UNIVERSIDADE FEDERAL DO PARANÁ

# Dirac Structures and their Homotopy Classification

Aline ZANARDINI

7 de Setembro de 2016

UNIVERSIDADE FEDERAL DO PARANÁ

DISSERTAÇÃO DE MESTRADO

# Dirac Structures and their Homotopy Classification

*Autor:*
Aline ZANARDINI

*Orientador:*
Llohann SPERANÇA
*Coorientador:*
Alexei KOTOV

*Dissertação apresentada como requisito parcial para a obtenção do grau de Mestre em Matemática*

*ao*

Programa de Pós-Graduação em Matemática
Departamento de Matemática

7 de Setembro de 2016

Zanardini, Aline
Dirac Structures and their homotopy classification / Aline Zanardini. – Curitiba, 2016.
76 f. : il.

Dissertação (mestrado) – Universidade Federal do Paraná, Setor de Ciências Exatas, Programa de Pós-Graduação em Matemática.
Orientador: Llohann Sperança
Coorientador: Alexei Kotov
Bibliografia: p.75-78

1. Dirac, Equações de. 2. Teoria da homotopia. 3. Álgebra. I. Sperança, Llohann. II. Kotov, Alexei. III. Título

CDD 530.124

MINISTÉRIO DA EDUCAÇÃO
UNIVERSIDADE FEDERAL DO PARANÁ
PRÓ-REITORIA DE PESQUISA E PÓS-GRADUAÇÃO
Setor CIÊNCIAS EXATAS
Programa de Pós Graduação em MATEMÁTICA
Código CAPES: 40001016041P1

## TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em MATEMÁTICA da Universidade Federal do Paraná foram convocados para realizar a arguição da Dissertação de Mestrado de **ALINE ZANARDINI**, intitulada: "**DIRAC STRUCTURES AND THEIR HOMOTOPY CLASSIFICATION**", após terem inquirido a aluna e realizado a avaliação do trabalho, são de parecer pela sua APROVAÇÃO.

Curitiba, 28 de Julho de 2016.

Prof OLEKSH KOTOV
Presidente da Banca Examinadora (UFPR)

Prof HENRIQUE BURSZTYN
Avaliador Externo (IMPA)

Prof MARCO ZAMBON
Avaliador Externo (KU LEUVE)

*Aos meus pais e a Paula.*

# *Agradecimentos*

Aos meus pais pelo amor e apoio incondicionais e por sempre terem me permitido seguir.

Àquela que torna meus dias mais leves. Paula, simplesmente OBRIGADA.

Aos meu orientadores por toda a dedicação dispensada. Em particular, *I am deeply thankful for Alexei's infinite patience in helping me throughout this work. I am also grateful for all the experience and knowledge he shared*.

Aos bons professores que tive ao longo de minha vida acadêmica e que me serviram de inspiração. De forma especial, aos professores Llohann, Durán e Eduardo por terem acreditado em mim.

Ao programa de pós-graduação em Matemática da UFPR pela oportunidade dada. Sobretudo aos professores Elizabeth e Jurandir e também a Cinthia pela eficiência e enorme prestatividade.

À CAPES pela bolsa de mestrado concedida através do PICME.

E, finalmente, aos membros da banca por terem aceitado o convite.

*"Por vezes sentimos que aquilo que fazemos não é senão uma gota de água no mar. Mas o mar seria menor se lhe faltasse uma gota. "*

Madre Teresa de Calcutá

UNIVERSIDADE FEDERAL DO PARANÁ

# *Abstract*

Departamento de Matemática

Mestre em Matemática

**Dirac Structures and their Homotopy Classification**

por Aline ZANARDINI

In the present thesis we are interested in studying Dirac structures. Our main goal is to classify such structures under homotopy. For that, we start by presenting the basics on the generalized geometry formalism. First, we introduce the linear counterpart and then we move on to manifolds, where we develop the most important concepts and constructions concerning Dirac structures. At last, we consider a generalization of the Chern-Weil map and we use an adequate algebraic model for equivariant cohomology in order to construct explicit secondary characteristic classes and thus obtain the homotopical invariants.

UNIVERSIDADE FEDERAL DO PARANÁ

# *Resumo*

Departamento de Matemática

Mestre em Matemática

**Dirac Structures and their Homotopy Classification**

por Aline ZANARDINI

Nesta dissertação estamos interessados em estudar estruturas de Dirac. Nosso objetivo maior é classificá-las por homotopia. Para tal, começamos apresentando o formalismo básico da geometria generalizada. Primeiro introduzimos a contraparte linear e então passamos para variedades, onde desenvolvemos os principais conceitos e construções que envolvem a noção de estrutura de Dirac. Por fim, consideramos uma generalização da construção de Chern-Weil e utilizamos um modelo algébrico adequado para a cohomologia equivariante para construir certas classes características secundárias obtendo assim os invariantes homotópicos desejados.

# Contents

# Introduction

The key ingredients for defining Dirac structures are found within the context of generalized geometry. The basic idea lie in replacing the tangent bundle $TM$ of a manifold $M$ by the Whitney sum $TM \oplus T^*M$ and the Lie bracket of vector fields by the so called Courant bracket. Moreover, on $TM \oplus T^*M$ one considers the natural symmetric bilinear form given by pairing $1-$forms and vector fields. The notion of a Dirac structure is then formalized as a maximally isotropic subbundle which is involutive with respect to the Courant bracket.

The main goal of the present work is to answer the following question

**How can one classify Dirac structures up to homotopy?**

The answer for it can be surprisingly found in the equivariant cohomology and it lies in the observation the homotopy class of a Dirac structure (more generally a maximally isotropic subbundle) only depends on the homotopy class of the corresponding section of a particular bundle. Such construction originally appeared in [1]. The merit here, however, is to exhibit the computations explicitly using a generalization of the Chern-Weil map suggested in [2] and the BRST model for equivariant cohomology [3].

The thesis is organized as follows.

In Chapter 1, we briefly describe some elementary facts about the theory of non-degenerate symmetric bilinear forms on vector spaces of finite dimension. A special attention is given to the theory of the so called split symmetric bilinear forms. Also, a contained discussion about Clifford algebras is given. The main references for this part are [4] and [5].

In Chapter 2, we present the basics about generalized geometry with some emphasis on Dirac structures. The main references here are [1],[6], and [7].

At last, in Chapter 3, we achieve the proposed goal. We obtain explicit secondary characteristic forms, the cohomology classes of which gives us a classification of Dirac structures up to homotopy. The main references are [2],[3] and [8].

It is worth to say the adopted framework is the $C^\infty$-category, that is, all manifolds, maps, vector bundles, etc we consider are assumed to be smooth.

As for the reader it is assumed basic knowledge on differential geometry and the text is intended to be an introductory reading for those who are interested in generalized geometry, Dirac structures and close topics. Hopefully it will be helpful to other students.

Finally, is valid mentioning the present work was vinculated to the National Scientific Initiation and Master's Program (PICME) and had the financial support from the Committee for the Betterment of Graduate Students (CAPES).

# Chapter 1

# Generalized Linear Algebra

*We briefly describe some elementary facts about the theory of non-degenerate symmetric bilinear forms on vector spaces of finite dimension. A special attention is given to the theory of the so called split symmetric bilinear forms. In particular, we introduce the canonical form on $E \oplus E^*$, where $E$ is a finite-dimensional (real) vector space and $E^*$ is its dual space. Also, a contained discussion about Clifford algebras is given. Throughout, $\mathbb{K}$ denotes a field of characteristic not two and $V$ is a $\mathbb{K}$ vector space of finite dimension.*

## 1.1 Symmetric Bilinear Forms

Given any bilinear form $B : V \times V \to \mathbb{K}$, consider the linear map $\varphi_B : V \to V^*$ defined by $u \mapsto B(u, \cdot)$. Recall a bilinear form $B$ on $V$ is said to be **symmetric** if $B(u, v) = B(v, u)$ for every $u, v \in V$. Since $V$ is of finite dimension this condition is equivalent to the map $\varphi_B$ being self-adjoint.

We define the **kernel** of $B$, and denote it by $\ker(B)$, as the kernel of the linear map $\varphi_B$, that is,

$$ker(B) \doteq \{u \in V | B(u, v) = 0 \quad \forall v \in V\}$$

The bilinear form $B$ is called **non-degenerate** if it has a trivial kernel. Thus, $B$ is non-degenerate if and only if $\varphi_B$ is an isomorphism. Analogously, if we consider the linear map $\psi_B : V \to V^*$ defined by $v \mapsto B(\cdot, v)$, then $\psi_B$ is an isomorphism precisely when $B$ is non-degenerate. This follows from the fact that $V^{**} \simeq V$ and $(\varphi_B)^* = \psi_B$. Notice that if in addition $B$ is symmetric, then $\varphi_B = \psi_B$. For the rest of this section let us assume $B$ is a non-degenerate symmetric bilinear form on $V$.

It is known that every bilinear form on $V$ is of the type $B(u, Av)$ for some (unique) endomorphism $A : V \to V$. In fact, the map $\Phi_A : V \times V \to \mathbb{K}$ given by $(u, v) \mapsto B(u, Av)$ is bilinear hence, $\Phi : A \mapsto \Phi_A$ is a map from $End(V)$ to the space $\mathcal{B}(V)$ of bilinear forms on $V$. This map is clearly linear. Moreover, since $B$ is non-degenerate, $\psi_B$ is injective and therefore so is $\Phi$. Since $End(V)$ and $\mathcal{B}(V)$ have the same dimension, we conclude the map $\Phi$ is an isomorphism.

In particular, if $\langle \cdot, \cdot \rangle$ is any (positive) inner product on $V$, then there exists a unique self-adjoint operator $A \in End(V)$ such that $B(u, v) = \langle Au, v \rangle, \forall u, v \in V$.

### 1.1.1 Isotropic Subspaces

Due to the polarization identity, we know that $B$ is uniquely determined by the associated quadratic form $Q_B(v) \doteq B(v,v)$. More precisely:

$$B(u,v) = \frac{1}{2}\Big(Q_B(u+v) - Q_B(u) - Q_B(v)\Big), \qquad \forall u, v \in V \tag{1.1}$$

**Definition 1.1.1** *A vector $v \in V$ is said to be **isotropic** if $B(v,v) = 0$ and **non-isotropic** if $B(v,v) \neq 0$.*

For a subspace $W \subset V$ we define its **orthogonal** subspace, and denote it by $W^\perp$, as the set

$$W^\perp \doteq \{v \in V | B(v,w) = 0 \quad \forall w \in W\}$$

Then, $\varphi_B(W^\perp) = W^\circ$, where $W^\circ$ is the annihilator of $W$. Since dim$(W)$+ dim$(W^\circ) = n$, it follows that

$$\text{dim}(W) + \text{dim}(W^\perp) = \text{dim}(V) \tag{1.2}$$

Let us call a basis of $\{e_1, \ldots, e_n\}$ of $V$ an **orthogonal basis** if $B(e_i, e_j) = 0, \forall i \neq j$. Since the restriction of $B$ to any subspace $W$ of $V$ has kernel $W \cap W^\perp$ one can easily show that $V$ admits an orthogonal basis. Furthermore, if $\mathbb{K} = \mathbb{R}$, then one can consider $B(e_i, e_j) = \pm 1$ depending on the signature of $B$.

The **orthogonal group** $O(V; B)$ or simply $O(V)$ is the group

$$O(V) \doteq \{T \in GL(V) | B(u,v) = B(Tu, Tv) \quad \forall u, v \in V\}$$

and it has a canonical structure of a Lie group. Note that if $N$ denotes the matrix of $B$ with respect to some basis of $V$, then $O(V)$ can be identified with the group of all $n \times n$ invertible matrices $M$ with coefficients in $\mathbb{K}$ such that $M^t N M = N$, where $^t$ denotes transposition.

If $\mathbb{K} = \mathbb{R}$ and the signature of $B$ is $(p,q), p+q = n$, then $O(V)$ is denoted by $O(p,q)$ and, up to isomorphism, it can be identified with the Lie group of real $n \times n$ matrices $M$ which satisfies

$$M^t I_{p,q} M = I_{p,q}, \quad \text{where} \quad I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

(and $I_r$ denotes the identity matrix of order $r$). Its Lie algebra, denoted by $\mathfrak{o}(p,q)$, consists of the set of $n \times n$ real matrices $X$ such that $X^t I_{p,q} = -I_{p,q} X$.

Note that in the particular case where $n$ is even and $p = q$, then $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ belongs to $O(p,p)$ if and only if

$$\begin{aligned} M_{11}M_{11}^t - M_{12}M_{12}^t &= I_p \\ M_{21}M_{21}^t - M_{22}M_{22}^t &= -I_p \end{aligned}$$

In this case, by setting

$$\begin{aligned} H_{11}(s) &= [(I_p + s^2 M_{12}M_{12}^t)(I_p + M_{12}M_{12}^t)^{-1}]^{1/2} M_{11} \\ H_{12}(s) &= sM_{12}, \quad H_{21}(s) = sM_{21} \\ H_{22}(s) &= [(I_p + s^2 M_{21}M_{21}^t)(I_p + M_{21}M_{21}^t)^{-1}]^{1/2} M_{22} \end{aligned}$$

we have that

$$O(p,p) \ni H(s) = \begin{pmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{pmatrix}, \quad \forall s \in [0,1],$$

$H(1) = M$ and $H(0)$ has only diagonal blocks which are elements of $O(p)$. As a consequence, we conclude that $O(p,p)$ can be contracted to its subgroup $O(p) \times O(p)$.

**Definition 1.1.2** *A subspace $W \subset V$ is called **isotropic** if $B\big|_{W\times W} \equiv 0$, that is, if $W \subset W^\perp$.*

The polarization identity (1.1) shows that a subspace $W \subset V$ is isotropic if and only if all of its vectors are isotropic. Moreover, from (1.2) it follows that if W is isotropic, then $2\dim(W) \leqslant \dim(V)$.

**Proposition 1.1.1** *Given any isotropic subspace $W \subset V$, there exists another isotropic subspace $W'$ such that $V = W^\perp \oplus W'$.*

*Proof.* Let $U$ be any complement of $W^\perp$. Then $0 = (U \oplus W^\perp)^\perp = U^\perp \cap W$. This implies that the map $\varphi_B\big|_W : W \to U^*$ defined by $w \mapsto B(w,\cdot)\big|_U$ is an isomorphism. Thus, if we consider the composition $\varphi : U \to W$ given by

$$u \mapsto B(u,\cdot)\big|_U \mapsto (\varphi_B\big|_W)^{-1}\left(B(u,\cdot)\big|_U\right)$$

we have that $B(\varphi x, y) = B(x,y), \forall x,y \in U$ and hence $B\left(x - \frac{1}{2}\varphi x, y - \frac{1}{2}\varphi y\right) = 0$ for every $x,y \in U$. Therefore, the subspace $W' \doteq \left\{u - \frac{1}{2}\varphi u | u \in U\right\}$ is isotropic. Now, $W'$ is a complement to $W^\perp$ because it is the graph of a map $U \to W \subset W^\perp$ and $U$ is a complement to $W^\perp$.

□

**Definition 1.1.3** *An isotropic subspace is called **maximal isotropic** if and only if it is not properly contained in any other isotropic subspace.*

**Proposition 1.1.2** *An isotropic subspace $W \subset V$ is maximal isotropic if and only if it contains all $v \in W^\perp$ such that $v$ is isotropic.*

*Proof.* Let $W \subset V$ be an isotropic subspace. Define $\tilde{W} \doteq \{w \in W^\perp | B(w,w) = 0\}$. It is clear that $W \subset \tilde{W}$.

($\Rightarrow$) Let us assume that there exists $w \in W^\perp \backslash W$ isotropic, that is, $W \subsetneq \tilde{W}$. Then $W$ is not maximal isotropic. Indeed, the subspace $W' \doteq W \oplus \operatorname{span}\{w\}$ is an isotropic subspace which properly contains $W$.

($\Leftarrow$) Conversely, suppose $W$ is not maximal isotropic. Then there exists $U$ an isotropic subspace such that $W \subsetneq U$ and therefore $U^\perp \subset W^\perp$. The later inclusion implies that $U \subset \tilde{W}$, since $U \subset U^\perp$. In particular, $W$ is properly contained in $\tilde{W}$ and hence $\tilde{W} \not\subset W$.

□

**Proposition 1.1.3** *Let $W, W' \subset V$ be maximal isotropic subspaces. Then dim$(W)$ =dim$(W')$.*

*Proof.* Let $U \doteq (W + W')/(W \cap W')$ and denote by $B_U$ the induced bilinear form and by $\pi : W + W' \to U$ the quotient map. Since $W$ and $W'$ are both maximal isotropic the previous proposition implies that $W^\perp \cap W' = W \cap W'$ and $W \cap (W')^\perp = W \cap W'$. Thus, from

$$\begin{aligned}(W + W') \cap (W + W')^\perp &= (W + W') \cap W^\perp \cap (W')^\perp \\ &= (W + (W' \cap W^\perp)) \cap (W')^\perp \\ &= (W \cap (W')^\perp) + (W' \cap W^\perp)\end{aligned}$$

it follows that the kernel of the restriction of $B$ to $W+W'$ is precisely $W \cap W'$ and therefore $B_U$ is non-degenerate. Moreover, $\pi(W)$ and $\pi(W')$ are clearly isotropic and a similar computation as above implies that $U = \pi(W) \oplus (\pi(W'))^\perp$.

Therefore, dim$V$ = dim$(\pi(W))$+dim$((\pi(W'))^\perp)$ = dim$(\pi(W))$+dim$(V)$−dim$(\pi(W'))$ and hence $\pi(W)$ and $\pi(W')$ have the same dimension. It follows that

$$\dim(W) = \dim(\pi(W)) + \dim(W \cap W') = \dim(\pi(W')) + \dim(W \cap W') = \dim(W')$$

□

**Definition 1.1.4** *The **Witt index** of the non-degenerate symmetric bilinear form $B$ is the dimension of a maximal isotropic subspace of $V$.*

**Proposition 1.1.4** *The Witt index of $B$ coincides with the maximum of the dimension of an isotropic subspace of $V$.*

*Proof.* If $d$ is the maximum possible dimension of an isotropic subspace of $V$, let $U$ be an isotropic subspace of dimension $d$.

It suffices to show that any isotropic subspace $W$ of dimension less than $d$ cannot be maximal isotropic. Since $W^\perp + U \subset (W \cap U)^\perp$ it follows that

$$\begin{aligned}\dim(W^\perp \cap U) &= \dim(W^\perp) + \dim(U) - \dim(W^\perp + U) \\ &\geqslant \dim(V) - \dim(W) + \dim(U) - \dim(W \cap U)^\perp \\ &= \dim(W \cap U) + \dim(U) - \dim(W) \\ &> \dim(W \cap U)\end{aligned}$$

Thus, there exists a nonzero isotropic vector $v \in (W^\perp \cap U)$ with $v \notin W$. The subspace $W \oplus \text{span}\{v\}$ is an isotropic subspace which properly contains $W$, so $W$ is not maximal.

□

**Corollary 1.1.1** *Any isotropic subspace $W \subset V$ is contained in a maximal isotropic subspace $L \subset V$ of dimension equals to the Witt index of $B$.*

**Proposition 1.1.5** *If $\mathbb{K} = \mathbb{R}$ and $B$ has signature $(p, q)$, then its Witt index equals $min\{p, q\}$.*

*Proof.* Without loss of generality, let us assume $p \leqslant q$. Then there exist vectors $u_1, \ldots, u_p$, $v_1, \ldots, v_p \in V$ such that $B(u_i, v_j) = 0$, $B(u_i, u_j) = \delta_{ij}$ and $B(v_i, v_j) = -\delta_{ij}$ for every $1 \leqslant i, j \leqslant p$. Next, consider $W \doteq span\{u_i + v_i | i = 1, \ldots, p\}$. Then $W$ is an isotropic subspace of dimension $p$. Now, let us assume there exists $U \subset V$ an isotropic subspace of dimension $d > p$. Since $p + q =$dim$(V)$, it follows that $U$ intersects any negative subspace of dimension $q$ (which exists), a contradiction.

□

## 1.2 Split Symmetric Bilinear Forms

In this section we shall assume $V$ is a real vector space of even dimension $2m$.

**Definition 1.2.1** *The non-degenerate symmetric bilinear form $B$ on $V$ is called **split** if its Witt index is equal to half of the dimension of $V$. In this case, maximal isotropic subspaces $L \subset V$ are characterized by the property $L = L^{\perp}$.*

Note that the existence of a maximal isotropic of dimension $m$ on $V$ is equivalent to $B$ having signature $(m, m)$. Throughout this section, let us consider that $B$ has Witt index equals $m$.

**Proposition 1.2.1** *Any maximal isotropic subspace $L \subset V$ admits a maximally isotropic complement $L'$. Moreover, any such complement is naturally isomorphic to $L^*$.*

*Proof.* Since $L = L^{\perp}$, from Proposition 1.1.1 we know that there exists $L' \subset V$ an isotropic subspace such that $L \oplus L' = V$. But, since $\dim(L) = m$ it follows that $L'$ is also an isotropic subspace of dimension $m$ and thus, maximal isotropic. Next, we define $\rho_{L,L'} : L' \to L^*$ given by $u \mapsto B(u, \cdot)\big|_L$ and observe that this linear map is an isomorphism. □

Given a maximal isotropic subspace $L \subset V$ we can exhibit a natural maximally isotropic complement as follows.

If $g : V \times V \to \mathbb{R}$ is any (positive) inner product on $V$, we can consider the unique symmetric operator $J : V \to V$ such that $B(u, v) = g(Ju, v)$ for every $u, v \in V$. Moreover, up to a different choice of the inner product, we can assume $J^2 = id$. Now, if $W \subset V$ is isotropic, then $J(W)$ is an isotropic subspace which is orthogonal to $W$, with respect to the inner product $g$. If $v \in J(W)$, then there exists $w \in W$ such that $v = J(w)$ and thus $g(v, u) = g(Jw, u) = B(w, u) = 0$ for every $u \in W$. Yet, since any two elements of $J(W)$ are of the form $Ju, Jv$ for some $u, v \in W$, it follows that $B(Ju, Jv) = g(JJu, Jv) = g(u, Jv) = 0$, that is, $J(W)$ is isotropic. We now observe that since $J$ is an isomorphism if $W$ is maximally isotropic, then $V = W \oplus J(W)$, with $J(W)$ also maximally isotropic .

Note that Proposition 1.2.1 tells us that given $L \subset V$ maximal isotropic if we choose a basis $l_1, \ldots, l_m$ of $L$ then we can complete it to a basis $l_1, \ldots, l_m, l^1, \ldots, l^m$ of $V$ such that

$B(l_i, l_j) = 0, B(l^i, l^j) = 0$ and $B(l_i, l^j) = \delta_{ij}$ for every $1 \leqslant i, j \leqslant m$. Clearly, the group $O(V)$ acts transitively on the set of such basis and therefore on the set of pairs of complementary maximally isotropic subspaces.

Also, as a consequence of Proposition 1.2.1, we conclude that $V$ is naturally isomorphic to $L \oplus L^*$, where $L \subset V$ is any maximal isotropic subspace. The bilinear form is given by the natural pairing:

$$B\Big((u, \xi), (v, \eta)\Big) = \langle \xi, v \rangle + \langle \eta, u \rangle = \xi(v) + \eta(u)$$

From now on let us fix a decomposition $V \simeq L \oplus L^*$, where $L \subset V$ is maximally isotropic, i.e., let us fix a pair $(L, L')$ of complementary maximally isotropic subspaces. Then, every orthogonal transformation $T \in O(V)$ can be written as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $T_{11} : L \to L, T_{12} : L^* \to L, T_{21} : L \to L^*$ and $T_{22} : L^* \to L^*$.

The following proposition implies that the subgroup of $O(V)$ which fixes $L$ pointwise consists of skew-adjoint maps $D : L^* \to L$ and thus can be identified with $\Lambda^2(L)$. Moreover, one can show that if $O(V)_L \subset O(V)$ is the subgroup that takes $L$ to itself, then $O(V)_L$ is an extension

$$1 \to \Lambda^2(L) \to O(V)_L \to GL(L) \to 1$$

For $\Lambda^2(L)$ can be embedded into $O(V)$ by considering the map $\omega \mapsto A_\omega$, where $A_\omega : v \mapsto v + \iota(\pi_{L^*}(v))\omega$ and $\pi_{L^*} : V \to L^*$ is the projection. Yet, if $T \in O(V)_L \subset O(V)$, then $T_{11} = T\big|_L \in GL(L)$. Moreover, such restriction equals identity if and only if $T$ fixes $L$ pointwise. To see that $O(V)_L \to GL(L)$ is surjective, let $A \in GL(L)$ and consider $\bar{A} \doteq A \oplus (A^{-1})^*$. Then $\bar{A} \in O(V)$ and $\bar{A}\big|_L = A$.

In particular, if $\bar{M}$ is the set of all pairs of complementary maximal isotropic subspaces, since $O(V)$ acts transitively on $\bar{M}$ it follows that $\bar{M} \simeq O(V)/GL(L)$. That is, the isotropy subgroup of $(L, L')$ under the action of $O(V)$ is isomorphic to $GL(L)$.

**Proposition 1.2.2** *Any other maximally isotropic complement to $L$ is in one-to-one correspondence with a skew-adjoint map $D : L^* \to L$.*

*Proof.* In fact, the correspondence is determined by the graph of $D$. Since the group $O(V)$ acts transitively on $\bar{M}$, it follows that maximally isotropic complements to $L$ are graphs of orthogonal maps which fixes $L$ pointwise.

Now, if $A \in GL(V)$ fixes $L$ pointwise, then $A(l, \xi) = (l + D(\xi), S(\xi))$ for some linear maps $D : L^* \to L$ and $S : L^* \to L^*$. If in addiction $A \in O(V)$, then

$$0 = B\Big(A(0, \xi), A(l, 0)\Big) - B\Big((0, \xi), (l, 0)\Big) = \langle S\xi - \xi, l \rangle \quad \forall l \in L, \forall \xi \in L^*$$

and hence $S = id$. Furthermore,

$$0 = B\Big(A(0, \xi), A(0, \eta)\Big) - B\Big((0, \xi), (0, \eta)\Big) = \langle \xi, D\eta \rangle + \langle \eta, D\xi \rangle, \quad \forall \xi, \eta \in L^*$$

and therefore $D = -D^*$.

Conversely, any map of the form $(l, \xi) \mapsto (l + D\xi, \xi)$, with $D : L^* \to L$ skew-adjoint, is orthogonal:

$$\begin{aligned} B\Big((l + D(\xi), \xi), (m + D(\eta), \eta)\Big) &= \langle \xi, m + D\eta \rangle + \langle \eta, l + D\xi \rangle \\ &= \langle \xi, m \rangle + \langle \xi, D\eta \rangle + \langle \eta, l \rangle + \langle \eta, D\xi \rangle \\ &= \langle \xi, m \rangle + \langle \eta, l \rangle \\ &= B\Big((l, \xi), (m, \eta)\Big) \quad \forall (l, \xi), (m, \eta) \in L \oplus L^* \end{aligned}$$

Thus, if we define $\tilde{L} \doteq \{l + \xi \in L \oplus L^* | l = D(\xi)\}$, where $D : L^* \to L$ is skew-adjoint, then $\tilde{L}$ is a maximally isotropic complement to $L$ and any maximally isotropic complement to $L$ can be obtained in this way. □

We now observe that Proposition 1.2.2 above implies that the set of maximal isotropic complementary subspaces to $L$ carries a canonical affine structure. It is an affine space whose associated vector space is isomorphic to $\Lambda^2(L)$. Moreover, if we let $M$ denote the set of all maximal isotropic subspaces of $V$, $\mathscr{L} \to M$ be the canonical vector bundle and $\pi : \bar{M} \to M$ denote the projection to the first component, then $\pi : \bar{M} \to M$ is an affine bundle modelled on the vector bundle $\Lambda^2(\mathscr{L})$.

We end this section by showing that $M$ is diffeomorphic to $O(m)$.

Recall we have fixed a pair $(L, L')$ of complementary maximally isotropic subspaces. Furthermore, we know that $L' \simeq L^*$. Thus, let $g : L \times L \to \mathbb{R}$ be the inner product on $L$ induced by the latter isomorphism and let us consider the dual inner product $g^*(\xi, \eta) \doteq g(R^{-1}\xi, R^{-1}\eta)$, $\forall \xi, \eta \in L^*$. Then, the map $R : L \to L^*$ given by $l \mapsto g(l, \cdot)$ is canonically an isometry and we can identify the image of $L$ through $R$ with $L$ itself. Via this identification, lets us define the sets $V_+ \doteq \{(l, l); l \in L\}$ and $V_- \doteq \{(l, -l); l \in L\}$, i.e., $V_\pm = \text{graph}(\pm R)$. The following result holds:

**Proposition 1.2.3** *The restriction of $B$ to $V_+$(resp. $V_-$) is positive (resp. negative).*

*Proof.* We begin by first noticing that under the identification $RL \simeq L$ we have that $V = V_+ \oplus V_-$. Clearly $V_+ \cap V_- = \{0\}$ so, let $(l, m) \in V \simeq L \oplus L$. Taking $l' = \frac{1}{2}\Big(l + m\Big)$ and $m' = \frac{1}{2}\Big(l - m\Big)$ it follows that $(l, m) = (l', l') + (m', -m')$ and therefore, $V = V_+ + V_-$.

