UNIVERSIDADE FEDERAL DO PARANÁ

Rodrigo Alexandre Siqueira

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Orientador: Prof. Dr. Pedro Danizete Damázio
Co-orientadora : Prof.ª Dra. Ana Leonor Silvestre

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Programa de Pós-Graduação MATEMÁTICA

**ATA Nº013**

# ATA DE SESSÃO PÚBLICA DE DEFESA DE DOUTORADO PARA A OBTENÇÃO DO GRAU DE DOUTOR EM MATEMÁTICA

No dia vinte e um de Fevereiro de dois mil e dezessete às 10:00 horas, na sala PC06, Rua Cel. Francisco H. dos Santos, 100 - Jardim das Américas, foram instalados os trabalhos de arguição do doutorando **RODRIGO ALEXANDRE SIQUEIRA** para a Defesa Pública de sua tese intitulada **SOBRE O SISTEMA DE NAVIER-STOKES QUÂNTICO PARA FLUÍDOS INCOMPRESSÍVEIS: RESULTADOS DE REGULARIDADE E UNICIDADE DE SOLUÇÕES FORTES E ANÁLISE DE ERRO PARA AS APROXIMAÇÕES SEMI-GALERKIN ESPECTRAIS**. A Banca Examinadora, designada pelo Colegiado do Programa de Pós-Graduação em MATEMÁTICA da Universidade Federal do Paraná, foi constituída pelos seguintes Membros: PEDRO DANIZETE DAMAZIO (UFPR), HIGIDIO PORTILLO OQUENDO (UFPR), YUAN JINYUN (UFPR), PATRICIA APARECIDA MANHOLI (UTFPR), JUAN AMADEO SORIANO PALOMINO (UEM). Dando início à sessão, a presidência passou a palavra ao discente, para que o mesmo expusesse seu trabalho aos presentes. Em seguida, a presidência passou a palavra a cada um dos Examinadores, para suas respectivas arguições. O aluno respondeu a cada um dos arguidores. A presidência retomou a palavra para suas considerações finais e, depois, solicitou que os presentes e o doutorando deixassem a sala. A Banca Examinadora, então, reuniu-se sigilosamente e, após a discussão de suas avaliações, decidiu-se pela APROVAÇÃO do aluno. O doutorando foi convidado a ingressar novamente na sala, bem como os demais assistentes, após o que a presidência fez a leitura do Parecer da Banca Examinadora. Nada mais havendo a tratar a presidência deu por encerrada a sessão, da qual eu, PEDRO DANIZETE DAMAZIO, lavrei a presente ata, que vai assinada por mim e pelos membros da Comissão Examinadora.

Curitiba, 21 de Fevereiro de 2017.

PEDRO DANIZETE DAMAZIO
Presidente da Banca Examinadora (UFPR)

HIGIDIO PORTILLO OQUENDO
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## TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em MATEMÁTICA da Universidade Federal do Paraná foram convocados para realizar a arguição da tese de Doutorado de **RODRIGO ALEXANDRE SIQUEIRA** intitulada: **SOBRE O SISTEMA DE NAVIER-STOKES QUÂNTICO PARA FLUÍDOS INCOMPRESSÍVEIS: RESULTADOS DE REGULARIDADE E UNICIDADE DE SOLUÇÕES FORTES E ANÁLISE DE ERRO PARA AS APROXIMAÇÕES SEMI-GALERKIN ESPECTRAIS**, após terem inquirido o aluno e realizado a avaliação do trabalho, são de parecer pela sua APROVAÇÃO.

Curitiba, 21 de Fevereiro de 2017.

PEDRO DANIZETE DAMAZIO
Presidente da Banca Examinadora (UFPR)

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*"Quando a situação for boa, desfrute-a. Quando a situação for ruim, transforme-a. Quando a situação não puder ser transformada, transforme-se."*

*Viktor Frankl*

*Dedico*
*À minha mãe Judite e à minha esposa Elaine que sempre me incentivam e me dão forças para que eu nunca desista.*

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E por fim, aos programas Capes-DS e Capes-PDSE, pelo apoio financeiro.

# Resumo

No presente trabalho estudaremos a existência e unicidade de solução forte e estimativas de erro para os casos local e global do Problema de Navier-Stokes Quântico para Fluidos Incompressíveis. Analisaremos o problema considerando o toro $\mathbb{T}^d$ com $d \leq 3$. Para garantirmos a existência e unicidade de solução forte local e global, usamos o método de Faedo-Galerkin semi-espectral.

**Palavras-chave:** Equação de Navier-Stokes quântica, fluidos incompressíveis, Solução forte, Local no tempo, Global no tempo, Estimativas de erro.

# Abstract

In this work we study the existence and uniqueness of strong solution and error estimates for the local and global cases of the Navier-Stokes problem for incompressible quantum fluids. We analyze the problem when considering the torus $\mathbb{T}^d$ with $d \leq 3$. To ensure the existence and uniqueness of local and global strong solution, we use the semi-spectral Faedo-Galerkin method.

**Key-words:** quantum Navier-Stokes equation, incompressible fluids, strong solution, Local in time, Global in time, error estimates.

# Conteúdo

# Notação

O produto escalar de vetores $\mathbf{a} = [a_1 \ a_2 \ ... \ a_n]$, $\mathbf{b} = [b_1 \ b_2 \ ... \ b_n]$ é denotado por;

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \sum_{i=1}^{n} a_i . b_i;$$

o produto escalar de matrizes $\mathbf{A} = [A_{i,j}]_{i,j=1}^{n}$, $\mathbf{B} = [B_{i,j}]_{i,j=1}^{n}$ é denotado por;

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = \sum_{i,j=1}^{n} A_{i,j} . B_{i,j};$$

o produto da matriz $\mathbf{A} = [A_{i,j}]_{i,j=1}^{n}$ com o vetor $\mathbf{b} = [b_1 \ b_2 \ ... \ b_n]$ é um vetor $\mathbf{A} \cdot \mathbf{b}$ com componentes dadas por;

$$[\mathbf{A}\mathbf{b}]_i = \sum_{j=1}^{n} A_{i,j} b_j \quad para \ i = 1, ..., n.$$

A transposta de uma matriz $\mathbf{A} = [A_{i,j}]_{i,j=1}^{n}$ é $\mathbf{A}^T = [A_{j,i}]_{i,j=1}^{n}$.

O traço da matriz $\mathbf{A} = [A_{i,j}]_{i,j=1}^{n}$ é $tr(\mathbf{A}) = \sum_{i=1}^{n} A_{i,i}$.

O símbolo $\mathbf{a} \otimes \mathbf{b}$ denota o produto tensorial dos vetores $\mathbf{a}$ e $\mathbf{b}$,

$$\mathbf{a} \otimes \mathbf{b} = [\mathbf{a}_i \cdot \mathbf{b}_j]_{i,j=1}^{n}.$$

O gradiente de uma função escalar $g : \Omega \longrightarrow \mathbb{R}$ é um vetor

$$\nabla g(x) = \left[ \frac{\partial g(x)}{\partial x_1} \ \frac{\partial g(x)}{\partial x_2} \ ... \ \frac{\partial g(x)}{\partial x_n} \right]$$

onde $x = (x_1, x_2, ..., x_n) \in \Omega$.

O gradiente de uma função vetorial $\mathbf{g}(x) = [g_1(x) \ g_2(x) \ ... \ g_n(x)]$ onde $g_i : \Omega \longrightarrow \mathbb{R}$, é uma função matricial,

$$\nabla \mathbf{g}(x) = \left[ \frac{\partial g_i(x)}{\partial x_j} \right]_{i,j=1}^{n}.$$

O divergente de uma função vetorial $\mathbf{g}(x) = [g_1(x) \ g_2(x) \ ... \ g_n(x)]$ onde $g_i : \Omega \longrightarrow \mathbb{R}$, é uma função escalar,

$$div \ \mathbf{g}(x) = \sum_{i=1}^{n} \frac{\partial g_i(x)}{\partial x_i}.$$

O divergente de uma função matricial $\mathbf{B} = [B_{ij}]_{i,j=1}^{n}$ onde $B_{ij} : \Omega \longrightarrow \mathbb{R}$, é uma função vetorial

$$[div\mathbf{B}(x)]_i = \sum_{j}^{n} \frac{\partial B_{ij}(x)}{\partial x_j}, \quad i = 1, ..., n.$$

O símbolo $\Delta$ denota o Operador Laplaciano

$$\Delta = div\nabla$$

e denotaremos $D(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$.
Por fim, denotaremos o símbolo $\mathbf{a} \cdot \nabla$ por

$$\mathbf{a} \cdot \nabla = a_1 \cdot \frac{\partial}{\partial x_1} + \cdots + a_n \cdot \frac{\partial}{\partial x_n}$$

No decorrer desta dissertação serão introduzidas outras notações.

# Introdução

Em 1952 quando o físico David Bohm [11], [12] redescobriu e generalizou a hipótese de Louis de Broglie [26] que propôs que a dualidade de onda-partícula seria uma propriedade geral dos objetos microscópicos, sugerindo que as partículas microscópicas, além de se comportarem como partículas materiais (com posição e momento definido a cada instante), também apresentavam características próprias de fenômenos ondulatórios. Haveria assim um novo tipo de onda em coexistência com o ponto material, a onda atuaria como um tipo de onda-piloto guiando a partícula. David Bohm parte da equação de Schrödinger,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi, \tag{1}$$

onde a função de onda $\psi(x,t)$ é uma função complexa de posição $x$ e instante de tempo $t$, a densidade de probabilidade $n(x,t)$ é uma função real definida por

$$n(x,t) = R^2(x,t) = |\psi(x,t)|^2.$$

Sem perda de generalidade, pode-se expressar a função de onda $\psi(x,t)$ em termos da densidade de probabilidade real $n(x,t)$ e uma função de fase $S(x,t)$ de variáveis reais, tais que,

$$\psi(x,t) = R(x,t)\exp\left(\frac{i}{\hbar}S(x,t)\right). \tag{2}$$

Substituindo (2) em (1) e separando as partes imaginária e real, obtém-se as seguintes equações:

$$\text{Parte Imaginária:} \qquad \frac{\partial}{\partial t}R^2(x,t) + div\left(R^2(x,t)\frac{\nabla S(x,t)}{m}\right), \tag{3}$$

$$\text{Parte Real:} \qquad \frac{\partial}{\partial t}S(x,t) = -\frac{(\nabla S(x,t))^2}{2m} + V + Q, \tag{4}$$

onde

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta R(x,t)}{R(x,t)} = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}}, \tag{5}$$

é o potencial quântico. O potencial quântico é o responsável por dar o comportamento quântico à partícula. Segundo Bohm, no limite clássico o potencial quântico $Q$ desaparece e a equação (4) se reduz à equação de Hamilton-Jacobi. Por essa razão, a função de fase $S(x,t)$ pode ser entendida como a ação mecânica e assim se define $\mathbf{u} = \nabla S/m$ como a velocidade quântica. As equações (3) e (4) fazem parte da teoria da Hidrodinâmica Quântica ou Fluidos Quânticos. Tais modelos podem ser utilizados para descrever superfluidos [30], semicondutores quânticos [13] e trajetórias quânticas da mecânica bohmiana [36].

Brenner [21] sugere o seguinte modelo de Navier-Stokes

$$n_t + div(n\mathbf{w}) = 0, \qquad (n\mathbf{u})_t + div(n\mathbf{u}\otimes\mathbf{w}) + \nabla p = divS,$$

o qual interpreta $\mathbf{u}$ e $\mathbf{w}$ como sendo a velocidade de volume e a velocidade de massa respetivamente, com a seguinte relação $\mathbf{u} = \mathbf{w} + \nu\nabla\log n$, onde $\nu > 0$ é constante. Em [2] e [3], sugere-se o seguinte modelo de Navier-Stokes Quântico

$$n_t + div(n\mathbf{u}) = \nu\Delta n, \tag{6}$$

$$(n\mathbf{u})_t + div(n\mathbf{u}\otimes\mathbf{u}) + \nabla p = n\mathbf{f} + \nu\Delta(n\mathbf{u}) + 2\varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right), \tag{7}$$

o qual considera o potencial quântico como um tensor de stress quântico (ver [36]).

Nos livros [36] e [4] é feita a interpretação física do $div(\mathbf{u})$ e, em particular, quando $div(\mathbf{u}) = 0$. Portanto, adicionando-se a hipótese de $div(\mathbf{u}) = 0$ no sistema (6)-(7) e considerando-se as seguintes identidades (que serão provadas na seção de Preliminar):

$$div(n\mathbf{u}\otimes\mathbf{u}) = (\mathbf{u}\cdot\nabla n)\mathbf{u} + (n\mathbf{u}\cdot\nabla)\mathbf{u},$$

$$\nu\Delta(n\mathbf{u}) = \nu n\Delta\mathbf{u} + 2\nu\nabla\mathbf{u}.\nabla n + \nu\mathbf{u}\Delta n,$$

$$div(n\mathbf{u}) = \mathbf{u}\cdot\nabla n,$$

obtém-se o modelo de Navier-Stokes Quântico para fluidos incompressíveis:

$$\begin{aligned}
&n_t + \mathbf{u}\cdot\nabla n = \nu\Delta n,\\
&(n\mathbf{u})_t + (n\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla n)\mathbf{u} + \nabla p = \nu n\Delta\mathbf{u} + 2\nu\nabla\mathbf{u}\cdot\nabla n + \nu\mathbf{u}\Delta n + n\mathbf{f}\\
&+2\varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right),\\
&div\ \mathbf{u} = 0.
\end{aligned}$$

Utilizando-se a primeira equação do modelo acima e seguinte identidade (que será provada na seção de Preliminar):

$$\begin{aligned}
&n_t = \nu\Delta n - \mathbf{u}\cdot\nabla n,\\
&2\varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = \varepsilon^2\nabla\Delta n - \varepsilon^2\frac{1}{n}\Delta n\nabla n - \varepsilon^2\frac{1}{n}(\nabla n\cdot\nabla)\nabla n + \varepsilon^2\frac{1}{n^2}(\nabla n\cdot\nabla n)\nabla n;
\end{aligned}$$

Obtém-se o seguinte modelo, o qual é o nosso objetivo de estudo:

**Problema de Navier-Stokes Quântico para fluidos incompressíveis:**

$$\begin{aligned}
&\text{Determinar as funções } \mathbf{u} : \mathbb{T}^d \times [0,T] \to \mathbb{R}^d; \quad n : \mathbb{T}^d \times [0,T] \to \mathbb{R};\\
&p : \mathbb{T}^d \times [0,T] \to \mathbb{R}; \text{ tais que}\\
&n_t + \mathbf{u} \cdot \nabla n = \nu \Delta n, && (8)\\
&n\mathbf{u}_t + (n\mathbf{u} \cdot \nabla)\mathbf{u} + \nu \left[-n\Delta \mathbf{u} - 2\nabla \mathbf{u} \cdot \nabla n\right] + \nabla p = n\mathbf{f}\\
&+\varepsilon^2 \left[ -\frac{1}{n}(\nabla n \cdot \nabla)\nabla n + \frac{1}{n^2}(\nabla n \cdot \nabla n)\nabla n - \frac{1}{n}\Delta n \nabla n + \nabla \Delta n \right], && (9)\\
&div\ \mathbf{u} = 0, && (10)\\
&\mathbf{u}(\cdot, 0) = \mathbf{u}_0, \quad n(\cdot, 0) = n_0, \quad 0 < \alpha \le n_0 \le \beta. && (11)
\end{aligned}$$

Considera-se a região de escoamento $\mathbb{T}^d$, o qual é, o toro $d$-dimensional ($d \le 3$), $\mathbf{u}(x,t) \in \mathbb{R}^d$ é a velocidade do fluido no ponto $x \in \mathbb{T}^d$ e no instante $t \in [0,T]$ , $\nu > 0$ é o coeficiente de viscosidade (o qual estamos considerando constante) e $\varepsilon > 0$ é a constante de Plank; a função $n(x,t) \in \mathbb{R}$ é a densidade do fluido no ponto $x \in \mathbb{T}^d$ e no instante $t \in [0,T]$, a função $p(x,t) \in \mathbb{R}$ é a pressão no ponto $x \in \mathbb{T}^d$ e no instante $t \in [0,T]$ e $\mathbf{f} : \mathbb{T}^d \times [0,T] \longrightarrow \mathbb{R}^d$ descreve as forças externas resultantes (por exemplo, de um campo elétrico).

A escolha do toro $\mathbb{T}^d$ como região de escoamento é feito de modo a evitar a dificuldade de definir precisamente a noção de “fronteira”de uma região na mecânica quântica. Além disso, a escolha do toro $\mathbb{T}^d$ permite tratar as condições de contorno como no caso periódico.

Jüngel [1] provou a existência de solução fraca global no tempo para o modelo de Navier-Stokes quântico barotrópico

$$\begin{aligned}
&n_t + div(n\mathbf{u}) = 0,\\
&(n\mathbf{u})_t + div(n\mathbf{u} \otimes \mathbf{u}) + \nabla p(n) - 2\varepsilon^2 n \nabla \left( \frac{\Delta\sqrt{n}}{\sqrt{n}} \right) - n\mathbf{f} = \nu div(n(\nabla \mathbf{u} + (\nabla \mathbf{u})^t)),
\end{aligned}$$

com viscosidade constante e menor do que a constante de Plank no $\mathbb{T}^d$ (toro $d$-dimencional $d \le 3$) . Posteriormente os resultados obtidos em [1] foram estendidos no caso da viscosidade ser igual a constante de Plank e a viscosidade ser maior que a constante de Plank respectivamente em [22] e [17].

Em [31] se faz a análise de um superfluido incompressível e irrotacional quando a função de densidade de probabilidade $n(x,t)$ é constante.

O sistema de equações (8)-(11) é semelhante às Equações de Movimento de fluidos Viscosos Incompressível com Fenômenos de Difusão. Assim trabalharemos de forma semelhante a [32], [33] e [18].

Este trabalho está organizado como segue:

No Capítulo 1, fixamos as notações a serem usadas e definimos os espaços funcionais sobre os quais trabalharemos. Enunciamos resultados teóricos que serão usados no desenvolvimento deste trabalho.

No Capítulo 2, utilizamos argumentos semelhantes de Damázio, Guillén-González, Gutiérrez-Santacreu e Rojar-Medar [32] (método de semi-Galerkin espectral) para obter uma formulação forte no tempo e fraca no espaço (no sentido de $L^2$) do problema, e definimos o problema aproximado. Encontraremos várias desigualdades diferenciais para a solução do problema aproximado, as quais nos auxiliarão a encontrar estimativas *a Priori*. Primeiramente provamos a existência e unicidade de solução forte local no tempo para o problema (8)-(11) quando $d = 2$ ou 3 sem assumir hipóteses sobre os dados iniciais (além de regularidade necessária) ou hipóteses sobre as constantes físicas damos resultados de regularidades melhores do que aqueles obtidos em [32]. No caso particular de $d = 2$, combinamos argumentos usados por [32] (método de semi-Galerkin espectral junto com exponenciais como funções peso) e [34] (utilizando desigualdades do tipo Ladyzhenskaya e de Gagliardo-Nirenberg), provamos a existência e unicidade de solução forte global no tempo sem assumir que a força externa seja suficientemente pequeno, mas com uma hipótese sobre as constantes físicas. No caso $d = 3$, com as hipóteses adicionais de os dados iniciais e a força externa serem suficientemente pequenos, sem pedir hipóteses sobre as constantes físicas provamos a existência e unicidade de solução global forte no tempo, e damos resultados de regularidades melhores do que obtidos em [32].

O Capítulo 3 desta tese é inteiramente dedicado a uma rigorosa análise de erro das aproximações semi-Galerkin espectrais; tal análise de erro tem o propósito de fornecer um sólido suporte teórico para futuras implementações computacionais para esta classe de problemas. Neste sentido, utilizando as regularidades obtidas no Capítulo anterior para a solução do problema (8)-(11) e com argumentos semelhantes a Damázio e Rojar-Medar [33] obtemos estimativas de erro locais para as aproximações da velocidade $\mathbf{u}$ no espaço $H^2$ e para a densidade $n$ no espaço $H^4$. Cabe notar que em [33] foram obtidas estimativas de erro locais para a velocidade $\mathbf{u}$ e densidade $\rho$ nos espaços $H^1$ e $H^2$ respetivamente; entretanto, devido à alta ordem de não-linearidades apresentadas nesse modelo de fluidos quânticos, é natural exigir-se alguma regularidade extra para os dados iniciais.

Os argumentos utilizados no Capítulo 3 para obter as estimativas de erro locais são facilmente adaptadas para o caso das estimativas de erro uniformes no tempo (caso global).

# Capítulo 1

# Preliminares

Esta seção foi pensada com o intuito de apresentar o maior número de conceitos e resultados, para que se possa ter uma melhor compreensão dos conteúdos abordados no restante deste trabalho.

Por simplicidade de notação, no decorrer desta dissertação será usada a letra C para designar qualquer constante positiva cuja dependência dos dados (iniciais e parâmetros físicos) do problema seja irrelevante.

## 1.1 Espaços Funcionais

Iniciamos construindo a região sobre a qual desenvolveremos nossos estudos (ver [20]), o toro $d$-dimensional $\mathbb{T}^d$ pode ser representado como o cubo

$$\mathbb{T}^d = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : |x_j| \leq \pi, \quad j = 1, \cdots, d\}$$

com os lados opostos identificados. Em outras palavras, $x, y \in \mathbb{R}^d$ são identificados $x \equiv y$ quando $x - y = 2\pi k$ para algum $k = (k_1, \cdots, k_d) \in \mathbb{Z}^d$. Claramente " $\equiv$ " é uma relação de equivalência, e a classe de equivalência de um elemento $x \in \mathbb{R}^d$ é dado por,

$$\begin{aligned} [x] &:= \{y \in \mathbb{R}^d : y - x = 2\pi k, \quad k \in \mathbb{Z}^d\} \\ &= \{y \in \mathbb{R}^d : y - x = z, \quad z \in 2\pi\mathbb{Z}^d\} \\ &= \{x + z, \quad z \in 2\pi\mathbb{Z}^d\} \\ &= x + 2\pi\mathbb{Z}^d. \end{aligned}$$

Assim, podemos identificar o toro $d$-dimensional com o espaço quociente, ou seja,

$$\mathbb{T}^d \cong \mathbb{R}^d / 2\pi\mathbb{Z}^d$$

munido da topologia quociente.

Podemos identificar de maneira natural, funções definidas sobre o $\mathbb{T}^d$ com funções $2\pi$-periódicas definidas sobre $\mathbb{R}^d$. Seja $f : \mathbb{T}^d \longrightarrow \mathbb{R}$ e definimos $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ por $g(x) = f([x])$. Note que $g$ está bem definida pois se $x \equiv y$ em $\mathbb{R}^d$ então,

$$g(y) = f([y]) = f([x]) = g(x).$$

A igualdade,

$$g(x + 2\pi k) = f([x + 2\pi k]) = f([x]) = g(x),$$

para todo $k \in \mathbb{Z}^d$ e para todo $x \in \mathbb{R}^d$, implica que a função $g$ é $2\pi$-periódica em $\mathbb{R}^d$. Ao logo deste trabalho não faremos distinção entre $f$ e $g$.

Por um multi-índice entendemos uma d-upla de números inteiros não-negativos $\alpha = (\alpha_1, ..., \alpha_d)$ e escreveremos $|\alpha| = \alpha_1 + \cdots + \alpha_d$; representamos por

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}},$$

o operador derivação parcial de ordem $\alpha$. No caso em que $\alpha = (0, 0, ..., 0)$, $D^\alpha$ denota o operador identidade.

Seja $m \in \mathbb{N} \cup \{0\}$, então,

$$C^m(\mathbb{T}^d) := \left\{ f : \mathbb{T}^d \longrightarrow \mathbb{R} : \sup_{x \in \mathbb{T}^d} |D^\alpha f| = \sup_{x \in [-\pi, \pi]^d} |D^\alpha f| < \infty, \quad \forall |\alpha| \leq m \right\}.$$

E definimos o espaço

$$C^\infty(\mathbb{T}^d) := \bigcap_{m \in \mathbb{N} \cup \{0\}} C^m(\mathbb{T}^d).$$

**Definição 1.1.1** *Seja uma sequencia* $\{\varphi_m\}_{m=1}^{\infty} \subset C^\infty(\mathbb{T}^d)$ *é dita convergente para* $\varphi \in C^\infty(\mathbb{T}^d)$ *se* $D^\alpha \varphi_m$ *converge para* $D^\alpha \varphi$*, uniformemente em* $\mathbb{T}^d$*, para todo multi-índice* $\alpha$*. Representamos por* $\mathcal{D}(\mathbb{T}^d)$*, o espaço* $C^\infty(\mathbb{T}^d)$ *munido da convergência definida acima.*

Seja $T$ um funcional linear sobre $\mathcal{D}(\mathbb{T}^d)$, então, para todo $\varphi \in \mathcal{D}(\mathbb{T}^d)$ denotaremos $T$ aplicado em $\varphi$ por $< T, \varphi >$ . Diremos que $T$ é um funcional linear e contínuo sobre $\mathcal{D}(\mathbb{T}^d)$ se $< T, \varphi_m > \longrightarrow < T, \varphi >$ com $m \longrightarrow \infty$, sempre que $\varphi_m \longrightarrow \varphi$ em $\mathcal{D}(\mathbb{T}^d)$.

**Definição 1.1.2** *Um funcional linear e contínuo sobre* $\mathcal{D}(\mathbb{T}^d)$ *é chamado de distribuição sobre* $\mathbb{T}^d$*. O conjunto de todas as distribuições sobre* $\mathbb{T}^d$ *é denotado por* $\mathcal{D}'(\mathbb{T}^d)$*.*

Seja $1 \leq p \leq \infty$, então os espaços $L^p(\mathbb{T}^d)$ são definidos como os espaços das (classes de) funções (Lebesgue-mensuráveis) $f : \mathbb{T}^d \longrightarrow \mathbb{R}$, munido com as seguintes normas;

$$\|f\|_{L^p(\mathbb{T}^d)} = \left(\int_{\mathbb{T}^d} |f(x)|^p dx\right)^{1/p} := \left(\int_{[-\pi,\pi]^d} |f(x)|^p dx\right)^{1/p}, \quad \text{se} \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\mathbb{T}^d)} = \sup_{x\in\mathbb{T}^d} \text{ess}\, |f(x)| := \sup_{[-\pi,\pi]^d} \text{ess}\, |f(x)| < \infty, \quad \text{se} \quad p = \infty.$$

Sejam $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ e $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ dois espaços vetoriais, sendo $\mathcal{A}$ um subespaço vetorial de $\mathcal{F}$. Dizemos que a inclusão $\mathcal{A} \subset \mathcal{F}$ é uma imersão contínua se a aplicação inclusão $I : \mathcal{A} \to \mathcal{F}$ definida por $Ix = x$ for contínua, ou seja, $\|Ix\|_{\mathcal{F}} \leq C\|x\|_{\mathcal{A}}$, $\forall x \in \mathcal{A}$. Denotamos este fato por

$$\mathcal{A} \hookrightarrow \mathcal{F};$$

se, além disso, a aplicação de inclusão for compacta, dizemos que a imersão $\mathcal{A} \hookrightarrow \mathcal{F}$ é compacta, denotaremos por

$$\mathcal{A} \overset{c}{\hookrightarrow} \mathcal{F}.$$

Em particular, se $(x_n)_{n\in\mathbb{N}}$ é uma sequência limitada de $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ então existe uma subsequência $(x_{n_j})_{j\in\mathbb{N}}$ convergente em $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$.

Um resultado importante é o que assegura que $\mathcal{D}(\mathbb{T}^d) \hookrightarrow L^p(\mathbb{T}^d) \hookrightarrow \mathcal{D}'(\mathbb{T}^d)$, e o espaço $\mathcal{D}(\mathbb{T}^d)$ é denso em $L^p(\mathbb{T}^d)$, para $1 \leq p < \infty$, (ver [20]).

**Definição 1.1.3** *Seja $f \in L^1(\mathbb{T}^d)$ e seja $\alpha$ um multi-índice. Então a função $f_\alpha \in L^1(\mathbb{T}^d)$ tal que*

$$\int_{\mathbb{T}^d} \varphi f_\alpha dx = (-1)^{|\alpha|} \int_{\mathbb{T}^d} f D^\alpha \varphi dx$$

*para todo $\varphi \in \mathcal{D}(\mathbb{T}^d)$ é chamada de* ***derivada fraca*** *de $f$ de ordem $\alpha$.*

**Definição 1.1.4** *Seja $T$ uma distribuição sobre $\mathbb{T}^d$ e $\alpha$ um multi-índice. A* **derivada distribucional** *de ordem $\alpha$ de $T$ é o funcional $T_\alpha$ definido sobre $\mathcal{D}(\mathbb{T}^d)$, dado por*

$$\langle T_\alpha, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle, \ \forall \varphi \in \mathcal{D}(\mathbb{T}^d).$$

**Definição 1.1.5** *Sejam $1 \leq p \leq \infty$ e $m \in \mathbb{N}$. O espaço de Sobolev de ordem $m$ sobre $\mathbb{T}^d$, denotado por $W^{m,p}(\mathbb{T}^d)$, é o espaço funcional das (classes de) funções em $L^p(\mathbb{T}^d)$ cujas derivadas distribucionais de ordem $\alpha$ pertencem a $L^p(\mathbb{T}^d)$, para todo multi-índice $\alpha$, com $|\alpha| \leq m$, ou seja,*

$$W^{m,p}(\mathbb{T}^d) = \{f \in L^p(\mathbb{T}^d);\ D^\alpha f \in L^p(\mathbb{T}^d),\ \forall \alpha \text{ tal que } |\alpha| \leq m\},$$

*munido com a seguinte norma,*

$$\|f\|_{W^{m,p}(\mathbb{T}^d)} := \left( \sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\mathbb{T}^d)} \right)^{1/p} \quad se \;\; 1 \leq p < \infty,$$

$$\|f\|_{W^{m,\infty}(\mathbb{T}^d)} := \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\mathbb{T}^d)} \quad se \;\; p = \infty.$$

Apenas no caso particular em que $p = 2$, o espaço $W^{m,2}(\mathbb{T}^d)$ é um espaço de Hilbert, o qual denotaremos por $H^m(\mathbb{T}^d)$.

Pode-se provar que o espaço $\mathcal{D}(\mathbb{T}^d)$ é denso em $W^{m,p}(\mathbb{T}^d)$, para $1 \leq p < \infty$, (ver [20]).

**Lema 1.1.1** *Seja* $\mathbf{u} \in H^1(\mathbb{T}^d)$, *então,*

$$(i) \qquad \int_{\mathbb{T}^d} div\ \mathbf{u}(x) dx = 0.$$

$$(ii) \qquad \int_{\mathbb{T}^d} \nabla \mathbf{u}(x) dx = 0.$$

**Demonstração:**

Inicialmente observe que $\mathbf{u}$ é $2\pi -$ periódica em $\mathbb{R}^d$. Para o item $(i)$ consideraremos o caso particular que $\mathbf{u}(x) \in \mathbb{R}^d$ para todo $x \in \mathbb{T}^d$. Então

$$\begin{aligned}
\int_{\mathbb{T}^d} \text{div } \mathbf{u}(x) dx &= \int_{[-\pi,\pi]^d} \text{div } \mathbf{u}(x) dx = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \text{div } \mathbf{u}(x_1, \cdots, x_d) dx_1 \cdots dx_d \\
&= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\partial \mathbf{u}_1}{\partial x_1}(x_1, \cdots, x_d) dx_1 \cdots dx_d + \cdots + \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\partial \mathbf{u}_d}{\partial x_d}(x_1, \cdots, x_d) dx_1 \cdots dx_d \\
&= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\partial \mathbf{u}_1}{\partial x_1}(x_1, \cdots, x_d) dx_1 \cdots dx_d \\
&\quad + \cdots + \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\partial \mathbf{u}_d}{\partial x_d}(x_1, \cdots, x_d) dx_d dx_1 \cdots dx_{d-1} \\
&= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (\mathbf{u}_1(\pi, x_2, \cdots, x_d) - \mathbf{u}_1(-\pi, x_2, \cdots, x_d)) dx_2 \cdots dx_d \\
&\quad + \cdots + \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (\mathbf{u}_d(x_1, \cdots, x_{d-1}, \pi) - \mathbf{u}_1(x_1, \cdots, x_{d-1}, -\pi)) dx_1 \cdots dx_{d-1} \\
&= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} 0 \ dx_2 \cdots dx_d + \cdots + \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} 0 \ dx_1 \cdots dx_{d-1} = 0.
\end{aligned}$$

A demonstração para o item $(ii)$ pode ser feita de modo análogo.

■

Considerando o Teorema da divergência, temos como consequência do Lema (1.1.1), que

$$\int_{\partial[-\pi,\pi]^d} \mathbf{u}.\overrightarrow{n}\, dS = \int_{[-\pi,\pi]^d} \text{div } \mathbf{u}(x)dx = \int_{\mathbb{T}^d} div\ \mathbf{u}(x)dx = 0,$$

portanto, nas Fórmulas de Green sobre $\mathbb{T}^d$ tem-se que o termo de fronteira é nulo.

A transformada de Fourier toroidal de uma função $f \in L^1(\mathbb{T}^d)$ é a sequência $\widehat{f} : \mathbb{Z}^d \longrightarrow \mathbb{R}$ definida por,

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^d} f(x)e^{-i\xi.x}dx, \quad \xi \in \mathbb{Z}^d.$$

E a fórmula inversa da transformada de Fourier toroidal para $f \in L^1(\mathbb{T}^d)$ e dado por;

$$f(x) = \sum_{\xi\in\mathbb{Z}^d} \widehat{f}(\xi)e^{i\xi.x} \quad \text{em} \quad L^1(\mathbb{T}^d).$$

Se $f \in L^p(\mathbb{T}^d)$, $1 \leq p \leq \infty$ então,

$$f(x) = \sum_{\xi\in\mathbb{Z}^d} \widehat{f}(\xi)e^{i\xi.x} \quad \text{em} \quad L^p(\mathbb{T}^d),$$

para mais detalhes ver [27]

**Teorema 1.1.1 (Desigualdade de Poincaré)** *Seja* $\mathbf{u} \in H^1(\mathbb{T}^d)$ *então,*

$$\|\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} \leq \left| \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathbf{u}(x)dx \right|^2 + \|\nabla\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}.$$

**Demonstração:**
Observe que

$$\widehat{\nabla\mathbf{u}}(\xi) = i\xi\widehat{\mathbf{u}}(\xi), \quad \xi \in \mathbb{Z}^d$$

onde

$$\widehat{\mathbf{u}}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathbf{u}(x)e^{-ix.\xi}dx.$$

Pela identidade de Parseval, temos que

$$\begin{aligned} \|\nabla\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} = \|\nabla\mathbf{u}\|^2_{L^2([-\pi,\pi]^d)} &= \sum_{\xi\in\mathbb{Z}^d} |i\xi|^2|\widehat{\mathbf{u}}(\xi)|^2 \\ &= \sum_{\xi\in\mathbb{Z}^d\setminus\{0\}} |i\xi|^2|\widehat{\mathbf{u}}(\xi)|^2. \end{aligned}$$

Como

$$\|\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} = \|\mathbf{u}\|^2_{L^2([-\pi,\pi]^d)} = \sum_{\xi\in\mathbb{Z}^d} |\widehat{\mathbf{u}}(\xi)|^2$$

$$
\begin{aligned}
&= |\widehat{\mathbf{u}}(0)|^2 + \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} |\widehat{\mathbf{u}}(\xi)|^2 \\
&\leq |\widehat{\mathbf{u}}(0)|^2 + \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} |i\xi|^2 |\widehat{\mathbf{u}}(\xi)|^2 \\
&= |\widehat{\mathbf{u}}(0)|^2 + \|\nabla \mathbf{u}\|^2_{L^2([-\pi,\pi]^d)} \\
&= \left| \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathbf{u}(x)dx \right|^2 + \|\nabla \mathbf{u}\|^2_{L^2([-\pi,\pi]^d)}.
\end{aligned}
$$

■

**Observação 1.1.1** *Sejam* $\mathbf{u} \in H^2(\mathbb{T}^d)$ *e o operador* $\nabla^2 = \nabla\nabla$, *então,*

$$
\begin{aligned}
\|\nabla^2 \mathbf{u}\|^2_{L^2(\mathbb{T}^d)} &= \int_{[-\pi,\pi]^d} |\nabla \mathbf{u}|^2 dx \\
&= \int_{[-\pi,\pi]^d} \nabla^2 \mathbf{u} : \nabla^2 \mathbf{u} dx \\
&= -\int_{[-\pi,\pi]^d} \nabla \mathbf{u}.div(\nabla^2 \mathbf{u}) dx \\
&= -\int_{[-\pi,\pi]^d} \nabla \mathbf{u}.\nabla \Delta \mathbf{u} dx \\
&= \int_{[-\pi,\pi]^d} div(\nabla \mathbf{u}) \Delta \mathbf{u} dx \\
&= \int_{[-\pi,\pi]^d} |\Delta \mathbf{u}|^2 dx \\
&= \|\Delta \mathbf{u}\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}
$$

**Observação 1.1.2** *Então, pela observação acima e pela Desigualdade de Poincaré (Teorema (1.1.1)) e o Lema (1.1.1), temos que;*

$$
\|\nabla f\|^2_{L^2(\mathbb{T}^d)} \leq \|\Delta f\|^2_{L^2(\mathbb{T}^d)}.
$$

*Assim, pode-se mostrar que*

$$
\|f\|^2_{H^m(\mathbb{T}^d)} := \|f\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^{m/2} \mathbf{u}\|^2_{L^2(\mathbb{T}^d)},
$$

*é uma norma nos espaços* $H^m(\mathbb{T}^d)$, *a partir de agora se considera a norma acima como a norma padrão do espaço* $H^m(\mathbb{T}^d)$. *Também temos a seguinte desigualdade;*

$$
\|\nabla f\|^2_{H^m(\mathbb{T}^d)} \leq \|\nabla \Delta^{m/2} f\|^2_{L^2(\mathbb{T}^d)}.
$$

**Teorema 1.1.2** *Sejam $d \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ e $1 \leq p \leq \infty$. Então os seguintes espaços são equivalentes:*

$$(i) \quad W^{m,p}(\mathbb{T}^d) = \left\{ f \in L^p(\mathbb{T}^d);\ D^\alpha f \in L^p(\mathbb{T}^d),\ \forall \alpha \text{ tal que } |\alpha| \leq m \right\}.$$

$$(ii) \quad W^{m,p}_{per}([-\pi,\pi]^d) = \left\{ f|_{[-\pi,\pi]^d} :\ f \in W^{m,p}_{loc}(\mathbb{R}^d) \ \text{ e } \ f \ \text{ é } \ 2\pi - \text{periódica em } \mathbb{R}^d \right\}.$$

$$(iii) \quad W^{m,p}_{\Gamma-per}((-\pi,\pi)^d) = \{ f \in W^{m,p}((-\pi,\pi)^d) :\ D^\alpha f|_{\Gamma_{j+d}} = D^\alpha f|_{\Gamma_j}, j = 1, ..., d,$$
$$\text{onde } \ \Gamma_1, ..., \Gamma_{2d} \ \text{ são as faces de } \ [-\pi,\pi]^d \ \text{ e } \ |\alpha| \leq m \}.$$

**Demonstração:** Ver [38]

■

Observe que $W^{m,p}_{\Gamma-per}((-\pi,\pi)^d) \subset W^{m,p}((-\pi,\pi)^d)$ onde $W^{m,p}((-\pi,\pi)^d)$ é o espaço de Sobolev sobre um aberto limitado do $\mathbb{R}^d$. Logo, toda a teoria clássica dos espaços de Sobolev $W^{m,p}((-\pi,\pi)^d)$ também é valida para $W^{m,p}_{\Gamma-per}((-\pi,\pi))$ pelo Teorema 1.1.2 tem-se que, toda teoria clássica dos espaços de Sobolev é válida para $W^{m,p}(\mathbb{T}^d)$. A partir de agora estaremos denotando por $W^{m,p}(\mathbb{T}^d)$ qualquer um dos espaços do Teorema 1.1.2.

**Proposição 1.1.1** *Seja $\Omega$ um aberto limitado do $\mathbb{R}^d$ com fronteira suficientemente suave e pelas observações acima também podemos ter $\Omega = \mathbb{T}^d$. Assim, temos as seguintes imersões:*

$$L^q(\Omega) \hookrightarrow L^p(\Omega) \quad \text{com} \quad 1 \leq p < q \leq +\infty.$$

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \text{ com } \begin{cases} 1/p* = 1/p - 1/d \text{ se } p < d, \\ p^* \in [1,\infty) \qquad \text{se } p = d, \\ p^* = +\infty \qquad \text{se } p > d, \end{cases}$$

$$W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) \text{ com } \begin{cases} 1 \leq q < \frac{dp}{d-p} & \text{se } p < d, \\ q \in [1,\infty) & \text{se } p = d; \end{cases}$$

$$W^{1,p}(\Omega) \overset{c}{\hookrightarrow} C^0(\overline{\Omega}) \quad \text{se} \quad p > d.$$

*Generalizando o resultado acima, temos*

$$W^{m,p}(\Omega) \hookrightarrow W^{n,q}(\Omega) \text{ com } \begin{cases} 1/q = 1/p - (m-n)/d \text{ se } (m-n)p < d, \\ q \in [1,\infty) \qquad \text{se } (m-n)p = d, \\ q = +\infty \qquad \text{se } (m-n)p > d; \end{cases}$$

*e se $d \geq 2$, então*

$$W^{m+1,p}(\Omega) \overset{c}{\hookrightarrow} W^{m,q}(\Omega) \text{ com } \begin{cases} 1 \leq q < \frac{dp}{d-p} & \text{se } p < d, \\ q \in [1,\infty) & \text{se } p = d, \end{cases}$$

*e se* $p > d$ *temos*

$$W^{m+1,p}(\Omega) \overset{c}{\hookrightarrow} C^m(\overline{\Omega}).$$

*Sejam* $mp > d$ *e* $K$ *o maior inteiro tal que* $0 \leq K < m - d/p$, *então*

$$W^{m,p} \overset{c}{\hookrightarrow} C^k(\bar{\Omega}).$$

**Demonstração:** Ver [8].

■

**Definição 1.1.6** *Dado um espaço de Banach* $X$, *se* $T > 0$ *é um numero real e* $1 \leq p < \infty$, *denotaremos por* $L^p(0,T;X)$, *o espaço vetorial das (classes de) funções vetoriais* $\varphi : (0,T) \to X$, *definidas em quase todo ponto em* $(0,T)$ *com valores em* $X$, *fortemente mensuráveis, e tais que a função* $t \longmapsto \|\varphi(t)\|_X$ *está em* $L^p(0,T)$. *Este espaço é de Banach quando consideramos a norma*

$$\|\varphi\|_{L^p(0,T;X)} = \left[\int_0^T \|\varphi(t)\|_X^p \; dt\right]^{1/p}, \text{ se } 1 \leq p < \infty.$$

*Quando* $q = \infty$ *o espaço* $L^\infty(0,T;X)$ *representa o espaço (das classes) de funções* $\varphi : [0,T] \longrightarrow X$ *mensuráveis e essencialmente limitadas. Este espaço é de Banach quando consideramos a norma*

$$\|\varphi\|_{L^\infty(0,T;X)} = \sup_{t\in(0,T)} \text{ess} \, \|\varphi(t)\|_X.$$

Sejam $p$ e $q$, tais que, $1/p + 1/q = 1$; um dos resultados fundamentais da teoria dos espaços $L^p(0,T;X)$, de demonstração bastante sofisticada, é aquele que estabelece a identificação do espaço dual topológico,

$$[L^p(0,T;X)]^* \cong L^q(0,T;X^*).$$

No caso em que $p = 1$, essa identificação fica

$$\left[L^1(0,T;X)\right]^* \cong L^\infty(0,T;X^*).$$

A dualidade entre esse espaços é dada na forma integral por

$$\langle v,u\rangle_{L^q(0,T;X^*),L^p(0,T;X)} = \int_0^T \langle v(t),u(t)\rangle_{X^*,X} \; dt.$$

Com esta identificação, os espaços $L^p(0,T;X)$ herdam as propriedades básicas do espaço de Banach $X$. Por exemplo, se $X$ é reflexivo então $L^p(0,T;X)$ será reflexivo, para $1 < p < \infty$. Se $X$ for separável então $L^p(0,T;X)$ também será separável, para $1 \leq p < \infty$ (ver [29]).

**Proposição 1.1.2** *Sejam $X$ e $Y$ espaços de Banach, e suponhamos que $X \hookrightarrow Y$. Se $1 \leq s \leq r \leq \infty$ então:*

$$L^r(0,T;X) \hookrightarrow L^s(0,T;Y).$$

**Demonstração:** Ver [29].

■

**Lema 1.1.2** *Se $f \in L^q(0,T;B)$ e $\partial f/\partial t \in L^q(0,T;B)$, para $1 \leq q \leq \infty$, então existe $f^* \in C([0,T];B)$ tal que $f = f^*$ q.t.p. em $[0,T]$.*

**Demonstração:** Ver [25].

■

**Lema 1.1.3 (Lema de Aubin-Lions)** *Sejam $B_0 \overset{c}{\hookrightarrow} B \hookrightarrow B_1$ espaços de Banach. Então, temos as seguintes imersões compactas:*

*(i)* $L^q(0,T;B_0) \cap \left\{\phi : \frac{\partial \phi}{\partial t} \in L^1(0,T;B_1)\right\} \overset{c}{\hookrightarrow} L^q(0,T;B)$ *se* $1 \leq q \leq \infty$,

*(ii)* $L^\infty(0,T;B_0) \cap \left\{\phi : \frac{\partial \phi}{\partial t} \in L^r(0,T;B_1)\right\} \overset{c}{\hookrightarrow} C(0,T;B)$ *se* $1 < r \leq \infty$.

**Demonstração:** Ver [23].

■

Introduziremos agora os espaços funcionais clássicos para o estudo das equações de Navier-Stokes. Definimos os seguintes espaços:

$$L_0^2 = \left\{\mathbf{u} \in L^2(\mathbb{T}^d) : \int_{\mathbb{T}^d} \mathbf{u}(x)dx = 0\right\},$$

$$H = \left\{\mathbf{u} \in L_0^2(\mathbb{T}^d) : \text{ div } \mathbf{u} = 0\right\},$$

$$H^\perp = \left\{\varphi : \varphi = \nabla p,\ p \in H^1(\mathbb{T}^d)\right\}$$

$$V = \left\{\mathbf{u} \in L_0^2 \cap H^1(\mathbb{T}^d) : \ div \ \mathbf{u} = 0\right\},$$

onde $\|\mathbf{u}\|_V^2 = (\nabla \mathbf{u}, \nabla \mathbf{u})$ é uma norma em $V$. Observe que $\|\Delta^{m/2}\mathbf{u}\|_{L^2(\mathbb{T}^d)}$ é uma norma no espaço $L_0^2(\mathbb{T}^d) \cap H^m(\mathbb{T}^d)$.

Temos que os espaços $H$ e $H^\perp$ são mutuamente ortogonais com relação ao produto interno usual de $L_0^2(\mathbb{T}^d)$ e então, a decomposição de Helmholtz nos fornece o fato de $L_0^2(\mathbb{T}^d) = H \oplus H^\perp$ (ver [37]).

Neste trabalho, denotaremos por $P$ o operador projeção ortogonal de $L_0^2(\mathbb{T}^d)$ sobre o subespaço $H$. Utilizando-nos da projeção $P$, teremos então, definido o operador de Stokes $A : D(A) \to H$, dado por $A = -P\Delta$ e cujo domínio $D(A)$ é o espaço $H \cap H^2(\mathbb{T}^d)$. Podemos escrever,

$$A\mathbf{u} = -\Delta\mathbf{u} \quad \text{para todo} \quad \mathbf{u} \in D(A) = H \cap H^2(\mathbb{T}^d)$$

É possível mostrar que $A$ é um operador auto-adjunto definido positivo caracterizado por

$$(A\mathbf{w}, \mathbf{v}) = (A^{1/2}\mathbf{w}, A^{1/2}\mathbf{v}) = (\nabla\mathbf{w}, \nabla\mathbf{v}) \quad \forall\ \mathbf{w} \in D(A), \quad \forall\ \mathbf{v} \in V.$$

Temos que $\|\mathbf{u}\|_{H^2(\mathbb{T}^d)}$ e $\|A\mathbf{u}\|_{L^2(\mathbb{T}^d)}$ são normas equivalentes em $D(A)$.

Denotaremos respectivamente por $\varphi_k$ e por $\lambda_k$ ($k \in \mathbb{N}$) a $k$-ésima autofunção e o $k$-ésimo autovalor do operador de Stokes definido sobre $H \cap H^2(\mathbb{T}^d)$. Prova-se que o conjunto de funções $\{\varphi_k\}_{k\in\mathbb{N}}$ é um conjunto ortogonal completo nos espaços $H$, $V$ e $V \cap H^2(\mathbb{T}^d)$ com respeito ao seus produtos internos usuais $(\mathbf{u}, \mathbf{v})$, $(\nabla\mathbf{u}, \nabla\mathbf{v})$ e $(A\mathbf{u}, A\mathbf{v})$, respectivamente.

Denotaremos por $V_k$ o espaço gerado pelas $k$ primeiras autofunções do operador de Stokes $A$, ou seja, $V_k = [\varphi_1, ..., \varphi_k]$ e por $P_k$ a projeção ortogonal de $L_0^2(\mathbb{T}^d)$ sobre $V_k$. Para a demonstração de tais fatos, sugerimos ver [10], [37].

Apresentamos em seguida, um resultado que ser-nos-á de grande utilidade na obtenção das taxas de convergência das aproximações semi-Galerkin espectral.

