vector that relates particle 1's position, $\mathbf{x}_1 = x_1\hat{\mathbf{x}}$, and particle 2's, $\mathbf{x}_2 = x_2\hat{\mathbf{x}}$, is $\mathbf{x}_{21} = \mathbf{x}_2 - \mathbf{x}_1 = -\mathbf{x}_{12}$. Actually, the very description $\mathbf{x}_1$ is already relational, as it is relative to the origin of the inertial frame (which may be the external and absolute frame α), $\mathbf{x}_0 = x_0\hat{\mathbf{x}} = \mathbf{0}$ say, in a way that $\mathbf{x}_1 = (x_1 - x_0)\hat{\mathbf{x}}$; remember the discussion in §4.1.2.

We want to demonstrate that we can also introduce reasonable relative quantities when one considers angular variables. Consider the situation regarding classical mechanics depicted in Fig. 4.3 (known as rigid rotor), for example, which consists of a body of mass m_1 and dynamical variables $\mathbf{r}_1 = r_1\,(\cos\theta_1, \sin\theta_1, 0)$ and $\mathbf{p}_1 = m_1\dot{\mathbf{r}}_1 = m_1 r_1\omega_1\,(-\sin\theta_1, \cos\theta_1, 0)$, with $\theta_1(t) = \delta_1 + \omega_1 t$ and $\omega_1 = \dot{\theta}_1$ the angular velocity; all of them written relative to some observer in an inertial frame. The body's angular momentum, $\boldsymbol{\ell}_1$, is simply $\boldsymbol{\ell}_1 = \mathbf{r}_1 \times \mathbf{p}_1 = (0, 0, m_1 r_1^2\omega_1)$. As a consequence of this model, we can state that $\boldsymbol{\ell}_1$ is parallel (or aligned) with the z axis established by the observer. If we consider the case $\mathbf{p}_1' = -\mathbf{p}_1$ instead, we would obtain $\boldsymbol{\ell}_1' = -\boldsymbol{\ell}_1$ and the new angular momentum would be anti parallel (or anti aligned) with the z axis. However, if we reverse the axes, $\{x', y', z'\} \to \{-x, -y, -z\}$, the conclusions would be interchanged, *i.e.*, the former (latter) angular momentum would be anti aligned (aligned) with the new z axis. The important feature to notice is if $\boldsymbol{\ell}_1$ is parallel or anti parallel with such axis.

Given the discussion above, we may ask: if there is a way for quantifying the relative position $\mathbf{r}$ between two particles, 1 and 2, positioned at $\mathbf{r}_1$ and $\mathbf{r}_2$, would there be not a way to quantify the *relative angular momentum* $\boldsymbol{\ell}$ between these particles, with $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ their corresponding angular momentum?

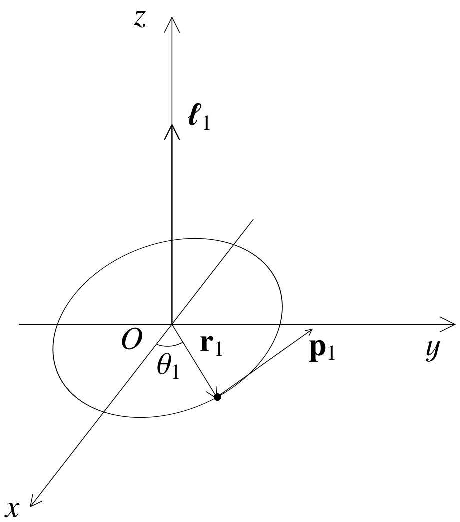

Figure 4.3: A body of mass m_1, position vector $\mathbf{r}_1$ and linear momentum $\mathbf{p}_1$ rotates in the xy plane around the z axis; where O is the origin's frame. In this image, the angular momentum $\boldsymbol{\ell}_1$ is aligned with the z axis.

To this end, consider the classical model illustrated in Fig. 4.4, an extension of the previous case, for it addresses two rigid rotors now: one rotating the z axis, as before, and another rotating the axis defined by $(x, y) = (0, a)$. The dynamics of this system is described by the set:

$$\begin{cases} \mathbf{r}_1 = r_1(\cos\theta_1, \sin\theta_1, 0), & \\ \mathbf{r}_2 = r_2(\cos\theta_2, \Delta + \sin\theta_2, 0), & (\theta_i = \delta_i + \omega_i t), \\ \mathbf{p}_i = m_i\dot{\mathbf{r}}_i = m_i r_i\omega_i(-\sin\theta_i, \cos\theta_i, 0), & (i = 1, 2), \end{cases} \tag{4.2.1}$$

where $\Delta \equiv a/r_2$ is a dimensionless parameter involving the distance between the axes of rotation. For their respective angular momentum, $\boldsymbol{\ell}_i = \mathbf{r}_i \times \mathbf{p}_i$, we get

$$\boldsymbol{\ell}_1 = (0, 0, m_1 r_1^2 \omega_1) = \ell_{1z}\hat{\mathbf{z}}, \qquad \text{and} \qquad \boldsymbol{\ell}_2 = (0, 0, [1 + \Delta \sin\theta_2] m_2 r_2^2 \omega_2) = \ell_{2z}\hat{\mathbf{z}}. \tag{4.2.2}$$

Through (4.2.1), we are able to calculate the relative (denoted by the lack of subscript and lowercase letters) and center of mass (denoted by capital letters) orientations, as well as their respective momenta, as

$$\begin{cases} \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \\ \mathbf{p} = \mu\left(\dfrac{\mathbf{p}_2}{m_2} - \dfrac{\mathbf{p}_1}{m_1}\right), \\ \boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}, \end{cases} \qquad \begin{cases} \mathbf{R} = \dfrac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \\ \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \\ \mathbf{L} = \mathbf{R} \times \mathbf{P}. \end{cases} \tag{4.2.3}$$

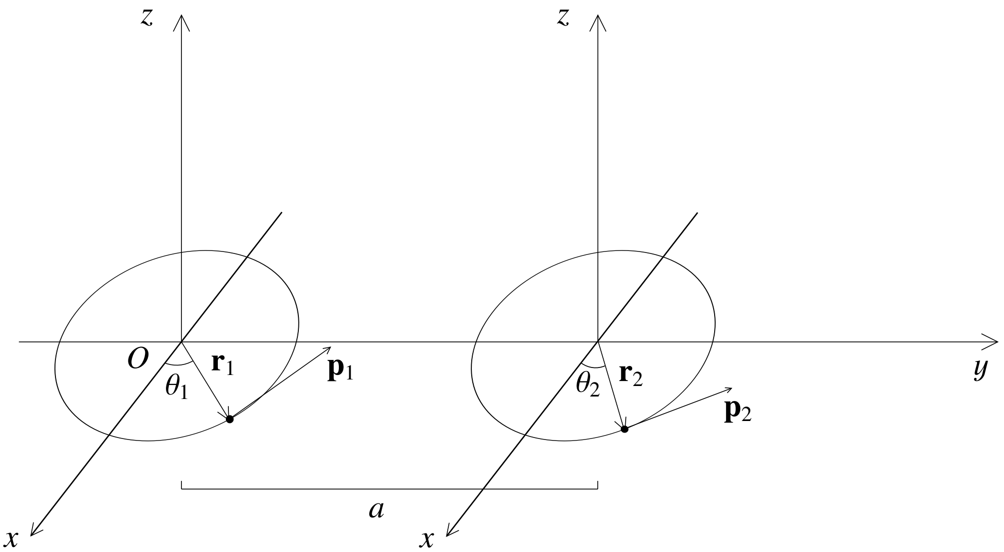

Figure 4.4: Illustration of two rigid rotors rotating in two distinct axes in the xy plane. The axes of rotation are a distance a apart. Although the linear momentum vectors have been drawn in a specific direction, consider them as arbitrary.

The total angular momentum given by the sum of each part, $\mathbf{L}_{\text{total}} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$, can also be taken as the sum from the relative part and the center of mass, $\mathbf{L}_{\text{total}} = \mathbf{L} + \boldsymbol{\ell}$. By direct calculation, we show that

$$\begin{aligned} \boldsymbol{\ell} &= \mu\left[r_1^2\omega_1 + r_2^2\omega_2 - r_1 r_2(\omega_1 + \omega_2)\cos(\theta_1 - \theta_2) - a\left(r_1\omega_1 \sin\theta_1 - r_2\omega_2 \sin\theta_2\right)\right]\hat{\mathbf{z}}, \\ \mathbf{L} &= \left[\frac{m_1^2 r_1^2 \omega_1 + m_2^2 r_2^2 \omega_2}{M} + \mu r_1 r_2(\omega_1 + \omega_2)\cos(\theta_1 - \theta_2) + \frac{m_2}{M} a\left(m_1 r_1 \omega_1 \sin\theta_1 + m_2 r_2 \omega_2 \sin\theta_2\right)\right]\hat{\mathbf{z}}, \end{aligned} \tag{4.2.4}$$

where $M = m_1 + m_2$; we see that if $r_1 = r_2 = r_0, \delta_1 = \delta_2, \omega_1 = \omega_2 = \omega$, and $\Delta = 0$, we obtain $\boldsymbol{\ell} = \mathbf{0}$ and $\mathbf{L} = M r_0^2 \omega \hat{\mathbf{z}}$, as expected. Henceforth, we will consider, for simplicity, $r_1 = r_2 = r_0$ and $m_1 = m_2 = m$, which yields some symmetry between the expressions:

$$\begin{aligned} \boldsymbol{\ell} &= m r_0^2(\omega_1 + \omega_2)\left[\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right) + \Omega_-(\theta_1, \theta_2, \Delta)\right]\hat{\mathbf{z}} = \ell_z \hat{\mathbf{z}}, \\ \mathbf{L} &= m r_0^2(\omega_1 + \omega_2)\left[\cos^2\left(\frac{\theta_1 - \theta_2}{2}\right) + \Omega_+(\theta_1, \theta_2, \Delta)\right]\hat{\mathbf{z}} = L_z \hat{\mathbf{z}}, \end{aligned} \tag{4.2.5}$$

with

$$\Omega_{\pm}(\theta_1, \theta_2, \Delta) \equiv \pm\Delta \frac{(\omega_1 \sin\theta_1 \pm \omega_2 \sin\theta_2)}{2(\omega_1 + \omega_2)}. \tag{4.2.6}$$

As we are not interested in the instantaneous angular momentum, evaluated at a particular time t, we can perform some kind of *trace* over the time variable, which consists of the integral

$$\langle \mathbf{M}(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \mathbf{M}(\tau) \mathrm{d}\tau, \tag{4.2.7}$$

and can be interpreted as the time average of the vector $\mathbf{M}$ over long periods of time. (Just as much, we could consider the trace over the initial phases δ_i.) The time averages for ℓ_z and L_z are then

$$\langle \ell_z \rangle = \langle L_z \rangle = m r_0^2 \bar{\omega}, \qquad \bar{\omega} = \frac{\omega_1 + \omega_2}{2}. \tag{4.2.8}$$

Finally, if we choose $\omega_1 = 2\omega$ and $\omega_2 = (\kappa - 1)2\omega$, with $\kappa \in \mathbb{R} - [0, 1]$, and defining $\mathcal{L}_0 \equiv |\kappa| m r_0^2 \omega$, we obtain

$$\frac{\langle \ell_z \rangle}{\mathcal{L}_0} = \frac{\langle L_z \rangle}{\mathcal{L}_0} = \operatorname{sgn} \kappa, \tag{4.2.9}$$

where sgn stands for the sign (or signum) function, defined as

$$\operatorname{sgn} x = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ +1, & \text{if } x > 0. \end{cases} \tag{4.2.10}$$

This result shows us that if, on one hand, $\omega > 0$ and $\kappa > 1$, we get $\omega_{1,2} > 0$, which corresponds of two angular momenta "+1" and relative angular momentum $\langle \ell_z \rangle / \mathcal{L}_0 = +1$. On the other hand, if $\kappa < 0$, then $\omega_1 > 0$ and $\omega_2 < 0$, corresponding to angular momenta "+1" and "−1", respectively, and relative angular momentum $\langle \ell_z \rangle / \mathcal{L}_0 = -1$.

Well, what do we know that, to within a proportionality factor, exhibits this feature of "±1" angular momentum? Apparently, this model treated hitherto can mimic particles with spin; or better, it can provide us with some physical intuition regarding the notion of *relative spins* for appropriate choices of the angular frequencies of the rigid rotors, *i.e.*, for pertinent choices of the quantum numbers.

4.2.2 Relative Spins

We now introduce the notion of relative spins following the foregone discussion. To begin, consider a spin 1/2 particle, labelled particle A, prepared in the state

$$|\psi\rangle_{\mathrm{L}} = |+\rangle^{\mathrm{A}}, \tag{4.2.11}$$

relative to a laboratory L, where $\sigma_{zR} |m_R\rangle^{\mathrm{R}} = m_R |m_R\rangle^{\mathrm{R}}$, with σ_{zR} the σ_z operator concerning system R (as there will be cases with more than one particle later, we give the generic expression already in order not to repeat it). Notice that we always treat a spin observable regarding some specific axis, x, y, z corresponding respectively to σ_x, σ_y and σ_z. But, who or what defines where the axes lie? Usually, a spin measurement is made with a given magnetic field $\mathbf{B}$ in some given direction and, if the spin is parallel (anti parallel) with such direction, we label it $|+\rangle$ ($|-\rangle$), which is an analogy to what we did in the classical case above: $|+\rangle$ ($|-\rangle$) can be interpreted as "+1" ("−1").

In spite of that, (4.2.11) does not contain any information about the axis (the "field's direction") to which it is being measured. For one to do it, one may consider an observer lying in an external (absolute) reference frame, represented by α, which is observing the system as a whole: (L + A). As the states $|\pm\rangle$ are relative to L's state, an angular momentum pointing in the positive z axis (relative to the external frame), we can write

$$|\Psi\rangle_\alpha = |\uparrow\rangle^{\mathrm{L}} |\uparrow\rangle^{\mathrm{A}} ; \tag{4.2.12}$$

where we have used $|\uparrow\downarrow\rangle$ to distinguish the states, for they are now being described relative to an absolute reference frame (whenever we are describing a state relative to the external frame α, a different notation shall be used). As a matter of relational physics, we may very well consider the superposition state

$$|\tilde{\Psi}\rangle_\alpha = \mu |\uparrow\rangle^{\mathrm{L}} |\uparrow\rangle^{\mathrm{A}} + \nu |\downarrow\rangle^{\mathrm{L}} |\downarrow\rangle^{\mathrm{A}} , \qquad |\mu|^2 + |\nu|^2 = 1, \tag{4.2.13}$$

for, in both terms, particle A's spin is parallel with L's angular momentum (which may also be some particle's spin). Note, however, that, to α, neither L's nor A's spin are defined, since they are in superposition.