Next, if $(l, l)$ and $(m, m)$ are elements in $V_+$, then

$$B\Big((l, l), (m, m)\Big) = g(l, m) + g(m, l) = 2g(l, m)$$

and since $g$ is positive, it follows that $B\big|_{V_+}$ is also positive. Analogously, if we take $(l, -l), (m, -m) \in V_-$, then

$$B\Big((l, -l), (m, -m)\Big) = -2g(l, m)$$

and we conclude that $B\big|_{V_-}$ is negative. □

**Corollary 1.2.1** *The map $R : L \to L^*$ determines a (positive) inner product on $V$ given by $B\big|_{V_+} - B\big|_{V_-}$.*

Finally, we show that maximally isotropic subspaces of $V$ are in one-to-one correspondence with graphs of maps $V_+ \to V_-$ determined by isometries $L \to L$.

**Proposition 1.2.4** *The set $M$, of maximal isotropic subspaces of $V$, is diffeomorphic to $O(m)$.*

*Proof.* We begin by observing that every element of $M$ is transversal to both $V_+$ and $V_-$ and hence it can be seen as a graph of a map $V_+ \to V_-$. Next, since $V_+ \simeq L$ and $V_- \simeq L$, given any map $\varphi : V_+ \to V_-$ it induces a map $A : L \to L$. Now, the graph of $\varphi$, which we will denote by $L''$, is obviously a linear complement to $L$. We will show that $L''$ is isotropic, thus maximally isotropic, if and only if $A$ belongs to $O(m)$, proving the statement. Indeed, elements of $L''$ can be written as $(l + Al, l - Al)$, where $l \in L$. Thus,

$$\begin{aligned} B\Big((l + Al, l - Al), (l + Al, l - Al)\Big) = 0 \quad &\iff \quad g(l - Al, l + Al) + g(l + Al, l - Al) = 0 \\ &\iff \quad 2g(l - Al, l + Al) = 0 \\ &\iff \quad 2(g(l, l) - g(Al, Al)) = 0 \\ &\iff \quad g(Al, Al) = g(l, l) \end{aligned}$$

□

Note that once we consider the inner product $g$ on $L$, if we denote by $L^-$ the subspace $L$ together with the opposite bilinear form $-g$, then we have a natural isometry from $L \oplus L^-$, with the split bilinear form $g \oplus (-g)$, into $L \oplus L^*$. Such an isometry, $\Phi$, is given by

$$(l, m) \mapsto \left(l + m, \frac{l - m}{2}\right),$$

Furthermore, if we define, for every $A \in O(L)$, the set $L_A \doteq \{\Phi(l, Al) | l \in L\}$, then $L_A$ is a maximal isotropic subspace of $V$ and we get a map $O(L) \to M$.

## 1.3 Clifford Algebras

In the present section we will follow the notation used earlier in Section 2.2. That is, $V$ denotes a finite dimensional vector space over some field $\mathbb{K}$ (of characteristic not two) and $B$ is a symmetric bilinear form on $V$[1] .

**Definition 1.3.1** *The **Clifford algebra** of $V$, denoted by $Cl(V; B)$ or simply $Cl(V)$, is defined as the quotient of the tensor algebra of $V$ by the two-sided ideal generated by $v \otimes v - B(v, v)1$ for all $v \in V$.*

[1]We may drop the assumption of non-degeneracy.

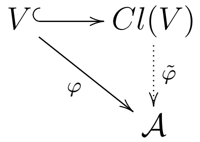
Note that we have a natural map $V \hookrightarrow Cl(V)$.

**Example 1.3.1** If $B \equiv 0$, then $Cl(V) = \Lambda^\bullet V$.

△

We now show that the Clifford algebra of $V$ is characterized by the following universal property:

**Proposition 1.3.1** (Universal Property) *Let $\mathcal{A}$ be a unital associative algebra with a linear map $\varphi : V \to \mathcal{A}$ such that $\varphi(v)^2 - B(v,v)1_{\mathcal{A}} = 0$ for every $v \in V$. Then, there exists a unique homomorphism of algebras $\tilde{\varphi} : Cl(V) \to \mathcal{A}$ making the diagram below commute*




(1.3)

*Proof.* From the universal property of the tensor algebra, $\mathcal{T}(V)$, we know that $\varphi$ extends to an algebra homomorphism $\bar{\varphi} : \mathcal{T}(V) \to \mathcal{A}$. Since $\varphi(v)^2 - B(v,v)1_{\mathcal{A}} = 0$ for every $v \in V$, it follows that $\bar{\varphi}$ vanishes on the ideal $\mathcal{I}(V) \doteq \langle v \otimes v - B(v,v)1 \rangle$. Thus, $\bar{\varphi}$ descends to the quotient. □

Moreover, if $\mathcal{C}$ is any unital associative algebra satisfying (1.3), then $Cl(V) \simeq \mathcal{C}$. That is, $Cl(V)$ is unique up to a unique isomorphism. Indeed, if we denote the embedding $V \hookrightarrow \mathcal{C}$ by $i$, then the isomorphism $V \to i(V)$ induces an algebra isomorphism $Cl(V) \to \mathcal{C}$.

Next, we observe that if we apply the defining relation for the Clifford algebra to a sum $u + v$ of two vectors in $V$ we obtain

$$u \otimes v + v \otimes u = 2B(u,v)1 \quad \forall u, v \in V \tag{1.4}$$

In particular, $Cl(V)$ is the unital associative algebra generated by $V$ and the relations (1.4) above.

**Example 1.3.2** If $V = \mathbb{R}^n$ with the bilinear form given by the opposite of the standard inner product [2], then we denote its Clifford algebra by $C(n)$.

Choosing $\{e_1, \ldots, e_n\}$ an orthonormal basis, with respect to the usual inner product, we see that $C(n)$ is the algebra generated by the $e_i's$ with relations

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

One easily checks that $C(1)$ is the two dimensional algebra over $\mathbb{R}$ generated by $\{1, e_1\}$ with relation ${e_1}^2 = -1$, that is, just the complex numbers, so $C(1) \simeq \mathbb{C}$.

[2] The appearance of this sign could be prevented by considering $v^2 = -B(v,v)1$ as the defining relation for the Clifford algebra. Both conventions can be found in the literature.

For $n = 2$, we have a four dimensional algebra over $\mathbb{R}$ generated by $\{1, e_1, e_2\}$ with relations

$$e_1^2 = e_2^2 = -1 \quad \text{and} \quad e_1 e_2 = -e_2 e_1$$

This is just the quaternions $\mathbb{H}$ under the identification $i = e_1$, $j = e_2$, $k = e_1 e_2$. Thus, $C(2) \simeq \mathbb{H}$.

△

**Corollary 1.3.1** (of Proposition 1.3.1) *Let $\mathbb{K} = \mathbb{R}$ and $h > 0$. Then $h \cdot B$ defines a new symmetric bilinear form on $V$. Let us denote by $Cl(V, h)$ the algebra generated by $V$ an the relations*

$$u \otimes v + v \otimes u = 2h \cdot B(u, v)1 \quad \forall u, v \in V$$

*Then $Cl(V, h) \simeq Cl(V)$.*

**Remark 1.3.1** If $h$ goes to zero, then we obtain the exterior algebra of $V$ so that, intuitively, we can think of $Cl(V)$ as a "deformation" of $\Lambda^\bullet V$. In fact, later on we will see how the exterior product can be deformed into the Clifford product.

◊

Note also, that as a consequence of Proposition 1.3.1, we can restate Corollary 1.3.1 in more general terms:

**Corollary 1.3.2** *If $B_1$ and $B_2$ are symmetric bilinear forms on the $\mathbb{K}$-vectors spaces $V_1$ and $V_2$, respectively, and $\varphi : V_1 \to V_2$ is a linear map such that*

$$B_2\Big(\varphi(u), \varphi(v)\Big) = B_1(u, v), \quad \forall u, v \in V_1$$

*Viewing $\varphi$ as a map into $Cl(V_2; B_2)$, the universal property provides a (unique) homomorphism of algebras $Cl(V_1; B_1) \to Cl(V_2; B_2)$.*

**Example 1.3.3** Suppose again that $(V_1, B_1)$ and $(V_2, B_2)$ are two vector spaces with symmetric bilinear forms. Then

$$Cl(V_1 \oplus V_2; B_1 \oplus B_2) \simeq Cl(V_1; B_1) \otimes Cl(V_2; B_2)$$

For, let $W \doteq V_1 \oplus V_2$ and let $i_1 : V_1 \to W, i_2 : V_2 \to W$ denote the inclusion maps. Since these maps are injective isometries they induce homomorphisms of the Clifford algebras, which we still denote by $i_1$ and $i_2$. That is, $i_1 : Cl(V_1) \to Cl(W)$ and $i_2 : Cl(V_2) \to Cl(W)$.

Now, let $j \doteq i_1 \otimes i_2$ and observe that if $u \in V_1$ and $v \in V_2$, then $i_1(u)$ and $i_2(v)$ are orthogonal, and hence

$$i_2(v) i_1(u) + i_1(u) i_2(v) = 0$$

So $j : Cl(V_1) \otimes Cl(V_2) \to Cl(V_1 \oplus V_2)$ is a morphism of super-algebras.

In fact $i_1$ and $i_2$ are even maps and thus they preserve parity. Moreover, for $a, b \in Cl(V_1)$ and $a', b' \in Cl(V_2)$ we have that

$$i_2(a') i_1(b) = (-1)^{|a'||b|} i_1(b) i_2(a')$$

and hence

$$\begin{aligned} j(a \otimes a')j(b \otimes b') &= i_1(a)i_2(a')i_1(b)i_2(b') \\ &= (-1)^{|a'||b|} i_1(a)i_1(b)i_2(a')i_2(b') \\ &= (-1)^{|a'||b|} i_1(ab)i_2(a'b') \\ &= (-1)^{|a'||b|} j(ab \otimes a'b') \\ &= j\Big((-1)^{|a'||b|} ab \otimes a'b'\Big) \\ &= j(a \otimes a' \cdot b \otimes b') \end{aligned}$$

To see $j$ is an isomorphism we exhibit its inverse. Let $k : V_1 \oplus V_2 \to Cl(V_1) \otimes Cl(V_2)$ be the map given by $u \oplus v \mapsto u \otimes 1_{V_2} + 1_{V_1} \otimes v$, where $1_{V_i}$ denotes the identity element in $Cl(V_i)$, $i = 1, 2$. Then

$$\begin{aligned} k(u \oplus v)^2 &= \Big(B_1(u,u) + B_2(v,v)\Big) 1_{V_1} \otimes 1_{V_2} \\ &= B_1 \oplus B_2(u \oplus v, u \oplus v) 1_{Cl(V_1) \otimes Cl(V_2)} \end{aligned}$$

Thus, $k$ extends to an algebra homomorphism $Cl(V_1 \oplus V_2) \to Cl(V_1) \otimes Cl(V_2)$.

Finally, an easy computation shows that the two maps we have constructed are inverses of each other. Their composition in both orders is the identity at the level of vectors and so is the identity map.

Indeed, for $u \in V_1$ and $v \in V_2$ we have that

$$\begin{aligned} k \circ j(u \otimes v) &= k\Big(i_1(u)i_2(v)\Big) \\ &= k(u \oplus 0) \cdot k(0 \oplus v) \\ &= (u \otimes 1_{V_2}) \cdot (1_{V_1} \otimes v) \\ &= u \otimes v \end{aligned}$$

Analogously, one can check that $j \circ k(u \oplus v) = u \oplus v$.

△

In particular, Corollary 1.3.2 implies we have a group homomorphism

$$O(V; B) \to Aut(Cl(V; B)); \quad g \mapsto \tilde{g},$$

where $\tilde{g}$ is the (unique) extension given by the universal property (1.3).

One automorphism of special interest is the so called *parity automorphism* which is obtained from $p : v \mapsto -v$ for every $v \in V$. Since $p^2 = 1$, it follows that $Cl(V)$ splits as $Cl^0(V) \oplus Cl^1(V)$, where $Cl^0(V) = ker(\tilde{p} - 1)$ and $Cl^1(V) = ker(\tilde{p} + 1)$. Moreover, multiplication in $Cl(V)$ behaves like

$$Cl^i(V) \otimes Cl^j(V) \to Cl^{i+j}(V)$$

where the indexes are taken modulo 2. As a consequence, $Cl(V)$ is naturally $\mathbb{Z}_2-$graded.

**Remark 1.3.2** The ideal $\mathcal{I}(V) \doteq \langle v \otimes v - B(v,v)1 \rangle$ is not an homogeneous ideal, so the quotient $\mathcal{T}(V)/\mathcal{I}(V)$ (which is just the Clifford algebra $Cl(V)$) does not inherit the $\mathbb{Z}-$grading from the tensor algebra. For instance, if $v \in V$, then $v$ is of degree one however, $v^2 \in \mathbb{K}$ has degree zero (instead of degree two).

$\diamond$

We now recall that the tensor algebra of $V$ has a natural filtration. If $\{v_1, \ldots, v_n\}$ is a basis of $V$ we can define

$$\tilde{F}^p \doteq span\{v_{i_1} \otimes \ldots \otimes v_{i_l}; l \leqslant p\}$$

Then, $\tilde{F}^0 \subset \tilde{F}^1 \subset \ldots \subset \mathcal{T}(V)$ and $\tilde{F}^p \otimes \tilde{F}^q \subset \tilde{F}^{p+q}$.

Setting $F^p \doteq q_{Cl}(\tilde{F}^p)$, where $q_{Cl} : \mathcal{T}(V) \to Cl(V)$ is the quotient map, we have that

$$F^0 \subset F^1 \subset \ldots \subset Cl(V) \quad \text{and} \quad F^p \cdot F^q \subset F^{p+q}. \tag{1.5}$$

In short, $Cl(V)$ is a filtered algebra.

Next, we observe that $F^0 = \mathbb{K}$, $F^1 = \mathbb{K} \oplus V$ and, in general, $F^p = span\{v_{i_1} \ldots v_{i_l}; l \leqslant p\}$. Thus we have a natural map $V \to F^1/F^0$. Given $v \in V$ let us denote by $\hat{v}$ its image in $F^1/F^0$ through such a map.

**Proposition 1.3.2** *The associated graded algebra to the filtration (1.5) is the exterior algebra.*

*Proof.* Let us define $Gr^i \doteq F^i/F^{i-1}$, then the associated graded algebra is $Gr\big(Cl(V)\big) \doteq \bigoplus Gr^i$. Now, if we consider the natural map $f : V \to Gr^2$, then the image of $v^2$ by $f$ is zero. On the other hand, $f(v^2)$ is just $\hat{v}^2$, thus $Gr^1 \simeq \Lambda^1 V$.

Since $\Lambda^1 V$ generates the whole exterior algebra, it follows that the associated graded algebra of $Cl(V)$ is isomorphic to $\Lambda^\bullet V$. Moreover, $Gr^p \simeq \Lambda^p V$. $\square$

To summarize, $Gr\big(Cl(V)\big)$ comes with a linear map $V \to F^1/F^0$ and all elements in the image square to zero. Thus, the universal property of the exterior algebra gives an algebra homomorphism of $\Lambda^\bullet V$ into $Gr\big(Cl(V)\big)$ $(*)$.

Also, we observe that we can associate to each $v \in V$ two linear endomorphisms of $\Lambda^\bullet V$. Namely, the exterior product $\lambda_v : \Lambda^{p-1} V \to \Lambda^p V$ and the interior product $\iota_v : \Lambda^p \to \Lambda^{p-1}$ given by $v_{i_1} \wedge \ldots \wedge v_{i_p} \mapsto \sum (-1)^{j+1} B(v, v_{i_j}) v_{i_1} \wedge \ldots \wedge \widehat{v_{i_j}} \wedge \ldots \wedge v_{i_p}$. Both maps square to zero and $\lambda_v \circ \iota_v + \iota_v \circ \lambda_v = B(v,v) \cdot Id$. Thus, the map $\varphi : v \mapsto \lambda_v + \iota_v$ of $V$ into $End(\Lambda^\bullet V)$ satisfies $(\varphi(v))^2 = B(v,v)1$. By the universal property (Proposition 1.3), this gives an algebra homomorphism $\tilde{\varphi} : Cl(V) \to End(\Lambda^\bullet V)$.

Furthermore, the map $\sigma : v \mapsto (\tilde{\varphi}(v))(1)$, $1 \in \mathbb{K} = \Lambda^0 V$ , which is known in literature as the *symbol map*, induces an isomorphism of graded super-algebras $Gr\big(Cl(V)\big) \to \Lambda^\bullet V$.

In particular, Clifford multiplication between $v \in V$ and $a \in Cl(V)$ can be written as

$$v \cdot a = v \wedge a + \iota_v a$$

so that $[v, a] = 2\iota_v a = 2B(v, a)$ defines a super Lie algebra structure in the vector space $F^1 = \mathbb{K} \oplus V$.

**Remark 1.3.3** As a consequence, if $B$ is non-degenerate, then the super-center of $Cl(V;B)$ consists precisely of the scalars $\mathbb{K}$.

$\lozenge$

To conclude, we give another proof for Proposition 1.3.2 by means of the following Lemma:

**Lemma 1.3.1** *The symbol map is an isomorphism of filtered super vector spaces.*

*Proof.* Let $\{e_1, \ldots, e_n\}$ be an orthogonal basis of $V$ and $I = (i_1, \ldots, i_k)$ an ordered subset of $\{1, \ldots, n\}$, i.e., such that $1 \leqslant i_1 < \ldots < i_k \leqslant n$. Denoting by $e_I$ the product $e_{i_1} \ldots e_{i_k}$, as $I$ ranges over all ordered subsets, the $e_I$ form a basis of $Cl(V)$.

Now, if we compute $(\tilde{\varphi}(e_I))(1)$, then all the interior products vanish:

$$\begin{aligned} \tilde{\varphi}(e_I)(1) &= \tilde{\varphi}(e_{i_1}) \ldots \tilde{\varphi}(e_{i_k})(1) \\ &= \left(\lambda_{e_{i_1}} + \iota_{e_{i_1}}\right) \ldots \left(\lambda_{e_{i_k}} + \iota_{e_{i_k}}\right)(1) \\ &= \lambda_{e_{i_1}} \ldots \lambda_{e_{i_k}}(1) \\ &= e_{i_1} \wedge \ldots \wedge e_{i_k} \end{aligned}$$

Thus, $\sigma(e_{i_1} \ldots e_{i_k}) = e_{i_1} \wedge \ldots \wedge e_{i_k}$, for all possible multi-index $I = (i_1, \ldots, i_k)$, which form a basis of $\Lambda^\bullet V$.

In particular, if $v \in Gr^p$, then $\sigma(v) \in \Lambda^p V$.

$\square$

Therefore, the associated graded map is an isomorphism of graded super vector spaces. To see that it is compatible with the products we must show that for $x \in Gr^p$ and $y \in Gr^q$ we have that $\sigma(xy) - \sigma(x)\sigma(y) \in \Lambda^{p+q-1}V$. But this follows from the fact that

$$\sigma(v_{i_1} \ldots v_{i_k}) = v_{i_1} \wedge \ldots \wedge v_{i_k} \mod \Lambda^{k-1}V$$

i.e., $\sigma \circ q_{Cl} : \otimes^k V \to \Lambda^\bullet V$ coincides with $q_\wedge : \otimes^k V \to \Lambda^\bullet V$ up to lower order terms, where $q_{Cl} : \mathcal{T}(V) \to Cl(V)$ and $q_\wedge : \mathcal{T}(V) \to \Lambda^\bullet V$ are the quotient maps.

The inverse map of the symbol map, which we denote by $q$, is called the *quantization map* and the associated graded map coincides with the map in $(*)$.

### 1.3.1 The Spin Group

In the next paragraphs we will turn our attention to the Lie algebra $q(\Lambda^2 V)$. Our aim is to relate the quadratic elements of the Clifford algebra $Cl(V;B)$ to the Lie algebra $\mathfrak{so}(V;B)$.

We have seen that associated to each vector $v \in V$ we have a derivation of $Cl(V;B)$ given by super commutator: $a \mapsto [v, a]$. Furthermore, for generators this agrees with the contraction by $2B(v, \cdot)$.

By contrast, we know that if $a \in F^i$ and $b \in F^j$, then their super commutator is computed as follows
$$[a, b] \doteq ab - (-1)^{ij} ba \in F^{i+j-2}$$
and, particularly, $F^2$ is closed under taking the bracket $[\cdot, \cdot]$.

Now, the quantization map restricts to an $O(V)-$equivariant map $\Lambda^2 V \to F^2$, so that the elements $q(\alpha), \alpha \in \Lambda^2 V$ span a Lie subalgebra of $Cl(V; B)$.

Moreover, for each such $\alpha$ the map $A_\alpha : V \to V$ given by $v \mapsto [q(\alpha), v]$ defines an element $A_\alpha \in \mathfrak{so}(V; B)$ and we have a morphism of Lie algebras:

$$\Lambda^2 V \mapsto \mathfrak{so}(V; B), \quad \alpha \mapsto A_\alpha \tag{1.6}$$

The bracket in $\Lambda^2 V$ is given by $\{\alpha, \beta\} \doteq \mathcal{L}_{A_\alpha} \beta = [q(\alpha), \beta]$.

**Remark 1.3.4** If $B$ is non-degenerate, then the map $\alpha \mapsto A_\alpha$ is an isomorphism.

◇

**Example 1.3.4** We know that the $\mathfrak{so}(n)$ consists of $n \times n$ real antisymmetric matrices and a basis for these is given by
$$A_{ij} = E_{ij} - E_{ji} \qquad (i < j)$$
for $i < j$, where $E_{ij}$ is the matrix whose unique non zero entry is the $ij$. Such matrices satisfy the following commutation relations:

$$[A_{ij}, A_{kl}] = -\delta_{il} A_{kj} + \delta_{ik} A_{lj} - \delta_{jl} A_{ik} + \delta_{jk} A_{il} \tag{1.7}$$

Now, the generators $e_i$ of the Clifford algebra $C(n)$ are such that $e_i e_j + e_j e_i = -2\delta_{ij}$and we can use these relations in order to show that $\frac{1}{2} e_i e_j$ verify analogous commutation relations as above in (1.7)

In particular, the vector space spanned by the quadratic elements in $C(n)$ of the form $e_i e_j$, $i < j$, as a Lie algebra, is isomorphic to $\mathfrak{so}(n)$.

△

**Definition 1.3.2** *The* ***Spin group****, denoted by* $Spin(V) \subset Cl(V)^\times$*, is the group obtained by exponentiating the quadratic elements of the Clifford algebra.*

Equivalently, one can define $Spin(V)$ in terms of the invertible elements of $Cl(V)$ which are even and that leave $V$ invariant under conjugation. Our goal is to show that we have a homomorphism $Spin(V) \to SO(V)$ whose kernel is central with order 2.

From now on, let us assume that $B$ is non-degenerate. Since $V$ can be thought of as a subspace of $Cl(V)$, it makes sense to consider the so called **Clifford Group**

$$\Gamma(V) \doteq \{a \in Cl(V)^\times; p(a) V a^{-1} = V\},$$

where $p : Cl(V) \to Cl(V)$ denotes the parity automorphism. Thus, the Clifford group has a natural representation $\Phi : \Gamma(V) \to GL(V)$ given by $a \mapsto \Phi_a : v \mapsto p(a) v a^{-1}$. Such representation, which is known in literature as **twisted adjoint representation**, takes values in

$O(V)$ and defines an exact sequence

$$1 \to \mathbb{K}^\times \to \Gamma(V) \to O(V) \to 1$$

Moreover, the elements of $\Gamma(V)$ are all products

$$a = v_1 \dots v_k, \tag{1.8}$$

where all the $v_1, \dots, v_k$ are non-isotropic and the corresponding element in $O(V)$ is a product of reflections $R_{v_1} \dots R_{v_k}$. In particular, every element in the Clifford group has a definite parity.

In fact, if $v \in V$, then $p(v) = -v$. Furthermore, if $v$ is non-isotropic, then it is invertible in $Cl(V)$, with $v^{-1} = \dfrac{v}{B(v,v)}$. Hence, for all $w \in V$ we have that

$$\Phi_v(w) = -vwv^{-1} = w - 2\frac{B(v,w)}{B(v,v)}v$$

That is, $\Phi_v$ is the reflection through the hyperplane orthogonal to $v$, which we will denote by $R_v$. This proves that whenever $a$ is of the form (1.8), then $\Phi_a = R_{v_1} \dots R_{v_k}$.

Now, the *Cartan-Dieudonné* theorem says that any map $A \in O(V)$ is a product of reflections, $A = R_{v_1} \dots R_{v_k}$, which implies the map $v \mapsto \Phi_v$ has $O(V)$ as its image and $\Gamma(V)$ is generated by non-isotropic vectors in $V$.

Next, we observe that $Cl(V)$ inherits the canonical anti-automorphism from the tensor algebra. Namely, if $v = v_1 \otimes v_2 \otimes \dots \otimes v_k \in \mathcal{T}^k(V)$, then the map $v \mapsto v^T$, given by

$$v^T \doteq v_k \otimes \dots \otimes v_2 \otimes v_1$$

is an anti-automorphism of $\mathcal{T}(V)$ which preserves the ideal $\mathcal{I}(V)$. Hence, it induces a well defined anti-automorphism on $Cl(V)$ which we will also denote by $v \mapsto v^T$ and refer to as *transposition*.

Thus, since every $a \in \Gamma(V)$ can be written in the form (1.8), it follows that $a^T a \in \mathbb{K}^\times$. This defines the *norm homomorphism* $N : a \mapsto a^T a$, which is a group morphism and has the property that $N(ta) = t^2 N(a)$ for every $t \in \mathbb{K}^\times$ and $a \in \Gamma(V)$.

**Remark 1.3.5** Note that if $v \in V$, then $N(v) = B(v,v)$.

◊

**Remark 1.3.6** Note also, that for an even element $a \in \Gamma(V)$ we have that $\Phi_a(v) = ava^{-1}$. So, if we take the derivative, we obtain a derivation of the Clifford algebra given by

$$\phi_a(v) \doteq [a, v]$$

**Proposition 1.3.3** *The Spin group is the intersection of the kernel of the norm homomorphism and those elements in the Clifford group which are even.*

*Proof.* It is sufficient to show that the Lie algebra of such group is $\gamma(\mathfrak{so}(V))$, where $\gamma \doteq q \circ \lambda$, $q$ denotes the quantization map and $\lambda : \mathfrak{so}(V) \to \Lambda^2 V$ is the inverse of the isomorphism given in (1.6).

By definition, for $A \in \mathfrak{so}(V)$ we have that $A(v) = [\gamma(A), v]$ for every $v \in V$. Thus,

$$exp(A)(v) = exp(\phi_{\gamma(A)})v$$

and using the identity $exp(x) \cdot y \cdot exp(-x) = exp(ad(x))y$, we obtain

$$exp(A)(v) = e^{\gamma(A)} v e^{-\gamma(A)}$$

Since the left hand side lies in $V$, it follows that $e^{\gamma(A)} \in \Gamma(V)$ and such element is even. Moreover, since $\gamma(A)^T = -\gamma(A)$ we have that $(e^{\gamma(A)})^T = e^{-\gamma(A)}$ and therefore, $N(e^{\gamma(A)}) = 1$. That is, $exp(\gamma(A)) \in Spin(V)$.

□

The normalization $N(a) = 1$ gives $a \in \Gamma(V)$ up to a sign, so we have an exact sequence

$$1 \to \mathbb{Z}_2 \to Spin(V) \to SO(V)$$

We see that given a non-isotropic vector $v$, then $R_{tv} = R_v$ for any non-zero scalar $t \in \mathbb{K}$ so one should be able to normalize any $a \in \Gamma(V)$ so that $N(a) = 1$. But the equation $t^2 = a$ may or may not be solvable in the field $\mathbb{K}$. Therefore, a sufficient condition for surjectivity $Spin(V) \to SO(V)$ is that every element in $\mathbb{K}$ admits a square root. In this case,

$$Spin(V) \doteq \{v_1 \dots v_r | v_i \in V, B(v_i, v_i) = 1 \quad \text{and} \quad r \quad \text{is even}\}$$

**Remark 1.3.7** If in $V$ we have non-zero isotropic vectors, then the condition that the equation $t^2 = a$ is always solvable in $\mathbb{K}$ is also a necessary condition.

◊

**Remark 1.3.8** If $\mathbb{K} = \mathbb{R}$, the Spin group is sometimes defined using the weaker condition $N(a) = \pm 1$ so that the map $Spin(V) \to SO(V)$ is surjective.

◊

### 1.3.2 The Clifford Product

Finally, we end this section by showing how the exterior product can be deformed into the Clifford product. We shall assume that the field $\mathbb{K}$ has characteristic zero, e.g., $\mathbb{K} = \mathbb{R}$.

Let $\mathcal{E} \doteq \{e_1, \dots, e_n\}$ be an orthogonal basis of $V$. Then, $B(e_i, e_j) = -\varepsilon_i \delta_{ij}$ for some $\varepsilon_i \in \mathbb{K}$. Also, let us consider the following operator in $\Lambda^\bullet V \otimes \Lambda^\bullet V$: $\hat{B} \doteq \sum_{i=1}^{n} \varepsilon_i e^i \otimes e^i$, where (as usual) $\{e^1, \dots, e^n\}$ denotes the dual basis of $\mathcal{E}$.

**Lemma 1.3.2** *Setting $\hat{B}_i \doteq \varepsilon_i e^i \otimes e^i$, for every $1 \leqslant i \leqslant n$, we have that $[\hat{B}_i, \hat{B}_j] = 0$ and hence $exp(\hat{B}) = \prod_i exp(\hat{B}_i)$.*

Next, we define the following product over $\Lambda^\bullet V$:

$$\mu : \Lambda^\bullet V \otimes \Lambda^\bullet V \xrightarrow{exp(\hat{B})} \Lambda^\bullet V \otimes \Lambda^\bullet V \xrightarrow{m_\wedge} \Lambda^\bullet V,$$

where $m_\wedge(\alpha \otimes \beta) \doteq \alpha \wedge \beta$ is the usual exterior product.