**Lema 1.1.4** *Se* $\mathbf{v} \in V$ *então*

$$\|\mathbf{v} - P_k\mathbf{v}\|^2_{L^2(\mathbb{T}^d)} \leq \frac{1}{\lambda_{k+1}}\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{T}^d)}.$$

*Se* $\mathbf{v} \in V \cap H^2(\mathbb{T}^d)$ *então*

$$\|\nabla\mathbf{v} - \nabla P_k\mathbf{v}\|^2_{L^2(\mathbb{T}^d)} \leq \frac{1}{\lambda_{k+1}}\|A\mathbf{v}\|^2_{L^2(\mathbb{T}^d)} \quad e \quad \|\mathbf{v} - P_k\mathbf{v}\|^2_{L^2(\mathbb{T}^d)} \leq \frac{1}{\lambda^2_{k+1}}\|A\mathbf{v}\|^2.$$

**Demonstração:** Ver [35].

■

**Lema 1.1.5** *Se* $\mathbf{v} \in V \cap H^3(\mathbb{T}^d)$ *então,*

$$\|\nabla \mathbf{v} - \nabla P_k \mathbf{v}\|^2_{L^2(\mathbb{T}^d)} \leq \frac{1}{\lambda^2_{k+1}} \|A^{3/2}\mathbf{v}\|^2_{L^2(\mathbb{T}^d)} \quad e \quad \|A\mathbf{v} - AP_k\mathbf{v}\|^2_{L^2(\mathbb{T}^d)} \leq \frac{1}{\lambda_{k+1}} \|A^{3/2}\mathbf{v}\|^2_{L^2(\mathbb{T}^d)}.$$

*Se* $\mathbf{v} \in V \cap H^4(\mathbb{T}^d)$ *então,*

$$\|A\mathbf{v} - AP_k\mathbf{v}\|^2_{L^2(\mathbb{T}^d)} \leq \frac{1}{\lambda^2_{k+1}} \|A^2\mathbf{v}\|^2_{L^2(\mathbb{T}^d)}.$$

**Demonstração:** Demonstra-se de forma análoga ao [35].

■

## 1.2 Outros Resultados Importantes

Nesta seção são apresentados resultados avulsos que serão usados nos capítulos posteriores.

**Lema 1.2.1 (Desigualdade de Hölder)** *Sejam* $f \in L^p(\Omega)$ *e* $g \in L^q(\Omega)$, *com* $1 \leq p < \infty$, *e q o expoente conjugado de p, isto é,* $\frac{1}{p} + \frac{1}{q} = 1$. *Então*

$$fg \in L^1(\Omega) \ e \ \|fg\|_{L^1(\Omega)} = \int_\Omega |f(x)g(x)|dx \leq \|f\|_{L^p(\mathbb{T}^d)} \|g\|_{L^q(\mathbb{T}^d)}$$

**Demonstração:** Ver [7].

■

**Lema 1.2.2 (Desigualdade de Hölder Generalizada)** *Sejam* $p_1, p_2, ..., p_k$ *números reais maiores que ou iguais a 1 e tais que* $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k} = 1$, *se* $p_i = 1$, *para algum* $i$, *então* $p_j = \infty$, $\forall j \neq i$. *Se* $f_i \in L^{p_i}(\Omega)$, *para* $i = 1, 2, ..., k$, *então* $f_1.f_2.....f_k \in L^1(\Omega)$ *e*

$$\|f_1 \cdot f_2 \cdot \dots \cdot f_k\|_{L^1(\Omega)} = \int_\Omega |f_1 \cdot f_2 \cdot \dots \cdot f_k| dx \leq \prod_{i=1}^{k} \|f_i\|_{L^{p_i}(\Omega)}.$$

**Demonstração:** Ver [7].

■

**Lema 1.2.3 (Desigualdade de Minkowsky)** *Se* $f, g \in L^p(\Omega)$, *com* $1 \leq p \leq +\infty$, *então*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

**Demonstração:** Ver [7].

■

**Lema 1.2.4 (Desigualdade de Interpolação)** *Seja* $f \in L^{q_1}(\mathbb{T}^d) \cap L^{q_2}(\mathbb{T}^d)$ *com* $1 \leq q_1 \leq q_2 \leq \infty$, *então* $f \in L^r(\mathbb{T}^d)$ *para todo* $q_1 \leq r \leq q_2$ *e*

$$\|f\|_{L^r(\mathbb{T}^d)} \leq \|f\|^k_{L^{q_1}(\mathbb{T}^d)} \|f\|^{1-k}_{L^{q_2}(\mathbb{T}^d)}$$

*onde* $\dfrac{1}{r} = \dfrac{k}{q_1} + \dfrac{1-k}{q_2}$ *com* $0 \leq k \leq 1$.

**Demonstração:** Ver [7].

■

Observa-se que para $d = 3$ e usando a imersão $H^1(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$, para $1 \leq q \leq 6$, obtém-se algumas situações que serão frequentemente usadas, dentre elas, podemos destacar:

$$\begin{aligned}
\|f\|_{L^3(\mathbb{T}^d)} &\leq \|f\|^{1/2}_{L^2(\mathbb{T}^d)} \|f\|^{1/2}_{L^6(\mathbb{T}^d)} \leq C\|f\|^{1/2}_{L^2(\mathbb{T}^d)} \|f\|^{1/2}_{H^1(\mathbb{T}^d)};\\
\|f\|_{L^4(\mathbb{T}^d)} &\leq \|f\|^{1/2}_{L^3(\mathbb{T}^d)} \|f\|^{1/2}_{L^6(\mathbb{T}^d)} \leq C\|f\|^{1/2}_{L^3(\mathbb{T}^d)} \|f\|^{1/2}_{H^1(\mathbb{T}^d)};\\
\|f\|_{L^4(\mathbb{T}^d)} &\leq \|f\|^{1/4}_{L^2(\mathbb{T}^d)} \|f\|^{3/4}_{L^6(\mathbb{T}^d)} \leq C\|f\|^{1/4}_{L^2(\mathbb{T}^d)} \|f\|^{3/4}_{H^1(\mathbb{T}^d)};\\
\|f\|_{L^6(\mathbb{T}^d)} &\leq \|f\|^{1/2}_{L^3(\mathbb{T}^d)} \|f\|^{1/2}_{L^\infty(\mathbb{T}^d)}.
\end{aligned}$$

**Lema 1.2.5** *Seja* $d = 2$, *então temos as seguintes desigualdades:*

$$\begin{aligned}
(i) &\quad \|f\|_{L^4(\mathbb{T}^2)} \leq C\|f\|^{1/2}_{L^2(\mathbb{T}^2)} \|f\|^{1/2}_{H^1(\mathbb{T}^2)},\\
(ii) &\quad \|\nabla f\|_{L^4(\mathbb{T}^2)} \leq C\|f\|^{1/2}_{L^\infty(\mathbb{T}^2)} \|f\|^{1/2}_{H^2(\mathbb{T}^2)},\\
(iii) &\quad \|f\|_{L^\infty(\mathbb{T}^2)} \leq C\|f\|^{1/2}_{L^2(\mathbb{T}^2)} \|f\|^{1/2}_{H^2(\mathbb{T}^2)},\\
(iv) &\quad \|f\|_{L^6(\mathbb{T}^2)} \leq C\|f\|^{1/3}_{L^2(\mathbb{T}^2)} \|f\|^{2/3}_{H^1(\mathbb{T}^2)}.
\end{aligned}$$

**Demonstração:** Ver [40], [16], [34] e [37].

■

**Lema 1.2.6 (Desigualdade de Young)** *Sejam* $1 < p, q < \infty$ *tais que* $\frac{1}{p} + \frac{1}{q} = 1$. *Então, para todos* $a, b \in \mathbb{R}$, *com* $a, b > 0$ *tem-se que*

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

**Demonstração:** Ver [7].

■

**Lema 1.2.7 (Desigualdade de Young generalizada)** *Sejam* $1 < p, q < \infty$ *tais que* $\frac{1}{p} + \frac{1}{q} = 1$ *e* $\varepsilon > 0$, *então, para todos* $a, b \in \mathbb{R}$, *com* $a, b > 0$ *tem-se que*

$$a \cdot b \leq \varepsilon a^p + C(\varepsilon) b^q,$$

*onde* $C(\varepsilon) = (\varepsilon p)^{-q/p}/q$.

**Demonstração:** Ver [7].

■

**Lema 1.2.8 (Desigualdade de Gronwall)** *Sejam* $\varphi \in L^\infty(0,T)$, *com* $\varphi(t) \geq 0$ *q.t.p. em* $[0,T]$ *e* $F \in L^1(0,T)$, $F(t) \geq 0$ *q.t.p. em* $[0,T]$ *tais que*

$$\varphi(t) \leq C + \int_0^t F(s)\varphi(s)\ ds,$$

*para quase todo* $t \in [0,T]$. *Então*

$$\varphi(t) \leq C\ exp\left(\int_0^t F(s)\right)\ ds,$$

*para quase todo* $t \in [0,T]$.

**Demonstração:** Ver [25].

■

**Lema 1.2.9 (Desigualdade de Gronwall Generalizada)** *Sejam* $\varphi$ *e* $\psi$ *funções contínuas não-negativas em* $[0,T]$, $a(t)$ *uma função absolutamente contínua com* $a(t) \geq 0$ *e* $F(t) \geq 0$ *integrável em* $[0,T]$, *tais que*

$$\varphi(t) + \int_0^t \psi(s)\ ds \leq a(t) + \int_0^t F(s)\varphi(s)\ ds, \quad \forall\ 0 \leq t \leq T.$$

*Então*

$$\varphi(t) + \int_0^t \psi(s)\ ds \leq a(t).\left(1 + \int_0^t F(s)\ ds\right) exp\left(\int_0^t F(s)\ ds\right),$$

*para todo* $t \in [0,T]$.

**Demonstração:** Ver [35].

■

A seguir, apresenta-se um resultado essencial sobre desigualdades diferenciais, que é fundamental para garantir a existência de um intervalo $[0,T]$ onde deverão estar definidas todas as soluções aproximadas do problema inicial.

**Lema 1.2.10** *Sejam* $g \in W^{1,1}(0,T)$ *e* $h \in L^1(0,T)$ *satisfazendo*

$$\frac{dg}{dt} \leq F(g) + h \quad em\ [0,T], \quad g(0) \leq g_0$$

*onde* $F : \mathbb{R} \to \mathbb{R}$ *é limitada em conjuntos limitados. Então para todo* $\varepsilon > 0$*, existe* $T_\varepsilon > 0$ *independente de* $g$ *tal que*

$$g(t) \leq g_0 + \varepsilon \quad \forall t \leq T_\varepsilon.$$

**Demonstração:** Ver [24].

■

**Teorema 1.2.1** *Seja* $\omega(t,x)$ *uma função contínua com domínio* $D \subset \mathbb{R}^2$ *e localmente Lipschitz. Seja* $x(t)$ *uma função diferenciavel em* $[t_0, T)$ *e tal que o seu gráfico esta contido em* $D$*. Suponha que* $x(t_0) \leq x_0$ *e* $\overline{x}(t)$ *é solução do problema abaixo;*

$$y_t = \omega(t, y(t)), \qquad y(t_0) = x_0.$$

*Se* $x(t)$ *é solução do seguinte problema;*

$$y_t \leq \omega(t, y(t)), \quad y(t_0) = x(t_0).$$

*Então,*

$$x(t) \leq \overline{x}(t),$$

*para todo* $t \geq t_0$*, tal que* $\overline{x}(t)$ *esta definido.*

**Demonstração:** Ver [9].

■

**Lema 1.2.11** *Sejam as funções* $\mathbf{a} : \Omega \to \mathbb{R}^d$ *e* $n : \Omega \to \mathbb{R}$*, onde* $\Omega$ *é um aberto do* $\mathbb{R}^d$ *e* $\mathbf{a}$*,* $n$ *suficientemente regulares. Então*

*(i)* $\nabla(\sqrt{n}) = \dfrac{1}{2\sqrt{n}}\nabla n$

*(ii)* $\Delta\nabla n = \nabla\Delta n$

*(iii)* $\nabla div(\mathbf{a}) = div([\nabla \mathbf{a}]^T)$

*(iv)* $n\nabla\left(\dfrac{1}{n}\right) = -\dfrac{1}{n}\nabla n$

**Demonstração:**

(i)

$$\begin{aligned}\nabla(\sqrt{n}) &= \left[\frac{1}{2\sqrt{n}}\frac{\partial n}{\partial x_1}, ..., \frac{1}{2\sqrt{n}}\frac{\partial n}{\partial x_d}\right] \\ &= \frac{1}{2\sqrt{n}}\left[\frac{\partial n}{\partial x_1}, ..., \frac{\partial n}{\partial x_d}\right] \\ &= \frac{1}{2\sqrt{n}}\nabla n.\end{aligned}$$

(ii)

$$\begin{aligned}\Delta\nabla n &= \Delta\left(\left[\frac{\partial n}{\partial x_1}, ..., \frac{\partial n}{\partial x_d}\right]\right) \\ &= \left[\Delta\left(\frac{\partial n}{\partial x_1}\right), ..., \Delta\left(\frac{\partial n}{\partial x_d}\right)\right] \\ &= \left[\frac{\partial(\Delta n)}{\partial x_1}, ..., \frac{\partial(\Delta n)}{\partial x_d}\right] \\ &= \nabla\Delta n.\end{aligned}$$

(iii)

$$\begin{aligned}\nabla div(\mathbf{a}) &= \left[\frac{\partial}{\partial x_1}div(\mathbf{a}), ..., \frac{\partial}{\partial x_d}div(\mathbf{a})\right] \\ &= \left[div\left(\frac{\partial \mathbf{a}}{\partial x_1}\right), ..., div\left(\frac{\partial \mathbf{a}}{\partial x_d}\right)\right] \\ &= div\left([\nabla \mathbf{a}]\right)^T.\end{aligned}$$

(iv)

$$n\nabla\left(\frac{1}{n}\right) = n\left[-\frac{1}{n^2}\frac{\partial n}{\partial x_1}, ..., -\frac{1}{n^2}\frac{\partial n}{\partial x_d}\right] = -\frac{1}{n}\nabla n.$$

■

**Lema 1.2.12** *Sejam as funções* $n : \Omega \to \mathbb{R}$, $\mathbf{a}, \mathbf{b} : \Omega \to \mathbb{R}^d$ *e* $\mathbf{B} : \Omega \to \mathbf{M}_{d\times d}(\mathbb{R})$, *onde* $\Omega$ *é um aberto contido em* $\mathbb{R}^d$ *e* $n, \mathbf{a}, \mathbf{b}, \mathbf{B}$ *suficientemente regulares. Então*

*(i)* $div(\mathbf{a} \otimes \mathbf{b}) = div(\mathbf{b})\mathbf{a} + \nabla \mathbf{a} \cdot \mathbf{b}$.

*(ii)* $div(n\mathbf{a}) = ndiv(\mathbf{a}) + \mathbf{a} \cdot \nabla n$.

*(iii)* $div(n\mathbf{B}) = ndiv(\mathbf{B}) + \mathbf{B} \cdot \nabla n$.

*(iv)* $(\mathbf{a} \otimes \nabla n) \cdot \mathbf{b} = (\mathbf{b} \cdot \nabla n)\mathbf{a}$.

*(v)* $(n\nabla \mathbf{a}) \cdot \mathbf{a} = n(\mathbf{a} \cdot \nabla)\mathbf{a}$.

**Demonstração:**

(i)

$$\begin{aligned}
div(\mathbf{a} \otimes \mathbf{b}) &= div\left([\mathbf{a}_i \mathbf{b}_j]_{i,j}^d\right) \\
&= \left[\sum_{j=1}^d \frac{\partial(\mathbf{a}_1 \mathbf{b}_j)}{\partial x_j}, ..., \sum_{j=1}^d \frac{\partial(\mathbf{a}_d \mathbf{b}_j)}{\partial x_j}\right] \\
&= \left[\sum_{j=1}^d \mathbf{a}_1 \frac{\partial \mathbf{b}_j}{\partial x_j} + \sum_{j=1}^d \mathbf{b}_j \frac{\partial \mathbf{a}_1}{\partial x_j}, ..., \sum_{j=1}^d \mathbf{a}_d \frac{\partial \mathbf{b}_j}{\partial x_j} + \sum_{j=1}^d \mathbf{b}_j \frac{\partial \mathbf{a}_d}{\partial x_j}\right] \\
&= \left[\sum_{j=1}^d \mathbf{a}_1 \frac{\partial \mathbf{b}_j}{\partial x_j}, ..., \sum_{j=1}^d \mathbf{a}_d \frac{\partial \mathbf{b}_j}{\partial x_j}\right] + \left[\sum_{j=1}^d \mathbf{b}_j \frac{\partial \mathbf{a}_1}{\partial x_j}, ..., \sum_{j=1}^d \mathbf{b}_j \frac{\partial \mathbf{a}_d}{\partial x_j}\right] \\
&= div(\mathbf{b})\mathbf{a} + \nabla \mathbf{a} \cdot \mathbf{b}.
\end{aligned}$$

As demonstrações dos casos (ii) e (v) são análogas.

■

**Teorema 1.2.2** *Sejam as funções* $n : \Omega \to \mathbb{R}$ *e* $\mathbf{a} : \Omega \to \mathbb{R}^d$, *onde* $\Omega$ *é um aberto do* $\mathbb{R}^d$ *e* $n$, $\mathbf{a}$ *suficientemente regulares. Então*

*(i)* $div(n\mathbf{a} \otimes \mathbf{a}) = div(\mathbf{a})n\mathbf{a} + (\mathbf{a}.\nabla n)\mathbf{a} + (n\mathbf{a}.\nabla)\mathbf{a}$.

*(ii)* $\Delta(n\mathbf{a}) = \mathbf{a}\Delta n + 2\nabla \mathbf{a}.\nabla n + n\Delta \mathbf{a}$.

*(iii)* $div(n\mathbf{a}) = div(\mathbf{a})n + \mathbf{a}.\nabla n$.

*(iv)* $2n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = \nabla\Delta n - \frac{1}{n}\Delta n \nabla n - \frac{1}{n}(\nabla n \cdot \nabla)\nabla n + \frac{1}{n^2}(\nabla n \cdot \nabla n)\nabla n.$

**Demonstração:**

(i)

$$\begin{aligned} div(n\mathbf{a}\otimes\mathbf{a}) &= div(\mathbf{a})n\mathbf{a}+\mathbf{a}.\nabla(n\mathbf{a}) \\ &= div(\mathbf{a})n\mathbf{a}+(\mathbf{a}.\nabla n)\mathbf{a}+(n\mathbf{a}.\nabla)\mathbf{a}. \end{aligned}$$

(ii)

$$\begin{aligned} \Delta(n\mathbf{a}) &= div(\nabla(n\mathbf{a})) \\ &= div(\mathbf{a}\otimes\nabla n+n\nabla\mathbf{a}) \\ &= div(\mathbf{a}\otimes\nabla n)+div(n\nabla\mathbf{a}) \\ &= div(\nabla n)\mathbf{a}+\nabla\mathbf{a}.\nabla n+div(\nabla\mathbf{a})n+\nabla\mathbf{a}.\nabla n \\ &= \mathbf{a}\Delta n+2\nabla\mathbf{a}.\nabla n+n\Delta\mathbf{a}. \end{aligned}$$

(iii)

$$div(n\mathbf{a})=div(\mathbf{a})n+\mathbf{a}.\nabla n.$$

(iv)

$$\begin{aligned} 2n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) &= 2n\nabla\left(\frac{1}{\sqrt{n}}div(\nabla\sqrt{n})\right) \\ &= 2n\nabla\left(\frac{1}{\sqrt{n}}div\left(\frac{1}{2\sqrt{n}}\nabla n\right)\right) \\ &= n\nabla\left(\frac{\Delta n}{n}+\frac{1}{\sqrt{n}}\nabla\left(\frac{1}{\sqrt{n}}\right).\nabla n\right) \\ &= n\nabla\left(\frac{\Delta n}{n}-\frac{1}{\sqrt{n}}\frac{1}{2n\sqrt{n}}\nabla n.\nabla n\right) \\ &= n\nabla\left(\frac{\Delta n}{n}-\frac{\nabla n.\nabla n}{2n^2}\right) \\ &= n\nabla\left(\frac{\Delta n}{n}\right)-\frac{n}{2}\nabla\left(\frac{\nabla n.\nabla n}{n^2}\right) \\ &= n\frac{(\nabla\Delta n)n-\Delta n(\nabla n)}{n^2}-\frac{n}{2}\frac{2n^2(\nabla n.\nabla)\nabla n-2n\nabla n(\nabla n.\nabla n)}{n^4} \\ &= \nabla\Delta n-\frac{1}{n}\Delta n\nabla n-\frac{1}{n}(\nabla n\cdot\nabla)\nabla n+\frac{1}{n^2}(\nabla n\cdot\nabla n)\nabla n \end{aligned}$$

■

# Capítulo 2

# Existência e Unicidade

Este capítulo é inteiramente destinado à investigação da existência e unicidade de soluções fortes Locais e Globais no tempo para o Problema de Navier-Stokes Quânticos para Fluidos Incompreensíveis descrito na introdução; também são analisadas questões relacionadas à unicidade bem como à regularidade de eventuais soluções.

Para provar os resultados de existência, usaremos as aproximações de Galerkin $\mathbf{u}^k$ para a velocidade, dadas em termos das autofunções do Operador de Stokes e para as aproximações da densidade $n^k$ utilizaremos as soluções infinito-dimensionais da equação de continuidade aproximada.

## 2.1 Formulação dos Problemas Variacional e Aproximado

Passemos assim, à formulação do "novo" problema. Multiplicando a equação (9) por $\mathbf{v} \in H$ e integrando sobre $\mathbb{T}^d$ obtemos;

$$(n\mathbf{u}_t, \mathbf{v}) + ((n\mathbf{u}.\nabla)\mathbf{u}, \mathbf{v}) + \nu[-(n\Delta\mathbf{u}, \mathbf{v}) - 2((\nabla n.\nabla)\mathbf{u}, \mathbf{v})] = (n\mathbf{f}, \mathbf{v})$$
$$+\varepsilon^2\left[ - \left(\frac{1}{n}(\nabla n.\nabla)\nabla n, \mathbf{v}\right) + \left(\frac{1}{n^2}(\nabla n.\nabla n)\nabla n, \mathbf{v}\right) - \left(\frac{1}{n}\Delta n\nabla n, \mathbf{v}\right) + (\nabla\Delta n, \mathbf{v})\right]$$

pois,

$$\begin{aligned}(\nabla p, \mathbf{v}) &= \int_{\partial[-\pi,\pi]^d} p.\mathbf{v} d\overrightarrow{S} - \int_{\mathbb{T}^d} p div(\mathbf{v}) dx \\ &= 0.\end{aligned}$$

Assim definimos a formulação Variacional do problema (8)-(11) como segue:

$$n_t + \mathbf{u}.\nabla n = \nu\Delta n, \tag{2.1}$$

$$(n\mathbf{u}_t, \mathbf{v}) + ((n\mathbf{u}.\nabla)\mathbf{u}, \mathbf{v}) + \nu[-(n\Delta\mathbf{u}, \mathbf{v}) - 2((\nabla n.\nabla)\mathbf{u}, \mathbf{v})] = (n\mathbf{f}, \mathbf{v})$$
$$+\varepsilon^2\Bigg[-\left(\frac{1}{n}(\nabla n.\nabla)\nabla n, \mathbf{v}\right) + \left(\frac{1}{n^2}(\nabla n.\nabla n)\nabla n, \mathbf{v}\right)$$
$$-\left(\frac{1}{n}\Delta n\nabla n, \mathbf{v}\right) + (\nabla\Delta n, \mathbf{v})\Bigg] \quad \forall\mathbf{v} \in H, \tag{2.2}$$

$$\mathbf{u}(\cdot, 0) = P\mathbf{u}_0, \quad n(\cdot, 0) = n_0, \quad 0 < \alpha \le n_0 \le \beta. \tag{2.3}$$

Definimos, para cada $k \in \mathbb{N}$, a aproximação semi-Galerkin espectral de $(\mathbf{u}, n)$ como sendo a solução $(\mathbf{u}^k, n^k) \in C^1([0, T^k]; H^2(\mathbb{T}^d) \cap V_k) \times C^1([0, T^k]; C^3(\mathbb{T}^d))$ do problema:

$$n^k_t + \mathbf{u}^k.\nabla n^k = \nu\Delta n^k, \tag{2.4}$$

$$(n^k\mathbf{u}^k_t, \mathbf{v}) + ((n^k\mathbf{u}.\nabla)\mathbf{u}^k, \mathbf{v}) + \nu[-(n^k\Delta\mathbf{u}^k, \mathbf{v}) - 2((\nabla n^k.\nabla)\mathbf{u}^k, \mathbf{v})] = (n^k\mathbf{f}, \mathbf{v})$$
$$+\varepsilon^2\Bigg[-\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \mathbf{v}\right) + \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{v}\right)$$
$$-\left(\frac{1}{n^k}\Delta n^k\nabla n^k, \mathbf{v}\right) + (\nabla\Delta n^k, \mathbf{v})\Bigg] \quad \forall\mathbf{v} \in V_k, \tag{2.5}$$

$$\mathbf{u}^k(\cdot, 0) = P_k\mathbf{u}_0, \quad n^k(\cdot, 0) = n_0, \quad 0 < \alpha \le n_0 \le \beta, \tag{2.6}$$

onde podemos tomar os dados iniciais no espaço $L^2$. Observamos que a expressão *"aproximações semi-Galerkin espectral"* se deve ao fato de estarmos fazendo aproximações finito-dimensional para a velocidade $\mathbf{u}$ e infinito-dimensional para a densidade $n$. Além disso, referir-nos-emos ao (2.4) -(2.6) como sendo o problema aproximado.

Vale lembrar que para todo $k \in \mathbb{N}$, o sistema (de EDO's) acima admite uma única solução $(\mathbf{u}^k, n^k)$ definida em $[0, T^k]$, com $0 < T^k \le T$ pelo Teorema de Carathéodory (ver [14]).

## 2.2 Desigualdades Diferenciais

Nesta seção iremos provar várias desigualdades diferenciais, as quais serão utilizadas nas próximas secções para obtermos estimativas *a Priori* Locais e Globais. No que se segue, imersões de Sobolev, desigualdade de Hölder, interpolação e desigualdade de Young são frequentemente aplicadas (ainda que de forma não-explícitas) para obter as desigualdades diferenciais desejadas.

**Observação 2.2.1** *Daremos agora uma lista das desigualdades utilizadas com frequência neste Capítulo:*

$$(i) \quad \|\nabla f\|^2_{H^m(\mathbb{T}^d)} \leq \|\nabla \Delta^{m/2} f\|^2_{L^2(\mathbb{T}^d)};$$

$$(ii) \quad \|f\|_{L^3(\mathbb{T}^d)} \leq C\|f\|^{1/2}_{L^2(\mathbb{T}^d)}\|f\|^{1/2}_{H^1(\mathbb{T}^d)};$$

*Para o caso particular $d = 2$:*

$$(iii) \quad \|f\|_{L^4(\mathbb{T}^2)} \leq C\|f\|^{1/2}_{L^2(\mathbb{T}^2)}\|f\|^{1/2}_{H^1(\mathbb{T}^2)};$$

$$(iv) \quad \|\nabla f\|_{L^4(\mathbb{T}^2)} \leq C\|f\|^{1/2}_{L^\infty(\mathbb{T}^2)}\|f\|^{1/2}_{H^2(\mathbb{T}^2)};$$

$$(v) \quad \|f\|_{L^\infty(\mathbb{T}^2)} \leq C\|f\|^{1/2}_{L^2(\mathbb{T}^2)}\|f\|^{1/2}_{H^2(\mathbb{T}^2)};$$

$$(vi) \quad \|f\|_{L^6(\mathbb{T}^2)} \leq C\|f\|^{1/3}_{L^2(\mathbb{T}^2)}\|f\|^{2/3}_{H^1(\mathbb{T}^2)}.$$

*Para mais detalhes sobre estas desigualdades ver Observação (1.1.2), Lema (1.2.4) e Lema (1.2.5)*

**Lema 2.2.1** *Seja $\mathbf{f} \in L^2(0,T;L^2(\mathbb{T}^d))$, então, para as soluções aproximadas $(\mathbf{u}^k, n^k)$ do problema (2.4)-(2.6) temos a seguinte desigualdade:*

$$\frac{d}{dt}\left(\frac{1}{2}\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\right) + \frac{\alpha}{2}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)}$$

$$+\frac{1}{2}\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2}{2}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2\alpha^3}{2\beta}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)}$$

$$\leq C(\nu,\varepsilon,\alpha,\beta)(\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + \|n^k\|^4_{H^2(\mathbb{T}^d)} + \|n^k\|^6_{H^2(\mathbb{T}^d)}$$

$$+\|n^k\|^8_{H^2(\mathbb{T}^d)}) + C(\nu,\alpha,\beta)\|n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}.$$

**Demonstração:**

Considerando a equação (2.4), temos para todo $k \in \mathbb{N}$, que $\alpha \leq n^k \leq \beta$ (ver Lema 3.1, pag. 54 de [15] e também ver [5]), em particular temos que $n^k \in L^\infty(0,T;L^\infty(\mathbb{T}^d))$, para $0 \leq T \leq \infty$. Agora, somando e subtraindo $n^k$ na equação (2.4), multiplicando por $n^k$ e integrando sobre $\mathbb{T}^d$ obtemos:

$$\frac{1}{2}\frac{d}{dt}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} = \|n^k\|^2_{L^2(\mathbb{T}^d)},$$

pois,

$$(\mathbf{u}^k \cdot \nabla n^k, n^k) = -(\text{div }\mathbf{u}^k, \frac{1}{2}|n^k|^2) = 0.$$

Observe que

$$0 < \alpha \leq \|n^k\|_{L^\infty(\mathbb{T}^d)} \leq \beta,$$

logo,

$$1 = \frac{\alpha^2}{\alpha^2} \leq \frac{1}{\alpha^2}\|n^k\|^2_{L^\infty(\mathbb{T}^d)} \leq C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}.$$

Portanto,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} &= \|n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C(\alpha)\|n^k\|^4_{H^2(\mathbb{T}^d)}. \qquad (2.7)
\end{aligned}$$

Pela equação (2.4), temos que:

$$\begin{aligned}
\|n^k_t\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\mathbf{u}^k.\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C\|\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^4(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + C\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + C(\nu,\alpha)\|n^k\|^4_{H^2(\mathbb{T}^d)}. \qquad (2.8)
\end{aligned}$$

Por outro lado fazendo $\mathbf{v} = \mathbf{u}^k_t$ na equação (2.5) temos,

$$(n^k\mathbf{u}^k_t, \mathbf{u}^k_t) + ((n^k\mathbf{u}.\nabla)\mathbf{u}^k, \mathbf{u}^k_t) + \nu[-(n^k\Delta\mathbf{u}^k, \mathbf{u}^k_t) - ((\nabla n^k.\nabla)\mathbf{u}^k, \mathbf{u}^k_t)]$$

$$= (n^k\mathbf{f}, \mathbf{u}^k_t) + \varepsilon^2\Bigg[-\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \mathbf{u}^k_t\right) + \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}^k_t\right)$$

$$-\left(\frac{1}{n^k}\Delta n^k\nabla n^k, \mathbf{u}^k_t\right) + \left(\nabla\Delta n^k, \mathbf{u}^k_t\right)\Bigg]$$

Observe que

$$(n^k\mathbf{u}^k_t, \mathbf{u}^k_t) = \|\sqrt{n^k}\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)} \geq \alpha\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)},$$

e também

$$\begin{aligned}
-\nu(n^k\Delta\mathbf{u}^k, \mathbf{u}^k_t) &= \nu(\nabla\mathbf{u}^k, \nabla(n^k\mathbf{u}^k_t)) \\
&= \nu(\nabla\mathbf{u}^k, n^k\nabla\mathbf{u}^k_t) + \nu(\nabla\mathbf{u}^k, \nabla n^k.\mathbf{u}^k_t)
\end{aligned}$$

em particular

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} &= \frac{1}{2}\frac{d}{dt}(\nabla\mathbf{u}^k, n^k\nabla\mathbf{u}^k) \\
&= \frac{1}{2}(\nabla\mathbf{u}^k_t, n^k\nabla\mathbf{u}^k) + \frac{1}{2}(\nabla\mathbf{u}^k, n^k_t\nabla\mathbf{u}^k) + \frac{1}{2}(\nabla\mathbf{u}^k, n^k\nabla\mathbf{u}^k_t) \\
&= (\nabla\mathbf{u}^k, n^k\nabla\mathbf{u}^k_t) + \frac{1}{2}(\nabla\mathbf{u}^k, n^k_t\nabla\mathbf{u}^k)
\end{aligned}$$

logo

$$\nu(\nabla\mathbf{u}^k, n^k\nabla\mathbf{u}^k_t) = \frac{\nu}{2}\frac{d}{dt}\|\sqrt{n}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} - \frac{\nu}{2}(\nabla\mathbf{u}^k, n^k_t\nabla\mathbf{u}^k)$$

portanto,

$$-\nu(n^k\Delta\mathbf{u}^k,\mathbf{u}_t^k)=\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}-\frac{\nu}{2}(\nabla\mathbf{u}^k,n_t^k\nabla\mathbf{u}^k)+\nu(\nabla\mathbf{u}^k,\nabla n^k.\mathbf{u}_t^k)$$

Assim, obtemos a seguinte desigualdade diferencial:

$$\begin{aligned}&\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\alpha\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\leq\frac{\nu}{2}(\nabla\mathbf{u}^k,n_t^k\nabla\mathbf{u}^k)-\nu(\nabla\mathbf{u}^k,\nabla n^k.\mathbf{u}_t^k)\\&-((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k,\mathbf{u}_t^k)-\nu[-2((\nabla n^k.\nabla)\mathbf{u}^k,\mathbf{u}_t^k)]+(n^k\mathbf{f},\mathbf{u}_t^k)\\&+\varepsilon^2\Bigg[-\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k,\mathbf{u}_t^k\right)+\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k,\mathbf{u}_t^k\right)\\&-\left(\frac{1}{n^k}\Delta n^k\nabla n^k,\mathbf{u}_t^k\right)+\left(\nabla\Delta n^k,\mathbf{u}_t^k\right)\Bigg].\end{aligned}$$

Agora, estimando os temos à direita da desigualdade acima, procederemos utilizando as desigualdades de Hölder, de Young, de interpolação e as imersões clássicas de Sobolev, como se segue:

$$\begin{aligned}\frac{\nu}{2}|(\nabla\mathbf{u}^k,n_t^k\nabla\mathbf{u}^k)| &\leq \frac{\nu}{2}\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|n_t^k\|_{L^6(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^3(\mathbb{T}^d)}\\&\overset{(i),(ii)}{\leq} C(\nu)\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|n_t^k\|_{H^1(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\\&\leq C(\nu,\delta_2,\delta_4)\|\nabla\mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)}+\delta_2\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_4\|n_t^k\|^2_{H^1(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|(\nabla\mathbf{u}^k,\nabla n^k.\mathbf{u}_t^k)| &\leq \nu\|\nabla\mathbf{u}^k\|_{L^3(\mathbb{T}^d)}\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\&\overset{(i),(ii)}{\leq} C(\nu)\|\nabla\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\&\leq C(\nu,\delta_1)\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\&\leq C(\nu,\delta_1,\delta_2)\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)}+\delta_2\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\&\leq C(\nu,\delta_1,\delta_2)\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}+C(\nu,\delta_1,\delta_2)\|\Delta n^k\|^8_{L^2(\mathbb{T}^d)}+\delta_2\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\&\quad+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(n^k\mathbf{u}^k.\nabla\mathbf{u}^k,\mathbf{u}_t^k)| &\leq \beta\|\mathbf{u}^k.\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\&\leq C(\beta)\|\mathbf{u}^k.\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\&\leq C(\beta,\delta_1)\|\mathbf{u}^k\|^2_{L^6(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^2_{L^3(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\&\leq C(\beta,\delta_1)\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\nabla^2\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\&\leq C(\beta,\delta_1)\|\nabla\mathbf{u}^k\|^3_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\&\leq C(\beta,\delta_1,\delta_2)\|\nabla\mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)}+\delta_2\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\varepsilon^2\left|\left(\nabla\Delta n^k,\mathbf{u}_t^k\right)\right| = \varepsilon^2|(\Delta n^k, div\ \mathbf{u}_t^k) = 0;$$

$$\begin{aligned}\left|(n^k\mathbf{f},\mathbf{u}_t^k)\right| &\leq \|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\delta_1)\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k,\mathbf{u}_t^k\right)\right| &\leq C(\varepsilon,\alpha)\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^3(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1)\|\nabla n^k\|^2_{L^6(\mathbb{T}^d)}\|\nabla^2 n^k\|^2_{L^3(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\ &\overset{(i),(ii)}{\leq} C(\varepsilon,\alpha,\delta_1)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^2(\mathbb{T}^d)}\|\nabla^3 n^k\|_{L^2(\mathbb{T}^d)}\\ &\quad+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1)\|\Delta n^k\|^3_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1,\delta_3)\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \delta_3\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\ &\quad+\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k,\mathbf{u}_t^k\right)\right| &\leq C(\varepsilon,\alpha)\|\nabla n^k\|^3_{L^6(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha)\|\Delta n^k\|^3_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1)\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{n^k}\Delta n^k\nabla n^k,\mathbf{u}_t^k\right)\right| &\leq C(\varepsilon,\alpha)\|\Delta n^k\|_{L^3(\mathbb{T}^d)}\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^3(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\ &\overset{(i),(ii)}{\leq} C(\varepsilon,\alpha,\delta_1)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1,\delta_3)\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \delta_3\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

Obtemos a seguinte desigualdade diferencial;

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \alpha\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\varepsilon,\alpha,\beta,\delta_1,\delta_2,\delta_3,\delta_4)(\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}\\ &+\|\nabla\mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^8_{L^2(\mathbb{T}^d)}) + 6\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + 2\delta_2\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ &+2\delta_3\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \delta_4\|n_t^k\|^2_{H^1(\mathbb{T}^d)} + C(\delta_1)\|n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}. \qquad (2.9)\end{aligned}$$

Agora, fazendo $\mathbf{v} = -\Delta\mathbf{u}^k$ na equação (2.5) temos;

$$\begin{aligned}&\nu(n^k\Delta\mathbf{u}^k,\Delta\mathbf{u}^k) = (n^k\mathbf{u}_t^k,\Delta\mathbf{u}^k) + ((n^k\mathbf{u}.\nabla)\mathbf{u}^k,\Delta\mathbf{u}^k) + 2\nu((\nabla n^k.\nabla)\mathbf{u}^k,\Delta\mathbf{u}^k)\\ &-(n^k\mathbf{f},\Delta\mathbf{u}^k) + \varepsilon^2\Bigg[\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k,\Delta\mathbf{u}^k\right) - \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k,\Delta\mathbf{u}^k\right)\end{aligned}$$

$$+\left(\frac{1}{n^k}\Delta n^k \nabla n^k, \Delta \mathbf{u}^k\right) - \big(\nabla \Delta n^k, \Delta \mathbf{u}^k\big)\Bigg]$$

Estimamos os termos à direita da equação acima, como se segue:

$$\varepsilon^2 \left|\big(\nabla \Delta n^k, \Delta \mathbf{u}^k\big)\right| = \varepsilon^2 |(\Delta n^k, div \Delta \mathbf{u}^k)| = \varepsilon^2 |(\Delta n^k, \Delta div\ \mathbf{u}^k)| = 0;$$

$$\begin{aligned}\left|(n^k \mathbf{u}_t^k, \Delta \mathbf{u}^k)\right| &\leq \beta \|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)} \|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\ &\leq \frac{\beta \mu_1}{2} \|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\beta}{2\mu_1} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(n^k \mathbf{u}^k.\nabla \mathbf{u}^k, \Delta \mathbf{u}^k)| &\leq \beta \|\mathbf{u}^k\|_{L^6(\mathbb{T}^d)} \|\nabla \mathbf{u}^k\|_{L^3(\mathbb{T}^d)} \|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C(\beta, \delta_2^{'}) \|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \|\nabla \mathbf{u}^k\|^2_{L^3(\mathbb{T}^d)} + \delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\ &\overset{(i),(ii)}{\leq} C(\beta, \delta_2^{'}) \|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} + \delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C(\beta, \delta_2^{'}) \|\nabla \mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)} + 2\delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}2\nu \left|\big((\nabla n^k.\nabla)\mathbf{u}^k, \Delta \mathbf{u}^k\big)\right| &\leq \nu \|\nabla n^k\|_{L^6(\mathbb{T}^d)} \|\nabla \mathbf{u}^k\|_{L^3(\mathbb{T}^d)} \|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C(\nu, \delta_2^{'}) \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \|\nabla \mathbf{u}^k\|^2_{L^3(\mathbb{T}^d)} + \delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\ &\overset{(i),(ii)}{\leq} C(\nu, \delta_2^{'}) \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} + \delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C(\nu, \delta_2^{'}) \|\Delta n^k\|^8_{L^2(\mathbb{T}^d)} + C(\nu, \delta_2^{'}) \|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + 2\delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(n^k \mathbf{f}, \Delta \mathbf{u}^k)| &\leq \|n^k\|_{L^\infty(\mathbb{T}^d)} \|\mathbf{f}\|_{L^2(\mathbb{T}^d)} \|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C(\delta_2^{'}) \|n^k\|^2_{L^\infty(\mathbb{T}^d)} \|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + \delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2 \left|\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \Delta \mathbf{u}^k\right)\right| &\leq C(\varepsilon, \alpha) \|\nabla n^k\|_{L^6(\mathbb{T}^d)} \|\nabla^2 n^k\|_{L^3(\mathbb{T}^d)} \|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\ &\overset{(i),(ii)}{\leq} C(\varepsilon, \alpha, \delta_2^{'}) \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \|\Delta n^k\|_{L^2(\mathbb{T}^d)} \|\nabla \Delta n^k\|_{L^2(\mathbb{T}^d)} \\ &\quad + \delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C(\varepsilon, \alpha, \delta_2^{'}, \delta_3^{'}) \|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \delta_3^{'} \|\nabla \Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\ &\quad + \delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2 \left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \Delta \mathbf{u}^k\right)\right| &\leq C(\varepsilon, \alpha) \|\nabla n^k\|^3_{L^6(\mathbb{T}^d)} \|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C(\varepsilon, \alpha) \|\Delta n^k\|^3_{L^2(\mathbb{T}^d)} \|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C(\varepsilon, \alpha, \delta_2^{'}) \|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \delta_2^{'} \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{n^k}\Delta n^k\nabla n^k,\Delta\mathbf{u}^k\right)\right| &\le C(\varepsilon,\alpha)\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\Delta n^k\|_{L^3(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\varepsilon,\alpha,\delta_2^{'})\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^3(\mathbb{T}^d)}+\delta_2^{'}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
&\overset{(i),(ii)}{\le} C(\varepsilon,\alpha,\delta_2^{'},\delta_3^{'})\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)}+\delta_3^{'}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\delta_2^{'}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

Obtemos,

$$\begin{aligned}
&\nu\alpha\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\le C(\nu,\varepsilon,\alpha,\beta,\delta_2^{'},\delta_3^{'})(\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}+\|\nabla\mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)}\\
&+\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)}+\|\Delta n^k\|^8_{L^2(\mathbb{T}^d)})+8\delta_2^{'}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+2\delta_3^{'}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\
&+C(\delta^{'})\|n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}+\frac{\beta\mu_1}{2}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\frac{\beta}{2\mu_1}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)},
\end{aligned}$$

fazendo $\mu_1 = \beta/\nu\alpha$, multiplicando por $\nu\alpha^2/2\beta^2$ e fazendo $\delta_2^{'} = 2\beta^2\delta_2/8\nu\alpha^2$ e $\delta_3^{'} = \beta^2\delta_3/\nu\alpha^2$ assim obtemos:

$$\begin{aligned}
&\frac{\nu^2\alpha^3}{4\beta^2}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\le C(\nu,\varepsilon,\alpha,\beta,\delta_2,\delta_3)(\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}+\|\nabla\mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)}\\
&+\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)}+\|\Delta n^k\|^8_{L^2(\mathbb{T}^d)})+\delta_2\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_3\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\
&+\frac{\alpha}{4}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+C(\nu,\alpha,\beta,\delta_2)\|n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}. \qquad (2.10)
\end{aligned}$$

Por outro lado, multiplicando a equação (2.4) por $-\Delta n_t^k+\nu\Delta^2 n^k$ e integrando sobre $\mathbb{T}^d$, obtemos:

$$\begin{aligned}
&-(n_t^k,\Delta n_t^k)+\nu(n_t^k,\Delta^2 n^k)-(\mathbf{u}^k.\nabla n^k,\Delta n_t^k)+\nu(\mathbf{u}^k.\nabla n^k,\Delta^2 n^k)\\
&=-\nu(\Delta n^k,\Delta n_t^k)+\nu^2(\Delta n^k,\Delta^2 n^k),
\end{aligned}$$

e fazendo integração por partes, temos:

$$\begin{aligned}
&\nu\frac{d}{dt}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}+\nu^2\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\
&=\nu(\nabla(\mathbf{u}^k.\nabla n^k),\nabla\Delta n^k)-(\nabla(\mathbf{u}^k.\nabla n^k),\nabla n_t^k)
\end{aligned}$$

Estimamos os termos à direita da equação acima, como se segue:

$$\begin{aligned}
\nu|(\nabla(\mathbf{u}^k.\nabla n^k),\nabla\Delta n^k)| &= \nu|(\nabla\mathbf{u}^k.\nabla n^k+\mathbf{u}^k.\nabla^2 n^k,\nabla\Delta n^k)|\\
&\le \nu\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^3(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)}\\
&\quad+\nu\|\mathbf{u}^k\|_{L^6(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^3(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)}\\
&\overset{(i),(ii)}{\le} C(\nu,\delta_3)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}+\delta_3\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\
&\quad+C(\nu,\delta_3)\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)}\\
&\quad+\delta_3\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}
\end{aligned}$$

$$\begin{aligned}
&\leq C(\nu,\delta_2,\delta_3)\|\Delta n^k\|^8_{L^2(\mathbb{T}^d)} + C(\nu,\delta_2,\delta_3)\|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}\\
&\quad + C(\nu,\delta_3)\|\nabla \mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + C\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \delta_2\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
&\quad + 3\delta_3\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(\nabla(\mathbf{u}^k.\nabla n^k),\nabla n^k_t)| &= |(\nabla \mathbf{u}^k.\nabla n^k + \mathbf{u}^k.\nabla^2 n^k, \nabla n^k_t)|\\
&\leq \|\nabla \mathbf{u}^k\|_{L^3(\mathbb{T}^d)}\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\nabla n^k_t\|_{L^2(\mathbb{T}^d)}\\
&\quad + \|\mathbf{u}^k\|_{L^6(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^3(\mathbb{T}^d)}\|\nabla n^k_t\|_{L^2(\mathbb{T}^d)}\\
&\overset{(i),(ii)}{\leq} C(\delta_4)\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \delta_4\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)}\\
&\quad + C(\delta_4)\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)} + \delta_4\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C(\delta_2,\delta_4)\|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + C(\delta_2,\delta_4)\|\Delta n^k\|^8_{L^2(\mathbb{T}^d)}\\
&\quad + C(\delta_3,\delta_4)\|\nabla \mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)}\\
&\quad + C(\delta_3,\delta_4)\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \delta_2\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_3\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\
&\quad + 2\delta_4\|n^k_t\|^2_{H^1(\mathbb{T}^d)};
\end{aligned}$$

Assim, temos:

$$\begin{aligned}
&\nu\frac{d}{dt}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu^2\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C(\nu,\delta_2,\delta_3,\delta_4)(\|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^4_{L^2(\mathbb{T}^d)}\\
&+\|\Delta n^k\|^8_{L^2(\mathbb{T}^d)}) + 2\delta_2\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + 4\delta_3\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + 2\delta_4\|n^k_t\|^2_{H^1(\mathbb{T}^d)}. \qquad (2.11)
\end{aligned}$$

Então, somando as desigualdades diferenciais (2.7), (2.8), (2.9), (2.10) e (2.11) e fazendo $\delta_1 = \alpha/24$, $\delta_2 = \nu^2\alpha^3/20\beta^2$, $\delta_3 = \nu^2/14$ e $\delta_4 = 1/6$ obtemos:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\sqrt{n^k}\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\right) + \frac{\alpha}{2}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)}\\
&+\frac{1}{2}\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2}{2}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2\alpha^3}{2\beta}\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C(\nu,\varepsilon,\alpha,\beta)(\|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + \|n^k\|^4_{H^2(\mathbb{T}^d)} + \|n^k\|^6_{H^2(\mathbb{T}^d)}\\
&+\|n^k\|^8_{H^2(\mathbb{T}^d)}) + C(\nu,\alpha,\beta)\|n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

■

**Lema 2.2.2** *Sejam* $\mathbf{u}_0 \in V \cap H^2(\mathbb{T}^d)$, $n_0 \in H^3(\mathbb{T}^d)$ *e* $\mathbf{f} \in L^2(0,T;H^1(\mathbb{T}^d))$, $\mathbf{f}_t \in L^2(0,T;L^2(\mathbb{T}^d))$. *Então o par* $(\mathbf{u}^k, n^k)$ *satisfaz a seguinte desigualdade diferencial:*

$$\frac{d}{dt}(\frac{1}{2}\|\sqrt{n^k}\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)})$$
$$+\frac{\nu\alpha}{2}\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu}{2}\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\varepsilon,\alpha,\beta)(\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}).(\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^4_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}) + C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|n_t\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}$$
$$+C\|n^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)}).$$

*Mais ainda,*

$$\|\mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\varepsilon,\alpha,\|\mathbf{u}_0\|_{H^2(\mathbb{T}^d)},\|n_0\|_{H^3(\mathbb{T}^d)}) \quad e$$
$$\|\nabla n_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\|\mathbf{u}_0\|_{H^2(\mathbb{T}^d)},\|n_0\|_{H^3(\mathbb{T}^d)}).$$

**Demonstração:**

Somando e subtraindo $n^k$ na equação (2.4), multiplicando por $n^k$ e integrando sobre $\mathbb{T}^d$ obtemos:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} &= \|n^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|n^k\|^2_{L^2(\mathbb{T}^d)} \qquad (2.12)\end{aligned}$$