We shall now attend to the irreality calculations for (4.2.11) first, and then (4.2.13). Thence, we need the corresponding density operators and their entropies. The density operator associated with this state is

$$\rho_{\mathrm{L}} = (|\psi\rangle \langle\psi|)_{\mathrm{L}} = |+\rangle \langle+| , \tag{4.2.14}$$

and we easily calculate that $S(\rho_{\mathrm{L}}) = 0$. To determine the map (2.2.41), we need an observable. Clearly, we are to choose $\sigma_{zA} \in \mathcal{E}_{\mathrm{A}} = \{|+\rangle , |-\rangle\}$; the corresponding set of projectors is $\{\Pi^{\mathrm{A}}\} = \{|+\rangle \langle+| , |-\rangle \langle-|\}$. Therefore, $\Phi_{\sigma_{zA}}(\rho_{\mathrm{L}})$ is given by

$$\begin{aligned} \Phi_{\sigma_{zA}}(\rho_{\mathrm{L}}) = \sum_{k=1}^{2} \Pi_k^{\mathrm{A}} \rho_{\mathrm{L}} \Pi_k^{\mathrm{A}} &= \Pi_1^{\mathrm{A}} \rho_{\mathrm{L}} \Pi_1^{\mathrm{A}} + \Pi_2^{\mathrm{A}} \rho_{\mathrm{L}} \Pi_2^{\mathrm{A}} \\ &= |+\rangle \langle+|\rho_{\mathrm{L}}|+\rangle \langle+| + |-\rangle \langle-|\rho_{\mathrm{L}}|-\rangle \langle-| \\ &= |+\rangle \langle+| , \end{aligned} \tag{4.2.15}$$

remembering that $\langle+|-\rangle = 0$ and $\langle+|+\rangle = 1$. Thus, $\Phi_{\sigma_{zA}}(\rho_{\mathrm{L}}) = \rho_{\mathrm{L}}$, $S\left(\Phi_{\sigma_{zA}}(\rho_{\mathrm{L}})\right) = 0$ and, as a result,

$$\Im(\sigma_{zA}|\rho_{\mathrm{L}}) = S\left(\Phi_{\sigma_{zA}}(\rho_{\mathrm{L}})\right) - S(\rho_{\mathrm{L}}) = 0; \tag{4.2.16}$$

which confirms our initial conclusion: to L, A's spin has a null irreality, being then well defined.

If we now take (4.2.13), with $\mu = \nu = 1/\sqrt{2}$ for simplicity, we must get a different result, since A's spin is now in superposition. The density operator takes the form

$$\tilde{\rho}_\alpha = (|\tilde{\Psi}\rangle \langle\tilde{\Psi}|)_\alpha = \frac{1}{2}\Big(|\uparrow\uparrow\rangle \langle\uparrow\uparrow| + |\uparrow\uparrow\rangle \langle\downarrow\downarrow| + |\downarrow\downarrow\rangle \langle\uparrow\uparrow| + |\downarrow\downarrow\rangle \langle\downarrow\downarrow| \Big) \tag{4.2.17}$$

such that $S(\tilde{\rho}_\alpha) = 0$. Caution must be taken as our state describes two particles, L and A, rather than one. This alters the way in which we act the projectors onto $\tilde{\rho}_\alpha$, for we must introduce the identity operator of the subspace that does not belong to the set of projectors concerning the observable of interest. As we are interested in σ_{zA}, with projectors $|\uparrow\rangle \langle\uparrow|$ and $|\downarrow\rangle \langle\downarrow|$, we need to introduce the identity of the L subspace, $\mathbb{1}_{\mathrm{L}} = |\uparrow\rangle \langle\uparrow| + |\downarrow\rangle \langle\downarrow|$, rendering us with

$$\mathbb{1}_{\mathrm{L}} \otimes \Pi_1^{\mathrm{A}} = \mathbb{1}_{\mathrm{L}} \otimes |\uparrow\rangle\langle\uparrow| = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|, \qquad \mathbb{1}_{\mathrm{L}} \otimes \Pi_2^{\mathrm{A}} = |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|. \tag{4.2.18}$$

Hence,

$$\begin{aligned} \Phi_{\sigma_{zA}}(\tilde{\rho}_\alpha) &= \sum_{k=1}^{2} \left(\mathbb{1}_{\mathrm{L}} \otimes \Pi_k^{\mathrm{A}}\right) \tilde{\rho}_\alpha \left(\mathbb{1}_{\mathrm{L}} \otimes \Pi_k^{\mathrm{A}}\right) \\ &= \frac{1}{2}\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right) \end{aligned} \tag{4.2.19}$$

with $S\left(\Phi_{\sigma_{zA}}(\tilde{\rho}_\alpha)\right) = \ln 2$. Consequently,

$$\mathfrak{I}(\sigma_{zA}|\tilde{\rho}_\alpha) = \ln 2, \tag{4.2.20}$$

i.e., A's spin is indefinite relative to α. The same analysis can be made for σ_{zL}: $\mathfrak{I}(\sigma_{zL}|\tilde{\rho}_\alpha) = \ln 2$.

To explicitly see where the contribution for the above irreality comes from, local coherence or quantum correlations, we use (2.2.47). We then need the local states:

$$\mathrm{Tr}_{\mathrm{A}}\,\tilde{\rho}_\alpha = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right), \tag{4.2.21a}$$

$$\mathrm{Tr}_{\mathrm{L}}\,\tilde{\rho}_\alpha = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right). \tag{4.2.21b}$$

The corresponding maps yield $\Phi_{\sigma_{zL}}(\mathrm{Tr}_{\mathrm{A}}\,\tilde{\rho}_\alpha) = \mathrm{Tr}_{\mathrm{A}}\,\tilde{\rho}_\alpha$ and $\Phi_{\sigma_{zA}}(\mathrm{Tr}_{\mathrm{L}}\,\tilde{\rho}_\alpha) = \mathrm{Tr}_{\mathrm{L}}\,\tilde{\rho}_\alpha$, so as to give us

$$\mathfrak{I}(\sigma_{zL}|\,\mathrm{Tr}_{\mathrm{A}}\,\tilde{\rho}_\alpha) = \mathfrak{I}(\sigma_{zA}|\,\mathrm{Tr}_{\mathrm{L}}\,\tilde{\rho}_\alpha) = 0, \tag{4.2.22}$$

meaning that the discordlike measures must be non-zero, as expected. For σ_{zL}, we have $D_{[\sigma_{zL}]}(\tilde{\rho}_\alpha) = I_{\mathrm{L:A}}(\tilde{\rho}_\alpha) - I_{\mathrm{L:A}}\left(\Phi_{\sigma_{zL}}(\tilde{\rho}_\alpha)\right)$, and

$$I_{\mathrm{L:A}}(\tilde{\rho}_\alpha) = \underbrace{S(\mathrm{Tr}_{\mathrm{A}}\,\tilde{\rho}_\alpha)}_{=\ln 2} + \overbrace{S(\mathrm{Tr}_{\mathrm{L}}\,\tilde{\rho}_\alpha)}^{=\ln 2} - \underbrace{S(\tilde{\rho}_\alpha)}_{=0} = \ln 4, \tag{4.2.23a}$$

whereas

$$I_{\mathrm{L:A}}\left(\Phi_{\sigma_{zL}}(\tilde{\rho}_\alpha)\right) = \underbrace{S\left(\mathrm{Tr}_{\mathrm{A}}\,\Phi_{\sigma_{zL}}(\tilde{\rho}_\alpha)\right)}_{=\ln 2} + \overbrace{S\left(\mathrm{Tr}_{\mathrm{L}}\,\Phi_{\sigma_{zL}}(\tilde{\rho}_\alpha)\right)}^{=\ln 2} - \underbrace{S\left(\Phi_{\sigma_{zL}}(\tilde{\rho}_\alpha)\right)}_{=\ln 2} = \ln 2. \tag{4.2.23b}$$

Therefore,

$$D_{[\sigma_{zL}]}(\tilde{\rho}_\alpha) = \ln 2. \tag{4.2.24}$$

In a completely analogous fashion, we also obtain $D_{[\sigma_{zA}]}(\tilde{\rho}_\alpha) = \ln 2$.

These calculations have demonstrated that the quantitative analysis indeed reproduces what is expected after a qualitative inspection: the spins of L and A are not defined when the state (4.2.13) is considered; that is, relative to α, the external frame.

Notwithstanding, we can find a frame where both A's and L's spin are defined thanks to the OO symmetry, which allows A to assign a vector state to L. For example, consider the map that yields the relative spin between U and R

$$|m_r\rangle_{\mathrm{R}}^{\mathrm{U}} \qquad \mapsto \qquad \left|\frac{1}{2}\,|(m_U - m_R)|\right\rangle_{\mathrm{R}}^{\mathrm{U}} . \tag{4.2.25}$$

(The factor 1/2 is just for us to maintain the unit value.) Applying it on some generic state gives us

$$|m_L\rangle_\alpha^{\mathrm{L}}\,|m_A\rangle_\alpha^{\mathrm{A}} \qquad \mapsto \qquad |m_L\rangle_\alpha^{\mathrm{L}}\,|m_r\rangle_{\mathrm{R}}^{\mathrm{U}} , \tag{4.2.26}$$

which consists of going from frame α to frame R while maintaining the L spin. If we take R = A, *i.e.*, moving to A's frame, we obtain

$$|\tilde{\Psi}\rangle_{\mathrm{A}} = \mu\,|\uparrow\rangle_\alpha^{\mathrm{L}}\,|0\rangle^{\mathrm{L}} + \nu\,|\downarrow\rangle_\alpha^{\mathrm{L}}\,|0\rangle^{\mathrm{L}} = \left(\mu\,|\uparrow\rangle_\alpha^{\mathrm{L}} + \nu\,|\downarrow\rangle_\alpha^{\mathrm{L}}\right)|0\rangle^{\mathrm{L}} , \tag{4.2.27}$$

where we have adopted that $|\uparrow\rangle$ $(|\downarrow\rangle) \doteq |{+1}\rangle$ $(|{-1}\rangle)$ for the relative part. This expression means that L's spin is defined, for it has been factorized, and the "z axis" is in superposition; it fits because in the original state, (4.2.13), A's spin is in superposition relative to α. Considering the reduced state $\rho_{\mathrm{A}} = \mathrm{Tr}_\alpha(|\tilde{\Psi}\rangle\langle\tilde{\Psi}|)_{\mathrm{A}}$, we get $\mathfrak{I}(\sigma_{zL}|\rho_{\mathrm{A}}) = 0$. It is also possible to show that $\mathfrak{I}\left(\sigma_{z\alpha}|(|\tilde{\Psi}\rangle\langle\tilde{\Psi}|)_{\mathrm{A}}\right) = \ln 2$, where $\sigma_{z\alpha}$ represents the "angular momentum", that defines the z axis, from α. Besides, note that $\mathscr{Q}_{\mathrm{L}} = \mathscr{Q}_{\mathrm{A}} = \ln 2$.

We can extend our procedure to two particles, A and B. For a more usual case, consider the singlet state

$$|\psi\rangle_{\mathrm{L}} = \frac{|+\rangle^{\mathrm{A}}\,|-\rangle^{\mathrm{B}} - |-\rangle^{\mathrm{A}}\,|+\rangle^{\mathrm{B}}}{\sqrt{2}} = \frac{|+-\rangle^{\mathrm{AB}} - |-+\rangle^{\mathrm{AB}}}{\sqrt{2}} \tag{4.2.28}$$

described by an observer in some laboratory L. Following the same steps, we now write it relative to an observer in α who sees the whole system, (L + A + B),

$$|\Psi\rangle_\alpha = |\uparrow\rangle^{\mathrm{L}}\left(\frac{|\uparrow\downarrow\rangle^{\mathrm{AB}} - |\downarrow\uparrow\rangle^{\mathrm{AB}}}{\sqrt{2}}\right); \tag{4.2.29}$$

consider Fig. 4.5 as an illustration of α's viewpoint.

Consequently, $\mathfrak{I}(\sigma_{zA}|\rho_\alpha) = \mathfrak{I}(\sigma_{zB}|\rho_\alpha) = \mathfrak{I}(\sigma_{zA}|\rho_{\mathrm{L}}) = \mathfrak{I}(\sigma_{zB}|\rho_{\mathrm{L}}) = \ln 2$, with $\rho_{\mathrm{L}} = \mathrm{Tr}_\alpha(|\Psi\rangle\langle\Psi|)_\alpha$, telling us that neither spin, relative to α or to L, is real and, as the singlet is an entangled state, the irreality comes from quantum correlations: $D_{[\sigma_{zA}]}(\rho_{\mathrm{L}}) = D_{[\sigma_{zB}]}(\rho_{\mathrm{L}}) = \ln 2$. Regardless of the spins of A and B not being real to α and L, we can find a frame in which they are: the frames of A and B. Take Eq. (4.2.25) once again so as to extend it to three particles:

$$|m_L\rangle_\alpha^{\mathrm{L}}\,|m_A\rangle_\alpha^{\mathrm{A}}\,|m_B\rangle_\alpha^{\mathrm{B}} \qquad \mapsto \qquad |m_L\rangle_\alpha^{\mathrm{L}}\,|m_r\rangle_{\mathrm{R}}^{\mathrm{U}}\,|m_r\rangle_{\mathrm{R}}^{\mathrm{V}} . \tag{4.2.30}$$

By moving to A's frame (R = A, U = L and V = B), we get

$$|\Psi\rangle_{\mathrm{A}} = |\uparrow\rangle_\alpha^{\mathrm{L}}\left(\frac{|01\rangle^{\mathrm{LB}} - |11\rangle^{\mathrm{LB}}}{\sqrt{2}}\right) = |\uparrow\rangle_\alpha^{\mathrm{L}}\left(\frac{|0\rangle^{\mathrm{L}} - |1\rangle^{\mathrm{L}}}{\sqrt{2}}\right)|1\rangle^{\mathrm{B}} , \tag{4.2.31}$$

where we recognize that A sees L in superposition whilst B's orientation is always anti parallel to A's, even with B in superposition relative to α and L. The same happens when we move to B's frame: B sees L in superposition whilst A's orientation is always anti parallel to B's. Notice once more that $\mathscr{Q}_{\mathrm{L}} = \mathscr{Q}_{\mathrm{A}} = \mathscr{Q}_{\mathrm{B}} = \ln 2$.

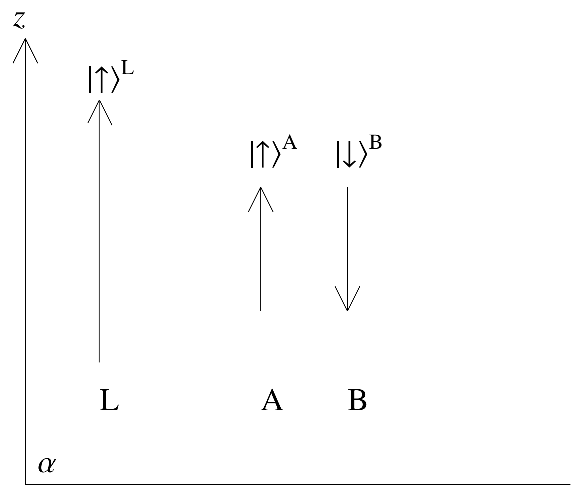

Figure 4.5: Illustration of the states relative to the z direction established by the external reference frame α. In this image, A's (B's) spin is parallel (anti parallel) with that of L.

This scheme was to demonstrate that even if σ_{zA} and σ_{zB} are not defined to α and L, their relative spin always is (A's spin is defined to B and vice-versa). However, even though the change of coordinates (4.2.25), and its generalization (4.2.30), implement the notion of relative physics we are seeking, it is not correct and must not be used. Before proceeding, we shall discuss the requirements that the change of coordinates must satisfy so as to completely describe the states, and exhibit some alternatives for the transformation, where only one will be chosen.

4.2.3 Change of Coordinates

CSCO

As is well known in quantum mechanics, in order to describe a physical situation, one must find a *complete set of commuting observables* (CSCO). The observables that form this set are responsible for removing the ambiguity that might arise in quantum states. When one measures every observable from this set and obtain the corresponding eigenvalues, the state of the system is completely characterized by the resulting eigenvector. This is an important feature when changing coordinates, for we must choose the new ones so as to form a CSCO as well.

For simplicity, we will only consider two observables, σ_{zL} and σ_{zA}, to demonstrate our results in this section. They can be extended so as to include σ_{zB} with no loss of generality. We see that σ_{zL} and σ_{zA} form a CSCO because

$$[\sigma_{zL}, \sigma_{zA}] = 0, \tag{4.2.32a}$$

and

$$(m_L, m_A) = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}. \tag{4.2.32b}$$

That is, the observables commute and the numbers (m_L, m_A) are sufficient to label all base states $|m_L\rangle\,|m_A\rangle$ of this four-dimensional space without ambiguity.

