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Magnetohydrodynamics of rotating plasmas under stationary equilibrium in spherical coordinates

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Em sessão pública iniciada às quatorze horas do dia 24 de maio de 2016, após um seminário sob o título acima e posterior arguição, esta banca examinadora decidiu APROVAR o candidato com o conceito global A.

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Abstract

An equation for MHD stationary equilibrium of rotating plasmas in the azimuthal direction is derived in the case of an spherical coordinate system, with a plasma description of a fluid, considered the entropy is a surface quantity. The equation obtained is solved using both an analytical and a semi-numerical approach for chosen profiles for the current and the pressure functions in terms of the flux function. Plots were made of the solution obtained by this method. The rotation affects only some components of the magnetic field and the current density, the flux function is also affected by the rotation.

Keywords: mhd. plasma. rotation. equilibrium. stationary.

Resumo

Foi derivada uma equação de equilíbrio estacionário MHD de plasmas em rotação na direcção azimutal, no caso de um sistema de coordenadas esféricas, describindo o plasma como um fluido, e considerando a entropia como uma quantidade de superfície. A equação obtida é resolvida utilizando uma abordagem analítica e numérica para os perfis escolhidos das funções de corrente e de pressão em termos da função de fluxo. Gráficos foram feitos da solução obtida por este método. A rotação afecta apenas alguns componentes do campo magnético, da densidade de corrente, a corrent de fluxo poloidal, assim como a função de fluxo também é afectada pela rotação.

palavras-chave: mhd. plasma. rotação. equilíbrio. estacionário

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“The goal of science
is to understand nature
not to memorize definitions.”
(Michael Seeds & Dana Backman)

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1 Introduction

When a gas is heated above a certain temperature, or it is subject to strong electromagnetic fields, it gets ionized, and a transition towards the so-called fourth state of matter, plasma state, is observed.

Being constituted by electrical charges, ions and electrons, a plasma responds to electric and magnetic fields including those that are produced by itself.

It actually turns out the plasma is an extremely complex medium characterized by non-linear phenomena that occur over a very wide range of temporal and spatial scales.

Understanding the plasma behavior is thus extremely challenging, and before turning our attention to the applications, it is necessary to disentangle, at least partially, this complexity uncovering together some of the basic phenomena that characterize the plasma state.

Plasma (from Greek $\pi\lambda\alpha\sigma\mu\alpha$[1], "anything formed") is one of the four fundamental states of matter, the others being solid, liquid, and gas. The word *plasma* means 'a moldable substance', 'jelly'. It was given by Irving Langmuir, one of the pioneers in the study of Plasma Physics. He was studying mercury arc discharges and found that this substance, the plasma, was diffusing in the vacuum chamber as a jelly in a mold. A plasma has properties unlike those of the other states.

If we increase the temperature of a solid then we can destroy the crystalline lattice structure and we can end up with a liquid. If we furthermore increase the temperature of the liquid, then the kinetic energy of the atoms increases and the distance between atoms also increases and what we end up with is a gas.

Figure 1: Aurora borealis in the northern hemisphere [a]

[a]Extracted from https://boredyoumustbe.wordpress.com/

Solid, liquid and gas are the three states of matter that we all know. But when we increase furthermore the temperature, then the kinetic energy of the atoms of the gas becomes so high

[1]used http://www.perseus.tufts.edu/hopper/

that when they collide, the electrons can be stripped away. Matter becomes a collection of ions and electrons. This is what we call *plasma*, the fourth state of matter. Plasma is therefore ionized gas, that can be reached if the temperature of the gas is sufficiently high that the electrons can be stripped away from the atoms. The difference between a neutral gas and a plasma, is that, in a neutral gas, particles interact only during a collision. In plasma, the particles interacts with a large number of other particles.

Plasmas show a simultaneous response of many particles to an external stimulus [10]. In this sense, plasmas show collective behavior, which means that the macroscopic result to an external stimulus is the cooperative response of many plasma particles. Mutual shielding of plasma particles or wave processes are examples of collective behavior [29].

We deal most of the time in our everyday life with gases, liquids, solids. Actually there are a number of examples of plasmas in nature and in the laboratory. Stars are made of plasmas [11]. As a matter of fact, most of the visible matter in the Universe is in the plasma state[2], in the form of an electrified gas with the atoms dissociated into positive ions and negative electrons. This estimate may not be very accurate, but it is certainly a reasonable one in view of the fact that stellar interiors and atmospheres, gaseous nebulae, and much of the interstellar hydrogen are plasmas [8].

There are also geophysical plasmas like the solar wind, the magnetosphere and the ionosphere as some examples [4]. But there are also examples of plasmas in nature on our planet. For example, the aurora Borealis as seen in figure (1) is a beautiful evidence of plasmas that can be found on our planet. And there are also man-made plasmas, neon lamps like the one in figure (2).

Figure 2: Neon lamps uses plasma to function are an example of plasmas [a]

[a]Extracted from http://www.keywordpicture.com/

Also in a number of laboratories worldwide matter is heated at very high temperature in order to reproduce the nuclear fusion reaction that occurs in a star. One of these devices that is used to heat up matter at such a high temperature, is the TCV tokamak in Lausanne, Switzerland, illustrated in figure (3).

The word 'tokamak' is derived from the Russian words, *toroidal'naya kamera s magnitnymi*

[2]see http://www.plasma-universe.com

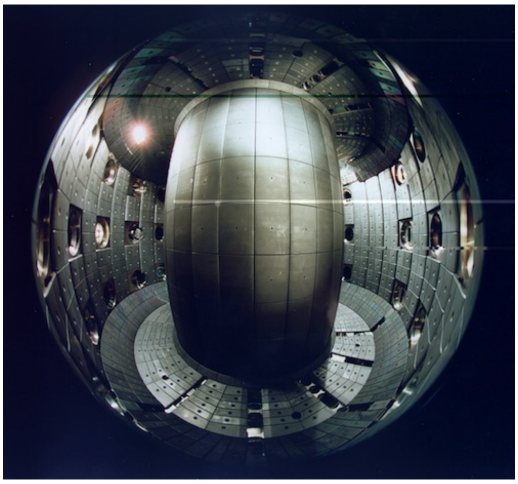

Figure 3: Variable Configuration Tokamak in Switzerland [a]

[a] Extracted from http://www.fusenet.eu/node/159

katushkami, meaning 'toroidal chamber with magnetic coils'. The device was invented in the Soviet Union, the early development took place in the late 1950s. The advantage of the tokamak comes from the increased stability provided by its larger toroidal magnetic field. The successful development of the tokamak was principally the result of the careful attention paid to the reduction of impurities and the separation of the plasma from the vacuum vessel by means of a limiter [43].

The idea is using on our planet the fusion reaction that occurs in a star to produce energy that is clean, sustainable, without emission of greenhouse gases, nor of long-lived radioactive waste [28].

We have mentioned stars, we have mentioned neon lamps. Clearly these are very different systems, characterized by very different temperatures, densities. In reality the plasma state spans a huge range of temperature and densities [35]. If we look at what is the state of matter as a function of density and temperature, density which we measure in particles per cubic meter, and temperature, which we measure in Kelvin, we observe that the solid, liquid and gaseous state occupy a very small portion of this huge parameter space (temperature and volume). All the rest is occupied by plasmas. At very high density and high temperature one can find that the plasma state is present in the solar core [11]. At slightly smaller density but a higher temperature one can find that plasmas that are found in machines that are used for obtaining the nuclear fusion reaction on our planet. And solar corona is at slightly smaller density and temperature [21]. Neon lamps are at similar density but lower temperature. And the temperature comparable to the solar corona but at lower density, one has the plasmas that are found in nebulas or in the solar wind [22]. And at even lower temperature, one can have plasmas that are found in the interstellar medium [17].

The earliest studies date to the 1920s, 1930s. A few scientists, among which Irving Langmuir, started to work on ionospheric plasmas because these were affecting radio transmission, and at

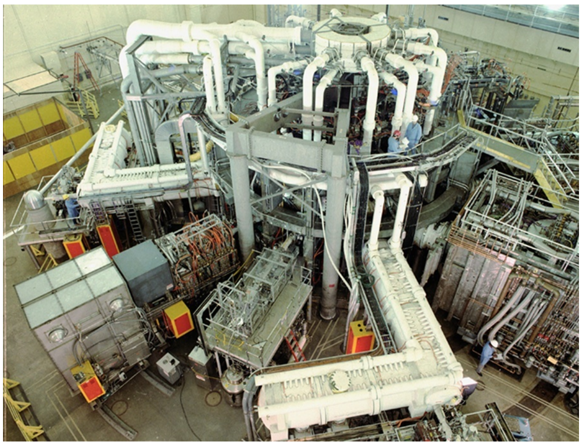

Figure 4: TFTR tokamak [a]

[a]Extracted from http://www.huffingtonpost.com/

the same time, on the gaseous electron tubes, these were devices that were used, for example, to rectify current in the era before semiconductors [37].

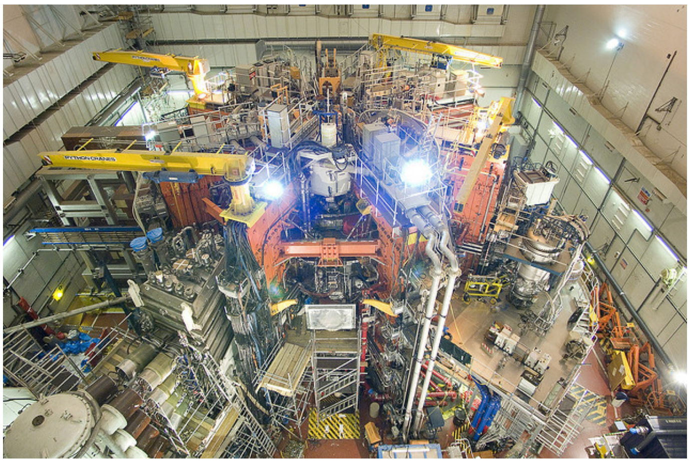

Figure 5: JET tokamak [a]

[a]Extracted from https://www.flickr.com/photos/fusionenergyvisual/

But the largest development of plasma physics came in the 1950s where few countries, the United States, the United Kingdom, Soviet Union, started to work on magnetic fusion research. Basically, on reproducing the reaction that occurs in the stars to produce energy. This was initially an offshoot of the nuclear weapons program. However the progress in this research was much slower than initially thought. And research, which up to that moment had been classified and carried out independently by these three countries, was unclassified during the Second UN Conference on Peaceful Uses of Atomic Energy that took place in Geneva in 1958 [7].

In general, the development of devices which could reproduce on our planet the fusion reaction that occurs in the stars has been the strongest drive to the study of plasma physics [44]. The research avenue that has been followed, is one of the construction of the tokamak device. Starting from the earliest device, a number of tokamak have been built in the world. For

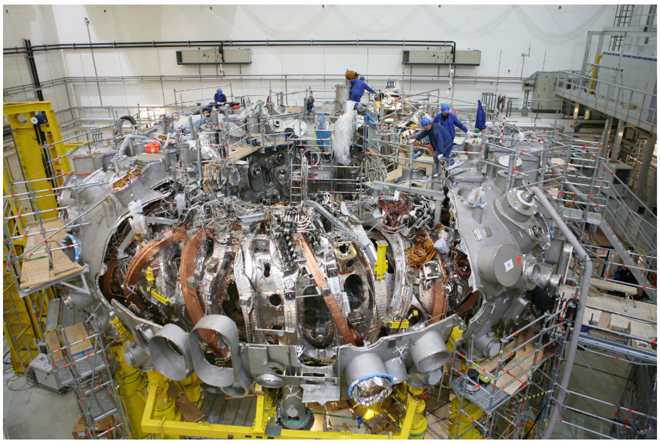

Figure 6: W7-X stellarator [a]

[a]Extracted from http://www.scilogs.de/formbar/wendelstein-w7-x-erstes-plasma/

example, the TFTR tokamak that was built in the United States figure (4), or the largest existing tokamak in the world, the JET tokamak, which is in the United Kingdom figure (5).

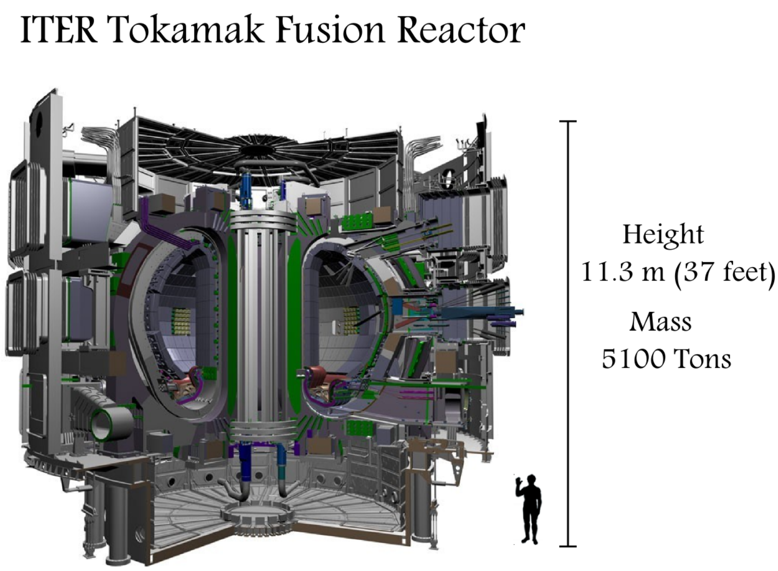

Figure 7: ITER tokamak dimensions [a]

[a]Extracted from http://www.universetoday.com/

Along with the mainstream tokamak approach, a number of other approaches have been followed in order to make energy from fusion a reality. Two of those are the W7-X Stellarator in Germany figure (6) and the National Ignition Facility in the United States. And nowadays, the ITER device figure (7). The ITER device is a tokamak, the largest experiment that is being built on our planet with the goal of showing that it is feasible to use the fusion reaction to produce energy on our planet [3].

And together with the development of fusion devices the study of space and astrophysical systems has been a strong drive for plasma physics[14]. Starting from the 60s, satellites have been launched and have provided us with a huge amount of data on the activity of the solar surface[30], of the solar wind [21] or the dynamics of the earth and magnetic fields[36]. And these have become key areas for plasma physicists. At the same time, plasma physicists have devoted their attention to the study of astrophysical systems [15]. Just to give an example, the astrophysical jets [2], the jets that are shot out from the stars, from active galactic nuclei or from black holes.

Since the 80s, industry has made massive use of technologies based on plasmas, where, in our everyday lives, plasma technology enters. Starting from the plasma TVs that we all know. The semiconductor industry makes a huge use of plasma physics. And also the plasma deposition technology is used on many of the materials that we encounter in our everyday life. Plasma medicine is a growing field of plasma physics [13].

The main drive of plasma physical research has been provided by fusion [12]. Fusion research [28], the quest for a source of energy which is clean, sustainable, with no production of greenhouse gases nor of long-lived radioactive waste [23]. But understanding plasma physics is essential to uncover the dynamics of astrophysical systems. And plasmas are playing a crucial role in industry today [31].

Studying plasma physics is extremely challenging [6]. From a theoretical point of view, dealing with plasma means dealing with electrically charged particles which move under the effect of the electromagnetic field that they have themselves produced. It means dealing with a complex, non-linear medium [42], and therefore, the most sophisticated numerical techniques, the most advanced computers that exist today in the world, advanced analytical techniques [14], they have all been used to uncover the dynamics of plasma. From an experimental point of view, taking measurement of a plasma is very difficult [18]. Its temperature is too hot. And therefore to infer what's going on inside a plasma, one has to use smarter ways, which typically require a multidisciplinary approach.

Plasma related definitions.

Beam injection.

Neutral atoms injected into a plasma travel in straight lines, being unaffected by the magnetic field. The atoms become ionized through collisions with the plasma particles and the resulting ions and electrons are then held by the magnetic field. Since the ions and electrons have the same velocity the energy is carried almost entirely by the more massive ions. Once ionized the ions have orbits in the magnetic field determined by their energy, angle of injection and point of deposition. The energy of the injected ions is gradually transferred to the plasma electrons and ions through Coulomb collisions. Thus the injected ions are initially slowed and then thermalized [43].

Beam heating.

Once the neutral beam particles entering the plasma have become ionized, the resulting fast ions are slowed down by Coulomb collisions. As the slowing down occurs energy is passed to the particles of the plasma, causing heating of both electrons and ions. At high injection velocity the electron heating is initially dominant. Then, as the beam ions slow down, the heating is transferred to the ions [43].

Magnetohydrodynamics or MHD.

The word *Magnetohydrodynamics* covers those phenomena, where, in an electrically conducting fluid, the velocity field $\boldsymbol{v}$ and the magnetic field $\boldsymbol{B}$ are coupled. In objects subject to solid body motion such as rotors, the velocity field is reduced to a change of the frame of reference, and its relation with the magnetic field is considerably simplified; such phenomena belong to electromagnetism. Similarly, the movement of an insulating fluid, which is insensitive to the presence of a magnetic field, belongs to fluid mechanics. However, as soon as these two vector fields $\boldsymbol{v}$ and $\boldsymbol{B}$ are dependent on each other, their description goes beyond these two independent disciplines, and calls for a more general formalism [24].

The mutual interaction of the magnetic field, $\boldsymbol{B}$, and the velocity field, $\boldsymbol{v}$, arises partially as a result of the laws of Faraday and Ampere, and partially because of the Lorentz force experienced by a current-carrying body [9]. It is convenient, to split this interaction into three parts:

First, the relative movement of a conducting fluid and a magnetic field causes an e.m.f. (of order $|\boldsymbol{v} \times \boldsymbol{B}|$) to develop in accordance with Faraday's law of induction. In general, electrical currents will ensue, the current density being of order $\sigma(\boldsymbol{v} \times \boldsymbol{B})$, σ being the electrical conductivity.

Second, these induced currents must, according to Ampere's law, give rise to a second, induced magnetic field. This adds to the original magnetic field and the change is usually such that the fluid appears to 'drag' the magnetic field lines along with it.

And third, the combined magnetic field (imposed plus induced) interacts with the induced current density, $\boldsymbol{J}$, to give rise to a Lorentz force (per unit volume), $\boldsymbol{J} \times \boldsymbol{B}$. This acts on the conductor and is generally directed so as to inhibit the relative movement of the magnetic field and the fluid.

The last two effects have similar consequences. In both cases the relative movement of fluid and field tends to be reduced. Fluids can 'drag' magnetic field lines (second effect) and magnetic fields can pull on conducting fluids (third effect). It is this partial 'freezing together' of the medium and the magnetic field which is the hallmark of MHD [9].

We study plasma rotations because the azimuthal rotation [34] in Tokamaks and other fusion machines is observed when confined plasma is subjected, for example, to neutral beam heating. The impacts of the beam particles with plasma electrons and ions amounts to a net momentum transfer with causes rotation in the toroidal direction.

A key problem in the theoretical study of azimuthal rotation [38] is whether such a plasma flow could coexist with a state of MHD (stationary) equilibrium. The answer turns to be positive provided some requirements are fulfilled by the system. If axisymmetry exists, field lines lie on magnetic flux surfaces with topology of tori and characterized by surface quantities, like the transversal magnetic flux. The set of ideal MHD equations allow us to derive a partial differential equation for it. Maschke and Perrin [20] obtained a MHD equilibrium equation for azimuthal plasma flows supposing that either the temperature or the entropy were surface quantities. They considered only cylindrical coordinates, having obtained exact analytical solutions for the transversal magnetic flux. Viana, Clemente and Lopes found a semi-analytical solution for spherical coordinates considering the temperature as a surface quantities [41].

The case where the plasma flow is adiabatic, demands the use of the entropy as a surface quantity. This is particularly important in the case of anisotropic plasmas. To apply the MHD equilibrium theory for realistic magnetic confinement schemes, is necessary an equilibrium equation in a general curvilinear coordinate system. An equilibrium equation for plasmas with azimuthal rotation was derived in [40], with the temperature as a surface quantity.

The objective of this work is to obtain MHD expressions in equilibria when the plasma flow is adiabatic and is rotating in the azimuthal, considering the confinement scheme as having a spherical symmetry.

The structure of this document is as follows: first, is presented the equations that describes the MHD status treating the plasma as fluid; second, are the equation for the MHD equilibrium status of the plasma in the ideal case with the description of the quantities used in the derivation for a system with axial symmetry. In the third chapter, are the equations for the MHD equilibrium of plasmas in rotation and also considering the entropy as a surface quantity, and specifying the use of a geometry like in spherical coordinates, also is presented the solution of the equation obtained with the corresponding analysis of the solution and its consequences. Finally there are the conclusions of this work, the appendix with useful information about generalized coordinates and curvilinear systems and the bibliography.

2 MHD Equations

The Magnetohydrodynamics combines the equations of fluid mechanics, the Maxwell equations for electromagnetism and some thermodynamical relations. The equations can be derived by two approaches, the microscopic approach and the macroscopic approach, the following is the latter one.

2.1 Equations for a fluid

In a plasma the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are not prescribed but are determined by the positions and motions of the charges themselves. One must find a set of particle trajectories and field patterns such that the particles will generate the fields as they move along their orbits and the fields will cause the particles to move in those exact orbits. If each of these particles follows a complicated trajectory predicting the plasma's behavior would be difficult. Fortunately, as much as 80% of plasma phenomena observed in real experiments can be explained by a model that is used in fluid mechanics, in which the identity of the individual particle is neglected, and only the motion of fluid elements is taken into account. In the case of plasmas, the fluid contains electrical charges [1]. In an ordinary fluid, frequent collisions between particles keep the particles in a fluid element moving together. That model works for plasmas [24], which generally have infrequent collisions [8].

2.1.1 Theory for a fluid

The plasma is considered like a mixture of two fluids, one electronic and other ionic. Using the subindices i for positive ions with charge $+Ze$ (being Z the atomic number), e for free electrons (with charge $-e$) and s for any of those species, we define the following quantities that depend on the position $\boldsymbol{\xi}$ and the time t by using n for the number densities, m for the masses and v for the velocities.

Density of mass ϱ:

$$\varrho(\boldsymbol{\xi}, t) = \sum_s m_s n_s = n_i m_i + n_e m_e, \tag{2.1}$$

where m_s is the mass of the species and n_s the number densities of that species.

Mean velocity of the fluid v:

$$v(\boldsymbol{\xi}, t) = \frac{1}{\varrho} \sum_s n_s m_s \boldsymbol{v}_s = \frac{n_i m_i \boldsymbol{v}_i + n_e m_e \boldsymbol{v}_e}{n_i m_i + n_e m_e}, \tag{2.2}$$

where v_s is the velocity of the species.

Electric charge density ρ_c:

$$\rho_c(\boldsymbol{\xi}, t) = \sum_s n_s q_s = e(n_i - n_e). \tag{2.3}$$

Electric current density J:

$$J(\boldsymbol{\xi}, t) = \sum_s n_s q_s \boldsymbol{v}_s = e(n_i \boldsymbol{v}_i - n_e \boldsymbol{v}_e), \tag{2.4}$$

where q_s is the electric charge of the species and e is the charge of an electron.

2.1.2 Mass conservation and the equations of Continuity

Continuity of mass.

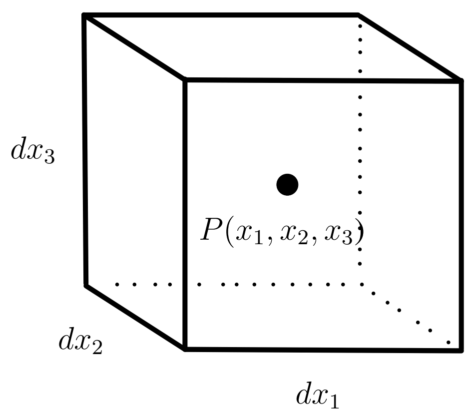

Figure 8: Eulerian volume element. Its boundaries are fixed in space.

The first thing is to start by looking at the mass flowing into and out of a physically infinitesimal volume element. From this, we will use the Eulerian viewpoint: a volume element is fixed in space in the laboratory frame of reference [32]. The volume element $dV = dx_1 dx_2 dx_3$ is shown in figure (8)

$P(x_1, x_2, x_3)$ is the centroid of the volume element. The sides of the volume element are fixed in space. Fluid can flow into and out of the volume element through the sides.

Let the mass density at $P(x_1, x_2, x_3)$ be $\varrho(x_1, x_2, x_3)$ (mass/volume). It is the average (and nearly uniform) mass density throughout dV. The total mass contained within dV is

$$M = \int \varrho dV = \int \varrho dx_1 dx_2 dx_3. \tag{2.5}$$

Assume that there are no sources or sinks of mass within dV. Then dM/dt is the rate at which mass enters or leaves through the surface dS.