Agora derivando a equação (2.4) em relação a variável temporal, temos:

$$n_{tt}^k + \mathbf{u}_t^k.\nabla n^k + \mathbf{u}^k.\nabla n_t^k = \nu\Delta n_t^k,$$

e multiplicando a equação acima por $n_t^k$ e integrando sobre $\mathbb{T}^d$, temos:

$$\frac{1}{2}\frac{d}{dt}\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} = -(\mathbf{u}_t^k.\nabla n^k, n_t^k) - (\mathbf{u}^k.\nabla n_t^k, n_t^k).$$

Estimando os termos à direita da equação acima:

$$\begin{aligned}|(\mathbf{u}_t^k.\nabla n^k, n_t^k)| &\leq \|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|n_t^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)}\|n_t^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C(\delta_2)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$|(\mathbf{u}^k.\nabla n^k, n_t^k)| \leq \|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla n_t^k\|_{L^2(\mathbb{T}^d)}\|n_t^k\|_{L^2(\mathbb{T}^d)}$$

$$\leq C(\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)};$$

Assim temos,

$$\frac{1}{2}\frac{d}{dt}\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)} \leq C(\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|n^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+C(\delta_2)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}. \tag{2.13}$$

Multiplicando a equação (2.4) por $-\Delta^2 n^k$, integrando sobre $\mathbb{T}^d$ e por integração por partes temos:

$$\frac{1}{2}\frac{d}{dt}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} = -(\nabla(\mathbf{u}^k.\nabla n^k), \nabla\Delta n^k),$$

estimando o termo da direita,

$$\begin{aligned}
|(\nabla(\mathbf{u}^k.\nabla n^k), \nabla\Delta n^k)| &= |(\nabla n^k.\nabla\mathbf{u}^k, \nabla\Delta n^k) + (\mathbf{u}^k.\nabla^2 n^k, \nabla\Delta n^k)| \\
&\leq \|\nabla\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\nabla n^k\|_{L^4(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)} \\
&\quad +\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)} \\
&\leq C(\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + 2\delta_1\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

Portanto, para $\delta_1 = \nu/4$ temos,

$$\frac{1}{2}\frac{d}{dt}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu}{2}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \tag{2.14}$$

Tirando a norma $L^2$ da equação (2.4), temos;

$$\begin{aligned}
\|n^k_t\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\mathbf{u}^k.\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}
\end{aligned} \tag{2.15}$$

Derivando a equação (2.4) em relação à variável temporal $t$, obtemos,

$$n^k_{tt} + \mathbf{u}^k_t.\nabla n^k + \mathbf{u}^k.\nabla n^k_t = \nu\Delta n^k_t, \tag{2.16}$$

multiplicando a equação (2.16) por $\Delta n^k_t$ e integrando sobre $\mathbb{T}^d$, temos,

$$\frac{1}{2}\frac{d}{dt}\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} = -(\mathbf{u}^k_t.\nabla n^k, \Delta n^k_t) - (\mathbf{u}^k.\nabla n^k_t, \Delta n^k_t),$$

e estimando os termos a direita da equação acima, como se segue:

$$\begin{aligned}
|(\mathbf{u}^k_t.\nabla n^k, \Delta n^k_t)| &\leq \|\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\Delta n^k_t\|_{L^2(\mathbb{T}^d)} \\
&\leq C(\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

$$|(\mathbf{u}^k.\nabla n^k_t, \Delta n^k_t)| \leq \|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla n^k_t\|_{L^2(\mathbb{T}^d)}\|\Delta n^k_t\|_{L^2(\mathbb{T}^d)}$$

$$\leq \quad C(\delta_1)\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)}.$$

Portanto, temos,

$$\frac{1}{2}\frac{d}{dt}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\delta_1)\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+C(\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + 2\delta_1\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)}. \tag{2.17}$$

Derivando a equação (2.5) em relação a variável $t$,

$$(n^k\mathbf{u}_{tt}^k, \mathbf{v}) - \nu(n^k\Delta\mathbf{u}_t^k, \mathbf{v}) = -(n_t^k\mathbf{u}_t^k, \mathbf{v}) + (n_t^k\Delta\mathbf{u}^k, \mathbf{v}) - ((n_t^k\mathbf{u}^k.\nabla)\mathbf{u}^k, \mathbf{v})$$
$$-((n^k\mathbf{u}_t^k.\nabla)\mathbf{u}^k, \mathbf{v}) - ((n^k\mathbf{u}^k.\nabla)\mathbf{u}_t^k, \mathbf{v}) - 2\nu[-((\nabla n_t^k.\nabla)\mathbf{u}^k, \mathbf{v}) - ((\nabla n^k.\nabla)\mathbf{u}_t^k, \mathbf{v})]$$
$$+(n_t^k\mathbf{f}, \mathbf{v}) + (n^k\mathbf{f}_t, \mathbf{v}) + \varepsilon^2\left[\left(\frac{1}{(n^k)^2}n_t^k(\nabla n^k.\nabla)\nabla n^k, \mathbf{v}\right) - \left(\frac{1}{n^k}(\nabla n_t^k.\nabla)\nabla n^k, \mathbf{v}\right)\right.$$
$$-\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n_t^k, \mathbf{v}\right) - \left(\frac{2}{(n^k)^3}n_t^k(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{v}\right) + \left(\frac{3}{(n^k)^2}(\nabla n_t^k.\nabla n^k)\nabla n^k, \mathbf{v}\right)$$
$$\left.+\left(\frac{1}{(n^k)^2}n_t^k\Delta n^k\nabla n^k, \mathbf{v}\right) - \left(\frac{1}{n^k}\Delta n_t^k\nabla n^k, \mathbf{v}\right) - \left(\frac{1}{n^k}\Delta n^k\nabla n_t^k, \mathbf{v}\right) + (\nabla\Delta n_t^k, \mathbf{v})\right]. \tag{2.18}$$

Fazendo $\mathbf{v} = \mathbf{u}_t^k$ na equação (2.18) e observando que,

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} = \frac{1}{2}\frac{d}{dt}(\mathbf{u}_t^k, n^k\mathbf{u}_t^k) = \frac{1}{2}(\mathbf{u}_{tt}^k, n^k\mathbf{u}_t^k) + \frac{1}{2}(\mathbf{u}_t^k, n_t^k\mathbf{u}_t^k)$$
$$+\frac{1}{2}(\mathbf{u}_t^k, n_t^k\mathbf{u}_{tt}^k) = (\mathbf{u}_{tt}^k, n^k\mathbf{u}_t^k) + \frac{1}{2}(\mathbf{u}_t^k, n_t^k\mathbf{u}_t^k)$$

segue que

$$(n^k\mathbf{u}_{tt}^k, \mathbf{u}_t^k) = \frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}(\mathbf{u}_t^k, n_t^k\mathbf{u}_t^k);$$

também observe,

$$-\nu(n^k\Delta\mathbf{u}_t^k, \mathbf{u}_t^k) = \nu(\nabla\mathbf{u}_t^k, \mathbf{u}_t^k.\nabla n^k) + \nu(\nabla\mathbf{u}_t^k, n^k\nabla\mathbf{u}_t^k)$$
$$= \nu\|\sqrt{n^k}\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu(\nabla\mathbf{u}_t^k, \mathbf{u}_t^k.\nabla n^k)$$
$$\geq \nu\alpha\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu(\nabla\mathbf{u}_t^k, \mathbf{u}_t^k.\nabla n^k).$$

Agora estimamos os seguintes termos como se segue:

$$\begin{aligned}
\frac{1}{2}|(\mathbf{u}_t^k, n_t^k\mathbf{u}_t^k)| &\leq \frac{1}{2}|(\mathbf{u}_t^k, \mathbf{u}^k.\nabla n^k\mathbf{u}_t^k)| + \frac{\nu}{2}|(\mathbf{u}_t^k, \Delta n^k\mathbf{u}_t^k)| \\
&\leq \frac{1}{2}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)} \\
&\quad +\frac{\nu}{2}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)} \\
&\leq C\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu, \delta_2)\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}
\end{aligned}$$

$$+\delta_2\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}\nu|(\nabla \mathbf{u}_t^k, \mathbf{u}_t^k.\nabla n^k)| &\leq \nu\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla \mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\nu,\delta_2)\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|(n_t^k\Delta \mathbf{u}^k, \mathbf{u}_t^k)| &\leq \nu\|n_t^k\|_{L^4(\mathbb{T}^d)}\|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C(\nu,\delta_2)\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(n_t^k\mathbf{f}, \mathbf{u}_t^k)| &\leq \|n_t^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{f}\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|n_t^k\|_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|_{H^1(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + C\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((n_t^k\mathbf{u}^k.\nabla)\mathbf{u}^k, \mathbf{u}_t^k)| &\leq \|n_t^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla \mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\varepsilon^2|(\nabla\Delta n_t^k, \mathbf{u}_t^k)| = \varepsilon^2|(\Delta n_t^k, div\mathbf{u}_t^k)| = 0;$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{n^k}\Delta n^k\nabla n_t^k, \mathbf{u}_t^k\right)\right| &\leq C(\varepsilon,\alpha)\|\Delta n^k\|_{L^4(\mathbb{T}^d)}\|\nabla n_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_2)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{n^k}\Delta n_t^k\nabla n^k, \mathbf{u}_t^k\right)\right| &\leq C(\varepsilon,\alpha)\|\Delta n_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{(n^k)^2}n_t^k\Delta n^k\nabla n^k, \mathbf{u}_t^k\right)\right| &\leq C(\varepsilon,\alpha)\|n_t^k\|_{L^6(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^6(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_2)\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{3}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n_t^k, \mathbf{u}_t^k\right)\right| &\leq C(\varepsilon,\alpha)\|\nabla n^k\|^2_{L^6(\mathbb{T}^d)}\|\nabla n_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^6(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_2)\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\varepsilon^2\left|\left(\frac{2}{(n^k)^3}n_t^k(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}_t^k\right)\right| \leq C(\varepsilon,\alpha)\|n_t^k\|_{L^4(\mathbb{T}^d)}\|\nabla n^k\|^3_{L^6(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}$$

$$
\begin{aligned}
&\le C(\varepsilon,\alpha)\|n_t^k\|_{H^1(\mathbb{T}^d)}\|\Delta n^k\|^3_{L^2(\mathbb{T}^d)}\|\nabla \mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\varepsilon,\alpha,\delta_2)\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)}\|n_t^k\|^2_{H^1(\mathbb{T}^d)}+\delta_2\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n_t^k,\mathbf{u}_t^k\right)\right| &\le C(\varepsilon,\alpha)\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\Delta n_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla n_t^k.\nabla)\nabla n^k,\mathbf{u}_t^k\right)\right| &\le C(\varepsilon,\alpha)\|\nabla n_t^k\|_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}\\
&\le C(\varepsilon,\alpha,\delta_2)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{(n^k)^2}n_t^k(\nabla n^k.\nabla)\nabla n^k,\mathbf{u}_t^k\right)\right| &\le C(\varepsilon,\alpha)\|n_t^k\|_{L^6(\mathbb{T}^d)}\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^6(\mathbb{T}^d)}\\
&\le C(\varepsilon,\alpha,\delta_2)\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)}\|n_t^k\|^2_{H^1(\mathbb{T}^d)}+\delta_2\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(n^k\mathbf{f}_t,\mathbf{u}_t^k)| &\le \|n^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{f}_t\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}\\
&\le C\|n^k\|_{H^1(\mathbb{T}^d)}\|\mathbf{f}_t\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\le C\|n^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(n^k\mathbf{u}_t^k.\nabla\mathbf{u}^k,\mathbf{u}_t^k)| &\le \beta\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}\\
&\le C(\beta,\delta_2)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(n^k\mathbf{u}^k.\nabla\mathbf{u}_t^k,\mathbf{u}_t^k)| &\le \beta\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\beta,\delta_2)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
2\nu\left|\left((\nabla n_t^k.\nabla)\mathbf{u}^k,\mathbf{u}_t^k\right)\right| &\le C(\nu)\|\nabla n_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}\\
&\le C(\nu,\delta_2)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
2\nu\left|\left((\nabla n^k.\nabla)\mathbf{u}_t^k,\mathbf{u}_t^k\right)\right| &\le C(\nu)\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\nu,\delta_2)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

Obtemos,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\nu\alpha\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\le C(\nu,\varepsilon,\alpha,\beta,\delta_1,\delta_2)(\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\|n_t^k\|^2_{L^2(\mathbb{T}^d)}\\
&+\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}+\|n^k\|^2_{L^2(\mathbb{T}^d)}+\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)})(\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\|n^k\|^2_{L^2(\mathbb{T}^d)}+\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}
\end{aligned}
$$

$$+\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}) + C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}$$
$$+C\|n^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)} + C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+2\delta_1\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} + 13\delta_2\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}. \tag{2.19}$$

Somando (2.12), (2.13), (2.14), (2.15), (2.17) e (2.19), fazendo $\delta_1 = \nu/10$, $\delta_2 = \nu\alpha/28$ obtemos

$$\frac{d}{dt}(\frac{1}{2}\|\sqrt{n^k}\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)})$$
$$+\frac{\nu\alpha}{2}\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+\nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu}{2}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} \le C(\nu,\varepsilon,\alpha,\beta)(\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)}).(\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^4_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}) + C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|n_t\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}$$
$$+C\|n^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)}).$$

Mostremos em seguida, que $\|\mathbf{u}^k_t(0)\|^2_{L^2(\mathbb{T}^d)}$ e $\|\nabla n^k_t(0)\|^2_{L^2(\mathbb{T}^d)}$ são limitados uniformemente em $k$. Fazendo $\mathbf{v} = \mathbf{u}^k_t$ em (2.5) obtemos:

$$\|\sqrt{n^k}\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} = -\Bigg((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k - \nu[-n^k\Delta\mathbf{u}^k - (\mathbf{u}^k.\nabla)\nabla n^k - (\nabla n^k.\nabla)\mathbf{u}^k + \mathbf{u}^k\Delta n^k]$$
$$+n^k\mathbf{f} + \varepsilon^2\Bigg[-\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k + \frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k - \frac{1}{n^k}\Delta n^k\nabla n^k\Bigg], \mathbf{u}^k_t\Bigg)$$
$$+\varepsilon^2(\nabla\Delta n^k, \mathbf{u}^k_t)$$
$$= (\Phi, \mathbf{u}^k_t) + \varepsilon^2(\nabla\Delta n^k, \mathbf{u}^k_t).$$

Observe que,

$$(\nabla\Delta n^k, \mathbf{u}^k_t) = -(\Delta n^k, (div(\mathbf{u}^k))_t) = 0,$$
$$\|\sqrt{n^k}\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} \ge \alpha\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)},$$

então,

$$\alpha\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} \le |(\Phi, \mathbf{u}^k_t)| \le \|\Phi\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}$$

o que implica,

$$\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} \le C(\alpha)\|\Phi\|^2_{L^2(\mathbb{T}^d)}.$$

Para todo $k \in \mathbb{N}$ temos que, $(\mathbf{u}^k, n^k) \in C^1([0,T^k]; H^2(\mathbb{T}^d) \cap V) \times C^2(\mathbb{T}^d \times [0,T^k])$; observe que por hipóteses $\mathbf{f} \in L^2(0,T; L^2(\mathbb{T}^d))$, $\mathbf{f}_t \in L^2(0,T; L^2(\mathbb{T}^d))$, portando pelo Lema (1.1.2) temos que $\mathbf{f} \in C([0,T]; L^2(\mathbb{T}^d))$. Logo, podemos tomar $t = 0$, assim,

$$
\begin{aligned}
&\|\mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \leq C(\alpha)\|\Phi(0)\|^2_{L^2(\mathbb{T}^d)} \\
&= C(\alpha)\Bigg\|(n^k(0)\mathbf{u}^k(0).\nabla)\mathbf{u}^k(0) - \nu[-n^k(0)\Delta\mathbf{u}^k(0) - (\mathbf{u}^k(0).\nabla)\nabla n^k(0) + n^k(0)\mathbf{f}(0) \\
&-(\nabla n^k(0).\nabla)\mathbf{u}^k(0) + \mathbf{u}^k(0)\Delta n^k(0)] + \varepsilon^2\Bigg[-\frac{1}{n^k(0)}(\nabla n^k(0).\nabla)\nabla n^k(0) \\
&+\frac{1}{(n^k(0))^2}(\nabla n^k(0).\nabla n^k(0))\nabla n^k(0) - \frac{1}{n^k(0)}\Delta n^k(0)\nabla n^k(0)\Bigg]\Bigg\|^2_{L^2(\mathbb{T}^d)},
\end{aligned}
$$

observe que $\mathbf{u}^k(0) = P_k\mathbf{u}_0$ e $n^k(0) = n_0$ e por hipótese $\mathbf{u}_0 \in V \cap H^2(\mathbb{T}^d)$, $n_0 \in H^3(\mathbb{T}^d)$, portanto mostra-se facilmente que,

$$
\|\mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \leq C(\alpha)\|\Phi(0)\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu, \varepsilon, \alpha, \|\mathbf{u}_0\|_{H^2(\mathbb{T}^d)}, \|n_0\|_{H^3(\mathbb{T}^d)}).
$$

Por outro lado, aplicando o operador $\nabla$ na equação (2.4), avaliando a norma $L^2(\mathbb{T}^d)$ (em ambos os lados) e fazendo $t = 0$ mostra-se facilmente que,

$$
\|\nabla n_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu, \|\mathbf{u}_0\|_{H^2(\mathbb{T}^d)}, \|n_0\|_{H^3(\mathbb{T}^d)}).
$$

■

**Lema 2.2.3** *Seja* $\mathbf{f} \in L^2(0,T; H^1(\mathbb{T}^d))$. *Então* $(\mathbf{u}^k, n^k)$ *satisfaz as seguintes desigualdades,*

$$
\begin{aligned}
(i)\quad & \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\alpha,\beta,\varepsilon)(\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)} \\
& +\|\nabla\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^8_{L^2(\mathbb{T}^d)}) \\
& +C(\nu,\varepsilon,\alpha,\beta)\|n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha,\beta)\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + C(\alpha)\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}. \\
(ii)\quad & \|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu)(\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}). \\
(iii)\quad & \|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\alpha,\beta)\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha)(\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)}) \\
& +C(\nu,\varepsilon,\alpha)(\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^8_{L^2(\mathbb{T}^d)}) \\
& +C(\nu,\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + C(\nu,\alpha)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha)\|\mathbf{u}_t^k\|^4_{L^2(\mathbb{T}^d)}.
\end{aligned}
$$

**Demonstração:**
$(i)$ Multiplicando a equação (2.4) por $\Delta^2 n^k$ e integrando sobre $\mathbb{T}^d$, temos:

$$
(n_t^k, \Delta^2 n^k) + (\mathbf{u}^k.\nabla n^k, \Delta^2 n^k) = \nu(\Delta n^k, \Delta^2 n^k);
$$

fazendo integrações por partes, obtemos:

$$
\nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} = (\nabla n_t^k, \nabla\Delta n^k) + (\nabla\mathbf{u}^k.\nabla n^k, \nabla\Delta n^k) + (\mathbf{u}^k.\nabla^2 n^k, \nabla\Delta n^k),
$$

e estimando os termos à direita da equação acima, como se segue:

$$
\begin{aligned}
|(\nabla n_t^k, \nabla\Delta n^k)| &\leq \|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} \|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)} \\
&\leq C(\delta_1)\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(\nabla\mathbf{u}^k.\nabla n^k, \nabla\Delta n^k)| &\leq \|\nabla\mathbf{u}^k\|_{L^3(\mathbb{T}^d)}\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)} \\
&\leq C(\delta_2)\|\nabla\mathbf{u}^k\|^2_{L^3(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\overset{(i),(ii)}{\leq} C(\delta_2)\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \delta\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C(\delta_1,\delta_2)\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + C(\delta_1,\delta_2)\|\Delta n^k\|^8_{L^2(\mathbb{T}^d)} + \delta_2\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
&\quad +\delta_1\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(\mathbf{u}^k.\nabla^2 n^k, \nabla\Delta n^k)| &\leq \|\mathbf{u}^k\|_{L^6(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^3(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)} \\
&\overset{(i),(ii)}{\leq} C(\delta_1)\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C(\delta_1)\|\nabla\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + C(\delta_1)\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + 2\delta_1\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

Assim, temos,

$$
\begin{aligned}
&\nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\delta_1,\delta_2)(\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^8_{L^2(\mathbb{T}^d)}) \\
&+4\delta_1\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C(\delta_1)\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}
$$

Somando a equação acima com (2.10) e fazendo $\delta_1 = \nu/10$, $\delta_2 = \nu^2\alpha^3/8\beta^2$ obtemos,

$$
\begin{aligned}
&\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\alpha,\beta,\varepsilon)(\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)} \\
&+\|\nabla\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^8_{L^2(\mathbb{T}^d)}) \\
&+C(\nu,\varepsilon,\alpha,\beta)\|n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha,\beta)\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + C(\alpha)\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}
$$

$(ii)$ Aplicando o operador $\Delta$ em (2.4), obtemos:

$$\Delta n_t^k + \Delta(\mathbf{u}^k.\nabla n^k) = \nu\Delta^2 n^k;$$

observe que

$$\Delta(\mathbf{u}^k.\nabla n^k) = \Delta\mathbf{u}^k.\nabla n^k + 2\nabla\mathbf{u}^k : \nabla^2 n^k + \mathbf{u}^k.\nabla\Delta n^k,$$

e portanto,

$$\nu\Delta^2 n^k = \Delta n_t^k + \Delta\mathbf{u}^k.\nabla n^k + \nabla\mathbf{u}^k : \nabla^2 n^k + \mathbf{u}^k.\nabla\Delta n^k,$$

avaliando a norma $L^2(\mathbb{T}^d)$ em ambos os lados da equação acima, segue que:

$$\nu\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k.\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}^k : \nabla^2 n^k\|^2_{L^2(\mathbb{T}^d)}$$

$$
\begin{aligned}
&+\|\mathbf{u}^k.\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\\
\leq\ & C(\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}+\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}+\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\\
&+\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\\
\leq\ & C(\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}+\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}).
\end{aligned}
$$

$(iii)$ Fazendo $\mathbf{v}=\Delta^2\mathbf{u}^k$ na equação (2.5) e integrando por partes, obtemos:

$$
\begin{aligned}
&\nu(n^k\nabla\Delta\mathbf{u}^k,\nabla\Delta\mathbf{u}^k)=-\nu(\nabla n^k.\Delta\mathbf{u}^k,\nabla\Delta\mathbf{u}^k)+(\nabla(n^k\mathbf{u}^k_t),\nabla\Delta\mathbf{u}^k)\\
&+(\nabla(n^k\mathbf{u}^k.\nabla\mathbf{u}^k),\nabla\Delta\mathbf{u}^k)+\nu[-2(\nabla(\nabla n^k.\nabla\mathbf{u}^k),\nabla\Delta\mathbf{u}^k)]-(\nabla(n^k\mathbf{f}),\nabla\Delta\mathbf{u}^k)\\
&+\Bigg[\left(\nabla\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k\right),\nabla\Delta\mathbf{u}^k\right)-\left(\nabla\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k\right),\nabla\Delta\mathbf{u}^k\right)\\
&+\left(\nabla\left(\frac{1}{n^k}\Delta n^k\nabla\right),\nabla\Delta\mathbf{u}^k\right)+(\nabla\Delta n^k,\Delta^2\mathbf{u}^k)\Bigg].
\end{aligned}
$$

Estimando os termos da direita:

$$
\begin{aligned}
\nu|(\nabla n^k.\Delta\mathbf{u}^k,\nabla\Delta\mathbf{u}^k)|\ \leq\ & \nu\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\
\leq\ & C(\nu,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
\leq\ & C(\nu,\delta_1)\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}+C(\nu,\delta_1)\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(\nabla(n^k\mathbf{u}^k_t),\nabla\Delta\mathbf{u}^k)|\ \leq\ & C(\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+C(\beta,\delta_1)\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}\\
&+2\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
\leq\ & C(\delta_1)\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}+C(\delta_1)\|\mathbf{u}^k_t\|^4_{L^2(\mathbb{T}^d)}\\
&+C(\beta,\delta_1)\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+2\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
&|(\nabla(n^k\mathbf{f}),\nabla\Delta\mathbf{u}^k)|\leq C(\delta_1)\|\nabla n^k\|^2_{L^4(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^4(\mathbb{T}^d)}+C(\delta_1)\|n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}\\
&+2\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
\leq\ & C(\delta_1)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}+C(\delta_1)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}+2\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
&|(\nabla(n^k\mathbf{u}^k.\nabla\mathbf{u}^k),\nabla\Delta\mathbf{u}^k)|\leq C(\delta_1)\|\nabla(n^k\mathbf{u}^k.\nabla\mathbf{u}^k)\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
\leq\ & C(\beta,\delta_1)\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}+C(\beta,\delta_1)\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
&+C(\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
\leq\ & C(\beta,\delta_1)\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}+(\beta,\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
&+C(\delta_1)\|\Delta\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)}+C(\delta_1)\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
\leq\ & C(\beta,\delta_1)\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}+C(\delta_1)\|\Delta\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)}+C(\delta_1)\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}
\end{aligned}
$$

$$+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}2\nu|(\nabla((\nabla n^k.\nabla)\mathbf{u}^k),\nabla\Delta\mathbf{u}^k)| &\leq C(\nu,\delta_1)\left\|\nabla\left((\nabla n^k.\nabla)\mathbf{u}^k\right)\right\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C(\nu,\delta_1)\|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)}\|\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\\ &\quad +C(\nu,\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C(\nu,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ &\quad +C(\nu,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C(\nu,\delta_1)\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}+C(\nu,\delta_1)\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}\\ &\quad +\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}&\varepsilon^2\left|\left(\nabla\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k\right),\nabla\Delta\mathbf{u}^k\right)\right|\leq C(\varepsilon,\delta_1)\left\|\nabla\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k\right)\right\|^2_{L^2(\mathbb{T}^d)}\\ &\quad +\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ \leq\ & C(\varepsilon,\alpha,\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\ &+C(\varepsilon,\alpha,\delta_1)\|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)}\\ &+C(\varepsilon,\alpha,\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla^3 n^k\|^2_{L^2(\mathbb{T}^d)}\\ &+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ \leq\ & C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^6_{L^2(\mathbb{T}^d)}+C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}\\ &+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}&\varepsilon^2\left|\left(\nabla\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k\right),\nabla\Delta\mathbf{u}^k\right)\right|\\ \leq\ & C(\varepsilon,\delta_1)\left\|\nabla\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k\right)\right\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ \leq\ & C(\varepsilon,\alpha,\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla n^k\|^6_{L^6(\mathbb{T}^d)}\\ &+3C(\varepsilon,\alpha,\delta_1)\|\nabla n^k\|^4_{L^\infty(\mathbb{T}^d)}\|\nabla^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\ &+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ \leq\ & C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^8_{L^2(\mathbb{T}^d)}+C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^6_{L^2(\mathbb{T}^d)}\\ &+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}&\varepsilon^2\left|\left(\nabla\left(\frac{1}{n^k}\Delta n^k\nabla n^k\right),\nabla\Delta\mathbf{u}^k\right)\right|\\ \leq\ & C(\varepsilon,\delta_1)\left\|\nabla\left(\frac{1}{n^k}\Delta n^k\nabla n^k\right)\right\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\end{aligned}$$

$$
\begin{aligned}
\leq \;& C(\varepsilon,\alpha,\delta_1)\|\nabla n^k\|^4_{L^\infty(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\
&+C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\\
&+C(\varepsilon,\alpha,\delta_1)\|\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)}\\
&+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
\leq \;& C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^6_{L^2(\mathbb{T}^d)}+C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}\\
&+\delta_1\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\varepsilon^2|(\nabla\Delta n^k,\Delta^2\mathbf{u}^k)|=\varepsilon^2|(\Delta n^k, div\Delta^2\mathbf{u}^k)|=0;
$$

Fazendo $\delta_1=\nu\alpha/24$ obtemos

$$
\begin{aligned}
&\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\leq C(\nu,\alpha,\beta)\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+C(\nu,\alpha)(\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}+\|\Delta\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)})\\
&+C(\nu,\varepsilon,\alpha)(\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k\|^6_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k\|^8_{L^2(\mathbb{T}^d)})\\
&+C(\nu,\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}+C(\nu,\alpha)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}+C(\nu,\alpha)\|\mathbf{u}^k_t\|^4_{L^2(\mathbb{T}^d)}.
\end{aligned}
$$

■

**Lema 2.2.4** *Sejam* $\mathbf{u}_0\in V\cap H^3(\mathbb{T}^d)$, $n_0\in H^4(\mathbb{T}^d)$ *e* $\mathbf{f}\in L^2(0,T;H^2(\mathbb{T}^d))$, $\mathbf{f}_t\in L^2(0,T;H^1(\mathbb{T}^d))$. *Então* $(\mathbf{u}^k,n^k)$ *satisfaz à seguinte desigualdade diferencial:*

$$
\begin{aligned}
&\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+\frac{\alpha}{2}\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)}+\frac{\nu^2\alpha^3}{4\beta^2}\|\Delta\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C(\nu,\beta)(\|n^k_t\|^2_{H^1(\mathbb{T}^d)}+\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}\\
&+C(\nu,\varepsilon,\alpha)(\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|n^k_t\|^2_{H^2(\mathbb{T}^d)}\\
&+C(\nu,\varepsilon,\alpha)(\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\|n^k_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}\\
&+\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+C(\nu,\varepsilon,\alpha)\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}\\
&+C(\nu,\varepsilon,\alpha)\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}+C(\nu,\varepsilon,\beta,\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)}
\end{aligned}
$$

*Mais ainda,*

$$
\|\nabla\mathbf{u}^k_t(0)\|^2_{L^2(\mathbb{T}^d)}\leq C(\nu,\varepsilon,\alpha,\beta,\|\mathbf{u}_0\|_{H^3(\mathbb{T}^d)},\|n_0\|_{H^4(\mathbb{T}^d)}).
$$

**Demonstração:**

Fazendo $\mathbf{v}=\mathbf{u}^k_{tt}$ na equação (2.18), obtemos:

$$
\begin{aligned}
&(n^k\mathbf{u}^k_{tt},\mathbf{u}^k_{tt})-\nu(n^k\Delta\mathbf{u}^k_t,\mathbf{u}^k_{tt})=-(n^k_t\mathbf{u}^k_t,\mathbf{u}^k_{tt})+(n^k_t\Delta\mathbf{u}^k,\mathbf{u}^k_{tt})-((n^k_t\mathbf{u}^k.\nabla)\mathbf{u}^k,\mathbf{u}^k_{tt})\\
&-((n^k\mathbf{u}^k_t.\nabla)\mathbf{u}^k,\mathbf{u}^k_{tt})-((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k_t,\mathbf{u}^k_{tt})-\nu[-2((\nabla n^k_t.\nabla)\mathbf{u}^k,\mathbf{u}^k_{tt})-2((\nabla n^k.\nabla)\mathbf{u}^k_t,\mathbf{u}^k_{tt})]\\
&+(n^k_t\mathbf{f},\mathbf{u}^k_{tt})+(n^k\mathbf{f}_t,\mathbf{u}^k_{tt})+\varepsilon^2\Bigg[\left(\frac{1}{(n^k)^2}n^k_t(\nabla n^k.\nabla)\nabla n^k,\mathbf{u}^k_{tt}\right)\\
&-\left(\frac{1}{n^k}(\nabla n^k_t.\nabla)\nabla n^k,\mathbf{u}^k_{tt}\right)-\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k_t,\mathbf{u}^k_{tt}\right)-\left(\frac{2}{(n^k)^3}n^k_t(\nabla n^k.\nabla n^k)\nabla n^k,\mathbf{u}^k_{tt}\right)\\
&\left(\frac{3}{(n^k)^2}(\nabla n^k_t.\nabla n^k)\nabla n^k,\mathbf{u}^k_{tt}\right)+\left(\frac{1}{(n^k)^2}n^k_t\Delta n^k\nabla n^k,\mathbf{u}^k_{tt}\right)-\left(\frac{1}{n^k}\Delta n^k_t\nabla n^k,\mathbf{u}^k_{tt}\right)\\
&-\left(\frac{1}{n^k}\Delta n^k\nabla n^k_t,\mathbf{u}^k_{tt}\right)+(\nabla\Delta n^k_t,\mathbf{u}^k_{tt})\Bigg]. \qquad (2.20)
\end{aligned}
$$

Observe que,

$$-\nu(n^k\Delta\mathbf{u}^k_t,\mathbf{u}^k_{tt})=\nu(\nabla\mathbf{u}^k_t,\mathbf{u}^k_{tt}.\nabla n^k)+\nu(\nabla\mathbf{u}^k_t,n^k\nabla\mathbf{u}^k_{tt})$$

e que,

$$
\begin{aligned}
\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} &= \frac{\nu}{2}\frac{d}{dt}(\nabla\mathbf{u}^k_t,n^k\nabla\mathbf{u}^k_t)\\
&= \frac{\nu}{2}(\nabla\mathbf{u}^k_{tt},n^k\nabla\mathbf{u}^k_t)+\frac{\nu}{2}(\nabla\mathbf{u}^k_t,n^k_t\nabla\mathbf{u}^k_t)+\frac{\nu}{2}(\nabla\mathbf{u}^k_{tt},n^k\nabla\mathbf{u}^k_t)\\
&= \nu(\nabla\mathbf{u}^k_t,n^k\nabla\mathbf{u}^k_{tt})+\frac{\nu}{2}(\nabla\mathbf{u}^k_t,n^k_t\nabla\mathbf{u}^k_t);
\end{aligned}
$$

portanto

$$-\nu(n^k\Delta\mathbf{u}^k_t,\mathbf{u}^k_{tt})=\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}-\frac{\nu}{2}(\nabla\mathbf{u}^k_t,n^k_t\nabla\mathbf{u}^k_t)+\nu(\nabla\mathbf{u}^k_t,\mathbf{u}^k_{tt}.\nabla n^k).$$

Também temos, que,

$$(n^k\mathbf{u}^k_{tt},\mathbf{u}^k_{tt})=\|\sqrt{n^k}\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)}\geq\alpha\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)}.$$

Estimamos os seguintes termos como se segue:

$$
\begin{aligned}
\frac{\nu}{2}|(\nabla\mathbf{u}^k_t,n^k_t\nabla\mathbf{u}^k_t)| &\leq \frac{\nu}{2}\|\nabla\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}\|n^k_t\|_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}^k_t\|_{L^4(\mathbb{T}^d)}\\
&\leq C(\nu,\delta_2)\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\Delta\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\nu|(\nabla\mathbf{u}^k_t,\mathbf{u}^k_{tt}.\nabla n^k)| &\leq \nu\|\nabla\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}^k_{tt}\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\\
&\leq C(\nu,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(n^k_t\mathbf{u}^k_t,\mathbf{u}^k_{tt})| &\leq \|\mathbf{u}^k_t\|_{L^4(\mathbb{T}^d)}\|n^k_t\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}^k_{tt}\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\delta_1)\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$\begin{aligned}
\nu|(n_t^k\Delta\mathbf{u}^k,\mathbf{u}_{tt}^k)| &\le \nu\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|n_t^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\nu,\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|n_t^k\|^2_{H^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|((n_t^k\mathbf{u}^k.\nabla)\mathbf{u}^k,\mathbf{u}_{tt}^k)| &\le \|n_t^k\|_{L^6(\mathbb{T}^d)}\|\mathbf{u}^k\|_{L^6(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^6(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\delta_1)\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n_t^k\mathbf{f},\mathbf{u}_{tt}^k)| &\le \|n_t^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{f}\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\delta_1)\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k\mathbf{u}_t^k.\nabla\mathbf{u}^k,\mathbf{u}_{tt}^k)| &\le \beta\|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\beta)\|\nabla\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\beta,\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k\mathbf{u}^k.\nabla\mathbf{u}_t^k,\mathbf{u}_{tt}^k)| &\le \beta\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\beta,\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k\mathbf{f}_t,\mathbf{u}_{tt}^k)| &\le \|n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}_t\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\delta_1)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
2\nu\left|\left((\nabla n_t^k.\nabla)\mathbf{u}^k,\mathbf{u}_{tt}^k\right)\right| &\le \nu\|\nabla n_t^k\|_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\nu,\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
2\nu\left|\left((\nabla n^k.\nabla)\mathbf{u}_t^k,\mathbf{u}_{tt}^k\right)\right| &\le \nu\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\nu,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{(n^k)^2}n_t^k(\nabla n^k.\nabla)\nabla n^k,\mathbf{u}_{tt}^k\right)\right| &\le C(\varepsilon,\alpha)\|n_t^k\|_{L^6(\mathbb{T}^d)}\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^6(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\varepsilon,\alpha,\delta_1)\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\
&\quad+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla n_t^k.\nabla)\nabla n^k,\mathbf{u}_{tt}^k\right)\right| &\le C(\varepsilon,\alpha)\|\nabla n_t^k\|_{L^4(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\\
&\le C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n_t^k,\mathbf{u}_{tt}^k\right)\right| \le C(\varepsilon,\alpha)\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla^2 n_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)}$$

$$\leq C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{2}{(n^k)^3}n^k_t(\nabla n^k.\nabla n^k)\nabla n^k,\mathbf{u}^k_{tt}\right)\right| &\leq C(\varepsilon,\alpha)\|n^k_t\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^3_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}^k_{tt}\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^6_{L^2(\mathbb{T}^d)}\|n^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{3}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k_t,\mathbf{u}^k_{tt}\right)\right| &\leq C(\varepsilon,\alpha)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla n^k_t\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}^k_{tt}\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{(n^k)^2}n^k_t\Delta n^k\nabla n^k,\mathbf{u}^k_{tt}\right)\right| &\leq C(\varepsilon,\alpha)\|n^k_t\|_{L^6(\mathbb{T}^d)}\|\Delta n^k\|_{L^6(\mathbb{T}^d)}\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\mathbf{u}^k_{tt}\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1)\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\ &\quad+\delta_1\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{n^k}\Delta n^k_t\nabla n^k,\mathbf{u}^k_{tt}\right)\right| &\leq C(\varepsilon,\alpha)\|\Delta n^k_t\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}^k_{tt}\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon\left|\left(\frac{1}{n^k}\Delta n^k\nabla n^k_t,\mathbf{u}^k_{tt}\right)\right| &\leq C(\varepsilon,\alpha)\|\Delta n^k\|_{L^4(\mathbb{T}^d)}\|\nabla n^k_t\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}^k_{tt}\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\varepsilon^2|(\nabla\Delta n^k_t,\mathbf{u}^k_{tt})|=\varepsilon^2|(\Delta n^k_t,div\mathbf{u}^k_{tt})|=0;$$

Assim obtemos,

$$\begin{aligned}&\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+\alpha\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)}\leq\delta_2\|\Delta\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+18\delta_1\|\mathbf{u}^k_{tt}\|^2_{L^2(\mathbb{T}^d)}\\ &+C(\nu,\beta,\delta_1,\delta_2)(\|n^k_t\|^2_{H^1(\mathbb{T}^d)}+\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}\\ &+C(\nu,\varepsilon,\alpha,\delta_1)(\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|n^k_t\|^2_{H^2(\mathbb{T}^d)}\\ &+C(\nu,\varepsilon,\alpha,\delta_1)(\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|n^k\|^2_{H^2(\mathbb{T}^d)}+\|n^k_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}\\ &+\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+C(\delta_1)\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}\\ &+C(\delta_1)\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}+C(\delta_1)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}\tag{2.21}$$

Agora, fazendo $\mathbf{v}=-\Delta\mathbf{u}^k_t$ na equação (2.18), temos,

$$\nu(n^k\Delta\mathbf{u}_t^k,\Delta\mathbf{u}_t^k) = (n^k\mathbf{u}_{tt}^k,\Delta\mathbf{u}_t^k) + (n_t^k\mathbf{u}_t^k,\Delta\mathbf{u}_t^k) - (n_t^k\Delta\mathbf{u}^k,\Delta\mathbf{u}_t^k) + ((n_t^k\mathbf{u}^k.\nabla)\mathbf{u}^k,\Delta\mathbf{u}_t^k)$$
$$+((n^k\mathbf{u}_t^k.\nabla)\mathbf{u}^k,\Delta\mathbf{u}_t^k) + ((n^k\mathbf{u}^k.\nabla)\mathbf{u}_t^k,\Delta\mathbf{u}_t^k) + \nu[-2((\nabla n_t^k.\nabla)\mathbf{u}^k,\Delta\mathbf{u}_t^k)$$
$$-2((\nabla n^k.\nabla)\mathbf{u}_t^k,\Delta\mathbf{u}_t^k)] - (n_t^k\mathbf{f},\Delta\mathbf{u}_t^k) - (n^k\mathbf{f}_t,\Delta\mathbf{u}_t^k)$$
$$-\varepsilon^2\Bigg[-\left(\frac{1}{n^k}(\nabla n_t^k.\nabla)\nabla n^k,\Delta\mathbf{u}_t^k\right) - \left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n_t^k,\Delta\mathbf{u}_t^k\right)$$
$$+\left(\frac{3}{(n^k)^2}(\nabla n_t^k.\nabla n^k)\nabla n^k,\Delta\mathbf{u}_t^k\right) + \left(\frac{1}{(n^k)^2}n_t^k\Delta n^k\nabla n^k,\Delta\mathbf{u}_t^k\right) - \left(\frac{1}{n^k}\Delta n_t^k\nabla n^k,\Delta\mathbf{u}_t^k\right)$$
$$-\left(\frac{2}{(n^k)^3}n_t^k(\nabla n^k.\nabla n^k)\nabla n^k,\Delta\mathbf{u}_t^k\right) - \left(\frac{1}{n^k}\Delta n^k\nabla n_t^k,\Delta\mathbf{u}_t^k\right)$$
$$+\left(\frac{1}{(n^k)^2}n_t^k(\nabla n^k.\nabla)\nabla n^k,\Delta\mathbf{u}_t^k\right) + (\nabla\Delta n_t^k,\Delta\mathbf{u}_t^k)\Bigg]. \tag{2.22}$$

Observe que

$$\begin{aligned}|(n^k\mathbf{u}_{tt}^k\Delta\mathbf{u}_t^k)| &\leq \beta\|\mathbf{u}_{tt}^k\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq \frac{\beta^2}{2\mu}\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\mu}{2}\|\Delta\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)},\end{aligned}$$

e fazendo as estimativas de forma totalmente análoga ao caso anterior, obtemos:

$$\nu\alpha\|\Delta\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} \leq \left(\frac{\mu}{2}+19\delta_2^{'}\right)\|\Delta\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\beta^2}{2\mu}\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+C(\nu,\beta,\delta_2^{'})(\|n_t^k\|^2_{H^1(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+C(\nu,\varepsilon,\alpha,\delta_2^{'})(\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|n_t^k\|^2_{H^2(\mathbb{T}^d)}$$
$$+C(\nu,\varepsilon,\alpha,\delta_2^{'})(\|n_t^k\|^2_{LH^1(\mathbb{T}^d)}\|n^k\|^2_{H^2(\mathbb{T}^d)} + \|n_t^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}$$
$$+\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C(\delta_2^{'})\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}$$
$$+C(\delta_2^{'})\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + C(\delta_2^{'})\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)}. \tag{2.23}$$

Fazendo $\delta_2^{'} = \nu\alpha/76$ e $\mu = \nu\alpha/2$ e multiplicando por $\nu\alpha^2/4\beta^2$ somando com (2.21), tomando $\delta_1 = \alpha/72$ e $\delta_2 = \nu^2\alpha^3/8\beta^2$ obtemos

$$\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\alpha}{2}\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2\alpha^3}{4\beta^2}\|\Delta\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}$$
$$\leq C(\nu,\beta)(\|n_t^k\|^2_{H^1(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+C(\nu,\varepsilon,\alpha)(\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|n_t^k\|^2_{H^2(\mathbb{T}^d)}$$
$$+C(\nu,\varepsilon,\alpha)(\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|n_t^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)}$$
$$+\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\varepsilon,\alpha)\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}$$
$$+C(\nu,\varepsilon,\alpha)\|n_t^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + C(\nu,\varepsilon,\beta,\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)}$$

Por último, considerando a equação (2.22), note que

$$\begin{aligned} -(n^k\mathbf{u}_t^k, \Delta\mathbf{u}_t^k) &= (n^k\nabla\mathbf{u}_t^k, \nabla\mathbf{u}_t^k) + (\mathbf{u}_t^k.\nabla n^k, \nabla\mathbf{u}_t^k) \\ &= \|\sqrt{n^k}\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + (\nabla n^k.\nabla\mathbf{u}_t^k, \mathbf{u}_t^k) \end{aligned}$$

e,

$$\nu(n^k\Delta\mathbf{u}^k, \Delta\mathbf{u}_t^k) = -\nu(n^k\nabla\Delta\mathbf{u}^k, \nabla\mathbf{u}_t^k) - \nu(\Delta\mathbf{u}^k.\nabla n^k, \nabla\mathbf{u}_t^k).$$

Assim temos,

$$\begin{aligned} \|\sqrt{n^k}\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} &= \Bigg( -\nabla((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k) - \nu n^k\nabla\Delta\mathbf{u}^k + \nu\Delta\mathbf{u}^k\nabla n^k + 2\nu\nabla((\nabla n^k.\nabla)\mathbf{u}^k) \\ &+\nabla(n^k\mathbf{f}) - \varepsilon^2\nabla\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k\right) + \varepsilon^2\nabla\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k\right) \\ &-\varepsilon^2\nabla\left(\frac{1}{n^k}\Delta n^k\nabla n^k\right) + \varepsilon^2\nabla(\nabla\Delta n^k), \nabla\mathbf{u}_t^k\Bigg) - (\nabla n^k.\nabla\mathbf{u}_t^k, \mathbf{u}_t^k) \\ &= (\Phi^k, \nabla\mathbf{u}_t^k) - (\nabla n^k.\nabla\mathbf{u}_t^k, \mathbf{u}_t^k) \leq |(\Phi^k, \nabla\mathbf{u}_t^k)| + |(\nabla n^k.\nabla\mathbf{u}_t^k, \mathbf{u}_t^k)| \\ &\leq \|\Phi^k\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)} + \|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C(\delta_1)\|\Phi^k\|^2_{L^2(\mathbb{T}^d)} + C(\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + 2\delta_1\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}. \end{aligned}$$

Observe que,

$$\|\sqrt{n^k}\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} \geq \alpha\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)},$$

fazendo $\delta_1 = \alpha/4$ obtemos

$$\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\alpha)\|\Phi\|^2_{L^2(\mathbb{T}^d)} + C(\alpha)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)},$$

fazendo estimativas totalmente análogas ao item $(iii)$ do Lema (2.2.3), obtemos:

$$\begin{aligned} \|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} &\leq C(\nu,\alpha)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha)\|\Delta\mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + C(\nu,\alpha)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\ &+C(\nu,\alpha)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha,\beta)\|\nabla\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + C(\alpha)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} \\ &+C(\nu,\varepsilon,\alpha)(\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^8_{L^2(\mathbb{T}^d)}) + C(\nu,\varepsilon,\alpha)\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}. \end{aligned}$$

Para todo $k \in \mathbb{N}$ temos que, $(\mathbf{u}^k, n^k) \in C^1([0,T^k]; H^2(\mathbb{T}^d) \cap V) \times C^2(\mathbb{T}^d \times [0,T^k])$, observe que por hipóteses $\mathbf{f} \in L^2(0,T;H^1(\mathbb{T}^d))$, $\mathbf{f}_t \in L^2(0,T;H^1(\mathbb{T}^d))$, portando pelo Lema (1.1.2) temos que $\mathbf{f} \in C([0,T];H^1(\mathbb{T}^d))$; logo, podemos tomar $t = 0$ na equação acima e pelo Lema (2.2.2) temos que $\|\mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\varepsilon,\alpha,\|\mathbf{u}_0\|_{H^2(\mathbb{T}^d)},\|n_0\|_{H^3(\mathbb{T}^d)})$ e assim concluímos que $\|\nabla\mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\varepsilon,\alpha,\beta,\|\mathbf{u}_0\|_{H^3(\mathbb{T}^d)},\|n_0\|_{H^4(\mathbb{T}^d)})$.