There may be a great number of possible transformations that yields the relative orientation between the spins of two objects U and V. But, as we cannot cover them all, we will only demonstrate three and stick with only one.

- This case is an analogous form of moving to the center of mass and relative coordinates of a system. We consider the set of observables regarding the total spin and the relative spin of a system composed of two particles, L and A. The former is the sum of their spins, $\sigma_{zT} = \frac{1}{2}(\sigma_{zL} + \sigma_{zA})$, and the latter is obtained via (4.2.25). This transformation consists of the map

$$(\sigma_{zL}, \sigma_{zA}) \quad \mapsto \quad (\sigma_{zT}, \sigma_{zr}) = (\sigma_{zL} + \sigma_{zA}, \sigma_{zA} - \sigma_{zL}), \tag{4.2.33a}$$

which renders a CSCO:

$$[\sigma_{zT}, \sigma_{zr}] = 0 \tag{4.2.33b}$$

and

$$(m_T, m_r) = \frac{1}{2}(m_L + m_A, m_A - m_L). \tag{4.2.33c}$$

Although σ_{zT} and σ_{zr} form a CSCO, they do not implement the relative physics we are seeking, for $|1, -1\rangle \mapsto |0, -1\rangle$ and $|-1, +1\rangle \mapsto |0, 1\rangle$. We are looking for variables that put in evidence the fact that these two states imply the same relative orientation between the spins. As a result, one might propose $\frac{1}{2}(\sigma_{zL} + \sigma_{zA}, |\sigma_{zA} - \sigma_{zL}|)$, but in this case $(m_T, m_r) = \{(1, 0), (0, 1), (0, 1), (-1, 0)\}$, yielding an incomplete set, since the states are not uniquely determined.

- A good transformation is

$$(\sigma_{zL}, \sigma_{zA}) \quad \mapsto \quad (\sigma_{zL}, |\sigma_{zA} - \sigma_{zL}|/2)\,, \tag{4.2.34a}$$

which gives

$$[\sigma_{zL}, |\sigma_{zA} - \sigma_{zL}|/2] = 0 \tag{4.2.34b}$$

and

$$(m_L, |m_A - m_L|/2) = \{(1, 0), (1, 1), (-1, 1), (-1, 0)\}. \tag{4.2.34c}$$

- Yet another good transformation is

$$(\sigma_{zL}, \sigma_{zA}) \quad \mapsto \quad (\sigma_{zL}, \sigma_{zA} \otimes \sigma_{zL}), \tag{4.2.35a}$$

since

$$[\sigma_{zL}, \sigma_{zA} \otimes \sigma_{zL}] = 0, \tag{4.2.35b}$$

$$(m_L, m_A m_L) = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}. \tag{4.2.35c}$$

This transformation will be the one we shall use, for reasons presented shortly.

Active and Passive Transformations

When one changes the set of coordinates used to describe some physical situation, there are two ways in which it can be done: **(i)** through an *active* transformation and **(ii)** through a *passive* transformation. We shall now discuss them briefly to highlight their conceptual difference and avoid ambiguity when we talk about the CNOT gate in the next section.

The active transformations, which form the active picture, transform the vector itself. That is, we maintain the same reference frame and we alter the vector. The passive transformations, which form the passive picture, leave the vector unaltered and change the reference frame. For example, consider the vector $\mathbf{P}_0 = x_0\hat{\mathbf{x}} + y_0\hat{\mathbf{y}}$ and the rotation matrix R, by an angle θ, in vector space $\mathbb{R}^2$,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4.2.36}$$

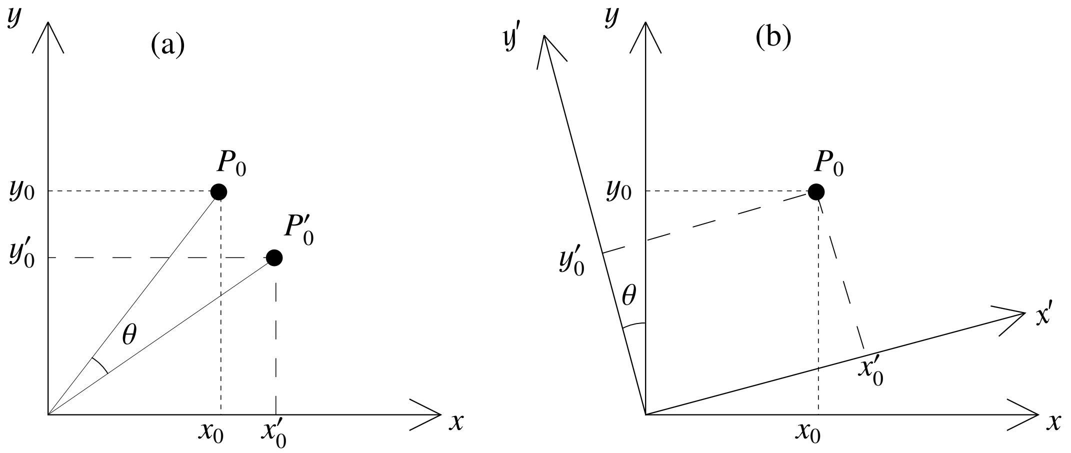

Figure 4.6: We have an active transformation in (a); the vector $\mathbf{P}_0 = x_0\hat{\mathbf{x}} + y_0\hat{\mathbf{y}}$ is transformed into a new one $\mathbf{P}'_0 = x'_0\hat{\mathbf{x}} + y'_0\hat{\mathbf{y}}$ through the application of a rotation matrix R, with $x'_0 = x_0\cos\theta - y_0\sin\theta$ and $y'_0 = x_0\sin\theta + y_0\cos\theta$. In (b), a passive transformation takes place; the vector $\mathbf{P}_0$ is left unchanged whereas the axes are rotated: $\hat{\mathbf{x}}' = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$ and $\hat{\mathbf{y}}' = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}$.

An active transformation consists in obtaining a new vector $\mathbf{P}'_0$ through

$$\mathbf{P}'_0 = R\mathbf{P}_0 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_0 & y_0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix} = \begin{pmatrix} x'_0 & y'_0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix}, \tag{4.2.37}$$

illustrated in Fig. 4.6(a). The passive transformation is the opposite, for

$$\mathbf{P}'_0 = R^{-1}\mathbf{P}_0 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_0 & y_0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix} = \begin{pmatrix} x_0 & y_0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \end{pmatrix}, \tag{4.2.38}$$

where R^{-1} is the inverse of R. This transformation is illustrated in Fig. 4.6(b).

Although we have used a classical vector, the same reasoning is applied within quantum mechanics involving quantum states: the rotation matrix would be a rotation operator and the vectors would be states.

4.2.4 The CNOT Gate

Active Picture

What is most interesting about the transformation (4.2.35), is that it corresponds, in the active picture, to a CNOT (controlled NOT) gate, as the transformation is such that

$$\begin{aligned} |+1,+1\rangle &\mapsto |+1,+1\rangle, \\ |+1,-1\rangle &\mapsto |+1,-1\rangle, \\ |-1,+1\rangle &\mapsto |-1,-1\rangle, \\ |-1,-1\rangle &\mapsto |-1,+1\rangle. \end{aligned} \tag{4.2.39}$$

The first element, σ_{zL}, is called the control qubit, whilst the second, σ_{zA}, is called the target qubit. If the control qubit is $+1$, the target qubit remains unchanged; if, on the other hand, the control qubit is -1, the target qubit flips, *i.e.*, passes through a NOT operation. Explicitly, we can obtain the matrix $\hat{U}_{\text{CNOT}}$ by making the identifications:

$$|+1,+1\rangle \doteq \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |+1,-1\rangle \doteq \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |-1,+1\rangle \doteq \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |-1,-1\rangle \doteq \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{4.2.40}$$

Thus,

$$\hat{U}_{\text{CNOT}} \doteq \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} = \begin{pmatrix}\mathbb{1}_{2\times 2} & 0_{2\times 2}\\ 0_{2\times 2} & \sigma_{xA}\end{pmatrix}, \tag{4.2.41}$$

with the properties: $\hat{U}_{\text{CNOT}} = \hat{U}^{-1}_{\text{CNOT}} = \hat{U}^{\dagger}_{\text{CNOT}}$, and $\hat{U}_{\text{CNOT}}\hat{U}^{\dagger}_{\text{CNOT}} = \hat{U}^{\dagger}_{\text{CNOT}}\hat{U}_{\text{CNOT}} = \mathbb{1}$.

For example, consider the state regarding L and A described by α

$$|\varphi_1\rangle_\alpha = |1\rangle^{\text{L}}\left(\mu\,|1\rangle^{\text{A}} + \nu\,|{-1}\rangle^{\text{A}}\right), \qquad |\mu|^2 + |\nu|^2 = 1. \tag{4.2.42}$$

By acting $\hat{U}_{\text{CNOT}}$ on it, we obtain the state relative to L:

$$\begin{aligned} \hat{U}_{\text{CNOT}}\,|\varphi_1\rangle_\alpha &= \mu\hat{U}_{\text{CNOT}}\,|11\rangle^{\text{LA}} + \nu\hat{U}_{\text{CNOT}}\,|1-1\rangle^{\text{LA}} \\ &= \mu\,|1\rangle^{\text{L}}_{\alpha}\,|1\rangle^{\text{A}}_{\text{L}} + \nu\,|1\rangle^{\text{L}}_{\alpha}\,|{-1}\rangle^{\text{A}}_{\text{L}} \\ &= |1\rangle^{\text{L}}_{\alpha}\left(\mu\,|1\rangle^{\text{A}}_{\text{L}} + \nu\,|{-1}\rangle^{\text{A}}_{\text{L}}\right) \\ &= |\varphi_1\rangle_{\text{L}} \end{aligned} \tag{4.2.43}$$

which is basically the same state[‡] as $|\varphi_1\rangle_\alpha$, because the z axis for L is the same z axis for α. Now, consider this other state instead:

$$|\varphi_2\rangle_\alpha = |{-1}\rangle^{\text{L}}\left(\mu\,|{-1}\rangle^{\text{A}} + \nu\,|1\rangle^{\text{A}}\right). \tag{4.2.44}$$

[‡]A comment is due here. We know that active transformations, when applied to a ket, do not alter the ket's space. That is, the resulting ket, after the application of the operator, belongs to the same space as the original ket. The change in subscripts and superscripts refers only to whom the ket is being described and not to a change in the kets' space.

By the action of $\hat{U}_{\text{CNOT}}$, we get

$$
\begin{aligned}
\hat{U}_{\text{CNOT}} |\varphi_2\rangle_\alpha &= \mu \hat{U}_{\text{CNOT}} |-1 - 1\rangle^{\text{LA}} + \nu \hat{U}_{\text{CNOT}} |-11\rangle^{\text{LA}} \\
&= \mu |-1\rangle^{\text{L}}_\alpha |1\rangle^{\text{A}}_{\text{L}} + \nu |-1\rangle^{\text{L}}_\alpha |-1\rangle^{\text{A}}_{\text{L}} \\
&= |-1\rangle^{\text{L}}_\alpha \left(\mu |1\rangle^{\text{A}}_{\text{L}} + \nu |-1\rangle^{\text{A}}_{\text{L}} \right) \\
&= |\varphi_2\rangle_{\text{L}} ,
\end{aligned}
\tag{4.2.45}
$$

and we see that the states of A have flipped, but is the same state as $|\varphi_1\rangle_{\text{L}}$: regardless of L being up or down, it sees $(\mu |1\rangle^{\text{A}}_{\text{L}} + \nu |-1\rangle^{\text{A}}_{\text{L}})$. In other words, $|\varphi_1\rangle_\alpha$ and $|\varphi_2\rangle_\alpha$ implement the same relative physics to L. Therefore, nothing prohibits us to consider the state

$$
|\phi\rangle_\alpha = \eta |1\rangle^{\text{L}} \left(\mu |1\rangle^{\text{A}} + \nu |-1\rangle^{\text{A}} \right) + \chi |-1\rangle^{\text{L}} \left(\mu |-1\rangle^{\text{A}} + \nu |1\rangle^{\text{A}} \right), \qquad |\eta|^2 + |\chi|^2 = 1, \tag{4.2.46}
$$

for it will give us

$$
\hat{U}_{\text{CNOT}} |\phi\rangle_\alpha = \left(\eta |1\rangle^{\text{L}}_\alpha + \chi |-1\rangle^{\text{L}}_\alpha \right) \left(\mu |1\rangle^{\text{A}}_{\text{L}} + \nu |-1\rangle^{\text{A}}_{\text{L}} \right). \tag{4.2.47}
$$

So far, this procedure has consisted of the active picture: by the application of the CNOT gate, $\hat{U}_{\text{CNOT}}$, into a initial state $|\alpha\rangle \in \mathcal{H}_{\text{R}}$, $\hat{U}_{\text{CNOT}} |\alpha\rangle$, we obtain another state, $\hat{U}_{\text{CNOT}} |\alpha\rangle = |\beta\rangle \in \mathcal{H}_{\text{R}}$. It is important to notice that this procedure can be carried out via physical interactions, in a laboratory, between two qubits. Now, let us rephrase it in the passive picture.

Passive Picture

It is evident from (4.2.40) that the quantum number $m_A m_L$ associated with the target qubit can be interpreted as "relative orientation". Indeed, by direct inspection we see that $m_A m_L = \cos\vartheta$, with ϑ the angle between the two spins in the z direction. For instance, $|-1, 1\rangle = |\downarrow\uparrow\rangle$ implies a relative angle of $\vartheta = \pi$ (see Fig. 4.5). Upon the CNOT, the resulting state is $|m_L, \cos\vartheta\rangle = |-1, -1\rangle$. Hence, the second subspace has indeed the meaning of relative orientation (remember, from the classical model, (4.2.9)). This legitimates us to consider the following passive transformation T:

$$
|m_L\rangle^{\text{L}}_\alpha |m_A\rangle^{\text{A}}_\alpha \quad \xrightarrow{T} \quad |m_L\rangle^{\text{L}}_\alpha |m_A m_L\rangle^{\text{A}}_{\text{L}} , \qquad T = \hat{U}_{\text{CNOT}}, \tag{4.2.48}
$$

where $|m_A m_L\rangle^{\text{A}}_{\text{L}}$ belongs to the Hilbert space $\mathcal{H}_{\text{AL}}$ used by the quantum frame L to describe the physics of A. If more than two particles are involved, the procedure is adapted as follows

$$
|m_L\rangle^{\text{L}}_\alpha |m_A\rangle^{\text{A}}_\alpha |m_B\rangle^{\text{B}}_\alpha \quad \xrightarrow{T} \quad |m_L\rangle^{\text{L}}_\alpha |m_A m_L\rangle^{\text{A}}_{\text{L}} |m_B m_L\rangle^{\text{B}}_{\text{L}} , \qquad T = \hat{U}^{(L,A)}_{\text{CNOT}} \hat{U}^{(L,B)}_{\text{CNOT}}, \tag{4.2.49}
$$

meaning that two CNOTs are required to yield the physics of both A and B relative to L. As mentioned in the previous section, we have that $\{\sigma_{zL}, \sigma_{zL} \otimes \sigma_{zA}, \sigma_{zL} \otimes \sigma_{zB}\}$ forms a CSCO because they mutually commute and $\{m_L, m_A, m_B\}$ is bijectively mapped onto $\{m_L, m_A m_L, m_B m_L\}$:

$$
\{m_L, m_A, m_B\} = \begin{Bmatrix} +1,+1,+1 \\ +1,+1,-1 \\ +1,-1,+1 \\ +1,-1,-1 \\ -1,+1,+1 \\ -1,+1,-1 \\ -1,-1,+1 \\ -1,-1,-1 \end{Bmatrix} \quad \longleftrightarrow \quad \begin{Bmatrix} +1,+1,+1 \\ +1,+1,-1 \\ +1,-1,+1 \\ +1,-1,-1 \\ -1,-1,-1 \\ -1,-1,+1 \\ -1,+1,-1 \\ -1,+1,+1 \end{Bmatrix} = \{m_L, m_A m_L, m_B m_L\}. \tag{4.2.50}
$$

Thereafter, the transformation is mathematically legal and yields the desired relative description. For a test, let us apply (4.2.49) on the singlet state (4.2.29) to obtain the description relative to L. Remembering that $|1\rangle = |\uparrow\rangle$ and $|-1\rangle = |\downarrow\rangle$, we have

$$|\Psi\rangle_\alpha = \frac{1}{\sqrt{2}}\Big(|11-1\rangle^{\mathrm{LAB}} - |1-11\rangle^{\mathrm{LAB}}\Big), \tag{4.2.51}$$

as to obtain

$$|\Psi\rangle_{\mathrm{L}} = |1\rangle^{\mathrm{L}}_\alpha \left(\frac{|1-1\rangle^{\mathrm{AB}} - |-11\rangle^{\mathrm{AB}}}{\sqrt{2}}\right), \tag{4.2.52}$$

where we see that the relative part is the same as (4.2.28), as expected.

