Universidade Federal do Paraná

Departamento de Informática

André Luiz Pires Guedes
Lilian Markenzon

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Directed Hypergraph Planarity

A. L. P. Guedes[1] and L. Markenzon[2]

[1] Universidade Federal do Paraná, PR, Brazil
andre@inf.ufpr.br
[2] Instituto Militar de Engenharia, RJ, Brazil
lilian@ime.eb.br

Abstract. Directed hypergraphs are generalizations of digraphs and can be used to model binary relations among subsets of a given set. Planarity of hypergraphs was studied by Johnson and Pollak; planarity of directed hypergraphs was studied by Mäkinen, being assumed a restricted definition. In this paper we extend the planarity concept to directed hypergraphs. It is well known that the planarity of a digraph relies on the planarity of its underlying graph. However, for directed hypergraphs, this property cannot be applied and we propose a new approach which generalizes the usual concept. We also show that the recognition of the planarity for directed hypergraphs is linear.

1 Introduction

Directed hypergraphs [1, 2] are a generalization of digraphs and can model binary relations among subsets of a given set. Such relationships appears in different areas of Computer Science such as database systems [1], expert systems [10], parallel programming [9] and scheduling [6, 3].

Planarity of hypergraphs was studied by Johnson and Pollak [5] and their paper yields our theoretical approach. Mäkinen [7] gives emphasis to the drawing of hypergraphs, where planarity plays an important role. He includes some remarks about directed hypergraph drawing, being assumed a restricted definition.

In this paper we extend the planarity concept to directed hypergraphs. It is well known that the planarity of a digraph relies on the planarity of its underlying graph. We show that, for directed hypergraphs, the property cannot be applied and we propose a whole new approach. In Section 2 some basic definitions about directed hypergraphs and hypergraph drawing are presented; in Section 3 the concept of hypergraph planarity is reviewed, and, in Section 4, the directed case is presented, showing that previous results are particular cases of a more general concept.

2 Basic Notions

This section introduces the hypergraph notation used throughout the paper. Basic graph concepts are assumed to be known and can be found in [8].

A directed hypergraph [2, 4] can be defined as follows:

Definition 1. *A* ***directed hypergraph*** $H = (V, A)$ *is a pair, where* V *is a (finite) set of vertices and* A *is a set of hyperarcs. A* ***hyperarc*** $a \in A$ *is an ordered pair* (X, Y) *where* X *and* Y *are (disjoint not empty) subsets of* V*. The set* X *is the the origin of* a *and the set* Y *is the destination of* a*, respectively,* $Org(a)$ *and* $Dest(a)$*.*

A directed hypergraph $H = (V, A)$ has size $|H| = \sum_{a=(X,Y)\in A} |X| + |Y|$.

Figure 1 shows a directed hypergraph, $H = (V, A)$, where $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and $A = \{a, b, c, d, e, f, g, h\}$. Two examples of hyperarcs are $a = (\{1, 2\}, \{3, 4\})$, and $c = (\{4\}, \{5, 8\})$.

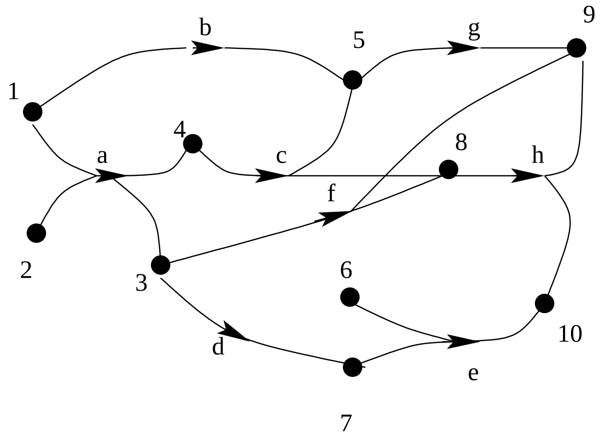

Fig. 1. Directed Hypergraph $H = (V, A)$

In the undirected version of hypergraphs each hyperarc is considered as a set instead of a pair of sets, and is named *hyperedge*.