■

**Lema 2.2.5** *Então* $(\mathbf{u}^k, n^k)$ *satisfaz às seguintes desigualdades:*

$$(i) \qquad \frac{d}{dt}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu)\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}.$$

$$\begin{aligned}(ii) \qquad \|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\ &+C(\nu)\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

$$\begin{aligned}(iii) \qquad \|\nabla\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\ &+\nu^2\|\nabla\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

**Demonstração:**
$(i)$ Aplicando o operador $\Delta^2$ à equação (2.4), multiplicando a equação resultante por $\Delta^2 n^k$ e integrando sobre $\mathbb{T}^d$, obtemos:

$$(\Delta^2 n^k_t, \Delta^2 n^k) - \nu(\Delta^2\Delta n^k, \Delta^2 n^k) = -(\Delta^2(\mathbf{u}^k.\nabla n^k), \Delta^2 n^k),$$

e fazendo integração por partes, temos:

$$\frac{1}{2}\frac{d}{dt}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\Delta^2\nu\|^2_{L^2(\mathbb{T}^d)} = (\nabla\Delta(\mathbf{u}^k.\nabla n^k), \nabla\Delta^2 n^k).$$

Estimamos os termos à direita da equação acima como se segue:

$$\begin{aligned}&|(\nabla\Delta(\mathbf{u}^k.\nabla n^k), \nabla\Delta^2 n^k)| \leq \|\nabla\Delta(\mathbf{u}^k.\nabla n^k)\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta^2 n^k\|_{L^2(\mathbb{T}^d)}\\ \leq\;& C(\gamma)\|\nabla\Delta(\mathbf{u}^k.\nabla n^k)\|^2_{L^2(\mathbb{T}^d)} + \gamma\|\nabla\Delta^2\|^2_{L^2(\mathbb{T}^d)}\\ \leq\;& C(\gamma)\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)} + C(\gamma)\|\Delta\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\\ &+C(\gamma)\|\nabla\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C(\gamma)\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\ &+\gamma\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\ \leq\;& C(\gamma)\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C(\gamma)\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \gamma\|\nabla\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

Assim, escolhendo o valor de $\gamma = \nu/2$, obtemos:

$$\frac{d}{dt}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu)\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}.$$

$(ii)$ Agora, aplicando o operador $\Delta$ à equação (2.4) e avaliando a norma $L^2(\mathbb{T}^d)$ nos termos da equação resultantes, temos:

$$\begin{aligned}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\Delta(\mathbf{u}^k.\nabla n^k)\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)} + C\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^\infty(\mathbb{T}^d)}\\ &\quad+C\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\ &\quad+C(\nu)\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

$(iii)$ Mais ainda, aplicando o operador $\nabla\Delta$ à equação (2.4) avaliando a norma $L^2(\mathbb{T}^d)$, temos:

$$\begin{aligned}
\|\nabla\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\nabla\Delta(\mathbf{u}^k.\nabla n^k)\|^2_{L^2(\mathbb{T}^d)} + \nu^2\|\nabla\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^\infty(\mathbb{T}^d)} \\
&\quad + C\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^4(\mathbb{T}^d)} + C\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^4(\mathbb{T}^d)} \\
&\quad + C\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \nu^2\|\nabla\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\quad + \nu^2\|\nabla\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

■

**Lema 2.2.6** *Sejam* $\mathbf{u}_0 \in V \cap H^3(\mathbb{T}^d)$, $n_0 \in H^4(\mathbb{T}^d)$ *e* $\mathbf{f} \in L^2(0,T;H^2(\mathbb{T}^d))$. *Então* $(\mathbf{u}^k, n^k)$ *satisfaz à seguinte desigualdade diferencial,*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu\alpha}{2}\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\Delta\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + C(\delta_1)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^2(\mathbb{T}^d)} \\
&+ C(\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + C(\nu,\delta_1)(\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \\
&+ \|n_t^k\|^2_{H^1(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\delta_1)\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&+ C(\varepsilon,\beta,\delta_1)(\|\Delta^2 n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta^2 n^k\|^6_{L^2(\mathbb{T}^d)} + \|\Delta^2 n^k\|^8_{L^2(\mathbb{T}^d)})
\end{aligned}$$

**Demonstração:**

Fazendo $\mathbf{v} = -\Delta^3\mathbf{u}^k$ na equação (2.5) e por integração por partes, obtemos;

$$\begin{aligned}
&(\nabla\Delta(n^k\mathbf{u}_t^k), \nabla\Delta\mathbf{u}^k) + \nu(\Delta(n^k\Delta\mathbf{u}^k), \Delta^2\mathbf{u}^k) = (\Delta(n^k\mathbf{u}^k.\nabla\mathbf{u}^k), \Delta^2\mathbf{u}^k) \\
&+\nu[-2(\Delta(\nabla n^k.\nabla\mathbf{u}^k), \Delta^2\mathbf{u}^k)] - (\Delta(n^k\mathbf{f}), \Delta^2\mathbf{u}^k) \\
&-\varepsilon^2\left[\left(\Delta\left(\frac{1}{n^k}(\nabla.\nabla)\nabla n^k\right), \Delta^2\mathbf{u}^k\right) + \left(\Delta\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k\right), \Delta^2\mathbf{u}^k\right)\right. \\
&\left. - \left(\Delta\left(\frac{1}{n^k}\Delta n^k\nabla n^k\right), \Delta^2\mathbf{u}^k\right) + (\nabla\Delta n^k, \Delta^3\mathbf{u}^k)\right].
\end{aligned}$$

Observe que,

$$\begin{aligned}
&(\nabla\Delta(n^k\mathbf{u}_t^k), \nabla\Delta\mathbf{u}^k) = \Big(\nabla(div(\mathbf{u}_t^k)\nabla n^k + (\mathbf{u}_t^k.\nabla)\nabla n^k + n^k\Delta\mathbf{u}_t^k + (\nabla n^k.\nabla)\mathbf{u}_t^k), \nabla\Delta\mathbf{u}^k\Big) \\
&= \Big((\nabla\mathbf{u}_t^k.\nabla)\nabla n^k + \mathbf{u}_t^k.\nabla^3 n^k + \nabla n^k.\Delta\mathbf{u}_t^k + n^k\nabla\Delta\mathbf{u}_t^k + (\nabla^2 n^k.\nabla)\mathbf{u}_t^k \\
&+\nabla n^k.\nabla^2\mathbf{u}_t^k, \nabla\Delta\mathbf{u}^k\Big).
\end{aligned}$$

Agora observe que,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} &= \frac{1}{2}\frac{d}{dt}(\nabla\Delta\mathbf{u}^k, n^k\nabla\Delta\mathbf{u}^k)\\ &= \frac{1}{2}(\nabla\Delta\mathbf{u}^k_t, n^k\nabla\Delta\mathbf{u}^k) + \frac{1}{2}(\nabla\Delta\mathbf{u}^k, n^k_t\nabla\Delta\mathbf{u}^k)\\ &\quad+\frac{1}{2}(\nabla\Delta\mathbf{u}^k, n^k\nabla\Delta\mathbf{u}^k_t)\\ &= (\nabla\Delta\mathbf{u}^k_t, n^k\nabla\Delta\mathbf{u}^k) + \frac{1}{2}(\nabla\Delta\mathbf{u}^k, n^k_t\nabla\Delta\mathbf{u}^k),\end{aligned}$$

ou seja,

$$(n^k\nabla\Delta\mathbf{u}^k_t, \nabla\Delta\mathbf{u}^k) = \frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}(\nabla\Delta\mathbf{u}^k, n^k_t\nabla\Delta\mathbf{u}^k).$$

Assim temos,

$$\begin{aligned}&(\nabla\Delta(n^k\mathbf{u}^k_t), \nabla\Delta\mathbf{u}^k) = \frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}(\nabla\Delta\mathbf{u}^k, n^k_t\nabla\Delta\mathbf{u}^k)\\ &+((\nabla\mathbf{u}^k_t.\nabla)\nabla n^k, \nabla\Delta\mathbf{u}^k) + (\mathbf{u}^k_t.\nabla^3 n^k, \nabla\Delta\mathbf{u}^k) + (\nabla n^k.\Delta\mathbf{u}^k_t, \nabla\Delta\mathbf{u}^k)\\ &+((\nabla^2 n^k.\nabla)\mathbf{u}^k_t, \nabla\Delta\mathbf{u}^k) + (\nabla n^k.\nabla^2\mathbf{u}^k_t, \nabla\Delta\mathbf{u}^k),\end{aligned}$$

para,

$$\begin{aligned}&\nu(\Delta(n^k\Delta\mathbf{u}^k), \Delta^2\mathbf{u}^k) = \nu(div(\Delta\mathbf{u}^k)\nabla n^k, \Delta^2\mathbf{u}^k) + \nu(\Delta\mathbf{u}^k\nabla^2 n^k, \Delta^2\mathbf{u}^k)\\ &+\nu(n^k\Delta^2\mathbf{u}^k, \Delta^2\mathbf{u}^k) + \nu(\nabla n^k.\nabla\Delta\mathbf{u}^k, \Delta^2\mathbf{u}^k) \geq \nu\alpha\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\ &+\nu(\Delta\mathbf{u}^k\nabla^2 n^k, \Delta^2\mathbf{u}^k) + \nu(\nabla n^k.\nabla\Delta\mathbf{u}^k, \Delta^2\mathbf{u}^k).\end{aligned}$$

Agora estimamos os seguintes termos como se segue:

$$\begin{aligned}\frac{1}{2}|(\nabla\Delta\mathbf{u}^k, n^k_t\nabla\Delta\mathbf{u}^k)| &\leq \|\nabla\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|n^k_t\|_{L^4(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C(\delta_1)\|n^k_t\|^2_{H^1(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\nabla\mathbf{u}^k_t.\nabla^2 n^k, \nabla\Delta\mathbf{u}^k)| &\leq \|\nabla\mathbf{u}^k_t\|_{L^4(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^4(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\mathbf{u}^k_t.\nabla^3 n^k, \nabla\Delta\mathbf{u}^k)| &\leq \|\mathbf{u}^k_t\|_{L^\infty(\mathbb{T}^d)}\|\nabla^3 n^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\nabla n^k.\Delta\mathbf{u}^k_t, \nabla\Delta\mathbf{u}^k)| &\leq \|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\Delta\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$|(\nabla^2 n^k.\nabla\mathbf{u}^k_t, \nabla\Delta\mathbf{u}^k)| \leq \|\nabla^2 n^k\|_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}^k_t\|_{L^4(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}$$

$$\leq \quad C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}
|(\nabla n^k.\nabla^2\mathbf{u}^k_t, \nabla\Delta\mathbf{u}^k)| &\leq \|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla^2\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\nu|(\Delta\mathbf{u}^k.\nabla^2 n^k, \Delta^2\mathbf{u}^k)| &\leq \nu\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^\infty(\mathbb{T}^d)}\|\Delta^2\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\leq C(\nu,\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\nu|(\nabla n^k.\nabla\Delta\mathbf{u}^k, \Delta^2\mathbf{u}^k)| &\leq \nu\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta^2\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\leq C(\nu,\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
&|(\Delta((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k), \Delta^2\mathbf{u}^k)| \leq \|\Delta((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k)\|_{L^2(\mathbb{T}^d)}\|\Delta^2\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
\leq\;& C(\delta_1)\|\Delta((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k)\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
\leq\;& C(\delta_1)\|\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)} \\
&+C(\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)} \\
&+C(\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
&+C(\beta,\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)} + C(\beta,\delta_1)\|\nabla\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
&+C(\beta,\delta_1)\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
\leq\;& C(\delta_1)\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C(\beta,\delta_1)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
&+\delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(\Delta(n^k\mathbf{f}), \Delta^2\mathbf{u}^k)| &\leq \|\Delta(n^k\mathbf{f})\|_{L^2(\mathbb{T}^d)}\|\Delta^2\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\leq C(\delta_1)\|\Delta(n^k\mathbf{f})\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C(\delta_1)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^4(\mathbb{T}^d)} + C(\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} \\
&\quad +C(\delta_1)\|n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\Delta\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C(\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + C(\delta_1)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^2(\mathbb{T}^d)} \\
&\quad +\delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
&2\nu\left|\left(\Delta\left((\nabla n^k.\nabla)\mathbf{u}^k\right), \Delta^2\mathbf{u}^k\right)\right| \leq \nu\left\|\Delta\left((\nabla n^k.\nabla)\mathbf{u}^k\right)\right\|_{L^2(\mathbb{T}^d)}\|\Delta^2\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
\leq\;& C(\nu,\delta_1)\left\|\Delta\left((\nabla n^k.\nabla)\mathbf{u}^k\right)\right\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
\leq\;& C(\nu,\delta_1)\|\nabla\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)} + C(\nu,\delta_1)\|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)} \\
&+C(\nu,\delta_1)\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\|^2_{L^2(\mathbb{T}^d)} \\
\leq\;& C(\nu,\delta_1)\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$
\begin{aligned}
& \varepsilon^2 \left| \left( \Delta \left( \frac{1}{n^k} (\nabla n^k.\nabla) \nabla n^k \right), \Delta^2 \mathbf{u}^k \right) \right| \\
\leq\ & \varepsilon^2 \left\| \Delta \left( \frac{1}{n^k} (\nabla n^k.\nabla) \nabla n^k \right) \right\|_{L^2(\mathbb{T}^d)} \|\Delta^2 \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
\leq\ & C(\varepsilon, \delta_1) \left\| \Delta \left( \frac{1}{n^k} (\nabla n^k.\nabla) \nabla n^k \right) \right\|^2_{L^2(\mathbb{T}^d)} + \delta_1 \|\Delta^2 \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
\leq\ & C(\varepsilon, \delta_1) \left\| \Delta \left( \frac{1}{n^k} \right) \right\|^2_{L^\infty(\mathbb{T}^d)} \|\nabla n^k\|^2_{L^4(\mathbb{T}^d)} \|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)} \\
& + C(\varepsilon, \delta_1) \left\| \nabla \left( \frac{1}{n^k} \right) \right\|^2_{L^4(\mathbb{T}^d)} \|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)} \|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)} \\
& + C(\varepsilon, \delta_1) \left\| \nabla \left( \frac{1}{n^k} \right) \right\|^2_{L^\infty(\mathbb{T}^d)} \|\nabla n^k\|^2_{L^4(\mathbb{T}^d)} \|\nabla^3 n^k\|^2_{L^4(\mathbb{T}^d)} + C(\varepsilon, \delta_1) \|\nabla \Delta n^k\|^2_{L^4(\mathbb{T}^d)} \|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)} \\
& + C(\varepsilon, \beta, \delta_1) \|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)} \|\nabla^3 n^k\|^2_{L^4(\mathbb{T}^d)} + C(\varepsilon, \beta, \delta_1) \|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)} \|\Delta \nabla^2 n^k\|^2_{L^2(\mathbb{T}^d)} \\
& + \delta_1 \|\Delta^2 \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
\leq\ & C(\varepsilon, \beta, \delta_1) \|\Delta^2 n^k\|^6_{L^2(\mathbb{T}^d)} + C(\varepsilon, \beta, \delta_1) \|\Delta^2 n^k\|^4_{L^2(\mathbb{T}^d)} + \delta_1 \|\Delta^2 \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
& \varepsilon^2 \left| \left( \Delta \left( \frac{1}{(n^k)^2} (\nabla n^k.\nabla n^k) \nabla n^k \right), \Delta^2 \mathbf{u}^k \right) \right| \\
\leq\ & \varepsilon^2 \left\| \Delta \left( \frac{1}{(n^k)^2} (\nabla n^k.\nabla n^k) \nabla n^k \right) \right\|_{L^2(\mathbb{T}^d)} \|\Delta^2 \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
\leq\ & C(\varepsilon, \delta_1) \left\| \Delta \left( \frac{1}{(n^k)^3} (\nabla n^k.\nabla n^k) \nabla n^k \right) \right\|^2_{L^2(\mathbb{T}^d)} + \delta_1 \|\Delta^2 \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
\leq\ & C(\varepsilon, \delta_1) \left\| \Delta \left( \frac{1}{(n^k)^2} \right) \right\|^2_{L^\infty(\mathbb{T}^d)} \|\nabla n^k\|^6_{L^6(\mathbb{T}^d)} \\
& + C(\varepsilon, \delta_1) \left\| \nabla \left( \frac{1}{(n^k)^2} \right) \right\|^2_{L^\infty(\mathbb{T}^d)} \|\nabla \Delta n^k\|^2_{L^6(\mathbb{T}^d)} \|\nabla n^k\|^4_{L^6(\mathbb{T}^d)} \\
& + C(\varepsilon, \delta_1) \left\| \nabla \left( \frac{1}{(n^k)^2} \right) \right\|^2_{L^\infty(\mathbb{T}^d)} \|\nabla^2 n^k\|^2_{L^6(\mathbb{T}^d)} \|\nabla n^k\|^4_{L^6(\mathbb{T}^d)} \\
& + C(\varepsilon, \beta, \delta_1) \|\nabla \Delta n^k\|^2_{L^4(\mathbb{T}^d)} \|\nabla n^k\|^4_{L^4(\mathbb{T}^d)} + \delta_1 \|\Delta^2 \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
\leq\ & C(\varepsilon, \beta, \delta_1) \|\Delta^2 n^k\|^8_{L^2(\mathbb{T}^d)} + C(\varepsilon, \beta, \delta_1) \|\Delta^2 n^k\|^6_{L^2(\mathbb{T}^d)} + \delta_1 \|\Delta^2 \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
& \left| \varepsilon^2 \left( \Delta \left( \frac{1}{n^k} \Delta n^k \nabla n^k \right), \Delta^2 \mathbf{u}^k \right) \right| \leq \varepsilon^2 \left\| \Delta \left( \frac{1}{n^k} \Delta n^k \nabla n^k \right) \right\|_{L^2(\mathbb{T}^d)} \|\Delta^2 \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
\leq\ & C(\varepsilon, \delta_1) \left\| \Delta \left( \frac{1}{n^k} \Delta n^k \nabla n^k \right) \right\|^2_{L^2(\mathbb{T}^d)} + \delta_1 \|\Delta^2 \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
\leq\ & C(\varepsilon, \delta_1) \left\| \Delta \left( \frac{1}{n^k} \right) \right\|^2_{L^\infty(\mathbb{T}^d)} \|\Delta n^k\|^2_{L^4(\mathbb{T}^d)} \|\nabla n^k\|^2_{L^4(\mathbb{T}^d)}
\end{aligned}
$$

$$
\begin{aligned}
&+C(\varepsilon,\delta_1)\left\|\nabla\left(\frac{1}{n^k}\right)\right\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^4(\mathbb{T}^d)}\\
&+C(\varepsilon,\delta_1)\left\|\nabla\left(\frac{1}{n^k}\right)\right\|^2_{L^\infty(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)}\\
&+C(\varepsilon,\beta,\delta_1)\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\\
&+C(\varepsilon,\beta,\delta_1)\|\nabla\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2 n^k\|^2_{L^4(\mathbb{T}^d)}\\
&+C(\varepsilon,\beta,\delta_1)\|\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\|\Delta\nabla n^k\|^2_{L^4(\mathbb{T}^d)}+\delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\\
\leq\ & C(\varepsilon,\beta,\delta_1)\|\Delta^2 n^k\|^4_{L^2(\mathbb{T}^d)}+C(\varepsilon,\beta,\delta_1)\|\Delta^2 n^k\|^6_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\varepsilon^2|(\nabla\Delta n^k,\Delta^3\mathbf{u}^k)|=\varepsilon^2|(\Delta n^k, div\Delta^3\mathbf{u}^k)|=0;
$$

Portanto, fazendo $\delta_1=\nu\alpha/12$ obtemos,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\frac{\nu\alpha}{2}\|\Delta^2\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\leq C\|\Delta\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+C(\delta_1)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^2(\mathbb{T}^d)}\\
&+C(\delta_1)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}+C(\nu,\delta_1)(\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\
&+\|n^k_t\|^2_{H^1(\mathbb{T}^d)}+\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)})\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+C(\nu,\delta_1)\|\Delta\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\\
&+C(\varepsilon,\beta,\delta_1)(\|\Delta^2 n^k\|^4_{L^2(\mathbb{T}^d)}+\|\Delta^2 n^k\|^6_{L^2(\mathbb{T}^d)}+\|\Delta^2 n^k\|^8_{L^2(\mathbb{T}^d)}).
\end{aligned}
$$

■

**Lema 2.2.7** *Então* $(\mathbf{u}^k, n^k)$ *satisfaz à seguinte desigualdade,*

$$
\begin{aligned}
&\|\nabla n^k_{tt}\|^2_{L^2(\mathbb{T}^d)}\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}\\
&+C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)}+\nu^2\|\nabla\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}
$$

**Demonstração:**

Derivando a equação (2.4) em relação a variável $t$, obtemos:

$$
n^k_{tt}+\mathbf{u}^k_t.\nabla n^k+\mathbf{u}^k.\nabla n^k_t=\nu\Delta n^k_t;
$$

multiplicando a equação acima por $-\Delta n^k_{tt}$ e integrando sobre $\mathbb{T}^d$, temos:

$$
-(n^k_{tt},\Delta n^k_{tt})-(\mathbf{u}^k_t.\nabla n^k,\Delta n^k_{tt})-(\mathbf{u}^k.\nabla n^k_t,\Delta n^k_{tt})=-\nu(\Delta n^k_t,\Delta n^k_{tt}),
$$

e fazendo integração por partes, temos,

$$
\begin{aligned}
&\|\nabla n^k_{tt}\|^2_{L^2(\mathbb{T}^d)}=-(\nabla\mathbf{u}^k_t.\nabla n^k,\nabla n^k_{tt})-(\mathbf{u}^k.\nabla^2 n^k_t,\nabla n^k_{tt})-(\nabla\mathbf{u}^k.\nabla n^k_t,\nabla n^k_{tt})\\
&-(\mathbf{u}^k_t.\nabla^2 n^k,\nabla n^k_{tt})+\nu(\nabla\Delta n^k_t,\nabla n^k_{tt}).
\end{aligned}
$$

Estimamos os termos à direita da equação acima, como se segue:

$$
|(\nabla\mathbf{u}^k_t.\nabla n^k,\nabla n^k_{tt})|\ \leq\ \|\nabla\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla n^k_{tt}\|_{L^2(\mathbb{T}^d)}
$$

$$\leq C(\gamma)\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \gamma\|\nabla n^k_{tt}\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}
|(\mathbf{u}^k.\nabla^2 n^k_t, \nabla n^k_{tt})| &\leq \|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla^2 n^k_t\|_{L^2(\mathbb{T}^d)}\|\nabla n^k_{tt}\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\gamma)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} + \gamma\|\nabla n^k_{tt}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(\nabla\mathbf{u}^k.\nabla n^k_t, \nabla n^k_{tt})| &\leq \|\nabla\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\nabla n^k_t\|_{L^4(\mathbb{T}^d)}\|\nabla n^k_{tt}\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\gamma)\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} + \gamma\|\nabla n^k_{tt}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(\mathbf{u}^k_t.\nabla^2 n^k, \nabla n^k_{tt})| &\leq \|\mathbf{u}^k_t\|_{L^4(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^4(\mathbb{T}^d)}\|\nabla n^k_{tt}\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\gamma)\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \gamma\|\nabla n^k_{tt}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(\nabla\Delta n^k_t, \nabla n^k_{tt})| &\leq \|\nabla\Delta n^k_t\|_{L^2(\mathbb{T}^d)}\|\nabla n^k_{tt}\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\gamma)\|\nabla\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)} + \gamma\|\nabla n^k_{tt}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

Então, escolhendo o valor de $\gamma = 1/10$ de forma conveniente, obtemos:

$$\|\nabla n^k_{tt}\|^2_{L^2(\mathbb{T}^d)} \leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+\nu^2\|\nabla\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)}.$$

■

## 2.3 Existência e Unicidade de Solução Local

Nesta seção provamos um resultado de existência e unicidade de solução forte local no tempo para o problema (8)-(11) quando $d = 2$ ou 3 sem assumir hipóteses adicionais sobre os dados iniciais (além de regularidade necessária) ou hipóteses sobre as constantes físicas e damos resultados de regularidades melhores do que obtidos em [32] os quais serão necessários para a análise de erro local para as soluções aproximadas da velocidade $\mathbf{u}^k$ e densidade $n^k$.

**Teorema 2.3.1** *Sejam $d = 2$ ou 3, e $\mathbf{u}_0 \in V$, $n_0 \in H^2(\mathbb{T}^d)$ e $\mathbf{f} \in L^2(0,T;L^2(\mathbb{T}^d))$. Então existem $T^*$ com $0 < T^* \leq T$, e uma única solução forte $(\mathbf{u}, n)$ para o problema (8)-(11) definida em $[0,T^*]$ que satisfaz:*

$$\begin{aligned}
\mathbf{u} &\in L^\infty(0,T^*;H^1(\mathbb{T}^d)), & n &\in L^\infty(0,T^*;H^2(\mathbb{T}^d)),\\
\mathbf{u} &\in L^2(0,T^*;H^2(\mathbb{T}^d)), & n &\in L^2(0,T^*;H^3(\mathbb{T}^d)),\\
\mathbf{u}_t &\in L^2(0,T^*;L^2(\mathbb{T}^d)), & n_t &\in L^2(0,T^*;H^1(\mathbb{T}^d)),\\
& & e \quad n_t &\in L^\infty(0,T^*;L^2(\mathbb{T}^d)).
\end{aligned}$$

*Além disso, temos que $\mathbf{u}(0) = \mathbf{u}_0$, $n(0) = n_0$ e existe $p \in L^2(0,T^*;H^1(\Omega))$ satisfazendo à equação (9) em quase todo ponto de $\mathbb{T}^d \times (0,T^*)$.*

**Demonstração:** A demonstração será dividida em quatro estágios:

*Parte 1: Estimativas a Priori.*
Inicialmente observe que,

$$\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} = \frac{2}{2}\left\|\frac{\sqrt{n^k}}{\sqrt{n^k}}\nabla\mathbf{u}^k\right\| \leq \frac{2}{2\alpha^{1/2}}\|\sqrt{n^k}\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \leq C(\alpha)\frac{1}{2}\|\sqrt{n^k}\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)},$$

e

$$\begin{aligned}
\|n^k\|^2_{H^2(\mathbb{T}^d)} &= \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&= \frac{2}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu}{\nu}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq \frac{C}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

Assim, podemos reescrever a desigualdade diferencial dado pelo Lema (2.2.1) da seguinte forma:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\right) + \frac{\alpha}{2}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} \\
&+\frac{1}{2}\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2}{2}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2\alpha^3}{2\beta}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C(\nu,\varepsilon,\alpha,\beta)\Bigg(\left(\frac{1}{2}\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\right)^2 + \left(\frac{1}{2}\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\right)^3 + \left(\frac{1}{2}\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\right)^4 \\
&+\left(\frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)}\right)^2 + \left(\frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)}\right)^3 + \left(\frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)}\right)^4 + \left(\nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\right)^2 \\
&+\left(\nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\right)^3 + \left(\nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\right)^4\Bigg) + C(\nu,\alpha,\beta)\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

Definimos

$$\begin{aligned}
\varphi(t) &= \frac{1}{2}\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}, \\
\chi(t) &= \frac{\alpha}{2}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)}, \\
\psi(t) &= \frac{\nu^2\alpha^3}{2\beta}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2}{2}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

Assim podemos escrever a seguinte desigualdade diferencial:

$$\begin{aligned}
\varphi^{'}(t) + \chi(t) + \psi(t) &\leq C(\nu,\varepsilon,\alpha,\beta)\varphi^2(t) + C(\nu,\varepsilon,\alpha,\beta)\varphi^3(t) + C(\nu,\varepsilon,\alpha,\beta)\varphi^4(t) \\
&+C(\nu,\alpha,\beta)\|\mathbf{f}(t)\|^2_{L^2(\mathbb{T}^d)}
\end{aligned}$$

definimos $C_1^* = C(\nu, \varepsilon, \alpha, \beta)$. Observe que,

$$\varphi(0) \leq \varphi_0 = \frac{1}{2}\|\sqrt{n_0}\nabla \mathbf{u}_0\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n_0\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta n_0\|^2_{L^2(\mathbb{T}^d)},$$

então, obtemos o seguinte problema de valor inicial:

$$\begin{cases} \varphi'(t) \leq C_1^*\varphi^2(t) + C_1^*\varphi^3(t) + C_1^*\varphi^4(t) + C(\nu,\alpha,\beta)\|\mathbf{f}(t)\|^2_{L^2(\mathbb{T}^d)} \\ \varphi(0) \leq \varphi_0. \end{cases}$$

Agora, usando o Lema (1.2.10), obtemos que para todo $\delta > 0$, existe $T_\delta$:

$$\varphi(t) \leq \varphi_0 + \delta, \quad \forall t \leq T_\delta,$$

e assim, existem $C(\delta, \|\mathbf{u}_0\|_{H^1(\mathbb{T}^d)}, \|n_0\|_{H^2(\mathbb{T}^d)}) = M > 0$ e $0 < T^* \leq T$ tais que

$$\varphi(t) \leq M, \quad \forall t \in [0, T^*].$$

Portanto,

$$\varphi'(t) + \chi(t) + \psi(t) \leq C_1^*M^2 + C_1^*M^3 + C_1^*M^4 + C(\nu,\alpha,\beta)\|\mathbf{f}(t)\|^2_{L^2(\mathbb{T}^d)},$$

e integrando a desigualdade diferencial acima de 0 a $t$, com $t \in [0, T^*]$, temos:

$$\begin{aligned} &\varphi(t) + \int_0^t \chi(s)ds + \int_0^t \psi(s)ds \\ &\leq \varphi(0) + (C_1^*M^2 + C_1^*M^3 + C_1^*M^4)\int_0^t ds + C(\nu,\alpha,\beta)\int_0^t \|\mathbf{f}(s)\|^2_{L^2(\mathbb{T}^d)}ds \\ &\leq C(\nu,\varepsilon,\alpha,\beta,\|\mathbf{u}_0\|_{H^1(\mathbb{T}^d)}, \|n_0\|_{H^2(\mathbb{T}^d)}, \|\mathbf{f}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}, T^*) = C_2^* \end{aligned}$$

Portanto obtemos que:

$$\begin{aligned} \mathbf{u}^k &\in L^\infty(0,T^*;H^1(\mathbb{T}^d)), & n^k &\in L^\infty(0,T^*;H^2(\mathbb{T}^d)), \\ \mathbf{u}^k &\in L^2(0,T^*;H^2(\mathbb{T}^d)), & n^k &\in L^2(0,T^*;H^3(\mathbb{T}^d)), \\ \mathbf{u}_t^k &\in L^2(0,T^*;L^2(\mathbb{T}^d)) \quad e & n_t^k &\in L^2(0,T^*;H^1(\mathbb{T}^d)). \end{aligned}$$

Pela desigualdade (2.8) dado pelo Lema (2.2.1) e pelas estimativas acima, temos que:

$$\|n_t^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} \leq C_2^*$$

logo $n_t^k \in L^\infty(0,T^*;L^2(\mathbb{T}^d))$.

Assim, pelas estimativas obtidas acima e juntamente com o Lema (1.1.3) implicam as seguintes convergências (passando a subsequências, se necessário):

$$(1) \quad \mathbf{u}^k \to \mathbf{u} \ \ em \ \ C([0,T^*];L^2(\mathbb{T}^d));$$

$$
\begin{array}{ll}
(2) & \mathbf{u}^k \to \mathbf{u} \;\; em \;\; L^q(0,T^*;L^2(\mathbb{T}^d)), \; para, \; 1 \le q \le \infty; \\
(3) & \mathbf{u}^k \to \mathbf{u} \;\; em \;\; L^2(0,T^*;H^1(\mathbb{T}^d)); \\
(4) & n^k \to n \;\; em \;\; C([0,T^*];H^1(\mathbb{T}^d)); \\
(5) & n^k \to n \;\; em \;\; L^q(0,T^*;H^1(\mathbb{T}^d)), \; para, \; 1 \le q \le \infty; \\
(6) & n^k \to n \;\; em \;\; L^2(0,T^*;H^2(\mathbb{T}^d)); \\
(7) & \mathbf{u}^k \rightharpoonup \mathbf{u} \;\; em \;\; L^q(0,T^*;H^1(\mathbb{T}^d)), \; para, \; 1 < q < \infty; \\
(8) & \mathbf{u}^k \rightharpoonup \mathbf{u} \;\; em \;\; L^2(0,T^*;H^2(\mathbb{T}^d)); \\
(9) & \mathbf{u}_t^k \rightharpoonup \mathbf{u}_t \;\; em \;\; L^2(0,T^*;L^2(\mathbb{T}^d)); \\
(10) & n^k \rightharpoonup n \;\; em \;\; L^q(0,T^*;H^2(\mathbb{T}^d)), \; para, \; 1 < q < \infty; \\
(11) & n^k \rightharpoonup n \;\; em \;\; L^2(0,T^*;H^3(\mathbb{T}^d)); \\
(12) & n_t^k \rightharpoonup n_t \;\; em \;\; L^2(0,T^*;H^1(\mathbb{T}^d)).
\end{array}
$$

*Parte 2: Passagem ao Limite.*

Para podermos passar o limite com $k \to \infty$ em (2.4)-(2.6) definimos,

$$
\mathbf{v} = \phi^m = \sum_{i=1}^{m} C_{im}(t)\varphi_i(x) \tag{2.24}
$$

onde $\varphi_i(x)$ é a i-ésima autofunção do operador de Stokes. Agora integrando a equação (2.5) sobre $[0,T^*]$, obtemos:

$$
\begin{aligned}
&\int_0^{T^*} (n^k \mathbf{u}_t^k, \phi^m)dt + \int_0^{T^*} ((n^k \mathbf{u}^k.\nabla)\mathbf{u}^k, \phi^m)dt + \nu \Bigg[ -\int_0^{T^*} (n^k \Delta \mathbf{u}^k, \phi^m)dt \\
&-2\int_0^{T^*} ((\nabla n^k.\nabla)\mathbf{u}^k, \phi^m)dt \Bigg] = \int_0^{T^*} (n^k \mathbf{f}, \phi^m)dt \\
&+\varepsilon^2 \Bigg[ -\int_0^{T^*} \left( \frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \phi^m \right) dt + \int_0^{T^*} \left( \frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \phi^m \right) dt \\
&-\int_0^{T^*} \left( \frac{1}{n^k}\Delta n^k \nabla n^k, \phi^m \right) dt + \int_0^{T^*} \left( \nabla \Delta n^k, \phi^m \right) dt \Bigg],
\end{aligned}
$$

e considerando $k > m$ passamos o limite com $k \to \infty$ em cada termo da equação acima, como se segue:

**(i)** $\displaystyle\int_0^{T^*} (\nabla \Delta n^k, \phi^m)dt \to \int_0^{T^*} (\nabla \Delta n, \phi^m)dt.$

De fato,

$$\begin{aligned}
&\left|\int_0^{T^*}\int_{\mathbb{T}^d}\nabla\Delta n^k.\phi^m dxdt - \int_0^{T^*}\int_{\mathbb{T}^d}\nabla\Delta n.\phi^m dxdt\right| \\
= &\left|\int_0^{T^*}\int_{\mathbb{T}^d}(\nabla\Delta n^k - \nabla\Delta n).\phi^m dxdt\right| \to 0,
\end{aligned}$$

pela convergência (11).

**(ii)** $\displaystyle\int_0^{T^*}(n^k\mathbf{u}_t^k,\phi^m)dt \to \int_0^{T^*}(n\mathbf{u}_t,\phi^m)dt.$

De fato,

$$\begin{aligned}
&\left|\int_0^{T^*}\int_{\mathbb{T}^d}(n^k\mathbf{u}_t^k - n\mathbf{u}_t).\phi^m dxdt\right| \\
= &\left|\int_0^{T^*}\int_{\mathbb{T}^d}(n^k\mathbf{u}_t^k - n\mathbf{u}_t^k + n\mathbf{u}_t^k - n\mathbf{u}_t^k).\phi^m dxdt\right| \\
\leq &\ \|n^k - n\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\mathbf{u}_t^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \\
&+\left|\int_0^{T^*}\int_{\mathbb{T}^d}(\mathbf{u}_t^k - \mathbf{u}_t)n\phi^m dxdt\right| \to 0,
\end{aligned}$$

pelas convergências (5) e (9).

**(iii)** $\displaystyle\int_0^{T^*}((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k,\phi^m)dt \to \int_0^{T^*}((n\mathbf{u}.\nabla)\mathbf{u},\phi^m)dt.$

De fato,

$$\begin{aligned}
&\left|\int_0^{T^*}\int_{\mathbb{T}^d}((n\mathbf{u}^k.\nabla)\mathbf{u}^k - (n\mathbf{u}.\nabla)\mathbf{u}).\phi^m dxdt\right| \\
= &\left|\int_0^{T^*}\int_{\mathbb{T}^d}((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k - (n^k\mathbf{u}^k.\nabla)\mathbf{u} + (n^k\mathbf{u}^k.\nabla)\mathbf{u} - (n^k\mathbf{u}.\nabla)\mathbf{u}\right. \\
&\left. +(n^k\mathbf{u}.\nabla)\mathbf{u} - (n\mathbf{u}.\nabla)\mathbf{u}).\phi^m dxdt\right| \\
\leq &\left|\int_0^{T^*}\int_{\mathbb{T}^d}(n^k\mathbf{u}^k.(\nabla\mathbf{u}^k - \nabla\mathbf{u})).\phi^m dxdt\right| + \left|\int_0^{T^*}\int_{\mathbb{T}^d}(n^k(\mathbf{u}^k - \mathbf{u}).\nabla\mathbf{u}).\phi^m dxdt\right| \\
&+\left|\int_0^{T^*}\int_{\mathbb{T}^d}((n^k - n)\mathbf{u}.\nabla\mathbf{u}).\phi^m dxdt\right| \\
\leq &\ \|n^k\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\|\mathbf{u}^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\nabla\mathbf{u}^k - \nabla\mathbf{u}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \\
&+\|n^k\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\|\mathbf{u}^k - \mathbf{u}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\nabla\mathbf{u}^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \\
&+\|n^k - n\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\mathbf{u}\|_{L^2(0,T^*;L^\infty(\mathbb{T}^d))}\|\nabla\mathbf{u}\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \to 0,
\end{aligned}$$

pelas convergências (3), (2) e (5).

**(iv)** $\displaystyle\int_0^{T^*}(n^k\Delta\mathbf{u}^k,\phi^m)dt \to \int_0^{T^*}(n\Delta\mathbf{u},\phi^m)dt.$

De fato,

$$\begin{aligned}
&\left|\int_0^{T^*}\int_{\mathbb{T}^d}(n^k\Delta\mathbf{u}^k - n\Delta\mathbf{u}).\phi^m dxdt\right| \\
= &\left|\int_0^{T^*}\int_{\mathbb{T}^d}(n^k\Delta\mathbf{u}^k - n\Delta\mathbf{u}^k + n\Delta\mathbf{u}^k - n\Delta\mathbf{u}).\phi^m dxdt\right| \\
\leq &\|n^k - n\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\Delta\mathbf{u}^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \\
&+\left|\int_0^{T^*}\int_{\mathbb{T}^d}(\Delta\mathbf{u}^k - \Delta\mathbf{u})n\phi^m dxdt\right| \to 0,
\end{aligned}$$

pelas convergências (5) e (8).

**(v)** $\displaystyle\int_0^{T^*}((\nabla n^k.\nabla)\mathbf{u}^k,\phi^m)dt \to \int_0^{T^*}((\nabla n.\nabla)\mathbf{u},\phi^m)dt.$

De fato,

$$\begin{aligned}
&\left|\int_0^{T^*}\int_{\mathbb{T}^d}(\nabla n^k.\nabla\mathbf{u}^k - \nabla n.\nabla\mathbf{u}).\phi^m dxdt\right| \\
\leq &\left|\int_0^{T^*}\int_{\mathbb{T}^d}(\nabla n^k.\nabla\mathbf{u}^k - \nabla n.\nabla\mathbf{u}^k + \nabla n.\nabla\mathbf{u}^k - \nabla n.\nabla\mathbf{u}).\phi^m dxdt\right| \\
\leq &\|\nabla n^k - \nabla\, n\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\nabla\mathbf{u}^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \\
&+\|\nabla\, n\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\nabla\mathbf{u}^k - \nabla\mathbf{u}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \to 0,
\end{aligned}$$

pelas convergências (5) e (3).

**(vi)** $\displaystyle\int_0^{T^*}(n^k\mathbf{f},\phi^m)dt \to \int_0^{T^*}(n\mathbf{f},\phi^m)dt.$

De fato,

$$\begin{aligned}
&\left|\int_0^{T^*}\int_{\mathbb{T}^d}(n^k\mathbf{f} - n\mathbf{f}).\phi^m dxdt\right| \\
\leq &\|n^k - n\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\mathbf{f}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \to 0,
\end{aligned}$$

pela convergência (5).

**(vii)** $\displaystyle\int_0^{T^*}\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k,\phi^m\right)dt \to \int_0^{T^*}\left(\frac{1}{n}(\nabla n.\nabla)\nabla n,\phi^m\right)dt.$

De fato,

$$\left|\int_0^{T^*}\int_{\mathbb{T}^d}\left(\frac{1}{n^k}\nabla n^k.\nabla^2 n^k - \frac{1}{n}\nabla n\nabla^2 n\right).\phi^m dxdt\right|$$

$$
\begin{aligned}
&= \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{1}{n^k} \nabla n^k \nabla^2 n^k - \frac{1}{n} \nabla n^k \nabla^2 n^k + \frac{1}{n} \nabla n^k \nabla^2 n^k - \frac{1}{n} \nabla n \nabla^2 n^k \right. \right. \\
&\quad \left. \left. + \frac{1}{n} \nabla n \nabla^2 n^k - \frac{1}{n} \nabla n \nabla^2 n \right) . \phi^m dx dt \right| \\
&\leq \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{n - n^k}{n^k n} \nabla n^k \nabla^2 n^k \right) . \phi^m dx dt \right| \\
&\quad + \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{1}{n} (\nabla n - \nabla n^k) . \nabla^2 n^k \right) . \phi^m dx dt \right| \\
&\quad + \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{1}{n} \nabla n . (\nabla^2 n - \nabla^2 n^k) \right) . \phi^m dx dt \right| \\
&\leq C \| n - n^k \|_{L^2(0,T^*;L^4(\mathbb{T}^d))} \| \nabla n^k \|_{L^\infty(0,T^*;L^4(\mathbb{T}^d))} \| \Delta n^k \|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \| \phi^m \|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \\
&\quad + C \| \nabla n - \nabla n^k \|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \| \Delta n^k \|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \| \phi^m \|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \\
&\quad + C \| \nabla n \|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \| \Delta n - \Delta n^k \|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \| \phi^m \|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \to 0,
\end{aligned}
$$

pelas convergências (5) e (6).

**(viii)** $\displaystyle\int_0^{T^*} \left( \frac{1}{(n^k)^2} (\nabla n^k . \nabla n^k) \nabla n^k, \phi^m \right) dt \to \int_0^{T^*} \left( \frac{1}{n^2} (\nabla n . \nabla n) \nabla n, \phi^m \right) dt.$

De fato,

$$
\begin{aligned}
&\left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{1}{(n^k)^2} (\nabla n^k . \nabla n^k) \nabla n^k - \frac{1}{n^2} (\nabla n . \nabla n) \nabla n \right) . \phi^m dx dt \right| \\
&= \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{1}{(n^k)^2} (\nabla n^k . \nabla n^k) \nabla n^k - \frac{1}{n^2} (\nabla n^k . \nabla n^k) \nabla n^k + \frac{1}{n^2} (\nabla n^k . \nabla n^k) \nabla n^k \right. \right. \\
&\quad - \frac{1}{n^2} (\nabla n . \nabla n^k) \nabla n^k + \frac{1}{n^2} (\nabla n . \nabla n^k) \nabla n^k - \frac{1}{n^2} (\nabla n . \nabla n) \nabla n^k + \frac{1}{n^2} (\nabla n . \nabla n) \nabla n^k \\
&\quad \left. \left. - \frac{1}{n^2} (\nabla n . \nabla n) \nabla n \right) . \phi^m dx dt \right| \\
&\leq \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{n - n^k}{n (n^k)^2} (\nabla n^k . \nabla n^k) \nabla n^k \right) . \phi^m dx dt \right| \\
&\quad + \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{n - n^k}{n^k n^2} (\nabla n^k . \nabla n^k) \nabla n^k \right) . \phi^m dx dt \right| \\
&\quad + \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{1}{n^2} ((\nabla n - \nabla n^k) . \nabla n^k) \nabla n^k \right) . \phi^m dx dt \right| \\
&\quad + \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{1}{n^2} (\nabla n . (\nabla n - \nabla n^k)) \nabla n^k \right) . \phi^m dx dt \right| \\
&\quad + \left| \int_0^{T^*} \int_{\mathbb{T}^d} \left( \frac{1}{n^2} (\nabla n . \nabla n) (\nabla n - \nabla n^k) \right) . \phi^m dx dt \right| \\
&\leq C \| n - n^k \|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \| \nabla n^k \|^3_{L^6(0,T^*;L^6(\mathbb{T}^d))} \| \phi^m \|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \\
&\quad + C \| n - n^k \|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \| \nabla n^k \|^3_{L^6(0,T^*;L^6(\mathbb{T}^d))} \| \phi^m \|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))} \\
&\quad + C \| \nabla n - \nabla n^k \|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \| \nabla n^k \|^2_{L^4(0,T^*;L^4(\mathbb{T}^d))} \| \phi^m \|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}
\end{aligned}
$$

$$+C\|\nabla n\|_{L^4(0,T^*;L^4(\mathbb{T}^d))}\|\nabla n-\nabla n^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\nabla n^k\|_{L^4(0,T^*;L^4(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}$$
$$+C\|\nabla n\|^2_{L^4(0,T^*;L^4(\mathbb{T}^d))}\|\nabla n-\nabla n^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\to 0,$$

pela convergência (5).

**(ix)** $\displaystyle\int_0^{T^*}\left(\frac{1}{n^k}\Delta n^k\nabla n^k,\phi^m\right)dt\to\int_0^{T^*}\left(\frac{1}{n}\Delta n\nabla n,\phi^m\right)dt.$

De fato,

$$\begin{aligned}
&\left|\int_0^{T^*}\int_{\mathbb{T}^d}\left(\frac{1}{n^k}\Delta n^k\nabla n^k-\frac{1}{n}\Delta n\nabla n\right).\phi^m dxdt\right|\\
=&\left|\int_0^{T^*}\int_{\mathbb{T}^d}\left(\frac{1}{n^k}\Delta n^k\nabla n^k-\frac{1}{n}\Delta n^k\nabla n^k+\frac{1}{n}\Delta n^k\nabla n^k-\frac{1}{n}\Delta n\nabla n^k+\frac{1}{n}\Delta n\nabla n^k\right.\right.\\
&\left.\left.-\frac{1}{n}\Delta n\nabla n\right).\phi^m dxdt\right|\\
\leq&\left|\int_0^{T^*}\int_{\mathbb{T}^d}\left(\frac{n-n^k}{nn^k}\Delta n^k\nabla n^k\right).\phi^m dxdt\right|\\
&+\left|\int_0^{T^*}\int_{\mathbb{T}^d}\left(\frac{1}{n}(\Delta n-\Delta n^k)\nabla n^k\right).\phi^m dxdt\right|\\
&+\left|\int_0^{T^*}\int_{\mathbb{T}^d}\left(\frac{1}{n}\Delta n(\nabla n-\nabla n^k\right).\phi^m dxdt\right|\\
\leq&\ C\|n-n^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\Delta n^k\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\|\nabla n^k\|_{L^2(0,T^*;L^\infty(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\\
&+C\|\Delta n-\Delta n^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\nabla n^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\\
&+C\|\Delta n\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\nabla n-\nabla n^k\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\|\phi^m\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\to 0,
\end{aligned}$$

pelas convergências (5) e (6).

Assim, obtemos:

$$\begin{aligned}
&\int_0^{T^*}(n\mathbf{u}_t,\phi^m)dt+\int_0^{T^*}((n\mathbf{u}.\nabla)\mathbf{u},\phi^m)dt+\nu\left[-\int_0^{T^*}(n\Delta\mathbf{u},\phi^m)dt\right.\\
&\left.-2\int_0^{T^*}((\nabla n.\nabla)\mathbf{u},\phi^m)dt\right]=\int_0^{T^*}(n\mathbf{f},\phi^m)dt\\
&+\varepsilon^2\left[-\int_0^{T^*}\left(\frac{1}{n}(\nabla n.\nabla)\nabla n,\phi^m\right)dt+\int_0^{T^*}\left(\frac{1}{n^2}(\nabla n.\nabla n)\nabla n,\phi^m\right)dt\right.\\
&\left.-\int_0^{T^*}\left(\frac{1}{n}\Delta n\nabla n,\phi^m\right)dt+\int_0^{T^*}(\nabla\Delta n,\phi^m)\,dt\right],
\end{aligned}\tag{2.25}$$

para toda $\phi^m$ dada por (2.24).

Além disso, temos:

$$\|n\mathbf{u}_t\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\leq\|n\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\|\mathbf{u}_t\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}$$

$$\leq C;$$

$$\begin{aligned}\|n\mathbf{u}.\nabla\mathbf{u}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} &\leq \|n\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\|\mathbf{u}\|_{L^2(0,T^*;L^\infty(\mathbb{T}^d))}\|\nabla\mathbf{u}\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\\ &\leq C;\end{aligned}$$

$$\begin{aligned}\|n\Delta\mathbf{u}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} &\leq \|n\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\|\Delta\mathbf{u}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\\ &\leq C;\end{aligned}$$

$$\begin{aligned}\|\nabla n.\nabla\mathbf{u}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} &\leq \|\nabla n\|_{L^2(0,T^*;L^\infty(\mathbb{T}^d))}\|\nabla\mathbf{u}\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\\ &\leq C;\end{aligned}$$

$$\begin{aligned}\|n\mathbf{f}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} &\leq \|n\|_{L^\infty(0,T^*;L^\infty(\mathbb{T}^d))}\|\mathbf{f}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))}\\ &\leq C;\end{aligned}$$

$$\begin{aligned}\left\|\frac{1}{n}\nabla n\nabla^2 n\right\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} &\leq C\|\nabla n\|_{L^2(0,T^*;L^\infty(\mathbb{T}^d))}\|\Delta n\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\\ &\leq C;\end{aligned}$$

$$\begin{aligned}\left\|\frac{1}{n^2}(\nabla n.\nabla n)\nabla n\right\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} &\leq C\|\nabla n\|^3_{L^\infty(0,T^*;L^6(\mathbb{T}^d))}\\ &\leq C\|\Delta n\|^3_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\\ &\leq C;\end{aligned}$$

$$\begin{aligned}\left\|\frac{1}{n}\Delta n\nabla n\right\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} &\leq C\|\Delta n\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\|\nabla n\|_{L^2(0,T^*;L^\infty(\mathbb{T}^d))}\\ &\leq C;\end{aligned}$$

$$\|\nabla\Delta n\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \leq C;$$

Isto implica que,

$$\begin{aligned}\mathbf{Lu} &= n\mathbf{u}_t + (n\mathbf{u}.\nabla)\mathbf{u} + \nu\left[-n\Delta\mathbf{u} - 2(\nabla n.\nabla)\mathbf{u}+\right] - n\mathbf{f}\\ &\quad -\varepsilon^2\left[-\frac{1}{n}(\nabla n.\nabla)\nabla n + \frac{1}{n^2}(\nabla n.\nabla n)\nabla n - \frac{1}{n}\Delta n\nabla n + \nabla\Delta n\right] \in L^2(0,T^*;L^2(\mathbb{T}^d)).\end{aligned}$$

Pelo fato das funções $\phi^m$ serem densas em $L^2(0,T^*;H)$, temos que (2.25) também é válido para $\phi \in L^2(0,T^*;H)$ e assim $\mathbf{Lu} \in L^2(0,T^*;H)^\perp$.