Now, how should we proceed if we want the description relative to A or B? The trivial answer is to take σ_{zU} $(U = A, B)$ as the control qubit of the CNOT gate. Then,

$$|m_L\rangle^{\mathrm{L}}_\alpha\,|m_A\rangle^{\mathrm{A}}_\alpha\,|m_B\rangle^{\mathrm{B}}_\alpha \quad \xrightarrow{T} \quad |m_L\rangle^{\mathrm{L}}_\alpha\,|m_L m_A\rangle^{\mathrm{L}}_{\mathrm{A}}\,|m_B m_A\rangle^{\mathrm{B}}_{\mathrm{A}}\,, \qquad T = \hat{U}^{(A,L)}_{\mathrm{CNOT}}\hat{U}^{(A,B)}_{\mathrm{CNOT}}, \tag{4.2.53}$$

and similarly for B. It can be verified that the new set $\{\sigma_{zL}, \sigma_{zU}\otimes\sigma_{zL}, \sigma_{zU}\otimes\sigma_{zV}\}$ $(U, V = A, B$ and $U \neq V)$ forms a CSCO as well, and

$$\{m_L, m_A, m_B\} = \begin{Bmatrix} +1,+1,+1 \\ +1,+1,-1 \\ +1,-1,+1 \\ +1,-1,-1 \\ -1,+1,+1 \\ -1,+1,-1 \\ -1,-1,+1 \\ -1,-1,-1 \end{Bmatrix} \quad \longleftrightarrow \quad \begin{Bmatrix} +1,+1,+1 \\ +1,+1,-1 \\ +1,-1,-1 \\ +1,-1,+1 \\ -1,-1,+1 \\ -1,-1,-1 \\ -1,+1,-1 \\ -1,+1,+1 \end{Bmatrix} = \{m_L, m_L m_A, m_B m_A\}. \tag{4.2.54}$$

Using it on (4.2.29) yields the state relative to A,

$$|\Psi\rangle_{\mathrm{A}} = |1\rangle^{\mathrm{L}}_\alpha \left(\frac{|1\rangle^{\mathrm{L}} - |-1\rangle^{\mathrm{L}}}{\sqrt{2}}\right)|-1\rangle^{\mathrm{B}}\,, \tag{4.2.55}$$

which, apart from the notation, is the same as (4.2.31). If we want the state relative to B, we use

$$|m_L\rangle^{\mathrm{L}}_\alpha\,|m_A\rangle^{\mathrm{A}}_\alpha\,|m_B\rangle^{\mathrm{B}}_\alpha \quad \xrightarrow{T} \quad |m_L\rangle^{\mathrm{L}}_\alpha\,|m_L m_B\rangle^{\mathrm{L}}_{\mathrm{B}}\,|m_A m_B\rangle^{\mathrm{A}}_{\mathrm{B}}\,, \qquad T = \hat{U}^{(B,L)}_{\mathrm{CNOT}}\hat{U}^{(B,A)}_{\mathrm{CNOT}}, \tag{4.2.56}$$

and

$$\{m_L, m_A, m_B\} = \begin{Bmatrix} +1,+1,+1 \\ +1,+1,-1 \\ +1,-1,+1 \\ +1,-1,-1 \\ -1,+1,+1 \\ -1,+1,-1 \\ -1,-1,+1 \\ -1,-1,-1 \end{Bmatrix} \quad \longleftrightarrow \quad \begin{Bmatrix} +1,+1,+1 \\ +1,-1,-1 \\ +1,+1,-1 \\ +1,-1,+1 \\ -1,-1,+1 \\ -1,+1,-1 \\ -1,-1,-1 \\ -1,+1,+1 \end{Bmatrix} = \{m_L, m_L m_B, m_A m_B\}. \tag{4.2.57}$$

Thus, (4.2.29), takes the form

$$|\Psi\rangle_{\mathrm{B}} = |1\rangle^{\mathrm{L}}_\alpha \left(\frac{|-1\rangle^{\mathrm{L}} - |1\rangle^{\mathrm{L}}}{\sqrt{2}}\right)|-1\rangle^{\mathrm{A}}\,. \tag{4.2.58}$$

Therefore, we see that the application of the CNOT gate yields the relative description in all cases. Moreover, as it is an unitary operator, this method can be tested experimentally to verify our results.

4.2.5 Quantumness Invariance

We are now in position to discuss the main question of the second part of this work: would it be possible to uphold any kind of invariance under a change of quantum reference frames?

Let us put the states (4.2.52), (4.2.55), and (4.2.58) together for a clearer comparison:

$$\begin{aligned} |\Psi\rangle_{\mathrm{L}} &= |1\rangle_{\alpha}^{\mathrm{L}}\left(\frac{|1-1\rangle^{\mathrm{AB}} - |-11\rangle^{\mathrm{AB}}}{\sqrt{2}}\right), \\ |\Psi\rangle_{\mathrm{A}} &= |1\rangle_{\alpha}^{\mathrm{L}}\left(\frac{|1\rangle^{\mathrm{L}} - |-1\rangle^{\mathrm{L}}}{\sqrt{2}}\right)|-1\rangle^{\mathrm{B}}, \\ |\Psi\rangle_{\mathrm{B}} &= |1\rangle_{\alpha}^{\mathrm{L}}\left(\frac{|-1\rangle^{\mathrm{L}} - |1\rangle^{\mathrm{L}}}{\sqrt{2}}\right)|-1\rangle^{\mathrm{A}}. \end{aligned} \tag{4.2.59}$$

We see that $|\Psi\rangle_{\mathrm{L}}$ indicates that A and B are in superposition and entangled, giving us

$$\mathfrak{I}\left(\sigma_{zA}|\rho_{\mathrm{L}}\right) = \mathfrak{I}\left(\sigma_{zB}|\rho_{\mathrm{L}}\right) = \mathfrak{D}_{[\sigma_{zA},\sigma_{zB}]} = \ln 2. \tag{4.2.60a}$$

$|\Psi\rangle_{\mathrm{A}}$ shows us that B's relative spin is defined and real, whereas L's spin is in superposition, agreeing with L, for A is in superposition relative to L; hence,

$$\mathfrak{I}\left(\sigma_{zB}|\rho_{\mathrm{A}}\right) = 0 \quad \text{and} \quad \mathfrak{I}\left(\sigma_{zL}|\rho_{\mathrm{A}}\right) = \mathfrak{I}\left(\sigma_{zL}|\operatorname{Tr}_{\mathrm{B}}\rho_{\mathrm{A}}\right) = \ln 2, \tag{4.2.60b}$$

where $\rho_{\mathrm{R}} = \operatorname{Tr}_{\alpha}(|\Psi\rangle\langle\Psi|)_{\mathrm{R}}$. $|\Psi\rangle_{\mathrm{B}}$, by its turn, shows that A's relative spin is also defined and real, agreeing with A, and that L's spin is in superposition, which agrees with A and L; thus,

$$\mathfrak{I}\left(\sigma_{zA}|\rho_{\mathrm{B}}\right) = 0 \quad \text{and} \quad \mathfrak{I}\left(\sigma_{zL}|\rho_{\mathrm{B}}\right) = \mathfrak{I}\left(\sigma_{zL}|\operatorname{Tr}_{\mathrm{A}}\rho_{\mathrm{B}}\right) = \ln 2. \tag{4.2.60c}$$

Indicating, once more, that the quantumness quantity is preserved when changing frames:

$$\mathscr{Q}_{\mathrm{L}} = \mathscr{Q}_{\mathrm{A}} = \mathscr{Q}_{\mathrm{B}} = \ln 2. \tag{4.2.61}$$

Note, however, that coherence or correlations are not preserved separately, only their sum is! Therefore, just like space and time end up to be on the same status by the Lorentz transformations, our observations suggest that coherence and correlations are, in this sense, equivalent.

We can extend our initial state of three particles, L, A, and B, to the most general one,

$$\begin{aligned} |\Psi\rangle_{\alpha} = \eta\,|1\rangle^{\mathrm{L}}&\left(a\,|11\rangle^{\mathrm{AB}} + b\,|1-1\rangle^{\mathrm{AB}} + c\,|-11\rangle^{\mathrm{AB}} + d\,|-1-1\rangle^{\mathrm{AB}}\right) \\ &\chi\,|-1\rangle^{\mathrm{L}}\left(a\,|-1-1\rangle^{\mathrm{AB}} + b\,|-11\rangle^{\mathrm{AB}} + c\,|1-1\rangle^{\mathrm{AB}} + d\,|11\rangle^{\mathrm{AB}}\right), \end{aligned} \tag{4.2.62}$$

where every object is in superposition and $|a|^2 + |b|^2 + |c|^2 + |d|^2 = |\eta|^2 + |\chi|^2 = 1$. This state denotes the most general entangled state of two particles, A and B, seen by L,

$$|\psi\rangle_{\mathrm{L}} = a\,|11\rangle^{\mathrm{AB}} + b\,|1-1\rangle^{\mathrm{AB}} + c\,|-11\rangle^{\mathrm{AB}} + d\,|-1-1\rangle^{\mathrm{AB}}, \tag{4.2.63}$$

considering yet that L is in superposition relative to the absolute reference frame α. If we move to each frame, L, A, and B, through the passive transformation of the CNOT gate, we obtain the set

$$
\begin{aligned}
|\Psi\rangle_{\mathrm{L}} &= \left(\eta\,|1\rangle_{\alpha}^{\mathrm{L}} + \chi\,|-1\rangle_{\alpha}^{\mathrm{L}}\right)\left[\left(a\,|1\rangle^{\mathrm{B}} + b\,|-1\rangle^{\mathrm{B}}\right)|1\rangle^{\mathrm{A}} + \left(c\,|1\rangle^{\mathrm{B}} + d\,|-1\rangle^{\mathrm{B}}\right)|-1\rangle^{\mathrm{A}}\right],\\
|\Psi\rangle_{\mathrm{A}} &= \left(\eta\,|1\rangle_{\alpha}^{\mathrm{L}} + \chi\,|-1\rangle_{\alpha}^{\mathrm{L}}\right)\left[\left(a\,|1\rangle^{\mathrm{L}} + d\,|-1\rangle^{\mathrm{L}}\right)|1\rangle^{\mathrm{B}} + \left(b\,|1\rangle^{\mathrm{L}} + c\,|-1\rangle^{\mathrm{L}}\right)|-1\rangle^{\mathrm{B}}\right],\\
|\Psi\rangle_{\mathrm{B}} &= \left(\eta\,|1\rangle_{\alpha}^{\mathrm{L}} + \chi\,|-1\rangle_{\alpha}^{\mathrm{L}}\right)\left[\left(a\,|1\rangle^{\mathrm{L}} + d\,|-1\rangle^{\mathrm{L}}\right)|1\rangle^{\mathrm{A}} + \left(c\,|1\rangle^{\mathrm{L}} + b\,|-1\rangle^{\mathrm{L}}\right)|-1\rangle^{\mathrm{A}}\right],
\end{aligned}
\tag{4.2.64}
$$

showing us that L's superposition relative to α is irrelevant (from the relative physics viewpoint), for it separates from the relative states. Further, we see that both kinds of elements, local coherence and correlations, are present in the same manner for all frames. *Ergo*,

$$
\mathscr{Q}_{\mathrm{L}} = \mathscr{Q}_{\mathrm{A}} = \mathscr{Q}_{\mathrm{B}}, \qquad \forall\ a, b, c, d, \eta \text{ and } \chi \tag{4.2.65}
$$

that respects the normalization condition.

4.2.6 Relativity and Incompatible Observables

The question we need to answer now is: does the invariance holds in other directions as, *e.g.*, for σ_x and σ_y? That is, if the relative spin between A and B in the z axis is definite, will it also be in the x or y axes? Classically, the answer would be yes, for if two vectors in $\mathbb{R}^3$ (or any dimension), $\mathbf{V}_1 = v_1\hat{\mathbf{z}}$ and $\mathbf{V}_2 = v_2\hat{\mathbf{z}}$, have only the z component relative to some observer, their relative description has only the z component, as well: $\mathbf{V}_i - \mathbf{V}_j = (v_i - v_j)\hat{\mathbf{z}}$, for $i, j = 1$ or 2. However, when spins are concerned, we know that the z component is a superposition of σ_x and σ_y eigenkets separately, such as

$$
|\pm 1\rangle = \frac{|1_x\rangle \pm |-1_x\rangle}{\sqrt{2}} = \frac{|1_y\rangle \pm \mathrm{i}\,|-1_y\rangle}{\sqrt{2}}, \tag{4.2.66}
$$

which means that, even if the relative spin between A and B in the z axis is real, their x and y counterparts, in general, will not be; contrasting clearly with the classical vectors.

To explicitly demonstrate this fact, we will analyse four cases considering the following procedure: **(i)** we start with the state described by the absolute reference frame, $|\Psi\rangle_\alpha$, written in the z basis; **(ii)** we use the passive transformation of the $\hat{U}_{\mathrm{CNOT}}$ to obtain the relative description for each frame, L, A, and B; **(iii)** we rewrite the initial state assuming it to be in the x basis, becoming $|\Psi_x\rangle_\alpha$; **(iv)** with the inverse of (4.2.66), we pass $|\Psi_x\rangle_\alpha$ to the z basis; **(v)** onto this new state, we reuse the passive transformation of $\hat{U}_{\mathrm{CNOT}}$ for each frame; **(vi)** and, finally, we use (4.2.66) to rewrite the state in the x basis and compare our results.