**Proposition 1.3.4** *The product $\mu$ is associative.*

*Proof.* We begin by observing that for each $1 \leqslant i \leqslant n$ we have that $\exp(\hat{B}_i) = id + \hat{B}_i$. Next, we show that

$$\mu(\mu \otimes id) = \mu(id \otimes \mu) \tag{1.9}$$

Firstly, let us denote by $\mu_i$ the composition $m_\wedge \circ exp(\hat{B}_i)$. Then, on the one hand,

$$\begin{aligned} \mu_i(\mu_i \otimes id) &= (m_\wedge + m_\wedge \circ \hat{B}_i)((m_\wedge + m_\wedge \circ \hat{B}_i) \otimes id) \\ &= m_\wedge(m_\wedge \otimes id) + m_\wedge \circ \hat{B}_i(m_\wedge \otimes id) + m_\wedge(m_\wedge \circ \hat{B}_i \otimes id) + m_\wedge \circ \hat{B}_i(m_\wedge \circ \hat{B}_i \otimes id) \quad (1.10) \\ &= m_\wedge(m_\wedge \otimes id) + m_\wedge \circ \hat{B}_i(m_\wedge \otimes id) + m_\wedge(m_\wedge \circ \hat{B}_i \otimes id) \quad (1.11) \end{aligned}$$

and on the other hand,

$$\mu_i(id \otimes \mu_i) = m_\wedge(id \otimes m_\wedge) + m_\wedge \circ \hat{B}_i(id \otimes m_\wedge) + m_\wedge(id \otimes m_\wedge \circ \hat{B}_i) \tag{1.12}$$

Note that the last term in (1.10) was dropped out because each $e^i$ acts by contraction and thus $\hat{B}_i = \varepsilon_i e^i \otimes e^i$ squares to zero.

Now, the expression in (1.11) can be written as

$$m_\wedge(m_\wedge \otimes id) \underbrace{\left(id \otimes id \otimes id + \varepsilon_i\Big((e^i \otimes id + id \otimes e^i) \otimes e^i\Big) + \varepsilon_i e^i \otimes e^i \otimes id\right)}_{A}$$

and, analogously, the expression in (1.12) is equivalent to

$$m_\wedge(id \otimes m_\wedge) \underbrace{\left(id \otimes id \otimes id + \varepsilon_i\Big(e^i \otimes (e^i \otimes id + id \otimes e^i)\Big) + \varepsilon_i id \otimes e^i \otimes e^i\right)}_{B}$$

Thus, since $A = B$ and the product $m_\wedge$ is associative, i.e., $m_\wedge(m_\wedge \otimes id) = m_\wedge(id \otimes m_\wedge)$, it follows that $\mu_i(\mu_i \otimes id) = \mu_i(id \otimes \mu_i)$.

Furthermore, if we compute $\mu_i \circ exp(\hat{B}_j)\Big(\mu_i \circ exp(\hat{B}_j) \otimes id\Big)$, with $i \neq j$, then we obtain the following

$$\mu_i(\mu_i \otimes id) + \underbrace{\mu_i \hat{B}_j(\mu_i \otimes id) + \mu_i(\mu_i \otimes id)(\hat{B}_j \otimes id)}_{*} \tag{1.13}$$

Yet, the expression $*$ is equivalent to

$$
\begin{aligned}
m_\wedge(m_\wedge \otimes id) \quad & \Big[\varepsilon_j\Big((e^j \otimes id + id \otimes e^j) \otimes e^j\Big) + \\
& + \varepsilon_j\varepsilon_i\Big((e^ie^j \otimes id + e^j \otimes e^i + e^i \otimes e^j + id \otimes e^ie^j) \otimes e^ie^j\Big) + \\
& + \varepsilon_j\varepsilon_i\Big((e^je^i \otimes e^i + e^i \otimes e^je^i) \otimes e^j\Big) \\
& + \varepsilon_j e^j \otimes e^j \otimes id + \varepsilon_j\varepsilon_i\Big((e^ie^j \otimes e^j + e^j \otimes e^ie^j) \otimes e^i\Big) + \varepsilon_j\varepsilon_i e^ie^j \otimes e^ie^j \otimes id\Big]
\end{aligned}
$$

which, in turn, is equivalent to $\mu_i\hat{B}_j(id \otimes \mu_i) + \mu_i(id \otimes \mu_i)(id \otimes \hat{B}_j)$. Thus, by checking that $\mu_i \circ exp(\hat{B}_j)\Big(id \otimes \mu_i exp(\hat{B}_j)\Big)$ equals to

$$\mu_i(id \otimes \mu_i) + \underbrace{\mu_i\hat{B}_j(id \otimes \mu_i) + \mu_i(id \otimes \mu_i)(id \otimes \hat{B}_j)}_{**}$$

it follows that

$$\mu_i \circ exp(\hat{B}_j)\Big(\mu_i \circ exp(\hat{B}_j) \otimes id\Big) = \mu_i \circ exp(\hat{B}_j)\Big(id \otimes \mu_i exp(\hat{B}_j)\Big) \quad \forall i \neq j$$

were we have used the associativity of each $\mu_i$. In other words, the expressions $*$ and $**$ are equal.

Note that, in order to show the equality between $*$ and $**$, what we have used, essentially, is that for all $1 \leqslant i, j \leqslant n$:

$$e^j\mu_i = \mu_i(e^j \otimes id + id \otimes e^j)$$

So, similar to the associativity of $\mu_i$, we can prove that $\mu_{(2)} \doteq \mu_1 \circ exp(\hat{B}_2)$ is also associative. Therefore, if we define $\mu_{(k)} \doteq m_\wedge \prod\limits_{i \leqslant k} exp(\hat{B}_i)$ and we proceed inductively over $k$, we conclude that the equality in (1.10) is valid.

In short, we have the following commutative diagram

$$
\begin{array}{ccccc}
\Lambda^\bullet V \otimes \Lambda^\bullet V \otimes \Lambda^\bullet V & \xrightarrow{id \otimes exp(\hat{B})} & \Lambda^\bullet V \otimes \Lambda^\bullet V \otimes \Lambda^\bullet V & \xrightarrow{id \otimes m_\wedge} & \Lambda^\bullet V \otimes \Lambda^\bullet V \\
\downarrow{\scriptstyle exp(\hat{B}) \otimes id} & & & & \downarrow{\scriptstyle exp(\hat{B})} \\
\Lambda^\bullet V \otimes \Lambda^\bullet V \otimes \Lambda^\bullet V & & & & \Lambda^\bullet V \otimes \Lambda^\bullet V \\
\downarrow{\scriptstyle m_\wedge \otimes id} & & & & \downarrow{\scriptstyle m_\wedge} \\
\Lambda^\bullet V \otimes \Lambda^\bullet V & \xrightarrow[exp(\hat{B})]{} & \Lambda^\bullet V \otimes \Lambda^\bullet V & \xrightarrow[m_\wedge]{} & \Lambda^\bullet V
\end{array}
\tag{1.14}
$$

□

Moreover,

**Proposition 1.3.5** *The product $\mu$ satisfies the Clifford relations.*

*Proof.* We first compute $exp(\hat{B})(e_k \otimes e_l)$, with $1 \leqslant k, l \leqslant n$:

$$\begin{aligned} exp(\hat{B})(e_k \otimes e_l) &= \left(id + \hat{B} + \frac{\hat{B}^2}{2!} + \dots\right)(e_k \otimes e_l) \\ &= (id + \hat{B})(e_k \otimes e_l) \\ &= e_k \otimes e_l + \sum_i \hat{B}_i(e_k \otimes e_l) \\ &= e_k \otimes e_l - \sum_i \varepsilon_i \delta_{ik} \otimes \delta_{il} \end{aligned}$$

Hence, $\mu(e_k \otimes e_l) = e_k \wedge e_l - \sum_i \varepsilon_i \delta_{kl}$ and, therefore, $\mu(e_k \otimes e_l) + \mu(e_l \otimes e_k) = 2B(e_k, e_l)$. □

At last,

**Proposition 1.3.6** *The "deformed" exterior algebra is generated by $V$.*

*Proof.* This follows from the following fact. If $\alpha = e_{i_1} \wedge \dots \wedge e_{i_p}$ and $\beta = e_{j_1} \wedge \dots \wedge e_{j_q}$ are forms such that the sets $\{i_n\}$ and $\{j_m\}$ are disjoint, then

$$\mu(\alpha \otimes \beta) = \alpha \wedge \beta$$

□

In particular, under the quantization map, the Clifford product and the "deformed" exterior product coincide. That is,

$$m_{Cl} \circ (q \otimes q) = q \circ m_\wedge \circ exp(\hat{B})$$

**Remark 1.3.9** Note that in the real case, if we had chosen $\tilde{B} = h \cdot \hat{B}$, where $h > 0$, then we could have considered a product $\tilde{\mu}$ given by $m_\wedge \circ exp(\tilde{B})$. Now, Corollary 1.3.1 implies that the algebras $(\Lambda^\bullet, \mu)$ and $(\Lambda^\bullet, \tilde{\mu})$ would be isomorphic. Thus, we can think of a family of parametrized products, $\mu(h) \doteq m_\wedge \circ exp(h \cdot \hat{B})$, such that $m_{Cl} \circ (q \otimes q) = q \circ \mu(h)$. Yet, if $h \to 0$, then $\mu(h) \to m_\wedge$.

◊

## 1.4 The Geometry of $E \oplus E^*$

In this section $E$ is a real vector space of dimension $m$ and $E^*$ denotes its dual space. In the space $E \oplus E^*$ we can consider the natural symmetric bilinear form given by

$$\langle X + \xi, Y + \eta \rangle \doteq \frac{1}{2}\Big(\xi(Y) + \eta(X)\Big) = \frac{1}{2}\begin{pmatrix} X & \xi \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Y \\ \eta \end{pmatrix},$$

where $X, Y \in E$ and $\xi, \eta \in E^*$. Such a pairing is non-degenerate and have signature $(m, m)$. Therefore, setting $V = E \oplus E^*$ and $B = \langle \cdot, \cdot \rangle$ the results we developed in Section 2.3 apply here.

Note that both $E$ and $E^*$ are null under the pairing $\langle \cdot, \cdot \rangle$ and since dim$(E)$ =dim$(E^*) = m$ these are examples of maximally isotropic subspaces. Our goal in this section is to describe the maximal isotropic subspaces of $E \oplus E^*$ using pure spinors.

Note also that the highest exterior power of $E \oplus E^*$ can be decomposed as follows

$$det(E \oplus E^*) = det(E) \otimes det(E^*).$$

Moreover, there is a natural pairing between $\Lambda^k V^*$ and $\Lambda^k V$ given by

$$(v^*, u) \doteq det(v_i^*(u_j))_{ij},$$

where $v^* = v_1^* \wedge \ldots \wedge v_k^* \in \Lambda^k V^*$ and $u = u_1 \wedge \ldots \wedge u_k \in \Lambda^k V$. Thus, $det(E \oplus E^*) \simeq \mathbb{R}$. The number $1 \in \mathbb{R}$ then defines a canonical orientation on $E \oplus E^*$ and the symmetry group of $E \oplus E^*$ naturally reduces to $SO(E \oplus E^*) \simeq SO(m, m)$.

Now, since we can identify $E \oplus E^*$ with its dual space, the Lie algebra of $SO(E \oplus E^*)$ is

$$\mathfrak{so}(E \oplus E^*) \doteq \{D | D = -D^*\} \simeq \Lambda^2(E \oplus E^*) \simeq \Lambda^2 E \oplus End(E) \oplus \Lambda^2 E^*$$

In particular, any element $D \in \mathfrak{so}(E \oplus E^*)$ can be written as

$$D = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix},$$

where $A \in End(E), \beta \in \Lambda^2 E$ and $B \in \Lambda^2 E^*$.

We now present how each part of the decomposition $\Lambda^2 E \oplus End(E) \oplus \Lambda^2 E^*$ acts on the space $E \oplus E^*$.

Given $A \in End(E)$, it corresponds to $D_A = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \in \mathfrak{so}(E \oplus E^*)$. Such an element acts on $E \oplus E^*$ by exponentiation as the linear map

$$\exp(D_A) = \begin{pmatrix} e^A & 0 \\ 0 & ((e^A)^*)^{-1} \end{pmatrix} \in SO(E \oplus E^*)$$

Note that the usual symmetries of the vector space $E$, that is $GL(E)$, can be regarded as part of the group $SO(E \oplus E^*)$. For the map

$$T \mapsto \begin{pmatrix} T & 0 \\ 0 & (T^*)^{-1} \end{pmatrix}$$

is an injection of $GL(E)$ into $SO(E \oplus E^*)$.

Next, let us consider a 2-form $B \in \Lambda^2 E^*$. Then $B$ corresponds to the element $D_B = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \in \mathfrak{so}(E \oplus E^*)$, which acts on $E \oplus E^*$ as the linear map

$$e^B \doteq \exp(D_B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \in SO(E \oplus E^*) \tag{1.15}$$

More explicitly, this is the map

$$\begin{pmatrix} X \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} X \\ \xi + \iota_X B \end{pmatrix},$$

for every $X \in E$ and $\xi \in E^*$.

In short, any 2-form $B \in \Lambda^2 E^*$ gives rise to a linear transformation which preserves the projection $\pi_E$ onto $E$, i.e., $\pi_E \circ e^B = \pi_E$. In fact, these are the unique elements of $SO(E \oplus E^*)$ satisfying this property.

**Proposition 1.4.1** *If $T \in SO(E \oplus E^*)$ preserves the projection onto $E$, then there exists $B \in \Lambda^2 E^*$ such that $T = e^B$.*

*Proof.* Since $T$ preserves $\pi_E$, it follows that $T(X + \xi) = X + (T_{21}X + T_{22}\xi)$ for some linear maps $T_{21} : E \to E^*$ and $T_{22} : E^* \to E^*$, that is,

$$T = \begin{pmatrix} 1 & 0 \\ T_{21} & T_{22} \end{pmatrix}.$$

Now, since $T$ is orthogonal,

$$0 = \langle T(0 + \xi), T(X + 0) \rangle - \langle 0 + \xi, X + 0 \rangle = (T_{22}\xi - \xi)(X) \quad \forall X \in E, \forall \xi \in E^*$$

and hence $T_{22} = id$. Moreover,

$$0 = \langle T(X + 0), T(Y + 0) \rangle - \langle X + 0, Y + 0 \rangle = T_{21}(X)(Y) + T_{21}(Y)(X) \quad \forall X, Y \in E$$

and therefore, $T_{21} = -T_{21}^*$.

Setting $B = T_{21}$ we have that $T = e^B$ with $B \in \Lambda^2 E^*$. □

Finally, a bivector $\beta \in \Lambda^2 E$ corresponds to the element $D_\beta = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \in \mathfrak{so}(E \oplus E^*)$, which acts on $E \oplus E^*$ as the linear map

$$e^\beta \doteq \exp(D_\beta) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in SO(E \oplus E^*)$$

Analogously to the case of 2-forms, this is the map

$$\begin{pmatrix} X \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} X + \iota_\xi \beta \\ \xi \end{pmatrix},$$

for every $X \in E$ and $\xi \in E^*$. That is, any bivector $\beta \in \Lambda^2 E$ gives rise to a linear transformation which preserves the projection $\pi_{E^*}$ onto $E^*$ and, similarly as above, these are the unique elements of $SO(E \oplus E^*)$ satisfying this property.

**Proposition 1.4.2** *If $T \in SO(E \oplus E^*)$ preserves the projection onto $E^*$, then there exists $\beta \in \Lambda^2 E$ such that $T = e^\beta$.*

The reader may observe that this has already been proved in Proposition 1.2.2.

We now turn our attention to the maximal isotropic subspaces of $E \oplus E^*$. In literature, such subspaces are also called *linear Dirac structures*.

**Example 1.4.1**

(i) $E$ and $E^*$ are clearly maximally isotropic.

(ii) Also, given $F$ a subspace of $E$, we have that $F \oplus F^\circ$ is a linear Dirac structure since it is obviously isotropic and $\dim(F^\circ) = m - \dim(F)$.

(iii) Yet, the set $e^\beta \cdot E^* \doteq \{\iota_\xi \beta + \xi | \xi \in E^*\}$ is maximally isotropic for every $\beta \in \Lambda^2 E$ and similarly, for any $B \in \Lambda^2 E^*$ the set $e^B \cdot E \doteq \{X + \iota_X B | X \in E\}$ is a maximal isotropic subspace of $E \oplus E^*$.

△

Further, given any subspace $F \subset E$ and $\varepsilon \in \Lambda^2 F^*$, if $j : F \hookrightarrow E$ denotes the inclusion, then

$$L(F, \varepsilon) \doteq \{X + \xi \in F \oplus E^* | j^* \xi = \iota_X \varepsilon\}$$

is maximally isotropic. If $X + \xi, Y + \eta \in L(F, \varepsilon)$, then $2\langle X + \xi, Y + \eta\rangle = \varepsilon(Y, X) + \varepsilon(X, Y) = 0$ and it follows that $L(F, \varepsilon)$ is isotropic. Now, the graph $\{X + \iota_X \varepsilon | X \in F\}$ has dimension equals $\dim(F)$ and there are $\dim(E^*) - \dim(F^*) = \dim(E) - \dim(F)$ possibilities for choosing $\xi \in E^*$ such that $j^* \xi = \iota_X \varepsilon$, thus $L(F, \varepsilon)$ is maximal.

Note that if $\varepsilon = 0$, then $L(F, 0) = F \oplus F^\circ$. Yet, $L(E, 0) = E$, $L(\{0\}, 0) = E^*$ and in general:

**Proposition 1.4.3** *Every maximal isotropic subspace of $E \oplus E^*$ is of the form $L(F, \varepsilon)$, for some $F \subset E$ and $\varepsilon \in \Lambda^2 E^*$.*

*Proof.* Let $L \subset E \oplus E^*$ be maximally isotropic. Setting $F \doteq \pi_E(L)$ we have that $L \cap E^* = F^\circ$ and we can define $\varepsilon : F \to F^*$ by $X \mapsto \pi_{E^*}(\pi_E^{-1}(X) \cap L)$. In other words, if $X \in F$, then there exists $\xi \in E^*$ such that $X + \xi \in L$ and we define $\varepsilon(X) \doteq [\xi] \in E^*/F^\circ \simeq F^*$.

Note that $\varepsilon(X)$ is well defined for every $X \in F$. If $\xi, \eta \in \varepsilon(X)$, then $X + \xi, X + \eta \in L$ and hence $\xi - \eta \in L$. Now, if $Y \in F$, then there exists $\zeta \in E^*$ such that $Y + \zeta \in L$ and therefore $0 = 2\langle \xi - \eta, Y + \zeta\rangle = \xi(Y) - \eta(Y)$, that is, $\xi - \eta \in F^\circ$.

By construction, every element of $L$ can be written as $X + \iota_X \varepsilon + \xi$, with $\xi \in F^\circ$. Thus, $L \subset L(F, \varepsilon)$ and since $L$ is maximal we have equality. □

The next result tells us that any maximal isotropic subspace of $E \oplus E^*$ can be seen as an action of a 2-form $B \in \Lambda^2 E^*$ on $L(F, 0)$, for some subspace $F \subset E$.

**Proposition 1.4.4** *Let $D \subset E, \theta \in \Lambda^2 D^*$ and $B \in \Lambda^2 E^*$. Then*

$$e^B(L(D,\theta)) = L(D, \theta + j^* B),$$

*where $j : D \hookrightarrow E$ denotes the inclusion.*

Thus, if $L \subset E \oplus E^*$ is any maximal isotropic subspace, then $L = L(F, \varepsilon)$, for some $F \subset E$ and $\varepsilon \in \Lambda^2 F^*$. Taking $B \in \Lambda^2 E^*$ such that $j^* B = \varepsilon$ we have that $L = e^B L(F, 0)$. The proof of the above proposition is straightforward.

### 1.4.1 Spinors

Recall (Section 1.3) for a general (real) vector space $W$ with a non-degenerate symmetric bilinear form $b$ its Clifford algebra is the associative unital algebra generated by the elements of $W$ satisfying the relation

$$ww' + w'w = 2b(w, w')1$$

Plus, one can easily see that as a vector space $Cl(W)$ is isomorphic to $\Lambda^\bullet(W)$. In fact, if $\{w_1, \ldots, w_n\}$ is an orthonormal basis of $W$, then any element of $Cl(W)$ is a linear combination of finite strings $w_{i_1} w_{i_2} \ldots$.

Now, using the relations $w_i w_j = -w_j w_i$, those strings can be put into a form where $i_1 < i_2 < \ldots$. Yet, since $w_i^2 = 1$, we conclude, just as for the exterior algebra, the $2^n$ elements below form a basis of $Cl(W)$.

$$\begin{array}{rr} & 1 \\ & w_i \\ w_i w_j & i < j \\ & \vdots \\ w_1 w_2 \ldots w_n & \end{array}$$

We now specialize to the case $W = E \oplus E^*$ and $b = \langle \cdot, \cdot \rangle$.

Let us consider the following natural action of $E \oplus E^*$ on the exterior algebra $\Lambda^\bullet E^*$. Given $X + \xi \in E \oplus E^*$ and $\rho \in \Lambda^\bullet E^*$, let

$$(X + \xi) \cdot \rho \doteq \iota_X \rho + \xi \wedge \rho \tag{1.16}$$

Next, we observe that this action satisfies

$$\begin{aligned} (X + \xi)^2 \cdot \rho &= \iota_X(\iota_X \rho + \xi \wedge \rho) + \xi \wedge (\iota_X \rho + \xi \wedge \rho) \\ &= (\iota_X \xi)\rho - \xi \wedge \iota_X \rho + \xi \wedge \iota_X \rho \\ &= \langle X + \xi, X + \xi \rangle \rho \end{aligned}$$

that is, $e^2 \cdot \rho = \langle e, e\rangle\rho$ for every $e \in E \oplus E^*$ and $\rho \in \Lambda^\bullet E^*$. Since this is the defining relation of the Clifford algebra $Cl(E \oplus E^*)$, it follows that we have a representation
$Cl(E \oplus E^*) \to End(\Lambda^\bullet E^*)$. Moreover, such representation is both faithful and irreducible.

**Remark 1.4.1** If $\mathcal{A}$ is an associative algebra with an irreducible representation acting on it and its center consists of the algebra of scalars $\mathbb{K}$, then by Wedderburn's theorem, $\mathcal{A}$ is the algebra of all endomorphisms of the vector space in question.

$\Diamond$

**Remark 1.4.2** From the discussion in the beginning of this section we already knew that $\dim(Cl(E \oplus E^*)) = 2^{2m} = (2^m)^2 = \dim(End(\Lambda^\bullet E^*))$.

$\Diamond$

In another view, if $Q$ denotes the associated quadratic form to the pairing $\langle \cdot, \cdot \rangle$, then $Q$ vanishes on any $L \subset E \oplus E^*$ maximal isotropic subspace. Therefore the subalgebra of $Cl(E \oplus E^*)$ generated by $L$ is naturally isomorphic to $\Lambda^\bullet L$. Since $E^*$ is maximally isotropic, it follows that $\mathcal{S} \doteq \Lambda^\bullet E^*$ is naturally embedded in $Cl(E \oplus E^*)$. The space $\mathcal{S}$ will be called the **space of spinors.**

Now, let us choose an orthonormal basis of $E \oplus E^*$, say $\{e_1 \pm e^1, \dots, e_m \pm e^m\}$.

**Proposition 1.4.5** *There exists a volume form $\omega \in Cl(E \oplus E^*)$ such that $\omega^2 = 1$ and thus $\mathcal{S}$ splits in $\mathcal{S}^+ \oplus \mathcal{S}^-$, where $\mathcal{S}^\pm \doteq ker(\omega \pm 1)$. Furthermore, such splitting corresponds to $\Lambda^{even} E^* \oplus \Lambda^{odd} E^*$.*

*Proof.* Define

$$\omega \doteq (-1)^{\frac{m(m-1)}{2}}(e_1 - e^1)\dots(e_m - e^m)(e_1 + e^1)\dots(e_m + e^m) \tag{1.17}$$

Then $\omega = \omega_1 \dots \omega_m$, where $\omega_i = (e_i - e^i)(e_i + e^i)$ and $1 \leqslant i \leqslant m$. Next, note that each $\omega_i$ squared equals one, thus $\omega^2 = 1$ follows easily. Moreover, $\omega_i = 1 - 2e^i e_i$ for every $1 \leqslant i \leqslant m$.

Finally, we claim that $\omega$ anti-commutes with elements of both $E$ and $E^*$ and also $w \cdot 1 = 1$. This implies the splitting $\mathcal{S}^+ \oplus \mathcal{S}^-$ corresponds to $\Lambda^{even} E^* \oplus \Lambda^{odd} E^*$.

We observe that since each $\omega_i$ is of even degree it is sufficient to check that $\omega_i e_i = -e_i \omega_i$ and, analogously, $\omega_i e^i = -e^i \omega_i$. In fact we have that each $\omega_i$ commutes with both $e_j$ and $e^j$ for $j \neq i$. $\Box$

**Remark 1.4.3** The volume form defined in (1.17) is such that $\omega\alpha = (-1)^p\alpha$ for every $\alpha \in \Lambda^p E^*$.

$\Diamond$

**Remark 1.4.4** Note that if $\tilde{e}_1 \pm \tilde{e}^1, \dots, \tilde{e}_m \pm \tilde{e}^m$ is any other orthonormal basis of $E \oplus E^*$, then $\tilde{e}_i \pm \tilde{e}^i = \sum_j g_{ij} e_j \pm e^j$, where $G = (g_{ij}) \in SO(E \oplus E^*)$. Therefore,

$$(\tilde{e}_1 - \tilde{e}^1)\dots(\tilde{e}_m - \tilde{e}^m)(\tilde{e}_1 + \tilde{e}^1)\dots(\tilde{e}_m + \tilde{e}^m) = (det(G))(e_1 - e^1)\dots(e_m - e^m)(e_1 + e^1)\dots(e_m + e^m)$$

and $\omega$ does not depend on the choice of a basis.

$\Diamond$

The spaces $\mathcal{S}^+$ and $\mathcal{S}^-$ are both irreducible representations of the spin group

$$Spin(E \oplus E^*) \doteq \{v_1 \dots v_r | v_i \in E \oplus E^*, \langle v_i, v_i \rangle = \pm 1 \quad \text{and} \quad r \quad \text{is even}\}$$

which is a double cover of $SO(E \oplus E^*)$ via the homomorphism

$$\begin{aligned} \Phi : Spin(E \oplus E^*) &\rightarrow SO(E \oplus E^*) \\ g &\mapsto \Phi(g) : v \mapsto gvg^{-1} \end{aligned}$$

By taking the derivative it induces a morphism on Lie algebras

$$d_e\Phi : \mathfrak{spin}(E \oplus E^*) \rightarrow \mathfrak{so}(E \oplus E^*)$$

given by $d_e\Phi(a)(v) = [a, v]$.

**Pure Spinors**

Our goal in this section is to associate to each maximally isotropic subspace of $E \oplus E^*$ a spinor (up to a non-zero scalar). Throughout, let us denote by $M$ (as in Section 2.3) the set of maximally isotropic subspaces of $E \oplus E^*$.

Next, let $\rho \in \mathcal{S}$ be a non-zero spinor and define its null space as

$$L_\rho \doteq \{X + \xi \in E \oplus E^*; (X + \xi) \cdot \rho = 0\}$$

In other words, consider the space of vectors in $E \oplus E^*$ which annihilate $\rho$ under the action $Cl(E \oplus E^*) \rightarrow End(\mathcal{S})$.

Note that $L_\rho$ depends equivariantly on $\rho$ under the spin representation. That is,

$$L_{g \cdot \rho} = \Phi(g) L_\rho$$

for every $g \in Spin(E \oplus E^*)$, where $\Phi : Spin(E \oplus E^*) \rightarrow SO(E \oplus E^*)$ as above. In particular, $L_{\lambda\rho} = L_\rho$ for every $\lambda \in \mathbb{R}^*$. Moreover,

**Proposition 1.4.6** *Given $\rho \in \mathcal{S} \backslash \{0\}$ its null space, $L_\rho$, is an isotropic subspace.*

*Proof.* Let $a = X + \xi$ e $b = Y + \eta$ be element in $L_\rho$. Since $E \oplus E^* \hookrightarrow Cl(E \oplus E^*)$, we have that $\langle a, b \rangle = \dfrac{1}{2}(ab + ba)$. Hence,

$$\langle a, b \rangle \rho = 0 \Rightarrow \langle a, b \rangle = 0$$

$\square$

**Definition 1.4.1** *A spinor $\rho \in \mathcal{S} \backslash \{0\}$ is called* ***pure*** *if, and only if, its null space is maximally isotropic.*

**Example 1.4.2** If $\rho = e^1 \wedge \ldots \wedge e^m$, then $L_\rho = E^*$ and $\rho$ is pure.

△

**Example 1.4.3** If $\rho = 1 \in \Lambda^0 V^*$, then $L_1 = V$ and $\rho$ is pure.

△

Finally, we prove that "lines" of pure spinors are in one-to-one correspondence with linear Dirac structures on $E \oplus E^*$.

**Proposition 1.4.7** *Given $L \subset E \oplus E^*$ a maximally isotropic subspace, there exists a pure spinor $\rho$ such that $L_\rho = L$.*

*Proof.* Proposition 1.4.3 tells us that $L = L(F, \varepsilon)$ for some $F \subset E$ and $\varepsilon \in \Lambda^2 E^*$. So, let us choose $\theta_1, \ldots, \theta_k$ such that they form a basis o f $E^\circ$ and $B \in \Lambda^2 E^*$ such that $j^* B = \varepsilon$, where $j : F \hookrightarrow E$ denotes the inclusion. Then, $\rho = e^{-B}\theta_1 \wedge \ldots \wedge \theta_k$ is pure and $L_\rho = L(F, \varepsilon)$.