Portanto, pelo Lema De Rham (ver [25]), existe $p \in L^2(0,T^*;H^1(\mathbb{T}^d))$ tal que

$$n\mathbf{u}_t + (n\mathbf{u}.\nabla)\mathbf{u} + \nu\left[-n\Delta\mathbf{u} - (\nabla n.\nabla)\mathbf{u}\right] - n\mathbf{f}$$
$$-\varepsilon^2\left[-\frac{1}{n}(\nabla n.\nabla)\nabla n + \frac{1}{n^2}(\nabla n.\nabla n)\nabla n - \frac{1}{n}\Delta n\nabla n + \nabla\Delta n\right] = \nabla p.$$

Agora, multiplicando a equação (2.4) por $\varphi \in L^2(0,T^*;L^2(\mathbb{T}^d))$ e integrando sobre $[0,T^*]\times\mathbb{T}^d$, obtemos:

$$\int_0^{T^*}\int_{\mathbb{T}^d}(n_t^k + \mathbf{u}^k.\nabla n^k - \nu\Delta n^k)\varphi dxdt = 0.$$

Vamos mostrar que a integral dupla acima converge para

$$\int_0^{T^*}\int_{\mathbb{T}^d}(n_t + \mathbf{u}.\nabla n - \nu\Delta n)\varphi dxdt = 0.$$

De fato,

**(i)**

$$\left|\int_0^{T^*}\int_{\mathbb{T}^d}(n_t^k - n_t)\varphi dxdt\right| \to 0,$$

pela convergência (12).

**(ii)**

$$\begin{aligned}
&\left|\int_0^{T^*}\int_{\mathbb{T}^d}(\mathbf{u}^k.\nabla n^k - \mathbf{u}.\nabla n)\varphi dxdt\right| \\
= &\left|\int_0^{T^*}\int_{\mathbb{T}^d}(\mathbf{u}^k.\nabla n^k - \mathbf{u}^k.\nabla n + \mathbf{u}^k.\nabla n - \mathbf{u}.\nabla n)\varphi dxdt\right| \\
\leq &\ \|\mathbf{u}^k\|_{L^\infty(0,T^*;L^4(\mathbb{T}^d))}\|\nabla n^k - \nabla n\|_{L^2(0,T^*;L^4(\mathbb{T}^d))}\|\varphi\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \\
&+\|\mathbf{u}^k - \mathbf{u}\|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))}\|\nabla n\|_{L^2(0,T^*;L^\infty(\mathbb{T}^d))}\|\varphi\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \to 0,
\end{aligned}$$

pelas convergências (2) e (5).

**(iii)**

$$\left|\int_0^{T^*}\int_{\mathbb{T}^d}(\Delta n^k - \Delta n)\varphi dxdt\right| \leq \|\Delta n^k - \Delta n\|_{L^2(0,T^*;L^\infty(\mathbb{T}^d))}\|\varphi\|_{L^2(0,T^*;L^2(\mathbb{T}^d))} \to 0,$$

pela convergência (6).

Usando o Lema de Du Bois Raymond (ver [39]), concluímos que:

$$n_t + \mathbf{u}.\nabla n = \nu\Delta n \quad \text{q.t.p. em} \quad [0,T^*]\times\mathbb{T}^d,$$

e, pelas estimativas anteriormente obtidas também obtemos que,

$$n_t + \mathbf{u}.\nabla n = \nu\Delta n \quad \text{em} \quad L^\infty(0,T^*;L^2(\mathbb{T}^d)).$$

*Parte 3: Verificação dos dados Iniciais.*

Temos pela convergência (1) que

$$\mathbf{u}^k \to \mathbf{u} \;\; \text{em} \;\; C([0,T^*];L^2(\mathbb{T}^d)),$$

em particular, temos que

$$\mathbf{u}^k(\cdot,0) \to \mathbf{u}(\cdot,0) \;\; \text{em} \;\; L^2(\mathbb{T}^d),$$

mas por outro lado,

$$\mathbf{u}_0^k \to \mathbf{u}_0 \;\; \text{em} \;\; L^2(\mathbb{T}^d),$$

pela unicidade de limite, concluímos que $\mathbf{u}(\cdot,0) = \mathbf{u}_0$ em $L^2(\mathbb{T}^d)$. De forma análoga se conclui que $n(\cdot,0) = n_0$ em $H^1(\mathbb{T}^d)$.

*Parte 4: Unicidade de Solução.*

Sejam $(\mathbf{u},n)$ e $(\mathbf{u}^1,n^1)$ duas soluções do problema (2.1)-(2.3), definimos $z = n - n^1$ e $\mathbf{w} = \mathbf{u} - \mathbf{u}^1$. Sejam

$$\begin{aligned} n_t + \mathbf{u}.\nabla n &= \nu\Delta n, \\ n_t^1 + \mathbf{u}^1.\nabla n^1 &= \nu\Delta n^1, \end{aligned}$$

de onde segue que:

$$z_t + \mathbf{u}.\nabla z + \mathbf{w}.\nabla n^1 = \nu\Delta z.$$

Multiplicando a equação acima por $\psi \in L^2(0,T^*;L^2(\mathbb{T}^d))$ e integrando sobre $\mathbb{T}^d$, obtemos;

$$(z_t,\psi) + (\mathbf{u}.\nabla z,\psi) + (\mathbf{w}.\nabla n^1,\psi) = \nu(\Delta z,\psi). \tag{2.26}$$

Em particular, fazendo $\psi = z$, teremos;

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(\mathbb{T}^d)} = \nu(\Delta z, z) - (\mathbf{u}.\nabla z, z) - (\mathbf{w}.\nabla n^1, z).$$

Observe que;

$$\begin{aligned} (\mathbf{u}.\nabla z, z) &= (\nabla z, z\mathbf{u}) \\ &= -(z, div(z\mathbf{u})) \\ &= -(z, z\; div(\mathbf{u})) - (z, \mathbf{u}.\nabla z) \\ &= -(\mathbf{u}.\nabla z, z), \end{aligned}$$

implica que $(\mathbf{u}.\nabla z, z) = 0$. Também temos que,

$$\nu(\Delta z, z) \;\; = \;\; -\nu(\nabla z, \nabla z)$$

$$= \quad -\nu\|\nabla z\|^2_{L^2(\mathbb{T}^d)}$$

$$\begin{aligned}(\mathbf{w}.\nabla n^1, z) &\leq \|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\|\nabla n^1\|_{L^\infty(\mathbb{T}^d)}\|z\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\delta_2)\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|z\|^2_{H^1(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

Portanto, obtemos;

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla z\|^2_{L^2(\mathbb{T}^d)} \leq C(\delta_2)\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|\nabla z\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}. \tag{2.27}$$

Por outro lado, fazendo $\psi = -\Delta z$ em (2.26), obtemos;

$$\frac{1}{2}\frac{d}{dt}\|\nabla z\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta z\|^2_{L^2(\mathbb{T}^d)} = (\mathbf{u}.\nabla z, \Delta z) + (\mathbf{w}.\nabla n^1, \Delta z).$$

Estimando os termos à direita da equação acima, como se segue:

$$\begin{aligned}|(\mathbf{u}.\nabla z, \Delta z)| &\leq \|\mathbf{u}\|_{L^\infty(\mathbb{T}^d)}\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\Delta z\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\delta_1)\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\nabla z\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta z\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\mathbf{w}.\nabla n^1, \Delta z)| &\leq \|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\|\nabla n^1\|_{L^\infty(\mathbb{T}^d)}\|\Delta z\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\delta_1)\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta z\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

Assim, obtemos que;

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla z\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta z\|^2_{L^2(\mathbb{T}^d)} \leq C(\delta_1)\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\nabla z\|^2_{L^2(\mathbb{T}^d)}\\ &+C(\delta_1)\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + 2\delta_1\|\Delta z\|^2_{L^2(\mathbb{T}^d)}.\end{aligned} \tag{2.28}$$

Além disso, considerando que;

$$\begin{aligned}&(n\mathbf{u}_t, \mathbf{v}) - \nu(n\Delta\mathbf{u}, \mathbf{v}) = ((n\mathbf{u}.\nabla)\mathbf{u}, \mathbf{v}) + 2\nu((\nabla n.\nabla)\mathbf{u}, \mathbf{v}) + (n\mathbf{f}, \mathbf{v})\\ &+\varepsilon^2\left[-\left(\frac{1}{n}(\nabla n.\nabla)\nabla n, \mathbf{v}\right) + \left(\frac{1}{n^2}(\nabla n.\nabla)\nabla n, \mathbf{v}\right) - \left(\frac{1}{n}\Delta n\nabla n, \mathbf{v}\right) + (\nabla\Delta n, \mathbf{v})\right],\end{aligned}$$

e,

$$\begin{aligned}&(n^1\mathbf{u}^1_t, \mathbf{v}) - \nu(n^1\Delta\mathbf{u}^1, \mathbf{v}) = ((n^1\mathbf{u}^1.\nabla)\mathbf{u}^1, \mathbf{v}) + 2\nu((\nabla n^1.\nabla)\mathbf{u}^1, \mathbf{v}) + (n^1\mathbf{f}, \mathbf{v})\\ &+\varepsilon^2\Bigg[-\left(\frac{1}{n^1}(\nabla n^1.\nabla)\nabla n^1, \mathbf{v}\right) + \left(\frac{1}{(n^1)^2}(\nabla n^1.\nabla)\nabla n^1, \mathbf{v}\right)\\ &-\left(\frac{1}{n^1}\Delta n^1\nabla n^1, \mathbf{v}\right) + (\nabla\Delta n^1, \mathbf{v})\Bigg],\end{aligned}$$

então, para $\mathbf{v} = \mathbf{w}$, segue que:

$$(n\mathbf{w}_t, \mathbf{w}) - \nu(n\Delta\mathbf{w}, \mathbf{w}) = -(z\mathbf{u}^1_t, \mathbf{w}) + \nu(z\Delta\mathbf{u}_1, \mathbf{w}) - ((z\mathbf{u}.\nabla)\mathbf{u}, \mathbf{w}) - ((n^1\mathbf{w}.\nabla)\mathbf{u}, \mathbf{w})$$

$$
\begin{aligned}
&-((n^1\mathbf{u}_1.\nabla)\mathbf{w},\mathbf{w}) - \nu[-2((\nabla z.\nabla)\mathbf{u},\mathbf{w}) - 2((\nabla n^1)\nabla\mathbf{w},\mathbf{w})] + (z\mathbf{f},\mathbf{w})\\
&+\varepsilon^2\Bigg[\left(\frac{z}{nn^1}(\nabla n.\nabla)\nabla n,\mathbf{w}\right) - \left(\frac{1}{n^1}(\nabla z.\nabla)\nabla n,\mathbf{w}\right) - \left(\frac{1}{n^1}(\nabla n^1.\nabla)\nabla z,\mathbf{w}\right)\\
&-\left(\frac{z(n^1+n)}{n^2(n^1)^2}(\nabla n.\nabla n)\nabla n,\mathbf{w}\right) + \left(\frac{1}{(n^1)^2}(\nabla z.\nabla n)\nabla n,\mathbf{w}\right) + \left(\frac{1}{(n^1)^2}(\nabla n^1.\nabla z)\nabla n,\mathbf{w}\right)\\
&+\left(\frac{1}{(n^1)^2}(\nabla n^1.\nabla n^1)\nabla z,\mathbf{w}\right) + \left(\frac{z}{nn^1}\Delta n\nabla n,\mathbf{w}\right) - \left(\frac{1}{n^1}\Delta z\nabla n,\mathbf{w}\right)\\
&-\left(\frac{1}{n^1}\Delta n^1\nabla z,\mathbf{w}\right) + (\nabla\Delta z,\mathbf{w})\Bigg].
\end{aligned}
$$

Observe que,

$$
\begin{aligned}
(n\mathbf{w}_t,\mathbf{w}) &= \frac{1}{2}\frac{d}{dt}\|\sqrt{n}\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}(n_t\mathbf{w},\mathbf{w})\\
&= \frac{1}{2}\frac{d}{dt}\|\sqrt{n}\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}((\mathbf{u}.\nabla n)\mathbf{w},\mathbf{w}) - \frac{\nu}{2}(\mathbf{w}\Delta n,\mathbf{w}),
\end{aligned}
$$

e

$$
\begin{aligned}
-\nu(n\Delta\mathbf{w},\mathbf{w}) &= \nu(n\nabla\mathbf{w},\nabla\mathbf{w}) + \nu(\mathbf{w}.\nabla n.\nabla\mathbf{w})\\
&\geq \nu\alpha\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \nu(\mathbf{w}.\nabla n,\nabla\mathbf{w}).
\end{aligned}
$$

Estimamos os seguintes termos como se segue:

$$
\begin{aligned}
\frac{1}{2}|((\mathbf{u}.\nabla n)\mathbf{w},\mathbf{w})| &\leq \frac{1}{2}\|\mathbf{u}\|_{L^6(\mathbb{T}^d)}\|\nabla n\|_{L^6(\mathbb{T}^d)}\|\mathbf{w}\|_{L^6(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\delta_2)\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C(\delta_2)\|\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\frac{\nu}{2}|(\Delta n\mathbf{w},\mathbf{w})| &\leq \frac{\nu}{2}\|\Delta n\|_{L^4(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^4(\mathbb{T}^d)}\\
&\leq C(\delta_2)\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\nu|(\mathbf{w}.\nabla n,\nabla\mathbf{w})| &\leq \nu\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\delta_2)\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(z\mathbf{u}^1_t,\mathbf{w})| &\leq \|z\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}^1_t\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^4(\mathbb{T}^d)}\\
&\leq C(\delta_2)\|\mathbf{u}^1_t\|^2_{L^2(\mathbb{T}^d)}\|z\|^2_{H^1(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\nu|(z\Delta\mathbf{u}^1,\mathbf{w})| &\leq \nu\|z\|_{L^4(\mathbb{T}^d)}\|\Delta\mathbf{u}^1\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^4(\mathbb{T}^d)}\\
&\leq C(\delta_2)\|\Delta\mathbf{u}_1\|^2_{L^2(\mathbb{T}^d)}\|z\|^2_{H^1(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|((z\mathbf{u}.\nabla)\mathbf{u},\mathbf{w})| &\leq \|z\|_{L^6(\mathbb{T}^d)}\|\mathbf{u}\|_{L^6(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^6(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|z\|_{H^1(\mathbb{T}^d)}\|\Delta\mathbf{u}\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|z\|^2_{H^1(\mathbb{T}^d)}+C\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(z\mathbf{f},\mathbf{w})| &\leq \|z\|_{L^4(\mathbb{T}^d)}\|\mathbf{f}\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^4(\mathbb{T}^d)}\\
&\leq C(\delta_2)\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)}\|\ z\|^2_{H^1(\mathbb{T}^d)}+\delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
|(\nabla\Delta z,\mathbf{w})| = |(\Delta z, div(\mathbf{w}))| = 0;
$$

$$
\begin{aligned}
\left|\left(\frac{1}{n^1}\Delta n^1\nabla z,\mathbf{w}\right)\right| &\leq \frac{1}{\alpha}\|\Delta n^1\|_{L^4(\mathbb{T}^d)}\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^4(\mathbb{T}^d)}\\
&\leq C(\delta_2)\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|\nabla z\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\left|\left(\frac{1}{n^1}\Delta z\nabla n,\mathbf{w}\right)\right| &\leq \frac{1}{\alpha}\|\Delta z\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\delta_1)\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta z\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\left|\left(\frac{z}{nn^1}\Delta n\nabla n,\mathbf{w}\right)\right| &\leq \frac{1}{\alpha^2}\|z\|_{L^6(\mathbb{T}^d)}\|\Delta n\|_{L^6(\mathbb{T}^d)}\|\nabla n\|_{L^6(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|z\|_{H^1(\mathbb{T}^d)}\|\nabla\Delta n\|_{L^2(\mathbb{T}^d)}\|\Delta n\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}+C\|\Delta n\|^2_{L^2(\mathbb{T}^d)}\|z\|^2_{H^1(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\left|\left(\frac{1}{(n^1)^2}(\nabla n^1.\nabla n^1)\nabla z,\mathbf{w}\right)\right| &\leq \frac{1}{\alpha^2}\|\nabla n^1\|_{L^\infty(\mathbb{T}^d)}\|\nabla n^1\|_{L^\infty(\mathbb{T}^d)}\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n^1\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^1\|_{L^2(\mathbb{T}^d)}\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|\nabla z\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\left|\left(\frac{1}{(n^1)^2}(\nabla n^1.\nabla z)\nabla n,\mathbf{w}\right)\right| &\leq C\|\nabla n^1\|_{L^\infty(\mathbb{T}^d)}\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n^1\|_{L^2(\mathbb{T}^d)}\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|\nabla z\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\left|\left(\frac{1}{(n^1)^2}(\nabla z.\nabla n)\nabla n,\mathbf{w}\right)\right| &\leq C\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\nabla z\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$\left|\left(\frac{z(n+n^1)}{n^2(n^1)^2}(\nabla n.\nabla n)\nabla n,\mathbf{w}\right)\right| \leq \frac{2\beta}{\alpha^4}\|z\|_{L^6(\mathbb{T}^d)}\|\nabla n\|^2_{L^6(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|z\|_{H^1(\mathbb{T}^d)}\|\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\Delta n\|^4_{L^2(\mathbb{T}^d)}\|z\|^2_{H^1(\mathbb{T}^d)} + C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};$$

$$\left|\left(\frac{1}{n^1}(\nabla n^1.\nabla)\nabla z,\mathbf{w}\right)\right| \leq \frac{1}{\alpha}\|\nabla n^1\|_{L^\infty(\mathbb{T}^d)}\|\nabla^2 z\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\delta_1)\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta z\|^2_{L^2(\mathbb{T}^d)};$$

$$\left|\left(\frac{z}{nn^1}(\nabla n.\nabla)\nabla n,\mathbf{w}\right)\right| \leq \frac{1}{\alpha^2}\|z\|_{L^6(\mathbb{T}^d)}\|\nabla n\|_{L^6(\mathbb{T}^d)}\|\nabla^2 n\|_{L^6(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|z\|_{H^1(\mathbb{T}^d)}\|\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\Delta n\|^2_{L^2(\mathbb{T}^d)}\|z\|^2_{H^1(\mathbb{T}^d)} + C\|\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};$$

$$\left|\left((\nabla n^1.\nabla)\mathbf{w},\mathbf{w}\right)\right| \leq \|\nabla n^1\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{w}\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\nabla\Delta n^1\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{w}\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\delta_2)\|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};$$

$$|((\nabla z.\nabla)\mathbf{u},\mathbf{w})| \leq \|\nabla z\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\mathbf{w}\|_{L^4(\mathbb{T}^d)}$$
$$\leq C\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\delta_2)\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\nabla z\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};$$

$$|(n^1\mathbf{u}^1.\nabla\mathbf{w},\mathbf{w})| \leq \beta\|\mathbf{u}^1\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{w}\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\Delta\mathbf{u}^1\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{w}\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\delta_2)\|\Delta\mathbf{u}^1\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};$$

$$|(n^1\mathbf{w}.\nabla\mathbf{u},\mathbf{w})| \leq \beta\|\mathbf{w}\|_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\nabla\mathbf{w}\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}\|_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\delta_2)\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};$$

$$\left|\left(\frac{1}{n^1}(\nabla z.\nabla)\nabla n,\mathbf{w}\right)\right| \leq C\|\nabla z\|_{L^2(\mathbb{T}^d)}\|\Delta n\|_{L^4(\mathbb{T}^d)}\|\mathbf{w}\|_{L^4(\mathbb{T}^d)}$$
$$\leq C(\delta_2)\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\nabla z\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)};$$

Obtemos,

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{n}\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \nu\alpha\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} \leq 12\delta_2\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + 2\delta_1\|\Delta z\|^2_{L^2(\mathbb{T}^d)}$$

$$+C(\delta_1)(\|\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}^1_t\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^1\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n\|^4_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^1\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)})(\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \|z\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z\|^2_{L^2(\mathbb{T}^d)}).$$

Somando a desigualdade diferencial acima com (2.27) e (2.28) escolhendo $\delta_2 = \nu\alpha/26$ e $\delta_1 = \nu/8$ obtemos,

$$\frac{1}{2}\frac{d}{dt}(\|\sqrt{n}\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \|z\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z\|^2_{L^2(\mathbb{T}^d)}) + \frac{\nu}{2}\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla z\|^2_{L^2(\mathbb{T}^d)}$$
$$+\frac{\nu}{2}\|\Delta z\|^2_{L^2(\mathbb{T}^d)} \leq C(\|\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}^1_t\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^1\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n\|^4_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^1\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)})(\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|z\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z\|^2_{L^2(\mathbb{T}^d)}). \tag{2.29}$$

Note que $\|\nabla\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}, \|\nabla z\|^2_{L^2(\mathbb{T}^d)}, \|\Delta z\|^2_{L^2(\mathbb{T}^d)} \geq 0$, e dai, a partir (2.29), integrando de 0 a $t$ e observando que $\|\sqrt{n}\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} \leq \alpha\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}$, obtem-se que:

$$\|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} + \|z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)}$$
$$\leq C\int_0^t \psi(s)(\|\mathbf{w}(s)\|^2_{L^2(\mathbb{T}^d)} + \|z(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(s)\|^2_{L^2(\mathbb{T}^d)})ds$$

onde

$$\begin{aligned}\psi(s) &= \|\Delta n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}^1_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^1(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\\ &\quad +\|\nabla\Delta n^1(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n(s)\|^4_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^1(s)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}(s)\|^2_{L^2(\mathbb{T}^d)}\end{aligned}$$

Agora, usando o Lema de Gronwall (1.2.8), obtemos;

$$\begin{aligned}&\|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} + \|z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)}\\ \leq\ & (\|\mathbf{w}_0\|^2_{L^2(\mathbb{T}^d)} + \|z_0\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z_0\|^2_{L^2(\mathbb{T}^d)}) \times \exp\left(C\int_0^{T^*}\psi(s)ds\right)\\ =\ & 0,\end{aligned}$$

pois $z_0 = n(0) - n^1(0) = 0$ e $\mathbf{w} = \mathbf{u}(0) - \mathbf{u}^1(0) = 0$. Logo $\mathbf{u} = \mathbf{u}^1$ e $n = n^1$.

Com respeito à unicidade da pressão $p$ associada ao problema, dado que o domínio em questão é conexo, a unicidade da pressão-solução $p$ é obtida em $L^2(0, T^*; H^1(\mathbb{T}^d/\mathbb{R}))$.

■

**Corolário 2.3.1** *Sejam $d = 2$ ou $3$, e $\mathbf{u}_0 \in V \cap H^2(\mathbb{T}^d)$, $n_0 \in H^3(\mathbb{T}^d)$, $\mathbf{f} \in L^2(0, T; H^1(\mathbb{T}^d))$ e $\mathbf{f}_t \in L^2(0, T; L^2(\mathbb{T}^d))$. Então a solução $(\mathbf{u}, n)$ dada pelo Teorema (2.3.1) verifica as seguintes regularidades:*

$$\begin{aligned}\mathbf{u} &\in L^\infty(0, T^*; H^2(\mathbb{T}^d)), & n &\in L^\infty(0, T^*; H^3(\mathbb{T}^d)),\\ \mathbf{u} &\in L^2(0, T^*; H^3(\mathbb{T}^d)), & n &\in L^2(0, T^*; H^4(\mathbb{T}^d)),\\ \mathbf{u}_t &\in L^\infty(0, T^*; L^2(\mathbb{T}^d)), & n_t &\in L^\infty(0, T^*; H^1(\mathbb{T}^d)),\\ \mathbf{u}_t &\in L^2(0, T^*; H^1(\mathbb{T}^d)) \quad e & n_t &\in L^2(0, T^*; H^2(\mathbb{T}^d)).\end{aligned}$$

**Demonstração:**

Pelo Lema (2.2.2) temos a seguinte desigualdade diferencial;

$$\frac{d}{dt}(\frac{1}{2}\|\sqrt{n^k}\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n_t^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)})$$
$$+\frac{\nu\alpha}{2}\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\|n_t^k\|^2_{L^2(\mathbb{T}^d)}+\|n^k\|^2_{L^2(\mathbb{T}^d)}+\nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)}+\nu\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\frac{\nu}{2}\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)}\leq C(\nu,\varepsilon,\alpha,\beta)(\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\|n^k\|^2_{L^2(\mathbb{T}^d)}+\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|n_t^k\|^2_{L^2(\mathbb{T}^d)}+\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}).(\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\|n^k\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\|\Delta n^k\|^4_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n^k\|^6_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)})+C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|n_t\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}$$
$$+C\|n^k\|^2_{H^1(\mathbb{T}^d)}\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)}).$$

Integrando a desigualdade diferencial acima de 0 a $t$, com $t\in[0,T^*]$, obtemos;

$$\frac{1}{2}\|\sqrt{n^k}\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n_t^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\frac{\nu\alpha}{2}\int_0^t\|\nabla\mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds+\int_0^t\|n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds+\int_0^t\|n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$+\nu\int_0^t\|\nabla n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds+\nu\int_0^t\|\nabla n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds+\nu\int_0^t\|\nabla\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$+\frac{\nu}{2}\int_0^t\|\Delta n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds\leq\frac{1}{2}\|\sqrt{n^k(0)}\mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n^k(0)\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|\Delta n^k(0)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\frac{1}{2}\|n_t^k(0)\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|\nabla n_t^k(0)\|^2_{L^2(\mathbb{T}^d)}+C(\nu,\varepsilon,\alpha,\beta)\int_0^t(\|\mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}+\|n^k(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)}+\|n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}+\|\nabla n_t^k(s)\|^2_{L^2(\mathbb{T}^d)})(\|\Delta\mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}+\|n^k(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)}+\|\Delta n^k(s)\|^4_{L^2(\mathbb{T}^d)}+\|\Delta n^k(s)\|^6_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\mathbf{f}(s)\|^2_{H^1(\mathbb{T}^d)}+\|\mathbf{f}_t(s)\|^2_{L^2(\mathbb{T}^d)})ds.$$

Agora, usando o Lema de Gronwall (1.2.9), obtemos;

$$\frac{1}{2}\|\sqrt{n^k}\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n_t^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\frac{\nu\alpha}{2}\int_0^t\|\nabla\mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds+\int_0^t\|n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds+\int_0^t\|n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$+\nu\int_0^t\|\nabla n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds+\nu\int_0^t\|\nabla n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds+\nu\int_0^t\|\nabla\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$+\frac{\nu}{2}\int_0^t\|\Delta n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds\leq\Bigg(\frac{1}{2}\|\sqrt{n^k(0)}\mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n^k(0)\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|\Delta n^k(0)\|^2_{L^2(\mathbb{T}^d)}$$

$$+\frac{1}{2}\|n_t^k(0)\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\nabla n_t^k(0)\|^2_{L^2(\mathbb{T}^d)}\Bigg).\Bigg(1 + C(\nu,\varepsilon,\alpha,\beta)\int_0^t (\|\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}$$

$$+\|\nabla\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k(s)\|^4_{L^2(\mathbb{T}^d)} + \|\Delta n^k(s)\|^6_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}(s)\|^2_{H^1(\mathbb{T}^d)}$$

$$+\|\mathbf{f}_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|n^k(s)\|^2_{L^2(\mathbb{T}^d)})ds\Bigg) \times \exp\Bigg(C(\nu,\varepsilon,\alpha,\beta)\int_0^t (\|\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|n^k(s)\|^2_{L^2(\mathbb{T}^d)}$$

$$+\|\nabla\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k(s)\|^4_{L^2(\mathbb{T}^d)} + \|\Delta n^k(s)\|^6_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)}$$

$$+\|\mathbf{f}(s)\|^2_{H^1(\mathbb{T}^d)} + \|\mathbf{f}_t(s)\|^2_{L^2(\mathbb{T}^d)})ds\Bigg);$$

como, pelo Lema (2.2.2) temos que

$$\|\nabla n_t^k(0)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \le C(\nu,\varepsilon,\alpha,\|\mathbf{u}_0\|_{H^2(\mathbb{T}^d)},\|n_0\|_{H^3(\mathbb{T}^d)})$$

obtem-se que:

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{n^k}\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)}\\
&+\frac{\nu\alpha}{2}\int_0^t \|\nabla \mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + \int_0^t \|n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + \int_0^t \|n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds\\
&+\nu\int_0^t \|\nabla n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + \nu\int_0^t \|\nabla n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + \nu\int_0^t \|\nabla\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds\\
&+\frac{\nu}{2}\int_0^t \|\Delta n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds\\
\le\;& C(\nu,\varepsilon,\alpha,\beta,\|\mathbf{u}_0\|_{H^2(\mathbb{T}^d)},\|n_0\|_{H^3(\mathbb{T}^d)},\|\mathbf{f}\|_{L^2(0,T^*;L^2(\mathbb{T}^d))},\|\mathbf{f}_t\|_{L^2(0,T^*;L^2(\mathbb{T}^d))},T^*) = C_2^*.
\end{aligned}$$

Portando, obtemos as seguintes regularidades,

$$\begin{aligned}
\mathbf{u}_t^k &\in L^\infty(0,T^*;L^2(\mathbb{T}^d)), & n_t^k &\in L^\infty(0,T^*;H^1(\mathbb{T}^d)),\\
\mathbf{u}_t^k &\in L^2(0,T^*;H^1(\mathbb{T}^d)) \quad e & n_t^k &\in L^2(0,T^*;H^2(\mathbb{T}^d)).
\end{aligned}$$

Observe que $\mathbf{u}^k \in L^\infty(0,T^*;H^1(\mathbb{T}^d))$, $n^k \in L^\infty(0,T^*;H^2(\mathbb{T}^d))$ e $\mathbf{f} \in C([0,T^*];L^2(\mathbb{T}^d))$ e então, pelo item $(i)$ do Lema (2.2.3), tem-se que;

$$\begin{aligned}
&\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \le C(\nu,\alpha,\beta,\varepsilon)(\|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k\|^6_{L^2(\mathbb{T}^d)}\\
&+\|\nabla \mathbf{u}^k\|^8_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^6_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^8_{L^2(\mathbb{T}^d)})\\
&+C(\nu,\varepsilon,\alpha,\beta)\|n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha,\beta)\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + C(\alpha)\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\\
&\le C_2^*;
\end{aligned}$$

portanto,

$$\mathbf{u} \in L^\infty(0,T^*;H^2(\mathbb{T}^d)) \quad e \quad n \in L^\infty(0,T^*;H^3(\mathbb{T}^d)).$$

Agora, integrando de 0 a $t$, com $t \in [0,T^*]$ o item $(ii)$ do Lema (2.2.3), obtemos;

$$\int_0^t \|\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C(\nu) \int_0^t (\|\Delta n_t^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \Delta n^k(s)\|^4_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta \mathbf{u}^k(s)\|^4_{L^2(\mathbb{T}^d)})ds \leq C_2^*,$$

logo,

$$n^k \in L^2(0,T^*;H^4(\mathbb{T}^d)).$$

Por fim, integrando de 0 a $t$, com $t \in [0,T^*]$ o item $(iii)$ do Lema (2.2.3), obtem-se que:

$$\int_0^t \|\nabla \Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \int_0^t \Bigg( C(\nu,\alpha,\beta)\|\nabla \mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha)(\|\Delta \mathbf{u}^k(s)\|^4_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta \mathbf{u}^k(s)\|^8_{L^2(\mathbb{T}^d)}) + C(\nu,\varepsilon,\alpha)(\|\nabla \Delta n^k(s)\|^4_{L^2(\mathbb{T}^d)} + \|\nabla \Delta n^k(s)\|^6_{L^2(\mathbb{T}^d)} + \|\nabla \Delta n^k(s)\|^8_{L^2(\mathbb{T}^d)})$$
$$+C(\nu,\alpha,\beta)\|\mathbf{f}(s)\|^2_{H^1(\mathbb{T}^d)} + C(\nu,\alpha)\|\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)}\|\nabla \mathbf{f}(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+C(\nu,\alpha)\|\mathbf{u}_t^k(s)\|^4_{L^2(\mathbb{T}^d)}\Bigg)ds \leq C_2^*,$$

e portanto, segue que

$$\mathbf{u}^k \in L^2(0,T^*;H^3(\mathbb{T}^d)).$$

■

**Corolário 2.3.2** *Sejam* $d = 2$ *ou* 3, *e* $\mathbf{u}_0 \in V \cap H^3(\mathbb{T}^d)$, $n_0 \in H^4(\mathbb{T}^d)$, $\mathbf{f} \in L^2(0,T;H^2(\mathbb{T}^d))$ *e* $\mathbf{f}_t \in L^2(0,T;H^1(\mathbb{T}^d))$. *Então a solução* $(\mathbf{u},n)$ *dada pelo Teorema (2.3.1) verifica as seguintes regularidades:*

$$\begin{aligned}
\mathbf{u} &\in L^\infty(0,T^*;H^3(\mathbb{T}^d)), & n &\in L^\infty(0,T^*;H^4(\mathbb{T}^d)),\\
\mathbf{u} &\in L^2(0,T^*;H^4(\mathbb{T}^d)), & n &\in L^2(0,T^*;H^5(\mathbb{T}^d)),\\
\mathbf{u}_t &\in L^\infty(0,T^*;H^1(\mathbb{T}^d)), & n_t &\in L^\infty(0,T^*;H^2(\mathbb{T}^d)),\\
\mathbf{u}_t &\in L^2(0,T^*;H^2(\mathbb{T}^d)), & n_t &\in L^2(0,T^*;H^3(\mathbb{T}^d)),\\
\mathbf{u}_{tt} &\in L^2(0,T^*;L^2(\mathbb{T}^d)) \quad e & n_{tt} &\in L^2(0,T^*;H^1(\mathbb{T}^d)).
\end{aligned}$$

**Demonstração:**

Considerando a desigualdade diferencial dada pelo Lema (2.2.4) e as regularidades obtidas no Corolário (2.3.1), tem-se válida a seguinte desigualdade diferencial;

$$\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2\alpha^3}{4\beta^2}\|\Delta \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\alpha}{2}\|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + C\|\mathbf{f}_t\|^2_{L^2(\mathbb{T}^d)} + C\|n_t^k\|^2_{H^2(\mathbb{T}^d)} + C;$$

integrando de 0 a $t$, com $t \in [0,T^*]$, obtem-se;

$$
\begin{aligned}
&\alpha\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\Delta \mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + \int_0^t \|\mathbf{u}_{tt}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \le C\|\nabla \mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \\
&+C\int_0^t \|\nabla \mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + C\int_0^t \|\nabla \mathbf{f}(s)\|^2_{L^2(\mathbb{T}^d)} ds \\
&+C\int_0^t \|\mathbf{f}_t(s)\|^2_{L^2(\mathbb{T}^d)} ds + C\int_0^t \|n_t^k(s)\|^2_{H^2(\mathbb{T}^d)} ds + \int_0^t C ds \\
&\le C_2^*.
\end{aligned}
$$

Assim, obtem-se que:

$$
\mathbf{u}_t^k \in L^\infty(0,T^*;H^1(\mathbb{T}^d)) \cap L^2(0,T^*;H^2(\mathbb{T}^d)) \ \text{ e } \ \mathbf{u}_{tt}^k \in L^2(0,T^*;L^2(\mathbb{T}^d)).
$$

Por sua vez, integrando-se de 0 a $t$, com $t \in [0,T^*]$ a desigualdade diferencial $(i)$ do Lema (2.2.5), tem-se que:

$$
\begin{aligned}
&\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \le \|\Delta^2 n_0^k\|^2_{L^2(\mathbb{T}^d)} + C\int_0^t \|\nabla\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \\
&+C\int_0^t \|\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \\
&\le C_2^*.
\end{aligned}
$$

Portando, segue que

$$
n^k \in L^\infty(0,T^*;H^4(\mathbb{T}^d)) \qquad e \qquad n^k \in L^2(0,T^*;H^5(\mathbb{T}^d)).
$$

Assim, pelas estimativas obtidas acima, tem-se pela desigualdade $(ii)$ do Lema (2.2.5), que:

$$
\begin{aligned}
&\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)} \le C\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\le C_2^*;
\end{aligned}
$$

logo,

$$
n_t^k \in L^\infty(0,T^*;H^2(\mathbb{T}^d)).
$$

E ainda, integrando-se de 0 a $t$, com $t \in [0,T^*]$ a desigualdade diferencial $(iii)$ do Lema (2.2.5), tem-se;

$$
\begin{aligned}
&\int_0^t \|\nabla\Delta n_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \le C\int_0^t \|\nabla\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + C\int_0^t \|\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \\
&+C\int_0^t \|\nabla\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \\
&\le C_2^*;
\end{aligned}
$$

e assim,

$$
n_t^k \in L^2(0,T^*;H^3(\mathbb{T}^d)).
$$

Agora, integrando-se de 0 a $t$, com $t \in [0, T^*]$ a desigualdade diferencial dada pelo Lema (2.2.6) e considerando-se as estimativas obtidas anteriormente, temos;

$$\alpha \|\nabla \Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\Delta^2 \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C_2^*.$$

Então, temos;

$$\mathbf{u}^k \in L^\infty(0, T^*; H^3(\mathbb{T}^d)) \qquad e \qquad \mathbf{u}^k \in L^2(0, T^*; H^4(\mathbb{T}^d)).$$

De forma análoga, integrando de 0 a $t$, com $t \in [0, T^*]$ a desigualdade diferencial dada pelo Lema (2.2.7 ) e considerando as estimativas obtidas anteriormente, temos;

$$\int_0^t \|\nabla n_{tt}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C_2^*,$$

e dai, conclui-se que

$$n_{tt}^k \in L^2(0, T^*; H^1(\mathbb{T}^d)).$$

■

## 2.4 Existência e Unicidade de Solução Global

Agora nesta seção vamos nos dedicar a investigação de resultados de existência e unicidade de solução forte global no tempo para o problema (8)-(11) analisando separadamente o caso Bidimensional do caso Tridimensional. Por fim obtemos resultados de regularidade que são necessários para a análise de erro Global.

### 2.4.1 Caso Bidimensional

Agora assumindo uma hipótese sobre as constantes físicas em vez dos dados iniciais suficientemente pequenos, provamos a existência e unicidade de solução forte global no tempo sem assumir que força externa é suficientemente pequeno, será obtida usando exponenciais como funções peso e desigualdades do tipo Ladyzhenskaya e de Gagliardo-Nirenberg, a qual é uma forma de trabalhar inspirada em [32] e [34]. Obtemos resultados de regularidade semelhantes ao caso local.

**Lema 2.4.1** *Sejam $d = 2$, $\mathbf{u}_0 \in V$, $n_0 \in H^2(\mathbb{T}^d)$, $\mathbf{f} \in L^\infty(0,\infty; L^2(\mathbb{T}^d))$ e $\varepsilon/\nu < \alpha/\beta C$, onde a constante $C > 0$ é proveniente de imersões de Sobolev. Então, para algum $\gamma^* > 0$ e para todo $0 < \gamma \leq \gamma^*$ e para todo $t \geq 0$, temos as seguintes estimativas para as soluções aproximadas $(\mathbf{u}^k, n^k)$ do problema (2.4)-(2.6):*

$$
\begin{aligned}
e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C_2, &\qquad e^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta n^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C_2,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|n^k_t(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C_2, &\qquad \|\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \leq C_2,\\
\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} \leq C_2 \quad e &\qquad \alpha \leq n^k(x,t) \leq \beta,
\end{aligned}
$$

*onde*

$$
C_2 = C_2(\nu, \varepsilon, \alpha, \beta, \|\mathbf{u}_0\|_{L^2(\mathbb{T}^d)}, \|\nabla n_0\|_{L^2(\mathbb{T}^d)}, \|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))})
$$

*em particular temos,*

$$
\begin{aligned}
\mathbf{u}^k \in L^\infty(0,\infty; L^2(\mathbb{T}^d)), &\qquad n^k \in L^\infty(0,\infty; H^1(\mathbb{T}^d)),\\
\mathbf{u}^k \in L^2_{loc}(0,\infty; H^1(\mathbb{T}^d)), &\qquad n^k \in L^2_{loc}(0,\infty; H^2(\mathbb{T}^d)),\\
n^k_t \in L^2_{loc}(0,\infty; L^2(\mathbb{T}^d)) \quad e &\qquad n^k \in L^\infty(0,\infty; L^\infty(\mathbb{T}^d)).
\end{aligned}
$$

*Além disso, $\|n^k_t(0)\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu, \|\nabla \mathbf{u}_0\|_{L^2(\mathbb{T}^d)}, \|\Delta n_0\|_{L^2(\mathbb{T}^d)})$.*

**Demonstração:**

De forma análoga ao Lema (2.2.1) se obtém que,

$$
0 < \alpha \leq n^k(x,t) \leq \beta \quad \text{q.t.p} \quad (x,t) \in \mathbb{T}^d \times [0,\infty],
$$

ou seja,

$$
n^k \in L^\infty(0,\infty; L^\infty(\mathbb{T}^d)).
$$

Também se obtém a seguinte igualdade diferencial:

$$
\frac{1}{2}\frac{d}{dt}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} = \nu\|n^k\|^2_{L^2(\mathbb{T}^d)}. \tag{2.30}
$$

Multiplicando a equação (2.4) por $-\Delta n^k$ e integrando-a sobre $\mathbb{T}^d$, por integração por partes obtem-se:

$$
\frac{1}{2}\frac{d}{dt}\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} = -(\nabla(\mathbf{u}^k.\nabla n^k), \nabla n^k),
$$

estimando o termo da direita,

$$
\begin{aligned}
|(\nabla(\mathbf{u}^k.\nabla n^k), \nabla n^k)| &= |(\nabla n^k.\nabla \mathbf{u}^k, \nabla n^k) + (\mathbf{u}^k.\nabla^2 n^k, \nabla n^k)|\\
&= \left|(\nabla n^k.\nabla \mathbf{u}^k, \nabla n^k) + \frac{1}{2}(\mathbf{u}^k, \nabla|\nabla n^k|^2)\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left|(\nabla n^k.\nabla \mathbf{u}^k, \nabla n^k) - \frac{1}{2}(div\mathbf{u}^k, |\nabla n^k|^2)\right| \\
&= \left|(\nabla n^k.\nabla \mathbf{u}^k, \nabla n^k)\right| \\
&\leq \|\nabla n^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\overset{(iv)}{\leq} C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|n^k\|_{H^2(\mathbb{T}^d)}\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\leq C\beta\|n^k\|_{H^2(\mathbb{T}^d)}\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\leq \frac{\nu}{2}\|n^k\|^2_{H^2(\mathbb{T}^d)} + \frac{2C^2\beta}{\nu}\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}
$$

Somando com (2.30) temos,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)}\Big) + \frac{\nu}{2}\|n^k\|^2_{H^2(\mathbb{T}^d)} &\leq \frac{C^2\beta^2}{\nu}\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
&\quad +\nu\|n^k\|^2_{L^2(\mathbb{T}^d)}. \qquad (2.31)
\end{aligned}
$$

Agora, fazendo $\mathbf{v} = \mathbf{u}^k$ em (2.5) e observando que,

$$
\begin{aligned}
(n^k\mathbf{u}^k_t, \mathbf{u}^k) &= \frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}(n^k_t\mathbf{u}^k, \mathbf{u}^k) \\
&= \frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}((\mathbf{u}^k.\nabla n^k)\mathbf{u}^k, \mathbf{u}^k) - \frac{\nu}{2}(\Delta n^k\mathbf{u}^k, \mathbf{u}^k),
\end{aligned}
$$

e,

$$
\begin{aligned}
-\nu(n^k\Delta \mathbf{u}^k, \mathbf{u}^k) &= \nu(n^k\nabla \mathbf{u}^k, \nabla \mathbf{u}^k) + \nu(\nabla n^k.\nabla \mathbf{u}^k, \mathbf{u}^k) \\
&= \nu\|\sqrt{n^k}\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \nu(\nabla n^k.\nabla \mathbf{u}^k, \mathbf{u}^k) \\
&\geq \nu\alpha\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \nu(\nabla n^k.\nabla \mathbf{u}^k, \mathbf{u}^k),
\end{aligned}
$$

portanto temos,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \nu\alpha\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \leq -\frac{1}{2}(\mathbf{u}^k.\nabla n^k.\mathbf{u}^k, \mathbf{u}^k) + \frac{\nu}{2}(\mathbf{u}^k\Delta n^k, \mathbf{u}^k) \\
&-\nu(\nabla n^k.\nabla \mathbf{u}^k, \mathbf{u}^k) - (n^k\mathbf{u}^k.\nabla \mathbf{u}^k, \mathbf{u}^k) + 2\nu(\nabla n^k.\nabla \mathbf{u}^k, \mathbf{u}^k) + (n^k\mathbf{f}, \mathbf{u}^k) \\
&+\varepsilon^2\Bigg[-\Big(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \mathbf{u}^k\Big) + \Big(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}^k\Big) \\
&-\Big(\frac{1}{n^k}\Delta n^k\nabla n^k, \mathbf{u}^k\Big) + \big(\nabla\Delta n^k, \mathbf{u}^k\big)\Bigg].
\end{aligned}
$$

Observe que,

$$
\begin{aligned}
\frac{\nu}{2}(\mathbf{u}^k\Delta n^k, \mathbf{u}^k) &= \frac{\nu}{2}\int_{\mathbb{T}^d}\Delta n^k\mathbf{u}^k.\mathbf{u}^k dx \\
&= -\frac{\nu}{2}\int_{\mathbb{T}^d}\nabla n^k.\nabla(\mathbf{u}^k.\mathbf{u}^k)dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\nu}{2}\int_{\mathbb{T}^d} 2\nabla n^k.\nabla \mathbf{u}^k.\mathbf{u}^k dx\\
&= -\nu(\nabla n^k.\nabla \mathbf{u}^k, \mathbf{u}^k),
\end{aligned}
$$

logo,

$$\frac{\nu}{2}(\mathbf{u}^k, \Delta n^k, \mathbf{u}^k) - \nu(\nabla n^k.\nabla \mathbf{u}^k, \mathbf{u}^k) + 2\nu(\nabla n^k.\nabla \mathbf{u}^k, \mathbf{u}^k) = 0.$$

Também observa-se que:

$$
\begin{aligned}
-\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \mathbf{u}^k\right) &= -\int_{\mathbb{T}^d} \frac{1}{n^k}\nabla n^k.\nabla^2 n^k.\mathbf{u}^k dx\\
&= -\int_{\mathbb{T}^d} \frac{1}{n^k}\mathbf{u}^k.\nabla^2.\nabla n^k dx\\
&= -\int_{\mathbb{T}^d} \nabla^2 n^k : \mathbf{u}^k \otimes \nabla\frac{1}{n^k} dx\\
&= \int_{\mathbb{T}^d} \nabla n^k.div\left(\mathbf{u}^k \otimes \nabla n^k \frac{1}{n^k}\right) dx\\
&= \int_{\mathbb{T}^d} \nabla n^k.\left(div\left(\frac{1}{n^k}\nabla n^k\right)\mathbf{u}^k + \nabla \mathbf{u}^k.\nabla\frac{1}{n^k}\right) dx\\
&= \int_{\mathbb{T}^d} \nabla n^k.\left(\frac{1}{n^k}\Delta n^k \mathbf{u}^k + \nabla\left(\frac{1}{n^k}\right).\nabla n^k \mathbf{u}^k + \nabla \mathbf{u}^k.\nabla n^k \frac{1}{n^k}\right) dx\\
&= \left(\frac{1}{n^k}\nabla n^k \Delta n^k, \mathbf{u}^k\right) - \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}^k\right)\\
&\quad + \left(\frac{1}{n^k}\nabla n^k.\nabla \mathbf{u}^k, \nabla n^k\right),
\end{aligned}
$$

e então,

$$
\begin{aligned}
&-\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \mathbf{u}^k\right) + \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}^k\right) - \left(\frac{1}{n^k}\Delta n^k \nabla n^k, \mathbf{u}^k\right)\\
= &\left(\frac{1}{n^k}\nabla n^k \Delta n^k, \mathbf{u}^k\right) - \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}^k\right) + \left(\frac{1}{n^k}\nabla n^k.\nabla \mathbf{u}^k, \nabla n^k\right)\\
&+ \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}^k\right) - \left(\frac{1}{n^k}\Delta n^k \nabla n^k, \mathbf{u}^k\right)\\
= &\left(\frac{1}{n^k}\nabla n^k.\nabla \mathbf{u}^k, \nabla n^k\right).
\end{aligned}
$$

Temos que,

$$
\begin{aligned}
-\frac{1}{2}((\mathbf{u}^k.\nabla n^k)\mathbf{u}^k, \mathbf{u}^k) &= -\frac{1}{2}\int_{\mathbb{T}^d} \sum_{i=1}^{2}\left(\mathbf{u}_i^k.\frac{\partial n^k}{\partial x_i}\right).\sum_{j=1}^{2}\mathbf{u}_j^k.\mathbf{u}_j^k dx\\
&= -\frac{1}{2}\int_{\mathbb{T}^d} \sum_{i=1}^{2}\left(\frac{\partial n^k}{\partial x_i}\right).\sum_{j=1}^{2}\mathbf{u}_i^k\mathbf{u}_j^k.\mathbf{u}_j^k dx\\
&= \frac{1}{2}\int_{\mathbb{T}^d} n^k \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_i}(\mathbf{u}_i^k\mathbf{u}_j^k.\mathbf{u}_j^k) dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{T}^d}\left(n^k\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial \mathbf{u}_i^k}{\partial x_i}\mathbf{u}_j^k.\mathbf{u}_j^k + 2n^k\sum_{i=1}^{2}\sum_{j=1}^{2}\mathbf{u}_i^k\frac{\partial \mathbf{u}_j^k}{\partial x_i}\mathbf{u}_j^k\right)dx \\
&= \frac{1}{2}((n^k div(\mathbf{u}^k)\mathbf{u}^k,\mathbf{u}^k) + 2(n^k\mathbf{u}^k.\nabla\mathbf{u}^k,\mathbf{u}^k)) \\
&= (n^k\mathbf{u}^k.\nabla\mathbf{u}^k,\mathbf{u}^k),
\end{aligned}
$$

portanto,

$$
-\frac{1}{2}((\mathbf{u}^k.\nabla n^k)\mathbf{u}^k\mathbf{u}^k) - (n^k\mathbf{u}^k.\nabla\mathbf{u}^k,\mathbf{u}^k) = (n^k\mathbf{u}^k.\nabla\mathbf{u}^k,\mathbf{u}^k) - (n^k\mathbf{u}^k.\nabla\mathbf{u}^k,\mathbf{u}^k) = 0.
$$