A surface element $d\boldsymbol{S}$ is shown in figure (9), the corresponding surface area is dS, and $\hat{\boldsymbol{n}}$ is a unit vector normal to the surface. When $d\boldsymbol{S}$ is a side of a volume element dV, $\hat{\boldsymbol{n}}$ is assumed to

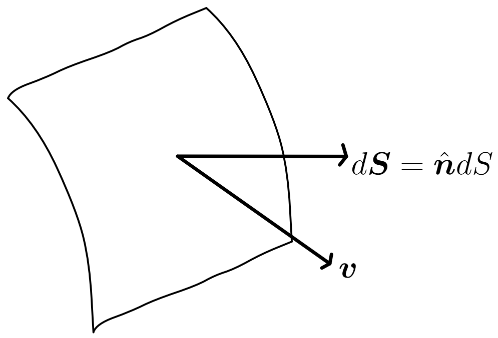

Figure 9: Eulerian surface element, showing the velocity vector $\boldsymbol{v}$ and the surface area element vector $d\boldsymbol{S}$

point out of the volume element (i.e., from inside to outside). The flux of mass passing through a surface is $\varrho\boldsymbol{v}$, where $\boldsymbol{v}$ is the fluid velocity. Then the mass per unit time flowing through $d\boldsymbol{S}$ is $\varrho\boldsymbol{v} \cdot d\boldsymbol{S} = \varrho\boldsymbol{v} \cdot \hat{\boldsymbol{n}} dS$, and the total rate of flow of mass out of the volume dV is

$$\sum_{faces} \varrho\boldsymbol{v} \cdot d\boldsymbol{S} \Rightarrow \oint_S \varrho\boldsymbol{v} \cdot d\boldsymbol{S} = \oint_S \varrho\boldsymbol{v} \cdot \hat{\boldsymbol{n}} dS, \tag{2.6}$$

where the integral is over the surface enclosing dV. Since this must be equal o $-dM/dt$, we have

$$\frac{dM}{dt} = \frac{d}{dt} \int_V \varrho dV = -\oint_S \varrho\boldsymbol{v} \cdot \hat{\boldsymbol{n}} dS. \tag{2.7}$$

For a fixed (Eulerian) surface, we can take the total time derivative inside the volume integral as a partial derivative:

$$\int_V \frac{\partial \varrho}{\partial t} dV = -\oint_S \varrho\boldsymbol{v} \cdot \hat{\boldsymbol{n}} dS, \tag{2.8}$$

applying the Gauss's theorem in the right hand side to transform the surface integral into a volume integral, gives

$$\oint_S \varrho\boldsymbol{v} \cdot \hat{\boldsymbol{n}} dS = \int_V \nabla \cdot (\varrho\boldsymbol{v}) dV, \tag{2.9}$$

rearranging terms and grouping integrals in dV gives

$$\int_V \left[\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho\boldsymbol{v}) \right] dV = 0. \tag{2.10}$$

This expression must hold for every arbitrarily shaped volume; the only way that it can be satisfied is if the integrand (the part between brackets) vanishes identically, or

$$\frac{\partial \varrho}{\partial t} = -\nabla \cdot (\varrho\boldsymbol{v}). \tag{2.11}$$

This is called the *continuity of mass equation*. It expresses *conservation of mass* in the

Eulerian frame of reference.

It must be noticed that the Eq. 2.11 has four dependent variables: ϱ and the three components of the velocity $\boldsymbol{v}$. If the velocity was known a priori, the system would be closed and we could solve Eq. 2.11 for the evolution of ϱ. Problems in which the velocity field is fixed, or specified in advance, are called *kinematic*. Problems where $\boldsymbol{v}$ is determined from other physical principles are called *dynamic*. We therefore have three more unknowns than we have equations; the problem is not closed. This problem of closure is of fundamental importance in MHD, and it will be discussed later.

Continuity of charge.

The instantaneous current i is defined as the time rate of change of charge transfer, in other words

$$i(t) = \frac{dq(t)}{dt}. \tag{2.12}$$

If charge with density $\rho_c(\boldsymbol{\xi}, t)$ is moving with velocity $\boldsymbol{v}(\boldsymbol{\xi}, t)$, the corresponding convective current density $\boldsymbol{J}$ is

$$\boldsymbol{J}(\boldsymbol{\xi}, t) = \rho_c(\boldsymbol{\xi}, t)\boldsymbol{v}(\boldsymbol{\xi}, t). \tag{2.13}$$

Noticing that if ρ_c is positive, then $\boldsymbol{J}$ is in the same direction as $\boldsymbol{v}$, but if ρ_c is negative, then $\boldsymbol{J}$ is in the opposite direction to $\boldsymbol{v}$.

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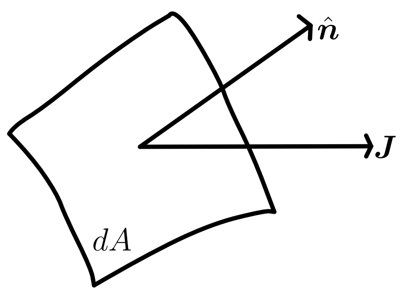


Figure 10: Surface element with unit normal vector $\hat{\boldsymbol{n}}$

Let dA be an infinitesimal element of a regular surface with unit normal vector $\hat{\boldsymbol{n}}$, as in figure (10).

If the current density $\boldsymbol{J}$ is evaluated at an interior point of dA, then the quantity $\boldsymbol{J} \cdot \hat{\boldsymbol{n}}dA$ is the rate at which charge flows across dA either into the region of space that $\hat{\boldsymbol{n}}$ is directed if ρ_c is positive, or into the region of space that $-\hat{\boldsymbol{n}}$ is directed if ρ_c is negative.

The net flow of charge per unit area per unit time, $\mathcal{F}_c$, across an arbitrarily oriented surface element $\hat{\boldsymbol{n}}dA$ is then given by

$$\mathcal{F}_c(\boldsymbol{\xi}, t) = \boldsymbol{J}(\boldsymbol{\xi}, t) \cdot \hat{\boldsymbol{n}}dA. \tag{2.14}$$

The total current I flowing across a regular surface S into the region of space that the unit surface normal $\hat{\boldsymbol{n}}$ to S is directed is then given by the surface integral

$$I(t) = \int\int_S \boldsymbol{J}(\boldsymbol{\xi}, t) \cdot \hat{\boldsymbol{n}} dA. \tag{2.15}$$

For a closed surface S, conservation of charge requires that

$$\oint_S \boldsymbol{J}(\boldsymbol{\xi}, t) \cdot \hat{\boldsymbol{n}} dA = -\frac{d}{dt} \int\int_V\int \rho_c(\boldsymbol{\xi}, t) d^3r, \tag{2.16}$$

where V is the volume enclosed by S.

If the surface S is fixed in space, Leibniz rule gives

$$\oint_S \boldsymbol{J}(\boldsymbol{\xi}, t) \cdot \hat{\boldsymbol{n}} dA = -\int\int_V\int \frac{\partial \rho_c(\boldsymbol{\xi}, t)}{\partial t} d^3r. \tag{2.17}$$

Application of the divergence theorem to this result then gives

$$\int\int_V\int \left(\nabla \cdot \boldsymbol{J}(\boldsymbol{\xi}, t) + \frac{\partial \rho_c(\boldsymbol{\xi}, t)}{\partial t} \right) d^3r. \tag{2.18}$$

Because this integral must vanish for an arbitrary region V, it is then necessary that the integrand itself be identically zero at all points of space, resulting in the equation of charge continuity

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho_c}{\partial t} = 0. \tag{2.19}$$

Not forgetting that the density current $\boldsymbol{J}$ and the charge density ρ_c both depend on the position vector and time $(\boldsymbol{\xi}, t)$.

2.1.3 Simplifying assumptions

The passage of the two fluids theory to the theory of one fluid is not entirely trivial, because it involves the use of a number of simplifying assumptions about the involved species (electrons, positive ions) as well as the characteristic scales of length and time involved.

Debye shielding.

The Debye length is an important physical parameter for the description of a plasma. It provides a measure of the distance over which the influence of the electric field of an individual charged particle (or of a surface at some nonzero potential) is felt by the other charged particles inside the plasma. The charged particles arrange themselves in such a way as to effectively shield any electrostatic fields within a distance of the order of the Debye length. This shielding of electrostatic fields is a consequence of the collective effects of the plasma particles [5].

The Debye length (λ_D) is directly proportional to the square root of the temperature (T) and inversely proportional to the square root of the electron number density (n_e) according to

$$\lambda_D = \left(\frac{\varepsilon_0 k_B T}{n_e e^2}\right)^{1/2}, \tag{2.20}$$

where $k_B = 1.38064852 \times 10^{-23} \frac{m^2 Kg}{s^2 K}$ is the Boltzmann's constant.

The Debye length can also be regarded as a measure of the distance over which fluctuating electric potentials may appear in a plasma, corresponding to a conversion of the thermal particle kinetic energy into electrostatic potential energy.

When a boundary surface is introduced in a plasma, the perturbation produced extends only up to a distance of the order of λ_D from the surface. In the neighborhood of any surface inside the plasma there is a layer of width of the order of λ_D, known as the plasma sheath, inside which the condition of macroscopic electrical neutrality may not be satisfied. Beyond the plasma sheath region there is the plasma region, where macroscopic neutrality is maintained.

Using 2.20 we can calculate the number N_D of particles in a "Debye sphere"

$$N_D = n\frac{4}{3}\pi\lambda_D^3 = 1.38 \times 10^6 T^{3/2}/n^{1/2}, \tag{2.21}$$

with the temperature T measured in Kelvin.

The Debye shielding effect is a characteristic of all plasmas, although it does not occur in every medium that contains charged particles. A necessary and obvious requirement for the existence of a plasma is that the physical dimensions of the system be large compared to λ_D. Otherwise there is just not sufficient space for the collective shielding effect to take place, and the collection of charged particles will not exhibit plasma behavior. If $\mathit{\Lambda}$ is a characteristic dimension of the plasma, a first criterion for the definition of a plasma is therefore

$$\mathit{\Lambda} \gg \lambda_D. \tag{2.22}$$

Quasi neutrality.

The requirement of Eq. 2.22 already implies in macroscopic charge neutrality if it is realized that deviations from neutrality can naturally occur only over distances of the order of λ_D. Sometimes is considered as a criterion for the existence of a plasma, although it is not an independent one, and can be expressed as

$$n_e = \sum_i n_i. \tag{2.23}$$

A charge in a plasma being shielded is envolved by a ”cloud” of charges with a characteristic length equal to the Debye length λ_D for a given species. For greater distances than Debye length we have, ideally, the neutrality of charges, in other words, the charge densities of electrons and ions are the equal, that is $q_e n_e = q_i n_i$, or $n_e = n_i$. For a plasma the Debye length is much smaller than the characteristic length of the system, so that this property is valid for all the system. However, the neutrality is not total at all, because the involved charges are in movement, so that is better to say that exists a quasi-neutrality, so

$$n_e \approx n_i. \tag{2.24}$$

A first consequence of the quasi neutrality of the plasma is that the electric charge density (Eq. 2.3) is approximately zero, or

$$\rho_c \approx 0, \tag{2.25}$$

in such a way that the equation of charge continuity (Eq. 2.19) becomes

$$\nabla \cdot \boldsymbol{J} = 0. \tag{2.26}$$

Then the electric current density (Eq. 2.4) can be expressed, in an approximate way, as

$$\boldsymbol{J} \approx en_i(\boldsymbol{v}_i - \boldsymbol{v}_e) \approx -en_e(\boldsymbol{v}_e - \boldsymbol{v}_i). \tag{2.27}$$

Mass ratio.

The mass ratio between electrons and positive ions is very small, if working with hydrogen, for example, we have

$$\frac{m_i}{m_e} = \frac{m_p}{m_e} \approx 1840 \ll 1. \tag{2.28}$$

Hence, we can neglect the ratio m_i/m_e when we are considering quantities from the fluid theory. As an example, the mass density can be written as

$$\varrho = n_i m_i \left(1 + \frac{n_e}{n_i}\frac{m_e}{m_i}\right) \approx n_i m_i \left(1 + \frac{m_e}{m_i}\right), \tag{2.29}$$

where we have used the quasi neutrality. Eventually, the ratio m_e/m_i is negligible compared with other involved terms, so that $\varrho \approx n_i m_i$ is acceptable.

2.1.4 Momentum equation - Euler fluid

The Euler equations describe how the velocity $\boldsymbol{v}$, pressure p and density ϱ of a moving fluid are related. The equations are a set of coupled differential equations and they can be solved for a given flow problem by using methods from calculus. Though the equations appear to be very complex, they are actually simplifications of the more general Navier-Stokes equations of fluid dynamics. The Euler equations neglect the effects of the viscosity of the fluid which are included in the Navier-Stokes equations. A solution of the Euler equations is therefore only an approximation to a real fluids problem.

Our world has three spatial dimensions (x, y, z), for convenience represented as (x_1, x_2, x_3)

and one time dimension t. In general, the Euler equations have a time-dependent continuity equation for conservation of mass and three time-dependent conservation of momentum equations.

We can express the total momentum $\mathit{\Gamma}$ in the x_i direction in the volume element dV as

$$\mathit{\Gamma}_i = \int \varrho v_i dV, \tag{2.30}$$

being $i = 1, 2, 3$.

The momentum rate of change is then

$$\frac{\partial \mathit{\Gamma}_i}{\partial t} = \int \frac{\partial(\varrho v_i)}{\partial t} dV. \tag{2.31}$$

We have assumed that there are no external forces being applied, like gravity, and no pressure (a fluid of non-interacting dust particles). We also assume viscous forces are negligible. Then, the only way a momentum-change can occur is by momentum flowing across the boundary surface S:

$$\frac{\partial \mathit{\Gamma}_i}{\partial t} = \int (\varrho v_i)\boldsymbol{v} \cdot d\boldsymbol{S} = \int (\varrho v_i) v_j dS_j. \tag{2.32}$$

Expressing the dot products using the Einstein summation convention: implied summation over repeated dummy indices.

Changing the surface integral into a volume integral using Green's Theorem, gives

$$\frac{\partial \mathit{\Gamma}_i}{\partial t} = -\int \nabla_j(\varrho v_i v_j) dV. \tag{2.33}$$

Comparing Eqs. 2.31 with 2.33 is clearly that

$$\frac{\partial \mathit{\Gamma}_i}{\partial t} = \int \frac{\partial(\varrho v_i)}{\partial t} dV = -\int \nabla_j(\varrho v_i v_j) dV, \tag{2.34}$$

or more importantly the integrands must be equal

$$\frac{\partial(\varrho v_i)}{\partial t} = -\nabla_j(\varrho v_i v_j). \tag{2.35}$$

This equations says that each component of the momentum-density ϱv_i (for each i separately) obeys a local conservation law.

The operator ∇_j is differentiating two velocities (v_i and v_j) only one of which undergoes dot-product summation over j.

Next, we consider the pressure of the mixture, it contributes a force on the particles in the control volume, specifically

$$f_i = \int p dS_i \tag{2.36}$$

and again using Green's theorem to transform it into a volume integral

$$f_i = -\int \nabla_i p dV. \tag{2.37}$$

Now we consider a uniform gravitational field, and it contributes another force equals to

$$f_i = \int \varrho g_i dV. \tag{2.38}$$

These forces contribute to changing the momentum, by the second law of motion:

$$\frac{d\Gamma_i'}{dt} = f_i. \tag{2.39}$$

The notation used (d/dt instead of $\partial/\partial t$ and Γ' instead of Γ) is interesting because reminds that the three laws of motion apply to particles, not to the control volume itself. The rate of change of P, the momentum in the control volume, contains the newtonian contributions (Eqs. 2.37 and 2.38) plus the flow contributions (Eq. 2.35).

Combining the contributions, the equation of motion for the fluid is

$$\frac{\partial(\varrho v_i)}{\partial t} + \nabla_j(\varrho v_i v_j) = -\nabla_i p + f_i, \tag{2.40}$$

where f_i is an external force per volume unit like gravity or any other force.

Expressing Eq. 2.40 in vectorial form, gives

$$\varrho\frac{\partial \boldsymbol{v}}{\partial t} + \varrho\nabla(\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \boldsymbol{f_{ext}}, \tag{2.41}$$

$\otimes$ is the outer product and $\boldsymbol{f_{ext}}$ is an external force applied on the system.

Expanding the left hand side of the Eq. 2.40 gives

$$\varrho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \varrho}{\partial t} + v_i\nabla_j(\varrho v_j) + \varrho v_j\nabla_j(v_i) = -\nabla_i p + f_i, \tag{2.42}$$

the second and third terms cancel because of conservation of mass (Eq. 2.11) leaving only

$$\varrho\frac{\partial v_i}{\partial t} + \varrho v_j\nabla_j(v_i) = -\nabla_i p + f_i, \tag{2.43}$$

or in vectorial form

$$\varrho\frac{\partial \boldsymbol{v}}{\partial t} + \varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \boldsymbol{f_{ext}}. \tag{2.44}$$

In fluid dynamics the relation between total and partial derivatives is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla,$$

also known as convective derivative, the left hand side expresses the rate of change of a quantity at a point moving with the fluid and the partial derivative at the right hand side expresses the rate of change of a quantity at a fixed point in space. So, in a more compact representation the equation becomes

$$\varrho \frac{d\boldsymbol{v}}{dt} = -\nabla p + \boldsymbol{f_{ext}}. \tag{2.45}$$

In our case the external force can be derived from the force on a charged particle q (can be electrons or ions) with velocity $\boldsymbol{v}$ under electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ respectively, or known as the Lorentz force $\boldsymbol{F}$ that is given by

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{2.46}$$

the total electromagnetic force per unit volume on electrons with charge $-e$ is

$$\boldsymbol{f_e} = -n_e e(\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{B}), \tag{2.47}$$

and for ions with charge e is

$$\boldsymbol{f_i} = n_i e(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B}), \tag{2.48}$$

where n_e and n_i are the electron and ion number densities respectively.

The total electromagnetic force per unit volume is the sum of Eq. 2.47 and 2.48

$$\boldsymbol{f} = \boldsymbol{f_e} + \boldsymbol{f_i} = e(n_i - n_e)\boldsymbol{E} + (en_i\boldsymbol{v_i} - en_e\boldsymbol{v_e}) \times \boldsymbol{B}, \tag{2.49}$$

as the plasma is considered as quasi-neutral, as explained before (i.e. $n_i \approx n_e = n$), the first term involving the electric field cancels, and this gives

$$\boldsymbol{f} = en(\boldsymbol{v_i} - \boldsymbol{v_e}) \times \boldsymbol{B}, \tag{2.50}$$

or by noting that the term $en(\boldsymbol{v_i} - \boldsymbol{v_e})$ is just the current density $\boldsymbol{J}$, we can say that

$$\boldsymbol{f} = en(\boldsymbol{v_i} - \boldsymbol{v_e}) \times \boldsymbol{B} = \boldsymbol{J} \times \boldsymbol{B}. \tag{2.51}$$

Hence by using this as an external force and putting into Eq. 2.45

$$\varrho \frac{d\boldsymbol{v}}{dt} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}. \tag{2.52}$$

Or expanding the convective derivative, the Eq. 2.52 gives

$$\varrho \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) \boldsymbol{v} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}. \tag{2.53}$$

Noting that $\boldsymbol{J} \times \boldsymbol{B}$ is the only change to fluid equations in MHD. An equation for the magnetic field and current density are needed to close the system.

2.2 Equations from thermodynamics

2.2.1 State equations

The plasma is considered as an ideal gas; then, the state equation $p = p(\varrho, T)$ for an ideal gas of density ϱ and temperature T is of the form

$$p = \varrho \bar{R} T = n k_B T, \tag{2.54}$$

where $\bar{R} = 8.31446 J/mol \cdot K$ is the Gas constant and $k_B = 1.38065 \times 10^{-23} m^2 Kg/s^2 \cdot K$ is the Boltzmann constant.

In this work we are considering an adiabatic process, so, the equation for that process in a system with volume V is

$$pV^\gamma = const., \tag{2.55}$$

and we can express it in terms of a new constant A that only depends on the entropy S. In an adiabatic process the entropy remains constant, so that constant $A(S)$ is also a constant, in such a way that we can write Eq. 2.55 in terms of the density of mass as

$$p = A(S)\varrho^\gamma, \tag{2.56}$$

using this equation to get $A(S)$ as $A(S) = p/\varrho^\gamma$ Then, as said before, $A(S)$ is constant in time so we can write

$$\frac{dA(S)}{dt} = \frac{d}{dt}\left(\frac{p}{\varrho^\gamma}\right) = 0. \tag{2.57}$$

But, using the definition of the convective derivative from Eq. 2.1.4 gives

$$\frac{dA(S)}{dt} = \frac{\partial A}{\partial t} + \boldsymbol{v} \cdot \nabla A = 0, \tag{2.58}$$

for the case of reversible processes.

In equilibrium, the partial derivative vanishes, giving the relation for the entropy S

$$\boldsymbol{v} \cdot \nabla A(S) = 0, \tag{2.59}$$

or

$$\boldsymbol{v} \cdot \nabla S = 0, \tag{2.60}$$

because the constant A is also a function of the entropy S.

2.3 Equations from electromagnetism

2.3.1 Maxwell's equations

The electromagnetic fields are $\boldsymbol{E}$, the electric field, and $\boldsymbol{B}$, the magnetic flux density or magnetic field, $\boldsymbol{J}$ is the electrical current density. Together, these must satisfy Maxwell's equations, but any temporal derivatives nulls because of the stationary nature:

Faraday's law:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{2.61}$$

Ampère's law:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{2.62}$$

Magnetic Gauss's law:

$$\nabla \cdot \boldsymbol{B} = 0. \tag{2.63}$$

These equations are written in MKS units. This system is the used in this work. In these units, the speed of light c is

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.99792458 \times 10^8 m/s. \tag{2.64}$$

The constant ε_0 is called the permitivity of free space and the constant μ_0 is called the permeability of free space. The dynamics of the electromagnetic fields and the fluid are coupled through Ohm's law [32].

The Gauss and Faraday laws suffer no changes in the context of MHD, but the electric Gauss and Ampère laws will experiment some changes.

2.3.2 Absence of the displacement current

The temporal term $\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$ appearing in Ampère's law, is known as the displacement current density, and can be neglected compared with the electrical current density $\boldsymbol{J}$ substituting the Faraday's law into Ampère's law gives

$$\nabla \times (\nabla \times \boldsymbol{E}) = \mu_0 \frac{\partial \boldsymbol{J}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \tag{2.65}$$

which, is well known, that will lead to wave equation for the electric field.

Let a plane wave with wave vector k and a frequency ω moving in the $\boldsymbol{\xi}$ direction:

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{\xi} - \omega t)}, \tag{2.66}$$

for which the associations are valid

$$\frac{\partial}{\partial t} \to -i\omega, \tag{2.67}$$

and

$$\nabla \to i\boldsymbol{k}, \tag{2.68}$$

and using them we can estimate the magnitude of their terms

$$|\nabla \times (\nabla \times \boldsymbol{E})| \sim k^2 |\boldsymbol{E}|, \tag{2.69}$$

$$\frac{1}{c^2}\left|\frac{\partial^2 \boldsymbol{E}}{\partial t^2}\right| \sim \frac{\omega^2}{c^2}|\boldsymbol{E}|, \tag{2.70}$$

then,

$$\frac{1}{c^2}\left|\frac{\partial^2 \boldsymbol{E}}{\partial t^2}\right| \ll |\nabla \times (\nabla \times \boldsymbol{E})|,$$

if ω is small, then the displacement current can be neglected, and turns the Ampère-Maxwell as

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}. \tag{2.71}$$

2.3.3 Ohm's Law

The dynamics of the electromagnetic fields and the fluid are coupled through Ohm's law,

$$\boldsymbol{E'} = \eta \boldsymbol{J}, \tag{2.72}$$

where η is the electrical resistivity, which is to be considered a material property of the fluid, and $\boldsymbol{E'}$ is the electric field as seen by a conductor moving with velocity $\boldsymbol{v}$.

According to the theory of relativity, this is given by

$$\boldsymbol{E'} = \frac{\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2.73}$$

where $\boldsymbol{E}$ is the electric field in the stationary frame.

Maxwell's equations and Ohm's law are Lorentz invariant, i.e., they are physically accurate to all orders of $\frac{v^2}{c^2}$. However, the fluid equations are Gallilean invariant; they are physically accurate only to $O(v/c)$. The two systems of equations are incompatible as presently formulated. So, we either need to make the fluid equations relativistic or need to render Maxwell's equations' Gallilean invariant. In MHD we will consider only low frequencies, i.e., $v^2/c^2 = (\omega L/c)^2 << 1$. We therefore choose the latter course and seek a form of Maxwell's equations that is only accurate through $O(v/c)$. Consider Ohm's law Eq. 2.72. From Eq. 2.73, when $v^2/c^2 << 1$ we can write the electric field in the moving frame as

$$\begin{aligned} \boldsymbol{E'} &= (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\left(1 - \frac{1}{2}\frac{v^2}{c^2} + \dots\right) && (2.74) \\ &= \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} + O\left(\frac{v^2}{c^2}\right). && (2.75) \end{aligned}$$

Ohm's law then becomes

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \frac{\boldsymbol{J}}{\sigma}, \tag{2.76}$$

where $\sigma = 1/\eta$ is the conductivity. This equation is the proper MHD form, it is sometimes called the resistive Ohm's law[32].

