

UNIVERSIDADE FEDERAL DO PARANÁ

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**FINITE-TIME STABILITY OF SWITCHED SYSTEMS WITH  
APPLICATION TO POWER SYSTEM STABILITY  
PROBLEMS**

CURITIBA

2016

BRUNA KRASOTA MATOS

**FINITE-TIME STABILITY OF SWITCHED SYSTEMS WITH  
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Dissertation submitted to the Graduate Program in Electrical Engineering from Federal University of Paraná, as partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering

Supervisor: Prof. Dr. Roman Kuiava

CURITIBA

2016

Matos, Bruna Krasota

Finite-time stability of switched systems with application to power system stability problems / Bruna Krasota Matos . – Curitiba, 2016.  
84 f. : il.; tabs.

Dissertation (degree of Master) – Federal University of Paraná.  
Sector Technology, Graduate Program in Electrical Engineering.  
Supervisor: Roman Kuiava  
Bibliography: p. 79-82

1. Estabilidade. 2. Sistemas dinâmicos diferenciais. 3. Inequações. I. Kuiava, Roman. II. Título.

CDD 629

## STATEMENT OF APPROVAL

BRUNA KRASOTA MATOS

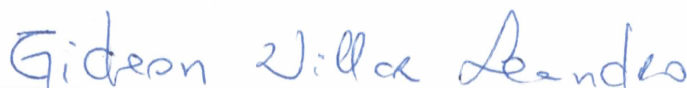
### Finite-time stability of switched systems with application to power system stability problems

Dissertation approved by the following examining committee, as partial fulfilment of the requirements for the degree of Master of Science in Electrical Engineering, of the Graduate Program in Electrical Engineering from the Federal University of Paraná.



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## **ACKNOWLEDGMENTS**

To Prof. Dr. Roman Kuiava who helped me patiently along this long road.

To all the professors who gave me their knowledge during the master's program, which will be very useful to my daily activities.

To my boyfriend who supported me all this time, especially during the most stressful moments.

To my Family for understanding why I was absent in various occasions.

## RESUMO

O conceito de estabilidade a tempo finito (ETF) foi criado em 1950. Sistemas dinâmicos cujas trajetórias convergem para o estado de equilíbrio em um tempo finito fazem parte desse conceito. Sistemas chaveados lineares não homogêneos também estão sendo considerados. Esses sistemas são oriundos de muitas aplicações de controle e para casos aonde sistemas físicos não são descritos por processos unicamente contínuos ou unicamente discretos. Essa dissertação está concentrada no problema de estabilidade a tempo finito de uma classe de sistemas chaveados lineares não homogêneos contínuos no tempo sob um sinal de chaveamento dependente do tempo seguindo um tempo de permanência  $T$ . Uma vez que a estabilidade a tempo finito é garantida, um dos principais resultados dessa dissertação garante que qualquer trajetória do sistema que comece em uma região  $\Omega_1$  do espaço de estados, permanecerá dentro de  $\Omega_2 \supset \Omega_1$  ao longo de um intervalo de tempo finito, e para qualquer sequência de chaveamento com tempo de estabelecimento  $\bar{T} \geq T$ . As condições de estabilidade a tempo finito obtidas na forma de inequações matriciais bilineares (BMIs), podem ser transformadas em inequações matriciais lineares (LMIs) por uma sequência de passos que incluem o cálculo dos conjuntos  $\Omega_1$  e  $\Omega_2$  por meio de um conhecimento prévio dos limites de operação do sistema. Dois exemplos ilustrativos do estudo de estabilidade em sistemas de potência são utilizados para apresentar a validade dos resultados.

Palavras-chave: estabilidade a tempo finito, sistemas chaveados não autônomos, inequações matriciais lineares

## ABSTRACT

The finite-time stability (FTS) concept was created in the 1950. Dynamical systems whose trajectories converge to an equilibrium state in finite time are involved in this concept. Switched non-homogeneous linear systems are being considered. These systems can result from many control applications and for cases where physical systems are not described by simply continuous or simply discrete processes. This dissertation is concerned with the finite-time stability problem of a class of linear continuous-time non-homogeneous switched systems under a time-dependent switching signal constrained by a dwell-time  $T$ . Once the finite-time stability is guaranteed, one of the main results of the dissertation guarantees that any system trajectory starting in a subset  $\Omega_1$  of the state-space will remain in  $\Omega_2 \supset \Omega_1$  over a finite time interval, and, for any switching sequence with a dwell-time  $\bar{T} \geq T$ . The finite-time stability conditions which provided in the form of bilinear matrix inequalities (BMIs), can be transformed to linear matrix inequalities (LMIs) by means of a step-by-step procedure that includes the computation of the sets  $\Omega_1$  and  $\Omega_2$  by the knowledge of the system's operating range. Two illustrative examples in power system stability study are used to show the validity of the results.

Keywords: finite-time stability, Non-autonomous switched systems, Linear matrix inequalities

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## LIST OF SYMBOLS

$\alpha$	Positive scalar
$\bar{\Omega}$	Closure of $\Omega$
$\bar{x}(t)$	Vector with the continuous state variables
$\beta$	Positive scalar
$\delta(t)$	Generator rotor angle
$\mathbb{I}_n$	$n$ -dimensional identity matrix
$\mathbb{R}^{n \times m}$	Set of $n \times m$ real
$\mathbb{R}^n$	$n$ -dimensional Euclidean space
$\mu$	Positive number
$\omega(t)$	Angular speed
$\sigma(t)$	Switching signal
$A_i$	Nonsingular matrix
$b_L(t)$	Imaginary part of the $i^{th}$ load admittance
$d_i$	Nonnegative scalar
$E'_q(t)$	Quadrature axis transient internal voltage of the generator
$E_{fd}(t)$	Voltage applied to the field circuit
$g_i$	Function of class $C^1$
$g_L(t)$	Real part of the $i^{th}$ load admittance
$K_a$	Gain constant of the AVR

$L_1$	Load 1
$L_2$	Load 2
$N$	Maximum number of switching times that can occur on the time interval
$P_i$	Positive matrix
$P_{0_i}$	Load active equivalent powers in steady-state
$Q_{0_i}$	Load reactive equivalent powers in steady-state
$T_a$	Time constant of the AVR
$T_D$	Dwell-time
$T_{L_i}$	Load time constant
$v(x(t))$	Real-valued Lyapunov function
$x^*$	Equilibrium point
$x_{zi}(t)$	Zero-input response
$x_{zs}(t)$	Zero-state response

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# CHAPTER 1

## INTRODUCTION

Switched systems are dynamic systems that consists of a family of subsystems and a switching law, that depending on the state and/or the time, selects which system should be activated for each time instant (LIBERZON; MORSE, 1999). Switched systems can be used to model many physical systems that cannot be described by purely continuous or purely discrete processes (WANG et al., 2012). The switching events of a system can arise from internal causes, like operating mode changes and failures (BALBIS et al., 2007). For example, in the power systems area, an autonomous switched affine system can be adopted to represent the electromechanical dynamics of a power distribution system with synchronous generators, which is subject to changes in the system's operating conditions due to fast varying loads (KUIAVA et al., 2014). Hence, the switching events of this system arise from load variations. Other practical examples of switched systems include traffic control (C; GJ; S, 1998), aerospace control (BALBIS et al., 2007) and switching power converters (CORONA et al., 2007).

The stability analysis and control synthesis of different classes of switched systems have been addressed by many authors, see (BEMPORAD; FERRARI-TRECATE; MORARI, 2000; JP, 2004; HESPANHA et al., 2005; GEROMEL; COLANERI, 2006; GEROMEL; COLANERI; BOLZERN, 2008; LIN; ANTSAKLIS, 2009; T, 2010; DEAECTO; GEROMEL; DAAFOUZ, 2011; VALENTINO et al., 2012; KUIAVA et al., 2013), as well as, the references therein. For autonomous switched systems with a common equilibrium point (typically the origin of the state-space), the asymptotic stability analysis at the origin can be established, for example, from an auxiliary scalar-valued Lyapunov function common to all the subsystems of the switched system (COLANERI; GEROMEL; ASTOLFI, 2008b; LIN; ANTSAKLIS, 2009). The existence of this common Lyapunov function ensures

that the origin is globally asymptotically stable for any arbitrary switching sequence. Unfortunately, a common Lyapunov function for all the subsystems may be difficult to find or a solution may not even exist. The search for this Lyapunov function can be simplified by adopting the invariance principle and its extensions to switched systems, as discussed, for example, in (HESPANHA et al., 2005; J; R, 2006; VALENTINO et al., 2012). Instead of using a common Lyapunov function for all the subsystems, the asymptotic stability of the origin can be verified by ensuring the subsystems are individually stable by the existence of piecewise Lyapunov functions and also guaranteeing that the Lyapunov function of the switched system, which is constituted by these piecewise functions, is uniformly decreasing for all the time (JP, 2004; LIN; ANTSAKLIS, 2009). In comparison to (JP, 2004; LIN; ANTSAKLIS, 2009), a less conservative result is given by (GEROMEL; COLANERI, 2006) and (COLANERI; GEROMEL; ASTOLFI, 2008b), where the above non-increasing condition on the Lyapunov functions of the subsystems is relaxed and replaced by a weaker condition where only the sequence of values of the Lyapunov functions in the switching times have to be uniformly decreasing as time goes on.

The subsystems of the autonomous switched system can share a common equilibrium point or not. When they do not share a common equilibrium point, stability can be investigated with respect to a set, rather than a particular point by using the finite-time and practical stability concepts. These concepts are investigated by (ZHAI; MICHEL, 2003, 2004; XU; ZHAI, 2005; XU; ZHAI; HE, 2007).

For Xu (2005), a hybrid, and switched system whose subsystems have no common equilibrium point is considered practical or finite-time stable under appropriate switching laws if its trajectory can keep within the bounds of a given point, in other words, these two stability concepts require that the system trajectory be confined in a certain subset of the state-space (region  $\Omega_2$ ) over a finite time interval (for finite-time stability) or an infinite time interval (for practical stability) given an initial state in a region  $\Omega_1$  such that  $\Omega_1 \subset \Omega_2$  (ZHAI; MICHEL, 2003, 2004; XU; ZHAI, 2005; XU; ZHAI; HE, 2007).

One example of the applicability of these concepts of stability is in power systems

stability analysis. The power system stability is related to the dynamical behavior of the system when disturbed from an equilibrium condition by a disturbance, such as short circuits, lightning, switchings and others. According to Kundur (1994), two categories should be considered when studying power system stability: small and large disturbances. This classification categorizes the stability analysis in terms of disturbance intensity and defines a mathematical approach to problem resolution.

The first category (large disturbances) considers the system's capacity to find an equilibrium operating condition after a severe disturbance. The second category (small disturbances) is related to the system's dynamic behavior when a small disturbance event occurs, such as load switching. For this last case, the stability problem can be solved by linearizing the power system model equations around the initial operation point, resulting in a set of linear differential equations, that can be resolved using the linear systems theory. This is a valid consideration, since after the small disturbance occurs, the system will oscillate around the initial operation point and return to it, or to an equilibrium point close to it, in cases that the system is considered stable.

Focusing on these concepts of stability, (KUIAVA et al., 2013) provides some sufficient conditions concerning practical stability and finite-time stability of nonlinear and affine continuous-time autonomous switched systems without a common equilibrium point and (KUIAVA et al., 2014) applies these theoretical results on the problem of small-signal stability in the power systems area. The class of switched systems studied by (KUIAVA et al., 2014) considers an independent term constant, limiting the applicability of the results for some physical systems.

## 1.1 Motivation

During the last decade, interest in systems with behavior that can be described mathematically mixing logic based switching and difference/differential equations increased. This happened because a lot of man-made and physical systems can be modeled using

such a framework, for example Multiple-Models, Switching and Tuning paradigm from adaptive control, Hybrid Control Systems and many other techniques in Driven Systems (SHORTEN et al., 2007).

Therefore, this dissertation focuses on extend the study of finite-time stability presented by Kuiava (2013,2014) considering the load dynamic in switched system model. For a practical viewpoint, the distribution network with synchronous generators should be working in a safety region (called of  $\Omega_2 \subset \mathbb{R}^n$ , that is obtained using an allowed range of values of state variables. So, in a realistic scenario, the problem of interest in this work is determine for which conditions the system trajectories will be in region  $\Omega_2$  for a determinate time interval, for a initial condition in a set  $\Omega_1 \subset \mathbb{R}^n$  such that  $\Omega_1 \subset \Omega_2$ .

The second point that this Masters's degree work is focused is on developing a methodology able to analyze the dynamical performance of distribution networks with distributed generation (synchronous generator) considering the finite-time stability in the presence of capacitor bank switching. When this kind of system is analyzed, the resultant model is a class of continuous-time non-homogeneous switched system. This kind of approach using finite-time stability theory to analyse a non-homogeneous system resulting of the power system model was never studied before. These results are important to analyze the the impacts of the capacitor bank switching in the power system quality and confiability.

## 1.2 Contributions of the dissertation

The first contribution of this dissertation is to extend the theoretical results on finite-time stability proposed by (KUIAVA et al., 2013, 2014) for a class of non-homogeneous switched systems, considering the load dynamical model in practical system modeling. The practical example used is the same used by Kuiava (2013,2014), but using the load dynamic model, increased the difficulty of problem resolution. The results obtained are presented in this dissertation.

The second and main contribution of this Master's work is to study the finite-time stability applicability into linear continuous-time non-homogeneous switched systems without a common equilibria under a time dependent switching signal. From a practical viewpoint, this class of switched systems allow us to study the dynamic behavior of, for example, AC-DC converters, while in the class of switched systems investigated by (KUIAVA et al., 2013, 2014) only CC-CC converters were studied. Sufficient conditions for finite-time stability are provided in the form of bilinear matrix inequalities (BMIs), which can be transformed to linear matrix inequalities (LMIs) by means of a step-by-step procedure that includes the computation of the sets  $\Omega_1$  and  $\Omega_2$  by the knowledge of the system's operating range.

### 1.2.1 Main Objective

The main objective of this Master's work is to study the finite-time stability applicability into linear continuous-time non-homogeneous switched systems without a common equilibria under a time dependent switching signal.

### 1.2.2 Specific Objectives

- Apply the existing results of finite-time stability proposed by Kuiava (2013,2014) in distributed generating systems considering the load dynamic
- Develop sufficient results of the finite-time stability of a class of non-homogeneous switched systems
- Develop a systematic proceeding based on LMIs resolution to apply results of finite-time stability
- Apply the results of finite-time stability in a illustrative system.

### 1.3 Dissertation outline

This dissertation comprises of four chapters and this is how it is organized:

Introduction: This is our current chapter where an introduction of this work is made, a literature review, motivations for the study of this topic, as well as an outline about its contributions;

Chapter 2 : In this chapter the theoretical fundamentals are presented. It starts by presenting the hybrid and switched systems concepts followed by a brief explanation of the Lyapunov stability, and then the direct method of Lyapunov is presented. After that, the stability analysis for switched systems and an illustrative definition of the finite-time stability of switched systems are presented;

Chapter 3 : It covers the finite-time stability of a class of continuous-time systems when they are submitted to load switching. Following, the main results on finite-time stability for the switched system with some subsystems are shown, and, then, some used proofs and theorems are demonstrated, based on what already was investigated by Kuiava (2013,2014). A distributed system with synchronous generations and load switching is implemented to exemplify the theory and its results, considering the load dynamic model, being the first contribution of this work;

Chapter 4 : It shows the finite-time stability of a class of continuous-time non-homogeneous switched systems, that is the main contribution of this work. Next, the main results on finite-time stability for the switched system with two or more subsystems are presented, as well as some proofs of theorems. In one section a similar system used in Chapter 3 is implemented to illustratively exemplify a real situation where this theory can be applied and it is followed by the presentation of a numerical example.

## 1.4 Notation

In this work,  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$  denotes the  $n$ -dimensional Euclidean space and the set of  $n \times m$  real matrices, respectively. For  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}^n$ ,  $\{a, b\}$  denotes the set constituted by only these two elements  $a$  and  $b$ , while  $[a, b]$  denotes the set containing all the points in the line segment between  $a$  and  $b$ . In addition,  $[a, b)$  denotes the set containing all the points of  $[a, b]$ , except the point  $b$ . Matrices are denoted by capital roman letters, such as  $P$ . The  $n$ -dimensional identity matrix is denoted by  $\mathbb{I}_n$ . For matrices and vectors,  $()'$  means transposition. When  $P$  is a symmetric matrix,  $P \succ 0$  (respectively,  $P \prec 0$ ) means that  $P$  is positive definite (respectively, negative definite). Positive (respectively, negative) semi-definiteness is denoted by  $P \succeq 0$  (respectively,  $P \preceq 0$ ). The Euclidean norm is denoted by  $\| \cdot \|$ . For a set  $\Omega \in \mathbb{R}^n$ ,  $\bar{\Omega}$  and  $\Omega^c$  denote the closure and the complement of  $\Omega$ , respectively.

## CHAPTER 2

# THEORETICAL FUNDAMENTALS OF SWITCHED SYSTEMS AND POWER SYSTEMS STABILITY

The first part of this chapter aims to present some basic concepts necessary to understand the process used in the dissertation. First, hybrid and switched systems concepts and a real example are presented. Next, in order to better understand the processes for the study of the switched systems stability analysis, Lyapunov's stability theory is presented, and the direct method of Lyapunov's definitions is addressed. The chapter also presents the typical representation of the continuous-time switched systems and an illustrative representation of the finite-time stability, that are the interest of this dissertation.

In second part are presented the power system stability definitions and the power system transients definitions. This knowledge will be important to the understanding of the dissertation.

### 2.1 Hybrid and Switched Systems

For Liberzon (2003), hybrid systems are all dynamical systems that describe an interaction between continuous and discrete dynamics. A similar definition was done by Shorten (2007) which considers that a hybrid system is a dynamical system described using a mixture of continuous/discrete dynamics and logic based switching. For classical theory, these systems evolve according to mode dependent continuous/discrete dynamics and the transactions between these modes are called events.