Assim obtemos,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \nu\alpha\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \leq (n^k\mathbf{f},\mathbf{u}^k) + \varepsilon^2\left(\frac{1}{n^k}\nabla n^k.\nabla\mathbf{u}^k,\nabla n^k\right) \\
+\varepsilon^2(\nabla\Delta n^k,\mathbf{u}^k).
\end{aligned}
$$

Estimando os termos a direita:

$$
\begin{aligned}
\varepsilon^2|(\nabla\Delta n^k,\mathbf{u}^k)| &= \varepsilon^2|(\Delta n^k, div\mathbf{u}^k)| \\
&= 0;
\end{aligned}
$$

$$
\begin{aligned}
|(n^k\mathbf{f},\mathbf{u}^k)| &\leq \|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}^k\|_{L^4(\mathbb{T}^d)} \\
&\leq C(\beta)\|\mathbf{f}\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\leq C(\nu,\alpha,\beta)\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu\alpha}{4}\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{n^k}\nabla n^k.\nabla\mathbf{u}^k,\nabla n^k\right)\right| &\leq \frac{\varepsilon^2}{\alpha}|(\nabla n^k.\nabla\mathbf{u}^k,\nabla n^k)| \\
&\leq \frac{\varepsilon^2}{\alpha}\|\nabla n^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\overset{(iv)}{\leq} \frac{\varepsilon^2 C}{\alpha}\|n^k\|_{L^\infty(\mathbb{T}^d)}\|n^k\|_{H^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\leq \frac{\varepsilon^2\beta C}{\alpha}\|n^k\|_{H^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)} \\
&\leq \frac{4\varepsilon^4\beta^2C^2}{\nu\alpha^3}\|n^k\|^2_{H^2(\mathbb{T}^d)} + \frac{\nu\alpha}{4}\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

Assim temos,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu\alpha}{2}\|\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} &\leq C(\nu,\alpha,\beta)\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} \\
&+\frac{4\varepsilon^4\beta^2C^2}{\nu\alpha^3}\|n^k\|^2_{H^2(\mathbb{T}^d)}. \qquad (2.32)
\end{aligned}
$$

Multiplicando a desigualdade diferencial (2.31) por $\alpha\nu^2/2\beta^2C^2$, impondo a seguinte hipótese;

$$\frac{\varepsilon}{\nu} < \frac{\alpha}{2^{1/4}\beta C}$$

e somando-a com (2.32), obtemos;

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\sqrt{n^k}\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\alpha\nu^2}{4\beta^2C}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\alpha\nu^2}{4\beta^2C}\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)}\right) + \frac{\nu\alpha}{2}\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \\
&+ \left(\frac{\alpha\nu^3}{2\beta^2C} - \frac{\varepsilon^4\beta^2C^2}{\nu\alpha^3}\right)\|n^k\|^2_{H^2(\mathbb{T}^d)} \\
\leq\ & C(\nu,\varepsilon,\alpha,\beta)\|n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\varepsilon,\alpha,\beta)\|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)} \\
\leq\ & C(\nu,\varepsilon,\alpha,\beta,\|\mathbf{f}\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}).
\end{aligned}$$

Definindo-se,

$$\begin{aligned}
\varphi(t) &= \frac{1}{2}\|\sqrt{n^k}\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\alpha\nu^2}{4\beta^2C}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\alpha\nu^2}{4\beta^2C}\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)}, \\
\psi(t) &= \frac{\nu\alpha}{2}\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \left(\frac{\alpha\nu^3}{2\beta^2C} - \frac{\varepsilon^4\beta^2C^2}{\nu\alpha^3}\right)\|n^k\|^2_{H^2(\mathbb{T}^d)}, \\
C_1 &= C(\nu,\varepsilon,\alpha,\beta,\|\mathbf{f}\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}),
\end{aligned}$$

tem-se a seguinte desigualdade diferencial,

$$\varphi'(t) + \psi(t) \leq C_1. \tag{2.33}$$

Observe que exite uma constante $C_2 > 0$ tal que $\varphi \leq C_2\psi$, então escolhendo $0 < \gamma \leq \gamma^* = 1/(2C_2)$, multiplicando (2.33) por $e^{\gamma t}$, temos:

$$\begin{aligned}
(e^{\gamma t}\varphi(t))' + (e^{\gamma t}\psi(t)) &\leq C_1e^{\gamma t} + \gamma e^{\gamma t}\varphi(t) \\
&\leq C_1e^{\gamma t} + \gamma^* e^{\gamma t}\varphi(t) \\
&\leq C_1e^{\gamma t} + \frac{1}{2}e^{\gamma t}\psi(t),
\end{aligned}$$

e então,

$$(e^{\gamma t}\varphi(t))' + \frac{1}{2}(e^{\gamma t}\psi(t)) \ \leq\ C_1e^{\gamma t},$$

integrando de 0 a $t$ e multiplicado por $e^{-\gamma t}$ obtemos;

$$\begin{aligned}
\varphi(t) + \frac{1}{2}e^{-\gamma t}\int_0^t e^{\gamma s}\psi(s)ds &\leq \varphi(0) + C_1e^{-\gamma t}\int_0^t e^{\gamma s}ds \\
&\leq C.
\end{aligned}$$

Logo,

$$\begin{array}{lcl} \mathbf{u}^k \in L^\infty(0,\infty;L^2(\mathbb{T}^d)), & & n^k \in L^\infty(0,\infty;H^1(\mathbb{T}^d)), \\ \mathbf{u}^k \in L^2_{loc}(0,\infty;H^1(\mathbb{T}^d)) & e & n^k \in L^2_{loc}(0,\infty;H^2(\mathbb{T}^d)). \end{array}$$

Aplicando a norma de $L^2$ à equação (2.4), obtemos;

$$\begin{aligned} \|n_t^k\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\mathbf{u}^k.\nabla n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C\|\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^4(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C\|\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|n^k\|_{L^\infty(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C(C_2)\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C(C_2)\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + C(C_5,\nu)\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}, \end{aligned} \tag{2.34}$$

e multiplicando tal desigualdade por $e^{\gamma t}$, integrando-a de 0 a $t$ e, em seguida multiplicando a desigualdade obtida por $e^{-\gamma t}$ concluí-se que,

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq C(C_2) \quad \forall t \geq 0,$$

e, portanto,

$$n_t^k \in L^2_{loc}(0,\infty;L^2(\mathbb{T}^d)).$$

Por último, aplicando novamente a norma de $L^2$ à equação (2.4) e fazendo $t=0$, obtemos;

$$\begin{aligned} \|n_t^k(0)\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\mathbf{u}^k(0).\nabla n^k(0)\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta n^k(0)\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C\|\nabla \mathbf{u}_0\|^2_{L^2(\mathbb{T}^d)}\|\Delta n_0\|^2_{L^2(\mathbb{T}^d)} + C(\nu)\|\Delta n_0\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C(\nu,\|\mathbf{u}_0\|_{\mathbf{V}},\|n_0\|_{H^2(\mathbb{T}^d)}) \\ &= \widehat{C}. \end{aligned}$$

■

**Lema 2.4.2** *Nas hipóteses do Lema (2.4.1), tem-se que* $(\mathbf{u}^k, n^k)$ *satisfazem à seguinte desigualdade diferencial:*

$$\frac{d}{dt}(\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\sqrt{n^k}\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}) + \nu\|n_t^k\|^2_{L^2(\mathbb{T}^d)}$$

$$+\nu\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2\alpha^3}{\beta^2}\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \alpha\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}$$

$$\leq C(\nu,\varepsilon,\alpha,\beta,C_2,\|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))})(\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{H^2(\mathbb{T}^d)} + \nu\|\sqrt{n^k}\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)})^2.$$

**Demonstração:**

Derivando-se a equação (2.4) em relação à variável temporal, multiplicando-se por $n_t^k$ e integrando-se sobre $\mathbb{T}^d$, obtem-se:

$$\frac{1}{2}\frac{d}{dt}\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} = -(\mathbf{u}_t^k.\nabla n^k, n_t^k) - (\mathbf{u}^k.\nabla n_t^k, n_t^k).$$

Estimando os termos à direita como segue:

$$\begin{aligned}
|(\mathbf{u}_t^k.\nabla n^k, n_t^k)| &\leq \|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^4(\mathbb{T}^d)}\|n_t^k\|_{L^4(\mathbb{T}^d)}\\
&\overset{(iv),(iii)}{\leq} C\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|n^k\|^{1/2}_{H^2(\mathbb{T}^d)}\|n^k\|^{1/2}_{L^\infty(\mathbb{T}^d)}\|n_t^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|n_t^k\|^{1/2}_{H^1(\mathbb{T}^d)}\\
&\leq C(\beta)\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|n_t^k\|^{1/2}\|n_t^k\|^{1/2}_{H^1(\mathbb{T}^d)}\|n^k\|^{1/2}_{H^2(\mathbb{T}^d)}\\
&\leq C(\beta,\delta_1,\delta_2)\|n_t^k\|^4_{L^2(\mathbb{T}^d)} + C(\beta,\delta_1,\delta_2)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|n_t^k\|^2_{H^1(\mathbb{T}^d)}.
\end{aligned}$$

$$|(\mathbf{u}^k.\nabla n_t^k, n_t^k)| = \frac{1}{2}|(\mathbf{u}^k, \nabla(n_t^k)^2)| = \frac{1}{2}|(div(\mathbf{u}^k), (n_t^k)^2)| = 0,$$

obtem-se:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} &\leq C(\beta,\delta_1,\delta_2)\|n_t^k\|^4_{L^2(\mathbb{T}^d)} + C(\beta,\delta_1,\delta_2)\|n^k\|^4_{H^2(\mathbb{T}^d)}\\
&\quad +\delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|n_t^k\|^2_{H^1(\mathbb{T}^d)}. \qquad (2.35)
\end{aligned}$$

Agora fazendo $\mathbf{v} = \mathbf{u}_t^k$ na equação (2.5), temos:

$$\begin{aligned}
&(n^k\mathbf{u}_t^k, \mathbf{u}_t^k) + ((n^k\mathbf{u}^k.\nabla)\mathbf{u}^k, \mathbf{u}_t^k) + \nu[-(n^k\Delta\mathbf{u}^k, \mathbf{u}_t^k) - 2((\nabla n^k.\nabla)\mathbf{u}^k, \mathbf{u}_t^k)]\\
&= (n^k\mathbf{f}, \mathbf{u}_t^k) + \varepsilon^2\Bigg[-\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \mathbf{u}_t^k\right) + \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}_t^k\right)\\
&-\left(\frac{1}{n^k}\Delta n^k\nabla n^k, \mathbf{u}_t^k\right) + \left(\nabla\Delta n^k, \mathbf{u}_t^k\right)\Bigg]
\end{aligned}$$

Observa-se que

$$(n^k\mathbf{u}_t^k, \mathbf{u}_t^k) = \|\sqrt{n^k}\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)} \geq \alpha\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)},$$

e também

$$\begin{aligned}
-\nu(n^k\Delta\mathbf{u}^k, \mathbf{u}_t^k) &= \nu(\nabla\mathbf{u}^k, \nabla(n^k\mathbf{u}_t^k))\\
&= \nu(\nabla\mathbf{u}^k, n^k\nabla\mathbf{u}_t^k) + \nu(\nabla\mathbf{u}^k, \nabla n^k.\mathbf{u}_t^k)
\end{aligned}$$

em particular

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} &= \frac{1}{2}\frac{d}{dt}(\nabla\mathbf{u}^k, n^k\nabla\mathbf{u}^k)\\
&= \frac{1}{2}(\nabla\mathbf{u}_t^k, n^k\nabla\mathbf{u}^k) + \frac{1}{2}(\nabla\mathbf{u}^k, n_t^k\nabla\mathbf{u}^k) + \frac{1}{2}(\nabla\mathbf{u}^k, n^k\nabla\mathbf{u}_t^k)\\
&= (\nabla\mathbf{u}^k, n^k\nabla\mathbf{u}_t^k) + \frac{1}{2}(\nabla\mathbf{u}^k, n_t^k\nabla\mathbf{u}^k);
\end{aligned}$$

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$$\nu(\nabla \mathbf{u}^k, n^k \nabla \mathbf{u}_t^k) = \frac{\nu}{2}\frac{d}{dt}\|\sqrt{n}\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} - \frac{\nu}{2}(\nabla \mathbf{u}^k, n_t^k \nabla \mathbf{u}^k)$$

e, portanto,

$$-\nu(n^k \Delta \mathbf{u}^k, \mathbf{u}_t^k) = \frac{\nu}{2}\frac{d}{dt}\|\sqrt{n}\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} - \frac{\nu}{2}(\nabla \mathbf{u}^k, n_t^k \nabla \mathbf{u}^k) + \nu(\nabla \mathbf{u}^k, \nabla n^k . \mathbf{u}_t^k).$$

Assim, obtemos a seguinte desigualdade diferencial:

$$\begin{aligned}
&\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \alpha\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} \leq \frac{\nu}{2}(\nabla \mathbf{u}^k, n_t^k \nabla \mathbf{u}^k) - ((n^k \mathbf{u}^k.\nabla)\mathbf{u}^k, \mathbf{u}_t^k)\\
&-\nu(\nabla \mathbf{u}^k.\nabla n^k, \mathbf{u}_t^k) - 2\nu(\nabla \mathbf{u}^k.\nabla n^k, \mathbf{u}_t^k) + (n^k \mathbf{f}, \mathbf{u}_t^k)\\
&+\varepsilon^2\Bigg[-\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \mathbf{u}_t^k\right) + \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}_t^k\right)\\
&-\left(\frac{1}{n^k}\Delta n^k \nabla n^k, \mathbf{u}_t^k\right) + \left(\nabla \Delta n^k, \mathbf{u}_t^k\right)\Bigg].
\end{aligned}$$

Estimando os termos a direita como segue:

$$\begin{aligned}
\frac{\nu}{2}|(\nabla \mathbf{u}^k, n_t^k \nabla \mathbf{u}^k)| &\leq \frac{\nu}{2}\|\nabla \mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|n_t^k\|_{L^2(\mathbb{T}^d)}\\
&\overset{(iii)}{\leq} C(\nu)\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|n_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\nu,\delta_3)\|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + C(\nu,\delta_3)\|n_t^k\|^4_{L^2(\mathbb{T}^d)} + \delta_3\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k \mathbf{u}^k.\nabla \mathbf{u}^k, \mathbf{u}_t^k)| &\leq \beta\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\overset{(v)}{\leq} C(\beta)\|\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta \mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\beta, C_2)\|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_3\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
3\nu|(\nabla \mathbf{u}^k.\nabla n^k, \mathbf{u}_t^k)| &\leq \|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\overset{(v),(i)}{\leq} C\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\nabla \Delta n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C(C_2)\|\nabla \mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\nabla \Delta n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C(C_2,\delta_1,\delta_4)\|\nabla \mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + \delta_4\|\nabla \Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

Observe-se que,

$$1 = \frac{\alpha}{\alpha} \leq \frac{1}{\alpha}\|n^k\|_{L^\infty(\mathbb{T}^d)} \leq C(\alpha)\|n^k\|_{H^2(\mathbb{T}^d)},$$

e então,

$$\begin{aligned}
|(n^k \mathbf{f}, \mathbf{u}_t^k)| &\leq \|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}
\end{aligned}$$

$$\leq \quad C(\alpha, \|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}, \delta_1)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \mathbf{u}^k_t\right)\right| \leq \frac{\varepsilon^2}{\alpha}\|\nabla n^k\|_{L^4(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}$$
$$\overset{(iv),(iii)}{\leq} C(\varepsilon,\alpha)\|n^k\|^{1/2}_{L^\infty(\mathbb{T}^d)}\|n^k\|^{1/2}_{H^2(\mathbb{T}^d)}\|\Delta n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\varepsilon,\alpha,\beta,\delta_1,\delta_4)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \delta_4\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \mathbf{u}^k_t\right)\right| &\leq \frac{\varepsilon^2}{\alpha^2}\|\nabla n^k\|^3_{L^6(\mathbb{T}^d)}\|\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}\\ &\overset{(vi)}{\leq} C(\varepsilon,\alpha)\|\nabla n^k\|_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\varepsilon,\alpha,C_2,\delta_1)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\varepsilon^2|(\nabla\Delta n^k, \mathbf{u}^k_t)| = \varepsilon^2|(\Delta n^k, div(\mathbf{u}^k_t))| = 0;$$

$$\varepsilon^2\left|\left(\frac{1}{n^k}\Delta n^k\nabla n^k, \mathbf{u}^k_t\right)\right| \leq \frac{\varepsilon^2}{\alpha}\|\nabla n^k\|_{L^4(\mathbb{T}^d)}\|\Delta n^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}$$
$$\overset{(iv),(iii)}{\leq} C(\varepsilon,\alpha)\|n^k\|^{1/2}_{L^\infty(\mathbb{T}^d)}\|n^k\|^{1/2}_{H^2(\mathbb{T}^d)}\|\Delta n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\varepsilon,\alpha,\beta,\delta_1,\delta_4)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_1\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \delta_4\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)};$$

Assim, obtem-se,

$$\frac{\nu}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \alpha\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} \leq C(\nu,\beta,C_2,\delta_1,\delta_3)\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + C(\nu,\delta_3)\|n^k_t\|^4_{L^2(\mathbb{T}^d)}$$
$$+C(\nu,\varepsilon,\alpha,C_2,\|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))},\delta_1,\delta_3,\delta_4)\|n^k\|^4_{H^2(\mathbb{T}^d)} + 6\delta_1\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + 2\delta_3\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+3\delta_4\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}. \tag{2.36}$$

Multiplicando a equação (2.4) por $\Delta^2 n^k$, integrando-a sobre $\mathbb{T}^d$ e fazendo-se integração por partes, obtem-se;

$$\frac{1}{2}\frac{d}{dt}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \quad = \quad (\nabla n^k.\nabla\mathbf{u}^k, \nabla\Delta n^k) + (\mathbf{u}^k.\nabla^2 n^k, \nabla\Delta n^k).$$

Estimando os termos à direita:

$$|(\nabla n^k.\nabla\mathbf{u}^k, \nabla\Delta n^k)| \leq \|\nabla\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\nabla n^k\|_{L^4(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)}$$
$$\overset{(iii),(vi)}{\leq} C\|\nabla\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|n^k\|^{1/2}_{L^\infty(\mathbb{T}^d)}\|n^k\|^{1/2}_{H^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\beta,\delta_3,\delta_4)\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + C(\beta,\delta_3,\delta_4)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_3\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\delta_4\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}|(\mathbf{u}^k.\nabla^2 n^k, \nabla\Delta n^k)| &\leq \|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)}\\ &\overset{(v)}{\leq} C\|\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta n^k\|_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(C_2,\delta_3,\delta_4)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_3\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_4\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

Portanto tem-se que,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} \leq C(\beta,C_2,\delta_3,\delta_4)\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}\\ &+C(\beta,C_2,\delta_3,\delta_4)\|n^k\|^4_{H^2(\mathbb{T}^d)} + 2\delta_3\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + 2\delta_4\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}. \qquad (2.37)\end{aligned}$$

Agora, fazendo-se $\mathbf{v} = -\Delta\mathbf{u}^k$ na equação (2.5) temos;

$$\begin{aligned}&\nu(n^k\Delta\mathbf{u}^k, \Delta\mathbf{u}^k) = (n^k\mathbf{u}^k_t, \Delta\mathbf{u}^k) + ((n^k\mathbf{u}.\nabla)\mathbf{u}^k, \Delta\mathbf{u}^k) + \nu((\nabla n^k.\nabla)\mathbf{u}^k, \Delta\mathbf{u}^k)\\ &-(n^k\mathbf{f}, \Delta\mathbf{u}^k) + \varepsilon^2\Bigg[\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \Delta\mathbf{u}^k\right) - \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \Delta\mathbf{u}^k\right)\\ &+\left(\frac{1}{n^k}\Delta n^k\nabla n^k, \Delta\mathbf{u}^k\right) - \left(\nabla\Delta n^k, \Delta\mathbf{u}^k\right)\Bigg]\end{aligned}$$

Estimando-se os termos à direita da equação acima, obtem-se:

$$\begin{aligned}|(n^k\mathbf{u}^k_t, \Delta\mathbf{u}^k)| &\leq \beta\|\mathbf{u}^k_t\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\ &\leq \frac{\beta\mu_1}{2}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{\beta}{2\mu_1}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(n^k\mathbf{u}^k.\nabla\mathbf{u}^k, \Delta\mathbf{u}^k)| &\leq \beta\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\ &\overset{(v)}{\leq} \beta\|\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\beta,C_2,\delta_5)\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + 2\delta_5\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|(\nabla n^k.\nabla\mathbf{u}^k, \Delta\mathbf{u}^k)| &\leq \nu\|\nabla n^k\|_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\ &\overset{(iv),(iii)}{\leq} C(\nu)\|n^k\|^{1/2}_{L^\infty(\mathbb{T}^d)}\|n^k\|^{1/2}_{H^2(\mathbb{T}^d)}\|\nabla\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\nu,\beta,\delta_5)\|n^k\|^4_{H^2(\mathbb{T}^d)} + C(\nu,\beta,\delta_5)\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)} + 2\delta_5\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

como

$$1 = \frac{\alpha}{\alpha} \leq \frac{1}{\alpha}\|n^k\|_{L^\infty(\mathbb{T}^d)} \leq C(\alpha)\|\Delta n^k\|_{L^2(\mathbb{T}^d)},$$

segue que:

$$\begin{aligned}|(n^k\mathbf{f}, \Delta\mathbf{u}^k)| &\leq \|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{f}\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C(\alpha)\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\mathbf{f}\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\end{aligned}$$

$$\leq \quad C(\alpha, \|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}, \delta_5)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_5\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k, \Delta\mathbf{u}^k\right)\right| \leq \frac{\varepsilon^2}{\alpha}\|\nabla n^k\|_{L^4(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^4(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}$$
$$\overset{(iv),(iii)}{\leq} C(\varepsilon,\alpha)\|n^k\|^{1/2}_{L^\infty(\mathbb{T}^d)}\|n^k\|^{1/2}_{H^2(\mathbb{T}^d)}\|\Delta n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^{1/2}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\varepsilon,\alpha,\beta,\delta_5,\delta_6)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_5\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_6\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2\left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k, \Delta\mathbf{u}^k\right)\right| \quad \leq \quad \frac{\varepsilon^2}{\alpha^2}\|\nabla n^k\|^3_{L^6(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}$$
$$\overset{(vi)}{\leq} \quad C(\varepsilon,\alpha)\|\nabla n^k\|_{L^2(\mathbb{T}^d)}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq \quad C(\varepsilon,\alpha,C_2,\delta_5)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_5\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2\left|\left(\frac{1}{n^k}\Delta n^k\nabla n^k, \Delta\mathbf{u}^k\right)\right| \leq \frac{\varepsilon^2}{\alpha}\|\nabla n^k\|_{L^4(\mathbb{T}^d)}\|\Delta n^k\|_{L^4(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}$$
$$\overset{(iv),(iii)}{\leq} C(\varepsilon,\alpha)\|n^k\|^{1/2}_{L^\infty(\mathbb{T}^d)}\|n^k\|^{1/2}_{H^2(\mathbb{T}^d)}\|\Delta n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^{1/2}_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C(\varepsilon,\alpha,\beta,\delta_5,\delta_6)\|n^k\|^4_{H^2(\mathbb{T}^d)} + \delta_5\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_6\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2|(\nabla\Delta n^k, \Delta\mathbf{u}^k)| = \varepsilon^2|(\Delta n^k, div\Delta\mathbf{u}^k)| = 0.$$

Por fim, tomando-se $\delta_5 = \nu\alpha/48$, $\delta_6 = 4\beta^2\nu/12\nu\alpha^2$ e $\mu_1 = 2\beta/\nu\alpha$ e multiplicando a desigualdade resultante por $\nu\alpha^2/4\beta^2$ obtemos:

$$\frac{\nu^2\alpha^3}{8\beta^2}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \leq \frac{\alpha}{4}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu}{4}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C(\nu,\alpha,\beta,C_2)\|\nabla\mathbf{u}^k\|^4_{L^2(\mathbb{T}^d)}$$
$$+C(\nu,\varepsilon,\alpha,\beta,C_2,\|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))})\|n^k\|^4_{H^2(\mathbb{T}^d)}. \tag{2.38}$$

Somando as desigualdades diferenciais (2.30), (2.34), (2.35), (2.36), (2.37) e (2.38) e fazendo $\delta_1 = \alpha/32$, $_2 = \nu/2$, $\delta_3 = \nu^2\alpha^3/16\beta^2$ e $\delta_4 = \nu/16$ assim obtemos:

$$\frac{d}{dt}(\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}) + \nu\|n^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+\nu\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu\|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2\alpha^3}{\beta^2}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \alpha\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$\leq C(\nu,\varepsilon,\alpha,\beta,C_2,\|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))})(\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{H^2(\mathbb{T}^d)} + \nu\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)})^2$$

■

**Teorema 2.4.1** *Sejam* $d = 2$, $\mathbf{u}_0 \in V$, $n_0 \in H^2(\mathbb{T}^d)$ *e* $\mathbf{f} \in L^\infty(0,\infty; L^2(\mathbb{T}^d))$ *e* $\varepsilon/\nu < \alpha/C\beta$. *Então existe uma única solução forte* $(\mathbf{u}, n)$ *para o problema (8)-(11), verificando as seguintes desigualdades, para todo* $\gamma > 0$ *e para todo* $t \geq 0$ :

$$\begin{aligned} \|\mathbf{u}(t)\|^2_{H^1(\mathbb{T}^d)} &\leq C, & \|n(t)\|^2_{H^2(\mathbb{T}^d)} &\leq C, \\ e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}(s)\|^2_{H^2(\mathbb{T}^d)} ds &\leq C, & e^{-\gamma t}\int_0^t e^{\gamma s}\|n(s)\|^2_{H^3(\mathbb{T}^d)} ds &\leq C, \\ e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)} ds &\leq C, & e^{-\gamma t}\int_0^t e^{\gamma s}\|n_t(s)\|^2_{H^1(\mathbb{T}^d)} ds &\leq C \\ & e & \|n_t(t)\|^2_{L^2(\mathbb{T}^d)} &\leq C. \end{aligned}$$

*Em particular,*

$$\begin{aligned} \mathbf{u} &\in L^\infty(0,\infty; H^1(\mathbb{T}^d)), & n &\in L^\infty(0,\infty; H^2(\mathbb{T}^d)), \\ \mathbf{u} &\in L^2_{loc}(0,\infty; H^2(\mathbb{T}^d)), & n &\in L^2_{loc}(0,\infty; H^3(\mathbb{T}^d)), \\ \mathbf{u}_t &\in L^2_{loc}(0,\infty; L^2(\mathbb{T}^d)), & n_t &\in L^2_{loc}(0,\infty; H^1(\mathbb{T}^d)), \\ & e & n_t &\in L^\infty(0,\infty; L^2(\mathbb{T}^d)). \end{aligned}$$

**Demonstração:**
Inicialmente trabalhamos no sentido de estabelecer *Estimativas a Priori* com o objetivo de garantir a existência de soluções. Com esse intuito, considerando-se,

$$\begin{aligned} \varphi(t) &= \|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}, \\ \psi(t) &= \nu\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu^2\alpha^3}{\beta^2}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}, \\ \chi(t) &= \alpha\|\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}, \\ C_1 &= C(\nu, \varepsilon, \alpha, \beta, C_2, \|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}), \end{aligned}$$

então, pelo Lema (2.4.2), tem-se,

$$\varphi'(t) + \chi(t) + \psi(t) \leq C_1\varphi^2(t). \tag{2.39}$$

Observe que,

$$0 < \alpha \leq \|n^k\|_{L^\infty(\mathbb{T}^d)} \leq C\|n^k\|_{H^2(\mathbb{T}^d)},$$

então,

$$0 < \alpha^2 \leq \varphi(t).$$

Como $\chi(t), \psi(t) \geq 0$, tem-se que:

$$\varphi'(t) \leq C_1\varphi^2(t),$$

e dividindo-se tal desigualdade por $\varphi(t)$ temos,

$$\frac{d}{dt}\ln\varphi(t) \leq C_1\varphi(t).$$

Multiplicando-se a desigualdade diferencial acima por $e^{\gamma t}$, com $\gamma > 0$, obtemos;

$$\frac{d}{dt}(e^{\gamma t}\ln\varphi(t)) \leq C_1 e^{\gamma t}\varphi(t) + \gamma e^{\gamma t}\ln\varphi(t).$$

Como,

$$\ln\varphi(t) \leq \varphi(t), \quad \text{para todo} \quad t \geq 0,$$

então,

$$\frac{d}{dt}(e^{\gamma t}\ln\varphi(t)) \leq C_1 e^{\gamma t}\varphi(t) + \gamma e^{\gamma t}\varphi(t),$$

e integrado-se a desigualdade diferencial acima de 0 a $t$, obtem-se que:

$$e^{\gamma t}\ln\varphi(t) - \ln\varphi_0 \leq C_1\int_0^t e^{\gamma s}\varphi(s)ds + \gamma\int_0^t e^{\gamma s}\varphi(s)ds,$$

multiplicando a desigualdade diferencial acima por $e^{-\gamma t}$, temos;

$$\ln\varphi(t) \leq e^{-\gamma t}\ln\varphi_0 + C_1 e^{-\gamma t}\int_0^t e^{\gamma s}\varphi(s)ds + \gamma e^{-\gamma t}\int_0^t e^{\gamma s}\varphi(s)ds,$$

pelo Lema (2.4.1) temos que,

$$\begin{aligned}\ln\varphi(t) &\leq e^{-\gamma t}\ln\varphi_0 + C_1 e^{-\gamma t}\int_0^t e^{\gamma s}\varphi(s)ds + \gamma e^{-\gamma t}\int_0^t e^{\gamma s}\varphi(s)ds \\ &\leq C,\end{aligned}$$

e, portanto, obtem-se que

$$\varphi(t) \leq e^C \leq C_3 \quad \forall t \geq 0.$$

Voltando para a desigualdade (2.39), multiplicando-a por $e^{\gamma t}$, integrando-a sobre 0 a $t$ e, em seguida, multiplicando-a por $e^{-\gamma t}$, temos;

$$\begin{aligned}\varphi(t) + e^{-\gamma t}\int_0^t e^{\gamma s}\chi(s)ds + e^{-\gamma t}\int_0^t e^{\gamma s}\psi(s)ds &\leq e^{-\gamma t}\int_0^t e^{\gamma s}C_3 ds \\ &\quad + \gamma e^{-\gamma t}\int_0^t e^{\gamma s}\varphi(s)ds + \varphi_0 \\ &\leq C.\end{aligned}$$

Com os resultados obtidos acima, pode-se fazer a *Passagem ao Limite* e *Verificação dos dados Iniciais* de forma totalmente análoga ao Teorema (2.3.1), concluindo que $(\mathbf{u}, n)$ é solução forte Global do problema (2.1) - (2.3).

O próximo passo será então, o de mostrar a unicidade de solução:

Trabalhando de forma análoga ao Teorema (2.3.1) obtem-se a seguinte desigualdade diferencial;

$$\frac{1}{2}\frac{d}{dt}(\|\sqrt{n}\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \|z\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z\|^2_{L^2(\mathbb{T}^d)}) + \frac{\nu}{2}\|\nabla \mathbf{w}\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla z\|^2_{L^2(\mathbb{T}^d)}$$
$$+\frac{\nu}{2}\|\Delta z\|^2_{L^2(\mathbb{T}^d)} \leq C(\|\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}^1_t\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^1\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^1\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n\|^4_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^1\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}\|^2_{L^2(\mathbb{T}^d)})(\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|z\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z\|^2_{L^2(\mathbb{T}^d)}).$$

Note-se que $\|\nabla \mathbf{w}\|^2_{L^2(\mathbb{T}^d)}, \|\nabla z\|^2_{L^2(\mathbb{T}^d)}, \|\Delta z\|^2_{L^2(\mathbb{T}^d)} \geq 0$ e integrando-se a desigualdade diferencial acima de $(m-1)T^*$ a $t$, com $t \in [(m-1)T^*, mT^*]$, para todo $m \in \mathbb{N}$, e observando-se que $\|\sqrt{n}\mathbf{w}\|^2_{L^2(\mathbb{T}^d)} \leq \alpha\|\mathbf{w}\|^2_{L^2(\mathbb{T}^d)}$, tem-se que:

$$\|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} + \|z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)} \leq C\|z((m-1)T^*)\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\|\nabla z((m-1)T^*)\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{w}((m-1)T^*)\|^2_{L^2(\mathbb{T}^d)} + C\int_{(m-1)T^*}^{t}(\|\Delta n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}^1_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^1(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^1(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n(s)\|^4_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^1(s)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}(s)\|^2_{L^2(\mathbb{T}^d)})(\|\mathbf{w}(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|z(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(s)\|^2_{L^2(\mathbb{T}^d)})ds.$$

Agora, usando o Lema de Gronwall (1.2.8), obtemos;

$$\|z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} \leq C(\|z((m-1)T^*)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla z((m-1)T^*)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{w}((m-1)T^*)\|^2_{L^2(\mathbb{T}^d)}) \exp\left(C\int_{(m-1)T^*}^{t}(\|\Delta n(s)\|^2_{L^2(\mathbb{T}^d)}\right.$$
$$+\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}^1_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^1(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^1(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$\left.+\|\Delta n(s)\|^4_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^1(s)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}(s)\|^2_{L^2(\mathbb{T}^d)})ds\right).$$

Dado que,

$$\int_{(m-1)T^*}^{t}(\|\Delta n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}^1_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^1(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\Delta n^1(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n(s)\|^4_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta \mathbf{u}^1(s)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}(s)\|^2_{L^2(\mathbb{T}^d)})ds < \infty$$

para todo $m \in \mathbb{N}$, segue-se que,

$$\|z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} \leq C(\|z((m-1)T^*)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla z((m-1)T^*)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{w}((m-1)T^*)\|^2_{L^2(\mathbb{T}^d)}).$$

Então, para $m = 1$ tem-se que

$$\|z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} \leq C(\|z(0)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(0)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\mathbf{w}(0)\|^2_{L^2(\mathbb{T}^d)}) = 0$$

logo,

$$\|z(t)\|^2_{L^2(\mathbb{T}^d)} = \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)} = \|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} = 0 \ \ \forall t \in [0, T^*].$$

Para $m = 2$ tem-se que

$$\|z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} \leq C(\|z(T^*)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla z(T^*)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\mathbf{w}(T^*)\|^2_{L^2(\mathbb{T}^d)}) = 0$$

e então,

$$\|z(t)\|^2_{L^2(\mathbb{T}^d)} = \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)} = \|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} = 0 \ \ \forall t \in [T^*, 2T^*].$$

Repetindo-se esse processo, conclui-se que

$$\|z(t)\|^2_{L^2(\mathbb{T}^d)} = \|\nabla z(t)\|^2_{L^2(\mathbb{T}^d)} = \|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^d)} = 0 \ \ \forall t \in [0, mT^*], \quad \forall m \in \mathbb{N}.$$

Portanto,

$$\mathbf{u} = \mathbf{u}^1 \ \text{ em } \ L^2(0,\infty; L^2(\mathbb{T}^d))$$
$$n = n^1 \ \text{ em } \ L^2(0,\infty; H^1(\mathbb{T}^d)).$$

■

**Corolário 2.4.1** *Sejam* $d = 2$, $\mathbf{u}_0 \in V \cap H^2(\mathbb{T}^d)$, $n_0 \in H^3(\mathbb{T}^d)$, $\mathbf{f} \in L^\infty(0,\infty; H^1(\mathbb{T}^d))$ *e* $\mathbf{f}_t \in L^\infty(0,\infty; L^2(\mathbb{T}^d))$. *Então a solução* $(\mathbf{u}, n)$ *dada pelo Teorema (2.4.1) verifica as seguintes desigualdades, para todo* $\gamma > 0$ *e para todo* $t \geq 0$ :

$$\begin{aligned}
\|\mathbf{u}(t)\|^2_{H^2(\mathbb{T}^d)} &\leq C, & \|n(t)\|^2_{H^3(\mathbb{T}^d)} &\leq C,\\
\|\mathbf{u}_t(t)\|^2_{L^2(\mathbb{T}^d)} &\leq C, & \|n^k_t(t)\|^2_{H^1(\mathbb{T}^d)} &\leq C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}(s)\|^2_{H^3(\mathbb{T}^d)}ds &\leq C, & e^{-\gamma t}\int_0^t e^{\gamma s}\|n(s)\|^2_{H^4(\mathbb{T}^d)}ds &\leq C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}_t(s)\|^2_{H^1(\mathbb{T}^d)}ds &\leq C \quad e & e^{-\gamma t}\int_0^t e^{\gamma s}\|n_t(s)\|^2_{H^2(\mathbb{T}^d)}ds &\leq C.
\end{aligned}$$

*Em particular,*

$$\begin{aligned}
\mathbf{u} &\in L^\infty(0,\infty; H^2(\mathbb{T}^d)), & n &\in L^\infty(0,\infty; H^3(\mathbb{T}^d)),\\
\mathbf{u} &\in L^2_{loc}(0,\infty; H^3(\mathbb{T}^d)), & n &\in L^2_{loc}(0,\infty; H^4(\mathbb{T}^d)),\\
\mathbf{u}_t &\in L^\infty(0,\infty; L^2(\mathbb{T}^d)), & n_t &\in L^\infty(0,\infty; H^1(\mathbb{T}^d)),\\
\mathbf{u}_t &\in L^2_{loc}(0,\infty; H^1(\mathbb{T}^d)) \quad e & n_t &\in L^2_{loc}(0,\infty; H^2(\mathbb{T}^d)).
\end{aligned}$$

**Demonstração:**

Pode-se reescrever a desigualdade diferencial dado pelo Lema (2.2.2) da seguinte forma:

$$\varphi'(t) + \chi(t) + \psi(t) \leq C_1\varphi(t)\beta(t) \tag{2.40}$$

onde,

$$\begin{aligned}
\varphi(t) &= \frac{1}{2}\|\sqrt{n^k}\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)},\\
\psi(t) &= \frac{\nu\alpha}{2}\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|n_t^k\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu}{2}\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)},\\
\chi(t) &= \nu\|\nabla n_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)},\\
\beta(t) &= \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^6_{L^2(\mathbb{T}^d)}\\
&\quad + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)},\\
C_1 &= C(\nu, \varepsilon, \alpha, \beta, \|\mathbf{f}\|^2_{L^\infty(0,\infty;H^1(\mathbb{T}^d))}, \|\mathbf{f}_t\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}).
\end{aligned}$$

Note-se que

$$0 < \alpha \leq \|n^k\|_{L^\infty(\mathbb{T}^d)} \leq C\|n^k\|_{H^2(\mathbb{T}^d)},$$

então,

$$0 < \alpha^2 \leq \varphi(t),$$

e

$$e^{-\gamma t}\int_0^t e^{\gamma s}\beta(s)ds \leq C$$

Então, de forma análoga ao Teorema (2.4.1) mostra-se primeiramente que

$$\varphi(t) \leq C \quad \forall t \geq 0.$$

E em seguida se obtém a seguinte desigualdade;

$$\begin{aligned}
\varphi'(t) + e^{-\gamma t}\int_0^t e^{\gamma s}\chi(s)ds + e^{-\gamma t}\int_0^t e^{\gamma s}\psi(s)ds &\leq Ce^{-\gamma t}\int_0^t e^{\gamma s}\beta(s)\\
&\quad + \gamma e^{-\gamma t}\int_0^t e^{\gamma s}\varphi(s)ds + \varphi_0\\
&\leq C, \quad \forall t \geq 0.
\end{aligned}$$

Assim, conclui-se que:

$$\begin{aligned}
&\mathbf{u}_t^k \in L^\infty(0,\infty;L^2(\mathbb{T}^d)), && n_t^k \in L^\infty(0,\infty;H^1(\mathbb{T}^d)),\\
&\mathbf{u}_t^k \in L^\infty(0,\infty;L^2(\mathbb{T}^d)), && n_t^k \in L^\infty(0,\infty;H^1(\mathbb{T}^d)),\\
&\mathbf{u}_t^k \in L^2_{loc}(0,\infty;H^1(\mathbb{T}^d)) \quad e && n_t^k \in L^2_{loc}(0,\infty;H^2(\mathbb{T}^d)).
\end{aligned}$$

Considerando a desigualdade $(i)$ do Lema (2.2.3), tem-se que

$$\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} \leq C(C_5) = C_7 \quad \forall t \geq 0;$$

logo,

$$\mathbf{u}^k \in L^\infty(0,\infty; H^2(\mathbb{T}^d)) \qquad e \qquad n^k \in L^\infty(0,\infty; H^3(\mathbb{T}^d)).$$

Pela desigualdade $(ii)$ do Lema (2.2.3), tem-se que

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C_7, \quad \forall t \geq 0,$$

em particular,

$$n^k \in L^2_{loc}(0,\infty; H^4(\mathbb{T}^d)).$$

E por fim, considerando a desigualdade $(iii)$ do Lema (2.2.3), tem-se que

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C_7, \quad \forall t \geq 0,$$

e, em particular,

$$\mathbf{u}^k \in L^2_{loc}(0,\infty; H^3(\mathbb{T}^d)).$$

■

**Corolário 2.4.2** *Sejam $d = 2$, $\mathbf{u}_0 \in V \cap H^3(\mathbb{T}^d)$, $n_0 \in H^4(\mathbb{T}^d)$, $\mathbf{f} \in L^\infty(0,\infty; H^2(\mathbb{T}^d))$ e $\mathbf{f}_t \in L^\infty(0,\infty; H^1(\mathbb{T}^d))$. Então a solução $(\mathbf{u}, n)$ dada pelo Teorema (2.4.1) verifica as seguintes desigualdades, para todo $\gamma > 0$ e para todo $t \geq 0$ :*

$$\begin{aligned}
\|\mathbf{u}(t)\|^2_{H^3(\mathbb{T}^d)} &\leq C, & \|n(t)\|^2_{H^4(\mathbb{T}^d)} &\leq C,\\
\|\mathbf{u}_t(t)\|^2_{H^1(\mathbb{T}^d)} &\leq C, & \|n^k_t(t)\|^2_{H^2(\mathbb{T}^d)} &\leq C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}(s)\|^2_{H^4(\mathbb{T}^d)} ds &\leq C, & e^{-\gamma t}\int_0^t e^{\gamma s}\|n(s)\|^2_{H^5(\mathbb{T}^d)} ds &\leq C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}_t(s)\|^2_{H^2(\mathbb{T}^d)} ds &\leq C, & e^{-\gamma t}\int_0^t e^{\gamma s}\|n_t(s)\|^2_{H^3(\mathbb{T}^d)} ds &\leq C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}(s)_{tt}\|^2_{L^2(\mathbb{T}^d)} ds &\leq C \quad e & e^{-\gamma t}\int_0^t e^{\gamma s}\|n(s)_{tt}\|^2_{H^1(\mathbb{T}^d)} ds &\leq C.
\end{aligned}$$

*Em particular,*

$$\begin{aligned}
\mathbf{u} &\in L^\infty(0,\infty; H^3(\mathbb{T}^d)), & n &\in L^\infty(0,\infty; H^4(\mathbb{T}^d)),\\
\mathbf{u} &\in L^2_{loc}(0,\infty; H^4(\mathbb{T}^d)), & n &\in L^2_{loc}(0,\infty; H^5(\mathbb{T}^d)),\\
\mathbf{u}_t &\in L^\infty(0,\infty; H^1(\mathbb{T}^d)), & n_t &\in L^\infty(0,\infty; H^2(\mathbb{T}^d)),\\
\mathbf{u}_t &\in L^2_{loc}(0,\infty; H^2(\mathbb{T}^d)), & n_t &\in L^2_{loc}(0,\infty; H^3(\mathbb{T}^d)),\\
\mathbf{u}_{tt} &\in L^2_{loc}(0,\infty; L^2(\mathbb{T}^d)) \quad e & n_{tt} &\in L^2_{loc}(0,\infty; H^1(\mathbb{T}^d)).
\end{aligned}$$

**Demonstração:**

Considerando a desigualdade diferencial dada pelo Lema (2.2.4) e as regularidades obtidas no Corolário (2.4.1), tem-se a seguinte desigualdade diferencial;

$$\frac{d}{dt}\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|n_t^k\|^2_{H^2(\mathbb{T}^d)} + C;$$

multiplicando-se tal desigualdade por $e^{\gamma t}$, com $\gamma > 0$, integrando-a de 0 a $t$, com $t \geq 0$, e por fim, multiplicando-se a desigualdade obtida por $e^{-\gamma t}$, segue-se que:

$$\begin{aligned}
&\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}_{tt}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + e^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta \mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq \|\nabla \mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)}\\
&+Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla \mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + Ce^{-\gamma t}\int_0^t e^{\gamma s}\|n_t^k(s)\|^2_{H^2(\mathbb{T}^d)}ds + e^{-\gamma t}\int_0^t Ce^{\gamma s}ds\\
&\leq C.
\end{aligned}$$

Assim obtemos a seguinte estimativa;

$$\mathbf{u}_t^k \in L^\infty(0,\infty; H^1(\mathbb{T}^d)), \quad \mathbf{u}_{tt}^k \in L^2_{loc}(0,\infty; L^2(\mathbb{T}^d)) \text{ e } \mathbf{u}_t^k \in L^2_{loc}(0,\infty; H^2(\mathbb{T}^d)).$$

Considerando-se a desigualdade diferencial $(i)$ do Lema (2.2.5), multiplicando-a por $e^{\gamma t}$, com $\gamma > 0$, integrando-se de 0 a $t$, com $t \geq 0$, e por fim, multiplicando-se a desigualdade obtida por $e^{-\gamma t}$, segue-se que:

$$\begin{aligned}
&\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq \|\Delta^2 n_0^k\|^2_{L^2(\mathbb{T}^d)}\\
&+Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds\\
&\leq C.
\end{aligned}$$

Portando, obtem-se que

$$n^k \in L^\infty(0,\infty; H^4(\mathbb{T}^d)) \quad e \quad n^k \in L^2_{loc}(0,\infty; H^5(\mathbb{T}^d)).$$

Assim, pelas estimativas obtidas acima, tem-se pela desigualdade $(ii)$ do Lema (2.2.5) que

$$\begin{aligned}
\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C \quad \forall t \geq 0,
\end{aligned}$$

e dai,

$$n_t \in L^\infty(0,\infty; H^2(\mathbb{T}^d)).$$

E ainda, considerando-se $(iii)$ do Lema (2.2.5), multiplicando-a por $e^{\gamma t}$, com $\gamma > 0$, integrando-se o resultado de 0 a $t$, com $t \geq 0$, e por fim, multiplicando este por $e^{-\gamma t}$, tem-se que:

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta n_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$

$$+Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$\leq C;$$

logo,

$$n_t^k \in L^2_{loc}(0,\infty; H^3(\mathbb{T}^d)).$$

Agora, multiplicando-se a desigualdade diferencial dada pelo Lema (2.2.6) por $e^{\gamma t}$, com $\gamma > 0$, integrando-se a desigualdade resultante de 0 a $t$, com $t \geq 0$, e por fim, multiplicando por $e^{-\gamma t}$ e considerando as estimativas obtidas anteriormente, tem-se que:

$$\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + e^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta^2\mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq C,$$

donde se conclui que

$$\mathbf{u}^k \in L^\infty(0,\infty; H^3(\mathbb{T}^d)) \qquad e \qquad \mathbf{u}^k \in L^2_{loc}(0,\infty; H^4(\mathbb{T}^d)).$$

De modo análogo, a partir da desigualdade dada pelo (2.2.7 ) obtem-se

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla n^k_{tt}(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq C,$$

e portanto,

$$n^k_{tt} \in L^2_{loc}(0,\infty; H^1(\mathbb{T}^d)).$$

■

### 2.4.2 Caso Tridimensional

Agora assumindo que os dados iniciais e a força externa são suficientemente pequenos e sem assumir hipóteses sobre as constantes físicas, provamos a existência e unicidade de solução forte global no tempo, este é obtido usando exponenciais como funções peso, a qual é uma forma de trabalhar inspirada em [32] e [34]. Obtemos resultados de regularidade semelhantes ao caso local.