In fact, for $X + \xi \in L(F, 0)$ one finds that

$$(X + \xi) \cdot \Omega = \iota_x \Omega + \xi \wedge \omega = 0,$$

where $\Omega \doteq \theta_1 \wedge \ldots \wedge \theta_k$. Hence, by maximality, $L(F, 0) = L_\Omega$. Now, if we pick $B \in \Lambda^2 E^*$ such that $j^* B = \varepsilon$, then $L(F, \varepsilon) = exp(-B)L(F, 0)$ (Proposition 1.4.4). Finally, by equivariance,

$$L(F, \varepsilon) = exp(-B)L(F, 0) = exp(-B)L_\Omega = L_\rho,$$

where $\rho = e^B \wedge \Omega$. □

In short, the representation of the Clifford group $\Gamma(E \oplus E^*)$ on the spinor space $\mathcal{S}$ restricts to a transitive action on the set of pure spinors, which we will denote by $\mathcal{S}_p$. The map $\mathcal{S}_p \to M$ given by $\rho \mapsto L_\rho$ is a $Spin(E \oplus E^*)-$equivariant surjection and has fibers $\mathbb{R}^*$.

### A Bilinear pairing on spinors

We now present a canonical bilinear form on $\mathcal{S}$ which behaves well under the spin representation.

To begin with, notice that the exterior algebras $\Lambda^\bullet E$ and $\Lambda^\bullet E^*$ are naturally subalgebras of $Cl(E \oplus E^*)$. So, in particular, we can apply the transposition anti-automorphism to elements in both $\Lambda^\bullet E$ and $\Lambda^\bullet E^*$. Moreover, $det(E) = \Lambda^m E$ is a distinguished line in the Clifford algebra since it is a minimal (left) ideal and therefore

$$Cl(E \oplus E^*) \cdot det(E) \subset Cl(E \oplus E^*)$$

Once we choose a generator $f \in det(E)$ every element of the ideal has a unique representation $\rho f$, where $\rho \in \mathcal{S}$. So the action of the Clifford algebra on the space of spinors satisfies

$$(\Phi(a)\rho)f = a\rho f, \quad a \in Cl(E \oplus E^*)$$

where $\Phi : Spin(E \oplus E^*) \to SO(E \oplus E^*)$.

**Definition 1.4.2** *We define the following bilinear form* $(\cdot,\cdot) : \Lambda^\bullet E^* \otimes \Lambda^\bullet E^* \to det(E^*)$ *given by*

$$(\alpha, \beta) \doteq [\alpha^T \wedge \beta]_{top},$$

*where* $[\cdot]_{top}$ *indicates taking only the top degree part of the form.*

Such pairing is used in the Proposition 1.4.8 below to classify whether or not linear Dirac structures intersect.

Now, we can express $(\cdot,\cdot)$ using any generator $f \in det(E)$ as follows

$$\begin{aligned}(\alpha, \beta) &= (\iota_f(\alpha, \beta))f \\ &= (\iota_f(\alpha^T \wedge \beta))f \\ &= f^T \alpha^T \beta f \\ &= (\alpha f)^T \beta f\end{aligned}$$

Hence, $(v \cdot \alpha, \beta) = (\alpha, v^T \cdot \beta)$. From this, it follows that $(v \cdot \alpha, v \cdot \beta) = \langle v, v\rangle(\alpha, \beta)$ for any $v = X + \xi \in E \oplus E^*$. Consequently, $(g \cdot \alpha, g \cdot \beta) = \pm(\alpha, \beta)$ for any $g \in Spin(E \oplus E^*)$.

**Proposition 1.4.8** *Given maximally isotropic subspaces* $L$ *and* $L'$ *they satisfy the condition* $L \cap L' = \{0\}$ *if, and only if, their associated pure spinors representatives* $\rho$ *and* $\rho'$ *(resp.) satisfy* $(\rho, \rho') \neq 0$.

*Proof.* First, let us assume that $(\rho, \rho') \neq 0$ and, by contradiction, there exists $0 \neq a = X + \xi \in L \cap L'$. Then, $a \cdot \rho = 0$ and $a \cdot \rho' = 0$, which imply $a \cdot (\rho, \rho') = 0$ as well. This means we can write $\rho = \xi \wedge \theta_1$ and $\rho' = \xi \wedge \theta_2$ and thus computing $(\rho, \rho')$ gives us zero, a contradiction.

For the converse, we begin by observing that $SO(E \oplus E^*)$ acts transitively over $M$, the set of maximally isotropic subspaces. Thus, for all $L'$ there exists $g \in Spin(E \oplus E^*)$ such that $L' = \Phi(g)E^*$. Now, $L \cap L' = \{0\}$ implies $\Phi(g^{-1})L \cap E^* = 0$ and therefore

$$0 \neq (\rho, \rho')$$

In fact, writing $\rho = exp^B \wedge \Omega$ (as in Proposition 1.4.7) and computing $(\rho, e^1 \wedge \ldots \wedge e^m)$ we see that this form is zero if $k > 0$ and equal to $\pm e^1 \wedge \ldots \wedge e^m \neq 0$ if $k = 0$. On the other hand, $\Omega$ spans $\left(\pi_E(L_\rho)\right)^\circ$ (proof of Proposition 1.4.3), so such pairing is non-zero if, and only if, $\left(\pi_E(L_\rho)\right)^\circ = \{0\}$. But, $L_\rho \cap E^* = \left(\pi_E(L_\rho)\right)^\circ$, which is a trivial intersection by hypothesis.

Finally, to conclude, we observe that the pairing $(\cdot,\cdot)$ is $Spin(E \oplus E^*)$ equivariant up to a sign. □

### 1.4.2 Linear Generalized Complex Structures

One of the aims of the next chapter is to introduce Generalized Complex Geometry. In order to do so we first start with the linear case $E \oplus E^*$ and then extend to $TM \oplus T^*M$, where $M$ is a smooth manifold of dimension $m$.

**Definition 1.4.3** *A* ***linear generalized complex structure*** *(LGCS) on $E$ is an endomorphism $\mathcal{J}$ of $E \oplus E^*$ such that it is both complex and symplectic, that is, $\mathcal{J}^2 = -Id$ and $\mathcal{J}^* = -\mathcal{J}$.*

**Proposition 1.4.9** *Equivalently, a LGCS on $E$ is a complex structure on $E \oplus E^*$ which is orthogonal with respect to the natural pairing.*

*Proof.* Straightforward. □

Interestingly, the usual concepts of complex and symplectic structures on $E$ are contained in the notion of a linear generalized complex structure as follows:

**Example 1.4.4** Let $J : E \to E$ be a complex structure on $E$ and $\omega \in \Lambda^2 E^*$ a symplectic form. Then the following transformations are linear generalized complex structures on $E$

$$\mathcal{J}_J \doteq \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} \qquad \mathcal{J}_\omega \doteq \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

△

**Example 1.4.5** If $\mathcal{J}$ is a linear generalized complex structure on $E$ and $A \in O(E \oplus E^*)$, then $A \circ \mathcal{J} \circ A^{-1}$ is also a LGCS on $E$. In particular, if $B \in \Lambda^2 E^*$, then $exp(B) \circ \mathcal{J} \circ exp(-B)$ is a LGCS.

△

**Proposition 1.4.10** *The vector space $E$ admits a linear generalized complex structure if, and only if, it has even dimension.*

The next result says that studying linear generalized complex structures on $E$ is the same as studying maximal isotropic subspaces in the complexification of $E \oplus E^*$ with zero real index.

**Proposition 1.4.11** *A linear generalized complex structure on $E$ is equivalent to the choice of a linear Dirac structure $L$ on $(E \oplus E^*) \otimes \mathbb{C}$ such that $L \cap \bar{L} = \{0\}$.*

*Proof.* Let $\mathcal{J}$ be a linear generalized complex structure on $V$. From the orthogonality condition, it follows that its $+i$ eigenspace $L \subset (E \oplus E^*) \otimes \mathbb{C}$ is isotropic. Hence, $L$ is maximal since it is half-dimensional. Now, since $\bar{L}$ is the $-i$ eigenspace of $\mathcal{J}$ we have $(E \oplus E^*) \otimes \mathbb{C} = L \oplus \bar{L}$. Conversely, if $L \subset (E \oplus E^*) \otimes \mathbb{C}$ is a maximal isotropic subspace such that $L \cap \bar{L} = \{0\}$, then one can define a LGCS on $E$ by letting $L$ and $\bar{L}$ be the $+i$ and $-i$ eigenspaces. □

# Chapter 2

# Dirac Geometry

*In this chapter we are interested in studying Dirac structures, a geometrical structure that unifies both Poisson and presymplectic geometries. For that reason, we introduce the Courant bracket on sections of $TM \oplus T^*M$, where $M$ is a smooth manifold of dimension $m$, and we present the more general concept of a Courant algebroid. The Ševera's classification of exact Courant algebroids is also given. We end by describing any Lagrangian (maximally isotropic) subbundle of an exact Courant algebroid.*

## 2.1 Preliminaries

In this section we present some results and definitions needed throughout the chapter.

### 2.1.1 Lie Algebroids

We begin by presenting the notion of a Lie algebroid.

**Definition 2.1.1** *A **Lie algebroid** is a vector bundle $A \to M$ equipped with a Lie bracket $[\cdot,\cdot]$ on sections of $A$ and a bundle map $a : A \to TM$, called the anchor. The anchor must induce a Lie algebra morphism $\Gamma(A) \to \mathfrak{X}(M)$*

$$a([X,Y]) = [a(X), a(Y)] \qquad \forall X, Y \in \Gamma(A) \tag{2.1}$$

*and the following Leibniz rule must be satisfied:*

$$[X, fY] = f[X,Y] + (a(X)f)Y \qquad \forall X, Y \in \Gamma(A) \text{ and } f \in C^\infty(M) \tag{2.2}$$

Thus, such notion generalizes the tangent bundle in the sense its sections act on functions via the anchor map and the bracket respects the Leibniz rule under multiplication by functions.

**Example 2.1.1** A Lie algebroid over a single point and with the zero anchor is simply a Lie algebra.

△

**Example 2.1.2** The tangent bundle is a Lie algebroid if we consider the identity map as the anchor. Such example is called the *standard algebroid*.

△

**Example 2.1.3** Any regular distribution $D \subset TM$ which is involutive defines a Lie algebroid over $M$ by considering the anchor map as the inclusion $D \hookrightarrow TM$ and the Lie bracket as the usual Lie bracket of vector fields.

△

**Example 2.1.4** (Atiyah algebroid) Given a principal bundle $G \circlearrowright P \xrightarrow{\pi} M$, let us consider $A \doteq TP/G$. Then, sections of $A$ are $G-$invariant vector fields on $P$. Such space is closed under the Lie bracket and since functions on $M$ can be identified with $G-$invariant functions on $P$ we have that $A \to M$ is naturally a Lie algebroid. In this example we have a surjective anchor $d\pi : A \to TM$ which defines an exact sequence of vector bundles on $M$:

$$0 \longrightarrow P \times_G \mathfrak{g} \longrightarrow A \longrightarrow TM \longrightarrow 0 \tag{2.3}$$

where $\mathfrak{g}$ denotes the Lie algebra of $G$.

△

**Example 2.1.5** (Action algebroid) Let $\mathfrak{g}$ be a Lie algebra acting on a manifold, i.e., we have a Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(M), X \mapsto \tilde{X}$. The trivial bundle $A \doteq \mathfrak{g} \times M$ can be turned into a Lie algebroid over $M$ by considering the anchor map as $(X, p) \mapsto \tilde{X}(p)$ and by defining the bracket on sections as

$$[\alpha, \beta](p) \doteq [\alpha(p), \beta(p)] + \widetilde{\beta(p)} \cdot \alpha - \widetilde{\alpha(p)} \cdot \beta$$

where sections of $A$ are identified with maps $M \to \mathfrak{g}$.

△

**Remark 2.1.1** The concept of a Lie algebroid can be complexified and we define a complex Lie algebroid by requiring $E$ to be a complex bundle and the anchor $a : E \to TM \otimes \mathbb{C}$ a complex map, satisfying complexified conditions (2.1) and (2.2).

⋄

Given a Lie algebroid $(A \to M, [\cdot, \cdot], \rho)$, we can define a degree one derivation, called the *Lie algebroid differential*, $d_A : \Gamma\left(\Lambda^k A^*\right) \to \Gamma\left(\Lambda^{k+1} A^*\right)$, which is given by a formula identical to the Cartan formula for the de Rham differential:

$$\begin{aligned} d_A\xi(X_1, \ldots, X_{k+1}) \doteq & \sum_{i=1}^{k} (-1)^i \rho(X_i)\xi(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) + \\ & + \sum_{i<j} (-1)^{i+j} \xi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \end{aligned} \tag{2.4}$$

where $\xi \in \Gamma\left(\Lambda^k A^*\right)$ and each $X_i$ is a section of $A$, for $i = 1, \ldots, k+1$. In particular, one can consider the Lie algebroid cohomology.

Conversely, any Lie algebroid $A \to M$ can be characterized as a differential graded algebra structure on $\Gamma(\Lambda^\bullet A^*)$, which may be interpreted in the supergeometric language as a homological vector field (also known as a $Q-$manifold structure) on $A[1]$. In this setting $\Gamma(\Lambda^\bullet A^*)$ is identified with the algebra of functions on $A[1]$.

More precisely, let $A \to M$ be a vector bundle. Given a degree one derivation $d_A : S^\bullet\Gamma(A[1])^* \to S^{\bullet+1}\Gamma(A[1])^*$, there exists a unique bundle map $\rho : \Gamma(A) \to \mathfrak{X}(M)$ and a unique graded skew-symmetric bilinear map $[\cdot,\cdot] : \Gamma(A) \otimes \Gamma(A) \to \Gamma(A)$, satisfying the Leibniz rule, such that the graded analogue of (2.4) holds. Moreover, the Jacobi identity is satisfied if and only if $d_A^2 = 0$.

In fact, if $X$ is a section of $A$, we identify it with $\iota_X$ and the bracket we consider is the derived bracket (see e.g. [9]), i.e.,

$$\iota_{[X,Y]}\xi \doteq [[\iota_X, d_A], \iota_Y]\xi$$

The anchor map is then uniquely defined by $\rho(X)f = [\iota_X, d_A]f$. Here, $f \in C^\infty(M)$, $\xi \in \Gamma(A^*)$ and $X, Y \in \Gamma(A)$. See e.g. [10] for more details.

**Example 2.1.6** (Chevalley-Eilenberg) In the particular case of a Lie algebroid over a point, i.e., a Lie algebra, the Lie algebroid differential is the *Chevalley-Eilenberg differential*, which we denote by $d_{CE}$:

$$d_{CE}\xi(X_1, \ldots, X_{k+1}) \doteq \sum_{i<j}(-1)^{i+j}\xi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \tag{2.5}$$

One can check (2.5) completely determines the Lie algebra structure, with the Jacobi identity being equivalent to $d_{CE}^2 = 0$. Note that the Lie bracket is a map $\Lambda^2\mathfrak{g} \to \mathfrak{g}$, so its dual is a map $S^1(g^*[1]) \to S^2(g^*[1])$ and since $S^\bullet(g^*[1])$ is free it determines $d_{CE}$.

△

**Definition 2.1.2** (Lie Bialgebroid) *Let $A \to M$ be a Lie algebroid and let us assume $A^*$ also has the structure of a Lie algebroid. We say $(A, A^*)$ is a **Lie bialgebroid** if the Lie algebroid differential $d_A$ is a derivation of the Schouten bracket on multi-sections of $A$, i.e.,*

$$d_A[X, Y] = [d_A X, Y] + [X, d_A Y]$$

**Example 2.1.7** The most simple example is $(TM, T^*M)$. We take the usual Lie algebroid structure on $TM$ and the trivial structure on $T^*M$.

△

**Example 2.1.8** More interesting, if $(M, \Pi)$ is a Poisson manifold (see discussion below), then $T^*M$ inherits a Lie algebroid structure with anchor map $\varphi_\Pi : T^*M \to TM$ as in (2.8), which satisfies $\varphi_\Pi(df) = X_f$, and Lie bracket on $\Omega^1(M)$ given by the *Koszul bracket*

$$[\xi, \eta] \doteq \mathcal{L}_{\varphi_\Pi(\xi)}\eta - \mathcal{L}_{\varphi_\Pi(\xi)}\beta - d\Pi(\xi, \eta)$$

△

**Remark 2.1.2** In general, any Lie algebroid can be viewed as a Lie bialgebroid by considering the trivial structure on the dual bundle, i.e., the zero anchor and the zero bracket.

◇

### 2.1.2 Poisson and Presymplectic Structures

As mentioned in the beginning of the chapter we are interested in studying Dirac structures. Moreover, such concept allows us to treat two types of "degenerate" symplectic geometry in a unified manner. In the next paragraphs we review these two types, namely Poisson and presymplectic geometries. A main idea in the next section then will be to understand both structures as certain subbundles of $TM \oplus T^*M$ given as graphs of the maps 2.6 and 2.8 below.

It is known that a symplectic structure on a manifold is given either by a non-degenerate closed $2-$form or by a non-degenerate Poisson bivector field. If we drop the non-degeneracy assumption then we are led by the first approach to the notion of a presymplectic structure, as the second leads us to Poisson structures (see e.g. [6]).

If $(M, \omega)$ is a symplectic manifold, that is, $M$ is a smooth manifold and $\omega \in \Omega^2_{cl}(M)$ is non-degenerate, then the bundle map

$$\varphi_\omega : TM \to T^*M,\ X \mapsto \iota_X \omega \tag{2.6}$$

is an isomorphism. Thus, given any function $f \in C^\infty(M)$ we can always consider its associated Hamiltonian vector field, $X_f$, which is the unique vector field satisfying the following condition

$$\iota_{X_f}\omega = df \tag{2.7}$$

Moreover, we have a bilinear map on functions $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ that measures the rate of change of a function $g$ along the flow of the Hamiltonian vector field of a function $f$:

$$\{f, g\} \doteq \omega(X_g, X_f) = \mathcal{L}_{X_f} g$$

Such map, known as the Poisson bracket, makes the pair $(C^\infty, \{\cdot,\cdot\})$ into a Poisson algebra, i.e., $\{\cdot,\cdot\}$ is a Lie bracket which is also compatible with the product on $C^\infty(M)$ via the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g$$

Now, if we consider $\Pi \in \Gamma(\Lambda^2 TM)$ to be the bivector field uniquely determined by

$$\Pi(df, dg) = \{f, g\}$$

then it defines a bundle map

$$\varphi_\Pi : T^*M \to TM,\ \xi \mapsto \iota_\xi \Pi \tag{2.8}$$

and we see that $\varphi_\Pi = (\varphi_\omega)^{-1}$.

More generally, given a bivector field $\Pi \in \Gamma(\Lambda^2 TM)$ we say it is non-degenerate if the map in (2.8) is an isomorphism. Furthermore, since

$$\frac{1}{2}[\Pi,\Pi](df,dg,dh) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\},$$

the associated bracket $\{f,g\} = \Pi(df,dg)$ makes $(C^\infty(M),\{\cdot,\cdot\})$ into a Lie algebra if and only if $[\Pi,\Pi] = 0$, where $[\cdot,\cdot] : \Gamma(\Lambda^p TM) \otimes \Gamma(\Lambda^q TM) \to \Gamma\left(\Lambda^{p+q-1} TM\right)$ denotes the Schouten bracket of multivector fields.

**Definition 2.1.3** *A bivector field $\Pi \in \Gamma(\Lambda^2 TM)$ satisfying $[\Pi,\Pi] = 0$ is called a* ***Poisson*** *structure on $M$. A pair $(M,\Pi)$, where $\Pi$ is a Poisson structure on $M$ is a* ***Poisson manifold****.*

To sum up, we have a one-to-one correspondence between non-degenerate bivector fields and non-degenerate $2-$forms on $M$ and a bivector field is Poisson if an only if the corresponding $2-$form is closed. Such correspondence is given by the relation

$$\Pi(df,dg) = \omega(X_f,X_g)$$

### Presymplectic manifolds

A $2-$form on a manifold is said to be *presymplectic* if it is closed, but not necessarily non-degenerate.

**Definition 2.1.4** *A* ***presymplectic manifold*** *is a manifold equipped with a presymplectic form.*

Note that still makes sense considering Hamiltonian vector fields on a presymplectic manifold $(M,\omega)$ defined via (2.7) however, because of the possible degeneracy of $\omega$, we can't ensure every function admits a Hamiltonian vector field. For instance, consider $f \in C^\infty(M)$ such that $df$ does not belong to the image of $\varphi_\omega$ as defined in (2.6).

**Example 2.1.9** On a symplectic manifold $(M,\omega)$, let us consider a submanifold $\iota : N \hookrightarrow M$. The $2-$form $\iota^*\omega$ on $N$ is always closed, but in general it fails to be non-degenerate.

△

More generally, any presymplectic form can be represented as the pullback of a symplectic form. In fact, recall we have a natural symplectic form $\Omega$ on the cotangent bundle $T^*M$, which is given by the negative of the de Rham differential of the so called tautological $1-$form on $T^*M$ [1]. Furthermore, if $B$ is any closed $2-$form, the form $\Omega_B \doteq \Omega + \tilde{\pi}^*B$ is also a symplectic form on $T^*M$, where $\tilde{\pi} : T^*M \to M$ denotes the projection[2]. Indeed, one can find local coordinates such that $\Omega_B$ is represented by the

[1]If $\varphi$ is a point on $T^*M$ (i.e., $\varphi : T_pM \to \mathbb{R}$ is linear, where $p = \tilde{\pi}(\varphi)$ and $\tilde{\pi} : T^*M \to M$), then the value of the tautological $1-$form, $\tau$, at $\varphi$ is defined by $\tau_\varphi \doteq \tilde{\pi}^*\varphi$.

[2]This symplectic form is also sometimes referred to as the *magnetic symplectic form* because $B$ has the physical interpretation of representing a homogeneous magnetic field.

following (constant) skew-symmetric matrix

$$\begin{pmatrix} B & Id \\ -Id & 0 \end{pmatrix}$$

which is nonsingular so that $\Omega_B$ is non-degenerate [3]. Plus, the form $\Omega_B$ is obviously closed since both $\Omega$ and $B$ are closed and the pullback commutes with exterior differentiation.

Hence, if $(M, \omega)$ is a presymplectic manifold, then $\omega$ is the pullback of the form $\Omega_\omega$ on $T^*M$ by its zero section, $z$. That is,

$$\omega = z^*\Omega_\omega = z^*(\Omega + \tilde{\pi}^*\omega)$$

Moreover, if a presymplectic form is of constant rank, then the following result due to $Gotay$ holds [11].

**Proposition 2.1.1** *Every presymplectic manifold $(M, \omega)$ (with $\omega$ of constant rank) can be coisotropically embedded into a symplectic manifold. Such construction is unique up to a local symplectomorphism in a neighborhood of $M$.*

For every $p \in M$ let $E_p$ be the kernel of $\omega_p$, i.e., $E_p \doteq \{X \in T_pM\,;\, \omega_p(X, \cdot) = 0\}$. Now, consider the so called *characteristic bundle* of $(M, \omega)$, which is the vector bundle $\pi : E \to M$ with fibers $E_p$, for $p \in M$. What Gotay showed is that the total space $E^*$ of its dual bundle carries a symplectic structure in a neighborhood $U$ of the zero section. Moreover, $(M, \omega)$ is coisotropically embedded in $(U, \omega)$ as the zero section and further, all coisotropic embeddings of $(M, \omega)$ into a symplectic manifold are *neighborhood equivalent*[4] to this embedding.

**Remark 2.1.3** A *coisotropic embedding* of $(M, \omega)$ into a symplectic manifold $(N, \Omega)$ is an embedding $i : M \hookrightarrow N$ such that $i^*\Omega = \omega$ and $i(M)$ is coisotropic as a submanifold of $N$.

◇

## 2.2 The Courant Bracket

The key ingredients for the precise definition of a Dirac structure lie in some canonical geometrical structures that are present in the generalized tangent bundle $TM \oplus T^*M$. One of these is known as the *Courant bracket* and we introduce it here.

We begin by recalling some known identities from the Cartan calculus that will be useful for us later on.

As usual, throughout the text $\Omega^\bullet(M)$ will denote the graded algebra of differential forms on $M$ and $\mathfrak{X}(M)$ will denote the space of sections of the tangent bundle $TM$, i.e., the space of vector fields on $M$.

[3]For non-degeneracy, one can also check that $\Omega_B^n$ is a volume form. This holds because $\Omega^k(\tilde{\pi}^*B)^{n-k} = 0$ for every $k < n$ and hence $\Omega_B^n = \Omega^n$.

[4]Two embeddings $i_1 : M \to (N_1, \Omega_1)$ and $i_2 : M \to (N_2, \Omega_2)$ are said to be neighborhood equivalent if there exist open neighborhoods $U_j$ of $i_j(M)$ in $N_j$ and a symplectomorphism $\psi : U_1 \to U_2$ such that $\psi \circ i_1 = i_2$

In $\Omega^\bullet(M)$ let us consider the following three endomorphisms: the exterior derivative, the contraction by a vector field and the Lie derivative along a vector field. Also, let us take their graded commutator.

For $X, Y \in \mathfrak{X}(M)$ we have that[5]

$$[d, d] = 0 \tag{2.9}$$

$$[d, \mathcal{L}_X] = 0 \tag{2.10}$$

$$[d, \iota_X] = \mathcal{L}_X \tag{2.11}$$

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} \tag{2.12}$$

$$[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]} \tag{2.13}$$

$$[\iota_X, \iota_Y] = 0 \tag{2.14}$$

where $d$ is the exterior derivative, $\iota_X$ is the contraction by $X$ and $\mathcal{L}_X$ is the Lie derivative along $X$; while $[\cdot, \cdot]$ indicates the graded commutator in $End(\Omega^\bullet(M))$, except from $[X, Y]$ in (2.12) and (2.13) which denotes the Lie bracket of the vector fields $X, Y$.

One consequence of equation (2.9), $[d, d] = 0$, is that we obtain a cochain complex:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}(M) \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \cdots$$

and hence we can consider the de Rham cohomology "groups":

$$H^P_{dR}(M) \doteq \frac{\text{Ker}(d|_{\Omega^p})}{\text{Im}(d|_{\Omega^{p-1}})}$$

Now, observe that from equation (2.13) it follows that

$$\iota_{[X,Y]} = [[d, \iota_X], \iota_Y]$$

Moreover, the latter equation can be used as an alternative definition for the Lie bracket on vector fields. The bracket $[X, Y]$ is the unique vector field which acts on forms in the following way

$$[X, Y] \cdot \varphi = [[d, \iota_X], \iota_Y]\varphi \qquad \forall \varphi \in \Omega^\bullet(M)$$

In short, under the identification $X \simeq \iota_X$ what is known in literature as a *derived bracket* (see e.g. [9]) is identified with the Lie bracket. Later on we will introduce another derived bracket on sections of $TM \oplus T^*M$ instead.

Now, let $X, Y \in \mathfrak{X}(M)$ be vector fields and $\xi, \eta \in \Omega^1(M)$ be 1-forms on $M$. Then, $X + \xi$ and $Y + \eta$ are sections of $TM \oplus T^*M$.

[5] Equation (2.11) is known as Cartan's magic formula.

**Definition 2.2.1** *The **Courant bracket**, $[\cdot,\cdot]_C$, on sections of $TM \oplus T^*M$ is defined by*

$$[X+\xi, Y+\eta]_C \doteq [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d\big(\iota_X\eta - \iota_Y\xi\big) \tag{2.15}$$

Note that on vector fields the Courant bracket reduces to the usual Lie bracket and, on the other hand, on 1-forms it vanishes. In particular, if $\pi : TM \oplus T^*M \to TM$ denotes the natural projection, then

$$\pi\big([\alpha,\beta]_C\big) = [\pi(\alpha), \pi(\beta)] \tag{2.16}$$

for every $\alpha, \beta \in \Gamma(TM \oplus T^*M)$.

Now, sections of $TM \oplus T^*M$ act on forms in $M$ by considering on each fiber the action given by the Spin representation 1.16 introduced in Chapter 1. Thus, one can define a bracket on sections of $TM \oplus T^*M$ by defining it through its action on $\Omega^\bullet(M)$.

**Definition 2.2.2** *The **Dorfman bracket**, $[\cdot,\cdot]_D$, is given by*

$$[\alpha,\beta]_D \cdot \varphi \doteq [[d, \alpha\cdot], \beta\cdot]\varphi \tag{2.17}$$

*for every $\varphi \in \Omega^\bullet(M)$ and $\alpha, \beta \in \Gamma(TM \oplus T^*M)$.*

**Lemma 2.2.1** *The Dorfman bracket satisfies*

$$[X+\xi, Y+\eta]_D = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi \tag{2.18}$$

*Moreover, it satisfies a Leibniz rule which resembles the Jacobi identity*

$$[\alpha, [\beta,\gamma]_D]_D = [[\alpha,\beta]_D, \gamma]_D + [\beta, [\alpha,\gamma]_D]_D \tag{2.19}$$

*for every $\alpha, \beta$ and $\gamma$ sections of $TM \oplus T^*M$.*

Note that the Dorfman bracket is not skew-symmetric:

$$[X+\xi, X+\xi]_D = \mathcal{L}_X\xi - \iota_X d\xi = d\iota_X\xi = d\langle X+\xi, X+\xi\rangle$$

But it differs from being skew-symmetric by exact terms.