By assuming a perfect conductivity (i.e. $\sigma \to \infty$) and with $\boldsymbol{J} \neq 0$ this gives

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0. \tag{2.77}$$

2.3.4 Ideal MHD equations

Putting in all together, then, the ideal MHD equations are

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{v}), \tag{2.78}$$

$$\rho \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) \boldsymbol{v} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}, \tag{2.79}$$

$$\frac{d}{dt} \left(\frac{p}{\varrho^\gamma} \right) = 0, \tag{2.80}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{2.81}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.82}$$

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}, \tag{2.83}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{2.84}$$

where p is the pressure, $\boldsymbol{J}$ the plasma current density, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{E}$ the electric field, $\boldsymbol{v}$ the velocity, ϱ the mass density and the γ is the adiabatic gas constant.

3 MHD equilibrium

3.1 Conditions for equilibria

In plasma confinement configurations, open (including adiabatic traps), closed or toroidal configurations, there are some terms that will be explained in the next paragraphs.

One condition considered to reach MHD equilibrium is that all the quantities do not depend on the time, or in other words, the time derivatives are equal to zero. The fluid is assumed to be spatially uniform with constant density ϱ. The magnetic Gauss's law $\nabla \cdot \boldsymbol{B} = 0$ is also satisfied, this is not a MHD equation but a necessary condition that must be satisfied by any magnetic field.

3.2 Ideal MHD equations under stationary equilibrium

Then, in stationary MHD equilibrium the (plasma) velocities are different from zero, i.e. $\boldsymbol{v} \neq 0$. In this ideal system the ideal MHD equations are expressed as

$$\nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{3.1}$$

$$\rho \left(\boldsymbol{v} \cdot \nabla\right) \boldsymbol{v} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}, \tag{3.2}$$

$$\frac{d}{dt}\left(\frac{p}{\varrho^{\gamma}}\right) = 0, \tag{3.3}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{3.4}$$

$$\nabla \times \boldsymbol{E} = 0, \tag{3.5}$$

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}, \tag{3.6}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{3.7}$$

3.3 Ideal MHD equations under static equilibrium

In static MHD equilibrium all the velocities are equal to zero, i.e. $\boldsymbol{v} = 0$ Also, the generalized Ohm's law $\boldsymbol{E} = \eta \boldsymbol{J}$ becomes $\boldsymbol{E} = 0$ (where η is the resistivity), this is because in an ideal MHD the resistivity is ideally zero, i.e. $\eta \to 0$. All these requirements must be met to achieve the MHD equilibrium.

In the study of magnetically confined plasmas, the most important requirement is the existence of an MHD equilibrium suitable to confinement. Under static equilibrium conditions the equilibrium velocity is considered zero, and throughout the fluid the magnetic induction $\boldsymbol{B}$ is constant [5]. This is defined by a stationary state without plasma flow (time derivatives are zero and $\boldsymbol{v} = 0$) [28].

The kinetic pressure of the plasma, which tends to make it expand, is counterbalanced by the joint action of magnetic forces. The latter is important, for example, in star balance, but it is negligible for laboratory and plasma fusion. For that reason the gravitational forces are ignored, so that the MHD equations for static *equilibrium* are determined by

$$\nabla p = \boldsymbol{J} \times \boldsymbol{B}, \tag{3.8}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{3.9}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{3.10}$$

3.4 Isobaric or magnetic surfaces

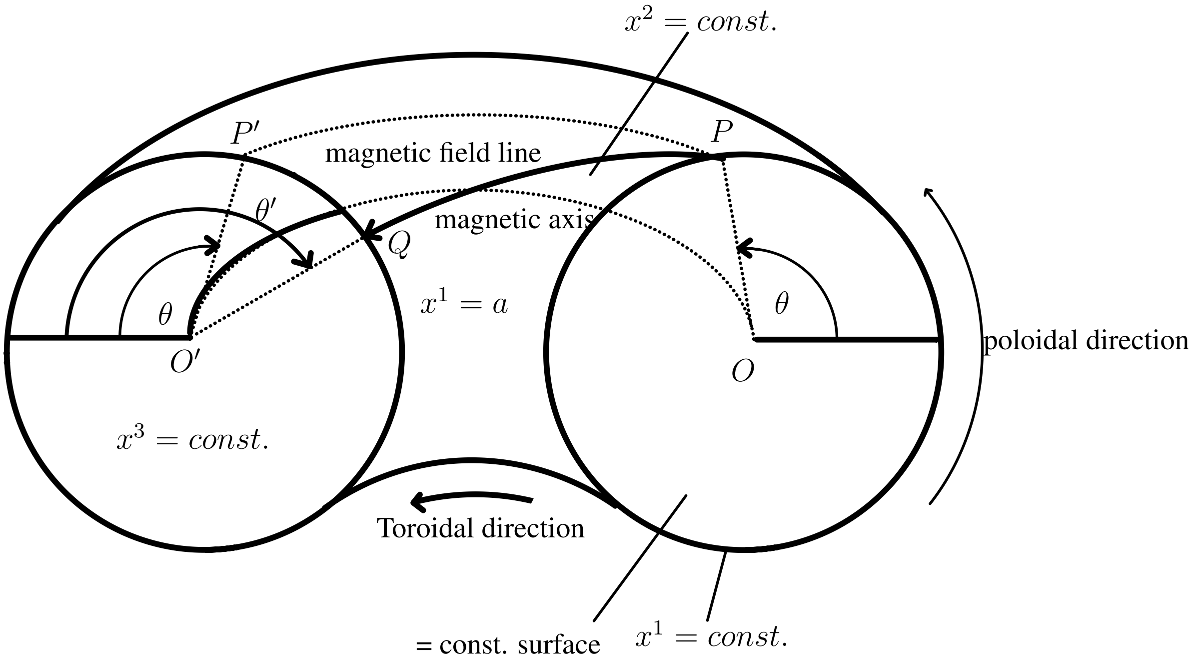


Figure 11: Sketch maps of the magnetic configuration of the tokamak. The magnetic field lines form a series of nested magnetic flux surfaces

A magnetic surface is represented as a function where the flux is constant, in toroidal configurations, the lines of force of the magnetic field lie on magnetic surfaces which form a system of nested tori, as shown in figure 11. That function that defines an equation for the magnetic surfaces, is called a magnetic surface function or, simply, a surface function.

We have from MHD condition for static equilibrium (that comes from the equation of movement 3.8) by doing the inner product of $\boldsymbol{B}$ with it

$$\boldsymbol{B} \cdot \nabla p = \boldsymbol{B} \cdot \boldsymbol{J} \times \boldsymbol{B} = 0. \tag{3.11}$$

Which means that the lines of force of the magnetic field and the lines of current lie on the surfaces of constant pressure or isobaric. Magnetic surfaces are also flux surfaces, the toroidal magnetic flux is independent of the position of the poloidal current.

Also by doing the inner product of the same MHD condition with the current $\boldsymbol{J}$

$$\boldsymbol{J} \cdot \nabla p = \boldsymbol{J} \cdot \boldsymbol{J} \times \boldsymbol{B} = 0. \tag{3.12}$$

So, the lines of current also lie on the magnetic surfaces. The magnetic surfaces only can be limited spatially if they have a nested tori topology. The existence of closed magnetic surfaces are a necessary condition but not a sufficient condition for magnetic confinement of fusion plasmas. The determination of the magnetic surfaces, in general, requires a configuration with a specific spatial symmetry, in this work that configuration is an sphere, so the symmetry involved is a spherical one.

3.4.1 Magnetic axis

The flux enclosed in a magnetic surface can be used as a label of the surfaces, another option is to use the volume inside the surface. The innermost surface has zero volume, it is called the magnetic axis. [10]

3.5 MHD equilibrium in systems with axial symmetry

We introduce the use of a curvilinear coordinate system with coordinates x^1 and x^2 are chosen in order to have the magnetic axis of the system coincident with a coordinate curve x^3 as shown in figure 11. The surface quantities over magnetic surfaces to work are the pressure, volume which are constant. These surface quantities do not depend on the third coordinate x^3, that is an ignorable coordinate, which means the system has axial symmetry. The magnetic axis, which is a degenerate surface, is a coordinate curve x^3, the first coordinate is represented as $x^1 = a$. S_2 is a segment of the coordinate surface $x^2 =$ *constant* from the magnetic axis to the coordinate x^3 [19]. The following periodicity

$$x^3 = L(x^1, x^2), \tag{3.13}$$

is assumed and $0 < x^3 < L$.

The poloidal flux function is the magnetic flux through S_2 per length unit in the x^3 direction, so

$$\Psi(x^1, x^2) = \frac{1}{L} \int\limits_{S_2} \boldsymbol{B} \cdot d\boldsymbol{S} = \frac{1}{L} \int_a^{x^1} dx'^1 \int_0^L dx^3 \sqrt{g} B^2, \tag{3.14}$$

where $g = \det(g_{ij})$ is the covariant metric tensor of the coordinate system (see Appendix), Ψ is a surface quantity that does not depend on the third coordinate x^3, then by doing the inner product of the the magnetic field and the gradient of Ψ gives

$$\boldsymbol{B}\cdot\nabla\Psi = 0, \tag{3.15}$$

using the magnetic Gauss's law where $\nabla\cdot\boldsymbol{B} = 0$

$$\frac{\partial\Psi}{\partial x^1} = \frac{1}{L}\int_a^{x^1} dx'^1 \int_0^L dx^3 \sqrt{g}B^2 = 0, \tag{3.16}$$

multiplying by dx^1 and integrating from $x^1 = a$ to any x^1 gives

$$B^1(x^1,x^2) = -\frac{1}{\sqrt{g}}\int_a^{x^1} dx'^1 \frac{\partial}{\partial x^2}(\sqrt{g}B^2), \tag{3.17}$$

multiplying by $\sqrt{g}dx^3$ ant integrating from $x^3 = 0$ to L we have

$$B^1 = -\frac{1}{\sqrt{g}}\frac{\partial\Psi}{\partial x^2}, \tag{3.18}$$

$$B^2 = \frac{1}{\sqrt{g}}\frac{\partial\Psi}{\partial x^1}. \tag{3.19}$$

Using the formulas, from the appendix, for the curvilinear coordinates (x^1, x^2, x^3) that relates their corresponding covariant and contravariant unit vectors $(\hat{\boldsymbol{e}}^1, \hat{\boldsymbol{e}}^2, \hat{\boldsymbol{e}}^3)$

$$\hat{\boldsymbol{e}}^1 \times \hat{\boldsymbol{e}}_3 = \frac{1}{\sqrt{g}}(g_{32}\hat{\boldsymbol{e}}_3 - g_{33}\hat{\boldsymbol{e}}_2), \tag{3.20}$$

$$\hat{\boldsymbol{e}}^2 \times \hat{\boldsymbol{e}}_3 = \frac{1}{\sqrt{g}}(g_{33}\hat{\boldsymbol{e}}_1 - g_{31}\hat{\boldsymbol{e}}_3), \tag{3.21}$$

$$\hat{\boldsymbol{e}}^3 \times \hat{\boldsymbol{e}}_3 = \frac{1}{\sqrt{g}}(g_{31}\hat{\boldsymbol{e}}_2 - g_{32}\hat{\boldsymbol{e}}_1), \tag{3.22}$$

gives

$$\frac{1}{g_{33}}\hat{\boldsymbol{e}}^3 \times \nabla\Psi = B^1\hat{\boldsymbol{e}}_1 + B^2\hat{\boldsymbol{e}}_2 - \left(\frac{g_{31}}{g_{33}}B^1 + \frac{g_{32}}{g_{33}}B^2\right)\hat{\boldsymbol{e}}_3, \tag{3.23}$$

multiplying by $\hat{\mathbf{e}}_3/g_{33}$ and because

$$\nabla\Psi = \frac{\partial\Psi}{\partial x^1}\hat{\boldsymbol{e}}^1 + \frac{\partial\Psi}{\partial x^2}\hat{\boldsymbol{e}}^2,$$

also,

$$\boldsymbol{B} = B^1\hat{\boldsymbol{e}}_1 + B^2\hat{\boldsymbol{e}}_2 + B^3\hat{\boldsymbol{e}}_3$$

then after writing in terms of $B_3 = g_{3j}B^j$ to lowering the indices we have

$$B_3\frac{\hat{\boldsymbol{e}}_3}{g_{33}} = \left(\frac{g_{31}}{g_{33}}B^1 + \frac{g_{32}}{g_{33}}B^2 + B^3\right)\hat{\boldsymbol{e}}_3 \tag{3.24}$$

we obtain the following representation for the magnetic field in terms of the poloidal flux function

$$\boldsymbol{B} = \frac{\hat{\boldsymbol{e}}_3}{g_{33}} \times \nabla\Psi + B_3\frac{\hat{\boldsymbol{e}}_3}{g_{33}}, \tag{3.25}$$

and with the previous expression we can verify that $\boldsymbol{B} \cdot \nabla\Psi = 0$ so that the magnetic surfaces are such that $\Psi = constant$ so they are also called flux surfaces.

3.5.1 Flux function and vector potential

Using $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and the Stoke's theorem in the definition of the poloidal flux function gives

$$\Psi(x^1, x^2) = \frac{1}{L}\int_{S_2} \nabla \times \boldsymbol{A} \cdot d\boldsymbol{S} = \frac{1}{L}\oint_{C_2} \boldsymbol{A} \cdot d\boldsymbol{l}, \tag{3.26}$$

where C_2 is a coordinate curve x^3. Then $d\boldsymbol{l} = -dx^3\hat{\boldsymbol{e}}_3$ and substituting in the previous equation gives

$$\Psi(x^1, x^2) = -\frac{1}{L}\int_0^L A_3 dx^3. \tag{3.27}$$

Because of the axial symmetry, A_3 does not depend on x^3 and we have

$$\Psi(x^1, x^2) = -A_3(x^1, x^2). \tag{3.28}$$

3.5.2 Poloidal flux current

The letter I represents the total electric current flowing through the surface S_2 by length unit in the x^3 direction, so

$$I(x^1, x^2) = I_{axis} + \frac{1}{L}\int_{S_2} \boldsymbol{J} \cdot d\boldsymbol{S} = I_{axis} + \frac{1}{L}\int_a^{x^1} dx'^1 \int_0^L dx^3\sqrt{g}J^2, \tag{3.29}$$

where I_{axis} is the current at the magnetic axis.

Proceeding in a similar way to the previous case and using the equation of charge continuity $\nabla \cdot \boldsymbol{J}$ we obtain the next relations

$$\mu_0 J^1 = -\frac{1}{\sqrt{g}}\frac{\partial I}{\partial x^2}, \tag{3.30}$$

$$\mu_0 J^2 = \frac{1}{\sqrt{g}}\frac{\partial I}{\partial x^1}, \tag{3.31}$$

where we have used the charge continuity equation $\nabla \cdot \boldsymbol{J} = 0$

3.5.3 Analogy between the flux and current functions

In the table 1 we can see some analogies between the equations for the magnetic field and the current density

magnetic field $\boldsymbol{B}$	current density $\boldsymbol{J}$
$\nabla \times \boldsymbol{A} = \boldsymbol{B}$	$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$
$\nabla \cdot \boldsymbol{B} = 0$	$\nabla \cdot \boldsymbol{J} = 0$
$B^1 = 0$ at the axis	$J^1 = 0$ at the axis
$\Psi(x^1, x^2) = \frac{1}{L}\int_{S_2} \boldsymbol{B} \cdot d\boldsymbol{S}$	$\mu_0 I(x^1, x^2) = I_{axis} + \frac{1}{L}\int_{S_2} \mu_0 \boldsymbol{J} \cdot d\boldsymbol{S}$

Table 1: Analogies between the expressions for the magnetic field and current density

So, we can conclude that Ψ is similar to $\mu_0 I$ and the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are also similar as well. With that analogy kept in mind, next, we can infer that

$$\mu_0 I(x^1, x^2) = -B_3(x^1, x^2), \tag{3.32}$$

where we have a representation for the current density in terms of the poloidal current function in a similar fashion of that used for the magnetic field

$$\boldsymbol{J}(x^1, x^2) = \frac{\hat{\boldsymbol{e}}_3}{g_{33}} \times \nabla I + J_3 \frac{\hat{\boldsymbol{e}}_3}{g_{33}}. \tag{3.33}$$

By doing $\boldsymbol{B} \cdot \nabla I$ we verify that Ψ and I are also surface quantities.

3.5.4 Grad-Shafranov equation

H. Grad [16] and VD Shafranov [33], derived an equilibrium equation in ideal magneto-hydrodynamics for a two dimensional plasma, starting from the expressions for the magnetic field

$$\boldsymbol{B} = \frac{\hat{\boldsymbol{e}}_3}{g_{33}} \times \nabla \Psi + B_3 \frac{\hat{\boldsymbol{e}}_3}{g_{33}}, \tag{3.34}$$

and for the current density

$$\boldsymbol{J}(x^1, x^2) = \frac{\hat{\boldsymbol{e}}_3}{g_{33}} \times \nabla I + J_3 \frac{\hat{\boldsymbol{e}}_3}{g_{33}}, \tag{3.35}$$

substituting these expressions in the MHD equilibrium condition $\nabla p = \boldsymbol{J} \times \boldsymbol{B}$ gives

$$\nabla p = \boldsymbol{J} \times \boldsymbol{B} = \frac{B_3}{g_{33}} \nabla I - \frac{J_3}{g_{33}} \nabla \Psi. \tag{3.36}$$

After multiplying by $\mu_0 g_{33}$ we obtain

$$\mu_0 J_3 \nabla \Psi = -\mu_0 g_{33} \nabla p - \mu_0^2 I \nabla I, \tag{3.37}$$

putting in Ampére's law $\mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B}$ gives

$$\mu_0 \boldsymbol{J} = \nabla \times \left(\frac{\hat{\boldsymbol{e}}_3}{g_{33}} \times \nabla \Psi \right) - \mu_0 \nabla \times \left(I \frac{\hat{\boldsymbol{e}}_3}{g_{33}} \right). \tag{3.38}$$

We expand it using the mathematical identity for two any vectors $\boldsymbol{A}$ and $\boldsymbol{B}$

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}, \tag{3.39}$$

we are interested in the component J_3 of $\boldsymbol{J}$ then $J_3 = \boldsymbol{J} \cdot \hat{\boldsymbol{e}}_3$.

Defining the Shafranov operator as

$$\Delta^* \Psi = g_{33} \nabla \cdot \left(\frac{\nabla \Psi}{g_{33}} \right) = \nabla^2 \Psi - \frac{\nabla \Psi}{g_{33}} \cdot \nabla g_{33}, \tag{3.40}$$

and using it in the expression obtained, gives

$$\mu_0 J_3 = \Delta^* \Psi - \mu_0 \left(\nabla \times I \frac{\hat{\boldsymbol{e}}_3}{g_{33}} \right) \cdot \hat{\boldsymbol{e}}_3. \tag{3.41}$$

It is necessary to use the vector identity for the rotational of any vector $\boldsymbol{A}$

$$\nabla \times \boldsymbol{A} = \frac{1}{\sqrt{g}} \varepsilon^{ijk} (\partial_i A_j) \hat{\boldsymbol{e}}_k, \tag{3.42}$$

then

$$\mu_0 \nabla \times I \frac{\hat{\boldsymbol{e}}_3}{g_{33}} = \frac{\mu_0}{\sqrt{g}} \varepsilon^{ijk} \partial_i \left(\frac{I g^{3j}}{g_{33}} \right) \hat{\boldsymbol{e}}_k. \tag{3.43}$$

Making the dot product with $\hat{\boldsymbol{e}}_3$

$$\mu_0 \nabla \times I \frac{\hat{\boldsymbol{e}}_3}{g_{33}} \cdot \hat{\boldsymbol{e}}_3 = \frac{\mu_0}{\sqrt{g}} \varepsilon^{ijk} \partial_i \left(I \frac{g_{j3}}{g_{33}} \right) g^{k3}, \tag{3.44}$$

expanding terms and simplifying, gives

$$\mu_0 \nabla \times I \frac{\hat{\boldsymbol{e}}_3}{g_{33}} \cdot \hat{\boldsymbol{e}}_3 = \frac{\mu_0 I g_{33}}{\sqrt{g}} \left[\partial_1 \left(\frac{g_{23}}{g_{33}} \right) - \partial_2 \left(\frac{g_{13}}{g_{33}} \right) \right]. \tag{3.45}$$

Putting in the expression obtained for $\mu_0 J_3$ it reads

$$\mu_0 J_3 = \Delta^* \Psi - \frac{\mu_0 I g_{33}}{\sqrt{g}} \left[\partial_1 \left(\frac{g_{23}}{g_{33}} \right) - \partial_2 \left(\frac{g_{13}}{g_{33}} \right) \right]. \tag{3.46}$$

After multiplying by $\nabla \Psi$ and using the equation $\mu_0 J_3 \nabla \Psi = -\mu_0 g_{33} \nabla p - \mu_0^2 I \nabla I$ we obtain the Grad-Shafranov equation

$$(\Delta^*\Psi)\nabla\Psi = -\mu_0 g_{33}\nabla p - \mu_0^2 I\nabla I + \mu_0 I\frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{23}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{13}}{g_{33}}\right)\right]\nabla\Psi. \tag{3.47}$$

Because I and p are surface quantities we can write them as

$$\nabla I = \frac{dI}{d\Psi}\nabla\Psi, \tag{3.48}$$

and

$$\nabla p = \frac{dp}{d\Psi}\nabla\Psi. \tag{3.49}$$

Dividing the Grad-Shafranov equation by $\nabla\Psi$ and rewriting it as

$$\Delta^*\Psi = -\mu_0 g_{33}p' - \frac{1}{2}\mu_0^2(I^2)' + \mu_0 I\frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{23}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_1}{g_{33}}\right)\right], \tag{3.50}$$

the prime represents the derivative with respect to Ψ. Noting that for orthogonal systems the metric tensor vanishes, i.e. $g_{ij} = 0$ for $i \neq j$, then

$$\Delta^*\Psi = -\mu_0 g_{33}p' - \frac{1}{2}\mu_0^2(I^2)', \tag{3.51}$$

where

$$\Delta^*\Psi = \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{\sqrt{g}g^{11}}{g_{33}}\frac{\partial\Psi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{\sqrt{g}g^{22}}{g_{33}}\frac{\partial\Psi}{\partial x^2}\right)\right]. \tag{3.52}$$

We need to specify the profiles for $p = p(\Psi)$ and $I = I(\Psi)$ with the objective of having a partial differential equation (elliptical) in Ψ (plus the boundary conditions).

By knowing $\Psi(x^1, x^2)$ we determine I and find the components of $\boldsymbol{B}$ and $\boldsymbol{J}$ using

$$B^1 = -\frac{1}{\sqrt{g}}\frac{\partial\Psi}{\partial x^2}, \tag{3.53}$$

$$B^2 = \frac{1}{\sqrt{g}}\frac{\partial\Psi}{\partial x^1}, \tag{3.54}$$

$$B^3 = -\mu_0 I, \tag{3.55}$$

$$\mu_0 J^1 = -\frac{1}{\sqrt{g}}\frac{\partial I}{\partial x^2}, \tag{3.56}$$

$$\mu_0 J^2 = \frac{1}{\sqrt{g}}\frac{\partial I}{\partial x^1}, \tag{3.57}$$

$$\mu_0 J^3 = \Delta^*\Psi. \tag{3.58}$$

All of these will be used after finding the expression for the flux function $\Psi(x^1, x^2)$.

4 MHD equilibrium of plasmas in rotation

4.1 Basic equations

We have defined the MHD stationary equilibrium state without an explicit dependency on time, then the set of MHD equations in the ideal case (remembering that it is when there is no resistivity) is:

$$
\begin{aligned}
\nabla \cdot (\varrho \boldsymbol{v}) &= 0, &(4.1)\\
\varrho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} &= -\nabla p + \boldsymbol{J} \times \boldsymbol{B}, &(4.2)\\
\nabla \cdot \left(p\varrho^{-\gamma+1}\right) &= 0 &(4.3)\\
\nabla \times \boldsymbol{E} &= 0 &(4.4)\\
\nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J}, &(4.5)\\
\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} &= \boldsymbol{0}, &(4.6)
\end{aligned}
$$

for $\boldsymbol{v} = \boldsymbol{0}$ we get the equations for static MHD equilibrium. If the plasma velocity is different from zero, like in the case of an uniform rotation, then, the above equations describe a stationary MHD equilibrium.