**Teorema 2.4.2** *Sejam* $d = 3$, $\mathbf{u}_0 \in V$, $n_0 \in H^2(\mathbb{T}^d)$ *e* $\mathbf{f} \in L^\infty(0,\infty; L^2(\mathbb{T}^d))$, *tais que* $\|\mathbf{u}_0\|^2_{H^1(\mathbb{T}^d)}$, $\|n_0\|^2_{H^2(\mathbb{T}^d)}$ *e* $\|\mathbf{f}\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}$ *são suficientemente pequenas. Então existe uma única solução forte* $(\mathbf{u}, n)$ *para o problema (8)-(11) em* $(0,\infty)$, *verificando as seguintes desigualdades, para todo* $\gamma > 0$ *e para todo* $t \geq 0$ :

$$\|\mathbf{u}(t)\|^2_{H^1(\mathbb{T}^d)} \leq C, \qquad \|n(t)\|^2_{H^2(\mathbb{T}^d)} \leq C,$$
$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}(s)\|^2_{H^2(\mathbb{T}^d)}ds \leq C, \qquad e^{-\gamma t}\int_0^t e^{\gamma s}\|n(s)\|^2_{H^3(\mathbb{T}^d)}ds \leq C,$$

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq C, \qquad e^{-\gamma t}\int_0^t e^{\gamma s}\|n_t(s)\|^2_{H^1(\mathbb{T}^d)}ds \leq C,$$
$$e \qquad \|n_t(t)\|^2_{L^2(\mathbb{T}^d)} \leq C.$$

*Em particular,*

$$\begin{aligned}
\mathbf{u} &\in L^\infty(0,\infty;H^1(\mathbb{T}^d)) & n &\in L^\infty(0,\infty;H^2(\mathbb{T}^d))\\
\mathbf{u} &\in L^2_{loc}(0,\infty;H^2(\mathbb{T}^d)) & n &\in L^2_{loc}(0,\infty;H^3(\mathbb{T}^d))\\
\mathbf{u}_t &\in L^2_{loc}(0,\infty;L^2(\mathbb{T}^d)) & n_t &\in L^2_{loc}(0,\infty;H^1(\mathbb{T}^d))\\
& & n_t &\in L^\infty(0,\infty;L^2(\mathbb{T}^d)).
\end{aligned}$$

**Demonstração:**

Pelo Lema (2.2.1) temos:

$$\varphi'(t)+\chi(t)+\psi(t) \quad \leq \quad C_1\varphi^2(t)+C_1\varphi^3(t)+C_1\varphi^4(t)+C_2\|\mathbf{f}(t)\|^2_{L^2(\mathbb{T}^d)} \tag{2.41}$$

onde,

$$\begin{aligned}
\varphi(t) &= \frac{1}{2}\|\sqrt{n^k}\nabla\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)}+\nu\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)},\\
\chi(t) &= \frac{\alpha}{2}\|\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+\nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|n^k_t\|^2_{L^2(\mathbb{T}^d)}+\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)},\\
\psi(t) &= \frac{\nu^2\alpha^3}{2\beta}\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\|n^k\|^2_{L^2(\mathbb{T}^d)}+\frac{\nu^2}{2}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)},\\
C_1 &= C_1(\nu,\varepsilon,\alpha,\beta),\\
C_2 &= C_2(\nu,\alpha,\beta).
\end{aligned}$$

Existe $C_3 = C_3(\nu,\varepsilon,\alpha,\beta) > 0$, tal que,

$$C_3\varphi(t) \leq \psi(t).$$

Definimos,

$$C_4 = C_2 \operatorname*{sup\,ess}_{t\geq 0}\|\mathbf{f}(t)\|^2_{L^2(\mathbb{T}^d)},$$

logo, temos a seguinte desigualdade diferencial:

$$\begin{aligned}
&\varphi'(t) \leq C_1(\varphi^2(t)+\varphi^3(t)+\varphi^4(t)) - C_3\varphi(t)+C_4\\
&\varphi(0) \leq \varphi_0.
\end{aligned}$$

Definimos a função:

$$G(y,C_4) = C_1(y^4+y^3+y^2)-C_3y+C_4.$$

Tem-se que a função $G$ é Localmente Lipschitz Continua. Seja $y(t)$ com $t \in [0,T)$ a solução do problema de valor inicial:

$$
\begin{aligned}
y^{'}(t) &= C_1(y^4(t) + y^3(t) + y^2(t)) - C_3 y(t) + C_4 = G(y, C_4) \\
y(0) &= \varphi_0.
\end{aligned} \tag{2.42}
$$

Pelo Teorema (1.2.1), temos que

$$\varphi(t) \leq y(t) \quad \forall t \in [0,T).$$

Agora considerando o caso particular da constante $C_4 = 0$ e encontrando as raízes da função $G(y,0)$ vemos que $y = 0$ e $y = C(C_1, C_3) > 0$ são as raízes de $G(y,0)$.

Considerando $C_4 > 0$ apenas irá transladar o gráfico da função $G$ para cima, assim considerando $C_4$ suficientemente pequeno tal que $\exists \delta > 0$ e $\alpha \leq \delta$, então,

$$y = \delta \quad \text{e} \quad y = C(C_1, C_3) - \delta > 0,$$

são as raízes de $G(y, C_4)$.

O sistema (2.42) é um problema autónomo, as raízes de $G(y, C_4)$ correspondem as soluções constantes do sistema dado,

$$y(t) = \delta \quad \text{e} \quad y(t) = C(C_1, C_3) - \delta > 0 \quad \forall t \in \mathbb{R}.$$

Como as trajetórias das soluções de problemas autónomos não se intersectam, as retas $y = \delta$ e $y = C(C_1, C_3) - \delta$ são assintotas horizontais dessas trajetórias. Estudando os sinais da função $G$, vemos que,

$$G(y, C_4) < 0 \iff \delta < y < C(C_1, C_3) - \delta,$$

$$G(y, C_4) > 0 \iff y < \delta \ \text{ ou } \ y > C(C_1, C3) - \delta.$$

Escolhendo os dados iniciais suficientemente pequenos tais que $\delta < \varphi_0 < C(C_1, C_3) - \delta$. Assim temos que a solução do problema dado satisfaz a seguinte desigualdade:

$$\alpha \leq \delta < y(t) < C(C_1, C_3) - \delta, \quad \forall t \geq 0.$$

Por fim concluímos que

$$\varphi(t) \leq y(t) \leq C(C_1, C_3) - \delta = C(\nu, \varepsilon, \alpha, \beta, \mathbb{T}^d, \|\mathbf{f}\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}), \quad \forall t \geq 0.$$

Em particular,

$$\mathbf{u}^k \in L^\infty(0,\infty; H^1(\mathbb{T}^d)) \quad n^k \in L^\infty(0,\infty; H^2(\mathbb{T}^d)).$$

Assim, pelas estimativas acima, temos que a desigualdade diferencial;

$$\varphi^{'}(t) + \chi(t) + \psi(t) \leq C_1\varphi^2(t) + C_1\varphi^3(t) + C_1\varphi^4(t) + C_4$$

$$\leq C(\nu, \varepsilon, \alpha, \beta, \mathbb{T}^d, \|\mathbf{f}\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d))})$$
$$= C_5$$

Multiplicando a desigualdade diferencial acima por $e^{\gamma t}$, onde $\gamma > 0$ e integrando de 0 a $t$, obtemos;

$$\begin{aligned}
&\int_0^t (e^{\gamma s}\varphi(s))' \, ds + \int_0^t e^{\gamma s}\chi(s)ds + \int_0^t e^{\gamma s}\psi(s)ds \\
&\leq C_5 \int_0^t e^{\gamma s}ds + \int_0^t \gamma e^{\gamma s}\varphi(s)ds \\
&\leq C_5 \int_0^t e^{\gamma s}ds + C_5 \int_0^t \gamma e^{\gamma s}ds \\
&\leq C_5 e^{\gamma t},
\end{aligned}$$

isto implica,

$$\begin{aligned}
&e^{\gamma t}\varphi(t) + \int_0^t e^{\gamma s}\chi(s)ds + \int_0^t e^{\gamma s}\psi(s)ds \\
&\leq \varphi(0) + C_5 e^{\gamma t},
\end{aligned}$$

multiplicando a desigualdade acima por $e^{-\gamma t}$, obtemos;

$$\begin{aligned}
&\varphi(t) + e^{-\gamma t}\int_0^t e^{\gamma s}\chi(s)ds + e^{-\gamma t}\int_0^t e^{\gamma s}\psi(s)ds \\
&\leq e^{-\gamma t}\varphi(0) + C_5 \\
&\leq \varphi(0) + C_5 \\
&\leq C(\nu, \varepsilon, \alpha, \beta, \mathbb{T}^d, \|\nabla \mathbf{u}_0\|^2_{L^2(\mathbb{T}^d)}, \|\Delta n_0\|^2_{L^2(\mathbb{T}^d)}, \|\mathbf{f}\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}) \\
&= C_6
\end{aligned}$$

Portanto,

$$\begin{aligned}
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}^k(s)\|^2_{H^2(\mathbb{T}^d)}ds \leq C_6 \qquad & e^{-\gamma t}\int_0^t e^{\gamma s}\|n^k(s)\|^2_{H^3(\mathbb{T}^d)}ds \leq C_6 \\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}^k_t(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq C_6 \qquad & e^{-\gamma t}\int_0^t e^{\gamma s}\|n^k_t(s)\|^2_{H^1(\mathbb{T}^d)}ds \leq C_6.
\end{aligned}$$

Em particular,

$$\begin{aligned}
\mathbf{u}^k \in L^2_{loc}(0,\infty; H^2(\mathbb{T}^d)) \qquad & n^k \in L^2_{loc}(0,\infty; H^3(\mathbb{T}^d)) \\
\mathbf{u}^k_t \in L^2_{loc}(0,\infty; L^2(\mathbb{T}^d)) \qquad & n^k_t \in L^2_{loc}(0,\infty; H^1(\mathbb{T}^d))
\end{aligned}$$

E finalmente, pela desigualdade (2.8) do Lema (2.2.1), temos que

$$n^k_t \in L^\infty(0,\infty; L^2(\mathbb{T}^d)).$$

Com os resultados obtidos acima, podemos fazer a *Passagem ao Limite* e *Verificação dos dados Iniciais* de forma totalmente análoga ao Teorema (2.3.1) e a unicidade é obtida de modo semelhante ao Teorema (2.4.1) e assim concluímos a demonstração.

■

**Corolário 2.4.3** *Sejam* $d = 3$, $\mathbf{u}_0 \in V \cap H^2(\mathbb{T}^d)$, $n_0 \in H^3(\mathbb{T}^d)$, $\mathbf{f} \in L^\infty(0,\infty; H^1(\mathbb{T}^d))$ *e* $\mathbf{f}_t \in L^\infty(0,\infty; L^2(\mathbb{T}^d))$. *Então a solução* $(\mathbf{u}, n)$ *dada pelo Teorema (2.4.2) verifica as seguintes desigualdades, para todo* $\gamma > 0$ *e para todo* $t \geq 0$ :

$$\begin{aligned}
\|\mathbf{u}(t)\|^2_{H^2(\mathbb{T}^d)} &\leq C, & \|n(t)\|^2_{H^3(\mathbb{T}^d)} &\leq C,\\
\|\mathbf{u}_t(t)\|^2_{L^2(\mathbb{T}^d)} &\leq C, & \|n^k_t(t)\|^2_{H^1(\mathbb{T}^d)} &\leq C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}(s)\|^2_{H^3(\mathbb{T}^d)}ds &\leq C, & e^{-\gamma t}\int_0^t e^{\gamma s}\|n(s)\|^2_{H^4(\mathbb{T}^d)}ds &\leq C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}_t(s)\|^2_{H^1(\mathbb{T}^d)}ds &\leq C \quad e & e^{-\gamma t}\int_0^t e^{\gamma s}\|n_t(s)\|^2_{H^2(\mathbb{T}^d)}ds &\leq C.
\end{aligned}$$

*Em particular,*

$$\begin{aligned}
\mathbf{u} &\in L^\infty(0,\infty; H^2(\mathbb{T}^d)), & n &\in L^\infty(0,\infty; H^3(\mathbb{T}^d)),\\
\mathbf{u} &\in L^2_{loc}(0,\infty; H^3(\mathbb{T}^d)), & n &\in L^2_{loc}(0,\infty; H^4(\mathbb{T}^d)),\\
\mathbf{u}_t &\in L^\infty(0,\infty; L^2(\mathbb{T}^d)), & n_t &\in L^\infty(0,\infty; H^1(\mathbb{T}^d)),\\
\mathbf{u}_t &\in L^2_{loc}(0,\infty; H^1(\mathbb{T}^d)) \quad e & n_t &\in L^2_{loc}(0,\infty; H^2(\mathbb{T}^d)).
\end{aligned}$$

**Demonstração:**

Podemos reescrever a desigualdade diferencial dado pelo Lema (2.2.2) da seguinte forma;

$$\varphi'(t) + \chi(t) + \psi(t) \leq C_1\varphi(t)\beta(t), \tag{2.43}$$

onde,

$$\begin{aligned}
\varphi(t) &= \frac{1}{2}\|\sqrt{n^k}\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)},\\
\psi(t) &= \frac{\nu\alpha}{2}\|\nabla \mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k_t\|^2_{L^2(\mathbb{T}^d)} + \frac{\nu}{2}\|\Delta n^k_t\|^2_{L^2(\mathbb{T}^d)},\\
\chi(t) &= \nu\|\nabla n^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla n^k\|^2_{L^2(\mathbb{T}^d)},\\
\beta(t) &= \|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^6_{L^2(\mathbb{T}^d)},\\
C_1 &= C_1(\nu,\varepsilon,\alpha,\beta,\|\mathbf{f}\|^2_{L^\infty(0,\infty;H^1(\mathbb{T}^d)}, \|\mathbf{f}_t\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d)}).
\end{aligned}$$

Como $\chi(t), \psi(t) > 0$, temos:

$$\varphi'(t) \leq C_1\varphi(t)\beta(t) \tag{2.44}$$

o que é equivalente á,

$$\frac{d}{dt}\ln\varphi(t) \leq C_1\beta(t).$$

Multiplicando a desigualdade diferencial acima por $e^{\gamma t}$, com $\gamma > 0$, obtemos;

$$\frac{d}{dt}(e^{\gamma t}\ln\varphi(t)) \leq C_1 e^{\gamma t}\beta(t) + \gamma e^{\gamma t}\ln\varphi(t).$$

Como,

$$\ln\varphi(t)\leq\varphi(t),\quad \text{para todo}\quad t\geq 0,$$

então,

$$\frac{d}{dt}(e^{\gamma t}\ln\varphi(t))\leq C_1e^{\gamma t}\beta(t)+\gamma e^{\gamma t}\varphi(t),$$

integrado a desigualdade diferencial acima de 0 a $t$, obtemos;

$$e^{\gamma t}\ln\varphi(t)-\ln\varphi_0\leq C_1\int_0^t e^{\gamma s}\beta(s)ds+\gamma\int_0^t e^{\gamma s}\varphi(s)ds,$$

multiplicando a desigualdade diferencial acima por $e^{-\gamma t}$, temos;

$$\ln\varphi(t)-e^{-\gamma t}\ln\varphi_0\leq C_1e^{-\gamma t}\int_0^t e^{\gamma s}\beta(s)ds+\gamma e^{-\gamma t}\int_0^t e^{\gamma s}\varphi(s)ds,$$

pelo Teorema (2.4.2) temos que, para todo $t\geq 0$,

$$e^{-\gamma t}\int_0^t e^{\gamma s}\beta(s)ds\leq C_2,$$
$$e^{-\gamma t}\int_0^t e^{\gamma s}\varphi(s)ds\leq C_2$$

onde,

$$C_2=C_2(\nu,\varepsilon,\alpha,\beta,\mathbb{T}^d,\|\mathbf{u}_0\|^2_{H^1(\mathbb{T}^d)},\|n_0\|^2_{H^2(\mathbb{T}^d)},\|\mathbf{f}\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d))},\|\mathbf{f}_t\|^2_{L^\infty(0,\infty;L^2(\mathbb{T}^d))}).$$

Então,

$$\ln\varphi(t)-\ln\varphi_0\leq C(C_1,C_2)=C_3.$$

Logo,

$$\ln\frac{\varphi(t)}{\varphi_0}\leq C_3,$$

o que implica,

$$\varphi(t)\leq\varphi_0e^{C_3}=C_4,\quad \forall t\geq 0.$$

Finalmente, concluímos que;

$$\mathbf{u}_t^k\in L^\infty(0,\infty;L^2(\mathbb{T}^d)),\qquad n_t^k\in L^\infty(0,\infty;H^1(\mathbb{T}^d)).$$

Agora, multiplicando a desigualdade diferencial (2.43) por $e^{\gamma t}$, integrando de 0 a $t$ e multiplicando agora por $e^{-\gamma t}$, obtem-se as seguintes regularidades:

$$\mathbf{u}_t^k\in L^\infty(0,\infty;L^2(\mathbb{T}^d)),\qquad n_t^k\in L^\infty(0,\infty;H^1(\mathbb{T}^d)),$$
$$\mathbf{u}_t^k\in L^2_{loc}(0,\infty;H^1(\mathbb{T}^d))\quad e\quad n_t^k\in L^2_{loc}(0,\infty;H^2(\mathbb{T}^d)).$$

Considerando os as estimativas obtidas no Teorema (2.4.2) e a desigualdade $(i)$ do Lema (2.2.3), temos que

$$\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\leq C\ \ \forall t\geq 0,$$

logo,

$$\mathbf{u}^k \in L^\infty(0,\infty;H^2(\mathbb{T}^d)) \qquad\qquad n^k \in L^\infty(0,\infty;H^3(\mathbb{T}^d)).$$

Pela desigualdade $(ii)$ do Lema (2.2.3), temos que

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \le C, \quad \forall t \ge 0,$$

em particular,

$$n^k \in L^2_{loc}(0,\infty;H^4(\mathbb{T}^d)).$$

E por fim, considerando a desigualdade $(iii)$ do Lema (2.2.3), temos que

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \le C, \quad \forall t \ge 0,$$

em particular,

$$\mathbf{u}^k \in L^2_{loc}(0,\infty;H^3(\mathbb{T}^d)).$$

■

**Corolário 2.4.4** *Sejam $d=3$, $\mathbf{u}_0 \in V \cap H^3(\mathbb{T}^d)$, $n_0 \in H^4(\mathbb{T}^d)$, $\mathbf{f} \in L^\infty(0,\infty;H^2(\mathbb{T}^d))$ e $\mathbf{f}_t \in L^\infty(0,\infty;H^1(\mathbb{T}^d))$. Então a solução $(\mathbf{u},n)$ dada pelo Teorema (2.4.2) verifica as seguintes desigualdades, para todo $\gamma > 0$ e para todo $t \ge 0$ :*

$$\begin{aligned}
\|\mathbf{u}(t)\|^2_{H^3(\mathbb{T}^d)} &\le C, & \|n(t)\|^2_{H^4(\mathbb{T}^d)} &\le C,\\
\|\mathbf{u}_t(t)\|^2_{H^1(\mathbb{T}^d)} &\le C, & \|n^k_t(t)\|^2_{H^2(\mathbb{T}^d)} &\le C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}(s)\|^2_{H^4(\mathbb{T}^d)}ds &\le C, & e^{-\gamma t}\int_0^t e^{\gamma s}\|n(s)\|^2_{H^5(\mathbb{T}^d)}ds &\le C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}_t(s)\|^2_{H^2(\mathbb{T}^d)}ds &\le C, & e^{-\gamma t}\int_0^t e^{\gamma s}\|n_t(s)\|^2_{H^3(\mathbb{T}^d)}ds &\le C,\\
e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}(s)_{tt}\|^2_{L^2(\mathbb{T}^d)}ds &\le C \quad e & e^{-\gamma t}\int_0^t e^{\gamma s}\|n(s)_{tt}\|^2_{H^1(\mathbb{T}^d)}ds &\le C.
\end{aligned}$$

*Em particular,*

$$\begin{aligned}
&\mathbf{u} \in L^\infty(0,\infty;H^3(\mathbb{T}^d)), && n \in L^\infty(0,\infty;H^4(\mathbb{T}^d)),\\
&\mathbf{u} \in L^2_{loc}(0,\infty;H^4(\mathbb{T}^d)), && n \in L^2_{loc}(0,\infty;H^5(\mathbb{T}^d)),\\
&\mathbf{u}_t \in L^\infty(0,\infty;H^1(\mathbb{T}^d)), && n_t \in L^\infty(0,\infty;H^2(\mathbb{T}^d)),\\
&\mathbf{u}_t \in L^2_{loc}(0,\infty;H^2(\mathbb{T}^d)), && n_t \in L^2_{loc}(0,\infty;H^3(\mathbb{T}^d)),\\
&\mathbf{u}_{tt} \in L^2_{loc}(0,\infty;L^2(\mathbb{T}^d)) \quad e && n_{tt} \in L^2_{loc}(0,\infty;H^1(\mathbb{T}^d)).
\end{aligned}$$

**Demonstração:**

Considerando a desigualdade diferencial dada pelo Lema (2.2.4) e as regularidades obtidas no Corolário (2.4.3), temos a seguinte desigualdade diferencial;

$$\frac{d}{dt}\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}_{tt}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)} + C,$$

multiplicando por $e^{\gamma t}$, com $\gamma > 0$, integrando de 0 a $t$, com $t \geq 0$, e por fim, multiplicando por $e^{-\gamma t}$, obtemos;

$$\begin{aligned}
&\|\nabla \mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)} + e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}_{tt}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + e^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta \mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \|\nabla \mathbf{u}_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \\
&+Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla \mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta n_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + e^{-\gamma t}\int_0^t Ce^{\gamma s} ds \\
&\leq C.
\end{aligned}$$

Assim obtemos a seguinte estimativa;

$$\mathbf{u}_t^k \in L^\infty(0,\infty; H^1(\mathbb{T}^d)), \quad \mathbf{u}_{tt}^k \in L^2_{loc}(0,\infty; L^2(\mathbb{T}^d)) \text{ e } \mathbf{u}_t^k \in L^2_{loc}(0,\infty; H^2(\mathbb{T}^d)).$$

Considerando a desigualdade diferencial $(i)$ do Lema (2.2.5), multiplicando por $e^{\gamma t}$, com $\gamma > 0$, integrando de 0 a $t$, com $t \geq 0$, e por fim, multiplicando por $e^{-\gamma t}$, obtemos;

$$\begin{aligned}
&\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \|\Delta^2 n_0^k\|^2_{L^2(\mathbb{T}^d)} \\
&+Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \\
&\leq C.
\end{aligned}$$

Portando obtemos

$$n^k \in L^\infty(0,\infty; H^4(\mathbb{T}^d)) \qquad e \qquad n^k \in L^2_{loc}(0,\infty; H^5(\mathbb{T}^d)).$$

Assim, pelas estimativas obtidas acima, temos pela desigualdade $(ii)$ do Lema (2.2.5) que

$$\begin{aligned}
\|\Delta n_t^k\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} \\
&\leq C \quad \forall t \geq 0,
\end{aligned}$$

logo,

$$n_t \in L^\infty(0,\infty; H^2(\mathbb{T}^d)).$$

E ainda, multiplicando por $e^{\gamma t}$, com $\gamma > 0$, integrando de 0 a $t$, com $t \geq 0$, e por fim, multiplicando por $e^{-\gamma t}$ a desigualdade diferencial $(iii)$ do Lema (2.2.5), temos;

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta n_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$

$$+Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + Ce^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\Delta^2 n^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$\leq C$$

logo,

$$n_t^k \in L^2_{loc}(0,\infty; H^3(\mathbb{T}^d)).$$

Agora, multiplicando por $e^{\gamma t}$, com $\gamma > 0$, integrando de 0 a $t$, com $t \geq 0$, e por fim, multiplicando por $e^{-\gamma t}$ a desigualdade diferencial dada pelo Lema (2.2.6) e considerando as estimativas obtidas anteriormente, temos;

$$\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + e^{-\gamma t}\int_0^t e^{\gamma s}\|\Delta^2\mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq C.$$

Então, temos;

$$\mathbf{u}^k \in L^\infty(0,\infty; H^3(\mathbb{T}^d)) \qquad e \qquad \mathbf{u}^k \in L^2_{loc}(0,\infty; H^4(\mathbb{T}^d)).$$

De forma análoga, multiplicando por $e^{\gamma t}$, com $\gamma > 0$, integrando de 0 a $t$, com $t \geq 0$, e por fim, multiplicando por $e^{-\gamma t}$ a desigualdade diferencial dada pelo Lema (2.2.7 ) e considerando as estimativas obtidas anteriormente, temos;

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla n_{tt}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq C,$$

logo,

$$n_{tt}^k \in L^2_{loc}(0,\infty; H^1(\mathbb{T}^d)).$$

■

# Capítulo 3

# Análise de Erro para as Aproximações semi-Galerkin Espectrais

Conforme mencionado na Introdução deste trabalho, este capítulo foi idealizado com o propósito de fornecer uma rigorosa e detalhada análise de erro das aproximações semi-Galerkin espectrais, que sirva de suporte teórico para futuras implementações computacionais.

Começamos por mencionar que em Damázio e Rojar-Medar [33] foram obtidas estimativas de erro locais para a velocidade $\mathbf{u}$ e densidade $\rho$ nos espaços $H^1$ e $H^2$ respectivamente para às Equações de Movimento de fluidos Viscosos Incompressível com Fenômenos de Difusão. Trabalhando de forma inspirada em [33], utilizamos as regularidades obtidas no Capítulo 2 para a solução do problema (8)-(11) e obtemos estimativas de erro locais para as aproximações da velocidade $\mathbf{u}$ no espaço $H^2$ e para a densidade $n$ no espaço $H^4$. Os argumentos utilizados para obter as estimativas de erro locais são facilmente adaptadas para o caso global.

## 3.1 Desigualdades Diferenciais

Esta seção foi elaborada com o propósito único de simplificar a demonstração dos resultados acerca das diferentes taxas de convergência, os quais são estabelecidos na próxima seção.

A projeção $P_k\mathbf{u}$ será usada como vetor intermediário entre a solução forte $\mathbf{u}$ do problema (8)-(11) e a solução aproximada $\mathbf{u}^k$ do problema aproximado de nível $k$. A principal razão de introduzir a projeção $P_k\mathbf{u}$ é porque pode-se decompor o erro em duas partes na forma: $\|\mathbf{u}-\mathbf{u}^k\| \leq \|\mathbf{u}-P_k\mathbf{u}\| + \|P_k\mathbf{u}-\mathbf{u}^k\|$ e são conhecidas estimativas de erro do primeiro termo do lado direito, graças aos Lemas (1.1.4) e (1.1.5) e restando apenas obter estimativas para o segundo termo no subespaço de dimensão finita $V_k$.

**Definição 3.1.1** *Sejam* $(\mathbf{u}, n)$ *a solução forte do problema (8)-(11) e* $(\mathbf{u}^k, n^k)$ *a solução do problema aproximado de nível* $k$*. Definimos então:*

$$\begin{aligned} &(i) \quad \theta^k = P_k\mathbf{u} - \mathbf{u}^k \\ &(ii) \quad \mathbf{E}^k = \mathbf{u} - P_k\mathbf{u} \\ &(iii) \quad \pi^k = n - n^k \end{aligned}$$

Em todos os Lemas desta seção é considerado as seguintes hipóteses:

$$(H1) \qquad \mathbf{u}_0 \in H^2(\mathbb{T}^d),\ n_0 \in H^3(\mathbb{T}^d),\ \mathbf{f} \in L^2(0,T^*;H^1(\mathbb{T}^d)) \text{ e } \mathbf{f}_t \in L^2(0,T^*;L^2(\mathbb{T}^d)).$$

**Lema 3.1.1** *Tem-se válida a seguinte desigualdade diferencial:*

$$\begin{aligned} &\frac{d}{dt}\big(\|\sqrt{n^k}\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\big) + \|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} \\ &+\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \le C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)} \\ &+C\beta(t)\big(\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\big), \end{aligned}$$

*onde*

$$\begin{aligned} \beta(t) = C\big(&\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)} \\ &+\|\nabla\Delta n^k\|^4_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} \\ &+\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + \|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}\big) \end{aligned}$$

**Demonstração:**

Multiplicando-se a equação (9) por $\mathbf{v}^k \in V_k$ e integrando-se a resultante sobre $\mathbb{T}^d$ tem-se que:

$$\begin{aligned} &\Big(n\mathbf{u}_t + (n\mathbf{u}.\nabla)\mathbf{u} + \nu\left[-n\Delta\mathbf{u} - 2(\nabla n.\nabla)\mathbf{u}\right] + \nabla p, \mathbf{v}^k\Big) = \Big(n\mathbf{f} \\ &+\varepsilon^2\left[-\frac{1}{n}(\nabla n.\nabla)\nabla n + \frac{1}{n^2}(\nabla n.\nabla n)\nabla n - \frac{1}{n}\Delta n\nabla n + \nabla\Delta n\right], \mathbf{v}^k\Big). \qquad (3.1) \end{aligned}$$

Considerando a equação (2.5),

$$\begin{aligned} &\Big(n^k\mathbf{u}^k_t + (n^k\mathbf{u}.\nabla)\mathbf{u}^k + \nu[-n^k\Delta\mathbf{u}^k - 2(\nabla n^k.\nabla)\mathbf{u}^k], \mathbf{v}^k\Big) = \Big(n^k\mathbf{f} \\ &+\varepsilon^2\left[-\frac{1}{n^k}(\nabla n^k.\nabla)\nabla n^k + \frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla n^k - \frac{1}{n^k}\Delta n^k\nabla n^k + \nabla\Delta n^k\right], \mathbf{v}^k\Big), \end{aligned}$$

e fazendo-se a diferença da equação (3.1) pela equação (2.5), obtem-se:

$$\Big(\pi^k\mathbf{u}_t + n^k\mathbf{E}_t^k + n^k\theta_t^k + (\pi^k\mathbf{u}.\nabla)\mathbf{u} + (n^k\mathbf{E}^k.\nabla)\mathbf{u} + (n^k\theta^k.\nabla)\mathbf{u} + (n^k\mathbf{u}^k.\nabla)\mathbf{E}^k$$
$$+(n^k\mathbf{u}^k.\nabla)\theta^k - \nu\big(\pi^k\Delta\mathbf{u} + n^k\Delta\mathbf{E}^k + n^k\Delta\theta^k + 2(\nabla\pi^k.\nabla)\mathbf{u}$$
$$+2(\nabla n^k.\nabla)\mathbf{E}^k + 2(\nabla n^k.\nabla)\theta^k\big) - \varepsilon^2\bigg( -\frac{\pi^k}{nn^k}(\nabla n.\nabla)\nabla n + \frac{1}{n^k}(\nabla\pi^k.\nabla)\nabla n$$
$$+\frac{1}{n^k}(\nabla n^k.\nabla)\nabla\pi^k + \frac{n+n^k}{(nn^k)^2}\pi^k(\nabla n.\nabla n)\nabla n - \frac{1}{(n^k)^2}(\nabla\pi^k.\nabla n)\nabla n$$
$$-\frac{1}{(n^k)^2}(\nabla n^k.\nabla\pi^k)\nabla n - \frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla\pi^k + \frac{\pi^k}{nn^k}\Delta n\nabla n + \frac{1}{n^k}\Delta\pi^k\nabla n$$
$$+\frac{1}{n^k}\Delta n^k\nabla\pi^k\bigg), \mathbf{v}^k\bigg) = \Big(\pi^k\mathbf{f} + \varepsilon^2\nabla\Delta\pi^k, \mathbf{v}^k\Big). \tag{3.2}$$

Em particular, para $\mathbf{v}^k = \theta^k$ na equação (3.2), o termo $-(n^k\Delta\theta^k, \theta^k)$ pode ser assim reescrito:

$$\begin{aligned} -(n^k\Delta\theta^k, \theta^k) &= (\nabla\theta^k, \nabla(n^k\theta^k)) \\ &= (\nabla\theta^k, n^k\nabla\theta^k) + ((\theta^k.\nabla)\theta^k, \nabla n^k) \\ &= \|\sqrt{n^k}\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + ((\theta^k.\nabla)\theta^k, \nabla n^k), \end{aligned}$$

e portanto, tem-se que:

$$(n^k\theta_t^k, \theta^k) + \nu\|\sqrt{n^k}\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} = \nu((\theta^k.\nabla)\theta^k, \nabla n^k) - (\pi^k\mathbf{u}_t, \theta^k) - (n^k\mathbf{E}_t^k, \theta^k)$$
$$-((\pi^k\mathbf{u}.\nabla)\mathbf{u}, \theta^k) - ((n^k\mathbf{E}^k.\nabla)\mathbf{u}, \theta^k) - ((n^k\theta^k.\nabla)\mathbf{u}, \theta^k) - ((n^k\mathbf{u}^k.\nabla)\mathbf{E}^k, \theta^k)$$
$$-((n^k\mathbf{u}^k.\nabla)\theta^k, \theta^k) + \nu\big((\pi^k\Delta\mathbf{u}, \theta^k) + (n^k\Delta\mathbf{E}^k, \theta^k)$$
$$+2((\nabla\pi^k.\nabla)\mathbf{u}, \theta^k) + 2((\nabla n^k.\nabla)\mathbf{E}^k, \theta^k) + 2((\nabla n^k.\nabla)\theta^k, \theta^k)\big)$$
$$+\varepsilon^2\bigg( - \Big(\frac{\pi^k}{nn^k}(\nabla n.\nabla)\nabla n, \theta^k\Big) + \Big(\frac{1}{n^k}(\nabla\pi^k.\nabla)\nabla n, \theta^k\Big)$$
$$+ \Big(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla\pi^k, \theta^k\Big) + \Big(\frac{n+n^k}{(nn^k)^2}\pi^k(\nabla n.\nabla n)\nabla n, \theta^k\Big)$$
$$- \Big(\frac{1}{(n^k)^2}(\nabla\pi^k.\nabla n)\nabla n, \theta^k\Big) - \Big(\frac{1}{(n^k)^2}(\nabla n^k.\nabla\pi^k)\nabla n, \theta^k\Big) + \Big(\frac{1}{n^k}\Delta\pi^k\nabla n, \theta^k\Big)$$
$$- \Big(\frac{1}{(n^k)}(\nabla n^k.\nabla n^k)\nabla\pi^k, \theta^k\Big) + \Big(\frac{\pi^k}{nn^k}\Delta n\nabla n, \theta^k\Big) + \Big(\frac{1}{n^k}\Delta n^k\nabla\pi^k, \theta^k\Big)$$
$$+(\nabla\Delta\pi^k, \theta^k)\bigg) + (\pi^k\mathbf{f}, \theta^k). \tag{3.3}$$

Note-se que,

$$\begin{aligned}(n^k\theta^k_t,\theta^k) &= \frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\theta^k\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}(n^k_t\theta^k,\theta^k)\\ &= \frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\theta^k\|^2_{L^2(\mathbb{T}^d)} - \frac{\nu}{2}(\Delta n^k\theta^k,\theta^k) + \frac{1}{2}(\mathbf{u}^k.\nabla n^k\theta^k,\theta^k),\end{aligned}$$

estimando-se os termos à direta da equação (3.3), obtem-se:

$$\begin{aligned}\frac{\nu}{2}|(\Delta n^k\theta^k,\theta^k)| &\leq C\|\Delta n^k\|_{L^3(\mathbb{T}^d)}\|\theta^k\|_{L^6(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\frac{1}{2}|(\mathbf{u}^k.\nabla n^k\theta^k,\theta^k)| &\leq C\|\Delta\mathbf{u}^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^6(\mathbb{T}^d)}\|\theta^k\|_{L^3(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|((\theta^k.\nabla)\theta^k,\nabla n^k)| &\leq C\|\theta^k\|_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\pi^k\mathbf{u}_t,\theta^k)| &\leq C\|\pi^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}_t\|_{L^3(\mathbb{T}^d)}\|\theta^k\|_{L^6(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(n^k\mathbf{E}^k_t,\theta^k)| &\leq C\|\mathbf{E}^k_t\|_{L^2(\mathbb{T}^d)}\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)} + C\|n^k\|^2_{H^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((\pi^k\mathbf{u}.\nabla)\mathbf{u},\theta^k)| &\leq C\|\pi^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\theta^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((n^k\mathbf{E}^k.\nabla)\mathbf{u},\theta^k)| &\leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\theta^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((n^k\theta^k.\nabla)\mathbf{u},\theta^k)| &\leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\theta^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2 + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((n^k\mathbf{u}^k.\nabla)\mathbf{E}^k,\theta^k)| &\leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$|((n^k\mathbf{u}^k.\nabla)\theta^k,\theta^k)| \leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^d)}\|\nabla\theta^k\|_{L^2(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}$$

$$\leq \quad C\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}\nu|(\pi^k\Delta\mathbf{u},\theta^k)| &\leq C\|\pi^k\|_{L^2(\mathbb{T}^d)}\|\Delta\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\theta^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|(n^k\Delta\mathbf{E}^k,\theta^k)| &= \nu|(\nabla\mathbf{E}^k,\nabla(n^k\theta^k))|\\ &\leq \nu|(\nabla\mathbf{E}^k,n^k\nabla\theta^k)| + \nu|(\theta^k.\nabla\mathbf{E}^k,\nabla n^k)|\\ &\leq C\|\nabla\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\theta^k\|_{L^2(\mathbb{T}^d)} + C\|\theta^k\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|((\nabla\pi^k.\nabla)\mathbf{u},\theta^k)| &\leq C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}.;\end{aligned}$$

$$\begin{aligned}\nu|((\nabla n^k.\nabla)\theta^k,\theta^k)| &\leq C\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\theta^k\|_{L^2(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|((\nabla n^k.\nabla)\mathbf{E}^k,\theta^k)| &\leq C\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\\ &\leq \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\pi^k\mathbf{f},\theta^k)| &\leq C\|\pi^k\|_{L^2(\mathbb{T}^d)}\|\mathbf{f}\|_{L^4(\mathbb{T}^d)}\|\theta^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\varepsilon^2|(\nabla\Delta\pi^k,\theta^k)| = \varepsilon^2|(\Delta\pi^k,div(\theta^k))| = 0;$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{\pi^k}{nn^k}(\nabla n\nabla)\nabla n,\theta^k\right)\right| &\leq C\|\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^6(\mathbb{T}^d)}\|\nabla^2 n\|_{L^6(\mathbb{T}^d)}\|\theta^k\|_{L^6(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla\pi^k.\nabla)\nabla n^k,\theta^k\right)\right| &\leq C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla^2 n^k\|_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_2\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla\pi^k,\theta^k\right)\right| &\leq C\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\Delta\pi^k\|_{L^2(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_3\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\varepsilon^2\left|\left(\frac{n+n^k}{(nn^k)^2}\pi^k(\nabla n.\nabla n)\nabla n,\theta^k\right)\right| \leq C\|\nabla n\|^3_{L^\infty(\mathbb{T}^d)}\|\pi^k\|_{L^2(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\nabla\Delta n\|^4_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2\left|\left(\frac{1}{(n^k)^2}(\nabla\pi^k.\nabla n)\nabla n,\theta^k\right)\right| \leq C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2\left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla\pi^k)\nabla n,\theta^k\right)\right| \leq C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2\left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla\pi^k,\theta^k\right)\right| \leq C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2\left|\left(\frac{\pi^k}{nn^k}\Delta n\nabla n,\theta^k\right)\right| \leq C\|\pi^k\|_{L^2(\mathbb{T}^d)}\|\Delta n\|_{L^\infty(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\varepsilon^2\left|\left(\frac{1}{n^k}\Delta\pi^k\nabla n,\theta^k\right)\right| \leq C\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\Delta\pi^k\|_{L^2(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_3\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}.$$

$$\varepsilon^2\left|\left(\frac{1}{n^k}\Delta n^k\nabla\pi^k,\theta^k\right)\right| \leq C\|\Delta n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_2\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}.$$

Assim sendo, obtem-se que:

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\theta^k\|^2_{L^2(\mathbb{T}^d)}+\nu\|\sqrt{n^k}\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}\leq C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+2\delta_3\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}+6\delta_2\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}+14\delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+C\big(\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}+\|\Delta n^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}+\|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)}+\|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta n\|^2_{L^4(\mathbb{T}^d)}$$
$$+\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\big)\|\theta^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\big(\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}+\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}+\|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}+\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}\big)\|\pi^k\|^2_{L^2(\mathbb{T}^d)}. \tag{3.4}$$

Agora, fazendo-se a diferença da equação (8) pela equação (2.4), obtem-se:

$$\pi_t^k + \theta^k.\nabla n + \mathbf{E}^k.\nabla n + \mathbf{u}^k.\nabla \pi^k - \nu\Delta\pi^k = 0, \tag{3.5}$$

multiplicando-se a equação (3.5) por $\pi^k$ e integrando-a sobre $\mathbb{T}^d$, obtem-se;

$$\frac{1}{2}\frac{d}{dt}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} = -(\theta^k.\nabla n, \pi^k) - (\mathbf{E}^k.\nabla n, \pi^k) \tag{3.6}$$

pois,

$$\int_{\mathbb{T}^d} \mathbf{u}^k.\nabla\pi^k\pi^k dx = \int_{\mathbb{T}^d} \mathbf{u}^k \frac{\nabla|\pi^k|^2}{2} dx = -\int_{\mathbb{T}^d} div(\mathbf{u}^k)\frac{|\nabla\pi^k|^2}{2} dx = 0.$$

Os temos à direita da equação (3.6) podem ser estimados como segue:

$$\begin{aligned} |(\theta^k.\nabla n, \pi^k)| &\leq \|\theta^k\|_{L^4(\mathbb{T}^d)}\|\nabla n\|_{L^4(\mathbb{T}^d)}\|\pi^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

$$\begin{aligned} |(\mathbf{E}^k.\nabla n, \pi^k)| &\leq \|\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\pi^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

Assim, tem-se que:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} \\ &\quad + \|\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}. \end{aligned} \tag{3.7}$$

Por outro lado, multiplicando-se a equação (3.5) por $-\Delta\pi^k$ e integrando-se o resultado sobre $\mathbb{T}^d$, obtemos;

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} = (\theta^k.\nabla n, \Delta\pi^k) + (\mathbf{E}^k.\nabla n, \Delta\pi^k) \\ &+(\mathbf{u}^k.\nabla\pi^k, \Delta\pi^k). \end{aligned} \tag{3.8}$$

Estimam-se os termos à direita da equação (3.8) como segue:

$$\begin{aligned} |(\theta^k.\nabla n, \Delta\pi^k)| &\leq \|\theta^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\Delta\pi^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_3\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

$$\begin{aligned} |(\mathbf{E}^k.\nabla n, \Delta\pi^k)| &\leq \|\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\Delta\pi^k\|_{L^2(\mathbb{T}^d)} \\ &\leq \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + \delta_3\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

$$\begin{aligned} |(\mathbf{u}^k.\nabla\pi^k, \Delta\pi^k)| &\leq \|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\Delta\pi^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_3\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}. \end{aligned}$$

Assim, tem-se que:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + 3\delta_3\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}. \tag{3.9}$$

Finalmente, somando-se as desigualdades diferenciais (3.4), (3.7) e (3.9), escolhendo-se de forma conveniente os valores de $\delta_1$, $\delta_2$ e $\delta_3$ obtem-se a desigualdade diferencial desejada:

$$\frac{d}{dt}\big(\|\sqrt{n^k}\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\big) + \|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}$$
$$\leq C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\beta(t)\big(\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\big),$$

onde

$$\beta(t) = C\big(\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^2 n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + \|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}\big)$$

■

**Lema 3.1.2** *Tem-se válida a seguinte desigualdade diferencial:*

$$\frac{d}{dt}\big(\|\sqrt{n^k}\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k\|^2_{H^1(\mathbb{T}^d)}\big) + \|\theta^k_t\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k_t\|^2_{L^2(\mathbb{T}^d)} \leq C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\beta(t)\big(\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k\|^2_{H^1(\mathbb{T}^d)}\big)$$

*onde*

$$\beta(t) = C\big(\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}\big)$$

**Demonstração:**

Fazendo $\mathbf{v}^k = \theta^k_t$ na equação (3.2) e observando que;

$$\begin{aligned}
-(n^k\Delta\theta^k, \theta^k_t) &= (\nabla\theta^k, \nabla(n^k\theta^k_t)) \\
&= (\nabla\theta^k, n^k\nabla\theta^k_t) + ((\theta^k_t.\nabla)\theta^k, \nabla n^k) \\
&= \frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + ((\theta^k_t.\nabla)\theta^k, \nabla n^k) - \frac{1}{2}(n^k_t\theta^k, \theta^k_t) \\
&= \frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + ((\theta^k_t.\nabla)\theta^k, \nabla n^k) - \frac{\nu}{2}(\Delta n^k\theta^k, \theta^k_t) \\
&\quad + \frac{1}{2}(\mathbf{u}^k.\nabla n^k.\theta^k, \theta^k_t)
\end{aligned}$$

portanto temos,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\sqrt{n^k}\theta^k_t\|^2_{L^2(\mathbb{T}^d)} = -((\theta^k_t.\nabla)\theta^k, \nabla n^k) \\
&+\frac{\nu}{2}(\Delta n^k\theta^k, \theta^k_t) - \frac{1}{2}(\mathbf{u}^k.\nabla n^k.\theta^k, \theta^k_t) - (\pi^k\mathbf{u}_t, \theta^k_t) - (n^k\mathbf{E}^k_t, \theta^k_t) \\
&-((\pi^k\mathbf{u}.\nabla)\mathbf{u}, \theta^k_t) - ((n^k\mathbf{E}^k.\nabla)\mathbf{u}, \theta^k_t) - ((n^k\theta^k.\nabla)\mathbf{u}, \theta^k_t) - ((n^k\mathbf{u}^k.\nabla)\mathbf{E}^k, \theta^k_t) \\
&-((n^k\mathbf{u}^k.\nabla)\theta^k, \theta^k_t) + \nu\big((\pi^k\Delta\mathbf{u}, \theta^k_t) + (n^k\Delta\mathbf{E}^k, \theta^k_t) \\
&+2((\nabla\pi^k.\nabla)\mathbf{u}, \theta^k_t) + 2((\nabla n^k.\nabla)\mathbf{E}^k, \theta^k_t) + 2((\nabla n^k.\nabla)\theta^k, \theta^k_t)\big) \\
&+\varepsilon^2\Bigg( - \left(\frac{\pi^k}{nn^k}(\nabla n.\nabla), \theta^k_t\right) + \left(\frac{1}{n^k}(\nabla\pi^k.\nabla)\nabla n, \theta^k_t\right) \\
&+ \left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla\pi^k, \theta^k_t\right) + \left(\frac{n+n^k}{(nn^k)^2}\pi^k(\nabla n.\nabla n)\nabla n, \theta^k_t\right) \\
&- \left(\frac{1}{(n^k)^2}(\nabla\pi^k.\nabla n)\nabla n, \theta^k_t\right) - \left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla\pi^k)\nabla n, \theta^k_t\right) + \left(\frac{1}{n^k}\Delta\pi^k\nabla n, \theta^k_t\right) \\
&- \left(\frac{1}{(n^k)}(\nabla n^k.\nabla n^k)\nabla\pi^k, \theta^k_t\right) + \left(\frac{\pi^k}{nn^k}\Delta n\nabla n, \theta^k_t\right) + \left(\frac{1}{n^k}\Delta n^k\nabla\pi^k, \theta^k_t\right) \\
&+(\nabla\Delta\pi^k, \theta^k_t)\Bigg) + (\pi^k\mathbf{f}, \theta^k_t) \qquad (3.10)
\end{aligned}
$$

Os termos a direita da equação acima, podem ser estimados como segue:

$$
\begin{aligned}
\nu|((\theta^k_t.\nabla)\theta^k, \nabla n^k)| &\leq C\|\theta^k_t\|_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)} \\
&\leq \|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\frac{\nu}{2}|(\Delta n^k\theta^k, \theta^k_t)| &\leq C\|\Delta n^k\|_{L^4(\mathbb{T}^d)}\|\theta^k\|_{L^4(\mathbb{T}^d)}\|\theta^k_t\|_{L^2(\mathbb{T}^d)} \\
&\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\frac{\nu}{2}|(\mathbf{u}^k.\nabla n^k\theta^k, \theta^k_t)| &\leq C\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^2(\mathbb{T}^d)}\|\theta^k_t\|_{L^2(\mathbb{T}^d)} \\
&\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
|(\pi^k\mathbf{f}, \theta^k_t)| &\leq C\|\pi^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{f}\|_{L^4(\mathbb{T}^d)}\|\theta^k_t\|_{L^2(\mathbb{T}^d)} \\
&\leq C\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)} + \delta_1\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2|(\nabla\Delta\pi^k, \theta^k_t)| &= \varepsilon^2|(\Delta\pi^k, div(\theta^k_t))| \\
&= \varepsilon^2|(\Delta\pi^k, (div(\theta^k))_t)| \\
&= 0;
\end{aligned}
$$

$$\begin{aligned}|(\pi^k\mathbf{u}_t,\theta_t^k)| &\leq C\|\pi^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}_t\|_{L^4(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(n^k\mathbf{E}_t^k,\theta_t^k)| &\leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{E}_t^k\|_{L^2(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\mathbf{E}_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((\pi^k\mathbf{u}.\nabla)\mathbf{u},\theta_t^k)| &\leq C\|\pi^k\|_{L^4(\mathbb{T}^d)}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((n^k\mathbf{E}^k.\nabla)\mathbf{u},\theta_t^k)| &\leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{E}^k\|_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((n^k\theta^k.\nabla)\mathbf{u},\theta_t^k)| &\leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\theta^k\|_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((n^k\mathbf{u}^k.\nabla)\mathbf{E}^k,\theta_t^k)| &\leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|((n^k\mathbf{u}.\nabla)\theta^k,\theta_t^k)| &\leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^d)}\|\nabla\theta^k\|_{L^2(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|(\pi^k\Delta\mathbf{u},\theta_t^k)| &\leq C\|\pi^k\|_{L^4(\mathbb{T}^d)}\|\Delta\mathbf{u}\|_{L^4(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|(n^k\Delta\mathbf{E}^k,\theta_t^k)| &\leq C\|n^k\|_{L^\infty(\mathbb{T}^d)}\|\Delta\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|((\nabla\pi^k.\nabla)\mathbf{u},\theta_t^k)| &\leq C\|\nabla\mathbf{u}\|_{L^\infty(\mathbb{T}^d)}\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\nu|((\nabla n^k.\nabla)\mathbf{E}^k,\theta_t^k)| &\leq \|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\nu|((\nabla n^k.\nabla)\theta^k,\theta_t^k)| \leq \nu\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\theta^k\|_{L^2(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}$$

$$
\begin{aligned}
&\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta_1^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{\pi^k}{nn^k}(\nabla n.\nabla)\nabla n, \theta_t^k\right)\right| &\leq C\|\pi^k\|_{L^4(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\Delta n\|_{L^4(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla\pi^k.\nabla)\nabla n, \theta_t^k\right)\right| &\leq C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla^2 n\|_{L^\infty(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla\pi^k, \theta_t^k\right)\right| &\leq C\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\Delta\pi^k\|_{L^2(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)}\\
&= C\|\pi_t^k + \theta^k.\nabla n + \mathbf{E}^k.\nabla n + \mathbf{u}^k.\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\pi_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\quad + C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{n+n^k}{(nn^k)^2}\pi^k(\nabla n.\nabla n)\nabla n, \theta_t^k\right)\right| &\leq C\|\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|^3_{L^\infty(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{(n^k)^2}(\nabla\pi^k.\nabla n)\nabla n, \theta_t^k\right)\right| &\leq C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla\pi^k)\nabla n, \theta_t^k\right)\right| &\leq C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\nabla n^k\|_{L^\infty(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla\pi^k, \theta_t^k\right)\right| &\leq C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2\left|\left(\frac{\pi^k}{nn^k}\Delta n^k\nabla n, \theta_t^k\right)\right| &\leq C\|\pi^k\|_{L^6(\mathbb{T}^d)}\|\Delta n\|_{L^6(\mathbb{T}^d)}\|\nabla n\|_{L^6(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)} + \delta_1\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}
$$

$$
\varepsilon^2\left|\left(\frac{1}{n^k}\Delta\pi^k\nabla n, \theta_t^k\right)\right| \leq C\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\Delta\pi^k\|_{L^2(\mathbb{T}^d)}\|\theta_t^k\|_{L^2(\mathbb{T}^d)}
$$

$$\begin{aligned}
&\leq C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)}\\
&= C\|\pi^k_t + \theta^k.\nabla n + \mathbf{E}^k.\nabla n + \mathbf{u}^k.\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\pi^k_t\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\quad + C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\varepsilon^2\left|\left(\frac{1}{n^k}\Delta n^k\nabla\pi^k, \theta^k_t\right)\right| &\leq C\|\Delta n^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)}\|\theta^k_t\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

E assim, obtem-se a seguinte desigualdade:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{n^k}\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \alpha\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)} \leq C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}\\
&+C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C^*\|\pi^k_t\|^2_{L^2(\mathbb{T}^d)} + 35\delta_1\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)} + C\big(\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\\
&+\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\big)\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\big(\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\\
&+\|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\\
&+\|\nabla\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}\big)\|\pi^k\|^2_{H^1(\mathbb{T}^d)}. \qquad (3.11)
\end{aligned}$$

Agora, multiplicando-se a equação (3.5) por $\pi^k_t$ e integrando-se o resultado sobre $\mathbb{T}^d$, obtem-se que:

$$\begin{aligned}
\frac{\nu}{2}\frac{d}{dt}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k_t\|^2_{L^2(\mathbb{T}^d)} &= (\theta^k.\nabla n, \pi^k_t) + (\mathbf{E}^k.\nabla n, \pi^k_t) + (\mathbf{u}^k.\nabla\pi^k, \pi^k_t)\\
&\leq C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}\\
&\quad +\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + 3\delta_2\|\pi^k_t\|^2_{L^2(\mathbb{T}^d)}. \qquad (3.12)
\end{aligned}$$

Escolhendo-se os valore de $\delta_1 = \alpha/70$ e $\delta_2 = 1/6$ e multiplicando-se a desigualdade diferencial (3.11) por $1/(4C^*)$ e somando-se as desigualdades diferenciais (3.7), (3.11) e (3.12) e definindo-se

$$\begin{aligned}
&\beta(t) = C\big(\|\nabla\Delta n^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}\\
&+\|\nabla\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}\big)
\end{aligned}$$

finalmente obtemos,

$$\begin{aligned}
&\frac{d}{dt}\big(\|\sqrt{n^k}\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k\|^2_{H^1(\mathbb{T}^d)}\big) + \|\theta^k_t\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k_t\|^2_{L^2(\mathbb{T}^d)} \leq C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}\\
&+C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\beta(t)\big(\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k\|^2_{H^1(\mathbb{T}^d)}\big).
\end{aligned}$$

■

**Lema 3.1.3** *São verificadas as seguintes desigualdades:*

$$(i)\qquad \|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \leq C\big(\|\pi^k_t\|^2_{L^2(\mathbb{T}^d)} + \|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\big)$$

$$(ii)\qquad \frac{d}{dt}\|\pi^k_t\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k_t\|^2_{L^2(\mathbb{T}^d)} \leq C\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\big(\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\mathbf{u}^k_t\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi^k_t\|^2_{L^2(\mathbb{T}^d)},$$

*e mais ainda,* $\|\pi^k_t(0)\|_{L^2(\mathbb{T}^d)} \leq C\|\mathbf{E}^k(0)\|_{L^2(\mathbb{T}^d)}$.

$$(iii)\quad \|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\nabla\pi^k_t\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}.$$

$$(iv)\qquad \frac{d}{dt}\|\nabla\pi^k_t\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\pi^k_t\|^2_{L^2(\mathbb{T}^d)} \leq C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)} + C\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\big(\|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\big)\|\nabla\pi^k_t\|^2_{L^2(\mathbb{T}^d)},$$

*e mais ainda,* $\|\nabla\pi^k_t(0)\|^2_{L^2(\mathbb{T}^d)} \leq C\|\nabla\mathbf{E}^k(0)\|^2_{L^2(\mathbb{T}^d)}$.