The next proposition tells us how the Courant and the Dorfman brackets are related.

**Proposition 2.2.1** *The Courant bracket is the skew-symmetrization of the Dorfman bracket. Moreover,*

$$[\cdot,\cdot]_C = [\cdot,\cdot]_D - d\langle\cdot,\cdot\rangle, \tag{2.20}$$

*where $\langle\cdot,\cdot\rangle$ is the natural pairing (considered fiberwise) introduced in the previous chapter, section 1.4.*

*Proof.* Let $\alpha = X+\xi$ and $\beta = Y+\eta$ be sections of $TM\oplus T^*M$. If we compute the difference between their Dorfman and Courant brackets, then we obtain

$$\begin{aligned}[\alpha,\beta]_D - [\alpha,\beta]_C &= \mathcal{L}_Y\xi - \iota_Y d\xi + \tfrac{1}{2}d\big(\iota_X\eta - \iota_Y\xi\big)\\ &= d(\iota_Y\xi) + \tfrac{1}{2}d(\iota_X\eta - \iota_Y\xi)\\ &= d\langle\alpha,\beta\rangle\end{aligned}$$

where in the second equality above we have used the Cartan's magic formula (2.11).

Direct computation shows the Courant bracket is the skew-symmetrization of the Dorfman bracket, i.e.,

$$\begin{aligned}\frac{1}{2}\Big([\alpha,\beta]_D - [\beta,\alpha]_D\Big) &= \frac{1}{2}\Big[\Big([X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi\Big) - \Big([Y,X] + \mathcal{L}_Y\xi - \iota_X d\eta\Big)\Big]\\ &= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d\big(\iota_X\eta - \iota_Y\xi\big)\\ &= [\alpha,\beta]_C\end{aligned}$$

where in the second equality above we have used the Cartan's magic formula (2.11) and the skew-symmetricity of the Lie bracket of vector fields. □

Although the Courant bracket is a skew-symmetric bracket it is not a Lie bracket since it does not satisfy the Jacobi identity. It is interesting though to measure its failure to satisfy the Jacobi identity.

If $\alpha,\beta$ and $\gamma$ are sections of $TM\oplus T^*M$ we define the Jacobiator to be the following trilinear operator

$$Jac(\alpha,\beta,\gamma) \doteq [[\alpha,\beta]_C,\gamma]_C + [[\beta,\gamma]_C,\alpha]_C + [[\gamma,\alpha]_C,\beta]_C$$

In the next Proposition we prove the Jacobiator can be expressed as the exterior derivative of a quantity we will call *Nijenhuis operator*. Thus, the Courant bracket satisfy the Jacobi identity up to an exact term.

**Proposition 2.2.2** *The following equality holds for sections of $TM\oplus T^*M$*

$$Jac(\alpha,\beta,\gamma) = d(Nij(\alpha,\beta,\gamma)), \tag{2.21}$$

*where*

$$Nij(\alpha,\beta,\gamma) \doteq \frac{1}{3}\Big(\langle[\alpha,\beta]_C,\gamma\rangle + \langle[\beta,\gamma]_C,\alpha\rangle + \langle[\gamma,\alpha]_C,\beta\rangle\Big)$$

*Proof.* Let $\alpha,\beta$ and $\gamma$ be sections of $TM\oplus T^*M$ and observe that

$$\begin{aligned}[[\alpha,\beta]_C,\gamma]_C &= [[\alpha,\beta]_C,\gamma]_D - d\langle[\alpha,\beta]_C,\gamma\rangle\\ &= [[\alpha,\beta]_D - d\langle\alpha,\beta\rangle,\gamma]_D - d\langle[\alpha,\beta]_C,\gamma\rangle\\ &= [[\alpha,\beta]_D,\gamma]_D - d\langle[\alpha,\beta]_C,\gamma\rangle\end{aligned} \tag{2.22}$$

On the other hand,

$$\begin{aligned}
[[\alpha,\beta]_C,\gamma]_C &= \frac{1}{2}\Big([[\alpha,\beta]_C,\gamma]_D - [\gamma,[\alpha,\beta]_C]_D\Big) \\
&= \frac{1}{4}\Big([[\alpha,\beta]_D,\gamma]_D - [[\beta,\alpha]_D,\gamma]_D - [\gamma,[\alpha,\beta]_D]_D + [\gamma,[\beta,\alpha]_D]_D\Big)
\end{aligned}$$

Next, introducing the notation $[\alpha,\beta]_D \doteq \alpha\circ\beta$ for simplicity, it follows from (2.19) that

$$[[\alpha,\beta]_C,\gamma]_C = \frac{1}{4}\Big(2\alpha\circ(\beta\circ\gamma) - 2\beta\circ(\alpha\circ\gamma) - \gamma\circ(\alpha\circ\beta) + \gamma\circ(\beta\circ\alpha)\Big)$$

Now, $Jac(\alpha,\beta,\gamma) = [[\alpha,\beta]_C,\gamma]_C + c.p.$, where $c.p.$ indicates cyclic permutations. Thus,

$$\begin{aligned}
Jac(\alpha,\beta,\gamma) &= \frac{1}{4}\Big(\alpha\circ(\beta\circ\gamma) - \beta\circ(\alpha\circ\gamma) + \gamma\circ(\alpha\circ\beta) + \\
&\quad -\gamma\circ(\beta\circ\alpha) + \beta\circ(\gamma\circ\alpha) - \alpha\circ(\gamma\circ\beta)\Big) \\
&= \frac{1}{4}\Big(\alpha\circ(\beta\circ\gamma) - \beta\circ(\alpha\circ\gamma) + \text{ c.p.}\Big) \\
&= \frac{1}{4}\Big((\alpha\circ\beta)\circ\gamma + \text{ c.p.}\Big)
\end{aligned}$$

and, therefore, using equation (2.22) above we find equality (2.21):

$$\begin{aligned}
Jac(\alpha,\beta,\gamma) &= \frac{1}{4}\Big([[\alpha,\beta]_C,\gamma]_C + d\langle[\alpha,\beta]_C,\gamma\rangle + \text{ c.p.}\Big) \\
&= \frac{1}{4}\Big(Jac(\alpha,\beta,\gamma) + 3dNij(\alpha,\beta,\gamma)\Big)
\end{aligned}$$

□

We now describe the failure of the Courant bracket to satisfy the second Lie algebroid (2.2) axiom.

**Proposition 2.2.3** *Let $f \in C^\infty(M)$. Then the Courant bracket satisfies*

$$[\alpha, f\beta]_C = f[\alpha,\beta]_C + (\pi(\alpha)f)\beta - \langle\alpha,\beta\rangle df \tag{2.23}$$

*Proof.* Let $\alpha = X + \xi$ and $\beta = Y + \eta$, then

$$\begin{aligned} [\alpha, f\beta]_C &= [X, fY] + \mathcal{L}_X f\eta - \mathcal{L}_{fY}\xi - \tfrac{1}{2}d\big(\iota_X(f\eta) - \iota_{fY}\xi\big) \\ &= f[X,Y] + (Xf)Y + f\mathcal{L}_X\eta + (Xf)\eta - f\mathcal{L}_Y\xi - (\iota_Y\xi)df + \\ &\quad -\tfrac{1}{2}fd\big(\iota_X\eta - \iota_Y\xi\big) - \tfrac{1}{2}\big((\iota_X\eta - \iota_Y\xi)df\big) \\ &= f[X+\xi, Y+\eta]_C + (Xf)Y + (Xf)\eta - (\iota_Y\xi)df - \tfrac{1}{2}\big(\iota_X\eta - \iota_Y\xi\big)df \\ &= f[X+\xi, Y+\eta] + (Xf)(Y+\eta) - \langle X+\xi, Y+\eta\rangle df \end{aligned}$$

as we claimed. □

In short, again an exact term is what differs $\pi : TM \oplus T^*M \to TM$ with the Courant bracket from being a Lie algebroid.

Note that both properties (2.21) and (2.23) tells us that the Courant bracket is closely related to the inner product $\langle\cdot,\cdot\rangle$. The next Proposition stresses this relationship.

**Proposition 2.2.4** *Differentiation of the natural pairing can be expressed in terms of the Courant bracket as follows*

$$\pi(\alpha)\langle\beta,\gamma\rangle = \langle[\alpha,\beta]_C + d\langle\alpha,\beta\rangle, \gamma\rangle + \langle\beta, [\alpha,\gamma]_C + d\langle\alpha,\gamma\rangle\rangle \tag{2.24}$$

*Proof.* Using the notation introduced in the proof of Proposition 2.2.2 what we want to show is that

$$\pi(\alpha)\langle\beta,\gamma\rangle = \langle\alpha\circ\beta,\gamma\rangle + \langle\beta,\alpha\circ\gamma\rangle.$$

Let $\alpha = X + \xi$, $\beta = Y + \eta$ and $\gamma = Z + \zeta$, then

$$\begin{aligned} \langle\alpha\circ\beta,\gamma\rangle + \langle\beta,\alpha\circ\gamma\rangle &= \frac{1}{2}\Big(\iota_{[X,Y]}\zeta + \iota_Z\big(\mathcal{L}_X\eta - \iota_Y d\xi\big) + \iota_{[X,Z]}\eta + \iota_Y\big(\mathcal{L}_X\zeta - \iota_X d\xi\big)\Big) \\ &= \frac{1}{2}\Big(\mathcal{L}_X\iota_Y\zeta + \mathcal{L}_X\iota_Z\eta\Big) \\ &= \mathcal{L}_X\left(\frac{1}{2}\Big(\iota_Y\zeta + \iota_Z\eta\Big)\right) \\ &= \pi(\alpha)\langle\beta,\gamma\rangle \end{aligned}$$

where in the second equality above we have used the formulas (2.13) and (2.14). □

In other words, in terms of the Dorfman bracket, $[\alpha,\cdot]$ is a derivation of the inner product.

**Remark 2.2.1** Note that (2.19) tell us that the adjoint endomorphism $[\alpha,\cdot]$ is also a derivation of the Dorfman bracket itself. Thus, over each fiber we have a (left) Loday algebra.

◇

The fundamental properties (2.16), (2.21), (2.23) and (2.24) make $(TM \oplus T^*M, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_C, \pi)$ into an example of a structure called a Courant algebroid as it was introduced in the paper [12] by Liu-Weinstein-Xu. There, the notion of a Courant algebroid was introduced in order to obtain a generalization of the double construction to Lie bialgebroids.

We present here, however, a simplification due to Roytenberg (see e.g. [13]) which uses the Dorfman bracket instead. It is worth mentioning yet, that such structure first appeared in the paper [14] by T.J. Courant.

**Definition 2.2.3** (Roytenberg,[13]) *A **Courant algebroid** is a quadruple $(E, (\cdot, \cdot), [\cdot, \cdot], \rho)$ where $\rho : E \to M$ is a vector bundle, $(\cdot, \cdot)$ is a non-degenerate symmetric bilinear form on the fibers of $E$, $[\cdot, \cdot]$ is a bilinear bracket on the space $\Gamma(E)$ and $\rho : E \to TM$ which is called the anchor and is a smooth bundle map, such that it satisfies the following properties for all $a, b, c \in \Gamma(E)$ and $f \in C^\infty(M)$.*

*(C1)* $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$

*(C2)* $\rho(a)(b, c) = ([a, b], c) + (b, [a, c])$

*(C3)* $[a, b] + [b, a] = \rho^*(d(a, b))$

*(C4)* $[a, fb] = f[a, b] + (\rho(a)f)b$

*(C5)* $\rho([a, b]) = [\rho(a), \rho(b)]$

*where $\rho^* : T^*M \to E^* \simeq E$ is the dual map. More precisely, we mean the composition $\kappa^{-1} \circ \rho^*$, where $\kappa : E \to E^*$ is the isomorphism induced by $(\cdot, \cdot)$.*

Sometimes, a bracket as in the definition above is referred to as a Courant bracket.

**Remark 2.2.2** Axioms $(iv)$ and $(v)$ can be deduced from the other three axioms [15]. In addition, condition $(iii)$ is equivalent to the following one: $[a, a] = \mathcal{D}\langle a, a \rangle$ for every $a \in \Gamma(E)$, where $\mathcal{D} \doteq 1/2(\rho^* \circ d)$.

◇

**Remark 2.2.3** The definition of $\mathcal{D}$ implies it satisfies the Leibniz rule,i.e.,

$$\mathcal{D}(fg) = f\mathcal{D}(g) + \mathcal{D}(f)g.$$

◇

**Remark 2.2.4** Note that the bracket $[\cdot, \cdot]$ in the above definition is not skew-symmetric but it satisfies the property in axiom 3. If it were skew-symmetric, then we would have a Lie algebroid. So, in particular, maximally isotropic subbundles which are closed under the Courant bracket gives us examples of Lie algebroids.

◇

**Example 2.2.1** When $M$ is just a single point a Courant algebroid is a quadratic Lie algebra $\mathfrak{g}$, i.e., on $\mathfrak{g}$ we have an invariant non-degenerate symmetric bilinear form.

△

**Example 2.2.2** Let $(A, A^*)$ be a pair of Lie algebroids in duality and denote the anchor by $a$ and $a^*$, respectively. Now, on $E \doteq A \oplus A^*$ let us define the following structure

$$\begin{aligned} (X+\xi, Y+\eta) &\doteq \xi(Y)+\eta(X) \\ [X+\xi, Y+\eta] &\doteq ([X,Y]_A + \mathcal{L}_\xi Y - \iota_\eta d_{A^*} X) + \\ &+ ([\xi,\eta]_{A^*} + \mathcal{L}_X \eta - \iota_Y d_A \xi) \\ \rho(X+\xi) &\doteq a(X) + a^*(\xi) \end{aligned}$$

If $(A, A^*)$ is a Lie bialgebroid, then $(E, (\cdot,\cdot), [\cdot,\cdot], \rho)$ is a Courant algebroid.

△

**Example 2.2.3** As a special case of the previous example recall Example 2.1.7, i.e., consider the standard Lie algebroid $TM$ and on $T^*M$ the zero anchor and zero bracket. Then the bracket as above reduces to the Dorfman bracket (2.18). See also Theorem 2.2.5 below.

△

**Definition 2.2.4** *Let $H \in \Omega^3_{cl}(M)$ be an arbitrary closed 3-form on $M$. Then the* ***twisted Dorfman bracket****, $[\cdot,\cdot]_H$ on $\Gamma(TM \oplus T^*M)$ is defined as*

$$[X+\xi, Y+\eta]_H \doteq [X+\xi, Y+\eta]_D - \iota_Y \iota_X H$$

**Remark 2.2.5** In the definition above we could have considered $H \in \Omega^3(M)$. The obtained bracket still makes sense, but one can show that $Jac \equiv 0$ if and only if $dH = 0$.

◇

The twisted Dorfman bracket is also a derived bracket. Recall (2.17):

$$[\alpha,\beta]_D \cdot \varphi \doteq [[d, \alpha\cdot], \beta\cdot]\varphi$$

Now, for $H \in \Omega^3_{cl}(M)$ we can consider the “differential” $d_H \doteq d + H \wedge \cdot$ which still satisfies $d_H^2 = 0$:

$$\begin{aligned} d_H^2 \cdot &= d_H(d + H \wedge \cdot) \\ &= d^2 \cdot + d(H \wedge \cdot) + H \wedge d \cdot + H \wedge H \wedge \cdot \\ &= dH \wedge \cdot - H \wedge d \cdot + H \wedge d \cdot \\ &= 0 \end{aligned}$$

Note, however, that $d_H$ is not of degree one and it is not a derivation, but it is still odd.

One can easily check that the twisted Dorfman bracket acts on forms as follows

$$[\alpha,\beta]_H \cdot \varphi = [[d_H, \alpha\cdot], \beta\cdot]\varphi$$

More interestingly,

**Theorem 2.2.5.** *The twisted Dorfman bracket turns $TM \oplus T^*M$ into a Courant algebroid with anchor map $\pi : TM \oplus T^*M \to TM$. Such structure will be denoted by $\mathbb{T}M^{(H)}$ (and in the particular case when $H \equiv 0$ it will be denoted simply by $\mathbb{T}M$).*

*Proof.* We start by checking (C2) and (C3). As before, let $\alpha = X + \xi, \beta = Y + \eta$ and $\gamma = Z + \zeta$. For axiom (C2) we proceed as in the proof of Proposition 2.2.4. For (C3):

$$[X + \xi, X + \xi]_H = [X, X] + \mathcal{L}_X\xi - \iota_X d\xi - \iota_X\iota_X H = d\iota_X\xi = \mathcal{D}\langle X + \xi, X + \xi\rangle$$

where we have used the Cartan's magic formula (2.11). Finally, we prove (C1):

$$\begin{aligned}
[[\alpha, \beta]_H, \gamma]_H + [\beta, [\alpha, \gamma]_H]_H &= [[X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi - \iota_Y\iota_X H, \gamma]_H + \\
&\quad [\beta, [X, Z] + \mathcal{L}_X\zeta - \iota_Z d\xi - \iota_Z\iota_X H]_H \\
&= [[X, Y], Z] + \mathcal{L}_{[X,Y]}\zeta - \iota_Z d\mathcal{L}_X\eta + \iota_Z d\iota_Y d\xi + \iota_Z d\iota_Y\iota_X H + \\
&\quad -\iota_Z\iota_{[X,Y]}H + [Y, [X, Z]] + \mathcal{L}_Y\mathcal{L}_X\zeta - \mathcal{L}_Y\iota_Z d\xi - \mathcal{L}_Y\iota_Z\iota_X H + \\
&\quad -\iota_{[X,Z]}d\eta - \iota_{[X,Z]}\iota_Y H \\
&= [X + \xi, [Y, Z] + \mathcal{L}_Y\zeta - \iota_Z d\eta - \iota_Z\iota_Y H]_H \\
&= [X + \xi, [Y + \eta, Z + \zeta]_H]_H
\end{aligned}$$

where in the third equation above we have used the formulas (2.11),(2.12) and (2.13), plus the fact that the Lie bracket of vector fields satisfies the Jacobi identity and also the closeness of $H$.

Note that remark 2.2.2 tells us we are done. □

**Definition 2.2.5** *A Courant algebroid is called* ***exact*** *if its underlying vector bundle fits into an exact sequence*

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0 \tag{2.25}$$

*where $\rho^* : T^*M \to E$ is defined by $(\rho^*\xi, e) = \xi(\rho e)$, for every $\xi \in T^*M$ and $e \in \Gamma(E)$.*

**Example 2.2.4** If $H \in \Omega^3_{cl}(M)$, then $\mathbb{T}M^{(H)}$ is obviously an exact Courant algebroid.

△

Note that for an exact Courant algebroid the inclusion $T^*M \subset E$ is always isotropic because

$$(\rho^*\xi, \rho^*\eta) = \xi(\rho\rho^*\eta)$$

and $\rho\rho^* \equiv 0$. Moreover, axiom (3) in Definition 2.2.3 implies $[\xi, \xi] = 0$, for every $\xi \in T^*M$. More generally, $[\cdot, \cdot]\big|_{T^*M} \equiv 0$. In fact, let $e \in \Gamma(E)$ and $\omega \in \Omega^1(M)$ then, for any $f \in \Gamma(E)$, we have that

$$\begin{aligned}
([e, \rho^*\omega], f) &= \rho(e)(\rho^*\omega, f) - (\rho^*\omega, [e, f]) \\
&= \rho(e)(\omega, \rho(f)) - (\omega, \rho[e, f]) \\
&= \rho(e)(\omega, \rho(f)) - (\omega, [\rho(e), \rho(f)]) \\
&= \rho(e)(\omega, \rho(f)) - \rho(e)(\omega, \rho(f)) + (\mathcal{L}_{\rho(e)}\omega, \rho(f)) \\
&= (\rho^*\mathcal{L}_{\rho(e)}\omega, f)
\end{aligned}$$

where we have used in the first equality above axiom (C2) and in the third axiom (C5).

In short, $[e, \rho^*\omega] = \rho^*\mathcal{L}_{\rho(e)}\omega$ and the result follows.

Also, such a vector bundle $E$ is always isomorphic to $TM \oplus T^*M$ by choosing an isotropic complement of $T^*M \simeq ker\rho$. In fact, if we choose any complement $F$ of $T^*M$ in $E$, then the map $T^*M \to F^*$ given by $\xi \mapsto (\xi, \cdot)\big|_F$ is an isomorphism. Now, since $F^* \simeq F$, we have a map $\varphi : F \to T^*M$ and setting $F'$ to be the graph of $-\frac{1}{2}\varphi$ we have that $F'$ is an isotropic complement of $T^*M$ (see Propositions 1.1.1 and 1.2.1 ).

In particular, if we have an exact Courant algebroid, then we can always split the associated exact sequence (2.25).

**Theorem 2.2.6** (Ševera, [16]). *A Courant algebroid $\pi : E \to M$ is exact if and only if it is isomorphic to $\mathbb{T}M^{(H)}$ for some $H \in \Omega^3_{cl}(M)$.*

*Proof.* Let us choose an isotropic splitting

$$0 \to T^*M \underset{s*}{\overset{\rho^*}{\rightleftarrows}} E \underset{s}{\overset{\rho}{\rightleftarrows}} TM \to 0$$

Then we can transport the Courant structure of $E$ to $TM \oplus T^*M$. First, the "connection" $s : TM \to E$ identifies $E$ with $TM \oplus T^*M$ with the canonical inner product (up to a $1/2$ factor). If $X, Y \in TM$ and $\xi, \eta \in T^*M$, then

$$(sX + \rho^*\xi, sY + \rho^*\eta) = \xi(\rho sY) + \eta(\rho sX) = \xi(Y) + \eta(X) = 2\langle X + \xi, Y + \eta\rangle$$

because $(sX, sY) = 0$ and $\rho\rho^* = 0$. Moreover,

$$[sX + \rho^*\xi, sY + \rho^*\eta] = [sX, sY] + [sX, \rho^*\eta] + [\rho^*\xi, sY] \tag{2.26}$$

where the second term on the right-hand side above has zero image under the anchor map $\rho$ so that it can be viewed as an element in $\Omega^1(M)$. Further, we can apply it on a vector field to obtain

$$\begin{aligned} [sX, \rho^*\eta](Z) &= ([sX, \rho^*\eta], sZ) \\ &= X(\rho^*\eta, sZ) - (\rho^*\eta, [sX, sZ]) \\ &= X\eta(Z) - \eta([X, Z]) \\ &= \iota_Z\mathcal{L}_X\eta \end{aligned}$$

where we have used in the second equality above axiom (C2). Therefore, $[sX, \rho^*\eta] = \mathcal{L}_X\eta$.

Now, similarly, the third term is given by

$$\begin{aligned} ([\rho^*\xi, sY], sZ) &= (\mathcal{D}(sY, \rho^*\xi) - [sY, \rho^*\xi], sZ) \\ &= \iota_Z d\iota_Y\xi - \iota_Z\mathcal{L}_Y\xi \\ &= -\iota_Z\iota_Y d\xi \end{aligned}$$

and so $[\rho^*\xi, sY] = -\iota_Y d\xi$. In the first equality above we have used axiom (C3) and in the third the Cartan's magic formula (2.11). Finally, if we look at the difference

$$[s(X), s(Y)] - s[X, Y] = \rho^* H(X, Y)$$

and using the properties of a Courant algebroid we conclude that $H \in \Omega^3(M)$, that is, $H$ is $C^\infty(M)-$linear and completely skew-symmetric. Note that $H$ is given by

$$H(X, Y) \doteq s^*[sX, sY]$$

Thus, the first term in (2.26) equals to $[X, Y] - \iota_Y \iota_X H$. Furthermore, from the Jacobi identity (C1) for the bracket $[\cdot,\cdot]$ on $E$ one deduces that $dH = 0$, which concludes the proof.

The Courant bracket becomes

$$[X + \xi, Y + \eta] = [X + \xi, Y + \eta]_H$$

□

Now, let us consider how the form $H$ changes if we change the splitting. Once a connection $s$ is chosen, any other one, differs from $s$ by the graph of a $2-$form. That is, the space of connections form an affine space over $\Omega^2(M)$.

In fact, let $B = s - s' : TM \to T^*M$, where $s, s' : TM \to E$ are two splittings (note that $\rho(s - s') = 0$). If $X \in TM$, then in the $s'$ splitting we have $s(X) = X + (s - s')X$. Using the fact the two splittings are isotropic it follows that

$$\begin{aligned}
(s - s')(X)(X) &= \langle X + b(X, \cdot), X + b(X, \cdot)\rangle \\
&= \langle X, X\rangle + \langle X, b(X, \cdot)\rangle + \langle b(X, \cdot), X\rangle + \langle b(X, \cdot), b(X, \cdot)\rangle \\
&= \langle X, b(X, \cdot)\rangle + \langle b(X, \cdot), X\rangle \\
&= 2b(X, X) = 0
\end{aligned}$$

where $b(\cdot,\cdot)$ is the bilinear form induced by $B$ and $\langle\cdot,\cdot\rangle$ is the natural pairing in $TM \oplus T^*M$. Therefore $B$ is skew-symmetric, i.e., $B \in \Lambda^2(T^*M)$.

Now, the corresponding "curvatures", $H'$ and $H$, are related simply by

$$H' = H + d\omega$$

Indeed, if we compute the bracket $[\cdot,\cdot]_H$ in the $s'$ splitting, we obtain

$$\begin{aligned}
[X + \iota_X\omega, Y + \iota_Y\omega]_H &= [X, Y] + \mathcal{L}_X\iota_Y\omega - \iota_Y d\big(\iota_X\omega\big) - \iota_Y\iota_X H \\
&= [X, Y] + \iota_{[X,Y]}\omega - \iota_Y\mathcal{L}_X\omega - \iota_Y d\big(\iota_X\omega\big) - \iota_Y\iota_X H \\
&= [X, Y] + \iota_{[X,Y]}\omega - \iota_Y\iota_X(H + d\omega)
\end{aligned}$$

In particular, the isomorphism classes of exact Courant algebroids are in one-to-one correspondence with a degree-3 class in the de Rham cohomology of M, which is known in literature as the Ševera class.

**Example 2.2.5** (Anton Alekseev, [17],[18]) Let us consider the case when $M = G$ is a Lie group whose Lie algebra $\mathfrak{g}$ is quadratic, i.e. we have a compatible (ad-invariant) non-degenerate symmetric bilinear form $B(\cdot,\cdot)$ on $\mathfrak{G}$. Then the exact Courant algebroid structure on $E = TG \oplus T^*G$ is given on constant sections by the following

$$\begin{array}{rcl} E & = & G \times (\mathfrak{g} \oplus \mathfrak{g}) \\ \rho(x,y) & = & g \cdot x - y \cdot g \\ \langle (x,x'),(y,y')\rangle & = & B(x,y) - B(x',y') \\ (x,x') \circ (y,y') & = & ([x,y],[x',y']) \end{array}$$

where we have trivialized both $TG$ and $T^*G$ as $G \times \mathfrak{g}$ by left translation.

Note the adjoint action of $G$ on itself extends to an action of $G \times G$ to $G$ as follows

$$G \times G \to Diff(G),\ (g,h) \mapsto L_h \circ R_{g^{-1}}$$

where $L_a(g) = a \cdot g$ and $R_a(g) = g \cdot a$. Thus, the projection $\rho : E \to TG$ is nothing but the infinitesimal action $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{X}(G)$ on constant sections.

Moreover, also note the natural twisted bracket on $E$ is simply the Lie bracket on the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$.

Now, recall $G$ has a canonical closed $3-$form, namely the Cartan $3-form\ C \doteq B([\theta^L,\theta^L],\theta^L)$, where $\theta^L$ denotes the left-invariant Maurer-Cartan form. One can show the Ševera class of such exact Courant algebroid is precisely $[C]$.

$\triangle$

### 2.2.1 Symmetries of the Dorfman bracket

It is well known that the Lie bracket of (smooth) vector fields on a manifold is invariant under diffeomorphisms. Moreover, these are the only symmetries of the tangent bundle preserving the Lie bracket.

**Lemma 2.2.2** *Let $(f,F)$ be an automorphism of the tangent bundle $\pi : TM \to M$ such that $F$ preserves the Lie bracket on vector fields. That is, the pair $(f,F)$ makes the following diagram commute*

$$\begin{array}{ccc} TM & \xrightarrow{F} & TM \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[f]{} & M \end{array}$$

*$F$ is linear over each fiber and $F([X,Y]) = [F(X),F(Y)]$.*

*Then $F$ must be the differential of $f$.*

In contrast, there are more symmetries of $TM \oplus T^*M$ which preserve the twisted Dorfman bracket. We have an additional symmetry which we call a $B-$*field transform*.

**Example 2.2.6** Let $f : M \to M$ be a diffeomorphism. Then $\tilde{f} \in Aut(TM \oplus T^*M)$ given by

$$\tilde{f} \doteq \begin{pmatrix} df & 0 \\ 0 & (f^*)^{-1} \end{pmatrix}$$

is an orthogonal transformation such that $[\tilde{f}\cdot, \tilde{f}\cdot]_{(f^*)^{-1}H} = \tilde{f}[\cdot,\cdot]_H$. So this does not give a symmetry unless $f^*H = H$.