The manual gearbox in an automobile is an example (SHORTEN et al., 2007). When a car is traveling along a fixed path two variables can be considered: velocity  $v$  and position

$s$  and two inputs: throttle angle  $u$  and engaged gear  $g$ . The velocity response is directly dependent on the throttle input of the engaged gear. For this situation, we can consider as hybrid nature the dynamic of the automobile, because in each mode the dynamic evolves in a continuous manner according to a differential equation. In Figure 2.1 it is possible to analyze that by driver interventions in the form of gear changes the transitions between modes are abrupt (SHORTEN et al., 2007).

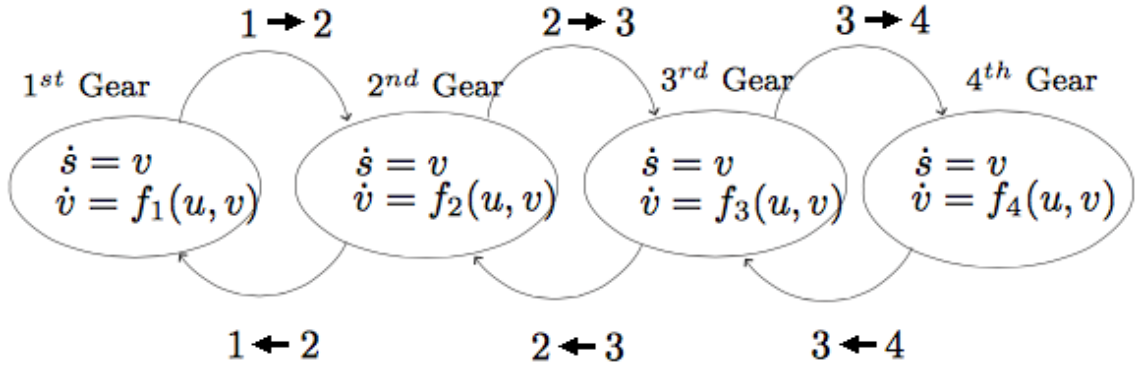


Figura 2.1: A hybrid model of a car with a manual gearbox. Source: (SHORTEN et al., 2007).

For the example presented above and all other examples presented by (SHORTEN et al., 2007), it is possible to see that hybrid systems theory can be applied to modelling a lot of complex dynamical systems. The complexity between modeling and analysis will be different and will be dependent on the methodology and theory used. For analysis, the main challenge is related to the fact that even simple hybrid dynamical systems can produce an extremely complicated nonlinear behavior. Therefore, the non resolved mathematical challenges increase with the study of switched systems, and most of them are a consequence of hybrid dynamical systems stability problems (SHORTEN et al., 2007).

For the most part of the applications, the main interest is on the continuous dynamic and its properties, the discrete dynamic has a secondary importance. So, instead of studying the discrete dynamic details, it is possible to consider that the system can be described by a set of continuous dynamics, and a switching logic is used for selecting one of those dynamics to study. These systems, when there is a family of continuous dynamics and a family of switchings between these dynamics, are called switched systems

(MAZANTI, 2011). To obtain a switched system, some details of the discrete behavior will be omitted so as to consider all possible switching patterns from a certain class. Doing that we will be moving away to hybrid systems, mainly at the analysis stage. The switching mechanisms are of greater importance in switching control design, but only essential properties of the discrete behavior are normally analyzed (SHORTEN et al., 2007).

Mathematically, it is possible to describe a switched system in  $\mathbb{R}^n$  by a family of applications  $f_k : \mathbb{R}^n \rightarrow \mathbb{R}^n, k \in S$ ,  $S$  being a set of indices, and a function  $\sigma : \mathbb{R}^+ \rightarrow S$  which is constant in parts, following the dynamic equation (MAZANTI, 2011):

$$\dot{x} = f_{\sigma(t)}(x(t)), \quad t \in \mathbb{R}^+ \quad (2.1)$$

The state  $x$  takes values in  $\mathbb{R}^d$  and the signal of the switching  $\sigma$  takes values in the set of indices  $S$  (MAZANTI, 2011).

To better understand the problems caused by switchings and for the study of stability, concepts and results from Lyapunov's stability theory are necessary, and they will be approached in the next subsection.

### 2.1.1 Lyapunov Stability

Consider a dynamical system which satisfies:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \quad x \in \mathbb{R}^n, \quad (2.2)$$

where  $f(x, t)$  satisfies the initial conditions for the existence and uniqueness of solutions, that is  $f(x, t)$  is Lipschitz continuous with respect to  $x$ , uniformly and piecewise continuous in  $t$ . The point  $x^* \in \mathbb{R}^n$  is an equilibrium point of (2.2) if  $f(x^*, t) \equiv 0$  for all  $t$ . The equilibrium point is locally stable if all solutions that start near  $x^*$  remain near  $x^*$  for all the time. If the equilibrium point  $x^*$  is locally stable and all solutions starting near  $x^*$  tend towards  $x^*$  as  $t \rightarrow \infty$ , so this equilibrium point is locally asymptotically stable

(MURRAY; LI; SASTRY, 1994).

It can be considered that the equilibrium point of interest occurs at  $x^* = 0$  when the shift occurs at the origin of the system. For the case where multiple equilibrium points exist, the stability will need to be studied for each one by appropriately shifting the origin (MURRAY; LI; SASTRY, 1994).

**Definition 1.** *Stability in the sense of Lyapunov*

*The equilibrium point  $x^* = 0$  of Equation 2.2 is stable in the sense of Lyapunov at  $t = t_0$  if for any  $\epsilon > 0$  there exists a  $\delta(t_0, \epsilon) > 0$  such that*

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \epsilon, \quad \forall t \geq t_0 \quad (2.3)$$

**Remark 1.** *On equilibrium points Lyapunov stability is a very mild requirement. It is not necessary that trajectories starting close to the origin tend to the origin asymptotically. Stability is defined at a time instant  $t_0$  (MURRAY; LI; SASTRY, 1994).*

**Definition 2.** *Asymptotic Stability (MURRAY; LI; SASTRY, 1994)*

*We can consider that an equilibrium point  $x^* = 0$  of Equation 2.2 is asymptotically stable at  $t = t_0$  if:*

- $x^* = 0$  is stable, and
- $x^* = 0$  is locally attractive; For example, there exists  $\delta(t_0)$  such that Equation 2.4 is respected.

$$\|x(t_0)\| < \delta \implies \lim_{t \rightarrow \infty} x(t) = 0 \quad (2.4)$$

Considering Definition 1 and Definition 2 are local definitions, we can say that they describe the behavior of the system near an equilibrium point. It is possible to consider that this equilibrium point is globally stable if it is stable for all initial conditions  $x_0 \in \mathbb{R}^n$  (MURRAY; LI; SASTRY, 1994).

### 2.1.2 The Direct Method of Lyapunov

With the direct method of Lyapunov, it is possible to study the stability of a system without explicitly integrating the differential equation (2.2). The method is a generalization of the idea that if there exists some "measure of energy" in a system, then we can study the rate of change of the energy of the system to ascertain stability. To obtain more precise results it is necessary to define what "measure of energy" means. Let  $B_\epsilon$  be a circumference of radius  $\epsilon$  around the origin,  $B_\epsilon = \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$  (MURRAY; LI; SASTRY, 1994).

**Definition 3.** *Locally positive definite functions(lpdf)*

A locally positive definite function is considered a continuous function  $V : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$  if for some  $\epsilon > 0$  and for some continuous, strictly increasing function  $\alpha : \mathbb{R}_+ \times \mathbb{R}$

$$V(0, t) = 0 \text{ and } V(x, t) \geq \alpha(\|x\|) \quad \forall x \in B_\epsilon, \forall t \geq 0 \quad (2.5)$$

a locally positive definite function is locally like an energy function. Functions which are globally like energy functions are called positive definite functions (MURRAY; LI; SASTRY, 1994).

**Definition 4.** *Positive definite functions(pdf)*

A positive definite function is considered a continuous function  $V : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$  if all conditions presented on Definition 3 are satisfied and, additionally,  $\alpha(p) \rightarrow \infty$  as  $p \rightarrow \infty$  (MURRAY; LI; SASTRY, 1994).

**Definition 5.** *Decrescent functions (MURRAY; LI; SASTRY, 1994)*

A decrescent function is considered a continuous function  $V : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$  if for some  $\epsilon > 0$  and some continuous, strictly increasing function  $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}$  (MURRAY; LI; SASTRY, 1994),

$$V(x, t) \leq \beta(\|x\|) \quad \forall x \in B_\epsilon, \forall t \geq 0 \quad (2.6)$$

Using the appropriate energy functions it is possible to determine the stability for a

system using these definitions and the theorem presented in (2.7) .

$$V(x, t) \leq \beta(\|x\|) \quad \forall x \in B_e, \forall t \geq t_0 \quad (2.7)$$

## 2.2 Switched Systems and Stability Analysis

In this dissertation, we are interested to study the stability of some classes of continuous-time switched systems. The dynamical system presented in (2.2) typically represents these systems whose general form is:

$$\dot{x} = f_{\sigma(\cdot)}(x(t)), \quad x(t_0) = x_0 \quad (2.8)$$

where  $x(t) \in \mathbb{R}^n$  defined for all  $t \geq 0$  is the state,  $\sigma(t)$  is a piecewise constant function called switching signal and  $t_0$  is the initial time. In this dissertation the switching signal is being considered time-dependent. For this case,  $\sigma$  is a function of time defined as  $\sigma(t) : I \rightarrow S$ , being  $i = [t_0, t_f)$  and  $t_f$  a finite constant. In addition  $S = \{1, \dots, N\}$  is a set of positive integers, where given a set of subsystems  $\{f_1, \dots, f_N\}$ , the switching signal is  $f_{\sigma(t)} \in \{f_1, \dots, f_N\}$  for each  $t \in I$ . It is clear that a discontinuity on  $f_{\sigma(t)}$  is naturally imposed by this model, since this matrix must jump instantaneously from  $f_i$  to  $f_j$  for some  $i \neq j = 1, \dots, N$  once switching occurs. For this case the instants of time where  $f_{\sigma(t)}$  is discontinuous are called switching times, this is,  $t_1, t_2, \dots, t_k, \dots \in I$  where  $t_0 < t_1 < t_2 < \dots < t_k < \dots$  (KUIAVA et al., 2013; GEROMEL; COLANERI, 2006).

The main difference between the common Lyapunov stability presented in the previous section and the stability that will be presented in this section is that earlier, the interval of interest was  $[t_0, \infty)$  and now is  $[t_0, t_f)$ .

Considering the class of switched systems presented in (2.8) and using an auxiliary scalar-valued Lyapunov function, it is possible to obtain some results of asymptotic stability of the origin from direct methods. The global asymptotic stability of the origin is guaranteed if, using direct methods, there is a common, continuously differentiable,

positive-definite real-valued Lyapunov function  $v(x(t))$  for all subsystems  $\{f_1, \dots, f_N\}$ , such that

$$\frac{\partial v}{\partial x} f_i < -\lambda_i v, \quad \lambda_i > 0, \quad \forall i = 1, \dots, N, \quad x \neq 0. \quad (2.9)$$

For this switched system, the equilibrium point  $x = 0$  has the global asymptotic stability guaranteed for any arbitrary switching signal if there is a common Lyapunov function for all subsystems  $\{f_1, \dots, f_N\}$ . For some systems' behaviors and dimensions, sometimes it can be very difficult to find a solution for inequality (2.9), or this solution may not exist, making it necessary to restrict the switching signals to obtain the stability. When we intend to study this kind of system, under restricted switching rules, the use of a common Lyapunov function for all subsystems can be replaced by the existence of a family of continuously differentiable, radially unbounded, positive-definite real-valued Lyapunov functions  $\{V_1(x(t)), \dots, V_N(x(t))\}$  such that (GEROMEL; COLANERI, 2006; JP, 2004):

$$\frac{\partial V_i}{\partial x} f_i < -\lambda_i V_i, \quad \lambda_i > 0, \quad \forall i = 1, \dots, N, \quad x \neq 0. \quad (2.10)$$

and,

$$V_{i_{k+1}}(x(t_k)) \leq V_{i_k}(x(t_k)). \quad (2.11)$$

where every switching time  $t_k \in I$  at which  $\sigma$  switches from  $i_k$  to  $i_{k+1}$ , where  $i_k, i_{k+1} \in S$ ,  $i_k \neq i_{k+1}$ . Different from condition (2.9), the conditions (2.10)-(2.11) allow some discontinuities in the Lyapunov function  $v(x(t)) = V_{\sigma(t)}(x(t))$  at the switching times, making them attractive for stability analysis of switched systems. For the situations where the subsystems  $\{f_1, \dots, f_N\}$  are individually stable (condition (2.10)) and  $v(x(t))$  is uniformly decreasing for all  $t \in I$  (condition (2.11)), the asymptotic stability of the origin is then verified (KUIAVA et al., 2013; GEROMEL; COLANERI, 2006; JP, 2004).

Geromel and Colaner (2006) give a less conservative result relaxing Lyapunov functions, and replacing them by a weaker condition which imposes that the sequence  $v(x(t_k))$ ,  $t_0, t_1, \dots, t_k \in I$  must converge uniformly to zero, being  $v(x(t_k)) = V_{i_k}(x(t_k))$  when  $\sigma$

switches to a mode  $i_k$  at the switching time  $t_k$ , requiring:

$$V_{i_{k+1}}(x(t_{k+1})) \leq V_{i_k}(x(t_k)), \quad i_k, i_{k+1} \in S, \quad i_k \neq i_{k+1}, \quad (2.12)$$

for all successive switching times  $k$  and  $k+1$ , being  $\sigma(t) = i_k \in S, \forall t \in [t_k, t_{k+1})$ . Figure 2.2 shows the difference between three stability conditions mentioned above. It is possible to see, that cases (a), (b) and (c) are exemplifying the expected behavior of Lyapunov functions when a solution is found for conditions (2.9), pair (2.10)-(2.11) and the pair (2.10)-(2.12) respectively. Analyzing Figure 2.2 (c), it is possible to see that the piecewise Lyapunov function allows that the current value be bigger than values in the past, for a determined period of time. So, it is necessary to ensure that the sequence  $v(x(t_k))$ , for all  $k = 1, 2, \dots, N$ , converges uniformly to zero when  $t \rightarrow \infty$ .

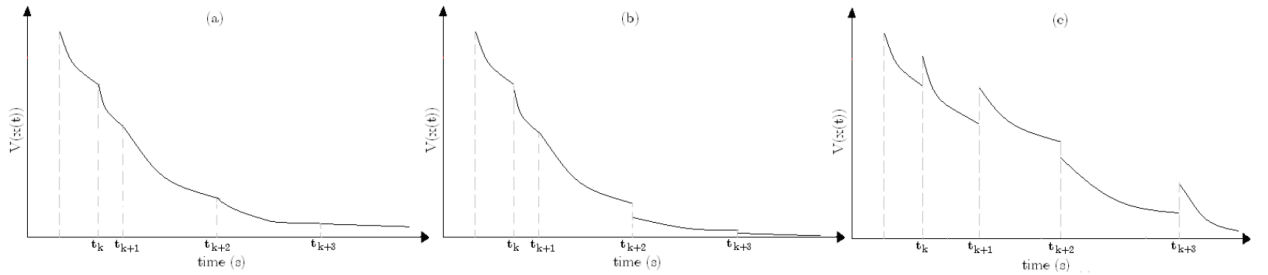


Figure 2.2: Lyapunov function for a found solution for (a) inequation (2.9), (b) inequations (2.10)-(2.11) and (c) inequations (2.10)-(2.12).

A common equilibrium is being considered by (BRANICKY, 1998; DECARLO et al., 2000; LIBERZON, 2003; LIBERZON; MORSE, 1999) for all subsystems in the hybrid and switched systems stability analysis, limiting the applicability of these systems. When subsystems are not in equilibrium or have a different equilibria, if appropriate switching laws are used, this system can produce interesting behaviors, its trajectory can be within a surface around a given point. This concept was formally called practical stability for ordinary differential equations (XU; ZHAI, 2005). Comparing with classical stability concepts, for example, Lyapunov stability and asymptotic stability, are all considering a system operating over an infinite interval of time. For the case of practical stability, they also have an infinite interval of time, but for the situation where the system is operating

with prescribed bounds and finite-time intervals, the term finite-time stability is used (DORATO, 2006).

The sufficient conditions for finite-time stability will be presented in the following.

### 2.2.1 Sufficient conditions of finite-time stability of switched affine systems

Let us consider the switched system presented in (2.8) as a switched affine system in the form:

$$\dot{x} = A_{\sigma(t)}x(t) + b_{\sigma(t)}, \quad x(t_0) = x_0, \quad (2.13)$$

where  $\sigma(t) = i_k \in S, \forall t \in [t_k, t_{k+1})$ , the finite-time stability definition will be presented in Figure 2.3. This figure shows an illustrative example of a second order dynamical system with ellipsoids representing the sets  $\Omega_1$  and  $\Omega_2$  (KUIAVA et al., 2014).

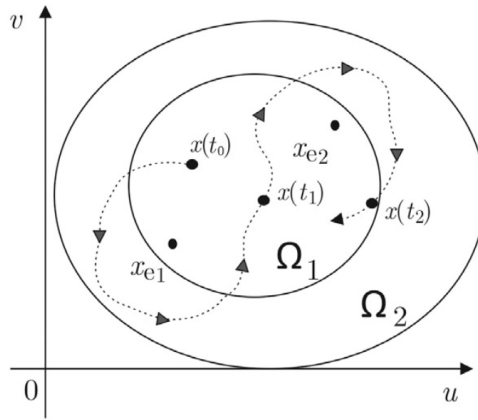


Figura 2.3: Illustrative example of the finite-time stability concept. Source:(KUIAVA et al., 2014)

**Definition 6.** *The switched affine system (2.13) is considered time-finite stable with respect to the sets  $\Omega_1 \subset \mathbb{R}^n$  and  $\Omega_2 \subset \mathbb{R}^n$  ( $\Omega_1 \subset \Omega_2$ ) in the time interval  $I = [t_0, t_f)$ , if  $x(t_0) \in \Omega_1$  implies  $x(t) \in \Omega_2$ , for all  $t \in I$  (KUIAVA et al., 2014).*

**Definition 7.** *The switched affine system (2.13) is considered finite-time unstable with respect to the sets  $\Omega_1 \subset \mathbb{R}^n$  and  $\Omega_2 \subset \mathbb{R}^n$  ( $\Omega_1 \subset \Omega_2$ ) in the time interval  $I = [t_0, t_f)$ , if there exist an instant  $\bar{t} \in I$  at which  $x(\bar{t}) \notin \Omega_2$  for  $x(t_0) \in \Omega_1$ .*

## 2.3 Power System Stability

Since 1920, power system stability has been considered an important problem for secure system operation. Proof of that are some blackouts caused by power system instability. Power systems are increasing continually and it is necessary to clarify our understanding of the stability of the power system, and, consequently, the necessity to review the definition and classification of power system stability. The stability of dynamic systems is similar to that of the power system and requires the consideration of rigorous mathematical theories. In this section there will be provided some physical motivation definitions of power system stability that can be used with the right mathematical definitions (KUNDUR et al., 2004).