A fundamental difference between the stationary equilibrium equations and the static ones, are, that now the density ϱ is present. For that reason, it is necessary to include in the theoretical description of plasmas in rotation, thermodynamical equations that specify the process involved. The Gibbs's equation can be a starting point from the thermodynamics

$$d\epsilon = Tds - pd\left(\frac{1}{\varrho}\right) = Tds + \frac{p}{\varrho^2}d\varrho, \tag{4.7}$$

where ϵ is the specific internal energy (per unit mass), s is the specific entropy, p is the kinetic pressure and T is the temperature of the plasma, pressure related through the state equation for an ideal gas

$$p = \varrho \bar{R} T = n k_B T \tag{4.8}$$

where k_B is the Boltzmann constant, $n = \varrho/(m_e + m_i)$ is the plasma particle density (equal for electrons and ions, because of quase neutrality), and $\bar{R}$ is the gas constant.

An important thermodynamical potential is the specific enthalpy h, that satisfies the relation

$$dp = \varrho(dh - Tds). \tag{4.9}$$

To close the set of thermodynamical equations of stationary MHD equilibrium, is necessary to specify a hypothesis for the thermodynamical process corresponding to the plasma rotation.

4.2 MHD equilibrium with azimuthal rotation

We are working with stationary equilibrium configurations where there is an ignorable coordinate (x^3), then the equilibrium is essentially bidimensional and we can use the same surface quantities defined before, just to remind:

- Transversal flux function: $\Psi(x^1, x^2)$, defined in (3.14);

- Transversal current function: $I(x^1, x^2)$, defined in (3.29),

with those, the magnetic field is represented as:

$$\boldsymbol{B}(x^1, x^2) = \frac{\hat{\boldsymbol{e}}_3}{g_{33}} \times \nabla\Psi(x^1, x^2) - \mu_0 I(x^1, x^2)\frac{\hat{\boldsymbol{e}}_3}{g_{33}}. \tag{4.10}$$

4.3 Ferraro's iso-rotation law

Another supposition used is to consider the rotation as being azimuthal, i.e., along the x^3 direction: $\boldsymbol{v} = v_3\hat{\boldsymbol{e}}_3$. This is compatible with the continuity equation, because

$$\nabla \cdot (\varrho \boldsymbol{v}) = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}(\sqrt{g}\varrho v^1) + \frac{\partial}{\partial x^2}(\sqrt{g}\varrho v^2)\right] = 0.$$

Assuming, also, a rigid rotation of plasma in that direction, defining an angular frequency of rotation Ω as

$$v_3 = \sqrt{g_{33}}\Omega \tag{4.11}$$

or

$$\boldsymbol{v} = \Omega\hat{\boldsymbol{e}}_3. \tag{4.12}$$

The angular frequency Ω is a surface quantity, that is, each magnetic surface rotates with a different angular velocity. This result is known as Ferraro's isorotation law. The way to prove this fact, is by eliminating the electric field through the generalized Ohm's law 4.6 and replacing in Faraday's law (4.4):

$$\nabla \times \boldsymbol{E} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{0}. \tag{4.13}$$

Substituting (4.11) and (4.10) in (4.13) gives

$$\begin{aligned} \nabla \times \left[\frac{\Omega}{g_{33}}\hat{\boldsymbol{e}}_3 \times (\hat{\boldsymbol{e}}_3 \times \nabla\Psi)\right] &= \boldsymbol{0} && (4.14)\\ -\nabla \times (\Omega\nabla\Psi) &= \boldsymbol{0} && (4.15)\\ \nabla\Omega \times \nabla\Psi &= \boldsymbol{0} && (4.16) \end{aligned}$$

that is, $\Omega = \Omega(\Psi)$ is, in fact, a surface quantity, so that

$$\nabla\Omega = \frac{d\Omega}{d\Psi}\nabla\Psi. \tag{4.17}$$

Using (4.11) and writing the velocity dependent term in the equation of stationary equilibrium (4.2) as

$$\begin{aligned} \varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= \varrho(\Omega\hat{\boldsymbol{e}}_3\cdot\nabla)(\Omega\hat{\boldsymbol{e}}_3) && (4.18)\\ &= \varrho\left(\Omega\hat{\boldsymbol{e}}_3\cdot\hat{\boldsymbol{e}}^j\partial_j\right)(\Omega\hat{\boldsymbol{e}}_3) && (4.19)\\ &= \varrho\Omega^2\partial_3\hat{\boldsymbol{e}}_3. && (4.20)\end{aligned}$$

In case the element g_{33} of the metric tensor does not depend on x^3 then the vector $\partial_3\hat{\boldsymbol{e}}_3\ \hat{\boldsymbol{e}}_3$, since

$$(\partial_3\hat{\boldsymbol{e}}_3)\cdot\hat{\boldsymbol{e}}_3 = \frac{1}{2}\partial_3 g_{33} = 0. \tag{4.21}$$

Because $\hat{\boldsymbol{e}}_3 = \sqrt{g}\hat{\boldsymbol{e}}^1\times\hat{\boldsymbol{e}}^2$ then, results that $\partial_3\hat{\boldsymbol{e}}_3$ is parallel to a linear combination of $\hat{\boldsymbol{e}}^1$ e $\hat{\boldsymbol{e}}^2$, in the form

$$\partial_3\hat{\boldsymbol{e}}_3 = a_1\hat{\boldsymbol{e}}^1 + a_2\hat{\boldsymbol{e}}^2. \tag{4.22}$$

The coefficients a_1 e a_2 can be calculated by doing the scalar product of (4.22) with $\hat{\boldsymbol{e}}_1$ e $\hat{\boldsymbol{e}}_2$, respectively:

$$a_1 = -\frac{1}{2}\partial_1 g_{33}, \qquad a_2 = -\frac{1}{2}\partial_2 g_{33}, \tag{4.23}$$

so that

$$\partial_3\hat{\boldsymbol{e}}_3 = -\frac{1}{2}\nabla g_{33}. \tag{4.24}$$

Substituting (4.24) in (A.8) the velocity dependent term in the equilibrium equation is written as

$$\varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{2}\varrho\Omega^2\nabla g_{33}. \tag{4.25}$$

Finally, using the isorotation law (4.17) we can write

$$\nabla\left(\frac{\Omega^2 g_{33}}{2}\right) = g_{33}\Omega\frac{d\Omega}{d\Psi}\nabla\Psi + \frac{1}{2}\Omega^2\nabla g_{33}, \tag{4.26}$$

and gives the expression

$$\varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -g_{33}\Omega\frac{d\Omega}{d\Psi}\nabla\Psi - \nabla\left(\frac{\Omega^2 g_{33}}{2}\right). \tag{4.27}$$

4.4 Equilibrium equation with azimuthal rotation

The magnetohydrostatic equilibrium condition $\nabla p = \boldsymbol{J} \times \boldsymbol{B}$ implies the following relation

$$\mu_0 J_3 \nabla\Psi = -\mu_0 g_{33} \nabla p - \mu_0^2 I \nabla I = -\mu_0 g_{33} \nabla p - \frac{1}{2}\mu_0^2 \frac{dI^2}{d\Psi} \nabla\Psi \tag{4.28}$$

so that $I = I(\Psi)$ because both are surface quantities. In the case of stationary MHD equilibrium, where the equation (4.2) is valid, we will have the condition (in absence of gravitational fields, i.e. $\Phi = 0$):

$$\varrho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla p = \boldsymbol{J} \times \boldsymbol{B}. \tag{4.29}$$

Then, to include the rotational azimuthal effect, we replace ∇p with the first member of the above relation in eq. (4.28):

$$\mu_0 J_3 \nabla\Psi = -\mu_0 g_{33} \left[\nabla p + \varrho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right] - \frac{1}{2}\mu_0^2 \frac{dI^2}{d\Psi} \nabla\Psi, \tag{4.30}$$

or else, using (4.22),

$$\left[\mu_0 J_3 + \frac{1}{2}\mu_0^2 \frac{dI^2}{d\Psi}\right] \nabla\Psi = -\mu_0 g_{33} \left(\nabla p - \frac{1}{2}\varrho\Omega^2 \nabla g_{33}\right). \tag{4.31}$$

As a consequence of Ampére law, we have

$$\mu_0 J_3 = \Delta^*\Psi + \mu_0 I \frac{g_{33}}{\sqrt{g}} \left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right], \tag{4.32}$$

where Δ^* is the Shafranov operator. Substituting (4.32) in (4.31) gives

$$\begin{aligned} \left\{\Delta^*\Psi + \mu_0 I \frac{g_{33}}{\sqrt{g}} \left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right] + \frac{1}{2}\mu_0^2 \frac{dI^2}{d\Psi}\right\} \nabla\Psi = \\ = -\mu_0 g_{33} \left(\nabla p - \frac{1}{2}\varrho\Omega^2 \nabla g_{33}\right). \end{aligned} \tag{4.33}$$

4.5 Adiabatic processes

In an adiabatic process the pressure and the density are related by Poisson's law

$$p = A(s)\varrho^\gamma, \tag{4.34}$$

where A is a constant that depends on the specific entropy s, that does not change with time. Using $ds = 0$ in Gibb's equation (4.7) gives

$$d\epsilon = \frac{p}{\varrho^2} d\varrho = A(s)\varrho^{\gamma-2} d\varrho, \tag{4.35}$$

by integrating, gives the specific internal energy as

$$\epsilon = \int d\epsilon = A(s)\frac{\varrho^{\gamma-1}}{\gamma-1} = \frac{h}{\gamma} \tag{4.36}$$

and the specific enthalpy is obtained from Eq. 4.9

$$h = \int dh = A(s)\gamma \int \varrho^{\gamma-2} d\varrho = A(s)\frac{\gamma}{\gamma-1}\varrho^{\gamma-1} = \gamma\epsilon. \tag{4.37}$$

From Gibbs's equation (4.7) the temperature is given by

$$T = \frac{\partial\epsilon}{\partial s} = \frac{\varrho^{\gamma-1}}{\gamma-1}\frac{dA}{ds}. \tag{4.38}$$

4.6 Entropy as a surface quantity

A key problem in the theoretical study of azimuthal rotation is whether such a plasma flow could coexist with a state of MHD under stationary equilibrium. The answer turns to be positive provided some requirements are fulfilled by the system [41]. If axissymetry exists, field lines lie on magnetic flux surfaces with topology of tori and characterized by surface quantities, like the transversal magnetic flux. The set of ideal MHD equations allows us to derive a partial differential equation for it. Maschke and Perrin [20] obtained a MHD equilibrium equation for azimuthal plasma flows supposing that either the temperature or the entropy were surface quantities. They considered only cylindrical coordinates, having obtained exact analytical solutions for the transversal magnetic flux. There are a few other solved cases in cylindrical and spherical geometries, but considering the temperature as a surface quantity. [39]

The case where the plasma flow is adiabatic, however, demands the use of the entropy as a surface quantity. This is particularly important in the case of anisotropic plasmas, where a double-adiabatic theory is necessary to describe the situation. To apply the MHD equilibrium theory for realistic magnetic confinement schemes, one would need an equilibrium equation in a general curvilinear coordinate system, that is provided in reference [41], and that is used next.

4.7 Maschke-Perrin equation for adiabatic processes

In the derivation of the equilibrium equation for azimuthal rotation (4.33) we did not any assumptions about the type of thermodynamic process represented by rotation. Assuming, therefore, that rotation is an adiabatic process, it will be Poisson's law

$$\frac{d}{dt}\left(\frac{p}{\varrho^{\gamma}}\right) = \frac{dA}{dt} = 0 \tag{4.39}$$

so that, as $A = A(s)$, the entropy will be constant:

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + (\boldsymbol{v}\cdot\nabla)s = 0, \tag{4.40}$$

where has been used the expression for the material derivative of a function of a moving plasma. As s does not depend explicitly of the time then follows the condition

$$\boldsymbol{v}\cdot\nabla s = 0. \tag{4.41}$$

If the rotation is purely toroidal, according to (4.11) this relation implies

$$\boldsymbol{v}\cdot\nabla s = \Omega\hat{\boldsymbol{e}}_3\cdot\nabla s = \Omega\frac{\partial s}{\partial x^3} = 0 \tag{4.42}$$

so that the plasma entropy will be a surface quantity, $s = s(x^1, x^2)$. Consequently

$$\boldsymbol{B}\cdot\nabla s = 0. \tag{4.43}$$

implying $\nabla\Psi\times\nabla s = 0$, so that $s = s(\Psi)$, such that

$$\nabla s = \frac{ds}{d\Psi}\nabla\Psi. \tag{4.44}$$

From (4.9) follows that $dp = \varrho(dh - Tds)$

$$\nabla p = \varrho(\nabla h - T\nabla s) = \varrho\nabla h - \varrho T\frac{ds}{d\Psi}\nabla\Psi. \tag{4.45}$$

Substituting this last expression in the general equation of equilibrium with azimuthal rotation (4.33):

$$\begin{aligned}\left\{\Delta^*\Psi + \mu_0 I\frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right] + \frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi}\right\}\nabla\Psi = \\ = -\mu_0 g_{33}\left(\varrho\nabla h - \varrho T\frac{ds}{d\Psi}\nabla\Psi - \frac{1}{2}\varrho\Omega^2\nabla g_{33}\right).\end{aligned} \tag{4.46}$$

Using

$$\frac{d\Omega}{d\Psi} = \frac{k_B T}{\Omega}\frac{d}{d\Psi}\left(\frac{\Omega^2}{2k_B T}\right) + \frac{\Omega}{2T}\frac{dT}{d\Psi} \tag{4.47}$$

to rewriting this equation takes the form

$$\begin{aligned}&\left\{\Delta^*\Psi + \mu_0 I\frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right] + \frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} - \mu_0 g_{33}\varrho T\frac{ds}{d\Psi} + \mu_0 g_{33}^2\varrho\Omega\frac{d\Omega}{d\Psi}\right\}\nabla\Psi \\ &= -\mu_0 g_{33}\varrho\nabla\left(h - \varrho T\frac{1}{2}\Omega^2 g_{33}\right) \\ &= -\mu_0 g_{33}\varrho\nabla\Theta,\end{aligned} \tag{4.48}$$

where we define the centrifugally corrected enthalpy as

$$\Theta = h - \frac{1}{2}\Omega^2 g_{33}, \tag{4.49}$$

since, in absence of rotation, $\Omega \to 0$ implies $\Theta \to h$.

Doing the cross product of (4.48) with $\nabla\Psi$ gives $\nabla\Theta \times \nabla\Psi = 0$, so that $\Theta = \Theta(\Psi)$ is also a surface quantity. Inserting

$$\nabla\Theta = \frac{d\Theta}{d\Psi}\nabla\Psi,$$

in (4.48) we obtain, for any $\nabla\Psi \neq 0$, the equation of Maschke-Perrin for adiabatic azimuthal rotations:

$$\begin{aligned}\Delta^*\Psi + \mu_0 I \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right] &= \\ = -\frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} - \mu_0 g_{33}\varrho\left[g_{33}\Omega\frac{d\Omega}{d\Psi} + \frac{d\Theta}{d\Psi} - T\frac{ds}{d\Psi}\right].\end{aligned} \tag{4.50}$$

In the case without rotation ($\Omega = 0$) the above equation reduces to

$$\Delta^*\Psi + \mu_0 I \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right] = -\frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} - \mu_0 g_{33}\frac{dp}{d\Psi}, \tag{4.51}$$

where we used (4.9) to write

$$\frac{dp}{d\Psi} = \varrho\left(\frac{dh}{d\Psi} - T\frac{ds}{d\Psi}\right). \tag{4.52}$$

Moreover, the temperature, given by (4.38), is

$$T = \frac{\varrho^{\gamma-1}}{\gamma-1}\frac{dA/d\Psi}{ds/d\Psi}, \tag{4.53}$$

since S and $A = A(S)$ are surface quantities. This relation can be used to eliminate $ds/d\Psi$ in (4.50):

$$\begin{aligned}\Delta^*\Psi + \mu_0 I \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right] &= \\ = -\frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} - \mu_0 g_{33}\varrho\left[g_{33}\Omega\frac{d\Omega}{d\Psi} + \frac{d\Theta}{d\Psi} - \frac{\varrho^{\gamma-1}}{\gamma-1}\frac{dA}{d\Psi}\right].\end{aligned} \tag{4.54}$$

Using (4.36) the density can be eliminated, so, the enthalpy is

$$h = \frac{\gamma}{\gamma-1}A(S)\varrho^{\gamma-1}$$

and gives the expression for the density as a function of surface quantities:

$$\varrho = \left[\frac{\gamma-1}{\gamma A(s)}\left(\Theta + \frac{1}{2}\Omega^2 g_{33}\right)\right]^{\frac{1}{\gamma-1}}. \tag{4.55}$$

that, inserted in (4.54), gives an equilibrium equation that involves only surface quantities:

$$\Delta^*\Psi + \mu_0 I \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right] = -\frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} - \tag{4.56}$$
$$-\mu_0 g_{33}\left[\frac{\gamma-1}{\gamma A(s)}\left(\Theta + \frac{1}{2}\Omega^2 g_{33}\right)\right]^{\frac{1}{\gamma-1}}\left\{g_{33}\Omega\frac{d\Omega}{d\Psi} + \frac{d\Theta}{d\Psi} - \frac{1}{\gamma A(s)}\left(\Theta + \frac{1}{2}\Omega^2 g_{33}\right)\frac{dA}{d\Psi}\right\}.$$

Writing for simplicity

$$\eta = \frac{\gamma}{\gamma - 1}. \tag{4.57}$$

Now, is possible to express (4.56) as

$$\Delta^*\Psi + \mu_0 I \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right] = -\frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} - \mu_0 g_{33}\left(1 + \frac{g_{33}\Omega^2}{2\Theta}\right)^{\eta-1}\times$$
$$\times\left\{\frac{d}{d\Psi}\left[\left(\frac{\Theta}{\eta}\right)^{\eta}A^{1-\eta}\right] + g_{33}\left[\frac{d}{d\Psi}\left(\frac{\Theta}{\eta}\right)^{\eta}A^{1-\eta}\frac{\Omega d\Omega/d\Psi}{d\Theta/d\Psi} + \left(\frac{\Theta}{\eta}\right)^{\eta}\frac{dA^{1-\eta}}{d\Psi}\frac{\Omega^2}{2\Theta}\right]\right\}. \tag{4.58}$$

This equation just can represents a closed system if we specify *a priori* the four functions $I(\Psi)$, $\Omega(\Psi)$, $\Theta(\Psi)$ e $A(\Psi)$. To simplify this task Maschke e Perrin [20] considered the particular case where the surface quantities Ω and Θ satisfy the next relation:

$$\frac{\Omega^2}{\Theta} = \frac{\omega^2}{\ell^2}, \tag{4.59}$$

where ω is a constant and ℓ is a characteristic length of the system. Thus

$$\frac{\Omega d\Omega/d\Psi}{d\Theta/d\Psi} = \frac{1}{2}\frac{d\Omega^2/d\Psi}{d\Theta/d\Psi} = \frac{1}{2}\frac{d\Omega^2}{d\Theta} = \frac{\omega^2}{2\ell^2}. \tag{4.60}$$

We introduce, also the following quantity

$$G(\Psi) = \left(\frac{\Theta}{\eta}\right)^{\eta}A^{1-\eta}, \tag{4.61}$$

that, when $\Omega \to 0$, it reduces to plasma kinetic pressure. Then G is a centrifugally corrected pressure. Substituting (4.60) and (4.61) in (4.7) gives

$$\Delta^*\Psi + \mu_0 I \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_{33}}\right)\right] = \\ = -\frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} - \mu_0 g_{33}\left(1+\frac{g_{33}\omega^2}{2\ell^2}\right)^{\eta}\frac{dG}{d\Psi}. \tag{4.62}$$

That has only two surface quantities whose profiles should be known, namely, I and G. This will be the basic equation, which we will consider possible analytical solutions. For completeness, in the case of orthogonal coordinate systems, we employ a simpler form of this equation:

$$\Delta^*\Psi + \frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} + \mu_0 g_{33}\left(1+\frac{g_{33}\omega^2}{2\ell^2}\right)^{\eta}\frac{dG}{d\Psi} = 0. \tag{4.63}$$

4.8 MHD equilibrium of plasmas in rotation in spherical coordinates

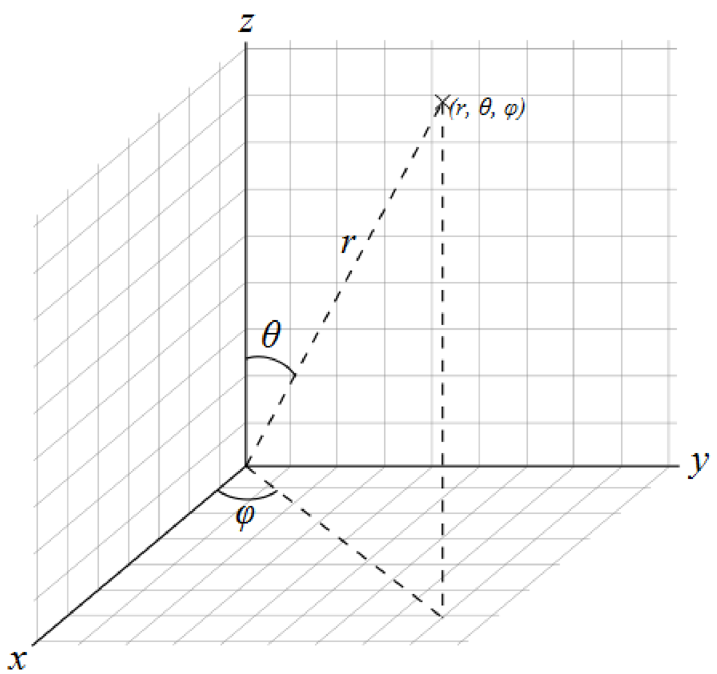


Figure 12: Spherical coordinates represented in a cartesian system

In this system (see figure 12) where $(x^1, x^2, x^3) = (r, \theta, \varphi)$ the ignorable coordinate is φ and $g_{33} = r^2\sin^2\theta$ (see appendix). The Shafranov operator is

$$\Delta^*\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2\Psi}{\partial\theta^2} - \cot\theta\frac{\partial\Psi}{\partial\theta}\right) \tag{4.64}$$

and the Maschke-Perrin equation with $l = r_0$ is

$$\Delta^* \Psi = -\frac{1}{2}\mu_0^2 \frac{dI^2}{d\Psi} - \mu_0 r^2 \sin^2\theta \left(1 + \frac{\omega^2 r^2 \sin^2\theta}{2r_0^2}\right)^\eta \frac{dG}{d\Psi}, \tag{4.65}$$

where the centrifugally corrected pressure and the poloidal flux current are given by adopting profiles for I^2 and G.

4.9 Solution to the Maschke-Perrin equation

The main problem addressing equation (4.65) is that the dependent variable is also an independent variable (read Ψ), so, to solve it, it is necessary the use of iterative methods, or another approach is by trying an ansatz for Ψ and after trying to solve the equation, this was done with no results, then, the following approach is by the choosing of profiles for the poloidal flux current I and the pressure parameter G and then trying to solve the equation by any known method. In this work, was assumed Ψ positive inside the plasma and vanishing at its boundary, so we opt for chosen profiles in the form

$$I^2(\Psi) = I_0^2 \pm \frac{2\lambda^2}{\mu_0^2}\Psi^2, \tag{4.66}$$

$$G(\Psi) = G_0 + \frac{\kappa^2}{\mu_0}\Psi, \tag{4.67}$$

where λ^2, κ^2, I_0^2 and G_0 are positively defined constants. The plus sign in front of λ^2 means that paramagnetic plasmas are considered, the minus sign refers to diamagnetic plasmas.

In a paramagnetic plasma, the poloidal currents increase the toroidal magnetic field relative to its value in the absence of the plasma. This reduces the allowable plasma kinetic pressure relative to the allowable value if the vacuum toroidal field was present. The converse is true for a diamagnetic plasma, in which the poloidal current decreases the toroidal field and increases the allowable pressure. [35]

Defining a typical length r_0, the spherical Maschke-Perrin equation can be simplified by introducing

$$x = \frac{r}{r_0} \tag{4.68}$$

in such a way that, according to assumption, it takes the form

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{x^2}\left(\frac{\partial^2 \Psi}{\partial \theta^2} - \cot\theta \frac{\partial \Psi}{\partial \theta}\right) \pm 2\lambda^2 r_0^2 \Psi + \kappa^2 r_0^4 x^2 \sin^2\theta \left(1 + \frac{\omega^2 x^2 \sin\theta}{2}\right)^\eta = 0. \tag{4.69}$$

Again, the variable Ψ is at the same time a dependent and independent variable, to solve the equation, after trying other known methods like separation of variables, changing of variables, etc., after no immediate analytical solution we use the same methodology introduced by Viana, et al., in [39] for a similar work but considering the temperature as a surface quantity. We start with an approximation by expanding the binomial term as

$$\left(1+\frac{\omega^2 x^2 \sin\theta}{2}\right)^\eta \approx 1+\frac{\eta}{2}\omega^2 x^2 \sin^2\theta + ..., \tag{4.70}$$

this is valid for values of

$$\frac{\omega^2 x^2 \sin\theta}{2} \ll 1,$$

or, in other words $\omega \ll \sqrt{2}$.