**Demonstração:**

**(i)** Considere a equação (3.5),

$$\nu\Delta\pi^k = \pi^k_t + \theta^k.\nabla n + \mathbf{E}^k.\nabla n + \mathbf{u}^k.\nabla\pi^k,$$

aplicando-se a norma $L^2(\mathbb{T}^d)$, a tal equação, tem-se:

$$\begin{aligned}
\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} &= \frac{1}{\nu}\|\pi^k_t + \theta^k.\nabla n + \mathbf{E}^k.\nabla n + \mathbf{u}^k.\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\big(\|\pi^{k2}_{\|}{}_{L^2(\mathbb{T}^d)} + \|\theta^k.\nabla n\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{E}^k.\nabla n\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{u}^k.\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\big)\\
&\leq C\big(\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla n\|^2_{L^\infty(\mathbb{T}^d)}\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla n\|^2_{L^\infty(\mathbb{T}^d)}\|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}\\
&\quad +\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\big)\\
&\leq C\big(\|\pi^k_t\|^2_{L^2(\mathbb{T}^d)} + \|\theta^k\|^2_{L^2(\mathbb{T}^d)} + \|\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\big).
\end{aligned}$$

**(ii)** Derivando-se a equação (3.5) em relação a variável temporal, multiplicando-a por $\pi^k_t$ e integrando-a sobre $\mathbb{T}^d$, obtem-se;

$$\frac{1}{2}\frac{d}{dt}\|\pi^k_t\|^2_{L^2(\mathbb{T}^d)} + \nu\|\nabla\pi^k_t\|^2_{L^2(\mathbb{T}^d)} = -(\theta^k_t.\nabla n, \pi^k_t) - (\theta^k.\nabla n_t, \pi^k_t) - (\mathbf{E}^k_t.\nabla n, \pi^k_t)$$
$$-(\mathbf{E}^k.\nabla n_t, \pi^k_t) - (\mathbf{u}^k_t.\nabla\pi^k, \pi^k_t) - (\mathbf{u}^k \cdot \nabla\pi^k_t, \pi^k_t),$$

note-se que $(\mathbf{u}^k \cdot \nabla\pi_t^k, \pi_t^k) = \frac{1}{2}(\mathbf{u}^k, \nabla|\pi_t^k|^2) = \frac{1}{2}(\text{div } \mathbf{u}^k, |\pi_t^k|^2) = 0$, estimando os termos a direita da equação acima, como se segue:

$$\begin{aligned} |(\theta_t^k.\nabla n, \pi_t^k)| &\leq \|\theta_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\pi_t^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi_t^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

$$\begin{aligned} |(\theta^k.\nabla n_t, \pi_t^k)| &\leq \|\theta^k\|_{L^4(\mathbb{T}^d)}\|\nabla n_t\|_{L^4(\mathbb{T}^d)}\|\pi_t^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\pi_t^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

$$\begin{aligned} |(\mathbf{E}_t^k.\nabla n, \pi_t^k)| &\leq \|\mathbf{E}_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\pi_t^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\mathbf{E}_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)}\|\pi_t^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

$$\begin{aligned} |(\mathbf{E}^k.\nabla n_t, \pi_t^k)| &\leq \|\mathbf{E}^k\|_{L^4(\mathbb{T}^d)}\|\nabla n_t\|_{L^4(\mathbb{T}^d)}\|\pi_t^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\pi_t^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

$$\begin{aligned} |(\mathbf{u}_t^k.\nabla\pi^k, \pi_t^k)| &\leq \|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}\|\nabla\pi^k\|_{L^4(\mathbb{T}^d)}\|\pi_t^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\|\pi_t^k\|^2_{L^2(\mathbb{T}^d)}. \end{aligned}$$

Assim, obtem-se que:

$$\begin{aligned} &\frac{d}{dt}\|\pi_t^k\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}_t^k\|^2_{L^2(\mathbb{T}^d)} \\ &C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\big(\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t\|^2_{L^2(\mathbb{T}^d)} \\ &+\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi_t^k\|^2_{L^2(\mathbb{T}^d)}. \end{aligned}$$

Além disso, a partir da equação (3.5) tem-se que,

$$\begin{aligned} \|\pi_t^k\|_{L^2(\mathbb{T}^d)} &\leq \|\mathbf{E}^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)} + \|\theta^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)} \\ &+\|\mathbf{u}^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)} + \nu\|\Delta\pi^k\|_{L^2(\mathbb{T}^d)} \\ &\leq C\|\mathbf{E}^k\|_{L^2(\mathbb{T}^d)} + C\|\theta^k\|_{L^2(\mathbb{T}^d)} + C\|\nabla\pi^k\|_{L^2(\mathbb{T}^d)} + C\|\Delta\pi^k\|_{L^2(\mathbb{T}^d)}, \end{aligned}$$

e dado que,

$$\|\theta^k(0)\|_{L^2(\mathbb{T}^d)} = \|\nabla\pi^k(0)\|_{L^2(\mathbb{T}^d)} = \|\Delta\pi^k(0)\|_{L^2(\mathbb{T}^d)} = 0.$$

então, conclui-se que

$$\|\pi_t^k(0)\|_{L^2(\mathbb{T}^d)} \leq C\|\mathbf{E}^k(0)\|_{L^2(\mathbb{T}^d)}.$$

**(iii)** Aplicando-se o operador $\nabla$ a equação (3.5) obtem-se:

$$\nu\nabla\Delta\pi^k = \nabla\pi_t^k + \nabla\theta^k.\nabla n + \theta^k.\nabla^2 n + \nabla\mathbf{E}^k.\nabla n + \mathbf{E}^k.\nabla^2 n + \nabla\mathbf{u}^k.\nabla\pi^k + \mathbf{u}^k.\nabla^2\pi^k,$$

tirando a norma de $L^2(\mathbb{T}^d)$ na equação acima,

$$\begin{aligned}
&\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\theta^k.\nabla n\|^2_{L^2(\mathbb{T}^d)} + \|\theta^k.\nabla^2 n\|^2_{L^2(\mathbb{T}^d)}\\
&+C\|\nabla\mathbf{E}^k.\nabla n\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}^k.\nabla^2 n\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{u}^k.\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{u}^k.\nabla^2\pi^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)} + C\|\theta^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2 n\|^2_{L^4(\mathbb{T}^d)}\\
&+C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)} + C\|\mathbf{E}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2 n\|^2_{L^4(\mathbb{T}^d)}\\
&+C\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^4(\mathbb{T}^d)} + C\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

**(iv)** Derivando a equação (3.5) em relação a variável temporal, aplicando o operador $\nabla$, multiplicando por $\nabla\pi_t^k$ e integrando sobre $\mathbb{T}^d$, obtemos;

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)} + \nu\|\Delta\pi_t^k\|^2_{L^2(\mathbb{T}^d)} = -(\nabla\theta_t^k.\nabla n, \nabla\pi_t^k) - (\theta_t^k.\nabla^2 n, \nabla\pi_t^k)\\
&-(\nabla\theta^k.\nabla n_t, \nabla\pi_t^k) - (\theta^k.\nabla n_t, \nabla\pi_t^k) - (\nabla\mathbf{E}_t^k.\nabla n, \nabla\pi_t^k) - (\mathbf{E}^k.\nabla^2 n_t, \nabla\pi_t^k)\\
&-(\nabla\mathbf{u}_t^k.\nabla\pi^k, \nabla\pi_t^k) - (\mathbf{u}_t^k.\nabla^2\pi^k, \nabla\pi_t^k) - (\nabla\mathbf{u}^k\cdot\nabla\pi_t^k, \nabla\pi_t^k) - (\mathbf{u}^k\cdot\nabla^2\pi_t^k, \nabla\pi_t^k)\\
&-(\mathbf{E}_t^k\cdot\nabla^2 n, \nabla\pi^k) - (\nabla\mathbf{E}^k\cdot\nabla n_t, \nabla\pi^k),
\end{aligned}$$

estimando os termos a direita da equação acima, como se segue:

$$(\mathbf{u}^k\cdot\nabla^2\pi_t^k, \nabla\pi_t^k) = \frac{1}{2}(\mathbf{u}^k, \nabla|\nabla\pi_t^k|^2) - \frac{1}{2}(\mathrm{div}\ \mathbf{u}^k, |\nabla\pi_t^k|^2) = 0;$$

$$\begin{aligned}
|(\nabla\mathbf{u}^k\cdot\nabla\pi_t^k, \nabla\pi_t^k)| &\leq \|\nabla\mathbf{u}^k\|_{L^4(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^4(\mathbb{T}^d)}\\
&\leq C\|\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\pi_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(\nabla\theta_t^k.\nabla n, \nabla\pi_t^k)| &= |(\theta_t^k, div(\nabla n\otimes\nabla\pi_t^k))|\\
&\leq |(\theta_t^k, \nabla n\Delta\pi_t^k)| + |(\theta_t^k, \nabla^2 n.\nabla\pi_t^k)|\\
&\leq \|\theta_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\Delta\pi_t^k\|_{L^2(\mathbb{T}^d)}\\
&\quad+\|\theta_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\pi_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(\theta_t^k.\nabla^2 n, \nabla\pi_t^k)| &\leq \|\theta_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla^2 n\|_{L^\infty(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}|(\nabla\theta^k.\nabla n_t,\nabla\pi_t^k)| &\leq \|\nabla\theta^k\|_{L^2(\mathbb{T}^d)}\|\nabla n_t\|_{L^4(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C\|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\pi_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\theta^k.\nabla^2 n_t,\nabla\pi_t^k)| &\leq C\|\theta^k\|_{L^4(\mathbb{T}^d)}\|\Delta n_t\|_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C\|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\pi_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\nabla\mathbf{E}_t^k.\nabla n,\nabla\pi_t^k)| &= |(\mathbf{E}_t^k, div(\nabla n\otimes\nabla\pi_t^k)|\\ &\leq |(\mathbf{E}_t^k,\nabla n\Delta\pi_t^k)|+|(\mathbf{E}_t^k,\nabla^2 n.\nabla\pi_t^k)|\\ &\leq \|\mathbf{E}_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla n\|_{L^\infty(\mathbb{T}^d)}\|\Delta\pi_t^k\|_{L^2(\mathbb{T}^d)}\\ &\quad+\|\mathbf{E}_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla^2 n\|_{L^4(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C\|\mathbf{E}_t^k\|^2_{L^2(\mathbb{T}^d)}+2\delta_1\|\Delta\pi_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\mathbf{E}^k.\nabla^2 n_t,\nabla\pi_t^k)| &\leq \|\mathbf{E}^k\|_{L^4(\mathbb{T}^d)}\|\Delta n_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^4(\mathbb{T}^d)}\\ &\leq C\|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\pi_t^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\nabla\mathbf{u}_t^k.\nabla\pi^k,\nabla\pi_t^k)| &\leq \|\nabla\mathbf{u}_t^k\|_{L^2(\mathbb{T}^d)}\|\nabla\pi^k\|_{L^\infty(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\quad+C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\mathbf{u}_t^k.\nabla^2\pi^k,\nabla\pi_t^k)| &\leq \|\mathbf{u}_t^k\|_{L^4(\mathbb{T}^d)}\|\Delta\pi^k\|_{L^4(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\mathbf{u}_t^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\quad+C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\nabla\mathbf{E}^k.\nabla n_t,\nabla\pi_t^k)| &\leq \|\nabla\mathbf{E}^k\|_{L^4(\mathbb{T}^d)}\|\nabla n_t\|_{L^4(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}+C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}|(\mathbf{E}_t^k.\nabla^2 n,\nabla\pi_t^k)| &\leq C\|\mathbf{E}_t^k\|_{L^2(\mathbb{T}^d)}\|\Delta n\|_{L^\infty(\mathbb{T}^d)}\|\nabla\pi_t^k\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|\mathbf{E}_t^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

Escolhendo-se o valor de $\delta_1$ de forma conveniente, obtem-se que;

$$\frac{d}{dt}\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}+\|\Delta\pi_t^k\|^2_{L^2(\mathbb{T}^d)}\leq C\|\mathbf{E}_t^k\|^2_{L^2(\mathbb{T}^d)}+C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}$$

$$+C\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla \mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\big(\|\Delta^2 n\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta \mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\big)\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)}.$$

Note-se que

$$\nabla\pi_t^k = \nu\nabla\Delta\pi^k - \nabla\theta^k.\nabla n - \theta^k.\nabla^2 n - \nabla\mathbf{E}^k.\nabla n - \mathbf{E}^k.\nabla^2 n - \nabla\mathbf{u}^k.\nabla\pi^k - \mathbf{u}^k.\nabla^2\pi^k,$$

e assim, aplicando-se a norma $L^2(\mathbb{T}^d)$ a tal equação, segue que:

$$\begin{aligned}
&\|\nabla\pi_t^k\|^2_{L^2(\mathbb{T}^d)} \le C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\theta^k.\nabla n\|^2_{L^2(\mathbb{T}^d)} + C\|\theta^k.\nabla^2 n\|^2_{L^2(\mathbb{T}^d)}\\
&+C\|\nabla\mathbf{E}^k.\nabla n\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}^k.\nabla^2 n\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{u}^k.\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{u}^k.\nabla^2\pi^k\|^2_{L^2(\mathbb{T}^d)}\\
&\le C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

Finalmente,

$$\|\nabla\Delta\pi^k(0)\|^2_{L^2(\mathbb{T}^d)} = \|\Delta\pi^k(0)\|^2_{L^2(\mathbb{T}^d)} = \|\nabla\theta^k(0)\|^2_{L^2(\mathbb{T}^d)} = 0,$$

pode-se concluir que

$$\|\nabla\pi_t^k(0)\|^2_{L^2(\mathbb{T}^d)} \le C\|\nabla\mathbf{E}^k(0)\|^2_{L^2(\mathbb{T}^d)}.$$

■

**Lema 3.1.4** *Tem-se válida a seguinte desigualdade:*

$$\begin{aligned}
\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)} \le\ & C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\big(\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)} + \|\nabla\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi^k\|^2_{H^1(\mathbb{T}^d)} + C\|\theta_t^k\|^2_{L^2(\mathbb{T}^d)}\\
&+C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\mathbf{E}_t^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\pi^k\|^2_{H^1(\mathbb{T}^d)}.
\end{aligned}$$

**Demonstração:**

Considerando-se $\mathbf{v}^k = -\Delta\theta^k$ na equação (3.2), isolando-se o termo $\nu(n^k\Delta\theta^k, \Delta\theta^k)$ e estimando-se os demais termos, obtem-se que:

$$|(\nabla\Delta\pi^k, \Delta\theta^k)| \le C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}
\left|\left(\frac{1}{(n^k)}\Delta n^k\nabla\pi^k, \Delta\theta^k\right)\right| &\le C\|\Delta n^k\|^2_{L^4(\mathbb{T}^d)}\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\le C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\left|\left(\frac{1}{(n^k)}\Delta\pi^k\nabla n, \Delta\theta^k\right)\right|^2 \le C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}$$

$$\leq C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};$$

$$\begin{aligned}\left|\left(\frac{\pi^k}{n^k n}\Delta n\nabla n, \Delta\theta^k\right)\right| &\leq C\|\pi^k\|^2_{L^4(\mathbb{T}^d)}\|\Delta n\|^2_{L^4(\mathbb{T}^d)}\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\pi^k\|^2_{H^1(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\left|\left(\frac{1}{(n^k)^2}(\nabla\pi^k.\nabla n)\nabla n, \Delta\theta^k\right)\right| &\leq C\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n\|^4_{L^\infty(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla\pi^k)\nabla n, \Delta\theta^k\right)\right| &\leq C\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\left|\left(\frac{1}{(n^k)^2}(\nabla n^k.\nabla n^k)\nabla\pi^k, \Delta\theta^k\right)\right| &\leq C\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n^k\|^4_{L^\infty(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\left|\left(\frac{n+n^k}{(nn^k)^2}\pi^k(\nabla n\nabla n)\nabla n, \Delta\theta^k\right)\right| &\leq C\|\pi^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n\|^6_{L^\infty(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\left|\left(\frac{1}{n^k}(\nabla n^k.\nabla)\nabla\pi^k, \Delta\theta^k\right)\right| &\leq C\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\left|\left(\frac{1}{n^k}(\nabla\pi^k.\nabla)\nabla n, \Delta\theta^k\right)\right| &\leq C\|\nabla\pi^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2 n\|^2_{L^4(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\left|\left(\frac{\pi^k}{nn^k}(\nabla n.\nabla)\nabla n, \Delta\theta^k\right)\right| &\leq C\|\pi^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla^2 n\|^2_{L^4(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\nabla\Delta n\|^4_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\pi^k\|^2_{H^1(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}\left|\left(\pi^k\mathbf{f}, \Delta\theta^k\right)\right| &\leq C\|\pi^k\|^2_{L^4(\mathbb{T}^d)}\|\mathbf{f}\|^2_{L^4(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\ &\leq C\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)} + \delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};\end{aligned}$$

$$\begin{aligned}
\big|\big((\nabla n^k.\nabla)\theta^k,\Delta\theta^k\big)\big| &\leq C\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\big|\big((\nabla n^k.\nabla)\mathbf{E}^k,\Delta\theta^k\big)\big| &\leq C\|\nabla n^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\big|\big((\nabla\pi^k.\nabla)\mathbf{u},\Delta\theta^k\big)\big| &\leq C\|\nabla\pi^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}\|^2_{L^4(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\big|\big(\pi^k\Delta\mathbf{u},\Delta\theta^k\big)\big| &\leq C\|\pi^k\|^2_{L^\infty(\mathbb{T}^d)}\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\pi^k\|^2_{H^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k\mathbf{u}^k.\nabla\theta^k,\Delta\theta^k)| &\leq C\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k\mathbf{u}^k.\nabla\mathbf{E}^k,\Delta\theta^k)| &\leq C\|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k\theta^k.\nabla\mathbf{u},\Delta\theta^k)| &\leq C\|\theta^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}\|^2_{L^4(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k\mathbf{E}^k.\nabla\mathbf{u},\Delta\theta^k)| &\leq C\|\mathbf{E}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\mathbf{u}\|^2_{L^4(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\big|\big((\pi^k\mathbf{u}.\nabla)\mathbf{u}\big)\big| &\leq C\|\pi^k\|^2_{L^4(\mathbb{T}^d)}\|\mathbf{u}\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\mathbf{u}\|^2_{L^4(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\pi^k\|^2_{H^1(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
\big|\big(\pi^k\mathbf{u}_t,\Delta\theta^k\big)\big| &\leq C\|\pi^k\|^2_{L^4(\mathbb{T}^d)}\|\mathbf{u}_t\|^2_{L^4(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}\\
&\leq C\|\nabla\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)}\|\pi^k\|^2_{H^1(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k\theta^k_t,\Delta\theta^k)| &\leq C\|\theta^k_t\|_{L^2(\mathbb{T}^d)}\|\Delta\theta^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)};
\end{aligned}$$

$$\begin{aligned}
|(n^k\mathbf{E}^k_t,\Delta\theta^k)| &\leq C\|\mathbf{E}^k_t\|_{L^2(\mathbb{T}^d)}\|\Delta\theta^k\|_{L^2(\mathbb{T}^d)}\\
&\leq C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)}+\delta_1\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

assim obtemos,

$$\begin{aligned}
\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}+C\big(\|\mathbf{f}\|^2_{H^1(\mathbb{T}^d)}+\|\nabla\mathbf{u}_t\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi^k\|^2_{H^1(\mathbb{T}^d)}+C\|\theta^k_t\|^2_{L^2(\mathbb{T}^d)}\\
&\quad+C\|\nabla\theta^k\|^2_{L^2(\mathbb{T}^d)}+C\|\mathbf{E}^k_t\|^2_{L^2(\mathbb{T}^d)}+C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}+C\|\pi^k\|^2_{H^1(\mathbb{T}^d)}.
\end{aligned}$$

■

**Lema 3.1.5** *Tem-se válida a seguinte desigualdade:*

$$\|\Delta^2\pi^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}.$$

**Demonstração:**

Aplicando-se o operador $\Delta$ à equação (3.5), obtem-se

$$\nu\Delta^2\pi^k = \Delta\pi^k_t + \Delta(\theta^k.\nabla n) + \Delta(\mathbf{E}^k.\nabla n) + \Delta(\mathbf{u}^k.\nabla\pi^k),$$

e então, aplicando-se a norma $L^2(\mathbb{T}^d)$ na equação acima e estimando-se os termos à direita segue-se que:

$$\begin{aligned} \|\Delta(\theta^k.\nabla n)\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)}\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\theta^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2 n\|^2_{L^4(\mathbb{T}^d)} \\ &\quad + C\|\theta^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

$$\begin{aligned} \|\Delta(\mathbf{E}^k.\nabla n)\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}\|\nabla n\|^2_{L^\infty(\mathbb{T}^d)} + C\|\nabla\mathbf{E}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2 n\|^2_{L^4(\mathbb{T}^d)} \\ &\quad + \|\mathbf{E}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta n\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

$$\begin{aligned} \|\Delta(\mathbf{u}^k.\nabla\pi^k)\|^2_{L^2(\mathbb{T}^d)} &\leq C\|\Delta\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla\pi^k\|^2_{L^4(\mathbb{T}^d)} + C\|\nabla\mathbf{u}^k\|^2_{L^4(\mathbb{T}^d)}\|\nabla^2\pi^k\|^2_{L^4(\mathbb{T}^d)} \\ &\quad + \|\mathbf{u}^k\|^2_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \\ &\leq C\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}; \end{aligned}$$

portanto, conclui-se que:

$$\|\Delta^2\pi^k\|^2_{L^2(\mathbb{T}^d)} \leq C\|\Delta\theta^k\|^2_{L^2(\mathbb{T}^d)} + C\|\Delta\mathbf{E}^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} + C\|\nabla\Delta\mathbf{u}^k\|^2_{L^2(\mathbb{T}^d)}\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)}.$$

■

## 3.2 Estimativas de Erro Local e Global

Nesta seção é apresentado um estudo detalhado e rigoroso tratando da análise de erro envolvendo as aproximações semi-Galerkin espectrais (e suas respectivas derivadas temporais). Conforme a ser mostrado na sequência, taxas de erro em diversas normas são obtidas para as aproximações da velocidade e da densidade com ordem de convergência 1/2, e impondo-se alguma regularidade extra aos dados iniciais do problema se obtêm ordem de convergência 1. Em particular obtemos uma convergência na norma $L^2$ da ordem

1. Segundo o nosso conhecimento, relativamente às clássicas equações de Navier-Stokes (caso particular do sistema de Navier-Stokes quântico, em que a densidade e constante), o melhor resultado obtido até o presente momento é uma convergência na norma $L^2$ da ordem 3/4, estabelecido por Boldrini e Rojas-Medar [6], para o sistema de Navier-Stokes não-homogêneo (que se aplica ao caso das equações clássicas).

É importante mencionar que os trabalhos de Damázio e Rojas-Medar para as equações de fluidos com fenômenos de difusão [32] , [33] e o trabalho para o sistema de Navier-Stokes não-homogêneo de Boldrini e Rojas-Medar [6] consideram condições de contorno de Dirichlet para a velocidade e de Neumann para a densidade as quais podem causar dificuldades adicionais.

Para enfatizar os efeitos dos dados sobre as taxas de convergência, consideram-se as seguintes hipóteses:

$$(H1)\ \mathbf{u}_0 \in H^2(\mathbb{T}^d),\ n_0 \in H^3(\mathbb{T}^d),\ \mathbf{f} \in L^2(0,T^*;H^1(\mathbb{T}^d)) \text{ e } \mathbf{f}_t \in L^2(0,T^*;L^2(\mathbb{T}^d));$$
$$(H2)\ \mathbf{u}_0 \in H^3(\mathbb{T}^d),\ n_0 \in H^4(\mathbb{T}^d),\ \mathbf{f} \in L^2(0,T^*;H^2(\mathbb{T}^d)) \text{ e } \mathbf{f}_t \in L^2(0,T^*;H^1(\mathbb{T}^d)).$$

**Teorema 3.2.1** *Sejam* $(\mathbf{u}, n)$ *a solução dado pelo Teorema (2.3.1) e* $(\mathbf{u}^k, n^k)$ *a solução aproximada; então, para todo* $t \in [0, T^*]$, *tem-se que*

$$\|\mathbf{u}(t)-\mathbf{u}^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|n(t)-n^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla(n-n^k)(t)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\int_0^t (\|\nabla(\mathbf{u}-\mathbf{u}^k)(s))\|^2_{L^2(\mathbb{T}^d)} + \|\nabla(n-n^k)(s)\|^2_{L^2(\mathbb{T}^d)} \tag{3.13}$$
$$+\|\Delta(n-n^k)(s)\|^2_{L^2(\mathbb{T}^d)})ds \leq \frac{C}{\lambda^j_{k+1}} \tag{3.14}$$

*com* $j = 1$, *considerando-se as hipóteses de* $(H1)$, *e* $j = 2$, *para as hipóteses de* $(H2)$.

**Demonstração:**

Integrando-se de 0 a $t$, com $t \in [0, T^*]$ a desigualdade diferencial dado pelo Lema (3.1.1), obtem-se que:

$$\|\sqrt{n^k(t)}\theta^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+\int_0^t \|\nabla\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + \int_0^t \|\Delta\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C\int_0^t \|\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+C\int_0^t \|\nabla\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + C\int_0^t \|\mathbf{E}^k_t(s)\|^2_{L^2(\mathbb{T}^d)} ds + C\int_0^t \beta(s)\big(\|\theta^k(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds.$$

Considerando-se as hipóteses de $(H1)$, tem-se, pelo Lema (1.1.4),

$$\|\theta^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(t)\|^2_{L^2(\mathbb{T}^d)} \|\nabla\pi^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + \int_0^t \|\nabla\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+ \int_0^t \|\Delta\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} \leq \frac{C}{\lambda_{k+1}} \int_0^t \big(\|\nabla\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)}\big) ds$$
$$+C \int_0^t \beta(s)\big(\|\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k(s)\|^2_{L^2(\mathbb{T}^d)}\big) ds$$
$$\leq \frac{C}{\lambda_{k+1}} + C \int_0^t \beta(s)\big(\|\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k(s)\|^2_{L^2(\mathbb{T}^d)}\big) ds.$$

Em seguida, aplicando-se o Lema de Gronwall (1.2.9), obtem-se que:

$$\|\theta^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+ \int_0^t \|\nabla\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + \int_0^t \|\Delta\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$\leq \frac{C}{\lambda_{k+1}} \left(1 + \int_0^t \beta(s) ds\right) \exp\left(\int_0^t \beta(s) ds\right) \leq \frac{C}{\lambda_{k+1}}.$$

Por sua vez, considerando-se as hipóteses $(H2)$, pelo Lema (1.1.5), segue-se que:

$$\|\theta^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+ \int_0^t \|\nabla\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + \int_0^t \|\Delta\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \frac{C}{\lambda^2_{k+1}} \int_0^t \big(\|\Delta\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\Delta\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)}\big) ds + C \int_0^t \beta(s)\big(\|\theta^k(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k(s)\|^2_{L^2(\mathbb{T}^d)}\big) ds \leq \frac{C}{\lambda^2_{k+1}} + C \int_0^t \beta(s)\big(\|\theta^k(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\pi^k(s)\|^2_{L^2(\mathbb{T}^d)}\big) ds,$$

e o resultado segue, usando-se o Lema de Gronwall (1.2.9).

Finalmente, dado que $\mathbf{u} - \mathbf{u}^k = \mathbf{E}^k + \theta^k$, então aplicando-se a desigualdade triangular e os Lemas (1.1.4) e (1.1.5) se conclui a demonstração.

$\blacksquare$

**Teorema 3.2.2** *Sejam* $(\mathbf{u}, n)$ *a solução dado pelo Teorema (2.3.1) e* $(\mathbf{u}^k, n^k)$ *a solução aproximada; então, para todo* $t \in [0, T^*]$, *tem-se que:*

$$\begin{aligned} &\|\nabla(\mathbf{u}-\mathbf{u}^k)(t)\|^2_{L^2(\mathbb{T}^d)} + \|(n-n^k)(t)\|^2_{H^1(\mathbb{T}^d)} \\ &+\int_0^t (\|(n_t-n_t^k)(s)\|^2_{L^2(\mathbb{T}^d)} + \|(\mathbf{u}_t-\mathbf{u}_t^k)(s)\|^2_{L^2(\mathbb{T}^d)})ds \le \frac{C}{\lambda^j_{k+1}} \end{aligned} \quad (3.15)$$

*com* $j = 1$, *considerando-se as hipóteses de* $(H1)$, *e* $j = 2$ *para as hipóteses de* $(H2)$.

**Demonstração:**

Integrando-se de 0 a $t$, com $t \in [0, T^*]$ a desigualdade diferencial dado pelo Lema (3.1.2), obtem-se que:

$$\begin{aligned} &\|\sqrt{n^k(t)}\nabla\theta^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(t)\|^2_{H^1(\mathbb{T}^d)} + \int_0^t \|\theta_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + \int_0^t \|\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \\ &\le C\int_0^t \|\mathbf{E}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \|\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \|\nabla\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \\ &+C\int_0^t \|\Delta\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \beta(s)\big(\|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(s)\|^2_{H^1(\mathbb{T}^d)}\big)ds, \end{aligned}$$

e dai,

$$\begin{aligned} &\|\nabla\theta^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(t)\|^2_{H^1(\mathbb{T}^d)} + \int_0^t \|\theta_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + \int_0^t \|\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \\ &\le C\int_0^t \|\mathbf{E}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \|\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \|\nabla\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \\ &+C\int_0^t \|\Delta\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \beta(s)\big(\|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(s)\|^2_{H^1(\mathbb{T}^d)}\big)ds. \end{aligned}$$

Sob as hipóteses de $(H1)$, e usando-se o Lema (1.1.4), segue-se que:

$$\begin{aligned} &\|\nabla\theta^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(t)\|^2_{H^1(\mathbb{T}^d)} + \int_0^t \|\theta_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + \int_0^t \|\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \\ &\le \frac{C}{\lambda_{k+1}}\int_0^t \big(\|\nabla\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds \\ &+C\int_0^t \beta(s)\big(\|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(s)\|^2_{H^1(\mathbb{T}^d)}\big)ds \\ &\le \frac{C}{\lambda_{k+1}} + C\int_0^t \beta(s)\big(\|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(s)\|^2_{H^1(\mathbb{T}^d)}\big)ds \end{aligned}$$

Uma aplicação do Lema de Gronwall (1.2.9), fornece o resultado para $j = 1$.

Por outro lado, utilizando-se as hipóteses $(H2)$, e o Lema (1.1.5), obtem-se que:

$$\|\nabla\theta^k(t)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(t)\|^2_{H^1(\mathbb{T}^d)} + \int_0^t \|\theta_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + \int_0^t \|\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$

$$\leq \frac{C}{\lambda_{k+1}^2}\int_0^t \big(\|\Delta \mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta \mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^2 \mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds$$
$$+C\int_0^t \beta(s)\big(\|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(s)\|^2_{H^1(\mathbb{T}^d)}\big)ds$$
$$\leq \frac{C}{\lambda_{k+1}^2} + C\int_0^t \beta(s)\big(\|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} + \|\pi^k(s)\|^2_{H^1(\mathbb{T}^d)}\big)ds.$$

Para concluir a prova, basta aplicar o Lema de Gronwall (1.2.9) e usando a desigualdade triangular para $\mathbf{u} - \mathbf{u}^k = \mathbf{E}^k + \theta^k$, e os Lemas (1.1.4) e (1.1.5).

■

**Teorema 3.2.3** *Sejam* $(\mathbf{u}, n)$ *a solução dado pelo Teorema (2.3.1) e* $(\mathbf{u}^k, n^k)$ *a solução aproximada; então, para todo* $t \in [0, T^*]$, *tem-se que:*

$$\|(n_t - n_t^k)(t)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta(n - n^k)(t)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\int_0^t \|\nabla(n_t - n_t^k)(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq \frac{C}{\lambda_{k+1}^j}. \tag{3.16}$$

*com* $j = 1$, *considerando-se as hipóteses de* $(H1)$, *e* $j = 2$ *para as hipóteses de* $(H2)$.

**Demonstração:**

Considerando-se a desigualdade diferencial $(ii)$ dada pelo Lema (3.1.3), integrando-se de 0 a $t$, com $t \in [0, T^*]$ segue-se que:

$$\|\pi_t^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq \|\pi_t^k(0)\|^2_{L^2(\mathbb{T}^d)} + C\int_0^t \|\theta_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$+C\int_0^t \|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \|\mathbf{E}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \|\nabla\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$+C\int_0^t \|\Delta\pi^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \big(\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds.$$

Em seguida, fazendo-se uso dos Teoremas (3.2.1) e (3.2.2) e Lema (1.1.4) e das hipóteses de $(H1)$, segue-se que:

$$\|\pi_t^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq \frac{C}{\lambda_{k+1}}\|\nabla\mathbf{u}_0\|^2_{L^2(\mathbb{T}^d)} + \frac{C}{\lambda_{k+1}}$$
$$+\frac{C}{\lambda_{k+1}}\int_0^t (\|\nabla\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}(s)\|\|^2_{L^2(\mathbb{T}^d)})ds + C\int_0^t \big(\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}_t^k(s)\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$

$$\leq \frac{C}{\lambda_{k+1}} + C\int_0^t \big(\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi^k_t(s)\|^2_{L^2(\mathbb{T}^d)} ds,$$

e aplicando-se o Lema de Gronwall (1.2.9), conclui-se que:

$$\|\pi^k_t(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\pi^k_t(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \frac{C}{\lambda_{k+1}}\Big(1 + C\int_0^t \big(\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds\Big) \exp\Big(C\int_0^t \big(\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds\Big)$$
$$\leq \frac{C}{\lambda_{k+1}}.$$

Além disso, a partir da desigualdade $(i)$ do Lema (3.1.3) obtem-se que:

$$\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \leq \frac{C}{\lambda_{k+1}},$$

concluindo-se o resultado para $j = 1$.

Ainda, sob as hipóteses de $(H2)$, temos pelos Teoremas (3.2.1) e (3.2.2) e o Lema (1.1.5), obtem-se a desigualdade

$$\|\pi^k_t(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\pi^k_t(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \frac{C}{\lambda^2_{k+1}}\|\Delta\mathbf{u}_0\|^2_{L^2(\mathbb{T}^d)} + \frac{C}{\lambda^2_{k+1}}$$
$$+\frac{C}{\lambda^2_{k+1}}\int_0^t (\|\Delta\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta\mathbf{u}(s)\|\|^2_{L^2(\mathbb{T}^d)})ds + C\int_0^t \big(\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi^k_t(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$\leq \frac{C}{\lambda^2_{k+1}} + C\int_0^t \big(\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi^k_t(s)\|^2_{L^2(\mathbb{T}^d)} ds,$$

e aplicando-se o Lema de Gronwall (1.2.9), obtem-se que:

$$\|\pi^k_t(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\pi^k_t(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \frac{C}{\lambda^2_{k+1}}\Big(1 + C\int_0^t \big(\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds\Big) \exp\Big(C\int_0^t \big(\|\nabla\Delta n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla \mathbf{u}^k_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds\Big)$$
$$\leq \frac{C}{\lambda^2_{k+1}}.$$

Finalmente, da desigualdade $(i)$ do Lema (3.1.3) segue-se que

$$\|\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \leq \frac{C}{\lambda^2_{k+1}}.$$

■

**Teorema 3.2.4** *Sejam* $(\mathbf{u}, n)$ *a solução dado pelo Teorema (2.3.1) e* $(\mathbf{u}^k, n^k)$ *a solução aproximada; então, para todo* $t \in [0, T^*]$, *tem-se que:*

$$\|\nabla(n_t - n_t^k)(t)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta(n - n^k)(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\nabla\Delta(n - n^k)(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+ \int_0^t \|\Delta(n_t - n_t^k)(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \frac{C}{\lambda^j_{k+1}}. \tag{3.17}$$

*com* $j = 1$, *considerando-se as hipóteses de* $(H1)$, *e* $j = 2$ *para as hipóteses de* $(H2)$.

**Demonstração:**

Utilizando-se a desigualdade diferencial $(iv)$ dada pelo Lema (3.1.3), e integrando-se tal desigualdade de 0 a $t$, com $t \in [0, T^*]$, segue-se que:

$$\|\nabla\pi_t^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\Delta\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq C\int_0^t \|\mathbf{E}_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + C\int_0^t \|\nabla\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+C\int_0^t \|\Delta\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + C\int_0^t \|\theta_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + C\int_0^t \|\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+C\int_0^t \|\Delta\pi^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + C\int_0^t \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} \|\nabla\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+C\int_0^t \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} \|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)} ds + \|\nabla\pi_t^k(0)\|^2_{L^2(\mathbb{T}^d)}$$
$$+C\int_0^t \left(\|\Delta^2 n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}\right) \|\nabla\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$

Nas condições enunciadas em $(H1)$, e considerando-se os Teoremas (3.2.1) e (3.2.2) e o Lema (1.1.4), segue-se que:

$$\|\nabla\pi_t^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\Delta\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \frac{C}{\lambda_{k+1}} \int_0^t \left(\|\nabla\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)}\right.$$
$$\left.+\|\Delta\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)}\right) ds + \frac{C}{\lambda_{k+1}} \int_0^t \|\nabla\Delta\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)} ds + \frac{C}{\lambda_{k+1}} + \frac{C}{\lambda_{k+1}} \int_0^t \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+\frac{C}{\lambda_{k+1}} \|\Delta\mathbf{u}_0\|^2_{L^2(\mathbb{T}^d)} + \frac{C}{\lambda_{k+1}} \int_0^t \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} ds$$
$$+C\int_0^t \left(\|\Delta^2 n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta\mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}\right) \|\nabla\pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)} ds$$

$$\le \frac{C}{\lambda_{k+1}} + C\int_0^t \big(\|\Delta^2 n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}\big)\|\nabla \pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$

Basta então, aplicar o Lema de Gronwall (1.2.9), para obter-se:

$$\|\nabla \pi_t^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\Delta \pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \le \frac{C}{\lambda_{k+1}}\bigg(1 + \int_0^t \big(\|\Delta^2 n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds\bigg)\exp\bigg(\int_0^t \big(\|\Delta^2 n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds\bigg)$$
$$\le \frac{C}{\lambda_{k+1}}.$$

Também, a partir da desigualdade $(iii)$ do Lema (3.1.3) segue-se que

$$\int_0^t \|\nabla\Delta\pi^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \le \frac{C}{\lambda_{k+1}} \quad \text{e} \quad \|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \le \frac{C}{\lambda_{k+1}}.$$

Ao se considerarem as hipóteses de $(H2)$, e o Teoremas (3.2.1), (3.2.2) e o Lema (1.1.5), obtem-se que:

$$\|\nabla \pi_t^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\Delta \pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \le \frac{C}{\lambda_{k+1}^2}\int_0^t \big(\|\Delta \mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\nabla\Delta \mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds + \frac{C}{\lambda_{k+1}^2}\int_0^t \|\Delta^2 \mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)}ds + \frac{C}{\lambda_{k+1}^2} + \frac{C}{\lambda_{k+1}^2}\int_0^t \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$+\frac{C}{\lambda_{k+1}^2}\|\nabla\Delta \mathbf{u}_0\|^2_{L^2(\mathbb{T}^d)} + \frac{C}{\lambda_{k+1}^2}\int_0^t \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$+C\int_0^t \big(\|\Delta^2 n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}\big)\|\nabla \pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$
$$\le \frac{C}{\lambda_{k+1}^2} + C\int_0^t \big(\|\Delta^2 n(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}\big)\|\nabla \pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$

O Lema de Gronwall (1.2.9), fornece então a estimativa

$$\|\nabla \pi_t^k(t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^t \|\Delta \pi_t^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \le \frac{C}{\lambda_{k+1}^2}\bigg(1 + \int_0^t \big(\|\Delta^2 n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds\bigg)\exp\bigg(\int_0^t \big(\|\Delta^2 n(s)\|^2_{L^2(\mathbb{T}^d)}$$
$$+\|\Delta n_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta \mathbf{u}^k(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds\bigg) \le \frac{C}{\lambda_{k+1}^2}.$$

Finalmente, da desigualdade $(iii)$ do Lema (3.1.3) segue-se que

$$\int_0^t \|\nabla\Delta\pi^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq \frac{C}{\lambda^2_{k+1}} \quad \text{e} \quad \|\nabla\Delta\pi^k\|^2_{L^2(\mathbb{T}^d)} \leq \frac{C}{\lambda^2_{k+1}}.$$

■

**Teorema 3.2.5** *Sejam* $(\mathbf{u}, n)$ *a solução dado pelo Teorema (2.3.1) e* $(\mathbf{u}^k, n^k)$ *a solução aproximada; então, para todo* $t \in [0, T^*]$, *nos temos*

$$\int_0^t \|\Delta(\mathbf{u} - \mathbf{u}^k)(s)\|^2 ds \leq \frac{C}{\lambda^j_{k+1}}. \tag{3.18}$$

*com* $j = 1$, *considerando-se as hipóteses de* $(H1)$, *e* $j = 2$ *para as hipóteses de* $(H2)$.

**Demonstração:**

A partir da desigualdade dada pelo Lema (3.1.4), e após uma integrando de 0 a $t$, com $t \in [0, T^*]$, obtem-se que

$$\int_0^t \|\Delta\theta^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq C\int_0^t \|\nabla\Delta\pi^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \big(\|\mathbf{f}(s)\|^2_{H^1(\mathbb{T}^d)}$$

$$+\|\nabla\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)\|\pi^k(s)\|^2_{H^1(\mathbb{T}^d)}ds + C\int_0^t \|\theta^k_t(s)\|^2_{L^2(\mathbb{T}^d)} + C\int_0^t \|\nabla\theta^k(s)\|^2_{L^2(\mathbb{T}^d)}ds$$

$$+C\int_0^t \|\mathbf{E}^k_t(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \|\Delta\mathbf{E}^k(s)\|^2_{L^2(\mathbb{T}^d)}ds + C\int_0^t \|\pi^k(s)\|^2_{H^1(\mathbb{T}^d)}ds.$$

Daí, considerando-se as hipóteses de $(H1)$, e fazendo-se uso dos Teoremas (3.2.1), (3.2.2), (3.2.4) e do Lema (1.1.4), obtem-se que:

$$\int_0^t \|\Delta\theta^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq \frac{C}{\lambda_{k+1}} + \frac{C}{\lambda_{k+1}}\int_0^t \big(\|\mathbf{f}(s)\|^2_{H^1(\mathbb{T}^d)} + \|\nabla\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds$$

$$+\frac{C}{\lambda_{k+1}}\int_0^t \big(\|\nabla\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\Delta\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds$$

$$\leq \frac{C}{\lambda_{k+1}}.$$

Por outro lado, considerando-se as hipóteses de $(H2)$, os Teoremas (3.2.1), (3.2.2), (3.2.4) e o Lema (1.1.5), segue-se que:

$$\int_0^t \|\Delta\theta^k(s)\|^2_{L^2(\mathbb{T}^d)}ds \leq \frac{C}{\lambda^2_{k+1}} + \frac{C}{\lambda^2_{k+1}}\int_0^t \big(\|\nabla\mathbf{f}(s)\|^2_{L^2(\mathbb{T}^d)} + \|\nabla\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds$$

$$+\frac{C}{\lambda^2_{k+1}}\int_0^t \big(\|\Delta\mathbf{u}_t(s)\|^2_{L^2(\mathbb{T}^d)} + \|\Delta^2\mathbf{u}(s)\|^2_{L^2(\mathbb{T}^d)}\big)ds$$

$$\frac{C}{\lambda^2_{k+1}}.$$

Para concluir-se a demonstração, basta utilizar a desigualdade desigualdade triangular e os Lemas (1.1.4) e (1.1.5).

■

**Teorema 3.2.6** *Sejam* $(\mathbf{u}, n)$ *a solução dado pelo Teorema (2.3.1) e* $(\mathbf{u}^k, n^k)$ *a solução aproximada, então, para todo* $t \in [0, T^*]$, *tem-se,*

$$\int_0^t \|\Delta^2(n - n^k)(s)\|^2_{L^2(\mathbb{T}^d)} ds \leq \frac{C}{\lambda^j_{k+1}}. \tag{3.19}$$

*com* $j = 1$, *considerando-se as hipóteses de* $(H1)$, *e* $j = 2$, *para as hipóteses de* $(H2)$.

**Demonstração:**

Integrando de 0 a $t$, com $t \in [0, T^*]$ a desigualdade dado pelo Lema (3.1.5), o resultado segue dos Teoremas (3.2.3) e (3.2.6) e dos Lemas (1.1.4) e (1.1.5).

■

**Observação 3.2.1** *Sendo* $(\mathbf{u}, n)$ *a solução dado pelo Teorema (2.4.1) ou (2.4.2) e* $(\mathbf{u}^k, n^k)$ *a solução aproximada, então considerando-se as hipóteses dos Corolários (2.4.1) e (2.4.2) e respectivamente os Corolários (2.4.3) e (2.4.4) pode-se obter todas as Estimativas de Erro desta seção de forma totalmente análoga mas definidas sobre o intervalo* $(0, \infty)$.

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