### 2.3.1 Definition of Power System Stability

Below there is a formal definition of power system stability which is easily understood and readily applicable by power system engineering.

**Definition 8.** *According to (KUNDUR et al., 2004), power system stability is the ability of an electric power system, for a given initial operating condition, to regain the state of operating equilibrium after being subjected to a physical disturbance, with most system variables bounded so that practically the entire system remains intact.*

This definition can be applied to an interconnected power system as a whole. Moreover, there are cases where the loads or generator groups stability will be of interest. Our power system is changing constantly when loads, generators and switches have its state altered. When this happens, the initial operating condition and the nature of the disturbance are important to stability analysis. Power systems should be able to maintain satisfactory operating conditions when submitted to disturbances, whether small (load changes), severe (short circuit on a transmission line or loss of a large generator) or large in nature (isolation of the faulty elements) (KUNDUR et al., 2004).

### 2.3.2 Classification of Power System Stability

The dynamic response of the high-order multivariable process comes from a modern power system change following characteristics and responses of its devices. When all opposing forces are balanced, the system is stable, but, an external action can cause an imbalance of this system, generating instability. As the power system is so complex, the classification of stability needs to be made detailing system representations and making an appropriate analysis. These analyses are made by finding the main factors that contribute to instability thus creating some methods to improve the stable operation (KUNDUR et al., 2004).

For the practical analysis and resolution of the stability problems, their classification is necessary. The following considerations are taken into account for categorization of power systems stability (KUNDUR; GRIGSBY, 2012):

- The main variable affected in the system where the instability was observed and its physical nature;
- The disturbance size, which will affect the prognostication and method of stability calculation;
- Time interval, devices and processes considered to assess stability.

A general idea about the power system stability problem is presented in Figure 2.4, where main categories and subcategories are identified. In the following there is a brief discussion about some of them (KUNDUR et al., 2004).

- Rotor Angle Stability: Related to the ability that synchronous machines of an interconnected power system have to keep in synchronism after a disturbance, or related to the capacity of each synchronous machine in the system to maintain or restore equilibrium between electromagnetic and mechanical torque. The instability effects are related to the increase of angular swings of some generators caused by the loss of synchronism with other generators. When a system is submitted to

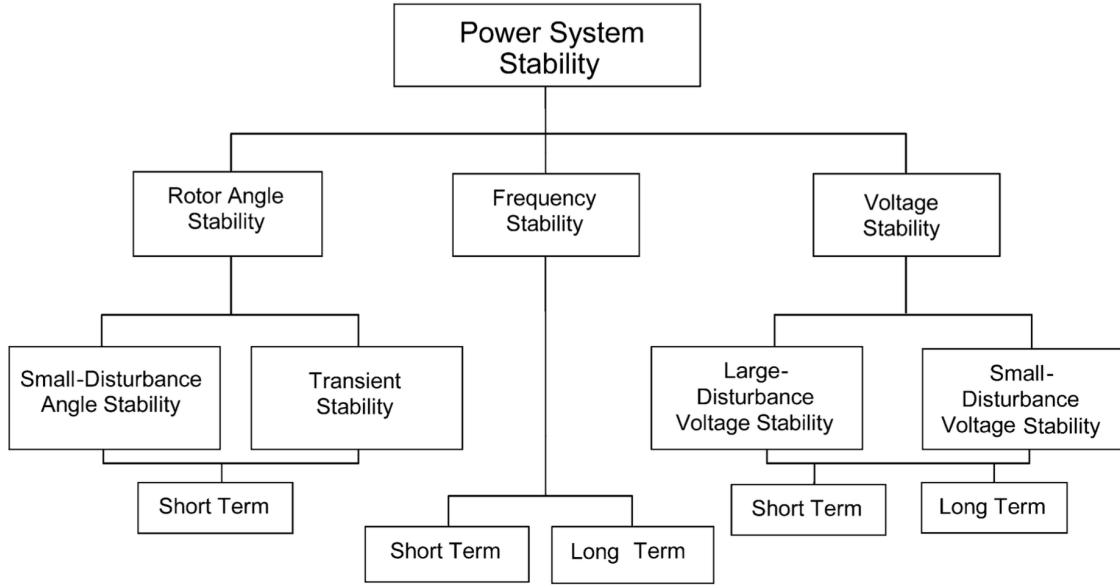


Figura 2.4: Classification of power system stability. Source: (KUNDUR et al., 2004).

a disturbance, its stability will depend on the deviations in angular positions of the rotors resulting in sufficient restoring torques. The synchronism loss can be from different causes, meaning that they can occur between groups of machines, with synchronism maintained within each group after separating from each other or between one machine and the rest of the system (KUNDUR; GRIGSBY, 2012).

According to (KUNDUR; GRIGSBY, 2012), for a better analysis and study of the nature of stability problems, rotor angle stability will be subdivided in terms of two subcategories:

- Small-disturbance (or small-signal) rotor angle stability: is related to the power system's ability to keep synchronism under small disturbances in rotor angle stability, usually associated with insufficient damping of oscillations, where the time frame is in the order of 10 to 20 seconds after the action;
- Large-disturbance rotor angle stability or transient stability: when a power system is able to maintain synchronism after a severe disturbance, such as a short circuit on a transmission line. Transient stability is mainly dependent on the severity of the disturbance and the initial operating state of the system, where the time frame studied is around 3 to 5 seconds after the disturbance.

A class of rotor angle stability can be found in the literature as dynamic stability, and, as shown in Figure 2.4 the transient stability and small-disturbance rotor angle stability are classified as short term phenomena (KUNDUR et al., 2004).

- Voltage Stability: This stability is related to the power system's ability to maintain all bus voltages for a given initial operating condition after being affected by a disturbance. It is directly related to the capacity to keep the equilibrium between load demand and load supply from the power system. Normally these disturbances are caused by loss of load in an area or when an action of transmission lines protection takes place. When synchronism of some generators is lost, it can cause outages or operating conditions that violate field current limits (VOURNAS, 1995).

According to (KUNDUR; GRIGSBY, 2012), similarly to the rotor angle stability case, voltage stability is subdivided into two subcategories:

- Large-disturbance voltage stability: Related to system's ability to keep voltage stability after large disturbances, such as loss of generation, system faults or circuit contingencies. It is related to the characteristics of the system and load, and to the interaction between continuous and discrete controls and protections. The time range of interest starts from a few seconds until tens of minutes;
- Small-disturbance voltage stability: Related to system's ability to keep voltage stability after small disturbances, such as incremental changes in system load. It is related to the load characteristics, and continuous and discrete controls at a given instant of time.

As presented in Figure 2.4, voltage stability can be classified as short-term or long-term, depending on the time frame. In (KUNDUR et al., 2004) it is possible to see its meaning, as presented below:

- Short-term voltage stability: related to the study of dynamic to the fast acting load components, such as electronically controlled loads, induction motors and HVDC converters, where the studied period is in the order of several seconds;

- Long-term voltage stability: related to the study of dynamic to the slower acting equipment, such as thermostatically controlled loads, tap-changing transformers and generator current limiters, where the studied period can extend to several or many minutes.
- Frequency Stability: Related to the power system's ability to keep frequency after a severe disturbance that will result in a significant imbalance between generation and load. Severe perturbations in the system usually cause large excursions of frequency, voltage, power flows and other system variables. It can be generated by the action of controls, processes and protections that are not modeled in conventional voltage stability or transient stability. The time period of interest for this kind of study ranges from fractions of seconds. The frequency value variation is mainly related to the abrupt voltage value variations and it can affect other components of the system (KUNDUR et al., 2004).

## 2.4 Power System Transients

Nowadays, power utilities, electric energy consumers and also the manufacturers of electric and electronic equipment are increasingly interested in the quality of voltage waveforms, that are expected to be a pure sinusoidal with a given frequency and amplitude. The waveform can be affected by different disturbances, and when it happens, the quality of the voltage supplied by the electrical power companies is affected too. Since most of the loads existent in power systems are inductive in nature, the reactive power needs to be supplied by the system. For that, normally capacitor banks are installed. However, the power quality is affected when capacitor banks are switched in electrical distribution systems creating transients (LOBOS; REZMER; KOGLIN, 2001). Next, there will be a quick presentation of the classification of power system transients and proposed results on finite-time stability to a physical system affected by some of these transient cases.

According to (BOLLEN; STYVAKTAKIS; GU, 2005), the term "transient" comes from

electric circuit theory, where it corresponds to the voltage and current components that appear during the transition from one steady-state to another steady-state. Differential equations can be used for electric circuit descriptions, where the solutions are the sum of a homogeneous solution and a particular solution. The steady-state of the system is found in particular solutions and the transient is in homogeneous solutions. A transient will always exist when we change the stable state of the system with a switch action. For power systems the transient has a slightly different meaning, considered as a phenomena in voltage and current with a short duration. For this case, the time limit is not so stiff, in a general mode transients are considered phenomena with a duration less than one cycle and are normally related to the correct operation of circuit breakers. But, for (DUGAN et al., 2004), transients need to be considered as a potential power-quality problem, making necessary new requirements on characterization and analysis of transient waveforms. Waveform characteristics and equipment performance need to be related, and transient waveforms need to have their information extracted using theoretical methods, and these methods need to be able to quantify site and system performance, transients need to be considered as a potential power-quality problem, making necessary new requirements on characterization and analysis of transient waveforms (BOLLEN; STYVAKTAKIS; GU, 2005).

One of the main challenges to the study of power system transients is a precise model at higher frequencies and the characterization of the measured values from the transient phenomena. These transients can be caused by lightning strokes to the wires in the power system or to the ground or by the equipment switch in the network (BOLLEN; STYVAKTAKIS; GU, 2005). For (STYVAKTAKIS, 2002), power system phenomena can be classified into three main classes:

- Events that can be classified by their fundamental frequency magnitude: in this case, voltage magnitude endures big changes for long time intervals. These kinds of transients are normally fault-induced events, transformer saturation, induction motor starting and others. The duration can be several seconds, to hours;

- Events that present significant changes in the fundamental frequency magnitude but of short duration: For this case, the voltage magnitude changes are worrisome. An example of this kind of transient is when self healing is activated;
- Events of very short duration (transients) for which the fundamental frequency magnitude does not offer important information.

In (BOLLEN; STYVAKTAKIS; GU, 2005) we have a power system transient classification that considers the waveform shapes. In this case, these events can be classified into "oscillatory transients" and "impulsive transients", a brief synthesis of this classification is shown in Table 2.1.

Tabela 2.1: Categorization of transients based on waveform shapes.

Mode Waveform-based classification	Event-based classification
impulsive transients	lightning
oscillatory transients	capacitor energizing restrike during capacitor de-energizing line or cable energizing
multiple transients	current chopping multiple restrikes repetitive switching actions

### 2.4.1 Impulsive Transients

Impulsive transients are abrupt changes in the stable condition of voltage, current or both. They do not propagate far from their source and are quickly damped by the resistive elements in the circuit. They have rise and decay time characteristics. Normally they are caused by lightning. We have an impulsive overvoltage induced when the lightning strike hits a transmission line (BOLLEN; STYVAKTAKIS; GU, 2005).

### 2.4.2 Oscillatory Transients

These transients correspond to the homogeneous solution of linear differential equations. They can present a damped frequency rate which can reach several megahertz. Since it

is possible to represent the electric power system by a set of linear differential equations, these are the "natural transients". A common cause to this kind of transient is the bank capacitor energization. Following is the main cause of oscillatory transients (BOLLEN; STYVAKTAKIS; GU, 2005):

- Capacitor energizing with magnification;
- Capacitor Energizing Without Magnification;
- Line Energizing.

### 2.4.3 Multiple Transients with a Single Cause

In real systems we normally will not have just one transient occurring with one single action, for example, in three-phase systems, when the switching actions occur, normally they do not happen at exactly the same time for all phases (BOLLEN; STYVAKTAKIS; GU, 2005). In this context, a good example case is the study by (STYVAKTAKIS, 2002), where events were considered separately from cases phase-to-phase and the phase-to-ground voltages.

## 2.5 Summary of the Chapter

This chapter has presented the theory necessary for a better understanding of this dissertation. The hybrid and switched systems definition was first presented, and next Lyapunov's stability theory was addressed in order to provide a better idea about the methods used to study the switched systems stability analysis. Moreover, an illustrative representation of the finite-time stability was presented.

Let us consider an example where system (2.13) describes a power distribution system. The set  $\Omega_2$  is the region where the system is operating with security guaranteed and the set  $\Omega_1$  will include the initial conditions created by a number of perturbations that can occur during the operation (KUIAVA et al., 2014). This real application of the finite-time

stability analysis is the main goal of this dissertation and will be addressed in next sections, where first the disturbance is created by load switching and in the second approach a class of continuous-time non-homogeneous switched system is studied.

## CHAPTER 3

# FINITE-TIME STABILITY OF A CLASS OF CONTINUOUS-TIME SWITCHED AFFINE SYSTEMS WITHOUT A COMMON EQUILIBRIA

In this chapter, the problem of finite-time stability of some classes of continuous-time switched systems is studied, and some sufficient conditions are presented concerning to finite-time stability of continuous-time switched nonlinear systems without a common equilibrium for all subsystems. In this class of switched system, the equilibrium point varies discontinuously according to a time-dependent switching signal. The stability is discussed with respect to a set, rather than a particular point. Using the stability preliminary result, sufficient conditions are presented in the form of linear matrix inequalities (LMIs) for finite-time stability of a particular class of switched affine systems without a common equilibria. In last part of this chapter is presented an illustrative example for one of the power system stability classes presented in previous chapter, showing the validity of the results.

### 3.1 Problem Statement and Assumptions

Consider the continuous-time switched system represented by equations (2.8) presented previously. In general, the main results in the analysis of the dynamic behavior of continuous-time switched systems assume that all subsystems of (2.8) share a common equilibrium (normally the origin  $x = 0$ , for example,  $f_i(0) = 0$  for all  $i = 1, \dots, N$ ), and hence the stability of (2.8) is actually the stability of this common equilibrium (KUIAVA et al., 2014; COLANERI; GEROMEL; ASTOLFI, 2008a; DECARLO et al., 2000; SHORTEN et al., 2007). In this dissertation we will not assume the existence of a common equilibria

for all subsystems  $\{f_1, \dots, f_N\}$ . The switching signal is defined as:

$$\sigma(t) = i_k \in S = \{1, \dots, N\}, \quad \forall t \in [t_k, t_{k+1}), \quad (3.1)$$

where  $t_k$  and  $t_{k+1}$  are two consecutive switching times that satisfy:

$$t_{k+1} - t_k \geq T_D \quad (3.2)$$

for all switching times  $t_1, t_2, \dots, t_k, t_{k+1}, \dots \in I$  and the index  $i_k \in S$  is arbitrarily selected at each of these switching times. For each,  $T_D$  is a positive number called dwell-time of the switching signal  $\sigma(t)$  (LIN; ANTSAKLIS, 2009).

**Definition 9.** A positive number  $T_D$  is called a dwell-time of the switching signal  $\sigma(t)$  if the time interval between any two consecutive switchings  $k$  and  $k+1$  is no smaller than  $T_D$ .

Once the switching signal  $\sigma(t)$  is only time-dependent and it is not considering a common equilibria for all subsystems, an asymptotic convergence of the system trajectories is not guaranteed to a specific equilibrium point. For this reason, stability with respect to a set, rather than a particular point, is studied in this dissertation in terms of finite-time stability (ZHAI; MICHEL, 2004).

**Definition 10.** The switched system (2.8) is considered finite-time stable with respect to the sets  $\Omega_1 \subset \mathbb{R}^n$  and  $\Omega_2 \subset \mathbb{R}^n$  ( $\Omega_1 \subset \Omega_2$ ) in the time interval  $I = [t_0, t_f)$ , if  $x(t_0) \in \Omega_1$  implies  $x(t) \in \Omega_2$ , for all  $t \in I$ .

**Remark 2.** If  $t_f$  is a finite scalar, that is,  $t_f < \infty$ , the concept of practical stability is also known as finite-time stability.

The first problem considered is:

**Problem 1.** Given the sets  $\Omega_1$  and  $\Omega_2$  ( $\Omega_1 \subset \Omega_2$ ) and a time interval  $I = [t_0, t_f)$ , determining a scalar  $T_D > 0$  such that the switched system (2.8) without a common

*equilibria is finite-time stable with respect to  $\Omega_1$  and  $\Omega_2$  in the time interval  $I$ , for every switching signal  $\sigma(t)$  satisfying (3.1) and (3.2) with a dwell-time  $T_D$  (KUIAVA et al., 2014).*

With the solution of Problem 1, it is possible to derive some results on finite-time stability for a particular class of switched systems without a common equilibria, for example, the switched affine system presented in (2.13), where the switching signal  $\sigma(t)$  is given by (3.1) and (3.2). Considering all matrices  $A_i$ ,  $i = 1, \dots, N$  as being nonsingular, each subsystem has a single equilibrium point at  $x_{e_i} = -A_i^{-1}b_i$ ,  $i = 1, \dots, N$  (KUIAVA et al., 2014).