1st Case

We begin with

$$
|\Psi\rangle_\alpha = |111\rangle^{\mathrm{LAB}}, \tag{4.2.67}
$$

denoting that all spins are up relative to α. This case will be done in more details to show the calculations. The CNOT transformation, in the passive picture to each frame, yields

$$
\begin{aligned}
|\Psi\rangle_{\mathrm{L}} &= |1\rangle_{\alpha}^{\mathrm{L}}\,|11\rangle^{\mathrm{AB}},\\
|\Psi\rangle_{\mathrm{A}} &= |1\rangle_{\alpha}^{\mathrm{L}}\,|11\rangle^{\mathrm{LB}},\\
|\Psi\rangle_{\mathrm{B}} &= |1\rangle_{\alpha}^{\mathrm{L}}\,|11\rangle^{\mathrm{LA}},
\end{aligned}
\tag{4.2.68}
$$

where we see that $(\mathscr{Q}_{\mathrm{L}})_z = (\mathscr{Q}_{\mathrm{A}})_z = (\mathscr{Q}_{\mathrm{B}})_z = 0$. Writing $|\Psi\rangle_\alpha$ in the x basis as $|\Psi_x\rangle_\alpha = |1_x 1_x 1_x\rangle^{\mathrm{LAB}} = |1_x\rangle^{\mathrm{L}} \otimes |1_x\rangle^{\mathrm{A}} \otimes |1_x\rangle^{\mathrm{B}}$, and using the inverse of (4.2.66) to transform each x ket into a z ket, we end up with

$$|\Psi_x\rangle_\alpha = 2^{-3/2}\left(|1\rangle^{\mathrm{L}} + |{-1}\rangle^{\mathrm{L}}\right)\left(|1\rangle^{\mathrm{A}} + |{-1}\rangle^{\mathrm{A}}\right)\left(|1\rangle^{\mathrm{B}} + |{-1}\rangle^{\mathrm{B}}\right). \tag{4.2.69}$$

The description for each frame is then

$$\begin{aligned}
|\Psi_x\rangle_{\mathrm{L}} &= 2^{-3/2}\left(|1\rangle^{\mathrm{L}}_\alpha + |{-1}\rangle^{\mathrm{L}}_\alpha\right)\left(|1\rangle^{\mathrm{A}} + |{-1}\rangle^{\mathrm{A}}\right)\left(|1\rangle^{\mathrm{B}} + |{-1}\rangle^{\mathrm{B}}\right),\\
|\Psi_x\rangle_{\mathrm{A}} &= 2^{-3/2}\left(|1\rangle^{\mathrm{L}}_\alpha + |{-1}\rangle^{\mathrm{L}}_\alpha\right)\left(|1\rangle^{\mathrm{L}} + |{-1}\rangle^{\mathrm{L}}\right)\left(|1\rangle^{\mathrm{B}} + |{-1}\rangle^{\mathrm{B}}\right),\\
|\Psi_x\rangle_{\mathrm{B}} &= 2^{-3/2}\left(|1\rangle^{\mathrm{L}}_\alpha + |{-1}\rangle^{\mathrm{L}}_\alpha\right)\left(|1\rangle^{\mathrm{L}} + |{-1}\rangle^{\mathrm{L}}\right)\left(|1\rangle^{\mathrm{A}} + |{-1}\rangle^{\mathrm{A}}\right).
\end{aligned} \tag{4.2.70}$$

Returning to the x basis through (4.2.66), we get

$$\begin{aligned}
|\Psi\rangle_{\mathrm{L}} &= |1_x\rangle^{\mathrm{L}}_\alpha\, |1_x 1_x\rangle^{\mathrm{AB}},\\
|\Psi\rangle_{\mathrm{A}} &= |1_x\rangle^{\mathrm{L}}_\alpha\, |1_x 1_x\rangle^{\mathrm{LB}},\\
|\Psi\rangle_{\mathrm{B}} &= |1_x\rangle^{\mathrm{L}}_\alpha\, |1_x 1_x\rangle^{\mathrm{LA}},
\end{aligned} \tag{4.2.71}$$

which is identical to the first set in the z basis, and $(\mathscr{Q}_{\mathrm{R}})_x = 0$ (R = L, A, B). In this case, we see invariance of quantumness in all frames and in both directions, x and z. Summarizing: we started with the spins of L, A, and B, well defined pointing upwards; the relative descriptions match, for each frame perceive the other particles pointing up; when we move to the x direction, all particles are in superposition of σ_z eigenkets; their relative descriptions also match because everyone sees each other in superposition; by returning to σ_x eigenkets, all particles point towards the positive direction of the x axis, coinciding with the initial state.

2nd Case

If we superpose L's state relative to α,

$$|\Psi\rangle_\alpha = \left(\frac{|1\rangle^{\mathrm{L}} + |{-1}\rangle^{\mathrm{L}}}{\sqrt{2}}\right)|11\rangle^{\mathrm{AB}}, \tag{4.2.72}$$

the relative descriptions are given by

$$\begin{aligned}
|\Psi\rangle_{\mathrm{L}} &= \left(|1\rangle^{\mathrm{L}}_\alpha |11\rangle^{\mathrm{AB}} + |{-1}\rangle^{\mathrm{L}}_\alpha |{-1}\;{-1}\rangle^{\mathrm{AB}}\right)/\sqrt{2},\\
|\Psi\rangle_{\mathrm{A}} &= \left(|1\rangle^{\mathrm{L}}_\alpha |1\rangle^{\mathrm{L}} + |{-1}\rangle^{\mathrm{L}}_\alpha |{-1}\rangle^{\mathrm{L}}\right)|1\rangle^{\mathrm{B}}/\sqrt{2},\\
|\Psi\rangle_{\mathrm{B}} &= \left(|1\rangle^{\mathrm{L}}_\alpha |1\rangle^{\mathrm{L}} + |{-1}\rangle^{\mathrm{L}}_\alpha |{-1}\rangle^{\mathrm{L}}\right)|1\rangle^{\mathrm{A}}/\sqrt{2},
\end{aligned} \tag{4.2.73}$$

giving us $(\mathscr{Q}_{\mathrm{R}})_z = \ln 2$. The next step yields

$$|\Psi_x\rangle_\alpha = \frac{1}{2}|1\rangle^{\mathrm{L}}\left(|1\rangle^{\mathrm{A}} + |{-1}\rangle^{\mathrm{A}}\right)\left(|1\rangle^{\mathrm{B}} + |{-1}\rangle^{\mathrm{B}}\right); \tag{4.2.74}$$

and the relative descriptions are

$$
\begin{aligned}
|\Psi_x\rangle_{\mathrm{L}} &= |1\rangle_\alpha^{\mathrm{L}}\left(|1\rangle^{\mathrm{A}} + |-1\rangle^{\mathrm{A}}\right)\left(|1\rangle^{\mathrm{B}} + |-1\rangle^{\mathrm{B}}\right)/2,\\
|\Psi_x\rangle_{\mathrm{A}} &= |1\rangle_\alpha^{\mathrm{L}}\left(|1\rangle^{\mathrm{L}} + |-1\rangle^{\mathrm{L}}\right)\left(|1\rangle^{\mathrm{B}} + |-1\rangle^{\mathrm{B}}\right)/2,\\
|\Psi_x\rangle_{\mathrm{B}} &= |1\rangle_\alpha^{\mathrm{L}}\left(|1\rangle^{\mathrm{L}} + |-1\rangle^{\mathrm{L}}\right)\left(|1\rangle^{\mathrm{A}} + |-1\rangle^{\mathrm{A}}\right)/2.
\end{aligned}
\tag{4.2.75}
$$

Returning to σ_x, we get

$$
\begin{aligned}
|\Psi\rangle_{\mathrm{L}} &= \left(|1_x\rangle_\alpha^{\mathrm{L}} + |-1_x\rangle_\alpha^{\mathrm{L}}\right)|1_x1_x\rangle^{\mathrm{AB}} / \sqrt{2},\\
|\Psi\rangle_{\mathrm{A}} &= \left(|1_x\rangle_\alpha^{\mathrm{L}} + |-1_x\rangle_\alpha^{\mathrm{L}}\right)|1_x1_x\rangle^{\mathrm{LB}} / \sqrt{2},\\
|\Psi\rangle_{\mathrm{B}} &= \left(|1_x\rangle_\alpha^{\mathrm{L}} + |-1_x\rangle_\alpha^{\mathrm{L}}\right)|1_x1_x\rangle^{\mathrm{LA}} / \sqrt{2},
\end{aligned}
\tag{4.2.76}
$$

with $(\mathscr{Q}_{\mathrm{R}})_x = \ln 2$. Summarizing: we started with L in superposition relative to α, and A and B well defined pointing upwards; their relative descriptions match, for if L's spin is up (down), he sees A and B parallel (anti parallel), which agrees with A's and B's description; going to the x axis with σ_z eigenkets, we see that L is defined whilst A and B are in superposition; their relative descriptions also match, for everyone sees everyone superposed; with σ_x eigenkets, L is superposed relative to α, and A and B are pointing in the positive x direction, coinciding with the initial state. Invariance of quantumness also occurs here.

3rd Case

We now use the singlet state relative to α:

$$
|\Psi\rangle_\alpha = |1\rangle^{\mathrm{L}}\left(\frac{|1-1\rangle^{\mathrm{AB}} - |-11\rangle^{\mathrm{AB}}}{\sqrt{2}}\right).
\tag{4.2.77}
$$

According to each frame, we have

$$
\begin{aligned}
|\Psi\rangle_{\mathrm{L}} &= |1\rangle_\alpha^{\mathrm{L}}\left(|1-1\rangle^{\mathrm{AB}} - |-11\rangle^{\mathrm{AB}}\right)/\sqrt{2},\\
|\Psi\rangle_{\mathrm{A}} &= |1\rangle_\alpha^{\mathrm{L}}\left(|1\rangle^{\mathrm{L}} - |-1\rangle^{\mathrm{L}}\right)|-1\rangle^{\mathrm{B}} / \sqrt{2},\\
|\Psi\rangle_{\mathrm{B}} &= |1\rangle_\alpha^{\mathrm{L}}\left(|1\rangle^{\mathrm{L}} - |-1\rangle^{\mathrm{L}}\right)|-1\rangle^{\mathrm{A}} / \sqrt{2},
\end{aligned}
\tag{4.2.78}
$$

and $(\mathscr{Q}_{\mathrm{R}})_z = \ln 2$. Next, it follows that

$$
|\Psi_x\rangle_\alpha = \frac{1}{2}\left(|1\rangle^{\mathrm{L}} + |-1\rangle^{\mathrm{L}}\right)\left(|1-1\rangle^{\mathrm{AB}} - |-11\rangle^{\mathrm{AB}}\right),
\tag{4.2.79}
$$

where we see that the singlet remains intact. The relative descriptions are

$$
\begin{aligned}
|\Psi_x\rangle_{\mathrm{L}} &= \left(|1\rangle_\alpha^{\mathrm{L}} - |-1\rangle_\alpha^{\mathrm{L}}\right)\left(|1-1\rangle^{\mathrm{AB}} - |-11\rangle^{\mathrm{AB}}\right)/2,\\
|\Psi_x\rangle_{\mathrm{A}} &= \left(|1\rangle_\alpha^{\mathrm{L}} - |-1\rangle_\alpha^{\mathrm{L}}\right)\left(|1\rangle^{\mathrm{L}} - |-1\rangle^{\mathrm{L}}\right)|-1\rangle^{\mathrm{B}}/2,\\
|\Psi_x\rangle_{\mathrm{B}} &= \left(|1\rangle_\alpha^{\mathrm{L}} - |-1\rangle_\alpha^{\mathrm{L}}\right)\left(|1\rangle^{\mathrm{L}} - |-1\rangle^{\mathrm{L}}\right)|-1\rangle^{\mathrm{A}}/2,
\end{aligned}
\tag{4.2.80}
$$

being the expected result. Now, with σ_x eigenkets, we get something new

$$
\begin{aligned}
|\Psi_x\rangle_{\mathrm{L}} &= |-1_x\rangle_\alpha^{\mathrm{L}} \left(|1_x - 1_x\rangle^{\mathrm{AB}} - |-1_x 1_x\rangle^{\mathrm{AB}} \right) / \sqrt{2}, \\
|\Psi_x\rangle_{\mathrm{A}} &= |-1_x\rangle_\alpha^{\mathrm{L}} |-1_x\rangle^{\mathrm{L}} \left(|1_x\rangle^{\mathrm{B}} - |-1_x\rangle^{\mathrm{B}} \right) / \sqrt{2}, \\
|\Psi_x\rangle_{\mathrm{B}} &= |-1_x\rangle_\alpha^{\mathrm{L}} |-1_x\rangle^{\mathrm{L}} \left(|1_x\rangle^{\mathrm{A}} - |-1_x\rangle^{\mathrm{A}} \right) / \sqrt{2},
\end{aligned} \tag{4.2.81}
$$

but $(\mathscr{Q}_{\mathrm{R}})_x = \ln 2$, nonetheless. The set shows us that the singlet between A and B remains, L points in the negative direction of x of α, and the spins of A and B are not defined relative to each other, despite of the singlet state, contradicting with the initial state in the z axis. Summarizing: we started with L pointing upwards and a singlet state; the relative descriptions match; going to x with σ_z eigenkets, we see that the singlet is left unaltered and L is in superposition; the relative descriptions also match, but there is a phase alteration on L's superposition; rewriting the states with σ_x eigenkets shows us a non-trivial description, for L sees A and B in a singlet state, but A and B sees L well defined and each other superposed.

4th Case

For a final case, take the state

$$
|\Psi\rangle_\alpha = |1 1\rangle^{\mathrm{LA}} \left(\frac{|1\rangle^{\mathrm{B}} + |-1\rangle^{\mathrm{B}}}{\sqrt{2}} \right). \tag{4.2.82}
$$

The passive transformation via CNOT leads us to

$$
\begin{aligned}
|\Psi\rangle_{\mathrm{L}} &= |1\rangle_\alpha^{\mathrm{L}} \left(|1\rangle^{\mathrm{B}} + |-1\rangle^{\mathrm{B}} \right) |1\rangle^{\mathrm{A}} / \sqrt{2}, \\
|\Psi\rangle_{\mathrm{A}} &= |1\rangle_\alpha^{\mathrm{L}} \left(|1\rangle^{\mathrm{B}} + |-1\rangle^{\mathrm{B}} \right) |1\rangle^{\mathrm{L}} / \sqrt{2}, \\
|\Psi\rangle_{\mathrm{B}} &= |1\rangle_\alpha^{\mathrm{L}} \left(|1 1\rangle^{\mathrm{LA}} + |-1-1\rangle^{\mathrm{LA}} \right) / \sqrt{2},
\end{aligned} \tag{4.2.83}
$$

with $(\mathscr{Q}_{\mathrm{R}})_z = \ln 2$. Changing basis yields

$$
|\Psi_x\rangle_\alpha = \frac{1}{2} \left(|1\rangle^{\mathrm{L}} + |-1\rangle^{\mathrm{L}} \right) \left(|1\rangle^{\mathrm{A}} + |-1\rangle^{\mathrm{A}} \right) |1\rangle^{\mathrm{B}}, \tag{4.2.84}
$$

and

$$
\begin{aligned}
|\Psi_x\rangle_{\mathrm{L}} &= \left(|1\rangle^{\mathrm{A}} + |-1\rangle^{\mathrm{A}} \right) \left(|1\rangle_\alpha^{\mathrm{L}} |1\rangle^{\mathrm{B}} + |-1\rangle_\alpha^{\mathrm{L}} |-1\rangle^{\mathrm{B}} \right) /2, \\
|\Psi_x\rangle_{\mathrm{A}} &= \left[\left(|1 1\rangle^{\mathrm{LB}} + |-1-1\rangle^{\mathrm{LB}} \right) |1\rangle_\alpha^{\mathrm{L}} + \left(|1-1\rangle^{\mathrm{LB}} + |-1 1\rangle^{\mathrm{LB}} \right) |-1\rangle_\alpha^{\mathrm{L}} \right] /2, \\
|\Psi_x\rangle_{\mathrm{B}} &= \left(|1\rangle_\alpha^{\mathrm{L}} |1\rangle^{\mathrm{L}} + |-1\rangle_\alpha^{\mathrm{L}} |-1\rangle^{\mathrm{L}} \right) \left(|1\rangle^{\mathrm{A}} + |-1\rangle^{\mathrm{A}} \right) /2.
\end{aligned} \tag{4.2.85}
$$

Rewriting it using σ_x, renders us with

$$
\begin{aligned}
|\Psi_x\rangle_{\mathrm{L}} &= \left(|1_x\rangle_\alpha^{\mathrm{L}} |1_x\rangle^{\mathrm{B}} + |-1_x\rangle_\alpha^{\mathrm{L}} |-1_x\rangle^{\mathrm{B}} \right) |1_x\rangle^{\mathrm{A}} / \sqrt{2}, \\
|\Psi_x\rangle_{\mathrm{A}} &= \left(|1_x\rangle_\alpha^{\mathrm{L}} |1_x 1_x\rangle^{\mathrm{LB}} + |-1_x\rangle_\alpha^{\mathrm{L}} |-1_x - 1_x\rangle^{\mathrm{LB}} \right) / \sqrt{2}, \\
|\Psi_x\rangle_{\mathrm{B}} &= \left(|1_x\rangle_\alpha^{\mathrm{L}} |1_x\rangle^{\mathrm{L}} + |-1_x\rangle_\alpha^{\mathrm{L}} |-1_x\rangle^{\mathrm{L}} \right) |1_x\rangle^{\mathrm{A}} / \sqrt{2},
\end{aligned} \tag{4.2.86}
$$

and $(\mathscr{Q}_{\mathrm{R}})_x = \ln 2$. Summarizing: we started with L's and A's spin defined and B's superposed; the relative point of view is the expected one; going to x using σ_z gives us the superposition of L's and A's spins, whereas B's one is defined; the relative description is the expected one, as well; rewriting it with σ_x eigenkets yields something unexpected, for their description do not match at all: L sees A's spin defined and B's in superposition, whereas A sees L and B superposed insofar as B sees L superposed and A defined, contradicting what the latter sees.