Definition 2. *Let* $H = (V, A)$ *be a directed hypergraph. The* ***underlying hypergraph*** *of* H *is the hypergraph* $H^u = (V, A^u)$ *where for every hyperarc* $a = (X, Y) \in A$ *there is a hyperedge* $e = X \cup Y \in A^u$*, and every hyperedge of* H^u *has a corresponding hyperarc in* H*.*

Subfamilies of directed hypergraphs, as defined in [2], can be associated with some earlier definitions, as the one presented in [1]. Such subfamilies can be defined as follows:

Definition 3. *Let $H = (V, A)$ be a directed hypergraph.*

1. *If every hyperarc $a \in A$ is such that $|Dest(a)| = 1$ than H is called a* ***B-graph****;*
2. *If every hyperarc $a \in A$ is such that $|Org(a)| = 1$ than H is called a* ***F-graph****;*
3. *If every hyperarc $a \in A$ is such that $|Dest(a)| = 1$ or $|Org(a)| = 1$ than H is called a* ***BF-graph****;*

A digraph is a particular case of BF-graphs, being $|Org(a)| = 1$ and $|Dest(a)| = 1$ for all arcs.

The visual representation of a hypergraph is as important as the same problem for graphs and digraphs. Mäkinen [7] presented some hypergraph drawing ideas based on methods for describing a hypergraph: the *subset standard* and the *edge standard*. The first one uses the fact that a hypergraph is a collection of subsets, which can be viewed as a Venn diagram, and in the second the vertices of a hyperedge are connected by curves.

The subset standard is not suitable to draw a directed hypergraph, because the vertices of the hyperarc are divided in two parts: origin and destination. The edge standard is the best choice for directed hypergraphs, and we can draw the hyperarcs as two sets connected by lines.

In fact this pictorial representation has been used by almost all papers related to the subject. The drawing shown in Fig. 1 is an example. In view of that, we can apply the well known concept of planarity to this structure, trying to avoid crossing lines when drawing.

3 Hypergraph Planarity

Planarity of (undirected) hypergraphs was studied by Johnson and Pollak [5], on a paper that presents three approaches of planarity. Two of these approaches, both introduced by the authors, are based on Venn diagrams. As this kind of representation is related to the subset standard, we will not develop these ideas. The third approach, based on Zykov planarity, is more convenient for the edge standard.

Zykov planarity associates hyperedges with faces (regions) of a planar subdivision. Let $H = (V, E)$ be a hypergraph. Each vertex of V is represented by a vertex and each hyperedge is represented by a face of the planar map. It can be observed that not every face represents a hyperedge and we are considering just one of many possible representations. Figure 2 shows an example of this representation for the hypergraph $H = (V, E)$, with $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ and $E = \{\{1, 2, 9, 10\}, \{2, 3, 10, 11\}, \{3, 4, 5, 11\}, \{5, 6, 7, 8\}, \{1, 7, 8, 9\}, \{5, 8, 11\}\}$.

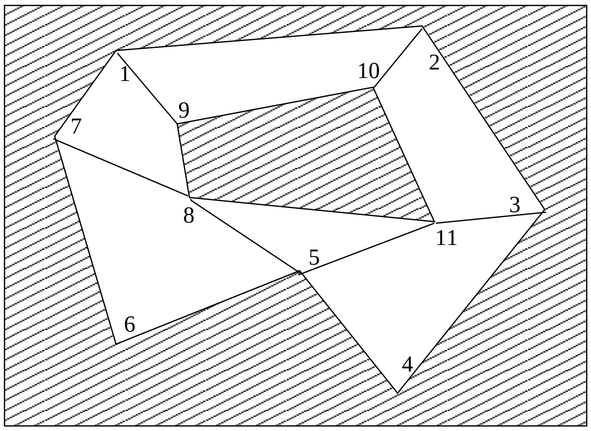

Fig. 2. Planar subdivision

Johnson and Pollak [5] define Zykov planarity using a bipartite graph associated with the hypergraph (see also [11]).