In chapter 4, there will be presented some sufficient results on finite-time stability for a more general class of switched system (and this is the main contribution of this dissertation), where the switched affine system (2.13) is a particular case.

Switched affine systems can arise naturally in many applications, for example in electric power systems. These systems can be affected by unexpected changes on their structures and operation, caused by the load demand and system topology for example, and, consequently, on their equilibrium points. In the next sections there will be presented examples in power system stability area, where the system modeling will be made by considering the switched affine systems (2.13).

The problem of interest is similar to the one studied by (KUIAVA et al., 2014), that determines the conditions (in terms of dwell-time  $T_D$ ) under which the system trajectories (starting on  $x(t_0) \in \Omega_1$ ) will be confined into the security region of operation  $\Omega_2$  for the interval  $I$ . So, the second considered problem is presented bellow.

**Problem 2.** *Given the sets  $\Omega_1$  and  $\Omega_2$  ( $\Omega_1 \subset \Omega_2$ ) and a time interval  $I = [t_0, t_f)$ , determining a scalar  $T_D > 0$  such that the switched affine system (2.13) is finite-time stable with respect to  $\Omega_1$  and  $\Omega_2$  in the time interval  $I$ , for every switching signal  $\sigma(t)$  satisfying (3.1) and (3.2) with a dwell-time  $T_D$  (KUIAVA et al., 2014).*

### 3.2 Main Results

Next, two theorems will be presented to provide sufficient results for Problems 1 and 2. These theorems were proposed in (KUIAVA et al., 2013).

**Theorem 1.** *Let the sets  $\Omega_1 \subset \mathbb{R}^n$  and  $\Omega_2 \subset \mathbb{R}^n$  ( $\Omega_1 \subset \Omega_2$ ) and a time interval  $I = [t_0, t_f]$  be given. If there exists a scalar  $T_D > 0$ , a family of radially unbounded, real-valued functions  $\{V_1, \dots, V_N\}$  satisfying local Lipschitz conditions in  $\Omega_2$  and a positive number  $\mu > 1$  such that*

$$(i) \quad \dot{V}_i(x(t)) \leq 0, \quad \forall i \in S, \quad \forall t \in I, \quad \forall x \in \Omega_2,$$

$$(ii) \quad V_{i_{k+1}}(x(t_{k+1})) \leq \mu V_{i_k}(x(t_{k+1} - T_D)), \quad \forall i_k, i_{k+1} \in S, \quad \forall t \in I, \quad i_k \neq i_{k+1}, \quad \forall x \in \Omega_2,$$

$$(iii) \quad \mu^{\bar{N}(t_0, t_f)} \max_{i \in S} \sup_{x \in \Omega_1} V_i(x(t_0)) < \min_{i \in S} \inf_{x \in \Omega_2^c} V_i(x(t)), \quad \forall t \in I,$$

then, the switched system (2.8) is finite-time stable with respect to  $\Omega_1$  and  $\Omega_2$  in the time interval  $I$  for every switching signal  $\sigma(t)$  satisfying (3.1)-(3.2) with a dwell-time  $T_D$ .

**Proof 1.** *The proof is by contradictions and can be checked on (KUIAVA et al., 2012, 2013).*

Let us consider the following set  $\Omega_1$  and  $\Omega_2$ :

$$\Omega_1 = \bigcap_{i=1}^N \Omega_{1_i}, \quad \Omega_2 = \bigcup_{i=1}^N \Omega_{2_i}, \quad (3.3)$$

where,

$$\Omega_{1_i} = \{x \in \mathbb{R}^n : (x - x_{e_i})' P_i (x - x_{e_i}) + d_i < \alpha\}, \quad (3.4)$$

$$\Omega_{2_i} = \{x \in \mathbb{R}^n : (x - x_{e_i})' P_i (x - x_{e_i}) + d_i < \beta\}$$

$\alpha$  and  $\beta$  being positive scalars that satisfy  $\alpha < \beta$ ,  $P_i \in \mathbb{R}^{n \times n}$  a positive matrix,  $d_i$  a nonnegative scalar,  $x_{e_i} = -A_i^{-1}b_i$  the equilibrium point of the subsystem  $i$ , where  $A_i$  is nonsingular and  $i = 1, \dots, N$  (KUIAVA et al., 2014). Considering the sets  $\Omega_1$  and  $\Omega_2$

given in the form of (3.3), the theorem presented bellow provides sufficient conditions for finite-time stability of the switched affine system (2.13). The proposed result was created applying Theorem 1 to Problem 2, assuming a piecewise Lyapunov function given by:

$$v(x(t)) = V_{\sigma(t)}(x(t)) = (x - x_{e_{\sigma(t)}})' P_{\sigma(t)} (x - x_{e_{\sigma(t)}}) + d_{\sigma(t)} \quad (3.5)$$

**Theorem 2.** *Let the sets  $\Omega_1$  and  $\Omega_2$  be given in the form of (3.3), the positive definite matrices  $P_1, \dots, P_N$ , the scalars  $d_1, \dots, d_N > 0$ ,  $\alpha$  and  $\beta$  ( $\alpha < \beta$ ) are known. If the condition  $A_i' P_i + P_i A_i \preceq 0$  is satisfied for all  $i \in S = \{1, \dots, N\}$ , and, there exist a scalar  $T_D > 0$  and a number  $\mu$  such as that the following matrix inequalities are satisfied:*

$$\begin{bmatrix} e^{A_i' T_D} P_j e^{A_i T_D} - \mu P_j & e^{A_i' T_D} P_j \Delta x_{e_{ij}} \\ * & \Delta x_{e_{ij}}' P_j \Delta x_{e_{ij}} + (d_j - \mu d_i) \end{bmatrix} \preceq 0, \quad \forall i, j \in S, i \neq j \quad (3.6)$$

$$1 < \mu < \sqrt[N(t_0, t_f)]{\frac{\beta}{\alpha}} \quad (3.7)$$

where  $\Delta x_{e_{ij}} = x_{e_i} - x_{e_j}$ , then the switched affine system (2.13) is finite-time stable with respect to  $\Omega_1$  and  $\Omega_2$  in the interval  $I$ , for every switching signal  $\sigma(t)$  satisfying (3.1) and (3.2) with a dwell-time  $T_D$  (KUIAVA et al., 2013).

**Proof 2.** *The proof can be checked on (KUIAVA et al., 2012, 2013).*

In the next section there will be presented an illustrative example in the power system stability area and its corresponding results, which demonstrate the effectiveness of the main result on finite-time stability for a switched system without a common equilibria and considering a time dependent switching control.

### 3.3 Numerical Example 1

In the last decades, the dominant stability problem on most systems has been the transient instability, being the focus of much of the industry's attention concerning system stability. This problem creates the necessity to clarify the understanding of different types

of instability, how they are interrelated, and their physical nature (KUNDUR et al., 2004). Below, there will be presented the definition and classification of power systems stability and the applicability of the proposed result on finite-time stability to a physical system affected by some of these instability cases.

### 3.3.1 Tests and Results

Considering the power system stability classification presented in section 2.3, below is presented a case related to rotor angle stability classification. Fig. 3.1 shows a typical arrangement of a distributed system with synchronous generation, usually viewed in cogeneration plants in the sugarcane industry of Brazil (KUIAVA et al., 2014). The same system was used by Kuiava (2014), but one contribution of this dissertation is the application of results presented in 3.2 considering the dynamic load modeling. The cogeneration plant ( $G$ ) is represented by a gas-turbine driven synchronous generator injecting 10MW to the grid. The synchronous generator is modeled by a third-order model

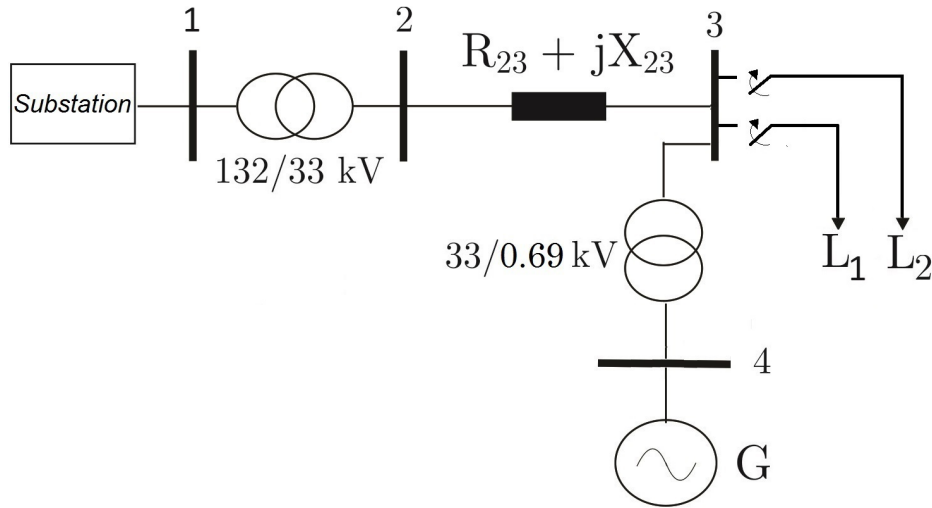


Figura 3.1: One-line diagram of the study system. Source: (KUIAVA et al., 2014).

given by (ARRIFANO et al., 2007):

$$\dot{\delta}(t) = w_s w(t) - w_s, \quad (3.8)$$

$$\dot{\omega}(t) = \frac{1}{2H} [P_{ref} - E'_q(t) I_d(t)], \quad (3.9)$$

$$\dot{E}'_q(t) = \frac{1}{\tau'_{d0}} [E_{fd}(t) - (X_d - X'_d) I_d(t) - E'_q(t)], \quad (3.10)$$

where  $\delta(t)$  and  $\omega(t)$  are the generator rotor angle and angular speed, respectively;  $E'_q(t)$  is the quadrature axis transient internal voltage of the generator and  $E_{fd}(t)$  is the voltage applied to the field circuit. Detailed information regarding this system model can be obtained in (ARRIFANO et al., 2007).

A set of local parallel loads ( $L_1$  and  $L_2$ ) is connected to bus 3. These loads can be active or not during an operating period of the system. The value of the actual equivalent local load depends on which of the local loads are active. This representation describes the operation of an industrial power plant in cogeneration scheme, at which the power excess can be sent to the bulk power system (ARRIFANO et al., 2007). In addition, considering the power system stability classification, this load switching is assumed to be a small disturbance and thus the rotor angle stability can be studied in accordance to the small-signal angle stability.

The synchronous generator is considered to be equipped with a first order model of an automatic voltage regulator (AVR) given by

$$\dot{E}_{fd}(t) = \frac{1}{T_a} (-E_{fd}(t) + K_a (U_{ref} - |U_t(t)|)), \quad (3.11)$$

where  $|U_t(t)|$  is the absolute value of the generator terminal voltage, while  $K_a$  and  $T_a$  are, respectively, the gain and time constant of the AVR.

In addition, the local load is represented by a dynamic load model given by

$$\dot{g}_L(t) = \frac{1}{T_{L_i}}(P_{0_i} - g_L(t)|U_3|^2), \quad (3.12)$$

$$\dot{b}_L(t) = \frac{1}{T_{L_i}}(Q_{0_i} - b_L(t)|U_3|^2), \quad (3.13)$$

where  $i = 1, 2$ . Also,  $g_L(t)$  and  $b_L(t)$  are, respectively, the real and imaginary parts of the  $i^{th}$  load admittance;  $T_{L_i}$  is the load time constant,  $P_{0_i}$  and  $Q_{0_i}$  are the load active and reactive equivalent powers in steady-state,  $|U_3|$  is the absolute value of the load terminal voltage, where  $I = [t_0, t_F]$  is the time interval of interest, being  $t_0$  and  $t_F$ , respectively, the initial and final instants of time. The operating modes are described in Table 3.1.

Tabela 3.1: Operating modes of the system.

Mode ( $i$ )	$L_1$	$L_2$	Equivalent active power	Equivalent reactive power
1	active	active	6.4 MW	2 MVar
2	active	inactive	3.2 MW	1 MVar
3	inactive	active	3.2 MW	1 MVar
4	inactive	inactive	0 MW	0 MVar

The nonlinear model of the system shown in Figure 3.1 was created using the equations (3.8)-(3.13), resulting in the following set of differential equations:

$$\begin{aligned}
\dot{\delta}(t) &= w_s w(t) - w_s, \\
\dot{\omega}(t) &= \frac{1}{2H} \left[ P_{ref} - E'_q(t)(E'_q(t)G_{22} - G_{21}\cos(\delta) - B_{21}\sin(\delta)) - D(w(t) - w_s) \right], \\
\dot{E}'_q(t) &= \frac{1}{\tau'_{d0}} \{ E_{fd}(t) - E'_q(t) + E'_q(t)(X_d - X'_d)(G_{21}\sin(\delta) - B_{21}\cos(\delta) + B_{22}) \} \\
&\quad [ (E'_q(t)\sin(\delta) - U_t\sin(\theta))\sin(\delta) + (E'_q(t)\cos(\delta) - U_t\cos(\theta))\cos(\delta) ], \\
\dot{E}_{fd}(t) &= \frac{1}{T_a} \left[ -E_{fd}(t) + K_a(U_{ref} - |U_t(t)\cos(\theta)|) \right], \\
\dot{g}_L(t) &= \frac{1}{T_{L_i}}(P_{0_i} - g_L(t)|U_3|^2), \\
\dot{b}_L(t) &= \frac{1}{T_{L_i}}(Q_{0_i} - b_L(t)|U_3|^2)
\end{aligned} \quad (3.14)$$

Parameters  $G_{22}$ ,  $G_{21}$ ,  $B_{21}$  and  $B_{22}$  are functions of operating conditions, where  $G_{22}$ ,  $G_{21}$ ,  $B_{21}$  and  $B_{22}$  are elements of the reduced admittance matrix, which in turn is obtained from

the expanded bus admittance matrix. This matrix reflects the topological characteristics of the distribution network, including the reactance and resistance of the line 2 – 3, the reactance of the transformers and the load impedances (KUIAVA et al., 2014). Detailed information about the equations of the presented model and their respective parameters can be obtained in (KUNDUR, 1994).

The focus of this system representation is to study the electromechanical transients of the generator due to load switching. For this purpose, the state-space model for the distribution system with synchronous generator shown in Figure 3.1 is a composition of equations (3.14), and can be written in the compact form

$$\bar{\Sigma}_A = \begin{cases} \dot{\bar{x}}(t) = \bar{f}(\bar{x}(t), \bar{p}(t)) \\ \dot{\bar{p}}(t) = 0 \\ \bar{x}(t_0) = \bar{x}_{t_0}, \bar{p}(t_0) = \bar{p}_{t_0}, \bar{p}(t_1) = \bar{p}_{t_1}, \dots \end{cases} \quad (3.15)$$

where  $t \in I$ ,  $\bar{x}(t) = [\delta(t) \ \omega(t) \ E'_q(t) \ E_{fd}(t) \ g_L(t) \ b_L(t)]'$  is the vector with the continuous state variables,  $\bar{f} : \mathbb{R}^6 \times \mathbb{R}^2 \rightarrow \mathbb{R}^6$  is a nonlinear vector function obtained from the right-hand side of (3.8)-(3.13),  $t_1 < t_2 < \dots < t_k < \dots \in I$ , are the time instants of load switching; so,  $\bar{p}(t) = \bar{p}_{t_k} \in \bar{S}$  for all  $t \in [t_k, t_{k+1})$ , where  $t_k$  and  $t_{k+1}$  are two consecutive load switching times that satisfy  $t_{k+1} - t_k \leq T_D$ , where  $T_D$  is a positive number.

The general form of the system model used for finite-time stability studies with load switching is a generalization of  $\bar{\Sigma}_A$  for power systems described by a n-dimensional state vector and a q-dimensional piecewise constant vector:

$$\Sigma_A = \begin{cases} \dot{x}(t) = f(x(t), p(t)) \\ \dot{p}(t) = 0 \\ x(t_0) = x_{t_0}, p(t_0) = p_{t_0}, p(t_1) = p_{t_1}, \dots \end{cases} \quad (3.16)$$

where  $t \in I$ ,  $t_0 < t_1 < \dots < t_k < \dots \in I$ , are the time instants of a load switching sequence. Also,  $x(t) \in \mathbb{R}^n$  is the vector with state variables,  $p(t) \in \mathbb{S}$  is the vector with the piecewise constant parameters,  $f : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^n$  is the nonlinear vector function.

The equilibrium point  $x_e$  of the system

$$\dot{x}(t) = f(x(t), p(t)) \quad (3.17)$$

is calculated, such that  $(f(x_e, p) = 0)$  for a specific operating condition, in this case, with different load levels. For the simulated case, four different load levels are considered, as shown in Table 3.1. So,  $x_{e1}$ ,  $x_{e2}$ ,  $x_{e3}$  and  $x_{e4}$  are calculated.

Considering a linearized representation of (3.17) in the vicinity of the equilibrium points  $x_{e1}$ ,  $x_{e2}$ ,  $x_{e3}$  and  $x_{e4}$ , the  $i$ th linearized model is given in the state-space form as (3.18) (KUIAVA et al., 2013).

$$\sum_i : \Delta \dot{x}_i(t) = A_i \Delta x_i(t) \quad (3.18)$$

where  $A_i \in \mathbb{R}^{6 \times 6}$  is the  $i$ th state matrix,  $\Delta x_i(t) = x(t) - x_{e_i}$ , and  $i = 1, 2, 3, 4$ . So,  $\sum_i$  is the linear approximation of (3.17) in the vicinity of its equilibrium point  $x_{e_i}$ . To transform  $\sum_i$  into an affine system, the process presented in (3.19) was made.

$$\begin{aligned} \sum_i : \Delta \dot{x}_i(t) &= A_i \Delta x_i(t) \Rightarrow \\ \dot{x}(t) - \dot{x}_{e_i} &= A_i(x(t) - x_{e_i}) \Rightarrow \\ \dot{x}(t) &= A_i x(t) - A_i x_{e_i} \Rightarrow \\ \sum_i : \dot{x}(t) &= A_i x(t) + b_i \end{aligned} \quad (3.19)$$

where  $b_i = -A_i x_{e_i}$  and  $i = 1, 2, 3, 4$  (KUIAVA et al., 2013). For the time interval of interest  $I = [t_0, t_f]$ , we modeled the set of affine systems  $\hat{\Sigma}_1$ ,  $\hat{\Sigma}_2$ ,  $\hat{\Sigma}_3$  and  $\hat{\Sigma}_4$  as switched affine system and used to study the finite-time stability of the system during the time interval  $I$ , where switchings can occur among the systems  $\hat{\Sigma}_2$ ,  $\hat{\Sigma}_3$  and  $\hat{\Sigma}_4$  (KUIAVA et al., 2013).