Commutation Relation for Relative Spins

These results enable us to conclude that, even tough the quantumness is kept invariant by a change of directions, the notion of relative spin in distinct directions does not hold. *Id est*, the usual spin algebra which arises from the commutation relation

$$\left[\sigma_i, \sigma_j\right] = 2\mathrm{i}\varepsilon_{ijk}\sigma_k, \tag{4.2.87}$$

and yields transformation (4.2.66), is not the same for the relative spin of two entities U and V in dissimilar axes. We can see that from

$$\begin{aligned}
\left[\sigma_{iU} \otimes \sigma_{iV}, \sigma_{jU} \otimes \sigma_{jV}\right] &= (\sigma_{iU} \otimes \sigma_{iV})(\sigma_{jU} \otimes \sigma_{jV}) - (\sigma_{jU} \otimes \sigma_{jV})(\sigma_{iU} \otimes \sigma_{iV}) \\
&= (\sigma_{iU}\sigma_{jU}) \otimes (\sigma_{iV}\sigma_{jV}) - (\sigma_{jU}\sigma_{iU}) \otimes (\sigma_{jV}\sigma_{iV}) \\
&= (2\mathrm{i}\varepsilon_{ijk}\sigma_{kU}) \otimes (2\mathrm{i}\varepsilon_{ijk}\sigma_{kV}) - (-2\mathrm{i}\varepsilon_{ijk}\sigma_{kU}) \otimes (-2\mathrm{i}\varepsilon_{ijk}\sigma_{kV}) \\
&= -4\sigma_{kU} \otimes \sigma_{kV} + 4\sigma_{kU} \otimes \sigma_{kV} \\
&= 0,
\end{aligned} \tag{4.2.88}$$

where we used the property $\varepsilon_{ijk}\varepsilon_{ijk} = 1$. As it is not the same as the usual spin relation (4.2.87), one cannot consider a transformation of relative spins such as (4.2.66) and, as a consequence, our procedure in the four cases above (and in any case concerning a change of axis regarding relative spins) is incorrect.

If one can find a set of operators, $\sigma_{x\mathrm{r}}$, $\sigma_{y\mathrm{r}}$ and σ_{zr}, which implement the notion of relative orientation between spins, and respects the property

$$\left[\sigma_{i\mathrm{r}}, \sigma_{j\mathrm{r}}\right] = 2\mathrm{i}\varepsilon_{ijk}\sigma_{k\mathrm{r}}, \tag{4.2.89}$$

then one is allowed to use transformation (4.2.66) with pertinent adaptations.

4.3 A Possible Approach: No External Reference Frame

In §4.1.2 and §4.2.5, we have seen that the introduction of the external frame's description does not alter the relative description, for it is separated from the latter and we can regard the physics as seen from the reduced state $\rho_{\mathrm{R}} = \mathrm{Tr}_{\alpha}(|\Psi\rangle\langle\Psi|)_{\mathrm{R}}$, with R the desired frame and $|\Psi\rangle$ the product state with one factor arising from the external description and the other from the relative one: $|\Psi\rangle = |\Psi_{\mathrm{ext}}\rangle \otimes |\Psi_{\mathrm{rel}}\rangle$ as, *e.g.*, the set (4.2.64).

Suppose there is an absolute reference frame α and the entire universe is described relative to it. Hypothetically, if an observer in this frame resolves to displace the latter by a determined distance Δ, would our physics change? Would we have the need of modifying our equations with some correction term $f(\Delta)$? If we stand for the notion that physics is a relative science, we must answer this question in the negative way, for our equations rely only on the relative description between the entities. In other words, it does not matter if particles (or observers) A and B lie in x_{A} and x_{B}, respectively, relative to α; their interaction/description will depend only on their relative distance $x_{\mathrm{B}} - x_{\mathrm{A}}$, which we may call d_{AB}: B is a distance d_{AB} apart from A, and vice-versa. Of course, our laws have been written relative to some inertial reference frame and are invalid in any other kind, but we always end up going to the relative coordinates of the system of interest, because only then we are able to measure and work with the fields and particles; record the discussion about the two-body problem and Faraday's law in §1.1.

Returning to §4.1.2: A Classical Argument, we may very well say that, while L assigns to A a position vector $\mathbf{x}_{\mathrm{A}} = x_{\mathrm{A}}\hat{\mathbf{x}}$ and writes his equations of motion as a function of $\mathbf{x}_{\mathrm{A}}$, A assigns to L a position vector $\mathbf{x}_{\mathrm{L}} = x_{\mathrm{L}}\hat{\mathbf{x}} = -x_{\mathrm{A}}\hat{\mathbf{x}}$ and writes his equations as a function of $\mathbf{x}_{\mathrm{L}}$. The distance between them is obviously the same[§], given by $|\mathbf{x}_{\mathrm{A}}| = |\mathbf{x}_{\mathrm{L}}| = x_{\mathrm{A}} = d_{\mathrm{AL}}$, and the respective equations of motion, (4.1.18), are functions only of the derivatives of their distances $d_{\mathrm{AL}} = q_1$, and $d_{\mathrm{AB}} = q_2$ when we include B.

In the ensuing section, A Quantum Argument, the same applies regarding the maps (4.1.23) and (4.1.25): the physics arises only from the relative descriptions; what the laboratory states about A and B, and what A and B state about the laboratory, for, in the end, we will trace over the external frame subspace and consider only the corresponding reduced states.

We can perform the same analysis concerning the relative angular momentum in §4.2. Consider the same situation of Fig. 4.3: a rigid rotor of mass m_1, position vector $\mathbf{r}_1$ and linear momentum $\mathbf{p}_1$; now, assume that particle 1 ascribes the vector $\mathbf{r}_0 = -\mathbf{r}_1$ to the system's origin O. Thus, $\mathbf{p}_0 = -\mu_0\dot{\mathbf{r}}_1$, where $\mu_0 = m_0 m_1/(m_0 + m_1)$ with m_0 the origin's mass. As $m_0 \gg m_1$, μ_0 reduces to m_1 and $\mathbf{p}_0 = -\mathbf{p}_1$. Therefore, $\boldsymbol{\ell}_0 = \mathbf{r}_0 \times \mathbf{p}_0 = m_1 r_1^2 \omega_1 \hat{\mathbf{z}} = \boldsymbol{\ell}_1$. That is, each observer, particle 1 and the origin, sees one another rotating with angular velocity ω_1 and angular momentum $\boldsymbol{\ell}_1$. One may object that this reasoning is invalid, for particle 1 is a non-inertial reference frame. This is true, indeed; in fact, if one evaluates the Lagrangian of this system and calculates the conjugate momentum associated with the origin's motion, one will find two terms: the first is due to the angular momentum $\boldsymbol{\ell}_1$, and the second due to fictitious forces (the non-inertiallity of particle 1). However, we are not concerned with the dynamics of the system, but rather with its kinematics; and, as angular momentum is a kinematic variable (as linear momentum), our results remain valid.

Concerning the spins discussion, it is irrelevant whether the laboratory is in superposition relative to α or not; it only matters the relative orientation of A and B with L (if they are parallel or anti parallel). Thus, our state of consideration, once more, is the reduced one (traced over α subspace). As an example, consider the state as seen by an observer in L,

$$\begin{aligned} |\psi\rangle_{\mathrm{L}} &= a\,|{+}{+}\rangle^{\mathrm{AB}} + b\,|{+}{-}\rangle^{\mathrm{AB}} + c\,|{-}{+}\rangle^{\mathrm{AB}} + d\,|{-}{-}\rangle^{\mathrm{AB}}, \\ &= \left(a\,|+\rangle^{\mathrm{B}} + b\,|-\rangle^{\mathrm{B}}\right)|+\rangle^{\mathrm{A}} + \left(c\,|+\rangle^{\mathrm{B}} + d\,|-\rangle^{\mathrm{B}}\right)|-\rangle^{\mathrm{A}}, \\ &= \left(a\,|+\rangle^{\mathrm{A}} + c\,|-\rangle^{\mathrm{A}}\right)|+\rangle^{\mathrm{B}} + \left(b\,|+\rangle^{\mathrm{A}} + d\,|-\rangle^{\mathrm{A}}\right)|-\rangle^{\mathrm{B}}. \end{aligned} \tag{4.3.1}$$

As we know, if A is in the state $|+\rangle$ ($|-\rangle$) relative to L, it means that their spin is parallel (anti parallel). With this reasoning, we can perform the passive transformation for A and B, in order to obtain, respectively,

$$\begin{aligned} |\psi\rangle_{\mathrm{A}} &= a\,|{+}{+}\rangle^{\mathrm{LB}} + b\,|{+}{-}\rangle^{\mathrm{LB}} + c\,|{-}{-}\rangle^{\mathrm{LB}} + d\,|{-}{+}\rangle^{\mathrm{LB}}, \\ &= \left(a\,|+\rangle^{\mathrm{B}} + b\,|-\rangle^{\mathrm{B}}\right)|+\rangle^{\mathrm{L}} + \left(d\,|+\rangle^{\mathrm{B}} + c\,|-\rangle^{\mathrm{B}}\right)|-\rangle^{\mathrm{L}}, \\ &= \left(a\,|+\rangle^{\mathrm{L}} + d\,|-\rangle^{\mathrm{L}}\right)|+\rangle^{\mathrm{B}} + \left(b\,|+\rangle^{\mathrm{L}} + c\,|-\rangle^{\mathrm{L}}\right)|-\rangle^{\mathrm{B}}, \end{aligned} \tag{4.3.2}$$

[§]An observer in frame R represents the frame's origin, O_{R}.

and

$$
\begin{aligned}
|\psi\rangle_{\mathrm{B}} &= a\,|{+}{+}\rangle^{\mathrm{LA}} + b\,|{-}{-}\rangle^{\mathrm{LA}} + c\,|{+}{-}\rangle^{\mathrm{LA}} + d\,|{-}{+}\rangle^{\mathrm{LA}}\,, \\
&= \Big(a\,|+\rangle^{\mathrm{A}} + c\,|-\rangle^{\mathrm{A}}\Big)\,|+\rangle^{\mathrm{L}} + \Big(d\,|+\rangle^{\mathrm{A}} + b\,|-\rangle^{\mathrm{A}}\Big)\,|-\rangle^{\mathrm{L}}\,, \\
&= \Big(a\,|+\rangle^{\mathrm{L}} + d\,|-\rangle^{\mathrm{L}}\Big)\,|+\rangle^{\mathrm{A}} + \Big(c\,|+\rangle^{\mathrm{L}} + b\,|-\rangle^{\mathrm{L}}\Big)\,|-\rangle^{\mathrm{A}}\,;
\end{aligned}
\tag{4.3.3}
$$

being the same as the relative part of (4.2.64). As we can see, this passive transformation yields the same relative result, the one we are actually concerned with, without the need of considering the external frame.

CHAPTER 5

CONCLUSIONS

We have begun our work defending that physics is a relative science. To illustrate this point, we have demonstrated that it is possible to treat the two-body problem of two particles of mass m_1 and m_2, (1.1.1), as only one "body" of mass μ, (1.1.4), when one moves to the center of mass and relative coordinates of the system. The former coordinate is written relative to some absolute (external) reference frame, whereas the latter is relative to one of the bodies, where, in this case, if one of the masses is much heavier than the other, μ approaches the lighter one and Newton's second law of motion becomes absolute, as stated in (1.1.5). That is, we need to summon an absolute reference frame to write our equations of motion, for they are only valid in inertial frames, and only then we are able to proceed to relative orientations. Nevertheless, it is meaningless to consider the description only from an inaccessible (a priori) absolute reference frame if our results cannot be measured in such frame. To this end, we have pursued the description of physical events regarding physical entities as reference frames, which can interact with the system.

The fact that a physical body can be identified as a reference frame leads us to conclude that the laws of physics must also be respected by this body as, for example, the conservation laws of momentum and energy. If such body possess a mass much greater than the rest of the system, its motion is not influenced by it. But, if this condition is no longer met, as physics cannot be relative only to infinite-mass bodies, our equations have to be altered (corrected, if we may say) in order to accommodate the body's motion. We have thus adapted the usual procedure to derive the time dilation effect as to analyse this possibility.

In Chapter 3, we have seen how the usual time contraction equation, (3.1.1), which can be obtained via the procedure in §2.1.3 (a moving train where a photon is emitted from the floor), receives a correction term arising from a dimensionless parameter defined in (3.2.12), $\varepsilon = h\nu_{\mathbb{S}'}/Mc^2$, when considering the substitution of the rigid train by two independent lightweight plates, each with mass M, yielding result (3.2.13); stating that the usual contraction, $\Delta t_{\mathbb{S}'}$, in relation to the external one, $\Delta t_{\mathbb{S}}$, is accentuated by the recoil of the plate. The correction term came directly from the relativistic energy and momentum conservation laws, which the momentum gained by the plate from the photon's emission is included, and not from the internal structure of the laboratory. It is straightforward to see that when heavy plates are involved, (3.2.13) approximates to (3.1.1). Even though the correction is negligible at extreme scenarios, it is always present whenever finite-mass laboratories are concerned. Additionally, the lightness of the mirror also implies a frequency alteration, $\nu_i \to \nu_r$, of the incident photon to the reflected photon; this alteration can occur even if the mirror is stationary relative to $\mathbb{S}$. These results give an interesting example of how a well-established effect of special relativity manifests itself in a scenario involving lightweight reference frames; concluding the first main part of this dissertation.

Our work towards quantum mechanics has started with the concepts introduced in §1.2.2 by Hugh Everett, which discuss how the wave function's evolution in time can be contradictory when more than one observer is present and the collapse notion is involved. This is so because each observer uses distinct processes regarding the wave function's change: a discontinuous one, brought by the observation of a quantity; and a continuous one, according to the Schrödinger equation (1.2.2). Consequently, in order to fix this contradiction, it must be assumed that wave mechanics is valid for all physical systems, including observers and measuring apparata. The inclusion of the observer (which can be a quantum particle) in the description lies on the fact that all correlations must be accounted, in every frame's description, to avoid paradoxes and correctly describe a physical phenomenon.