Definition 4. *A hypergraph $H = (V, E)$ is* ***Zykov-planar*** *if and only if the bipartite graph $H_B = (U, F)$ is planar, where $U = V \cup E$ and $F = \{\{v, e\} | e \in E$ and $v \in e\}$.*

Figure 3 shows the graph H_B of the hypergraph H above. The white vertices a, b, c, d, e, f represent hyperedges.

The planar representation of the bipartite graph H_B can be seen as a refinement of the planar subdivision used on the original definition from Zykov, as the vertices representing the hyperedges can lie just inside the faces.

The recognition of a hypergraph $H = (V, E)$ as a Zykov-planar hypergraph is equivalent to the recognition of the bipartite graph $H_B = (U, F)$ as a planar graph and it can be done in linear time. It is important to observe that an ordinary graph is planar under Definition 4 (when viewed as a hypergraph) if and only if it is planar in the ordinary sense. So, Zykov planarity is a true generalization of the planarity concept to hypergraphs.

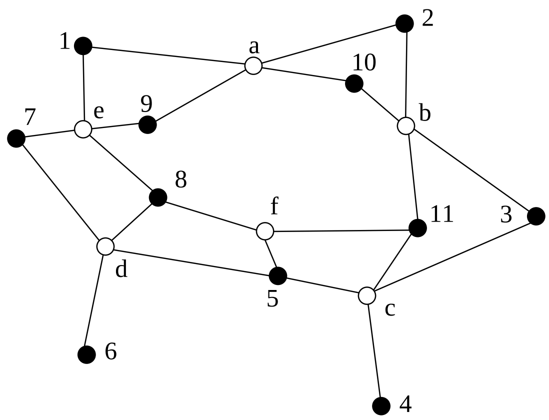

Fig. 3. Graph H_B

4 Directed Hypergraph Planarity

As it was mentioned earlier in this paper, testing the planarity of a digraph is the same as testing the planarity of the underlying graph. So, it would be nice if the solution presented in Sect. 3 could also be extended to directed hypergraphs.

Let us try to adapt Definition 4. Each hyperedge generates a new vertex for the bipartite graph H_B. So, it can be established that the vertex which represents the hyperarc lies in a central point, connected to the original vertices by arcs. However as we can see in Fig. 4(a), the drawing of the hyperarc $a = (\{1, 2, 3\}, \{4, 5\})$ mix the vertices of the origin with the vertices of

the destination of the hyperarc. So it is not enough to add direction when generating the graph, it is necessary to group the vertices of each set just like it is shown in Fig. 4(b).

A solution that forces such grouping to happen is to use two new vertices, instead of just one, to represent the hyperarc. One of these vertices is used to group the origin and the other to group the destination of the hyperarc. Figure 4(c) shows the result for this example. With this transformation the vertices of the origin and destination will not be mixed in a planar representation and the drawing of the hyperarc obeys the edge standard presented on Sect. 2.

The directed hypergraph planarity can now be defined. First, we need to define a new transformation of the directed hypergraph.