So, in order to study the finite-time stability of distributed system with synchronous generator under consideration, the switched affine system presented in (2.13) was adopted,

where matrixes  $A_{\sigma(t)}$  and  $b_{\sigma(t)}$  are presented in (3.20)-(3.23).

$$A_1 = \begin{bmatrix} 0 & 377.0 & 0 & 0 & 0 & 0 \\ -0.3 & -0.7 & -0.8 & 0 & -0.2 & 0.1 \\ 0 & 0 & -0.3 & 0.2 & -0.1 & -0.1 \\ -142.5 & 0 & -2304.1 & -100 & 106.9 & 299.3 \\ -10.5 & 0 & -107.2 & 0 & -96.6 & 19.9 \\ -3.3 & 0 & -33.5 & 0 & 1.9 & -96.4 \end{bmatrix} \quad b_1 = \begin{bmatrix} 377 \\ -1.8 \\ -0.2 \\ -2315.8 \\ -168.2 \\ -52.6 \end{bmatrix} \quad (3.20)$$

$$A_2 = \begin{bmatrix} 0 & 377 & 0 & 0 & 0 & 0 \\ -0.3 & -0.7 & -0.7 & 0 & -0.3 & 0.1 \\ 0 & 0 & -0.3 & 0.2 & -0.1 & -0.1 \\ -47.9 & 0 & -1999.7 & -100 & 81.9 & 268.7 \\ -2.5 & 0 & -54.7 & 0 & -102.9 & 10.4 \\ -0.8 & 0 & -17.1 & 0 & 0.8 & -102.2 \end{bmatrix} \quad b_2 = \begin{bmatrix} 377 \\ -1.7 \\ -0.2 \\ -1942.7 \\ -83.9 \\ -26.2 \end{bmatrix} \quad (3.21)$$

$$A_3 = \begin{bmatrix} 0 & 377 & 0 & 0 & 0 & 0 \\ -0.3 & -0.7 & -0.7 & 0 & -0.3 & 0.1 \\ 0 & 0 & -0.3 & 0.2 & -0.1 & -0.1 \\ -47.9 & 0 & -1999.7 & -100 & 81.9 & 268.7 \\ -2.5 & 0 & -54.7 & 0 & -102.9 & 10.4 \\ -0.8 & 0 & -17.1 & 0 & 0.8 & -102.2 \end{bmatrix} \quad b_3 = \begin{bmatrix} 377 \\ -1.7 \\ -0.2 \\ -1942.7 \\ -83.9 \\ -26.2 \end{bmatrix} \quad (3.22)$$

$$A_4 = \begin{bmatrix} 0 & 0.377 & 0 & 0 & 0 & 0 \\ -0.3 & -0.7 & -0.7 & 0 & -0.3 & 0.1 \\ 0 & 0 & -0.3 & 0.2 & -0 & -0.1 \\ 17.3 & 0 & -1719.5 & -100 & 60.3 & 237.5 \\ 0 & 0 & 0 & 0 & -107.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -107.4 \end{bmatrix} \quad b_4 = \begin{bmatrix} 377 \\ -1.7 \\ -0.2 \\ -1604.2 \\ 0 \\ 0 \end{bmatrix} \quad (3.23)$$

With values of matrixes  $A_i$  and vectors  $b_i$ , it is possible to calculate the equilibrium point for each subsystem by  $x_{e_i} = -A_i^{-1}b_i$ . The numerical values of the parameters used are:  $\omega = 379.99 \text{ rad/s}$ ,  $H = 1.5 \text{ s}$ ,  $D = 0.2$ ,  $|U_1| = 1 \text{ p.u.}$ ,  $R_{23} = 0.751 \text{ p.u.}$ ,  $X_{23} = 0.242 \text{ p.u.}$ ,  $X_{T1} = X_{T2} = 0.05 \text{ p.u.}$ ,  $P_m = 0.1 \text{ p.u.}$ . So the four equilibrium points calculated are presented on (3.24):

$$x_{e1} = \begin{bmatrix} 0.6762 \\ 1.0000 \\ 0.9774 \\ 0.9240 \\ 0.6238 \\ 0.1949 \end{bmatrix} \quad x_{e2} = \begin{bmatrix} 0.9618 \\ 1.0000 \\ 0.9379 \\ 0.7148 \\ 0.3034 \\ 0.0948 \end{bmatrix} \quad x_{e3} = \begin{bmatrix} 0.9618 \\ 1.0000 \\ 0.9379 \\ 0.7148 \\ 0.3034 \\ 0.0948 \end{bmatrix} \quad x_{e4} = \begin{bmatrix} 1.2136 \\ 1.0000 \\ 0.9116 \\ 0.5764 \\ 0 \\ 0 \end{bmatrix} \quad (3.24)$$

Let  $T_D = 6 \text{ s}$ ,  $t_0 = 0 \text{ s}$  and  $t_f = 24 \text{ s}$ . In (3.25) is shown the allowable operating range of the system.

$$\begin{aligned} \bar{X} &:= \{[\delta(t) \quad \omega(t) \quad E'_q(t) \quad E_{fd}(t) \quad g_L(t) \quad b_L(t)]' \in \mathbb{R}^6 : \\ &\quad -2 \leq \delta \leq 2; 0.95 \leq \omega \leq 1.05; 0.9 \leq E'_q \leq 1.1; \\ &\quad 0.7 \leq E_{fd} \leq 1.1; 0.2 \leq g_L \leq 0.8; 0.05 \leq b_L \leq 0.3\} \end{aligned} \quad (3.25)$$

In the sequence, there was specified a set of points  $\hat{x}_1, \dots, \hat{x}_m$  that should be in  $\Omega_1$  and a set of points  $\bar{x}_1, \dots, \bar{x}_m$  that should be in  $\Omega_2$ , where the used points are presented in (3.26)

and (3.27) respectively.

$$\hat{x}_1 = [2.0; 1.05; 1.1; 1.1; 0.8; 0.3]'; \quad (3.26)$$

$$\hat{x}_2 = [-2.0; 0.95; 0.9; 0.7; 0.2; 0.05]';$$

$$\bar{x}_1 = [2.2; 1.155; 1.21; 1.21; 0.88; 0.33]'; \quad (3.27)$$

$$\bar{x}_2 = [2.4; 1.26; 1.32; 1.32; 0.96; 0.36]';$$

Theorem 2 was used to study the finite-time stability of the switched system (2.13). The parameter  $\alpha$  was specified as 0.85 and  $\beta$  as 0.86. We specified  $\mu = 0.99\bar{\mu}$ , where  $\bar{\mu}$  is the upper bound of  $\mu$ . The calculated sets  $\Omega_1$  and  $\Omega_2$  are shown in Figure 3.2 where both of them are constituted by border and inside. This plot corresponds to the cut of the actual estimate of the sets  $\Omega_1$  and  $\Omega_2$  in the hyperplane defined by the system states  $\delta$  and  $\omega$ . The set  $\Omega_2$  represents a realistic operating region of a power system where the rotor speed can vary depending on the rotor angle value, in other words,  $\Omega_2$  is the region where the system can operate with safety.

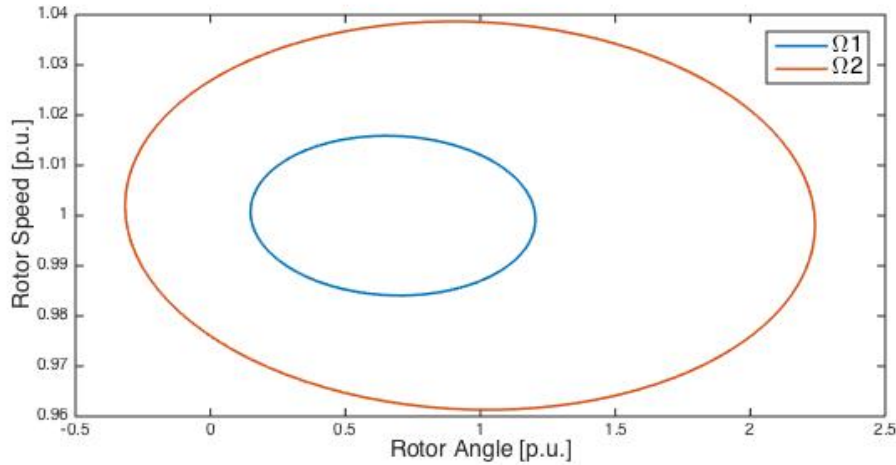


Figure 3.2: The sets  $\Omega_1$  and  $\Omega_2$  representing a realistic operating region of the power distribution system.

In Figure 3.3 we can see the rotor speed behavior during the studied interval. It is

possible to check that in switching times the curve is altered, meaning that the proposed model is responding to the load variations.

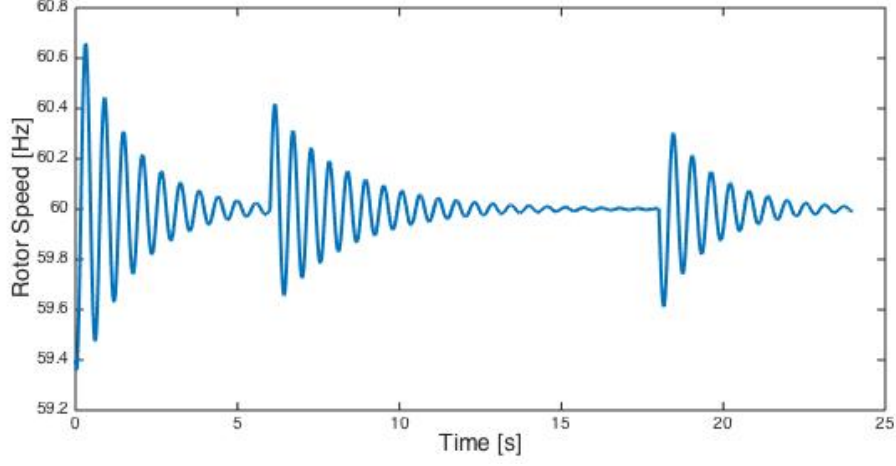


Figure 3.3: Rotor speed of the switched affine system for  $T_D = 6s$ .

The same happens to the generator rotor angle, as can be checked in Figure 3.4.

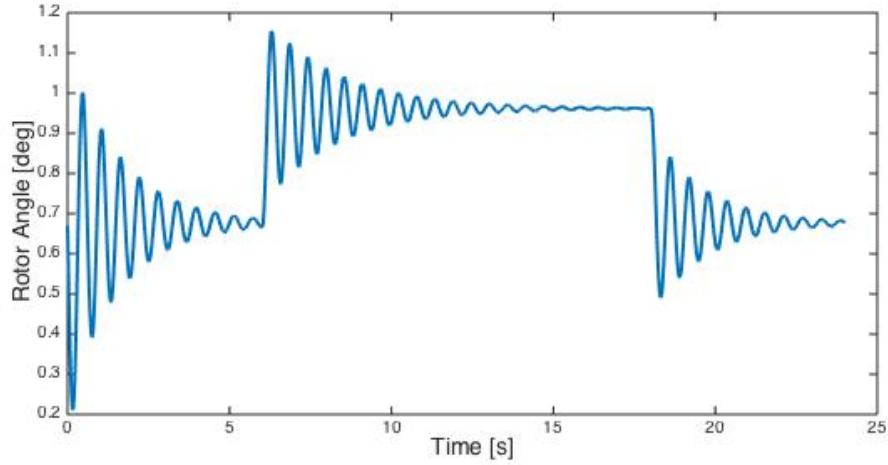


Figure 3.4: Rotor angle of the switched affine system for  $T = 6s$ .

The system trajectory for an initial condition in  $\Omega_1$  and for a switching signal with a dwell-time  $T_D = 6s$  is shown in Figure 3.5. It can be seen that the trajectory remains confined in  $\Omega_2$  for all  $t \in I$ , so we can conclude that this is stable in terms of finite-time stability.

Figure 3.6 shows the value of the Lyapunov function for this same system trajectory.

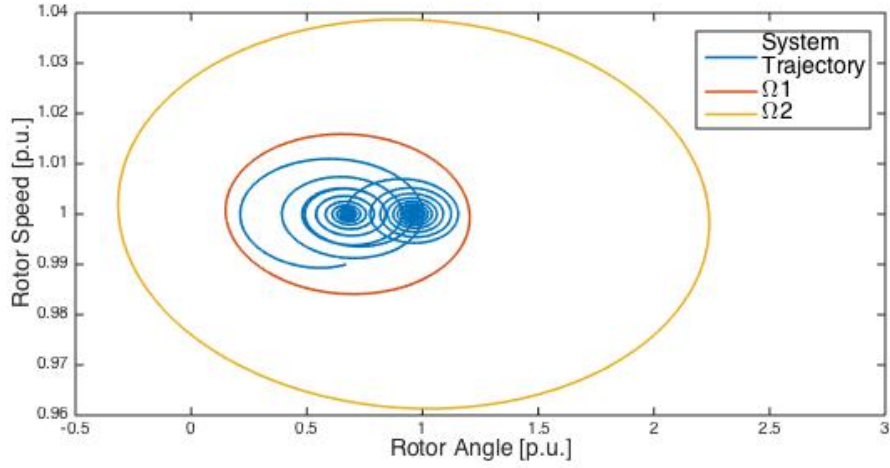


Figura 3.5: System trajectory for an initial condition  $x_0 \in \Omega_1$  and a dwell-time  $T_D = 6s$ .

Analyzing this figure and considering that  $\mu$  is lower than 1, it is possible to check that the Lyapunov function has an expected dynamic, meaning that its initial value is the biggest value, and for the following switchings, the peak values are lower than the previous switch and bigger than the latter switch value.

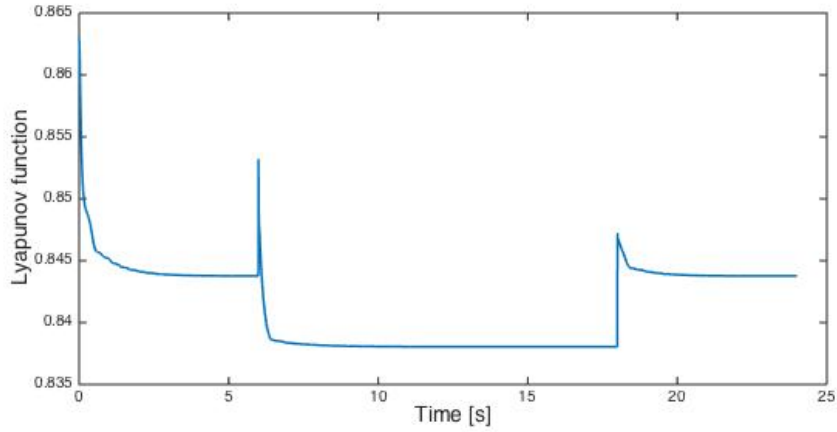


Figura 3.6: Lyapunov function for a system trajectory starting in  $\Omega_1$  and with dwell-time  $T_D = 6s$ .

Figure 3.7 shows a system trajectory starting outside of  $\Omega_1$  and not confined in  $\Omega_2$  for all the time, representing an instability case in terms of finite-time stability. It was caused by an abrupt change to the initial values, meaning that the trajectory is not starting close to the equilibrium value, in this case the  $\delta$  equilibrium changed from 0.6762 to 1.3525.

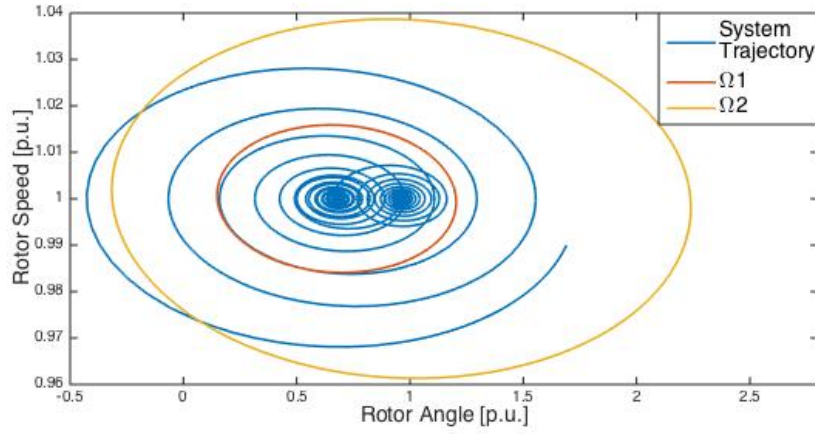


Figure 3.7: System trajectory starting for an unstable situation for  $T_D = 6s$ .

### 3.4 Summary of the Chapter

This chapter has presented the problem of finite-time stability of some classes of continuous-time switched systems. For the finite-time stability analysis, we considered continuous-time switched nonlinear systems without a common equilibrium for all subsystems. The example used to implement the finite-time stability analysis presented in this chapter was the load switching in a power system. This is a different approach of the stability theory and it can be proved that it is a valid method that can complement the stability analysis of the power systems submitted to disturbances of such nature.

In the next section the finite-time stability will be studied for the cases where non-homogeneous switched systems appear.