Indeed, $\tilde{f}$ is orthogonal:

$$\begin{aligned}
\left\langle \tilde{f}(X+\xi), \tilde{f}(Y+\eta) \right\rangle &= \left\langle df(X) + (f^*)^{-1}(\xi), df(Y) + (f^*)^{-1}(\eta) \right\rangle \\
&= \frac{1}{2}\Big( \big((f^*)^{-1}(\xi)\big)((df(Y)) + \big((f^*)^{-1}(\eta)\big)((df(X)) \Big) \\
&= \frac{1}{2}\Big( \xi(Y) + \eta(X) \Big) \\
&= \langle X+\xi, Y+\eta \rangle
\end{aligned}$$

Moreover, since $\iota_{df(X)}(f^*)^{-1} = (f^*)^{-1}\iota_X$ and the pullback commutes with the de Rham differential it follows that

$$[\tilde{f}(X+\xi), \tilde{f}(Y+\eta)]_{(f^*)^{-1}H} = \tilde{f}[X+\xi, Y+\eta]_H$$

△

**Example 2.2.7** If $B \in \Omega^2(M)$ is a 2-form on $M$ we can view it as a map $TM \to T^*M$ and we have an automorphism $(id, exp(B))$ of $TM \oplus T^*M$ which is, fiberwise, given by (1.15). Such an automorphism is an orthogonal transformation and it preserves the twisted Dorfman bracket if and only if it is closed, i.e., $dB = 0$. Indeed in this case it preserves the Courant bracket as well.

In fact, for $X+\xi, Y+\eta \in TM \oplus T^*M$ we have that

$$\begin{aligned}
\left[e^B(X+\xi), e^B(Y+\eta)\right]_H &= \left[X+\xi+\iota_X B, Y+\eta+\iota_Y B\right]_H \\
&= [X,Y] + \mathcal{L}_X(\eta+\iota_Y B) - \iota_Y d(\xi+\iota_X B) - \iota_Y\iota_X H \\
&= [X+\xi, Y+\eta]_H + \iota_{[X,Y]}B + \iota_Y\mathcal{L}_X B - \iota_Y d\iota_X B \\
&= [X+\xi, Y+\eta]_H + \iota_{[X,Y]}B + \iota_Y\iota_X dB \\
&= e^B\big([X+\xi, Y+\eta]_H\big) + \iota_Y\iota_X dB \\
&= e^B\big([X+\xi, Y+\eta]_{H-dB}\big)
\end{aligned}$$

where in the third equality above we have used equation (2.13), while in the fourth, the Cartan's magic formula (2.11).

△

We now prove that every symmetry of the Dorfman bracket arrives in some sense from the two given examples.

Given $H \in \Omega^3_{cl}(M)$, let us denote by $Diff_{[H]}$ the subgroup of diffeomorphisms of $M$ preserving the cohomology class $[H] \in H^3_{dR}(M)$. Then,

**Proposition 2.2.7** *The group of orthogonal automorphisms of an exact Courant algebroid is a semidirect product of $Diff_{[H]}$ with $\Omega^2_{cl}(M)$ for the corresponding cohomology class $[H]$.*

*Proof.* We know that if $\rho : E \to M$ is an exact Courant algebroid, then $E \simeq \mathbb{T}M^{(H)}$ for some $H \in \Omega^3_{cl}(M)$. Thus, let $F$ be an orthogonal automorphism of $\mathbb{T}M^{(H)}$ covering the diffeomorphism $f : M \to M$, with $f^*H = H$. Then, $F = \tilde{f} \circ exp(B)$, for some $B \in \Omega^2_{cl}(M)$. In fact, let $G = \tilde{f}^{-1} \circ F$ and $g \in C^\infty(M)$, then $G[\alpha, g\beta]_H = [G(\alpha), G(g\beta)]_H$ and from axiom $(C4)$ it follows that

$$\begin{aligned} G\big(g[\alpha,\beta]_H + (\pi(\alpha)g)\beta\big) &= g[G(\alpha), G(\beta)]_H + (\pi(G(\alpha))g)G(\beta) \\ \iff (\pi(\alpha)g)G(\beta) &= (\pi(G(\alpha))g)G(\beta) \\ \iff \pi &= \pi \circ G \end{aligned}$$

Thus, $G$ is an orthogonal automorphism of $\mathbb{T}M^{(H)}$ which preserves the projection $\pi : TM \oplus T^*M \to TM$ and hence from Proposition 1.4.1 there exists $B \in \Omega^2(M)$ such that $G = exp(B)$. Moreover, since $G$ preserves the bracket, $B$ must be closed. □

In particular, we have a short exact sequence

$$0 \to \Omega^2_{cl} \to Sym(E) \to Diff_{[H]} \to 0$$

where $Sym(E)$ denotes the group of symmetries of an exact Courant algebroid $E \to M$.

**Remark 2.2.6** If we consider the Courant bracket $[\cdot,\cdot]_C$ defined in (2.15), then we found that the group of orthogonal automorphisms of the Courant bracket is $\Omega^2_{cl}(M) \rtimes Diff(M)$.

◇

## 2.3 Dirac Structures

So far we have seen that the Courant bracket $[\cdot,\cdot]_C$ fails to be a Lie algebroid because of exact terms involving the inner product $\langle \cdot,\cdot \rangle$. Our goal in this section is to determine whether we can find a subbundle $L \subset TM \oplus T^*M$ that is closed under the Courant bracket and isotropic as well so that the exact terms would vanish and $(L, [\cdot,\cdot]_C, \pi)$ would be a Lie algebroid. Such subbundles which are maximally isotropic are the so called *Dirac structures* and we will see that achieving the proposed goal can be expressed by the vanishing of an element in $\Gamma(\Lambda^3 L^*)$.

Recall in Chapter 1 we have considered real linear Dirac structures (maximally isotropic subspaces). Now, we present Dirac structures on manifolds. In what follows, we will discuss some examples of such structures and show how these, as previously stated, unifies presymplectic and Poisson geometries.

**Definition 2.3.1** *A real maximally isotropic subbundle $L \subset TM \oplus T^*M$ is called an **almost Dirac structure**. If in addition $L$ is closed under the Courant bracket (see Rmk.2.3.1 below), then we say $L$ is **integrable** or a **Dirac structure**. Similarly, an involutive and maximally isotropic subbundle $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ is called a **complex Dirac structure**.*

**Remark 2.3.1** Note that for the integrability condition it does not matter the type of bracket we are considering, i.e, Dorfman or Courant. Since $L$ is assumed to be isotropic and equation (2.20) holds, that is, $[\alpha,\beta]_C = [\alpha,\beta]_D + d\langle\alpha,\beta\rangle$, the brackets coincide on $L$. Furthermore, Theorem 2.26 implies that in the Definition 2.3.1 above we could have replaced $TM \oplus T^*M$ by an exact Courant algebroid $E$ with a chosen splitting of zero curvature (this implies the cohomology class $[H]$ is trivial).

◇

**Example 2.3.1**

(i) From our discussion $T^*M$ is always a Dirac structure.

(ii) Note that if we split $(E, [\cdot,\cdot]_H)$, where $E$ is an exact Courant algebroid, then $TM$ is involutive if and only if $H = 0$.

(iii) Again, suppose we have a splitting $(E, [\cdot,\cdot]_H)$, then any maximally isotropic complement to $T^*M$ is the graph of $e^B$ for some $B \in \Omega^2(M)$. One can check that $[\Gamma_B, \Gamma_B] \subset \Gamma_B$ if and only if $[TM, TM]_{H-dB} \subset TM$, where $\Gamma_B$ indicates the graph of $e^B$ (see Proposition 2.2.7). Thus, $\Gamma_B$ is a Dirac structure when and only when $H = dB$ (see also Example 2.3.2).

△

**Remark 2.3.2** Recall from the previous section if $B \in \Omega^2(M)$, then the so called $B-$*field transform* $e^B$ is an orthogonal automorphism of an exact Courant algebroid whenever $B$ is closed. Therefore, if we apply a $B-$field transform on a Dirac structure, with $dB = 0$, then what we obtain is another Dirac structure.

◇

**Proposition 2.3.1** *If $L \subset TM \oplus T^*M$ is closed under the Courant bracket, then $L$ must either be an isotropic subbundle, or a bundle of the type $\Delta \oplus T^*M$ for some nontrivial involutive subbundle $\Delta \subset TM$.*

*Proof.* Let $L \subset TM \oplus T^*M$ be involutive and let us assume $L$ is not isotropic. Then there exists $X + \xi \in \Gamma(L)$ and a point $p \in M$ such that $\xi(X) \neq 0$ at $p$. Thus, from (2.23):

$$[X+\xi, f(X+\xi)]_C = f[X+\xi, X+\xi]_C + (Xf)(X+\xi) - \xi(X)df \qquad \forall f \in C^\infty(M)$$

and the involutivity of $L$ it follows that $df_p \in L$ for all $f \in C^\infty(M)$. In particular, $T_p^*M \subset L_p$. Since $T_p^*M$ is isotropic, the inclusion must be proper and hence $L_p = \Delta_p \oplus T_p^*M$, where $\Delta_p \doteq ker\left(\pi_{T^*M}\big|_L\right) : L_p \to T_p^*M$. Moreover, the rank of $L$ must be bigger than the maximal dimension of an isotropic subbundle, which is in this case $m$, the dimension of $M$ and so $L$ is not isotropic at every point of $M$. Thus, $\Delta \doteq ker\left(\pi_{T^*M}\big|_L\right) : L \to T^*M$ is an involutive subbundle of $TM$ and $L = \Delta \oplus T^*M$. □

Moreover, for maximally isotropic subbundles we have the following equivalences:

**Proposition 2.3.2** *Let $L \subset TM \oplus T^*M$ be a Lagrangian subbundle. Then the following are equivalent:*

*(i) $L$ is involutive,*

*(ii) $Nij\big|_L \equiv 0$,*

*(iii) $Jac\big|_L \equiv 0$.*

*Proof.* The equivalences $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear. Thus, it is sufficient to show that $(iii)$ implies $(i)$. By contradiction, suppose $L$ is not involutive, then there exists $\alpha, \beta \in \Gamma(L)$ such that $[\alpha, \beta]_C \notin L$. The maximality of $L$ then implies it must exist $\gamma \in \Gamma(L)$ such that $\langle [\alpha, \beta]_C, \gamma \rangle \neq 0$. Then, for any function $f \in C^\infty(M)$ we have that

$$\begin{aligned} 0 &= Jac(\alpha, \beta, f\gamma) \\ &= dNij(\alpha, \beta, f\gamma) \\ &= Jac(\alpha, \beta, \gamma)f + Nij(\alpha, \beta, \gamma)df \\ &= Nij(\alpha, \beta, \gamma)df \\ &= \langle [\alpha, \beta]_C, \gamma \rangle df \end{aligned}$$

a contradiction.

Note that second equality above follows from Proposition 2.2.2, while in the third equality we have used the fact $Nij\big|_L$ is tensorial (see Rmk. 2.3.3 below) and the last equality is a consequence of axiom C2 as well $L$ being isotropic. □

**Remark 2.3.3** If $L \subset TM \oplus T^*M$ is isotropic, then the restriction $Nij\big|_L$ is tensorial and it belongs to the space of sections of $\Lambda^3 L^*$. Thus, Proposition above tells us that the integrability condition for a Dirac structure is given by the vanishing of a $3-$tensor. As a consequence, any almost Dirac structure on a two dimensional surface is always integrable.

For, let $\alpha, \beta, \gamma \in \Gamma(L)$ and $f \in C^\infty(M)$, then:

$$\begin{aligned} Nij(\alpha, f\beta, \gamma) &= \langle [\alpha, f\beta], \gamma \rangle + \text{ c.p.} \\ &= \langle f[\alpha, \beta] + \rho(\alpha)f\beta, \gamma \rangle + \text{ c.p.} \qquad \text{(axiom C4)} \\ &= f\langle [\alpha, \beta], \gamma \rangle + \text{ c.p.} \\ &= fNij(\alpha, \beta, \gamma) \end{aligned}$$

Analogously, $Nij(f\alpha, \beta, \gamma) = Nij(\alpha, \beta, f\gamma) = fNij(\alpha, \beta, \gamma)$.

◇

We now give some more examples of Dirac structures.

**Example 2.3.2** (Presymplectic geometry) Let $\omega \in \Omega^2(M)$ be a $2-$form considered as a skew-symmetric map $TM \to T^*M$. Then its graph

$$e^\omega \doteq \{X + \iota_X \omega\,;\ X \in TM\}$$

is a maximally isotropic subbundle whose projection to $TM$ is non-degenerate. Moreover, any maximally isotropic subbundle which has a non-degenerate projection to $TM$ is a graph of a $2-$form (see Proposition 1.4.1). Now, Proposition 2.2.7 tells us $e^{\omega}$ is a Dirac structure if and only if $\omega$ is closed. In particular, presymplectic geometry can be described by a Dirac structure.

In other words, presymplectic structures are identified with Dirac structures $L \subset TM \oplus T^*M$ satisfying the condition $L \cap T^*M = \{0\}$. Moreover, the Lie algebroid structure of $L$ is isomorphic to the canonical one in $TM$.

△

**Example 2.3.3** (Poisson geometry) Let $\Pi \in \Gamma(TM \otimes TM)$ be a tensor field considered as a map $T^*M \to TM$. Then its graph

$$e^{\Pi} \doteq \{\iota_{\xi}\Pi + \xi\,;\ \xi \in T^*M\}$$

is maximally isotropic if and only if $\Pi \in \Gamma(\Lambda^2(TM))$. In this case the projection to $T^*M$ is non-degenerate and, conversely, any maximally isotropic subbundle which has a non-degenerate projection to $T^*M$ is a graph of a bivector field (see Proposition 1.4.2). One can check that $e^{\Pi}$ is a Dirac structure if and only if $\Pi$ is Poisson, i.e., $[\Pi, \Pi] = 0$.

In fact, since $Nij\big|_{e^{\Pi}}$ is tensorial, it suffices to check the integrability condition for sections of the form $\iota_{\xi}\Pi + \xi$, where $\xi = df$ for some $f \in C^{\infty}(M)$. The bivector $\Pi$ defines a bracket on functions given by $\{f, g\} \doteq \Pi(df, dg)$ such that

$$Nij(\iota_{df}\Pi + df, \iota_{dg}\Pi + dg, \iota_{dh}\Pi + dh) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}$$

which implies the Nijenhuis tensor vanishes on $e^{\Pi}$ if and only if the bracket $\{\cdot, \cdot\}$ satisfies the Jacobi identity, i.e., if and only if $\Pi$ is a Poisson structure.

△

**Remark 2.3.4** If $E = TM \oplus T^*M$ has non-trivial cohomology class $[H]$ one needs to replace the Poisson structure $\Pi$ by a twisted one, i.e., such that $[\Pi, \Pi] = -\Pi^* H$ (see e.g. [19]).

◇

**Example 2.3.4** (Foliated geometry) Let $\Delta \subset TM$ be a smooth distribution of constant rank, i.e., a subbundle. Then $L \doteq \Delta \oplus \Delta^{\circ}$ is maximally isotropic (see Example 1.4.1 in the previous chapter, section 1.4) and since the Courant bracket vanishes on $T^*M$ it follows that $L$ is involutive (with respect to the Courant bracket) if and only if $\Delta$ is Lie involutive, i.e., if and only if $\Delta$ is integrable. Thus, in this setting, the Frobenius theorem allows us to describe regular foliations as Dirac structures.

△

**Example 2.3.5** (Complex geometry) Given an almost complex structure $J$ on $M$, we can consider its $-i-$eigenbundle $T^{0,1}M \subset TM \otimes \mathbb{C}$ and the corresponding maximal isotropic subbundle:

$$L_J \doteq T^{0,1}M \oplus (T^{0,1}M)^{\circ} = T^{0,1}M \oplus (T^{1,0}M)^*$$

The same argument used in the previous example tells us involutivity of $L_J$ implies Lie involutivity of $T^{0,1}M$ and hence $J$ is integrable[6]. Conversely, if $J$ is a complex structure and $X+\xi, Y+\eta \in \Gamma(L_J)$, then

$$\begin{aligned}[X+\xi, Y+\eta] &= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(\iota_X\eta - \iota_Y\xi) \\ &= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi \\ &= [X,Y] + \iota_X d\eta - \iota_Y d\xi \\ &= [X,Y] + \iota_X(\partial+\bar{\partial})\eta - \iota_Y(\partial+\bar{\partial})\xi \\ &= [X,Y] + \iota_X\bar{\partial}\eta - \iota_Y\bar{\partial}\xi \qquad \in \Gamma(L_J)\end{aligned}$$

where we have used the definition of $L_J$ plus in the third equality above, the Cartan's magic formula (2.11), while in the fourth, the fact $J$ is integrable.

Therefore, $L_J$ is a Dirac structure if and only if $J$ is integrable and we see that complex geometry can be described by Dirac structures.

△

**Example 2.3.6** (Cartan-Dirac, see e.g. [17],[19]) Recall Example 2.2.5. In that setting we can define the Lagrangian subbundle

$$L_G \doteq \left\{ e_a = a^L - a^R \oplus \frac{1}{2}(a^L + a^R) \mid X \in \mathfrak{g} \right\}$$

where $a^L$ and $a^R$ denote the left and right invariant vector fields determined by $a$, respectively. One can prove that $L_G$ is closed with respect to the twisted Dorfman bracket, with $H$ equals the Cartan $3-$form on $G$. That is, computing the ordinary Courant bracket of sections $e_a$ and $e_b$ gives us an expression that differs from $e_{[a,b]}$ by a term in the $1-$form component that vanishes if we add $-\dfrac{1}{2}B([a,b],\cdot)$.

△

**Remark 2.3.5** Note that the map $a \mapsto e_a$ is an isomorphism to each fiber of $L_G \to G$. In particular, we conclude the Lie algebroid $L_G$ is isomorphic to the action Lie algebroid $\mathfrak{g} \times G$, where the action is simply the adjoint action (see Example 2.2.5). Thus, the (twisted) corresponding presymplectic leaves are the connected components of the conjugacy classes in $G$.

◇

**Example 2.3.7** We may construct other examples of Dirac structures from the observation that $\Omega^2_{cl}(M)$ acts on $TM \oplus T^*M$ preserving both the inner product $\langle\cdot,\cdot\rangle$ and the bracket $[\cdot,\cdot]$. See Remark 2.3.2.

[6] An almost complex structure on a manifold $M$ is integrable if and only if the associated Nijenhuis tensor, $N : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by

$$N(X,Y) \doteq [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

vanishes. This happens precisely when sections of $T^{1,0}M$ (equiv. $T^{0,1}M$) are closed under the Lie bracket.

△

### 2.3.1 Some Properties

We now describe the main geometrical properties Dirac structures have. Given a Dirac structure $L \subset TM \oplus T^*M$ on $M$ we have seen the vector bundle $L \to M$ inherits a canonical Lie algebroid structure $\rho : L \to TM$. Now, every Lie algebroid induces an integrable smooth distribution called *characteristic distribution* (possibly of non-constant rank) which is given by the image of the anchor map (see e.g. [14] Theorem 2.1.3). It turns out in the case of a Lie algebroid coming from a Dirac structure, such distribution possesses extra data: the corresponding foliation has presymplectic leaves.

In fact, $L$ being fixed, let us denote by $\mathcal{S}_\alpha$ each leaf of the described foliation. Then $\mathcal{S}_\alpha$ inherits a closed $2-$form $\Omega_{L,\alpha} : \rho(L) \to \rho(L)^*$, which is defined at each $x$ by

$$\Omega_{L,\alpha}(x)(X, Y) = \xi(Y)$$

where $X, Y \in T_x\mathcal{S}_\alpha = \rho(L)\big|_x$ and $\xi \in T_x^*M$ is such that $X + \xi \in L_x$. The fact $L$ is isotropic guarantees each $\Omega_{L,\alpha}$ does not depend on the choice of $\xi$ and hence it is well defined. Closedness follows from the integrability of $L$.

Thus, if we regard the Dirac structure $L$ as a (singular) presymplectic foliation $\{(\mathcal{S}_\alpha, \Omega_{L,\alpha})\}_\alpha$ and assume connectedness of $M$, then

(i) $L$ corresponds to a presymplectic structure as in Example 2.3.2 if and only if the foliation consists of a single leaf.

(ii) $L$ corresponds to a regular foliation as in Example 2.3.4 if and only if $\Omega_{L,\alpha} = 0$ for all $\alpha$.

(iii) $L$ corresponds to a Poisson structure as in Example 2.3.3 if and only if each $\Omega_{L,\alpha}$ is non-degenerate, i.e., a symplectic structure on $\mathcal{S}_\alpha$.

Moreover, one can show all the leaves of $L$ have the same parity.

#### Hamiltonian vector fields

Since $L$ is completely determined by the corresponding presymplectic foliation it becomes interesting to define what the Hamiltonian vector field of a function on $M$ is, just as in the symplectic or Poisson cases.

We say a function $f \in C^\infty(M)$ is *admissible* if there exists a vector field $X_f$ on $M$ such that $X_f \oplus df$ is a section of $L$. Such vector field is referred to as a *Hamiltonian* vector field of $f$. Note that a Hamiltonian of a function is not uniquely determined since we can add any vector field tangent to the distribution $L \cap TM$.

Now, if $f$ and $g$ are two admissible functions, we can define their bracket as follows

$$\{f, g\} \doteq X_f \cdot g = dg(X_f)$$

and one can check $\{f, g\}$ depends only on $g$ and not on $X_g$. In particular, the set of admissible functions with the above bracket is turned into a Poisson algebra, [14].

### 2.3.2 Expressing the Integrability Condition

We end our discussion on Dirac structures by describing all Lagrangian subbundles of an exact Courant algebroid $\rho : E \to TM$ with a result that generalizes Proposition 1.2.4 in the previous chapter. Such result allows us to express the integrability condition of a Dirac structure by means of the Levi-Civita connection on $M$ once a Riemannian metric is fixed.

Thus, let us choose any splitting $\sigma$, so that $E = TM \oplus T^*M$, and also fix an arbitrary Riemannian metric $g$ on $M$. Then $g$ can be thought of as a non-generate symmetric map $TM \to T^*M$ and, similarly, the dual metric, $g^*$, is acting from $T^*M$ to $TM$[7]. Combining such maps we obtain an involution $\tau : E \to E$ given by $X + \xi \mapsto g^*(\xi) + g(X, \cdot)$, which clearly when squared equals $Id$. Thus, $E = E_+ \oplus E_-$, where $E_\pm \doteq Ker(\tau \mp Id)$.

**Proposition 2.3.3** ([14], [18]) *There exists a one-to-one correspondence between maximally isotropic subbundles $L \subset E$ and sections $\mathcal{O} \in \Gamma(O(TM))$.*

*Proof.* The proof is analogous to the one of Proposition 1.2.4. First, we observe that the restriction of the natural pairing $\langle \cdot, \cdot \rangle$ to $E_+$ (resp. $E_-$) defines a positive (resp. negative) metric and $\langle E_+, E_- \rangle \equiv 0$ so that any Lagrangian subbundle of $E$ is transversal to $E_\pm$. Hence, given $L$ maximally isotropic, it follows that $L$ is a graph of some map $\varphi : E_+ \to E_-$. It remains to check that using the splitting we can identify both $E_\pm$ with $TM$ and $\varphi$ uniquely corresponds to an orthogonal transformation of $TM$. Thus,

$$\Gamma(L) \simeq \{(Id + \mathcal{O})X + ((Id - \mathcal{O})X)^* \, ; \; X \in \mathfrak{X}(M)\} \tag{2.27}$$

for some $\mathcal{O} \in \Gamma(O(TM))$. □

**Remark 2.3.6** Note that any graph of a map $E_+ \to E_-$ can be thought of as a linear complement to $TM$ by means of $\sigma$, so if such graph is isotropic, then it is maximally isotropic.

◇

By means of expression (2.27) we now have a parametrization of any section of a Lagrangian subbundle $L \subset E$. The next result uses such parametrization and gives us the integrability condition for $L$ to be a Dirac structure in terms of the Levi-Civita connection $\nabla$ on $M$.

**Proposition 2.3.4** *Let $L \subset E$ be any Lagrangian subbundle and $\mathcal{O}$ the orthogonal operator which represents it as the set $\{(Id + \mathcal{O})X + ((Id - \mathcal{O})X)^*\}$. Then $L$ is a Dirac structure if and only if the following property holds for every $X_i \in \mathfrak{X}(M)$:*

$$\sum_{\sigma \in \mathbb{Z}_3} g\left(\mathcal{O}^{-1}\nabla_{(Id+\mathcal{O})X_{\sigma(1)}}(\mathcal{O})X_{\sigma(2)}, X_{\sigma(3)}\right) = -\frac{1}{2}H\big((Id + \mathcal{O})X_1, (Id + \mathcal{O})X_2, (Id + \mathcal{O})X_3\big) \tag{2.28}$$

[7] The value of $g^*$ on a 1−form $\eta$ is given by the Riesz representation theorem, i.e., is defined as the unique vector field $Y$ such that $\eta = g(Y, \cdot)$

*Proof.* Recall the bracket on $E$ is given by

$$x \circ y \doteq [X,Y] + \mathcal{L}_X \eta - \iota_Y d\xi - \iota_Y \iota_X H$$

where $x \simeq X + \xi, y \simeq Y + \eta$ are sections of $E \simeq TM \oplus T^*M$. Thus, using the Levi-Civita connection $\nabla$, which is torsion free, we find

$$\begin{aligned} [X,Y] &= \nabla_X Y - \nabla_Y X = \nabla_{\rho(x)}\rho(y) - \nabla_{\rho(y)}\rho(x) && (2.29)\\ \mathcal{L}_X \eta &= \langle \eta, \nabla X\rangle + \nabla_X \eta = \langle \eta, \nabla X\rangle + \nabla_{\rho(x)}\eta && (2.30)\\ -\iota_Y d\xi &= d\iota_Y \xi - \mathcal{L}_Y \xi = \langle \nabla \xi, Y\rangle - \nabla_{\rho(y)}\xi && (2.31)\\ -\iota_Y \iota_X H &= -H(X,Y,\cdot) = -H(\rho(x),\rho(y),\cdot) && (2.32) \end{aligned}$$

where in Equation (2.31) we have used the Cartan's magic formula (2.11) and the compatibility of $\nabla$ with the metric $g$.

Now, checking that $\langle \nabla x, y\rangle = \langle \nabla \xi, Y\rangle + \langle \nabla X, \eta\rangle$ and combining Equations (2.29),(2.30), (2.31) and (2.32) above, we conclude

$$x \circ y = \nabla_{\rho(x)} y - \nabla_{\rho(y)} x + \langle \nabla x, y\rangle - H(\rho(x),\rho(y),\cdot) \qquad (2.33)$$

In short, we have rewritten the bracket in terms of the connection. Let us now use such expression to derive the bracket between sections of $L$. Let $l_i \in \Gamma(L)$ for $i = 1,2,3$, then

$$l_i = (Id + \mathcal{O})X_i + ((Id - \mathcal{O})X_i)^*$$

and, therefore,

$$\begin{aligned} l_1 \circ l_2 &= (Id + \mathcal{O})\left(\nabla_{\rho(l_1)}X_2 - \nabla_{\rho(l_2)}X_1\right) + \Bigg((Id - \mathcal{O})\left(\nabla_{\rho(l_1)}X_2 - \nabla_{\rho(l_2)}X_1\right)\Bigg)^* \\ &\quad + \nabla_{\rho(l_1)}(\mathcal{O})X_2 - \nabla_{\rho(l_2)}(\mathcal{O})X_1 + \Bigg(-\nabla_{\rho(l_1)}(\mathcal{O})X_2 + \nabla_{\rho(l_2)}(\mathcal{O})X_1\Bigg)^* \\ &\quad - 2g(\mathcal{O}^{-1}\nabla(O)X_1, X_2) - H\big((Id + \mathcal{O})X_1, (Id + \mathcal{O})X_2, \cdot\big) \qquad (2.34) \end{aligned}$$

Next, our final goal is to express the 3−tensor $Nij(l_1,l_2,l_3)$ using what we have done so far. Recall Proposition 2.3.2 which states its vanishing is the integrability condition for $L$ being a Dirac structure.

Now, observe that $Nij(l_1,l_2,l_3) = \langle l_1 \circ l_2, l_3\rangle$. Also, note the sum of the first two terms in (2.34) belongs to $L$ so its product with $l_3$ vanishes.