We have the equation in the form

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{x^2}\left(\frac{\partial^2\Psi}{\partial\theta^2}-\cot\theta\frac{\partial\Psi}{\partial\theta}\right) = -(\pm 2\lambda^2) r_0^2\Psi - \kappa^2 r_0^4 x^2\sin^2\theta - \frac{\eta}{2}\kappa^2 r_0^4\omega^2 x^4\sin^4\theta, \tag{4.71}$$

The solution of Eq. (4.71) can be written in the form

$$\Psi(x,\theta) = \Psi_p(x,\theta)+\Psi_g(x,\theta), \tag{4.72}$$

where the subscripts refer to the particular and general solutions, respectively. By analyzing the right hand side of the inhomogeneous equation (4.71), we test for a particular solution having the same order of the second and third terms with arbitrary coefficients, namely

$$\Psi_p(x,\theta) = \mathcal{A}x^2\sin^2\theta + \mathcal{B}x^4\sin^4\theta, \tag{4.73}$$

with the coefficients given by

$$\mathcal{A} = \frac{\kappa^2\left(2\eta\omega^2-\lambda^2 r_0^2\right)}{\pm 2\lambda^4}, \mathcal{B} = -\frac{\eta\kappa^2 r_0^2\omega^2}{\pm 4\lambda^2}, \tag{4.74}$$

and the general solution of the homogeneous equation will be

$$\Psi_g(x,\theta) = \mathcal{C}f_1^\pm(x)\sin^2(\theta) + \mathcal{D}f_2^\pm(x)\left[1-\frac{5\sin^2(\theta)}{4}\right]\sin^2(\theta), \tag{4.75}$$

where $\mathcal{C},\mathcal{D}$ are coefficients to be determined by boundary conditions, and the $(\pm)$ sign refers to the sign of λ in the profile for I^2. When the plus sign is adopted, $f_{1,2}^+$ are

$$f_1^+(x) = \frac{\sin(\lambda R x)}{\lambda R x} - \cos(\lambda R x) \tag{4.76}$$

$$f_2^+(x) = \frac{\sin(\lambda R x)}{\lambda R x}\left[\frac{15}{(\lambda R x)^2}-6\right] - \cos(\lambda R x)\left[\frac{15}{(\lambda R x)^2}-1\right], \tag{4.77}$$

and when the minus sign is adopted, $f_{1,2}^-$ are

$$f_1^-(x) = \frac{\sinh(\lambda R x)}{\lambda R x} - \cosh(\lambda R x) \tag{4.78}$$

$$f_2^-(x) = \frac{\sinh(\lambda R x)}{\lambda R x}\left[\frac{15}{(\lambda R x)^2} + 6\right] - \cosh(\lambda R x)\left[\frac{15}{(\lambda R x)^2} + 1\right], \tag{4.79}$$

A spherical boundary condition that allows us to define $\mathcal{C}$ and $\mathcal{D}$ corresponds to the case in which the plasma is enclosed by a conducting spherical shell of radius $r_0 = R$. In this case I_0^2 must vanish and the plus sign in front of λ^2 must be chosen. Correspondingly,

$$\mathcal{C} = \frac{\kappa^2 R\left[20\eta\omega^2 - \lambda^2 R^2(2\eta\omega^2 + 5)\right]}{5\lambda^3\left[\lambda R\cos(\lambda R) - \sin(\lambda R)\right]} \tag{4.80}$$

$$\mathcal{D} = -\frac{2\eta\kappa^2\lambda R^5\omega^2}{5\left[3(5 - 2\lambda^2 R^2)\sin(\lambda R) + \lambda R(\lambda^2 R^2 - 15)\cos(\lambda R)\right]}. \tag{4.81}$$

One important aspect to mention is the limit as the parameter ω approximates to zero, namely, when $\omega = 0$ then we are in the static case ($\Omega = 0$), that was described by Morikawa in 1968 [25] and 1969 [26]. The solution for the static case becomes

$$\Psi(x,\theta) = \frac{\kappa^2 R^2}{\lambda^2}\left[\frac{\sin(\lambda R x/\lambda R x) - \cos(\lambda R x)}{\sin(\lambda R/\lambda R) - \cos(\lambda R)} - x^2\right]\sin^2\theta, \tag{4.82}$$

which is the same solution obtained by Morikawa in 1969, a difference to mention is that he considered a thick force-free shell between a plasma sphere of radius $r = \Delta$ and a conducting shell at $r = R$.

A force-free magnetic field is a magnetic field that arises when the plasma pressure is so small, relative to the magnetic pressure, that the plasma pressure may be ignored, and so only the magnetic pressure is considered. For a force free field, the electric current density is either zero or parallel to the magnetic field. The name "force-free" comes from being able to neglect the force from the plasma. [27]

The imposed boundary conditions on the plasma and force-free regions constrain the possible values of the λ and κ parameters.

We restrict the cases where Ψ has only one maximum inside the imposed boundary, the values of λ^2 are restricted (see [25]) by the condition

$$\frac{\sin\lambda R}{\lambda R} - \cos\lambda R = 0 \tag{4.83}$$

or

$$\tan\lambda R = \lambda R. \tag{4.84}$$

Equation (4.84) is satisfied by discrete values of λR, we are interested in the first ones, then

$$0 \leq \lambda R \leq 4.493\cdots, \tag{4.85}$$

$\lambda R = 0$ corresponds to a spherical rotating field-reversed configuration without toroidal magnetic field. $\lambda R = 4.493...$ corresponds to a force-free magnetic field configuration in which the centripetal force is balanced by the pressure gradient (i.e. the pressure increases monotonously with $r\sin\theta$) [39]. In this case, care has to be taken in evaluating the constant $\mathcal{C}$ since κ also vanishes.

To illustrate these features it is convenient to write the solution at $\theta = \pi/2$:

$$\Psi(x, \pi/2) = -\left(\mathcal{A} + \frac{4\mathcal{B}}{5}\right)\frac{f_1^{\pm}(x)}{f_1^{\pm}(1)} - \frac{\mathcal{B}}{5}\frac{f_2^{\pm}(x)}{f_2^{\pm}(1)} + \mathcal{A}x^2 + \mathcal{B}x^4. \tag{4.86}$$

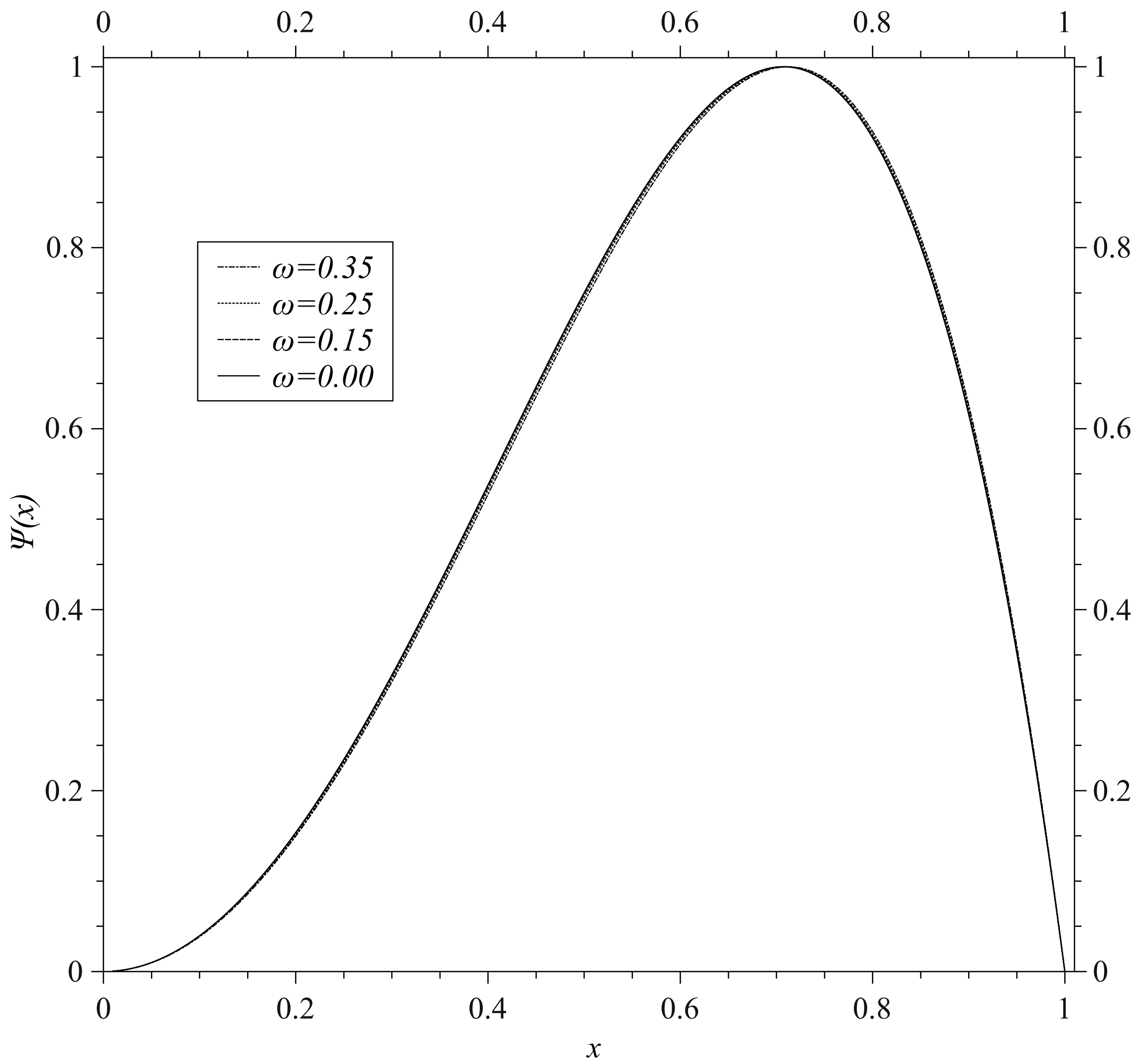


Figure 13: Flux Ψ as a function of x for ω values of 0 (no rotation), 0.15, 0.25 and 0.35 for $\lambda R = 0.50$

After finding the expressions for the flux function in terms of (x, θ) plots were made for different values of ω in the equatorial plane (i.e. $\theta = \pi/2$) with a value of $\lambda R = 0.50$; as can be seen in figure (13), the maximum of the flux function increases with increasing values of ω.

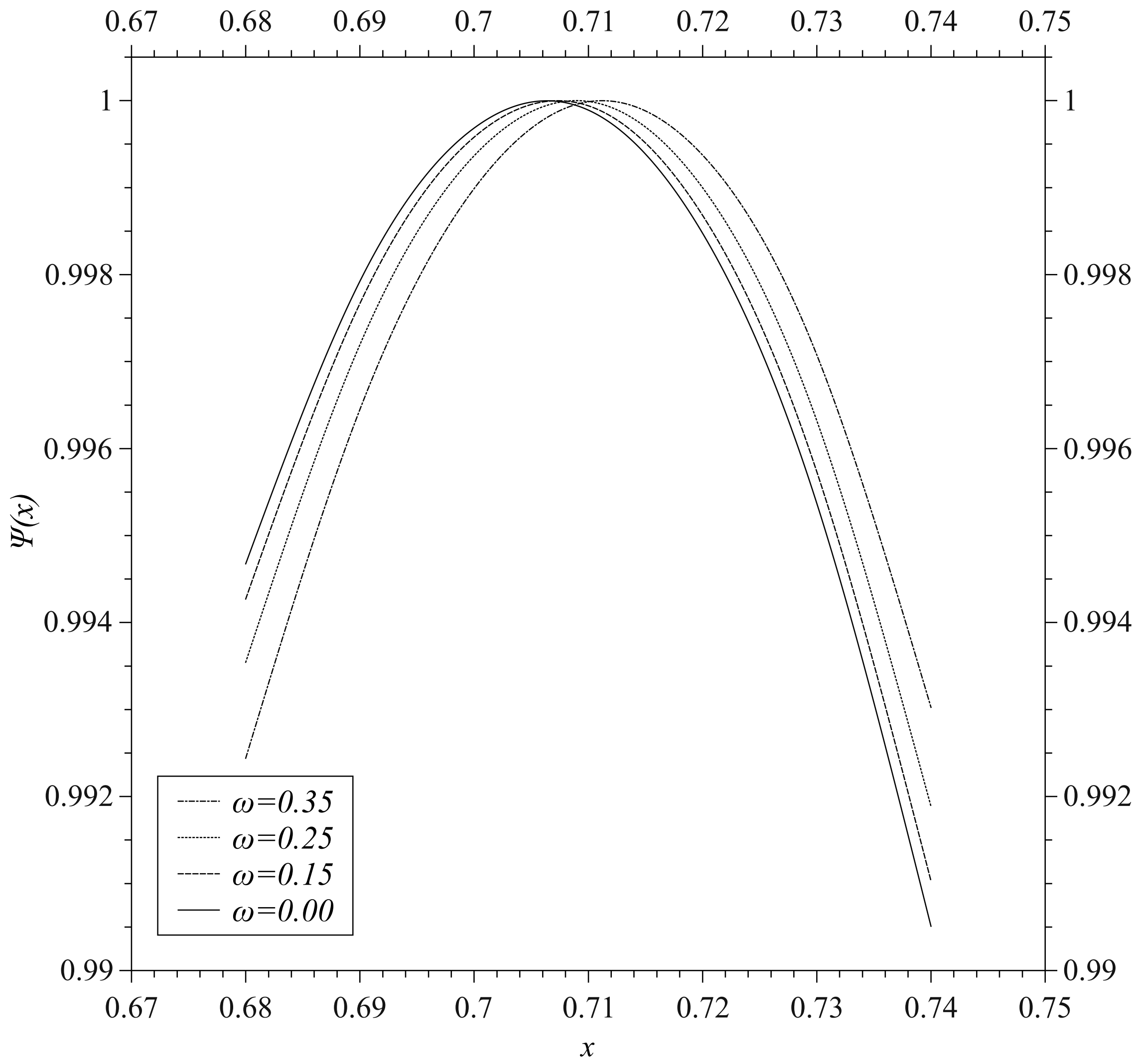


Figure 14: Flux Ψ as a function of x for ω values of 0 (no rotation), 0.15, 0.25 and 0.35 for $\lambda R = 0.50$

The maximum of the flux function experiments a displacement because of the rotation, figure () shows the effect of the rotation near the maximum of the flux function.

Is notorious that the flux function Ψ has a maximum at a certain position $x = x^*$. Indicating this maximum as $\Psi_{max} = (x^*, \pi/2)$ and assume it to be a fixed parameter. x^* is a function of λR and it is shown in figure (15) for different values of ω. As can be seen, rotational effects are more evident for lower values of λR, resulting in an outwards shift of x^* (i.e. shifting of the magnetic axis).

Then, the constant κ^2 appearing in the definition of G can be related to Ψ_{max} [39]. The non-dimensional parameter $\kappa^2 R^4/\Psi_{max}$ is a function of $x^*(\lambda R)$ and in figure (16) it has been

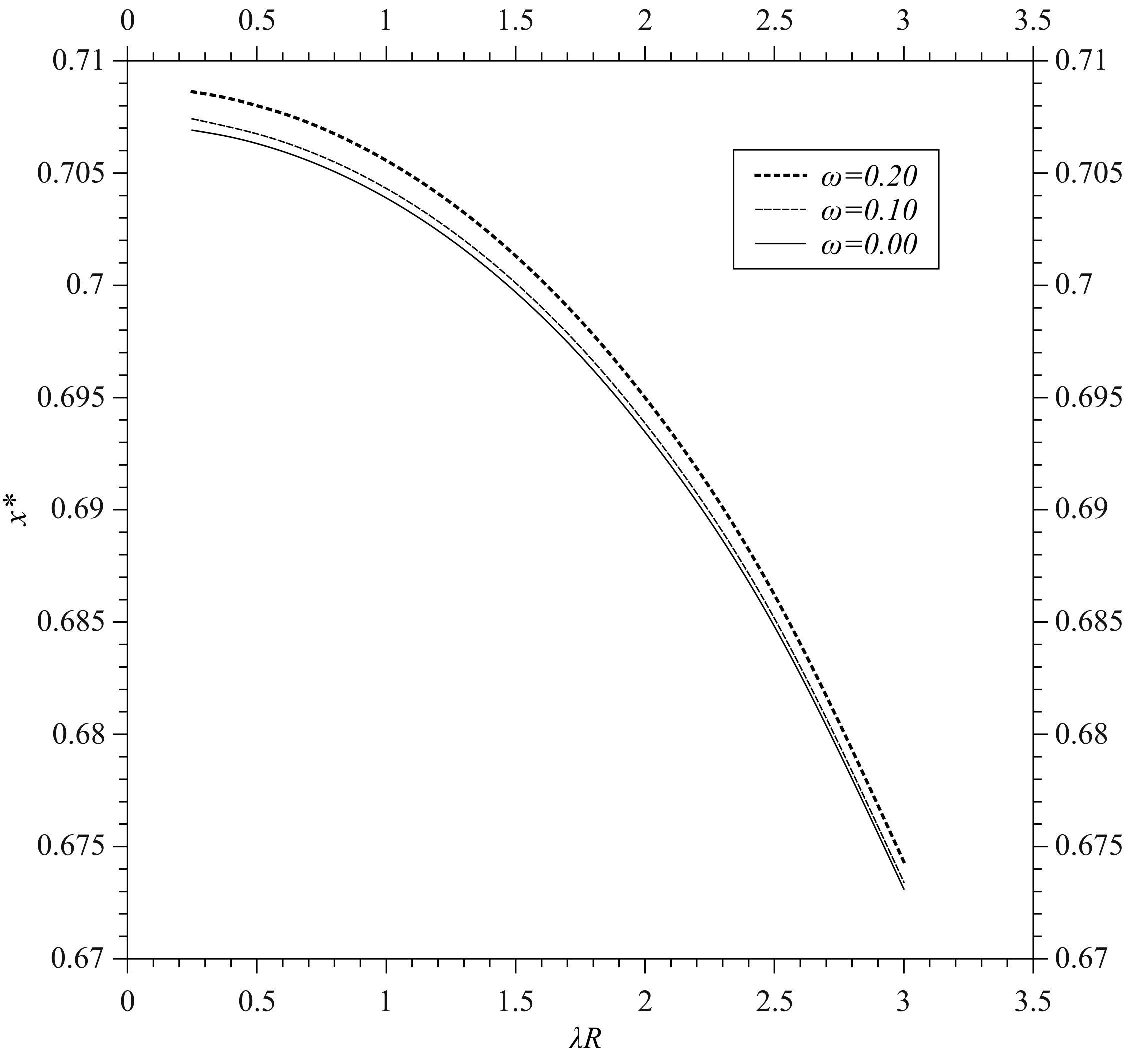


Figure 15: Magnetic axis radial location as a function of λR for ω values of 0 (no rotation), 0.10 and 0.20

plotted as a function of λR for different values of ω. As can be seen, κ^2 decreases when the rotation is increased; once again the effect is more evident at small λR.

The flux function was also plotted as a function of the dimensionless parameter x and the variable angle θ through

$$\Psi(x,\theta) = \mathcal{A}x^2 \sin^2\theta + \mathcal{B}x^4 \sin^4\theta + \mathcal{C}f_1^{\pm}(x)\sin^2\theta + \mathcal{D}f_2^{\pm}(x)\left(1 - \frac{5\sin^2\theta}{4}\right)\sin^2\theta, \quad (4.87)$$

using a value of $\omega = 0.15$ and $\lambda R = 0.50$, the result is shown if figure (17), this kind of plot is also known as contour lines, level curves or level set of the flux function, and it shows the plasma edge as well the different magnetic surfaces in the plasma.

Figure (18) shows the comparison, for exaggeration, between the contour lines of the plasma in figure (18a) with no rotation, i.e. $\omega = 0$ and, in figure (18b) with the maximum rotation studied, $\omega = 0.35$. It is evident the difference in the magnetic surfaces, and it is more evident

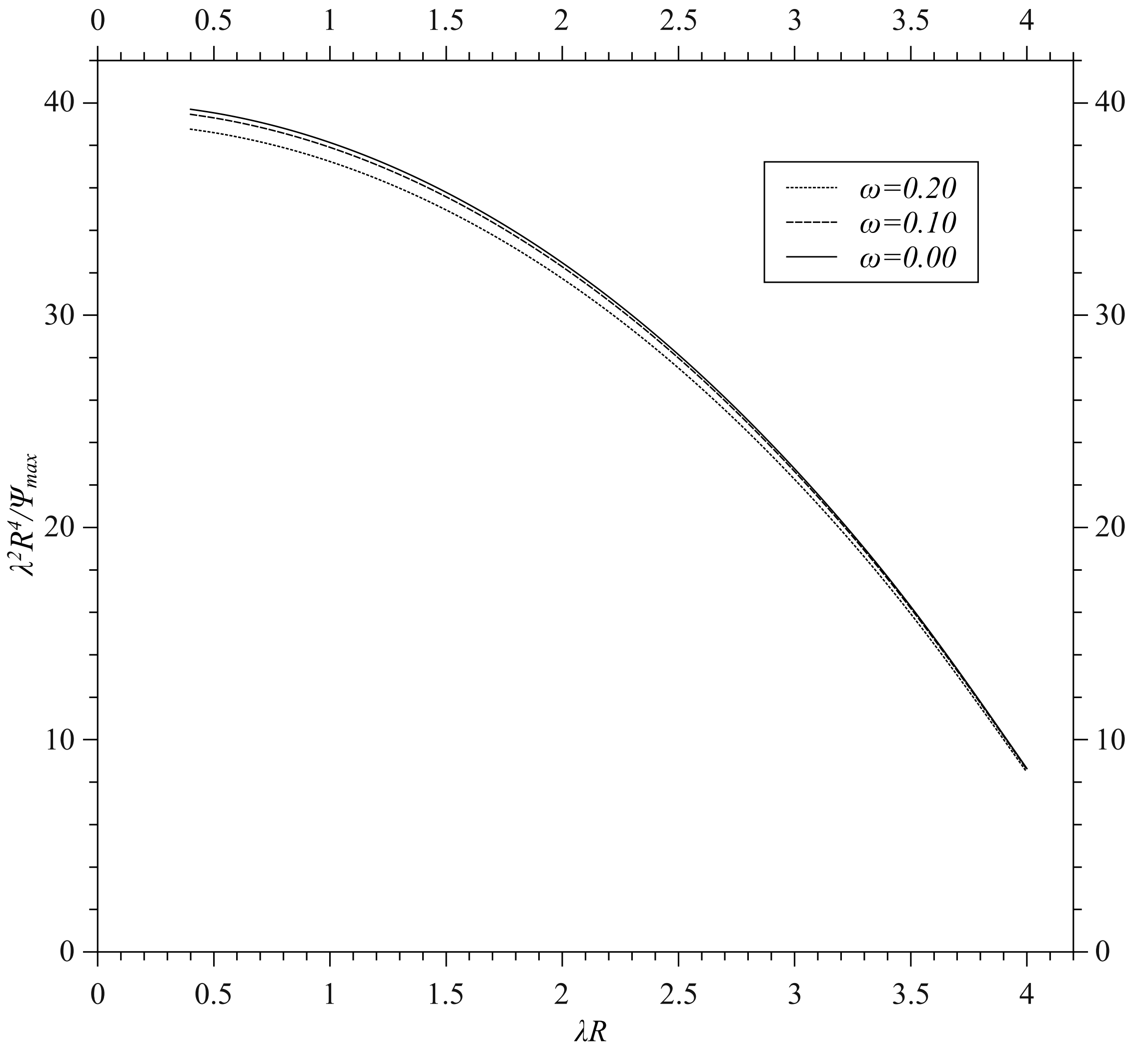


Figure 16: Non-dimensional pressure parameter as a function of λR for different values of ω

near the magnetic axis where the flux reaches the maximum value than in the edge of the plasma.

Near the magnetic axis, additional plots were made, as shown in figure (19) containing figures (19a) and (19b) showing the deformation of the level curve near the maximum of the flux function, related to the rotation effect.