## CHAPTER 4

# FINITE-TIME STABILITY OF A CLASS OF CONTINUOUS-TIME NON-HOMOGENEOUS SWITCHED SYSTEMS

In this chapter the main results of this dissertation are presented. The chapter is concerned with the finite-time stability problem of a class of linear continuous-time non-homogeneous switched systems under a time-dependent switching signal constrained by a dwell-time  $T_D$ , similar to the results presented in the last chapter. Once the finite-time stability is guaranteed, one of the main results is guaranteeing that any system trajectory starting in a subset  $\Omega_1$  of the state-space will remain in  $\Omega_2 \supset \Omega_1$  over a finite time interval and for any switching sequence with a dwell-time  $T_D$ . The finite-time stability conditions are provided in the form of bilinear matrix inequalities (BMIs), which can be transformed to linear matrix inequalities (LMIs) by means of a step-by-step procedure that includes the computation of the sets  $\Omega_1$  and  $\Omega_2$  by the knowledge of the system's operating range. The illustrative example is used to show the validity of the results.

### 4.1 Problem Statement and Assumptions

Consider  $I = [t_0, t_f)$  a finite time interval, where  $t_0$  is the initial time and  $t_f$  is a finite and positive number such that  $t_f > t_0$ . Hence, since  $t_f$  is finite, our interest is on finite time stability. In this chapter, our interest is on the class of continuous-time non-homogeneous switched systems, described by a set of subsystems  $\Sigma_i$  in the general form

$$\Sigma_i : \quad \dot{x}(t) = A_i x(t) + g_i(t), \quad x(t_0) = x_0, \quad (4.1)$$

where  $A_i \in \mathbb{R}^{n \times n}$  is an invertible and Hurwitz matrix,  $g_i : I \rightarrow \mathbb{R}^n$  is a known function of class  $C^1$ ,  $i \in S = \{1, \dots, N\}$ , where  $N$  is the number of subsystems of the switched system and  $x_0$  is the initial state. It is important to emphasize that the main difference between (4.1) and the switched system class presented in Chapter 3 is the independent term, that before was constant and now is time-varying. This allows us to study, for example, dynamic systems with time-varying input signals.

The active subsystem at each time instant  $t \in I$  is regulated by a switching sequence  $\sigma$  over  $I$  that belongs to the set

$$D_{T,I} := \{ \sigma = ((t_0, i_0), \dots, (t_{k-1}, i_{k-1}), (t_k, i_k), \dots) : i_{k-1}, i_k \in S, t_k - t_{k-1} \geq T, \\ t_{k-1}, t_k \in I, k = 1, 2, \dots \},$$

where  $T_D$  is the dwell-time of  $\sigma$  and the instants  $t_1, t_2, \dots \in I$  are the switching times. The switching sequence adopted here is the same as the one adopted in the last chapter (see (3.1) and (3.2)). The difference here is only the definition of the set  $D_{T,I}$  composed by all the possible switching sequences for a given  $T_D$  and time interval  $I$ .

Thus, the subsystem  $\Sigma_{i_{k-1}}$  is assumed to be active for all  $t \in [t_{k-1}, t_k)$  and at  $t_k$  the system is switched to  $\Sigma_{i_k}$ , where  $i_{k-1}, i_k \in S$ ,  $i_{k-1} \neq i_k$  and  $k = 1, 2, \dots$ . This means that the trajectory  $x(t)$  of the switched system is the trajectory of the subsystem  $\Sigma_{i_{k-1}}$  for all  $t \in [t_{k-1}, t_k)$ . Basically, the index  $i_k \in S$  does not depend on the state, only on the time. So, the family of switching sequences in  $D_{T,I}$  is time-dependent one.

Also, for a switching sequence  $\sigma \in D_{T,I}$ , the number of switching times is finite and equal to  $N_{D_{T,I}}(\sigma)$ .

**Remark 3.** For a certain time interval  $I$  and constant  $T_D$ , the maximum number of switching times that a switching sequence can possess is  $\bar{N}_{D_{T,I}} = \max_{\sigma \in D_{T,I}} N_{D_{T,I}}(\sigma)$ .

For the finite-time stability analysis of the switched system (4.1), notice that for the

time interval  $[t_{k-1}, t_k)$ , the solution of (4.1) can be written as

$$x(t) = x_{zi}(t) + x_{zs}(t) = e^{A_{i_{k-1}}(t-t_{k-1})}x(t_{k-1}) + \int_{t_{k-1}}^t e^{A_{i_{k-1}}(t-\tau)}g(\tau)d\tau, \quad (4.2)$$

where  $t \in [t_{k-1}, t_k)$ ,  $x_{zi}(t)$  and  $x_{zs}(t)$  are, respectively, the zero-input and zero-state responses (CHEN, 1999).

We consider that the zero-state response  $x_{zs}(t)$  for  $t \in [t_{k-1}, t_k)$  can be written as

$$x_{zs}(t) = \int_{t_{k-1}}^t e^{A_{i_{k-1}}(t-\tau)}g(\tau)d\tau = h_{i_{k-1}}(t) - e^{A_{i_{k-1}}(t-t_{k-1})}h_{i_{k-1}}(t_{k-1}), \quad (4.3)$$

where  $h_{i_{k-1}}(t)$  is such that  $\dot{h}_{i_{k-1}}(t) = A_{i_{k-1}}h_{i_{k-1}}(t) + g_{i_{k-1}}(t)$ . This allows us to write the solution of (4.1) for the time interval  $[t_{k-1}, t_k)$  in the form

$$x(t) = e^{A_{i_{k-1}}(t-t_{k-1})} \left( x(t_{k-1}) - h_{i_{k-1}}(t_{k-1}) \right) + h_{i_{k-1}}(t). \quad (4.4)$$

This form of writing the solution of (4.1) has two terms, one describing the transient behavior of  $x(t)$ , and the other, that is,  $h_{i_{k-1}}$ , the steady-state of  $x(t)$ .

In this dissertation, we are particularly interested in those switched systems where the functions  $h_i(t)$ ,  $i = 1, \dots, N$ , are limit cycles or equilibrium points of their respective subsystems. Also, our interest is not in studying the stability of these equilibrium points and limit cycles, but the asymptotic behavior of (4.1). In order to clarify this point, let us consider the following example.

**Example 1.** Consider the switched system with two subsystems given by

$$\Sigma_i : \dot{x}(t) = a_i x(t) + g_i(t), \quad x(t_0) = x_0, \quad (4.5)$$

where  $x(t) \in \mathbb{R}$ ,  $a_i$  is a negative real scalar and  $i \in S = \{1, 2\}$ . Also,  $g_1(t) = A_m \cos(\omega t)$  and  $g_2(t) = 0$  for all  $t \in I$ , being  $A_m$  and  $\omega$ , respectively, the amplitude and frequency of the sinusoidal function. Let us assume the active subsystem at each time instant  $t \in I$  is regulated by a switching sequence  $\sigma \in D_{T,I}$ , where  $T_D$  is a known positive constant. This

switched system can arise, for example, in AC-DC power inverters. Consider that the subsystem  $\Sigma_1$  is active for all  $t \in [t_{k-1}, t_k)$ . So, the solution of (4.5) for this time interval can be written as

$$x(t) = x_{zi}(t) + x_{zs}(t) = e^{a_1(t-t_{k-1})}x(t_{k-1}) + x_{zs}(t),$$

where the zero-state response  $x_{zs}(t)$  can be written as

$$x_{zs}(t) = \int_{t_{k-1}}^t e^{a_1(t-\tau)} A_m \cos(\omega\tau) d\tau = h_1(t) - e^{a_1(t-t_{k-1})}h_1(t_{k-1}),$$

where

$$h_1(t) = \frac{A_m}{\sqrt{a_1^2 + \omega^2}} \cos(\omega t + \phi), \quad (4.6)$$

being  $\phi = \pi + \tan^{-1}\left(\frac{\omega}{a_1}\right)$ . The function  $h_1(t)$  is then a limit cycle of the subsystem  $\Sigma_1$ . On the other hand,  $h_2 = 0$ , which means that  $h_2$  is an equilibrium point of  $\Sigma_2$ .

Notice that, for a positive scalar  $\tau = \frac{\pi}{\omega}$  s, we have that

$$\|h_1(t) - h_2(t)\| \leq \|h_1(t - \tau) - h_2(t - \tau)\|$$

for all  $t \in [t_0 + \tau, t_f)$ . The existence of this positive scalar  $\tau$  is important for the application of the results on finite-time stability for the switched system (4.1) with two subsystems, as they will be presented in Theorem 4.

Given a time interval of interest  $I$ , an initial state  $x_0$  and a switching sequence  $\sigma \in D_{T,I}$ , the focus of this dissertation is to study the asymptotic behavior of the solutions of (4.5), that is, the behavior of  $\tilde{x}(t) = x(t) - h_i(t)$ , being the value of the index  $i \in S$  at each time instant  $t \in I$  regulated by  $\sigma$ .

Assuming, for example, that subsystem  $\Sigma_1$  is active for all  $t \in [t_{k-1}, t_k)$ , then the

asymptotic behavior of the solution of (4.5) for this time interval is

$$\tilde{x}(t) = e^{a_1(t-t_{k-1})} (x(t_{k-1}) - h_1(t_{k-1})).$$

In subsection (4.4.2) will be presented results for this example using the results on finite-time stability proposed further below.

The concept of finite-time stability to the system (4.7) is also based on behavior of switched system solutions in sets  $\Omega_1$  and  $\Omega_2$  as discussed in previous chapter. So, the extension of the concept for (4.7) is given as:

**Definition 11.** *Given a time interval of interest  $I = [t_0, t_f)$ , a positive scalar  $T_D$ , a switching sequence  $\sigma \in D_{T,I}$  and a pair of subsets  $\Omega_1$  and  $\Omega_2$  of  $\mathbb{R}^n$  such that  $\Omega_1 \subset \Omega_2$ , the system (4.1) under the switching sequence  $\sigma$  is said to be finite-time stable with respect to  $(I, \Omega_1, \Omega_2)$  if  $\tilde{x}(t_0) \in \Omega_1$  implies  $\tilde{x}(t) \in \Omega_2$  for all  $t \in I$ , where  $\tilde{x}(t) = x(t) - h_i(t)$ , being the value of the index  $i \in S$  at each time instant  $t \in I$  regulated by  $\sigma$ .*

The main results on finite-time stability for this class of switched systems are presented in the next section.

## 4.2 Main Results

Let us first consider the class of nonlinear continuous-time non-homogeneous switched systems described by a set of subsystems  $\hat{\Sigma}_i$  in the general form

$$\hat{\Sigma}_i : \quad \dot{x}(t) = f_i(x(t)) + g_i(t), \quad (4.7)$$

where  $x \in \mathbb{R}^n$  is the state,  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a nonlinear function of  $x$ ,  $g_i : I \rightarrow \mathbb{R}^n$  is a known time-varying function of class  $C^1$ . The proposed results on finite-time stability from (4.7) are useful for the development of results on finite-time stability of the system

(4.1).

The next theorem provides sufficient conditions of finite time stability for the switched system (4.7).

**Theorem 3.** *Let the sets  $\Omega_1 \subset \mathbb{R}^n$  and  $\Omega_2 \subset \mathbb{R}^n$  ( $\Omega_1 \subset \Omega_2$ ), a time interval  $I = [t_0, t_f)$ , a positive scalar  $T_D$  and a switching sequence  $\sigma \in D_{T,I}$  be given. If there exists a family of radially unbounded, real-valued functions  $\{V_1, \dots, V_N\}$  satisfying local Lipschitz conditions in  $\Omega_2$  and a scalar  $\mu > 1$  such that*

$$\begin{aligned} (i) \quad & \dot{V}_i(x(t), t) \leq 0, \\ (ii) \quad & V_{i_k}(x(t_k), t_k) \leq \mu V_{i_{k-1}}(x(t_k - T_D), t_k - T_D), \\ (iii) \quad & \mu^{\bar{N}_{D_{T,I}}} \max_{i \in S} \sup_{x \in \Omega_1} V_i(x(t_0), t_0) < \min_{i \in S} \inf_{x \in \partial\Omega_2} V_i(x(t), t), \end{aligned}$$

$\forall i_{k-1}, i_k \in S, \forall t \in I, i_{k-1} \neq i_k, \forall x \in \Omega_2$ , then the switched system (4.7) governed by the switching sequence  $\sigma$  is finite-time stable with respect to  $(I, \Omega_1, \Omega_2)$ .

**Proof 3.** *The proof is by contradiction. Let us consider  $x(t)$  as being the corresponding trajectory of (4.7) under a switching sequence  $\sigma \in D_{T,I}$  for an initial time  $t_0$  and state  $x(t_0) = x_0 \in \Omega_1$ . Let us assume that there exists a  $\bar{t} \in I$  such that  $x(\bar{t}) \notin \Omega_2$  for the first time. Let the switching times be denoted beforehand. For a switching sequence  $\sigma \in D_{T,I}$ , we have that  $f_{i_{k-1}}$  is active for all  $t \in [t_{k-1}, t_k)$ ,  $i_{k-1} \in S$ ,  $k = 1, \dots, m$ , where  $t_k = t_{k-1} + T_k$  with  $T_k \geq T$ .*

Let  $v(x(t), t) = V_i(x(t), t)$ ,  $i \in S$ , be a piecewise Lyapunov function for the switched system (4.7), where  $V_i$  is switched among the elements of the set  $\{V_1, \dots, V_N\}$  in accordance with the switching sequence  $\sigma$ , so  $v(\cdot) = V_{i_{k-1}}(\cdot)$ ,  $i_{k-1} \in S$ , for all  $t \in [t_{k-1}, t_k)$ . Since  $V_i$ ,

$i \in S$ , satisfies local Lipschitz conditions, we have

- $\sigma(t) = i_m \in S, \quad \forall t \in [t_m, \bar{t}],$

$$v(x(\bar{t}), t) = V_{i_m}(x(\bar{t}), \bar{t}) = V_{i_m}(x(t_m), t_m) + \int_{t_m}^{\bar{t}} \dot{V}_{i_m}(x(\tau), \tau) d\tau$$

- $\sigma(t) = i_{m-1} \in S, \quad \forall t \in [t_{m-1}, t_m),$

$$v(x(t_m^-), t_m^-) = V_{i_{m-1}}(x(t_m^-), t_m^-) = V_{i_{m-1}}(x(t_{m-1}), t_{m-1}) + \int_{t_{m-1}}^{t_m} \dot{V}_{i_{m-1}}(x(\tau), \tau) d\tau$$

$\vdots$

- $\sigma(t) = i_0 \in S, \quad \forall t \in [t_0, t_1),$

$$v(x(t_1^-), t_1^-) = V_{i_0}(x(t_1^-), t_1^-) = V_{i_0}(x(t_0), t_0) + \int_{t_0}^{t_1} \dot{V}_{i_0}(x(\tau), \tau) d\tau,$$

where  $\dot{V}_i(x(t), t) = \frac{\partial V_i(x(t), t)}{\partial x(t)} \frac{dx(t)}{dt} + \frac{dV_i(x(t), t)}{dt}$  denotes the derivative of  $\dot{V}_i(x(t), t)$  along  $x(t)$  while subsystem  $i \in S$  is active. Using the inequality (i) of Theorem 3 in the above sequence, we have

$$v(x(\bar{t}), \bar{t}) = V_{i_m}(x(\bar{t}), \bar{t}) \leq V_{i_m}(x(t_m), t_m) \quad (4.8)$$

$$V_{i_{m-1}}(x(t_m^-), t_m^-) \leq V_{i_{m-1}}(x(t_{m-1}), t_{m-1}) \quad (4.9)$$

$\vdots$

$$V_{i_0}(x(t_1^-), t_1^-) \leq V_{i_0}(x(t_0), t_0) = v(x(t_0), t_0) \quad (4.10)$$

Now, we show that  $v(x(t_k), t_k) \leq \mu v(x(t_{k-1}), t_{k-1})$  for any two consecutive switching

times. For that, using the inequality (ii) of Theorem 4, we have

$$\begin{aligned}
v(x(t_k), t_k) &= V_{i_k}(x(t_k), t_k) \\
&\leq \mu V_{i_{k-1}}(x(t_k - T_D), t_k - T_D) \\
&\leq \mu \left( V_{i_{k-1}}(x(t_{k-1}), t_{k-1}) + \int_{t_{k-1}}^{t_k - T_D} \dot{V}_{i_{k-1}}(x(\tau), \tau) d\tau \right) \\
&\leq \mu V_{i_{k-1}}(x(t_{k-1}), t_{k-1}) = \mu v(x(t_{k-1}), t_{k-1}),
\end{aligned} \tag{4.11}$$

where the second inequality maintains once that for every  $t_k - T_D \geq 0$  it is true that  $\dot{V}_{i_{k-1}}(x(t), t) \leq 0$ .

Now, using inequalities (4.11) and (4.8)-(4.10) it follows that

$$\begin{aligned}
v(x(\bar{t}), \bar{t}) &\leq V_{i_m}(x(t_m), t_m) \leq \mu V_{i_{m-1}}(x(t_{m-1}), t_{m-1}) \leq \dots \leq \mu^m V_{i_0}(x(t_0), t_0) = \\
\mu^m v(x(t_0), t_0) &\Rightarrow v(x(\bar{t}), \bar{t}) \leq \mu^m v(x(t_0), t_0),
\end{aligned} \tag{4.12}$$

where  $m$  is the number of switching times on the time interval  $[t_0, \bar{t}]$ .

Let us consider the maximum number of switching times  $N_{D_{T,I}}$  that a switching sequence of  $D_{T,I}$  can possess (which satisfies  $N_{D_{T,I}} \geq m$ ). Hence, it follows from the inequality in (4.12) and condition (iii) that

$$\begin{aligned}
v(x(\bar{t}), \bar{t}) &\leq \mu^m v(x(t_0), t_0) \\
&\leq \mu^m \max_{i \in S} \sup_{x \in \Omega_1} V_i(x(t_0), t_0) \\
&\leq \mu^{\bar{N}_{D_{T,I}}} \max_{i \in S} \sup_{x \in \Omega_1} V_i(x(t_0), t_0) \\
&< \min_{i \in S} \inf_{x \in \partial \Omega_2} V_i(x(\bar{t}), \bar{t}),
\end{aligned}$$

for a  $\mu > 1$ . So, we have that  $v(x(\bar{t}), \bar{t}) < \min_i \inf_{x \in \partial \Omega_2} V_i(x(\bar{t}), \bar{t})$ , which implies that  $x(\bar{t}) \notin \Omega_2^c$ . This leads to a contradiction with the original assumption. So, there is not a  $\bar{t} \in [t_0, t_f]$  such that  $x(\bar{t}) \in \Omega_2^c$ , which means that  $x(t) \in \Omega_2, \forall t \in [t_0, t_f]$ .