To conclude, $\langle l_1 \circ l_2, l_3 \rangle = (I) + (II) + (III)$, where

$$
\begin{aligned}
(I) &= \langle \nabla_{\rho(l_1)}(\mathcal{O})X_2 - (\nabla_{\rho(l_1)}(\mathcal{O})X_2)^*, (Id+\mathcal{O})X_3 + ((Id-\mathcal{O})X_3)^* \rangle - \\
&\quad \langle \nabla_{\rho(l_2)}(\mathcal{O})X_1 - (\nabla_{\rho(l_2)}(\mathcal{O})X_1)^*, (Id+\mathcal{O})X_3 + ((Id-\mathcal{O})X_3)^* \rangle \\
&= g(\nabla_{\rho(l_1)}(\mathcal{O})X_2, (Id-\mathcal{O})X_3) - g(\nabla_{\rho(l_1)}(\mathcal{O})X_2, (Id+\mathcal{O})X_3) \\
&\quad -g(\nabla_{\rho(l_2)}(\mathcal{O})X_1, (Id-\mathcal{O})X_3) + g(\nabla_{\rho(l_2)}(\mathcal{O})X_1, (Id+\mathcal{O})X_3) \\
&= g(\nabla_{\rho(l_1)}(\mathcal{O})X_2, X_3) - g(\nabla_{\rho(l_1)}(\mathcal{O})X_2, \mathcal{O}X_3) - g(\nabla_{\rho(l_1)}(\mathcal{O})X_2, X_3) \\
&\quad -g(\nabla_{\rho(l_1)}(\mathcal{O})X_2, \mathcal{O}X_3) - g(\nabla_{\rho(l_2)}(\mathcal{O})X_1, X_3) + g(\nabla_{\rho(l_2)}(\mathcal{O})X_1, \mathcal{O}X_3) \\
&\quad +g(\nabla_{\rho(l_2)}(\mathcal{O})X_1, X_3) + g(\nabla_{\rho(l_2)}(\mathcal{O})X_1, \mathcal{O}X_3) \\
&= -2g(\mathcal{O}^{-1}\nabla_{\rho(l_1)}(\mathcal{O})X_2 - \mathcal{O}^{-1}\nabla_{\rho(l_2)}(\mathcal{O})X_1, X_3)
\end{aligned}
$$

$$
(II) = -2g(\mathcal{O}^{-1}\nabla_{\rho(l_3)}(\mathcal{O})X_1, X_2)
$$

$$
(III) = -H\big((Id+\mathcal{O})X_1, (Id+\mathcal{O})X_2, (Id+\mathcal{O})X_3\big)
$$

Therefore, $Nij(l_1, l_2, l_3) = 0$ if and only if (2.28) holds. □

# Chapter 3

# Some Characteristic Classes

*We have seen the notion of Dirac structures encompasses both Poisson and presymplectic geometries in a natural way by considering these as graphs of bundle maps $T^*M \to TM$ and $TM \to T^*M$, respectively. Interestingly, there are other examples of Dirac structures on exact Courant algebroids which are not necessarily projectable, either to $TM$ or $T^*M$. In this chapter we are interested in the problem of classification, up to homotopy, of Dirac structures. We use the generalized Chern-Weil map, [2], and the BRST model for equivariant cohomology, [3], to obtain explicit characteristic forms, the cohomology classes of which gives us homotopical invariants solving the proposed problem.*

## 3.1 Characteristic Classes associated to Dirac Structures

We begin this final chapter by recalling Proposition 2.3.3 which establishes (once a Riemannian metric, $g$, is chosen on $M$) a one-to-one correspondence between Lagrangian subbundles $L \subset \mathbb{T}M = TM \oplus T^*M$ and orthogonal maps acting pointwise on $TM$. The idea is to consider such correspondence for constructing some characteristic forms, the cohomology classes of which will give us homotopical invariants for $L$. In other words, to classify $L$ under homotopy it is enough to obtain homotopical invariants for the corresponding section $\mathcal{S} : M \to O(TM)$: the homotopy class of $\mathcal{S}$ depends only on the homotopy class of $L$.

To make it more precise, first note that any section $\mathcal{S} \in \Gamma(O(TM))$ can be viewed as an $O(n)-$equivariant map, $P \to O(n)$, where $P$ is the bundle of orthogonal frames and $O(n)$ acts on itself by conjugation. Moreover, if $\alpha : M \to BO(n)$ denotes the classifying map, then it induces a map $\tilde{\alpha} : P \times_{O(n)} O(n) = O(TM) \to EO(n) \times_{O(n)} O(n)$ so that we have the following commutative diagram

$$\begin{array}{ccc} P \times_{O(n)} O(n) & \xrightarrow{\tilde{\alpha}} & EO(n) \times_{O(n)} O(n) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & BO(n) \end{array} \tag{3.1}$$

In particular, $(\mathcal{S} \circ \tilde{\alpha})^*$ produces a map from the equivariant cohomology of $O(n)$ to the cohomology of $M$. Consequently, if we restrict our attention to Dirac structures, we end up with a characteristic map from the product of the set of homotopy classes of Dirac structures with $H_{O(n)}(O(n))$ to the cohomology ring of $M$. Our task now is in the next paragraphs to make such construction explicit by means of the machinery presented in [2] which gives us a generalization of the Chern-Weil map and also by a more geometrical approach which will lead us to some natural and explicit secondary characteristic forms.

## 3.2 Generalized Chern-Weil Theory

Recall in the usual Chern-Weil theory we have a basic construction known as the Chern-Weil homomorphism which goes from $G$ principal bundles to the de Rham cohomology of the base manifold by means of the choice of a connection form and further evaluation of its curvature in the invariant polynomials of the Lie algebra to finally produce the so called curvature characteristic form. Moreover, when the group $G$ is compact, such construction gives us a map from the cohomology of the classifying space $BG$ to the de Rham cohomology ring.

We begin this section by describing the Chern-Weil homomorphism as a map from the Weil algebra to the algebra of differential forms on the total space of the principal bundle. This provides us with an algebraic framework for constructing characteristic classes. In fact this can be generalized to any $\mathfrak{g}-$differential algebra (see Definition 3.2.1 below) and such approach was first introduced by H. Cartan in [20].

### 3.2.1 $\mathfrak{g}-$Differential Algebras

The first step then is to introduce the notion of a $\mathfrak{g}-$differential algebra (or simply $\mathfrak{g}$-da), the algebraic counterpart of a principal bundle.

**Definition 3.2.1** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. A **$\mathfrak{g}$-differential algebra** is a graded commutative algebra $A = \bigoplus_k A^k$ endowed with graded derivations $d^A, L^A, \iota^A$ of degrees $1, 0, -1$, respectively, such that $(A, d^A)$ is a differential algebra, $L^A$ gives a representation of $\mathfrak{g}$ in $A$*[1] *and, moreover, the following graded commutation relations are satisfied*

$$[d^A, d^A] = 0, \;\; [d^A, L^A_X] = 0, \;\; [d^A, \iota^A_X] = L^A_X$$
$$[L^A_X, L^A_Y] = L^A_{[X,Y]}, \;\; [L^A_X, \iota^A_Y] = \iota^A_{[X,Y]}, \;\; [\iota^A_X, \iota^A_Y] = 0$$

*where $X, Y \in \mathfrak{g}$.*

Equivalently, such structure is determined by a linear mapping $X \mapsto \iota^A_X$ of $\mathfrak{g}$ into $Der^{-1}(A)$ (anti-derivations of degree $-1$ of $A$) such that if one defines the derivation $L^A_X$ of degree $0$ by $L^A_X = [d^A, \iota^A_X]$, then one has $[L^A_X, \iota^A_Y] = \iota^A_{[X,Y]}$ and all other graded commutation relations as above can be derived from these two.

**Remark 3.2.1** Sometimes in literature a $\mathfrak{g}-$da is also referred to as a $\mathfrak{g}-$operation or an operation of $\mathfrak{g}$ in a dg-algebra.

[1] In fact a Lie algebra morphism of $\mathfrak{g}$ into $Der^0(A)$ (derivations of degree $0$ of $A$).

◇

In short, a $\mathfrak{g}-$differential algebra $A$ is a dg-algebra $(A, d^A)$ with algebraic counterparts of the Lie derivative and contraction operations for an action of the Lie algebra $\mathfrak{g}$.

For later purposes it is useful to consider some distinguished elements in a $\mathfrak{g}-$da.

**Definition 3.2.2** *Given any $\mathfrak{g}-$da, $A$, we say an element is **horizontal** if it lies in the kernel of $\iota_X^A$ for every $X \in \mathfrak{g}$, i.e., the horizontal subalgebra is $\bigcap_X ker(\iota_X^A)$. Similarly, we define the **invariant** subalgebra as $\bigcap_X ker(L_X^A)$ and, finally, the **basic** elements are those which are both horizontal and invariant.*

Such terminology, for some clear reasons, is inspired by the theory of differential forms on principal bundles. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\pi : P \to M$ be a $G-$principal bundle. In particular,$G$ acts freely on $P$ and $M \simeq P/G$. Then the projection $\pi$ lifts to a map $d\pi : TP \to TM$ and hence induces by duality an injective homomorphism $\pi^* : \Omega(M) \to \Omega(P)$. The elements in the image $\pi^*(\Omega(M))$ denoted by $\Omega_B(P)$ are called basic differential forms on $P$. Moreover, we say a tangent vector to $P$ is vertical if it lies in the kernel of $d\pi$ and it is known we have a well defined map (in fact a morphism of Lie algebras) from $\mathfrak{g}$ to the set of vertical vector fields which associates to each element $X \in \mathfrak{g}$ the so called fundamental vector field denoted by $X^{\#}$[2]. Also, the horizontal forms on $P$ are those for which the inner derivations with vertical vector fields vanish and, finally, forms on $P$ which are invariant under the action of the structure group $G$ are said to be invariant. To conclude, if we denote by $\iota_{X^{\#}}$ the inner derivation of $\Omega(P)$ by the fundamental vector field corresponding to $X \in \mathfrak{g}$, then one can check the map $X \mapsto \iota_{X^{\#}}$ turns $\Omega(P)$ into a $\mathfrak{g}-$differential algebra. Furthermore, the usual notions of basicity, horizontality and invariance correspond to the ones as in Definition 3.2.2 above.

Now, on any $\mathfrak{g}-$da we can define the concept of an algebraic connection.

**Definition 3.2.3** *A **connection** on a $\mathfrak{g}-$differential algebra is an invariant element $\theta \in A^1 \otimes \mathfrak{g}$ satisfying the property $\iota_X^A\theta = X$. We define its **curvature** as the element $F_\theta$ in $A^2 \otimes \mathfrak{g}$ given by*

$$F_\theta \doteq d^A\theta + \frac{1}{2}[\theta, \theta]$$

### 3.2.2 The Weil Algebra

We now turn our attention to one particular example of a $\mathfrak{g}-$da with a connection, namely the Weil algebra. Furthermore, we shall see that in some sense, such example is unique.

In one hand, the notion of Weil algebra, usually defined for Lie algebras, is commonly presented, for a finite-dimensional Lie algebra $\mathfrak{g}$, as the graded exterior algebra generated from $\mathfrak{g}^*$ and another copy of it shifted by one in degree

$$W(\mathfrak{g}) \doteq \Lambda^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1]) \simeq \Lambda^\bullet\mathfrak{g}^* \otimes S^\bullet\mathfrak{g}^* \tag{3.2}$$

[2]More precisely, $X^{\#} : p \mapsto \left.\frac{d}{dt}\right|_{t=0} \varphi(exp(tX), p)$, where $\varphi : G \times P \to P$ denotes the action.

So elements in $W(\mathfrak{g})$ correspond to functions on $T[1]\mathfrak{g}[1]$. In fact we can endow it with a derivation making it into a dg-algebra, such that restricted to $\mathfrak{g}^*$ this derivation agrees with the Chevalley-Eilenberg differential.

On the other hand, if we return to our discussion on $\mathfrak{g}-$differential algebras and algebraic connections one can understand $W(\mathfrak{g})$ as the algebraic counterpart of the classifying bundle $EG$ and we define the Weil algebra by the following universal property:

**Theorem 3.2.1** ([21]). *There exists a $\mathfrak{g}-$da $W(\mathfrak{g})$ with connection $\theta_W$ such that if $A$ is any other $\mathfrak{g}-$da with connection $\theta$, then there is a unique algebra morphism $c : W(\mathfrak{g}) \to A$ that sends $\theta_W$ to $\theta$.*

*Proof.* First, it is clear that if such $W(\mathfrak{g})$ exists, then it is unique up to isomorphism by the universal property. Moreover, the presentation above in (3.2) indeed gives us an explicit construction. Let $\{e_a\}$ be a basis of $\mathfrak{g}$ and denote by $e^a$ the dual basis. Also, let $\{\eta^a\}$ be the corresponding generators of $\Lambda^\bullet\mathfrak{g}^*$ and $\phi^a \simeq d\eta^a$ the generators of $S^\bullet\mathfrak{g}^*$. Then, on $\Lambda^\bullet\mathfrak{g}^* \otimes S^\bullet\mathfrak{g}^*$ define the differential $d^{W(\mathfrak{g})} \doteq d + d_{CE}$ and the contraction by

$$\iota_{e_a}^{W(\mathfrak{g})}\eta^b = \delta_a^b, \ \ \iota_{e_a}^{W(\mathfrak{g})}\phi^b = 0$$

Finally, consider $\theta_W \doteq \eta^a \otimes e_a$. □

**Remark 3.2.2** One can check the differential $d^{W(\mathfrak{g})}$ is completely determined by

$$\begin{aligned} d^{W(\mathfrak{g})}\eta^a &= \phi^a - \frac{1}{2}C^a_{bc}\eta^b\eta^c \\ d^{W(\mathfrak{g})}\phi^a &= -C^a_{bc}\phi^b\eta^c \end{aligned}$$

where $C^a_{bc}$ are the structure constants of $\mathfrak{g}$, i.e., the constants given by $[e_b, e_c] = C^a_{bc}e_a$.

◇

In particular, we can consider $A$ as the algebra of differential forms on the total space of a $G$ principle bundle $P \to M$. Then, the restriction of the characteristic homomorphism $c : W(\mathfrak{g}) \to \Omega(P)$ to the basic subalgebra of $W(\mathfrak{g})$, which consists precisely of the invariant elements in $S^\bullet\mathfrak{g}^*$, does not depend on the choice of the connection form on $P$, i.e., if we choose two different connections then their corresponding characteristic morphisms are $\mathfrak{g}-$homotopic, meaning we have a chain homotopy interchanging contractions and Lie derivatives.

As a consequence, we obtain the Chern-Weil map

$$S^\bullet(\mathfrak{g}^*)^{\mathfrak{g}} \to H(\Omega_B(P)) \simeq H(\Omega(M))$$

**Remark 3.2.3** Recall every principal $G-$bundle $P \to M$ is the pullback of the universal bundle $EG \to BG$ by some map $\alpha : M \to BG$. If $G$ is compact, then by the discussion above we may understand the Weil algebra as a model for $EG$[3]. In fact, its cohomology

[3]If $EG$ were a manifold $\alpha$ would induce a morphism of $\mathfrak{g}$-da's $\Omega(EG) \to \Omega(P)$.

is trivial since $\mathbb{R} \to W(\mathfrak{g})$ is a homotopy equivalence and this corresponds to the contractibility of $EG$. Moreover, the cohomology of the basic subcomplex of $W(\mathfrak{g})$ agrees with $H(BG)$ (for compact $G$).

◇

### 3.2.3 The 'Q-category'

Our next goal is to reformulate the previous construction using the language of $Q-$manifolds as suggested in [2].

To begin with, recall we have begun Chapter 2 with a brief discussion on Lie algebroids and have seen the usual "anchor-bracket" definition on a vector bundle $A \to M$ may be interpreted super-geometrically as a homological vector field $Q$ on $A[1]$ (the Lie algebroid $A$ with the degree of fibers shifted by one). This is one of the basic examples of $Q-$manifolds which appear in literature.

**Definition 3.2.4** *By definition, a $Q-$**manifold**, also known as a dg-manifold, is a supermanifold $M$ equipped with a degree one vector field, $Q$, satisfying the integrability condition $[Q, Q] = 0$, i.e., a homological vector field.*

**Example 3.2.1** In the basic example of a Lie algebroid, $A \to M$, $Q$ is simply the usual Lie algebroid differential, $d_A$ (see (2.4)), and the algebra of functions on $A[1]$ is modeled on the exterior algebra of $A-$valued forms. Thus, in the particular case of a Lie algebra, $\mathfrak{g}$, it is considered an odd manifold of degree one and denoted by $\mathfrak{g}[1]$. The algebra of functions is identified with $\Lambda^\bullet \mathfrak{g}^*$, by declaring the product of two elements in $\mathfrak{g}^*$ to be anti-commuting and elements in $\mathfrak{g}^*$ to have degree one. The $Q$ field is given by the Chevalley-Eilenberg differential, $d_{CE}\xi(X, Y) = -\xi([X, Y])$, for $\xi \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$(see (2.5)). This construction will be generalized later on to the case where the Lie algebra acts on a manifold $M$ by vector fields.

△

**Example 3.2.2** For the standard Lie algebroid, $T[1]M$ is known as the odd tangent bundle. The algebra of functions is canonically isomorphic to the algebra of differential forms on $M$ with the $Q-$structure being determined by the de Rham operator.

△

**Example 3.2.3** In general, we can endow the tangent bundle of a $Q-$manifold with a canonical $Q-$structure by considering the Lie derivative along $Q$.

△

**Definition 3.2.5** *A morphism of $Q-$manifolds (or $Q - morphism$), $(M_1, Q_1)$ and $(M_2, Q_2)$, is a degree-preserving map $\varphi : M_1 \to M_2$ which induces a morphism of differential algebras, i.e., its pullback $\varphi^* : C^\infty(M_2) \to C^\infty(M_1)$ commutes with the corresponding homological vector fields, viewed as super-derivations on functions. In short, the following chain property holds:*

$$Q_1\varphi^* - \varphi^*Q_2 = 0$$

Since the composition of two $Q-$morphisms is again a $Q-$morphism one can consider the well-defined category of $Q-$manifolds.

**Definition 3.2.6** *A $Q-$**bundle** is a fiber bundle in the category of $Q-$manifolds.*

**Remark 3.2.4** Not every section of a $Q-bundle$ is a section in the category of $Q-$manifolds, i.e., not necessarily a $Q-$morphism (see e.g. [2]).

$\diamond$

**Example 3.2.4** Every vector bundle can be regarded as a $Q-$bundle by taking the trivial differential.

$\triangle$

**Example 3.2.5** If we consider two $Q-$manifolds their product is again a $Q-$manifold. By considering the projections to each factor we produce trivial $Q-$bundle structures.

$\triangle$

**Example 3.2.6** (Atiyah Algebroid) Given a $G$ principal bundle $\pi : P \to M$, let us consider the corresponding Atiyah algebroid, see Example 2.1.4. Any connection on $P$ allows us to lift tangent vectors on $M$ to tangent vectors on $P$. Equivariance with respect to the $G-$action then gives a bundle map $TM \to TP/G$ and hence a section of the $Q-$bundle $\pi_* : T[1]P/G \to T[1]M$. Such section is not a $Q-$morphism in general and it is, precisely when the connection is flat (see e.g. [2], Section 2).

$\triangle$

Now, for any degree preserving map between two $Q-$manifolds $\varphi : M_1 \to M_2$ the difference $F \doteq Q_1\varphi^* - \varphi^* Q_2$ is of particular interest since, by definition, it measures the failure of $\varphi$ being a morphism in the $Q-$category. Moreover, it can be interpreted as a curvature since the Maurer-Cartan equation is a particular example of the chain map property, [2].

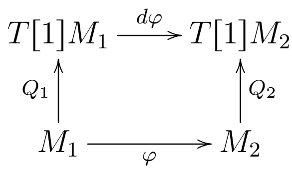
Now, the operator $F$, called the *field strength* [22], is a degree one derivation on $C^\infty(M_2)$ that takes values in $C^\infty(M_1)$ satisfying the following property

$$\begin{aligned} F(gh) &= (Q_1\varphi^* - \varphi^* Q_2)(gh) \\ &= Q_1(\varphi^* g \varphi^* h) - \varphi^*((Q_2 g)h) - (-1)^{|g|}\varphi^*(g(Q_2 h)) \\ &= (Q_1\varphi^*)(g)\varphi^*(h) + (-1)^{|g|}\varphi^*(g)(Q_1\varphi^*)(h) - (\varphi^* Q_2)(g)\varphi^*(h) - (-1)^{|g|}\varphi^*(g)(\varphi^* Q_2)(h) \\ &= (Q_1\varphi^* - \varphi^* Q_2)(g)\varphi^* h + (-1)^{|g|}\varphi^* g(Q_1\varphi^* - \varphi^* Q_2)(h) \\ &= F(g)\varphi^*(h) + (-1)^{|g|}\varphi^*(g)F(h) \end{aligned} \tag{3.3}$$

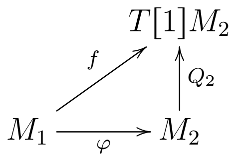
where $|g|$ denotes the degree of $g$.

The next step then is to identify such operator with a degree-preserving map covering $\varphi$ $f : M_1 \to T[1]M_2$ so that the Leibniz-type property in (3.3) is expressed as a morphism of algebras. Moreover, under a suitable choice of a $Q-$field this map $f$ becomes a $Q-$morphism.

In order to do so, let us consider the $Q-$fields as maps from the underlying manifolds to the odd tangent bundles and hence the following non-commutative diagram, where the maps are morphisms of graded manifolds:




By noticing both ways from $M_1$ to $T[1]M_2$ end in the same fiber over $M_2$ it makes sense considering the difference $f \doteq d\varphi Q_1 - Q_2\varphi$:




Note that $d\varphi \in \Gamma(Hom(TM_1, \varphi^*(TM_2)))$ and thus $f$ is a section of the pullback bundle $\varphi^*(TM_2)$. Moreover, $f^* : \Omega(M_2) \to C^\infty(M_1)$[4] satisfies

$$f^*(h) = \varphi^*(h), \quad f^*(dh) = F(h), \text{ and } f^*(\alpha\beta) = f^*(\alpha)f^*(\beta) \tag{3.4}$$

for $h \in C^\infty(M_2)$[5] and $\alpha, \beta \in \Omega(M_2)$.

In fact, let $x \in M_1$, then $f(x) \in T_{\varphi(x)}M_2$ and we have that

$$\begin{aligned} f^*(dh)(x) &= \langle f(x), dh\big|_{\varphi(x)}\rangle \\ &= \langle d\varphi Q_1(x), dh\big|_{\varphi(x)}\rangle - \langle \varphi^* Q_2(\varphi(x)), dh\big|_{\varphi(x)}\rangle \\ &= Q_1(\varphi^* h)(x) - \varphi^*(Q_2 h)(x) \\ &= (Q_1\varphi^* - \varphi^* Q_2)(h)(x) \\ &= F(h)(x) \end{aligned}$$

The remaining properties are straightforward.

Now, let us present, as previously stated, how one can endow $T[1]M_2$ with a $Q-$field, which we will denote by $Q_{tot}$, such that

$$Q_1 f^* - f^* Q_{tot} = 0 \tag{3.5}$$

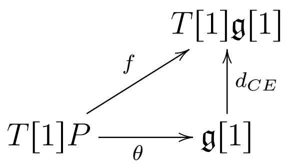
The homological vector field we are looking for is given by the sum of two canonical $Q-$structures. We consider $Q_{tot} = d + \mathcal{L}_{Q_2}$, where $d$ is the super-version of the de Rham differential and $\mathcal{L}_{Q_2}$ is the super Lie derivative along $Q_2$ (see footnote number 4).

[4]One can describe the algebra of functions on $T[1]M$ as the algebra of super-differential forms $\Omega(M)$ according to the Bernstein Leites sign convention, see e.g. [23] .

[5]Note that $\Omega(M_2)$ is generated by $h, dh$ for $h \in C^\infty(M_2)$.

For checking that (3.5) indeed holds, it is enough to apply the left-hand side of it on the generators of $\Omega(M_2)$ and then use the properties of $f^*$ listed in (3.4).

**Example 3.2.7** (Weil Algebra, [2]) Let $P \to M$ be a $G$ principal bundle, with the Lie algebra of $G$ being denoted by $\mathfrak{g}$. Take $M_1$ as the odd tangent bundle $T[1]P$ with $Q_1$ as the de Rham differential and take $M_2$ as $\mathfrak{g}[1]$ with $Q_2$ as the Chevalley-Eilenberg differential. As $\varphi$ we choose any connection one-form on $P$ , say $\theta$. We want to describe our chain map $f$




in this particular case.

Note that the dual map $f^*$ goes from the Weil algebra, $W(\mathfrak{g}) \simeq C^\infty(T[1]\mathfrak{g}[1]) \simeq \Omega(\mathfrak{g}[1])$, to the algebra of forms on $P$, $\Omega(P) \simeq C^\infty(T[1]P)$.

Now, it is known that one can produce a chain map from $W(\mathfrak{g})$ to any dg commutative algebra $A$ as an extension of a graded morphism $\psi : \Lambda^\bullet\mathfrak{g}^* \to A$. To our purposes, let $A$ be the algebra of forms $\Omega(P)$. Since the exterior algebra $\Lambda^\bullet\mathfrak{g}^*$ is generated by $\mathfrak{g}^*$, $\psi$ can be identified with $\theta$, which is an element $\Omega^1(P) \otimes \mathfrak{g}$. Since its curvature $F_\theta \doteq d\theta + \frac{1}{2}[\theta, \theta]$ belongs to $\Omega^2(P) \otimes \mathfrak{g}$, the required map is given by

$$\eta \otimes \phi \mapsto \eta(\theta, \ldots, \theta)\phi(F_\theta, \ldots, F_\theta) \tag{3.6}$$

where $\eta \in \Lambda^q(\mathfrak{g}^*)$ and $\phi \in S^p(\mathfrak{g}^*)$.

In fact, let us fix a basis $\{e_a\}$ on $\mathfrak{g}$ as in the proof of Theorem 3.2.1. Using the notation employed there it suffices to check the following

$$\begin{aligned} f^*(\eta^a) &= \eta^a(\theta) \\ f^*(d\eta^a) &= (d\theta^* - \theta^* d_{CE})(\eta^a) \end{aligned}$$

The first equality is clear. And as for the second, we compute:

$$\begin{aligned} (d\theta^* - \theta^* d_{CE})(\eta^a) &= \theta^*\left(d\eta^a + \frac{1}{2}C^a_{bc}\eta^b\eta^c\right) \\ &= d\theta_a + \frac{1}{2}C^a_{bc}\theta_b\theta_c \\ &= {F_\theta}^a = \phi^a(F_\theta) = f^*(\phi^a) \end{aligned}$$

and we are done, since $\phi^a = d\eta^a$.

Next, note that on the odd tangent bundle $T[1]P$ we have a natural action of the Lie algebra $\mathfrak{g}$, which is simply the infinitesimal counterpart of the group action. Furthermore, we also have an action of $\mathfrak{g}[1]$, which is given by

$$\mathfrak{g} \ni X \mapsto \iota_X^\# \in \mathfrak{X}(P)$$

where, as before, $X^{\#}$ denotes the so called fundamental vector field.

Now, by definition, the connection one-form satisfies the following

$$\theta(X^{\#}) = X \text{ and } (R_g)^*\theta = Ad_{g^{-1}}\theta$$

If connectedness of $G$ is assumed[6], then such properties are equivalent, respectively, to:

$$\iota_{X^{\#}}\theta = X \text{ and } \mathcal{L}_{X^{\#}}\theta = [\theta, X]$$

And, thus we can view $\theta$ as a $\bar{\mathfrak{g}}-$equivariant map $T[1]P \to \mathfrak{g}[1]$, where $\bar{\mathfrak{g}} \doteq \mathfrak{g}[1] \oplus \mathfrak{g}$.

**Remark 3.2.5** Recall the Maurer-Cartan form, $\theta^{MC}$, gives us a way of trivializing $TG$. More precisely, for each $g \in G$ we have an isomorphism $\theta^{MC}(g) : T_gG \to \mathfrak{g}$, given by the following. If $V \in TG$ is such that $V_e$ equals $X \in \mathfrak{g}$ (here, $V_e$ denotes the vector field evaluated at the identity), then $\theta^{MC}(g)(V) = X$. Thus, since the fibers of $P$ are naturally copies of $G$, equivariance with respect to the odd part of $\bar{\mathfrak{g}}$ tells us $\theta$ restricted to each fiber of $P$ is nothing but $\theta^{MC}$. As a consequence, the curvature form $F_\theta$ vanishes when restricted to fibers.

◇

In particular, the map $f^*$ in (3.6) reproduces the usual Chern-Weil map.

In fact, we have a $\bar{\mathfrak{g}}$ action on $W(\mathfrak{g})$ such that for the generators $\phi^a$:

$$\iota_X^{W(\mathfrak{g})}\phi^a = 0 \quad \forall X \in \mathfrak{g}$$

Thus, the horizontal subalgebra of $W(\mathfrak{g})$ is precisely $S^\bullet\mathfrak{g}^*$. Moreover, if an element $\omega \in S^\bullet\mathfrak{g}^*$ is $\mathfrak{g}-$ equivariant, then $L_X^{W(\mathfrak{g})}\omega = 0$ for every $X \in \mathfrak{g}$ and hence the basic subalgebra is $S^\bullet(\mathfrak{g}^*)^{\mathfrak{g}}$. Finally, one observes that the $\bar{\mathfrak{g}}-$invariant forms on $P$ are, by definition, the basic ones. That is,

$$f^* : S^\bullet(\mathfrak{g}^*)^{\mathfrak{g}} \to \Omega(M)$$

To see such map does not depend on the choice of connection and hence it maps to the de Rham cohomology of $M$ one considers the $Q-$bundle given by the anchor map $\rho : A[1] \to T[1]M$ of the associated Atiyah algebroid, $A$, to the $G$ principal bundle $P \to M$. Then $\theta$ can be viewed as a section of $\rho$, i.e., a splitting of the Atiyah sequence (2.3), and we apply the construction of characteristic classes suggested on Theorem 3.4 in [2].

△

Finally, we conclude this section by showing how one may apply the formalism employed so far to the computation of *equivariant cohomology* (see Section 3.3). For that, we shall consider the following.

[6]For the implications $\theta(X^{\#}) = X \Rightarrow \iota_{X^{\#}}\theta = X$ and $(R_g)^*\theta = Ad_{g^{-1}}\theta \Rightarrow \mathcal{L}_{X^{\#}}\theta = [\theta, X]$ the group $G$ does not need to be connected.

**Example 3.2.8** (Equivariant Cohomology) Let $\pi : P \to B$ be a $G-$principal bundle and let $M$ be a $G-$space[7], so we can define the associated bundle $\alpha : P \times_G M \to B$. Now, let us choose a connection $1-$form $\theta$ on $P$ and a section of $\alpha$, which we identify with a $G-$equivariant map $s : P \to M$. Then $(\theta, s)$ can be viewed as a $\bar{\mathfrak{g}}-$equivariant map $T[1]P \to A[1]$, where $A \doteq \mathfrak{g} \times M$ is the action Lie algebroid, with $\mathfrak{g}$ denoting the Lie algebra of G. Here the $\bar{\mathfrak{g}}$ action is slightly modified and it is determined in terms of the operators Kalkman introduces for the BRST model of equivariant cohomology (see [3]), in the next section we shall make this precise.