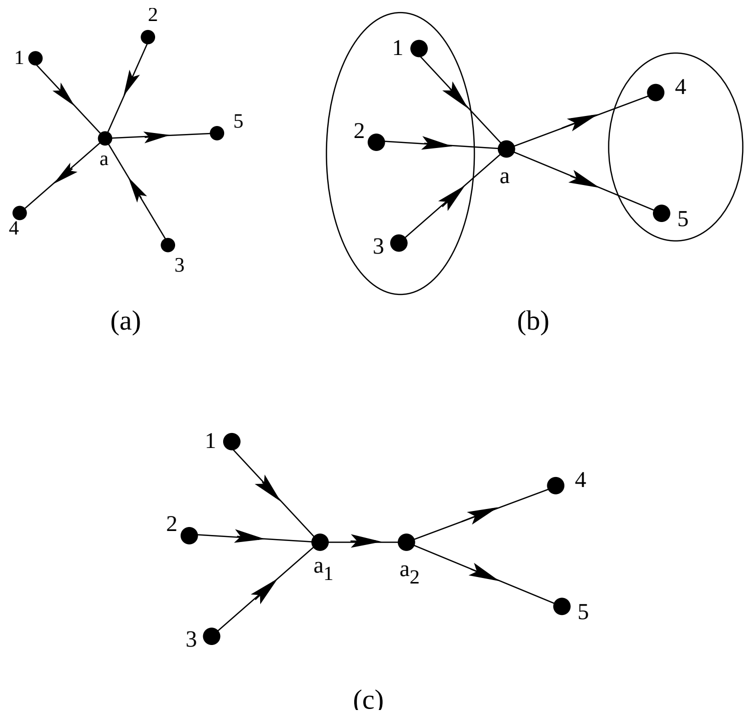

Fig. 4. Transformation of a hyperarc

Definition 5. *Let $H = (V, A)$ be a directed hypergraph. The* ***structure graph*** *associated with H is the digraph $H_S = (V \cup U, B)$, where $U = A \times \{1, 2\}$, and the elements of U are denoted by a_i, with $a \in A$ and $i = 1, 2$; and $B = B_O \cup B_D \cup \{(a_1, a_2) | a \in A\}$, where B_O and B_D are defined as*

$$B_O = \{(v, a_1) | a \in A \text{ and } v \in Org(a)\}, \text{ and}$$

$$B_D = \{(a_2, v) | a \in A \text{ and } v \in Dest(a)\}.$$

Definition 6 (Planarity). *A directed hypergraph H is* ***planar*** *if and only if its structure graph H_S is planar.*

The relation between this concept of planarity and the Zykov planarity can now be establish.

Given a graph $G = (V, E)$, the *contraction* of an edge is defined as the operation of removing $e = (x, y) \in E$ from G and identifying x and y (with a single new vertex xy) so that every edge (other than (x, y)) originally incident with either x or y becomes incident with xy. By the Contraction

Form of Kuratowski Theorem [8], we know that this operation preserves planarity, that is, if G is planar than the resulting contracted graph is also planar.

Lemma 1. *Let $H = (V, A)$ be a directed hypergraph. If H is planar then the underlying hypergraph of H is Zykov-planar.*

Proof. If H is planar then the structure graph associated with H, H_S, is planar and obviously, also its underlying graph, H_S^u, is planar.

Let H_B be the bipartite graph of Definition 4 applied to H^u, the underlying hypergraph of H.

Let a be a hyperarc of H; (a_1, a_2) is an edge of H_S^u. The contraction of each one of these edges generates a new graph, isomorphic to H_B. Since the contraction preserves planarity, if H_S^u is planar, so is H_B. Finally, as H_B is planar, by Definition 4, the underlying hypergraph of H is Zykov-planar. □

The converse of Lemma 1 is not true; Fig. 5(a) shows a directed hypergraph H that is a counter-example. Fig. 5(b) presents the bipartite planar graph, constructed for the underlying hypergraph of H; Fig. 5(c) shows H_S^u, the the underlying graph of the structure graph H_S associated with H, which is clearly not planar, since it is homeomorphic to $K_{3,3}$.

The planarity concept for directed hypergraphs is more restrictive when compared with the same concept for hypergraphs. This restriction is abolished for some important particular cases.

Theorem 1. *Let H be BF-graph. Then H is planar if and only if its underlying hypergraph is Zykov-planar.*

Proof. Every hyperarc a of H has $|Org(a)| = 1$ or $|Dest(a)| = 1$. Let us suppose, without loss of generality, that the hyperarc a has only one vertex at its destination.