Furthermore, with EPR's criteria about the reality of some determined observable $\mathcal{O}_1$ given a pure state ρ, we have seen that is possible to quantify it using quantum information theory and the Bilobran-Angelo quantifier (2.2.41), which is an extension of such criteria, for it is also able to give us the reality of mixed states. The map $\Phi_{\mathcal{O}_1}(\rho)$ represents the best possible description one can give to the system after it has been measured. As a measurement took place, the observable is real and the probabilities regarding the outcomes reflect only our subjective ignorance about the value of $\mathcal{O}_1$. With these tools, we have used (2.2.45) to determine how far a state ρ, pure or mixed, is from $\Phi_{\mathcal{O}_1}(\rho)$ as an entropic distance. We have learnt that the irreality measure, $\Im(\mathcal{O}_1|\rho)$, from (2.2.45), can be written as a contribution from two terms: a local irreality $\Im(\mathcal{O}_1|\rho_1)$, which denotes local coherence, and the discordlike measure $D_{[\mathcal{O}_1]}(\rho)$, which in turn denotes correlations between subspaces, as given in (2.2.47). These definitions enabled us to seek an invariant quantity within quantum mechanics when changing reference frames.

Beginning in Chapter 4, regarding continuous variables in §4.1, we have discussed the concept of Galilean boost, which is the application, in the active picture, of the shift operator (by a parameter ξ) onto a ket written in the position basis in frame S, displacing the mean position by $-\xi$, as in (4.1.5). This view is indistinguishable from that of the passive picture, wherein a reference frame S′ is displaced by a distance ξ relative to frame S and nothing happens with the ket state. The passive picture makes direct contact with the usual Galilean scenarios where an inertial reference frame is involved, by identifying $\xi = ut$ with u the velocity of S′ relative to S. The change of perspective when concerning two particles, from S to S′ (or vice-versa), does not alter the amount of correlations present in the system; it only displaces the corresponding wave functions in space by $-\xi$. Additionally, this conclusion remains as long as the reference frames, S and S′, are treated classically: with position and momentum well defined at all instant of times. We can then state that the amount of correlations present in a system is invariant under classical boosts.

In §4.1.2, we have postulated the Observer-Observable Symmetry as a principle of relationality: if an observer describes the motion of an object (the observable), the object (now observer) has every right to describe the motion of the observer (now observable). Initially, two examples were given to support this postulate: a two particle system, A and B, described by an observer L with classical mechanics, and another two particle system described with quantum mechanics. The former gave us the same equation of motion from the two-body problem when A looks at B and a very reasonable result when A looks at L; L sees A moving under the influence of a force $\mathbf{F}$, whilst A sees L moving under the influence of a force $-\mathbf{F}$, as seen in (4.1.18). This procedure was achieved considering an external observer which ascribes to L, A, and B, the respective position vectors: $\mathbf{x}_{\mathrm{L}}$, $\mathbf{x}_{\mathrm{A}}$, and $\mathbf{x}_{\mathrm{B}}$. The ensuing quantum mechanical example was an analogy to the classical one; an observer L in a laboratory assigns a superposition state regarding the states of position $|u\rangle^{\mathrm{R}}$, which denotes wavepackets centred at u, from A and B. By direct inspection, we have utilized the irreality measures to show that the position of A and B are indeed not real (agreeing with the fact that they are in superposition and hence, not defined), according to the state given by L. Performing a change of coordinates, which takes us to the frame of either A or B, we see that the position of L relative to them is also not real, for it is in a superposition, agreeing with what L states; thus, all observers agree upon observations. This reasoning is only possible if one includes the observer in the description, as also demonstrated in the floating-slit experiment solution: no paradox emerges if the observer attached with the slit includes the external observer in his state of consideration. This argumentation concludes the first half of the second main part of this work.

In §4.2, we have started a discussion about relative angular momentum: being familiar with the concept of relative position, how can we build, in an equivalent fashion, the relative angular momentum between two rotating bodies? To this end, we have come up with a classical model where two rigid rotors rotate around two parallel and separated axes in the same plane. With (4.2.3) and some assumptions, we have arrived at (4.2.5), which shows the relative and center of mass angular momenta of the two rotors. By further manipulating the former, such as considering a time average of it, expression (4.2.9) was attained, yielding $+1$ (-1) if the bodies rotate in the same (opposite) direction; one can think of it as $+1$: parallel angular momenta; and -1: anti parallel angular momenta. Hence, within a choice of parameters, like the angular frequencies, this model can mimic two particles with spin, where the labels can be interpreted as: $+1 \rightarrow |+\rangle$ and $-1 \rightarrow |-\rangle$.

This has led us to the notion of relative spin orientation, where we could see that even if an object's spin is not defined (real) to some observer (*e.g.*, being in a superposition state), it can be to some other observer. The singlet state (4.2.28) denotes an entangled superposition of the spins of two particles A and B relative to L, where neither of their spins is defined. However, when we move to the reference frame of A or B, we see that their relative spin is real and that of L is in superposition, as in (4.2.31). After analysing some possible transformation of coordinates, we have chosen (4.2.35), for it corresponds, in the active picture, to a CNOT gate, explicitly shown in (4.2.41). We have showed that the passive picture can be interpreted as a relative orientation between the control qubit and the target qubit, so as to obtain the transformation (4.2.48). Considering the most possible entangled state regarding the spin of two particles, (4.2.62), we have demonstrated that all observers, L, A, and B, also agree upon observations, represented by the set (4.2.64); indicating that both kinds of elements, local coherence and correlations, are present in the same manner for all frames.

In consequence thereof, we have proposed the quantity $\mathscr{Q}_\mathrm{S}$ (denoting the total *quantumness* of the system S), defined in (4.1.30), as an *invariant* one in quantum mechanics. This proposal was supported by the fact that, in all situations considered in this work, it remains the same (although the contributions, coherences and quantum correlations, vary) for all observers (reference frames); *i.e.*, given that an observer in frame S assigns the state ρ_S to the system under consideration and obtains the value $\mathscr{Q}_\mathrm{S}$, we have that $\mathscr{Q}_\mathrm{L} = \mathscr{Q}_\mathrm{A} = \mathscr{Q}_\mathrm{B}$.

The results for $\mathscr{Q}_\mathrm{S}$ above were made regarding only one direction and observer: σ_{zR}, the σ_z operator for object R. Our task then was to investigate if $\mathscr{Q}_\mathrm{S}$ could be kept invariant under a change of directions; for σ_x or σ_y, for instance. A procedure had been followed and we may answer that, although the quantumness evaluated in the x axis, $(\mathscr{Q}_\mathrm{S})_x$, is invariant for all frames and the same as the one evaluated in the z axis, $(\mathscr{Q}_\mathrm{S})_z$, one cannot use expression (4.2.66) as a transformation of relative spins, for they do not obey the same algebra as the usual spins; this can be seen from the fact that the commutation relation from the former, (4.2.88), is not the same as the latter's, (4.2.87).

To close this work, we have argued that it might not be strictly necessary to consider the external and absolute reference frame, labelled as α, for our equations rely only on the relative quantities, like distance, velocity, angular momentum and so on, of the system under investigation. Evidently, if we are interested in the dynamics of such system, we have to account for the eventual non-inertiallity of the reference frame, resulting in extra terms in the laws of motion. Nevertheless, as far as kinematics is concerned, the line of reasoning presented in §4.3 proved to be just as effective as the usual method wherein the external frame is employed.

Our results, although preliminaries and based upon simple models, point to physical concepts that deserve deeper investigation, redirecting us towards an interesting research program to the future.

Appendices

APPENDIX A

APPENDIX

A.1 Rigid-Body Motion in Special Relativity

The concept of rigid body in relativity is still not completely accepted and clear. Some works [57, 101–105] have studied it and the main cause of "confusion" seems to be the real interpretation of the length contraction. We focus this section in the discussion due to Franklin's papers [101, 102], where he defines a "relativistic rigid body" and derives the necessary conditions for this notion to be valid.

We start with the analysis of rigid body motion in classical mechanics. Within this framework, the motion of a rigid-body is assumed, by definition, to preserve the body's dimension during its dynamics. However, there are two problems with this definition: **(i)** any actual physical object will have elastic properties, so there must be some distortion during accelerated motion; and **(ii)** due to the finite velocity of sound in any real object, one end of a rigid-body will not move until a short time has passed since the other end has been struck. But, still, these difficulties can be generally dispensed by assuming that the body is so rigid that the elastic deformation can be ignored, and the speed of sound is so fast that the initial delay in the motion of the other end can also be neglected.

Nevertheless, when we move to relativistic dynamics, these definitions are no longer valid due to the fact that the length of a moving body changes as its velocity increases, as seen in §2.1.1. Hence, we need a new definition. Franklin then proposes that: *"a rigid body retains its rest frame dimensions while in translational motion."*. Requiring a moving rigid-body to change its length in any frame in which it is moving.

To conclude his work, he analyses the motion of a rigid rod of length L as seen from two observers in relative motion, $\mathbb{S}$ and $\mathbb{S}'$, and shows that in order to keep a constant length in its rest frame, the front and back ends of the rod must have different constant accelerations, a'_f and a'_b, respectively, in the rest system. The relation between them is

$$a'_b = \frac{a'_f}{1 - a'_f L/c^2}. \tag{A.1.1}$$

Further, although the acceleration varies throughout the rod, there will be no strain or stress involved because this varying acceleration preserves the rest frame dimensions of the body. This reasoning was also presented by [57, 103].

Now, relativity taught us that signals cannot travel faster than the speed of light. As a result, once a signal has been emitted at a point A towards point B, it will take a time L/c, where L is the separation between the points, to arrive at B. Hence, if an end of the rod receives a force and starts moving, the other end cannot move at the same instant if it wants to respect relativity principles. Therefore, even in the rod's rest frame, the rod's length cannot remain constant. In addition, given that different parts of the body can display different states of motion, one may fairly wonder whether one can conceive a meaningful notion of rest frame.

A.1.1 Heuristic Model of an Elastic Body

To support this argument, we propose an heuristic model of an *elastic body*. Consider it to be composed of two interacting particles, labelled as 1 and 2, with its length $x_2(t) - x_1(t) = L(t)$ lying in the x axis of an inertial reference frame $\mathbb{S}$. Initially, the system is at rest in this frame, *i.e.*, the particles' velocities are $v_1 = v_2 = 0$. Moreover, we suppose that $x_1(0) = 0$, $x_2(0) = L$ and $L(0) = L$. At $t = 0$, an impulsive force

$$F_{\text{ext}} = F_0 \delta(t) \tag{A.1.2}$$

acts upon particle 1, where F_0, measured in N · s, designates a constant impulse (positive or negative) and $\delta(t)$ is Dirac's delta function. Our proposal for the dynamics of this system is given by the expressions

$$x_2(t) = L\left[1 + f(t)\right], \tag{A.1.3a}$$

and

$$x_2(t) - x_1(t) = Lg(t). \tag{A.1.3b}$$

Here, $f(t)$ and $g(t)$ are yet to be determined functions, but we know that $g(0) = 1$, for $x_2(0) - x_1(0) = L$. To implement the notion of local causality, let us consider that $f(t)$ has a behaviour similar to

$$f(t) \sim \Theta(t - \tau)h(t) \qquad h(\tau) = 0, \tag{A.1.4}$$

where Θ is the step function and $h(t)$ a smooth function. τ denotes a time scale in which particle 2 will receive the information that particle 1 has started moving (one can think of it as the "shock wave" emanated from particle 1). Before this interval, particle 2 does not move at all. We can imagine that $\tau = L/v_s$, with $v_s \ll c$ the sound's velocity within the medium of the elastic body.

We will now use the equations of dynamics from relativity to see what they tell us about our model. Relativistic momentum will be written as $p_j = m(v_j)v_j = m_0 c\gamma_j\beta_j$, where m_0 is the particles' rest masses, $\beta_j = v_j/c$, and $\gamma_j = 1/\sqrt{1-\beta_j^2}$. Through $\mathrm{d}p_k/\mathrm{d}t = \sum_i F_{ki}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0 c\gamma_1\beta_1) = F_{21} + F_{\text{ext}}(t), \tag{A.1.5a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0 c\gamma_2\beta_2) = -F_{21}. \tag{A.1.5b}$$

The sum produces $\mathrm{d}\left(m_0 c\gamma_1\beta_1 + m_0 c\gamma_2\beta_2\right)/\mathrm{d}t = F_{\text{ext}}(t)$ (an analog expression for the particles' center of mass), resulting in

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1\beta_1 + \gamma_2\beta_2) = \beta_0\delta(t), \qquad \beta_0 \equiv \frac{v_0}{c} = \frac{F_0}{m_0 c}. \tag{A.1.6}$$

Integrating with the condition that $\beta_k(0) = v_k(0)/c = 0$ ($k = 1, 2$), yields

$$\gamma_1\beta_1 + \gamma_2\beta_2 = \beta_0, \tag{A.1.7}$$

where now it should be implicit that $\beta_k = \beta_k(t)$; the same applying for γ_k. This relation, along with (A.1.3), (A.1.4) and $c\beta_k(t) = \dot{x}_k$ defines our elastic body model. Notice that we are still not interested in changing to another frame. For now, our only intention is to solve the dynamics of an elastic body in an inertial frame $\mathbb{S}$.

Solution for Short Time Intervals

We start with a simplification: for short times, $t \ll \tau$, we get $x_2 = L$ and $\beta_2 = 0$. From (A.1.7), we have $\gamma_1\beta_1 = \beta_0$, which gives us $\beta_1 = \beta_0/\sqrt{1+\beta_0^2}$. Because $c\beta_1 = \dot{x}_1$, it follows that $x_1(t) = c\beta_0 t/\sqrt{1+\beta_0^2}$. Once returned to (A.1.3b), we obtain $g(t)$. Summarizing, the results for $t \ll \tau$ are

$$\begin{cases} x_1(t) = \dfrac{c\beta_0 t}{\sqrt{1-\beta_0^2}}, \\ x_2(t) = L, \\ g(t) = 1 - \dfrac{(c\beta_0/L)t}{\sqrt{1-\beta_0^2}}, \end{cases} \tag{A.1.8}$$

being clear that, in this domain, the rod's length decreases linearly with time. Additionally, we can rewrite the set above with dimensionless parameters; defining $\mathrm{x}_k \equiv x_k/L$ and $\mathrm{t} \equiv t/\tau$, with $L = v_s\tau = c\beta_s\tau$, we have

$$\begin{cases} \mathrm{x}_1(\mathrm{t}) = \dfrac{(v_0/v_s)\mathrm{t}}{\sqrt{1-\beta_0^2}}, \\ \mathrm{x}_2(\mathrm{t}) = 1, \\ g(\mathrm{t}) = 1 - \dfrac{(v_0/v_s)\mathrm{t}}{\sqrt{1-\beta_0^2}}. \end{cases} \tag{A.1.9}$$

Observe that, even in the nonrelativistic limit ($\beta_0 \ll 1$), the length's variation is governed by the ratio v_0/v_s between the velocity associated with the impulse provided by the external force and the velocity of the shock wave.

A.1.2 Conclusions

We see that if one defines a rigid body as a body whose dimensions are always constant while in translational motion, one gets in contradiction with the concept of causality and relativity. In order to respect them, we cannot have a body whose length remains constant at all times t. Our model shows that, at the very beginning of the impulse provided by the external force at one end of the rod, the rod's length decreases linearly with t until a time $t \sim \tau = L/v_s$; only at this time the other end will start moving. One might argue that, at some later time ($t \gg \tau$), the rod's length obeys $L(t \gg \tau) = L =$ constant, in frame $\mathbb{S}$, and the rod is indeed a rigid body. However, the assumption that the length of the body remains constant at all times and, hence, is rigid at all times, is not true if one wishes to respect the concept of local causality.