Now, this initial data gives a chain map

$$\begin{array}{ccc} & & T[1]A \\ & \overset{f}{\nearrow} & \uparrow{\scriptstyle d_{CE}+d} \\ T[1]P & \xrightarrow[(\theta,s)]{} & A[1] \end{array} \tag{3.7}$$

where under the previous notations we take as $M_1$ the odd tangent bundle $T[1]P$ with the de Rham differential and as $M_2$ the action algebroid $A[1]$ with shifted fibers. In particular, the dual acts from $W(\mathfrak{g}) \otimes \Omega(M)$ to the algebra of forms on $P$ and restricted to basic elements we end up, we obtain a map

$$(S^{\bullet}(\mathfrak{g}^*) \otimes \Omega(M))^{\mathfrak{g}} \to H(B) \tag{3.8}$$

which is precisely the Cartan model of equivariant cohomology (see next section).

An important question one might ask is the following:

**How does $f^*$ acts?**

Clearly, if $\phi \in S^k(\mathfrak{g}^*)$, then $f^*(\phi \otimes 1)$ agrees with the map in (3.6), i.e., the usual Chern-Weil map.

△

## 3.3 Equivariant Cohomology

In what follows we are interested in studying differential forms (or better cohomology) on the space of orbits $M/G$ for $G$ a compact and connected Lie group and $M$ a space where $G$ acts, but not necessarily a $G-$space, i.e., $M/G$ may not be a manifold. For that, the main idea is to replace $M$ by a $G-$space which is homotopically equivalent to $M$.

Let $EG \to BG$ be the universal or classifying bundle for $G$. We define the *homotopy quotient* as the total space of the associated bundle $EG \times_G M \to BG$ (where $EG \times_G M \equiv (EG \times M)/G$) and we denote it by $M_G$. Note that, such construction agrees up to homotopy with the quotient $M/G$ provided the action of $G$ on $M$ is free. More precisely, let us consider $E \doteq EG \times M$. Then $G$ acts freely on $E$ and because the action of $G$ on $EG$ is

[7]By $M$ being a$G-$space we mean that we have a $G-$action on $M$ so that the space of orbits is a manifold.

fiberwise we have that map $EG \times_G M \to BG$ which sends the orbit through $(e,m)$ to the orbit through $e$ is well defined and has as typical fiber $M$. Now, if in addition the action of $G$ on $M$ is free, then such map descends to a well-defined map on the quotient with typical fiber $EG$ and since $EG$ is contractible we see in this case the homotopy quotient has the homotopy type of $M/G$. In short, we have the following diagram (Borel's diagram)

$$\begin{array}{ccccc} EG & \longleftarrow & EG \times M & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ BG & \longleftarrow & EG \times_G M & \longrightarrow & M/G \end{array}$$

By definition, the equivariant cohomology of $M$, denoted as $H_G(M)$, is the usual cohomology ring of $M_G$. In literature this is referred to as the Borel model of equivariant cohomology. One can check that $H_G(M)$ is a module over $H(BG)$.

**Example 3.3.1** (A point) Clearly, if $M$ is a single point, then $H_G(M) = H_G(pt) = H(BG)$.

△

**Example 3.3.2** (Principal $G-$bundles) Let $E \to B$ be a principal $G-$bundle. Then, $E_G$ can be viewed as a bundle over $B$ and since the fibers of such bundle are contractible we have a homotopy equivalence between the homotopy quotient and the base manifold. Thus, $H_G(E) = H(B)$.

△

**Example 3.3.3** ($G-$spaces) If $M$ is a $G-$space, then $M_G$ can be viewed as a bundle over the quotient $M/G$ with fibers the contractible space $EG$ and hence $H_G(M) = H(M/G)$.

△

**Example 3.3.4** ($G$ acting on itself by conjugation) If we take $M = G$ itself and consider the adjoint action, then one can prove $H_G(G) = H(BG) \otimes H(G)$. This example is of particular interest to our purposes.

△

As we are assuming $G$ is compact and connected there are other two nice models for the equivariant cohomology, namely the Weil and the Cartan models.

The Weil model considers the basic subalgebra of $\mathcal{A} \doteq W(\mathfrak{g}) \otimes \Omega(M)$ with the natural differential $D = d^{W(\mathfrak{g})} + d$. More precisely, one considers the following intersection

$$\left( \bigcap_a ker(\iota_{e_a}^{W(\mathfrak{g})} \otimes 1 + 1 \otimes \iota_{e_a}) \right) \cap \left( \bigcap_b ker(L_{e_b}^{W(\mathfrak{g})} \otimes 1 + 1 \otimes \mathcal{L}_{e_b}) \right)$$

elements in the left being the horizontals while in the right, the invariants.

In contrast, the Cartan model is given by the algebra morphism

$$(W(\mathfrak{g}) \otimes \Omega(M))_{\text{basic}} \simeq (S^\bullet \mathfrak{g}^* \otimes \Omega(M))^G$$

induced from the map $\eta^a \mapsto 0$. Here, the upper $G$ means the infinitesimal $G-$invariant subalgebra.

### 3.3.1 The BRST Model

Furthermore, in the paper [3] it is presented an explicit relation between the BRST differential algebra and the two mentioned models of equivariant cohomology. We summarize here the main ideas.

The BRST differential algebra is defined as nothing but the algebra $\mathcal{A} \doteq W(\mathfrak{g}) \otimes \Omega(M)$ with the differential

$$\delta \doteq D + \eta^a \mathcal{L}_{e_a} - \phi^b \iota_{e_b} \tag{3.9}$$

where we have maintained the previous notation, i.e., $\eta^a$ and $\phi^a$ are the generators of $W(\mathfrak{g})$ of degree 1 and 2, respectively, and $\{e_a\}$ is a basis of $\mathfrak{g}$. Moreover, its basic subalgebra is defined in such way[8] it agrees with the algebra in the Cartan model. The laborious part is to show the map $\psi = exp(-\eta^a \iota_{e_b})$ makes the following diagram commute

$$\begin{array}{ccc} W(\mathfrak{g}) \otimes \Omega(M) & \xrightarrow{\psi} & W(\mathfrak{g}) \otimes \Omega(M) \\ \delta \downarrow & & \downarrow D \\ W(\mathfrak{g}) \otimes \Omega(M) & \xrightarrow[\psi]{} & W(\mathfrak{g}) \otimes \Omega(M) \end{array} \tag{3.10}$$

where $D = d^{W(\mathfrak{g})} + d$.

One needs to check the derivation $\psi^{-1} D \psi$ agrees with $\delta$ on generators and the next step then is to compute the image of the basic subalgebra (as in the Weil model) under $\psi^{-1} = exp(\eta^a \iota_{e_a})$. It turns out that for the horizontal elements one can look only to the contractions on the Weil algebra which yields on the BRST algebra the subalgebra $S^\bullet(\mathfrak{g}^*) \otimes \Omega(M)$. Moreover, one observes that the $G$ action is the same in both sides of the diagram (3.10) above so that in the end we obtain $(S^\bullet(\mathfrak{g}^*) \otimes \Omega(M))^G$.

The restriction of $\psi$ to the 'basic subalgebras' on each side of (3.10) agrees with the map $\eta^a \to 0$, the same as before.

### 3.3.2 Closed Equivariant Extensions

In the last paragraphs we have been discussing the equivariant cohomology $H_G(M)$ and have seen there are algebraic models for computing it. Following the ideas in [8] our next step is to present how one can obtain explicit representatives for the equivariant cohomology.

Recall in the usual Chern-Weil construction given a connection form on the total space of some principal bundle by evaluating its curvature on invariant polynomials we obtain the so called curvature characteristic form. Out of such forms one can produce some secondary characteristic forms namely the Chern-Simons forms. Now, in the equivariant case one may ask if equivariant characteristic classes in the Cartan model can be constructed in a similar way, i.e., out of the curvature of some connection. The answer is

[8] Recall example 3.2.8. There, the $\bar{\mathfrak{g}}$ action is defined is such a way the horizontal subalgebra of $\mathcal{A}$ is given by $\bigcap_a (\iota_{e_a}^{W(\mathfrak{g})} \otimes 1)$ and the invariant elements are those in the intersection $\bigcap_a (L_{e_a}^{W(\mathfrak{g})} \otimes 1 + 1 \otimes \mathcal{L}_{e_a})$.

indeed positive as we shall see and it will be explained in the next paragraphs where we make the later statements more precise.

To start, let $\theta_1$ and $\theta_2$ be two connection forms on $M$, i.e., $\theta_i \in \Omega^1(M, \mathfrak{g})$, with $i = 1, 2$, and let $\phi \in (S^k(\mathfrak{g}^*))^G$ be an invariant polynomial of degree $k$. We define the following form $\omega^k(\theta_1, \theta_2) \in \Omega^{2k-1}(M)$

$$\omega^k(\theta_1, \theta_2) \doteq \int_0^1 \phi(F_{\tilde{\theta}}) \tag{3.11}$$

where $\tilde{\theta} = (1-t)\theta_1 + t\theta_2$ for $t \in [0,1]$ is viewed as a form on $M \times [0,1]$ and $F_{\tilde{\theta}}$ denotes its curvature form.

In particular, if $M = G$ and $G$ acts on itself by conjugation, then we consider $\omega^k(\theta^L) \doteq \omega^k(0, \theta^L)$, where $\theta^L$ denotes the left invariant Maurer-Cartan form. For compact $G$ it is known the cohomology classes $[\omega^k(\theta^L)]$ generate $H(G)$.

Note that the forms $\omega^k(\theta^L)$ are in fact closed, since

$$d\omega^k(\theta^L) = \phi(F_{\theta^L}) - \phi(F_0) = 0 \tag{3.12}$$

Now, to obtain equivariant extensions of the forms in (3.11) , let us consider a $G-$principal bundle $P \to M$ and recall in our discussion on equivariant cohomology we were assuming $G$ is both compact and connected. Thus, starting with a connection on $P$ we can use the averaging trick over the Haar measure on $G$ to obtain a $G-$invariant connection. Therefore, let us consider the Cartan model for $H_G(P)$ and let us choose $\theta_1, \theta_2$ two $G-$invariant connections on $P$. The $G$ action on $P$ induces an infinitesimal action of $\mathfrak{g}$: $\mathfrak{g} \ni X \mapsto X^{\#} \in \Omega(P)$ and we define the *moment map* $\mu \in (C^\infty(P) \otimes \mathfrak{g})^G \otimes \mathfrak{g}^*$ by

$$X \mapsto \mu(X) \doteq -\iota_{X^\#}\theta \in \mathfrak{g}$$

for $\theta \in \Omega(P, \mathfrak{g})$. That is, $\mu(X) : P \to \mathfrak{g}$ is the vertical action of $X$ determined by $\theta$.

Next, we observe that an element in $(S^k(\mathfrak{g}^*) \otimes \Omega(P))^G$ can be viewed as a polynomial map $\phi : \mathfrak{g} \to \Omega(P)$ which is $G-$equivariant. So that, given an invariant polynomial $\phi \in S^k(\mathfrak{g}^*)$, the corresponding equivariant characteristic form on $P$ has a representative in the Cartan model given by [9]

$$\phi(F_{\tilde{\theta}} + \mu)$$

Then, by considering the form $\tilde{\theta} = (1-t)\theta_1 + t\theta_2$ for $t \in [0,1]$ as before, we obtain the following equivariantly closed form:

$$\omega_G^k(\theta_1, \theta_2) \doteq -\int_0^1 \phi(F_{\tilde{\theta}} + \mu) \tag{3.13}$$

These are equivariantly closed extensions of the forms in (3.11) and their cohomology class are the pullback of classes in $H_G(M)$.

[9]The term $F_{\tilde{\theta}} + \mu$ is known in literature as the equivariant curvature and in fact corresponds to the curvature of some superconnection on $P$, see e.g. [8].

Restricting to the particular case where $M = G$ and the action is by conjugation, we find that $\mu(X)\big|_g = -Ad_{g^{-1}}(X)$ and in particular,

$$d_G\omega_G^k(0, \theta^L) = \phi(Ad_{g^{-1}}(X)) - \phi(X) = 0$$

## 3.4 Secondary Characteristic Forms

We end the present thesis by presenting, on this final section, a geometrical approach to the construction in (3.1).

First, we start quite generally. Let $P \to M$ be a $G-$principal bundle, with compact and connected $G$, and let us consider the action of $G$ on itself by conjugation. Then $P \times G \to E$ is also a principal bundle, where $E$ is the associated bundle $P \times_G G$. The main idea will be to consider some particular basic forms on the total space of this bundle so that given a section of $E \to M$ they pull back to forms on the base manifold $M$. Moreover, such construction when restricted to forms on $G$ gives us precisely the forms in (3.13) and which were developed in the paper [8].

The ingredients we need are the following. Starting with a connection form on $P$, say $\theta$, the key point is to consider a connection form on the trivial bundle $P \times G$, say $\theta'$, such that, viewed as forms $P \times G$, $\theta$ and $\theta'$ are conjugated as well as their curvatures.

We define $\theta' = Ad_{g^{-1}}(\theta) + \theta^L$, where (as before) $\theta^L$ denotes the left-invariant Maurer-Cartan form of $G$. Then, $\theta' = Ad_{g^{-1}}(\theta + d)$ and for every Ad-invariant polynomial $\phi \in (S^k(\mathfrak{g}^*))^{\mathfrak{g}}$ we have that

$$0 = \phi(F_{\theta'}) - \phi(F_\theta) = dCS(\phi, \theta, G) \tag{3.14}$$

where $CS(\phi, \theta, G) \in \Omega^{2k-1}(P \times G)$ is some secondary characteristic form which is basic and hence pulls back to a form on $E$. That is, we have a map

$$S^\bullet(\mathfrak{g}^*)^G \to H^{odd}(P \times_G G)$$

In the case $P = G$, we call such map *transgression* and the elements on its image are called transgressive. For compact and connected $G$ it is known the cohomology ring of $G$ is generated as an $\mathbb{R}-$algebra by the transgressive elements, see also remark 3.4.1 below.

More explicitly, the secondary characteristic form we obtain is given by

$$CS(\phi, \theta, G) = \int_0^1 \phi(F_{\tilde{\theta}}) \tag{3.15}$$

where $\tilde{\theta}$ is a connection form on the total space of $p : (P \times G) \times [0, 1] \to P \times G$ so that

$$\tilde{\theta}\left(\frac{\partial}{\partial t}\right) = 0 \quad , \quad \tilde{\theta}\Big|_{P\times\{t\}} = p^*\bar{\theta}(t) \tag{3.16}$$

and $\bar{\theta}(t)$ is the family of connections $(1 - t)\theta + t\theta'$.

Note that by setting $A \doteq \theta' - \theta$ we have that $\bar{\theta}(t) = \nabla + tA$ and thus (3.15) can be rewritten as

$$CS(\phi, \theta, G) = \int_0^1 \phi(dt \wedge A + F_{\bar{\theta}(t)}) \tag{3.17}$$

Further, by pulling back either the form in (3.15) or the form in (3.17) we obtain a form on $E$ and hence a form on $M$ whose cohomology class only depends on the homotopy class of some section $\mathcal{S} \in \Gamma(E)$:

$$\begin{array}{ccc} P \times G & \longrightarrow & P \times_G G \\ & & \downarrow \uparrow \mathcal{S} \\ & & M \end{array}$$

Moreover, one can prove that if $\theta = 0$, then the equivariant form on the associated bundle $E$ which corresponds to $CS(\phi, \theta, G)$ is precisely $\omega_G(\theta^L)$, as in (3.13). In other words, if we don't look to the data on $P$, then the forms we obtain are just equivariant extensions of forms on $G$.

Now, this construction is universal in the sense that as described above the data $(\theta, S)$ produces a map $H_G(G) \to H(M)$.

Recall on a slightly different approach we have seen this may also lead us to the generalized Chern-Weil map (3.8), i.e., a chain map

$$\begin{array}{ccc} & & T[1](\mathfrak{g} \times G) \\ & \overset{f}{\nearrow} & \uparrow d_{CE}+d \\ T[1]P & \xrightarrow[(\theta,\mathcal{S})]{} & \mathfrak{g}[1] \times G \end{array}$$

To state more precisely, if we look at the algebraic counterpart of the construction (3.14) above, the ingredients we face with are the following: First, forms on $P \times G$ correspond to the Weil model $W(\mathfrak{g}) \times \Omega(G)$. Secondly, on the $\mathfrak{g}-$differential algebra, $W(\mathfrak{g})$, we consider the (algebraic) connection $\eta = \theta_W \doteq \eta^a \otimes e_a$ as in the proof of Theorem 3.2.1 and, moreover, on the product we consider $\eta' = Ad_{g^{-1}}(\eta) + \theta^L$ so that, combining these, we obtain some closed and equivariant basic form whose differential is given by

$$\phi(F_{\eta'}) - \phi(F_\eta)$$

In fact, one can check that $F_{\eta'} = Ad_{g^{-1}} F(\eta)$ and we may apply the map $\psi^{-1} = exp(\eta^a \iota_{e_a})$ as in (3.10), which is the analogue of the Mathai and Quillen map [24] described by Kalkman on [3], to obtain an expression for $\eta'$ on the BRST model.

Furthermore, by means of $\mathcal{S}^*$, such secondary characteristic forms correspond to a form on the base manifold $M$. In particular, either ways somehow we obtain the same results, i.e, we obtain a map

$$H_G(G) \to H(M)$$

induced by the data $(\theta, \mathcal{S})$.

**Remark 3.4.1** Note that in this setting, (see e.g. Example 3.3.4) $H_G(G)$ can be decomposed as $H(BG) \otimes H(G)$. Plus, an algebraic way of computing $H(BG)$ is given by the Weil algebra and when restricted to its basic elements in fact $f^*$ recovers the usual Chern-Weil map (3.6). Thus, the information we are really interested in lie in $H(G)$ and, for instance, how $f^*$ acts on $\Omega(G)$. In fact, one may apply the following theorem due to Hopf, see e.g. [25]:

**Theorem 3.4.1** (Hopf). *If $G$ is a compact and connected Lie group of rank $n$, then the cohomology ring $H(G, \mathbb{R})$ is an exterior algebra on $n$ generators (called primitives) of odd degrees, i.e., there exist $P_1, \ldots, P_n$ of odd degrees such that*

$$H(G, \mathbb{R}) \simeq \Lambda[P_1, \ldots, P_n]$$

so that it is sufficient to know how $f^*$ acts on each $P_i$.

$\diamond$

At last, we observe that for fullfilling our task it only remains to specialize to the case of our interest, i.e., we must take as $G$ the orthogonal group $O(n)$[10] so that $\mathcal{S}$ corresponds to a Lagrangian subbundle $L \subset TM \oplus T^*M$.

In order to do so, recall that for stablishing the correspondence between lagrangian subbundles and orthogonal maps we have picked a Riemannian metric $g$ on $M$, hence we can consider the Levi-Civita connection on the orthogonal frame bundle $P$ so that the corresponding horizontal distribution at some orthogonal basis $\beta$ of $T_pM$ is given by the derivatives of all curves $\beta(t)$ in $P$ that are parallel along $p(t) \in M$. In particular, the Levi-Civita connection, which we shall denote by $\nabla$, uniquely corresponds to a connection form on $P$ (that we still denote by $\nabla$) and one can check the secondary characteristic forms we obtain look as follows.

$$CS(\phi, \nabla, SO(n)) \doteq \int_0^1 \phi(F_{\tilde{\theta}})$$

where $\tilde{\theta}$ is defined as in (3.16) but the family of connections we consider here is given by $(1-t)\nabla + t(g^{-1}\nabla g) + \theta^L)$, with $t \in [0,1]$.

Now, such form, which corresponds to a form on the associated bundle $E$, can be pulled-back to a form on the base manifold $M$ by means of any section $\mathcal{S} \in \Gamma(E)$ so that we obtain [1]

$$c_\phi(\mathcal{S}, g) \doteq \int_0^1 \phi(F_{\tilde{\theta}}) \tag{3.18}$$

where, anagously to (3.17), $\tilde{\theta}$ is defined on $M \times [0,1]$ in terms of $\bar{\theta}(t) \doteq \nabla + t\mathcal{A}$, and $\mathcal{A} \doteq \mathcal{S}^{-1}\nabla(\mathcal{S})$. Note that here $\nabla$ is denoting the Levi-Civita on $M$ compatible with the chosen metric $g$.

[10] In fact, its connected components

We immediately see that if $\mathcal{S}$ is covariantly constant, then $c_\phi(\mathcal{S}, g) = \phi(F_\nabla)$ and our construction restricts to the usual Chern-Weil map. In some sense there is no new information and as for our chain map (3.8), it vanishes on $\Omega(G)$.

Now, we can make (3.18) more explicit by observing the following. If $A$ belongs to the Lie algebra $\mathfrak{so}(n)$, i.e., $A$ is skew-symmetric, then

$$det(\lambda - A) = det(\lambda - A^t) = det(\lambda + A)$$

so that the characteristic polynomial of A may be written as

$$det(\lambda - A) = \lambda^n + \phi_1(A)\lambda^{n-2} + \phi_2(A)\lambda^{n-4} + \dots$$

for some polynomials $\phi_i$ of degree $2i$ that are known to generate the ring $S^\bullet(\mathfrak{so}(n)^*)^{SO(n)}$ for odd $n$, [25].

For instance, if $\lambda_1, \dots, \lambda_n$ are the $n$ eigenvalues of $A$, then by Vieta's formulas

$$\begin{aligned} \sum_{i=1}^n \lambda_i &= -\phi_1(A) \\ \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant n} \lambda_{i_1}\lambda_{i_2} &= \phi_2(A) \\ &\vdots \\ \sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_k \leqslant n} \lambda_{i_1}\dots\lambda_{i_k} &= (-1)^k \phi_k(A) \\ &\vdots \\ \lambda_1\lambda_2\dots\lambda_n &= (-1)^n\phi_n(A) = -\phi_n(A) \quad (\text{odd } n) \end{aligned}$$

and we see that, up to a sign, $\phi_1(A) = Tr(A)$ and $\phi_n(A) = det(A)$.

Alternatively, we have a correspondence between the symmetric polynomials $\phi_i$ and the so called *Newton polynomials*, $p_k(A) \doteq Tr(A^k) = \sum_{i=1}^n {\lambda_i}^k$. Such correspondence is given by the Newton formula:

$$k\phi_k = \sum_{i=1}^k (-1)^{i-1} p_i \phi_{k-i}$$

Now, in the even case we have an extra invariant which is not a polynomial in the $\phi_i$ called the Pfaffian and which is defined as follows: To each $A \in \mathfrak{so}(n)$ we may associate the alternating bilinear form $\omega_A$ given by

$$\omega_A(u, v) \doteq (Au, v)$$

for every $u, v \in \mathbb{R}^{2k}$, where $(\cdot, \cdot)$ denotes the inner product. The Pfaffian of $A$, denoted by Pfaff$(A)$, is then defined by

$$\frac{1}{n!}\omega_A^n = \text{Pfaff}(A) vol,$$

where $vol \doteq e^1 \wedge \ldots \wedge e^{2k}$, for some oriented orthonormal basis $\{e_i\}$ and as usual $\{e^i\}$ is denoting the dual basis.

In fact, one can find an orthonormal basis relative to which $A$ have blocks along its main diagonal of the type

$$\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$$

and hence $\text{Pfaff}(A) = \lambda_1 \ldots \lambda_n$.

On the other hand, since $det(A) = \lambda_1^2 \ldots \lambda_n^2$, it follows that Pfaff $^2 = det$ and thus

$$\left(\frac{1}{2\pi^n}\text{Pfaff}\right)^2 = \phi_n$$

In short, all characteristic classes (3.18) can be obtained from the $\phi_i$'s (and possibly Pfaff depending on the parity of $n$).

The main profit of obtaining such characteristic forms (3.18), as described above, is that now we can use it to distinguish different Dirac structures (actually maximally isotropic spaces in general). The most significant results to our interests are summarized next.

**Theorem 3.4.2** ([1]). *Let $D$ be a Dirac structure on an exact Courant algebroid $E$. Then the projection of $D$ to $TM$ (resp. $T^*M$) is non-degenerate, if and only if, the corresponding section $\mathcal{S}$ is homotopic to $1$ (resp. $-1$).*

As a consequence, we have that a Dirac structure is the graph of a presymplectic form (resp. a Poisson bivector field) if and only if its corresponding section is homotopic to the identity section (resp. minus the identity).

Moreover,

**Theorem 3.4.3** ([1]). *Let $D, D'$ be a pair of Dirac structures and denote the corresponding sections by $\mathcal{S}$ and $\mathcal{S}'$, respectively. Then, $D \cap D' = \{0\}$ if and only if $\mathcal{S}^{-1}\mathcal{S}'$ can be deformed to the minus identity section.*

In particular, the above result gives us a homotopical invariant for Lie bialgebroids since these can be defined as a pair of two Lie algebroids being transverse Dirac structures in a Courant algebroid, see e.g. [12].

# Bibliography


[1] A. Kotov and T. Strobl. "Generalizing geometry-algebroids and sigma models". In: *Handbook of Pseudo-Riemannian Geometry and Supersymmetry*. Ed. by V. Cortés. Vol. 16. IRMA Lectures in Mathematics and Theoretical Physics. 2010.

[2] A. Kotov and T. Strobl. "Characteristic classes associated to Q-bundles". In: *International Journal of Geometric Methods in Modern Physics* 12 (2015).

[3] J. Kalkman. "BRST Model for Equivariant Cohomology and Representatives for the Equivariant Thom Class". In: *Communications in Mathematical Physics* 153 (1993), pp. 447–463.

[4] E. Meinrenken. *Clifford Algebras and Lie Theory*. 1st ed. Springer-Verlag Berlin Heidelberg, 2013.

[5] H.B. Lawson. *Spin Geometry*. Princeton University Press, 1989.

[6] H. Bursztyn. "A Brief Introduction to Dirac Manifolds". In: *Lectures at the School on Geometric and Topological Methods for Quantum Field Theory* (2009).

[7] M. Gualtieri. "Generalized Complex Geometry". PhD Thesis. Oxford, 2003.

[8] L. Jeffrey. "Group cohomology construction of the cohomology of moduli spaces of flat connections on 2-manifolds". In: *Duke Math. J.* 77 (1995), pp. 407–429.

[9] Y. Kosmann-Schwarzbach. "Derived Brackets". In: *Letters in Mathematical Physics* 69 (2004), pp. 61–87.

[10] A. Y. Vaintrob. "Lie algebroids and homological vector fields". In: *Russian Mathematical Surveys* 52.2 (1997).

[11] M.J. Gotay. "On Coisotropic Imbeddings of Presymplectic Manifolds". In: *Proc. of the American Mathematical Society* 84 (1982), pp. 111–114.

[12] Z.-J. Liu, A. Weinstein, and P. Xu. "Manin Triples for Lie Bialgebroids". In: *Journal of Differential Geometry* 45 (1997), pp. 547–574.

[13] D. Roytenberg. "Courant algebroids, derived brackets and even symplectic supermanifolds". PhD thesis. University of California at Berkeley, 1999.

[14] T.J. Courant. "Dirac Manifolds". In: *Transactions of the American Mathematical Society* 319 (1990), pp. 631–661.

[15] K. Uchino. "Remarks on the Definition of a Courant Algebroid". In: *Letters in Mathematical Physics* 60 (2002), pp. 171–175.

[16] P. Ševera. *How Courant algebroids appear in 2-dim variational problems (or maybe in string theory).* A letter to Alan Weinstein. http://sophia.dtp.fmph.uniba.sk/~severa/letters/no1.ps. 1998.

[17] A. Alekseev, H. Bursztyn, and E. Meinrenken. "Pure Spinors on Lie groups". In: *Astèrisque* 327 (2009), pp. 131–199.

[18] A. Kotov, P. Schaller, and T. Strobl. "Dirac Sigma Models". In: *Communications in Mathematical Physics* 260 (2005), pp. 455–480.

[19] A. Weinstein and P. Ševera. "Poisson Geometry with a 3-form Background". In: *Prog.Theor.Phys.Suppl.* 144 (2001), pp. 145–154.

[20] H. Cartan. *Cohomologie réelle d'un espace fibré principal différentiable. I: notions d'algèbre différentielle, algèbre de Weil d'un groupe de Lie*. http://www.numdam.org/item?id=SHC_1949-1950__2__A18_0. 1949-1950.

[21] E. Meinrenken. "Equivariant cohomology and the Cartan model". In: *Encyclopedia of mathematical physics*. Ed. by J.-P. Françoise, G.L. Naber, and T.S. Tsun. 2006.

[22] M. Bojowald, A. Kotov, and T. Strobl. "Lie algebroid morphisms, Poisson sigma models and off-shell closed gauge symmetries". In: *Journal of Geometry and Physics* 54 (2005), pp. 400–426.

[23] D A Leites. "Introduction to the theory of supermanifolds". In: *Russian Mathematical Surveys* 35.1 (1980), p. 1.

[24] V. Mathai and D. Quillen. "Superconnections, Thom Classes and equivariant differential forms". In: *Topology* 25.1 (1986), pp. 85–110.

[25] V. W. Guillemin and S. Sternberg. *Supersymmetry and Equivariant de Rham Theory*. Springer-Verlag Berlin Heidelberg, 1999.

[26] A. Alekseev and E. Meinrenken. "Lie Theory and the Chern-Weil Homomorphism". In: *Ann.Scient. Éc. Norm. Sup.* 38 (2005), pp. 303–338.

[27] H. Cartan. *Cohomologie réelle d'un espace fibré principal différentiable. II: transgression dans un groupe de Lie et dans un espace fibré principal; recherche de la cohomologie de l'espace de base*. http://www.numdam.org/item?id=SHC_1949-1950__2__A19_0. 1949-1950.