When constructing H_S (by Definition 6), vertex a_2 has degree 2 (as in Fig. 6(b)). We must recognize if H_S^u is planar. Vertex a_2 plus its incident edges can be replaced by a single edge. A similar operation can be performed on all the vertices with degree 2 generated by the hyperarcs of H. In other words, we construct a homeomorphic graph with a smaller number of vertices. After such operations, the resulting graph is isomorphic to H_B. So, H_S^u is homeomorphic to H_B. As homeomorphism does not interfere with planarity, H_S is planar if and only if H_B is planar. Consequently, H is planar if and only if its underlying hypergraph is Zykov-planar. □

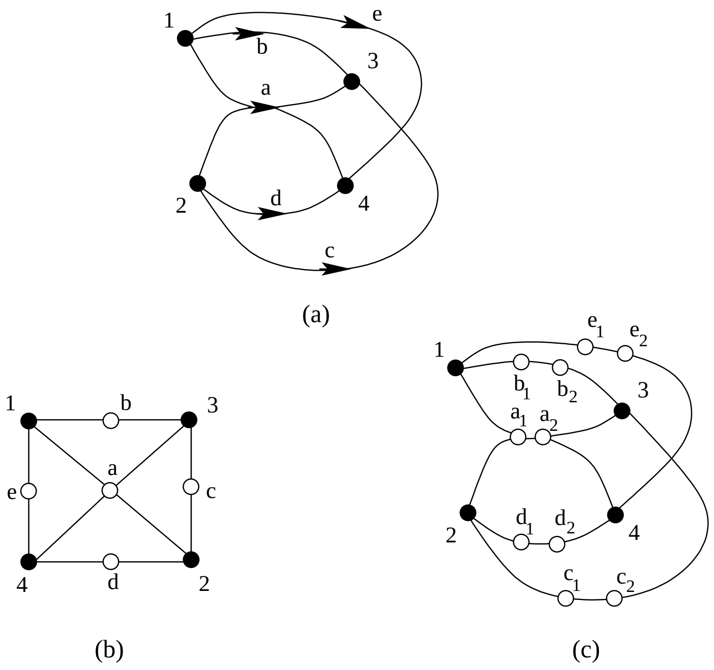

Fig. 5. H^u is Zykov-planar but H is not planar

Corollary 1. *If G is a digraph then G is planar if and only if its underlying graph is Zykov-planar.*

Proof. G is a BF-graph. □

It is interesting to highlight the complexity of the recognition of planarity for directed hypergraphs.

Let $H = (V, A)$ be a directed hypergraph. The number of vertices of H_S, n_S, is equal to $|V| + 2|A|$. As $|V| \leq |H|$ and $2|A| \leq |H|$, then $n_S \leq 2|H|$.

As the construction of the structure graph H_S can be done in linear time in the size of the hypergraph ($|H|$), and testing the planarity of a digraph can be done in linear time in its number of vertices (n_S), the test whether a directed hypergraph is planar can be done in linear time in its size ($\mathcal{O}(|H|)$).

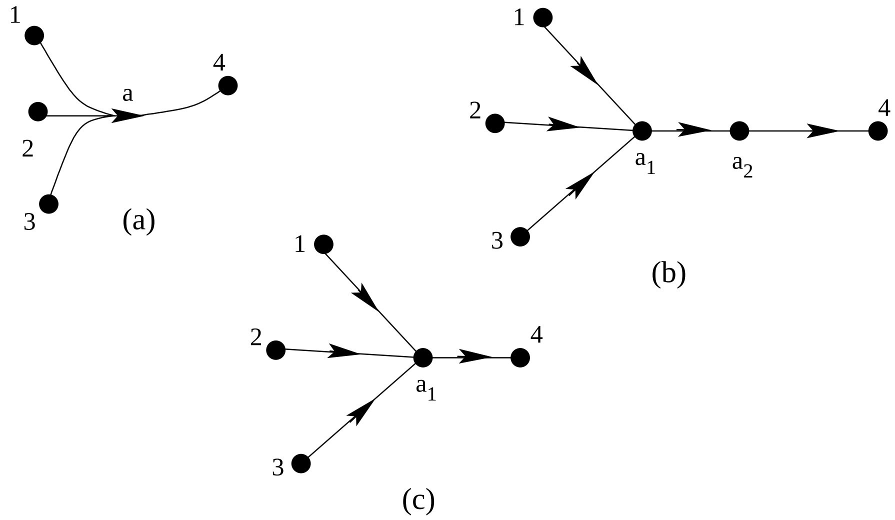

Fig. 6. The transformation of an hyperarc with one vertex at destination

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