BIBLIOGRAPHY

[1] H. Goldstein *et al*, *Classical Mechanics*, (3rd edition, Addison-Wesley), 2001

[2] J. B. Marion and S. T. Thornton, *Classical Dynamics of Particles and Systems*, (5th edition, Brooks/Cole - Thomson Learning, Belmont), 2003

[3] W. Greiner, *Classical Mechanics: Point Particles and Relativity*, Springer, 2004

[4] A. Einstein, "On the Electrodynamics of Moving Bodies", *Ann. der Physik* **17**:891, (1905)

[5] J. Jackson, *Classical Electrodynamics*, (3rd edition, Wiley), 1998

[6] D. Griffiths, *Introduction to Electrodynamics*, (3rd edition, Prentice Hall of India), 1998

[7] J. Reitz and F. Milford, *Foundations of Electromagnetic Theory*, (4th edition, Addison-Wesley), 2008

[8] A. Michelson and E. Morley, "On the Relative Motion of the Earth and the Luminiferous Ether", *Amer. J. of Science* **34**: 333-345 (1887)

[9] R. Resnick, *Introduction to Special Relativity*, (John Wiley & Sons), 1968

[10] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, (John Wiley & Sons), 1972

[11] C. Møller, *The Theory of Relativity*, (Oxford University Press, 2nd edition), 1976

[12] H. Hertz, "Über einen Einfluss des Ultravioletten Lichtes auf die Electrische Entlaudung" (On an Effect of Ultra-Violet Light Upon the Electrical Discharge), *Ann. der Physik* **267**:983-1000, (1887)

[13] J. Franck and G. Hertz, "Über Zusammenstöße zwischen Elektronen und Molekülen des Quecksilberdampfes und die Ionisierungsspannung desselben" (On the Collisions Between Electrons and Molecules of Mercury Vapor and the Ionization Potential of the Same), *Verh. Deutsche Phys. Ges.* **16**, 457-467 (1914)

[14] C. Davisson and L. Germer, "Reflection of Electrons by a Crystal of Nickel", *Proc. Natl. Acad. Sci U.S.A.* **14**(**4**), 317-322 (1928)

[15] W. Greiner, *Quantum Mechanics: An Introduction*, (4th edition, Springer-Verlag Berlin Heidelberg) 2001

[16] E. Merzbacher, *Quantum Mechanics*, (2nd edition, John Wiley & Sons), 1970

[17] L. Landau and E. Lifshitz, *Course of Theoretical Physics, Vol. 3, Quantum Mechanics: Non-Relativistic Theory*, (3rd edition, Pergamon Press) 1977

[18] C. Cohen-Tannoudji *et al*, *Quantum Mechanics* Vol. 1, (1st edition, Wiley) 1991

[19] J. Sakurai, *Modern Quantum Mechanics*, (Revised edition, Addison-Wesley Publishing Company) 1994

[20] D. Griffiths, *Introduction to Quantum Mechanics*, (1st edition, Prentice Hall) 1995

[21] A. Phillips, *Introduction to Quantum Mechanics*, (1st edition, John Wiley & Sons) 2003

[22] H. Everett, "Relative States Formulation of Quantum Mechanics", *Rev. Mod. Phys.* **29**, 454 (1957)

[23] Y. Aharonov and L. Susskind, "Charge Superselection Rule", *Phys. Rev.* **155**, 1428 (1967)

[24] Y. Aharonov and L. Susskind, "Observability of the Sign Change of Spinors Under 2π Rotations", *Phys. Rev.* **158**, 1237 (1967)

[25] Y. Aharonov and G. Carmi, "Quantum Aspects of the Equivalence Principle", *Found. Phys.* **3**, 493 (1973)

[26] Y. Aharonov and G. Carmi, "Quantum-Related Reference Frames and the Local Physical Significance of Potentials", *Found. Phys.* **4**, 75 (1974)

[27] Y. Aharonov and T. Kaufherr, "Quantum Frames of Reference", *Phys. Rev.* D. **30**, 368 (1984)

[28] S. Massar and S. Popescu, "Optimal Extraction of Information from Finite Quantum Ensembles", *Phys. Rev. Lett.* **74**, 1259 (1995)

[29] N. Gisin and S. Popescu, "Spin Flips and Quantum Information for Antiparallel Spins", *Phys. Rev. Lett.* **83**, 492 (1999)

[30] A. Peres and P. Scudo, "Entangled Quantum States as Direction Indicators", *Phys. Rev. Lett.* **86**, 4160 (2001)

[31] E. Bagan *et al*, "Optimal Encoding and Decoding of a Spin Direction", *Phys. Rev.* A. **63**, 052309 (2001)

[32] D. Poulin and J. Yard, "Dynamics of a Quantum Reference Frame", *New. J. Phys.* **9**, 156 (2007)

[33] S. Barlett, T. Rudolph and R. Spekkens, "Reference Frames, Superselection Rules and Quantum Information", *Rev. Mod. Phys.* **79**, 555 (2007)

[34] J. Boileau *et al*, "Quantum Reference Frames and the Classification of Rotationally-Invariant Maps", *J. Math. Phys.* **49**, 032105 (2008)

[35] F. Girelli and D. Poulin, "Quantum Reference Frames and Deformed Symmetries", *Phys. Rev.* D. **77**, 104012 (2008)

[36] G. Gour and R. Spekkens, "The Resource Theory of Quantum Reference Frames", *New. J. Phys.* **10**, 033023 (2008)

[37] G. Gour, I. Marvian and R. Spekkens, "Measuring the Quality of a Quantum Reference Frame: The Relative Entropy of Frameness", *Phys. Rev.* A. **80**, 012307 (2009)

[38] J. Oppenheim and B. Reznik, "Fundamental Destruction of Information and Conservation Laws", arXiv:0902.2361v1 [hep-th] (2009)

[39] S. Bartlett *et al*, "Quantum Communication Using a Bounded-Size Quantum Reference Frame", *New. J. Phys.* **11**, 063013 (2009)

[40] F. Costa *et al*, "Entanglement Detection with Bounded Reference Frames", *New. J. Phys.* **11**, 123007 (2009)

[41] Y. Liang *et al*, "Nonclassical Correlations from Randomly Chosen Local Measurements", *Phys. Rev. Lett.* **104**, 050401 (2010)

[42] L. Ioannou and M. Mosca, "Public-Key Cryptography Based on Bounded Quantum Reference Frames", arXiv:0903.5156v3 [quant-ph] (2011)

[43] R. M. Angelo *et al*, "Physics Within a Quantum Reference Frame", *J. Phys. A: Math. Theor.* **44**, 145304 (2011)

[44] R. M. Angelo and A. Ribeiro, "Kinematics and Dynamics in Noninertial Quantum Frames of Reference", *J. Phys. A: Math. Theor.* **45**, 465306 (2012)

[45] M. Palmer, F. Girelli and S. Bartlett, "Changing Quantum Reference Frames", *Phys. Rev.* A. **89**, 052121 (2014)

[46] S. Pereira and R. M. Angelo, "Galilei Covariance and Einstein's Equivalence Principle in Quantum Reference Frames", *Phys. Rev.* A. **91**, 022107 (2015)

[47] J. Pienaar, "A Relational Approach to Quantum Reference Frames for Spins", arXiv:1601.07320v2 [quant-ph] (2016)

[48] A. Einstein, B. Podolsky and N. Rosen, "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?", *Phys. Rev.* **47**, 777 (1935)

[49] V. N. Strel'tsov, "On the Relativistic Length", *Found. Phys.* **6**:293 (1976)

[50] G. Spavieri, "Nonequivalence of Ether Theories and Special Relativity", *Phys. Rev.* A. **34**, 1708 (1986)

[51] C. W. Sherwin, "New Experimental Test of Lorentz's Theory of Relativity", *Phys. Rev.* A. **35**, 3650 (1987)

[52] G. P. Sastry, "Is Length Contraction Really Paradoxical?", *Am. J. Phys.* **55**, 943 (1987)

[53] W. Rindler, "Length Contraction Paradox", *Am. J. Phys.* **29**, 365 (1991)

[54] A. Laub *et al*. “Lorentz Contraction of Flux Quanta Observed in Experiments with Annular Josephson Tunnel Junctions”, *Phys. Rev. Lett.* **75**, 1372 (1995)

[55] J. R. Graham, “Special Relativity and Length Contraction”, *Am. J. Phys.* **63**, 637 (1995)

[56] M. Pardy, “Cerenkov Effect and the Lorentz Contraction”, *Phys. Rev.* A. **55**, 1647 (1997)

[57] H. Nikolic, “Relativistic Contraction of an Accelerated Rod”, *Am. J. Phys.* **67**, 1007 (1999)

[58] C. Barceló and G. Jannes, “A Real Lorentz-FitzGerald Contraction”, *Found. Phys.* **38**:191 (2008)

[59] R. Chamseddine, “Graphical Representations for the Successive Lorentz Transformations. Application: Lorentz Contraction on Thomas Rotation”, *Found. Phys.* **46**:428 (2016)

[60] M. F. Guasti and C. Zagoya, “How to Obtain the Lorentz Space Contraction Formula for a Moving Rod from Knowledge of the Positions of its Ends at Different Times”, *Eur. J. Phys.* **30**, 253 (2009)

[61] J. C. Hafele and R. E. Keating, “Around-the-World Atomic Clocks: Predicted Relativistic Time Gains”, *Science* **177** (4044), 168 (1972)

[62] J. Bailey *et al.*, “Measurements of Relativistic Time Dilatation for Positive and Negative Muons in a Circular Orbit”, *Nature* **268**, 301 (1977)

[63] R. W. McGowan *et al.*, “New Measurement of the Relativistic Doppler Shift in Neon”, *Phys. Rev. Lett.* **70**, 251 (1993)

[64] G. Saathoff *et al.*, “Improved Test of Time Dilation in Special Relativity”, *Phys. Rev. Lett.* **91**, 190403 (2003)

[65] S. Reinhardt *et al.*, “Test of Relativistic Time Dilation with Fast Optical Atomic Clocks at Different Velocities”, *Nature Phys.* **3**, 861 (2007)

[66] C. Lämmerzah, “Special Relativity: A Matter of Time”, *Nature Phys.* **3**, 831 (2007)

[67] B. Najjari, A. Surzhykov and A. B. Voitkiv, “Relativistic Time Dilation and the Spectrum of Electrons Emitted by 33-TeV Lead Ions Penetrating Thin Foils”, *Phys. Rev.* A. **77**, 042714 (2008)

[68] P. Dubé *et al.*, “High-Accuracy Measurement of the Differential Scalar Polarizability of a $^{88}Sr^{+}$ Clock Using the Time-Dilation Effect”, *Phys. Rev. Lett.* **112**, 173002 (2014)

[69] B. Botermann *et al.*, “Test of Time Dilation Using Stored Li^{+} Ions as Clocks at Relativistic Speed”, *Phys. Rev. Lett.* **113**, 120405 (2014)

[70] K. T. Bainbridge, “The Equivalence of Mass and Energy”, *Phys. Rev.* **44**, 123 (1933)

[71] W. F. G. Swann, “The Mass-Energy Relation”, *Phys. Rev.* **74**, 200 (1948)

[72] R. C. Cornelius, “Rest Mass Energy of the Negative Pion”, *Phys. Rev.* **92**, 1583 (1953)

[73] W. F. G. Swann, “Mass-Energy in Quantum Theory”, *Phys. Rev.* **109**, 998 (1958)

[74] E. Eriksen and K. Vøyenli, "The Classical and Relativistic Concepts of Mass", *Found. Phys.* **6**:115 (1976)

[75] D. J. Steck and F. Rioux, "An Elementary Development of Mass-Energy Equivalence", *Am. J. Phys.* **51**, 461 (1983)

[76] E. Bitsakis, "Mass, Matter, and Energy. A Relativistic Approach", *Found. Phys.* **21**:63 (1991)

[77] L. B. Okun, "Mass Versus Relativistic and Rest Masses", *Am. J. Phys.* **77**, 430 (2009)

[78] E. Hecht, "Einstein on Mass and Energy", *Am. J. Phys.* **77**, 799 (2009)

[79] S. Re Fiorentin, "A Re-Interpretation of the Concept of Mass and of the Relativistic Mass-Energy Relation", *Found. Phys.* **39**:1394 (2009)

[80] M. Zhang, "Weak Values Obtained from Mass-Energy Equivalence", *Phys. Rev.* A. **95**, 012114 (2017)

[81] R. K. Pathria and P. D. Beale, *Statistical Mechanics*, (3rd edition, Academic Press), 2011

[82] J. von Neumann, "Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik", *Göttinger Nachrichten* **1**: 245-272 (1927)

[83] C. E. Shannon, "A Mathematical Theory of Communication", *Bell System Technical Journal* **27**, 379 (1948)

[84] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, (10th Anniversary Edition, Cambridge University Press), 2010

[85] A. Wehrl, "General Properties of Entropy", *Rev. Mod. Phys.* **50**, 221 (1978)

[86] H. Ollivier and W. H. Zurek, "Quantum Discord: A Measure of the Quantumness of Correlations", *Phys. Rev. Lett.* **88**, 017901 (2001)

[87] L. Henderson and V. Vedral, "Classical, Quantum and Total Correlations", *J. Phys.* A. **34**, 6899 (2001)

[88] D. Bru, "Characterizing Entanglement", *J. Math. Phys.* **43**, 4237 (2002)

[89] F. Mintert, C. Viviescas and A. Buchleitner, *Entanglement and Decoherence*, vol. 768 on Lecture Notes in Physics, Ch. 2 pg 61 (Springer Berlin Heidelberg, Berlim, 2009)

[90] A. Bilobran and R. Angelo, "A Measure of Physical Reality", *Eur. Lett.* **112**, 40005 (2015)

[91] R. M. Angelo and A. Ribeiro, "Complementarity as Corollary", arXiv:1304.2286 (2013)

[92] T. Baumgratz, M. Cramer and M. B. Plenio, "Quantifying Coherence", *Phys. Rev. Lett.* **113**, 140401 (2014)

[93] P. Ehrenfest, "Bemerkung über die angenärte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik", *Zeitschrift für Physik* **45**, 455 (1927)

[94] A. Bolivar, "Teorema de Ehrenfest e o Limite Clássico da Mecânica Quântica" (Ehrenfest Theorem and the Classical Limit of Quantum Mechanics), *Rev. Bras. Ensino Fis.* **23**, n. 2, 190-195 (2001)

[95] D. M. Greenberger, "Inadequacy of the Usual Galilean Transformation in Quantum Mechanics", *Phys. Rev. Lett.* **87**, 100405 (2011)

[96] A. Gjurchinovski, "Reflection of Light from an Uniformly Moving Mirror", *Am. J. Phys.* **72**, 1316 (2004)

[97] A. L. O. Bilobran, "Uma Medida de Realidade Física" (A Measure of Physical Reality), MSc Thesis, UFPR (2015)

[98] N. Bohr, "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?", *Phys. Rev.* **48**, 696 (1935)

[99] L. P. Schmidt *et al.*, "Momentum Transfer to a Free Floating Double-Slit: Realization of a Thought Experiment from the Einstein-Bohr Debates", *Phys. Rev. Lett.* **111**, 103201 (2013)

[100] Xiao-Jing Liu *et al.*,, "Einstein-Bohr Recoiling Double-Slit Gedanken Experiment Performed at the Molecular Level", *Nature Photonics* **9**, 102-125 (2015)

[101] J. Franklin, "Lorentz Contraction, Bell's Spaceships and Rigid Body Motion in Special Relativity", *Eur. J. Phys.* **31**, 291 (2010)

[102] J. Franklin, "Rigid Body Motion in Special Relativity", *Found. Phys.* **43**:1489 (2013)

[103] G. Cavalleri and A. G. Spinella, "Does a Rod, Pushed by a Force, Accelerate Less than the Same Rod Pulled by the Same Force?", *Nuovo Cimento* B **66**, 11 (1970)

[104] V. Petkov, "Accelerating Spaceships Paradox and Physical Meaning of Length Contraction", arXiv:0903.5128v1 (2009)

[105] F. M. Paiva and A. F. F. Teixeira, "Oscillation of a Rigid Rod in Special Relativity", arXiv:1201.0670 (2011)