Obviously, for general switched systems in the form (4.7), finding a family of functions

$V_i(x(t))$ ,  $i \in S$ , satisfying the conditions of Theorem 3 may be difficult, once the constraints (i)-(ii) must be checked along the time interval  $I$ . However, for the special class of non-homogeneous switched systems in the form of (4.1), more applicable results on finite-time stability can be obtained from the general result presented in Theorem 3. These results on finite-time stability for the switched systems in the form of (4.1) are presented in Theorems 4 and 5.

#### 4.2.1 Results on finite-time stability for the switched system (4.1) with two subsystems, $\Sigma_1$ and $\Sigma_2$ .

Let us consider the switched system (4.1) with only two subsystems,  $\Sigma_1$  and  $\Sigma_2$ ,  $S = \{1, 2\}$ , and also that there exists a positive scalar  $\tau$  such that  $\|h_1(t) - h_2(t)\| \leq \|h_1(t - \tau) - h_2(t - \tau)\|$  for all  $t \in [t_0 + \tau, t_f]$ . This type of switched system can appear, for example, in power converters.

Based on the idea of studying the asymptotic behavior of the solutions of (4.1), let us consider the sets  $\Omega_1$  and  $\Omega_2$  in the following form

$$\Omega_1 = \Omega_{1_1} \cap \Omega_{1_2}, \quad \Omega_2 = \Omega_{2_1} \cup \Omega_{2_2}, \quad (4.13)$$

where,

$$\begin{aligned} \Omega_{1_i} &= \left\{ x \in \mathbb{R}^n : (x - z_i)' P_i (x - z_i) + d_i \|z_i - z_j\|^2 < \alpha, \quad z_i \in H_i, \quad z_j \in H_j, \quad j \neq i \in S \right\}, \\ \Omega_{2_i} &= \left\{ x \in \mathbb{R}^n : (x - z_i)' P_i (x - z_i) + d_i \|z_i - z_j\|^2 < 1, \quad z_i \in H_i, \quad z_j \in H_j, \quad j \neq i \in S \right\}, \end{aligned}$$

and  $i = 1, 2$ , being  $\alpha$  a positive scalar satisfying  $\alpha < 1$ ,  $P_i \in \mathbb{R}^{n \times n}$ ,  $d_i \in \mathbb{R}$  a positive scalar and

$$H_i = \{z_i \in \mathbb{R}^n : z_i = h_i(t), \quad t \in I\}.$$

**Theorem 4.** Consider the switched system (4.1) with two subsystems,  $\Sigma_1$  and  $\Sigma_2$ . Let

the time interval  $I = [t_0, t_f)$ , a positive scalar  $T_D$  such that

$$\|h_1(t) - h_2(t)\| \leq \|h_1(t - T_D) - h_2(t - T_D)\|$$

for all  $t \in [t_0 + T_D, t_f)$  and a switching sequence  $\bar{\sigma} \in D_{T,I}$  be given. If there exist positive definite matrices  $P_1$  and  $P_2$ , positive scalars  $d_1$  and  $d_2$ , as well as,  $\varphi, \alpha$  and  $\mu$  such that

$$A'_i P_i + P_i A_i \preceq -\varphi P_i, \quad (4.14)$$

$$\begin{bmatrix} e^{A'_i T_D} P_j e^{A_i T_D} - \mu P_i & e^{A'_i T_D} P_j \\ * & P_j + (d_j - \mu d_i) \mathbb{I}_n \end{bmatrix} \preceq 0, \quad (4.15)$$

$$\mu^{\bar{N}_{D_{T,I}}} < \frac{1}{\alpha}, \quad (4.16)$$

$$d_i \leq \min_{t \in I} \left\{ \varphi / \left( \varphi \|h_i - h_j\|^2 + 2(\dot{h}_i - \dot{h}_j)'(h_i - h_j) \right) \right\} \quad (4.17)$$

$i, j = 1, 2, i \neq j$ , then the switched system (4.1) with two subsystems under the switching sequence  $\sigma$  is finite-time stable with respect to  $(I, \Omega_1, \Omega_2)$ .

**Proof 4.** For the proof, we can show that the pair of conditions (4.14)-(4.17), as well as, conditions (4.15) and (4.16) are equivalent, respectively, to the conditions (i), (ii) and (iii) of Theorem 3.

Let us consider a switching sequence  $\sigma \in D_{T,I}$  and also that the subsystem  $\Sigma_i$  is active  $\forall t \in [t_{k-1}, t_k)$ , where  $t_k = t_{k-1} + T_k$  with  $T_k \geq T$ , and at  $t = t_k$ , the system switches to  $\Sigma_j$ , for some  $i, j = 1, 2, i \neq j$ .

Consider the piecewise Lyapunov function given by  $v(x(t), t) = V_i(x(t), t)$ , where

$$V_i(x(t), t) = (x(t) - h_i(t))' P_i (x(t) - h_i(t)) + d_i \|h_i(t) - h_j(t)\|^2, \quad (4.18)$$

and  $h_i(t)$  is such that  $\dot{h}_i(t) = A_i h_i(t) + g_i(t)$  (according to (4.3)), where  $i, j = 1, 2, i \neq j$ . For convenience, the time-dependency of  $x$  and  $h$  will be omitted in the next developments.

For all  $t \in [t_{k-1}, t_k)$ , the time derivative of the Lyapunov function (4.18) along an

arbitrary trajectory of the system (4.1) is given by

$$\dot{v}(x(t), t) = (x - h_i)' (A_i' P_i + P_i A_i) (x - h_i) + 2d_i(\dot{h}_i - \dot{h}_j)'(h_i - h_j),$$

From (4.14), (4.13) and (4.17), one can see that for all  $t \in [t_{k-1}, t_k)$ , the time derivative of the Lyapunov function (4.18) along an arbitrary trajectory of the system (4.1) holds

$$\begin{aligned} \dot{v}(x, t) &= (x - h_i)' (A_i' P_i + P_i A_i) (x - h_i) + 2d_i(\dot{h}_i - \dot{h}_j)'(h_i - h_j) \\ &\leq -\varphi (x - h_i)' P_i (x - h_i) + 2d_i(\dot{h}_i - \dot{h}_j)'(h_i - h_j) \\ &\leq -\varphi \left(1 - d_i \|h_i - h_j\|^2\right) + 2d_i(\dot{h}_i - \dot{h}_j)'(h_i - h_j) \\ &\leq 0, \end{aligned}$$

for all  $\tilde{x} = x - h_i \in \Omega_2$ , which satisfies the condition (i) of Theorem 3. Now, let us rewrite condition (4.15) as

$$\begin{bmatrix} e^{A_i' T_D} P_j e^{A_i T_D} & e^{A_i' T_D} P_j \\ * & P_j + d_j \mathbb{I}_n \end{bmatrix} \preceq \mu \begin{bmatrix} P_i & 0 \\ * & d_i \mathbb{I}_n \end{bmatrix} \quad (4.19)$$

Multiplying (4.19) on the right and the left by

$$\begin{bmatrix} e^{A_i(T_k - T_D)} & 0 \\ * & \mathbb{I}_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e^{A_i'(T_k - T_D)} & 0 \\ * & \mathbb{I}_n \end{bmatrix},$$

respectively, we have that

$$\begin{bmatrix} e^{A_i' T_k} P_j e^{A_i T_k} & e^{A_i' T_k} P_j \\ * & P_j + d_j \mathbb{I}_n \end{bmatrix} \preceq \mu \begin{bmatrix} e^{A_i'(T_k - T_D)} P_i e^{A_i(T_k - T_D)} & 0 \\ * & d_i \mathbb{I}_n \end{bmatrix}$$

Using this last inequality and also that  $\|h_i(t) - h_j(t)\| \leq \|h_i(t - T_D) - h_j(t - T_D)\|$  for

all  $t \in [t_0 + T_D, t_f)$ , we have

$$\begin{aligned}
& V_j(x(t_k), t_k) \\
&= (x(t_k) - h_j(t_k))' P_j (x(t_k) - h_j(t_k)) + d_j \|h_j(t_k) - h_i(t_k)\|^2 \\
&= \left( e^{A_i T_k} (x(t_{k-1}) - h_i(t_{k-1})) + h_i(t_k) - h_j(t_k) \right)' P_j (*) + d_j \|h_i(t_k) - h_j(t_k)\|^2 \\
&= \begin{bmatrix} x(t_{k-1}) - h_i(t_{k-1}) \\ h_i(t_k) - h_j(t_k) \end{bmatrix}' \begin{bmatrix} e^{A_i' T_k} P_j e^{A_i T_k} & e^{A_i' T_k} P_j \\ * & P_j + d_j \mathbb{I}_n \end{bmatrix} \begin{bmatrix} x(t_{k-1}) - h_i(t_{k-1}) \\ h_i(t_k) - h_j(t_k) \end{bmatrix} \\
&\leq \mu \begin{bmatrix} x(t_{k-1}) - h_i(t_{k-1}) \\ h_i(t_k) - h_j(t_k) \end{bmatrix}' \begin{bmatrix} e^{A_i' (T_k - T_D)} P_i e^{A_i (T_k - T_D)} & 0 \\ * & d_i \mathbb{I}_n \end{bmatrix} \begin{bmatrix} x(t_{k-1}) - h_i(t_{k-1}) \\ h_i(t_k) - h_j(t_k) \end{bmatrix} \\
&\leq \mu \left[ \left( e^{A_i (T_k - T_D)} (x(t_{k-1}) - h_i(t_{k-1})) + h_i(t_k - T_D) - h_i(t_k - T_D) \right)' P_i (*) + \right. \\
&\quad \left. d_i \|h_i(t_k) - h_j(t_k)\|^2 \right] \\
&\leq \mu \left[ ((x(t_k - T_D) - h_i(t_k - T_D))' P_i (*) + d_i \|h_i(t_k) - h_j(t_k)\|^2) \right] \\
&\leq \mu \left[ ((x(t_k - T_D) - h_i(t_k - T_D))' P_i (*) + d_i \|h_i(t_k - T) - h_j(t_k - T)\|^2) \right] \\
&\leq \mu [V_i(x(t_k - T_D), t_k - T_D)],
\end{aligned}$$

which leads to the condition (ii) of Theorem 3. Now, from the sets  $\Omega_1$  and  $\Omega_2$  given by (4.13), we have that

$$\max_{i \in S} \sup_{x \in \Omega_1} V_i(x(t_0), t_0) = \alpha, \quad \min_{i \in S} \inf_{x \in \Omega_2^c} V_i(x(t), t) = 1,$$

where  $S = \{1, 2\}$ . So, it is easy to notice that constraint  $\mu^{\bar{N}_{DT,I}} < \frac{1}{\alpha}$  leads to the condition (iii) of Theorem 3.

**Remark 4.** Since the finite-time stability is guaranteed by Theorem 4, any system trajectory  $\tilde{x}$  starting in a subset  $\Omega_1$  of the state-space will remain in  $\Omega_2 \supset \Omega_1$  over a finite time interval and for any switching sequence  $\sigma \in D_{T,I}$ .

### 4.2.2 Results on finite-time stability for the switched system (4.1) with $N$ subsystems, $\Sigma_1, \dots, \Sigma_2$ .

Let us consider the switched system (4.1) with  $N$  subsystems, so  $S = \{1, \dots, N\}$ . Let us also consider the sets  $\Omega_1$  and  $\Omega_2$  in the following form

$$\Omega_1 = \bigcap_{i=1}^N \Omega_{1_i}, \quad \Omega_2 = \bigcup_{i=1}^N \Omega_{2_i}, \quad (4.20)$$

where

$$\begin{aligned} \Omega_{1_i} &= \{x \in \mathbb{R}^n : (x - z)'P_i(x - z) + d_i < \alpha, \ z \in H_i\}, \\ \Omega_{2_i} &= \{x \in \mathbb{R}^n : (x - z)'P_i(x - z) + d_i < 1, \ z \in H_i\}, \end{aligned}$$

being  $\alpha$  a positive scalar satisfying  $\alpha < 1$ ,  $d_i$  a positive scalar,  $P_i \in \mathbb{R}^{n \times n}$ ,  $H_i = \{z \in \mathbb{R}^n : z = h_i(t), \ t \in I\}$  and  $i \in S$ .

**Theorem 5.** *Let the time interval  $I = [t_0, t_f)$ , a positive scalar  $T_D$  and a switching sequence  $\sigma \in D_{T,I}$  be given. If there exists a set of positive definite matrices  $P_1, \dots, P_N$  and positive scalars  $d_1, \dots, d_N$ ,  $\alpha$  and  $\mu$  such that*

$$A_i'P_i + P_iA_i \preceq 0, \quad (4.21)$$

$$\begin{bmatrix} e^{A_i'T_D}P_j e^{A_iT_D} - \mu P_i & e^{A_i'T_D}P_j \Delta h_{ij}(t_k) \\ * & \Delta h_{ij}(t_k)'P_j \Delta h_{ij}(t_k) + d_j - \mu d_i \end{bmatrix} \preceq 0, \quad (4.22)$$

$$\mu^{\tilde{N}_{D_{T,I}}} < \frac{1}{\alpha}, \quad (4.23)$$

$\forall i, j \in S, \ i \neq j, \ \forall k = 1, 2, \dots$ , such that  $t_k \in I$ , where  $\Delta h_{ij}(t_k) = h_i(t_k) - h_j(t_k)$ , then the switched system (4.1) under the switching sequence  $\sigma$  is finite-time stable with respect to  $(I, \Omega_1, \Omega_2)$ .

**Proof 5.** *For the proof of Theorem 5, we only need to show that conditions (4.21), (4.22) and (4.23) lead, respectively, to the conditions (i), (ii) and (iii) of Theorem 3.*

Let us assume a piecewise Lyapunov function given by

$$v(x(t), t) = V_i(x(t), t) = (x(t) - h_i(t))' P_i (x(t) - h_i(t)) + d_i. \quad (4.24)$$

Now, let us consider a load switching sequence  $\sigma \in D_{T,I}$  in which  $\sigma = i \in S \ \forall t \in [t_{k-1}, t_k)$ , where  $t_k = t_{k-1} + T_k$  with  $T_k \geq T_D$ , and at  $t = t_k$ ,  $\sigma$  jumps to  $j \in S$ ,  $i \neq j$ . From (4.21), one can see that for all  $t \in [t_{k-1}, t_k)$ , the time derivative of the Lyapunov function (4.24) along an arbitrary trajectory of the system (4.1) holds

$$\dot{v}(x(t), t) = (x(t) - h_i(t))' (A_i' P_i + P_i A_i) (x(t) - h_i(t)) \leq 0,$$

for all  $x \in \Omega_2$ , which satisfies the condition (i) of Theorem 3. Now, let us rewrite condition (4.22) as

$$\begin{bmatrix} e^{A_i' T_D} P_j e^{A_i T_D} & e^{A_i' T_D} P_j \Delta h_{ij}(t_k) \\ * & \Delta h_{ij}(t_k)' P_j \Delta h_{ij}(t_k) + d_j \end{bmatrix} \preceq \mu \begin{bmatrix} P_i & 0 \\ * & d_i \end{bmatrix}, \quad (4.25)$$

where  $\Delta h_{ij}(t_k) = h_i(t_k) - h_j(t_k)$ . Multiplying (4.25) on the right and the left by  $\text{diag}(e^{A_i'(T_k - T)}, 1)'$  and  $\text{diag}(e^{A_i'(T_k - T)}, 1)$ , respectively, we have that

$$\begin{bmatrix} e^{A_i' T_k} P_j e^{A_i T_k} & e^{A_i' T_k} P_j \Delta h_{ij}(t_k) \\ * & \Delta h_{ij}(t_k)' P_j \Delta h_{ij}(t_k) + d_j \end{bmatrix} \preceq \mu \begin{bmatrix} e^{A_i'(T_k - T_D)} P_i e^{A_i(T_k - T_D)} & 0 \\ * & d_i \end{bmatrix}.$$

Using this last inequality we have

$$\begin{aligned} & V_j(x(t_k), t_k) \\ &= (x(t_k) - h_j(t_k))' P_j (x(t_k) - h_j(t_k)) + d_j \\ &= \left( e^{A_i T_k} (x(t_{k-1}) - h_i(t_{k-1})) + \Delta h_{ij}(t_k) \right)' P_j (*) + d_j \\ &= \begin{bmatrix} x(t_{k-1}) - h_i(t_{k-1}) \\ 1 \end{bmatrix}' \begin{bmatrix} e^{A_i' T_k} P_j e^{A_i T_k} & e^{A_i' T_k} P_j \Delta h_{ij}(t_k) \\ * & \Delta h_{ij}(t_k)' P_j \Delta h_{ij}(t_k) + d_j \end{bmatrix} (*) \end{aligned}$$

$$\begin{aligned}
&\leq \mu \begin{bmatrix} x(t_{k-1}) - h_i(t_{k-1}) \\ 1 \end{bmatrix}' \begin{bmatrix} e^{A_i'(T_k - T_D)} P_i e^{A_i(T_k - T_D)} & 0 \\ * & d_i \end{bmatrix} \begin{bmatrix} x(t_{k-1}) - h_i(t_{k-1}) \\ 1 \end{bmatrix} \\
&\leq \mu \left( e^{A_i(T_k - T_D)} (x(t_{k-1}) - h_i(t_{k-1})) + h_i(t_k - T_D) - h_i(t_k - T_D) \right)' P_i (*) + d_i \\
&\leq \mu ((x(t_k - T_D) - h_i(t_k - T_D))' P_i (*) + d_i \\
&\leq \mu V_i(x(t_k - T_D), t_k - T_D),
\end{aligned}$$

which leads to the condition (ii) of Theorem 3. Now, from the sets  $\Omega_1$  and  $\Omega_2$  given by (4.13), we have that

$$\max_{i \in S} \sup_{x \in \Omega_1} V_i(x(t_0), t_0) = \alpha, \quad \min_{i \in S} \inf_{x \in \Omega_2} V_i(x(t), t) = 1,$$

so, it is easy to notice that constraint  $\mu^{\bar{N}_{D_{T,I}}} < \frac{1}{\alpha}$  leads to the condition (iii) of Theorem 3.