Plots were also made in the equatorial plane for each of the components of the magnetic field B, the poloidal flux current I and for the poloidal current density J using the expressions from Eqs. (3.53) to (3.58). In spherical coordinates we have $g_{11} = 1$, $g_{22} = r^2$ and $g_{33} = r^2 \sin^2\theta$, then, the metric gives

$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = r^4 \sin^2\theta \tag{4.88}$$

and Eqs. (3.53) to (3.58) becomes

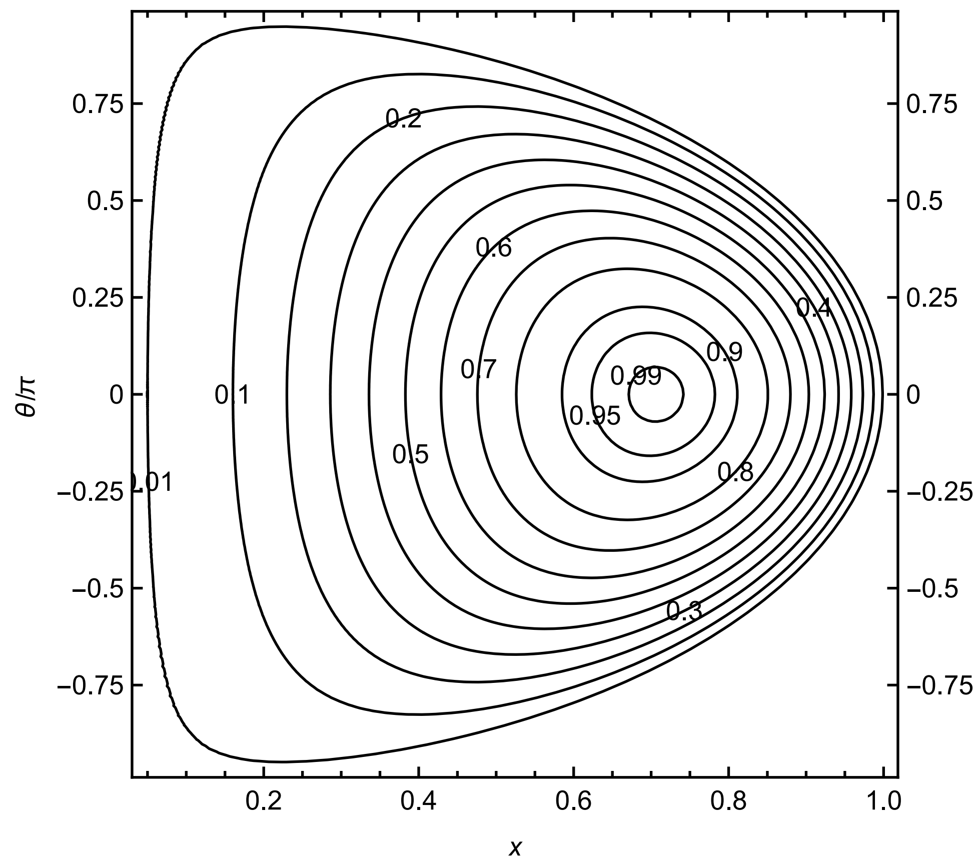
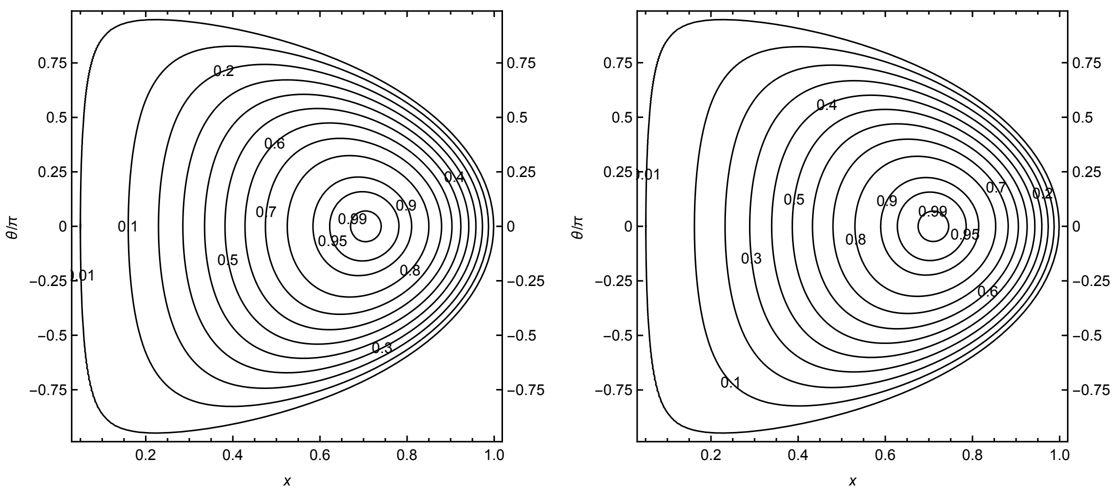


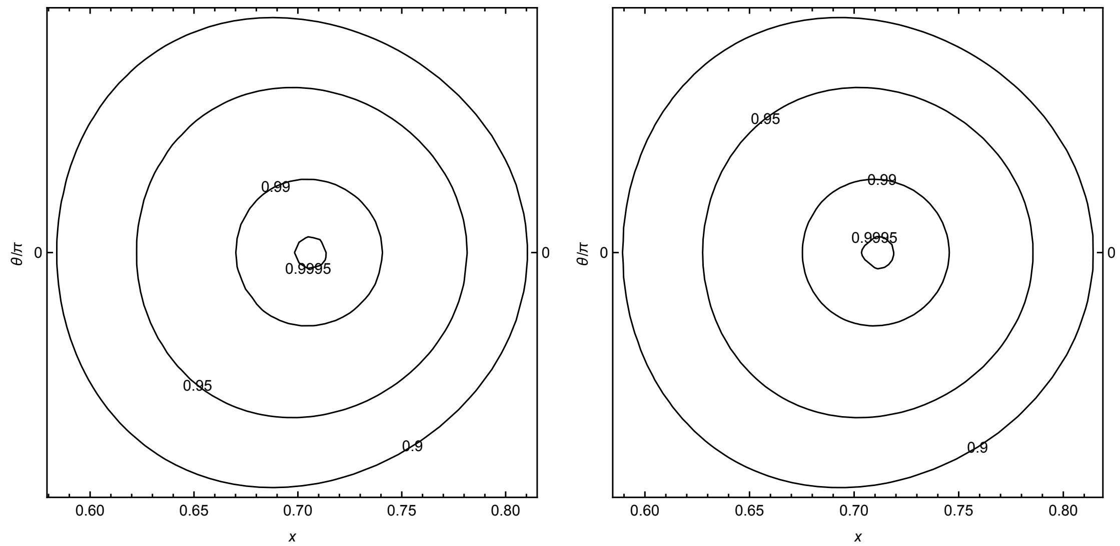
Figure 17: Plot as contour lines of the flux function Ψ as a function of x and θ with $\omega = 0.15$



(a) Contour lines of the flux function Ψ as a function of x and θ with $\omega = 0.00$ (no rotation)

(b) Contour lines of the flux function Ψ as a function of x and θ with $\omega = 0.35$

Figure 18: Magnetic contour lines for $\lambda R = 0.50$



(a) Contour lines of the flux function Ψ as a function of x and θ with $\omega = 0.00$ (no rotation)

(b) Contour lines of the flux function Ψ as a function of x and θ with $\omega = 0.35$

Figure 19: Magnetic contour lines for $\lambda R = 0.50$ showing deformation of the contour line near the maximum

$$B_r = -\frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}, \tag{4.89}$$

$$B_\theta = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial r}, \tag{4.90}$$

$$B_\varphi = -\mu_0 I, \tag{4.91}$$

$$\mu_0 J_r = -\frac{1}{r^2 \sin\theta}\frac{\partial I}{\partial \theta}, \tag{4.92}$$

$$\mu_0 J_\theta = \frac{1}{r^2 \sin\theta}\frac{\partial I}{\partial r}, \tag{4.93}$$

$$\mu_0 J_\varphi = \Delta^* \Psi. \tag{4.94}$$

So, we obtain the corresponding expressions for each of the component and made the corresponding plots of each quantity.

The figure (20) represents the behavior with rotation of the magnetic field B for the φ component, which reads

$$B_\varphi(x, \pi/2) = \pm\sqrt{\frac{2\lambda^2 x^2}{\mu_0^2}\left[\frac{\mathcal{A}}{x^2} + \mathcal{B} + \mathcal{C}\frac{f_1^{\pm}(x)}{x^4} - \mathcal{D}\frac{f_2^{\pm}(x)}{4x^4}\right] + \frac{I_0^2}{x^2}}, \tag{4.95}$$

the φ component of the magnetic field B is affected by the rotation, the effect is greater at the magnetic axis where B_φ has its maximum absolute value, and also is greater as the rotation also increases in its value.

The r component of the magnetic field B is not affected by the rotation and it remains with

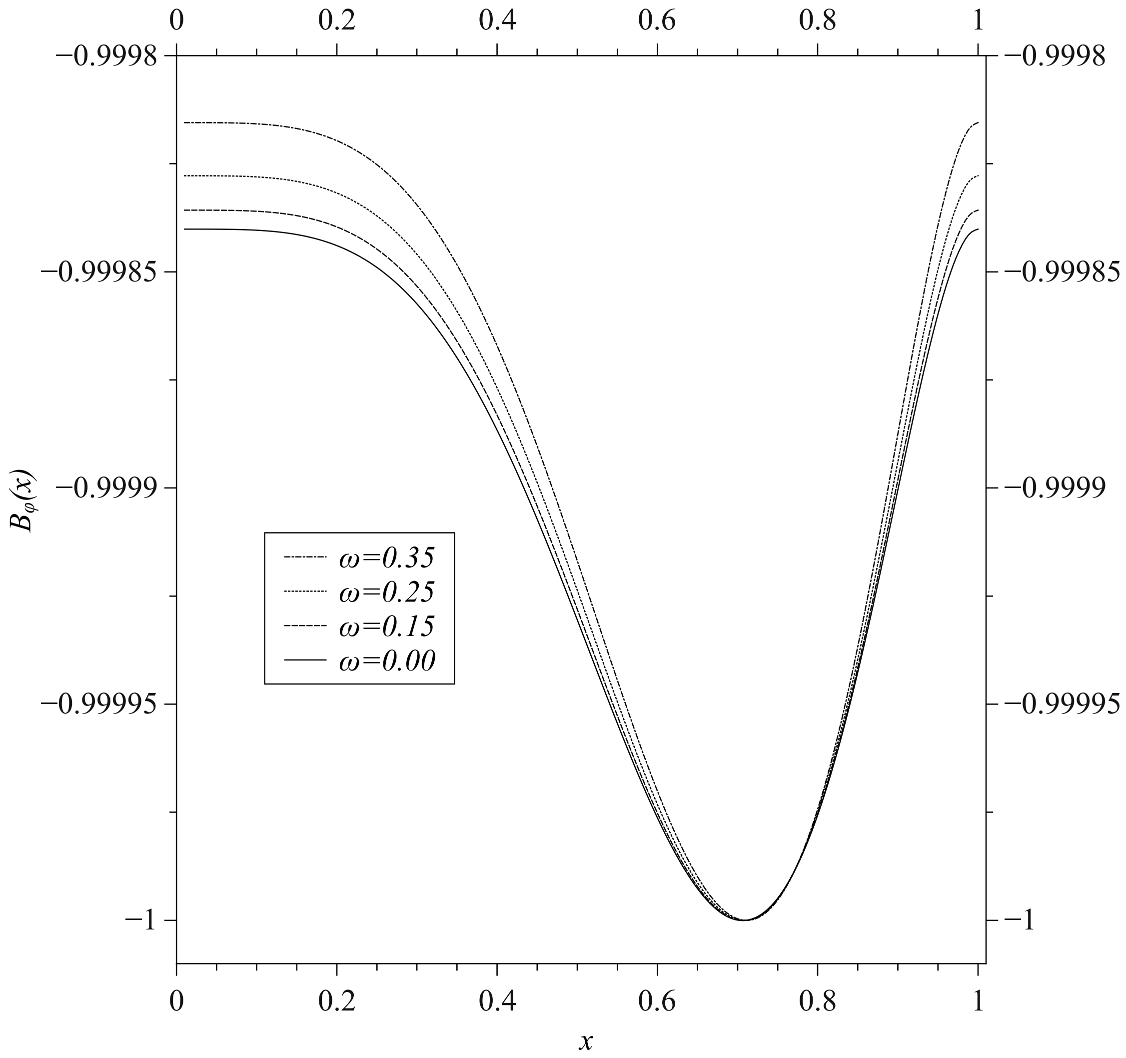


Figure 20: Plot in the equatorial plane of the φ component of the magnetic field B for ω values of 0 (no rotation), 0.15, 0.25 and 0.35

a value of zero along the x parameter, i.e., $B_r = 0$

The θ component of the magnetic field is plotted in figure (21), with the corresponding expression

$$B_\theta(x, \pi/2) = -2\mathcal{A} - 4\mathcal{B}x^2 - \mathcal{C}\frac{f_1^{\pm}(x)}{x} + \mathcal{D}\frac{f_2^{\pm}(x)}{4x}. \tag{4.96}$$

B_θ begins very strong for small x and then rapidly decreases until approaching to zero near the end at $x = 1.0$. It is difficult to note the effects of rotation because of the scale, so, a new plot was made showing the plot near the magnetic axis the plot is represented in figure (22), and now, the effect of rotation is more evident, also, from the graphic, we see, first, that the component has an inversion of its value, from positive to negative, and second, we observe an intersection of the curves at a different position of the magnetic axis.

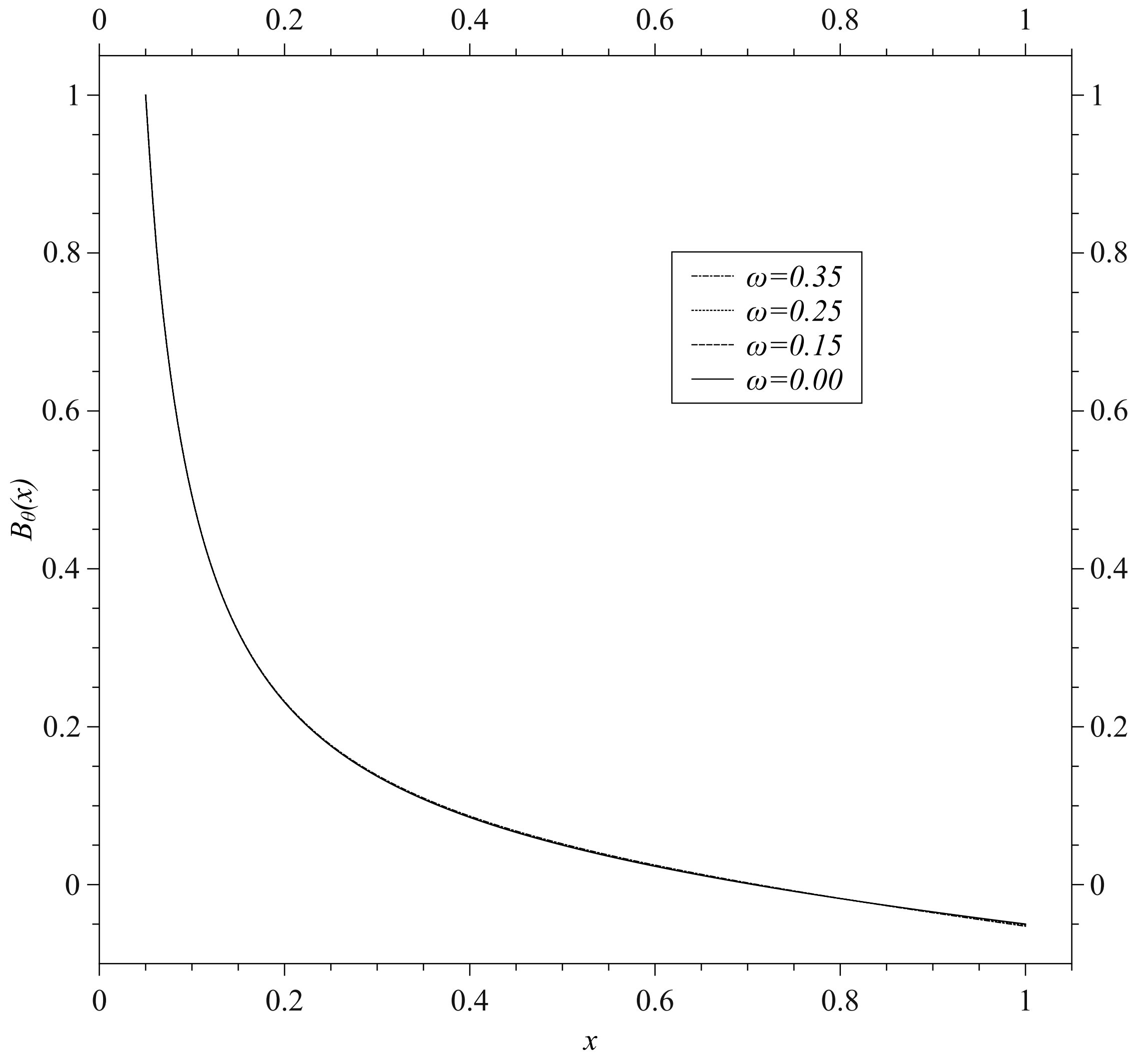


Figure 21: Plot in the equatorial plane of the θ component of the magnetic field B for ω values of 0 (no rotation), 0.15, 0.25 and 0.35

The plot for the poloidal flux current computed through the equation

$$I_(x,\theta) = \sqrt{I_0^2 + \frac{2\lambda^2}{\mu_0^2}\Psi(x,\theta)}, \tag{4.97}$$

and evaluated in the equatorial plane gives

$$I_(x,\pi/2) = \sqrt{I_0^2 + \frac{2\lambda^2}{\mu_0^2}\left[-\left(\mathcal{A}+\frac{4\mathcal{B}}{5}\right)\frac{f_1^{\pm}(x)}{f_1^{\pm}(1)} - \frac{\mathcal{B}}{5}\frac{f_2^{\pm}(x)}{f_2^{\pm}(1)} + \mathcal{A}x^2 + \mathcal{B}x^4\right]}, \tag{4.98}$$

and is shown in figure (23) where it shows the strong rotational effects located at the magnetic axis, the poloidal flux current starts with its initial value, then changes because of the rotation and then returns to its initial value at the end in $x = 1.00$.

There is no changes for the current density J in the φ component, because of the rotation,

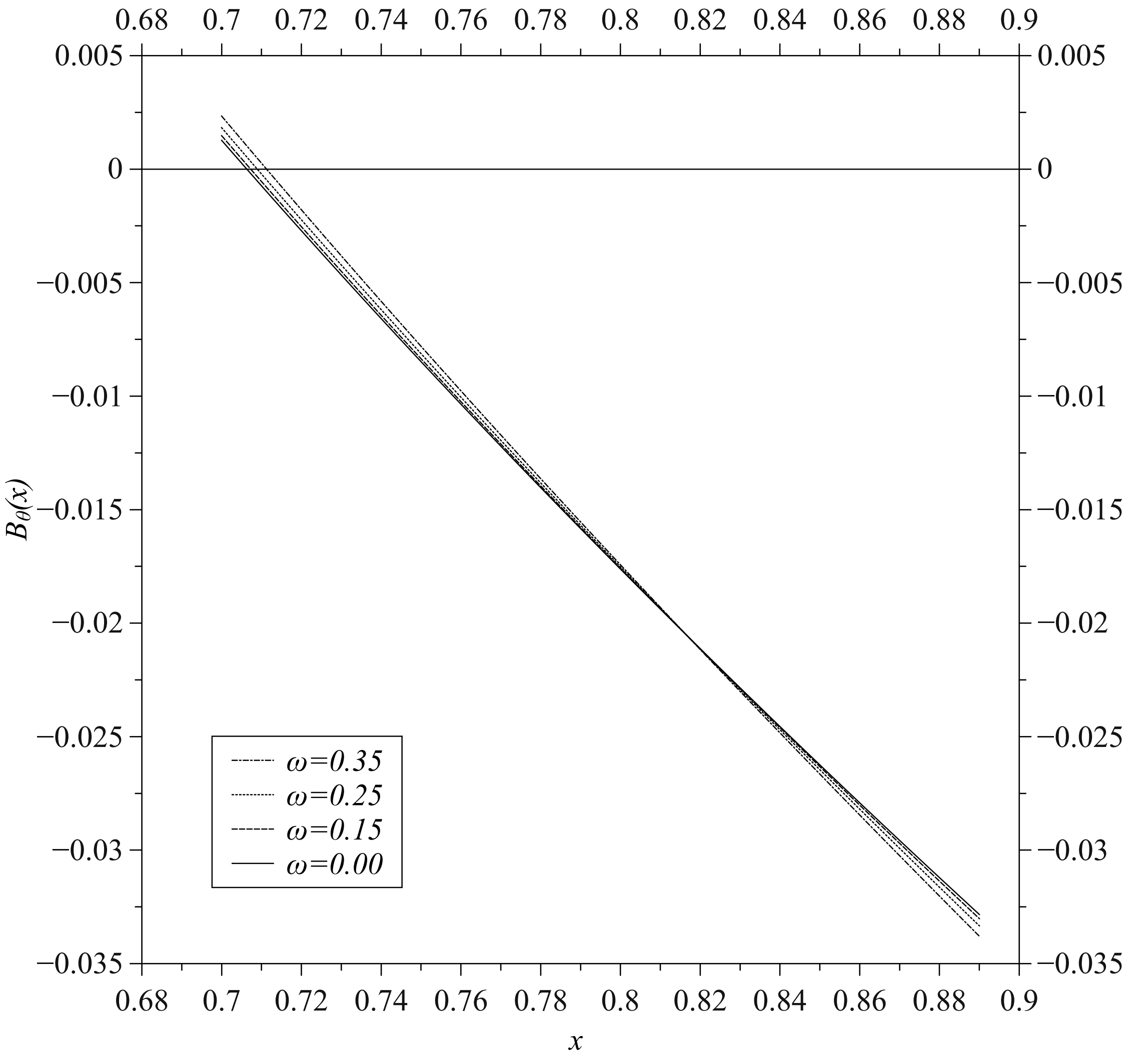


Figure 22: Plot in the equatorial plane of the θ component of the magnetic field B only for ω values of 0 (no rotation) and 0.35 (rotation)

the plotted figure (24), shows no sign of rotational effects. The plot was made through the expression

$$J_\phi(x,\theta) = \frac{\kappa}{\mu_0} x \sin\theta + \frac{2\lambda^2}{\mu_0^2}\frac{\Psi(x,\theta)}{x\sin\theta}, \tag{4.99}$$

then evaluating it in the equatorial plane it simplifies to

$$J_\phi(x,\pi/2) = \frac{\kappa}{\mu_0} x + \frac{2\lambda^2}{\mu_0^2}\frac{1}{x}\left[-\left(\mathcal{A}+\frac{4\mathcal{B}}{5}\right)\frac{f_1^\pm(x)}{f_1^\pm(1)} - \frac{\mathcal{B}}{5}\frac{f_2^\pm(x)}{f_2^\pm(1)} + \mathcal{A}x^2 + \mathcal{B}x^4\right]. \tag{4.100}$$

J_φ starts with an initial value of zero, then decreases until it reaches a minimum at the end in $x = 1.00$

The r component does not experiment any changes in the current density component, that is,

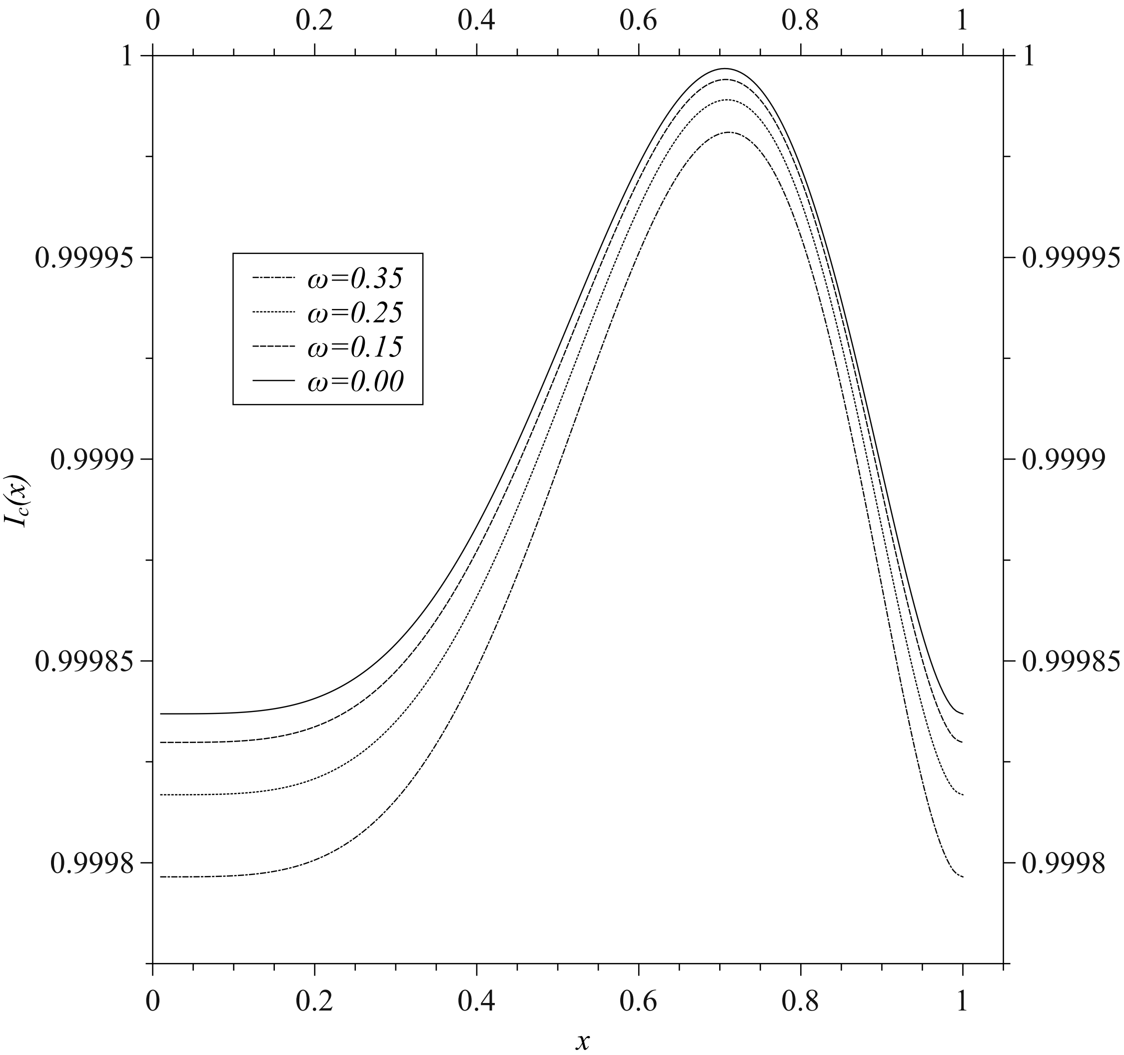


Figure 23: Plot in the equatorial plane of the poloidal flux current for ω values of 0 (no rotation), 0.15, 0.25 and 0.35

$J_r = 0$. The figure (25) represents the θ component of the current density J with

$$J_\theta(x,\pi/2) = -\frac{2\lambda^2}{\mu_0^2 x}[\mathcal{A} + \mathcal{B}x^4 + \mathcal{C}f_1^\pm(x) - \frac{1}{4}\mathcal{D}f_2^\pm(x)]\times \\ \times \frac{[2\mathcal{A}x + 4\mathcal{B}x^3 + \mathcal{C}f_1^\pm(x) - \frac{1}{4}\mathcal{D}f_2^\pm(x)]}{\sqrt{I_0^2 + \frac{2\lambda^2}{\mu_0^2}\left[\mathcal{A}x^2 + \mathcal{B}x^4 + \mathcal{C}f_1^+(x) - \frac{1}{4}\mathcal{D}f_2^+(x)\right]^2}}. \tag{4.101}$$

This is also an interesting plot, because B_θ has one positive value starting from a zero value, then exhibits rotational effects with a maximum at about half the magnetic axis distance, the rotational effect remains until the magnetic axis where it reaches again the zero value and after the magnetic axis it changes to a negative value that is also affected by the rotation, with a minimum between the magnetic axis position and the end, finally at the end reaches the initial value of zero.