**Remark 5.** Due to the time-dependency of the inequalities (4.22), the finite-time stability of the switched system (4.1) with  $N$  subsystems is guaranteed only for the switching sequence  $\sigma \in D_{T,I}$ , unless the functions  $h_i(t)$ ,  $i = 1, \dots, N$ , are constants for all  $t \in I$ . In this case, the finite-time stability is guaranteed for any switching sequence in  $D_{T,I}$ .

### 4.3 Step-by-step procedure to using Theorems 4 and 5 to assess the finite-time stability of the switched system (4.1)

The step-by-step procedure proposed below systematizes the use of Theorems 4 and 5 to assess finite-time stability of the switched system (4.1). This procedure includes the computation of the sets  $\Omega_1$  and  $\Omega_2$ .

**Step 1: Initialization** - (i) calculate the functions  $h_i(t)$ ,  $i = 1, \dots, d_N$ , of each subsystem of the switched system; (ii) from a practical knowledge of the study system, specify a set of  $m$  points,  $\hat{x}_1, \dots, \hat{x}_m$ , that should be in  $\Omega_1$  and a set of  $q$  points,  $\bar{x}_1, \dots, \bar{x}_q$ , that should not be in  $\Omega_2$ ; (iii) specify a time interval of interest  $I = [t_0, t_f]$ .

**Step 2: Computing the sets  $\Omega_1$  and  $\Omega_2$**  - (i) specify a value for  $\alpha$  such that  $\alpha < 1$ ; (ii) solve the following optimization problem on the variables  $P_1, \dots, P_N$ , as well as,  $d_1, \dots, d_N, \lambda, \varphi > 0$ :

minimize  $\lambda$

subject to

**If  $N=2$**

$$0 \preceq P_i \preceq \lambda I, \quad i = 1, 2 \quad (4.26)$$

$$\max_{t \in I} \|\hat{x}_k - h_i(t)\|^2 \lambda + \max_{t \in I} \|h_i(t) - h_j(t)\|^2 d_i < \alpha, \quad i, j = 1, 2, \quad i \neq j, \quad k = 1, \dots, m \quad (4.27)$$

$$\min_{t \in I} \|\bar{x}_k - h_i(t)\|^2 \lambda + \min_{t \in I} \|h_i(t) - h_j(t)\|^2 d_i > 1, \quad i, j = 1, 2, \quad i \neq j, \quad k = 1, \dots, q \quad (4.28)$$

$$A'_i P_i + P_i A_i \preceq -\varphi P_i, \quad i = 1, 2 \quad (4.29)$$

$$d_i \leq \min_{t \in I} \left\{ \varphi / \left( \varphi \|h_i - h_j\|^2 + 2(\dot{h}_i - \dot{h}_j)'(h_i - h_j) \right) \right\}, \quad i = 1, 2 \quad (4.30)$$

**Else**

$$0 \preceq P_i \preceq \lambda I, \quad i = 1, 2, \dots, N \quad (4.31)$$

$$\max_{t \in I} \|\hat{x}_k - h_i(t)\|^2 \lambda + d_i < \alpha, \quad i, j = 1, 2, \dots, N, \quad i \neq j, \quad k = 1, \dots, m \quad (4.32)$$

$$\min_{t \in I} \|\bar{x}_k - h_i(t)\|^2 \lambda + d_i > 1, \quad i, j = 1, 2, \dots, N, \quad i \neq j, \quad k = 1, \dots, q \quad (4.33)$$

$$A'_i P_i + P_i A_i \preceq 0, \quad i = 1, 2 \quad (4.34)$$

**Step 3: Checking the finite-time stability for a specific  $T_D$**  - (i) specify a value for  $T_D$ ; (ii) compute the maximum number of switchings  $N_{D,T,I}$  that can occur within the time interval  $I$ ; (iii) solve the following feasibility problem on the variable  $\mu$  (using  $P_1, \dots, P_N, d_1, \dots, d_N, \alpha$  computed in the previous step):

find  $\mu$   
subject to

$$\begin{aligned} & \text{If } N=2 \\ & \begin{bmatrix} e^{A'_i T_D} P_j e^{A_i T_D} - \mu P_i & e^{A'_i T_D} P_j \\ * & P_j + (d_j - \mu d_i) \mathbb{I}_n \end{bmatrix} \preceq 0, \quad i, j = 1, 2, \quad i \neq j \quad (4.35) \\ & \mu^{\bar{N}_{D_{T,I}}} < \frac{1}{\alpha}, \quad (4.36) \end{aligned}$$

$$\begin{aligned} & \text{Else} \\ & \begin{bmatrix} e^{A'_i T_D} P_j e^{A_i T_D} - \mu P_i & e^{A'_i T_D} P_j \Delta h_{ij}(t_k) \\ * & \Delta h_{ij}(t_k)' P_j \Delta h_{ij}(t_k) + d_j - \mu d_i \end{bmatrix} \preceq 0, \quad (4.37) \\ & \mu^{\bar{N}_{D_{T,I}}} < \frac{1}{\alpha}, \quad (4.38) \end{aligned}$$

$$\forall i, j = 1, \dots, N, \quad i \neq j, \quad k = 1, 2, \dots, \text{ such that } t_k \in I.$$

**Remark 6.** The computation of the sets  $\Omega_1$  and  $\Omega_2$  via Step 2 requires a practical knowledge of the study system. In electric power systems, for example, the set  $\Omega_2$  may be viewed as a representation of an operating security region, which means that the allowable range for the system variables is known and this information is essential for Step 2. On the other hand, the set  $\Omega_1$  includes the initial conditions created by a number of perturbations to which the system may be subject during its operation (these perturbations are usually known in advance as the results of the process of contingency screening).

## 4.4 Numerical and Illustrative Examples

### 4.4.1 Power System Transient Illustrative Example

In Chapter 3 we showed a typical arrangement of a distributed system with synchronous generation presented by (KUIAVA et al., 2014). To exemplify a real case of power systems where the finite-time stability of a class of continuous-time in non-homogeneous switched systems theory can be applied, a capacitor was inserted in parallel to the load. This capacitor is in series with a switch as presented in Figure 4.1 and will be switched depending to the necessity of the system. Considering the power system transient classification presented in section 2.4, when it happens, that is within oscillatory transient classification.

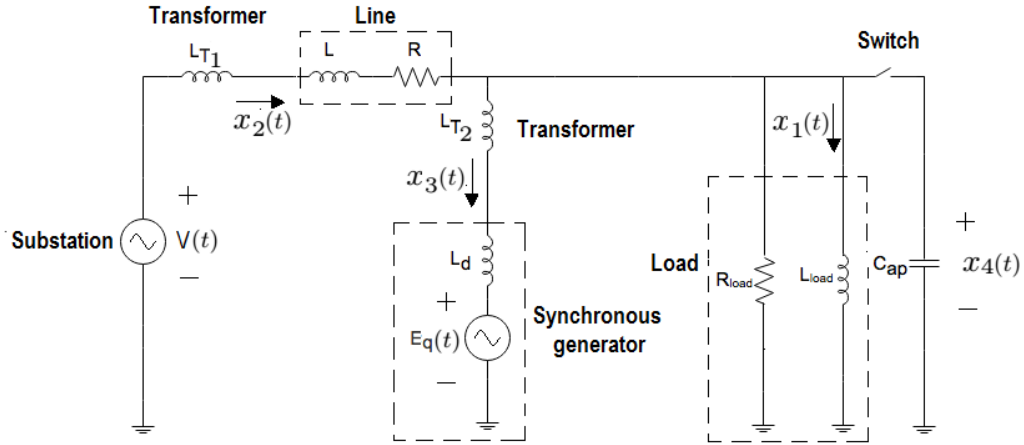


Figura 4.1: One-line diagram of the study system with switched capacitor.

To model system intention the synchronous generator is represented by an ideal AC voltage source ( $E_q(t)$ ) connected to an inductance ( $L_d$ ). The substation is represented by an ideal AC voltage source ( $V(t)$ ). Transformers are represented by inductances ( $L_{T1}$  and  $L_{T2}$ ), while the line and the load by resistances and inductances as shown in Figure 4.1. The focus of this system representation is to study the current and voltage transients of the network stimulated by switching events of the capacitor.

The state-space model of the system shown in Figure 4.1 can be obtained by nodal

analysis and it can be written in the form

$$\Sigma_i : \dot{x}(t) = A_i x(t) + g_i(t), \quad x(t_0) = x_0, \quad (4.39)$$

by considering a linear description of each element of the network, where  $x(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t)]'$ ,  $t \in I = [t_0, t_f)$ ,  $A_i$  is an invertible matrix,  $g_i(t) \in \mathbb{R}^4$  is function of  $t$  and  $i = 1, 2$ . The states  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$  are currents, while  $x_4(t)$  is the capacitor voltage, as it is shown in Figure 4.1. The subsystems  $\Sigma_1$  and  $\Sigma_2$  describe the operating modes at which the switch is open and closed, respectively. The active subsystem at each time instant  $t \in I$  is regulated by a switching sequence  $\sigma$  over  $I$  that belongs to  $D_{T,I}$ , where  $T_D$  is the minimum time elapsed between two consecutive capacitor switchings.

This is just an illustrative example, so there are no numerical results created for this case. A numerical example is presented below.

#### 4.4.2 Numerical Example

Let us consider again the switched system (4.5) of Example 1 presented in Section 4.1, where the parameters  $a_1$  and  $a_2$ , as well as, the function  $g_1(t)$  are given by, respectively,  $-10$ ,  $-5$  and  $5 \cos(100t)$ . The function  $h_1(t)$  is given by (4.6) and  $h_2 = 0$  for all  $t \in I$ , where  $I = (0, 1]$ , which means that  $t_0 = 0$ s and  $t_f = 1$ s. The step-by-step procedure proposed in Subsection 4.3 was adopted to use the Theorem 4 to assess the finite-time stability of this switched system.

For this switched system, let us consider a dwell-time  $T_D = \frac{\pi}{w}$ s, which satisfies the condition  $\|h_1(t) - h_2(t)\| \leq \|h_1(t - T_D) - h_2(t - T_D)\|$  for all  $t \in [t_0 + T_D, t_f)$ . With this value of dwell-time, we have that  $N_{D_{T,I}} = 120$  (in accordance to Remark 1) and the upper bound for parameter  $\mu$  (in (4.16)) is 1.0058, by considering  $\alpha = 0.5$ .

Some points for the state  $x$  satisfying  $|x| < 6$  and  $|x| > 8$  were adopted to, respectively, build the constraints (4.27) and (4.28). Parameter  $\varphi$  was chosen to be equal to 2 (this value was easily chosen from the fact that the eigenvalues of subsystems 1 and 2 are,

respectively,  $-10$  and  $-5$ ).

Figure 4.2 shows the system trajectory for an initial condition in  $\Omega_1$  and for a switching signal with a dwell-time  $T = \frac{\pi}{w}$ s.

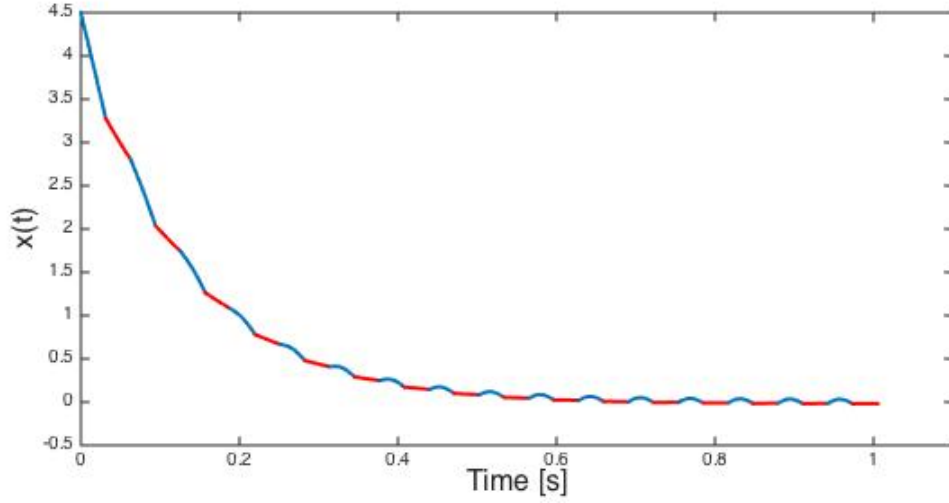


Figura 4.2: System trajectory for an initial condition in  $\Omega_1$  and with a dwell-time  $T = \frac{\pi}{w}$ s, where the blue and red lines are the solution of the switched system when subsystems 1 and 2 are active, respectively.

Figure 4.3 shows the value of the Lyapunov function for this same system trajectory. It can be seen that the trajectory remains confined in  $\Omega_2$  for all  $t \in I$ .

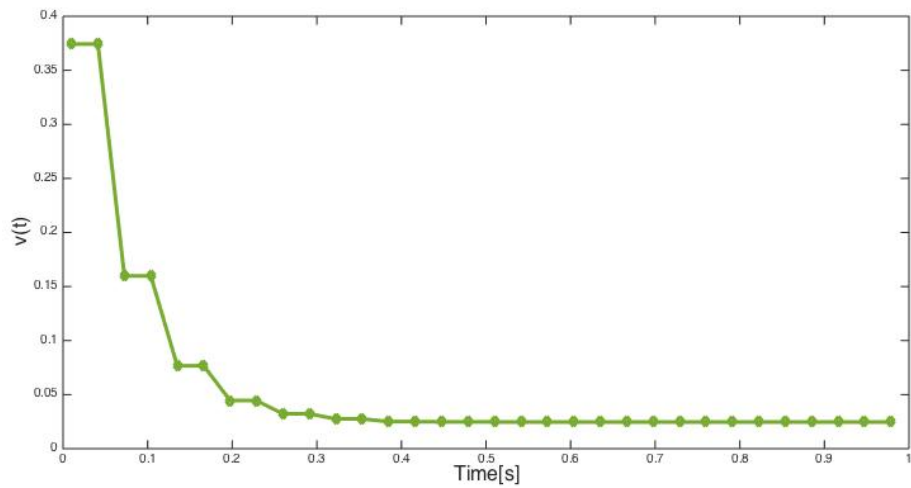


Figura 4.3: Lyapunov function for a system trajectory starting in  $\Omega_1$  and with a dwell-time  $T = \frac{\pi}{w}$ .

## 4.5 Summary of the Chapter

This chapter presented the main contribution of this dissertation. To study the finite-time stability problems of a class of linear continuous-time non-homogeneous switched systems under a time-dependent switching signal some theorems were proposed and a step-by-step procedure on using them. Next, a numerical example showed the application of the theory presented, and an illustrative real example was shown where the switching of one capacitor is able to create a scenario in power systems where finite-time stability can be studied applying the results obtained in this chapter.

## CHAPTER 5

### CONCLUDING REMARKS AND FUTURE STUDY

In this chapter a summary of the entire work is presented, together with some comments about the achieved results.

#### 5.1 Concluding Remarks

The contextualization of this dissertation was presented, how it can contribute to the academic society and how the theory of switched systems can be applied in a real situation. It started with an explanation about hybrid and switched systems, presenting some definitions and how they are classified. Next, the Lyapunov stability theory and mathematical definitions, including the direct method of Lyapunov, were presented considering time-dependent switched systems. This was necessary because the largest part of definitions used subsequently in the document were based on Lyapunov stability.

Another topic was the continuous-time switched systems representation and how the stability analysis can be done in these systems, considering the general idea that there exists a common equilibria for all subsystems. Considering cases where there is not a common equilibrium point, it was necessary to present the finite-time stability theory with definitions and theorems.

In the first contribution of this dissertation, where in addition to what was studied and proposed by (KUIAVA et al., 2013, 2014), the system modelling was done considering the load dynamic and consequently, increasing the state space representation of the studied system.

As the power system stability was being studied, a definition was presented for it and the main categories and subcategories, considering affected variables, disturbance size and

time interval. Next, a system with rotor angle stability affected by load switchings was studied. The nonlinear model was presented and the linearization process too, considering that all four subsystems do not have the same equilibrium point.

The proposed results provide means to evaluate the conditions in which the system trajectories of a switched system will be confined within a certain region in the state-space (region  $\Omega_2$ ). The set  $\Omega_2$  may represent a realistic operating region of a power system. The obtained results can be used, for example, to evaluate the power quality delivered to power suppliers in power distribution networks.

The main contribution of this dissertation was presented, namely some sufficient results on finite-time stability for a class of non-homogeneous switched systems. Since the type of system studied in this chapter resulted from power system transients, a brief theory about that was presented too and its categorization of transients based on waveform shapes.

The proposed results provide means to evaluate the conditions in which the system trajectories of a switched system will be confined within a certain region in the state-space (region  $\Omega_2$ ). It is important to observe that the construction of constraints (4.27) and (4.28), or alternatively, (4.32) and (4.33), requires a practical knowledge of the study system.

The application of the proposed results on finite-time stability in the power system stability area by considering impulsive transients in the network will be the next steps in this research.

The main contributions of this work can be summarized as follows:

- Extended the results presented by Kuiava (2014) considering the load switching in system modeling;
- A new methodology to analyze the finite-time stability of linear continuous-time non-homogeneous switched system;
- Use the proposed theory to study the finite-time stability of power systems, repre-

senting a new applicability for this method.

## 5.2 Further Study

This work has shown the illustrative example of the finite-time stability of a class of continuous-time non-homogeneous switched systems, as further study, this example can be applied considering the three theorems presented in Chapter 4.

The application of the proposed results on finite-time stability in the power converts analysis and design is one direction of our research. Another direction is to extend the proposed results to a class of switched systems with switching governed by a finite state Markov chain, as well as, a switching governed by a state dependent signal.

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