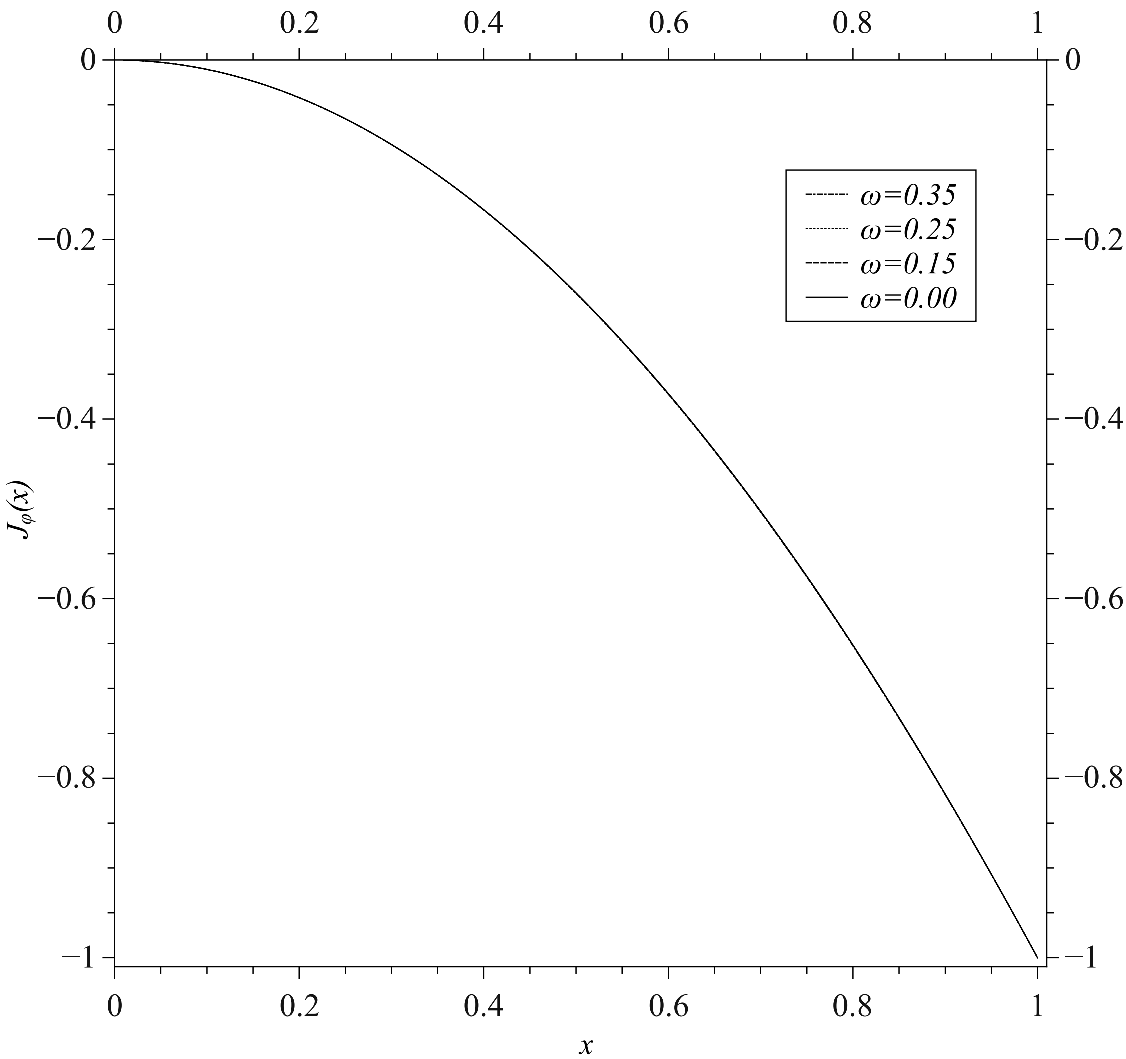


Figure 24: Plot in the equatorial plane of the ϕ component for the current density for ω values of 0 (no rotation), 0.15, 0.25 and 0.35

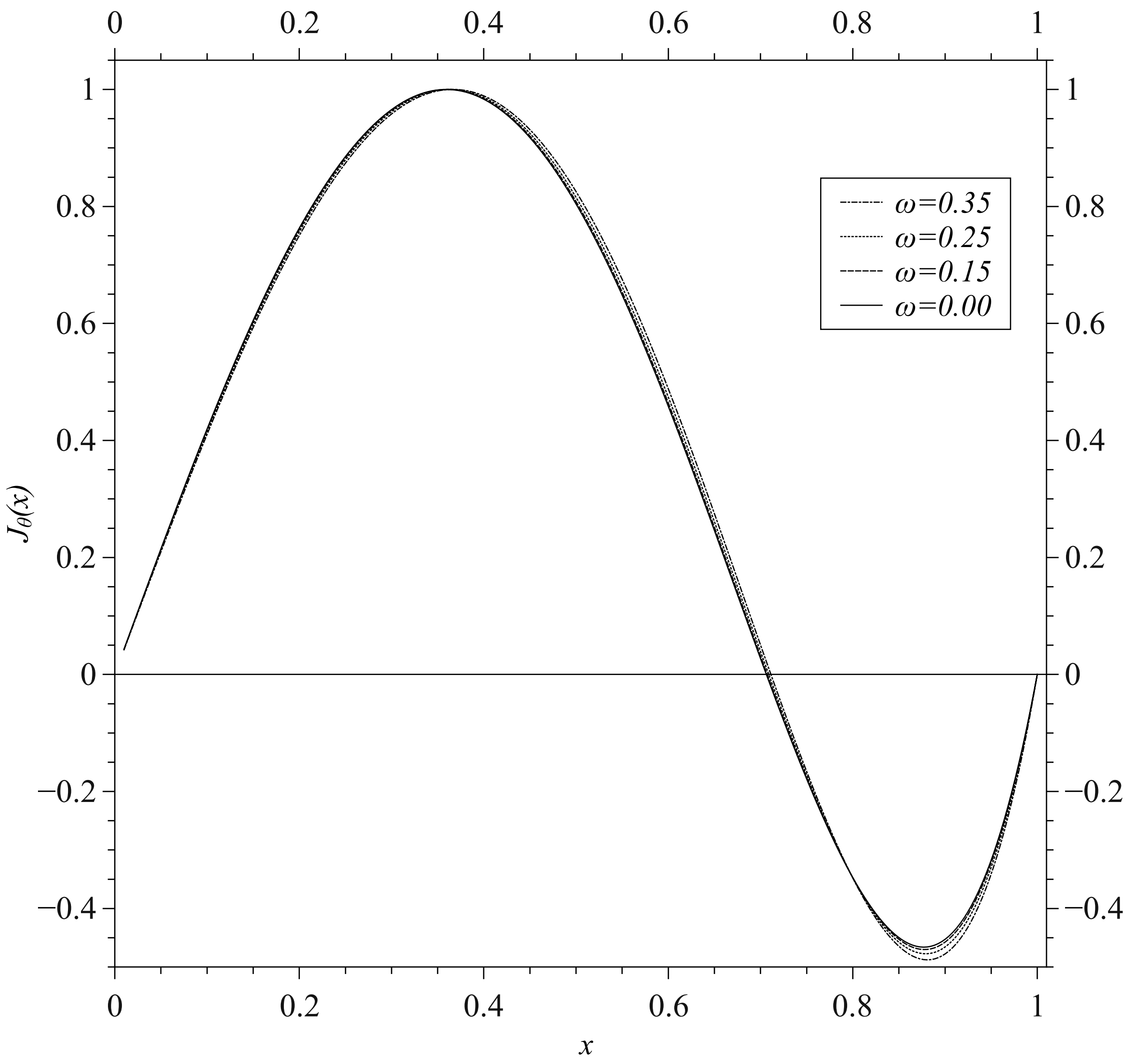


Figure 25: Plot in the equatorial plane of the θ component for the current density for ω values of 0 (no rotation), 0.15, 0.25 and 0.35

5 Conclusions

Spherically symmetric MHD adiabatic equilibria with azimuthal rotation find some applications in fusion and astrophysical problems. We have obtained a pressure equilibrium equation through a procedure introduces by Maschke and Perrin (1980) for this kind of geometry.

This equation was solved through choosing of pressure and current profiles linear in the magnetic flux, supposing an adiabatic system and the assumption of small rotation velocities. The solution of the rotating equilibrium equation was written in a closed form, dependent on the current λ and pressure κ parameters. The solutions obtained are valid for paramagnetic and diamagnetic plasmas.

The configuration studied in this work consists of a plasmoid contained in a spherical conducting shell. In the limit of no rotation the solution reduces to results obtained by Morikawa in 1969. For angular velocities different of zero we show the behavior of λ^2 and κ^2 with the rotation parameter, also the behavior of magnetic flux and pressure surfaces, as well as the behavior of the components of the current densities, the magnetic fields and the current. We have restricted our analysis to an interval for λR whose limits are the first two non-negative eigenvalues of $\tan(\lambda R) = \lambda R$. These two limiting cases are a rotating field-reversed configuration and a force-free configuration, respectively.

We are extending this solution to the configurations proposed, in the static case, by Morikawa (1969) and Morikawa and Rebhan (1970). The former includes a force-free shell between the plasma sphere and the conducting shell. The resulting boundary conditions limit the possible values for λ and κ^2.

An observation of our solution is that the magnetic axis position, located at the equatorial plane, is shifted outwards as the rotation velocity increases, and this effect is more evident for lower values of λR. Considering the value at the magnetic axis to be a constant parameter, we observe a decrease in the pressure parameter κ^2 when rotation is increased, this effect is also more pronounced for small values of λR.

The magnetic surface of the plasma exhibits changes because of the rotation, the effect being greater surrounding the magnetic axis than at the ends.

The components of B are all affected by the rotation except for the B_r that suffers no changes and remains constant. In B_θ there are no rotational effects at the magnetic axis, but at other distances. In B_φ the rotational effect is more evident at the magnetic axis.

The poloidal flux current exhibits rotational effects being greater at the magnetic axis.

The components of J are affected by the rotation, but J_θ which exhibits greater rotational effects located at two different positions. J_φ, which is variable through the distance does not experiment rotational changes and J_r remains constant with a value equal to zero.

At the moment of writing, this is the second solution known of the Grad-Shafranov equation in spherical coordinates by considering the plasma as an adiabatic system with the entropy as a surface quantity.

A Curvilinear coordinate systems

The MHD equations presented in this work used arbitrary coordinates. For that reason, we used curvilinear coordinates, for which the basic definitions and properties are the objectives of this appendix.

This appendix address two commonly curvilinear coordinates systems, cartesian (for clarity) and spherical, both are orthogonal and found in MHD equilibrium configurations.

A.1 Coordinates and base vectors

The position vector can be written as a function of the contravariant coordinates (x^1, x^2, x^3) as

$$\boldsymbol{r} = \boldsymbol{r}(x^1, x^2, x^3), \tag{A.1}$$

a coordinate surface x^i is defined by the condition

$$x^i(\boldsymbol{r}) = c^i = const., \qquad (i = 1, 2, 3). \tag{A.2}$$

Defining covariant base vectors $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ as

$$\hat{\boldsymbol{e}}_i = \frac{\partial \boldsymbol{r}}{\partial x^i}, \quad (i = 1, 2, 3), \tag{A.3}$$

such that the base vectors $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ are tangent to the coordinate surface $x^1 = const.$ at point $\boldsymbol{r}$.

Also defining contravariant base vectors $(\hat{\boldsymbol{e}}^1, \hat{\boldsymbol{e}}^2, \hat{\boldsymbol{e}}^3)$ by

$$\hat{\boldsymbol{e}}^i = \nabla x^i, \quad (i = 1, 2, 3), \tag{A.4}$$

such that the base vector $\hat{\boldsymbol{e}}^i$ is perpendicular to the coordinate surface $x^i = const.$. Results that $\hat{\boldsymbol{e}}^1$ is perpendicular to the vectors $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$, etc.

In general, the covariant base vector $\hat{\boldsymbol{e}}_i$ is tangent to the intersection curve of the coordinate surfaces $x^j = const.$ and $x^k = const.$, with $i \neq j \neq k$, and (i, j, k) are in a cyclical permutation (even) of indices $(1, 2, 3)$. Therefore $\hat{\boldsymbol{e}}_i$ is perpendicular to vectors $\hat{\boldsymbol{e}}^j$ and $\hat{\boldsymbol{e}}^k$, so that we can write the following relations between the co- and contravariant base vectors:

$$\hat{\boldsymbol{e}}_1 = c_1 \hat{\boldsymbol{e}}^2 \times \hat{\boldsymbol{e}}^3, \tag{A.5}$$

$$\hat{\boldsymbol{e}}_2 = c_2 \hat{\boldsymbol{e}}^3 \times \hat{\boldsymbol{e}}^1, \tag{A.6}$$

$$\hat{\boldsymbol{e}}_3 = c_3 \hat{\boldsymbol{e}}^1 \times \hat{\boldsymbol{e}}^2, \tag{A.7}$$

where c_1, c_2 e c_3 are coefficients to be determined. Generalizing we have

$$\hat{\boldsymbol{e}}_i = c_i \hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^k. \tag{A.8}$$

Similarly, writing relations for the contravariant base vectors as function of the covariant ones, with coefficients to be determined:

$$\hat{\boldsymbol{e}}^1 = c^1 \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3, \tag{A.9}$$
$$\hat{\boldsymbol{e}}^2 = c^2 \hat{\boldsymbol{e}}_3 \times \hat{\boldsymbol{e}}_1, \tag{A.10}$$
$$\hat{\boldsymbol{e}}^3 = c^3 \hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2, \tag{A.11}$$

or simply

$$\hat{\boldsymbol{e}}^i = c^i \hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k. \tag{A.12}$$

The contravariant base vectors help us to write the gradient of a scalar function $\Phi(\boldsymbol{r})$:

$$\nabla\Phi = \frac{\partial\Phi}{\partial x^i}\nabla x^i = \frac{\partial\Phi}{\partial x^i}\hat{\boldsymbol{e}}^i. \tag{A.13}$$

Noting that the use of the sum convention for curvilinear coordinates involves a that: one of the two summed indices must be contravariant (up) and the other one covariant (down), or vice-versa. Consistent with this convention, the partial derivative $\partial\Phi/\partial x^i$ acts as if it had a covariant index, then

$$\partial_i\Phi = \frac{\partial\Phi}{\partial x^i}, \tag{A.14}$$

such that

$$\nabla\Phi = \partial_i\Phi\hat{\boldsymbol{e}}^i.$$

Combining (A.3) e (A.13)

$$d\Phi = \nabla\Phi\cdot\boldsymbol{dl} = \frac{\partial\Phi}{\partial x^i}dx^j\hat{\boldsymbol{e}}^i\cdot\hat{\boldsymbol{e}}_j. \tag{A.15}$$

Comparing with the formula of total differential

$$d\Phi = \frac{\partial\Phi}{\partial x^i}dx^i, \tag{A.16}$$

results that

$$dx^i = dx^j\hat{\boldsymbol{e}}^i\cdot\hat{\boldsymbol{e}}_j,$$

where we obtain the orthonormality relation:

$$\hat{\boldsymbol{e}}^i\cdot\hat{\boldsymbol{e}}_j = \delta^i_j, \tag{A.17}$$

where δ^i_j is the Kronecker delta.

Substituting (A.8) and (A.12) in (A.17) we have, in the case $i \neq j$, that

$$c^i = \frac{1}{\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k}, \tag{A.18}$$
$$c_i = \frac{1}{\hat{\boldsymbol{e}}^i \cdot \hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^k}. \tag{A.19}$$

The jacobian of the transformation (x, y, z) to (x^1, x^2, x^3) is denoted by $\sqrt{g}$:

$$\sqrt{g} = \frac{\partial(x, y, z)}{\partial(x^1, x^2, x^3)} = \begin{vmatrix} \frac{\partial x}{\partial x^1} & \frac{\partial x}{\partial x^2} & \frac{\partial x}{\partial x^3} \\ \frac{\partial y}{\partial x^1} & \frac{\partial y}{\partial x^2} & \frac{\partial y}{\partial x^3} \\ \frac{\partial z}{\partial x^1} & \frac{\partial z}{\partial x^2} & \frac{\partial z}{\partial x^3} \end{vmatrix} \tag{A.20}$$

Using the expression for the mixed cross product we obtain

$$\sqrt{g} = \frac{\partial \boldsymbol{r}}{\partial x^1} \cdot \frac{\partial \boldsymbol{r}}{\partial x^2} \times \frac{\partial \boldsymbol{r}}{\partial x^3} = \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3, \tag{A.21}$$

that, because of the cyclical property of the mixed product, lead us to

$$c^1 = c^2 = c^3 = \frac{1}{\sqrt{g}}, \tag{A.22}$$

Now, let be the inverse transformation:

$$x^1 = x^1(x, y, z), x^2 = x^2(x, y, z), x^3 = x^3(x, y, z),$$

whose jacobian is, by analogy,

$$\frac{\partial(x^1, x^2, x^3)}{\partial(x, y, z)} = \nabla x^1 \cdot \nabla x^2 \cdot \nabla x^3 = \hat{\boldsymbol{e}}^1 \cdot \hat{\boldsymbol{e}}^2 \times \hat{\boldsymbol{e}}^3.$$

As this jacobian is the inverse of the previous one, we will have

$$\frac{1}{\sqrt{g}} = \hat{\boldsymbol{e}}^1 \cdot \hat{\boldsymbol{e}}^2 \times \hat{\boldsymbol{e}}^3, \tag{A.23}$$

and, furthermore,

$$c_1 = c_2 = c_3 = \sqrt{g}. \tag{A.24}$$

From (A.22) and (A.24), the relations (A.8) and (A.12) provide

$$\hat{\boldsymbol{e}}_i = \sqrt{g}\hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^k, \tag{A.25}$$
$$\hat{\boldsymbol{e}}^i = \frac{1}{\sqrt{g}}\hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k, \tag{A.26}$$

where (i, j, k) are in a cyclical permutation of $(1, 2, 3)$.

A.2 Line elements, area and volume

The vectorial line element is written as

$$d\boldsymbol{l} = \frac{\partial \boldsymbol{r}}{\partial x^i} dx^i = dx^i \hat{\boldsymbol{e}}_i. \tag{A.27}$$

Be, now, a coordinate surface $x^i = const.$. Two line elements parallel to that surface in a given point are

$$d\boldsymbol{l}_1 = dx^j \hat{\boldsymbol{e}}_j \tag{A.28}$$
$$d\boldsymbol{l}_2 = dx^k \hat{\boldsymbol{e}}_k. \tag{A.29}$$

The vectorial area element is given by

$$d\boldsymbol{S} = d\boldsymbol{l}_1 \times d\boldsymbol{l}_2 = dx^j dx^k \hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k.$$

Using (A.26) we have

$$d\boldsymbol{S}^{(i)} = \sqrt{g} dx^j dx^k \hat{\boldsymbol{e}}^i, \tag{A.30}$$

where the superindex (i) indicates that $d\boldsymbol{S}$ points in the direction perpendicular to the surface coordinate $x^i = const.$: $d\boldsymbol{S}^{(i)} = dS\hat{\boldsymbol{e}}^i$.

The volume element in cartesian coordinates is $dV = dxdydz$. In the coordinate transformation the jacobian of the transformation is given by (A.20), so

$$dxdydz = \sqrt{g} dx_1 dx_2 dx_3.$$

Then, the volume element in curvilinear coordinates is

$$dV = \sqrt{g} dx^1 dx^2 dx^3. \tag{A.31}$$

A.3 The metric tensor

The square of the line element can be written in a quadratic way in contravariant coordinates of the differential, namely, the double sum

$$d\ell^2 = g_{ij} dx^i dx^j, \tag{A.32}$$

where g_{ij} are the elements of the covariant metric tensor. Using (A.27) we have

$$d\ell^2 = d\boldsymbol{l} \cdot d\boldsymbol{l} = dx^i d^j \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j \tag{A.33}$$

so that

$$g_{ij} = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j, \tag{A.34}$$

as $g_{ji} = g_{ij}$ the covariant metric tensor is symmetric.

The square modulus of the gradient of a scalar function may also be expressed as a quadratic form of partial derivatives, as

$$(\nabla\Phi)^2 = g^{ij}\partial_i\Phi\partial_j\Phi, \tag{A.35}$$

where we used the shorthand notation (A.14), e g^{ij} are the elements of the contravariant metric tensor. Using (A.13) follows that

$$g^{ij} = \hat{\boldsymbol{e}}^i \cdot \hat{\boldsymbol{e}}^j. \tag{A.36}$$

The contravariant metric tensor is the inverse of the covariant metric tensor, because the product of both is equal to the identity tensor

$$g^{ik}g_{kj} = \delta^i_j. \tag{A.37}$$

To show that relation, we use the respective definition (A.34) and (A.36) as well as the following property of tensorial calculus

$$(\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}) = \boldsymbol{c}\cdot(\boldsymbol{a}\otimes\boldsymbol{b})\cdot\boldsymbol{d} = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}).$$

Taking the square of (A.21) and using the determinant multiplication rule

$$\begin{aligned} g &= (\sqrt{g})(\sqrt{g}) = (\hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}_2\times\hat{\boldsymbol{e}}_3)(\hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}_2\times\hat{\boldsymbol{e}}_3) && \text{(A.38)}\\ &= \begin{vmatrix} \hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}_3 \\ \hat{\boldsymbol{e}}_2\cdot\hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_2\cdot\hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_2\cdot\hat{\boldsymbol{e}}_3 \\ \hat{\boldsymbol{e}}_3\cdot\hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_3\cdot\hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_3\cdot\hat{\boldsymbol{e}}_3 \end{vmatrix} && \text{(A.39)}\\ &= \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}, && \text{(A.40)} \end{aligned}$$

and therefore,

$$g = \det(g_{ij}). \tag{A.41}$$

Doing the same, now, for (A.23) we obtain

$$g = \frac{1}{\det(g^{ij})}. \tag{A.42}$$

A curvilinear coordinate system is said orthogonal if their metric tensor is diagonal, that is, the elements outside the diagonal are zero: $g_{ij} = 0$ if $i \neq j$. For them

$$g = g_{11}g_{22}g_{33} = \frac{1}{g^{11}g^{22}g^{33}}, \tag{A.43}$$

where

$$g_{11} = \frac{1}{g^{11}}, \qquad g_{22} = \frac{1}{g^{22}}, \qquad g_{33} = \frac{1}{g^{33}}. \tag{A.44}$$

In literature, the diagonal elements of the metric tensor of an orthogonal system are also called metric coefficients:

$$h_i = g_{ii}, \qquad h^i = g^{ii} = \frac{1}{g_{ii}} = \frac{1}{h_i}. \tag{A.45}$$

A.4 Vector components

A vector-valued of curvilinear coordinates can be expressed both in terms of their contravariant and covariant basis vectors:

$$\boldsymbol{A} = A^i\hat{\boldsymbol{e}}_i = A_i\hat{\boldsymbol{e}}^i, \tag{A.46}$$

where the contravariant and covariant componentes are given, respectively, by

$$A^i = \boldsymbol{A}\cdot\hat{\boldsymbol{e}}^i, \qquad A_i = \boldsymbol{A}\cdot\hat{\boldsymbol{e}}_i. \tag{A.47}$$

The scalar product (or inner product) of two vectors, using the relation of orthonormality (A.17) is

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_iB^i = A^iB_i, \tag{A.48}$$

and the square module of a vector

$$A^2 = \boldsymbol{A}\cdot\boldsymbol{A} = A_iA^i = A^iA_i. \tag{A.49}$$

The vector cross product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is obtained using (A.47), (A.25) and (A.26):

$$\boldsymbol{A}\times\boldsymbol{B} = \frac{1}{\sqrt{g}}\left[(A_2B_3 - A_3B_2)\hat{\boldsymbol{e}}_1 + (A_3B_1 - A_1B_3)\hat{\boldsymbol{e}}_2 + (A_1B_2 - A_2B_1)\hat{\boldsymbol{e}}_3\right]. \tag{A.50}$$

The i-th contravariant component of the cross product is

$$(\boldsymbol{A}\times\boldsymbol{B})^i = \frac{1}{\sqrt{g}}(A_jB_k - A_kB_j), \tag{A.51}$$

where (i, j, k) are in a cyclic permutation of the indices $(1, 2, 3)$.

A shorthand notation for this expression draw on the Levi-Civita symbol, defined as

$$\varepsilon_{ijk} = \varepsilon^{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ are in an even permutation of índices } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ are in an odd permutation of indices } (1,2,3), \\ 0 & \text{if there are repeated indices,} \end{cases} \tag{A.52}$$

so that,

$$\boldsymbol{A} \times \boldsymbol{B} = \frac{1}{\sqrt{g}} \varepsilon^{ijk} A_j B_k \hat{\boldsymbol{e}}_i = \sqrt{g} \varepsilon_{ijk} A^j B^k \hat{\boldsymbol{e}}^i. \tag{A.53}$$

The contravariant and covariant metric tensors are useful to raise or lowering indices of vectorial components. For example

$$A^i = \boldsymbol{A} \cdot \hat{\boldsymbol{e}}^i = A_j \hat{\boldsymbol{e}}^j \cdot \hat{\boldsymbol{e}}^i = A_j g^{ji}$$

As the metric tensors are symmetric,

$$A^i = g^{ij} A_j, \tag{A.54}$$

allowing to lowering the index. Similarly, is possible to raise the indices by

$$A_i = g_{ij} A^j. \tag{A.55}$$

In addition, we can use these expressions to raise or lower indices of covariant and contravariant base vectors. For example, the following cross products of basis vectors will be useful

$$\hat{\boldsymbol{e}}^1 \times \hat{\boldsymbol{e}}_3 = \tfrac{1}{\sqrt{g}} \left(-g_{33} \hat{\boldsymbol{e}}_2 + g_{32} \hat{\boldsymbol{e}}_3 \right) \tag{A.56}$$

$$\hat{\boldsymbol{e}}^2 \times \hat{\boldsymbol{e}}_3 = \tfrac{1}{\sqrt{g}} \left(g_{33} \hat{\boldsymbol{e}}_1 - g_{31} \hat{\boldsymbol{e}}_3 \right) \tag{A.57}$$

$$\hat{\boldsymbol{e}}^3 \times \hat{\boldsymbol{e}}_3 = \tfrac{1}{\sqrt{g}} \left(-g_{32} \hat{\boldsymbol{e}}_1 + g_{31} \hat{\boldsymbol{e}}_2 \right). \tag{A.58}$$

The squared modulus of a vector is a quadratic form of their contravariant or covariant components

$$|\boldsymbol{A}|^2 = \boldsymbol{A} \cdot \boldsymbol{A} = g_{ij} A^i A^j = g^{ij} A_i A_j. \tag{A.59}$$

In an orthogonal coordinate system these sums include only the diagonal elements, so that

$$|\boldsymbol{A}|^2 = g_{11} A^1 A^1 + g_{22} A^2 A^2 + g_{33} A^3 A^3 = g^{11} A_1 A_1 + g^{22} A_2 A_2 + g^{33} A_3 A_3. \tag{A.60}$$

Remembering that the vector $\boldsymbol{A}$ will commonly represent some physical quantity. In general, due to the factor $\sqrt{g}$, the dimensions of the contravariant and covariant components will not be the same as the vector itself. To remedy this problem we have defined only in orthogonal systems, the so called physical components

$$A_{<i>} = \sqrt{g_{ii}} A^i = \sqrt{g^{ii}} A_i, \tag{A.61}$$

where the index i, although repeated, does not sum. So we have

$$\begin{aligned} |\boldsymbol{A}|^2 &= (\sqrt{g_{11}} A^1)(\sqrt{g_{11}} A^1) + (\sqrt{g_{22}} A^2)(\sqrt{g_{22}} A^2) + (\sqrt{g_{33}} A^3)(\sqrt{g_{33}} A^3) & \text{(A.62)} \\ &= A^2_{<1>} + A^2_{<2>} + A^2_{<3>}. & \text{(A.63)} \end{aligned}$$

In general, the contravariant and covariant basis vectors are not mutually orthogonal nor have unit module, unlike the cartesian vectors $(\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z)$. To remedy this problem, we introduced also orthonormal basis vectors, defined as

$$\hat{\boldsymbol{e}}_{<i>} = \sqrt{g_{ii}} \hat{\boldsymbol{e}}^i = \sqrt{g^{ii}} \hat{\boldsymbol{e}}_i, \tag{A.64}$$

where again the index i does not sum despite being repeated. We can show the orthonormality relation for them:

$$\hat{\boldsymbol{e}}_{<i>} \cdot \hat{\boldsymbol{e}}_{<j>} = \delta_{ij}. \tag{A.65}$$

With these basis vectors and physical components an arbitrary vector is written as

$$\boldsymbol{A} = A_{<i>} \hat{\boldsymbol{e}}_{<i>}, \tag{A.66}$$

with summation over i.

A.5 Vector differential operators

The gradient of a scalar function $\Phi(\boldsymbol{r})$ has been defined in (A.13). Sometimes it is helpful to write it in terms of orthonormal basis vectors (A.64):

$$\nabla \Phi = \frac{\partial \Phi}{\partial x^i} \frac{1}{\sqrt{g_{ii}}} \hat{\boldsymbol{e}}_{<i>}. \tag{A.67}$$

The divergent of a vector function $\boldsymbol{A}(\boldsymbol{r})$ is

$$\nabla \cdot \boldsymbol{A} = \nabla \cdot (A^i \hat{\boldsymbol{e}}_i) = \hat{\boldsymbol{e}}_i \cdot \nabla A^i + A^i (\nabla \cdot \hat{\boldsymbol{e}}_i). \tag{A.68}$$

Using (A.13) and (A.17)

$$\hat{\boldsymbol{e}}_i \cdot \nabla A^i = \partial_j A^i \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}^j = \partial_j A^i \delta_i^j = \partial_i A_i. \tag{A.69}$$

From (A.25) we obtain

$$\nabla \cdot \hat{\boldsymbol{e}}_i = \nabla \cdot \left(\sqrt{g}\hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^k\right) \tag{A.70}$$

$$= (\hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^k) \cdot \nabla\sqrt{g} + \sqrt{g}\nabla \cdot \left(\hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^k\right) \tag{A.71}$$

$$= \left(\hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^k\right) \cdot \partial_i\sqrt{g}\hat{\boldsymbol{e}}^i, \tag{A.72}$$

where we used the identity

$$\nabla \cdot (\hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^k) = \nabla \cdot (\nabla x^j \times \nabla x^i) = 0.$$

Then

$$\nabla \cdot \hat{\boldsymbol{e}}_i = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i}, \tag{A.73}$$

which, when inserted into (A.68) provides

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A^i}{\partial x^i} + \frac{1}{\sqrt{g}}A^i\frac{\partial\sqrt{g}}{\partial x^i} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^i\right). \tag{A.74}$$

The Laplacian of a scalar function, according to (A.74), is given by

$$\nabla^2\Phi = \nabla \cdot \nabla\Phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}(\nabla\Phi)^i\right), \tag{A.75}$$

where the contravariant components of the gradient are

$$(\nabla\Phi)^i = \nabla\Phi \cdot \hat{\boldsymbol{e}}_i = \partial_j\Phi\hat{\boldsymbol{e}}^j \cdot \hat{\boldsymbol{e}}^i = g^{ij}\partial_j\Phi$$

hence,

$$\nabla^2\Phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial\Phi}{\partial x^j}\right). \tag{A.76}$$

The rotational of a vector function is

$$\begin{aligned}
\nabla \times \boldsymbol{A} =& \nabla \times (A_i\hat{\boldsymbol{e}}^i) \\
=& A_i\nabla \times \hat{\boldsymbol{e}}^i + (\nabla A^i) \times \hat{\boldsymbol{e}}^k \\
=& A_i\underbrace{\nabla \times \nabla x^i}_{=0} + \partial_j A^i\hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^i \\
=& \partial_j A^i\hat{\boldsymbol{e}}^j \times \hat{\boldsymbol{e}}^i,
\end{aligned} \tag{A.77}$$

where we used (A.4). Expanding the above double sum and using (A.25) we collect the similar terms to write

$$\nabla \times \boldsymbol{A} = \frac{1}{\sqrt{g}}(\partial_2A^3 - \partial_3A^2)\hat{\boldsymbol{e}}_1 + \frac{1}{\sqrt{g}}(\partial_3A^1 - \partial_1A^3)\hat{\boldsymbol{e}}_2 + \frac{1}{\sqrt{g}}(\partial_1A^2 - \partial_2A^1)\hat{\boldsymbol{e}}_3, \tag{A.78}$$

which can be rewritten using the Levi-Civita symbol as

$$\nabla \times \boldsymbol{A} = \frac{1}{\sqrt{g}} \varepsilon^{ijk} \partial_i A_j \hat{\boldsymbol{e}}_k, \tag{A.79}$$

A.6 Cartesian or rectangular coordinates

1. Contravariant coordinates

$$x^1 = x, \qquad x^2 = y, \qquad x^3 = z. \tag{A.80}$$

2. Surface coordinates

   - $x^1 = const.$: perpendicular planes to the x axis;
   - $x^2 = const.$: perpendicular planes to the y axis;
   - $x^3 = const.$: perpendicular planes to the z axis

3. Contravariant base vectors

$$\hat{\boldsymbol{e}}_1 = \frac{\partial \boldsymbol{r}}{\partial x^1} = \hat{\boldsymbol{e}}_x \tag{A.81}$$

$$\hat{\boldsymbol{e}}_2 = \frac{\partial \boldsymbol{r}}{\partial x^2} = \hat{\boldsymbol{e}}_y \tag{A.82}$$

$$\hat{\boldsymbol{e}}_3 = \frac{\partial \boldsymbol{r}}{\partial x^3} = \hat{\boldsymbol{e}}_z \tag{A.83}$$

4. Covariant metric tensor: $g_{ij} = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}$

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.84}$$

$$g = \det g_{ij} = 1 \tag{A.85}$$

5. Contravariant base vectors

$$\hat{\boldsymbol{e}}^1 = \nabla x^1 = \frac{\partial x}{\partial x} \hat{\boldsymbol{e}}_x = \hat{\boldsymbol{e}}_x \tag{A.86}$$

$$\hat{\boldsymbol{e}}^2 = \nabla x^2 = \frac{\partial y}{\partial y} \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_y \tag{A.87}$$

$$\hat{\boldsymbol{e}}^3 = \nabla x^3 = \frac{\partial z}{\partial z} \hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_z \tag{A.88}$$

6. Contravariant metric tensor: $g^{ij} = \hat{\boldsymbol{e}}^i \cdot \hat{\boldsymbol{e}}^j = \delta^{ij}$

$$(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.89}$$

7. Physical components

$$A_{<1>} = A^1 = A_x, \tag{A.90}$$
$$A_{<2>} = A^2 = A_y, \tag{A.91}$$
$$A_{<3>} = A^3 = A_z, \tag{A.92}$$

8. Orthonormal basis vectors

$$\hat{\boldsymbol{e}}_{<1>} = \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_x, \tag{A.93}$$
$$\hat{\boldsymbol{e}}_{<2>} = \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_y, \tag{A.94}$$
$$\hat{\boldsymbol{e}}_{<3>} = \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_z, \tag{A.95}$$

9. Gradient

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{\boldsymbol{e}}_x + \frac{\partial\Phi}{\partial y}\hat{\boldsymbol{e}}_y + \frac{\partial\Phi}{\partial z}\hat{\boldsymbol{e}}_z. \tag{A.96}$$

10. Divergent

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{A.97}$$

11. Laplacian

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}. \tag{A.98}$$

12. Rotational

$$\nabla \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{e}}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{e}}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{e}}_z \tag{A.99}$$

13. Differential operator

$$\begin{aligned}(\boldsymbol{A} \cdot \nabla)\boldsymbol{B} = {} & \left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right)\hat{\boldsymbol{e}}_x + \\ & \left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right)\hat{\boldsymbol{e}}_y + \\ & \left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)\hat{\boldsymbol{e}}_z\end{aligned} \tag{A.100}$$

A.7 Spherical coordinates

1. Contravariant coordinates

$$x^1 = r, \qquad (0 \le R \le a) \tag{A.101}$$
$$x^2 = \theta, \qquad (0 \le \theta \le \pi) \tag{A.102}$$
$$x^3 = \phi, \qquad (0 \le \phi < 2\pi) \tag{A.103}$$

2. Coordinate surfaces

   - $x^1 = const.$: spheres of radius r;
   - $x^2 = const.$: cones with vertices at the origin
   - $x^3 = const.$: planes containing the axis z;

3. Relation with the cartesian coordinates

$$x = r\sin\theta\cos\phi \tag{A.104}$$
$$y = r\sin\theta\sin\phi \tag{A.105}$$
$$z = r\cos\theta. \tag{A.106}$$

4. Covariant basis vectors

$$\hat{\boldsymbol{e}}_1 = \sin\theta\cos\phi\hat{\boldsymbol{e}}_x + \sin\theta\sin\phi\hat{\boldsymbol{e}}_y + \cos\theta\hat{\boldsymbol{e}}_z \tag{A.107}$$
$$\hat{\boldsymbol{e}}_2 = r\cos\theta\cos\phi\hat{\boldsymbol{e}}_x + r\cos\theta\sin\phi\hat{\boldsymbol{e}}_y - r\sin\theta\hat{\boldsymbol{e}}_z \tag{A.108}$$
$$\hat{\boldsymbol{e}}_3 = -r\sin\theta\sin\phi\hat{\boldsymbol{e}}_x + r\sin\theta\cos\phi\hat{\boldsymbol{e}}_y \tag{A.109}$$

5. Covariant metric tensor

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{A.110}$$

$$g = r^4\sin^2\theta, \tag{A.111}$$

6. Contravariant metric tensor

$$(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{A.112}$$

7. Physical components

$$
\begin{aligned}
A_{<1>} &= A_r = A^1, && \text{(A.113)}\\
A_{<2>} &= A_\theta = rA^2, && \text{(A.114)}\\
A_{<3>} &= A_\phi = r\sin\theta A^3, && \text{(A.115)}
\end{aligned}
$$

8. Orthonormal basis vectors

$$
\begin{aligned}
\hat{\boldsymbol{e}}_{<1>} &= \hat{\boldsymbol{e}}_r = \hat{\boldsymbol{e}}_1 && \text{(A.116)}\\
\hat{\boldsymbol{e}}_{<2>} &= \hat{\boldsymbol{e}}_\theta = \frac{1}{r}\hat{\boldsymbol{e}}_2, && \text{(A.117)}\\
\hat{\boldsymbol{e}}_{<3>} &= \hat{\boldsymbol{e}}_\phi = \frac{1}{r\sin\theta}\hat{\boldsymbol{e}}_3, && \text{(A.118)}
\end{aligned}
$$

9. Gradient

$$
\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\hat{\boldsymbol{e}}_\phi. \qquad \text{(A.119)}
$$

10. Divergent

$$
\nabla\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{\partial A_\phi}{\partial\phi}\right] \qquad \text{(A.120)}
$$

11. Laplacian

$$
\begin{aligned}
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + && \text{(A.121)}\\
&\quad \frac{1}{r\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}. && \text{(A.122)}
\end{aligned}
$$

12. Rotational

$$
\begin{aligned}
\nabla\times\boldsymbol{A} = &\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta)A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{e}}_r + && \text{(A.123)}\\
&\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\right]r\hat{\boldsymbol{e}}_\theta + && \text{(A.124)}\\
&\frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{e}}_\theta. && \text{(A.125)}
\end{aligned}
$$

Bibliography

[1] D. J. Acheson and R. Hide. Hydromagnetics of rotating fluids. *Reports on Progress in Physics*, 36(2):159, 1973.

[2] S. Appl and M. Camenzind. The structure of relativistic MHD jets: a solution to the nonlinear Grad-Shafranov equation. *AAP*, 274:699, July 1993.

[3] Marco Ariola and Alfredo Pironti. *Magnetic Control of Tokamak Plasmas*. Springer, Cham, 2016.

[4] Wolfganf Baumjohann and Rudolf A. Treumann. *Basic space plasma physics*. Imperial College Press, London, reprinted edition, 2006.

[5] J. A. Bittencourt. *Fundamentals of Plasma Physics*. Springer New York, New York, NY, 2004.

[6] T. J. M Boyd and J. J Sanderson. *The physics of plasmas*. Cambridge University Press, Cambridge, UK; New York, 2003.

[7] C. M Braams and P. E Stott. *Nuclear fusion half a century of magnetic confinement fusion research.* Institute of Physics Pub, Bristol, 2002.

[8] Francis F. Chen and Francis F. Chen. *Introduction to plasma physics and controlled fusion*. Plenum Press, New York, 2nd ed edition, 1984.

[9] Peter Alan Davidson. *An introduction to magnetohydrodynamics*. Cambridge texts in applied mathematics. Cambridge Univ. Press, Cambridge, 2001.

[10] A. Dinklage, editor. *Plasma physics: confinement, transport and collective effects*. Number 670 in Lecture notes in physics. Springer, Berlin ; New York, 2005.

[11] Peter Foukal. *Solar astrophysics*. Wiley-VCH, Weinheim, 2nd rev. ed edition, 2004.

[12] Jeffrey P Freidberg. *Plasma physics and fusion energy*. Cambridge University Press, Cambridge, 2007.

[13] Alexander A Fridman and Gary Friedman. *Plasma medicine*. John Wiley & Sons, Chichester, West Sussex, UK, 2013.

[14] J. P Goedbloed, Rony Keppens, and Stefaan Poedts. *Advanced magnetohydrodynamics with applications to laboratory and astrophysical plasmas*. Cambridge University Press, New York, 2010.

[15] Marcel Goossens. *An introduction to plasma astrophysics and magnetohydrodynamics*. Kluwer Academic Publishers, Dordrrecht; Boston, 2003.

[16] Harold Grad and Hanan Rubin. Hydromagnetic equilibria and force-free fields. *Journal of Nuclear Energy (1954)*, 7(3):284–285, 1958.

[17] Donald A. Gurnett and A. Bhattacharjee. *Introduction to plasma physics: with space and laboratory applications*. Cambridge University Press, Cambridge, UK ; New York, 2005.

[18] I. H. Hutchinson. *Principles of plasma diagnostics*. Cambridge University Press, Cambridge ; New York, 2nd ed edition, 2002.

[19] Mutsuko Y. Kucinski and Iberê L. Caldas. MHD equilibrium equation in symmetric systems. *arXiv preprint arXiv:1103.5063*, 2011.

[20] E. K. Maschke and H. Perrin. Exact solutions of the stationary MHD equations for a rotating toroidal plasma. *Plasma Physics*, 22(6):579, 1980.

[21] Nicole Meyer-Vernet. *Basics of the solar wind*. Cambridge University Press, Cambridge, 2007.

[22] Mari Paz Miralles and Jorge Sánchez Almeida, editors. *The Sun, the Solar Wind, and the Heliosphere*. Number 4 in IAGA special sopron book series. Springer, Dordrecht, 2011.

[23] Kenro Miyamoto. *Plasma physics and controlled nuclear fusion*. Number 38 in Springer series on atomic, optical, and plasma physics. Springer, Berlin ; New York, 2005.

[24] René Moreau. *Magnetohydrodynamics*, volume 3 of *Fluid Mechanics and Its Applications*. Springer Netherlands, Dordrecht, 1990.

[25] G. K. Morikawa. Double-Toroidal Hydromagnetic-Equilibrium Configurations within a Perfectly Conducting Sphere. *Physics of Fluids*, 12(8):1648, 1969.

[26] G. K. Morikawa and E. Rebhan. Toroidal Hydromagnetic Equilibrium Solutions with Spherical Boundaries. *Physics of Fluids (1958-1988)*, 13(2):497–500, February 1970.

[27] Paul Murdin, editor. *Encyclopedia of astronomy and astrophysics*. Institute of Physics Pub. ; Nature Pub. Group, Bristol ; Philadelphia : London ; New York, 2001.

[28] K Nishikawa and Masahiro Wakatani. *Plasma Physics Basic Theory with Fusion Applications*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2000.

[29] Alexander Piel. Definition of the Plasma State. In *Plasma Physics*, pages 29–43. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[30] E. R. Priest. *Magnetohydrodynamics of the Sun*. Cambridge University Press, New York, NY, 2014.

[31] J. Reece Roth. *Industrial plasma engineering. Vol. 2: Applications to nonthermal plasma processing*. Inst. of Physics Publ, Bristol, 2001.

[32] Dalton D. Schnack. *Lectures in Magnetohydrodynamics*, volume 780 of *Lecture Notes in Physics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[33] V. D. Shafranov. Plasma Equilibrium in a Magnetic Field. *Reviews of Plasma Physics*, 2:103, 1966.

[34] V. D. Shafranov. Closed magnetic plasma traps with a zero angle of rotational conversion. *Soviet Atomic Energy*, 22(5):449–456, 1967.

[35] Weston M. Stacey. *Fusion plasma physics*. Wiley-VCH ; John Wiley [distributor], Weinheim : Chichester, 2005.

[36] Peter A. Sturrock. *Plasma physics: an introduction to the theory of astrophysical, geophysical, and laboratory plasmas*. Cambridge University Press, Cambridge [England] ; New York, 1994.

[37] C. Guy Suits, C. Guy Suits, and Harold E Way. *The collected works of Irving Langmuir. with contributions in memoriam including a complete bibliography of his works Volume 12, Volume 12,*. 1962.

[38] E. P. Velikhov. The problem of plasma rotation. *Atomic Energy*, 14(6):597–598, 1964.

[39] R. L. Viana, R. A. Clemente, and S. R. Lopes. Spherically symmetric stationary MHD equilibria with azimuthal rotation. *Plasma physics and controlled fusion*, 39(1):197, 1997.

[40] Ricardo L. Viana. MHD Equilibrium Equation with Azimuthal Rotation in a Curvilinear Coordinate System. *International journal of theoretical physics*, 37(10):2657–2668, 1998.

[41] Ricardo L. Viana. Adiabatic plasma rotations in orthogonal coordinate systems. *Brazilian Journal of Physics*, 31:58–64, 2001.

[42] Darryl D. Holm; Jerrold E. Marsden; Tudor Ratiu; Alan Weinstein. Nonlinear stability of fluid and plasma equilibria. *Physics Reports*, 123(1-2), 1985.

[43] John Wesson and D. J. Campbell. *Tokamaks*. Number 118 in Oxford science publications. Clarendon Press ; Oxford University Press, Oxford : New York, 3rd ed edition, 2004.

[44] Hartmut Zohm. *Magnetohydrodynamic stability of tokamaks*. Wiley-VCH, Weinheim, 2015.